DECOMPOSABILITY AND STRUCTURE OF BANDS OF NONNEGATIVE OPERATORS

By

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To Dr. Heydar Radjavi and my parents

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Abstract

The Invariant Subspace Problem is one of the most intriguing problems in Hilbert Space Theory. Attempts to solve it have led to other interesting related problems in Operator Theory. In the past few years extensive research has been done to find conditions under which a semigroup of operators (*i.e.*, a collection of operators closed under multiplication) can be shown to have a common nontrivial invariant subspace. Such a semigroup is called *reducible*.

The present thesis focuses on semigroups of (functionally) nonnegative operators and in particular, semigroups of nonnegative idempotents called nonnegative *bands* on a finite or infinite-dimensional Hilbert space and obtains necessary and sufficient conditions for the existence of special kind of invariant subspaces for these semigroups which are termed standard subspaces. (An $n \times n$ matrix with nonnegative entries is an example of a nonnegative operator on $\mathbb{C}^n$ and the span of a subset of the standard basis $\{e_1, e_2, ..., e_n\}$ of $\mathbb{C}^n$ is a standard subspace of $\mathbb{C}^n$). A semigroup with a common nontrivial standard invariant subspace is said to be *decomposable*. It is proved that a nonnegative band with each member having rank greater than one and containing at least one finite-rank operator is decomposable. An example of an indecomposable nonnegative band in $\mathcal{B}(l^2)$ with constant infinite rank is given and it is shown that finiteness of such a band makes it decomposable. Further, the structure of constant finite-rank bands is studied. Under a special condition of *fullness*, maximal nonnegative bands of constant rank r are shown to be the direct sum of r maximal rank-one indecomposable nonnegative bands. Finally, a geometric characterization of maximal, rank-one, indecomposable nonnegative bands is obtained, which in view of the result stated above, gives a geometric characterization of maximal, finite-rank, indecomposable, nonnegative bands.

Introduction

One of the most longstanding problems in Operator Theory is the *Invariant Subspace Problem*: Does every bounded linear operator on an infinite-dimensional Hilbert space have a nontrivial closed invariant subspace? This problem has been solved for a few special classes of operators. One of the most significant results, due to Lomonosov, is the existence of nontrivial hyperinvariant subspaces for any nonzero compact operator [9, 13].

Although the solution to the problem of finding a nontrivial invariant subspace for any bounded linear operator on a Hilbert space remains elusive, it has not deterred interested mathematicians from looking for common invariant subspaces for collections of operators satisfying certain properties. One related problem is the *Transitive Algebra Problem*: If $\mathcal{A}$ is a transitive operator algebra on a Hilbert space $\mathcal{H}$, must $\mathcal{A}$ be pointwise dense in $\mathcal{B}(\mathcal{H})$? (A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called transitive if the only closed invariant subspaces for $\mathcal{A}$ are $\{0\}$ and $\mathcal{H}$). An affirmative answer to the transitive algebra problem would imply that every operator which is not a multiple of the identity has a nontrivial hyperinvariant subspace (cf.[13], chap.8).

A multiplicative semigroup of operators in $\mathcal{B}(\mathcal{H})$ is a collection of operators which is closed under multiplication. Note that any algebra in $\mathcal{B}(\mathcal{H})$ is a multiplicative semigroup. By a semigroup of operators in $\mathcal{B}(\mathcal{H})$, we shall always mean a multiplicative semigroup. It is easily seen that the algebra generated by such a semigroup is just its linear span. A semigroup $\mathcal{S}$ is said to be *reducible* if its members have a common nont 'vial invariant subspace. Thus a semigroup is irreducible if and only if the algebra it generates is transitive. It has been the endeavour of several mathematicians in the past few years to find sufficient conditions under which a semigroup can be reduced. The next step would be to see if these conditions are strong enough to give (simultaneous) triangularizability for the semigroup. This means the existence of a chain $\mathcal{C}$ of closed subspaces of $\mathcal{H}$ such that

(a) $\mathcal{C}$ is maximal (as a chain of closed subspaces of $\mathcal{H}$), and

(b) every member of $\mathcal{C}$ is invariant for $\mathcal{S}$.

I.Kaplansky [5, 6] proved that a semigroup of $n \times n$ matrices over a field of characteristic zero having constant trace is simultaneously triangularizable. This was a unification of Kolchin's Theorem [7] that a semigroup of unipotent matrices, *i.e.* matrices of the form $I + N$, with N nilpotent is simultaneously triangularizable and Levitzki's Theorem [8] that a semigroup of nilpotent matrices can be put in a simultaneous triangular form. H.Radjavi [11] proved an extension of these theorems (in finite and infinite dimensions) stating that a necessary and sufficient condition for a semigroup of trace class operators to be triangularizable is that their trace be permutable. We say that trace is *permutable* on a semigroup $\mathcal{S}$ if for every k, every word $A_1A_2\dots A_k$ in $\mathcal{S}$, and every permutation s of $\{1, 2, \dots, k\}$, the equation

$$tr(A_{s(1)}A_{s(2)}\dots A_{s(k)}) = tr(A_1A_2\dots A_k)$$

holds. It is easy to see that this is the case if and only if

$$tr(ABC) = tr(CBA) \text{ for all } A, B, C \text{ in } \mathcal{S}.$$

As a corollary to this, it is obtained that a semigroup of compact idempotents on $\mathcal{H}$

is triangularizable. In fact, this is true of a semigroup of idempotents containing at least one member of finite rank. (Note that a compact idempotent has finite rank).

We use the term *band* to designate a semigroup consisting of idempotents. In this thesis, we shall examine bands of nonnegative operators *i.e.*, (operators which map nonnegative vectors to nonnegative vectors) under conditions which would imply the existence of special kinds of invariant subspaces for these bands called *standard subspaces*. (The precise definition of nonnegative operators and nonnegative vectors is given in the text. But to have an idea, nonnegative operators in finite dimensions are finite square matrices with nonnegative entries). A semigroup which has a nontrivial standard invariant subspace will be called *decomposable*. A *standard subspace* of a finite-dimensional vector space $\mathcal{V}$ with a fixed basis is a subspace spanned by a subset of the basis vectors. Thus decomposability can be understood as permutation-reducibility, *i.e.*, a matrix A is decomposable if there exists a permutation matrix P such that $P^{-1}AP$ has the form $\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where B, D are square matrices.

Semigroups of $n \times n$ matrices with nonnegative entries were studied in [12] and conditions were obtained to give reducibility for them. Also it has been proved [12] that submultiplicativity of the spectral radius on the members of a semigroup of compact operators represented by matrices with nonnegative entries results in the reducibility of the semigroup, although it may not yield decomposability. Under certain conditions, semigroups of nonnegative quasinilpotent operators have been proved to be not only decomposable but simultaneously triangularizable with a maximal subspace chain consisting of standard subspaces [1]. Even in finite dimensions, where we know that a band is triangularizable, the structure of bands is still not at all well understood. Some attempts have been made to study the structure of bands, *e.g.* in [2] and [3]. Thus it is worthwhile to study semigroups for reducibility or decomposability under the extra condition of nonnegativity. In the present thesis, we focus our attention on bands of nonnegative operators. By a nonnegative semigroup (or a nonnegative band), we shall mean a semigroup (or a band) of nonnegative operators.

We start by considering nonnegative semigroups and in particular, nonnegative bands on a finite-dimensional vector space in Chapters 1 and 2, and devote the rest of the thesis to studying nonnegative bands on an infinite-dimensional Hilbert space. In Chapter 1, necessary and sufficient conditions for a semigroup of nonnegative-entried $n \times n$ matrices to be decomposable are given. It is shown among other results that a nonnegative band in which every member has rank greater than one is decomposable.

In Chapter 2, starting with the general form of a nonnegative band of constant rank one, it is proved that a maximal nonnegative band of constant rank r under the special condition of fullness is a direct sum of r maximal rank-one nonnegative indecomposable bands. In addition, the structure of any maximal nonnegative band of constant rank r is exhibited.

Chapter 3 presents the infinite-dimensional analogues of the results obtained in Chapters 1 and 2. In this case, the possibility that operators in a band can have infinite rank gives new perspective to the study of their decomposability. It is proved that a nonnegative band with each member having rank greater than one and containing at least one finite-rank operator is decomposable. The question whether a band of infinite-rank operators on an infinite-dimensional Hilbert space is reducible is still unsolved. Here we present a negative answer to this problem as regards decomposability through an example of a nonnegative band in $\mathcal{B}(l^2)$ with constant infinite rank which is not decomposable. Further, it is shown that under the additional hypothesis of finiteness, an infinite-rank nonnegative band is decomposable.

Lastly, in Chapter 4, a geometric characterization of a maximal, nonnegative, indecomposable rank-one bands is obtained. This result completely determines the structure of maximal, nonnegative, indecomposable, finite-rank bands by what has been proved in the earlier chapters. It is shown that a maximal, nonnegative, indecomposable, rank-one band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is of the form $\mathcal{U} \otimes \mathcal{V}$, $\mathcal{U}$, $\mathcal{V}$ subsets of the nonnegative cone of $\mathcal{L}^2(\mathcal{X})$; $\mathcal{U}$ is a translation by a positive vector of a space containing only mixed vectors (*i.e.*, vectors having both positive and negative parts) and $\mathcal{V}$ is the orthogonal complement of this space consisting of vectors having inner product 1 with the positive vector.

Chapter 1

Decomposability in finite dimensions

1.1 Preliminaries

The contents of this chapter deal with the existence of a special kind of invariant subspace for a single operator or a collection of operators on a vector space $\mathcal{V}$ with dimension $n > 1$ over the complex field $\mathbb{C}$. Thus we shall be considering $\mathcal{V}$ as $\mathbb{C}^n$ with the standard basis $\{e_1, e_2, \ldots, e_n\}$. The one-to-one correspondence between $\mathcal{B}(\mathcal{V})$ and $\mathcal{M}_n(\mathbb{C})$ allows us to identify linear operators on $\mathcal{V}$ with their matrix representations with respect to the fixed basis. We would like to say at the outset that all the results given in this thesis hold true if the field of scalrs $\mathbb{C}$ is replaced with $\mathbf{R}$. We begin with some definitions .

Definition 1.1.1 *An operator $T \in \mathcal{B}(\mathcal{V})$ is called* **nonnegative (resp. positive)** *if $T(x) \geq 0$ (resp. $T(x) > 0$) whenever $x \geq 0$ (resp. $0 \neq x \geq 0$) in $\mathcal{V}$. We write $x = (x_i) \geq 0$ (resp. $x > 0$) if $x_i \geq 0$ (resp. $x_i > 0$) for all i, in which case x is called a nonnegative (resp. positive) vector.*

Let $A = (a_{ij})$ be an $n \times n$ matrix. It is called **nonnegative (resp. positive)** *if $a_{ij} \geq 0$ (resp. $a_{ij} > 0$) for $i, j = 1, 2, \ldots, n$.*

It is easily seen that an operator is nonnegative (resp. positive) if and only if its matrix is nonnegative (resp. positive).

Throughout the chapter, we shall be dealing with nonnegative (or positive) linear transformations and matrices.

Definition 1.1.2 *A subspace of $\mathcal{V}$ is called a* **standard subspace** *if it is the span of a subset of $\{e_1, e_2, \ldots, e_n\}$. It is nontrivial if it is different from $\{0\}$ and $\mathcal{V}$.*

Definition 1.1.3 *A linear transformation T on $\mathcal{V}$ is called* **decomposable** *if there is a nontrivial standard subspace invariant under T, otherwise, it is* **indecomposable.** *Equivalently, for an $n \times n$ matrix A, decomposability means the existence of a proper subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ such that*

$$\bigvee\{Ae_{i_1}, Ae_{i_2}, \ldots, Ae_{i_k}\} \subseteq \bigvee\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}.$$

(For any set of vectors $\{v_1, v_2, \ldots\}$, $\bigvee\{v_1, v_2, \ldots\}$ denotes the (closed) linear span of the vectors $\{v_1, v_2, \ldots\}$).

The composition of a permutation matrix and a diagonal matrix with positive diagonal entries will be called a **generalized permutation matrix.**

In the following two simple propositions, we prove an equivalent condition for decomposability of a nonnegative $n \times n$ matrix which will be used throughout the sequel.

Proposition 1.1.4 *An $n \times n$ matrix $A = (a_{ij})$ is decomposable if and only if there exists a permutation matrix P such that*

$$P^{-1}AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B and D are square matrices.

Proof. If A is decomposable, then

$$\bigvee\{Ae_{i_1}, Ae_{i_2}, \ldots, Ae_{i_k}\} \subseteq \bigvee\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$$

for some proper subset $K = \{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$.

Let $\mathcal{K}$ denote the index set $\{i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l\}$ where $K^c = \{j_1, j_2, \ldots, j_l\}$ and consider the new basis $\mathcal{C} = \{e_i\}_{i \in \mathcal{K}}$. Note that in $\mathcal{C}$, the basis elements have been rearranged only. Thus the transition matrix P is a permutation matrix such that

$$P^{-1}AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$

Conversely, suppose there exists a permutation matrix such that

$$P^{-1}AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$

In other words, there exists an ordering $\{i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l\}$ of $\{1, 2, \ldots, n\}$ such that

$$\begin{aligned} a_{i_\alpha j_\beta} &= 0 \quad (\alpha = 1, 2, \ldots, k\,;\ \beta = 1, 2, \ldots, l) \\ i.e., \ \left\langle Ae_{j_\beta}, e_{i_\alpha} \right\rangle &= 0 \quad \text{for all } \alpha \text{ and } \beta. \end{aligned}$$

This shows that

$$\bigvee\{Ae_{j_1}, Ae_{j_2}, \ldots, Ae_{j_l}\} \subseteq \bigvee\{e_{j_1}, e_{j_2}, \ldots, e_{j_l}\}.$$

Thus A is decomposable. □

Proposition 1.1.2 *An $n \times n$ matrix $A = (a_{ij})$ is decomposable if and only if there exists a generalized permutation matrix P such that*

$$P^{-1}AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B and D are square matrices.

Proof. The proof is the same as for the preceding proposition. □

1.2 Decomposability of nonnegative semigroups

By a semigroup in $\mathcal{B}(\mathcal{V})$ (or $\mathcal{M}_n(\mathbb{C})$), we mean a collection of operators (or matrices) which is closed under multiplication. In this section, we shall be exclusively concerned with semigroups containing nonnegative matrices in $\mathcal{M}_n(\mathbb{C})$. Such semigroups will be called nonnegative semigroups. Whenever we consider semigroups in $\mathcal{M}_n(\mathbb{C})$, it is with the tacit understanding that the matrices are operators with respect to the given fixed basis.

The definition of decomposability of a single matrix can be extended to a semigroup of matrices in the obvious manner. Thus a semigroup $\mathcal{S} \subseteq \mathcal{M}_n(\mathbb{C})$ is decomposable if $\mathcal{L}at\mathcal{S}$, the lattice of subspaces of $\mathbb{C}^n$ which are left invariant by all operators in $\mathcal{S}$, contains a nontrivial standard subspace. Equivalently, $\mathcal{S}$ is decomposable if and only if there exists a permutation matrix P such that

$$P^{-1}SP = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \text{ for all } S \in \mathcal{S},$$

where S_{11} and S_{22} are square and of fixed sizes r and $n-r$ respectively.

A semigroup $\mathcal{S}$ in $\mathcal{B}(\mathcal{V})$ is said to be reducible if it has a common nontrivial invariant subspace. Observe that decomposability implies reducibility but the converse may not be true. A simple example to illustrate this is the cyclic permutation matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which is obviously reducible but is indecomposable as $Ae_1 = e_2, Ae_2 = e_3, Ae_3 = e_1$.

Definition 1.2.1 *A subset $\mathcal{J}$ of a semigroup $\mathcal{S}$ is called an ideal if JS and SJ belong to $\mathcal{J}$ for all $J \in \mathcal{J}$ and for all $S \in \mathcal{S}$.*

It is a well known result that every nonzero ideal of an irreducible semigroup of operators is irreducible [12]. In Proposition 1.2.3, we prove its counterpart for indecomposable semigroups of $n \times n$ matrices with nonnegative entries. In a later chapter, this result will be proved in a more general setting.

It is easy to see that any decomposable semigroup in $\mathcal{M}_n(\mathbb{C})$ has a common zero entry. The following lemma proves the converse of this result for nonnegative semigroups in $\mathcal{M}_n(\mathbb{C})$.

Lemma 1.2.2 *If a nonnegative semigroup $\mathcal{S} \subseteq \mathcal{M}_n(\mathbb{C})$ has a common zero entry, that is, if for some fixed i and j, the (i,j) entry of every member of $\mathcal{S}$ is zero, then $\mathcal{S}$ is decomposable.*

Proof. We can distinguish two cases :

(i) The common zero is not on the diagonal of each matrix in $\mathcal{S}$.

(ii) The common zero lies on the diagonal of each matrix in $\mathcal{S}$.

We first consider the case when the common zero does not lie on the diagonal and then show that the other case reduces to this. By permuting the basis, we can assume with no loss of generality, that this zero lies in the $(2,1)$ slot of the matrix $A = (a_{ij})$ of every A in $\mathcal{S}$, *i.e.*,

$$a_{21} = 0 \quad \text{for all } A \in \mathcal{S}.$$

Now, for every pair A, B in $\mathcal{S}$, the $(2,1)$ entry of AB is zero, *i.e.*,

$$\Sigma_{k=1}^{n} a_{2k} b_{k1} = 0.$$

Since the matrices are nonnegative, this implies that

$$a_{2k} b_{k1} = 0 \;\text{ for all } k = 1, 2, \ldots, n \text{ and for all } A, B \in \mathcal{S}.$$

Define a set

$$\mathcal{U} = \{k \in \{1, \ldots, n\} : \exists\; B^{(k)} = (b_{ij}^{(k)}) \in \mathcal{S} \text{ such that } b_{k1}^{(k)} \neq 0\}.$$

If $\mathcal{U} = \phi$, then $b_{k1} = 0$ for all k and for all $B \in \mathcal{S}$, which implies that $\bigvee\{e_1\} \in \mathcal{L}at\mathcal{S}$ and so $\mathcal{S}$ is decomposable. Therefore, we can assume that $\mathcal{U}$ is nonempty. Also as $2 \notin \mathcal{U}$, $\mathcal{U} \neq \{1, \ldots, n\}$. Consider $\mathcal{M} = \bigvee\{e_k : k \in \mathcal{U}\}$; then $\mathcal{M}$ is a nontrivial standard subspace of $\mathbb{C}^n$. We claim that $\mathcal{M} \in \mathcal{L}at\mathcal{S}$. It suffices to prove that for any $A \in \mathcal{S}$,

$$a_{pk} = 0 \quad \text{for all} \quad k \in \mathcal{U} \text{ and } \text{ for all } p \notin \mathcal{U}.$$

Let $p \notin \mathcal{U}$, then $b_{p1} = 0$ for all $B \in \mathcal{S}$. Since $\mathcal{S}$ is a semigroup, the $(p, 1)$ entry of AB is zero for all A, B in $\mathcal{S}$, *i.e.*, $\Sigma_{k=1}^{n} a_{pk} b_{k1} = 0$. This implies that $a_{pk} b_{k1} = 0$ for all k and for all $A, B \in \mathcal{S}$ (the matrices being nonnegative). If $k \in \mathcal{U}$, then there exists $B^{(k)} \in \mathcal{S}$ such that $b_{k1}^{(k)} \neq 0$. Thus $a_{pk} = 0$ for all $A \in \mathcal{S}$ and for all $k \in \mathcal{U}$, $p \notin \mathcal{U}$ which proves our claim.

Next, if the common zero of $\mathcal{S}$ is a diagonal entry, then by permuting the basis, we can bring it to the $(1, 1)$ slot. Now, if the first row is zero for every A in $\mathcal{S}$, we are done for then $\mathcal{S}$ is decomposable ($\bigvee\{e_2, e_3, \ldots, e_n\}$ being the nontrivial standard invariant subspace). Otherwise, $a_{1i} \neq 0$ for some $i_0 \neq 1$ and for some $A \in \mathcal{S}$. Now for any $B \in \mathcal{S}$,

$$\begin{aligned} 0 = (AB)_{11} &= \Sigma_{i=1}^{n} a_{1i} b_{i1} \\ \Rightarrow \quad & a_{1i} b_{i1} = 0 \quad \text{for all } i \text{ and } \text{ for all } B \in \mathcal{S} \\ \Rightarrow \quad & b_{i_0 1} = 0 \quad \text{for all } B \in \mathcal{S} \end{aligned}$$

i.e., a nondiagonal entry is permanently zero in $\mathcal{S}$ which reduces the problem to the previous case. □

Proposition 1.2.3 *If $\mathcal{S}$ is an indecomposable semigroup of $n \times n$ nonnegative matrices, then so is every nonzero ideal of $\mathcal{S}$.*

Proof. Let $\mathcal{J}$ be a nonzero ideal of $\mathcal{S}$ and suppose that it is decomposable. Then after a permutation of basis, every member J of $\mathcal{J}$ assumes the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A, C are square matrices.

Pick a nonzero J of this form and let

$$S = \begin{pmatrix} X & Y \\ W & Z \end{pmatrix}$$

be an arbitrary element of $\mathcal{S}$. Then

$$SJ = \begin{pmatrix} XA & XB + YC \\ WA & WB + ZC \end{pmatrix}.$$

$\mathcal{J}$ being an ideal, $SJ \in \mathcal{J}$ and therefore we must have $WA = 0$.

Now, if $A \neq 0$, then W and hence $\mathcal{S}$ will have a permanently zero slot (if $a_{ij} \neq 0$ in A, then on multiplying the k th row $(w_{k1}, w_{k2}, \ldots, w_{kn})$ of W by the j th column $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{nj} \end{pmatrix}$ of A, we get $w_{ki}a_{ij} = 0$ which implies that $w_{ki} = 0$, in other words, the i th column of W is zero; we pick a single entry from this zero column and observe that it is zero in all the members of $\mathcal{S}$). By Lemma 1.2.2, $\mathcal{S}$ is decomposable which is a contradiction. Therefore, assume that $A = 0$ for all $J \in \mathcal{J}$. Again

$$\begin{aligned} JS &= \begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix} \begin{pmatrix} X & Y \\ W & Z \end{pmatrix} \\ &= \begin{pmatrix} BW & BZ \\ CW & CZ \end{pmatrix} \in \mathcal{J}. \end{aligned}$$

Thus $BW = 0 = CW$. If either of B or C is nonzero, then by the same reasoning as above, we shall find $\mathcal{S}$ to be decomposable. Therefore, we must have $B = 0 = C$, in other words, $J = 0$, a contradiction. Hence every nonzero ideal of $\mathcal{S}$ must be indecomposable. $\square$

Definition 1.2.4 *By a nonnegative (resp. positive) linear functional f on $\mathbb{C}^n$, we mean a linear transformation from $\mathbb{C}^n$ into $\mathbb{C}$ satisfying $f(x) \geq 0$ (resp. $f(x) > 0$) whenever $x \geq 0$ (resp. $0 \neq x \geq 0$) in $\mathbb{C}^n$.*

We include the proof of the following fundamental result which will be required in our next proposition.

Lemma 1.2.5 *Let f be a nonnegative linear functional on $\mathcal{M}_n(\mathbb{C})$. Then there exists a nonnegative matrix B in $\mathcal{M}_n(\mathbb{C})$ such that $f(A) = tr(BA)$ for all $A \in \mathcal{M}_n(\mathbb{C})$.*

Proof. We know that the collection $\{E_{ij}\}, i,j = 1,2,,\ldots,n$ where the (i,j) entry in E_{ij} is 1 and the remaining entries are zero, forms a basis for $\mathcal{M}_n(\mathbb{C})$. Thus for any $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{C})$, we can write

$$A = \Sigma_{i,j} a_{ij} E_{ij}.$$

Then

$$f(A) = \Sigma_{i,j} a_{ij} f(E_{ij}).$$

Write $f(E_{ij}) = \alpha_{ji}$ and define $B = (\alpha_{ji})$.

Then it can be easily verified that $f(A) = tr(BA)$. Further, if f is nonnegative, then $\alpha_{ji} \geq 0$ for all i,j and thus B is nonnegative. $\square$

Proposition 1.2.6 *Let $\mathcal{S}$ be a semigroup in $\mathcal{M}_n(\mathbb{C})$ with nonnegative matrices and f a nonzero, nonnegative linear functional on $\mathcal{M}_n(\mathbb{C})$ whose restriction to $\mathcal{S}$ is zero. Then $\mathcal{S}$ is decomposable.*

Proof. By Lemma 1.2.5, there exists a nonnegative matrix B such that

$$f(A) = tr(BA) \text{ for all } A \in \mathcal{M}_n(\mathbb{C}).$$

By our assumption, $tr(BA) = 0$ for all $A \in \mathcal{S}$. Also f nonzero implies that B is nonzero. Suppose b_{ij} is a nonzero entry in B. Since the entries in BA are nonnegative and $tr(BA) = 0$ for all $A \in \mathcal{S}$, all the diagonal entries of BA are zero for each $A \in \mathcal{S}$, in particular, the (i, i) entry is zero. Thus

$$b_{i1}a_{1i} + b_{i2}a_{2i} + \cdots + b_{ij}a_{ji} + \cdots + b_{in}a_{ni} = 0.$$

Each summand in the above sum being zero, we have

$$b_{ij}a_{ji} = 0 \Rightarrow a_{ji} = 0 \text{ as } b_{ij} \neq 0.$$

This shows that if the (i, j) entry of B is nonzero, then the (j, i) entry of each A in $\mathcal{S}$ is zero. Hence by Lemma 1.2.2, $\mathcal{S}$ is decomposable. □

Remark 1.2.7 The analogue of the above result for reducible semigroups is as follows [12]:

If $\mathcal{S}$ is a semigroup in $\mathcal{M}_n(\mathbb{C})$ and f a nonzero functional on $\mathcal{M}_n(\mathbb{C})$ such that the restriction of f to $\mathcal{S}$ is zero, then $\mathcal{S}$ is reducible. The proof of this result is an easy consequence of Burnside's Theorem.

Consider the algebra $\mathcal{A}$ generated by $\mathcal{S}$. If $\mathcal{S}$ is irreducible, then so is $\mathcal{A}$ and by Burnside's Theorem, $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$. But $f|_{\mathcal{S}} \equiv 0$ implies $f|_{\mathcal{A}} \equiv 0$ which is a contradiction. Thus $\mathcal{S}$ must be reducible.

Furthermore, in case of reducibility, if f is a nonzero functional on $\mathcal{B}(\mathcal{V})$ which is permutable on any collection $\mathcal{S}$ in $\mathcal{B}(\mathcal{V})$, then $\mathcal{S}$ is reducible. As a corollary to this, we have that if a nonzero functional is multiplicative or constant on a semigroup in $\mathcal{B}(\mathcal{V})$, then the semigroup is reducible, (cf. [12]).

The same hypothesis does not give decomposability in case of nonnegative semigroups. For example, consider the one-element semigroup

$$\mathcal{S} = \left\{ \begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix} \right\}.$$

Then the trace functional is such that it is constant on $\mathcal{S}$ and thus permutable but $\mathcal{S}$ is obviously not decomposable.

We list a few equivalent conditions for decomposability of nonnegative semigroups in $\mathcal{M}_n(\mathbb{C})$.

Theorem 1.2.8 *For a semigroup $\mathcal{S}$ in $\mathcal{M}_n(\mathbb{C})$ with nonnegative matrices, the following are equivalent*

(i) $\mathcal{S}$ is decomposable.

(ii) There exists a nonzero, nonnegative functional on $\mathcal{M}_n(\mathbb{C})$ whose restriction to $\mathcal{S}$ is zero.

(iii) $\mathcal{S}$ has a common zero entry.

(iv) $\mathcal{S}$ has a common nondiagonal entry which is zero.

(v) There exist A, B in $\mathcal{M}_n(\mathbb{C})$, both nonzero and nonnegative such that $A\mathcal{S}B = \{0\}$.

Proof. $(i) \Rightarrow (ii)$ If $\mathcal{S}$ is decomposable, then after a permutation of basis, every member S of $\mathcal{S}$ is of the form $\begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$, where S_{11}, S_{22} are square matrices. Define a linear functional f on $\mathcal{M}_n(\mathbb{C})$ by $f(A) = a_{ij}$ where a_{ij} is the fixed (i,j) entry in the matrix representation of A with respect to the permuted basis from the block A_{21}. Clearly f is a nonzero, nonnegative functional on $\mathcal{M}_n(\mathbb{C})$ such that $f|_{\mathcal{S}} \equiv 0$.

$(ii) \Rightarrow (iii)$ This has been proved in Proposition 1.2.6.

$(iii) \Rightarrow (iv)$ Suppose the common zero entry in $\mathcal{S}$ lies on the diagonal of each member of $\mathcal{S}$. Then, as proved in the last part of the Lemma 1.2.2, we shall obtain that a nondiagonal entry is commonly zero in $\mathcal{S}$.

$(iv) \Rightarrow (v)$ Let $s_{jk} = 0$ for all $S \in \mathcal{S}$ for some $j \neq k$. Construct an $n \times n$ matrix A such that $a_{i_0 j} > 0$ for some i_0 and the remaining entries are zero. Similarly, let $B \in \mathcal{M}_n(\mathbb{C})$ be such that $b_{kl_0} > 0$ for some l_0 and the remaning entries are zero. Then A, B are nonzero, nonnegative matrices and it can be easily verified that $A\mathcal{S}B = \{0\}$.

$(v) \Rightarrow (i)$ We have $A\mathcal{S}B = \{0\}$ for some nonzero, nonnegative A, B in $\mathcal{M}_n(\mathbb{C})$. If a_{ij} and b_{kl} are nonzero entries in A and B respectively, then it is easy to see that the (j,k) entry in each $S \in \mathcal{S}$ is zero. This makes use of the fact that A, B and S are nonnegative-entried matrices. By Lemma 1.2.2, $\mathcal{S}$ is decomposable. $\square$

Remark 1.2.9 Clearly, if $\mathcal{S}$ is decomposable, it has a common nondiagonal zero entry but decomposability may not give a common diagonal zero entry.

For example,

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right\}$$

is a singleton semigroup which is decomposable but no permutation of the basis will produce a zero on the diagonal.

1.3 Decomposability of nonnegative bands

Definition 1.3.1 *A band in $\mathcal{B}(\mathcal{V})$ (resp. $\mathcal{M}_n(\mathbb{C})$) is a multiplicative semigroup of idempotents i.e., operators (resp. matrices) E such that $E = E^2$.*

In this section, we confine our attention to bands in $\mathcal{M}_n(\mathbb{C})$ with nonnegative-entried matrices and prove their decomposability under certain conditions. We start with a singleton nonnegative band.

Lemma 1.3.2 *Let E be a nonnegative $n \times n$ idempotent with rank $r > 1$. Then E is decomposable.*

Proof. We first show that if $r > 1$, then the range of E contains a nonzero (column) vector z with nonnegative entries and at least one zero entry. Pick any two nonnegative linearly independent elements x and y in the range of E. Then $Ex = x$ and $Ey = y$. If either x or y has a zero entry, we are done. Otherwise, let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

and let $\frac{y_j}{x_j} = max\{\frac{y_i}{x_i} : i = 1, 2, \ldots n\}$. Then the vector $z = y_j x - x_j y$ is nonzero, has nonnegative entries, and its j th entry is zero. Since $Ez = z$, it is the desired vector. With no loss of generality, we can assume that z is the vector with a minimal number of nonzero entries. After a permutation of the basis, we can assume that the entries (z_i) of z satisfy

$$z_1 \geq \cdots \geq z_k > z_{k+1} = \cdots = z_n = 0.$$

Then the equation $Ez = z$, together with the nonnegativity of entries in E and z, implies that the (i, j) entry of E is zero whenever $i \geq k + 1$ and $j \leq k$. Thus the span of the first k basis vectors is invariant under E, *i.e.*, E is decomposable. □

Remark 1.3.3 The above result can also be obtained using the Perron-Frobenius Theorem (Theorem 5.5.1(i) in [10], p.124) part of which says that an $n \times n$ nonnegative indecomposable matrix has a real positive eigenvalue, say r, which is a simple root of its characteristic equation. Thus if E is indecomposable, then since an idempotent has only 0 and 1 as eigenvalues, the eigenvalue 1 will occur only once in its spectrum and so the trace of E is 1. But for an idempotent, its rank equals its trace and therefore, $\operatorname{rank}(E) = 1$, which is a contradiction. Thus E must be decomposable.

We use the convention $\mathcal{L}at'\mathcal{S}$ to denote the lattice of all standard subspaces which are invariant under every member of $\mathcal{S}$, where $\mathcal{S}$ is a collection of operators on any Hilbert space with finite or infinite dimension. It can be shown by simple induction in the finite-dimensional case (and by Zorn's Lemma in infinite dimensions) that for any semigroup $\mathcal{S}$, $\mathcal{L}at'\mathcal{S}$ has a maximal chain. This chain may be nontrivial or trivial according as $\mathcal{S}$ has a nontrivial standard subspace or not. Each chain in $\mathcal{L}at'\mathcal{S}$ gives rise to a block triangularization for $\mathcal{S}$ and since the members in the chain are standard subspaces, we shall call it a **standard block triangularization**. Evidently, to say that $\mathcal{S}$ has a standard block triangularization is equivalent to saying that there exists a permutation matrix P such that for each S in $\mathcal{S}$, $P^{-1}SP$ has the upper block triangular form.

Suppose $\mathcal{C}$ is a chain in $\mathcal{L}at'\mathcal{S}$ and $\mathcal{M}, \mathcal{N}$ are two successive elements in $\mathcal{C}$ such that $\mathcal{M} \subset \mathcal{N}$, then $\mathcal{N} \ominus \mathcal{M}$ is called a gap in the chain. If P is the orthogonal projection onto $\mathcal{N} \ominus \mathcal{M}$, then the restriction of $P\mathcal{S}P$ to the range of P is called the compression of $\mathcal{S}$ to $\mathcal{N} \ominus \mathcal{M}$. Note that every compression corresponds to a diagonal block in the block triangularization of $\mathcal{S}$.

Theorem 1.3.4 *Let E be an $n \times n$ idempotent of rank $r > 1$ with nonnegative entries. Then*

1. *any maximal standard block triangularization of E has the two properties*
   - (a) *each diagonal block is either zero or a positive idempotent of rank one.*
   - (b) *there are exactly r nonzero diagonal blocks.*

2. *there exists a standard block triangularization of E with properties (a) and (b) such that no two consecutive diagonal blocks are zero (so that the total number of diagonal blocks is $\leq 2r+1$).*

Proof. By Lemma 1.3.2, E is decomposable. Let $\mathcal{C}$ be a maximal chain in $\mathcal{L}at'E$ resulting in a maximal standard block triangularization of E. If $\mathcal{M}$ and $\mathcal{N}$ are in $\mathcal{C}$ such that $\mathcal{N} \ominus \mathcal{M}$ is a gap, then if the compression of E to $\mathcal{N} \ominus \mathcal{M}$ is nonzero, it is an indecomposable idempotent for otherwise, if it has an invariant subspace $\mathcal{K}$ of the desired kind, then $\mathcal{M} \oplus \mathcal{K}$ is a standard subspace, invariant under E which lies strictly between $\mathcal{M}$ and $\mathcal{N}$ and is comparable with every member of $\mathcal{C}$, thus contradicting the maximality of $\mathcal{C}$. Therefore, every nonzero compression (or diagonal block) is indecomposable and of rank one by Lemma 1.3.2. Since the rank of an idempotent equals its trace, it is apparent that the number of nonzero diagonal blocks is exactly r. (Observe that in any block triangularization of an idempotent, the diagonal blocks or the compressions are idempotents).

It is easy to see that an indecomposable rank-one matrix cannot have any zeros in it. A zero entry would lead to a zero row (or a zero column) which after a permutation of basis can be brought to the position of the last row (or first column), thus rendering the matrix decomposable. Therefore, a nonzero diagonal block is a positive idempotent of rank one.

Lastly, the fact that a 2×2 block matrix whose $(1,1)$, $(2,1)$ and $(2,2)$ blocks are all zero is an idempotent if and only if it is zero proves part 2 of the theorem. □

We now study the decomposability of a nonnegative band with more than a single member.

Theorem 1.3.5 *Suppose $\mathcal{S}$ is a band in $\mathcal{M}_n(\mathbb{C})$ with nonnegative matrices such that rank $(S) > 1$ for all $S \in \mathcal{S}$. Then $\mathcal{S}$ is decomposable.*

Proof. Let $m = \min\{\operatorname{rank}(S); S \in \mathcal{S}\}$. Select a P in $\mathcal{S}$ of rank m. For an arbitrary $S \in \mathcal{S}$, consider PSP. This is an idempotent whose range is contained in the range of P and whose null space contains the null space of P. Since $\operatorname{rank}(PSP) = \operatorname{rank}(P) = m$, we obtain $PSP = P$. Thus $P\mathcal{S}P = \{P\}$.

Further, since $\operatorname{rank}(P) = m > 1$, by Theorem 1.3.4, we can see that P has the form $\begin{pmatrix} P_1 & A \\ 0 & P_2 \end{pmatrix}$ with respect to some permutation of basis where both P_1 and P_2 are nonzero. Let $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ be the representation of an arbitrary S in $\mathcal{S}$ with respect to this permuted basis. Then $PSP = P$ implies that $P_2 S_{21} P_1 = 0$. As in the proof of Theorem 1.2.8 ($(v) \Rightarrow (i)$), we can show the existence of a zero entry in S_{21}. Since S is arbitrary and P fixed, this zero will occur commonly in each S_{21} and hence in $\mathcal{S}$. By Lemma 1.2.2, $\mathcal{S}$ is decomposable which proves the theorem. □

Remark 1.3.6 In the proof of the theorem above, if we consider $\mathcal{J}$ to be the collection of all rank m elements in $\mathcal{S}$, then for any $J \in \mathcal{J}$ and $S \in \mathcal{S}$,

$$\operatorname{rank}(JS) \leq \min\{\operatorname{rank}(J), \operatorname{rank}(S)\} = \operatorname{rank}(J) = m.$$

By minimality of m, we get $\operatorname{rank}(JS) = m$ ($JS \neq 0$ as $\operatorname{rank}(S) > 1$ for all S in $\mathcal{S}$); therefore $JS \in \mathcal{J}$. Similarly, it can be shown that $SJ \in \mathcal{J}$. Thus $\mathcal{J}$ is a nonzero ideal of $\mathcal{S}$. By Proposition 1.2.3, $\mathcal{S}$ is decomposable if and only if $\mathcal{J}$is decomposable. Thus, with no loss of generality, we can assume that $\mathcal{S}$ is a nonnegative band of constant rank m.

Theorem 1.3.7 *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{M}_n(\mathbb{C})$ such that rank $(S) > 1$ for all S in $\mathcal{S}$. Then any maximal standard block triangularization of $\mathcal{S}$ has the property that each nonzero diagonal block is a nonnegative band with at least one element of rank one in it.*

Proof. By Theorem 1.3.5, $\mathcal{S}$ is decomposable. Let $\mathcal{C}$ be a maximal chain in $\mathcal{L}at'\mathcal{S}$ resulting in a standard block triangularization of $\mathcal{S}$. If $\mathcal{M}$ and $\mathcal{N}$ are in $\mathcal{C}$, $\mathcal{M} \subset \mathcal{N}$ such that $\mathcal{N} \ominus \mathcal{M}$ is a gap and if the compression of $\mathcal{S}$ to $\mathcal{N} \ominus \mathcal{M}$ is nonzero, it clearly forms a nonnegative band. Further, it must be indecomposable, for otherwise, if it has a standard invariant subspace $\mathcal{K}$, then $\mathcal{M} \oplus \mathcal{K}$ is in $\mathcal{L}at'\mathcal{S}$, lies strictly between $\mathcal{M}$ and $\mathcal{N}$ and is comparable with every member of $\mathcal{C}$, thus contradicting the maximality of $\mathcal{C}$. Thus, every nonzero compression (or diagonal block) constitutes an indecomposable band and hence by Theorem 1.3.5 it must contain at least one element of rank one. $\square$

Chapter 2

Structure of constant-rank nonnegative bands in finite dimensions

In the previous chapter, we saw in (Remark 1.3.6) that the question of decomposability for a nonnegative band reduces to the case of a constant-rank ideal in it. This fact shows the significance of constant-rank nonnegative bands and motivates us to study their structure. We are still dealing with nonnegative bands in $\mathcal{M}_n(\mathbb{C})$.

Lemma 2.1.1 *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{M}_n(\mathbb{C})$ of constant rank one. Then there exists a permutation matrix P such that for each $S \in \mathcal{S}, P^{-1}SP$ has the block-triangular form*

$$\begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix},$$

where the diagonal block $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ constitutes a rank-one indecomposable band and X and Y are nonnegative matrices of suitable size.

Proof. As usual, we view the members of $\mathcal{S}$ as matrices of operators relative to a fixed basis $\mathcal{B}$. Let $\mathcal{B}_1$ consist of the elements of $\mathcal{B}$ which are in $\ker \mathcal{S}$ and $\mathcal{B}_3$ consist of those elements of $\mathcal{B}$ which are in $\ker \mathcal{S}^*$ but not in $\ker \mathcal{S}$. Let $\mathcal{B}_2$ be the complement of $\mathcal{B}_1 \cup \mathcal{B}_3$ in $\mathcal{B}$. Then the arrangement $\mathcal{B}_1 \dot{\cup} \mathcal{B}_2 \dot{\cup} \mathcal{B}_3$ of the basis $\mathcal{B}$ gives rise to the permutation matrix P such that for each S in $\mathcal{S}$, $P^{-1}SP$ has the matrix form

$$\begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix},$$

where X, Y, Z are matrices of suitable size.

The equations $E^2 = E$, $X = XE$, $Y = EY$ and $Z = XEY$ are obtained using the fact that each matrix in $\mathcal{S}$ is an idempotent. Lastly, the diagonal block $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ forms a rank-one band because $\mathcal{S}$ is a rank-one band. It is easily checked that $\mathcal{S}_0$ is indecomposable, for otherwise, a zero entry in $\mathcal{S}_0$ will lead to a common zero row or a common zero column (using the fact that the rank of $\mathcal{S}$ is one), which is not possible as all the zero rows and zero columns have already been taken out. $\square$

Lemma 2.1.2 *If $\mathcal{S}$ is a nonnegative band in $\mathcal{M}_n(\mathbb{C})$ with constant rank r, then $\mathcal{S}$ has a standard block triangular form with exactly r nonzero diagonal blocks, each constituting an indecomposable band of rank one. Furthermore, this can be done so that no two diagonal blocks are consecutively zero. Therefore, if k be the total number of diagonal blocks, then $k \leq 2r + 1$.*

Proof. We shall prove the lemma by induction on r. The case $r = 1$ is dealt with in Lemma 2.1.1. Suppose $r > 1$; then we know by Theorem 1.3.5 that $\mathcal{S}$ is decomposable. Therefore, after a permutation of basis, every $S \in \mathcal{S}$ is of the form

$$\begin{pmatrix} S_1 & X \\ 0 & S_2 \end{pmatrix},$$

where S_1, S_2 are square matices. Consider the two diagonal blocks, $\mathcal{S}_1 = \{S_1 : S \in \mathcal{S}\}$ and $\mathcal{S}_2 = \{S_2 : S \in \mathcal{S}\}$. Clearly, $\mathcal{S}_1$ and $\mathcal{S}_2$ form nonzero, nonnegative bands. We now prove that $\mathcal{S}_1$ and $\mathcal{S}_2$ are constant-rank bands.

Let

$$S = \begin{pmatrix} S_1 & X \\ 0 & S_2 \end{pmatrix} \text{ and } T = \begin{pmatrix} T_1 & Y \\ 0 & T_2 \end{pmatrix}$$

be two elements in $\mathcal{S}$ such that rank $(S_1) = m_1$ and rank $(T_1) = m_2$. Let us assume that $m_1 \leq m_2$. Then since the rank of S and T is r, rank $(S_2) = r - m_1$ and rank $(T_2) = r - m_2$. Consider

$$ST = \begin{pmatrix} S_1 & X \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} T_1 & Y \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} S_1T_1 & S_1Y + XT_2 \\ 0 & S_2T_2 \end{pmatrix}.$$

Now

$$\text{rank } (S_1T_1) \leq \min \{ \text{ rank } (S_1), \text{ rank } (T_1) \} = \min \{m_1, m_2\} = m_1$$

and

$$\text{rank } (S_2T_2) \leq \min \{ \text{ rank } (S_2), \text{ rank } (T_2) \} = \min \{r - m_1, r - m_2\} = r - m_2$$

But then,

$$\text{rank } (ST) = \text{ rank } (S_1T_1) + \text{ rank } (S_2T_2) \leq m_1 + r - m_2 < r,$$

which impllies that $m_1 = m_2$. Therefore $\mathcal{S}_1$ has constant rank and by the same argument so does $\mathcal{S}_2$. Also since $\mathcal{S}_1$ and $\mathcal{S}_2$ are nonzero bands, their ranks are less than r. Thus induction applies and we obtain the desired result.

Lastly, the fact that a 2×2 block matrix all of whose blocks except $(1, 2)$ are zero is an idempotent if and only if it is zero justifies the assertion that no two diagonal blocks are consecutively zero. □

Definition 2.1.3 *A semigroup $\mathcal{S}$ in $\mathcal{M}_n(\mathbb{C})$ of nonnegative matrices will be called a* **full semigroup** *if $\mathcal{S}$ has no common zero row and no common zero column.*

Lemma 2.1.4 *Let $\mathcal{S}$ be a full band of nonnegative matrices in $\mathcal{M}_n(\mathbb{C})$ with constant rank one. Then $\mathcal{S}$ is indecomposable.*

Proof. Suppose $\mathcal{S}$ is decomposable. Then after a permutation of basis, each $S \in \mathcal{S}$ can be assumed to have the form

$$\begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}.$$

Now

$$\begin{aligned} & \text{rank } S = \text{ rank } (S_{11}) + \text{ rank } (S_{22}) = 1 \\ \Rightarrow\quad & \text{either rank } (S_{11}) = 0 \text{ or rank } (S_{22}) = 0 \\ \Rightarrow\quad & \text{either } S_{11} = 0 \text{ or } S_{22} = 0. \end{aligned}$$

With no loss of generality (*i.e.*, by considering $\mathcal{S}^*$ if necessary), we can assume that $S_{11} = 0$. Therefore, this particular S has the form

$$S = \begin{pmatrix} 0 & S_{12} \\ 0 & S_{22} \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

be an arbitrary element in $\mathcal{S}$. We claim that $T_{11} = 0$. Assume not; then $T_{22} = 0$, in which case

$$ST = \begin{pmatrix} 0 & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which is not possible as each member of $\mathcal{S}$ has rank one. Thus, we have $T_{11} = 0$ which implies that any operator S in $\mathcal{S}$ has the representation

$$S = \begin{pmatrix} 0 & S_{12} \\ 0 & S_{22} \end{pmatrix},$$

but this contradicts the fact that $\mathcal{S}$ is full. Hence $\mathcal{S}$ must be indecomposable. $\square$

Theorem 2.1.5 *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{M}_n(\mathbb{C})$ with constant rank r.*

(i) If $\mathcal{S}$ is full, then there exists a permutation matrix P such that for any $S \in \mathcal{S}$, $P^{-1}SP$ has the block diagonal form

$$\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix},$$

where each $\mathcal{S}_i = \{\ S_i : \ S \in \mathcal{S}\}$ is an indecomposable band of rank-one matrices.

(ii) In general, there is a permutation matrix Q such that for each $S \in \mathcal{S}$, $Q^{-1}SQ$ has the upper block triangular form

$$\begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix},$$

where matrices X, Y are of appropriate size and $\mathcal{S}_0 = \{E : \ S \in \mathcal{S}\}$ is as in case (i).

Proof. (i) If the rank r of $\mathcal{S}$ is one, then the result is true by Lemma 2.1.4. We shall prove the theorem by induction on r. Let $r > 1$, then by Lemma 2.1.2, each S in $\mathcal{S}$ can be assumed to have the form

$$\begin{pmatrix} S_1 & X_1 \\ 0 & S_2 \end{pmatrix},$$

where the diagonal blocks $\mathcal{S}_1 = \{S_1 : S \in \mathcal{S}\}$ and $\mathcal{S}_2 = \{S_2 : S \in \mathcal{S}\}$ form nonzero bands of constant rank less than r. Also then, by the fullness of $\mathcal{S}$, $\mathcal{S}_1$ has no common zero column and $\mathcal{S}_2$ has no common zero row.

Let

$$E = \begin{pmatrix} E_1 & X \\ 0 & E_2 \end{pmatrix}$$

be arbitrary but fixed in $\mathcal{S}$.

Let

$$F = \begin{pmatrix} F_1 & Y \\ 0 & F_2 \end{pmatrix} \text{ and } G = \begin{pmatrix} G_1 & Z \\ 0 & G_2 \end{pmatrix}$$

be arbitrary members in $\mathcal{S}$. Then

$$\begin{aligned} GEF &= \begin{pmatrix} G_1 & Z \\ 0 & G_2 \end{pmatrix} \begin{pmatrix} E_1 & X \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} F_1 & Y \\ 0 & F_2 \end{pmatrix} \\ &= \begin{pmatrix} G_1E_1F_1 & G_1E_1Y + G_1XF_2 + ZE_2F_2 \\ 0 & G_2E_2F_2 \end{pmatrix} \end{aligned}$$

The fact that GEF is an idempotent implies that

$$G_1E_1F_1(G_1E_1Y + G_1XF_2 + ZE_2F_2) + (G_1E_1Y + G_1XF_2 + ZE_2F_2)G_2E_2F_2 = G_1E_1Y + G_1XF_2 + ZE_2F_2.$$

Premultiplying the above equation by $G_1E_1F_1$ and postmultiplying by $G_2E_2F_2$, we obtain

$$\begin{aligned} & G_1E_1F_1(G_1E_1Y + G_1XF_2 + ZE_2F_2)G_2E_2F_2 = 0 \\ \Rightarrow\ & G_1E_1F_1G_1E_1YG_2E_2F_2 + G_1E_1F_1G_1XF_2G_2E_2F_2 + G_1E_1F_1ZE_2F_2G_2E_2F_2 = 0. \end{aligned}$$

Since all the matrices are nonnegative, this gives

$$G_1E_1F_1G_1XF_2G_2E_2F_2 = 0. \tag{1}$$

Now $G_1, E_1F_1 \in \mathcal{S}_1$ and $F_2, G_2E_2 \in \mathcal{S}_2$ both of which have constant rank. Therefore,

$$G_1E_1F_1G_1 = G_1 \text{ and } F_2G_2E_2F_2 = F_2.$$

Thus (1) reduces to

$$G_1XF_2 = 0. \qquad (2).$$

Since $G_1 \in \mathcal{S}_1$ and $F_2 \in \mathcal{S}_2$ are arbitrary, (2) reduces to

$$\mathcal{S}_1X\mathcal{S}_2 = 0.$$

But $\mathcal{S}_1$ has no common zero column, therefore $X\mathcal{S}_2 = 0$ and the fact that $\mathcal{S}_2$ has no common zero implies that $X = 0$. Thus

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}.$$

This shows that any general element S in $\mathcal{S}$ is of the form

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix},$$

where $\mathcal{S}_1 = \{S_1 : S \in \mathcal{S}\}$ and $\mathcal{S}_2 = \{S_2 : S \in \mathcal{S}\}$ are nonnegative full bands with constant rank less than r. Hence induction applies and $\mathcal{S}$ is of the desired form.

(ii) In the general case, we consider the following arrangement of the basis $\mathcal{B}$ relative to which the matrices are expressed. Let $\mathcal{B}_1$ be the vectors in $\mathcal{B}$ which are in $\ker \mathcal{S}$ and $\mathcal{B}_3$ be those basis elements which are in $\ker \mathcal{S}^*$ but not in $\ker \mathcal{S}$ and let the remaining vectors in $\mathcal{B}$ be denoted by $\mathcal{B}_2$. Then with respect to this permutation of basis (*viz.*, $\mathcal{B} = \mathcal{B}_1 \dot\cup \mathcal{B}_2 \dot\cup \mathcal{B}_3$), every element S of $\mathcal{S}$ assumes the form

$$S = \begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $S^2 = S$, we have

$$E^2 = E,\ X = XE,\ Y = EY \text{ and } Z = XEY.$$

These equations imply that $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ cannot have a common zero row or a common zero column. Thus $\mathcal{S}_0$ is a full nonnegative band of constant rank r and hence is of the form given in (i) above. □

Remark 2.1.6 1. It is easily verified that the product of any two block matrices of the form exhibited in part (ii) of Theorem 2.1.5 is again of the same form.

2. If in the statement of the theorem above, $\mathcal{S}$ is taken to be a maximal band, then it is readily observed that the bands $\mathcal{S}_i$ must be maximal. In part (ii), $\mathcal{S}_0$ and the collection of all X, Y are maximal too.

3. In Theorem 2.1.9, we show that the converse of part (i) of Theorem 2.1.5 is also true in case the bands $\mathcal{S}_i$ are maximal. To prove this, we we shall need a couple of lemmas, of which Lemma 2.1.8 may be of independent interest.

Lemma 2.1.7 *Let $\mathcal{S}$ be an indecomposable, nonnegative semigroup in $\mathcal{M}_n(\mathbb{C})$ and e_i be any basis vector. Then $\bigvee\{\mathcal{S}e_i\}$ contains a positive vector.*

Proof. Since $\mathcal{S}$ is indecomposable, no entry in the members of $\mathcal{S}$ is permanently zero. Therefore, for each $k = 1, 2, \ldots, n$, there exists $A^{(k)} \in \mathcal{S}$ such that its (k, i) entry is nonzero. It is evident that then $(A^{(1)} + A^{(2)} + \cdots + A^{(n)})e_i$ is the desired positive vector.

□

Lemma 2.1.8 *Let $\mathcal{S}$ be a direct sum of r nonnegative, indecomposable semigroups $\mathcal{S}_1, \ldots, \mathcal{S}_r$, so that each member of $\mathcal{S}$ has block diagonal representation*

$$\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix},$$

where $S_i \in \mathcal{S}_i, i = 1, 2, \ldots, r$, with respect to a fixed decomposition $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_r$ of $\mathcal{V}$ into standard subspaces. Then every $\mathcal{M} \in \mathcal{L}at'\mathcal{S}$ is of the form $\mathcal{M} = \oplus_{i=1}^{r} \epsilon_i \mathcal{M}_i$, where each ϵ_i is either 0 or 1.

Proof. It is obvious that each $\mathcal{M}_i$ belongs to $\mathcal{L}at'\mathcal{S}$. Also, each $\mathcal{S}_i$ being indecomposable, $\mathcal{M}_i$ is a minimal standard subspace in $\mathcal{L}at'\mathcal{S}$ in the sense that $\mathcal{S}$ has no standard invariant subspace properly contained in it. Now let $\mathcal{M} \in \mathcal{L}at'\mathcal{S}$. We define $\epsilon_i = 1$ if $\mathcal{M}_i \cap \mathcal{M}$ contains a basis vector and $\epsilon_i = 0$ otherwise. To prove the desired result, it is enough to show that if $e_j \in \mathcal{M}_i$ is such that $e_j \in \mathcal{M}$, then $\mathcal{M}_i \subseteq \mathcal{M}$. We write e_j with respect to the given decomposition of the space and suppose the resulting vector is $\begin{pmatrix} 0 \\ \vdots \\ x_i \\ \vdots \\ 0 \end{pmatrix}$ where the column vector x_i has 1 at the appropriate place and zero elsewhere. Consider $\mathcal{S}e_j$. Then $\mathcal{S}e_j = \begin{pmatrix} 0 \\ \vdots \\ \mathcal{S}_i x_i \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}$. Since $\mathcal{S}_i$ is a nonnegative, indecomposable semigroup, by Lemma 2.1.7, we obtain a positive vector y_i in $\mathcal{M}_i$ which is a nonnegative linear combination of $\{\mathcal{S}_i x_i\}$. Consider $y = \begin{pmatrix} 0 \\ \vdots \\ y_i \\ \vdots \\ 0 \end{pmatrix}$; y is a positive linear combination of all the basis vectors which span $\mathcal{M}_i$. Also $y \in \mathcal{M}$ and $\mathcal{M}$ being a standard subspace, it is spanned by a subset of basis vectors. Expressing y as a linear combination of the basis vectors that span $\mathcal{M}$, we observe by the linear independence of the basis vectors that there cannot be any basis vector which is in $\mathcal{M}_i$ but not in $\mathcal{M}$. Hence we must have $\mathcal{M}_i \subseteq \mathcal{M}$ which proves the lemma. □

Theorem 2.1.9 *A direct sum of r maximal, indecomposable, nonnegative rank-one bands is a maximal band of constant rank r.*

Proof. For $r = 1$, the result is obvious. Therefore, let $r > 1$. Suppose $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_r$ are r maximal indecomposable, nonnegative rank-one bands and consider their direct sum. Every member S of $\mathcal{S}$ is of the form

$$\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix}$$

where $S_i \in \mathcal{S}_i$, $i = 1, 2, \cdots, r$.

If $\mathcal{S}$ is not maximal, then let $\mathcal{S}' \supseteq \mathcal{S}$ be a band with constant rank r. Now observe that $\mathcal{S}$ is a full band. Therefore, $\mathcal{S}'$ is full too. By part (i) of Theorem 2.1.5, $\mathcal{S}'$ is a direct sum of r rank-one indecomposable, nonnegative bands, say, $\mathcal{S}'_1, \mathcal{S}'_2, \cdots, \mathcal{S}'_r$. Now $\mathcal{L}at'\mathcal{S}' \subseteq \mathcal{L}at'\mathcal{S}$. By the previous lemma, the cardinality of both $\mathcal{L}at'\mathcal{S}$ and $\mathcal{L}at'\mathcal{S}'$ is the same which is 2^r. Therefore, we must have $\mathcal{L}at'\mathcal{S} = \mathcal{L}at'\mathcal{S}'$. Thus, after permuting the basis if necessary, we obtain $\mathcal{S}_i \subseteq \mathcal{S}'_i$. But since the bands $\mathcal{S}_i$ are maximal, we have $\mathcal{S}_i = \mathcal{S}'_i$. Hence $\mathcal{S}$ is maximal. □

Theorem 2.1.9 and Remark 2.1.6 can be summed up to give the following characterization of maximal nonnegative bands of constant rank.

Theorem 2.1.10 *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{M}_n(\mathbb{C})$ of constant rank r.*

(i) If $\mathcal{S}$ is full, then $\mathcal{S}$ is maximal if and only if

$$\mathcal{S} = \left\{ \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix} : S_i \in \mathcal{S}_i,\ i = 1, 2, \cdots, r \right\},$$

where $\mathcal{S}_i$ is a maximal rank-one indecomposable band for each i.

(ii) In general, if $\mathcal{S}$ is maximal, then

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix} : E \in \mathcal{S}_0, X \in \mathcal{X}, Y \in \mathcal{Y} \right\},$$

where $\mathcal{S}_0$ is a direct sum as in part (i) and $\mathcal{X}$ and $\mathcal{Y}$ are the entire sets of nonnegative matrices of suitable size.

In chapter 4, we shall give a geometric characterization of maximal bands of constant finite rank.

Chapter 3

Nonnegative bands on $\mathcal{L}^2$-spaces

3.1 Preliminary definitions and results

Let $\mathcal{X}$ be a separable, locally compact Hausdorff space and μ a Borel measure on $\mathcal{X}$. We write $\mathcal{L}^2(\mathcal{X})$ for the Hilbert space of (equivalence classes of) complex-valued measurable functions on $\mathcal{X}$ which are square-integrable relative to μ. We also assume for simplicity that $\mu(\mathcal{X}) < \infty$. This is not a great restriction and almost all our considerations will be valid for the case of a σ-finite measure with obvious modifications. We denote by $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, the space of all bounded linear operators on $\mathcal{L}^2(\mathcal{X})$.

In this chapter, we shall study the infinite-dimensional analogues of some of the results which led to the decomposability of the nonnegative semigroups and in particular nonnegative bands in $\mathcal{M}_n(\mathbb{C})$. It will also highlight the main difference between the concept of decomposability for nonnegative bands in finite and infinite dimensions. We start with some definitions.

Definition 3.1.1 *A function $f \in \mathcal{L}^2(\mathcal{X})$ is said to be nonnegative (resp. positive), written $f \geq 0$ (resp. $f > 0$) if*

$$\mu\{x \in \mathcal{X} : f(x) < 0\} = 0 \ (\textit{resp. } \mu\{x \in \mathcal{X} : f(x) \leq 0\} = 0).$$

Definition 3.1.2 *Let* $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *be Borel subsets of* $\mathcal{X}$. *An operator* A *from* $\mathcal{L}^2(\mathcal{X}_1)$ *to* $\mathcal{L}^2(\mathcal{X}_2)$ *is called* **nonnegative** *if*

$$Af \geq 0 \text{ whenever } f \geq 0 \text{ in } \mathcal{L}^2(\mathcal{X}_1).$$

Similarly, A *is called* **positive** *if*

$$Af > 0 \text{ whenever } 0 \neq f \geq 0 \text{ in } \mathcal{L}^2(\mathcal{X}_1).$$

Definition 3.1.3 *A subspace of* $\mathcal{L}^2(\mathcal{X})$ *is a norm-closed linear manifold in* $\mathcal{L}^2(\mathcal{X})$. *A* **standard subspace** *of* $\mathcal{L}^2(\mathcal{X})$ *is a subspace of the form*

$$\mathcal{L}^2(U) = \{f \in \mathcal{L}^2(\mathcal{X}) : f = 0 \text{ a.e. on } U^c\}$$

for some Borel subset U *of* $\mathcal{X}$. *This space is nontrivial if* $\mu(U).\mu(U^c) > 0$.

Definition 3.1.4 *An operator* $A \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ *is said to be decomposable if there exists a nontrivial standard subspace of* $\mathcal{L}^2(\mathcal{X})$ *invariant under* A.

Definition 3.1.5 *For any function* f, *we define the support of* f *as*

$$\text{supp } f = \{x \in \mathcal{X} : f(x) \neq 0\}.$$

If f *is a member of* $\mathcal{L}^2(\mathcal{X})$, *then supp* f *is defined up to a null set (i.e., a set of measure zero).*
When no confusion is likely to arise, we simply write supp f *for any* $f \in \mathcal{L}^2(\mathcal{X})$ *to mean supp* f_0, *where* f_0 *is a function representing* f.

We shall be using the following propositions repeatedly throughout the chapter.

Proposition 3.1.6 *For any two nonnegative functions* f, g *in* $\mathcal{L}^2(\mathcal{X})$,

$$\langle f, g \rangle = 0 \text{ if and only if } \mu\{\text{supp } f \cap \text{supp } g\} = 0.$$

Proof. Observe that

$$\begin{aligned} & \langle f, g\rangle = 0 \\ \Leftrightarrow\ & \int_{\mathcal{X}} f(x)\ g(x)\ \mu(dx) = 0 \\ \Leftrightarrow\ & f(x)\ g(x) = 0 \ a.e. \text{ on } \mathcal{X} \ (\text{ because } fg \text{ is nonnegative }) \\ \Leftrightarrow\ & \mu\{x : f(x) \neq 0 \ \text{ and } \ g(x) \neq 0\} = 0 \\ \Leftrightarrow\ & \mu\{supp\ f \cap supp\ g\} = 0. \end{aligned}$$

Hence the proposition. □

Proposition 3.1.7 *For any* $f \in \mathcal{L}^2(\mathcal{X})$,

$$f \geq 0 \Leftrightarrow \langle f, g\rangle \geq 0, \ \textit{for all } g \geq 0 \textit{ in } \mathcal{L}^2(\mathcal{X}).$$

Proof. If $f \geq 0$ and $g \geq 0$, then $\langle f, g\rangle = \int_{\mathcal{X}} f(x)\ g(x)\ \mu(dx) \geq 0$.

Suppose $\langle f, g\rangle \geq 0$ for all $g \geq 0$. To show that $f \geq 0$. Let $E = \{x : f(x) < 0\}$. If $\mu(E) > 0$, then

$$\begin{aligned} \langle f, \chi_E\rangle &= \int_{\mathcal{X}} f(x)\ \chi_E(x)\ \mu(dx) \\ &= \int_E f(x)\ \mu(dx) < 0, \end{aligned}$$

contrary to the hypothesis. Therefore $\mu(E) = 0$. □

Proposition 3.1.8 *For any* A *in* $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, $A \geq 0 \Leftrightarrow A^* \geq 0$.

Proof. We have

$$\begin{aligned} A \geq 0 &\Leftrightarrow Af \geq 0 \text{ whenever } f \geq 0 \text{ in } \mathcal{L}^2(\mathcal{X}) \\ &\Leftrightarrow \langle Af, g\rangle \geq 0 \text{ for all } f \geq 0 \text{ and for all } g \geq 0 \text{ in } \mathcal{L}^2(\mathcal{X}) \\ &\Leftrightarrow \langle f, A^*g\rangle \geq 0 \text{ for all } f, g \geq 0 \\ &\Leftrightarrow \langle A^*g, f\rangle \geq 0 \text{ for all } f, g \geq 0 \\ &\Leftrightarrow A^*g \geq 0 \text{ for all } g \geq 0 \\ &\qquad\qquad (\text{by Proposition 3.1.7}) . \end{aligned}$$

Hence the proposition. □

Proposition 3.1.9 *Let S be a nonnegative operator on $\mathcal{L}^2(\mathcal{X})$ and U, V be any Borel subsets of $\mathcal{X}$. Then $\langle S\chi_U, \chi_V\rangle = 0$ if and only if $\langle Sf, g\rangle = 0$ for all $f \in \mathcal{L}^2(U)$ and for all $g \in \mathcal{L}^2(V)$.*

Proof. If $\langle Sf, g\rangle = 0$ for all $f \in \mathcal{L}^2(U)$ and for all $g \in \mathcal{L}^2(V)$, then in particular, $\langle S\chi_U, \chi_V\rangle = 0$.

Conversely, suppose that $\langle S\chi_U, \chi_V\rangle = 0$. We first prove that

$$\langle Sf, g\rangle = 0 \text{ for all } f \geq 0 \text{ in } \mathcal{L}^2(U) \text{ and for all } g \geq 0 \text{ in } \mathcal{L}^2(V).$$

Now $\langle S\chi_U, \chi_V\rangle = 0$ implies that $\mu\{supp\ S\chi_U \cap V\} = 0$ (by Proposition 3.1.6). Thus for any nonnegative g in $\mathcal{L}^2(V)$, $\mu\{supp\ S\chi_U \cap supp\ g\} = 0$. Thus

$$\begin{aligned} & \langle S\chi_U, g\rangle = 0 \\ \Rightarrow\ & \langle \chi_U, S^*g\rangle = 0 \ \text{ for all } g \geq 0 \text{ in } \mathcal{L}^2(V) \\ \Rightarrow\ & \mu\{U \cap supp\ S^*g\} = 0 \ \text{ if } g \geq 0 (\text{ Proposition 3.1.6 }). \end{aligned}$$

Thus for any $f \geq 0$ in $\mathcal{L}^2(U)$ and $g \geq 0$ in $\mathcal{L}^2(V)$, $\mu\{supp\ f \cap supp\ S^*g\} = 0$. Hence

$$\begin{aligned} & \langle f, S^*g\rangle = 0 \\ \Rightarrow\ & \langle Sf, g\rangle = 0 \text{ for all } f \geq 0 \text{ in } \mathcal{L}^2(U) \\ & \text{and for all } g \geq 0 \text{ in } \mathcal{L}^2(V). \end{aligned} \tag{3.1}$$

Further, any $f \in \mathcal{L}^2(U)$ can be written as $f = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$ where $f_1 = \text{Re } f$, $f_2 = \text{Im } f$, $f_1^+ = \frac{1}{2}(|f_1| + f_1)$, $f_2^+ = \frac{1}{2}(|f_2| + f_2)$ and $f_1^- = \frac{1}{2}(|f_1| - f_1)$, $f_2^- = \frac{1}{2}(|f_2| - f_2)$ denote the positive and negative parts of f_1, f_2 respectively. Since the positive and negative parts are all nonnegative, (3.1) gives that

$$\langle Sf, g\rangle = 0 \text{ for all } f \in \mathcal{L}^2(U) \text{ and for all } g \in \mathcal{L}^2(V).$$

This concludes the proof. □

Proposition 3.1.10 *A nonnegative operator S on $\mathcal{L}^2(\mathcal{X})$ is decomposable if and only if there exists a Borel subset U of $\mathcal{X}$ with $\mu(U).\mu(U^c) > 0$ such that*

$$\langle S\chi_U, \chi_{U^c}\rangle = 0.$$

Proof. If S is decomposable, then by definition there exists a Borel subset U of $\mathcal{X}$ with $\mu(U).\mu(U^c) > 0$ such that $S(\mathcal{L}^2(U)) \subseteq \mathcal{L}^2(U)$. Now $\chi_U \in \mathcal{L}^2(U)$. Therefore $S\chi_U \in \mathcal{L}^2(U)$ which implies that $\mu\{supp\ S\chi_U \cap U^c\} = 0$. From Proposition 3.1.6, we get $\langle S\chi_U, \chi_{U^c}\rangle = 0$ (here note that $S\chi_U \geq 0$ and $\chi_{U^c} \geq 0$).

Conversely, suppose $\left\langle S\chi_U, \chi_U^c\right\rangle = 0$ for some Borel subset U of $\mathcal{X}$ with

$$\mu(U).\mu(U^c) > 0.$$

By Proposition 3.1.9, $\langle Sf, \chi_{U^c}\rangle = 0$ for all $f \geq 0$ in $\mathcal{L}^2(U)$. By Proposition 3.1.6, for any $f \geq 0$ in $\mathcal{L}^2(U)$, $\mu\{supp\ Sf \cap U^c\} = 0$ which implies that $Sf \in \mathcal{L}^2(U)$. Decompose $f = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$ where $f_1 = \operatorname{Re} f$, $f_2 = \operatorname{Im} f$. Then by what we have proved above, we obtain $Sf_1^+, Sf_1^-, Sf_2^+, Sf_2^- \in \mathcal{L}^2(U)$ and thus $Sf \in \mathcal{L}^2(U)$. Hence $S(\mathcal{L}^2(U)) \subseteq \mathcal{L}^2(U)$, which proves our claim. □

3.2 Decomposability of nonnegative semigroups

In this section, we shall study the decomposability of semigroups of nonnegative operators in $B(\mathcal{L}^2(\mathcal{X}))$. A semigroup $\mathcal{S}$ in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is said to be decomposable if there is a nontrivial standard subspace of $\mathcal{L}^2(\mathcal{X})$ invariant under every member of $\mathcal{S}$. We note here that our assumptions on $\mathcal{X}$ make $\mathcal{L}^2(\mathcal{X})$ separable. Also, we would like to remark that all the propositions proved above for a single nonnegative operator hold true for semigroups of nonnegative operators on $\mathcal{L}^2(\mathcal{X})$.

We saw in the finite-dimensional case how the existence of a common zero entry in a semigroup of nonnegative matrices leads to its decomposability. Lemma 3.2.5

below is the analogue of this fact for nonnegative semigroups in $B(\mathcal{L}^2(\mathcal{X}))$.

We need a couple of simple propositions.

Proposition 3.2.1 *Let $B : \mathcal{L}^2(\mathcal{X}) \to \mathcal{L}^2(\mathcal{Y})$ be a nonnegative operator such that $Bf_0 = 0$ for some $f_0 > 0$ in $\mathcal{L}^2(\mathcal{X})$. Then $B = 0$.*

Proof.

$$\begin{aligned} Bf_0 = 0 \ &\Rightarrow\ \langle Bf_0, g\rangle = 0 \text{ for all } g \geq 0 \text{ in } \mathcal{L}^2(\mathcal{Y}) \\ &\Rightarrow\ \langle f_0, B^*g\rangle = 0 \text{ for all } g \geq 0 \text{ in } \mathcal{L}^2(\mathcal{Y}) \\ &\Rightarrow\ \int_{\mathcal{X}} (B^*g)(x) f_0(x)\ \mu(dx) = 0 \\ &\Rightarrow\ (B^*g)(x) f_0(x) = 0 \ a.e. \text{ on } \mathcal{X}. \end{aligned}$$

But $f_0(x) > 0$ *a.e.* Therefore,

$$\begin{aligned} & (B^*g)(x) = 0 \ a.e. \text{ for all } g \geq 0 \\ \Rightarrow\ & B^*g = 0 \text{ for all } g \geq 0 \\ \Rightarrow\ & B^* = 0 \\ \Rightarrow\ & B = 0. \end{aligned}$$

This concludes the proof. □

Corollary 3.2.2 *Let B be a nonnegative operator in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$. If h is a nonzero, nonnegative vector in $\mathcal{L}^2(\mathcal{X})$ which belongs to the kernel of B, then B is decomposable.*

Proof. Consider *supp* h. Clearly, $h > 0$ on its support. By the hypothesis, $Bh = 0$ for $h > 0$ in $\mathcal{L}^2(supp\ h)$. By proposition 3.2.1, $B = 0$ on $\mathcal{L}^2(supp\ h)$. Hence B is decomposable. □

Proposition 3.2.3 *Let $\mathcal{S}$ be a semigroup of nonnegative operators on $\mathcal{L}^2(\mathcal{X})$. Then $\mathcal{S}$ is decomposable if and only if $\mathcal{S}^*$ is decomposable.*

Proof. This is apparent from Proposition 3.1.10 and the fact that for any Borel subset E of $\mathcal{X}$ and for any $S \in \mathcal{S}$,

$$\langle S^* \chi_{E^c}, \chi_E \rangle = \langle \chi_{E^c}, S\chi_E \rangle .$$

□

Lemma 3.2.4 *Let $\mathcal{A}$ be a collection of nonnegative vectors in $\mathcal{L}^2(\mathcal{X})$. Then there exists a minimal Borel subset $\mathcal{G}$ in $\mathcal{X}$ (defined up to a null set) such that all the vectors in $\mathcal{A}$ vanish on $\mathcal{G}^c$.*

Proof. Since $\mathcal{L}^2(\mathcal{X})$ is a separable metric space, so is $\mathcal{A}$. Let $\mathcal{M}$ be a countable dense subset of $\mathcal{A}$. Suppose $\mathcal{M} = \{f_1, f_2, \cdots\}$ where $f_1, f_2, \cdots$ are chosen representatives of the equivalence classes of functions in $\mathcal{M}$. Consider

$$\mathcal{G} = \bigcup_i \mathit{supp}\ f_i.$$

Let $f \in \mathcal{A}$, then $\bar{\mathcal{M}} = \mathcal{A}$ implies that there exists a subsequence $\{f_{n_k}\}$ in $\mathcal{M}$ such that $f_{n_k} \to f$ pointwise *a.e.* (cf. [15], p.68, Theorem 3.12).

Let

$$\mathcal{G}_0 = \bigcup_k \mathit{supp}\ f_{n_k} \subseteq \mathcal{G}.$$

If $x \in \mathcal{G}_0^c$, then $f_{n_k}(x) = 0$ for all k implies $f(x) = 0$. Thus

$$f \in \mathcal{L}^2(\mathcal{G}_0) \subseteq \mathcal{L}^2(\mathcal{G}).$$

This shows that $\mathcal{A} \subseteq \mathcal{L}^2(\mathcal{G})$. Also $\mathcal{G}$ has no subset of positive measure on which all the vectors in $\mathcal{A}$ vanish, for then the vectors f_i will all vanish on that subset which is not possible by the construction of $\mathcal{G}$. Thus $\mathcal{G}$ is the minimal subset of $\mathcal{X}$, up to a null set, on whose complement all the vectors in $\mathcal{A}$ vanish. □

Lemma 3.2.5 *Let $\mathcal{S}$ be a semigroup of nonnegative operators on $\mathcal{L}^2(\mathcal{X})$ with the property that $\langle A\chi_E, \chi_F\rangle = 0$ for all $A \in \mathcal{S}$, where E, F are Borel subsets of $\mathcal{X}$ with $\mu(E).\mu(F) > 0$. Then $\mathcal{S}$ is decomposable.*

Proof. We distinguish two cases

(i) $\mu(E \cap F) = 0$

(ii) $\mu(E \cap F) > 0$

We prove case (i) and show that the second case can be reduced to the first. In case (i), we can assume with no loss of generality that $E \cap F = \phi$. Thus, we can write

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F) \oplus \mathcal{L}^2(G),$$

where E, F, G can be assumed mutually disjoint with $\mu(G) > 0$ (if $\mu(G) = 0$, then $\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(E^c)$ and the hypothesis $\langle A\chi_E, \chi_{E^c}\rangle = 0$ for all $A \in \mathcal{S}$ gives that $\mathcal{L}^2(E)$ is a nontrivial standard invariant subspace for $\mathcal{S}$). Then, with respect to some choice of bases for $\mathcal{L}^2(E)$, $\mathcal{L}^2(F)$ and $\mathcal{L}^2(G)$, every $A \in \mathcal{S}$ has the matrix representation

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where $A_{21} = 0$, by hypothesis and Proposition 3.1.9.

Let $A \in \mathcal{S}$ be arbitrary and $B \in \mathcal{S}$ be fixed, where

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}.$$

Then $BA \in \mathcal{S}$ implies that $(BA)_{21} = 0$, and thus

$$B_{23}A_{31} = 0 \text{ for all } A \in \mathcal{S}. \tag{3.2}$$

Consider the set

$$\mathcal{A} = \{A_{31}(f):\ A \in \mathcal{S},\ f \in \mathcal{L}^2(E),\ f \geq 0\}.$$

If $A_{31}(f) = 0$ for all A and for all $f \geq 0$ in $\mathcal{L}^2(E)$, then $A_{31} = 0$ for all A, and so $\mathcal{L}^2(E)$ is a standard invariant subspace for $\mathcal{S}$. Therefore, we can assume that there exists at least one $A \in \mathcal{S}$ and some $f \in \mathcal{L}^2(E), f \geq 0$ such that $A_{31}(f) \neq 0$.

Consider the closed linear span $\hat{\mathcal{A}}$ of $\mathcal{A}$. It is a proper subspace of $\mathcal{L}^2(G)$ (for otherwise from (3.2), $B_{23} = 0$ for all $B \in \mathcal{S}$ and then $\mathcal{L}^2(E) \oplus \mathcal{L}^2(G)$ is a standard invariant subspace for $\mathcal{S}$). By Lemma 3.2.4, we can find a minimal subset G_0 of G, up to a null set, on whose complement all the vectors in $\hat{\mathcal{A}}$ and hence in $\mathcal{A}$ vanish, or equivalently

$$\left\langle A\chi_{E}, \chi_{G\setminus G_0}\right\rangle = 0 \text{ for all } A \in \mathcal{S}.$$

Thus, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F) \oplus \mathcal{L}^2(G_0) \oplus \mathcal{L}^2(G\setminus G_0),$$

the matrix representation of any $A \in \mathcal{S}$ is given by

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & A_{42} & A_{43} & A_{44} \end{pmatrix}.$$

Consider the new matrix of B with respect to the decomposition above. Using the facts that $BA \in \mathcal{S}$ for all $A \in \mathcal{S}$ and that $(BA)_{21} = 0$, we get

$$\begin{aligned} & B_{23}A_{31} = 0 \text{ for all } A \in \mathcal{S} \text{ where } A_{31} : \mathcal{L}^2(E) \to \mathcal{L}^2(G_0) \\ \Rightarrow\ & B_{23}A_{31}(\mathcal{L}^2(E)) = 0 \\ \Rightarrow\ & B_{23}(\mathcal{A}) = 0 \\ \Rightarrow\ & B_{23}(\hat{\mathcal{A}}) = 0. \end{aligned} \tag{3.3}$$

The minimality of G_0 implies that every Borel subset of G_0 of positive measure is the support of some vector from $\hat{\mathcal{A}}$, in particular, there exists $f \in \hat{\mathcal{A}}$ such that *supp* $f = G_0$; in other words, $f > 0$ on G_0. (In fact, the vector $f = \Sigma_{j=1}^{\infty} \frac{1}{2^{j-1}} \frac{f_j}{\|f_j\|}$ in $\hat{\mathcal{A}}$ is such that its support is G_0, *i.e.*, $f > 0$ on G_0).

From (3.3), we get

$$B_{23}(f) = 0 \text{ where } f > 0 \text{ in } \mathcal{L}^2(G_0).$$

By Proposition 3.2.1, $B_{23} = 0$. This is true for all $B \in \mathcal{S}$. Further, using the fact that $(BA)_{41} = 0$ for all $A \in \mathcal{S}$, we obtain $B_{43}A_{31} = 0$ for all $A \in \mathcal{S}$, and by the same argument as above, we get $B_{43} = 0$ for all $B \in \mathcal{S}$. This shows that $\mathcal{L}^2(E) \oplus \mathcal{L}^2(G_0) \in \mathcal{L}at\mathcal{S}$ and hence $\mathcal{S}$ is decomposable.

(ii) Next, consider the case when $\mu(E \cap F) > 0$. This gets subdivided into two cases according as $\mu(E \Delta F)$ is zero or positive, where $E\Delta F = (E \backslash F)\dot{\cup}(F \backslash E)$.

(a) If $\mu(E\Delta F) = 0$, then $E = F$ with no loss of generality and we can write

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(E^c).$$

Since $\langle A\chi_E, \chi_E \rangle = 0$ for all $A \in \mathcal{S}$, every $A \in \mathcal{S}$ has a representation

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with respect to the decomposition above. For a fixed $B \in \mathcal{S}$,

$$(BA)_{11} = 0 \Rightarrow B_{12}A_{21} = 0,$$

where $A_{21} : \mathcal{L}^2(E) \to \mathcal{L}^2(E^c)$.

Again by Lemma 3.2.4, applied to the set

$$\mathcal{A}_1 = \{A_{21}(f) : \ A \in \mathcal{S}, \ f \in \mathcal{L}^2(E), \ f \geq 0\},$$

we can find a minimal subset $\mathcal{N}$ of E^c having positive measure such that

$$\left\langle A\chi_{E}, \chi_{E^c \setminus \mathcal{N}} \right\rangle = 0 \text{ for all } A \in \mathcal{S}$$

where $\mathcal{N}$ is the union of the supports of all vectors in a countable dense subset of the closed linear span $\hat{\mathcal{A}}_1$ of $\mathcal{A}_1$. Then with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(\mathcal{N}) \oplus \mathcal{L}^2(E^c \setminus \mathcal{N}),$$

any $A \in \mathcal{S}$ has the matrix representation

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}.$$

For a fixed $B \in \mathcal{S}$,

$$(BA)_{11} = 0 \Rightarrow B_{12}A_{21} = 0 \text{ and } (BA)_{31} = 0 \Rightarrow B_{32}A_{21} = 0,$$

where $A_{21} : \mathcal{L}^2(E) \to \mathcal{L}^2(\mathcal{N})$.

Now $B_{12}(\hat{\mathcal{A}}_1) = 0 = B_{32}(\hat{\mathcal{A}}_1)$. Following the argument in case (i), by the minimality of $\mathcal{N}$ (or otherwise), we show the existence of a vector g in $\hat{\mathcal{A}}_1$ such that $g > 0$ on $\mathcal{N}$. Therefore, $B_{12}(g) = 0 = B_{32}(g)$.

By Proposition 3.2.1, $B_{12} = 0 = B_{32}$. This is true for all $B \in \mathcal{S}$. Thus $\mathcal{L}^2(E) \oplus \mathcal{L}^2(\mathcal{N}) \in \mathcal{L}at\mathcal{S}$ and hence $\mathcal{S}$ is decomposable.

(b) Next, suppose that $\mu(E \Delta F) > 0$, in which case either $E \setminus F$ or $F \setminus E$ must have positive measure. By considering $\mathcal{S}^*$, if necessary, we can assume with no loss of generality that $\mu(F \setminus E) > 0$. Then, we can write

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F \setminus E) \oplus \mathcal{L}^2(\mathcal{X} \setminus (E \cup F)),$$

where we have $\left\langle A\chi_{E}, \chi_{F\setminus E}\right\rangle = 0$ for all $A \in \mathcal{S}$. With respect to this decomposition, any $A \in \mathcal{S}$ has a matrix representation

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

This reduces to case (i) and hence $\mathcal{S}$ is decomposable.

Proposition 3.2.6 *If $\mathcal{S}$ is an indecomposable semigroup of nonnegative operators in $\mathcal{L}^2(\mathcal{X})$, then so is every nonzero ideal of $\mathcal{S}$.*

Proof. Let $\mathcal{J}$ be a nonzero ideal of $\mathcal{S}$ and suppose that it is decomposable. Then there exists a Borel subset U of $\mathcal{X}$ with $\mu(U).\mu(U^c) > 0$ such that

$$\mathcal{L}^2(U) = \{f \in \mathcal{L}^2(X) : f = 0 \text{ } a.e. \text{ on } U^c\}$$

is invariant under every member of $\mathcal{J}$. This is equivalent to saying that

$$\langle J\chi_{U}, \chi_{U^c}\rangle = 0 \quad \text{for all } J \in \mathcal{J}.$$

Thus with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(U) \oplus \mathcal{L}^2(U^c), \tag{3.4}$$

every member J of $\mathcal{J}$ assumes the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Pick a nonzero J of this form and let

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

be an arbitrary element of $\mathcal{S}$ with respect to the decomposition (3.4). Then

$$SJ = \begin{pmatrix} S_{11}A & S_{11}B + S_{12}C \\ S_{21}A & S_{21}B + S_{22}C \end{pmatrix}.$$

Since $\mathcal{J}$ is an ideal, $SJ \in \mathcal{J}$ and therefore, we must have

$$S_{21}A = 0. \tag{3.5}$$

If A is nonzero, then A being a nonnegative operator on $\mathcal{L}^2(U)$, there exists a nonzero, nonnegative function in its range. Let us call this element f_0. There must exist some $\epsilon > 0$ for which the set $E = \{x \in U : f_0(x) \geq \epsilon\}$ has positive measure. Then χ_E is a nonzero characteristic function in $\mathcal{L}^2(U)$ and is such that

$$\begin{aligned} f_0(x) &\geq \epsilon\chi_E(x) \quad \text{for all } x \in U \\ \textit{i.e.,} \quad \chi_E &\leq \alpha f_0 \ , \ \alpha = \frac{1}{\epsilon} > 0. \end{aligned}$$

From equation (3.5), $S_{21}\chi_E \leq \alpha S_{21} f_0 = 0$ *i.e.*, $S_{21}\chi_E = 0$. Thus $\langle S_{21}\chi_E, \chi_F\rangle = 0$ for any Borel subset F in U^c of positive measure. Therefore, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F) \oplus \mathcal{L}^2(G),$$

where $G = (U \backslash E)\ \dot{\cup}\ (U^c \backslash F)$, any $S \in \mathcal{S}$ has the following representation

$$S = \begin{pmatrix} S'_{11} & S'_{12} & S'_{13} \\ 0 & S'_{22} & S'_{23} \\ S'_{31} & S'_{32} & S'_{33} \end{pmatrix}.$$

Thus $\langle S\rangle \quad \chi_F\rangle = 0$ for all $S \in \mathcal{S}$, which implies by Lemma 3.2.5 that $\mathcal{S}$ is decomposable which is a contradiction.

Thus assume that $A = 0$ for all $J \in \mathcal{J}$. Then

$$JS = \begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} BS_{21} & BS_{22} \\ CS_{21} & CS_{22} \end{pmatrix} \in \mathcal{J}.$$

We have $BS_{21} = 0 = CS_{21}$. Since $\mathcal{S}$ is indecomposable, we can pick an element S in $\mathcal{S}$ for which $S_{21} \neq 0$. Then S_{21}^* is also a nonzero, nonnegative operator from $\mathcal{L}^2(U^c)$ into $\mathcal{L}^2(U)$ and we consider $(BS_{21})^* = S_{21}^* B^*$. As argued above for A, if B^* is nonzero, we can find a nonzero characteristic function, say χ in $\mathcal{L}^2(U)$ such that $S_{21}^*\chi = 0$ which would imply decomposability of $\mathcal{S}^*$ and consequently of $\mathcal{S}$. This contradiction leads to B^* and thus B being zero. By a similar reasoning $C = 0$, in other words, $J = 0$, a contradiction. Hence every nonzero ideal of $\mathcal{S}$ must be indecomposable. □

Proposition 3.2.7 *Let $\mathcal{S}$ be a collection of nonnegative operators from $\mathcal{L}^2(\mathcal{X})$ into $\mathcal{L}^2(\mathcal{Y})$. Let A and B be nonzero, nonnegative operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{Y}))$ and $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ respectively, satisfying $A\mathcal{S}B = \{0\}$. Then there exist Borel subsets $E \subseteq \mathcal{X}$ and $F \subseteq \mathcal{Y}$ with positive measures such that $\langle S\chi_{_E}, \chi_{_F}\rangle = 0$, for all $S \in \mathcal{S}$.*

Proof. The hypothesis $A\mathcal{S}B = \{0\}$ gives that

$$\begin{aligned} &\langle ASBf, g\rangle = 0 \quad \text{for all } f \in \mathcal{L}^2(\mathcal{X}) \text{ and for all } g \in \mathcal{L}^2(\mathcal{Y}) \\ \Rightarrow\ &\langle SBf, A^*g\rangle = 0 \quad \text{for all } f \in \mathcal{L}^2(\mathcal{X}) \text{ and for all } g \in \mathcal{L}^2(\mathcal{Y}). \end{aligned} \tag{3.6}$$

Now, since B is nonnegative and nonzero, its range must contain a nonzero, nonnegative element, say f_0. The same is true for A^* since $A^* \geq 0$ and $A^* \neq 0$ (because $A \geq 0$ and $A \neq 0$). Therefore, there exists a nonzero, nonnegative function g_0 in the range of A^*.

Further, f_0 nonnegative and nonzero implies that there exists some $\epsilon > 0$ such that the set $\{x \in \mathcal{X} : f_0(x) \geq \epsilon\}$ has positive measure. Denote this set by E and consider $\chi_{_E}$. Then $\chi_{_E}$ is a nonzero characteristic function in $\mathcal{L}^2(\mathcal{X})$ and is such that $f_0(x) \geq \epsilon\chi_{_E}(x)$ for all $x \in \mathcal{X}$ *i.e.*, $\chi_{_E} \leq \alpha f_0, \alpha = \dfrac{1}{\epsilon}$. Similarly, we can find a Borel subset F in $\mathcal{Y}$ of positive measure such that $\chi_{_F} \leq \beta g_0$ for some positive scalar β.

For any $S \in \mathcal{S}$, since S is a nonnegative operator, we have $S\chi_{_E} \leq \alpha S f_0$. By the property of monotonicity for integrals,

$$\langle S\chi_{_E}, \chi_{_F}\rangle \leq \langle \alpha S f_0, \beta g_0\rangle = \alpha\beta\,\langle S f_0, g_0\rangle = 0 \text{ for all } S \in \mathcal{S} \quad (\text{ from } (3.6))$$

which proves the proposition. □

Corollary 3.2.8 *A nonnegative semigroup of operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is decomposable if and only if there exist nonzero, nonnegative operators A and B on $\mathcal{L}^2(\mathcal{X})$, not necessarily in $\mathcal{S}$ such that $A\mathcal{S}B = \{0\}$.*

Proof. By the preceding proposition, the condition $A\mathcal{S}B = \{0\}$ implies

$$\langle S\chi_E, \chi_F\rangle = 0 \text{ for all } S \in \mathcal{S}$$

for Borel subsets E, F of $\mathcal{X}$ with $\mu(E).\mu(F) > 0$. This gives decomposability of $\mathcal{S}$ by Lemma 3.2.5.

Conversely, suppose $\mathcal{S}$ is decomposable. Then there exists a Borel subset U of $\mathcal{X}$ with $\mu(U).\mu(U^c) > 0$ such that with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(U) \oplus \mathcal{L}^2(U^c), \tag{3.7}$$

every $S \in \mathcal{S}$ has the following matrix representation

$$S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}.$$

If with respect to the decomposition (3.7), we define two nonzero, nonnegative operators

$$A = \begin{pmatrix} 0 & A_{12} \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

then it is easily verified that $ASB = 0$ for all $S \in \mathcal{S}$ *i.e.*, $A\mathcal{S}B = \{0\}$. □

3.3 When is a nonnegative band decomposable?

This section is devoted to studying the decomposability of nonnegative bands in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$. We shall first establish the decomposability of a single nonnegative idempotent which is already a proven result (cf. Zhong [16]). We are including the proof here for the sake of completeness and also because it has a slightly different approach from Zhong's and works for a more general class of nonnegative idempotents.

Lemma 3.3.1 *Let A be a nonnegative idempotent on $\mathcal{L}^2(\mathcal{X})$ and let f be a nonnegative element in the range of A. Fix a nonnegative representative of f (and still denote it by f). If*

$$U = supp\ f = \{x : f(x) > 0\},$$

then $\mathcal{L}^2(U) \in \mathcal{L}at A$.

Proof. It suffices to prove that $\langle A\chi_U, \chi_{U^c}\rangle = 0$. By hypothesis,

$$\begin{aligned}
& \langle Af, \chi_{U^c}\rangle = 0 \text{ (as } Af = f \text{ and } supp\ f = U) \\
\Rightarrow\ & \langle f, A^*\chi_{U^c}\rangle = 0 \\
\Rightarrow\ & \int_U (A^*\chi_{U^c})\ (x)\ f(x)\ \mu(dx) = 0 \\
\Rightarrow\ & (A^*\chi_{U^c})\ (x)\ f(x) = 0 \ a.e. \text{ on } U \text{ as } A^* \geq 0
\end{aligned}$$

But $f(x) > 0$ *a.e.* on U. Therefore,

$$\begin{aligned}
& (A^*\chi_{U^c})(x) = 0 \text{ for almost all } x \in U \\
\Rightarrow\ & \langle A^*\chi_{U^c}, \chi_U\rangle = 0 \\
\Rightarrow\ & \langle A\chi_U, \chi_{U^c}\rangle = 0
\end{aligned}$$

which proves the lemma. □

Lemma 3.3.2 *Let A be as in the preceding lemma. If an element f in the range of A is real, then there exists a nonnegative element h in $\mathcal{L}^2(\mathcal{X})$ such that $Ah = 0$ and $f^+ + h, f^- + h$ are in the range of A.*

Proof. For the proof, see [16]. □

Lemma 3.3.3 *If an element f in $\mathcal{L}^2(\mathcal{X})$ belongs to the range of a nonnegative idempotent A, then the real part $Re\ f$ and the imaginary part $Im\ f$ of f are also in the range of A.*

Proof. Observe that $Re\ f + iIm\ f = f = Af = A(Re\ f) + iA(Im\ f)$. Since A is nonnegative, it sends real valued functions to real valued functions. Therefore, comparing the real and imaginary parts in the equation above, we obtain $A(Re\ f) = Re\ f$ and $A(Im\ f) = Im\ f$. □

Definition 3.3.4 *By* $ker\mathcal{A}$*, for any collection* $\mathcal{A}$ *of operators in* $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$*, we mean*

$$\{f \in \mathcal{L}^2(\mathcal{X}) : Sf = 0 \textit{ for all } S \in \mathcal{A}\}.$$

Theorem 3.3.5 *Let* A *be a nonnegative idempotent in* $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ *of rank at least two. Then* A *is decomposable.*

Proof. If $Ah = 0$ for some nonzero, nonnegative h, then A is zero on $\mathcal{L}^2(supp\ h)$ and is thus decomposable (by Corollary 3.2.2). Therefore assume that $kerA$ contains no nonzero, nonnegative element. By Lemma 3.3.3, if an element is in the range of A, then so are its real and imaginary parts. Thus, we can obtain a basis of the range of A consisting of real elements. Further, with our assumption together with Lemma 3.3.2, we can obtain a basis of the range of A containing nonnegative elements.

Since rank $A \geq 2$, A has at least two nonnegative, nonzero linearly independent elements in its range, say f and g. If either of them is zero on a set of positive measure, we are done by Lemma 3.3.1. Therefore, assume that both f and g are positive. We shall prove that some linear combination of f and g has to be mixed *i.e.*, it has positive and negative parts with supports of positive measure.

Consider the following subsets of reals

$$S_1 = \{r : f - rg > 0\}$$
$$S_2 = \{r : f - rg < 0\}$$

Now S_1 is nonempty as zero belongs to it. Also S_2 cannot be empty for then $f > rg$ for all $r \in \mathbf{R}$ which is not possible.

Let $r_0 = inf\ S_1$ and $s_0 = sup\ S_2$. Observe that if r_0 and s_0 are finite, then we cannot have $r_0 = s_0$, for then f and g would be linearly dependent which is not true. Therefore, we have $r_0 < s_0$ (note that since S_1 is not empty, r_0 cannot be equal to infinity if s_0 is infinite). We can pick a number p such that $r_0 < p < s_0$. Since $s_0 < p, p \notin S_2$ and therefore, $f - pg \not> 0$. Similarly, as $p < r_0, p \notin S_1$ and thus $f - pg \not< 0$. Hence $f - pg$ is a mixed vector *i.e.*, it has nonzero positive and negative parts and also it is clearly in the range of A. Existence of such a vector in the range of A gives decomposability of A, for if u is such that $Au = u, u = u^+ - u^-, u^+, u^-$ nonzero, then by Lemma 3.3.2, we can find $h \geq 0$ in $\mathcal{L}^2(\mathcal{X}), Ah = 0$ such that $u^+ + h$ and $u^- + h$ are in the range of A. But by our assumption, $h = 0$. Therefore, u^+, u^- are in the range of A. Consider the vector u^+. Then by Lemma 3.3.1, $\mathcal{L}^2(supp\ u^+)$ is a nontrivial standard invariant subspace for A. Hence A is decomposable. □

Having established the decomposability of a single nonnegative idempotent with rank at least two, we now prove that it has a very special standard block triangularization. This will require a couple of lemmas and some definitions.

Lemma 3.3.6 *An indecomposable, nonnegative rank-one operator on $\mathcal{L}^2(\mathcal{X})$ is positive.*

Proof. Let A be an indecomposable, nonnegative rank-one operator on $\mathcal{L}^2(\mathcal{X})$. Then we know that $A = u \otimes v$, where u, v are nonzero, nonnegative vectors in $\mathcal{L}^2(\mathcal{X})$, so that $Af = \langle f, v \rangle u$ for all $f \in \mathcal{L}^2(\mathcal{X})$.

Suppose A is not positive. Then there exists a nonzero, nonnegative vector f in $\mathcal{L}^2(\mathcal{X})$ for which Af is not positive. In other words, the set $E = \{x \in \mathcal{X} : (Af)(x) = 0\}$ has positive measure. Also, if $Af = 0$, then $A \equiv 0$ on $\mathcal{L}^2(supp\ f)$ and is thus decomposable (Corollary 3.2.2) which is not possible. Therefore $Af \neq 0$. Now

$$\begin{aligned} E &= \{x \in \mathcal{X} : \langle f, v \rangle\, u(x) = 0\} \\ &= \{x \in \mathcal{X} : u(x) = 0\} \quad (\text{ because } Af \neq 0). \end{aligned}$$

Since $\mu(E) > 0$, χ_E is a nonzero, nonnegative vector and is such that

$$\begin{aligned} A^*\chi_E &= (v \otimes u)\chi_E = \langle \chi_E, u \rangle\, v \\ &= \left(\int_E u(x)\ \mu(dx) \right) v \\ &= 0. \end{aligned}$$

This implies that $A^* \equiv 0$ on $\mathcal{L}^2(E)$ (by Proposition 3.2.1). Thus A^* and consequently A is decomposable which is a contradiction. Hence A must be positive. □

Definition 3.3.7 *A nonnegative semigroup $\mathcal{S}$ in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ will be called a* **full semigroup** *if neither ker $\mathcal{S}$ nor ker $\mathcal{S}^*$ has a nonzero, nonnegative vector. A single nonnegative operator is called full if the semigroup generated by it is full.*

Definition 3.3.8 *A chain of subspaces of $\mathcal{L}^2(\mathcal{X})$ is called* **maximal** *if it is not properly contained in any other chain of subspaces of $\mathcal{L}^2(\mathcal{X})$.*

If $\mathcal{C}$ is any chain of subspaces and $\mathcal{M} \in \mathcal{C}$, then we define $\mathcal{M}_-$ to be the closed linear span of all those members of $\mathcal{C}$ which are properly contained in $\mathcal{M}$. It is not difficult to see [14] that a subspace chain is maximal if and only if

(i) $\mathcal{C}$ is closed under arbitrary spans and intersections,

(ii) for each $\mathcal{M}$ in $\mathcal{C}$, $\mathcal{M} \ominus \mathcal{M}_-$ is at most one-dimensional.

A maximal chain $\mathcal{C}$ is said to be **continuous** if $\mathcal{M} = \mathcal{M}_-$ for each $\mathcal{M}$ in $\mathcal{C}$, in other words, $\mathcal{C}$ has no gaps in it.

Definition 3.3.9 *A collection of operators $\mathcal{S}$ in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is said to have a* **continuous standard triangularization** *if* *(i) $\mathcal{L}atS$ contains a continuous maximal chain, say $\mathcal{C}$,*

(ii) each member of $\mathcal{C}$ is a standard subspace.

Example 3.3.10 For $t \in [0,1]$, the multiplication operator $M : \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]$ defined by

$$(Mf)(t) = tf(t)$$

is a nonnegative operator. For any $\alpha \in [0,1]$, define

$$\mathcal{M}_\alpha = \{f \in \mathcal{L}^2[0,1] : f(t) = 0 \ \forall \, t \geq \alpha\}.$$

Then $\{\mathcal{M}_\alpha : \alpha \in [0,1]\}$ is a maximal subspace chain which is continuous and consists of standard invariant subspaces for M. Thus M has a continuous standard triangularization and since $M = M^*$, so does M^*.

Example 3.3.11 Let $\mathcal{H} = \mathcal{L}^2[0,1] \oplus \mathcal{L}^2[0,1]$ and define $E : \mathcal{H} \to \mathcal{H}$ by

$$E = \begin{pmatrix} M & M \\ I-M & I-M \end{pmatrix},$$

where M is the multiplication operator in the preceding example and $I - M : \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]$ is the multiplication operator by $1 - t$. Then E is a nonnegative idempotent and

$$E^* = \begin{pmatrix} M & I-M \\ M & I-M \end{pmatrix}.$$

Let $\mathcal{N}_\alpha = \mathcal{M}_\alpha \oplus \mathcal{M}_\alpha$ for $\alpha \in [0,1]$. Then using the fact that $\{\mathcal{M}_\alpha : \alpha \in [0,1]\}$ is maximal, it is not hard to prove that $\{\mathcal{N}_\alpha : \alpha \in [0,1]\}$ is a maximal subspace chain in $\mathcal{H}$ which is continuous. Also it consists of standard invariant subspaces for E and E^*. Thus E and E^* have a simultaneous continuous standard triangularization.

Lemma 3.3.12 *Let A in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ and B in $\mathcal{B}(\mathcal{L}^2(\mathcal{Y}))$ be nonzero, nonnegative operators such that neither ker A nor ker B^* has a nonzero, nonnegative vector. If $S : \mathcal{L}^2(\mathcal{Y}) \to \mathcal{L}^2(\mathcal{X})$ is a nonnegative operator such that $ASB = 0$, then $S = 0$.*

Proof. Suppose SB is nonzero. Then $SB : \mathcal{L}^2(\mathcal{Y}) \to \mathcal{L}^2(\mathcal{X})$ is a nonzero, nonnegative operator. Therefore, there exists a nonzero, nonnegative vector f in $\mathcal{L}^2(\mathcal{Y})$ such that SBf is nonzero, nonnegative. Write $g = SBf$. Then $Ag = 0$ which implies that $ker\ A$ has a nonzero, nonnegative vector, a contradiction. Therefore, we must have $SB = 0$ which gives that $B^*S^* = 0$. If S is nonzero, then S^* is a nonzero, nonnegative operator and thus its range contains a nonzero, nonnegative vector, say h but that would imply $B^*(S^*h) = 0$, contrary to the fact that $ker\ B^*$ has no nonzero, nonnegative vector. Hence, we have $S = 0$. □

Theorem 3.3.13 *(a) Let A be a nonnegative idempotent on $\mathcal{L}^2(\mathcal{X})$ with rank r which is full.*

(i) If r is finite, then there exists a decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \cdots \oplus \mathcal{L}^2(\mathcal{X}_r)$$

with respect to which

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix},$$

where each $A_i : \mathcal{L}^2(\mathcal{X}_i) \longrightarrow \mathcal{L}^2(\mathcal{X}_i)$ is a positive idempotent of rank one.

(ii) If $r = \infty$, then with respect to some direct sum decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{Y}_1) \oplus \mathcal{L}^2(\mathcal{Y}_2),$$

$$A = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix},$$

where E and F have the following descriptions: If $E \neq 0$, then $\mathcal{L}^2(\mathcal{Y}_1) = \oplus_{i=1}^{N} \mathcal{L}^2(\mathcal{Z}_i)$ for some $N \leq \infty$, where $\mathcal{L}^2(\mathcal{Z}_i)$ are standard subspaces of $\mathcal{L}^2(\mathcal{X})$ which are reducing *under A, and $E : \mathcal{L}^2(\mathcal{Y}_1) \to \mathcal{L}^2(\mathcal{Y}_1)$ has the block diagonal form*

$$\begin{pmatrix} E_1 & & & & \\ & E_2 & & & \\ & & \ddots & & \\ & & & E_i & \\ & & & & \ddots \end{pmatrix}$$

with each $E_i : \mathcal{L}^2(\mathcal{Z}_i) \to \mathcal{L}^2(\mathcal{Z}_i)$ being a positive idempotent of rank one.

If $F \neq 0$, then F and F^* have a simultaneous continuous standard triangularization.

(b) In general, if A is not full, then there exists a decomposition of $\mathcal{L}^2(\mathcal{X})$, say

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{W}_1) \oplus \mathcal{L}^2(\mathcal{W}_2) \oplus \mathcal{L}^2(\mathcal{W}_3),$$

where $\mathcal{L}^2(\mathcal{W}_i)$ $(i = 1, 2, 3)$ are standard invariant subspaces of $\mathcal{L}^2(\mathcal{X})$ such that with respect to this decomposition

$$A = \begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix},$$

where $E : \mathcal{L}^2(\mathcal{W}_2) \to \mathcal{L}^2(\mathcal{W}_2)$ is an idempotent of the form in (i) or (ii) according as rank of A is finite or infinite.

Proof. (a) (i) When r is finite, we prove the result by induction on r. If $r = 1$, we know by Lemma 3.3.6, that A is a positive idempotent of rank one. Let $r > 1$, then we know that A is decomposable and therefore, there exists a Borel subset $U \subseteq \mathcal{X}$ with $\mu(U).\mu(U^c) > 0$ such that with respect to

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(U) \oplus \mathcal{L}^2(U^c),$$

$$A = \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix},$$

where with no loss of generality, we can assume that A_1 and A_2 are nonzero. Now $A^2 = A$ implies that $A_1X + XA_2 = X$. Premultiplying by A_1 and postmultiplying by A_2, we obtain $A_1XA_2 = 0$. Since A is full, $ker\ A_1$ and $ker\ A_2^*$ have no nonzero, nonnegative vector. Therefore, by Lemma 3.3.12, $X = 0$. Thus

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Since A_1 and A_2 are nonzero, their ranks are less than r and both are full because A is full. Hence induction applies and we obtain the desired result.

(ii) If r is infinite, A is certainly decomposable. Let $\mathcal{C}$ be a maximal chain in $\mathcal{L}at'A$. Our first claim is that each gap in the chain is reducing for A. Let $\mathcal{N} \ominus \mathcal{M}$ be a gap where $\mathcal{M} \subset \mathcal{N}$ in $\mathcal{C}$. We wish to show that $\mathcal{N} \ominus \mathcal{M}$ is invariant under both A and A^*. Consider the block triangularization of A with respect to the following decomposition of $\mathcal{L}^2(\mathcal{X})$,

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{M} \oplus (\mathcal{N} \ominus \mathcal{M}) \oplus (\mathcal{L}^2(\mathcal{X}) \ominus \mathcal{N}),$$

and

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}.$$

If we first regard A as the 2×2 block matrix

$$\begin{pmatrix} A_0 & X_0 \\ 0 & B_0 \end{pmatrix},$$

where

$$A_0 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \ X_0 = \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}, \ B_0 = A_{33},$$

then as shown in part (i), the fullness of A gives $X_0 = 0$, *i.e.*, $A_{13} = 0 = A_{23}$.

Similarly, considering A as the 2×2 block matrix

$$\begin{pmatrix} A_{00} & X_{00} \\ 0 & B_{00} \end{pmatrix},$$

where

$$A_{00} = (A_{11}),\ X_{00} = (A_{12}\ \ A_{13}),\ B_{00} = \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix},$$

we shall obtain $X_{00} = 0$, *i.e.*, $A_{12} = 0 = A_{13}$. Therefore

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix}.$$

This shows that $\mathcal{N} \ominus \mathcal{M}$ is reducing which proves our claim.

Also the maximality of $\mathcal{C}$ implies that the compression of A to each gap, if nonzero, must be an indecomposable (and thus positive) idempotent of rank one. Further, because of separability of $\mathcal{L}^2(\mathcal{X})$, there can only be countably many reducing gaps. Thus, after a permutation of basis, we can obtain a decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{Y}_1) \oplus \mathcal{L}^2(\mathcal{Y}_2)$$

with respect to which

$$A = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix},$$

where $\mathcal{L}^2(\mathcal{Y}_1) = \oplus_{i=1}^{N}\mathcal{L}^2(\mathcal{Z}_i)$, $\{\mathcal{L}^2(\mathcal{Z}_i)\}_{i=1}^{N}, N \leq \infty$ being a collection of reducing subspaces of A and $E : \mathcal{L}^2(\mathcal{Y}_1) \to \mathcal{L}^2(\mathcal{Y}_1)$ has the block diagonal form as mentioned in the statement of the theorem. The fullness of A makes both $\mathcal{L}^2(\mathcal{Y}_1)$ and $\mathcal{L}^2(\mathcal{Y}_2)$ reducing standard subspaces. Further, since all the gaps have been absorbed in $\mathcal{L}^2(\mathcal{Y}_1)$, the operators F and F^* are continuously triangularizable and since this triangularization results from a maximal chain of standard subspaces, we can say that F and F^* have a simultaneous continuous standard triangularization.

(b) Here, we consider the general case when A is not full.

Suppose $\mathcal{A}$ is the collection of all nonzero, nonnegative vectors in $ker\mathcal{A}$. By Lemma 3.2.4, we can find a minimal subset G in $\mathcal{X}$, defined upto a null set, on whose complement all the vectors in $\mathcal{A}$ vanish. This gives the existence of a vector f in $\mathcal{A}$

such that $G = supp\ f$. But this means that $Af = 0$ for some $f > 0$ in $\mathcal{L}^2(G)$ which implies that $A \equiv 0$ on $\mathcal{L}^2(G)$.

Similarly, we can find a set G^* of positive measure such that $A^* \equiv 0$ on $\mathcal{L}^2(G^*)$. Then, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(G) \oplus \mathcal{L}^2(\mathcal{X}\backslash(G \cup G^*)) \oplus \mathcal{L}^2(G^*),$$

A has the representation

$$\begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix}$$

where $E^2 = E$, $X = XE$, $Y = EY$ and $Z = XEY$.

Renaming $\mathcal{L}^2(G) = \mathcal{L}^2(\mathcal{W}_1)$, $\mathcal{L}^2(\mathcal{X}\backslash(G \cup G^*)) = \mathcal{L}^2(\mathcal{W}_2)$ and $\mathcal{L}^2(G^*) = \mathcal{L}^2(\mathcal{W}_3)$, we obtain the representation of A as described in part (b) of the theorem. Also, these equations show that E is full and hence it is of the form described in part (a) of the theorem. □

From a single nonnegative idempotent, we now move on to analyze a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ with more than one element in it. As in the discrete case, we shall find that if a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ with rank of each member being > 1 has even a single member of finite rank, it is decomposable.

Theorem 3.3.14 *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ having at least one element of finite rank and with rank $(S) > 1$ for all S in $\mathcal{S}$. Then $\mathcal{S}$ is decomposable.*

Proof. Let $m =$ min { rank (S) : $S \in \mathcal{S}$ }; then $m > 1$. Let $\mathcal{J}$ be the set of all elements of rank m in $\mathcal{S}$. For any $S \in \mathcal{S}$ and $J \in \mathcal{J}$,

$$\begin{aligned} & \text{rank } (SJ) \leq \text{ min } \{ \text{ rank } (S), \text{ rank } (J) \} = \text{ rank } (J) = m \\ \Rightarrow \quad & SJ = 0 \text{ or rank } (SJ) = m. \end{aligned}$$

But $SJ \neq 0$ as all members of $\mathcal{S}$ have rank greater than one. Therefore, rank $(SJ) = m$ which implies that $SJ \in \mathcal{J}$. Similarly, it can be shown that $JS \in \mathcal{J}$ for all $J \in \mathcal{J}$ and for all $S \in \mathcal{S}$. Thus $\mathcal{J}$ is a nonzero ideal of $\mathcal{S}$.

Now $\mathcal{S}$ is decomposable if and only if $\mathcal{J}$ is decomposable. Therefore, we can assume with no loss of generality that $\mathcal{S} = \mathcal{J}$ so that $\mathcal{S}$ has constant rank m.

Select a $P \in \mathcal{S}$. Let S be an arbitrary element of $\mathcal{S}$ and consider PSP. This is an idempotent whose range is contained in the range of P and whose null space contains the null space of P and since rank $(PSP) = m =$ rank (P), we have $PSP = P$. Thus $P\mathcal{S}P = \{P\}$.

Since $m > 1$, by Theorem 3.3.13, we can find a Borel subset U of $\mathcal{X}$ with positive measure such that with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(U) \oplus \mathcal{L}^2(U^c), \tag{3.8}$$

P has the matrix representaion

$$\begin{pmatrix} P_1 & X \\ 0 & P_2 \end{pmatrix},$$

where both P_1 and P_2 are nonzero.

Pick an arbitrary S in $\mathcal{S}$ and let its matrix representation with respect to (3.8) be $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$. Then $PSP = P$ implies that $P_2 S_{21} P_1 = 0$. By Proposition 3.2.7, there exist Borel subsets E, F in U and U^c respectively having positive measures such that

$$\langle S_{21}\chi_{E}, \chi_{F}\rangle = 0.$$

Finally, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F) \oplus \mathcal{L}^2(G),$$

where $G = (U \backslash E) \cup (U^c \backslash F)$, every $S \in \mathcal{S}$ has the following matrix representation

$$S = \begin{pmatrix} S'_{11} & S'_{12} & S'_{13} \\ 0 & S'_{22} & S'_{23} \\ S'_{31} & S'_{32} & S'_{33} \end{pmatrix}.$$

This shows that $\langle S\chi_{E}, \chi_{F}\rangle = 0$ for all $S \in \mathcal{S}$. Hence, by Lemma 3.2.5, $\mathcal{S}$ is decomposable. □

Theorem 3.3.15 *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ such that rank $(S) > 1$ for all S in $\mathcal{S}$ and $\mathcal{S}$ has at least one element of finite rank. Then any maximal standard block triangularization of $\mathcal{S}$ has the property that the compression of $\mathcal{S}$ to each nonzero gap constitutes a nonnegative band with at least one element of rank one in it.*

Proof. Same as in the finite-dimensional case (refer Theorem 1.3.7). □

In the Theorem3.3.14, we saw that the decomposability of a band in which every member has rank > 1 and which has at least one finite-rank member reduced to the decomposability of a constant-rank band. The most pertinent question to be asked after this is:

Question 3.3.16 *Is every constant-rank nonnegative band decomposable?*

Let us answer this question systematically. We start by considering such bands in $\mathcal{M}_n(\mathbb{C})$. The answer to the question above is in the negative if the rank is one. A simple example to substantiate this is the band $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$. If the constant rank is greater than one, then we know that the band is decomposable (see the proof of Theorem 1.3.5). This completes our analysis of the problem in finite dimensions. For a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, we have seen in the proof of Theorem 3.3.14 that with constant finite rank greater than one, the band is decomposable. Now, the natural question which occurs is whether a constant infinite-rank nonnegative band is decomposable? The answer is a resounding no as we illustrate through a counter example in $\mathcal{B}(l^2)$.

Example 3.3.17 There exists an indecomposable nonnegative band in $\mathcal{B}(l^2)$ in which every member has infinite rank.

Proof. For each integral i, define an operator S_i as follows

$$S_i = \begin{pmatrix} T_i & & \\ & T_i & \\ & & \ddots \end{pmatrix},$$

where T_i is a $2^i \times 2^i$ block with each entry equal to $\frac{1}{2^i}$. Let $\mathcal{S} = \{S_0, S_1, S_2, \cdots\}$. It is easily verified that

$$\text{for } i \leq j,\ S_iS_j = S_j \text{ and } S_jS_i = S_j.$$

Thus $\mathcal{S}$ is a nonnegative band where each S_i is of infinite rank. We claim that $\mathcal{S}$ is indecomposable. It suffices to prove that $\mathcal{S}$ has no common zero entry. Suppose on the contrary, that $\mathcal{S}$ has a common zero entry, say

$$(S_i)_{\alpha\beta} = 0 \text{ for all } S_i \in \mathcal{S}.$$

Now, we can find i and j such that $\alpha \leq 2^i$ and $\beta \leq 2^j$. With no loss of generality, we can assume that $i \leq j$. But then S_j will have the entry $(S_i)_{\alpha\beta}$ in its first diagonal block T_j which is positive. Thus $\mathcal{S}$ cannot have a common zero entry and hence is indecomposable. □

3.4 The structure of nonnegative, constant finite-rank bands

We saw in the previous section that constant-rank bands play a significant role in ascertaining the decomposability of nonnegative bands. It would be therefore interesting to study their structure completely which will be our task in this section. It is a generalization of the same in the finite-dimensional case. We already know that an infinite-rank nonnegative band may not be decomposable; therefore we shall restrict ourselves to nonnegative bands with constant finite rank.

Lemma 3.4.1 *If $\mathcal{S}$ is a band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ of nonnegative operators with constant finite rank r, then $\mathcal{S}$ has a standard block triangularization with r nonzero diagonal blocks, each block constituting an indecomposable band of rank-one operators. Furthermore, no two consecutive diagonal blocks are zero. Therefore, if k is the total number of diagonal blocks, then $k \leq 2r+1$.*

Proof. The proof runs exactly on the same lines as for the finite-dimensional case (see Lemma 2.1.2). □

Lemma 3.4.2 *Let $\mathcal{S}$ be a nonnegative full band of rank-one operators. Then $\mathcal{S}$ is indecomposable.*

Proof. Same as that of Lemma 2.1.4 in the finite-dimensional case. □

Theorem 3.4.3 *Let $\mathcal{S}$ be a band of nonnegative operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ with constant finite rank r.*

(i) If $\mathcal{S}$ is full, then there exists a decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \oplus \cdots \oplus \mathcal{L}^2(\mathcal{X}_r),$$

with respect to which every member S of $\mathcal{S}$ is of the form

$$\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix},$$

where each $\mathcal{S}_i = \{S_i \in \mathcal{L}^2(\mathcal{X}_i) : S \in \mathcal{S}\}$ is an indecomposable band of rank-one operators.

(ii) In general, there exists a decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1') \oplus \mathcal{L}^2(\mathcal{X}_2') \oplus \mathcal{L}^2(\mathcal{X}_3'),$$

with respect to which every member S of $\mathcal{S}$ is of the form

$$\begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix},$$

where X, Y are nonnegative operators on suitable spaces. Furthermore, the diagonal blocks in $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ constitute a band of the form in case (i).

Proof. (i) The proof is exactly as in the finite-dimensional case (refer Theorem 2.1.5 (i)).

(ii) Now, let us consider the general case. Suppose $\mathcal{A}$ is the collection of all the nonzero, nonnegative vectors in $ker\mathcal{S}$. Just as in the proof of Theorem 3.3.13 (b), we can find a set G of positive measure such that $\mathcal{S} \equiv 0$ on $\mathcal{L}^2(G)$ and also, we can find a set G^* of positive measure such that $\mathcal{S}^* \equiv 0$ on $\mathcal{L}^2(G^*)$.

Then, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(G) \oplus \mathcal{L}^2(\mathcal{X} \backslash (G \cup G^*)) \oplus \mathcal{L}^2(G^*),$$

every member S in $\mathcal{S}$ has the form

$$\begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix},$$

where $E^2 = E$, $X = XE$, $Y = EY$, and $Z = XEY$.

These equations show that the set $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ of the middle diagonal blocks is such that neither $\mathcal{S}_0$ nor $\mathcal{S}_0^*$ have any nonzero, nonnegative vectors in their null spaces and thus $\mathcal{S}_0$ is of the form in part (i) of the theorem. □

Remark 3.4.4 If in the statement of the theorem above, $\mathcal{S}$ is taken to be a maximal band, then it is readily observed that the bands $\mathcal{S}_i$ must be maximal. In part (ii), $\mathcal{S}_0$ and the collection of all X, Y are maximal too.

In Theorem 3.4.7, we prove the converse of part (i) of the preceding Theorem to obtain a characterization of maximal, nonnegative, constant-rank bands which are full. This will require a couple of lemmas, which may also be of independent interest.

Lemma 3.4.5 *Let $\mathcal{S}$ be a nonnegative, indecomposable semigroup in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ and f be a nonzero, nonnegative vector in $\mathcal{L}^2(\mathcal{X})$. Let $\mathcal{A}$ be the set of all nonnegative linear combinations of the members of $\{Sf : S \in \mathcal{S}\}$. Then $\bar{\mathcal{A}}$ contains a positive vector in $\mathcal{L}^2(\mathcal{X})$.*

Proof. Since $\mathcal{L}^2(\mathcal{X})$ is separable, so is the set $\mathcal{S}f$. Therefore, let $\mathcal{M} = \{S_1 f, S_2 f, \ldots\}$ be a countable dense subset of $\mathcal{S}f$, where $S_1 f, S_2 f, \ldots$ are the chosen representatives of the equivalence classes of functions in $\mathcal{M}$. Write

$$U = \bigcup_i \, supp \; S_i f.$$

Then the function g defined by

$$g = \frac{S_1 f}{\|S_1 f\|} + \frac{1}{2}\frac{S_2 f}{\|S_2 f\|} + \frac{1}{2^2}\frac{S_3 f}{\|S_3 f\|} + \cdots$$

is a nonegative vector in $\bar{\mathcal{A}}$ with support U; in other words, $g > 0$ in $\mathcal{L}^2(U)$. We shall prove that g is the desired positive vector in $\mathcal{L}^2(\mathcal{X})$, for which we need to show that $U = \mathcal{X}$ (up to a null set).

Now, by the construction of g, $g = 0$ *a.e.* on U^c. This implies that $S_i f = 0$ *a.e.* on U^c for every i, since each $S_i f$ is nonnegative. By the density of $\mathcal{M}$ in $\mathcal{S}f$, we further obtain that $Sf = 0$ *a.e.* on U^c for every $S \in \mathcal{S}$, and thus

$$\begin{aligned} Sg &= \frac{SS_1 f}{\|S_1 f\|} + \frac{1}{2}\frac{SS_2 f}{\|S_2 f\|} + \frac{1}{2^2}\frac{SS_3 f}{\|S_3 f\|} + \cdots \\ &= 0 \text{ a.e. on } U^c \text{ for every } S \in \mathcal{S}. \end{aligned} \tag{3.9}$$

Our claim is that $\mathcal{L}^2(U)$ is invariant under $\mathcal{S}$. Since $\mathcal{S}$ is indecomposable, this will prove that $\mathcal{L}^2(U) = \mathcal{L}^2(\mathcal{X})$. We prove this considering two possibilities: (i) g is bounded below on U, and (ii) g is not bounded below on U.

In case (i), there exists a nonnegative, nonzero scalar α such that $g(x) \geq \alpha$ *a.e.* on U. Let $E = \{x \in U : \ g(x) \geq \alpha\}$, then $\mu(E^c \cap U) = 0$. Also we have

$$\begin{aligned} g(x) &\geq \alpha\chi_E(x) \text{ for all } x \in U \\ \text{i.e. } \chi_E &\leq \frac{1}{\alpha} g. \end{aligned}$$

For any $S \in \mathcal{S}$, $S\chi_E \leq \frac{1}{\alpha} Sg$. Using (3.9), we obtain $S\chi_E = 0$ *a.e.* on U^c for all $S \in \mathcal{S}$, *i.e.* $\langle S\chi_U, \chi_{U^c}\rangle = 0$ for all $S \in \mathcal{S}$, *i.e.*, $\mathcal{L}^2(U)$ is invariant under $\mathcal{S}$.

If g is not bounded below on U, we can write U as a disjoint union of the sets U_n, where

$$U_n = \left\{x \in U : \frac{1}{n+1} < g(x) \leq \frac{1}{n}\right\}$$

Now g is bounded below on each U_n. Just as in case (i), we shall obtain $\left\langle S\chi_{U_n}, \chi_{U^c}\right\rangle = 0$ for all $S \in \mathcal{S}$. But $\chi_U = \Sigma\chi_{U_n}$. This will give $\langle S\chi_U, \chi_{U^c}\rangle = 0$ and we are in case (i). □

Lemma 3.4.6 *Suppose $\mathcal{S}$ is a direct sum of r nonnegative, indecomposable semigroups $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r$ so that each member of $\mathcal{S}$ has a block diagonal representation*

$$\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix},$$

where $S_i \in \mathcal{S}_i$, $i = 1, \ldots, r$ with respect to some decomposition of $\mathcal{L}^2(\mathcal{X})$, say

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \cdots \oplus \mathcal{L}^2(\mathcal{X}_r).$$

Then every $\mathcal{M} \in \mathcal{L}at'\mathcal{S}$ is of the form $\mathcal{M} = \oplus_{i=1}^{r} \epsilon_i \mathcal{L}^2(\mathcal{X}_i)$ where each ϵ_i is either 0 or 1.

Proof. Obviously, $\mathcal{L}^2(\mathcal{X}_i) \in \mathcal{L}at'\mathcal{S}$ for every $i = 1, \ldots, r$. Further, each $\mathcal{S}_i$ being indecomposable, $\mathcal{L}^2(\mathcal{X}_i)$ is a minimalstandard subspace in $\mathcal{L}at'\mathcal{S}$ in the sense that $\mathcal{S}$ has no nonzero standard invariant subspace properly contained in it. Now let $\mathcal{M} \in \mathcal{L}at'\mathcal{S}$. We first show that if a nonzero, nonnegative f is in $\mathcal{M}$ such that *supp* $f = \mathcal{X}_i$ for some i, then $\mathcal{L}^2(\mathcal{X}_i) \subseteq \mathcal{M}$. Suppose $\mathcal{M} = \mathcal{L}^2(U)$ for some Borel subset U of $\mathcal{X}$ of positive measure. It is enough to prove that $\mathcal{X}_i \subseteq U$ upto a null set, or equivalently, $\mu(U^c \cap \mathcal{X}_i) = 0$. Suppose not, in which case $\mu(U^c \cap \mathcal{X}_i) > 0$. Now $f \in \mathcal{M}$ implies that $f = 0$ *a.e.* on U^c, and in particular, $f = 0$ *a.e.* on $U^c \cap \mathcal{X}_i$ which is contained in $\mathcal{X}_i$ *i.e.*, f is zero *a.e* on a subset of $\mathcal{X}_i$ of positive measure which is not possible as *supp* $f = \mathcal{X}_i$. Therefore, we must have $\mu(U^c \cap \mathcal{X}_i) = 0$ and this proves the desired result.

Next, observe that we can write

$$\mathcal{M} = \mathcal{L}^2(U_1) \oplus \cdots \oplus \mathcal{L}^2(U_r),$$

where $U_i = U \cap \mathcal{X}_i$. Let $f_i = \chi_{U_i}$, then the vector $f = \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}$ and by our assumption $\mathcal{S}f \in \mathcal{M}$ where

$$\mathcal{S}f = \left\{ \begin{pmatrix} 0 \\ \vdots \\ Sf_i \\ \vdots \\ 0 \end{pmatrix} : S \in \mathcal{S}_i \right\}.$$

Define

$$\epsilon_i = 0 \quad \text{if } f_i \text{ is zero}$$
$$\epsilon_i = 1 \quad \text{if } f_i \text{ is nonzero .}$$

To complete the proof, we must show that whenever $\epsilon_i = 1$, we have $\mathcal{L}^2(\mathcal{X}_i) \subseteq \mathcal{M}$. Now $\mathcal{S}_i$ is a band acting on $\mathcal{L}^2(\mathcal{X}_i)$ and $f_i \in \mathcal{L}^2(U_i)$. By Lemma 3.3.14, we obtain a positive vector, say g_i, in $\mathcal{L}^2(\mathcal{X})$ which is also a limit of nonnegative linear combinations of the members of $\{\mathcal{S}_i f_i\}$. Consider the vector $g = \begin{pmatrix} 0 \\ \vdots \\ g_i \\ \vdots \\ 0 \end{pmatrix}$. Then $g \in \mathcal{M}$ and *supp* $g = \mathcal{X}_i$. Therefore, by what we have proved above, we obtain $\mathcal{L}^2(\mathcal{X}_i) \subseteq \mathcal{M}$.
□

Theorem 3.4.7 *A direct sum of r maximal, indecomposable, nonnegative rank-one bands is a maximal band of constant rank r.*

Proof. For $r = 1$, the result is obvious. Therefore let $r > 1$. Suppose $S_1, S_2, \cdots, S_r$ are r maximal, indecomposable, nonnegative rank-one bands and consider their direct sum. Every member S of $\mathcal{S}$ is of the form

$$\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix},$$

where $S_i \in \mathcal{S}_i$, $i = 1, 2. \cdots, r$. Also suppose that this representation of the members of $\mathcal{S}$ is with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \cdots \oplus \mathcal{L}^2(\mathcal{X}_r),$$

where $\mathcal{X}_1, \cdots, \mathcal{X}_r$ are Borel subsets of $\mathcal{X}$ of positive measure.

If $\mathcal{S}$ is not maximal, then let $\mathcal{S}'$ be a band properly containing $\mathcal{S}$ and having constant rank r. Now observe that $\mathcal{S}$ is a full band. Therefore, $\mathcal{S}'$ is full too. By part (i) of Theorem 3.4.3, $\mathcal{S}'$ is a direct sum of r rank-one, indecomposable, nonnegative bands, say, $\mathcal{S}'_1, \mathcal{S}'_2, \cdots, \mathcal{S}'_r$. Now $\mathcal{L}at'\mathcal{S}' \subseteq \mathcal{L}at'\mathcal{S}$. By the previous lemma, the cardinality of both $\mathcal{L}at'\mathcal{S}$ and $\mathcal{L}at'\mathcal{S}'$ is the same which is 2^r. Therefore, we must have $\mathcal{L}at'\mathcal{S} = \mathcal{L}at'\mathcal{S}'$. Thus we can rearrange the spaces $\mathcal{L}^2(\mathcal{X}_i)$ in the direct sum above to obtain a new decomposition of $\mathcal{L}^2(\mathcal{X})$ so that $\mathcal{S}_i \subseteq \mathcal{S}'_i$. But since the bands $\mathcal{S}_i$ are maximal, we have $\mathcal{S}'_i = \mathcal{S}_i$ for each i. Hence $\mathcal{S}$ is maximal. □

Theorem 3.4.3 and the Remark 3.4.4 can be combined to give the following characterization of maximal nonnegative bands of constant finite rank.

Theorem 3.4.8 *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ of constant finite rank r.*

(i) If $\mathcal{S}$ is full, then $\mathcal{S}$ is maximal if and only if

$$\mathcal{S} = \left\{ \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix} : S_i \in \mathcal{S}_i, i = 1, 2, \ldots, r \right\},$$

where $\mathcal{S}_i$ is a maximal rank-one indecomposable band for each i.

(ii) In general, if $\mathcal{S}$ is maximal, then

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix} : E \in \mathcal{S}_0, X \in \mathcal{X}, Y \in \mathcal{Y} \right\},$$

where $\mathcal{S}_0$ is a direct sum as in part (i) and $\mathcal{X}, \mathcal{Y}$ are the entire sets of nonnegative operators on appropriate spaces.

We shall see in Theorem 3.5.6 in the next section that in special cases, a nonnegative band with constant infinite rank is decomposable.

3.5 Some conditions leading to decomposability of infinite-rank, nonnegative bands

Definition 3.5.1 *Suppose $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ and $\{\mathcal{N}_j\}_{j\in\mathcal{J}}$ are collections of mutually orthogonal subspaces of $\mathcal{L}^2(\mathcal{X})$ whose direct sum equals $\mathcal{L}^2(\mathcal{X})$. Then $\{\mathcal{M}_i\}_i$ is said to be a* **refinement** *of $\{\mathcal{N}_j\}_j$ if each $\mathcal{N}_j$ can be expressed as a direct sum of a (finite or infinite) subcollection of $\{\mathcal{M}_i\}_i$.*

In the definition above, $\{\mathcal{N}_j\}_j$ is called a **coarsening** *of $\{\mathcal{M}_i\}_i$*

Definition 3.5.2 *A nonnegative operator A in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ will be called* **nondegenerate** *if A is full and there is no continuous part in any maximal chain in $\mathcal{L}at'A$.*

Lemma 3.5.3 *Let A be a full nonnegative idempotent in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, and C any maximal chain in $\mathcal{L}at'A$. Then there cannot be any nontrivial gaps in C with corresponding compressions of A equal to zero. If A is nondegenerate, then it can be expressed as a direct sum of countably many positive idempotents of rank one.*

Proof. Let A and C be as described in the statement. It was shown in the proof of Theorem 3.3.13 that each nontrivial gap in $\mathcal{C}$ is a reducing subspace for A and thus the compression to any such gap cannot be zero for this will contradict the fullness of A. In fact each nonzero compression to a gap is a positive idempotent of rank one. Again by Theorem 3.3.13, if A is nondegenerate, then it is a direct sum of positive idempotents of rank one which are countable because of the separability of $\mathcal{L}^2(\mathcal{X})$. □

Lemma 3.5.4 *If A, B are positive operators on $\mathcal{L}^2(\mathcal{X})$ and S is a nonzero, nonnegative operator on $\mathcal{L}^2(\mathcal{X})$, then ASB is positive.*

Proof. Let f be a nonzero, nonnegative vector in $\mathcal{L}^2(\mathcal{X})$. Since B is positive, $Bf > 0$. Also, S being nonzero and nonnegative, $SBf \neq 0$ (by Proposition 3.2.1). Thus $0 \neq SBf \geq 0$ because $S \geq 0$. But A is positive. Therefore, $A(SBf) > 0$ which implies that ASB is positive. □

Lemma 3.5.5 *Let A be a nondegenerate idempotent on $\mathcal{L}^2(\mathcal{X})$ such that with respect to some decomposition*

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \oplus \mathcal{L}^2(\mathcal{X}_3) \oplus \cdots,$$

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where each A_{ij} is either zero or positive. Then A has a block diagonalization with positive diagonal blocks with respect to some decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{W}_1) \oplus \mathcal{L}^2(\mathcal{W}_2) \oplus \cdots,$$

where the collection $\{\mathcal{L}^2(\mathcal{W}_i)\}_i$ is a coarsening of the collection $\{\mathcal{L}^2(\mathcal{X}_i)\}_i$.

Proof. If rank(A)=1, then the fullness of A implies that A is positive and therefore $\{\mathcal{L}^2(\mathcal{X}_i)\}_i$ itself is the required coarsening. Therefore, assume that rank $(A) > 1$ in which case A is decomposable. Thus it has a nontrivial invariant standard subspace, say $\mathcal{L}^2(\mathcal{Y})$, where $\mathcal{Y}$ is a Borel subset of $\mathcal{X}$ such that $\mu(\mathcal{Y}).\mu(\mathcal{Y}^c) > 0$. We can assume, with no loss of generality, that the sets $\mathcal{X}_i$ are disjoint so that $\mathcal{X} = \dot{\cup}_i \mathcal{X}_i$. Now we can write

$$\mathcal{Y} = \mathcal{Y}_1 \,\dot{\cup}\, \mathcal{Y}_2 \,\dot{\cup}\, \ldots$$

where $\mathcal{Y}_i = \mathcal{Y} \cap \mathcal{X}_i$. Let $J = \{j \in \mathbb{N} : \mu(\mathcal{Y}_j) > 0\}$. Then J is nonempty, for otherwise $\mathcal{L}^2(\mathcal{Y}) = \{0\}$. We rearrange $\{\mathcal{X}_i\}$ to obtain

$$\mathcal{X} = (\dot{\cup}_{j \in J} \mathcal{X}_j) \ \cup\ (\dot{\cup}_{j \notin J} \mathcal{X}_j).$$

Suppose

$$A = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

with respect to

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\dot{\cup}_{j \in J} \mathcal{X}_j) \oplus \mathcal{L}^2(\dot{\cup}_{j \notin J} \mathcal{X}_j). \tag{3.10}$$

We shall prove that $G = 0$. Clearly, any vector in $\mathcal{L}^2(\mathcal{Y})$ is of the form $\begin{pmatrix} f \\ 0 \end{pmatrix}$, for some $f \in \mathcal{L}^2(\dot{\cup}_{j \in J} \mathcal{X}_j)$ with respect tc (3.10). Since for each $i \in J$, $\mu(\mathcal{Y}_i) > 0$, we can select a nonzero, nonnegative function f_i in $\mathcal{L}^2(\mathcal{X}_i)$ with *supp* $f_i = \mathcal{Y}_i$ such that $\begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}$ is a vector in $\mathcal{L}^2(\dot{\cup}_{j \in J} \mathcal{X}_j) = \oplus_{j \in J} \mathcal{L}^2(\mathcal{X}_j)$. Write $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}$. Now

$$A \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} Ef \\ Gf \end{pmatrix} \in \mathcal{L}^2(\mathcal{Y})$$

by the invariance of $\mathcal{L}^2(\mathcal{Y})$. The form of vectors in $\mathcal{L}^2(\mathcal{Y})$ gives that $Gf = 0$. Let

$$\begin{pmatrix} G_{11} & G_{12} & \cdots \\ G_{21} & G_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

be the block matrix form of $G : \oplus_{j\in J}\mathcal{L}^2(\mathcal{X}_j) \to \oplus_{j\notin J}\mathcal{L}^2(\mathcal{X}_j)$. Now $Gf = 0$ implies that

$$G_{i1}f_1 + G_{i2}f_2 + \cdots = 0 \text{ for each } i = 1, 2, \ldots,$$

which by nonnegativity of G_{ij} further implies that

$$G_{ij}f_j = 0 \text{ for each } i, j = 1, 2, \ldots$$

If G_{ij} is nonzero for some (i, j), then it is positive and f_j being nonzero, nonnegative, we shall obtain $G_{ij}f_j > 0$ which is not true. Therefore

$$G_{ij} = 0 \text{ for every } i, j$$

and hence $G = 0$. Thus $A = \begin{pmatrix} E & F \\ 0 & H \end{pmatrix}$. This shows that $\oplus_{j\in J}\mathcal{L}^2(\mathcal{X}_j)$ is invariant under A. Since A is full, we have $F = 0$. We now claim that $\mathcal{L}^2(\mathcal{Y}) = \oplus_{j\in J}\mathcal{L}^2(\mathcal{X}_j)$. Work'ng with the same f as above, we have

$$A\begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & H \end{pmatrix}\begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} Ef \\ 0 \end{pmatrix} \in \mathcal{L}^2(\mathcal{Y}).$$

Suppose $E : \oplus_{j\in J}\mathcal{L}^2(\mathcal{X}_j) \to \oplus_{j\in J}\mathcal{L}^2(\mathcal{X}_j)$ has the block matrix form

$$\begin{pmatrix} E_{11} & E_{12} & \cdots \\ E_{21} & E_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}.$$

Then

$$Ef = \begin{pmatrix} E_{11} & E_{12} & \cdots \\ E_{21} & E_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} E_{11}f_1 + E_{12}f_2 + \cdots \\ E_{21}f_1 + E_{22}f_2 + \cdots \\ \vdots \end{pmatrix}.$$

Since A is full, each of its rows contains at least one positive block. This coupled with the fact that each f_i is a nonzero, nonnegative function in $\mathcal{L}^2(\mathcal{X}_i)$, implies that each component of Ef is a positive function in $\mathcal{L}^2(\mathcal{X}_i)$; in other words, *supp* $Ef = \dot{\cup}_{j\in J}\mathcal{X}_j$. But $Ef \in \mathcal{L}^2(\mathcal{Y})$. Therefore, we must have

$$\mathcal{L}^2(\mathcal{Y}) = \mathcal{L}^2(\dot{\cup}_{j\in J}\mathcal{X}_j) = \oplus_{j\in J}\mathcal{L}^2(\mathcal{X}_j).$$

As $\mathcal{L}^2(\mathcal{Y})$ is nontrivial, $\oplus_{j\notin J}\mathcal{L}^2(\mathcal{X}_j)$ is nontrivial *i.e.*, J is a proper subset of $\mathbb{N}$.

Since A is nondegenerate, by Lemma 3.5.3, there exists a decomposition of $\mathcal{L}^2(\mathcal{X})$, say

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{W}_1) \oplus \mathcal{L}^2(\mathcal{W}_2) \oplus \cdots,$$

with respect to which A has a block diagonal form

$$\begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \end{pmatrix},$$

where each $A_i : \mathcal{L}^2(\mathcal{W}_i) \to \mathcal{L}^2(\mathcal{W}_i)$ is a positive idempotent of rank one. Clearly, each $\mathcal{L}^2(\mathcal{W}_i)$ is a standard subspace invariant under A. Therefore, by what we have proved above, each $\mathcal{L}^2(\mathcal{W}_i)$ is a direct sum of a subcollection of $\{\mathcal{L}^2(\mathcal{X}_i)\}_i$. Hence $\{\mathcal{L}^2(\mathcal{W}_i)\}_i$ is a coarsening of $\{\mathcal{L}^2(\mathcal{X}_i)\}_i$ such that with respect to

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{W}_1) \oplus \mathcal{L}^2(\mathcal{W}_2) \oplus \cdots,$$

A has a block diagonalization with positive diagonal blocks. □

Theorem 3.5.9 below answers Question 3.3.16 affirmatively under the additional hypothesis of finiteness; Example 3.3.17 shows the necessity of this hypothesis. But we first consider a finite, nonnegative infinite-rank band whose members are nondegenerate and prove that under this special condition of nondegeneracy, the band has a block diagonalization.

Theorem 3.5.6 *A nonnnegative finite band in which every member is nondegenerate and has infinite rank is decomposable. Furthermore, it has infinitely many mutually orthogonal standard invariant subspaces whose direct sum is $\mathcal{L}^2(\mathcal{X})$; equivalently, the band is block diagonalizable.*

Proof. Let $\mathcal{S}$ be a band with k elements, say $S_1, S_2, \cdots, S_k$ such that each S_i is nondegenerate and is of infinite rank. Consider S_1. By Lemma 3.5.3, there is a collection $\{\mathcal{M}_i^{(1)}\}_{i=1}^{\infty}$ of standard subspaces of $\mathcal{L}^2(\mathcal{X})$ such that with respect to

$$\mathcal{L}^2(\mathcal{X}) = \oplus_{i=1}^{\infty}\ \mathcal{M}_i^{(1)},$$

$$S_1 = \begin{pmatrix} S_{11}^{(1)} & & & & \\ & S_{22}^{(1)} & & & \\ & & \ddots & & \\ & & & S_{ii}^{(1)} & \\ & & & & \ddots \end{pmatrix},$$

where each $S_{ii}^{(1)} : \mathcal{M}_i^{(1)} \to \mathcal{M}_i^{(1)}$ is a positive idempotent of rank one.

Next, consider $S_1 S_2 S_1$ where

$$S_2 = \begin{pmatrix} S_{11}^{(2)} & S_{12}^{(2)} & \cdots \\ S_{21}^{(2)} & S_{22}^{(2)} & \cdots \\ \vdots & \vdots & \cdots \end{pmatrix}$$

with respect to the decomposition $\mathcal{L}^2(\mathcal{X}) = \oplus_{i=1}^{\infty}\ \mathcal{M}_i^{(1)}$. Then

$$S_1 S_2 S_1 = \begin{pmatrix} S_{11}^{(1)} S_{11}^{(2)} S_{11}^{(1)} & S_{11}^{(1)} S_{12}^{(2)} S_{22}^{(1)} \cdots \\ S_{22}^{(1)} S_{21}^{(2)} S_{11}^{(1)} & S_{22}^{(1)} S_{22}^{(2)} S_{22}^{(1)} \cdots \\ \vdots & \vdots \end{pmatrix}.$$

By Lemma 3.5.4, since each $S_{ii}^{(1)}$ is positive, an arbitrary block $S_{jj}^{(1)} S_{jk}^{(2)} S_{kk}^{(1)}$ in $S_1 S_2 S_1$ is zero or positive according as $S_{jk}^{(2)}$ is zero or nonzero. Now, by hypothesis $S_1 S_2 S_1$ is nondegenerate. Therefore, by Lemma 3.5.5, there exists a coarsening of $\{\mathcal{M}_i^{(1)}\}_{i=1}^{\infty}$ which we denote by $\{\mathcal{M}_i^{(2)}\}_{i=1}^{\infty}$ such that with respect to the decomposition $\mathcal{L}^2(\mathcal{X}) = \oplus_{i=1}^{\infty} \mathcal{M}_i^{(2)}$, $S_1 S_2 S_1$ is a direct sum of positive rank-one idempotents. Since $\{\mathcal{M}_i^{(2)}\}_i$ is a coarsening of $\{\mathcal{M}_i^{(1)}\}_i$, S_1 is a direct sum of idempotents which are full (because each is a direct sum of positive idempotents) with respect to $\mathcal{L}^2(\mathcal{X}) = \oplus_i \mathcal{M}_i^{(2)}$. Suppose

$$S_1 = \begin{pmatrix} S'_{11} & & \\ & S'_{22} & \\ & & \ddots \end{pmatrix}$$

and

$$S_2 = \begin{pmatrix} S''_{11} & S''_{12} & S''_{13} & \cdots \\ S''_{21} & S''_{22} & S''_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

with respect to $\mathcal{L}^2(\mathcal{X}) = \oplus_i \mathcal{M}_i^{(2)}$ so that

$$S_1 S_2 S_1 = \begin{pmatrix} S'_{11} S''_{11} S'_{11} & S'_{11} S''_{12} S'_{22} & \cdots \\ S'_{22} S''_{21} S'_{11} & S'_{22} S''_{22} S'_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}.$$

Then we know that the nondiagonal blocks are zero and the diagonal blocks are positive idempotents. But any nondiagonal block is of the form $S'_{ii} S''_{ij} S'_{jj}$ for $i \neq j$. Since S'_{ii} and S'_{jj} are full, therefore $S'_{ii} S''_{ij} S'_{jj} = 0$ implies that $S''_{ij} = 0$. Thus both S_1 and S_2 are diagonal with respect to the decomposition $\mathcal{L}^2(\mathcal{X}) = \oplus_i \mathcal{M}_i^{(2)}$.

Next, consider $(S_1 S_2 S_1) S_3 (S_1 S_2 S_1)$. As reasoned above, there shall exist a coarsening $\{\mathcal{M}_i^{(3)}\}_i$ of $\{\mathcal{M}_i^{(2)}\}_i$ such that with respect to the decomposition $\mathcal{L}^2(\mathcal{X}) = \oplus_i \mathcal{M}_i^{(3)}$, S_1, S_2, and S_3 are diagonal. Proceeding like this, after k steps, we shall arrive at a direct sum decomposition $\oplus_{i=1}^{\infty} \mathcal{M}_i^{(k)}$ of $\mathcal{L}^2(\mathcal{X})$ with respect to which each S_i has zero nondiagonal blocks . This proves that $\mathcal{S}$ is decompo~able, in fact it is block diagonalizable with respect to $\mathcal{L}^2(\mathcal{X}) = \oplus_i \mathcal{M}_i^{(k)}$. □

Next, we prove the result that a nonnegative finite band with constant infinite rank is decomposable. For this, we need a couple of lemmas.

Lemma 3.5.7 *If a band $\mathcal{S}$ has more than one member, then there exists $P \in \mathcal{S}$ such that $P\mathcal{S}P$ is a proper subset of $\mathcal{S}$.*

Proof. Suppose there does not exist any P in $\mathcal{S}$ satisfying the required condition. Then $P\mathcal{S}P = \mathcal{S}\ \forall\ P \in \mathcal{S}$. We claim that this implies $PSP = S$ for all P and S in $\mathcal{S}$. If $S \in \mathcal{S}$, then $S = PS_1P$ for some $S_1 \in \mathcal{S}$, *i.e.*, $PSP = PS_1P =$ This further gives that $PSPS = S$ and $SPSP = S$, *i.e.*, $PS = S = SP$ for all P and S in $\mathcal{S}$. Thus

$S = P$ for all P, S in $\mathcal{S}$. Hence $\mathcal{S}$ is a singleton which contradicts the hypothesis. Therefore, there exists some $P \in \mathcal{S}$ such that $P\mathcal{S}P$ is properly contained in $\mathcal{S}$. □

Lemma 3.5.8 *If a collection $\mathcal{S} \subseteq \mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ contains a member P which is a full idempotent such that $P\mathcal{S}P$ is decomposable, then so is $\mathcal{S}$.*

Proof. Since $P\mathcal{S}P$ is decomposable, there exists a decomposition of $\mathcal{L}^2(\mathcal{X})$, say

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2)$$

with respect to which every member T of $P\mathcal{S}P$ has the block matrix form

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}.$$

As P is a member of $P\mathcal{S}P$, it also has a block matrix form with respect to the above decomposition, say

$$\begin{pmatrix} P_1 & X \\ 0 & P_2 \end{pmatrix}.$$

But since P is a full idempotent, by Lemma 3.3.12, we get $X = 0$. Now for any $S \in \mathcal{S}$, let $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ be the block matrix form of S with respect to the given decomposition. Then

$$\begin{aligned} PSP &= \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \\ &= \begin{pmatrix} P_1 S_{11} P_1 & P_1 S_{12} P_2 \\ P_2 S_{21} P_1 & P_2 S_{22} P_2 \end{pmatrix}. \end{aligned}$$

By decomposability of $P\mathcal{S}P$, we have $P_2 S_{21} P_1 = 0 \ \forall\ S \in \mathcal{S}$. But P_1 and P_2 are full because P is full and therefore, by Lemma 3.3.12 we get $S_{21} = 0 \ \forall\ S \in \mathcal{S}$. Hence $\mathcal{S}$ is decomposable. □

Theorem 3.5.9 *A nonnegative finite band in which every member has infinite rank is decomposable.*

Proof. Let $\mathcal{S}$ be a nonnegative finite band with constant infinite rank. We shall prove the theorem by induction on $|\mathcal{S}|$, the cardinality of $\mathcal{S}$. Suppose $|\mathcal{S}| = n$. Assume that every nonnegative band with constant infinite rank which has cardinality less than n is decomposable.

Consider $\mathcal{S}$. If $\mathcal{S}$ is a singleton, then by Theorem 3.3.5, it is decomposable. Therefore, assume that $|\mathcal{S}| > 1$ By Lemma 3.5.7, there exists $P \in \mathcal{S}$ such that $P\mathcal{S}P$ is a proper subset of $\mathcal{S}$. By Theorem 3.3.13 (b), P has a block matrix form

$$\begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to some decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \oplus \mathcal{L}^2(\mathcal{X}_3),$$

where $E : \mathcal{L}^2(\mathcal{X}_2) \to \mathcal{L}^2(\mathcal{X}_2)$ is full. For any $S \in \mathcal{S}$, let

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$$

be its block matrix representation with respect to the above-mentioned decomposition of the space. Then

$$PSP = \begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & XE(S_{21}X + YS_{31}X + S_{22} + YS_{32})E & XE(S_{21}X + YS_{31}X + S_{22} + YS_{32})EY \\ 0 & E(S_{21}X + YS_{31}X + S_{22} + YS_{32})E & E(S_{21}X + YS_{31}X + S_{22} + YS_{32})EY \\ 0 & 0 & 0 \end{pmatrix}.$$

Let

$$\mathcal{T} = \{E(S_{21}X + YS_{31}X + S_{22} + YS_{32})E : PSP \in P\mathcal{S}P\}.$$

Observe that $\mathcal{T}$ is a nonnegative band such that $|\mathcal{T}| \leq |P\mathcal{S}P| < |\mathcal{S}|$. Then by the inductive hypothesis, $\mathcal{T}$ is decomposable. Therefore, there exist Borel subsets E, F of $\mathcal{X}$ with $\mu(E).\mu(F) > 0$ such that $\langle T\chi_E, \chi_F\rangle = 0 \ \forall\, T \in \mathcal{T}$.

$$\Rightarrow \ \langle E(S_{21}X + YS_{31}X + S_{22} + YS_{32})E\chi_E, \chi_F\rangle = 0 \ \forall\, S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \in \mathcal{S}$$

$$\begin{aligned} \Rightarrow \ \langle ES_{21}XE\chi_E, \chi_F\rangle + \langle EYS_{31}XE\chi_E, \chi_F\rangle + \langle ES_{22}E\chi_E, \chi_F\rangle + \\ \langle EYS_{32}E\chi_E, \chi_F\rangle = 0 \ \forall\, S \in \mathcal{S}. \end{aligned}$$

Since all the operators are nonnegative, this gives that

$$\langle ES_{22}E\chi_E, \chi_F\rangle = 0,$$

in other words, the collection $\{ES_{22}E : S \in \mathcal{S}\}$ is decomposable. Also this collection contains E which is a full idempotent. Therefore, by Lemma 3.5.8, the collection

$$\left\{ S_{22} : S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \in \mathcal{S} \right\}$$

is decomposable. Just as in the proof of Theorem 3.3.14, we conclude that $\mathcal{S}$ is decomposable. □

Corollary 3.5.10 *A finitely generated nonnegative band in which every member has infinite rank is decomposable.*

Proof. This is a consequence of the interesting result on abstract bands due to Green and Rees [4]: every finitely generated band is finite. □

Corollary 3.5.11 *Every finitely generated nonnegative infinite-rank band $\mathcal{S}$ has the property that any maximal standard block triangularization of $\mathcal{S}$ is such that the compression of $\mathcal{S}$ to each nonzero gap constitutes a nonnegative finite band with at least one element of rank one in it.*

Proof. Same as in the finite-dimensional case (cf. Theorem 1.3.7). □

Chapter 4

A geometric characterization of maximal, nonnegative, indecomposable bands of constant finite rank

We shall borrow the notation and the terminology from the preceding chapter to define our Hilbert space $\mathcal{L}^2(\mathcal{X})$, the space $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ and all other terms used in this chapter.

We have proved in Theorem 3.4.8 that every maximal, indecomposable, nonnegative band with constant finite rank r, say which is full is the direct sum of maximal, indecomposable, nonnegative rank-one bands. Thus the structure of such bands is completely determined if the structure of maximal, constant rank-one bands is known. In this chapter, we shall obtain a geometric characterization of maximal, indecomposable, nonnegative constant rank-one bands.

Before we embark on this task, we would like to mention for reasons which will be apparent later that our field of scalars $\mathbf{C}$ will be replaced with $\mathbf{R}$.

We know that a nonzero, nonnegative rank-one operator in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is of the form $u \otimes v$, where u, v are nonzero, nonnegative functions in $\mathcal{L}^2(\mathcal{X})$ and $(u \otimes v)f = \langle f, v \rangle u$ for all $f \in \mathcal{L}^2(\mathcal{X})$. Further, for $u \otimes v$ to be an idempotent, u, v must satisfy the equation $\langle u, v \rangle = 1$.

Thus, if $\mathcal{S}$ is a nonnegative band of rank-one operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, then we can find sets $\mathcal{U}, \mathcal{V}$ in the nonnegative cone of $\mathcal{L}^2(\mathcal{X})$, *viz.* $\mathcal{K}$, so that $\mathcal{S} \subseteq \mathcal{U} \otimes \mathcal{V}$, where

$$\mathcal{U} \otimes \mathcal{V} = \{u \otimes v : u \in \mathcal{U},\ v \in \mathcal{V}\}$$

and

$$\langle u, v \rangle = 1 \text{ for all } u \in \mathcal{U} \text{ and for all } v \in \mathcal{V}.$$

(By the nonnegative cone of $\mathcal{L}^2(\mathcal{X})$, we mean the set $\mathcal{K} = \{f \in \mathcal{L}^2(\mathcal{X}) : f \geq 0\}$).

Further, if $\mathcal{S}$ is maximal, then we must have $\mathcal{S} = \mathcal{U} \otimes \mathcal{V}$ for some $\mathcal{U}, \mathcal{V}$ of the kind mentioned above. We wish to find the general form of $\mathcal{U}$ and $\mathcal{V}$ for a maximal, nonnegative, indecomposable band $\mathcal{S}$ of rank-one operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$.

We observe that if $u_1, u_2 \in \mathcal{U}$, then $\langle tu_1 + (1-t)u_2, v \rangle = 1$ for $0 \leq t \leq 1$ and for all $v \in \mathcal{V}$. Thus for a maximal $\mathcal{U} \otimes \mathcal{V}$, $\mathcal{U}$ must contain all the convex combinations of its members too. Furthermore, it is clear that $\mathcal{U}$ is closed (in norm). Also, we cannot have every member of $\mathcal{U}$ equal to zero *a.e.* on any Borel subset of $\mathcal{X}$ with positive measure, for if, there were such a set, say $W \subseteq \mathcal{X}$ such that $u = 0$ *a.e.* in W for every $u \in \mathcal{U}$, then for any $u \in \mathcal{U}$ and $v \in \mathcal{V}$,

$$\begin{aligned} \langle (u \otimes v)f, \chi_W \rangle &= \int_W (u \otimes v) f(x)\ \mu(dx) \\ &= \int_W \langle f, v \rangle u(x)\ \mu(dx) \\ &= \langle f, v \rangle \int_W u(x)\ \mu(dx) = 0 \text{ for all } f \in \mathcal{L}^2(\mathcal{X}) \end{aligned}$$

which by Lemma 3.2.5 implies that $\mathcal{U} \otimes \mathcal{V}$ is decomposable. This together with the fact that $\mathcal{U}$ is closed and convex allows us to assume with no loss of generality that

$\mathcal{U}$ has a positive element. Let us pick one such element in $\mathcal{U}$, say u_0, *i.e.* $u_0 > 0$ *a.e.* on $\mathcal{X}$.

Now, any $u \in \mathcal{U}$ satisfies $\langle u, v\rangle = 1$ for all $v \in \mathcal{V}$. In particular, $\langle u_0, v\rangle = 1$ for all $v \in \mathcal{V}$. Thus, for any $u \in \mathcal{U}$,

$$\begin{aligned} & \langle u, v\rangle = \langle u_0, v\rangle \text{ for all } v \in \mathcal{V} \\ \Rightarrow\ & \langle u - u_0, v\rangle = 0 \text{ for all } v \in \mathcal{V} \\ \Rightarrow\ & u - u_0 \in \mathcal{V}^{\perp} \\ \Rightarrow\ & u \in u_0 + \mathcal{V}^{\perp} \text{ for all } u \in \mathcal{U} \\ \Rightarrow\ & \mathcal{U} \subseteq u_0 + \mathcal{V}^{\perp}. \end{aligned}$$

Also, if $v' \in \mathcal{V}^{\perp}$, then for any $v \in \mathcal{V}$,

$$\langle u_0 + v', v\rangle = \langle u_0, v\rangle = 1.$$

Thus, by the maximality of $\mathcal{S}$, we obtain

$$\mathcal{U} = \{u_0 + \mathcal{V}^{\perp}\} \cap \mathcal{K}. \tag{4.1}$$

By the same reasoning, we can find a positive vector v_0 in $\mathcal{V}$ and obtain

$$\mathcal{V} = \{v_0 + \mathcal{U}^{\perp}\} \cap \mathcal{K}. \tag{4.2}$$

Next, we show that if $\mathcal{U}$ and $\mathcal{V}$ are given as in (4.1) and (4.2) respectively, for some positive u_0, v_0 and subspaces $\mathcal{W}$, $\mathcal{Z}$, *i.e.*,

$$\mathcal{U} = \{u_0 + \mathcal{W}\} \cap \mathcal{K} \tag{4.3}$$

$$\mathcal{V} = \{v_0 + \mathcal{Z}\} \cap \mathcal{K} \tag{4.4}$$

where $\langle u_0, v_0\rangle = 1$, $\mathcal{W} = \{v_0 + \mathcal{Z}\}^{\perp}$ and $\mathcal{Z} = \{u_0 + \mathcal{W}\}^{\perp}$, then $\mathcal{S} = \mathcal{U} \otimes \mathcal{V}$ is a maximal band of nonnegative rank-one operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$. It is easy to see that

$\mathcal{S}$ forms a nonnegative band of rank-one operators. Suppose $\mathcal{S}$ is contained in a band $\mathcal{S}_0$ of rank-one operators, where

$$\mathcal{S}_0 \subset \mathcal{U}' \otimes \mathcal{V}' = \{u' \otimes v' : u' \in \mathcal{U}', v' \in \mathcal{V}'\},$$

for some sets $\mathcal{U}', \mathcal{V}' \subseteq \mathcal{K}$ and $\langle u', v' \rangle = 1$ for all $u' \in \mathcal{U}'$ and $v' \in \mathcal{V}'$.
Let $S = p \otimes q \in \mathcal{S}_0$. Since $\mathcal{S}_0$ is a semigroup, $u \otimes v \cdot p \otimes q \in \mathcal{S}_0$ for all $u \otimes v \in \mathcal{S}$. Therefore, for any $f \in \mathcal{L}^2(\mathcal{X})$,

$$\begin{aligned}
(u \otimes v)(p \otimes q)f &= (u \otimes v)\langle f, q\rangle p \\
&= \langle f, q\rangle ((u \otimes v)p) \\
&= \langle f, q\rangle \langle p, v\rangle u \\
&= \langle p, v\rangle (\langle f, q\rangle u) \\
&= \langle p, v\rangle (u \otimes q)f \\
&= (\langle p, v\rangle u \otimes q)f
\end{aligned}$$

i.e. $(u \otimes v)(p \otimes q) = \langle p, v\rangle u \otimes q$. Thus $(u \otimes v)(p \otimes q)$ is an idempotent if and only if $\langle \langle p, v\rangle u, q\rangle = 1$, *i.e.*, if and only if $\langle p, v\rangle \langle u, q\rangle = 1$. With no loss of generality, we can assume that $\langle p, v\rangle = 1$ and $\langle u, q\rangle = 1$ (for if, $\langle p, v\rangle = \alpha(\neq 1)$, then $\langle u, q\rangle = \frac{1}{\alpha}$, so that we can write $s = \frac{1}{\alpha}p \otimes \alpha q = p' \otimes q'$ where $p' = \frac{1}{\alpha}p$, $q' = \alpha q$ and $\langle p', v\rangle = 1$, $\langle u, q'\rangle = 1$).
Now

$$\begin{aligned}
& \langle u, q\rangle = 1 \text{ for all } u \in \mathcal{U} \\
\Rightarrow\ & \langle u_0, q\rangle = 1 \text{ and } \langle u_0 + w, q\rangle = 1\ \forall\ w \in \mathcal{W} \\
\Rightarrow\ & \langle u_0, q\rangle = 1 \text{ and } \langle w, q\rangle = 0\ \forall\ w \in \mathcal{W} \\
\Rightarrow\ & \langle u_0 + w, q - v_0\rangle = 0\ \forall\ w \in \mathcal{W} \\
\Rightarrow\ & q - v_0 \in \{u_0 + \mathcal{W}\}^{\perp} = \mathcal{Z} \\
\Rightarrow\ & q \in v_0 + \mathcal{Z} = \mathcal{V}
\end{aligned}$$

Similarly, we can show that $p \in \mathcal{U}$. Thus $p \otimes q \in \mathcal{U} \otimes \mathcal{V} = \mathcal{S}$ which implies that $\mathcal{S}_0 \subseteq \mathcal{S}$. Hence $\mathcal{S}$ is maximal.

Next, we would like to see which subspaces $\mathcal{W}$ and $\mathcal{Z}$ give rise to maximal indecomposable bands as in (4.3) and (4.4). Suppose there is some $w \in \mathcal{W}$ such that

$w \geq 0$ or $w \leq 0$. Consider the case when $w \geq 0$ and the support of w is a Borel subset of positive measure. Then

$$\begin{aligned} & \langle w, v\rangle = 0 \ \forall\, v \in \mathcal{V} \\ \Rightarrow \ & \int_{\mathcal{X}} w(x)v(x)\ \mu(dx) = 0 \ \forall\, v \in \mathcal{V} \\ \Rightarrow \ & w(x)v(x) = 0 \text{ } a.e. \text{ on } \mathcal{X} \ \ \forall\, v \in \mathcal{V} \text{ (as } w, v \geq 0) \end{aligned}$$

Let $\mathcal{N} = supp\ w$, then $v = 0$ *a.e.* on $\mathcal{N}\ \forall\, v \in \mathcal{V}$. By the same argument given once before, this will yield decomposability of $\mathcal{S}$ which is not true. Similarly, if $w \leq 0$ with positive-measured support, we shall find $\mathcal{S}$ to be decomposable. This shows that every vector of $\mathcal{W}$ must necessarily be a "mixed" vector *i.e.*, a vector having positive and negative parts with supports of positive measure. In other words, the space $\mathcal{W}$ intersects $\mathcal{K}$ trivially. Following the same argument, we conclude that $\mathcal{Z} \cap \mathcal{K} = \{0\}$. (We shall call such a space a mixed space).

We summarize the discussion above in the following theorem.

Theorem 4.1.1 *Let $\mathcal{S}$ be a maximal, nonnegative, indecomposable band of rank-one operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$. Denote the positive cone of $\mathcal{L}^2(\mathcal{X})$ by $\mathcal{K}$. Then there exist positive vectors u_0, v_0 in $\mathcal{K}$ with $\langle u_0, v_0\rangle = 1$ and there exist mixed subspaces $\mathcal{W}$, $\mathcal{Z}$ of $\mathcal{L}^2(\mathcal{X})$ with $\mathcal{W} = \{v_0 + \mathcal{Z}\}^\perp$, $\mathcal{Z} = \{u_0 + \mathcal{W}\}^\perp$ such that $\mathcal{S} = \mathcal{U} \otimes \mathcal{V}$, where*

$$\begin{aligned} \mathcal{U} &= \{u_0 + \mathcal{W}\} \cap \mathcal{K} \\ \mathcal{V} &= \{u_0 + \mathcal{Z}\} \cap \mathcal{K}. \end{aligned}$$

Since we would like to conclude with an example in finite dimensions, let us see what form a nonnegative, maximal, indecomposable rank-one band assumes in $\mathcal{M}_n(\mathbf{R})$.

A nonzero rank-one operator in $\mathbf{R}^n$ is of the form xy^* for some nonzero $x, y \in \mathbf{R}^n$. It will be an idempotent if and only if its trace equals 1 *i.e.*, if and only if $y^*x = 1$. If we denote the nonnegative cone of $\mathbf{R}^n$ by $\mathbf{R}^n_+$, then by what we have obtained above, a maximal, nonnegative, indecomposable band of rank-one matrices in $\mathcal{M}_n(\mathbf{R})$ is of the form $\mathcal{X}\mathcal{Y}^*$, where

$$\begin{aligned}\mathcal{X} &= \{a+\mathcal{W}\} \cap \mathbf{R}^n_+\\ \mathcal{Y} &= \{b+\mathcal{Z}\} \cap \mathbf{R}^n_+\end{aligned}$$

for some positive vectors a, b in $\mathbf{R}^n$ and mixed subspaces $\mathcal{W}$, $\mathcal{Z}$.

Further, we observe that the positive vector $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ can be replaced with the vector $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ if the whole band is transformed by an appropriate similarity L, where L is given by

$$\begin{pmatrix} \frac{1}{a_1} & & & \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ & & & \frac{1}{a_n} \end{pmatrix}.$$

Then instead of working with $\mathcal{X}\mathcal{Y}^*$, we work with $L\mathcal{X}\mathcal{Y}^*L^{-1}$ which is again of the form $\mathcal{X}'\mathcal{Y}'^*$ where $\mathcal{X}' = L\mathcal{X}$ and $\mathcal{Y}' = ((L^{-1})^*\mathcal{Y})^*$.

A special case is when $\mathcal{Z} = \{0\}$, *i.e.*, when $\mathcal{X}$ is a singleton. In this case, $\mathcal{X}\mathcal{Y}^*$ is similar (upto a diagonal similarity with positive diagonal entries) to

$$\left\{ \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} : x_1 + \cdots + x_n = 1, x_i \geq 0 \right\}.$$

Bibliography

[1] M. D. Choi, E. A. Nordgren, H. Radjavi, P. Rosenthal, and Y. Zhong, *Triangularizing semigroups of quasinilpotent operators with non-negative entries*, Indiana University Mathematics Journal **42** (1993), no. 1, 15–25.

[2] P. Fillmore, G. Macdonald, M. Radjabalipour, and H. Radjavi, *Towards a classification of maximal unicellular bands*, Semigroup Forum, Springer-Verlag New York Inc. **49** (1994), 195–215.

[3] ———, *On principal-ideal bands*, preprint.

[4] J. A. Green and D. Rees, *On semigroups in which $x^r = x$*, Proc. Cambridge Philos. Soc. **48** (1952), 35–40.

[5] I. Kaplansky, *Fields and rings*, University of Chicago Press, Chicago, 1972.

[6] ———, *The Engel-Kolchin theorem revisited*, Contributions to Algebra (1977).

[7] E. Kolchin, *On certain concepts in the theory of algebraic matric groups*, Annals of Mathematics **49** (1948), 774–789.

[8] J. Levitzki, *Uber nilpotente unterringe*, Math.Ann. **105** (1931), 620–627.

[9] V. J. Lomonosov, *Invariant subspaces for operators commuting with compact operators*, Functional Anal.Appl. **7** (1973), 213–214.

[10] M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, 1964.

[11] H. Radjavi, *A trace condition equivalent to simultaneous triangularizability*, Canadian Journal of Mathematics **38** (1986), no. 2, 376–386.

[12] ———, *On reducibility of semigroups of compact operators*, Indiana University Mathematics Journal **39** (1990), no. 2, 499–515.

[13] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag, 1973.

[14] J. R. Ringrose, *Compact non-self-adjoint operators*, Van Nostrand, Princeton, 1971.

[15] W. Rudin, *Real and complex analysis*, third ed., McGraw-Hill, 1986.

[16] Y. Zhong, *Functional positivity and invariant subspaces of semigroups of operators*, Houston Journal of Mathematics **2** (1993), 239–261.