DEGENERATE KUNDT SPACETIMES AND THE EQUIVALENCE PROBLEM

by

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Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

at

Dalhousie University
Halifax, Nova Scotia

March 2013

DALHOUSIE UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

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DALHOUSIE UNIVERSITY

DATE: March 20, 2013

AUTHOR: David McNutt

TITLE: DEGENERATE KUNDT SPACETIMES AND THE EQUIVALENCE PROBLEM

DEPARTMENT OR SCHOOL: Department of Mathematics and Statistics

DEGREE: Ph.D. CONVOCATION: May YEAR: 2013

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Abstract

This thesis is mainly focused on the equivalence problem for a subclass of Lorentzian manifolds: the degenerate Kundt spacetimes. These spacetimes are not defined uniquely by their scalar curvature invariants. To prove two metrics are diffeomorphic, one must apply Cartan's equivalence algorithm, which is a non-trivial task: in four dimensions Karlhede has adapted the algorithm to the formalism of General Relativity and significant effort has been spent applying this algorithm to particular subcases. No work has been done on the higher dimensional case. First, we study the existence of a non-spacelike symmetry in two well-known subclasses of the N dimensional degenerate Kundt spacetimes: those spacetimes with constant scalar curvature invariants (CSI) and those admitting a covariant constant null vector ($CCNV$). We classify the CSI and $CCNV$ spacetimes in terms of the form of the Killing vector giving constraints for the metric functions in each case. For the rest of the thesis we fix $N = 4$ and study a subclass of the CSI spacetimes: the CSI_Λ spacetimes, in which all scalar curvature invariants vanish except those constructed from the cosmological constant. We produce an invariant characterization of all CSI_Λ spacetimes. The Petrov type N solutions have been classified using two scalar invariants. However, this classification is incomplete: given two plane-fronted gravitational waves in which both pairs of invariants are similar, one cannot prove the two metrics are equivalent. Even in this relatively simple subclass, the Karlhede algorithm is non-trivial to implement. We apply the Karlhede algorithm to the collection of vacuum Type N VSI ($CSI_{\Lambda=0}$) spacetimes consisting of the vacuum PP-wave and vacuum Kundt wave spacetimes. We show that the upper-bound needed to classify any Type N vacuum VSI metric is four. In the case of the vacuum PP-waves we have proven that the upper-bound is sharp, while in the case of the Kundt waves we have lowered the upper-bound from five to four. We also produce a suite of invariants that characterize each set of non-equivalent metrics in this collection. As an application we show how these invariants may be related to the physical interpretation of the vacuum plane wave spacetimes.

Acknowledgements

I would like to thank my supervisors Alan Coley and Robert Milson for their guidance in selecting the topic, their patience and the many discussions which helped make all of this possible.

Particular thanks are needed for Nicos Pelavas for his expertise and the numerous hours spent working with me, and Jiri Podolsky for his excellent advice and encouragement. I am indebted to Ian Anderson and his cooperation with Maplesoft to produce the Differential Geometry package for Maple.

Many thanks to my family and friends for their support and understanding throughout my studies. A special thank you must be given to Andy Sears, without her contributions the final chapters of this thesis would not have existed. Most importantly, I must thank Marie-Michelle Thibodeau for her support and patience.
This thesis is partially based on the following published papers:

- D. McNutt, A. Coley, N. Pelavas (2009). Isometries in higher-dimensional CCNV Spacetimes. *IJGMMP*, Vol 6, Issue 3, pp 419-450.

- D. McNutt, N. Pelavas, A. Coley (2010). Killing vectors in higher-dimensional spacetimes with constant scalar curvature invariants. *IJGMMP*, Vol 7, Issue 8, pp 1349-1369.

- A. Coley, R. Milson, D. McNutt (2012). Vacuum Plane Wave Spacetimes: Cartan Invariants and Physical Interpretation. *CQG* Vol 29, Issue 23, pp 235023-235034.

- D. McNutt, R. Milson, A. Coley (2013). Vacuum Kundt Waves. *CQG* Vol 30, Issue 5, 055010-055039.

- R. Milson, D. McNutt, A. Coley (2013). Invariant classification of vacuum PP-waves. . *JMP* Vol 54, Issue 2, pp 022502-022531.

- D. McNutt (2013). Spacetimes with all scalar curvature invariants in terms of the Cosmological Constant. *IJMPD* Vol 22, Issue 2, pp 1350003-1350014.

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Chapter 1

Introduction

How do we determine when two given Riemannian manifolds describe the same geometric object, so that they are equivalent under some coordinate transformation? This is an interesting question with a complicated answer. Without going into technical details at this point, the answer lies in the components of the derived tensors arising from the metric, such as the components of the Riemann tensor and its covariant derivatives. These quantities are invariant functions, and so any invariant combination of these functions will retain the same form regardless of the coordinate system.

The difficulty arises from the sheer number of invariants produced and how the freedom of the frame bundle can be used to simplify these invariant quantities. To cope with these problems algorithms have been developed to calculate these invariants and their functional relationships. However, for particular manifolds these approaches may still be computationally infeasible or non-algorithmic at some later stage. For example, if one wishes to determine the exact coordinate transformation used, one must solve an extensive list of equations and some of these equations may be transcendental.

If one is interested in showing the inequivalence of two manifolds, one may avoid such heavy mathematical machinery by examining the necessary conditions for the two manifolds to be the same. If any of these conditions differ it may be concluded the manifolds are distinct. The existence of symmetries like curvature collineations, conformal motions, homothetic motions and Killing vectors are helpful as these are geometric objects which exist independent of the coordinate system used. In the same manner algebraic considerations like Segre type for the Riemann and Ricci tensor or any other covariant formed from products of the Riemann tensor and its derivatives with other tensors may be used to partially classify these spaces. As an example, in the Lorentzian case one may use the Newman-Penrose formalism to work with the covariants of the Weyl spinor, Ψ_{ABCD} with a spinor ξ^A to reproduce the Petrov classification.

The covariants of the Riemann tensor and its derivatives are an effective means of showing the inequivalence of two metrics. One particular set of covariants, the polynomial scalar curvature invariants, are of particular importance (especially in the context of Lorentzian geometry). These are produced by contracting the Riemann tensor and its covariant derivatives in combinations that produce polynomial functions in terms of the curvature and its derivative. They are remarkable because they allow for a partial answer to the question of equivalence. In four dimensions it has been shown that a Lorentzian manifold is either a degenerate Kundt spacetime or the polynomial scalar curvature invariants are unique to the space [94, 92], namely they are *I-non-degenerate*.

In higher dimensions it is believed this property will persist. Either the space will be uniquely described by the scalar polynomial curvature invariants or the Lorentzian manifold belongs to the class of higher-dimensional degenerate Kundt metrics where the kinematic and curvature null frames are aligned so that the curvature tensor and its covariant derivatives have vanishing positive boost-weight components [92]. If the conjecture holds true, then the polynomial scalar curvature invariants provide a straightforward answer to the equivalence problem for almost all spacetimes, except the degenerate Kundt metrics. The study of degenerate Kundt spacetimes and the equivalence problem will be important themes in this thesis.

For these spacetimes, the scalar curvature invariants provide necessary conditions for equivalence but not sufficient conditions. As an example, the metrics describing Minkowski space and a PP-wave spacetime both have vanishing scalar curvature invariants and yet are inequivalent as metrics. Of course by comparing the isometry group of these two metrics we can show inequivalence by noting Minkowski space admits a group with ten isometries while at most the PP-wave metrics may admit an isometry group of dimension six - in the case of the plane-waves. How can we distinguish two PP-wave metrics with the same number of isometries, or more generally, how do we distinguish between two degenerate Kundt spacetimes with the same geometric quantities? To provide an answer one must work with Cartan invariants and an appropriate algorithm suited to the geometry under study, namely the Karlhede algorithm.

Admittedly the full invariant classification of the degenerate Kundt spacetimes in any dimension is too general of a question to answer. We instead concentrate on a significant subclass of the degenerate Kundt metrics, the CSI spacetimes, where all polynomial scalar curvature invariants are constant. Unlike in the Riemannian case, where any CSI Riemannian metric is automatically a locally homogeneous space [46], the CSI Lorentzian metrics are either locally homogeneous spaces (H) or degenerate Kundt CSI spacetimes (CSI_K) [94]. However, for every CSI_K spacetime with a particular set of constant invariants there exists at least one homogeneous spacetime with the same constant invariants. Furthermore, it was conjectured that the CSI spacetimes could be built from locally homogeneous spaces and the vanishing scalar invariant (VSI) spacetimes (i.e. the CSI_R conjecture, this conjecture has been shown to be true in three dimensions [89]).

From the perspective of differential geometry and invariant classification these properties and the simplicity of the condition on the invariants make the CSI spacetimes an ideal subclass to study. In addition, the CSI spacetimes have applications to mathematical physics [91, 102]. The CSI spacetimes admit a subclass of spacetimes which are solutions of supergravity, [85], including $AdS \times S$ spacetimes [23], generalizations of $AdS \times S$ based on different VSI seeds [85] and the AdS gyratons [75]. The conditions these CSI spacetimes must satisfy in order to preserve some supersymmetry is not fully known. A necessary (but not sufficient) condition for supersymmetry to be preserved is that the spacetime admits a Killing spinor, and hence a null or timelike Killing vector.

Although the invariant classification of the CSI spacetimes is incomplete, those CSI spacetimes which are candidates for solutions to supergravity admit an isometry, and hence can be classified coarsely. Towards this end, in chapter (4) we use the Killing equations to determine the form of all n-dimensional CSI spacetimes admitting at least one Killing vector. With this class of metrics written explicitly, we determine the subset of these that admit a covariantly constant vector. If we are given two of these CSI metrics, we may determine inequivalence by comparing the constant scalar curvature invariants, the form of the Killing vector, and the existence of a covariantly constant null vector.

The existence of a covariantly constant null vector is relevant for the higher-dimensional VSI solutions of type IIB supergravity [84]. Those VSI spacetimes in which some supersymmetry is preserved belong to the set of spacetimes admitting a covariantly constant null vector. In general the $CCNV$ metrics are of Ricci type N and Weyl type III or higher; the Weyl type N solutions are a well-know class of metrics, the PP-waves. These were originally presented as a gravitational equivalent of the electromagnetic plane-fronted waves [9], describing a non-expanding, non-rotating gravitational wave in vacuum or admitting a null radiation as a matter-field. As such, in General relativity, they are interpreted as gravitational radiation propagating at great distances from the source; they are generally treated as toy models due to the simplicity of their geometry and interpretation.

The condition for a null vector to be covariantly constant, $\ell_{a;b} = 0$, implies the metrics are geodesic, shear-free, non-expanding and non-twisting; they are a particular subclass of the Kundt metrics, and belong to the class of degenerate Kundt metrics. It is known that the CSI metrics admit solutions which are $CCNV$ as well, one would expect that there are other degenerate Kundt $CCNV$ metrics outside of the CSI class. The invariant classification of the N-dimensional degenerate $CCNV$ spacetimes (like the CSI spacetimes) is incomplete, and so one must resort to comparing geometric objects to show inequivalence.

In the four dimensional case, Kundt and Ehlers exhaustively listed the admissible symmetry groups for all vacuum PP-waves along with a corresponding list of canonical forms for these metrics [11]. Presumably one could continue this process for the remaining $CCNV$ spacetimes to produce a broad equivalence classes based on the number of elements in the isometry group and their commutator relations. In application this is a difficult thing to do in N-dimensions due to the generality involved. For example, a particular choice of two-dimensional transverse space may offer additional spacelike isometries.

Towards this end, we examine the existence of N-dimensional $CCNV$ spacetimes admitting a second non-spacelike Killing vector in chapter (5), we produce an exhaustive list of canonical forms and the resulting Killing Lie algebras. Noting that the $CCNV$ metrics belong to the degenerate Kundt class, and hence contains the CSI metrics, we extend the work done in chapter (4) by determining the subclass of CSI

$CCNV$ spacetimes admitting a second Killing vector. The chapter concludes with an exhaustive list of all $CCNV$ spacetimes admitting a null or timelike Killing vector on the entire manifold.

In the subsequent chapter, (6), we provide explicit examples of $CCNV$ VSI metrics with applications to supersymmetric supergravity. Supersymmetric supergravity solutions are of interest in the context of the AdS/CFT conjecture, the microscopic properties of black hole entropy, and in the search for a deeper understanding of string theory dualities [35, 44, 43]. In five dimensions, solutions preserving various fractions of supersymmetry of $N = 2$ gauged supergravity have been studied. The Killing spinor equations imply that supersymmetric solutions preserve $2, 4, 6$ or 8 of the supersymmetries. For each supersymmetry the space must admit a Killing spinor and hence a null or timelike Killing vector, consequently the study of $CCNV$ CSI metrics with multiple non-spacelike Killing vectors is necessary.

Comparing symmetry groups only yields so much information. To provide more necessary conditions for equivalence, one must produce further invariants. This is a non-issue as there are a wealth of Cartan invariants to work with. However, if one wishes to have a finer classification for inequivalence we need only pick a few invariants. To illustrate this, we examine the four dimensional spaces with all non-zero invariants expressed as polynomials in terms of a non-zero cosmological constant, the CSI_Λ spacetimes in chapter (7). We determine the conditions on the spin-coefficients and curvature components as an analogue to the VSI theorem in [55]. Here we may use Λ to differentiate between solutions while the remaining curvature scalars $\Phi_{12},\ \Phi_{22}$ and Ψ_4 may be compared to provide necessary conditions for equivalence.

The CSI_Λ spaces contain all plane-fronted gravitational waves: the PP-waves and Kundt waves as $CSI_0 = VSI$ spacetimes along with their generalizations with non-zero Λ as well all of the Type N CSI_Λ spacetimes. Originally these were defined for spaces in which the Ricci scalar vanished. In 1984, Ozsvath, Robinson and Rozga (O.R.R.) [33] studied the plane-fronted gravitational waves in spaces with cosmological constant. In 1999 Bicak and Podolsky formally outlined the classification introduced by O.R.R. [33] into six canonical subclasses depending on the sign of Λ and the sign of an additional constant κ': the sole component of the double Lie derivative of the metric in the direction of ℓ.

According to this classification the rotating plane-fronted gravitational waves correspond to $\Lambda = 0$ and $\kappa' = 1$, denoted as the Kundt waves. Similarly the PP-wave metric correspond to $\Lambda = \kappa' = 0$. For non-zero Λ the situation is somewhat different: if $\Lambda > 0$ there is only one class of plane-fronted waves. However, if $\Lambda < 0$ there are three possibilities: κ' less than, equal to or greater than zero; corresponding to the generalized PP-waves, Kundt waves and Siklos waves, respectively [50].

To complete Chapter (7) we analyze the conditions for Type N CSI_Λ spacetimes, using the coordinates of [94], to reproduce all plane-fronted gravitational waves in spacetimes with non-zero cosmological constant [33]. Using the six canonical forms for the metrics we relate this classification to the one using Petrov and Segre type in [55].

These comparisons between two CSI_Λ metrics can only prove inequivalence. Presumably if one used enough invariants, one could show equivalence between two metrics. In order to determine the smallest set of invariants to prove equivalence one must invariantly classify these spaces using the Karlhede algorithm. As in the CSI case the CSI_Λ classification is incomplete; with the exception of the vacuum Petrov type D [37, 28] and the Petrov type O [34, 48, 38, 39] spacetimes with particular matter conditions, one cannot determine the equivalence of two CSI_Λ metrics. In fact the Karlhede algorithm has yet to be fully implemented in the case of the PP-paves.

As an illustration of the utility of the Karlhede algorithm, in the remaining chapters we will invariantly classify all of the vacuum Type N VSI spacetimes, the plane-fronted gravitational waves. Although the zeroth order invariants are the same for vacuum PP-waves and vacuum Kundt waves after normalization, i.e., $\Psi_4 = 1$, the analysis must be split into two cases depending on the vanishing of the Cartan invariant $\tau = 1$ relative to the new coframe. Due to the relative simplicity of the PP-waves and their familiarity in the literature these are analyzed first by setting $\tau = 0$.

In Chapter (8) we choose a special coordinate system adapted to the double ζ derivatives of the metric function $f(\zeta, u)$ to better study the functional dependence of the Cartan invariants that arise in the Karlhede classification. We accomplish this by studying the degenerate cases of the algorithm where none, one, or two invariants appear at the first iteration (the case where three invariants appear provides no information for general analysis and hence must be treated as the non-degenerate case in the classification algorithm). Comparing our work with the symmetry classification [11], we sub-classify the PP-wave metrics according to their symmetry groups and invariant structure. It is also shown that the upper-bound on the number of iterations of the Karlhede algorithm applied to the vacuum PP-waves is sharp.

The PP-waves contain a subclass providing many simple and non-trivial applications of the invariant classification, the plane waves. These were introduced by Einstein and Rosen [3] relative to a coordinate system with a metric singularity (Rosen coordinates), due to this they were ignored as unphysical until it was revealed that a change of coordinates could make the metric components regular (Brinkmann coordinates). Although physically reasonable, they are still toy models, but the plane waves allow us to explore the relationship between the invariants and the physical interpretation of the space. In Chapter (9) we restrict our classification to the plane waves, which correspond to the PP-wave solutions with the $\alpha = 0$ relative to the first order coframe of the Karlhede algorithm. Unlike the PP-waves with $\alpha \neq 0$, the first order coframe is not a canonical frame; however, it is not unique as a null rotation about ℓ leaves $\Psi_4 = 1$ and γ invariant.

For a physical interpretation of these spaces, we review the formalism developed to study the geodesic deviation equations [51] for an arbitrary timelike geodesic. This is achieved by determining a unique coframe which is covariantly constant along any timelike geodesic. This describes the gravitational field a timelike observer would experience and may be decomposed into transverse, longitudinal and coulomb components [15]. For Type N spacetimes, the physical situation is simpler as an arbitrary timelike observer would only experience transverse waves. With these tools, we invariantly classify the weak-field circularly polarized plane waves in Rosen coordinates, and discuss the physical interpretation of these spaces using Cartan invariants. Relative to Brinkmann coordinates, the metric form for the plane waves, $f(\zeta, u) = A(u)\zeta^2$, give rise to a simple form for the equations of geodesic of deviation. By imposing conditions on the Cartan invariants we may produce plane wave spacetimes whose physical interpretation in terms of polarization and magnitude of the wave may be related to the Cartan invariants used. Chapter (9) concludes with some simple examples of this approach.

In Chapter (10) we examine the vacuum Kundt waves, which make up the remainder of the type N vacuum VSI spaces. These spaces lack a covariantly constant null vector and hence an isometry; in general to classify these spaces one will need four invariants as essential coordinates. This is reflected in the Karlhede algorithm where the upper-bound on the number of iterations to classify the vacuum Kundt waves is strictly less than six, but potentially greater than four [36, 47]. By examining the conditions for which a Kundt wave in the Karlhede algorithm admits one, two, three or four functionally independent invariants in the components of the curvature and the its first order derivatives, we derive invariant conditions to differentiate between these four cases. In fact, in the cases where only one or two invariants appear at first order, i.e. with invariant counts $(0, 1, ...)$ and $(0, 2, ...)$, we may integrate to produce canonical forms for the metrics.

Within this formalism we reproduce the result that the vacuum Kundt waves upper-bound in the Karlhede algorithm is strictly less than six; this was originally proven symbolically using the GHP formalism [47]. Although our approach requires the choice of coordinates, the explicit metric forms for the vacuum Kundt waves with invariant counts $(0, n, ...)$ $n = 1, 2$ allow us to lower the upper-bound once more, implying the vacuum Kundt waves require at most four covariant derivatives to invariantly classify them. This is done by exhaustively listing the possible scenarios of the Karlhede algorithm which admit $q = 5$, namely $(0, 1, 2, 3, 4, 4)$ or $(0, 2, 2, 3, 4, 4)$, and showing that these metrics cannot exist. For example, those spaces with invariant counts starting with $(0, 1, 2, ...)$ must have $(0, 1, 2, 2)$. This class of spacetimes is notable as we recover the known Kundt wave admitting two symmetries [27], along with a new Kundt wave with a distinct two dimensional symmetry group.

As the upper-bound for the Kundt waves is now the same as the PP-waves, we address the question of sharpness. This can only occur for the vacuum Kundt waves with invariant counts $(0, 1, 3, 4, 4)$, $(0, 2, 3, 4, 4)$ and $(0, 3, 3, 4, 4)$. By working with the Cartan invariants and their frame derivatives it is shown that the latter two cases do not occur and that the vacuum Kundt waves with $(0, 1, 3, 4, 4)$ are unique in that they are the only Kundt waves require the fourth order derivatives of curvature in the algorithm. The chapter concludes with an invariant classification of the vacuum Kundt waves as a byproduct of the investigation of the upper-bound. As in the case of the vacuum PP-waves, we provide a table of distinct canonical forms for the vacuum Kundt wave metrics along with the necessary invariants to classify the spaces.

We examine the symmetries of the vacuum Kundt waves with permitted matter-fields using methods independent of invariants. Instead of examining the Killing equations, we work with an invariant coframe and calculate the symmetries by requiring the Lie derivatives of these one-forms vanish for a particular vector field. The invariant coframe produced in the Karlhede algorithm is not well-suited to this analysis, and so we use an alternative invariant coframe where the parameters for spins and null rotations about ℓ are zero, i.e., $\theta = B = \bar{B} = 0$. These two frames will differ due to normalizing different components of the curvature tensor and its derivatives; however, this fact ensures the new coframe is invariant. Using this invariant coframe we recover the results of [27], illustrating the utility of invariant coframes.

Chapter 2

Differential Geometry

2.1 Non-coordinate Bases

Given a manifold equipped with a metric, (M, g), there is a unique connection for which the metric is covariantly constant, the Levi-Civita connection, that simplifies the calculation of the connection components and the curvature and torsion tensors. Cartan introduced a method [4] which not only calculates the connection components but also yields the torsion and curvature tensors as well. To begin, instead of the usual coordinate basis for T_pM, consider a basis for T_pM:

$$e_a = e_a{}^{\alpha}\partial_{x^\alpha} \quad e_a{}^{\alpha} \in GL(N, \mathbb{R}),$$

such that $\{e_a{}^{\alpha}\}$ is orthonormal with respect to the metric:

$$g(e_a, e_b) = e_a{}^{\alpha} e_b{}^{\beta} g_{\alpha\beta} = \eta_{ab}.$$

Using this basis, we can express the original metric components, $g_{\alpha\beta}$ in terms of $e^a{}_{\alpha}$, and its inverse:

$$g_{\alpha\beta} = e^a{}_{\alpha} e^b{}_{\beta} \eta_{ab}.$$

Choosing this as our coordinate basis we can re-express any vector $V \in T_pM$, in the usual way:

$$V^\alpha = V^a e_a{}^{\alpha} \qquad V^a = e^a{}_{\alpha} V^\alpha$$

To find a basis for T_p^*M, let $\{e^a\}$ be a set of N linearly independent one-forms with the property that $< e_a, e^b >= \delta^a{}_b$, after a little thought we see that the basis covectors must have the form: $e^a = e^a{}_{\alpha} dx^\alpha$. Given an arbitrary one-form $\omega \in T_p^*M$,

$$\omega_\alpha = \omega_a e^a{}_{\alpha} \qquad \omega_a = e_a{}^{\alpha} \omega_\alpha,$$

the metric expressed in this basis can be written in much simpler terms:

$$g = \eta_{ab} e^a \otimes e^b.$$

In general this choice of basis for the tangent space will have a non-vanishing commutator;

$$[e_a, e_b] = C^c{}_{ba} e_c$$

We call $\{e_a\}$ a non-coordinate basis or frames for $T_p M$ and we will call the indices: $a, b, c, d, i, j, k, l, m, n, p, q$ frame indices while $\alpha,\ \beta,\ \gamma,\ \delta, e, f, g, h, r, s, t, w$ will be the coordinate indices. The metric will be used to raise and lower indices of the same type; for example, $e_{a\alpha} = g_{ab} e^a{}_\alpha = \eta_{ab} e^a{}_\alpha$ while $e_a^\alpha = g^{\alpha\beta} e_{a\beta}$.

To determine the frame connection components relative to this new basis, consider:

$$\nabla_a e_b = \Gamma^c{}_{ba} e_c,$$

but each e_a is just a vector field and so,

$$\nabla_a e_b = e_a^\alpha \nabla_\alpha (e_b{}^\alpha \partial_{x^\alpha}). \tag{2.1}$$

Thus the frame connection components can be expressed as:

$$\Gamma^c{}_{ba} = e^c{}_\gamma e_a{}^\alpha (\partial_\alpha e_b{}^\beta + e_b{}^\beta \Gamma^\gamma{}_{\beta\alpha}),$$

and the components of T and R in this basis become:

$$T^c{}_{ab} = \Gamma^c{}_{[ab]} - C^c{}_{ab} \tag{2.2}$$

$$R^d{}_{cba} = e_b[\Gamma^d{}_{ca}] - e_a[\Gamma^d{}_{cb}] + \Gamma^\epsilon{}_{ca}\Gamma^d{}_{\epsilon b} - \Gamma^\epsilon{}_{cb}\Gamma^d{}_{\epsilon a} - C^\epsilon_{ab}\Gamma^d{}_{\epsilon c}$$

Now introduce a matrix-valued one-form called the connection one-form $\{\omega^a{}_b\}$:

$$\omega^a{}_b = \Gamma^a{}_{bc} e^c$$

which satisfies the Cartan Structure equations,

$$de^a + \omega^a{}_b \wedge e^a = T^a \tag{2.3}$$

$$d\omega^a{}_b + w^a{}_c \wedge \omega^c{}_b = R^a{}_b, \tag{2.4}$$

where T^a and $R^a{}_b$ are the torsion-two form and curvature two-form respectively,

$$T^a = T^a_{bc} e^b \wedge e^c, \;\; R^a{}_b = R^a{}_{bcd} e^c \wedge e^d.$$

Taking the exterior derivative of the structure equations gives the non-coordinate basis versions of the Bianchi identities:

$$dT^a + w^a{}_b \wedge T^b = R^a{}_b \wedge e^b, \;\; dR^a{}_b + w^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0$$

The non-coordinate basis and the resulting form for the metric is preserved only under certain transformations. To see this consider at a point p in the manifold, a new set of frames $\hat{e}^a(p) = \Lambda^a{}_b e^b(p)$ with the property that the simpler form of the metric is preserved in the new basis. This implies

$$g = \eta_{ab} e^a \otimes e^b = \lambda^c{}_a g_{cd} \lambda^d{}_b e^a \otimes e^b,$$

in order to satisfy the above we must have $\Lambda \in SO(N-1,1)$. Thus the coordinate indices transform under coordinate changes $GL(N,\mathbb{R})$ while the frame indices transform under $SO(N-1,1)$. As an example, the frame vectors transform in the following way:

$$\hat{e}_a = \Lambda^b_a \hat{e}_b.$$

Therefore an arbitrary tensor with respect to the frames transforms in the normal way under orthogonal rotations. Both the vector valued torsion two-form and matrix valued curvature two-form transform homogeneously, this can be seen by comparing the transformation rule for the usual formulation of the two tensors T^a_{bc} and $R^a{}_{bcd}$ and comparing the results with their respective two form representation, we see that:

$$\hat{T}^a = \Lambda^a{}_b T^b, \quad \hat{R}^a{}_b = \Lambda^a{}_c R^c{}_d \Lambda_b{}^d. \tag{2.5}$$

Using (2.5) and (2.3), we know that

$$d\hat{e}^a + \hat{\omega}^a{}_b \wedge \hat{e}^b = de^a + \omega^a{}_b \wedge e^b,$$

substituting $\hat{e}^a = \Lambda^a{}_b e^b$ and noting that $d(\Lambda^a{}_c)\Lambda_b{}^c + \Lambda^a{}_c d(\Lambda_b{}^c) = 0$ we find:

$$\hat{\omega}^a{}_b = \Lambda^a{}_c \omega^c{}_d \Lambda^d{}_b + \Lambda^a{}_c d(\Lambda^{-1})^c{}_b.$$

So far the connection has been left arbitrary. By choosing to use the Levi-Civita connection the torsion tensor vanishes, and so we have:

$$\Gamma^c{}_{ab} - \Gamma^c{}_{ba} = C^c{}_{ab}.$$

The connection components with respect to the non-coordinate basis are no longer symmetric with respect to the last two indices due to (2.1); instead the metric compatibility condition implies that

$$\Gamma_{abc} = -\Gamma_{bac}.$$

where $\Gamma_{abc} = \eta_{ad}\Gamma^d{}_{bc}$ are called the Ricci Rotation Coefficients. Hence the matrix valued connection one-form is also anti-symmetric:

$$\omega^a{}_b = -\omega^b{}_a.$$

Lastly the Cartan Structure Equations now become:

$$de^a = -\omega^a{}_b \wedge e^b$$
$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b.$$

To find the connection components expand out (2.3):

$$de^a = -[\Gamma^a{}_{bc} - \Gamma^a{}_{cb}]m^c \wedge m^b$$

By explicitly exterior differentiating the frame one-forms we find:

$$de^a = m^a{}_{i,j}dx^j \wedge dx^i = [m^a{}_{i,j}m_b{}^i m_c{}^j - m^a{}_{i,j}m_c{}^i m_b{}^j]m^c \wedge m^b.$$

Equating the coefficients of each $m^c \wedge m^b$ we find at most $\frac{N^3-N^2}{2}$ linear equations for the $\frac{N^3-N^2}{2}$ potential connection components; thus it is possible to solve for the $\Gamma^a{}_{bc}$'s, and thus $\omega^a{}_b$. With the Cartan connection one-form known, we can use the second Cartan Equation (2.4) to find the matrix valued curvature tensor, simply by calculating $d\omega^a{}_b$ and adding the wedge product $\omega^a{}_c \wedge \omega^c{}_b$, then by comparing the coefficients of $m^a \wedge m^b$ the components of R_{ab} are found easily.

2.2 The Lorentz Group

In N-dimensional Minkowski space, the Lorentz group, $\mathbb{L}$, is the isotropy subgroup of the Poincaré group; that is, the Lorentz group is the subgroup of all isometries in Minkowski space that leave the origin fixed. This is of particular interest because once one moves away from a flat-spacetime, i.e., Minkowski space, to a spacetime that is curved, it may not be sensible to consider translations from one point in the spacetime to another, we are left with the isometries that leave a point fixed. Since it is a Lie group, we can talk about its topological properties as a manifold; the Lorentz group is a six-dimensional non-compact Lie group whose four connected components are not simply connected (given a loop at a point $p \in \mathbb{L}$, it is not homotopic to the identity map). The four components of the Lorentz group are:

1. The elements that reverse the direction of time-like vectors.
2. The elements that reverse the orientation of the spatial vectors.
3. The elements that reverse both the direction of time and orientation of the spatial vectors.
4. Those elements that do not reverse the direction of the vector basis for T_pM.

If a Lorentz transformation preserves the direction of time we call it orthochronous; on the other hand if a Lorentz transformation preserves the orientation of the spatial vectors we call it proper. The component of the Lorentz group containing all of the proper, orthochronous Lorentz transformations, will be called the Restricted Lorentz group. We will ignore those transformations that are not proper and orthochronous. The restricted Lorentz group is generated by ordinary spatial rotations, Lorentz boosts which can be seen as hyperbolic rotations in a plane that includes a time-like vector and a space-like vector. As an example, in four dimensions (t, x_1, x_2, x_3) a spatial rotation in the $x_1 - x_2$ plane is of the form:

$$\begin{pmatrix} \hat{t} \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) & 0 \\ 0 & -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \theta \in [0, 2\pi).$$

While a boost in the $t - x_1$ plane is given by:

$$\begin{pmatrix} \hat{t} \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} cosh(\theta) & -sinh(\theta) & 0 & 0 \\ -sinh(\theta) & cosh(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \theta \in [0, \infty).$$

There is a much easier way to describe the Lorentz transformations and in particular the Lorentz boosts. By adopting a null frame, that is a basis for T_pM consisting of two null vectors ℓ, $\mathbf{n}$ and N-2 spatial vectors m^i, a Lorentz boosts now becomes:

$$\hat{\ell} = \lambda \ell, \quad \hat{\mathbf{n}} = \lambda^{-1}\mathbf{n}, \quad \hat{m}^i = m^i, \tag{2.6}$$

while spatial rotations or spins are now:

$$\hat{\ell} = \ell, \quad \hat{\mathbf{n}} = \mathbf{n}, \quad \hat{m}^i = X^i{}_j m^j \quad X^i{}_j \in O(N-2). \tag{2.7}$$

Furthermore we can now rotate about the null vectors, for example a null rotation about the $\mathbf{n}$ axis is:

$$\hat{\ell} = \ell + z_i m^i - \frac{1}{2} z^i z_i \, \mathbf{n}, \quad \hat{\mathbf{n}} = \mathbf{n}, \quad \hat{m}^i = m^i - z_i \mathbf{n} \tag{2.8}$$

while a rotation about the ℓ-axis is of a similar form, except with $\mathbf{n}$ and ℓ interchanged.

2.3 Killing Vectors

Given a Riemannian manifold with metric g, suppose we are given a vector field X with the property that its flow is a one-parameter group of isometries, or more simply by choosing coordinates, x^α at a point $p \in M$, the infintesimal change from x^α to $x^\alpha + \epsilon X^\alpha(p)$ preserves the metric:

$$\frac{\partial(x^\alpha + \epsilon X^\alpha)}{\partial x^\beta} \frac{\partial(x^\gamma + \epsilon X^\gamma)}{\partial x^\delta} g_{\alpha\gamma}(x + \epsilon X) = g_{\beta\delta}(x)$$

Then we say the vector field X is a Killing Vector field, while expanding out the above equation, we find that $g_{\alpha\beta}$ and X satisfy the following:

$$X^\alpha \partial_\alpha g_{\beta\gamma} + \partial_\beta X^\alpha g_{\alpha\gamma} + \partial_\gamma X^\alpha g_{\beta\alpha} = 0$$

This is called the Killing equation, and it can be written in the more compact form:

$$\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0. \tag{2.9}$$

This could be done in frame indices as well assuming one had the connection components calculated with respect to the frame.

Returning to the matter of the integral curves of X, (2.9) tells us that the local geometry does not change as we move along them; in this sense the Killing vector fields represent the direction of the symmetries of the manifold. There may be more Killing vector fields than the dimension of the manifold the maximum number of Killing vectors is dependent on dimension. For example, flat N-dimensional Minkowski space has $\frac{N(N+1)}{2}$ Killing vector fields, N of which generate translations, (N-1) are boosts and $\frac{(N-1)(N-2)}{2}$ spatial rotations. Flat Minkowski and other spacetimes which have $\frac{N(N+1)}{2}$ Killing vector fields are called Maximally symmetric spaces.

2.4 Classification Methods

We can classify exact solutions to Einstein's field equations, by examining and classifying the corresponding Ricci and Weyl tensors which arise from the the metric describing the exact solution. By doing so we can summarize many facts about a spacetime into a compact form; for example, all those four dimensional spacetimes that are vacuum solutions and admit gravitational radiation can be described as a spacetime for which the Ricci tensor vanishes and the Weyl tensor is of a particular canonical form. There are a variety of methods towards a classification for these two tensors in four dimensional spacetimes however many do not generalize easily to higher dimensions.

As such they will not be helpful in this instance and instead, we will focus on dimension independent methods to classifying the Ricci and Weyl tensors. It is important to note that all of these methods reviewed in this thesis are in a neighborhood of a point in the spacetime, as such, a spacetime may have a Weyl tensor of a certain type in one region and an entirely different Weyl tensor at a point in another part of the manifold. In the case of the characterization of geometries using invariants, equivalence will be purely local in nature. Given two metrics which cover disjoint regions of a single analytic space it will be impossible to determine whether one is an analytic continuation of the other with these approaches alone. Alternatively topological considerations can complicate matters by producing metrics which are locally equivalent but globally inequivalent. We additionally assume that all classifying quantities, discrete or continuous are respectively constant or sufficiently smooth in open open neighborhoods.

2.4.1 Classification of the Weyl Tensor in N-Dimensions

The Petrov classification describes the possible invariant algebraic type of the Weyl tensor at a point in a Lorentzian manifold. This classification was developed by Petrov [56] and extended in [8]. If a spacetime has the same Petrov type at all points then we say that the spacetime has a certain Petrov type. Petrov's original classification originated by treating the Weyl tensor as a symmetric matrix mapping bivectors into bivectors and looking at the eigenvalues of this symmetric matrix. There are several equivalent methods of obtaining the classification; for example, there is an approach introduced by Penrose [8, 31] using two-spinors to decompose the Weyl tensor into a totally symmetric spinor Ψ_{ABCD}, which then can be decomposed in terms of four principal spinors α_A, β_B, γ_C, δ_D

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}.$$

Petrov types now correspond to various multiplicities of principal spinors and the following possibilities occur

$$I = \{1111\} : \qquad \Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}, \tag{2.10}$$

$$II = \{211\} : \qquad \Psi_{ABCD} = \alpha_{(A}\alpha_B\gamma_C\delta_{D)}, \tag{2.11}$$

$$D = \{22\} : \qquad \Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\beta_{D)}, \tag{2.12}$$

$$III = \{31\} : \qquad \Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}, \tag{2.13}$$

$$N = \{4\} : \qquad \Psi_{ABCD} = \alpha_A\alpha_B\alpha_C\alpha_D. \tag{2.14}$$

$$O = \{-\} : \qquad \Psi_{ABCD} = 0 \tag{2.15}$$

All Petrov types except that of type I are called algebraically special.

We have seen previously that in four dimensions every spinor gives rise to a null vector, thus every principal spinor can be assigned to a null vector via $v^\alpha = \xi^A\bar{\xi}^{A'}$;

these will be called *principal null vectors* and the corresponding directions of the vectors *principal null directions* (PNDs). Thus there are at most four distinct PNDs at each point of a spacetime. In the tensorial form, PNDs corresponding to a vector ℓ satisfies three equivalent conditions [31], [22] given in Table (2.4.1 for the various Petrov types.

$\ell^b\ell^c\ell_{[e}C_{a]bc[d}\ell_{f]} = 0$	ℓ at least simple PND (I)	$\Psi_0 = 0,\ \Psi_1 \neq 0$
$\ell^b\ell^c C_{abc[d}\ell_{e]} = 0$	ℓ at least double PND (II,D)	$\Psi_0 = \Psi_1 = 0,\ \Psi_2 \neq 0$
$\ell^c C_{abc[d}\ell_{e]} = 0$	ℓ at least triple PND (III)	$\Psi_0 = \Psi_1 = \Psi_2 = 0,\ \Psi_3 \neq 0$
$\ell^c C_{abcd} = 0$	ℓ quadruple PND (N)	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0,$ $\Psi_4 \neq 0$

Table 2.1: Each line contains three equivalent conditions for various Petrov types in four dimensions. $\Psi_0 \ldots \Psi_4$ are complex components of the Weyl tensor in the Newman-Penrose formalism. The third condition means a frame can be chosen in which given components of the Weyl tensor vanish.

In some exact solutions there is a correspondence between the symmetries of the solution and the Petrov type. For example, a static spherically symmetric spacetime will be of type D with two double PND's defining radially ingoing and outgoing null congruences whereas the Friedmann-Robertson-Walker spacetime is of type O and it is isotropic and does not contain any preferred null direction. In fact, in order to find exact solutions of the Einstein equations it is usually necessary to assume some symmetries or to assume a specific algebraically special type of a spacetime; by doing so the field equations can also be considerably simplified and solved. Remember that the Petrov type is defined at point, and so it is possible for a spacetime to have a different Petrov type at distinct points or regions of spacetime, for example in four dimensions, black holes of type I can have isolated or Killing horizons of type II [77].

While there are several distinct methods that lead to the Petrov Classification in four dimensions only a few generalize to classifying the Weyl tensor in higher dimensions. Given the recent interest in higher dimensional manifolds in gravity theories, the classification of these manifolds is of particular importance and we would like an dimension independent method that gives the Petrov classification in four dimensions and in higher dimensions a partial classification for the Weyl tensor. The alignment method introduced in [76] satsifies these two requirements; it is dimension independent and in four dimensions it reproduces the Petrov classification.

To introduce the alignment method, consider an N-dimensional Lorentzian manifold and define a null frame: $m^0 = \mathbf{n}$, $m^1 = \ell$ and m^i, with two null vectors $\mathbf{n}$, ℓ and $N-2$ spacelike vectors m^i:

$$\ell^a \ell_a = \mathbf{n}^a \mathbf{n}_a = 0, \quad \ell^a n_a = 1,$$
$$m^{ia} m^j_a = \delta_{ij}, \quad m^{ia}\ell_a = 0 = m^{ia} n_a.$$

The metric now has the form

$$g_{ab} = 2\ell_{(a}n_{b)} + \delta_{ij} m^i_a m^j_b, \tag{2.16}$$

which remains unchanged under Lorentz transformations consisting of null rotations (2.8), spins (2.7), and boosts (2.6).

We say that a tensor-component $T^{a_1,\ldots,a_m}{}_{b_1,\ldots,b_m}$ has a *boost weight* b if it transforms under boost according to $\hat{T}^{a_1,\ldots,a_n}{}_{b_1,\ldots,b_n} = \lambda^b T^{a_1,\ldots,a_n}{}_{b_1,\ldots,b_n}$. Furthermore we call the *boost order* of a tensor $\mathbf{T}$ the maximum boost weight of its frame components. The boost order of a tensor depends only on the choice of a null direction ℓ and thus spins and boost do not affect it [76]. We will denote the boost order of a given tensor $\mathbf{T}$ as $b(\ell)$ to indicate the dependence on the choice of ℓ, while the identity of $\mathbf{T}$ should be clear from the context.

Denote the maximum value of $b(\boldsymbol{k})$ taken over all null vectors $\boldsymbol{k}$ as $b_{\max}$. Then we say a null vector $\boldsymbol{k}$ is *aligned* with the tensor $\mathbf{T}$ whenever $b(\boldsymbol{k}) < b_{\max}$. Then the integer $b_{\max} - b(\mathbf{k}) - 1$ will be defined as the *order of alignment*. The Weyl tensor aligned null vectors represent a natural generalization of the PND's, so we will call these higher-dimensional null vectors WANDs (Weyl aligned null directions). A classification of the Weyl tensor in higher dimensions then depends on the existence of WANDs of various orders of alignment. The Weyl tensor in an arbitrary dimension has in general components with boost weights $-2 \leq b \leq 2$ and thus the order of alignment of a WAND cannot exceed 3.

We will call the *primary alignment type* of the Weyl tensor G if there are no WANDs and I,II,III, or N if the maximally aligned null vector has order of alignment 0, 1, 2, 3, respectively. Once a certain ℓ is chosen as a WAND with maximal order of alignment, it is possible to search for another vector $\mathbf{n}$ with maximal order of alignment subject to the constraint $\mathbf{n} \cdot \ell = 1$. If such a $\mathbf{n}$ is found we can similarly

N>4 dimensions		4 dimensions
Weyl type	alignment type	Petrov type
G	G	
I	(1)	
I_i	(1,1)	I
II	(2)	
II_i	(2,1)	II
D	(2,2)	D
III	(3)	
III_i	(3,1)	III
N	(4)	N

Table 2.2: Classification of the Weyl tensor in four and higher dimensions. Note that in four dimensions alignment type (1) is necessarily equivalent to the type (1,1), (2) to (2,1) and (3) to (3,1) and since there is always at least one PND, type G does not exist.

define *secondary alignment type.* Alignment type is a pair consisting of primary and secondary alignment types. Possible alignment types are summarized in Table 2. We also introduce *Weyl type* with notation emphasizing the link with the four dimensional Petrov classification.

To express the Weyl tensor in terms of its components in the ℓ, $\mathbf{n}$, m^i frame, introduce the operation { } such that:

$$w_{\{a}x_b y_c z_{d\}} \equiv \frac{1}{2}(w_{[a}x_{b]}y_{[c}z_{d]} + w_{[c}x_{d]}y_{[a}z_{b]}). \tag{2.17}$$

Now the Weyl tensor in arbitrary dimension can be written as

$$\begin{aligned} C_{abcd} = {} & \overbrace{4C_{1i1j}\,\mathbf{n}_{\{a}m^i_{\ b}\mathbf{n}_c m^j_{\ d\}}}^{2} + \overbrace{8C_{121i}\,\mathbf{n}_{\{a}\ell_b\mathbf{n}_c m^i_{\ d\}} + 4C_{1ijk}\,\mathbf{n}_{\{a}m^i_{\ b}m^j_{\ c}m^k_{\ d\}}}^{1} \\ & + \overbrace{4C_{1212}\;\mathbf{n}_{\{a}\ell_b\mathbf{n}_c\ell_{\ d\}} + C_{12ij}\;\mathbf{n}_{\{a}\ell_b m^i_{\ c}m^j_{\ d\}}}^{0} \\ & + \overbrace{8C_{1i2j}\;\mathbf{n}_{\{a}m^i_{\ b}\ell_c m^j_{\ d\}} + C_{ijkl}\;m^i_{\{a}m^j_{\ b}m^k_{\ c}m^{(l)}_{\ d\}}}^{0} \\ & + \overbrace{8C_{212i}\,\ell_{\{a}\mathbf{n}_b\ell_c m^i_{\ d\}} + 4C_{2ijk}\,\ell_{\{a}m^i_{\ b}m^j_{\ c}m^k_{\ d\}}}^{-1} + \overbrace{4C_{2i2j}\,\ell_{\{a}m^i_{\ b}\ell_c m^j_{\ d\}}}^{-2} \end{aligned} \tag{2.18}$$

where summation over i, j, k, l indices is implicitly assumed, and the numbers over the brackets indicate boost order.

The number of independent frame components of various boost weights are,

$$\overbrace{2\left(\frac{(m+2)(m-1)}{2}\right)}^{2,-2}+\overbrace{2\left(\frac{(m+1)m(m-1)}{3}\right)}^{1,-1}+\overbrace{\frac{m^2(m^2-1)}{12}+\frac{m(m-1)}{2}}^{0},$$

with $m = N - 2$. This is in agreement with the number of independent components of the Weyl tensor

$$\frac{(N+2)(N+1)N(N-3)}{12}. \tag{2.19}$$

Specializing to four dimensions and using the standard complex null tetrad (ℓ, $\mathbf{n}$, $\boldsymbol{m}$, $\bar{\boldsymbol{m}}$) [31, 22] the Weyl tensor has five complex components:

- $\Psi_0 = -C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \ell^\gamma m^\delta$
- $\Psi_1 = -C_{\alpha\beta\gamma\delta}\ell^\alpha \mathbf{n}^\beta \ell^\gamma m^\delta$
- $\Psi_2 = -C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \bar{m}^\gamma \mathbf{n}^\delta$
- $\Psi_3 = -C_{\alpha\beta\gamma\delta}\ell^\alpha \mathbf{n}^\beta \bar{m}^\gamma \mathbf{n}^\delta$
- $\Psi_4 = -C_{\alpha\beta\gamma\delta}\mathbf{n}^\alpha \bar{m}^\beta \mathbf{n}^\gamma \bar{m}^\delta$

We notice that the components of the Weyl tensor Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4 have boost weight -2,-1,0,1,2, respectively. When ℓ coincides with a PND then Ψ_0 vanishes, other components can be transformed away only in algebraically special cases. Namely, as can be seen from the Table 1, for types I, II, III, N one can transform away components with boost weight greater or equal to 2,1,0,-1, respectively. Type D is a special subcase of type II in which all components with non-zero boost weight can be eliminated.

2.4.2 Segre Classification for Rank Two Tensors

It was shown in [76] that Alignment allows for a partial classification of the Ricci tensor in four dimensions, however alignment alone does not provide a fine enough classification for the Ricci tensor; instead an older method called the Segre classification will be discussed. Using the fact that the Ricci tensor can be interpreted as a symmetric matrix we can classify any Ricci tensor by looking at its corresponding eigenvalues and eigenvectors. Given any rank two tensor F on the N-dimensional manifold with signature (r,s) s.t $r + s = N$, the components F_{ij} form a matrix of functions from the manifold to $\mathbb{R}$; where at an arbitrary point in M, the elements of T_pM and/or the elements of T_p^*M act as vectors for the matrix, depending on the type of the rank two tensor. It will be convenient to be able to classify rank two tensors, so the question arises: How do we distinguish between a pair of arbitrary rank two tensors F, and G?

By treating F and G as matrices at an arbitrary point p in the manifold this problem is resolved using their respective Jordan Canonical Form. If instead of the real numbers, an algebraically closed field like the complex numbers, $\mathbb{C}$ are used, then there exists basis transformations P and Q such that $F_J = PFP^{-1}$ and $G_J = QGQ^{-1}$, where F_J and G_J have the eigenvalues $\lambda_{f,i}$ and $\lambda_{g,j}$ of F and G respectively along the diagonal and potentially 1's along the off-diagonal [68] or equivalently both F and G can be expressed as the matrix sum $D_\lambda + Nil$ where D_λ is a $N \times N$ diagonal matrix containing the eigenvalues, and Nil is a $N \times N$ nilpotent matrix.

In order to find the eigenvalues of a matrix F, the characteristic polynomial arising from $det[F - \lambda I_s^r]$ must be solved; in a real vector space with signature (r,s) this can cause problems because the characteristic polynomial may have irreducible factors and since the real numbers are not algebraically closed the characteristic polynomial may factor into irreducible polynomials of degree greater than one. If this happens the Jordan Canonical for a real-valued matrix cannot be expressed without complexifying the vector space. If the complex numbers are used every complex valued matrix can be transformed into its Jordan canonical form because the characteristic polynomial can always be factored into a product of degree one polynomials.

If the eignenvalues are distinct then the vector space can be divided into invariant one dimensional subspaces, or eigenspaces. If an eigenvalue, λ_i is repeated p-times, then depending on the dimension of the nullspace of $F - \lambda_i Id$, there will be p eigenvectors or q eigenvectors and $p - q$ pseudo-vectors. Given a real valued matrix with at least one complex eigenvalue this defines an invariant two-dimensional real subspace, which arise from the real and imaginary part of the complex eigenvector. Using these ideas we will classify the matrices by the number and size of the invariant subspaces.

To describe this better, the particular form of the Jordan Canonical form must be examined. We will assume that V is a vector space over the complex numbers, since real valued matrices are included and their complexifications record information about the matrix in a convenient way. If F has $M \leq N$ eigenvalues the eigenvectors and associated pseudo-vectors of a matrix F are linearly independent and span the vector space, so this can be used as a basis for the matrix F; doing so the matrix is re-expressed as:

$$F = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \dots & \\ & & & A_m. \end{pmatrix}$$

where for $i \in [1, M]$, each A_i is a $a_i \times a_i$ diagonal block matrix made up of smaller matrices, called Simple Jordan blocks, B_{ij}, for $P \leq a_i$:

$$A_i = \begin{pmatrix} B_{i1} & & & \\ & B_{i2} & & \\ & & \dots & \\ & & & B_{iP}. \end{pmatrix}$$

Where the B_{ij} are $b_{ij} \times b_{ij}$ matrices with λ_i down the diagonal and possibly one's along the above-off-diagonal. The b_{ij} satisfy $b_{i1} + ... + b_{iP} = a_i$, and are called the geometric multiplicity, they correspond to the number of independent eigenvectors associated with an eigenvalue. Each a_i, $i \in [1, N]$ from the factorization of the characteristic polynomial denotes the algebraic multiplicity of the eigenvalue, which in a sense tells us the dimension of the entire eigenspace associated with an eigenvalue. Using the Segre Classification this can be recorded in a more compact form:

$$\{(b_{11}, ..., b_{1,P_1}), ..., (b_{M,1}, ..., b_{M,P_M})\}$$

Where each round bracket is associated with a particular algebraic multiplicity and, the b_{ij} are the geometric multiplicities. Naturally the Segre type of a matrix is unique up to ordering of the algebraic and geometric multiplicities.

At times when discussing higher dimensional spacetimes it will be helpful nonetheless to use the alignment method with respect to the Ricci tensor in order to get a broad classification, whence the Segre type is used for finer classification. In general the Ricci tensor is of the form:

$$R_{\alpha\beta} = \overbrace{\hat{R}_{22}\ell_{\{\alpha}\ell_{\ \beta\}}}^{-2} + \overbrace{\hat{R}_{i2}m^{i}{}_{\{\alpha}\ell_{\ \beta\}}}^{-1} + \overbrace{R_{12}n_{\ \{\alpha}\ell_{\ \beta\}} + R_{ij}m^{i}{}_{\{\alpha}m^{j}{}_{\beta\}}}^{0}$$
$$+ \overbrace{R_{i1}m^{i}{}_{\{\alpha}n_{\ \beta\}}}^{1} + \overbrace{R_{11}n_{\ \{\alpha}n_{\ \beta\}}}^{2}$$

We will say a Ricci Tensor is of type I, II, III, N, or O respectively, if there exists an aligned null vector ℓ along which the Ricci tensor the highest boost order is 1,0,-1,-2 or all components vanish. We will call this rough classification, the Ricci type.

2.4.3 The Karlhede Classification Algorithm

The problem of determining whether or not two given spacetimes are locally equivalent is vital to the classification of exact solutions in General Relativity. One approach to show the inequivalence of two spacetimes is achieved by taking the Riemann tensor and its covariant derivatives for each spacetime and contracting them to produce scalar curvature invariants. If their invariants differ the spacetimes are distinct; in the case that the invariants match can one conlude that the spacetimes are equivalent?

Consider any two vacuum PP-wave spacetimes, their scalar curvature invariants both vanish, however if one picks a sufficiently distinct $f(\zeta, u)$ and $f_o(\zeta, u)$ it is likely that the spacetimes will not be diffeomorphic. The PP-wave spacetimes belong to a class of spacetimes with vanishing curvature invariants, VSI, which are part of a larger class of constant curvature invariants, CSI, which contains the (locally) homogeneous spacetimes as well [94], [94]. For all of these spacetimes, we cannot use the scalar curvature invariant approach to differentiate between them.

As an alternative we adapt a method introduced by Cartan, applicable to the equivalence of sets of differential forms on manifolds under appropriate transformation groups. [42]. Assuming the metric is smooth enough, this produces a unique local characterization, well-suited for comparing metrics. To do so, we take construct coframes from their respective metrics and calculate the exterior derivative of the coframe for both respectively:

$$d\omega^i = C^i{}_{jk}\omega^j \wedge \omega^k.$$

If the metrics are equivalent, both of the first set of Cartan structure equations will agree. Of course, this is necessary but not sufficient for showing equivalence.

To provide sufficient conditions one must take the exterior derivative repeatedly, starting with $dC^{i}{}_{jk}$, until no new functionally independent functions appear. The procedure terminates when differentiation produces no such new functions arise, as any further derivatives will depend on the previous invariants. As these are invariants any relationship between independent and dependent invariants will be the same regardless of which coordinate system used. The number of functionally independent invariants, k, is called the *rank*, and it will be assumed to be constant in a neighborhood whose points are called *regular*. As the dimension of the manifold is N, $k \leq N$ (where inequality occurs if the space admits symmetries), this procedure must end at some finite step.

Finally one must equate the quantities in both manifolds and determine whether the relations arising from this can be solved, although this may be formally undecidable. For manifolds admitting a metric, the Cartan equations show that the repeated differentiation of the structure equations is equivalent to the repeated covariant differentiation of the Riemann tensor viewed as members of $\mathbb{F}(M)$ And so, we may characterize a metric uniquely by the components of the curvature and its covariant derivatives. Denoting $\mathbb{R}^q$ as the set $\{R_{abcd}, R_{abcd;e}, \quad R_{abcd;e_1 \ldots e_q}\}$ of the components of the curvature tensor and its derivatives up to the q-th order. If p is the last derivative at which any new functionally independent quantities arise, we must calculate $\mathbb{R}^{p+1}$ to determine all classifying functions and we say p is the order.

Christoffel was the first to work on the equivalence problem for Riemannian manifolds [1] using the full coordinate frame bundle of dimension $N(N+1)$. This approach dealt only with metrics admitting no symmetries and implied that the twentieth derivatives of the curvature would be required to determine equivalence of four dimensional spacetimes; this is still computationally impractical. In 1946, Cartan reduced the upper-bound significantly using frames with constant metric components, in which case the dimension of the space is reduced to $\frac{1}{2}N(N+1)$, and so the maximum order for four dimensional spacetimes will be 10. Cartan also showed how this approach could be expanded to metrics admitting symmetries, this was completed by Sternberg [12]. For $k < \frac{1}{2}N(N+1)$ there is an isometry group of dimension $N(N+1) - k$. Following this work, we have the following result [25]

Theorem 2.4.1. *Let M and $\bar{M}$ be spacetimes of differentiability class C^{13}, x be a* *regular point of M and E be a frame at x and similarly for $\bar{M}$. Then there is an isometry which maps (x, E) to $(\bar{x}\bar{E})$ if and only if $\mathbb{R}^{p+1}$ for M is such that:*

1. *The set of indices of the corresponding components of $\mathbb{R}^p$, $\mathbb{A}$ indexes quantities $\bar{I}^\alpha$ which are functionally independent in $\mathbb{F}(M)$,*

2. *$I^\alpha(x, E) = \bar{I}^\alpha(\bar{x}, \bar{E})$ for $\alpha = 1, ..., k$, and*

3. *the functions giving all other components of $\mathbb{R}^{p+1}$ in terms of the I^α and $\bar{I}^\alpha$ are the same for M and $\bar{M}$.*

The above theorem has been applied practically in [13, 19] where in the first used a scheme using canonical forms chosen by lexicography of bases, while the later considered canonical forms of the Weyl tensor at the first step. This is similar to the algorithm introduced by Karlhede and implemented by Aman and others [21, 24, 41]. The Karlhede algorithm for classifying geometries has become a prefered method for classification of spacetimes.

In this section we briefly review the various steps of the algorithm. We start by calculating the Riemann tensor in a particular frame and its higher derivatives up to a particular order, using the various frame transformations to simplify the components at each order of differentiation until a complete classification of the geometry has been obtained. We denote the frame components of the Riemann tensor and its covariant derivatives up to the qth order by R^q, with this the algorithm may be written as:

1. Let $q = 0$.

2. Compute R^q.

3. Fix the frame as much as possible using frame transformations (spins, boosts, rotations, null-rotations).

4. Find the invariance group H^q of the frame which leaves R^q invariant.

5. Find the number of functionally independent components t^q amongst the set R^q.

6. If $t^q \neq t^{q-1}$ or $dim(H^q) \neq dim(H^{q-1})$ then set $q = q + 1$ and go to step 2.

7. Otherwise the set $\{H^p, t^p, R^p\},\ p = 1, ..., q$ classifies the solution and the dimension of the isometry group I of the metric follows from the result [4] $dim(I) = dim(H^q) + N - t^q$.

   If we wish to compare two metrics g and g_o for equivalence, we start by completing the above classification for each metric. The remainder of the algorithm is summarized in the next two steps.

8. If the two sequences $(H^0, t^0; H^1, t^1; ...; H^q, t^q)$ for g and g_o differ, then so do the metrics.

9. If the set of simultaneous algebraic or transcendental equations $R^0 = R^0_o, R1 = R^1_o, ..., R^q = R^q_o$ admit a coordinate trnasformation $x^i_o = x^i_o(x^j)\ i, j = 1, ..., N$ as a solution then the metrics are equivalent, otherwise they are inequivalent. We note that this step is not algorithmic, as there is no constructive procedure for solving simultaneous algebraic equations.

2.5 General Relativity in Brief

General Relativity is the theory of spacetime and gravitation, introduced in 1915 by Albert Einstein, which unified Special Relativity and Newton's Law of Universal gravitation. Unlike classical mechanics gravity is not treated as another force affecting objects, but rather as the curvature of space and time caused by matter and/or energy being present; so that an object affected by gravity is not being pushed towards the ground, but rather is moving along the path of "least resistance" in the curved geometry of space and time. This leads to another difference between Newtonian physics and General Relativity. Before we treated time as a distinct parameter tracing curves in a three dimensional Riemannian manifold; time was an absolute, independent of the observers velocity or acceleration and so two simultaneous events happen at time $t = t_0$ for all observers. On the other hand Special Relativity treats space and time in a single Lorentzian manifold called a Minkowski space, it is worth noting that we no longer have an idea of absolute time. General Relativity goes further by allowing curved Lorentzian manifolds in lieu of Minkowski space as models for our spacetime. Exactly how the spacetime becomes curved is described by the

Einstein Field equations:

$$G_{\alpha\beta} = (8\pi C)T_{\alpha\beta}$$

where c is the speed of light, $G_{uv} = R_{uv} - \frac{1}{2}Rg_{uv}$ is called the Einstein tensor which is made up of the Ricci tensor R_{uv}, the Ricci Scalar R and the metric of the spacetime g_{uv}, and T_{uv} is the stress-energy tensor. The stress-energy tensor is a symmetric rank two tensor that essentially describes the density and flux of energy and momentum in the spacetime; the most common stress-energy tensors used in General Relativity, are those for perfect fluids, pure radiation fields, electromagnetic fields and vacuum fields (A better summary of these stress-energy tensors and their properties is given in [22].). Due to the symmetries of the tensors involved this gives rise to in general ten non-linear partial differential equations which can be very difficult to solve. Often in order to solve these solutions, assumptions must be made to simplify the equations. The resulting solutions to the field equations are called Exact Solutions, and they have played an important role in the discussion of physical problems. For example Friedmann's solution contributed to Cosmology by providing an argument for the big bang, while the Kerr and Schwarzchild solutions gave theoretical evidence for the existence of black holes.

Chapter 3

Degenerate Kundt Spacetimes

3.1 Kundt Spacetimes

In this section we introduce a complete local description of the general N-dimensional Kundt spacetimes. This is done using a kinematic description, where the Kundt metrics are defined as those admitting a null vector, ℓ which is geodesic, non-expanding, shear-free and non-twisting. In four dimensions one recovers a well-known theorem [92]

Theorem 3.1.1. *A space M defines (locally) a Kundt geometry in 4 dimensions if and only if there exists (locally) a family of frames $\{e_a\} = \{\ell, \mathbf{n}, m, \bar{m}\}$, defined up to null rotations about ℓ and characterized by the conditions that*

$$\kappa = \rho = \sigma = 0$$

for the spin connection coefficients. Also, the generic line element is given, in a local coordinate system $\{x^\alpha\} = \{u, v\zeta, \bar{\zeta}\}$:

$$ds^2 = 2du(Hdu + dv + Wd\zeta + \bar{W}d\bar{\zeta}) - 2P^{-2}d\zeta d\bar{\zeta}, \quad P_{,v} = 0$$

In higher dimension the conditions involving the spin connection coefficients may be replaced with the following invariants derived from the covariant derivative of ℓ

$$\ell^i \ell_{j;i} = \ell^i{}_{;i} = \ell^{i;j}\ell_{(i;j)} = \ell^{i;j}\ell_{[i;j]} = 0.$$

The existence of such a null vector ℓ implies that there is a local coordinate system (u, v, x^i) such that [65, 92]

$$ds^2 = 2du\,(dv + Hdu + W_e dx^e) + \tilde{g}_{ef}(u, x^g)dx^e dx^f \tag{3.1}$$

where $H = H(v, u, x^e)$, $W_e = W_e(v, u, x^f)$. The coordinate transformations that preserve this metric are:

1. $(v', u', x'^e) = (v, u, f^e(u; x^f))$, and $J^e{}_f \equiv \frac{\partial f^e}{\partial x^f}$.

$$\begin{aligned} H' &= H + g_{ef} f^e_{,u} f^f_{,u} - W_f \left(J^{-1}\right)^f{}_e f^e_{,u}, \quad W'_e = W_f \left(J^{-1}\right)^f{}_e - g_{ef} f^f_{,u} \\ g'_{ef} &= g_{gl} \left(J^{-1}\right)^g{}_e \left(J^{-1}\right)^l{}_f \end{aligned} \tag{3.2}$$

2. $(v', u', x'^e) = (v + h(u, x^g), u, x^e)$

$$H' = H - h_{,u}, \quad W'_e = W_e - h_{,e} \quad g'_{ef} = g_{ef} \tag{3.3}$$

3. $(v', u', x'^e) = (v/g_{,u}(u), g(u), x^e)$

$$H' = \frac{1}{g^2_{,u}} \left(H + v \frac{g_{,uu}}{g_{,u}} \right) \quad W'_e = \frac{1}{g_{,u}} W_e, \quad g'_{ef} = g_{ef} \tag{3.4}$$

The Kundt spacetimes are of particular relevance to the equivalence problem, as they contain a subclass of metrics which are not uniquely determined by their polynomial scalar curvature invariants. To elaborate on this point we introduce two definitions:

Definition 3.1.2. For a spacetime (M, g), a metric deformationm, $\hat{g}_\tau$,$\tau \in [0, \epsilon)$, is a family of smooth metrics on M such that

1. $\hat{g}_\tau$ is continuous in τ,

2. $\hat{g}_0 = g$, and

3. $\hat{g}_\tau$ for $\tau > 0$ is not diffeomorphic to g.

For any spacetime we define the set of all scalar polynomial curvature invariants

$$I = \{R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\alpha\beta}C^{\mu,\nu,\alpha\beta}, R_{\mu\nu\alpha\beta;\gamma}R^{\mu\nu\alpha\beta;\gamma}, ...\}.$$

In a sense the set of invariants acts a function of the metric and its derivatives. It is of interest when this function has an inverse and to what extent.

Definition 3.1.3. Consider a spacetime (M, g) with a set of invariants I. If there does not exist a metric deformation of g having the same set of invariants as g we call the set of invariants non-degenerate, and the metric g will be called I-non-degenerate.

Thus a metric which is I-non-degenerate will be uniquely characterized by the set of invariants in I. Those metrics which are not I-non-degenerate cannot be classified by their scalar curvature invariants, for these spacetimes one must compare Cartan invariants using the Karlhede algorithm. In the Kundt class, those metrics for which the kinematic frame and null curvature frame are aligned. That is, relative to the kinematic frame, the Riemann tensor and its covariant derivatives have vanishing positive boost-weight components; imposing this condition we find that $W_{,vv} = 0$ and $H_{,vvv} = 0$ are necessary conditions for the degenerate Kundt metrics,

$$ds^2 = 2du\,(dv + Hdu + W_e dx^e) + \tilde{g}_{ef}(u, x^g)dx^e dx^f, \quad W_{,vv} = H_{,vvv} = 0. \tag{3.5}$$

Using the invariant and covariant quantities [92]

$$I_0 = R^{abcd} R_{a\ c}^{\ e\ f} \mathbb{L}_\ell \mathbb{L}_\ell g_{bd} \mathbb{L}_\ell \mathbb{L}_\ell g_{ef},, \quad K_{ab} = \mathbb{L}_\ell \mathbb{L}_\ell g_{ab}$$

one may differentiate between the degenerate Kundt metrics and the remainder of the Kundt class by combining Theorem 4.2 and Proposition 6.1 in [92]:

Proposition 3.1.4. *Within the Kundt class, if I_0 and K_{ab} are identically zero, the metric belongs to the degenerate Kundt class.*

In four dimensions it has been shown that a spacetime is either I-non-degenerate, a locally homogeneous space or a degenerate Kundt spacetime [94].

3.2 CSI Spacetimes

There is a class of spacetimes for which all polynomial scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant[1], called CSI spacetimes; those Lorentzian manifolds for which all curvature invariants vanish identically are called VSI spacetimes, with $VSI \subset CSI$. The set of all locally homogeneous spacetimes (denoted by H) are a subset of the class of CSI spacetimes (i.e., $H \subset CSI$) as well. Let us denote by CSI_R all reducible CSI spacetimes that can be built from VSI and H by (i) warped products (ii) fibered products, and (iii) tensor sums. Similarly we denote by CSI_F those spacetimes for which there exists a frame with a null vector ℓ such that all components of the Riemann tensor and its covariant derivatives

[1] For brevity we will call these curvature invariants

in this frame have the property that (i) all positive boost weight components (with respect to ℓ) are zero and (ii) all zero boost weight components are constant. Finally, let us denote by CSI_K, those CSI spacetimes that belong to the (higher-dimensional) Kundt class (the form of its metric is given by (3.1)); the so-called Kundt CSI spacetimes. By construction CSI_R is at least of Weyl type II (i.e.,, of type II, III, N or O [66]), and by definition CSI_F and CSI_K are at least of Weyl type II (more precisely, at least of Riemann type II).

A Riemannian manifold with constant scalar invariants is immediately homogeneous, ($CSI \equiv H$). This is not true for Lorentzian manifolds, there are examples of CSI spacetimes which consists of homogeneous spaces and a certain subclass of the Kundt spacetimes [79]. Interestingly, for every CSI spacetime there is a homogeneous spacetime (not necessarily unique) with precisely the same constant invariants. This suggests that CSI spacetimes can be constructed from homogeneous spacetimes H and VSI spacetimes using warped and fibered products (e.g., CSI_R). In particular, the relationships between CSI_R, CSI_F, CSI_K and especially with $CSI \backslash H$ were studied in arbitrary dimensions and considered in more detail for the four dimensional case in [79, 94]. A CSI spacetimes is either:

- (A) of Petrov (P)-type I or Plebanski-Petrov (PP)-type I, or
- (B) of P-type II and PP-type II.

It is plausible that in case A, since the spacetime is not of P-type II and PP-type II, it is necessarily completely backsolvable (CB) [59]. Furthermore due to all of the zeroth order scalar curvature invariants being constant, there exists a frame in which the components of the Riemann tensor are all constant, and the spacetime is curvature homogeneous of order 0, CH_0. If the spacetime is also CH_1, then it is necessarily locally homogeneous H [78]; therefore, if it is not locally homogeneous, it cannot be CH_1. Then by considering differential scalar invariants, we can obtain information on the spin coefficients in the CSI spacetime. Therefore, in this case there are severe constraints on those spacetimes that are not locally homogeneous. Necessarily the spacetime is of P-type I or PP-type I (and does *not* belong to CSI_K, CSI_F, or CSI_R), it belongs to $CSI_0 \equiv CH_0$, but not CH_1 with a number of further constraints arising from the non-CB conditions and the differential constraints; and it is plausible that there are no such spacetimes.

In case B, we have that the spacetime is necessarily of P-type II and PP-type II. It then follows from Theorem 7.1 in [79], that all boost weight zero terms are necessarily constant. The spacetime is then either CB, in which case all of the results in case A apply, or they are NCB and a number of further conditions apply (these conditions are very severe [59]). In either case, there are a number of different classes characterized by their P-type and PP-type (in each case at least of type II), and in each class there are a number of further restrictions. By investigating the differential scalar invariants we then find conditions on the spin coefficients, and it was conjectured that all of these spacetimes are necessarily CSI_K. In four dimensions the CSI_R, CSI_F and CSI_K spacetimes are closely related, and it is plausible that $CSI\backslash H$ is at most of Weyl type II.

Motivated by these results, the authors of [79] proposed higher-dimensional analogues of the CSI_F, CSI_R and CSI_K conjectures. These conjectures have all been proven in three and four dimensions [89, 94]. In higher dimensions, CSI_F conjecture has been proven as a corollary to the more general result for tensors whose scalar curvature invariants do not uniquely characterize them [103, 104].

Theorem 3.2.1 (CSI_F Theorem)**.** *A spacetime is CSI if and only if there exists a null frame in which the Riemann tensor and its derivatives can be brought into one of the following forms: either*

1. *The Riemann tensor and its derivatives are constant, in which case we have a locally homogeneous space, or*

2. *The Riemann tensor and its derivatives are of boost order zero with constant boost weight zero components at each order. This implies that the Riemann tensor is of type II or less.*

Assuming that there exists such a preferred null frame, then

Conjecture 3.2.2 (CSI_K conjecture)**.** *If a spacetime is CSI, then the spacetime is either locally homogeneous or belongs to the higher dimensional Kundt CSI class.*

And lastly,

Conjecture 3.2.3 (CSI_R conjecture)**.** *If a spacetime is CSI , then it can be constructed from locally homogeneous spaces and VSI spacetimes.*

This construction can be done by means of fibering, warping and tensor sums. From the results above and these conjectures, it is plausible that for CSI spacetimes that are not locally homogeneous, the Weyl type is II, III, N or O, and that all boost weight zero terms are constant. In four dimensions, these results have been verified so that every CSI spacetime is immediately a member of the degenerate Kundt class of metrics.

If a particular Kundt spacetime possesses a frame $\{\ell, \mathbf{n}, m^i{}_e\}$ such that the components of the Riemann are constant $R_{\alpha\beta\delta\gamma} = R_{abcd} m^a{}_\alpha m^b{}_\beta m^c{}_\delta m^d{}_\gamma$ and all of its covariant derivatives $R_{\alpha\beta\delta\gamma;\epsilon_1,\ldots\epsilon_n} = R_{abcd;e_1\ldots e_n} m^a{}_\alpha \ldots m^d{}_{de} m^{e_1}_{\epsilon_1} \ldots m^{e_n}_{\epsilon_n}$ $\forall n \in \mathbb{Z}$ are constant, then this spacetime lies in $CSI \subset CSI_F \bigcap CSI_K$. Furthermore there exists (locally) a coordinate transformation such that the metric form is preserved and

$$\tilde{g}_{ef} \equiv \tilde{g}'_{gl} \frac{\partial f^g}{\partial x^e} \frac{\partial f^l}{\partial x^f}, \quad \tilde{g}'_{ef,u'} = 0 \tag{3.6}$$

In the case of VSI spacetimes, from [79] we are assured of the existence of a coordinate change $(v', u', x'^i) = (v, u, F^i(u; x^k))$ such that the transverse metric components take the form:

$$\tilde{g}_{ef} \equiv \tilde{g}'_{kl} \frac{\partial F^k}{\partial x^e} \frac{\partial F^l}{\partial x^f}, \quad \tilde{g}'_{ef} = \delta_{ef} \tag{3.7}$$

This agrees with (3.17), the canonical metric for VSI spacetimes. Moreover, in the more general case of CSI spacetimes, Theorem 4.1 in [79], assures us that the transverse space is locally homogeneous.

Given a Kundt metric satisfying (3.6) then it is CSI_0 and the following Riemann components are constant: with $m^i_e m_{if} = g_{ef}$, the non-zero frame connection components are:

$$\Gamma_{21i} = \tfrac{D_1 W_i}{2}, \quad \Gamma_{212} = D_1 H, \quad \Gamma_{2i2} = D_i H - D_2 W_i, \tag{3.8}$$

$$\Gamma_{i12} = \tfrac{D_1 W_i}{2}, \quad \Gamma_{i21} = \tfrac{D_1 W_i}{2}, \quad \Gamma_{i2j} = \tfrac{A_{ij}}{2}, \quad \Gamma_{ij2} = \tfrac{A_{ij}}{2}, \tag{3.9}$$

$$\Gamma_{ijk} = -\tfrac{1}{2} \left(D_{ijk} + D_{jki} - D_{kij} \right) \tag{3.10}$$

where the tensors involved are written in terms of

$$m_{ie}, \quad D_{ijk} = 2 m_{ie,f} m_{[j}{}^e m_{k]}{}^f, \quad A_{ij} = D_{[j} W_{i]} - D_{kji} W^k = 2 W_{[i;j]}.$$

The linearly independent components of the Riemann tensor with boost weight 1 and 0 may be written as:

$$
\begin{aligned}
R_{121i} &= -\frac{1}{2}W_{i,vv} \\
R_{1212} &= -H_{,vv} + \frac{1}{4}\left(W_{i,v}\right)\left(W^{i,v}\right), \\
R_{12ij} &= W_{[i}W_{j],vv} + W_{[i;j],v}, \\
R_{1i2j} &= \frac{1}{2}\left[-W_j W_{i,vv} + W_{i;j,v} - \frac{1}{2}\left(W_{i,v}\right)\left(W_{j,v}\right)\right], \\
R_{ij\hat{i}\hat{j}} &= \tilde{R}_{ij\hat{i}\hat{j}}.
\end{aligned}
$$

The spacetime will be CSI_0 if there exists a frame $\{\ell, n, m^i\}$, a constant σ, anti-symmetric matrix $\mathsf{a}_{\hat{i}\hat{j}}$, and symmetric matrix $\mathsf{s}_{\hat{i}\hat{j}}$ such that:

$$
W_{\hat{i},vv} = 0, \tag{3.11}
$$

$$
H_{,vv} - \frac{1}{4}\left(W_{\hat{i},v}\right)\left(W^{\hat{i},v}\right) = \sigma, \tag{3.12}
$$

$$
W_{[\hat{i};\hat{j}],v} = \mathsf{a}_{\hat{i}\hat{j}}, \tag{3.13}
$$

$$
W_{(\hat{i};\hat{j}),v} - \frac{1}{2}\left(W_{\hat{i},v}\right)\left(W_{\hat{j},v}\right) = \mathsf{s}_{\hat{i}\hat{j}}, \tag{3.14}
$$

and the components $\tilde{R}_{ij\hat{i}\hat{j}}$ are all constants. That is, the tranverse metric dS_H^2 is curvature homogeneous. Integrating the above constraints gives the following form for the metric functions:

$$
W_i(v,u,x^e) = vW_i^{(1)}(u,x^e) + W_i^{(0)}(u,x^e), \tag{3.15}
$$

$$
H(v,u,x^e) = \frac{v^2}{8}\left[4\sigma + (W_i^{(1)})(W^{(1)i})\right] + vH^{(1)}(u,x^e) + H^{(0)}(u,x^e). \tag{3.16}
$$

If we further require that that our CSI_0 Kundt spacetime is also CSI_1 we have that

$$
\begin{aligned}
R_{121i;2} &= -\frac{1}{2}\left[\sigma W_{i,v} - \frac{1}{2}(\mathsf{s}_{ji} + \mathsf{a}_{ji})W^{j,v}\right] \\
R_{1ijk;2} &= -\frac{1}{2}\left[W^{n,v}\tilde{R}_{nijk} - W_{i,v}\mathsf{a}_{jk} + (\mathsf{s}_{i[j} + \mathsf{a}_{i[j})W_{k],v}\right]
\end{aligned}
$$

are constant, that is,

$$
\begin{aligned}
\sigma W_{i,v} - \frac{1}{2}(\mathsf{s}_{ji} + \mathsf{a}_{ji})W^{j,v} &= \alpha_i \\
(\mathsf{s}_{ij} + \mathsf{a}_{ij})_{;k} - (\mathsf{s}_{ik} + \mathsf{a}_{ik})_{;f} &= \beta_{ijk}
\end{aligned}
$$

where the Ricci identity has been used to rewrite the latter condition.

In the case where $\sigma = 0$ & $\mathsf{a}_{\hat{e}\hat{f}} = \mathsf{s}_{\hat{e}\hat{f}} = 0$, then the spacetime belongs to the VSI_0 class, and it will be of Ricci and Weyl type III, N, O. VSI_1 spacetimes will arise from the vanishing of certain linear combinations of the scalars derived from $\ell_{\alpha;\beta}$; these spacetimes have been studied in four dimensions in [78]. Less is known about these spacetimes in higher dimensions, but it is plausible that those spacetimes that are properly VSI_1 (i.e., zero and first order invariants vanish but spacetime is not VSI) are of Weyl type N, Ricci-Type N or O and admit an aligned geodesic null congruence.

3.3 VSI Spacetimes

There is more known about the subclass of CSI spacetimes in which all curvature invariants vanish, the VSI spacetimes. From [65] we have the following theorem:

Theorem 3.3.1 (VSI Theorem)**.** *All curvature invariants of all orders vanish in an N-dimensional Lorentzian spacetime if and only if there exists an aligned, non-expanding, non-twisting, shear-free, geodesic null direction ℓ^α along which the Riemann tensor has negative boost order.*

To be precise, there exists a null vector ℓ such that

$$\ell_{\alpha;\beta} = L_{11}\ell_\alpha\ell_\beta + L_{1i}\ell_\alpha m^i{}_\beta + L_{i1}m^i{}_\alpha\ell_\beta$$

While if we choose a frame including ℓ as a basis vector, the Riemann tensor will be of type III, N or O, this implies that all VSI spacetimes belong to the Generalized Kundt class. In fact it was shown in [80] that there is a canonical form for the metrics of VSI spacetimes:

$$ds^2 = 2du(dv + Hdu + W_e dx^e) + \delta_{ef}dx^e dx^f, \tag{3.17}$$

where the metric functions satisfy certain constraints that will be discussed in the next subsection. The authors of [80] go further by classifying all higher-dimensional VSI spacetimes according to their Weyl type, Ricci-type and whether W_1 has v-dependence or not.

3.4 $CCNV$ Metrics

As was noted in [80], in the case of a Kundt spacetime the aligned, repeated, null vector of (3.17) is also a null Killing vector if and only if all of the metric functions are independent of $\ell = \frac{\partial}{\partial v}$; that is, the metric takes the form:

$$ds^2 = 2du\left(dv + H(u,x^e)du + W_e(u,x^f)dx^e\right) + \tilde{g}_{ef}(u,x^g)dx^e dx^f \tag{3.18}$$

This forces all of the components of L_{ab} to vanish, which implies ℓ is a covariantly constant null vector, i.e., $\ell_{a;b} = 0$. We will say that a spacetime admitting a covariantly constant null vector is a $CCNV$ spacetime. From [80] all of the $CCNV$ VSI spacetimes were listed as special cases of the canonical Kundt metric (3.17), and there are also $CCNV$ CSI spacetimes. All $CCNV$ VSI spacetimes are of Weyl type III and Ricci Type III. In general, if there exists an ℓ that is a $CCNV$ then from the Ricci identity $\ell^\alpha R_{\alpha\beta\gamma\delta} = 0$, we expect in CSI spacetimes the Riemann tensor to be of type II or less.

To calculate the connection coefficients for the above $CCNV$ metric, we define the useful tensor

$$B_{ij} = m_{ie,u} m_j{}^e. \tag{3.19}$$

Then with the other two tensors given as A_{ij} and D_{ijk} the connection coefficients are:

$$\Gamma_{2i2} = (H_{,e} - W_{e,u}) m_i{}^e \tag{3.20}$$

$$\Gamma_{2ij} = -\frac{1}{2}A_{ij} - B_{(ij)} \tag{3.21}$$

$$\Gamma_{ij2} = \frac{1}{2}A_{ij} - B_{[ij]} \tag{3.22}$$

$$\Gamma_{ijk} = -\frac{1}{2}[D_{ijk} + D_{jki} + D_{kji}] \tag{3.23}$$

the non-vanishing curvature components are then:

$$R_{2ij2} = \Gamma_{2i2,e} m_j{}^e - (\Gamma_{2ik} m^k{}_e)_{,u} m_j{}^e - \Gamma_{2a2}\Gamma_{aij} + \Gamma_{2aj}\Gamma_{ai2} \tag{3.24}$$

$$R_{2ijk} = (\Gamma_{2il} m^l{}_{[e}),_{f]} (m_j{}^f m_k{}^e + \Gamma_{2aj}\Gamma_{aik} - \Gamma_{2ak}\Gamma_{aij} \tag{3.25}$$

$$R_{ijkl} = (\Gamma_{ij\hat{i}} m^{\hat{i}}{}_{[\hat{j}}),_{\hat{k}]} m_k{}^{\hat{k}} m_l{}^{\hat{j}} - \Gamma_{\hat{i}\hat{i}l}\Gamma_{\hat{i}jk} - \Gamma_{\hat{i}\hat{i}k}\Gamma_{\hat{i}jl} \tag{3.26}$$

By calculating the curvature tensor of the metric (3.18), and its covariant derivatives, we note that $R_{ijkl} = \widetilde{R}_{ijkl}$, where the tilde refers to curvature tensors with respect to the spatial metric $\tilde{g}_{ef}(u, x^g)$. This metric is Riemannian and depends on the 'parameter' u. In particular, R_{ijkl} is the Riemann tensor of the 'transverse space'.

The non-zero components of the Ricci tensor are:

$$R_{22} = R_{i2i2} = -R_{2ii2} \tag{3.27}$$

$$R_{2i} = R_{j2ji} = -R_{2jji} \tag{3.28}$$

$$R_{ij} = R_{kikj} \tag{3.29}$$

We can express (3.28) in a convenient form by expanding (3.25) and replacing the directional derivatives in terms of covariant derivatives we obtain

$$R_{2ijk} = \Gamma_{2ik|j} - \Gamma_{2ij|k} + \Gamma_{2il}(\Gamma_{lkj} - \Gamma_{ljk} + C_{ljk} - C_{lkj})$$

where $_{|i} = m_i{}^e\nabla_e$ and $C_{ijk} = m_{ij_*,l}m^l_j m^{j_*}_k$, this becomes

$$R_{2ijk} = \Gamma_{2ik|j} - \Gamma_{2ij|k}. \tag{3.30}$$

Therefore the boost weight -1 components of the Ricci tensor can be expressed in terms of a covariant derivative and a divergence,

$$R_{2i} = \Gamma_{2jj|i} - \Gamma_{2jil|j}$$

From the definition of B_{ij} follows the identity

$$B_{ij} + B_{ji} = m_{ie}m_{jf}\partial_u g^{ef} \tag{3.31}$$

by (3.21) this yields $\Gamma_{2ii} = -B_{ii} = -\frac{1}{2}g^{ij}\partial_u g_{ij}$ and consequently

$$2R_{2i} = -(g^{jk}\partial_u g_{jk})_{|i} - A_{il|l} + (B_{il} + B_{li})_{|l} \tag{3.32}$$

We note that by contracting (3.31) with $m^i{}_e m^{jf}$ gives $B_i{}^j + B^j{}_i = g^{ik}\partial_u g_{ki}$ in terms of coordinate indices. This can be used to express the last term of (3.32) in terms of a divergence over the coordinate index j.

3.4.1 Criteria for a $CCNV$ Metric to be CSI or VSI

It has been shown that the line-element (3.18) for a spacetime admitting a covariantly constant null vector has a Riemann curvature tensor with the following boost weight decomposition:

$$R_{\alpha\beta\gamma\delta} = \overbrace{R_{ijkl}m^i{}_{\{\alpha}m^j{}_\beta m^k{}_\gamma m^l{}_{\delta\}}}^{0} + \overbrace{R_{2jkl}\ell_{\{\alpha}m^j{}_\beta m^k{}_\gamma m^l{}_{\delta\}}}^{-1} + \overbrace{R_{2j2l}\ell_{\{\alpha}m^j{}_\beta \ell_\gamma m^l{}_{\delta\}}}^{-2} \tag{3.33}$$

$$\begin{aligned}
R_{2ij2} &= \Gamma_{2i2,e}m_j{}^e - (\Gamma_{2ik}m^k{}_{e})_{,u}m_j{}^e - \Gamma_{2a2}\Gamma_{aij} + \Gamma_{2aj}\Gamma_{ai2} \\
R_{2ijk} &= (\Gamma_{2il}m^l{}_{[e})_{,f]}(m_j{}^f m_k{}^e + \Gamma_{2aj}\Gamma_{aik} - \Gamma_{2ak}\Gamma_{aij} \\
R_{ijkl} &= (\Gamma_{ij\hat{\imath}}m^{\hat{\imath}}{}_{[\hat{\jmath}})_{,\hat{k}]}m_k{}^{\hat{k}}m_l{}^{\hat{\jmath}} - \Gamma_{\hat{\imath}\hat{\imath}l}\Gamma_{\hat{\imath}jk} - \Gamma_{\hat{\imath}\hat{\imath}k}\Gamma_{\hat{\imath}jl}
\end{aligned} \tag{3.34}$$

where all of the boost weight 0 components arise from the curvature of the Riemannian transverse space. In order to take the covariant derivative of (3.33), we first consider the covariant derivative of the frame components R_{abcd}:

$$\nabla_\epsilon R_{abcd} = \ell\,{}_\epsilon D_2 R_{abcd} + m^i{}_\epsilon D_i R_{abcd} \tag{3.35}$$

where we have used the v-independence of (3.18) to set $D_1 R_{abcd} = 0$. The covariant derivatives of the frame one-forms, m_i & ℓ are $\nabla_\epsilon \ell_\alpha = 0$ and

$$\nabla_\epsilon m_{i\alpha} = \Gamma_{lin}m^l{}_\alpha m^n{}_\epsilon + \Gamma_{li2}m^l{}_\alpha \ell_\epsilon + \Gamma_{2il}\ell_\alpha m^l{}_\epsilon + \Gamma_{2i2}\ell_\alpha \ell_\epsilon\,. \tag{3.36}$$

Thus (3.35) and (3.36) imply that ∇ does not raise boost weight because covariant differentiation does not introduce the null vector $\mathbf{n}$ into the expressions; as a result $\nabla_\epsilon R_{\alpha\beta\gamma\delta}$ will contain frame components whose highest boost weight is 0 and these will correspond only to the covariant derivative of the curvature of the Riemannian transverse space. Using an inductive argument, this can be shown to hold for any number of covariant derivatives of the Riemann tensor. Let the k^{th} covariant derivative of (3.33) be represented symbolically as $\nabla^k R$, then we can say that $\nabla^k R$ has frame components whose highest boost weight is zero and ℓ contracted with any index of $\nabla^k R$ vanishes; i.e., $\nabla^k R \cdot \ell = 0$.

It now follows that all curvature invariants of (3.18) will be completely equivalent to the curvature invariants of the transverse space. Therefore, if we impose the CSI condition on (3.18), we are requiring the transverse space to be a CSI Riemannian metric. From a theorem of [F. Prufer from 1996] we conclude that the transverse metric is locally homogeneous, which establishes the following,

Lemma 3.4.1. *A generalized Kundt metric admitting a CCNV is CSI if and only if the transverse metric is locally homogeneous.*

Now, consider the Ricci invariant $r_2 = R_{ab}R^{ab} = R_{ij}R^{ij}$, where the second equality follows from the form of the Ricci tensor (3.28)-(3.29), which shows that the boost weight 0 components arise solely from the transverse metric. Since $r_2 = \sum_{i,j}(R_{ij})^2$

is a sum of squares, we have that if $r_2 = 0$ then $R_{ij} = 0$. A theorem of [D.V. Alekseevski 1975] states that a homogeneous Riemannian space that is Ricci-flat is flat. Therefore, combining these with Lemma (3.4.1) gives the result:

Proposition 3.4.2. *If a generalized Kundt metric admitting a $CCNV$ is CSI and $R_{ab}R^{ab} = 0$ then the metric is VSI.*

Although $R_{\alpha\beta\gamma\delta}\ell^\alpha = 0 = R_{\alpha\beta}\ell^\alpha$, it does not follow that $C_{\alpha\beta\gamma\delta}\ell^\alpha = 0$. More precisely, if we consider the decomposition of the Riemann tensor into its trace and trace-free parts we obtain

$$C_{1bcd} = \frac{1}{2}(\eta_{1d}R_{bc} - \eta_{1c}R_{bd}) + \frac{1}{6}(\eta_{1c}\eta_{bd} - \eta_{1d}\eta_{bc})R\,. \tag{3.37}$$

It is clear that there will exist boost weight 0 and -1 components of Weyl with projections along ℓ that do not vanish, namely

$$C_{1212} = -\frac{1}{6}R, \quad C_{1i2j} = -\frac{1}{2}R_{ij} + \frac{1}{6}\delta_{ij}R, \quad C_{12i2} = \frac{1}{2}R_{ij} \tag{3.38}$$

Assuming the conditions in proposition (3.4.2) are satisfied then the Weyl components in (3.38) vanish and also $C_{ijkl} = 0$ from [17], therefore the remaining non-vanishing Weyl components of the VSI metric are C_{2jkl} and C_{2j2l}.

3.5 Kundt Spacetimes and Supergravity

A VSI spacetime admits an aligned shear-free, non-expanding, non-twisting, geodesic null direction ℓ^a along which the Riemann tensor has negative boost order [65]. These spacetimes can be classified according to their Weyl type (III, N, O), Ricci type (III,N,O) and the vanishing or non-vanishing spin coefficients.

One particular subset of Ricci type N VSI spacetimes, the higher-dimensional pp-wave spacetimes, have been studied extensively in the literature, and are known to be exact solutions in string theory [35, 44, 43], in type IIB supergravity with an R-R five-form [57], and with NS-NS form fields as well [58]. The pp-wave spacetimes are of Weyl type N. However there are Ricci type N solutions of Weyl type III, like the string gyratons and in fact all Ricci Type N VSI spacetimes are solutions to supergravity [84].

Moreover in [84] it was shown that there are VSI spacetime solutions of type IIB supergravity which are of Ricci Type III, assuming appropriate source fields are provided. In order for a VSI spacetime to be of Ricci type III, the dilation must be a non-constant function of the light-cone coordinate, u and the metric will have v-dependence. However it was shown in [84] that no null or timelike killing vectors can exist in a VSI spacetime if the metric is dependent on v, thus the Ricci Type III spacetimes do not preserve supersymmetry. Furthermore, while the Ricci type N, Weyl type III solutions can be reduced to Weyl type N, the Ricci type III solution can only have Weyl type III.

It has been shown that the plane wave spacetimes have the fascinating property that all quantum corrections vanish in these spacetimes, [18, 16]; four dimensional spacetimes which exhibit this property are called *universal*. The universal spacetimes give important insights into the quantum theory, despite having actual knowledge of this theory. It has been shown that all four-dimensional universal spacetimes are actually CSI spacetimes [91, 102]. For these reasons the study of the subclass of universal CSI spacetimes will be of great help in the pursuit of a theory of a more general gravity theory.

In four dimensions VSI spacetimes are known to be exact string solutions to all orders in the string tension α', even in the presence of additional fields [54]. Similarly it can be shown that higher-dimensional supergravity solutions supported by the proper fields (for example, the dilation scalar field, Kalb-Ramond field, and form fields) are also exact solutions in string theory using arguments from [35, 57, 58]. Thus it can be analogously argued that the VSI supergravity spacetimes are exact string solutions to all orders in the string tension α' in the presence of the appropriate fields, and so it is to be expected that the special VSI supergravity solution introduced in [84] will be as well. From this we conclude that the Ricci type III solution may be of relevance to string theory.

Since the Ricci Type III IIB-supergravity solutions do not preserve supersymmetry, this leads to the question of what are the necessary conditions to preserve supersymmetry? In a number of supergravity theories (e.g. D= 11 [73], type IIB [69]), in order to preserve some supersymmetry it is a necessary, but not sufficient, that the spacetime involved admit a Killing spinor ϵ^A which then yields a null or timelike Killing vector from its Dirac current, $x^\alpha = \bar{\epsilon}^{A'}\gamma^\alpha_{AA'}\epsilon^A$ where the γ^α are the higher dimensional analogues to the four-dimensional gamma matrices.

In the case of N-dimensional VSI spacetimes, the existence of Killing vectors depends on whether the components of the metric function are independent of the light-cone coordinate v. This requirement leads to the conclusion that all VSI spacetime solutions to type IIB supergravity preserving some supersymmetry are of Ricci type N, Weyl type III(a) or N [80] Such spacetimes include not only pp-waves but also spacetimes of Weyl type III(a), an example of which is the string gyratons [82]. Weyl type III(a) spacetimes like the vacuum solution or NS-NS solutions only preserve some of the supersymmetry.

It is known that $AdS_d \times S^{(D-d)}$ (in short $AdS \times S$) is an exact solution of supergravity (and preserves the maximal number of supersymmetries) for certain values of (D,d) and for particular ratios of the radii of curvature of the two space forms. Such spacetimes (with $d = 5, D = 10$) are supersymmetric solutions of IIB supergravity (there are analogous solutions in $D = 11$ supergravity) [63]: $AdS \times S$ is an example of a CSI spacetime [79]. There are a number of other CSI spacetimes known to be solutions of supergravity and admit supersymmetries; namely, generalizations of $AdS \times S$ (for example, see [81]) and (generalizations of) the chiral null models [44]. The AdS gyraton (which is a CSI spacetime with the same curvature invariants as pure AdS) [75] is a solution of gauged supergravity [83] (the AdS gyraton can be cast in the Kundt form [85]).

Other known CSI spacetimes may be investigated to determine whether they contain solutions of supergravity. For example, we can consider the product manifolds of the form $M \times K$, where M is an Einstein space with negative constant curvature and K is a (compact) Einstein-Sasaki spacetime. The warped product of AdS_3 with an 8-dimensional compact (Einstein-Kahler) space M_8 with non-vanishing 4-form flux are supersymmetric solutions of D=11 supergravity [81], while in [45] supersymmetric solutions of $D = 11$ supergravity, where M is the squashed S^7, were given.

A class of CSI spacetimes which are solutions of supergravity and preserve supersymmetries, were built from a VSI seed and locally homogeneous (Einstein) spaces by warped products fibered products, and tensor sums [79], yielding generalizations of $AdS \times S$ or AdS gyratons [85]. In particular, solutions obtained by restricting attention to $CCNV$ and Ricci type N spacetimes were considered, some explicit examples of CSI supergravity spacetimes were constructed by taking a homogeneous (Einstein) spacetime, $(\mathcal{M}_{\text{Hom}}, \tilde{g})$ of Kundt form and generalizing to inhomogeneous spacetimes, $(\mathcal{M}, g)$ by including arbitrary Kundt metric functions (by construction, the curvature invariants of $(\mathcal{M}, g)$ will be identical to those of $(\mathcal{M}_{\text{Hom}}, \tilde{g})$); a number of 5D examples were given, in which ds^2_{hom} was taken to be Euclidean space or hyperbolic space [64].

The question then is whether these CSI solutions preserve any supersymmetry. It is known that for many supergravity theories, if the spacetime admits a Killing spinor, it then admits a null or timelike Killing vector. We will be particularly interested in spacetimes that admit a null covariantly constant vector.

Chapter 4

CSI Spacetimes with a Non-spacelike Isometry

This chapter is based on: D. McNutt, N. Pelavas, A. Coley (2010). Killing vectors in higher-dimensional spacetimes with constant scalar curvature invariants. *IJGMMP*, Vol 7, Issue 8, pp 1349-1369.

4.1 The Killing Equations

Let $\zeta = \zeta_1 n + \zeta_2 \ell + \zeta_i m^i$ be a Killing vector field in a CSI Kundt spacetime; it satisfies the Killing equations for $a, b \in [1, N]$

$$\zeta_{a,b} + \zeta_{b,a} - 2\Gamma^c{}_{(ab)}\zeta_c = 0.$$

To simplify the analysis of these equations, we choose new coordinates where one of the Killing vectors of the transverse space, Y, has been rectified so that locally it behaves as a translation; i.e., $Y = A\frac{\partial}{\partial x^3}$. In this coordinate system g_{33} will be constant, and so it is possible to pick a coframe with an upper-triangular matrix $m^i{}_e$ and $m^3{}_3$ constant [20]. This choice of coframe causes Γ_{3ij} and $\Gamma_{3(ij)}$ $\forall i, j \in [3, N]$ to vanish. Rotating the frame so that the spatial component of ζ is locally aligned with m^3, ζ takes the form $\zeta = \zeta_1 n + \zeta_2 \ell + \zeta_3 m^3$

The components ζ_1 and ζ_3 may be partially integrated from the equations with indices (11), (13), ($3i$):

$$\zeta_1 = \quad \zeta_1(u, x^3), \quad \zeta_3 = -D_3(\zeta_1)v + \zeta_3^{(0)}(u, x^e), \tag{4.1}$$

where $\zeta_3^{(0)}$ satisfies the following differential equations from ($3i$):

$$D_i\zeta_3^{(0)} + W_i^{(0)}D_3(\zeta_1) = 0, \quad D_iD_3\zeta_1 - W_i^{(1)}D_3\zeta_1 = 0 \tag{4.2}$$

The tensors $\Gamma_{2i2} = D_iH - D_2W_i$ and $A_{mn} = D_{[n}W_{m]}$ may be express in orders of v, where $\sigma^* \equiv 4\sigma + W_i^{(1)}W^{(1)i}$ and the metric functions H and $W_i = m_i{}^e w_e$ are of the form (3.15) and (3.16).

$$\Gamma_{2i2} = \overbrace{(D_i\sigma^* - \sigma^* W_i^{(1)})}^{\Gamma_i^{(2)}}\frac{v^2}{8} + \overbrace{(D_i H^{(1)} - \frac{1}{4}W_i^{(0)}\sigma^* - D_2 W_i^{(1)})}^{\Gamma_i^{(1)}}v$$
$$+ \overbrace{D_i H^{(0)} - W_i^{(0)} H^{(1)} - D_2 W_i^{(0)} + H^{(0)} W_i^{(1)}}^{\Gamma_i^{(0)}}, \tag{4.3}$$
$$A_{ij} = \overbrace{2D_{[j}W_{i]}^{(1)}}^{A_{ij}^{(1)}} v + \overbrace{2D_{[j}W_{i]}^{(0)} - 2W_{[j}^{(0)}W_{i]}^{(1)}}^{A_{ij}^{(0)}}.$$

Substituting these into the equation with indices (21) yields ζ_2 in orders of v:

$$\zeta_2 = \overbrace{(\frac{\sigma^*\zeta_1}{4} - W_3^{(1)} D_3(\zeta_1))}^{\zeta_2^{(2)}}\frac{v^2}{2} + \overbrace{(W_3^{(1)}\zeta_3^{(0)} - D_2\zeta_1 + H^{(1)}\zeta_1)}^{\zeta_2^{(1)}}v + \zeta_2^{(0)}(u, x^e). \tag{4.4}$$

Our primary interest are those CSI spacetimes which do not admit covariantly constant null vectors, since the existence of Killing vectors in $CCNV$ spacetimes was considered in [100]. The analysis will be restricted to non-spacelike Killing vectors, $|\zeta| \leq 0$. Using the definition of the vector components given above the magnitude is expanded into orders of v:

$$\tfrac{-\sigma^*}{4}(\zeta_1)^2 + W_3^{(1)} D_3(\zeta_1)\zeta_1 + (D_3(\zeta_1))^2 \leq 0 \tag{4.5}$$
$$\zeta_1(W_3^{(1)}\zeta_3^{(0)} - D_2\zeta_1 + H^{(1)}\zeta_1) + D_3(\zeta_1)\zeta_3^{(0)} = 0 \tag{4.6}$$
$$(\zeta_3^{(0)})^2 - 2\zeta_1\zeta_2^{(0)} \leq 0. \tag{4.7}$$

The remaining Killing equations, with indices 22, 23 and $2n$ are now expanded into orders of v, giving the following set of equations:

$$\Gamma_3^{(2)} D_3(\zeta_1) = 0, \tag{4.8}$$
$$D_2\zeta_2^{(2)} + \tfrac{1}{4}\sigma^*\zeta_2^{(1)} - H^{(1)}\zeta_2^{(2)} - \tfrac{1}{4}\Gamma_3^{(1)} D_3(\zeta_1) + \tfrac{1}{4}\Gamma_3^{(2)}\zeta_3^{(0)} = 0, \tag{4.9}$$
$$D_2\zeta_2^{(1)} + \tfrac{1}{4}\sigma^*\zeta_2^{(0)} - H^{(0)}\zeta_2^{(2)} - \Gamma_3^{(0)} D_3(\zeta_1) + \Gamma_3^{(1)}\zeta_3^{(0)} = 0, \tag{4.10}$$
$$D_2\zeta_2^{(0)} - H^{(0)}\zeta_2^{(1)} + H^{(1)}\zeta_2^{(0)} + \Gamma_3^{(0)}\zeta_3^{(0)} = 0, \tag{4.11}$$

$$\tfrac{1}{4}\sigma^* D_3(\zeta_1) + D_3\zeta_2^{(2)} - W_3^{(1)}\zeta_2^{(2)} - \tfrac{1}{4}\Gamma_3^{(2)}\zeta_1 = 0, \tag{4.12}$$
$$D_2 D_3(\zeta_1) - H^{(1)} D_3(\zeta_1) - D_3\zeta_2^{(1)} + W_3^{(0)}\zeta_2^{(2)} + \Gamma_3^{(1)}\zeta_1 = 0, \tag{4.13}$$
$$D_2\zeta_3^{(0)} + H^{(0)} D_3(\zeta_1) + D_3\zeta_2^{(0)} - W_3^{(0)}\zeta_2^{(1)} - \Gamma_3^{(0)}\zeta_1 + W_3^{(1)}\zeta_2^{(0)} = 0, \tag{4.14}$$

$$D_n\zeta_2^{(2)} - W_n^{(1)}\zeta_2^{(2)} - \tfrac{1}{4}\Gamma_n^{(2)}\zeta_1 + A_{3n}^{(1)}D_3(\zeta_1) = 0, \tag{4.15}$$

$$D_n\zeta_2^{(1)} - W_n^{(0)}\zeta_2^{(2)} - \Gamma_n^{(1)}\zeta_1 - A_{3n}^{(1)}\zeta_3^{(0)} + A_{3n}^{(0)}D_3(\zeta_1) = 0, \tag{4.16}$$

$$D_n\zeta_2^{(0)} - W_n^{(0)}\zeta_2^{(1)} - \Gamma_n^{(0)}\zeta_1 + W_n^{(1)}\zeta_2^{(0)} - A_{3n}^{(0)}\zeta_3^{(0)} = 0. \tag{4.17}$$

The analysis splits into subcases arising from (4.8) where either $D_3(\zeta_1)$ or $\Gamma_3^{(2)}$ are assumed to vanish seperately.

4.2 Implications of $\zeta_{[a;b]} = 0$

Before each case is analyzed it will be beneficial to examine the anti symmetrization of $\zeta_{a;b} = 0$ to determine the set of CSI spacetimes admitting a covariantly constant non-spacelike vector. Non-spacelike Killing vectors in $CCNV$ CSI spacetimes has already been studied in [100] as such if a CSI spacetime is shown to be $CCNV$ it may be disregarded in the current analysis. Conversely it is of interest to determine when a CSI spacetime admits a Killing vector but cannot admit a covariantly constant vector.

Using the form of ζ given above, the vanishing of $\zeta_{[a;b]}$, yields the following equations:

$$D_2\zeta_1 - D_1\zeta_2 - \Gamma^1{}_{12}\zeta_1 = 0, \tag{4.18}$$

$$D_3\zeta_1 - D_1\zeta_3 - 2\Gamma^1{}_{[13]}\zeta_1 = 0, \tag{4.19}$$

$$2\Gamma^1{}_{[1n]}\zeta_1 = 0, \tag{4.20}$$

$$D_3\zeta_2 - D_2\zeta_3 + \Gamma^1{}_{32}\zeta_1 = 0, \tag{4.21}$$

$$D_n\zeta_2 + \Gamma^1{}_{n2}\zeta_1 = 0, \tag{4.22}$$

$$D_n\zeta_3 - 2\Gamma^1{}_{[3n]}\zeta_1 = 0, \tag{4.23}$$

$$\Gamma^1{}_{[nm]}\zeta_1 = 0. \tag{4.24}$$

Assuming $\zeta_1 \neq 0$ and expanding (4.20) implies $\Gamma^1{}_{[1n]} = W_n^{(1)} = 0$. Similarly $2\Gamma^1{}_{[13]} = W_3^{(1)}$ and so equation (4.19) gives $W_3^{(1)} = 2D_3 ln(\zeta_1)$. Equation (4.24) implies that A_{nm} must vanish. Using (4.3) we may summarize these observations as

Lemma 4.2.1. *For those spacetimes admitting a vector ζ such that $\zeta_{(a;b)} = 0$ and*

$\zeta_{[a;b]} = 0$ *it is necessary that the metric functions* W_i *satisfy the following:*

$$W_3^{(1)} = 2D_3 ln(\zeta_1), \quad W_n^{(1)} = 0,$$
$$A_{nm} = 2D_{[m}W_{n]}^{(0)} = 0.$$

The remaining equations are:

$$D_2\zeta_1 - D_1\zeta_2 - \Gamma_{212}\zeta_1 = 0, \tag{4.25}$$
$$D_3\zeta_2 - D_2\zeta_3 + \Gamma_{232}\zeta_1 = 0, \tag{4.26}$$
$$D_n\zeta_2 + \Gamma_{2n2}\zeta_1 = 0, \tag{4.27}$$
$$D_n\zeta_3 - A_{3n}\zeta_1 = 0. \tag{4.28}$$

These will be studied once the analysis of the Killing equations has been completed.

4.3 Case 1: $D_3(\zeta_1) = 0$

Setting $D_3(\zeta_1)$ equal to zero we obtain

$$\zeta_1 = \quad \zeta_1^{[0]}(u), \quad \zeta_3 = \zeta_3^{(0)}(u) \tag{4.29}$$

$$\zeta_2 = (\overbrace{\frac{\sigma^*\zeta_1}{4}}^{\zeta_2^{(2)}})\,\tfrac{v^2}{2} + \overbrace{(W_3^{(1)}\zeta_3 - D_2\zeta_1 + H^{(1)}\zeta_1)}^{\zeta_2^{(1)}}\,v + \zeta_2^{(0)}(u, x^e). \tag{4.30}$$

The non-spacelike conditions are now

$$-\sigma^*(\zeta_1)^2 \leq 0, \quad \zeta_1(W_3^{(1)}\zeta_3 - D_2\zeta_1 + H^{(1)}\zeta_1) = 0, \quad (\zeta_3)^2 - \zeta_1\zeta_2^{(0)} \leq 0 \tag{4.31}$$

so either ζ_1 vanishes and ζ is a null Killing vector or $\zeta_1 \neq 0$ and $\sigma^* \geq 0$.

4.3.1 Case 1.1 : $\zeta_1 = 0$

If ζ_1 is allowed to vanish, the remaining non-spacelike conditions imply that $\zeta_3 = 0$ and so the Killing vector is of the form $\zeta = \zeta_2\ell$. In light of the special form of ζ_2 it must be a function of only u and the spatial coordinates x^e. The remaining Killing equations are

$$\sigma^*\zeta_2^{(0)} = 0 \tag{4.32}$$
$$D_2\zeta_2^{(0)} + H^{(1)}\zeta_2^{(0)} = 0 \tag{4.33}$$
$$D_3\zeta_2^{(0)} + W_3^{(1)}\zeta_2^{(0)} = 0 \tag{4.34}$$
$$D_n\zeta_2^{(0)} + W_n^{(1)}\zeta_2^{(0)} = 0. \tag{4.35}$$

The vanishing of σ^* in the first term (4.32) implies $W^{(1)}_i W^{(1)i} = -4\sigma$ where $W^{(1)}_i = m_i{}^e w^{(1)}_e$ and hence

$$W^{(1)}_i W^{(1)i} = g^{ef} W^{(1)}_e W^{(1)}_f = -4\sigma. \tag{4.36}$$

Since the transverse metric is Riemannian, it is positive-definite and restricts the value of σ to be less than or equal to zero.

4.3.2 Case 1.1.1:

If $\sigma = 0$, this implies $W^{(1)}_e = 0$ for all $e \in [3, N]$. The vector component ζ_2 will be a function of u only and the remaining equation (4.33) determines the metric function

$$H^{(1)}(u) = -D_2 ln(\zeta_2). \tag{4.37}$$

One may always make a coordinate transform of the form (3.4) to set $H^{(1)} = 0$, so the metric is independent of the null coordinate v and $\zeta = \ell = \frac{\partial}{\partial v}$, implying that ζ is a covariantly constant null vector.

4.3.3 Case 1.1.2:

If $\sigma < 0$, one may solve for the metric functions $W^{(1)}_i$ and $H^{(1)}$ in terms of $\zeta(u, x^e)$:

$$H^{(1)}(u, x^e) = -D_2 ln(\zeta_2), \quad W^{(1)}_i(u, x^e) = -D_i ln(\zeta_2).$$

These CSI spacetimes do not admit a covariantly constant vector. To see this, assume $\sigma < 0$ and consider equations (4.25) - (4.27); the first two are automatically satsified while the last implies that ζ_2 is a function of u only. This forces the $W^{(1)}_i$ to all vanish, leading to the contradiction: $0 = \sigma^* = \sigma < 0$, hence these spacetimes do not admit a $CCNV$.

Given a null vector of the form, $\zeta = \zeta(u, x^e)\ell$, it will be a Killing vector for the CSI spacetime with a locally homogeneous transverse space and metric functions:

$$H = -(ln\zeta)_{,u} v + H^{(0)}(u, x^e), \quad W_e = -(ln\zeta)_{,e} v + W^{(0)}_e(u, x^f). \tag{4.38}$$

The vanishing of the function σ^* leads to one last condition for the CSI spacetime. Since $W^{(1)}_e = -(ln\zeta)_{,e}$, the only constraint on the function ζ arises from (3.14).

$$\sum_{i=3}^{N} [D_i ln(\zeta_2)]^2 = -4\sigma, \qquad \sigma < 0 \tag{4.39}$$

$$g^{ef}W_e^{(1)}W_f^{(1)} = -4\sigma.$$

The left-hand-side must be positive, and so it is necessary that $\sigma = R_{1212}$ is a negative real number.

4.3.4 Case 1.2 :

The remaining conditions from $|\zeta| \leq 0$ are

$$D_2\zeta_1 - H^{(1)}\zeta_1 = W_3^{(1)}\zeta_3 \tag{4.40}$$

$$(\zeta_3)^2 \leq \zeta_1\zeta_2^{(0)} \quad . \tag{4.41}$$

Expanding ζ_2 and $\Gamma_3^{(1)}$, we find that the $O(v^2)$ terms, (4.12) and (4.15) are automatically satisfied, while (4.9) and using (4.40) yield the following differential equation for $\sigma^*(u, x^e)$:

$$\zeta_1 D_2\sigma^* + \zeta_3 D_3\sigma^* = 0. \tag{4.42}$$

Using a coordinate transformation of the form (3.4), coordinates are chosen so that $\zeta_1 = 1$ and the non-spacelike condition (4.40) determines a part of H

$$H^{(1)} = -W_3^{(1)}\zeta_3.$$

We can apply another coordinate transform of type (3.3) to eliminate $H^{(0)}$ as well. In this coordinate system the Killing equations are:

$$\sigma^*(\zeta_2^{(0)} - \zeta_3 W_3^{(0)}) = 0, \tag{4.43}$$

$$D_2\zeta_2^{(0)} + \zeta_3 D_3\zeta_2^{(0)} + \zeta_3 D_2\zeta_3 = 0, \tag{4.44}$$

$$D_2 W_i^{(1)} + \zeta_3 D_3 W_i^{(1)} = 0, \tag{4.45}$$

$$D_2 W_3^{(0)} - \zeta_3 W_3^{(0)} W_3^{(1)} = -D_2\zeta_3 - D_3\zeta_2^{(0)} - W_3^{(1)}\zeta_2^{(0)}, \tag{4.46}$$

$$D_2 W_n^{(0)} + \zeta_3 D_3 W_n^{(0)} = (W_3^{(0)} W_n^{(1)} + D_n W_3^{(0)})\zeta_3 - W_n^{(1)}\zeta_2^{(0)} - D_n\zeta_2^{(0)}, \tag{4.47}$$

If ζ_3 is non-zero, equation (4.44) simplifies the differential equation for $W_3^{(0)}$ in (4.46). Thus two subcases must be considered in which ζ vanishes or not.

4.3.5 Case 1.2.1:

Setting ζ_3 equal to zero causes $H^{(1)}$ to vanish while (4.44) and (4.45) imply

$$D_2 W_i^{(1)} = D_2 \zeta_2^{(0)} = 0. \tag{4.48}$$

The remaining equations give constraints for the remaining metric functions:

$$\sigma^*(\zeta_2^{(0)}) = 0, \tag{4.49}$$

$$D_2 W_3^{(0)} = -D_3 \zeta_2^{(0)} - W_3^{(1)} \zeta_2^{(0)}, \tag{4.50}$$

$$D_2 W_n^{(0)} = -W_n^{(1)} \zeta_2^{(0)} - D_n \zeta_2^{(0)}. \tag{4.51}$$

Thus there are two minor subcases to consider arising from (4.49).

4.3.6 Case 1.2.1a

Assuming $\sigma^* \neq 0$, $\zeta_2^{(0)}$ vanishes and the set of spacetimes with metric functions:

$$H(v, x^e) = \sigma^* \frac{v^2}{8}, \quad W_i(v, x^e) = W_i^{(1)}(x^e) v + W_i^{(0)}(x^e) \tag{4.52}$$

are $CCNV$ spacetimes with $\frac{\partial}{\partial u}$ as a covariantly constant null vector admitting a Killing vector of the form:

$$\zeta = n + \frac{\sigma^* v^2}{8} \ell.$$

If we suppose ζ is a covariantly constant vector; Lemma (4.2.1) and equations (4.26) and (4.27) force the metric functions $W_i^{(1)}$ and $W_n^{(0)}$ to vanish. However a contradiction arises from (4.25) as it requires $\sigma^* = 0$ but we have assumed that $\sigma^* \neq 0$ and so the above spacetime cannot admit a covariantly constant vector.

Case 1.2.1b

For the other subcase, σ^* is equal to zero, and the positive-definite signature of the transverse metric restricts $\sigma \leq 0$. For arbitrary $\zeta_2^{(0)}(x^e)$, and any choice of $W_i^{(1)}(x^e)$ satisfying (4.36) with $\sigma = R_{1212} \leq 0$, the CSI Kundt spacetime with a locally homogeneous transverse space and metric functions:

$$H = 0, \quad W_i(u, v, x^e) = W_i^{(1)} v - (D_i \zeta_2^{(0)} + W_i^{(1)} \zeta_2^{(0)}) u + w_i(x^e) \tag{4.53}$$

admit a Killing vector of the form:

$$\zeta = n + \zeta_2^{(0)}\ell$$

To preserve the non-spacelike requirement $\zeta_2^{(0)}$ must always be greater than or equal to zero. If this killing vector is covariantly constant, $W_i^{(1)} = 0$ and hence $\sigma = 0$, equation (4.28) implies $A_{ij} = 0$, and the remaining equations (4.26) and (4.27) force $\zeta_2^{(0)}$ to be constant. Thus ζ is the sum of the $CCNV$'s ℓ and n.

4.3.7 Case 1.2.2: $\zeta_3 \neq 0$

Divide by ζ_3 in (4.44) and substitute the result into (4.46) to simplify the differential equation for $W_3^{(0)}$:

$$D_2 W_3^{(0)} - \zeta_3 W_3^{(0)} W_3^{(1)} = \frac{D_2 \zeta_2^{(0)}}{\zeta_3} - W_3^{(1)} \zeta_2^{(0)} \tag{4.54}$$

then by multiplying the above by $E(u, x^e) = e^{-[\int W_3^{(1)} \zeta_3 du]}$, integration by parts gives the solution

$$W_3^{(0)} = \frac{\zeta_2^{(0)}}{\zeta_3} + e^{[\int W_3^{(1)} \zeta_3 du]} \int \frac{\zeta_2^{(0)} D_2 \zeta_3}{(\zeta_3)^2 e^{[\int W_3^{(1)} \zeta_3 du]}} du.$$

From (4.43) there are two minor subcases to consider, depending upon whether σ^* vanishes or not.

4.3.8 Case 1.2.2a :

Supposing that σ^* does indeed vanish, the functions $W_3^{(1)}(x^e)$ and $W_n^{(1)}$ must satisfy (4.36) with $\sigma \leq 0$. For arbitrary $\zeta_3(u)$ and any solution of the following differential equation

$$D_2 \zeta_2^{(0)} + \zeta_3 D_3 \zeta_2^{(0)} = -\zeta_3 D_2 \zeta_3 \tag{4.55}$$

the Kundt CSI spacetime with a locally homogeneous transverse space and

$$\begin{gathered} H = -W_3^{(1)} \zeta_3 v \\ W_3(u, v, x^e) = W_3^{(1)}(u, x^e) v + \frac{\zeta_2^{(0)}}{\zeta_3} + \frac{1}{E} \int \frac{E \zeta_2^{(0)} D_2 \zeta_3}{(\zeta_3)^2} du, \qquad E = e^{\int H^{(1)} du} \\ W_n(u, v, x^e) = W_n^{(1)}(u, x^e) v + W_n^{(0)}(u, x^e) \end{gathered} \tag{4.56}$$

satisfying the following differential equations:

$$D_2W_i^{(1)} + \zeta_3 D_3 W_i^{(1)} = 0 \tag{4.57}$$

$$D_2W_n^{(0)} + \zeta_3 D_3 W_n^{(0)} = \frac{\zeta_3 W_n^{(1)}}{E} \int \frac{E\zeta_2^{(0)}}{(\zeta_3)^2} du + D_n[\frac{\zeta_3}{E} \int \frac{E\zeta_2^{(0)} D_2\zeta_3}{(\zeta_3)^2} du] \tag{4.58}$$

admits a Killing vector of the form

$$\ell + \zeta_2^{(0)}(u, x^e)n + \zeta_3(u)m^3$$

Requiring ζ to be a $CCNV$, the $W_i^{(1)}$ must vanish, causing $H = 0$ and $\sigma = 0$. This is an example of a $CCNV$ metric, with $\ell = \frac{\partial}{\partial v}$ as the $CCNV$, where ζ will be a second $CCNV$. The additional constraints (4.25) - (4.28) imply $A_{ij} = 0$ while the remaining equations lead to two possible subcases for Kundt spacetimes admitting a covariantly constant vector, either $D_n\zeta_2 = 0$ or $D_2\zeta_3 = 0$. The first case leads to the following form for ζ and the metric functions

$$\zeta = n + [-\zeta_3^2]\ell + \zeta_3(u)m^3, \quad 3\zeta_3^2 \leq 0$$

$$H = 0, \quad W_3(u, x^e) = -\zeta_3 + w_3(x^e), \quad W_n(x^e) = \int D_n w_3 dx^3 + w_n(x^r).$$

The non-spacelike condition $3\zeta_3^2 \leq 0$ eliminates the above case, as we've assumed $\zeta_3 \neq 0$ this case is not admissible. In the second case ζ_3 must be constant, scaling x^3 so that $\zeta_3 = 1$,

$$\zeta = n + \zeta_2(x^r)\ell + m^3, \quad 1 \leq 2\zeta$$

$$H = 0, \quad W_3(x^e) = w_3(x^e), \quad W_n(x^e) = \int D_n(w_3)dx^3 - 2D_n(\zeta_2)x^3 + w_n(x^r),$$

The vanishing of $A_{3j} = D_{[j}W_{3]}^{(0)}$ implies that $D_n\zeta_2^{(0)} = 0$ and so $\zeta_2^{(0)}$ must be constant. If ζ is timelike, the constant $\zeta_2^{(0)} > \frac{1}{2}$ while if ζ is null $\zeta_2^{(0)} = \frac{1}{2}$. These spacetimes will automatically be $CCNV$ spacetimes with ℓ as another covariantly constant null vector.

4.3.9 Case 1.2.2b:

If σ^* is non-zero, it satisfies the differential equation (4.42)

$$D_2\sigma^* + \zeta_3 D_3\sigma^* = 0$$

and the identity $\zeta_2^{(0)} = \zeta_3 W_3^{(0)}$ may be derived from (4.43), which causes (4.54) to simplify, implying $\zeta_2^{(0)} D_2\zeta_3 = 0$. Letting $\zeta_2^{(0)} = 0$, the differential equation (4.55) for $\zeta_2^{(0)}$ forces $D_2\zeta_3 = 0$. In either case ζ_3 must be constant and henceforth will be set to one. For any solution $\zeta_2^{(0)}$ to the differential equation

$$D_2\zeta_2^{(0)} + D_3\zeta_2^{(0)} = 0, \tag{4.59}$$

the vector

$$n + [\frac{\sigma^*}{8}v^2 + \zeta_2^{(0)}]\ell + m^3$$

will be a Killing vector for any CSI Kundt spacetime of the form

$$H = \tfrac{\sigma^*}{8}v^2 - W_3^{(1)}\zeta_3 v, \tag{4.60}$$

$$W_3(u, v, x^e) = W_3^{(1)}(u, x^e)v + \zeta_2^{(0)}, \quad W_n(u, v, x^e) = W_n^{(1)}(u, x^e)v + W_n^{(0)}(u, x^e)$$

where the $W_i^{(1)}$ and $W_i^{(0)}$ satisfy the following equations:

$$\begin{aligned} D_2W_i^{(1)} + D_3W_i^{(1)} &= 0, &(4.61)\\ D_2W_i^{(0)} + D_3W_i^{(0)} &= 0. &(4.62)\end{aligned}$$

If ζ is required to be covariantly constant, a contradiction arises from (4.25) as it requires $\sigma^* = 0$ despite the fact that we have assumed $\sigma^* \neq 0$. Thus there are no $CCNV$ spacetimes of the form (4.60).

4.4 Case 2 : $\Gamma_3^{(2)} = 0$

For the remainder of this case we shall assume $D_3\zeta_1 \neq 0$ to avoid the previous subcases. Supposing $W_3^{(1)} = 0$, this implies that $D_3D_3\zeta_1 = 0$ and $\sigma^* = \sigma$. This causes a contradiction to arise between the Killing equation (4.12) and the non-spacelike condition (4.5):

$$2\sigma D_3(\zeta_1) = 0, \quad (D_3\zeta_1)^2 \leq \sigma(\zeta_1)^2.$$

The first implies that $\sigma = 0$ as we have assumed $D_3\zeta_1 \neq 0$; however, by the second inequality the vanishing of σ implies $D_3\zeta_1 = 0$ which contradicts our original assumption. Thus $D_3D_3\zeta_1$ is always non-zero, and using this fact we may derive another identity for $\sigma^* = 4\sigma + (W_3^{(1)})^2$ in terms of ζ_1 from the vanishing of $\Gamma_3^{(2)}$:

$$\sigma^* = \frac{D_3\sigma^*}{W_3^{(1)}} = \frac{2W_3^{(1)}D_3W_3^{(1)}}{W_3^{(1)}} = 2D_3D_3(lnD_3\zeta_1). \tag{4.63}$$

Using a coordinate transform of type (3.4) with $g(u) = \frac{u}{\sqrt{|\sigma|}}$, we may rescale σ in (3.12) so that it equals $\sigma = -1, 0, 1$ depending on it's sign. Doing so will scale all of the metric functions and Killing vector components by a constant value, but otherwise will leave them unchanged.

Dropping the primes and substituting (4.63) into the original identity for σ^* yields another differential equation for $D_3\zeta_1$:

$$D_3D_3ln(D_3\zeta_1) - \frac{1}{2}(D_3ln(D_3\zeta_1))^2 = 2\sigma.$$

Multiplication by $exp(-\frac{1}{2}\int D_3(lnD_3\zeta_1)dx^3) = (D_3\zeta_1)^{-\frac{1}{2}}$ leads to the simpler equation

$$D_3D_3[(D_3\zeta_1)^{-\frac{1}{2}}] = -\sigma(D_3\zeta_1)^{-\frac{1}{2}}. \tag{4.64}$$

There are three possible solutions to this equation depending on whether σ is positive, negative or zero:

$$\begin{aligned}
\sigma = -1 \quad &: \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1(u)cosh(x^3) + c_2(u)sinh(x^3),\\
\sigma = 0 \quad &: \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1'(u)x^3 + c_2',\\
\sigma = 1 \quad &: \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1''(u)cos(x^3) + c_2''(u)sin(x^3).
\end{aligned}$$

Ignoring these facts for a moment, we recall that the metric functions W_i may be expressed in terms of $\zeta_3^{(0)}$ and ζ_1 using (4.2) :

$$W_i^{(0)} = \frac{-D_i\zeta_3^{(0)}}{D_3\zeta_1}, \quad W_i^{(1)} = D_iln(D_3\zeta_1).$$

In this case, it is possible to set all but $W_3^{(1)}$ to zero by making a coordinate transform of type (3.3) with $h = -\frac{\zeta_3^{(0)}}{D_3\zeta_1}$. In these new coordinates, the metric functions take the form:

$$W_3 = D_3ln(D_3\zeta_1)v, \quad W_n = 0. \tag{4.65}$$

The following coefficient functions of H change in the new coordinate system:

$$H^{(1)} = H'^{(1)} + \frac{\zeta_3^{(0)}\sigma^*}{4D_3\zeta_1}, \quad H^{(0)} = H'^{(0)} + \frac{\zeta_3^{(0)}H'^{(1)}}{D_3\zeta_1} + D_2\left(\frac{\zeta_3^{(0)}}{D_3\zeta_1}\right)\frac{\sigma^*(\zeta_3^{(0)})^2}{8(D_3\zeta_1)^2},$$

$$H^{(1)} = H'^{(1)} + \frac{\zeta_3^{(0)}\sigma^*}{4D_3\zeta_1}, \quad H^{(0)} = H'^{(0)} + \frac{\zeta_3^{(0)}H'^{(1)}}{D_3\zeta_1} + D_2\left(\frac{\zeta_3^{(0)}}{D_3\zeta_1}\right)\frac{\sigma^*(\zeta_3^{(0)})^2}{8(D_3\zeta_1)^2},$$

where primed functions denote the functions in the previous coordinate system.

As the original $H'^{(1)}$ and $H'^{(0)}$ were arbitrary functions of u and the spatial coordinates, we may ignore the special form the v-coefficients take in this coordinate system and treat them simply as new arbitrary functions. In this coordinate system the tensor A_{3n} given in (4.3) vanishes, and the connection coefficients Γ_{2i2} are of the form:

$$\Gamma_{2i2} = \overbrace{(D_iH^{(1)} - D_2W_i^{(1)})}^{\Gamma_i^{(1)}}\, v + \overbrace{D_iH^{(0)} + H^{(0)}W_i^{(1)}}^{\Gamma_i^{(0)}}.$$

This choice of coordinate system simplifies the Killing equations considerably; for example, the other two covector components are now

$$\zeta_2 = \overbrace{(\frac{\sigma^*\zeta_1}{4} - D_3D_3\zeta_1)}^{\zeta_2^{(2)}}\frac{v^2}{2} + \overbrace{(H^{(1)}\zeta_1 - D_2\zeta_1)}^{\zeta_2^{(1)}}\, v + \zeta_2^{(0)}(u, x^e),$$
$$\zeta_3 = -D_3(\zeta_1)v.$$

Taking the magnitude of the vector and invoking the non-spacelike conditions yield

$$D_3D_3ln[(D_3\zeta_1)^{-\frac{1}{2}}] + D_3(ln(D_3\zeta_1))D_3ln(\zeta_1) + (D_3ln(\zeta_1))^2 \leq 0, \tag{4.66}$$

$$\zeta_1(H^{(1)}\zeta_1 - D_2\zeta_1) = 0, \tag{4.67}$$

$$\zeta_1\zeta_2^{(0)} \geq 0. \tag{4.68}$$

Thus $\zeta_2^{(1)}$ must vanish and we may solve for $H^{(1)}$ in terms of ζ_1,

$$H^{(1)} = D_2ln(\zeta_1).$$

Further constraints on H involving $H^{(0)}$ may be found by taking those Killing equations involving the spatial derivatives of $\zeta_2^{(0)}$; i.e., (4.14) and (4.17) and considering integrability conditions. In this coordinate system (4.14) and (4.17) are

$$D_3\zeta_2^{(0)} + H^{(0)}D_3\zeta_1 - \zeta_1D_3H^{(0)} - \zeta_1H^{(0)}D_3ln(D_3\zeta_1) + \zeta_2^{(0)}D_3ln(D_3\zeta_1) = 0$$
$$D_n\zeta_2^{(0)} - \zeta_1D_nH^{(0)} = 0$$

We note that the commutator applied to any function independent of v vanishes (i.e. $[D_3, D_n]f(u, x^e) = 0$); thus differentiating the first equation by D_n and the latter by D_3 and subtracting the result gives the following constraint

$$2D_n(H^{(0)})D_3\zeta_1 = 0.$$

Hence $H^{(0)}$ and $\zeta_2^{(0)}$ are actually functions of u and the spatial coordinate x^3.

In light of this fact the Killing equations (4.15) - (4.17) are automatically satisfied. Similarly, equation (4.12) may be ignored as it gives the identity $\sigma^* = 2D_3D_3ln(D_3\zeta_1)$, which arose from the vanishing of $\Gamma_3^{(2)}$. The remaining Killing equations are now:

$$D_2\sigma^* = 4D_2\left(\tfrac{D_3D_3\zeta_1}{\zeta_1}\right) - \tfrac{1}{2}D_2[(D_3ln(\zeta_1))^2], \tag{4.69}$$

$$D_3H^{(0)} = \tfrac{\sigma^*}{4D_3\zeta_1}(\zeta_2^{(0)} - H^{(0)}\zeta_1), \tag{4.70}$$

$$D_2\zeta_2^{(0)} = -\zeta_2^{(0)}D_2ln(\zeta_1), \tag{4.71}$$

$$2D_2D_3ln(\zeta_1) = D_2D_3ln(D_3\zeta_1), \tag{4.72}$$

$$D_3(\zeta_2^{(0)}D_3\zeta_1) = \zeta_1^2D_3[H^{(0)}D_3ln(\zeta_1)]. \tag{4.73}$$

Differentiating (4.72) and using the fact that $[D_3, D_2]f(u, x^e) = 0$, one finds the following expression for $D_2\sigma^* = 2D_2D_3D_3ln(D_3\zeta_1)$:

$$D_2\sigma^* = 4D_2\left[\frac{D_3D_3\zeta_1}{\zeta_1} - \left(\frac{D_3\zeta_1}{\zeta_1}\right)^2\right].$$

Subtracting this from (4.69) yields the following constraint

$$D_2\left(D_3ln(\zeta_1)\right)^2 = 0$$

implying that ζ_1 must take the form:

$$\zeta_1 = e^{A(x^3)}e^{B(u)}. \tag{4.74}$$

Apply a coordinate transform of type (3.4) with $g = \int e^{-B(u)}du$ will remove the u dependence from ζ_1. Rewriting (4.71) in terms of $\zeta'^{(0)}_2 = \zeta_2^{(0)}e^B$, it is easily shown that this implies $D_2\zeta'^{(0)}_2 = 0$. Denoting $\zeta'_1 = e^{A(x^3)}$ the Killing vector $\zeta = e^Ae^Bn + \zeta_2\ell + \zeta_3m^3$ becomes:

$$\zeta = \zeta'_1n' + \left[\left(\frac{\sigma^*\zeta'_1}{4} - D_3D_3(\zeta'_1)\right)\frac{v'^2}{2} + \zeta'^{(0)}_2(x^3)\right]\ell' + [-D_3(\zeta'_1)v']m^3.$$

In the remaining Killing equations, (4.70) and (4.73), the function $H^{(0)}$ in the new coordinate system becomes $H'^{(0)} = e^{2B}H^{(0)}$ and so we may remove e^B entirely from these two equations.

Dropping the primes and combining (4.70) with (4.73) yields the following algebraic equation for $H^{(0)}$:

$$H^{(0)}\left(D_3D_3ln(\zeta_1) - \frac{\sigma^*}{4}\right) = \frac{D_3(D_3(\zeta_1)\zeta_2^{(0)})}{\zeta_1^2} - \frac{\sigma^*\zeta_2^{(0)}}{4\zeta_1}$$

The coefficient of $H^{(0)}$ cannot vanish, as the non-spacelike condition (4.66) would imply

$$2(D_3 ln(\zeta_1))^2 \leq 0.$$

It is assumed that $D_3\zeta_1 \neq 0$ so the above constraint is impossible. Simplifying the above expression $H^{(0)}$ may be written as

$$H^{(0)} = \frac{D_3(D_3(\zeta_1)\zeta_2^{(0)}) + D_3 D_3 ln((D_3\zeta_1)^{-\frac{1}{2}})\zeta_2^{(0)}\zeta_1}{\zeta_1^2 D_3 D_3 ln(\zeta_1(D_3\zeta_1)^{-\frac{1}{2}})} \tag{4.75}$$

Having exhausted the Killing equations, we look to the remaining non-spacelike conditions (4.66) and (4.68).

4.4.1 Case 2.1: Null Killing Vectors

If ζ is required to be null $\zeta_2^{(0)}$ must be zero, forcing $H^{(0)}$ to vanish as well. Using (4.66) and (4.64) we find the following expression

$$D_3(A) = D_3 ln[(D_3\zeta_1)^{-\frac{1}{2}}] \pm \sqrt{2[D_3 ln((D_3\zeta_1)^{-\frac{1}{2}})]^2 + \sigma}. \tag{4.76}$$

Combining this with the solution to (4.64) for a particular $\sigma = -1, 0, 1$:

$$\sigma = -1 \quad : \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1 cosh(x^3) + c_2 sinh(x^3) \tag{4.77}$$

$$\sigma = 0 \quad : \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1 x^3 + c_2 \tag{4.78}$$

$$\sigma = 1 \quad : \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1 cos(x^3) + c_2 sin(x^3) \tag{4.79}$$

we may algebraically solve for ζ_1 by noting that $D_3\zeta_1 = D_3(A)e^A = D_3(A)\zeta_1$:

$$\sigma = -1 : \zeta_1 = \frac{(c_1 cosh(x^3)+c_2 sinh(x^3))^{-1}}{c_1 sinh(x^3)+c_2 cosh(x^3)\pm\sqrt{c_1^2+c_2^2+(c_1 sinh(x^3)+c_2 cosh(x^3))^2}} \tag{4.80}$$

$$\sigma = 0 : \zeta_1 = \frac{1}{c_1(1\pm\sqrt{2})(c_1 x^3+c_2)} \tag{4.81}$$

$$\sigma = 1 : \zeta_1 = \frac{(c_1 cos(x^3)+c_2 sin(x^3))^{-1}}{-c_1 sin(x^3)+c_2 cos(x^3)\pm\sqrt{c_1^2+c_2^2+(-c_1 sin(x^3)+c_2 cos(x^3))^2}} \tag{4.82}$$

Supposing that ζ is covariantly constant, the constraint in Lemma (4.2.1) on $W_3^{(1)}$ along with the identity (4.2) yields

$$D_3 ln(\zeta_1) = -D_3 ln[(D_3\zeta_1)^{\frac{1}{2}}]. \tag{4.83}$$

Since $ln(\zeta_1) = A$, the above simplifies (4.76) in the null case, giving

$$2D_3 ln[(D_3\zeta_1)^{-\frac{1}{2}}] \pm \sqrt{2[D_3 ln((D_3\zeta_1)^{-\frac{1}{2}})]^2 + \sigma} = 0$$

Multiplying both roots together the result must vanish

$$2[D_3 ln[(D_3\zeta_1)^{-\frac{1}{2}}]]^2 - \sigma = 0 \tag{4.84}$$

Substituting the three posibilities of $(\zeta_1)^{\frac{1}{2}}$ gives the constraint:

$$\begin{gathered} 3[c_1^2 - c_2^2]sinh^2(x^3) + 6c_1c_2 sinh(x^3)cosh(x^3) + 2c_2^2 + c_1^2 = 0 \\ \tfrac{c_1^2}{(c_1x^3+c_2)^2} = 0 \\ \sigma = 1 : 3[c_1^2 - c^2]sin^2(x^3) - 6c_1c_2 sin(x^3)cos(x^3) + 2c_2^2 - c_1^2 = 0 \end{gathered}$$

In each case this identity will only hold if $c_1 = c_2 = 0$; however, this will imply that $D_3\zeta_1 = 0$, which cannot happen. Thus the null killing vector ζ cannot be covariantly constant.

4.4.2 Case 2.2: Timelike Killing Vectors

If we require ζ to be timelike, equation (4.68) along with the fact that $\zeta_1 = e^A$ forces $\zeta_2^{(0)}$ to be greater than or equal to zero for all values of x^3. To find ζ_1 we integrate each of the three solutions to (4.64) given above

$$\sigma = -1 : \zeta_1 = \tfrac{sinh(x^3)}{c_1(c_1cosh(x^3)+c_2sinh(x^3))} + c_3 \tag{4.85}$$

$$\sigma = 0 : \zeta_1 = \tfrac{-1}{c_1(c_1x^3+c_2)} + c_3 \tag{4.86}$$

$$\sigma = 1 : \zeta_1 = \tfrac{sin(x^3)}{c_1(c_1cos(x^3)+c_2sin(x^3))} + c_3. \tag{4.87}$$

The inequality (4.66) restricts the choice of c_3 depending on the choice of c_1 and c_2:

$$\begin{gathered} \sigma = -1 : [c_1^2 + c_2^2]\zeta_1^2 - 2\left(\tfrac{c_1sinh(x^3)+c_2cosh(x^3)}{c_1cosh(x^3)+c_2sinh(x^3)}\right)\zeta_1 + \tfrac{1}{(c_1cosh(x^3)+c_2sinh(x^3))^2} < 0 \\ \sigma = 0 : -c_1^2\zeta_1^2 - 2\left(\tfrac{c_1}{c_1x^3+c_2}\right)\zeta_1 + \tfrac{1}{(c_1x^3+c_2)^2} < 0 \\ \sigma = 1 : -[c_1^2 + c_2^2]\zeta_1^2 - 2\left(\tfrac{-c_1sin(x^3)+c_2cos(x^3)}{c_1cos(x^3)+c_2sin(x^3)}\right)\zeta_1 + \tfrac{1}{(c_1cos(x^3)+c_2sin(x^3))^2} < 0 \end{gathered} \tag{4.88}$$

Notice in both the null and timelike case, the value of σ restricts the domain of x^3. When $\sigma = 1$, the domain of x^3 is limited to a finite interval, $x^3 \in (x_0^3, x_0^3 + \pi)$, as the value $x_0^3 = arctan(-\frac{c_1}{c_2})$ will cause $(D_3\zeta_1)^{-\frac{1}{2}}$ to vanish. When $\sigma = 0$, $x^3 \geq -\frac{c_2}{c_1}$

to avoid singularities. In the case with $\sigma = -1$, $x^3 > x^3_0 = arctanh(-\frac{c_1}{c_2})$ when $c_1/c_2 \leq 1$, otherwise ζ_1 is regular on the whole of the real line. Requiring ζ to be covariantly constant, equation (4.83) may be rewritten as a function set to zero in terms of ζ and $(D_3\zeta_1)^{\frac{1}{2}}$ for the three subcases with $\sigma = -1, 0, 1$ respectively:

$$[c_1 + 2c_1^2c_2c_3]sinh^2(x^3) + [c_2 + c_1^3c_3 + c_1c_2^2c_3]cosh(x^3)sinh(x^3) + [c_1^2c_2c_3 + c_1]$$

$$c_1c_3(c_1x^3 + c_2)$$

$$-[c_1 + 2c_1^2c_2c_3]sin^2(x^3) + [c_2 - c_1^3c_3 + c_1c_2^2c_3]cos(x^3)sin(x^3) + [c_1^2c_2c_3 + c_1]$$

In both cases where $\sigma = -1, 1$ the vanishing of the first and third equation will hold only if c_1 and c_2 both vanish, which violates the assumption $D_3\zeta_1 \neq 0$, and so there are no timelike covariantly constant vectors in either of these two cases. When $\sigma = 0$, setting the second equation to zero implies $c_3 = 0$, the Killing vector of the form (4.75) with $\zeta_1 = -1/(c_1^2x^3 + c_1c_2)$ satsifies the condition in (4.83). A problem arises from the inequality (4.66)

$$-c_1^2\left(\frac{1}{c_1^2(c_1x^3+c_2)^2}\right) - 2\left(\frac{c_1}{(c_1x^3+c_2)}\right)\left(\frac{-1}{c_1(c_1x^3+c_2)}\right) + \frac{1}{(c_1x^3+c_2)^2} < 0;$$

simplifying the above leads to the inequality $2 < 0$ which is clearly impossible. We conclude there are no covariantly constant timelike vectors in the spacetimes belonging to Case 2.

4.5 Conclusions and Summary

To determine the subset of Kundt CSI spacetimes admitting a null or timelike isometry, several choices were made to simplify the Killing equations. Local coordinates were chosen so that one of the spacelike Killing vectors, Y, belonging to the (locally) homogeneous transverse space has been rectified to act locally as a translation in the x^3 direction, i.e., $Y = A\frac{\partial}{\partial x^3}$. The frame was then rotated so that the frame vector m^3 was aligned with the spatial part of ζ and, moreover, that the matrix m_{ie} was upper-triangular with $m_{33} = 1$. This causes the connection components Γ_{3ij} and Γ_{ij3} to vanish, simplifying the Killing equations considerably.

In this coordinate system we determined the special form for the components of ζ in terms of arbitrary functions and in terms of H and the W_e; i.e., (4.1) and (4.4). All of the functions involved (metric or otherwise) are expressed as polynomials in v with coefficient functions of u and x^e. These are substituted into the remaining Killing equations which are rearranged into the various orders of v to give (4.8) - (4.17), while the non-spacelike conditions yield (4.5) - (4.7). The highest order equation (4.8) gives two major subcases, either $D_3\zeta_1 = 0$ or $\Gamma_3^{(2)} = 0$ in (4.3).

It is known that all VSI spacetimes admitting a non-spacelike isometry are $CCNV$ spacetimes with ℓ as the covariantly constant vector [84]. As an analogue to this result, the equations arising from $\nabla_{[a}\zeta_{b]} = 0$ were examined to determine which CSI Kundt spacetimes admit a covariantly constant vector and which cannot.

The results of the analysis are summarized below:

Case 1.1.1: $\zeta = \ell$

In this case $R_{1212} = \sigma = 0$, the metric functions in (3.1) takes the form: $H(u, x^k)$ and $W_i(u, x^k)$. All CSI spacetimes in this subcase are clearly $CCNV$ spacetimes with $\ell = \frac{\partial}{\partial v}$ covariantly constant

Case 1.1.2: $\zeta = \zeta_2(u, x^e)\ell$

With $R_{1212} = \sigma < 0$, the metric functions H and $W_i = m_i^e W_e$ will be of the form (4.38) while ζ_2 must satisfy the further constraint (4.39). These CSI spacetimes do not admit a covariantly constant vector.

Case 1.2.1a : $\zeta = n + \frac{\sigma^* v^2}{2}\ell$

The metric (3.1) with H and W_i take the form (4.52), R_{1212} may be any value in $\mathbb{R}$. There are no $CCNV$ spacetimes belonging to this subset of CSI spacetimes.

Case 1.2.1b : $\zeta = n + \zeta_2(x^e)\ell$**,** $\zeta_2 \geq 0\ \forall x^e$**.**

For any $\zeta_2^{(0)}(x^e) > 0,\ \forall x^e$, and any choice of $W_i^{(1)}(x^e)$ satisfying (4.36) with $\sigma = R_{1212} \leq 0$; the CSI Kundt spacetime with H and W_i given in (4.53) will admit a timelike Killing vector. If $\zeta_2^{(0)}(x^e) > 0,\ \forall x^e$, $\zeta = n$ will be a covariantly constant null vector.

If this Killing vector is covariantly constant, $W_i^{(1)} = 0$ and hence $\sigma = 0$, equation (4.28) and Lemma (4.2.1) imply $A_{ij} = 0$, and the remaining equations (4.26) and

(4.27) force $\zeta_2^{(0)}$ to be constant. Thus ζ is the sum of the $CCNV$'s ℓ and n.

Case 1.2.2a : $\zeta = \ell + \zeta_2(u, x^e)n + \zeta_3(u)m^3$

For any ζ_3, and a particular choice of ζ_2 such that it satisfies the inequality $\zeta_3^2 \leq 2\zeta_2$ and the differential equation (4.55), the vector ζ will be a Killing vector for the CSI spacetime with metric functions given in (4.56) where $W_n^{(1)}$ and $W_n^{(0)}$ are, respectively, solutions (4.57) and (4.58). Due to the vanishing of $\sigma^* = 4\sigma + W_i^{(1)}W^{(1)i}$, the $W_i^{(1)}$ must also satisfy (4.36)

Requiring ζ to be a $CCNV$, the $W_i^{(1)}$ must vanish, causing $H = 0$ and $\sigma = 0$ as well; this is an example of a $CCNV$ metric with $\ell = \frac{\partial}{\partial v}$ as the $CCNV$ and ζ acting as a second $CCNV$. The additional constraints (4.25) - (4.28) require that $A_{ij} = 0$ along with the following simplification of ζ and the metric functions:

$$\zeta = n + \zeta_2\ell + m^3, \quad \zeta_2 \in \mathbb{R}$$

$$H = 0, \quad W_3(x^e) = w_3(x^e), \quad W_n(x^e) = \int D_n(w_3)dx^3 + w_n(x^r),$$

If ζ is timelike, then $\zeta_2 > \frac{1}{2}$, while if ζ is null, $\zeta_2 = \frac{1}{2}$. All of the CSI spacetimes belonging to this subcase are automatically $CCNV$ with ℓ as another covariantly constant vector.

Case 1.2.2b : $n + [\frac{\sigma^* v^2}{2} + \zeta_2(u, x^e)]\ell + m^3$

For a particular choice of ζ_2 satisfying (4.59) the vector ζ is a Killing vector for the CSI spacetime with the metric functions in (4.60) where the $W_n^{(1)}$ and $W_n^{(0)}$ satisfy (4.61) and (4.62). The magnitude condition requires $\sigma^* > 0$ implying that $R_{1212} = \sigma > 0$. If ζ is now covariantly constant, a contradiction arises from (4.25), as it requires $D_1H = \sigma v = 0$ despite the fact that we have assumed $\sigma \neq 0$. Thus the subset of CSI spacetimes associated with this subcase are never $CCNV$.

Case 2

Using a coordinate transform of type (3.4) with $g(u) = \frac{u}{\sqrt{|\sigma|}}$, σ in equation (3.12) is rescaled so that it equals $\sigma = -1, 0, 1$. Another coordinate transform of type (3.3) with $h = -\frac{\zeta_3^{(0)}}{D_3\zeta_1}$ causes all but one component to vanish:

$$W_3(u, x^3) = D_3 ln(D_3\zeta_1)v, \quad W_n = 0 \quad .$$

The other Killing vector components may be expressed entirely in terms of ζ_1

$$\zeta_2 = \overbrace{(\frac{\sigma^*\zeta_1}{4} - D_3D_3\zeta_1)}^{\zeta_2^{(2)}}\frac{v^2}{2} + \overbrace{(H^{(1)}\zeta_1 - D_2\zeta_1)}^{\zeta_2^{(1)}}v + \zeta_2^{(0)}(u, x^e),$$
$$\zeta_3 = -D_3(\zeta_1)v.$$

Making one final coordinate transform of type (3.4) with $g = \int e^{-B(u)}du$ removes the u dependence from ζ_1 and, in fact, removes all u dependence from the other components of the Killing vector and the Killing equations, (i.e., (4.70) and (4.73)) involving $H^{(0)}$. Solving these yields the following algebraic equation for $H^{(0)}$

$$H^{(0)} = \frac{D_3(D_3(\zeta_1)\zeta_2^{(0)}) + D_3D_3ln((D_3\zeta_1)^{-\frac{1}{2}})\zeta_2^{(0)}\zeta_1}{\zeta_1^2 D_3D_3ln(\zeta_1(D_3\zeta_1)^{-\frac{1}{2}})}$$

With all of the Killing equations satisfied, the non-spacelike conditions (4.68) and (4.66) give two subcases depending on whether ζ is a null or timelike Killing vector.

Case 2.1: $\zeta = \zeta_1 n + \left[\left(\frac{\sigma^*\zeta_1}{4} - D_3D_3(\zeta_1)\right)\frac{v^2}{2}\right]\ell + [-D_3(\zeta_1)v]m^3$

If ζ is null, ζ_1 takes the following form, depending on the sign of σ:

$$\sigma = -1 : \zeta_1 = \frac{(c_1cosh(x^3)+c_2sinh(x^3))^{-1}}{c_1sinh(x^3)+c_2cosh(x^3)\pm\sqrt{c_1^2+c_2^2+(c_1sinh(x^3)+c_2cosh(x^3))^2}}$$
$$\sigma = 0 : \zeta_1 = \frac{1}{c_1(1\pm\sqrt{2})(c_1x^3+c_2)}$$
$$\sigma = 1 : \zeta_1 = \frac{(c_1cos(x^3)+c_2sin(x^3))^{-1}}{-c_1sin(x^3)+c_2cos(x^3)\pm\sqrt{c_1^2+c_2^2+(-c_1sin(x^3)+c_2cos(x^3))^2}}$$

There are no covariantly constant null vectors in this subcase as the constraint in Lemma (4.2.1) on $W_3^{(1)}$ along with the identity (4.2) lead to a contradition with the given form of ζ_1.

Case 2.2: $\zeta = \zeta_1 n + \left[\left(\frac{\sigma^*\zeta_1}{4} - D_3D_3(\zeta_1)\right)\frac{v^2}{2} + \zeta_2^{(0)}(x^3)\right]\ell + [-D_3(\zeta_1)v]m^3$

If ζ is to be timelike, depending on the sign of σ, ζ_1 takes the form:

$$\sigma = -1 : \zeta_1 = \frac{sinh(x^3)}{c_1(c_1cosh(x^3)+c_2sinh(x^3))} + c_3$$
$$\sigma = 0 : \zeta_1 = \frac{-1}{c_1(c_1x^3+c_2)} + c_3$$
$$\sigma = 1 : \zeta_1 = \frac{sin(x^3)}{c_1(c_1cos(x^3)+c_2sin(x^3))} + c_3.$$

The inequality (4.66) restricts the choice of c_3 depending on the choice of c_1 and c_2:

$$\sigma = -1 : [c_1^2 + c_2^2]\zeta_1^2 - 2\left(\frac{c_1 sinh(x^3)+c_2 cosh(x^3)}{c_1 cosh(x^3)+c_2 sinh(x^3)}\right)\zeta_1 + \frac{1}{(c_1 cosh(x^3)+c_2 sinh(x^3))^2} < 0$$
$$\sigma = 0 : -c_1^2\zeta_1^2 - 2\left(\frac{c_1}{c_1 x^3+c_2}\right)\zeta_1 + \frac{1}{(c_1 x^3+c_2)^2} < 0$$
$$\sigma = 1 : -[c_1^2 + c_2^2]\zeta_1^2 - 2\left(\frac{-c_1 sin(x^3)+c_2 cos(x^3)}{c_1 cos(x^3)+c_2 sin(x^3)}\right)\zeta_1 + \frac{1}{(c_1 cos(x^3)+c_2 sin(x^3))^2} < 0$$

There are no timelike covariantly constant vectors in CSI spacetimes admitting ζ as a Killing vector. Notice in both the null and timelike case, the value of σ restricts the domain of x^3. When $\sigma = 1$ the domain of x^3 limited to a finite interval, $x^3 \in (x_0^3, x_0^3+\pi)$, as the value $x_0^3 = arctan(-\frac{c_1}{c_2})$ will cause $(D_3\zeta_1)^{-\frac{1}{2}}$ to vanish. When $\sigma = 0$, $x^3 \geq -\frac{c_2}{c_1}$ to avoid singularities. In the case with $\sigma = -1$ $x^3 > x_0^3 = arctanh(-\frac{c_1}{c_2})$ when $c_1/c_2 \leq 1$, otherwise ζ_1 is regular on the whole of the real line.

Chapter 5

$CCNV$ Spacetimes Admitting a Two Dimensional Isometry Group

This chapter is based on: D. McNutt, A. Coley, N. Pelavas (2009). Isometries in higher-dimensional CCNV Spacetimes. *IJGMMP*, Vol 6, Issue 3, pp 419-450.

5.1 The Killing Equations

The Killing equations for $X = X_1 n + X_2 \ell + X_i m^i$ are:

$$D_1 X_1 = 0 \tag{5.1}$$

$$D_2 X_1 + D_1 X_2 = 0 \tag{5.2}$$

$$D_3 X_1 + D_1 X_3 = 0 \tag{5.3}$$

$$D_m X_1 = 0 \tag{5.4}$$

$$D_2 X_2 + \sum_i J_i X_i = 0 \tag{5.5}$$

$$D_i X_2 + D_2 X_i - J_i X_1 - \sum_j (A_{ji} + B_{ij}) X_j = 0 \tag{5.6}$$

$$D_j X_i + D_i X_j + 2B_{(ij)} X_1 - 2\sum_k \Gamma_{k(ij)} X_k = 0 \quad . \tag{5.7}$$

To start, we make a coordinate transformation to eliminate $\hat{W}_3$ in (3.18)

$$(u', v', x'^i) = (u, v + h(u, x^k), x^i), \quad h = \int \hat{W}_3 dx^3. \tag{5.8}$$

This choice of coordinates will be useful in order to write down the metric functions and Killing covector components themselves or determining equations for them. For any point in the manifold we may rotate the frame, setting $X_3 \neq 0$ and $X_m = 0$.

This can be done by taking the spatial part of the Killing form $X = X_1 n + X_2 \ell + X_i m^i$ and choosing

$$m^3 = \frac{1}{\chi} X_i m^i \qquad \chi = \sqrt{\sum_i {X_i}^2}. \tag{5.9}$$

Using Gram-Schmidt orthonormalization it is possible to determine the remaining vectors for the frame basis. This is a local orthogonal rotation so the form of our metric remains unchanged while X is now $X = X_1 n + X_2 \ell + \chi m^3$. Henceforth it will be assumed that the matrix m_{ie} is upper-triangular, due to the QR decomposition.

The frame derivatives are

$$\begin{aligned} \ell &= D_1 = \partial_v \\ n &= D_2 = \partial_u - H\partial_v \\ m_i &= D_i = m_i{}^e(\partial_e - \hat{W}_e \partial_v) \end{aligned} \tag{5.10}$$

Thus (5.1) – (5.4) imply that the Killing vector components are of the form:

$$\begin{aligned} X_1 &= F_1(u, x^e) \\ X_2 &= -D_2(X_1)v + F_2(u, x^e) \\ X_3 &= -D_3(X_1)v + F_3(u, x^e). \end{aligned} \tag{5.11}$$

The remaining Killing equations ((5.5)- (5.7) involve A_{ij} and J_i, if we define $W_i = m_i{}^e \hat{W}_e$, we may write $\Gamma_{2i2} = J_i$ and A_{ij} in terms of frame derivatives

$$J_i = D_i H - D_2 W_i - B_{ji} W^j \tag{5.12}$$

$$A_{ij} = D_{[j} W_{i]} + D_{k[ij]} W^k. \tag{5.13}$$

From the commutation relations

$$\begin{aligned} [D_1, D_a] &= 0 \\ [D_2, D_j] &= J_j D_1 - \sum_i B_{ij} D_i \\ [D_k, D_j] &= A_{kj} D_1 + 2\sum_i \Gamma_{i[kj]} D_i \,. \end{aligned} \tag{5.14}$$

applied to the Killing equations the following can be derived:

$$D_a D_3 X_1 = \Gamma_{3na} D_3 X_1 = 0, a = 2, 3, \ldots N \tag{5.15}$$

This leads to two cases either $D_3 X_1 = 0$ or $\Gamma_{3n2} = \Gamma_{3n3} = \Gamma_{3nm} = 0$ and X_1 is linear in x^3. Supposing there exists another Killing vector X we will find further constraints on its components X_a as well as the metric functions W_e and m_{ie} in the ensuing subcases.

5.1.1 Case 1: $D_3 X_1 = 0$

Equation (5.4) implies X_1 must be independent of all spacelike coordinates. Using equation (5.5) and the definition of F_2 from (5.11), we have that X_1 must be of the form

$$X_1 = c_1 u + c_2. \tag{5.16}$$

If $c_1 \neq 0$ we may always use a type (3) coordinate transform from Chapter 3 to set $X_1 = u$, while if $c_1 = 0$ we may choose $c_2 = 1$ by scaling all coordinates by c_2 in both cases the functions F_2, F_3, H and W_e in the new coordinate system are just the original functions multiplied by constants.

Equations (5.15) are identically satisfied, and (5.5)-(5.7) reduce to:

$$\begin{aligned}
c_1 H + D_2 F_2 + J_3 F_3 &= 0 && (5.17)\\
-J_3 X_1 + D_3 F_2 + D_2 F_3 - B_{33} F_3 &= 0 && (5.18)\\
c_1 W_n - J_n X_1 + D_n F_2 - (A_{3n} + B_{n3}) F_3 &= 0 && (5.19)\\
B_{33} X_1 + D_3 F_3 &= 0 && (5.20)\\
2B_{(3n)} X_1 + D_n F_3 - \Gamma_{3n3} F_3 &= 0 && (5.21)\\
2B_{(nm)} X_1 - 2\Gamma_{3(nm)} F_3 &= 0\,. && (5.22)
\end{aligned}$$

Setting $c_1 = c_2 = F_3 = 0$, X reduces to a scalar multiple of the known Killing covector ℓ. We must consider the possibility where F_3 vanishes.

5.1.2 Subcase 1.1: $F_3 = 0$

Setting $F_3 = 0$ in equations (5.20)-(5.22) imply that $B_{(ij)} = 0$. Rewriting this as $B_{(ij)} = m_i{}^e m_j{}^f g_{ef,u}$, the metric is independent of u. By virtue of the upper-triangular form of m_{ie} we see it must be independent of u also. Then assuming $c_1 \neq 0$, we make the appropriate coordinate transformation to set $X_1 = u$, equation (5.17) yields H algebraically:

$$H = -D_2 F_2.$$

Solving the resulting differential equation from (5.19), W_m is expressed as:

$$W_m = \frac{1}{u}\left[\int -D_m(u D_2 F_2 + F_2) du + B_m(x^e)\right].$$

Taking (5.18) with $J_3 = D_3 H$ we see that

$$D_2 D_3(uF_2) = 0,$$

implying that F_2 must be of the form

$$F_2 = \frac{f_2(x^e)}{u} + \frac{g_2(u)}{u}.. \tag{5.23}$$

We rewrite the equations of H and W_m in terms of these two functions

$$\begin{aligned} H &= \frac{f_2(x^e)}{u^2} - \frac{g_2'(u)}{u} + \frac{g_2(u)}{u^2} && (5.24) \\ W_m &= \frac{B_m(x^e)}{u} && (5.25) \end{aligned}$$

where g' denotes the derivative of g with respect to u

If $c_1 = 0$, F_2 must be independent of u, we rescale our coordinates so that $X_1 = 1$, the equations for H and W_n are

$$\begin{aligned} H &= F_2(x^e) + A_0(u, x^r) && (5.26) \\ W_n &= \int D_n A_0 du + C_n(x^e). && (5.27) \end{aligned}$$

In either case, the only requirement on the transverse metric is that it be independent of u. The arbitrary functions in this case are F_2 and the functions arising from integration.

5.1.3 Subcase 1.2: $F_3 \neq 0$

As a consequence of the upper triangular form of m_{ie} the system of equations (5.17) – (5.22) decouples in the following order. Beginning with equation (5.20), we may reduce this to an equation for m_{33} in terms of F_3

$$\frac{m_{33,u}}{m_{33}} = -\frac{1}{X_1} D_3 F_3, \tag{5.28}$$

whose solution is

$$m_{33} = -\int \frac{1}{X_1} F_{3,3} du + A_1(x^3, x^r). \tag{5.29}$$

Next, consider the diagonal components of (5.22) followed by the off-diagonal components to find the determining equation for m_{nr}

$$m_{nr,u} = -m_{nr,3} \frac{F_3}{m_{33} X_1}, \tag{5.30}$$

while equation (5.21) reduces to

$$m_{3r,u} = -\frac{F_{3,r}}{X_1} - \frac{m_{3[r,3]}{m_3}^3 F_3}{X_1}. \tag{5.31}$$

With the transverse metric now determined and assuming $c_1 \neq 0$, we again choose coordinates so that $u' = c_1 u + c_2$, equations (5.17) and (5.18) lead to the form of H

$$H = -D_2 F_2 - \frac{D_2(F_3^2)}{2u} - \frac{F_3 D_3 F_2}{u} - \frac{F_3 D_3(F_3^2)}{2u^2}. \tag{5.32}$$

The form of A_{ij} expressed in frame derivatives (5.13), along with equations (5.21) and (5.22) simplify (5.19) to become the determining equation for the W_n

$$D_2(uW_n) + F_3 D_3 W_n + D_n(F_2 - uH) = 0\,. \tag{5.33}$$

Given $F_2(u, x^e)$ and $F_3(u, x^e)$, we treat the equations (5.32) and (5.33) as constraining equations for H and the W_n.

If c_1 vanishes, rescale to make $c_2 = 1$, from (5.17) and (5.18) F_2 satisfies the equation

$$D_2 F_2 + F_3 D_3 F_2 + \frac{1}{2} D_2(F_3^2) + \frac{1}{2} F_3 D_3(F_3^2) = 0. \tag{5.34}$$

The metric function H may be written as

$$H \quad = \quad \int m_{33} D_2 F_3 dx^3 + F_2 + \frac{1}{2} F_3^2 + A_2(u, x^r). \tag{5.35}$$

The only equation for W_n is

$$F_3 D_3 W_n + D_2 W_n = D_n(H). \tag{5.36}$$

If $c_2 \neq 0$ the equation for m_{ie} holds, however if $X_1 = 0$ they simplify. The equations (5.28) and (5.30) become

$$F_{3,3} = 0, , \quad m_{nr,3} = 0. \tag{5.37}$$

A constraint on the function m_{33} arises from equation (5.18)

$$D_2 log(m_{33}) = -\frac{D_3 F_2}{F_3} - D_2 log(F_3). \tag{5.38}$$

From (5.19) W_n is found

$$W_n = -\int \frac{m_{33} D_n F_2}{F_3} dx^3 + E_n(u, x^r), \tag{5.39}$$

while equation (5.17) gives H

$$H = -\int \frac{m_{33}D_2F_2}{F_3}dx^3 + A_3(u, x^r). \tag{5.40}$$

There are two further subcases to consider, expanding and simplifying equation (5.21)

$$\frac{m_{3r,3}}{F_3} = \left(\frac{m_{33}}{F_3}\right)_{,r} \tag{5.41}$$

and so, either m_{3r} is a function of x^3 or not. If $m_{33,r} \neq 0$ we may integrate (5.41) for m_{3r}

$$m_{3r} = \int \left(\frac{m_{33}}{F_3}\right)_{,r} F_3 dx^3 + G_r(u, x^s). \tag{5.42}$$

Thus the above along with (5.37) and the requirement that $m_{mr,3} = 0$ are the only conditions on the matrix m_{ie}. If m_{3r} is independent of x^3 then we must have $D_n\left(\frac{m_{33}}{F_3}\right) = 0$. implying

$$m_{33,r} = 0. \tag{5.43}$$

Substituting $m_{33}(u, x^3)$ into (5.38) yields a differential equation, whose solution is

$$m_{33} = \frac{1}{F_3}\int -F_{2,3}du + A_4(x^3). \tag{5.44}$$

5.1.4 Case 2: $\Gamma_{3ia} = 0$

To investigate what constraints these requirements give, we expand the expressions for the connection coefficients in question:

$$\begin{aligned}
\Gamma_{3n2} &= W_{[3,f]}m_3{}^3m_n{}^f - m_{3e,u}m_n{}^e + m_{n3,u}m_3{}^3, \\
\Gamma_{3ni} &= \frac{-1}{2}m_{3[e,f]}m_n{}^em_i{}^f + m_{n[f,3]}m_i{}^fm_3{}^3 + m_{i[f,3]}m_n{}^fm_3{}^3.
\end{aligned}$$

These constraints lead to the following facts about the metric functions

Lemma 5.1.1. *The vanishing of Γ_{3ia} imply the metric functions of* (3.18) *must satisfy the following constraints*

$$\hat{W}_{[3,r]} = m_{3[3}m^3{}_{r],u} \tag{5.45}$$

$$m_{3[3,r]} = 0 \tag{5.46}$$

$$m_{3[r,s]} = 0 \tag{5.47}$$

$$g_{rs,3} = 0. \tag{5.48}$$

Proof. To begin the proof consider Γ_{3n2}, using the upper-triangular form for m_{ie} this simplifies to be

$$\hat{W}_{[3,r]} m_3{}^3 m_n{}^r - m_{3e,u} m_n{}^e = 0.$$

Multiplying through by $m^n{}_s$, we note that $m_n{}^e m^n{}_s = \delta^e{}_s - m_3{}^e m^3{}_s$ and so the above equation becomes

$$\hat{W}_{[3,s]} m_3{}^3 = m_{3s,u} - m_{33,u} m_{3s} m_3{}^3.$$

Dividing through by $m_3{}^3 = m_{33}^{-1}$ leads to the desired identity

$$\hat{W}_{[3,s]} = m_{3s,u} m_{33} - m_{33,u} m_{3s}.$$

To show the next identity take Γ_{3n3} , the upper-triangular form leads to the simpler expression

$$m_{3[r,3]} m_n{}^r m_3{}^3 = 0.$$

Since $m_n{}^r$ is invertible, (5.46) follows from this identity. Finally, taking Γ_{3np}

$$m_{3[e,f]} m_n{}^e m_p{}^f + m_{nr,3} m_p{}^r m_3{}^3 + m_{pr,3} m_n{}^r m_3{}^3 = 0.$$

From the above identity $m_{3[3,r]} = 0$ this simplifies to be

$$m_{3[r,s]} m_n{}^r m_p{}^s = -(m_{nf,3} m_p{}^f + m_{pf,3} m_n{}^f) m_3{}^3$$

but $m_{nf,3} m_p{}^f + m_{pf,3} m_n{}^f = m_n{}^r m_p{}^s g_{rs,3}$. Substituting this into the left-hand side we have

$$m_{3[r,s]} m_n{}^r m_p{}^s = -m_n{}^r m_p{}^s g_{rs,3} m_3{}^3.$$

The matrix $m_{3[r,s]}$ is anti-symmetric and g_{rs} is symmetric, by symmetrizing the above we find (5.47) and (5.48) hold. □

Equations (5.46) and (5.47) imply that $m_{3e} = M_{,e}$ for some $M(u, x^k)$ which will be at least a function of the spatial coordinate x^3, otherwise the component m_{33} vanishes and the matrix m_{ie} is no longer invertible. Interestingly, the matrix components m_{nr}

will be independent of x^3 due to (5.48) and the fact that $m_{nr,3}m_p{}^r$ is upper triangular, since for $n < p$ we have

$$m_n{}^r m_p{}^s g_{rs,3} = m_{nr,3}m_p{}^r = 0.$$

In addition, the requirement that $g_{rs,3} = m_{nr,3} = 0$ give further constraints on M ; by expanding the metric $g_{rs} = m_{ir}m^i{}_s$ and differentiating we see that

$$g_{rs,3} = (M_{,r}M_{,s})_{,3} + (m_{nr}m^n{}_s)_{,3} = 0.$$

Choosing $s = r$ this becomes $M_{,r3}M_{,r} = 0$ and so M must be a function of x^3 and possibly coordinate u. The vanishing of $m_{3r} = M_{,r}$ along with (5.45) and (5.13) imply that $A_{3n} = 0$. This equation will be particularly helpful in the subsequent cases as an equation for the $W_n = m_n^e \hat{W}_e$, which in expanded form is

$$-D_{[3}W_{n]} + W^k D_{k3n} = 0. \tag{5.49}$$

However, by looking at the definition of D_{k3n} we see that it vanishes.

Collecting the above results we have the following proposition

Proposition 5.1.2. *The vanishing of* Γ_{3ia} *imply the upper-triangular matrix* m_{ie} *arising from the transverse metric of* (3.18) *takes the form,*

$$m_{33} = M_{,3}(u, x^3), \quad m_{3r} = 0, \quad m_{nr} = m_{nr}(u, x^r). \tag{5.50}$$

While W_n *must satisfy*

$$D_3(W_n) = 0. \tag{5.51}$$

The remaining Killing equations are then:

$$D_2X_2 + J_3X_3 = 0 \tag{5.52}$$

$$D_3X_2 + D_2X_3 - J_3X_1 - B_{33}X_3 = 0 \tag{5.53}$$

$$D_nX_2 - J_nX_1 = 0 \tag{5.54}$$

$$D_3X_3 + B_{(33)}X_1 = 0 \tag{5.55}$$

$$D_nX_3 + 2B_{(3n)}X_1 = 0 \tag{5.56}$$

$$B_{(mn)}X_1 = 0. \tag{5.57}$$

From equation (5.57) we have two subcases to consider. Either $X_1 = 0$ or $B_{(mn)} = 0$.

5.1.5 Case 2.1: $X_1 = 0$, $B_{(mn)} \neq 0$

If $F_3 = 0$, F_2 becomes a constant and so X is some scaling of ℓ, we will assume that $F_3 \neq 0$. From the third equation (5.54)

$$D_n F_2 = m_n{}^e F_{2,e} = 0.$$

Multiplying by $m^n{}_f$ so that $m_n{}^e m^n{}_f = \delta^e{}_f - m_3{}^e m^3{}_f$

$$F_{2,r} = m_3{}^3 m^3{}_r F_{2,3}. \tag{5.58}$$

By Proposition (5.1.2) we see that $m^3{}_r = 0$ and so the left hand side vanishes implying that F_2 is independent of all spacelike coordinates except possibly x^3. Thus the remaining components of X will be the following arbitrary functions,

$$X_2 = F_2(u, x^3) \tag{5.59}$$

$$X_3 = F_3(u). \tag{5.60}$$

Expanding equation (5.53) we find that the constraining equation for m_{33} is

$$\frac{m_{33,u}}{m_{33}} = \frac{-D_3 F_2 - D_2 F_3}{F_3}. \tag{5.61}$$

While from (5.52) we have

$$H = -\int \frac{m_{33} D_2 F_2}{F_3} dx^3 + A_5(u, x^r). \tag{5.62}$$

Thus, (5.62) and (5.61) are equations for H and m_{33}. The only constraint given for the W_n comes from Proposition (5.1.2), i.e, they are all independent of x^3. This is just Case 1.2 with $X_1 = 0$ and the additional constraints in Proposition (5.1.2).

5.1.6 Case 2.2: $B_{(mn)} = 0$, $X_1 \neq 0$

By Proposition (5.1.2), we may repeat a similar calculation as Case 1.1 except with $B_{(np)}$ to show that for $n < p$ the vanishing of $B_{(np)}$ implies

$$m_{nr,u} = 0. \tag{5.63}$$

Furthermore, by proposition (5.1.2), the special form of m_{ie} implies that $m_r{}^3 = 0$, the only non-zero component of the tensor B is B_{33}. Since $v \in (-\infty, \infty)$ we may expand the Killing equations into orders of v, using (5.11) and the definition of the frame derivatives (5.10), to find a system of equations for F_1.

$$D_2 D_2 F_1 + J_3 D_3 F_1 = 0 \tag{5.64}$$

$$D_3 D_2 F_1 + D_2 D_3 F_1 - B_{33} D_3 F_1 = 0 \tag{5.65}$$

$$D_n D_2 F_1 = 0 \tag{5.66}$$

$$D_3 D_3 F_1 = 0 \tag{5.67}$$

$$D_n D_3 F_1 = 0. \tag{5.68}$$

Along with another system of equations involving F_2 and F_3

$$H D_2 F_1 + D_2 F_2 + J_3 F_3 = 0 \tag{5.69}$$

$$H D_3 F_1 + D_3 F_2 + D_2 F_3 - J_3 F_1 - B_{33} F_3 = 0 \tag{5.70}$$

$$D_n F_2 + W_n D_2 F_1 - J_n F_1 = 0 \tag{5.71}$$

$$D_3 F_3 + B_{33} F_1 = 0 \tag{5.72}$$

$$W_n D_3 F_1 + D_n F_3 = 0. \tag{5.73}$$

To begin, the special form of m_{ie} from Proposition (5.1.2) along with equation (5.68) lead to the conclusion that F_1 must be independent of x^3 or x^r . Note that if $F_{1,3} = 0$ then we have Case 1, with the added constraints $\Gamma_{3n2} = \Gamma_{3mi} = 0$. The analysis is not difficult, Case 1.23 may be omitted since $m_{3r} = 0$ while in Case 1.21 and 1.22, equation (5.30) is satisfied immediately, (5.31) now implies that $F_{3,r} = 0$. Only equation (5.29) still holds, these cases are given in the table at the end of this section

It will be assumed that $F_{1,3} \neq 0$, then by expanding equation (5.67) the following relation between F_1 and m_{33} is found

$$\frac{m_{33,3}}{m_{33}} = \frac{F_{1,33}}{F_{1,3}}. \tag{5.74}$$

Rewriting the term $D_2 D_3(F_1)$ in (5.65) using the commutation relations (5.14) with $i = 3$ as

$$D_2 D_3 F_1 = D_3 D_2 F_1 - B_{33} D_3 F_1 \tag{5.75}$$

this is substituted into (5.65) yielding

$$2(D_3 D_2 F_1 - B_{33} D_3 F_1) = 0. \tag{5.76}$$

The expanded form of (5.76) gives another relation between m_{33} and F_1

$$\frac{m_{33,u}}{m_{33}} = \frac{F_{1,3u}}{F_{1,3}}. \tag{5.77}$$

Thus $m_{33}(u, x^3)$ is entirely defined by F_1.

We may solve for H and the W_n algebraically from (5.70) and (5.73)

$$H = \frac{D_3 D_2 F_1}{D_3(F_1)^2} F_3 - \frac{D_2^2 F_1}{D_3(F_1)^2} F_1 - \frac{2D_{(2}F_{3)}}{D_3 F_1} \tag{5.78}$$

$$W_n = -\frac{D_n F_3}{D_3 F_1}. \tag{5.79}$$

Notice that by integrating (5.72), F_3 is of the form:

$$F_3 = \int \frac{m_{33} F_1 D_3 D_2 F_1}{D_3 F_1} dx^3 + A_6(u, x^r) \tag{5.80}$$

Substituting (5.79) and (5.78) into (5.69) and (5.71) yields several equations for F_2

$$D_2 F_2 - \frac{D_2 F_1 D_3 F_2}{D_3 F_1} = D_2\left(\frac{F_3 D_2 F_1}{D_3 F_1}\right) - \frac{F_3 D_3 (D_2 F_1)^2}{2(D_3 F_1)^2} + \frac{F_1 D_2 (D_2 F_1)^2}{2(D_3 F_1)^2} \tag{5.81}$$

$$D_3(F_1) D_3(F_1 D_n F_2) = D_3(F_1 D_2 F_1) D_n F_3. \tag{5.82}$$

Hence $F_2(u, x^e)$ must depend on the choice of the arbitrary functions $F_1(u, x^3)$ and $A_6(u, x^r)$.

5.1.7 Summary of Results

We have considered the possibility of an additional Killing form in a $CCNV$ spacetime, where the metric functions H, $\hat{W}_i$ and g_{ef} are determined by the Killing vector. Given the arbitrary form of the $CCNV$ metric in equation (3.18), we used a coordinate transformation (5.8) to eliminate $\hat{W}_3$; this is done to simplify the constraints on the metric functions. Next the null frame was rotated so that m^3 is parallel with the spatial part of X. Due to the QR decomposition it is always possible to treat the matrix, m_{ie} as an upper-triangular matrix, this is assumed through-out the paper.

The first four equations (5.1) – (5.4) imply that the components of the Killing co-vector are given by (5.11). By applying the commutator relations for the frame derivatives (5.10) to the Killing equation splits the analysis into two simpler cases, depending on whether $D_3 X_1 = 0$ or $\Gamma_{3m2} = \Gamma_{3mj} = 0$. While the first case requires that X_1 is independent of x^3, the implications of $\Gamma_{3m2} = \Gamma_{3mj} = 0$ lead to the constraints on the W_n and the matrix m_{ie} given in Proposition (5.1.2). Both cases are summarized in the tables below.

Case	X_1	F_2	F_3	m_{ie}	H	W_n
1.11	u	(5.23)	0	$m_{ie,u} = 0$	(5.24)	(5.25)
1.12	1	$F_{2,u} = 0$	0	$m_{ie,u} = 0$	(5.26)	(5.27)
1.21	u	F_2	F_3	(5.29) – (5.31)	(5.32)	(5.33)
1.22	1	(5.34)	F_3	(5.29) – (5.31)	(5.35)	(5.36)
1.23 $m_{3r,3} \neq 0$	0	F_2	F_3	(5.37), (5.38) (5.42)	(5.40)	(5.39)
1.24 $m_{3r,3} = 0$	0	F_2	F_3	(5.37), (5.44)	(5.40)	(5.39)

Table 5.1: Summary of analysis in Case 1

5.1.8 Killing Lie Algebra

In (5.1.7) we found that there are only three particular forms for the Killing vector in those $CCNV$ spacetimes admitting an additional Killing vector, depending on the choice of X_1. The three cases depend on whether X_1 is linear in u, X_1 is a constant or X_1 is a function of u and x^3. The remaining functions involved with X_2 and X_3 are functions of u and x^e, satisfying the appropriate equations in the above two tables . We will label those spacetimes admitting an additional Killing vector by its type; using (5.11) we may write the three possible types for the Killing vector X as

$$\begin{aligned}
X_A &= cn + F_2(u, x^e)\ell + F_3(u, x^e)m^3 \\
X_B &= un + [F_2(u, x^e) - v]\ell + F_3(u, x^e)m^3 \\
X_C &= F_1(u, x^3)n + [F_2(u, x^e - D_2F_1v]\ell + [F_3 - D_3F_1v]m^3.
\end{aligned}$$

To see if these spacetimes admit even more Killing vectors we will examine each case and consider the commutator with ℓ. Using the frame formalism, the commutator of two vector-fields $X = X^a e_a$, $Y = Y^b e_b$ is

$$[X, Y] = X^a e_a(Y^b) - Y^a e_a(X^b) + 2X^a Y^c \Gamma^b_{[ac]} \tag{5.83}$$

When $Y = \ell$ we have that $Y^c = \delta^c_1$ and from (5.14) $\Gamma^b_{1a} = 0$ so the commutator is

$$[X, \ell] = -\ell(X^b).$$

Case	X_1	F_2	F_3	m_{ie}	H	W_n
2.1	0	$F_{2,r} = 0$	$F_{3,e} = 0$	(5.38)	(5.40)	(5.39)
2.21	u	(5.23)	0	$m_{ie,u} = 0$	(5.24)	(5.25)
2.22	1	$F_{2,u} = 0$	0	$m_{ie,u} = 0$	(5.26)	(5.27)
2.23	u	F_2	$F_{3,r} = 0$	(5.29), (5.63)	(5.32)	(5.33)
2.24	1	(5.34)	$F_{3,r} = 0$	(5.29), (5.63)	(5.35)	(5.36)
2.25	0	F_2	$F_{3,e} = 0$	(5.38), (5.63)	(5.40)	(5.39)
2.26	$F_{1,r} = 0$	(5.81) (5.82)	(5.80)	(5.77), (5.74), (5.63)	(5.78)	(5.79)

Table 5.2: Summary of Case 2, where Proposition (5.1.2) implies $D_3(W_n) = 0$ and m_{ie} takes the special form (5.50)

Thus in the Type A spacetimes there are no other required Killing vectors except ℓ and X. Similarly in the Type B spacetimes, the commutator of ℓ and X is

$$[X_B, \ell] = -\ell$$

this is just a scaling of a known vector so we may conclude in general that Type B Spacetimes contain no additional Killing vectors other than ℓ and X.

The most general case is more interesting because the commutator of X and ℓ yields a new Killing vector

$$Y_C = [\ell, X_C] = D_2 F_1 \ell + D_3 F_1 m_3. \tag{5.84}$$

Clearly this will be a space-like Killing vector for all choices of F since its magnitude is $|Y_C| = (D_3 F_1)^2 > 0$. The commutator of Y_C with ℓ vanishes because F_1 is a function of u and x^3, however the commutator of Y_C and X_C cannot in general be set to zero. A quick calculation gives $Z_C = [X_C, Y_C]$

$$\begin{aligned} Z_C &= [F_3 D_3 D_2 F_1 - D_3 F_1 D_3 F_2 + (D_2 F_1)^2]\ell \\ &\quad -(D_3 F_1)^2 n + [D_2 F_1 D_3 F_1] m_3 \end{aligned} \tag{5.85}$$

Thus because we assumed $F_1 \neq 0$ and $D_3 F_1 \neq 0$ we may never set $Z_C = 0$ due to the coefficients of n. The Type C spacetimes admit at least one additional spacelike Killing vector.

5.2 CSI $CCNV$ Spacetimes Possessing an Additional Killing Vector

In section 3.4.1 it was shown that if a $CCNV$ spacetime has constant scalar curvature invariants to all orders its transverse metric g_{ef} must be locally homogeneous. Applying this result we may write down the constraints for a CSI $CCNV$ spacetime to admit an additional Killing vector by choosing an appropriate locally homogeneous Riemannian manifold for the transverse space g_{ef}. This choice may affect the components of the Killing vector X.

Due to the local homogeneity of g_{ef} one may perform a coordinate transformation so that the matrix m_{ie} is independent of u. Looking at the tables in section 5.1.7 we note that Cases 1.11 - 1.13 and 2.21 - 2.23 already require that m_{ie} be independent of u and there are no constraints on the Killing vector components involving m_{ie}. Therefore the CSI spacetimes are the subcases of these cases where the transverse metric is a locally homogeneous. The remaining cases are more interesting since they involve a non-zero spatial component of X.

5.2.1 Case 1

In Case 1.2 equation (5.28) implies that

$$F_{3,3} = 0 \tag{5.86}$$

while from (5.30)

$$m_{nr,3} = 0. \tag{5.87}$$

The remaining equations (5.30), (5.28), (5.31) arose from (5.21), this may be rewritten as a differential equation for F_3

$$D_n(logF_3) = \Gamma_{3n3}. \tag{5.88}$$

It is possible to derive even more constraints on the transverse space through the commutation relations $[D_i, D_n]$ applied to $F_3(u, x^r)$. To start, we note that because of (5.14) and the u-independence of m_{ie}, $[D_2, D_n](logF_3)$ becomes

$$D_nD_2(logF_3) = 0. \tag{5.89}$$

Next we consider $[D_3, D_n]$, since $D_3(logF_3) = 0$ and $D_3({m_n}^r) = 0$ the commutator is

$$[D_3, D_n](logF_3) = D_3(\Gamma_{3n3}) = 0.$$

However using (5.14) with $k = 3$ we find that

$$[D_3, D_n](logF_3) = {\Gamma^m}_{[3n]} D_m(logF_3).$$

Thus we find two new constraints on the connection coefficients arising from the transverse metric

$$D_3(\Gamma_{3n3}) = 0 \tag{5.90}$$

$${\Gamma^m}_{[3n]}\Gamma_{3m3} = 0. \tag{5.91}$$

With $k, j > 3$ in (5.14) we find that

$$[D_n, D_m](logF_3) = 2{\Gamma^p}_{[nm]}\Gamma_{3p3}$$

whereas from (5.88) we have that

$$[D_n, D_m](logF_3) = D_n\Gamma_{3m3} - D_m\Gamma_{3n3}.$$

Equating the two gives another constraint on the transverse space,

$${\Gamma^p}_{[nm]}\Gamma_{3p3} = D_n\Gamma_{3m3} - D_m\Gamma_{3n3}. \tag{5.92}$$

Thus if we require the Killing vector to have a spatial component the connection coefficients arising from the transverse metric must satisfy the equations (5.87), (5.90), (5.91) and (5.92). We will assume such a transverse metric has been found in order to continue with the analysis.

Reconsidering (5.31) and solving for $m_{3r,3}$ leads to another differential equation

$$m_{3r,3} = F_3 \left(\frac{m_{33}}{F_3}\right)_{,r} \tag{5.93}$$

and two possibilities, either $m_{3r,3}$ vanishes or not. If $m_{3r,3} \neq 0$ we may integrate the above equation to find an expression for m_{3r}

$$m_{3r} = \int \left(\frac{m_{33}}{F_3}\right)_{,r} F_3 dx^3 + B_r(x^s). \tag{5.94}$$

Differentiating with respect to u we must have

$$\left(\frac{F_{3,r}}{F_3}\right)_{,u} = 0. \tag{5.95}$$

So that F_3 is of the form

$$log(F_3) = g_3(u) + f_3(x^r). \tag{5.96}$$

The form of $log(F_3)$ above agrees with the differential equation given in (5.88) and (5.89), where we expect f_3 is determined by the Γ_{3n3}. In this case, (5.87) and (5.94) are the only equations for the transverse metric so far. The remaining constraints on the metric functions will vary for each subcase depending on the choice of $X_1 = c_1 u + c_2$.

If $c_1 \neq 0$, we see that H is given by

$$H = -\frac{1}{u}(uD_2F_2 + F_3D_3F_2 + F_3D_2F_3). \tag{5.97}$$

While the W_n satisfy the determining equation

$$D_2(uW_n) + F_3D_3W_n + D_n(F_2 - uH) = 0\,. \tag{5.98}$$

If $c_1 = 0$, F_2 is no longer arbitrary it must satisfy the following equation

$$D_2F_2 + F_3D_3F_2 + F_3D_2F_3 = 0 \tag{5.99}$$

H may be written as

$$H \;=\; F_2 + \int m_{33} D_2F_3 dx^3 + A_1(u, x^r), \tag{5.100}$$

and the equation for W_n is now

$$D_2W_n + F_3D_3W_n = D_nF_2 - D_nH. \tag{5.101}$$

If $c_1 = c_2 = 0$ then X_1 vanishes, we know that this turns (5.29) – (5.31) into the same set of differential equations (5.86), (5.87) and (5.88).

The remaining metric functions H and the W_n are

$$H = -\int \frac{m_{33}D_2F_2}{F_3} dx^3 + A_2(u, x^r), \tag{5.102}$$

$$W_n = -\int \frac{m_{33}D_nF_2}{F_3} dx^3 + E_n(u, x^r). \tag{5.103}$$

If $m_{3r,3} = 0$ we find the following expression for m_{33}

$$D_n\left(\frac{m_{33}}{F_3}\right) = 0. \tag{5.104}$$

. This leads to the following equality

$$D_n(logm_{33}) = D_n(logF_3) = \Gamma_{3n3} \tag{5.105}$$

Expanding this out we find the constraint,

$$m_{33,3}m^3{}_r = (m_{33}^2)_{,r} \tag{5.106}$$

The equations for the metric functions follow as above in the various cases arising from $X_1 = c_1 u + c_2$.

5.2.2 Case 2

If $m_{ie,u}$ vanishes, Case 2.1 and Case 2.26 are now the same case. Due to Proposition 5.1.2, equations (5.30) (5.28) (5.31) imply that

$$F_{3,e} = 0. \tag{5.107}$$

From (5.38) we find the familiar equation for F_2:

$$F_{2,3} = -m_{33}F_{3,u}. \tag{5.108}$$

The metric function H is given by equation (5.62) and W_n may be arbitrary functions of u and x^r.

In Cases 2.24 - 2.26 the vanishing of $m_{ie,u}$ causes (5.29) and (5.31) to imply

$$F_{3,e} = 0. \tag{5.109}$$

So F_3 is only a function of u. With this in mind the equations for the remaining metric functions are the same as in (5.2.1), $m_{3r,3} = 0$, with the additional constraints from Proposition (5.1.2).

In Case 2.27, equation (5.77) now implies

$$F_{1,u3} = 0. \tag{5.110}$$

The function F_1 must be of the form

$$F_1 = g_1(u) + f_1(x^3) \tag{5.111}$$

g_1 is an arbitrary function of u however, f_1 satisfies the differential equation in (5.74)

$$D_3(logm_{33}) = D_3log(f_{1,3}).$$

Letting c_3 be arbitrary constant we have by integrating that

$$f_1 = m_{33} + c_3 \tag{5.112}$$

Combining (5.111) and (5.112) yields

$$F_1 = m_{33} + g_1 + c_3 \tag{5.113}$$

From (5.80), F_3 must be a function of u and x^r, the equations (5.81) and (5.82) are

$$D_2F_2 - \tfrac{D_2F_1}{D_3F_1}D_3F_2 = (\tfrac{D_2^2F_1}{D_3F_1})F_3 + \tfrac{D_2F_1}{D_3F_1}D_2F_3 + \tfrac{F_1D_2F_1D_2^2F_1}{D_3F_1^2}, \tag{5.114}$$
$$D_3(F_1D_nF_2) = D_2F_1D_nF_3. \tag{5.115}$$

Assuming $g_1' \neq 0$, dividing through (5.114) by $\frac{g_1'}{f_1}$ it is possible to solve for D_2F_3, while dividing from g_1' in (5.115) we have D_nF_3, hence it is possible to solve for F_3 entirely in terms of F_2 and g_1'.

Simplifying (5.78) and (5.79) we find that,

$$H = -\frac{D_2^2F_1}{D_3(F_1)^2}F_1 - \frac{D_2F_3}{D_3F_1} - \frac{D_3F_2}{D_3F_1} \tag{5.116}$$
$$W_n = -\frac{D_nF_3}{D_3F_1}. \tag{5.117}$$

If $D_2F_1 \neq 0$, dividing (5.114) by this then substituting this into the equation for H gives,

$$H = (\frac{D_2^2F_1}{D_3F_1})F_3 - D_2F_2. \tag{5.118}$$

We note that F_2 may be entirely an arbitrary function of u and x^3.

5.2.3 Summary of Constraints

As in the previous section we summarize our results for the existence of an additional null Killing covector in a CSI $CCNV$ spacetime. In order for a $CCNV$ spacetime to be CSI, we found in section 3.4.1 that the transverse space must be locally homogeneous. This allows one to choose a coordinate chart locally such that

$$m_{ie,u} = 0$$

so that many of the differential equations given in the previous section are simpler. The results of this analysis are summarized below in the two tables. In all cases the transverse metric is locally homogeneous, although it should be noted that in subsection (5.2.1), if $F_3 \neq 0$ then the transverse space must satisfy the following constraints

$$\begin{aligned} m_{nr,3} &= 0 \\ D_3(\Gamma_{3n3}) &= 0 \\ \Gamma^m{}_{[3n]}\Gamma_{3m3} &= 0 \\ \Gamma^p{}_{[nm]}\Gamma_{3p3} &= D_n\Gamma_{3m3} - D_m\Gamma_{3n3}. \end{aligned} \tag{5.119}$$

We also remind the reader that if $m_{3r,3} = 0$ equation (5.106) holds, so that m_{33} is independent of x^r. In the second case the matrix m_{ie}, related to the locally homogeneous transverse metric, must satisfy (5.50) in Proposition (5.1.2).

Case	X_1	F_2	F_3	H	W_n
1.11	u	(5.23)	0	(5.24)	(5.25)
1.12	1	$F_{2,u} = 0$	0	(5.26)	(5.27)

Table 5.3: Summary of Killing equations analysis in Case 1 for a CSI $CCNV$ spacetime, when $F_3 = 0$

5.3 Application: Non-spacelike Isometries

The metric for $CCNV$ spacetimes (3.18) must be independent of v, varying this coordinate value leaves the metric unchanged. With regards to the set of $CCNV$

Case	X_1	F_2	F_3	H	W_n
1.21a	u	F_2	(5.96)	(5.97)	(5.98)
1.22a	1	(5.99)	(5.96)	(5.100)	(5.101)
1.23a	0	F_2	F_3	(5.102)	(5.103)
$m_{3r,3} \neq 0$			(5.94)		

Table 5.4: Summary of Killing equations analysis in Case 1 for a CSI $CCNV$ spacetime, when $F_3 \neq 0$ and $m_{3r,3} \neq 0$

Case	X_1	F_2	F_3	H	W_n
1.21b	u	F_2	$F_{3,3} = 0$	(5.97)	(5.98)
1.22b	1	(5.99)	$F_{3,3} = 0$	(5.100)	(5.101)
1.23b	0	F_2	$F_{3,3} = 0$	(5.102)	(5.103)

Table 5.5: Summary of Killing equations analysis in Case 1 for a CSI $CCNV$ spacetime, when $F_3 \neq 0$ and $m_{3r,3} = 0$

spacetimes admitting an additional Killing vector, a good question to ask is: which of these spacetimes admit a non-spacelike Killing vector for all values v? In [90] this was considered for CSI $CCNV$ spacetimes, however the approach taken differs from the one presented in this paper.

While the frame was rotated so that the Killing covector X has one spatial component X_3 and the matrix m_{ie} is upper-triangular, a coordinate transformation was made to eliminate H instead of W_3. Regardless of these coordinate changes, the equations (5.1) – (5.4) lead to the same form for the Killing covector components given in (5.11). The non-spacelike requirement for the Killing vector field maybe written as

$$D_3(X_1)^2 v^2 + 2(D_2(X_1)X_1 - D_3(X_1)F_3)v + {F_3}^2 - 2X_1F_2 \leq 0$$

Since $v \in (-\infty, \infty)$ this implies that $D_3(X_1)$ must vanish and either X_1 is independent of u or $X_1 = 0$.

Thus either X_1 is constant or it vanishes entirely. This requirement along with the u independence of m_{ie} from the CSI condition lead to a simpler set of equations for the remaining components of X, as such the commutator relations were ignored in [90] and the analysis was done using the coordinate basis instead of the frame

Case	X_1	F_2	F_3	H	W_n
2.1	0	$F_{2,r}=0$	$F_{3,e}=0$	(5.40) $m_{ie,u}=0$	(5.39)
2.21 $F_{2,3}\neq 0$	u	(5.23)	0	(5.24)	(5.25)
2.22	1	$F_{2,u}=0$	0	(5.26)	(5.27)
2.23	u	F_2	$F_{3,e}=0$	(5.97)	(5.98)
2.24	1	(5.99)	$F_{3,e}=0$	(5.100)	(5.101)
2.25	(5.113)	(5.114) (5.115)	$F_{3,3}=0$	(5.116)	(5.117)

Table 5.6: Summary of Case 2 for a CSI $CCNV$ spacetime - Proposition (5.1.2) implies $D_3(W_n)=0$ and m_{ei} takes the special form (5.50)

formalism. As such the results of [90] agree with the results given in this paper, but only as special subcases of 1.1 and 1.2 in the Table 5.3.2 given below.

Instead of the labour intensive approach given in [90] we may use the result of the previous section to find an answer to the question of non-spacelike Killing vectors in $CCNV$ spacetimes. Since X_1 must be constant from the non-spacelike requirement we have that Cases 1.11, 1.21, 2.21, 2.23 and 2.26 are no longer admissible. In the remaining cases the only constraint left is for F_2 and F_3 is

$$F_3{}^2 - 2X_1F_2 \leq 0. \tag{5.120}$$

Hence we will divide the analysis into two cases depending on whether the vector is timelike or null.

5.3.1 Timelike Killing Vector Fields

If we allow X to be a timelike Killing vector field, we have the constraint that

$$F_3^2 < 2c_2F_2 \tag{5.121}$$

and so the cases with $X_1=0$ (1.23,1.24,2.1 and 2.25) are no longer valid since F_3 is a real-valued function and with $F_3^2<0$. which is impossible and so these cases will be disregarded. In the remaining cases (1.12, 1.22, 2.22 and 2.24) equation (5.121) is an additional constraint on F_3 and F_2. Thus the Killing vector field X will always be of the form

$$n + F_2(u, x^e)\ell + F_3(u, x^r)m^3. \tag{5.122}$$

The requirement that $F_3^2 < 2F_2$ does not affect the equations in the various cases.

5.3.2 Null Killing Vector Fields

If X is null, and $c_2 \neq 0$ we can rescale n so that (5.120) implies that $2F_2 = {F_3}^2$, from which we naturally find the helpful identity

$$D_a(F_2) = D_a(F_3)F_3. \tag{5.123}$$

If F_3 vanishes as in Case 1.12, F_2 must vanish as well, so X takes the form

$$X = n. \tag{5.124}$$

the remaining equations for the metric functions are now

$$H = A_0(u, x^r) \tag{5.125}$$

$$W_n = \int D_n(A_0)du + C_n(x^e). \tag{5.126}$$

The transverse metric is unaffected by (5.120).

In Case 1.22, X is now

$$X = n + \frac{F_3^2}{2}\ell + F_3 m^3 \tag{5.127}$$

taking equation (5.34) we find a differential equation for F_3

$$D_2(F_3) + D_3(F_3)F_3 = 0. \tag{5.128}$$

This allows us to rewrite H as

$$H = A_2(u, x^r). \tag{5.129}$$

The constraining equation for the W_n is

$$D_2(W_n) + D_3(W_n)F_3 = D_n(A_2). \tag{5.130}$$

We may rewrite (5.28) in a simpler form using (5.128)

$$\frac{m_{33,u}}{m_{33}} = \frac{D_2(F_3)}{F_3}. \tag{5.131}$$

This may be integrated to find m_{33} in terms of F_3, assuming $m_{33,u} = 0$, however this is more useful as a differential equation. The remaining two equations (5.30) and (5.31) are unchanged. If $c_2 = 0$, then X_1 vanishes entirely and (5.120) implies that

$$F_3^2 = 0. \tag{5.132}$$

If $X_1 = F_3 = 0$ then the Killing equations (5.17) – (5.22) implies F_2 must be constant. That is, our Killing covector is a scalar multiple of ℓ, so we will disregard this as well as Case 2.1 and 2.25.

The remaining cases 2.22 and 2.24 are just a repetition of the above equations with the added constraints that Proposition (5.1.2) holds and $m_{mn,u} = 0$. In the first case where $F_3 = 0$, this changes the W_n

$$W_n = \int D_n(A_0)du + C_n(x^r). \tag{5.133}$$

No other metric functions are affected. When $F_3 \neq 0$, the additional constraints imply that (5.30) is satisfied trivially and (5.31) becomes

$$F_{3,r} = 0. \tag{5.134}$$

Lastly since $D_3(W_n) = 0$, equation (5.130) implies

$$W_n = -\int D_n(A_2)du + A_7(x^r). \tag{5.135}$$

We summarize these results in the following table.

If we wish to find *CSI CCNV* spacetimes admitting Killing vectors which are non-spacelike for all values of v, the above table will be helpful. The *CSI CCNV* spacetimes are the subcases of the above cases, where the transverse space is locally homogenous, allowing for a choice of coordinates where m_{ie} is independent of u.

In case 1.1 and 2.1, none of the equations are affected by the vanishing of $m_{ie,u}$, while in case 1.2 and 2.2 equation (5.131) is no longer applicable, instead we look to (5.29) which implies that

$$F_{3,3} = 0. \tag{5.136}$$

Case	X_1	F_2	F_3	m_{ie}	H	W_n
1.1	1	0	0	$m_{ie,u}=0$	(5.125)	(5.126)
1.2	1	$\frac{1}{2}F_3^2$	F_3	(5.131), (5.31) (5.30)	(5.129)	(5.130)
2.1	1	0	0	$m_{ie,u}=0$ (5.50)	(5.125)	(5.133)
2.2	1	$\frac{1}{2}F_3^2$	$F_{3,r}=0$	(5.131), (5.50) $m_{nr,u}=0$	(5.129)	(5.135)

Table 5.7: Constraints on $CCNV$ metric in order to allow null isometry.

Unlike the previous cases, where no other function is affected by our choice of coordinates, in case 1.2 (5.31) implies

$$m_{mr,3}=0. \tag{5.137}$$

From (5.30) we find the following differential equation for F_3

$$D_n(logF_3)=\Gamma_{3n3}. \tag{5.138}$$

The remaining constraints on the metric functions follows as in section 5.2 where the Killing vector X is of type A with some minor modifications due to $F_2=\frac{1}{2}F_3^2$. We will do a simple example to illustrate.

5.3.3 A Simple Example

To simplify matters, we will assume that the transverse space is locally homogeneous and that Γ_{3n3}, $m_{3r,3}$ both vanish. By Lemma 3.4.1 this will be a CSI $CCNV$ spacetime and since (5.87), (5.90), (5.91) and (5.92) are all satisfied, it will also admit a null Killing vector X of the form

$$X=n+\frac{F_3^2}{2}\ell+F_3m^3 \tag{5.139}$$

For brevity we will only consider the simpler subcase where it is assumed that the components m_{3r} are independent of x^3. Expanding Γ_{3n3} in terms of the transverse space frame matrix

$$\Gamma_{3n3}=m_{3[r,3]}m_n{}^r m_3{}^3=-m_{33,r}m_n{}^r m_3{}^3 \tag{5.140}$$

Then by multiplying this by ${m^n}_r$ we find that

$$m_{33,r} = 0. \tag{5.141}$$

This and equation (5.106) then implies that

$$m_{33,3}{m^3}_r = 0. \tag{5.142}$$

So that either $m_{33,3}$ or ${m^3}_r$ must vanish. If ${m^3}_r = 0$, the matrix m_{ie} will take the form

$$\begin{aligned} m_{33} &= m_{33}(x^3) \\ m_{3r} &= 0 \\ m_{nr} &= m_{nr}(x^r). \end{aligned} \tag{5.143}$$

On the other hand, if $m_{33,3} = 0$, it will be a constant, say M_{33}, and then the matrix m_{ie} is of the form

$$\begin{aligned} m_{33} &= M_{33} \\ m_{3r} &= m_{3r}(x^r) \\ m_{nr} &= m_{nr}(x^r). \end{aligned} \tag{5.144}$$

In either case, the choice does not affect the remaining metric functions and Killing vector components. Noting (5.86) in (5.88), we may multiply by ${m^n}_s$ to see that F_3 is at most a function of u. However, from equation (5.131) we see that it must be a constant. By requiring X to be null, we obtain $F_2 = \frac{F_3^2}{2}$, and so (5.99) gives no new information. The Killing vector may then be written as

$$n + \frac{1}{2}\ell + m^3, \tag{5.145}$$

subtracting the known Killing vector $\frac{1}{2}\ell$ we find the spacelike Killing vector, $Y = n + m^3$. The metric function H is found to be an arbitrary function of u and x^r by (5.129), while W_n is determined by the linear partial differential equation in (5.130)

$$D_3 W_n + D_2 W_n = D_n H. \tag{5.146}$$

Rewritting the above in coordinate form

$$\hat{W}_{r,3} + \hat{W}_{r,u} = H_{,r}. \tag{5.147}$$

Applying the method of characteristics, the solution is written as

$$\hat{W}_r = \frac{1}{\sqrt{2}} \int_L H(u, x^r) ds + g(x^3 - u) \tag{5.148}$$

where g is an arbitrary function of one variable and L is the characteristic line segment from the u-axis to an arbitrary point (x_0^3, u_0).

Thus we have found that in the subset of CSI $CCNV$ spacetimes where $m_{33,r}$ and Γ_{3n3} both vanish, there are no null Killing vectors other than ℓ. However, these spacetimes always admit the space-like Killing vector

$$Y = n + m^3. \tag{5.149}$$

Chapter 6

$CCNV$ Spacetimes and (Super)symmetries

This chapter is based on: D. McNutt, N. Pelavas (2013). CCNV spacetimes and (Super) Symmetries . *Submitted to MPAG*. Copyright 2013, with kind permission from Springer Science+Business Media B.V.

6.1 Supergravity and Supersymmetries

Supersymmetric supergravity solutions are of interest in the context of the AdS/CFT conjecture, the microscopic properties of black hole entropy, and in a search for a deeper understanding of string theory dualities. For example, in five dimensions solutions preserving various fractions of supersymmetry of $N = 2$ gauged supergravity have been studied. The Killing spinor equations imply that supersymmetric solutions preserve $2, 4, 6$ or 8 of the supersymmetries. The AdS_5 solution with vanishing gauge field strengths and constant scalars preserves all of the supersymmetries. Half supersymmetric solutions in gauged five dimensional supergravity with vector multiplets possess two Dirac Killing spinors and hence two time-like or null Killing vectors. These solutions have been fully classified, using the spinorial geometry method, in [87]. Indeed, in a number of supergravity theories [73], in order to preserve some supersymmetry it is necessary that the spacetime admits a Killing spinor which then yields a null or timelike Killing vector from its Dirac current. Therefore, a necessary (but not sufficient) condition for supersymmetry to be preserved is that the spacetime admits a null or timelike Killing vector .

In this chapter we investigate the existence of additional Killing vectors in the class of higher-dimensional Kundt spacetimes admitting a covariantly constant null vector ($CCNV$) [100], which is of interest in the study of supergravity solutions preserving a non-minimal fraction of supersymmetries. $CCNV$ spacetimes belong to the Kundt class because they contain a null Killing vector which is geodesic, non-expanding, shear-free and non-twisting. The existence of an additional Killing vector puts constraints on the metric functions and the vector components. Killing vectors that are null or timelike locally or globally (for all values of the coordinate v) are of particular importance. As an illustration we present two explicit examples.

The subset of $CCNV$ spacetimes which are also CSI or VSI are of particular interest. Indeed, it has been shown previously that the higher-dimensional VSI spacetimes with fluxes and dilaton are solutions of type IIB supergravity [84]. A subset of Ricci type N VSI spacetimes, the higher-dimensional Weyl type N pp-wave spacetimes, are known to be solutions in type IIB supergravity with an R-R five-form or with NS-NS form fields [44, 43]. In fact, all Ricci type N VSI spacetimes are solutions to supergravity and, moreover, there are VSI spacetime solutions of type IIB supergravity which are of Ricci type III, including the string gyratons, assuming appropriate source fields are provided [84]. It has been argued that the VSI supergravity spacetimes are exact string solutions to all orders in the string tension.

Those VSI spacetimes in which supersymmetry is preserved admit a $CCNV$. Higher-dimensional VSI spacetime solutions to type IIB supergravity preserving some supersymmetry are of Ricci type N, Weyl type III(a) or N [80]. It is also known that $AdS_d \times S^{(D-d)}$ spacetimes are supersymmetric CSI solutions of IIB supergravity. There are a number of other CSI spacetimes known to be solutions of supergravity and admit supersymmetries [79], including generalizations of $AdS \times S$ [81], of the chiral null models [44], and the string gyratons [75]. Some explicit examples of CSI $CCNV$ Ricci type N supergravity spacetimes have been constructed [85].

6.2 Additional Isometries

Let us choose the coframe $\{m^a\}$

$$m^1 = n = dv + Hdu + \hat{W}_e dx^e, \quad m^2 = \ell, \quad m^i = m^i{}_e dx^e, \tag{6.1}$$

where $m^i{}_e m_{if} = g_{ef}$ and $m_{ie} m_j{}^e = \delta_{ij}$. The frame derivatives are given by

$$\ell = D_1 = \partial_v, \quad n = D_2 = \partial_u - H\partial_v, \quad m_i = D_i = m_i{}^e(\partial_e - \hat{W}_e\partial_v).$$

The Killing vector can be written as $X = X_1 n + X_2 \ell + X_i m^i$.A coordinate transformation can be made to eliminate $\hat{W}_3$ in (3.18) and we may rotate the frame in order to set $X_3 \neq 0$ and $X_m = 0$ [100]. X is now given by

$$X = X_1 n + X_2 \ell + \chi m^3.$$

Henceforth it will also be assumed that the matrix m_{ie} is upper-triangular.

The Killing equations can then be written as:

$$X_{1,v} = 0, \quad X_{1,u} + X_{2,v} = 0, \quad m_3{}^e X_{1,e} + X_{3,v} = 0, \quad m_n{}^e X_{1,e} = 0, \tag{6.2}$$

which imply

$$X_1 = F_1(u, x^e), \quad X_2 = -D_2(X_1)v + F_2(u, x^e), \quad X_3 = -D_3(X_1)v + F_3(u, x^e),$$

and

$$D_2 X_2 + \sum_i J_i X_i = 0 \tag{6.3}$$

$$D_i X_2 + D_2 X_i - J_i X_1 - \sum_j (A_{ji} + B_{ij}) X_j = 0 \tag{6.4}$$

$$D_j X_i + D_i X_j + 2B_{(ij)} X_1 - 2 \sum_k \Gamma_{k(ij)} X_k = 0, \tag{6.5}$$

where $B_{ij} = m_{ie,u} m_j{}^e$, $W_i = m_i{}^e \hat{W}_e$, and $J_i \equiv \Gamma_{2i2} = D_i H - D_2 W_i - B_{ji} W^j$, $A_{ij} \equiv D_{[j} W_{i]} + D_{k[ij]} W^k$, $D_{ijk} \equiv 2 m_{ie,f} m_{[j}{}^e m_{k]}{}^f$. Further information can be found by taking the Killing equations and applying the commutation relations, which leads to two cases; (1) $D_3 X_1 = 0$, or (2) $\Gamma_{3n2} = \Gamma_{3n3} = \Gamma_{3nm} = 0$.

6.2.1 Case 1: $D_3 X_1 = 0$

Using equation (6.3) and the definition of F_2 from (5.11), we have that $X_1 = c_1 u + c_2$. If $c_1 \neq 0$ we may always choose coordinates to set $X_1 = u$, while if $c_1 = 0$ we may choose $c_2 = 1$.

Subcase 1.1: $F_3 = 0$. (i) $c_1 \neq 0$, $X_1 = u$; F_2 must be of the form given in (5.23):

$$F_2 = \frac{f_2(x^e)}{u} + \frac{g_2(u)}{u}.$$

H and W_m are given in terms of these two functions which arise from ((5.24) and (5.25)) (where $g' \equiv \frac{dg}{du}$)

$$H = \frac{f_2(x^e)}{u^2} - \frac{g_2'(u)}{u} + \frac{g_2(u)}{u^2}, \quad W_m = \frac{B_m(x^e)}{u}.$$

(ii) $c_1 = 0$, $X_1 = 1$; $F_{2,u} = 0$, and H and W_n in ((5.26) and (5.27)) become

$$H = F_2(x^e) + A_0(u, x^r), \quad W_n = \int D_n A_0 du + C_n(x^e).$$

In either case, the only requirement on the transverse metric is that it be independent of u. The arbitrary functions in this case are F_2 and the functions arising from integration.

Subcase 1.2: $F_3 \neq 0$. The transverse metric is now determined by the constraints (5.29) and (5.31):

$$\begin{aligned} m_{33} &= -\int \tfrac{1}{X_1} F_{3,3} du + A_1(x^3, x^r). \\ m_{nr,u} &= -m_{nr,3} \tfrac{F_3}{m_{33} X_1}, \\ m_{3r,u} &= -\tfrac{F_{3,r}}{X_1} - \tfrac{m_{3[r,3]} {m_3}^3 F_3}{X_1} \end{aligned} .$$

(i) $c_1 \neq 0$, $X_1 = u$; $F_i(u, x^e)$ $(i = 1, 2)$ are arbitrary functions, from (5.32) H is given by

$$H = -D_2 F_2 - \frac{D_2(F_3^2)}{2u} - \frac{F_3 D_3 F_2}{u} - \frac{F_3 D_3(F_3^2)}{2u^2},$$

and W_n is determined by equation (5.33)

$$D_2(u W_n) + F_3 D_3 W_n + D_n(F_2 - uH) = 0. \tag{6.6}$$

(ii) $c_1 = 0$, $(c_2 \neq 0)$ $X_1 = 1$; F_2 and F_3 must satisfy (5.34)

$$D_2 F_2 + F_3 D_3 F_2 + \frac{1}{2} D_2(F_3^2) + \frac{1}{2} F_3 D_3(F_3^2) = 0.$$

From equation (5.35) H may be written as

$$H \quad = \quad \int m_{33} D_2 F_3 dx^3 + F_2 + \frac{1}{2} F_3^2 + A_2(u, x^r).$$

The only equation for W_n is (5.36)

$$F_3 D_3 W_n + D_2 W_n = D_n(H).$$

(iii) $X_1 = 0$:

$$F_{3,3} = 0, \quad m_{nr,3} = 0, \quad D_2 log(m_{33}) = -\frac{D_3 F_2}{F_3} - D_2 log(F_3). \tag{6.7}$$

$$W_n = -\int \frac{m_{33} D_n F_2}{F_3} dx^3 + E_n(u, x^r), H = -\int \frac{m_{33} D_2 F_2}{F_3} dx^3 + A_3(u, x^r). \tag{6.8}$$

There are two further subcases depending upon whether $m_{33,r} = 0$ or not, whence we may further integrate to determine the transverse metric.

6.2.2 Case 2: $\Gamma_{3ia} = 0$

This implies the upper-triangular matrix m_{ie} takes the form: $m_{33} = M_{,3}(u, x^3), m_{3r} = 0, m_{nr} = m_{nr}(u, x^r)$, while the W_n must satisfy $D_3(W_n) = 0$. The remaining Killing equations then simplify. In particular, $B_{(mn)}X_1 = 0$, leading to two subcases: (1) $X_1 = 0$, or (2) $B_{(mn)} = 0$.

Case 2.1: $X_1 = 0$**,** $B_{(mn)} \neq 0$**.** $F_{2,r} = 0$, $F_{3,e} = 0$; m_{ie}, H, W_n given by (6.7) and (5.39).

Case 2.2: $B_{(mn)} = 0$**,** $X_1 \neq 0$**.** This case is similar to the subcases dealt with in Case 1.1 (see equations (5.23)-(5.29), (5.36)-(5.39)). For $n < p$ the vanishing of $B_{(np)}$ implies $m_{nr,u} = 0$, the special form of m_{ie} implies that $m_r{}^3 = 0$, and the only non-zero component of the tensor B is B_{33}.

If we assume that $F_{1,3} \neq 0$ and F_1 is independent of x^r then it is of the form (5.74):

$$\frac{m_{33,3}}{m_{33}} = \frac{F_{1,33}}{F_{1,3}}, \quad \frac{m_{33,u}}{m_{33}} = \frac{F_{1,3u}}{F_{1,3}}.$$

Thus $m_{33}(u, x^3)$ is entirely defined by F_1. We may solve for H and the W_n:

$$H = \frac{D_3 D_2 F_1}{D_3(F_1)^2} F_3 - \frac{D_2^2 F_1}{D_3(F_1)^2} F_1 - \frac{2D_{(2}F_{3)}}{D_3 F_1}, \quad W_n = -\frac{D_n F_3}{D_3 F_1}.$$

F_3 is of the form:

$$F_3 = \int \frac{m_{33} F_1 D_3 D_2 F_1}{D_3 F_1} dx^3 + A_6(u, x^r)$$

There are differential equations for F_2 in terms of the arbitrary functions $F_1(u, x^3)$ and $A_6(u, x^r)$. These solutions are summarized in Table 2 in [100].

Killing Lie Algebra: There are three particular forms for the Killing vector in those $CCNV$ spacetimes admitting an additional isometry:

$$(A) \qquad X_A = cn + F_2(u, x^e)\ell + F_3(u, x^e)m^3$$

$$(B) \qquad X_B = un + [F_2(u, x^e) - v]\ell + F_3(u, x^e)m^3$$

$$(C) \qquad X_C = F_1(u, x^3)n + [F_2(u, x^e) - D_2 F_1 v]\ell + [F_3 - D_3 F_1 v]m^3.$$

To determine if these spacetimes admit even more Killing vectors we examine the commutator of X with ℓ in each case. In case (A), $[X_A, \ell] = 0$ and in case B

$[X_B, \ell] = -\ell$, and thus there are no additional Killing vectors. In the most general case $Y_C \equiv [X_C, \ell]$ can yield a new Killing vector; $Y_C = D_2 F_1 \ell + D_3 F_1 m_3$. However, this will always be spacelike since $(D_3 F_1)^2 > 0$. Note that $[Y_C, \ell] = 0$, while, in general, $[Y_C, X_C] \neq 0$.

Non-spacelike isometries: Let us consider the set of $CCNV$ spacetimes admitting an additional non-spacelike Killing vector, so that

$$D_3(X_1)^2 v^2 + 2(D_2(X_1)X_1 - D_3(X_1)F_3)v + {F_3}^2 - 2X_1 F_2 \leq 0$$

If the Killing vector field is non-spacelike for all values of v, then $D_3(X_1)$ must vanish and X_1 is constant. Therefore, various subcases discussed above are excluded. In the remaining cases equation (5.120) applies, that is:

$${F_3}^2 - 2X_1 F_2 \leq 0.$$

In the timelike case, the subcases with $X_1 = 0$ are no longer valid since ${F_3}^2 < 0$. In the case that X is null and $c_2 \neq 0$ we can rescale n so that $2F_2 = {F_3}^2$. We can then integrate out the various cases: If $F_3 = 0$, F_2 must vanish as well and $X = n$. The remaining metric functions are now $H = A_0(u, x^r)$ and $W_n = \int D_n(A_0)du + C_n(x^e)$. The transverse metric is unaffected.

If $F_3 \neq 0$, $H = A_2(u, x^r)$, $D_2(W_n) + D_3(W_n)F_3 = D_n(A_2)$, and $(log m_{33})_{,u} = D_2(log F_3)$. If $c_2 = 0$, F_2 must be constant, and the Killing vector is a scalar multiple of ℓ and can be disregarded. The remaining cases are just a repetition of the above with added constraints. The CSI $CCNV$ spacetimes admitting Killing vectors which are non-spacelike for all values of v are the subcases of the above cases where the transverse space is locally homogenous.

6.3 Explicit Examples

I: We first present an explicit example for the case where $X_1 = u$ and $F_3 \neq 0$. Assuming that $F_3(u, x^i) = \epsilon u m_{33}$ and ϵ is a nonzero constant, we obtain

$$m_{is,u} + \epsilon m_{is,3} = 0 \tag{6.9}$$

and the transverse metric is thus given by

$$m_{is} = m_{is}(x^3 - \epsilon u, x^n)\,. \tag{6.10}$$

We have the algebraic solution

$$\hat{W}_3 = -\frac{1}{\epsilon}(H + F_{2,u}) - F_{2,3} - \epsilon {m_{33}}^2, \tag{6.11}$$

where $F_2(u, x^i)$ is an arbitrary function and H is given by

$$H(u, x^i) = \frac{1}{u}\left[-\int^u S(z, x^3 - \epsilon u + \epsilon z, x^n)dz + A(x^3 - \epsilon u, x^n)\right], \tag{6.12}$$

where A is an arbitrary function and S is given by

$$S(u, x^3, x^n) = (uF_{2,u})_u + \epsilon u F_{2,3u} + \epsilon^2 u({m_{33}}^2)_u\,. \tag{6.13}$$

Furthermore, the solution for $\hat{W}_n$, $n = 4, \ldots, N$ is

$$\hat{W}_n(u, x^i) = \frac{1}{u}\left[-\int^u T_n(z, x^3 - \epsilon u + \epsilon z, x^m)dz + B_n(x^3 - \epsilon u, x^m)\right] \tag{6.14}$$

where B_n are arbitrary functions and T_n is given by

$$T_n(u, x^3, x^m) = \left[(uF_2)_u + \epsilon u F_{2,3} + \epsilon^2 u {m_{33}}^2\right]_{,n} + \epsilon m_{3n} m_{33}\,. \tag{6.15}$$

In this example, the Killing vector and its magnitude are given by

$$X = u\mathbf{n} + (-v + F_2)\boldsymbol{\ell} + \epsilon u m_{33}\mathbf{m}^3, \quad X_a X^a = -2uv + 2uF_2 + (\epsilon u m_{33})^2\,. \tag{6.16}$$

Clearly, the causal character of X will depend on the choice of $F_2(u, x^i)$, and for any fixed (u, x^i) X is timelike or null for appropriately chosen values of v. Moreover, (6.16) is an example of case (B); therefore the commutator of X and $\boldsymbol{\ell}$ gives rise to a constant rescaling of $\boldsymbol{\ell}$ and, in general, there are no more Killing vectors. The additional Killing vector is only timelike or null locally (for a restricted range of coordinate values). However, the solutions can be extended smoothly so that the Killing vector is timelike or null on a physically interesting part of spacetime. For example, a solution valid on $u > 0$, $v > 0$ (with $F_2 < 0$), can be smoothly matched across $u = v = 0$ to a solution valid on $u < 0$, $v < 0$ (with $F_2 > 0$), so that the Killing vector is timelike on the resulting coordinate patch.

As an illustration, suppose the m_{3s} are separable as follows

$$m_{3s} = (x^3 - \epsilon u)^{p_s} h_s(x^n) \tag{6.17}$$

and F_2 has the form

$$F_2 = -\frac{\epsilon}{2p_3+1}(x^3-\epsilon u)^{2p_3+1}{h_3}^2 + g(u,x^n), \tag{6.18}$$

where the p_s are constants and h_s, g arbitrary functions. Thus, from (6.12)

$$H = -\epsilon^2(x^3-\epsilon u)^{2p_3-1}[x^3-\epsilon(p_3+1)u]{h_3}^2 - g_{,u} + u^{-1}A(x^3-\epsilon u, x^n), \tag{6.19}$$

and hence from (6.11)

$$\hat{W}_3 = -\epsilon^2 p_3 u(x^3-\epsilon u)^{2p_3-1}{h_3}^2 - (\epsilon u)^{-1}A(x^3-\epsilon u, x^n). \tag{6.20}$$

Last, equation (6.14) gives

$$\hat{W}_n = \epsilon(x^3-\epsilon u)^{p_3}h_3\left\{\frac{2(x^3-\epsilon u)^{p_3}}{2p_3+1}\left[x^3-\epsilon\left(p_3+\frac{3}{2}\right)u\right]h_{3,n}\right.$$
$$\left. -(x^3-\epsilon u)^{p_n}h_n\right\} - g_{,n} + u^{-1}B_n(x^3-\epsilon u, x^m)\,. \tag{6.21}$$

II: A second example corresponding to the distinct subcase where $X_1 = 1$ and assuming $F_3(u,x^i) = \epsilon m_{33}$ gives the same solutions (6.10) for the transverse metric (although, in this case, the additional Killing vector is globally timelike or null). In addition, we have

$$\hat{W}_3 = \int H_{,3}du + \epsilon^{-1}(F_2+f) \tag{6.22}$$

where $H(u,x^i)$, $F_2(x^3-\epsilon u, x^n)$ and $f(x^i)$ are arbitrary functions. Last, the metric functions $\hat{W}_n$ are

$$\hat{W}_n(u,x^i) = \int^u L_n(z, x^3-\epsilon u+\epsilon z, x^m)dz + E_n(x^3-\epsilon u, x^m), \tag{6.23}$$

with E_n arbitrary and L_n given by

$$L_n(u,x^3,x^m) = H_{,n} + \epsilon\int H_{,3n}du + f_{,n}\,. \tag{6.24}$$

The Killing vector and its magnitude is

$$X = \mathbf{n} + F_2\boldsymbol{\ell} + \epsilon m_{33}\mathbf{m}^3, \qquad X_aX^a = 2F_2 + (\epsilon m_{33})^2\,. \tag{6.25}$$

Since F_2 and m_{33} have the same functional dependence there always exists F_2 such that X is everywhere timelike or null. The Killing vector (6.25) is an example of case

(A) and thus X and $\boldsymbol{\ell}$ commute and hence no additional Killing vectors arise. For instance, suppose $H = H(x^3 - \epsilon u, x^n)$ and f is analytic at $x^3 = 0$ (say) then (6.22) and (6.23) simplify to give

$$\hat{W}_3 = -\epsilon^{-1}(H - F_2 - f), \tag{6.26}$$

$$\hat{W}_n = \epsilon^{-1}\sum_{p=0}^{\infty} \partial_n {\partial_3}^p f(0, x^m)\frac{(x^3)^{p+1}}{(p+1)!} + E_n(x^3 - \epsilon u, x^m)\,. \tag{6.27}$$

This explicit solution is an example of a spacetime admitting 2 global null or timelike Killing vectors, and may be of importance in the study of supergravity solutions preserving a non-minimal fraction of supersymmetries.

Chapter 7

Plane-fronted Gravitational Waves as CSI_Λ Spaces

This chapter is based on: D. McNutt (2013). Spacetimes with all scalar curvature invariants in terms of the Cosmological Constant. *IJMPD* Vol 22, Issue 2, pp 1350003-1350014. Copyright 2013, World Scientific Publishing.

7.1 The CSI_Λ Property

The plane-fronted waves in general relativity were originally derived by Kundt [9] in 1961 with vanishing cosmological constant. At the time, this was a reasonable constraint as it produced the simplest pure radiation solutions admitting a twist-free and non-expanding null congruence. Although the plausibility of a non-vanishing cosmological constant had been addressed [5], it was not until the 1981 that the question of the existence of Petrov type N solutions with cosmological constant was investigated [[26],[27]]. and in particular whether there were plane-fronted gravitational waves in spacetimes with cosmological constant [33] by Ozvath, Robinson and Rozga (O.R.R.).

In the O.R.R. paper, the resulting class of $KN(\Lambda)[\alpha,\beta]$ metrics were classified by the sign of the cosmological constant $\Lambda \neq 0$ and another invariant $\kappa' = \frac{1}{3}\Lambda\alpha^2 + 2\beta\bar{\beta}$ arising from the metric,

$$\begin{aligned} ds^2 &= -2q^2p^{-2}du\left(\left(-\frac{\kappa'}{2}v^2 + (lnq)_{,u}v + S(u,\zeta,\bar{\zeta})\right)du + dv\right) + 2p^{-2}d\zeta d\bar{\zeta}, \\ p &= 1 + \frac{\Lambda}{6}\zeta\bar{\zeta}, \\ q &= (1 - \frac{\Lambda}{6}\zeta\bar{\zeta})\alpha(u) + \zeta\bar{\beta}(u) + \zeta\beta(u). \end{aligned} \tag{7.1}$$

Excluding, the $\lambda = 0$ cases, this produces four canonical classes, which were assumed to have a canonical form by setting α and β to be particular values. It was not until 1991 that Bicak and Podolsky in their paper [50] provided the coordinate transforms needed to produce canonical forms for the metric:

- $\kappa' > 0$, $\Lambda > 0 : KN(\Lambda^+)[0,1]$

- $\kappa' > 0,\ \Lambda < 0 : KN(\Lambda^{-})[0,1]$
- $\kappa' < 0,\ \Lambda < 0 : KN(\Lambda^{-})[1,0]$
- $\kappa' = 0,\ \Lambda < 0 : KN(\Lambda^{-})[1, \sqrt{\frac{-\lambda}{6}} e^{i\omega(u)}]$

The plane wave spacetimes with $\Lambda = 0$ belong to the VSI class of spacetimes [55], by adding a non-vanishing cosmological constant these spacetimes are now CSI. In light of the results of [95] and [94] we may classify the above solutions by examining the Segre type and comparing to the metric forms in [94]. The goal of this section will be to derive a general characterization for the class of CSI spacetimes with all non-zero scalar curvature invariants expressed in terms of the cosmological constant $\Lambda \neq 0$, as a parallel to the result in [55]. In general, these spacetimes will be of Petrov type III or higher, however we will restrict our interests to the Petrov Type N case and derive the metric of (7.1) in the Kundt coordinates used in [55],[95] and [94].

7.1.1 The CSI_{Λ} Theorem

Our interest is to provide a general criteria for spacetimes in which the Ricci Scalar is constant, and the only curvature invariants which are non-zero are the zeroth order invariants expressed as various polynomials of the cosmological constant Λ.

Theorem 7.1.1. *Given a spacetime, all invariants constructed from the traceless Ricci tensor, Weyl tensor and their covariant derivatives vanish, if and only if the following conditions are satisfied:*

1. *The spacetime possesses a non-diverging, shear-free geodesic null congruence.*
2. *Relative to this congruence, the Ricci Scalar is constant and all other curvature scalars with non-negative boost-weight vanish.*

These spacetimes belong to the CSI class of spacetimes and we will say they are CSI_{Λ} spacetimes.

We choose the tangent vector to the null congruence to be ℓ^a and a spin basis so that $o^A \bar{o}^{\dot{A}} \leftrightarrow \ell^a$. The analytic conditions of (7.1.1) (1) for this spin basis is

$$\kappa = \sigma = \rho = 0, \tag{7.2}$$

and the second condition of (7.1.1) may be expressed as

$$\Psi_0 = \Psi_1 = \Psi_2 = 0, \tag{7.3}$$

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{11} = 0 \tag{7.4}$$

$$\Lambda \equiv constant \tag{7.5}$$

Following the work done for VSI spacetimes, the definitions and results given in the Necessity and Sufficiency proof of [55] may be used to generalize the case where $\Lambda \neq 0$ is constant.

7.1.2 Sufficiency of the Conditions

To prove this direction of Theorem (7.1.1) we will use the Newmann-Penrose (NP) and the compacted (GHP) formalisms [29]. Throughout this paper we use a normalized spin basis $\{o^A, \iota^A\}$ such that $o^A\iota_A = 1$ and $o^A o_A = \iota^A\iota_A = 0$. From this we may build the corresponding tetrad:

$$\ell^a \leftrightarrow o^A\bar{o}^{\dot{A}}, \quad n^a \leftrightarrow \iota^A\bar{\iota}^{\dot{A}}, \quad m^a \leftrightarrow o^A\bar{\iota}^{\dot{A}}, \quad \bar{m}^a \leftrightarrow \iota^A\bar{o}^{\dot{A}}, \tag{7.6}$$

with the usual non-zero scalar products $-\ell_a n^a = m^a\bar{m}_a = 1$. The spinorial form of the Riemann tensor R_{abcd} is

$$\begin{aligned} R_{abcd} &\leftrightarrow \chi_{ABCD}\bar{\epsilon}_{\dot{A}\dot{B}}\bar{\epsilon}_{\dot{C}\dot{D}} + \bar{\chi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\epsilon_{AB}\epsilon_{CD} \\ &+ \Phi_{AB\dot{C}\dot{D}}\bar{\epsilon}_{\dot{A}\dot{B}}\epsilon_{CD} + \bar{\Phi}_{\dot{A}\dot{B}CD}\epsilon_{AB}\bar{\epsilon}_{\dot{C}\dot{D}} \end{aligned} \tag{7.7}$$

where

$$\chi_{ABCD} = \Psi_{ABCD} + \Lambda(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BD}) \tag{7.8}$$

and $\Lambda = R/24$ with R the Ricci scalar. The Weyl spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$ is related to the Weyl tensor by

$$C_{abcd} = \Psi_{ABCD}\bar{\epsilon}_{\dot{A}\dot{B}}\bar{\epsilon}_{\dot{C}\dot{D}} + \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\epsilon_{AB}\epsilon_{CD}. \tag{7.9}$$

Taking projections of this tensor onto the basis spinors o^A, ι^A give five complex scalar quantities $\Psi_i, i \in [0, 4]$. Similarly the Ricci Spinor $\Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})} = \bar{\Phi}_{\dot{A}\dot{B}CD}$ is connected to the traceless Ricci tensor $S_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}$

$$\Phi_{AB\dot{C}\dot{D}} \leftrightarrow -\frac{1}{2}S_{ab}. \tag{7.10}$$

We denote the projections of $\Phi_{AB\dot{C}\dot{D}}$ onto o^A, ι^A by $\Phi_{00} = \bar{\Phi}_{00}$, $\Phi_{01} = \bar{\Phi}_{10}$, $\Phi_{02} = \bar{\Phi}_{20}$, $\Phi_{11} = \bar{\Phi}_{11}$, $\Phi_{12} = \bar{\Phi}_{21}$ and $\Phi_{22} = \bar{\Phi}_{22}$

The analytic expressions of Theorem (7.1.1) (1), (2) imply

$$\Psi_{ABCD} = \Psi_4 o_A o_B o_C o_D - 4\Psi_3 o_{(A} o_B o_C \iota_{D)}, \tag{7.11}$$

$$\Phi_{AB\dot{C}\dot{D}} = \Phi_{22} o_A o_B \bar{o}_{\dot{C}} \bar{o}_{\dot{D}} - 2\Phi_{12} \iota_{(A} o_{B)} \bar{o}_{(\dot{C}} \bar{o}_{\dot{D})} - 2\Phi_{21} o_{(A} o_{B)} \bar{\iota}_{(\dot{C}} \bar{o}_{\dot{D})}. \tag{7.12}$$

Following the convention used in [55] we will say a scalar η is a weighted quantity of type $\{p, q\}$ if for every non-vanishing scalar field λ, a transformation of the form

$$o^A \mapsto \lambda o^A, \iota^A \mapsto \lambda^{-1} \iota^A,$$

representing a boost in the $\ell^a - n^a$ plane and a spatial rotation in the $m^a - \bar{m}^a$ plane transforms η in the following manner

$$\lambda^p \bar{\lambda}^q \eta$$

The boost weight, b, of a weighted quantity is defined by $b = \frac{1}{2}(p+q)$.

The frame derivatives are defined as

$$D = \ell^a \nabla_a = o^A \bar{o}^{\dot{A}} \nabla_{A\dot{A}}, \delta = m^a \nabla_a = o^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}}$$
$$D' = n^a \nabla_a = \iota^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}}, \delta' = \bar{m}^a \nabla_a = \iota^A \bar{o}^{\dot{A}} \nabla_{A\dot{A}}$$

and so the covariant derivative may be expressed in terms of the frame,

$$\nabla^a = \nabla^{A\dot{A}} = \iota^A \bar{\iota}^{\dot{A}} D + o^A \bar{o}^{\dot{A}} D' - \iota^A \bar{o}^{\dot{A}} \delta - o^A \bar{\iota}^{\dot{A}} \delta'.$$

The GHP formalism introduces new derivative operators $\eth, \text{þ}, \eth'$ and $\text{þ}'$ which are additive and obey the Leibniz rule. They act on scalars, spinors and tensors η of type $\{p, q\}$ as follows: þ

$$\text{þ} = (D + p\gamma' + q\bar{\gamma}')\eta, \eth = (\delta + p\beta + q\bar{\beta}')\eta \tag{7.13}$$
$$\text{þ}' = (D' - p\gamma - q\bar{\gamma})\eta, \eth' = (\delta' + p\beta' + q\bar{\beta})\eta.$$

To show the sufficiency conditions we assume the analytic conditions of Theorem (7.1.1) hold along with the requirement that o^A, ι^A are parallely propogated along ℓ^a as well. Due to (7.2) we have the following relations on the spin coefficients

$$\gamma' = 0, \text{ } and \text{ } \tau' = 0. \tag{7.14}$$

The spin-coefficient equations, the Bianchi identities and commutator relations [29] are greatly simplified by imposing (7.4), (7.3), (7.5). The non-trivial relations that apply to proving the theorem are:

$$\begin{aligned}
þ\tau &= 0, && (7.15)\\
þ\sigma' &= 0, && (7.16)\\
þ\rho' &= -2\Lambda, && (7.17)\\
þ\kappa' &= \tau þ' + \tau\sigma' - \Psi_3 - \Phi_{21}, && (7.18)\\
þ\Psi_3 &= 0, && (7.19)\\
þ\Phi_{21} &= 0, && (7.20)\\
þ\Phi_{22} &= ð'\Phi_{21} + (ð' - 2\tau)\Psi_3, && (7.21)\\
þ\Psi_4 &= ð'\Psi_3 + (ð' - 2\bar{\tau})\Phi_{21}, && (7.22)\\
þþ' - þ'þ &= \bar{\tau}ð + \tau ð' + p\Lambda + q\Lambda, && (7.23)\\
þð - ðþ &= 0. && (7.24)
\end{aligned}$$

To proceed we analyze the boost weights of the quantities involved in these relations. In particular we will use the idea of a balanced scalar.

Definition 7.1.2. Given a weighted scalar η with boost-weight b, we shall say it is balanced if $þ^{-b}\eta = 0$ for $b < 0$ and $\eta = 0$ for $b \geq 0$.

Many of the Lemmas as given in [55] follow without change despite Λ's non-vanishing. The proof of Lemma 4 requires some modification due to (7.17). For that reason, we will state each lemma leading to the main result without proof, unless there is some required change due to $\Lambda \neq 0$:

Lemma 7.1.3. *If η is a balanced scalar then $\bar{\eta}$ is also balanced.*

Lemma 7.1.4. *If η is a balanced scalar then,*

$$\tau\eta,\ \rho'\eta,\ \sigma'\eta,\ \kappa'\eta$$
$$þ\eta,\ ð\eta,\ ð'\eta,\ þ'\eta$$

are all balanced as well.

	p	q	b		p	q	b
o^A	1	0	$\frac{1}{2}$	ι^A	-1	0	$-\frac{1}{2}$
κ	3	1	2	κ'	-3	-1	-2
σ	3	-1	1	σ'	-3	1	-1
ρ	1	1	1	ρ'	-1	-1	-1
τ	1	-1	0	τ'	-1	1	0
Þ	1	1	1	Þ$'$	-1	-1	-1
ð	1	-1	0	ð$'$	-1	1	0
Ψ_r	4-2r	0	2-r	Φ_{rt}	2-2r	2-2t	2-r-t
				Λ	0	0	0

Table 7.1: Boost weights of weighted quantities

Proof. Let b be the boost-weight of a balanced scalar η. Using Table (7.1) it is clear that the scalars listed in the first row have boost-weights $b, b-1, b-1, b-2$, respectively.To show these are balanced we must prove that the following must vanish:

$$þ^{-b}(\tau\eta),\ þ^{1-b}(\rho'\eta), þ^{1-b}(\sigma'\eta),\ þ^{2-b}(\kappa'\eta).$$

while for the second row we require that four more quantities vanish to match their boost-weight:

$$þ^{-(b+1)}(þ\eta),\ þ^{-b}(ð\eta),\ þ^{-b}(ð'\eta),\ þ^{-(b-1)}(þ'\eta).$$

As the equations (7.17) and (7.23) are the only that differ from the VSI case, we must only check to see if two conditions still hold

$$þ^{1-b}(\rho'\eta) = þ^{1-b}(þ'\eta) = 0$$

and the remaining six conditions hold automatically. The first condition follows using the Leibniz rule and equations (7.17) and the fact that $þ^2\rho' = 0$. Since we may expand this as

$$þ^{1-b}(\rho'\eta) = þ\rho' þ^{-b}\eta + \rho' þ(þ^{-b}\eta),$$

Since η is a balanced scalar for which $b < 0$, these last two terms vanish. To prove the second condition, we use the commutator relation (7.23) and the constancy of Λ

to get

$$\text{þ}^{1-b}(\text{þ}'\eta) \;=\; \text{þ}^{-b}(\text{þ}'\text{þ}\eta) + \bar{\tau}(\text{þ}^{-b}\eth\eta) + \tau(\text{þ}^{-b}\eth\eta) + \text{þ}^{-b}(p\Lambda\eta + q\Lambda\eta)$$
$$= \text{þ}^{-b}(\text{þ}'\text{þ}\eta)$$

Using induction one may show that $\text{þ}^{1-b}\text{þ}'\eta = \text{þ}'\text{þ}^{1-b}\eta = 0$. □

Lemma 7.1.5. *If η_1, η_2 are balanced scalars both of type $\{p, q\}$ then $\eta_1 + \eta_2$ is a balanced scalar of type $\{p, q\}$ as well.*

Lemma 7.1.6. *If η_1 and η_2 are balanced scalars then $\eta_1\eta_2$ is also balanced.*

Definition 7.1.7. A balanced spinor is a weighted spinor of type $\{0, 0\}$ whose components are all balanced scalars.

Lemma 7.1.8. *If S_1 and S_2 are balanced spinors then S_1S_2 is also a balanced spinor*

Lemma 7.1.9. *A covariant derivative of an arbitrary order of a balanced sinpor S is again a balanced spinor*

Proof. Applying the covariant derivative to a balanced spinor S,

$$\nabla^a = \nabla^{A\dot{A}} = \iota^A\bar{\iota}^{\dot{A}}D + o^A\bar{o}^{\dot{A}}D' - \iota^A\bar{o}^{\dot{A}}\delta - o^A\bar{\iota}^{\dot{A}}\delta'.$$

From table 1 in [55] it follows that $\nabla^{A\dot{A}}S$ is a weighted spinor of type $\{0, 0\}$. By virtue of how $\text{þ}, \eth, \text{þ}'$ and $\eth'$ act on the basis vectors, the components may be shown to be balanced scalars using Lemmas (7.1.3), (7.1.4) and (7.1.5). □

Lemma 7.1.10. *A scalar constructed as a contraction of a balanced spinor is equal to zero.*

From table 1 in [55], and equations (7.19)- (7.22) it follows that the Weyl spinor and Ricci spinor and their complex conjugates are balanced spinors (lemma (7.1.3)). Their product and covariant derivatives of arbitrary orders are balanced spinors as well (Lemmas (7.1.8) and (7.1.9)). At this point to prove the sufficiency of the conditions of Theorem (7.1.1) we must state two more results:

Lemma 7.1.11. *The product of a balanced spinor and a weighted constant of type $\{0, 0\}$ is a balanced spinor.*

Lemma 7.1.12. *A scalar constructed as a contraction from the product of a balanced spinor, ϵ_{AB}, ϵ^{AB} and their conjugates is equal to zero.*

With these observations, and equations (7.7), (7.8), (7.9) and (7.10) imply that any contraction of the product of N copies of the Riemann tensor with itself must vanish except for the contraction of the term built exclusively out of the product of N copies of

$$\Lambda(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BD}\bar{\epsilon}_{\dot{A}\dot{B}}\bar{\epsilon}_{\dot{C}\dot{D}}.$$

Lemma 7.1.11 ensures all other terms are the products of balanced spinors, ϵ's and $\bar{\epsilon}$'s; these terms must vanish when contracted by Lemmas 7.1.10 and 7.1.12. To show that all non-zero curvature invariants appear at zeroth order, we note that the n^{th} covariant derivative of the Riemann tensor is a balanced spinor for $n > 0$, as $\nabla\epsilon_{AB} = 0$ and Λ is a constant. Thus any product of the Riemann tensor with its n^{th} covariant derivative must vanish upon contraction by Lemma 7.1.12, while any contraction of the product of the n^{th} and m^{th} covariant derivative of the Riemann tensor must vanish necessarily by Lemma 7.1.10.

7.1.3 Necessity of the Conditions

To show that these conditions are necessary follows by repeating the proof from [55] vertabim, this can be done because the particular Newman Penrose equations used and the Bianchi Identities do not involve Λ, or the derivatives of Λ - since they vanish if Λ is constant. Thus by requiring that all invariants vanish except those constructed as polynomials of Λ which are assumed to be constant, one may prove conditions (1) and (2) of Theorem (7.1.1) hold.

7.2 CSI_Λ Spacetimes of Petrov Type N

To conform to the theme of plane-fronted waves, we will ignore those CSI_Λ spacetimes of Petrov type III, and instead focus on the Type N spacetimes. As a step towards a metric form, as these CSI_Λ spacetimes admit a non-diverging, shear-free and geodesic null congruence, their corresponding metrics must be of Kundt form. As in [94], we may choose coordinates $\zeta, \bar{\zeta}, u, v$ such that the metric is of the form:

$$ds^2 = 2\frac{d\zeta d\bar{\zeta}}{P^2} - 2du[dv + Hdu + Wd\zeta + \bar{W}d\bar{\zeta}] \tag{7.25}$$

where $P(\zeta, \bar{\zeta}, u)$, and $H(\zeta, \bar{\zeta}, u, v)$ are real-valued functions and $W(\zeta, \bar{\zeta}, u, v)$ is complex valued. To work with this metric we choose the following complex null coframe:

$$m_a = \tfrac{d\zeta}{P} - Wdu,\ \ell_a = du,\ n_a = dv + [H + P^2 W\bar{W}]du, \tag{7.26}$$

expressed in this coframe the metric becomes

$$2m_{(a}\bar{m}_{b)} - 2\ell_{(a}n_{b)}.$$

Only certain coordinate transformations, (31.10)$(a), (b), (c)$ in [22], can be performed which preserves the Kundt form and corresponding null coframe.

The second part of Theorem (7.1.1) dictates that these spacetimes must be of Petrov type III or higher, and Plebanksi Petrov type N or higher. Despite limiting the scope of this paper to the plane-fronted waves, which belong to the class of non-expanding pure radiation solutions [33] - Petrov type N and Plebanski-Petrov type O, we will derive the metric form for all CSI_Λ metrics inevitably. In these coordinates it is not possible, in general, to simultaneously simplify the forms of the Ricci spinor components in PP-types N and O by a boost and null and spatial rotations. In most cases it is possible to specialize the solution form using an appropriate choice of coordinates, further limiting the range of allowed coordinate transformations.

7.2.1 Plebanski-Petrov Type N, $\Phi_{12} \neq 0$ and $\Phi_{22} \neq 0$

- Petrov type III:

  By requiring that the appropriate curvature scalars to vanish so that the spacetime is of PP-type N, P-type III we produce equations for the functions H, W and P. In these coordinates Φ_{00} and Ψ_0 vanish automatically. The scalars Ψ_1 and Φ_{01} are equal and so requiring one to vanish, eliminates the other, and forces the requirement that W must be linear in v

$$\Phi_{01} = \Psi_1 = \frac{1}{4}PW_{,vv} = 0. \tag{7.27}$$

The equations arising from $\Phi_{11} = 0$ and $\Phi_{02} = 0$ simplify to be

$$-\tfrac{1}{2}P_{,\bar{\zeta}}P_{,\zeta} + \tfrac{1}{2}P_{,\bar{\zeta}\zeta}P + \tfrac{1}{8}P^2W_{,v}\bar{W}_{,v} + \tfrac{1}{4}H_{,vv} = 0, \tag{7.28}$$

$$\tfrac{1}{4}P(4P_{,\zeta}W_{,v} - PW_{,v}^2 + 2PW_{,v\zeta}) = 0. \tag{7.29}$$

The vanishing of Ψ_2 gives another equation

$$\tfrac{1}{3}P_{,\bar{\zeta}}P_{,\zeta} - \tfrac{1}{3}P_{,\bar{\zeta}\zeta}P + \tfrac{1}{3}P^2W_{,\bar{\zeta}v} - \tfrac{1}{6}P^2\bar{W}_{,v\zeta} + \tfrac{1}{6}H_{,vv} = 0. \tag{7.30}$$

Setting the equation for Λ equal to a constant, say $\lambda/6$ in the NP-formalism, where λ is the cosmological constant (Note: For now we will just call $\Lambda = \lambda/6$) we find

$$-\frac{P_{,\bar{\zeta}}P_{,\zeta}}{6} + \frac{P_{,\bar{\zeta}\zeta}P}{6} - \frac{P^2W_{,v}\bar{W}_{,v}}{8} + \frac{P^2W_{,\bar{\zeta}v}}{12} + \frac{P^2\bar{W}_{,v\zeta}}{12} - \frac{H_{,vv}}{12} = \Lambda. \tag{7.31}$$

Adding (7.30) and its conjugate together gives a differential relation between W and $\bar{W}$

$$W_{,v\bar{\zeta}} = \bar{W}_{,v\zeta} \tag{7.32}$$

Simplifying (7.30) and solving for $H_{,vv}$,

$$H_{,vv} = -2P_{,\bar{\zeta}}P_{,\zeta} + 2P_{,\bar{\zeta}\zeta}P - P^2\bar{W}_{,v\zeta}, \tag{7.33}$$

we may substitute the result into (7.28) to find the following differential equation for $W_{,v}$ and its conjugate

$$-P_{,\bar{\zeta}}P_{,\zeta} + P_{,\bar{\zeta}\zeta}P + \tfrac{1}{8}P^2W_{,v}\bar{W}_{,v} - \tfrac{1}{4}P^2\bar{W}_{,v\zeta} = 0, \tag{7.34}$$

this equation along with $\Phi_{02} = 0$ will give the necessary constraints on $W_{,v}$. For now we will only use this to solve for $P^2W\bar{W}$ and substitute it into (7.31) to get a simple differential equation for P

$$-P_{,\bar{\zeta}}P_{,\zeta} + P_{,\bar{\zeta}\zeta}P = \Lambda. \tag{7.35}$$

Using a type (I) transformation we may choose $\zeta, \bar{\zeta}$ such that P takes the form [33],

$$P(\zeta, \bar{\zeta}) = 1 + \Lambda\zeta\bar{\zeta}. \tag{7.36}$$

This choice of coordinates restricts the Type (I) transformations to the following form [50]:

$$\zeta' = \frac{\bar{b}(u)+a(u)\zeta}{\bar{a}(u)-\Lambda b(u)\zeta}$$

where a and b are arbitrary complex valued functions of retarded time u.

To solve for W, look at the simpler forms of (7.29) and (7.34),

$$2(P^2 W_{,v})_{,\zeta} = P^2 W_{,v}^2, \tag{7.37}$$

$$2P^2 \bar{W}_{,v\zeta} = 8\Lambda + P^2 W_{,v} \bar{W}_{,v} \tag{7.38}$$

From (7.37) we may integrate for the function $1/P^2 W_{,v}$,

$$\frac{1}{P^2 W_{,v}} = \frac{1}{2\Lambda\bar{\zeta}P} + \frac{1}{\bar{\zeta}w(\bar{\zeta},u)} \tag{7.39}$$

where $w(\bar{\zeta}, u)$ is a holomorphic function. To determine further constraints on $\bar{w}(\zeta, u)$ we solve for $W_{,v}$ from above and substitute into (7.38). After much simplification, $\bar{w}$ becomes,

$$\bar{w}(\zeta, u) = \frac{-2\Lambda(2P^2 w_{,\bar{\zeta}}\Lambda\bar{\zeta} - 8P^2\Lambda^2 - 2P^2\Lambda w - 4wP\Lambda - w^2)}{(2Pw_{,\bar{\zeta}}\Lambda\bar{\zeta} - 8P\Lambda^2 - 4wP\Lambda - 2w\Lambda - w^2}. \tag{7.40}$$

As the right hand side is both a function of ζ and $\bar{\zeta}$ we take a first order Taylor series expansion about $\zeta = 0$,

$$\bar{w}(z, u) - 2\Lambda + \left(\frac{4\Lambda^3\bar{\zeta}(\bar{\zeta}w_{,\bar{\zeta}} - 2w - 4\Lambda)}{2w_{,\bar{\zeta}}\Lambda\bar{\zeta} - 6w\Lambda - 8\Lambda^2 - w^2}\right)\zeta + O(\zeta^2)$$

then by requiring that the ζ-linear piece is some complex valued function $K'(u)$ we may rationalize both sides to find a differential equation for w. After some simplification this equation takes the form,

$$(2\Lambda^2\bar{\zeta} + K')\bar{\zeta}w_{,\bar{\zeta}} - (4\Lambda^2\bar{\zeta} + 3K')w - \frac{K'w^2}{2\Lambda} - 8\Lambda^3\bar{\zeta} - 4K'\Lambda = 0. \tag{7.41}$$

We know already that at $\zeta = 0$, $w = -2\Lambda$ and so setting $\zeta = 0$ in the above equation we find it is satisfied. Supposing $\zeta \neq 0$ or$-K'$ we divide the above equation by $(2\Lambda^2\bar{\zeta} + K')$ to get

$$w_{,\bar{\zeta}} = \left(\frac{K'}{2\Lambda\bar{\zeta}(2\Lambda^2\bar{\zeta}+K')}\right)w^2 + \left(\frac{4\Lambda^2\bar{\zeta}+3K')}{\bar{\zeta}(2\Lambda^2\bar{\zeta}+K')}\right)w + \frac{4\Lambda}{\bar{\zeta}}.$$

This belongs to a special case of the Riccatti equations, 9, with

$$f(\zeta) = \frac{K'}{2\Lambda\bar{\zeta}(2\Lambda^2\bar{\zeta}+K')}, \quad g(\zeta) = \frac{4\Lambda^2\bar{\zeta}+3K')}{\bar{\zeta}(2\Lambda^2\bar{\zeta}+K'}, \quad and \quad a = -2\Lambda,$$

following the prescription given in [61], the solution may be written as

$$w(\bar{\zeta}, u) = \frac{-2\Lambda((-C + 2K'\bar{\zeta} + \Lambda\bar{\zeta}^2)}{-C + K'\bar{\zeta})}$$

where C is an arbitrary complex valued function of u. Dividing the numberator by K', denoting $k' = 1/K'$ and absorbing the $1/K'$ term into C gives the simpler form

$$w(\bar{\zeta}, u) = \frac{-2\Lambda(-C+2\bar{\zeta}+k'\Lambda\bar{\zeta}^2)}{(-C+\zeta)}.$$

To determine C we substitute the above into (7.40), after much simplification we find,

$$\bar{w} = \frac{-2\Lambda(-k' + 2\bar{\zeta} + C\Lambda\bar{\zeta}^2)}{(-k + \bar{\zeta})},$$

thus $C = \bar{k}'$, and so w takes the final form

$$w(\bar{\zeta}, u) = \frac{-2\Lambda(-\bar{k}'+2\bar{\zeta}+k'\Lambda\bar{\zeta}^2)}{(-\bar{k}'+\zeta)}. \tag{7.42}$$

To calculate the form W substitute (7.42) into (7.39) and solve algebraically, W may be written as,

$$W = \left(\frac{2(-k'+2\Lambda\bar{\zeta}+\bar{k}'\Lambda\bar{\zeta}^2)}{P(2+\bar{k}'\bar{\zeta}+k'\zeta-P)}\right) v + W^{(0)}(\zeta, \bar{\zeta}, u)$$

Defining the function

$$Q = 2 + k'\zeta + \bar{k}'\bar{\zeta} - P(\zeta, \bar{\zeta}), \tag{7.43}$$

we may write W simply,

$$W = ln\,(P^2/Q^2))_{,\zeta}\, v + W^{(0)}. \tag{7.44}$$

In a similar manner H is found via (7.33):

$$H(\zeta, \bar{\zeta}, u, v) = \left(\frac{-2(\Lambda+k'\bar{k}')P^2}{(2+\bar{k}'\bar{\zeta}+\zeta k'-P)^2}\right)\frac{v^2}{2} + H^{(1)}(\zeta, \bar{\zeta}, u)v + H^{(0)}(\zeta, \bar{\zeta}, u). \tag{7.45}$$

To relate this to the notation in [33] and [50], we have found the form of $q(\zeta,\bar{\zeta})$ in the coordinate system where $\alpha(u)=1$ and $\beta(u)=K'$, depending on the sign of κ' and Λ it may be possible to set $\alpha'=0$ so that $Q'=K'\bar{\zeta}+\bar{K}'\zeta$. The remaining coordinate freedom preserving the form of $P(\zeta,\bar{\zeta})$ is now

$$\begin{gathered}\zeta'=f(\zeta,u)=\tfrac{\bar{b}(u)+a(u)\zeta}{\bar{a}(u)-\Lambda b(u)\zeta}\\ P'^2=P^2 f_{,\zeta}\bar{f}_{,\bar{\zeta}}\ ,\ W'=\tfrac{W}{f_{,\zeta}}-\tfrac{\bar{f}_{,u}}{P^2 f_{,\zeta}\bar{f}_{,\bar{\zeta}}}\ ,\\ H'=H-\tfrac{1}{f_{,\zeta}\bar{f}_{,\bar{\zeta}}}\left(\tfrac{f_{,u}\bar{f}_{,u}}{P^2}+Wf_{,u}\bar{f}_{,\bar{\zeta}}+\bar{W}f_{,\zeta}\bar{f}_{,u}\right);\end{gathered}$$

$$\begin{gathered}v'=v+g(\zeta,\bar{\zeta},u)\ ,\\ W'^{(0)}=W^{(0)}-g_{,\zeta}-gW_{,v}\ ,\ H'^{(1)}=H^{(1)}-2H_{,vv}g\ ,\\ H'^{(0)}=H^{(0)}-H^{(1)}g+H_{,vv}g^2-g_{,u};\end{gathered}\tag{7.46}$$

$$\begin{gathered}u'=h(u)\ ,\ v'=v/h_{,u}\ ,\\ W'^{(0)}=\tfrac{W^{(0)}}{h_{,u}}\ ,\ H'^{(1)}=\tfrac{H^{(1)}}{h_{,u}}\ ,\ H'^{(0)}=\tfrac{H^{(0)}}{h_{,u}^2}-\tfrac{h_{,uu}}{h_{,u}}.\end{gathered}$$

The non-vanishing Ricci spinor components are:

$$\begin{aligned}\Phi_{12} &= \frac{P}{4}\left(\left(Q^2\left[\left(\frac{P^2W^{(0)}}{Q^2}\right)_{,\bar{\zeta}}-\left(\frac{P^2\bar{W}^{(0)}}{Q^2}\right)_{,\zeta}\right]\right)_{,\zeta}\right)\\ &\quad -\bar{W}^{(0)}P^3 ln(Q)_{,\zeta\zeta}+\bar{W}^{(0)}P(PP_{,\zeta})_{,\zeta}+W^{(0)}P(PP_{,\bar{\zeta}})_{,\zeta}\\ &\quad -2PP_{,\zeta}P_{,\bar{\zeta}}W^{(0)}-2P^2\bar{W}^{(0)}P_{,\zeta}ln(Q)_{,\zeta}-\frac{P^2}{2}QW^{(0)}_{,\bar{\zeta}}(\frac{P}{Q})_{,\zeta}\\ &\quad +\frac{P^2}{2}Q\bar{W}^{(0)}_{,\zeta}(\frac{P}{Q})_{,\zeta}+\frac{P}{2}H^{(1)}_{,\zeta}-\frac{P}{4}W_{,vu} \qquad (7.47)\\ \Phi_{22} &= P^2H^{(0)}_{,\bar{\zeta}\zeta}+\frac{1}{2}W_{,v}H^{(0)}_{,\bar{\zeta}}+\frac{1}{2}\bar{W}_{,v}H^{(0)}_{,\zeta}\\ &\quad +v\left(P^2H^{(1)}_{,\bar{\zeta}\zeta}+\frac{1}{2}W_{,v}H^{(1)}_{,\bar{\zeta}}+\frac{1}{2}\bar{W}_{,v}H^{(1)}_{,\zeta}\right)\\ &\quad +v\left(H^{(1)}(H_{,vv}-2P_{,\zeta\bar{\zeta}})-\frac{P^2}{2}(W_{,v\bar{\zeta}u}+\bar{W}_{,v\zeta u})\right)\end{aligned}$$

The non-vanishing Weyl spinor components are

$$\begin{aligned}
\Psi_3 &= \frac{P}{4}\left(\left[Q^2\left(\left(\frac{P^2W^{(0)}}{Q^2}\right)_{,\bar{\zeta}} - \left(\frac{P^2\bar{W}^{(0)}}{Q^2}\right)_{,\zeta}\right)\right]_{,\bar{\zeta}}\right) \\
&\quad -\bar{W}^{(0)}P^3 ln(Q)_{,\zeta\bar{\zeta}} + \frac{P}{2}H^{(1)}_{,\bar{\zeta}} - \frac{P}{4}\bar{W}_{,vu} \\
\Psi_4 &= \frac{P^2(H^{(0)}\bar{W}_{,v})_{,\bar{\zeta}\bar{\zeta}}}{\bar{W}_{,v}} + \frac{6\Lambda\zeta(H^{(0)}P^2\bar{W}_{,v})_{,\bar{\zeta}}}{P^2\bar{W}} - 3(P^2)_{,\bar{\zeta}\bar{\zeta}}H^{(0)} \qquad (7.48) \\
&\quad +v\left(\frac{P^2(H^{(1)}\bar{W}_{,v})_{,\bar{\zeta}\bar{\zeta}}}{\bar{W}_{,v}} + \frac{6\Lambda\zeta(H^{(1)}P^2\bar{W}_{,v})_{,\bar{\zeta}}}{P^2\bar{W}} - 3(P^2)_{,\bar{\zeta}\bar{\zeta}}H^{(1)}\right) \\
&\quad -v\left(\frac{1}{2}H^{(1)}\bar{W}^2_{,v}P^2 + (P^2\bar{W})_{,vu\bar{\zeta}}\right)
\end{aligned}$$

It will be worthwhile to know when it is possible to set $W^{(0)}$ to zero when we specialize to the Petrov type N, Plebanski-Petrov type O.

Lemma 7.2.1. *The metric function $W^{(0)}$ may be set to zero if and only if $W'^{(0)}_{,\bar{\zeta}} = \bar{W}'^{(0)}_{,\zeta}$.*

Proof. Using a type (II) transformation, $v' = v + g(\zeta, \bar{\zeta}, u)$ such that

$$(\frac{P^2}{Q^2}g)_{,\zeta} = \frac{P^2W^{(0)}}{Q^2}, \quad (\frac{P^2}{Q^2}g)_{,\bar{\zeta}} = \frac{P^2\bar{W}^{(0)}}{Q^2}$$

Simplifying this gives

$$(\frac{P^2}{Q^2}g)_{,\zeta} = \frac{P^2}{Q^2}W^{(0)}, \quad (\frac{P^2}{Q^2}g)_{,\bar{\zeta}} = \frac{P^2}{Q^2}\bar{W}^{(0)}$$

The integrability conditions hold for $g(z, bz, u)$ if and only if

$$W^{(0)}_{,\bar{\zeta}} = \bar{W}^{(0)}_{,\zeta}$$

which was to be shown. □

- Petrov type N and O: In light of the differing form of Φ_{12} and Ψ_3 when $W^{(0)} \neq 0$, we expect that there will be spacetimes of PP-type N and P-Type N or higher. In this case we neglect extensive analysis and focus on the simpler case where $W^{(0)}_{,\bar{\zeta}} = W^{(0)}_{,\zeta}$, as our interest lie in the plane-fronted waves and these may be seen as precursors to these solutions. we may set $W^{(0)} = 0$ and so $\Psi_3 = \Phi_{12}$,

Corollary 7.2.2. *For the metric with functions W and H written in terms of P and Q [(7.36), (7.43), (7.44) and (7.45)], if $W^{(0)}_{,\bar{\zeta}} = \bar{W}^{(0)}_{,\zeta}$, those spacetimes of Petrov Type N will be of Plebanski-Petrov type O or higher.*

7.2.2 Plebanski-Petrov Type O, $\Phi_{12} = 0$ and $\Phi_{22} \neq 0$

- Petrov type III:

  In general $W^{(0)}$ will be non-zero, and so the equation $\Phi_{12} = 0$, (7.47), does not readily give informtion about these spaces. Instead we state a simple result about the pre-plane-fronted waves, where coordinates may be found in which $W^{(0)} = 0$

  Corollary 7.2.3. *For the metric with functions W and H written in terms of P and Q [(7.36), (7.43), (7.44) and (7.45)], and $W^{(0)} = 0$ all spacetimes with Plebanksi-Petrov Type O will be of Petrov type N or higher.*

- Petrov type N: By calculating $\Phi_{12} - \bar{\Psi}_3 = 0$, we produce a very complicated expression which may be simplified to find the expression:

$$\left[\left(\frac{q^3}{p}\right)\left(\left(\frac{P^2 W^{(0)}}{Q^2}\right)_{,\bar{\zeta}} - \left(\frac{P^2 \bar{W}^{(0)}}{Q^2}\right)_{,\zeta}\right)\right]_{,\bar{\zeta}} = 0$$

  Denoting $\omega' = \left(\frac{q^3}{p}\right)\left(\left(\frac{P^2 W^{(0)}}{Q^2}\right)_{,\bar{\zeta}} - \left(\frac{P^2 \bar{W}^{(0)}}{Q^2}\right)_{,\zeta}\right)$, it is subject to a transformation law under the Type (I) transformations. In these coordinates, the transformation law is rather complicated, we defer to the proof in [33]

  Corollary 7.2.4. *In Kundt coordinates, the metric function $W^{(0)}$ may be set to zero for any Petrov type N, Plebanski-Petrov type O spacetime with the CSI_Λ property.*

  Proof. Consider the coframe,

$$e_1 = \frac{d\zeta'}{P}, 2_2 = \frac{d\bar{\zeta}'}{P}, e_3 = du', e_4 = \frac{q^2}{p^2}dv' + \bar{Z}d\zeta + Zd\bar{\zeta} - Fdu. \qquad (7.49)$$

  with metric functions

$$P = 1 + \Lambda\zeta'\bar{\zeta}', Q = (1 - \Lambda\zeta'\bar{\zeta}')\alpha(u') + \bar{\beta}(u')\bar{\zeta}' + \beta(u')\zeta'$$

$$Z(,\zeta,\bar{\zeta},u), \quad \bar{Z}_{,\bar{\zeta}} = Z_{,\zeta}$$

$$F = \tfrac{1}{2}\kappa\frac{Q^2}{P^2}v^2 - \frac{(Q^2)_{,u}}{P^2}v - \frac{Q}{P}H,\ \kappa = 2(\Lambda\alpha + \beta\bar{\beta})$$

with the additional assumption that $\alpha = 1$ and $\beta = k'(u)$. This may always be done using the coordinate transformation given in [50], in this coordinate system $q = Q$. To transform the metric to Kundt form, keep $\zeta' = \zeta, \bar{\zeta}' = \bar{\zeta}, u' = u$ and

$$v' = \frac{P^2}{Q^2}v.$$

In these new coordinates, examining e_4 and noting $ln(Q^{-2})_{,u} = \frac{Q^2}{P^2}(\frac{P^2}{Q^2})_{,u}$, we find the transformation law for the metric components

$$W^{(1)} = \tfrac{Q^2}{P^2}(\tfrac{P^2}{Q^2})_{,\zeta}, \quad H^{(2)} = -\kappa\tfrac{P^2}{Q^2},$$
$$W^{(0)} = \tfrac{Q^2}{P^2}\bar{Z}\ , \ H^{(1)} = \iota + ln(Q^{-2})_{,u}. \tag{7.50}$$

With the various components in [33] identitied, ω' is easily expressed and the transformation law given by equation (4.26) in this paper. From this it is clear for any choice of $\omega'(u), \alpha(u)$ and $\beta(u)$ one may set $\omega' = 0$. Transforming back to the Kundt coordinates this implies,

$$\left(\tfrac{P^2 W^{(0)}}{Q^2}\right)_{,\bar{\zeta}} - \left(\tfrac{P^2 \bar{W}^{(0)}}{Q^2}\right)_{,\zeta} = 0,$$

as P and Q are real-valued, this implies $W^{(0)}_{,\bar{\zeta}} = \bar{W}^{(0)}_{,\zeta}$ and so we may apply lemma (7.2.1), thus $W^{(0)}$ may be set to zero by the coordinate transformation.

□

In these coordinates, the metric functions are now

$$W = \left(\tfrac{2(-k+2\Lambda\bar{\zeta}+\bar{k}\Lambda\bar{\zeta}^2)}{P(2+\bar{k}\bar{\zeta}+\zeta k-P)}\right)v, \tag{7.51}$$
$$H(\zeta,\bar{\zeta},u,v) = \left(\tfrac{-2(\Lambda+k\bar{k})P^2}{(2+\bar{k}\bar{\zeta}+\zeta k-P)^2}\right)\tfrac{v^2}{2} + H^{(1)}(\zeta,\bar{\zeta},u)v + H^{(0)}(\zeta,\bar{\zeta},u) \tag{7.52}$$

With these simplifications the remaining coordinate freedom is restricted to type (III). The non-vanishing Ricci spinor components are:

$$\Phi_{12} = H^{(1)}_{,\zeta} - \frac{1}{2}W_{,vu} \tag{7.53}$$
$$\begin{aligned}\Phi_{22} = {} & P^2 H^{(0)}_{,\bar{\zeta}\zeta} + \frac{1}{2}W_{,v}H^{(0)}_{,\bar{\zeta}} + \frac{1}{2}\bar{W}_{,v}H^{(0)}_{,\zeta} \\ & +v\left(P^2 H^{(1)}_{,\bar{\zeta}\zeta} + \frac{1}{2}W_{,v}H^{(1)}_{,\bar{\zeta}} + \frac{1}{2}\bar{W}_{,v}H^{(1)}_{,\zeta}\right) \\ & +v\left(H^{(1)}(H_{,vv} - 2P_{,\zeta\bar{\zeta}}) - \frac{P^2}{2}(W_{,v\bar{\zeta}u} + \bar{W}_{,v\zeta u})\right)\end{aligned} \tag{7.54}$$

The non-vanishing Weyl spinor components are

$$\Psi_3 = H^{(1)}_{,\bar{\zeta}} - \frac{1}{2}\bar{W}_{,vu} \tag{7.55}$$

$$\begin{aligned}\Psi_4 = {} & \frac{P^2(H^{(0)}\bar{W}_{,v})_{,\bar{\zeta}\bar{\zeta}}}{\bar{W}_{,v}} + \frac{6\Lambda\zeta(H^{(0)}P^2\bar{W}_{,v})_{,\bar{\zeta}}}{P^2\bar{W}} - 3(P^2)_{,\bar{\zeta}\bar{\zeta}}H^{(0)} \\ & +v\left(\frac{P^2(H^{(1)}\bar{W}_{,v})_{,\bar{\zeta}\bar{\zeta}}}{\bar{W}_{,v}} + \frac{6\Lambda\zeta(H^{(1)}P^2\bar{W}_{,v})_{,\bar{\zeta}}}{P^2\bar{W}} - 3(P^2)_{,\bar{\zeta}\bar{\zeta}}H^{(1)}\right) \\ & -v\left(\frac{1}{2}H^{(1)}\bar{W}^2_{,v}P^2 + (P^2\bar{W})_{,vu\bar{\zeta}}\right)\end{aligned} \tag{7.56}$$

To integrate the equation arising from $\Phi_{21} = \Psi_3 = 0$, take (7.53) and use (7.51) to rewrite the lefthand side

$$H^{(1)}_{,\zeta} = \frac{1}{2}ln\left(\frac{P^2}{Q^2}\right)_{,\zeta u}$$

integrating with respect to z and expanding the left hand side yields

$$H^{(1)} = \frac{-k_{,u}(\zeta+\bar{\zeta})}{1+k(\zeta+\bar{\zeta})-\Lambda\zeta\bar{\zeta}} + h_1(u).$$

where h_1 is an arbitrary function of retarded time u. This may be removed by making a type (III) transformation, exhausting the coordinate freedom given in (7.46),

$$H^{(1)} = \frac{-k_{,u}(\zeta+\bar{\zeta})}{1+k(\zeta+\bar{\zeta})-\Lambda\zeta\bar{\zeta}}.. \tag{7.57}$$

Notice that in the P-Type N and PP-Type O case in the O.R.R. coordinates [33], $Z = 0$ and $\iota = ln(Q)_{,u} = -\frac{1}{2}ln(Q^{-2})_{,u}$, $H^{(1)}$ becomes

$$H^{(1)} = \frac{1}{2}ln(Q^{-2})_{,u}$$

which agrees with (7.53) and (7.55). The remaining curvature scalars are

$$\Phi_{22} = P^2H^{(0)}_{,\bar{\zeta}\zeta} + \frac{1}{2}W_{,v}H^{(0)}_{,\bar{\zeta}} + \frac{1}{2}\bar{W}_{,v}H^{(0)}_{,\zeta} \tag{7.58}$$

$$\Psi_4 = \frac{P^2(H^{(0)}\bar{W}_{,v})_{,\bar{\zeta}\bar{\zeta}}}{\bar{W}_{,v}} + \frac{6\Lambda\zeta(H^{(0)}P^2\bar{W}_{,v})_{,\bar{\zeta}}}{P^2\bar{W}} - 3(P^2)_{,\bar{\zeta}\bar{\zeta}}H^{(0)} \tag{7.59}$$

- Petrov type O:

Due to the choice of $H^{(1)}$, those terms linear in v in Ψ_4 vanish. To analyze the case when $\Psi_4 = 0$ we consider a gauge transformation $-e^{f(\zeta,\bar{\zeta},u)}H'^{(0)} = H^{(0)}$

and solve the first order differential equation arising from the vanishing of the coefficient of $H^{(0)}_{,\zeta}$.

$$f(\zeta, \bar{\zeta}, u) = -2ln(P) + ln(2 + k\zeta + \bar{k}\bar{\zeta} - P).$$

Thus $H^{(0)} = \frac{QH'^{(0)}}{P^2}$ where $H'^{(0)}$ satisfies $\Psi_4 = 0$:

$$-H'^{(0)}_{,\bar{\zeta}\bar{\zeta}}Q = 0, \tag{7.60}$$

Implying that this function is linear in ζ and $\bar{\zeta}$:

$$H'^{(0)} = A(u)\zeta\bar{\zeta} + B(u)\zeta + \bar{B}(u)\bar{\zeta} + C(u). \tag{7.61}$$

The remaining curvature scalar Φ_{22} is

$$\Phi_{22} = \frac{-W(H'^{(0)}_{,\bar{\zeta}\zeta}P - P_{,\bar{\zeta}}H'^{(0)}_{,\zeta} - P_{,\zeta}H'^{(0)}_{,\bar{\zeta}} + H'^{(0)}P_{,\bar{\zeta}\zeta})}{P}, \tag{7.62}$$

or simpler yet

$$\Phi_{22} = \frac{(\Lambda C + A)Q}{P}. \tag{7.63}$$

7.2.3 Vacuum Solutions

For those CSI_Λ spacetimes where $W^{(0)}$ cannot be set to zero there will be non-trivial solutions to the equations $\Phi_{12} = \Phi_{22} = 0$ of Petrov type III. As these spaces are not of interest in the present work we neglect them. In the case that $W^{(0)}$ vanishes in a particular coordinate system, these are automatically of type Petrov type N. In the next section we will examine the vacuum solution as a particular subcase of the Plebanski-Petrov type O.

7.3 Plane-fronted Waves in Vacuum or Admitting Either a Null Einstein-Maxwell or Pure Radiation Field.

In terms of the Ricci spinor, these are expressed as $\Phi_{12} = 0$ and either $\Phi_{22} = 0, \Phi_{22} = \frac{f\bar{f}p}{q}$, or, $\Phi_{22} = P^2\Phi(\zeta, \bar{\zeta}, u)$. These spacetimes will be of P-type N or higher and so the function $H^{(1)}$ is given by (7.57).

- Petrov type N :

If Ψ_4 does not vanish, we make another gauge transformation to simplify the equation $\Phi_{22} = 0$. From our original H (7.52), we assume $-e^{f'(\zeta,\bar{\zeta},u)}H'^{(0)} = H^{(0)}$ and solve the first order differential equation arising from the coefficients of $H^{(0)}_{,\zeta}$ being set to zero to find,

$$f'(\zeta, \bar{\zeta}, u) = -ln(P) + ln(-2 - k\zeta - \bar{k}\bar{\zeta} + P).$$

Thus $H^{(0)} = \frac{QH'^{(0)}}{P}$ where $H'^{(0)}$ satisfies the following constraint,

$$H'^{(0)}_{,\bar{\zeta}\zeta} + \frac{2H'^{(0)}}{P^2} = 0. \tag{7.64}$$

Alternatively in the case of a null Einstein Maxwell field or pure radiation, the same gauge transformation gives a non-homogenous version of (7.64)

$$H'^{(0)}_{,\bar{\zeta}\zeta} + \tfrac{2H'^{(0)}}{P^2} = f\bar{f}p/q \tag{7.65}$$

$$H'^{(0)}_{,\bar{\zeta}\zeta} + \tfrac{2H'^{(0)}}{P^2} = P^2\Phi(\zeta, \bar{\zeta}, u). \tag{7.66}$$

To solve the vacuum case, we substitute our conjugate coordinates for polar coordinates r, θ, assume $H'^{(0)}$ is seperable, and takes the form, $H'^{(0)} = R(r)[e^{\frac{n}{2}i\theta} + e^{-\frac{n}{2}i\theta}]$. Simplifying the above equation yields a remaining equation for R

$$r^2R_{,rr} + rR_{,r} + R(\frac{8\Lambda r^2}{P^2} - n^2) = 0$$

using Maple we find the general solution,

$$R(r) = C_1(\frac{2}{P} + n - 1)r^n + C_2(\frac{2}{P} - n - 1)r^{-n}$$

where C_1 and C_2 are real-valued functions of u. Substituting this into the form of $H'^{(0)}$

$$\begin{aligned} H'^{(0)} \quad = \quad & C_1[\frac{2\zeta^n}{P} - \zeta^n + \zeta(\zeta^n)_{,\zeta} + \frac{2\bar{\zeta}^n}{P} - \bar{\zeta}^n + \bar{\zeta}(\bar{\zeta}^n)_{,\bar{\zeta}}] \\ & + C_2[\frac{2\zeta^{-n}}{P} - \zeta^{-n} + \zeta(\zeta^{-n})_{,\zeta} + \frac{2\bar{\zeta}^{-n}}{P} - \bar{\zeta}^{-n} + \bar{\zeta}(\bar{\zeta}^{-n})_{,\bar{\zeta}} \end{aligned}$$

Due to the linear nature of the above differential equation we may multiply a solution by any complex constant and add arbitrary many solutions together.

Thus for any analytic function, $\Phi(\zeta, u) = \sum_n a_n \zeta^n$ the combination,

$$H'^{(0)} = \tfrac{2\Phi}{P} - \Phi + \zeta\Phi_{,\zeta} + \tfrac{2\bar{\Phi}}{P} - \bar{\Phi} + \bar{\zeta}\bar{\Phi}_{,\bar{\zeta}}$$

is a valid solution of (7.64). To match the expression in [33], let $\Phi = \frac{\phi}{\zeta}$, $H'^{(0)}$ becomes

$$H'^{(0)} = \phi_{,\zeta} - \frac{2\Lambda\bar{\zeta}\phi}{P} + \bar{\phi}_{,\bar{\zeta}} - \frac{2\Lambda\zeta\bar{\phi}}{P} \tag{7.67}$$

In the case of a null Einstein-Maxwell field, the solutions given in [33] is the best approach. When $k \neq 0$ the general solution may be found by taking a function $\mu(\zeta, \bar{\zeta})$, and supposing the combination $H'^{(0)} = \mu_{,\zeta} - \frac{2\Lambda\bar{\zeta}\mu}{P}$ satisfies

$$H'^{(0)}_{,\zeta\bar{\zeta}} + \frac{2\Lambda H'^{(0)}}{P^2} = \frac{1}{2} f \bar{f} \frac{P}{Q}.$$

Integrating we find

$$\mu = \frac{1}{2}\int^{\bar{\zeta}} p^2 \int^{\zeta} p^{-2} \int^{\zeta'} \frac{f\bar{f}p}{q} d\zeta'' d\zeta' d\bar{\zeta},$$

where ζ and $\bar{\zeta}$ are treated as independent complex variables involved in the contour integrals, and the relationship between the two are incorporated into the final formula for $H'^{(0)}$ The end result is given as equation (7.7) in [33].

When $k(u)$ may be set to zero, f is a polynomial function of ζ and ζ^{-1} with coefficients dependent on u, [33] provides an alternative approach by choosing a new variable $t = \Lambda\zeta\bar{\zeta}$. Exploiting the linearity of the above differential equation one may simplify the problem to a particular solution of equation (7.66) with the right-hand side equal to $(p/q)\bar{\zeta}^n\zeta^{n+k},\ n, k \in \mathbb{Z}$. The substitution of $H = \frac{\zeta^k Y(t)}{\Lambda^{n+1}}$ into (7.66) gives a second order differential equation for Y:

$$tY'' + (k+1)Y' + \frac{2Y}{(1+t)^2} = \frac{1+t}{1-t}t^n.$$

Given two independent solutions of the homogeneous problem, a solution to the above equation may be found using the method of variations. The authors of [33] graciously provide the two independent solutions to the homogeneous equation in the Appendix, and explore the simplest cases where $f = \zeta^n,\ n \in \mathbb{Z}$

- Petrov type O:

  If $\Psi_4 = 0$ we may make a gauge transformation as in the PP-type O, P-type O case so that $H^{(0)}$ takes the form (7.61). In these coordinates, the simpler form of (7.63) implies that $A = \Lambda C$ in a vacuum and hence

$$H'^{(0)} = (1 - \Lambda\zeta\bar{\zeta})C + \bar{B}\bar{\zeta} + B\zeta. \tag{7.68}$$

  Imposing the conditions of a null Einstein-Maxwell field, (7.63) implies

$$\frac{(\Lambda C + A)Q}{P} = f\bar{f}\frac{P}{Q}. \tag{7.69}$$

  Then solving for $f\bar{f} = (\Lambda C + A)\frac{Q^2}{P^2}$ and noting that that $log(f\bar{f})$ is harmonic we find a contradition as $ln(\frac{Q^2}{P^2})$ is not harmonic for any choice of k.

7.4 Canonical Forms of the Type N CSI_Λ Metrics

From [33] and [50] we know that all P-Type N and PP-Type O spacetimes may be classified by Λ and the sign of the sole component of the second Lie derivative of the metric with respect to ℓ, L'. It will be helpful to transform these metric into our preferred coordinate system so the metric is of Kundt form. The aim of such a coordinate transform will be to relate these spacetimes to the paper [94], where a general metric form for the CSI spacetimes has been introduced and organized by sign of the curvature of the transverse space and the Ricci tensor's Segre type[71].

Case I, $\Lambda > 0,\ L' > 0$: Metrics in this class are equivalent to the ORR metric with $\alpha = 0,\ \beta = 1$ and hence in Kundt form, the coframe member n_a becomes

$$\begin{aligned} n &= dv + v\left(ln\left(\frac{(1+\Lambda\zeta\bar{\zeta})^2}{(\zeta+\bar{\zeta})^2}\right)_{,\zeta}\right)d\zeta + v\left(ln\left(\frac{(1+\Lambda\zeta\bar{\zeta})^2}{(\zeta+\bar{\zeta})^2}\right)_{,\bar{\zeta}}\right)d\bar{\zeta} \\ &+ \left(\frac{-(1+\Lambda\zeta\bar{\zeta})^2}{(\zeta+\bar{\zeta})^2}v^2 + H^{(0)}\right)du. \end{aligned}$$

Opting for the dimensionless complex coordinate $\zeta' = \sqrt{2\Lambda}\zeta$, choosing

$$\zeta' = \sqrt{2}tan(\frac{x}{2})e^{iy}$$

yields a metric with the tranverse space is the usual form for the two dimensional sphere,

$$\frac{ds^2 = dx + sin^2(x)dy}{2\Lambda} \tag{7.70}$$

the covector n transforms to become,

$$n = dv - 2cot(x)vdx + 2tan(y)vdy + \left(\left(-\frac{tan^2(\frac{x}{2})}{4cos^2(y)sin^4(\frac{x}{2})}\right)v^2 + H^{(0)}\right)du$$

These correspond to the Type N CSI spacetimes with Segre type $\{(1,111)\}$ or $\{(2,11)\}$ for which the transverse space has the geometry of a sphere S^2, and a one-form $\mathbf{W}^{(1)}$ of type 2 [94].

Case II, $\Lambda < 0,\ L' > 0:$ Metrics in this class are equivalent to the ORR metric with $\alpha = 0,\ \beta = 1$ and hence in Kundt form, the coframe member n_a becomes

$$\begin{aligned} n &= dv + v\left(ln\left(\frac{(1+\Lambda\zeta\bar{\zeta})^2}{(\zeta+\bar{\zeta})^2}\right)_{,\zeta}\right)d\zeta + v\left(ln\left(\frac{(1+\Lambda\zeta\bar{\zeta})^2}{(\zeta+\bar{\zeta})^2}\right)_{,\bar{\zeta}}\right)d\bar{\zeta} \\ &\quad + \left(\left(\frac{-(1+\Lambda\zeta\bar{\zeta})^2}{(\zeta+\bar{\zeta})^2}\right)v^2 + H^{(0)}\right)du \end{aligned}$$

Opting for the dimensionless complex coordinate $\zeta' = \sqrt{2\Lambda}\zeta$, choosing

$$\zeta' = \sqrt{2}tanh(\frac{x}{2})e^{iy}$$

yields a metric with the tranverse space is the usual form,

$$\frac{ds^2 = dx + sinh^2(x)dy}{2\Lambda} \tag{7.71}$$

the covector n transforms to become,

$$n = dv - 2coth(x)vdx + 2tan(y)vdy + \left(\left(-\frac{sec^4(\frac{x}{2})}{4cos^2(y)tanh^2(\frac{x}{2})}\right)v^2 + H^{(0)}\right)du$$

These correspond to the Type N CSI spacetimes with Segre type $\{(1,111)\}$ or $\{(2,11)\}$ for which the transverse space has the geometry of the hyperbolic plane H^2, and a one-form $\mathbf{W}^{(1)}$ of a type not listed in the literature [94]. By calculating the coefficient of the v^2 term in Kundt coordinates [94], we expect these to belong to the class where the one-form $\mathbf{W}^{(1)}$ is of type (5), as these have $\tilde{\sigma} > 0$ for all values of x and y relative to the coordinates in which the metric takes the form

$$dx^2 + cosh(x)^2dy^2.$$

Case III, $\Lambda < 0,\ L' < 0:$ Metrics in this class are equivalent to the ORR metric with $\alpha = 1,\ \beta = 0$ and hence in Kundt form, the coframe member n_a becomes

$$\begin{aligned} n &= dv + v\left(ln\left(\frac{(1+\Lambda\zeta\bar{\zeta})^2}{(1-\Lambda\zeta\bar{\zeta})^2}\right)_{,\zeta}\right)d\zeta + v\left(ln\left(\frac{(1+\Lambda\zeta\bar{\zeta})^2}{(1-\Lambda\zeta\bar{\zeta})^2}\right)_{,\bar{\zeta}}\right)d\bar{\zeta} \\ &\quad + \left(\left(\frac{-\Lambda(1+\Lambda\zeta\bar{\zeta})^2}{(1-\Lambda\zeta\bar{\zeta})^2}\right)v^2 + H^{(0)}\right)du \end{aligned}$$

Opting for the dimensionless complex coordinate $\zeta' = \sqrt{2\Lambda}\zeta$, choosing

$$\zeta' = \sqrt{2}tanh(\frac{x}{2})e^{iy}$$

yields a metric with the tranverse space is the usual form,

$$\frac{ds^2 = dx + sinh^2(x)dy}{2\Lambda} \tag{7.72}$$

the covector n transforms to become,

$$n = dv - 2vtanh(x)dx + \left(\left(-\frac{sec^4(\frac{x}{2})}{(1+tanh^2(\frac{x}{2}))^2}\right)v^2 + H^{(0)}\right)du$$

These correspond to the Type N CSI spacetimes with Segre type $\{(1,111)\}$ or $\{(2,11)\}$ for which the transverse space has the geometry of the hyperbolic plane H^2, and a one-form $\mathbf{W}^{(1)}$ of type 3 [94].

Case $IV_0,\ \Lambda < 0,\ L' = 0:$ Metrics in this class are equivalent to the ORR metric with $\alpha = 0,\ \beta = \lambda = \sqrt{-\Lambda}$ and hence in Kundt form, the coframe member n_a becomes

$$n = dv + v\left(ln\left(\frac{(1+\Lambda\zeta\bar{\zeta})^2}{((1+\lambda\zeta)(1+\lambda\bar{\zeta}))^2}\right)_{,\zeta}\right)d\zeta + v\left(ln\left(\frac{(1+\Lambda\zeta\bar{\zeta})^2}{((1+\lambda\zeta)(1+\lambda\bar{\zeta}))^2}\right)_{,\bar{\zeta}}\right)d\bar{\zeta} + H^{(0)}du$$

Opting for the dimensionless complex coordinate $\zeta' = \sqrt{2\Lambda}\zeta$, choosing

$$\zeta' = \sqrt{2}tanh(\frac{x}{2})e^{iy}$$

yields a metric with the tranverse space is the usual form,

$$\frac{ds^2 = dx + sinh^2(x)dy}{2\Lambda} \tag{7.73}$$

the covector n transforms to become,

$$n = dv - 2v\left(\frac{cosh(x)cos(y)+sinh(x)}{sinh(x)cos(y)+cosh(x)}\right)dx + 2v\left(\frac{sin(y)sinh(x)}{sinh(x)cos(y)+cosh(x)}\right)dy + H^{(0)}du$$

These correspond to the Type N CSI spacetimes with Segre type $\{(1,111)\}$ or $\{(2,11)\}$ for which the transverse space has the geometry of the hyperbolic plane H^2, and a one-form $\mathbf{W}^{(1)}$ not listed in [94]. However, by calculating the coefficient of v^2, we expect these to correspond to the first type with $\epsilon = 0$ and tranverse metric

$$dx^2 + e^{2x}dy^2$$

Case $IV_1,\ \Lambda < 0,\ L' = 0:$ Metrics in this class are equivalent to the ORR metric with $\alpha = 0,\ \beta = \lambda e^{iW(u)}$, where $\lambda = \sqrt{-\Lambda}$ and hence in Kundt form, the coframe member n_a becomes

$$v(ln(\tfrac{(1+\Lambda\zeta\bar{\zeta})^2}{((1+\lambda\zeta e^{iW(u)})(1+\lambda\bar{\zeta}e^{-iW(u)}))^2}),_{\zeta})d\zeta + v(ln(\tfrac{(1+\Lambda\zeta\bar{\zeta})^2}{((1+\lambda\zeta e^{iW(u)})(1+\lambda\bar{\zeta}e^{-i\lambda u}))^2}),_{\bar{\zeta}})d\bar{\zeta}$$
$$+(vln(\tfrac{(1+\Lambda\zeta\bar{\zeta})^2}{((1+\lambda\zeta e^{iW(u)})(1+\lambda\bar{\zeta}e^{-iW(u)}))^2}),_u + H^{(0)})du$$

Opting for the dimensionless complex coordinate $\zeta' = \sqrt{2\Lambda}\zeta$, choosing

$$\zeta' = \sqrt{2}tanh(\frac{x}{2})e^{iy}$$

yields a metric with the tranverse space is the usual form,

$$\frac{ds^2 = dx + sinh^2(x)dy}{2\Lambda} \tag{7.74}$$

the covector n transforms to become,

$$\begin{aligned} n \quad = \quad & dv - 2v\left(\frac{cosh(x)cos(y-W(u)) + sinh(x)}{sinh(x)cos(y-W(u)) + cosh(x)}\right)dx \\ & +2v\left(\frac{sin(y-W(u))sinh(x)}{sinh(x)cos(yW(u)) + cosh(x)}\right)dy \\ & + \left(\left(v\frac{2\sqrt{|\Lambda|}W_{,u}tanh(\frac{x}{2})sin(y-W)}{1+2|\Lambda|tanh^2(\frac{x}{2}) + 2\sqrt{|\Lambda|}tanh(\frac{x}{2})cos(y-W(u))}\right) + H^{(0)}\right)du. \end{aligned}$$

These correspond to the Type N CSI spacetimes with Segre type $\{(1,111)\}$ or $\{(2,11)\}$ for which the transverse space has the geometry of the hyperbolic plane H^2, and a one-form $\mathbf{W}^{(1)}$ not listed in [94]. However, by calculating the coefficient of v^2, we expect these to correspond to the first type with $\epsilon = 0$ and tranverse metric

$$dx^2 + e^{2x}dy^2$$

Notice that by translating the y coordinate all u dependence in $\mathbf{W}^{(1)}$ may be eliminated, recovering the form given in case IV_0.

Chapter 8

The Karlhede Classification of the Vacuum PP-waves

This chapter is based on: R. Milson, D. McNutt, A. Coley (2013). Invariant classification of vacuum PP-waves. . *JMP* Vol 54, Issue 2, pp 022502-022531. Copyright 2013, AIP Publishing LLC.

8.1 Introduction

In this paper we provide an invariant approach to characterizing the vacuum PP-wave spacetimes, which may be summarized in a flow chart broken into three figures: (8.1), (8.2) and (8.3). In these flowcharts, each end-node is a subclass of the PP-wave spacetimes, with labels corresponding to those in the first column of the tables (8.2) - (8.4). For each subclass these tables give a canonical form for $f(\zeta, u)$, the independent invariants at each order, and the corresponding numbered lemma listing the essential functionally dependent invariants used in sub-classification. Following the work of Collins [36]; where an analysis of the Karlhede algorithm for Type N vacuum spacetimes, a coframe is chosen with $\Psi_4 = 1$ and for which the Cartan invariants are expressed in terms of spin-coefficients. By examining the invariants at each order we may express all possible sub-cases for the Karlhede algorithm and determine when all isotropy may be eliminated. In this manner upper-bounds for all vacuum Type N metrics were given, in particular the upper-bound for the PP-wave spacetimes was $q = 4$. By choosing coordinates and analyzing the cases where $q = 4$ we produce four classes of metrics, proving the upper-bound is sharp. The following theorem summarizes the work done in section (8.3):

Theorem 8.1.1. *The metrics with $f(\zeta, u)$ taking either of the following canoncial forms,*

- $\frac{Clog\zeta + ke^{iZ(u)}\zeta}{u^2}$, $C, k \in \mathbb{R}$, *and $Z(u)$ is real valued,*

- $(k\zeta)^{2ik_0} + k_1 e^{iZ(u)(1-2ik_0)}\zeta, \quad k, k_0, k_1 \in \mathbb{R}$ *and* $Z(u)$ *is real valued,*

- $f(\zeta, u) = e^{k_0\zeta} + e^{ik_1}e^{Z(u)}\zeta,\ k_0, k_1 \in \mathbb{R}$ *and* $Z(u)$ *is real valued,*

- $f(\zeta, u) = -e^{ik_0}ln(\zeta) + k_1 e^{iZ(u)}\zeta,\ k_0, k_1 \in \mathbb{R}$ *and* $Z(u)$ *is real valued,*

require the fourth order covariant derivatives of the Weyl tensor to be fully classified by the Karlhede algorithm. (8.2)*.*

We also explore all of the G_2 subcases arising as degenerate subcases of the Karlhede algorithm. These are all specific subcases of the G_2 I, II and III cases arising from the classification of symmetry groups having the same symmetry group but distinct geometric structure [11]. The following theorem and flowcharts summarize our work.

Theorem 8.1.2. *For those PP-wave spacetime admitting a two-dimensional isometry group, the second Killing vector field V annihilates the invariant coframe and all Cartan invariants. If $d\alpha \wedge d\bar{\alpha} \neq 0$ the spacetime belongs to G_2- I,II or III depending on the form of the invariants, $\bar{\delta}\alpha$ and μ expressed in terms of α, $\bar{\alpha}$ and μ:*

$$\begin{aligned} G_2 - I \quad &: \quad \bar{\delta}\alpha = \alpha^2; \\ G_2 - II \quad &: \quad \bar{\delta}\alpha \neq \alpha^2, \quad Re(\mu\alpha^{-2}\bar{\delta}\alpha - \bar{\mu}) = CIm(\mu\bar{\delta}\alpha - \bar{\mu}); \\ G_2 - III \quad &: \quad \bar{\delta}\alpha \neq \alpha^2, \quad Re(\mu\alpha^{-2}\bar{\delta}\alpha - \bar{\mu}) = 0; \\ G_2 - III_{k=0} \quad &: \quad \bar{\delta}\alpha \neq \alpha^2, \quad Re(\mu\alpha^{-2}\bar{\delta}\alpha - \bar{\mu}) = 0, \Delta(\nu) = 0. \end{aligned}$$

If $d\alpha \wedge d\bar{\alpha} = 0$, the spacetime belongs to one of the special subclasses given in table (8.3)*. Further conditions for determining which subclass is given in the following decision tree.*

8.2 The Vacuum PP-wave Spacetimes

Following the work of [36] we choose the normalized dyad $\{o^A, \iota^A\}$ satisfying, $o_A\iota^A = 1$ and define the generic symbols ζ_a^A for the dyad:

$$\zeta_0^A = o^A, \quad \zeta_1^A = \iota^A, \quad \bar{\zeta}_{0'}^{A'} = \bar{o}^{A'}, \quad \bar{\zeta}_{1'}^{A'} = \bar{\iota}^{A'}.$$

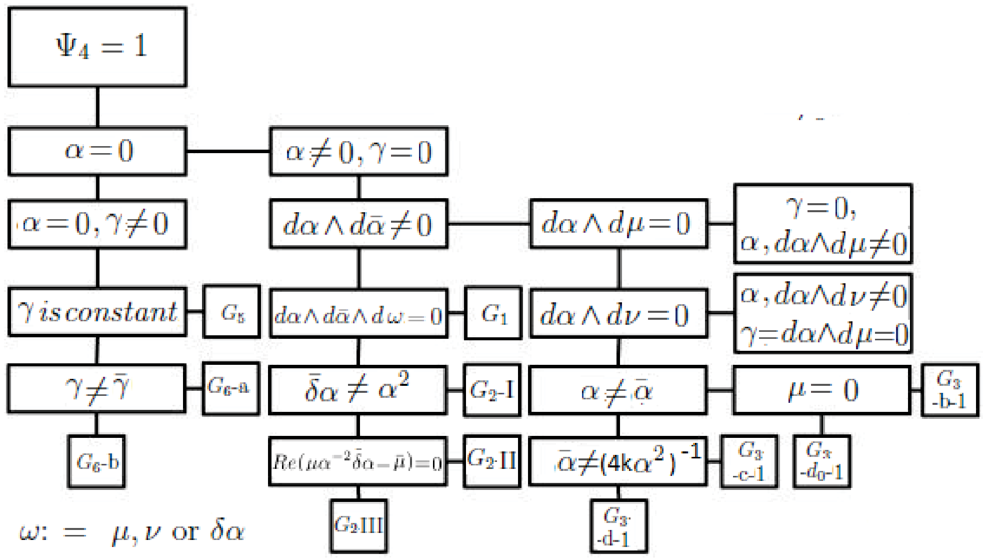

Figure 8.1: The decision-tree for PP-wave spacetimes. Here $\omega = \mu, \nu$ or $\bar{\delta}\alpha$. Two branches have been cut off and written explicitely in Figures (8.2) and (8.3)

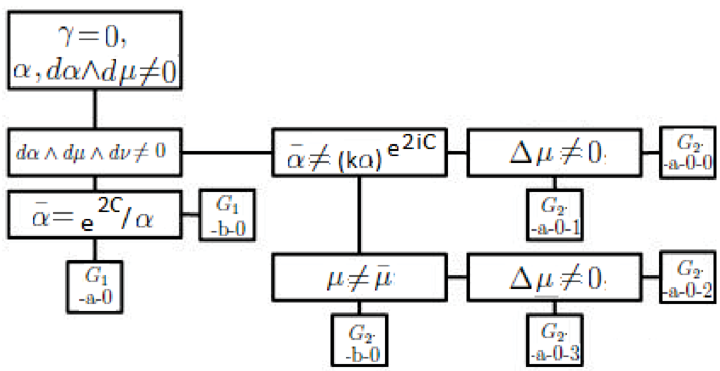

Figure 8.2: A branch of the decision tree in Figure (8.1)

Using the Newman-Penrose formalism [29], the type N vacuum spacetimes have $\Phi_{ij} = 0$ for all $i, j \in [0, 2]$ and $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ and $\Psi_4 = 1$. Applying a boost and a spatial rotation we may always set $\Phi_4 = 1$. The non-trivial Bianchi identities yield:

$$\kappa = \sigma = 0, \quad \rho = 4\epsilon, \quad \tau = 4\beta. \tag{8.1}$$

The first covariant derivative is then defined by:

$$(D\Psi)_{\mu f'} = \Psi_{ABCD;EF'}\zeta^A_a \zeta^B_b \zeta^C_c \zeta^D_d \zeta^E_e \bar{\zeta}^{F'}_{f'}$$

where μ of the unprimed dyad vectors are ζ^A_1's. The non-vanishing components are:

$$(D\Psi)_{40'} = \rho, \quad (D\Psi)_{50'} = 4\alpha, \quad (D\Psi)_{41'} = \tau, \quad (D\Psi)_{51'} = 4\gamma.$$

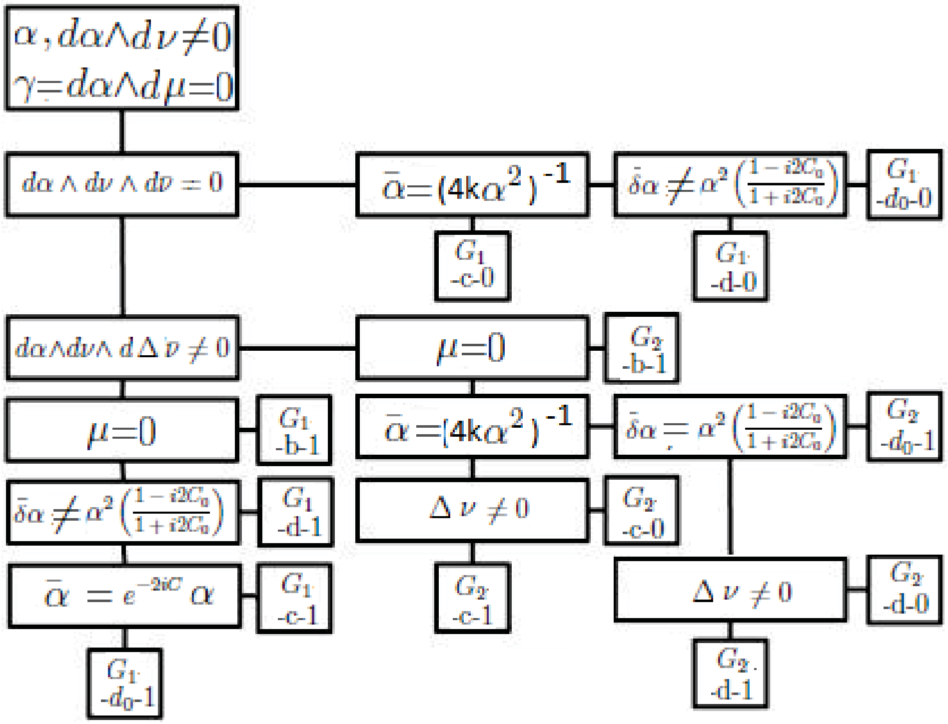

Figure 8.3: A branch of the decision tree in Figure (8.1)

Where (8.1) was used to express the spin coefficients in terms of α, γ, ρ and τ. Calculating the second covariant derivative of the Weyl tensor, denoted as

$$(D^2\Psi)_{\mu f';gh'} \quad (\mu = 0,1,2,3,4,5; a,b,c = 0,1).$$

In [36], it may be shown that the non-vanishing components will be for $i = 3,4,5$:

$$(D^2\Psi)_{i0';00'}, (D^2\Psi)_{i0';01'}, (D^2\Psi)_{i0';10'}, (D^2\Psi)_{i0';11'},$$
$$(D^2\Psi)_{i1';00'}, (D^2\Psi)_{i1';01'}, (D^2\Psi)_{i1';10'}, (D^2\Psi)_{i1';11'}.$$

Imposing the conditions for a pp-wave spacetime the spin-coefficients κ, σ, τ and ρ vanish by definition. Applying the Bianchi identities (8.1) we see that ϵ and β are zero as well. The remaining first order derivatives of the Weyl Tensor, Ψ for the vacuum Type N spacetimes are

$$(D\Psi)_{50'} = 4\alpha, \quad (D\Psi)_{51'} = 4\gamma.$$

There are two cases to consider at this point, depending on whether or not α vanishes. If $\alpha \neq 0$, we may use a null rotation about o^A, to set $\gamma = 0$

$$\alpha' = \alpha, \quad \gamma' = \gamma + B\alpha.$$

- Vacuum PP-Wave spacetimes with $\alpha \neq 0$: The first order covariant derivative of Ψ has only one component, as $\gamma = 0$,

$$(D\Psi)_{50'} = 4\alpha.$$

The NP field equations require that $\pi = \lambda = 0$, and so the non-trivial NP-eqations become:

$$\begin{gathered} D\alpha = 0,\ \delta\alpha = \alpha\bar{\alpha},\ \Delta\alpha = -\bar{\mu}\alpha, \\ D\mu = 0,\ \bar{\delta}\mu = -\alpha\mu, \\ D\nu = 0,\ \bar{\delta}\nu = -3\alpha\nu + 1,\ \Delta\mu - \delta\nu = -\mu^2 + \bar{\alpha}\nu. \end{gathered} \tag{8.2}$$

The non-vanishing components of the second order derivative of Ψ from the general case are simply,

$$\begin{gathered} (D^2\Psi)_{50';00'} = 4D\alpha, \quad (D^2\Psi)_{50';01'} = 4\delta\alpha - 4\bar{\alpha}\alpha, \\ (D^2\Psi)_{50';10'} = 4\bar{\delta}\alpha + 20\alpha^2, \quad (D^2\Psi)_{50';11'} = 4\Delta\alpha, \\ (D^2\Psi)_{51';10'} = -4\bar{\mu}\alpha, \ \ (D^2\Psi)_{51';11'} = -4\bar{\nu}\alpha, \ \ (D^2\Psi)_{41';11'} = -\bar{\nu}\alpha. \end{gathered}$$

Further information may be obtained by using the NP field equations in conjunction with the commutator identities if necessary.

- Vacuum PP-Wave spacetimes with $\alpha = 0$: The non-zero component of the first covariant derivative of the Weyl tensor Ψ is

$$(D\Psi)_{51'} = 4\gamma.$$

The relevant Newman-Penrose equations for γ are

$$D\gamma = 0,\ \delta\gamma = 0,\ \bar{\delta}\gamma = 0, \tag{8.3}$$

and the second order covariant derivative of Ψ has only one non-zero component

$$(D^2\Psi)_{51';11'} \ = \ 4\Delta\gamma + 20\gamma^2 + 4\bar{\gamma}\gamma.$$

These metrics will in general have $q = 2$ due to the simple form of the sole Cartan invariant γ as a single variable function of one coordinate.

8.3 Vacuum PP-Wave Spacetimes - Degenerate Subcases of the Karlhede Algorithm with q=4.

We are interested in the degenerate cases of the Karlhede algorithm, in particular the case where the algorithm needs to calculate the fourth covariant derivative of the Weyl tensor. From the previous section, we know that $\alpha \neq 0$ allows for γ to be set to zero, and that the isotropy group has dimension 0. Thus the $q = 4$ bound can only be reached if and only if, one functionally independent invariant appears at each iteration. At first order the only invariants are α and its conjugate.

To construct a solution where only one new real-valued invariant appears at each order, i.e., $(0, 1, 2, 3, 3)$, $\bar{\alpha}$ must be functionally dependent on α and so the wedge product of their differentials vanishes. This yields three equations, one of which is the complex conjugate of the other:

$$\delta\alpha\bar{\delta}\bar{\alpha} - \bar{\delta}\alpha\delta\bar{\alpha} = 0, \quad \delta\alpha\Delta\bar{\alpha} - \delta\bar{\alpha}\Delta\alpha = 0, \quad \bar{\delta}\alpha\Delta\bar{\alpha} - \bar{\delta}\bar{\alpha}\Delta\alpha = 0. \tag{8.4}$$

There are two cases to consider, depending on whether or not $\Delta\bar{\alpha}$ is non-zero. For the moment let us assume that it is non-zero, implying that $\mu \neq 0$ via the NP field equations. Solving for $\bar{\delta}\alpha$ from (8.4)-B we find

$$\bar{\delta}\alpha = \frac{\bar{\mu}}{\mu}\alpha^2. \tag{8.5}$$

Looking at this relation and the NP field equations, we see that all frame derivatives of α may be written as functions of α, ν and μ, and so we need only inspect ν, μ and their conjugates. Applying the commutators to these three invariants yield further relations:

$$\delta\mu = -\bar{\alpha}\mu, \quad \tfrac{\Delta\mu}{\mu} = \tfrac{\Delta\bar{\mu}}{\bar{\mu}}.$$

Now let us consider the case where $\Delta\bar{\alpha} = 0$, the NP field equations become

$$\bar{\delta}\nu = -3\alpha\nu + 1, \; \delta\nu = -\bar{\alpha}\nu, \quad \delta\alpha = \alpha\bar{\alpha}, \; D\alpha = 0, D\nu = 0. \tag{8.6}$$

The only equation from (8.4) is

$$\delta\alpha\bar{\delta}\bar{\alpha} - \bar{\delta}\alpha\delta\bar{\alpha} = 0,$$

and the non-vanishing commutator relations yield

$$\Delta\bar{\delta}\alpha = 0, \quad \delta\bar{\delta}\alpha = -2\bar{\alpha}\delta\alpha, \quad \delta\Delta\nu = -2\bar{\alpha}\Delta\nu, \quad \bar{\delta}\Delta\nu = -4\alpha\Delta\nu.$$

8.3.1 Coordinate forms of the Cartan Invariants

To analyze the equations given by (8.4), we choose a different set of coordinates than the usual $(\zeta, \bar{\zeta}, u, v)$ coordinates to derive the spin-coefficients, using the second ζ derivative of $f(\zeta, u)$ instead:

$$a(\zeta, u) = \frac{1}{4} ln(f_{,\zeta\zeta}). \tag{8.7}$$

We use $a(\zeta, u)$ and its conjugate as spatial coordinates, letting $\zeta = \zeta(a, u)$ the arbitrary analytic function f becomes $f'(a, u) = f(\zeta, u)$. The relation $a = \frac{1}{4}ln(f_{,\zeta\zeta})$ gives an important equation for $f'_{,aa}$ in these coordinates,

$$\left(\frac{f'_{,a}}{\zeta_{,a}}\right)_{,a} = \zeta_{,a} e^{4a}. \tag{8.8}$$

The first order invariant α changes to

$$\alpha = \frac{e^{a-\bar{a}}}{\bar{\zeta}_{,\bar{a}}} \tag{8.9}$$

and the dual of the coframe with $\Psi_4 = 1$ becomes:

$$\delta = \bar{\alpha}\left(\frac{\partial}{\partial a} - \bar{\zeta}_{,u}\zeta_{,a}\frac{\partial}{\partial v}\right), \quad D = e^{a+\bar{a}}\frac{\partial}{\partial v}, \quad \Delta = e^{-a-\bar{a}}\left(\frac{\partial}{\partial u} - (Re(f') - \zeta_{,u}\bar{\zeta}_{,u})\frac{\partial}{\partial v}\right).$$

In both subcases, it is possible that $\zeta_{,aa} = 0$. If this happens the second equation in (8.4) vanishes; in the $\Delta\alpha \neq 0$ case the solution to $\zeta_{,a}$ has the linear case as a special subcase, while in $\Delta\alpha = 0$ it must be explicitely considered.

Lemma 8.3.1. *For those spacetimes with $d\alpha \wedge d\bar{\alpha} = 0$, the function $\zeta_{,a}$ has two possible forms. Supposing $\Delta\alpha \neq 0$,*

$$\zeta_{,a} = (e^{iC_2} e^{R(u)})^{e^{iC_1}} e^{-(e^{2iC_1}+1)a}, \quad C_1, C_2 \in \mathbb{R}. \tag{8.10}$$

While if $\Delta\alpha = 0$,

$$\zeta_{,a} = c_4 e^{\frac{-2\epsilon a}{1-i2C_3}}, \quad C_3, C_4 \in \mathbb{R}, \epsilon = 0, 1. \tag{8.11}$$

Proof. Assuming for a moment that $\Delta\alpha$ and $\zeta_{,aa} \neq 0$, the equations (8.4) yields the following separable equations

$$\frac{\bar{\zeta}_{,\bar{a}}}{\zeta_{,u\bar{a}}} + \frac{\zeta_{,a}}{\zeta_{,ua}} + \frac{\zeta_{,aa}}{\zeta_{,ua}} = 0, \quad \frac{\bar{\zeta}_{,\bar{a}}}{\zeta_{,\bar{a}\bar{a}}} + \frac{\zeta_{,a}}{\zeta_{,aa}} + 1 = 0. \tag{8.12}$$

The first implies

$$\frac{\zeta_{,a}}{\zeta_{,ua}} = -\boldsymbol{k}_1(u), \quad \frac{\zeta_{,a}}{\zeta_{,ua}} + \frac{\zeta_{,aa}}{\zeta_{,ua}} = k_1(u),$$

combining these equations we have

$$ln(\zeta_{,a})_{,a} = -\frac{k_1}{\boldsymbol{k}_1} - 1.$$

It is easily shown that (8.12)-B is automatically satisfied in this case. To summarize, the function $\zeta(a,u)$ satisfies

$$\zeta_{,a} = k_2(u)e^{-\left(\frac{k_1}{\boldsymbol{k}_1}+1\right)a}, \quad ln(\zeta_{,a})_{,u} = -\frac{1}{\boldsymbol{k}_1}. \tag{8.13}$$

We may write $k_1 = r(u)e^{i\theta(u)}$ and so

$$ln(\zeta_{,a})_{,u} = -a(e^{i\theta}+1)_{,u} + ln(k_2)_{,u} = -\frac{1}{\boldsymbol{k}_1}.$$

If this holds, $\theta_{,u} = 0$ and $ln(k_2')_{,u} = -\frac{1}{\boldsymbol{k}_1}$. Rewriting $k_1 = r(u)e^{iC_1}$, $C_1 \in \mathbb{R}$, the second equation from (8.13) yields

$$ln(k_2) = -\int \frac{e^{iC_1}}{r} du + c_2, \quad c_2 \in \mathbb{C}.$$

As k_2 is arbitrary, let us suppose $k_2 = (k_2')^{e^{iC_1}}$, then we may divide through by e^{iC_1} and absorb the real part of the constant of integration by calling $R(u) = \int \frac{1}{r} du + C'$ to write

$$k_2 = (e^{iC_2}e^{R(u)})^{e^{iC_1}}$$

with this piece known we may write down the derivative of ζ,

$$\zeta_{,a} = (e^{iC_2}e^{R(u)})^{e^{iC_1}} e^{-(e^{2iC_1}+1)a}, \quad C_1, C_2 \in \mathbb{R}.$$

We note when $C_1 = \pi/2, 3\pi/2 \mod 2\pi$ causes $\zeta(a,u)$ to be linear in a. Now to consider the form of $\zeta(a,u)$ when $\Delta\alpha = 0$. In the linear case, $\zeta_{,aa} = 0$, the wedge product of $d\alpha$ with its conjugate automatically vanishes, and so $\zeta_{,a} = c_4$. Next we assume $\zeta_{,aa} \neq 0$, in this case we only get (8.12)-B, as this is separable we find two constraints,

$$\frac{\zeta_{,aa}}{\zeta_{,a}} = \frac{1}{c_3}, \quad c_3 + \bar{c}_3 = -1.$$

The second constraint implies $c_3 = -\frac{1}{2} + iC_3$ and so integrating we find

$$\zeta_{,a} = c_4 e^{\frac{-2a}{1-i2C_3}}, \quad C_3 \in \mathbb{R},\ c_4 \in \mathbb{C}.$$

Introducing $\epsilon = 0, 1$ we may write a general form for $\zeta(a, u)$ to differentiate between the linear and exponential case,

$$\zeta_{,a} = c_4 e^{\frac{-2\epsilon a}{1-i2C_3}}, \quad C_3 \in \mathbb{R},\ c_4 \in \mathbb{C}, \epsilon = 0, 1.$$

□

In the $\Delta\alpha = 0$ case, the second order invariant $\bar{\delta}\alpha$ cannot be expressed using the NP field equations or the commutators. However the above lemma yields a simple expression for this invariant in terms of first order invariants

Corollary 8.3.2. *If $\alpha \neq 0$ and $\Delta\alpha = 0$ in a particular PP-wave spacetime, then*

$$\bar{\delta}\alpha = -\alpha^2 \left(1 - \frac{\epsilon}{\frac{1}{2} + iC_3}\right). \tag{8.14}$$

Proof. Using (8.11) and (8.9) we may explicitely calculate $\bar{\delta}\alpha$,

$$\bar{\delta}\alpha = -\alpha^2 \left(1 - \frac{\epsilon}{\frac{1}{2} + iC_3}\right).$$

□

To continue we will calculate the second order invariants $\Delta\alpha$ and ν for the two possible forms for $\zeta_{,a}$.

Corollary 8.3.3. *In the case that $d\alpha \wedge d\bar{\alpha} = 0$ coordinates may always be chosen in which the spin-coefficients $\Delta\alpha$ and ν take the following form:*

$$\Delta\alpha = e^{iC_1} R_{,u} e^{-a-\bar{a}} \alpha,$$
$$\nu = -\frac{1}{\alpha(e^{-2iC_1} - 3)} - \frac{(\zeta - Z_1)}{\zeta_{,a}} \left(\frac{\mu^2}{\bar{\alpha}} + \frac{\Delta\mu}{\bar{\alpha}}\right) - Z_{1,uu} e^{-a-3\bar{a}}; \tag{8.15}$$
$$\Delta\alpha = 0, \quad \nu = \frac{\frac{1}{2} + iC_3}{\alpha(2 - \epsilon + i4C_3)} - Z_{2,uu} e^{-a-3\bar{a}} \tag{8.16}$$

where $Z_1(u), Z_2(u)$ are arbitrary functions arising from integrating (8.10) and (8.11) respectively.

Proof. From the NP field equations $\Delta\alpha = \mu\alpha$, so we may restrict our attention to μ and ν. In the current coordinate system the remaining spin-coefficients are

$$\mu = \frac{\zeta_{,ua}}{\zeta_{,a}} e^{-a-\bar{a}}, \nu = \left(\frac{\bar{f}'_{,\bar{a}}}{\bar{\zeta}_{,\bar{a}}} + \zeta_{,uu} \right) e^{-a-3\bar{a}}. \tag{8.17}$$

As we must integrate $\zeta_{,a}$ to determine ν, we define a new constant dependent on the other constant C_1 - in order to differentiate between the exponential and linear case in (8.10)

$$\begin{aligned} \dot{C}(C_1) &= -1 \quad if\ C_1 = \pi/2,\ 3\pi/2\ =\ mod\ 2\Pi \\ &= e^{2iC_1} + 1 \quad otherwise. \end{aligned} \tag{8.18}$$

Integrating the conjugate of (8.8), we may write $\bar{f}'_{,\bar{a}}/\bar{\zeta}_{,\bar{a}}$ in a simpler form

$$\frac{\bar{f}'_{,\bar{a}}}{\bar{\zeta}_{,\bar{a}}} = -\frac{\bar{\zeta}_{,\bar{a}} e^{4\bar{a}}}{(e^{2iC_1} - 3)} + f'_1(u), \quad or \quad \frac{\bar{f}'_{,\bar{a}}}{\bar{\zeta}_{,\bar{a}}} = \bar{\zeta}_{,\bar{a}} e^{4\bar{a}} \left(\frac{\frac{1}{2} + iC_3}{2 - \epsilon + i4C_3} \right) + f'_2(u);$$

We note that in the $(\zeta, \bar{\zeta}, u, v)$ coordinate system the arbitrary functions of integration, f'_i $i = 1, 2$ are the coefficients of the ζ-linear term in $f(\zeta, u)$ and hence may be set to zero by the following coordinate transformation:

$$\zeta' = \zeta + F_i,\ v' = v + 2Re(\bar{F}_{i,u}\zeta) + G(u), \quad H' = H - \bar{f}_i\zeta - f_i\bar{\zeta} \tag{8.19}$$
$$F_i = \int\int \bar{f}'_i(u) du du, \quad G(u) = \int | \int f'_i du|^2 du$$

this is built out of (8.37) and (8.39). As this coordinate transformation does not affect $f_{,\zeta\zeta}$ the alternative coordinate system $(a, \bar{a}, u, v)$ is still valid, except that $f'(a, u)$ will have $F_{i,u}\bar{F}_{i,u}$ added to it - which will not affect the equations needed for analysis. The arbitrary function of integration, Z_i, may be redefined to absorb $F_1(u)$.

$$\frac{\bar{f}'_{,\bar{a}}}{\bar{\zeta}_{,\bar{a}}} = -\frac{\bar{\zeta}_{,\bar{a}} e^{4\bar{a}}}{(e^{2iC_1} - 3)}, \quad or \quad \frac{\bar{f}'_{,\bar{a}}}{\bar{\zeta}_{,\bar{a}}} = \bar{\zeta}_{,\bar{a}} e^{4\bar{a}} \left(\frac{\frac{1}{2} + iC_3}{2 - \epsilon + i4C_3} \right), \tag{8.20}$$

and so the remaining spin coefficients may be written in terms of invariants where the last invariant is $\mathbf{V} = z_{,uu} e^{-a-3\bar{a}}$. □

Due to the NP field equations (8.2) and the commutator relations arising from $d\alpha \wedge d\bar{\alpha} \neq 0$ and equations [(8.6), (8.5)] or [(8.6), (8.14)] we may exclude all derivatives of α as functionally independent invariants if we treat μ and ν as potential functionally independent invariants. Of course, for the degenerate case of the Karlhede algorithm with $q = 4$ we must have only one new invariant at this iteration, hence $d\alpha \wedge d\mu \wedge d\nu = d\alpha \wedge d\nu \wedge \bar{\nu} = 0$ at least, although there are further constraints.

Lemma 8.3.4. *Those PP-wave metrics with $d\alpha \wedge d\bar{\alpha} = 0$, $\Delta\alpha \neq 0$ and $d\alpha \wedge d\mu \neq 0$ require at most the third order derivatives of the Weyl tensor to be classified invariantly.*

Proof. To start we will consider the linear case when $C_1 = \pi/2 or 3\pi/2$, the two-form we will use in terms of α and μ is

$$\frac{-d\alpha\wedge d\mu}{e^{-iC_1}\alpha\mu} = i2sin(C_1)da \wedge d\bar{a} + 2Re\left(\left[R_{,u} - e^{iC_1}\frac{R_{,uu}}{R_{,u}}\right] da \wedge du\right). \tag{8.21}$$

Instead of ν we pick only a piece of it which cannot be expressed in terms of previous invariants,

$$\begin{aligned} V &= -\frac{e^{iC_1}(\zeta - Z_i)\dot{C}}{\zeta_{,a}\mu^2}\left(\Delta\mu + \mu^2\right) - Z_{1,uu}\bar{\alpha}e^{-a-3\bar{a}}\frac{e^{iC_1}\dot{C}}{\mu^2} \\ &= \left[-\frac{(\zeta - Z_i)\dot{C}}{\zeta_{,a}}\left(\frac{R_{,uu}}{R_{,u}^2} + e^{iC_1}\right) - \frac{\dot{C}}{e^{iC_1}}\frac{Z_{1,uu}}{\zeta_{,a}R_{,u}^2}\right]. \end{aligned} \tag{8.22}$$

Where $\dot{C}(C_1) = -1$ if $C_1 = \pi/2,\ 3\pi/2 \bmod 2\pi$ and $\dot{C}(C_1) = e^{2iC_1} + 1$ otherwise. It will be helpful to pick out the subcase when $C_1 = \pi/2,\ 3\pi/2 \bmod 2\pi$, as $\frac{\dot{C}(\zeta - Z_i)}{\zeta_{,a}} = a$. In the case of $C_1 = \pi/2 \bmod 2\pi$, taking the wedge product of the above two-form with the differential of V yields the following equation:

$$-2aR_{,u}^2\left(\frac{R_{,uu}}{R_{,u}^2}\right)_{,u} - \left(R_{,u}^4 + R_{,uu}^2 - 2e^{C_2 - iR}Z_{1,uu}R_{,u}^2 + 2ie^{C_2 - iR}R_{,u}^2\left(\frac{Z_{1,uu}}{R_{,u}^2}\right)\right).$$

This must vanish if we require V to be functionally dependent on α and μ. Notice that by setting this to zero we get two separate equations, the coefficient of the a-linear term gives a differential equation for $R(u)$ while the remaining piece gives constraints on Z_1. Solving for $R(u)$ yields two possibilities, depending on whether $R_{,uu}$ vanishes or not:

$$R_a(u) = C_5 u, \quad R_b(u) = C_5 ln(u), \quad C_5 \in \mathbb{R}. \tag{8.23}$$

If $Z_{1,uu} = 0$ it follows that $C_5 = 0$ which cannot happen as we have assumed $R_{,u} \neq 0$. Plugging each possibility into the remaining equation gives a special form for Z_1 respectively,

$$Z_{1a} = -(c_6 - \tfrac{iC_5 u}{2})\tfrac{e^{iC_5 u}}{e^{C_2}}, \quad Z_{1b} = -(c_6 - \tfrac{(1+iC_5)}{2}ln(u))\tfrac{u^{iC_5}}{e^{C_2}}, c_6 \in \mathbb{C} \tag{8.24}$$

The case with $C_1 = 3\pi/2 \bmod 2\pi$ is similiar to the above except the constants now have a negative sign. These both correspond to known G_2 metrics given in table [22], the first case belonging to $f(\zeta, u) = Ce^{4e^{C_2}(\zeta+c)e^{-iC_5 u}}$ $C \in \mathbb{C}$, and the second corresponding to $f(\zeta, u) = Ce^{4e^{C_2}(\zeta+c)u^{-iC_5}}/u^2$. At the third iteration of the Karhlede algorithm the only potentially functionally independent invariant is $\Delta\mu/\mu^2$, which is constant:

$$\left(\frac{\Delta\mu}{\mu^2}\right)_a = 0, \quad \left(\frac{\Delta\mu}{\mu^2}\right)_b = iC_5^{-1}.$$

Assuming $C_1 \neq \pi/2,\ 3\pi/2 \bmod 2\pi$, we take the wedge product of (8.21) with the differential of (8.22) and require that this must vanish. Doing so we find a complicated expression from which one may obtain,

$$sin(C_1)e^{-(e^{2iC_1}+1)a}\,(R_{,u}R_{,uuu} - 2R_{,uu}) = F(u) \tag{8.25}$$

where $F(u)$ is a complicated expression involving the functions $R(u)$, $Z_{1,uu}(u)$ and their derivatives. This can only happen when either $C_1 = 0$ or $\pi \bmod 2\pi$ or R is either of the form

$$R_a(u) = C_7 u \ \ or \ \ R_b(u) = ln(C_7 u) + C_8, \quad C_7 \in \mathbb{R}. \tag{8.26}$$

Let us assume $C_1 = 0 \bmod 2\pi$, the right hand side of (8.25) gives a useful constraint,

$$Z_{1,uu}(R_{,u}^2 - R_{,uu}) = 0$$

If $Z_{1,uu} = 0$ we may transform the coordinates to set $Z_1 = 0$; this subcase corresponds to the known G_2 metric with $f(\zeta, u) = -e^{4R}ln(\zeta)$. Instead, if we assume that the differential equation for R vanishes, we find a simple form for $R(u)$:

$$R(u) = -ln(C_7 u), \quad C_7 \in \mathbb{R}.$$

Notice that in this case (8.21) vanishes, implying that α and μ are functionally dependent. This cannot happen as we have assumed that $d\alpha \wedge d\mu \neq 0$. All of these spacetimes have at most two functionally independent invariants and hence have $q = 3$ in the Karlhede algorithm. In the case that $C_1 \neq 0 \text{mod } 2\pi$, R is of the above form (8.26), and the vanishing of the left hand side of (8.25) gives a differential equation for $Z_{1,uu}$:

$$R_a \quad : \quad Z_{1,uuu} + \frac{-C_7}{sin(C_1)} Z_{1,uu} = 0 \tag{8.27}$$

$$R_b \quad : \quad uZ_{1,uuu} + \frac{C_7 + e^{iC_1} + isin(C_1)}{isin(C_1)} Z_{1,uu} \tag{8.28}$$

These are G_2 spacetimes since $\Delta\mu = 0$ in the first case and $\Delta\mu = -C_7^{-1}\mu^2$ in the second case, so that at third order there are no new functionally independent invariants. $\square$

As a byproduct we have proved another lemma that will help narrow down the class of $\Delta\alpha \neq 0$ pp-wave spacetimes for which $q = 4$ in the Karlhede algorithm

Lemma 8.3.5. *In the case that $\Delta\alpha \neq 0$ the wedge product $d\alpha \wedge d\mu \wedge d\nu$ vanishes if and only if $C_1 = 0$ or π modulo 2π in equation* (8.10) *and $R(u)$ is of the form*

$$R(u) = e^{iC_1} ln(C_7 u), \quad C_7 \in \mathbb{R}. \tag{8.29}$$

In general, $d\alpha \wedge d\nu \wedge d\bar{\nu} \neq 0$ for arbitrary complex-valued $Z_{1,uu}(u)$.

Using this result, we examine the lone coefficient of the triple wedge product of the differentials of α, ν and $\bar{\nu}$, to pin down those spacetimes with $\Delta\alpha \neq 0$ and $q = 4$ by requiring this to vanish. Doing so we find the following differential equation

$$Z_{1,uu}\bar{Z}_{1,uu} + u(Z_{1,uu}\bar{Z}_{1,uuu} + Z_{1,uuu}\bar{Z}_{1,uu} = 0,$$

substituting $Z_{1,uu} = e^{R(u)}e^{ih(u)}$ we find that $R(u) = -2ln(u) + C$ and that there are no conditions on $h(u)$:

$$Z_{1,uu} = \frac{e^C e^{ih(u)}}{u^2}$$

In general these spacetimes will have α, ν and $\Delta\nu$ as three functionally independent invariants, however for particular $h(u)$ only two will be functionally independent. To determine the form for $h(u)$ we take the triple wedge product of the differentials of these three invariants and look at the only non-zero coefficient:

$$h_{,uu}u + h_{,u} = 0.$$

Solving this simple ordinary differential equation we find that if $Z_{1,uu}$ is of the form

$$Z_{1,uu} = Cu^{-2+iC_0}e^{iC_1},$$

these will be G_2 spacetimes. Summarizing these results in a lemma we have proven one-fourth of Theorem (8.1.1) by taking (8.10) and solving for $f(\zeta, u)$.

Lemma 8.3.6. *Those spacetimes with* $\Delta\alpha \neq 0$ *and* $q = 4$ *in the Karlhede algorithm have the following form:*

$$f(\zeta, u) = \frac{1}{u^2}(ln(z) + Ce^{ih(u)}\zeta), \quad h(u) \neq C_0 ln(u) + C_1, C \in \mathbb{R}$$

Next, suppose that $\mu = 0$, the vanishing of μ leaves ν and its conjugate as possible candidates for the second functionally independent invariant. To ensure there is only one invariant at this stage we require that $d\alpha \wedge d\nu \wedge d\bar{\nu} = 0$. Yet again, we pick off the piece of ν that cannot be written in term of previous invariants, V

$$V = Z_{2,uu}e^{-a-3\bar{a}}.$$

With this invariant we will be able to prove the final lemma from which Theorem (8.1.1) follows trivially:

Lemma 8.3.7. *Those PP-wave spacetimes with* $d\alpha \wedge d\bar{\alpha} = \mu = 0$*, with coordinates chosen such that* ν *takes the form given in* (8.15)*, will have three functionally dependent invariants* $\{\alpha, V, \Delta V\}$ *for an arbitrary choice of a real-valued function* $B(u)$ *if and only if* $Z_{2,uu} \neq 0$ *and*

$$C_3 \neq 0, \quad \epsilon = 1 : Z_{2,uu} = ce^{B(u)(-4C_3+I)}B(u) \neq \tfrac{ln(u)}{2C_3}, \tag{8.30}$$

$$C_3 = 0, \quad \epsilon = 1 : Z_{2,uu} = e^{iB(u)}, \quad Z_{2,uu} \neq ce^{iCu}, \quad C \in \mathbb{R}, \tag{8.31}$$

$$\epsilon = 0 : Z_2 = B(u), \quad Z_{2,uu} \neq Cu^{-2}. \tag{8.32}$$

Proof. Using this invariant, we construct a purely imaginary two-form by scaling $dV \wedge d\bar{V}$,

$$\frac{dV \wedge d\bar{V}}{V\bar{V}} = -8da \wedge d\bar{a} - 2iIm(3[ln(Z_{2,uu})]_{,u} - [ln(\bar{Z}_{2,uu})]_{,u})da \wedge du)$$

and next we wedge product this with the one-form,

$$\frac{d\alpha}{\alpha} = da - \left(1 - \frac{\epsilon}{\frac{1}{2} + iC_3}\right) d\bar{a},$$

then by denoting $Z' = ln(Z_{2,uu})$, the above becomes

$$\epsilon\left(2Z'_{,u} + 2iIm(Z'_{,u})\right) - 4iIm(Z'_{,u}) + 8C_3 Im(Z'_{,u}) = 0.$$

There are two cases to consider here, depending on $Z_{2,uuu}$. If this vanishes, the above equation is satisfied and these spacetimes admit G_2's since $\Delta(V)$ is the only invariant that could be functionally independent and it vanishes. These belong to the third entry in the table 24.2 in [22]. Choosing $Z' = A(u) + iB(u)$, we find that in the case with $\epsilon = 0$ the imaginary piece of the equation implies $B(u)$ is a constant and hence

$$Z_{2,uu} = e^{A(u)}.$$

Supposing that $\epsilon = 1$ we find that

$$A = -4C_3 B, \quad and \quad B = B$$

and we may write this constraint as

$$Z_{2,uu} = c(e^{B(u)})^{-4C_3+i}, c \in \mathbb{C}.$$

To determine the functional independence of α, V and ΔV, in the case that $\epsilon = 1$, we will work with a different third order invariant

$$W = \frac{\Delta V}{V^2(1-4C_3)} = B^{4C_3-1-i} B_{,u} e^{2\bar{a}}$$

. Taking the triple wedge product of α, V and W we wish to find and avoid all B such that this 3-form vanishes, i.e. all B satisfying,

$$(1 + i4C_3)B_{,uu} + (2C_3 + i8C_3^2)B_{,u} = 0 \tag{8.33}$$

When $C_3 \neq 0$ we may solve this equation to find the following form for B

$$B(u) = \frac{ln(u)}{2C_3}, \quad C_4 \in \mathbb{R},$$

by translating and scaling u, the form of $Z_{2,uu}$ is then

$$Z_{2,uu} = c(u)^{-2+\frac{I}{2C_3}}. \tag{8.34}$$

Those spacetimes in this class with $C_3 = 0$ admit a subclass of G_2 spacetimes. Setting $C_3 = 0$ in (8.33)

$$B_{,uu} = 0. \tag{8.35}$$

Solving this equation, and noting that $Z_{2,uu} = (e^{B(u)})^i$ we find that

$$Z_{2,uu} = ce^{iC_{11}u}, \quad C_{11} \in \mathbb{R}.$$

Spacetimes satisfying the above constraints are G_2's since the wedge product of $d\alpha$, $d\nu$ and $d\Delta(\nu)$ vanishes automatically. These correspond to the third entry of the Kundt-Ehlers' table with $\kappa = iC_{11}$. Thus if we allow $Z_2(u)$ to satisfy (8.31), these spacetimes will be $G1$'s with the possibility of $q = 4$ in the Karlhede algorithm. Now to consider the $\epsilon = 0$ case. There is a subcase containing G_2's, supposing the triple wedge products of $d\alpha$, $d\nu$, and $d\Delta(\nu)$ vanish, Z_2 satisfies

$$2Z_{2,uuuu}Z_{2,uu} = 3Z_{2,uuu}^2 \tag{8.36}$$

integrating yields

$$Z_{2,uu} = C_{12}u^{-2}.$$

These correspond to the second entry in the Kundt-Ehlers' table with $\kappa = 0$. Again, by restricting Z_2 to be any other class of functions, the spacetimes described by (8.32) will admit a G_1, and they will have $q = 4$ in the Karlhede algorithm. $\square$

8.4 The Invariant Classification of All Vacuum PP-wave Spacetimes

During the investigation of the G_1 spacetime with $q = 4$ in the Karlhede algorithm we have calculated a list of Cartan invariants for the degenerate subcases of the vacuum PP-wave spacetimes with $\alpha \neq 0$. To list these we define notation, $(t_1, t_2, t_3, ...t_n)$, where t_i denotes the number of functionally independent invariants at i-th iteration of the Karlhede algorith, and n denotes the last iteration of the algorithm required to determine the canonical form for the spacetime. Listing all possibilities for the various isometry groups admitted by the PP-wave spacetimes, we denote those found in the previous analysis by a star:

- G_1 spacetimes: $(0, 2, 3, 3)$, $(0, 1, 3, 3)^*$ and $(0, 1, 2, 3, 3)^*$;
- G_2 spacetimes: $(0, 2, 2)$ and $(0, 1, 2, 2)$;
- G_3 spacetimes: $(0, 1, 1)$;

- G_5 spacetimes: $(0, 1, 1)$;
- G_6 spacetimes: $(0, 0)$;

These invariants are the spin-coefficients of a particular coframe (8.15), found by boosting and spatially rotating the standard null coframe so that the Weyl curvature component $\Psi_4 = 1$ and then determining which spin-coefficient α or γ may be set to zero. In the cases dealt with in section (8.3) α was non-zero and by null rotating about o^A we may always set $\gamma = 0$. This approach may be applied to the remaining subcases of the Karlhede classification of PP-wave spacetimes with $\alpha \neq 0$ and $\alpha = 0$ G_n, $n > 3$.

8.4.1 $\alpha \neq 0$ Case:

In this case, one may simplify the components of the first and second derivatives of the Weyl tensor to only a few spin-coefficients:

$$D_1\Psi : (\alpha, \bar{\alpha}), \quad D_2\Psi : (\mu, \bar{\mu}, \nu, \bar{\nu}, \bar{\delta}\alpha, \delta\bar{\alpha}).$$

As all spin coefficients are used in the first and second derivatives of Ψ, we conclude that for all $n > 3$, the n-th derivatives will only involve the spin coefficients and their frame derivatives up to m-th order $m < n$. It will be helpful to pick coordinates, $(\zeta, \bar{\zeta}, u, v)$ to express the frame derivatives and their corresponding spin-coefficients. The class of coordinate transformations that preserve the form of this metric are:

$$\zeta' = e^{i\alpha'}\zeta + h(u), v' = v + 2Re(h_{,uu}\bar{\zeta}) \quad H' = H - 2Re(h_{,uu}\bar{\zeta}) + \bar{h}_{,u}h_u; \tag{8.37}$$
$$u' = \frac{(u+u_0)}{a'}, \; v' = a'v, \qquad H' = {a'}^2 H; \tag{8.38}$$
$$v \; = v + g(u), \qquad H' = H - g_{,u} \tag{8.39}$$

where a', $u_0 \in \mathbb{R}$. Denoting $a = \frac{1}{4}ln(f_{,\zeta\zeta})$, we apply the following boost and spatial rotation to the standard Kundt coframe,

$$m' = e^{a-\bar{a}}m, \quad \ell' = e^{a+\bar{a}}\ell, \quad n' = e^{-a-\bar{a}}n, \tag{8.40}$$

in this coframe the first order invariants α and γ become:

$$\alpha = e^{a-\bar{a}}\bar{a}_{,\bar{\zeta}}, \quad \gamma = e^{-a-\bar{a}}\bar{a}_{,u}. \tag{8.41}$$

Let us first suppose that $\alpha \neq 0$, then by making a null rotation with $E = e^{-2\bar{a}}\frac{a_{,u}}{a_{\zeta}}$

$$m'' = m' + \bar{E}\ell', \quad \ell'' = \ell', \quad n'' = n' + E\bar{m}' + \bar{E}m' + E\bar{E}\ell' \quad , \tag{8.42}$$

we may eliminate γ and leave α unchanged. In this coframe the frame derivatives and the remaining spin-coefficients may be written in terms of α, f and a, and their conjugates:

$$\begin{gathered} \delta = \tfrac{1}{\alpha}\left(\bar{a}_{\bar{\zeta}}\tfrac{\partial}{\partial\zeta} - \bar{a}_{,u}\tfrac{\partial}{\partial v}\right), D = e^{a+\bar{a}}\tfrac{\partial}{\partial v}, \\ \Delta = \tfrac{e^{-a-\bar{a}}}{|\alpha|^2}\left(|a_{,\zeta}|^2\tfrac{\partial}{\partial u} - a_{,u}\bar{a}_{,\bar{\zeta}}\tfrac{\partial}{\partial\zeta} - \bar{a}_{,u}a_{,\zeta}\tfrac{\partial}{\partial\bar{\zeta}} + (|a_{,u}|^2 - (f+\bar{f})|a_{,\zeta}|^2)\tfrac{\partial}{\partial v}\right). \end{gathered} \tag{8.43}$$

The second order invariants may now be written as

$$\begin{aligned} \mu &= \frac{e^{-2a}}{\bar{\alpha}}\left(a_{,u\zeta} - \frac{a_{,u}a_{,\zeta\zeta}}{a_{,\zeta}}\right), \\ \nu &= \frac{e^{-4a}}{\bar{\alpha}^3}\left(\bar{f}_{,\bar{\zeta}}(a_{,\zeta})^3 + 2a_{,u}a_{,\zeta}a_{,\zeta u} - a_{,uu}(a_{,\zeta})^2 - a_{,\zeta\zeta}(a_{,u})^2\right), \\ \bar{\delta}\alpha &= \bar{\alpha}^{-1}(-|a_{,\zeta}|^2\alpha + a_{,\zeta}\bar{a}_{,\bar{\zeta}\bar{\zeta}}e^{a-\bar{a}}). \end{aligned} \tag{8.44}$$

All non-zero spin coefficients and $\bar{\delta}\alpha$ are now involved as invariants. Thus at each iteration of the Karlhede algorithm we may reduce this set to all of the n-th order frame derivatives of α, μ, ν, $\bar{\delta}\alpha$ and their conjugates. By taking the arbitrary form of the vacuum PP-wave spacetime and picking which invariants are functionally dependent at each iteration of the Karlhede algorithm it is possible to recreate the various entries in table 24.2 in [22] along with special subcases of the G_2 metrics and G_1 metrics. The results of this analysis is summarized in tables (8.2) - (8.4), where the work involved has been relegated to the sections (8.7), (8.8) and (8.9).

8.4.2 $\alpha = 0$ Case:

In a similar manner those vacuum PP-wave spacetimes with $\alpha = 0$ may be classified. By choosing the same boosted and spatial rotated null coframe in which $\Psi_4 = 1$, the components of the covariant derivatives of the Weyl tensor Ψ will be expressed in terms of the spin-coefficient γ and its invariants:

$$D_1\Psi : (\gamma, \bar{\gamma}), \quad D_2\Psi : (\bar{\delta}\gamma, \delta\bar{\gamma}, \Delta\gamma).$$

	$f(\zeta,u)$	1^{st} order	2^{nd} order	Classifying Functions
G_1	$f(\zeta,u)$	$(\alpha,\bar{\alpha})$	ω: $=$ μ,ν or $\bar{\delta}\alpha$	(8.87)
G_1-I-0	$g_1(u)ln\zeta + g_2(u)\zeta$	$(\alpha,\bar{\alpha})$	**V**	(8.90)
G_1-II-0	$F(g_1(u)^{ik}\zeta)g_1(u)^2 + g_2(u)\zeta$	$(\alpha,\bar{\alpha})$	**X**, (8.50)	(8.88)
G_1-II-1	$F(u^{ik}\zeta)u^2 + g_1(u)\zeta$	$(\alpha,\bar{\alpha})$	**X**, (8.50)	(8.91), (8.92)
G_1-III-0	$F(e^{ig_1(u)}\zeta) + g_2(u)\zeta$	$(\alpha,\bar{\alpha})$	**X**, (8.50)	(8.89)
G_1-III-1a	$F(e^{iu}\zeta) + g_1(u)\zeta$	$(\alpha,\bar{\alpha})$	**X**, (8.50)	(8.93), (8.94)
G_1-III-1b	$F(\zeta) + g_1(u)\zeta$	$(\alpha,\bar{\alpha})$	**X**, (8.50)	(8.95)
G_2-I	$g(u)ln(\zeta)$	$(\alpha,\bar{\alpha})$;		(8.109)
G_2-II	$u^{-2}F(\zeta u^{ik})$	$(\alpha,\bar{\alpha})$;		(8.110), (8.111)
G_2-IIIa	$F(\zeta e^{iku})$	$(\alpha,\bar{\alpha})$;		(8.112), (8.113)
G_2-IIIb	$f(\zeta)$	$(\alpha,\bar{\alpha})$;		(8.114), (8.115)

Table 8.1: Summary of analysis in Case with $\alpha \neq 0$ and $d\alpha \wedge d\bar{\alpha} \neq 0$. Here $a, k \in \mathbb{R}$ and $A(u)$ is a complex valued function.

It may be shown by direct calculation or manipulation of the NP equations that γ must be a function of the retarded time coordinate u. As no new functions are introduced in covariant differentiation of the Weyl tensor $q = 2$ at most in the Karhlede algorithm. in fact there are only two cases [1] $(0, 1, 1)$ and $(0, 0)$. We revert to the coframe found by boosting and spatially rotating the Kundt coframe, (8.42). The vanishing of α in (8.41) requires that $\bar{a}_{,\bar{\zeta}} = 0$, so that $\bar{f}_{,\bar{\zeta}\bar{\zeta}\bar{\zeta}} = 0$, giving a solution of the form

$$f(\zeta, u) = A(u)\zeta^2. \tag{8.45}$$

Rewriting γ in (8.41),

$$\gamma = \frac{1}{4\sqrt{A\bar{A}}} ln(\bar{A})_{,u}$$

[1] We note that only in the case that $\alpha = 0$ is $(0, 0)$ possible.

	$f(\zeta,u)$	1^{st} order	2^{nd} order	3^{rd} order	Classifying Invariants
G_1-a-0	$e^{\frac{e^C e^{iR'(u)}}{(\zeta - z(u))}}$	α	$(\mathbf{W},\mathbf{X})$		(8.96)
G_1-b-0	$(C_0 e^{iR'(u)}(\zeta - z(u)))^{2iC_1}$	α	$(\mathbf{U},\mathbf{X})$		(8.97)
G_1-c-0	$e^{C(\zeta - z(u))}$	α	$(\mathbf{W}_0,\mathbf{X})$		(8.98)
G_1-d-0	$[C(\zeta - z(u))]^{4iC_0}$	α	$(\mathbf{R},\mathbf{P})$		(8.99)
G_1-d_0-0	$e^{iC} ln(\zeta - z(u))$	α	$(\mathbf{R}_0,\mathbf{P})$		(8.100)
G_1-b-1	$\frac{C^2 e^{iC_0}}{u^2} ln\zeta - \frac{e^{C_1} e^{iZ(u)}}{u^2}\zeta$	α	$\mathbf{W}_1$	$\mathbf{Y}$	(8.101)
G_1-c-1	$e^{C\zeta} + e^{iC_0} e^{R(u)}\zeta$	α	$\mathbf{U}_0$	$\mathbf{Y}_0$	(8.102)
G_1-d-1	$(C\zeta)^{4iC_0} + c_1 e^{(-4C_0+I)Z(u)}\zeta$	α	$\mathbf{V}$	$\mathbf{W}_2$	(8.103)
G_1-d_0-1	$e^{iC} ln(\zeta) + e^{C_0} e^{iR(u)}\zeta$	α	$\mathbf{Z}$	$\mathbf{Y}_1$	(8.106)

Table 8.2: Summary of all G_1 spacetimes arising in the case, $\alpha \neq 0$ and $d\alpha \wedge d\bar{\alpha} = 0$. These are in addition to the NP equations (8.2) and equations ((8.6), (8.5)) or ((8.6), (8.14)).

and supposing that $\gamma(u) = \gamma_1(u) + i\gamma_2(u)$ with $A = r(u)e^{i\theta(u)}$ we may solve for A in terms the invariant real-valued functions in γ,

$$\frac{r_{,u}}{r^2} - i\frac{\theta_{,u}}{r} = 4\gamma_1 + 4i\gamma_2.$$

By integrating we find the expressions for r and θ which may be summarized as a lemma.

Lemma 8.4.1. *For any pp-wave spacetime expressed in terms of a canonical coframe with $\alpha = 0$ and $\gamma = \gamma_1 + i\gamma_2, \quad \neq 0$ we may express the canonical form for $f(\zeta,u)$ as*

$$A = re^{i\theta}; \quad r(u) = [C_0 - \textstyle\int 4\gamma_1 du]^{-1}, \quad \theta(u) = -\textstyle\int 4r\gamma_2 du + C_1, \quad C_0, C_1 \in \mathbb{R} \tag{8.46}$$

Here, γ is the only functionally independent invariant and the essential classifying functions are $\bar{\gamma}(u)$ and $\Delta\gamma(u)$ expressed in terms of γ. If γ is constant, there are two possibilities for $A(u)$ depending on where γ lies in the complex plane, these are given in (8.5).

To show equivalence for two metrics in this class $g, \ g_o$ we just need to examine the classifying manifolds $(\gamma_o; \bar{\gamma}_o, \Delta\gamma_o)$ and $(\gamma; \bar{\gamma}, \Delta\gamma)$, if they overlap the spacetimes must

	$f(\zeta, u)$	1^{st} order	2^{nd} order	Classifying Invariants
G_2-a-0-0	$(C_0\zeta e^{iu})^{2iC} + C_1 e^{iu}\zeta$	α	μ	(8.116)
G_2-a-0-1	$(C_0 u^{-iC}\zeta)^{2iC_1} + C_4 u^{-iC}\zeta)u^{-2}$	α	μ	(8.117)
G_2-a-0-2	$e^{C(\zeta+e^{iC_0})e^{iu}}$	α	μ	(8.118)
G_2-a-0-3	$u^{-2}e^{4e^{C_0}(\zeta+c)u^{iC_1}}$	α	μ	(8.119)
G_2-b-0	$-e^{R(u)} ln(\zeta)$	α	μ	(8.120)
G_2-b-1	$u^{-2}(Ce^{iC_0} ln\zeta - c_1 u^{-iC_2}\zeta)$	α	$\mathbf{V}$	(8.121)
G_2-c-0	$e^{C\zeta} + c_0\zeta$	α	$\mathbf{V}$	(8.122)
G_2-c-1	$e^{C\zeta} + c_0 u^{-2}\zeta$	α	$\mathbf{U}_1$	(8.123)
G_2-d-0	$(C\zeta)^{2iC_1} + c_0\zeta$	α	$\mathbf{W}_4$	(8.124)
G_2-d-1	$(C\zeta)^{2iC_1} + c_0 u^{-2-\frac{i}{2C_1}}\zeta$	α	$\mathbf{V}$	(8.125)
G_2-d_0-1	$-e^{iC} ln(\zeta) - \zeta e^{iC_0 u}$	α	$\mathbf{W}_5$	(8.126)

Table 8.3: Summary of all G_2 spacetimes arising in the case, $\alpha \neq 0$ and $d\alpha \wedge d\bar{\alpha} = 0$. These are in addition to the NP equations (8.2) and equations ((8.6), (8.5)) or ((8.6), (8.14)).

be equivalent, and inequivalent otherwise. Any two equivalent Kundt metric with $\alpha = 0$, γ non-constant, may be transformed using the coordinate transformations (8.37) and (8.39). The only freedom left are rotations of ζ and rescaling and translating u. Supposing $u' = \frac{1}{c}(u+d)$, $\zeta' = e^{ia}\zeta$ we find in the new coordinates that $A'(u') = (e^{ia}c)^2 A(cu'-d)$ and

$$\gamma' = \frac{1}{|c|}\gamma(cu'-d) \tag{8.47}$$

Thus if two vacuum Kundt metrics with $\alpha = 0$ are equivalent, we may equate the invariant γ for both spacetimes and determine c and d. This can be used for diffeomorphisms that do not preserve Kundt form, like the one used in equation (24.49) in [22] to switch from Kundt form to Rosen form, our invariant approach gives equivalence and provides a complete integration of the metric function $A(u)$. In light of the results in [[98], [99]], where a general formalism is introduced for studying arbitrary polarization states of pp-wave spacetimes with $\alpha = 0$ expressed in Rosen-form.

	$f(\zeta, u)$	1^{st} order	Classifying Invariants
G_3-b-0	$u^{-2}e^{C}e^{2iC_0}ln\zeta$	α	(8.126)
G_3-c-0	$e^{2C\zeta}$	α	(8.127)
G_3-d-0	$(C\zeta)^{iC_1}$	α	(8.128)
G_3-d_0-0	$e^{iC}ln\zeta$	α	(8.130)

Table 8.4: Summary of all G_3 spacetimes arising in the case, $\alpha \neq 0$ and $d\alpha \wedge d\bar{\alpha} = 0$. These are in addition to the NP equations (8.2) and equations ((8.6), (8.5)) or ((8.6), (8.14)).

	$f(\zeta, u)$	1	2
G_5	$\frac{(A(u))^4}{2}\zeta^2$, (8.46)	γ; (8.47)	$\Delta\gamma$
G_6-a	$\frac{u^{\frac{iC_1}{C_0}-1}}{16C_0^2}\zeta^2$	$\gamma = C_0 + iC_1$	
G_6-b	$e^{iC_1 u}\zeta^2$	$\gamma = i\frac{C_1}{4}$	

Table 8.5: Summary of cases with $\alpha = 0$. Here, $C_0, C_1 \in \mathbb{R}$, $A(u)$ is a complex valued function.

Lemma (8.4.1) allows for any novel solution found by this formalism to be expressed in Kundt coordinates.

8.5 Symmetry Methods for PP-wave Spacetimes with $d\alpha \wedge d\bar{\alpha} \neq 0$

For this class of spacetimes, an analysis of the vanishing of the necessary wedge products does not readily produce tractable equations. However if one considers the fact that a killing vector must annihilate all invariants, and that all invariants may be expressed in this subcase in terms of α and $\bar{\alpha}$: the normalization $\hat{\Delta}\alpha \to 0$ via a null rotation about ℓ will be a helpful choice as it will then be a linear combination of Killing vectors.

$$A = \frac{\bar{\delta}\alpha}{\alpha^2}, \tag{8.48}$$
$$B = \mu A - \bar{\mu}, \tag{8.49}$$
$$X = B/(A\bar{A} - 1), \tag{8.50}$$
$$Y = (3 - A)\nu - 1/\alpha + (\Delta\mu + \mu^2)/\bar{\alpha}, \tag{8.51}$$
$$\hat{\nu} = \nu + X(\mu + 2\bar{X})/\bar{\alpha}, \quad A\bar{A} \neq 1, \tag{8.52}$$
$$\hat{\Upsilon} = \Delta(\hat{\nu}/\bar{X}) - 2\hat{\nu} + 1/\alpha - 4iXX_2/\bar{\alpha}, \tag{8.53}$$
$$\hat{\Delta} = \Delta + \frac{X}{\bar{\alpha}}\delta + \frac{\bar{X}}{\alpha}\bar{\delta} + \frac{|X|^2}{|\alpha|^2}D \tag{8.54}$$

Using the coordinates $a = \frac{1}{4}lnf_{,\zeta\zeta}$, we will specify the form of invariants and the invariant coframe

Proposition 8.5.1. *Suppose that $\alpha \neq 0$. If the normalization $\Psi_4 = 1$ and $\gamma = 0$ holds then*

$$\alpha = \frac{e^{a-\bar{a}}}{\bar{Z}_{\bar{a}}}, \quad \zeta = Z(a,u), \tag{8.55}$$
$$\mu = e^{-a-\bar{a}}L_u, \quad L = logZ_{,a}, \tag{8.56}$$
$$M = \frac{e^{-2\bar{a}}}{\bar{Z}_{,\bar{a}}}L_u, \tag{8.57}$$
$$\nu = e^{-a-3\bar{a}}\left(Z_{,uu} + \frac{\bar{\Phi}_{\bar{a}}}{\bar{Z}_{\bar{a}}}\right), \quad \Phi(a,u) = f(\zeta,u) \tag{8.58}$$
$$A = -1 - \bar{L}_{\bar{a}}, \tag{8.59}$$
$$\omega^1 = \frac{da}{\bar{\alpha}} \tag{8.60}$$
$$\omega^2 = e^{a+\bar{a}}du, \tag{8.61}$$
$$\omega^4 = e^{-a-\bar{a}}((f + \bar{f} + Z_{,u}\bar{Z}_{,u})du + dv - Z_{,u}d\bar{\zeta} + \bar{Z}_{,u}d\zeta); \tag{8.62}$$

we also have the relations

$$\mu = X\bar{A} + \bar{X}, \tag{8.63}$$
$$\delta A = 0. \tag{8.64}$$

Given $Q(a,u)$ then

$$\delta Q = \bar{\alpha}Q_{,a}, \quad \Delta Q = e^{-a-\bar{a}}Q_{,u} \tag{8.65}$$

For our purposes we will study those spacetimes where $A\bar{A} \neq 1$ $B \neq 0$ Expressing our vector field relative to a frame

$$V = V^1\delta + \bar{V}\bar{\delta} + V^3\Delta + V^4 D, \quad \bar{V}^3 = V^3, \bar{V}^4 = V^4 \quad ;$$

the following proposition gives the constraints for the invariant coframe with $\Delta\alpha = 0$ additionally.

Proposition 8.5.2. *Suppose that $\mathbf{A}\bar{\mathbf{A}} \neq 1$, then every vector field satisfying*

$$\mathfrak{L}_V \alpha = \mathfrak{L}_V \bar{\alpha} = 0, \;\; V^3 \neq 0. \tag{8.66}$$

has the form $V = a\hat{\Delta} + bD$, $a \neq 0$. If $\mathbf{A}\bar{\mathbf{A}} = 1$, but $\boldsymbol{B} \neq 0$ then (8.66) *does not have a solution. If $\mathbf{A}\bar{\mathbf{A}} = 1$ and $\boldsymbol{B} = 0$, then there is a 1-paramater family of solutions to* (8.66)*.*

Proof. Applying a null rotation, $\hat{\Delta} = \Delta + \bar{E}\delta + E\bar{\delta} + E\bar{E}D$, we produce the following linear system:

$$\begin{bmatrix} \mathbf{A}\alpha & \bar{\alpha} \\ \alpha & \bar{\mathbf{A}}\bar{\alpha} \end{bmatrix} \begin{pmatrix} E \\ \bar{E} \end{pmatrix} = \begin{pmatrix} \mu \\ \bar{\mu} \end{pmatrix}. \tag{8.67}$$

In the case that $\mathbf{A}\bar{\mathbf{A}} \neq 1$, the solution is

$$E = \frac{\bar{\mathbf{A}}\bar{\mu} - \mu}{(\mathbf{A}\bar{\mathbf{A}} - 1)\alpha} = \frac{\bar{X}}{\alpha}.$$

If $\mathbf{A}\bar{\mathbf{A}} = 1$, the system has rank 1, the system will be consistent if and only if,

$$\begin{vmatrix} \mathbf{A}\alpha & \bar{\mu} \\ \alpha & \mu \end{vmatrix} = \alpha\mathbf{B} = 0$$

$\square$

To study the existence of Killing vectors distinct from $\ell = \frac{\partial}{\partial v}$, we use the fact that along such a vector the invariant coframe $\omega^a = \{m, \bar{m}, \ell, n\}$ is covariantly constant, that is $V^1 \neq 0$, $V^3 \neq 0$ and

$$\mathfrak{L}_V \omega^1 = \mathfrak{L}_V \omega^2 = 0 \tag{8.68}$$

$$\mathfrak{L}_V \omega^4 = 0 \tag{8.69}$$

To continue we first study the implications of (8.68), and then relate this fact back to the vanishing of wedge products:

Proposition 8.5.3. *The vanishing of the Lie derivatives* $\mathfrak{L}_V\omega^1 = \mathfrak{L}_V\omega^2 = 0$ *will occur if and only if* $C = \bar{\alpha}V^1$ *is a constant while* V^3 *satisfies*

$$V^3\bar{\mu} = C + \bar{C}A, \tag{8.70}$$

$$\delta V^3 = \bar{\alpha}V^3, \tag{8.71}$$

$$\Delta V^3 = -C - \bar{C}. \tag{8.72}$$

Proof. To prove this, consider the Lie derivatives

$$\begin{aligned}
\mathfrak{L}_V\alpha &= \alpha(C + \bar{C}A - V^3\bar{\mu}), &(8.73)\\
\mathfrak{L}_V(\bar{\alpha}\omega^1) &= d(\mathfrak{L}_V da) = dC, &(8.74)\\
\mathfrak{L}_V\omega^3 &= (\delta V^3 - \bar{\alpha} - \bar{\alpha}V^3)\omega^1 + (\bar{\delta}V^3 - \alpha V^3)\omega^2 + (\Delta V^3 + C + \bar{C})\omega^3. &(8.75)
\end{aligned}$$

The vanishing of these quantities lead to the above identities as the coframe is linearly independent. □

Proposition 8.5.4. *If* $B \neq 0$ *then* (8.68) *is equivalent to the conjunction of*

$$d\alpha \wedge d\bar{\alpha} \wedge d\mu = 0, \tag{8.76}$$

and the condition

$$d(B/\bar{B}) = 0. \tag{8.77}$$

Proof. If we require (8.68), then $\mathfrak{L}_V\alpha = \mathfrak{L}_V\bar{\alpha} = \mathfrak{L}_V\mu = 0$, as they are involved in the structure equations:

$$d\omega^1 = \alpha\omega^1 \wedge \omega^2 - \mu\omega^1 \wedge \omega^3, \tag{8.78}$$

$$d\omega^3 = (\bar{\alpha}\omega^1 + \alpha\omega^2) \wedge \omega^3, \tag{8.79}$$

$$d\omega^4 = (\bar{\mu} - \mu)\omega^1 \wedge \omega^2 + (\bar{\nu}\omega^1 + \nu\omega^2) \wedge \omega^3 - (\bar{\alpha}\omega^1 + \alpha\omega^2) \wedge \omega^4. \tag{8.80}$$

So, if 3 functions on a 4 dimensional manifold are annihilated by 2 independent vector fields, they must be functionally dependent and hence (8.76) holds. To show the constancy of the ratio, consider (8.70) and its conjugate. This function is real valued, equating the two yields

$$C(\mu - \bar{A}\bar{\mu}) = \bar{C}(A\mu - \bar{\mu}). \tag{8.81}$$

from this fact it follows that $B/\bar{B}$ is constant. Conversely suppose that (8.76) and (8.77) hold. By assumption $C/\bar{C} = B/\bar{B} = X\bar{X}$ for some constant C. Hence C/X is real and by setting $V = C/X\hat{\Delta}$ (where $\hat{\Delta}$ comes from Proposition (8.5.2)) we obtain a real vector field such that $\mathfrak{L}_V\alpha = 0$ and so $\mathfrak{L}_V\mu = 0$. For this choice $V^1 = C/\bar{\alpha}$ and so by Propostion (8.5.3) $\mathfrak{L}_V\omega^1 = 0$, applying this fact to the differential of ω^1:

$$0 = \mathfrak{L}_V d\omega^1 = -\mu\omega^1 \wedge \mathfrak{L}_V\omega^3.$$

Thus $\mathfrak{L}_V\omega^3$ must vanish as $d\mathfrak{L}_V\omega^1 = \mathfrak{L}_V d\omega^1 = 0$. □

8.5.1 The G_1 Spacetimes

To continue we study the conditions imposed by the form of the vector field.

Proposition 8.5.5. *There exists a vector field such that*

$$\mathfrak{L}_V\omega^1 = \mathfrak{L}_V\omega^3 = 0, \quad V^1 = 0, V^3 \neq 0; \tag{8.82}$$

if and only if $\mu = 0$. This class of metrics has the form $f(\zeta, u) = F(\zeta) + g\zeta$.

Proof. Suppose that (8.82) holds, by (8.73): $\mathfrak{L}_V\alpha = -V^3\alpha\mu = 0$. Thus $\mu = 0$. To prove the converse consider the component $V^3 = e^{a+\bar{a}}$ by (8.65) the relations (8.71) and (8.72) follow. To prove the second part, consider the form of μ above in (8.56), if this were to vanish $Z_{ua} = 0$, and so $Z = F(a)$ for some arbitrary function of ζ. Solving for a and integrating yields the desired form: □

Proposition 8.5.6. *There exists a vector field such that*

$$\mathfrak{L}_V\omega^1 = \mathfrak{L}_V\omega^3 = 0, \quad V^1 \neq 0, V^3 = 0; \tag{8.83}$$

if and only if $A = 1$.

Proof. Suppose that (8.83) holds, by (8.73), $C + \bar{C} - 0$ and hence $C = \bar{\alpha}V^1$ is imaginary. Applying (8.70), $C + \bar{C}A = 0$ implying $A = 1$. Alternatively if $A = 1$, (8.70)-(8.73) to hold, it suffice to set $V^1 = i/\bar{\alpha}$, $V^3 = 0$. □

In the more general cases where V^1 and V^3 are both non-zero, the above analysis will not give up results so easily. Instead we will first consider the G_1 spacetimes where either $A = 1$ and $B = 0$, $A\bar{A} \neq 1$ and $B_1 = 0$, or $A\bar{A} = 1$ and $B/\bar{B}$ is constant.

Proposition 8.5.7. *Suppose that $B_1 \neq 0$ and $\bar{A}A \neq 1$, the following are equivalent: (i) $\frac{B_2}{B_1} = k$ is a real constant and (ii) $f(\zeta, u) = F(h^{ik}\zeta)h^2 + g\zeta$*

Proof. Let $C = 1 + ik$ then

$$\frac{B}{\bar{B}} = \frac{1 + i\frac{B_2}{B_1}}{1 - i\frac{B_2}{B_1}} \tag{8.84}$$

implying that B/C is real valued. Supposing that (i) holds, the equations (8.49), (8.56) and (8.59) this becomes

$$\tfrac{C}{B}(A\bar{A} - 1) = \tfrac{C}{\bar{\mu}}\left(1 + \tfrac{\bar{B}}{B}A\right) = \tfrac{C + \bar{C}A}{\bar{\mu}} = (\tfrac{-2k - CL_{,a}}{L_{,u}})e^{a+\bar{a}}.$$

The piece inside brackets is real and holomorphic in a, thus it must be independent of this complex coordinate,

$$L_{,u} + (1 + ik)h_1 L_{,a} = -2ikh_1, \tag{8.85}$$

where $h_1 = h_1(u) \neq 0$ is real-valued. Conversely, (8.85) with $h_1 \neq 0$ implies condition (i). Solving for $f_{,\zeta\zeta}$ and integrating yields:

$$\begin{gathered} L = F(a - (1 + ik)h_2) - 2ikh_2, \quad h_2' = h_1 \\ Z = F(a - (1 + ik)h_2)e^{-2ikh_2} \\ a = (1 + ik)h_2 + F(e^{2ikh_2}\zeta) \\ f_{,\zeta\zeta} = h^{2+2ik}F(h^{ik}\zeta), \quad h = e^{2h_2} \\ f = F(h^{ik}\zeta)h^2 + g\zeta. \end{gathered}$$

□

Proposition 8.5.8. *Suppose that $A \neq 1$, the following are equivalent (i) $B_1 = 0$ and (ii) $f(\zeta, u) = F(e^{ih}\zeta) + g\zeta$.*

Proof. By (8.49), (8.56) and (8.59), condition (i) is equivalent to

$$e^{a+\bar{a}}((L_{,a} + 2)\bar{L}_{,u} + L_{,u}(\bar{L}_{,\bar{a}} + 2)) = 0.$$

where $L_{,a} \neq -2$ by assumption. To satisfy this, it must equal some real function multiplied by i, simplification gives

$$L_{,u} i h_1 L_{,a} = 2h_1$$

Solving for $f_{,\zeta\zeta}$ and integrating twice yields the desired form:

$$L = F(a + ih_2) + 2ih_2, h_2' = h_1$$
$$Z = F(a + ih_2)e^{2ih_2}$$
$$a = -ih_2 + F(e^{-2ih_2}\zeta)$$
$$f_{,\zeta\zeta} = e^{-4ih_2}F(\zeta e^{-i2h_2})$$
$$f = \tilde{F}(\zeta e^{-2ih_2}) + g\zeta, \quad \tilde{F}'' = F.$$

□

Proposition 8.5.9. *The following are equivalent: (i) $A = 1$ and (ii) $f(\zeta, u) = g_1 log\zeta + g_2\zeta$*

Proof. By (8.59) condition (i) is equivalent to $L = -2a + g$, hence

$$Z = e^{-2a+g}, \quad -2a = log\zeta - g, \quad f_{,\zeta\zeta} = e^{2g}\zeta^{-2}, \quad f = g_1 log\zeta + g_2\zeta.$$

in this case $B = \mu - \bar{\mu}$, hence B_1 automatically vanishes in this case. □

Proposition 8.5.10. *Suppose that $B_1 \neq 0$, $A\bar{A} \neq 1$, the following are equivalent: (i) condition* (8.68) *holds, (ii) $B_2/B_1 = k$, $\Delta X_1 = 2X_1^2$, and (iii) $f(\zeta, u) = F(u^{ik}\zeta)u^{-2} + g\zeta$.*

Proof. Suppose that (i) holds, then the reality of V^3 in (8.5.3)

$$C\bar{B} = \bar{C}B$$

as $B \neq 0$ we have $\mu \neq 0$ as well. Solving for the ratio and factoring out B_1 we find

$$\frac{C}{\bar{C}} = \frac{1 + i\frac{B_2}{B_1}}{1 - i\frac{B_2}{B_1}}.$$

Furthermore since $X/\bar{X} = B/\bar{B}$ it follows by (8.70)

$$\frac{1}{X_1} = \frac{C}{X} = \frac{C(B + \bar{B}A)}{B\bar{\mu}} = \frac{C + \bar{C}A}{\bar{\mu}} = V^3$$

condition (ii) follows by using (8.72) on this expression. To show (ii) implies (iii), we know by assumption that $f(\zeta, u) = F(u^{ik}\zeta)u^{-2} + g\zeta$ belongs to a particular G_1 class given in proposition (8.5.7) the equation below (8.84) yields

$$\frac{1}{X_1} = \frac{e^{a+\bar{a}}}{h_1}$$

where h_1 is real valued, applying Δ to this yields

$$\Delta(\tfrac{1}{X_1}) = \left(\tfrac{1}{h_1}\right)'$$
$$(1/h_1)' + 2 = 0$$
$$h_1 = \tfrac{-1}{2u}$$

Therefore,

$$L_{,u} - \left(\frac{1+ik}{2u}\right) L_{,a} = \frac{ik}{u}$$

by following the same method in proposition (8.5.7) we find $h = u^{-1}$ which specializes the form of the metric in proposition (8.5.7). Lastly we show (iii) leads to (i), to do this, set $V^1 = C/\bar{\alpha}$ with $C = 1 + ik$ while for V^3,

$$V^3 = \frac{1}{X_1} = -2ue^{a+\bar{a}}$$

Conditions (8.70) and (8.71) follow by (8.65). □

Proposition 8.5.11. *Suppose that $B_1 = 0$ $\mu \neq 0$ and $A\bar{A} \neq 1$. The following are equivalent: (i)* (8.68) *holds, (ii) $\Delta X_2 = 0$, and (iii) $f(\zeta, u) = F(e^{iu}\zeta) + g\zeta$.*

Proof. To show (i) leads to (ii), notice that$C = \bar{\alpha}V^1 \neq 0$ is a constant such that $Im(B/C) = 0$.Since $B_1 = 0$ we have $C = i$, without loss of generality. Thus $V^3 = \frac{1}{X_2}$ and $\Delta X_2 = 0$ by (8.71). Next, (ii) implies (iii), by assumption $f(\zeta, u) = F(e^{iu}\zeta) + g\zeta$ belongs to the class given in proposition (8.5.8), since $B = iB_2$ we have

$$\frac{-i}{X_2} = \frac{1}{\bar{X}} = \frac{\bar{B} + A\bar{B}}{\bar{B}\mu} = \frac{1 - \bar{A}}{\mu}.$$

By proposition (8.5.8)

$$e^{-a-\bar{a}}/X_2 = -i(2 + L_{,a})/L_{,u} = -1/h_1$$

where $h_1 = -2h' \neq 0$ is real. Since $\Delta X_2 = 0$ this implies that h_1 is a constant. Lastly to show (iii) implies (i), set $V^1 = C/\bar{\alpha}$ where $C = i$ and $V^3 = \frac{1}{X_2} = -2ke^{a+\bar{a}}$, conditions (8.70) and (8.71) follow by (8.65). □

Proposition 8.5.12. *Suppose that $A\bar{A} \neq 1$, the following are equivalent: (i)* (8.68) *holds with $V^1 = 0$, (ii) $\Delta X_2 = 0$ and $\mu = 0$, and (iii) $f(\zeta, u) = F(\zeta) + g\zeta$.*

Proof. To show (i) leads to (ii), we invoke (8.5.5) and note that $\mu = 0$ in this case; next notice that $C = \bar{\alpha}V^1 \neq 0$ is a constant such that $Im(B/C) = 0$.Since $B_1 = 0$ we have $C = i$, without loss of generality. Thus $V^3 = \frac{1}{X_2}$ and $\Delta X_2 = 0$ by (8.71). Next, (ii) implies (iii), by assumption $f(\zeta, u) = F(\zeta) + g\zeta$ belongs to the class given in proposition (8.5.8), since $B = iB_2$ we have

$$\frac{-i}{X_2} = \frac{1}{\bar{X}} = \frac{\bar{B} + A\bar{B}}{\bar{B}\mu} = \frac{1 - \bar{A}}{\mu}.$$

By proposition (8.5.8)

$$e^{-a-\bar{a}}/X_2 = -i(2 + L_{,a})/L_{,u} = -1/h_1$$

where $h_1 = -2h' \neq 0$ is real. Since $\Delta X_2 = 0$ this implies that h_1 is a constant. The vanishing of μ implies $L_{,u} = 0$ so that by integrating for L in the proof of (8.5.8) give the required form for $f\zeta, u)$. Lastly to show (iii) implies (i), set $V^1 = C/\bar{\alpha}$ where $C = i$ and $V^3 = e^{a+\bar{a}}$, conditions (8.70) and (8.71) follow by (8.65). $\square$

8.5.2 The G_2 Spacetimes

To recover the solutions introduced by Kundt and Ehlers, we impose the condition (8.69), which will be equivalent to the vanishing of the last triple wedge product $d\alpha \wedge d\bar{\alpha} \wedge d\nu$. By Proposition (8.82) the (0,2,...) solution with $V^3 = 0$ belongs to some particular form of the metric given in Proposition (8.5.9), while the $V^1 = 0$ vector field is given by the metric given in Proposition (8.5.12). The remaining (0,2) solutions are special cases of the metrics in Propositions (8.5.10), (8.5.11). This specialiation arises from the vanishing of the invariants Υ and $\hat{\Upsilon}$ which are defined above.

Proposition 8.5.13. *The class of metrics of the form* $g_1 log\zeta + g_2\zeta$ *have* $d\alpha \wedge d\bar{\alpha} \wedge d\nu = 0$ *if and only if: (i)* $Y = 0$ *and (ii)* $g_2 = 0$

Proof. By Proposition (8.5.6), $V = Im(\alpha^{-1}\bar{\delta})$ annihilates ω^1, ω^2 α and μ. Hence the vanishing of the wedge product is equivalent to $\mathfrak{L}_V \nu = 0$. By direct calculation (8.51)

$$(\delta\nu)/\alpha - (\bar{\delta}\nu)/\bar{\alpha} = -1/\alpha + 2\nu + (\mu^2 + \Delta\mu)/\bar{\alpha} = Y$$

thus (i) follows from the vanishing of the wedge product. A direct calculation yields

$$\alpha\bar{Y} = -g_2(g_1\bar{g}_1)^{-1/2}$$

proving that (i) is equivalent to (ii). □

Proposition 8.5.14. *The class of metrics with* $f(\zeta,u) = F(u^{-ik}\zeta)u^{-2} + g^{-2-ik}\zeta$, $k \neq 0$ *have* $d\alpha \wedge d\bar{\alpha} \wedge d\nu = 0$ *if and only if: (i)* $\hat{\Upsilon} = 0$ *and (ii)* $g' = 0$.

Proof. By proposition (8.5.7) $V = X_1^{-1}\hat{\Delta}$ annihilates $\omega^1, \omega^3, \alpha$ and μ the Newman-Penrose equations imply

$$dlog\alpha = \bar{\alpha}\omega^1 + A\alpha\omega^2 - \mu\omega^3. \tag{8.86}$$

Thus $\mathfrak{L}_V A = \mathfrak{L}_V X = 0$ as well. Let $\hat{\nu}$ be the invariant defined in (8.52), by proposition (8.5.3) and Newman-Penrose equations:

$$\delta X = -\bar{\alpha}X,\ \bar{\delta}X = -\alpha X,\ \Delta X = 2XX_1,$$
$$\delta(\hat{\nu}/\bar{X} = -4iX_2, \bar{\delta}(\hat{\nu}/\bar{X}) = (1-2\hat{\nu}\alpha)/\bar{X}$$
$$\hat{\Delta}(\hat{\nu}/\bar{X}) = \Delta(\hat{\nu}/\bar{X}) - 4iXX_2/\bar{\alpha} + (1-2\hat{\nu}\alpha)/\alpha = \hat{\Upsilon} \quad .$$

where the latter is the invariant defined in (8.53). This proves the equivalence of the vanishing of the wedge product and (i). A direct calculation shows the final equivalence of (i) and (ii):

$$\bar{\hat{\Upsilon}} = 4u\frac{X_1^2}{X}g_1'F''(u^{-ik}\zeta)^{-1/2}$$

Notice that if $g' = 0$ this a constant and the term, $Cu^{-2-ik}\zeta$ may be absobed into $F(u^{-ik}\zeta)$. Furthermore $\hat{\nu}$ arises as the transformed invariant made by a null rotation to set $\hat{\Delta}\alpha = 0$. □

Proposition 8.5.15. *The class of metrics with* $f(\zeta,u) = F(e^{iu}\zeta) + ge^{iu}\zeta$ *have* $d\alpha \wedge d\bar{\alpha} \wedge d\nu = 0$ *if and only if (i)* $\hat{\Upsilon} = 0$ *and (ii)*$g' = 0$.

Proof. By proposition (8.5.8) $V = X_1^{-1}\hat{\Delta}$ annihilates $\omega^1, \omega^3, \alpha$ and μ the Newman-Penrose equations imply

$$d(log\alpha) = \bar{\alpha}\omega^1 + A\alpha\omega^2 - \mu\omega^3.$$

Thus $\mathfrak{L}_V A = \mathfrak{L}_V X = 0$ as well. Let $\hat{\nu}$ be the invariant defined in (8.52), by proposition (8.5.3) and Newman-Penrose equations:

$$\delta X = -\bar{\alpha}X,\ \bar{\delta}X = -\alpha X,\ \Delta X = 0,$$
$$\delta(\hat{\nu}/\bar{X} = -4iX_2, \bar{\delta}(\hat{\nu}/\bar{X}) = (1-2\hat{\nu}\alpha)/\bar{X}$$
$$\hat{\Delta}(\hat{\nu}/\bar{X}) = \Delta(\hat{\nu}/\bar{X}) - 4iXX_2/\bar{\alpha} + (1-2\hat{\nu}\alpha)/\alpha = \hat{\Upsilon} \quad .$$

where the latter is the invariant defined in (8.53). This proves the equivalence of the vanishing of the wedge product and (i). A direct calculation shows the final equivalence of (i) and (ii):

$$\bar{\hat{\Upsilon}} = 4u\bar{X}g_1'F''(e^{iu}\zeta)^{-1/2}$$

$\square$

Proposition 8.5.16. *The class of metrics of the form $f(\zeta, u) = F(\zeta) + g(u)\zeta$ have $d\alpha \wedge d\bar{\alpha} \wedge d\nu = 0$ if and only if: (i) $\Delta\nu = 0$ and (ii) $g' = 0$.*

Proof. By proposition (8.5.5) a multiple of Δ annihilates $\omega^1, \omega^2, \alpha$ and μ. Thus the vanishing of the triple wedge product is equivalent to (i). Using the metric form given in (8.5.5), a direct calculation of $\Delta\nu$ gives

$$\Delta\bar{\nu} = e^{-2a}g'/F''$$

showing the equivalence (i) and (ii). $\square$

8.6 Conclusions

In our search for those vaccuum PP-wave spacetimes in which the fourth-order covariant derivatives of Ψ are required to classify them entirely we have produced an invariant classification of the vacuum PP-wave spacetimes, which will be finer than the analysis of each spacetime's isometry group alone. The summary of this invariant approach to classification is given in tables (8.1), (8.2) - (8.4) and (8.5) which relate the invariant classes determined in Figure (8.1) to an appropriate lemma in the sections (8.7), (8.8) and (8.9). Each lemma gives a canonical form for the metric along with the functionally independent invariants arising at each order and the essential functionally dependent invariants that are required for the sub-equivalence problem. Although a particular coordinate system has been chosen for the lemmas and their proofs to express the functionally independent invariants at each order, their classifying functions - the essential and non-essential functionally dependent invariants given in the proof - will hold regardless of the coordinate system chosen as long as the canonical coframe is used.

To find the canonical coframe for any PP-wave metric with arbitrary $f_o(\zeta, u)$, one chooses a null coframe, then by applying an appropriate spin and boost one may always set $\Psi_4 = 1$. At zeroeth order this reduces the dimension of the isotropy group, $dim(\mathbf{H}_0) = 2$ and eliminates all functionally independent invariants available from the Weyl tensor Ψ. At second order, the covariant derivatives of Ψ involve the spin-coefficients $\{\alpha, \gamma\}$ and their conjugates. There are two cases here depending on α vanishing or not.

- If $\alpha \neq 0$ we may use a null rotation to set $\gamma = 0$, this reduces $dim(\mathbf{H}_0)$ to zero and this will be the canonical coframe. Using the decision tree in Figure (8.1) one may use the wedge product on the differentials of the spin-coefficients to determine which invariant subclass the metric belongs to.

- The vanishing of α produces a differential equation for $f_o(\zeta, u)$, so that it must be of the form (8.45). As null-rotations about ℓ will not affect the invariant structure, we note the dimension of the isotropy group will be two, $dim(\mathbf{H}_0) = 2$. Thus we have many choices for our canonical frame in this case. In general these spacetimes will only require up to second order for the Karlhede algorithm, while in the special case that $\gamma \equiv \ Constant$ only first order is required, as depicted in Figure (8.1).

With the invariant subclass of $f_o(\zeta, u)$ found one may use tables [(8.1), (8.4)] for those spacetimes with $\alpha \neq 0$ and table (8.5) when $\alpha = 0$. To find the canonical form of the metric and determine the coordinate transformation used to switch between it and the original f_o. Here are some simple examples that illustrate the approach:

1. Consider $f_o(\zeta, u) = e^{4e^{C_0}(\zeta + c)e^{iCu}}$ and another metric $f(\zeta, u) = f(\zeta e^{iku})$ where $f(Z),\ Z = \zeta e^{iku}$ is some analytic function belonging strictly to $G_2 - III$, i.e., when $\bar{\delta} \neq \pm\alpha^2$. By inspection of either table 24.2 in [22] or table (8.1) it is apparent that f_o belongs to this class as well. Is there some coordinate transform that changes f_o to f? To answer this question we need only look at the first order invariants α_0 and α in both coordinate systems. From (8.112) and (8.118): $\bar{\alpha}_o = \alpha_o$ while $\bar{\alpha} \neq \alpha$, implying that the spacetimes are distinct.

2. As we require $f(Z)$ in $G_2 - III$ to be analytic we may always expand this function as a power series where the coefficients a_n of $Z^n,\ n \geq 0$ are functions

of u. In light of this fact, consider the almost trivial example $f(Z) = (\zeta e^{iku})^n$ for $n \geq 0$.

- If $n = 0$ or 1, it is always possible to set $f(Z)$ to zero using the coordinate transformations (8.37) - (8.39). This would violate the constraint that $\Psi_4 \neq 0$ and so these cases cannot occur.
- If $n = 2$ this metric belongs to the $G_6 - b$ class with $C_1 = 2k$, these spacetimes have no functionally independent invariants and only two essential constants $\Phi_4 = 1$, $\gamma = \frac{ik}{2}$.
- If $n = m + 2,\ m \geq 0$, we see by inspection that this belongs to $G_2 - III$. Using (8.112) we may calculate α

  $$\alpha = \frac{m}{4} Z^{\frac{m}{4}} \bar{Z}^{-\frac{m}{4}-1}$$

  Taking α and its conjugate we construct simpler invariants

  $$A = \frac{m^2}{16}(\alpha\bar{\alpha})^{-1} = Z\bar{Z}, \quad B = (\alpha/\bar{\alpha})^{\frac{4}{2m+4}} = Z/\bar{Z}$$

  we see that Z may be written as

  $$Z = \left(\frac{m^2}{16} \alpha^{\frac{-2m}{2m+4}} \bar{\alpha}^{\frac{-2(2m+4)}{2m+4}} \right)^{\frac{1}{2}}.$$

  Solving for $f(Z)$ in terms of α and its conjugate the second order invariants μ and ν in (8.112) imply k and m are essential constants.

3. Consider another metric from $G_2 - III$, with the analytic function $f(Z) = Z^{m+2} + cZ$, $c \in \mathbb{C}$, and $Z = \zeta e^{iku}$. These spacetimes have exactly the same form for α and its conjugate. Naturally $Z(\alpha, \bar{\alpha})$ will be the same, implying m is an essential constant. Only by looking at the second order invariant ν does one find that k and c are essential constants.

4. For any contrived example constructed by taking a canonical form for $f(\zeta, u)$ and applying the permitted Kundt-coordinate transformations in some arbitrary way, we may always use the above flow-chart to determine its original canonical form in some coordinate system. And then by comparing the Cartan invariants in either system determine the constant and functions involved in the coordinate transformation.

8.7 G_1 Spacetimes

By classifying the G_1 spacetimes according to Cartan invariants instead of the isometry groups, we find several interesting examples. In particular four subclasses of the PP-wave spacetimes require the fourth covariant derivatives of the Weyl tensor. Proving that the upper-bound introduced by Collins [36] for vacuum PP-wave spacetimes is sharp.

The largest and most general G_1 case, $(0,2,3,3)$, where $f(\zeta,u)$ is any analytic function not listed below has the following invariant structure. The first order invariants, α and its conjugate are functionally independent, at second order there can be at most one new functionally independent invariant. We find one real-valued function and two complex-valued classifying functions by writing the remaining second order invariants in term of the three invariants. At third order, regardless of which invariant is chosen, there will only be one new classifying function not arising from the NP field equations (8.2) at second and third order respectively:

$$\begin{gathered} D\alpha = 0,\ \Delta\alpha = -\bar{\mu}\alpha,\ \delta\alpha = \alpha\bar{\alpha}, \\ D\mu = 0,\ \bar{\delta}\mu = -\alpha\mu,\ \Delta\mu - \delta\nu = -\mu^2 + \bar{\alpha}\nu. \\ D\nu = 0,\ \bar{\delta}\nu = -3\alpha\nu + 1. \end{gathered}$$

Thus, at third order the only new information comes from $\delta\mu$ or $\Delta\nu$ or $\bar{\delta}\bar{\delta}\alpha$ respectively if μ, ν or $\bar{\delta}\alpha$, is chosen to be the third functionally independent invariant. In total these spacetimes require at most seven non-trivial real-valued classifying functions in terms of the three invariants. For each of the possible triples we find the following classification functions which are not generic to type N vacuum spacetimes:

$$\begin{gathered} (\alpha,\bar{\alpha},\mu) : (\bar{\mu},\nu,\bar{\delta}\alpha,\delta\mu), \\ (\alpha,\bar{\alpha},\nu) : (\bar{\nu},\mu,\bar{\delta}\alpha,\Delta\nu), \\ (\alpha,\bar{\alpha},\bar{\delta}\alpha) : (\delta\bar{\alpha},\mu,\nu,\bar{\delta}\bar{\delta}\alpha). \end{gathered} \tag{8.87}$$

For spacetimes with $(0,1,3,3)$, there are two cases to consider, with two subcases each. These arise from $\Delta\alpha$ being zero or otherwise, looking at (8.10) and (8.11), we note that the subcases of each differentiate between a linear or an exponential form for $\zeta(a,u)$. This will be reflected in the classifying functions.

8.7.1 $(0,2,3,3)$; $\Delta\alpha \neq 0$ case:

Lemma 8.7.1. *The PP-wave metric belonging to the $G_1 - II - 0$ class will have the canonical form*

$$f(\zeta,u) = F(g_1(u)^{ik}\zeta)g_1(u)^2 + g_2(u)\zeta$$

where g_1 and g_2 are arbitrary complex functions. The functionally independent invariants are α, its conjugate and any one of the second order invariants $\bar{\delta}\alpha$, μ, ν and their conjugates - we will work with $X = \mu\bar{\delta}\alpha\alpha^{-2} - \bar{\mu}$ from (8.50)*. These invariants are expressed in the $(a,\bar{a})$ coordinates,*

$$\alpha = -\tfrac{1}{2}\tfrac{e^{a-\bar{a}}}{Z_{,a}}, \quad X = e^{-a-\bar{a}}\left(\tfrac{L_{,u}+L_{,u}\bar{L}_{,\bar{a}}+\bar{L}_{,u}}{L_{,a}+\bar{L}_{,\bar{a}}+L_{,a}\bar{L}_{,\bar{a}}}\right)$$

where $\zeta = Z(a,u) = (F'')^{-1}(a - \frac{(1+ik)}{2}ln(g_1))g_1^{-2ik}$ and $L = ln(Z_{,a})$.The essential classifying functions are the remaining second order invariants

$$\begin{aligned}\mu &= e^{-a-\bar{a}}log(Z_{,a})_{,u},\\ \bar{\delta}\alpha &= \bar{\alpha}\left(-\frac{1}{2}\frac{e^{a-\bar{a}}}{Z_{,a}}\right)_{,a},\\ \nu &= e^{-a-3\bar{a}}\left(Z_{,uu} + \frac{\bar{\Phi}_{\bar{a}}}{\bar{Z}_{\bar{a}}}\right), \quad \Phi(a,u) = f(\zeta,u),\end{aligned} \tag{8.88}$$

and the third order invariants $\delta X, \bar{\delta}X$ and ΔX.

Proof. See the proof to Proposition (8.5.7). □

Lemma 8.7.2. *The PP-wave metric belonging to the $G_1 - III - 0$ class will have the canonical form*

$$f(\zeta,u) = F(e^{ig_1(u)}\zeta) + g_2(u)\zeta$$

where g_1 and g_2 are arbitrary complex functions. The functionally independent invariants are α, its conjugate and any one of the second order invariants $\bar{\delta}\alpha$, μ, ν and their conjugates - we will work with $X = \mu\bar{\delta}\alpha\alpha^{-2} - \bar{\mu}$ from (8.50)*. These invariants are expressed in the $(a,\bar{a})$ coordinates,*

$$\alpha = -\tfrac{1}{2}\tfrac{e^{a-\bar{a}}}{Z_{,a}}, \quad X = e^{-a-\bar{a}}\left(\tfrac{L_{,u}+L_{,u}\bar{L}_{,\bar{a}}+\bar{L}_{,u}}{L_{,a}+\bar{L}_{,\bar{a}}+L_{,a}\bar{L}_{,\bar{a}}}\right)$$

where $\zeta = Z(a,u) = F(a - ig_1)e^{-2ig_1}$ *and* $L = ln(Z_{,a})$. *The essential classifying functions are the remaining second order invariants*

$$\begin{aligned}
\mu &= e^{-a-\bar{a}} log(Z_{,a})_{,u}, \\
\bar{\delta}\alpha &= \bar{\alpha}\left(-\frac{1}{2}\frac{e^{a-\bar{a}}}{Z_{,a}}\right)_{,a}, \\
\nu &= e^{-a-3\bar{a}}\left(Z_{,uu} + \frac{\bar{\Phi}_{\bar{a}}}{\bar{Z}_{\bar{a}}}\right), \quad \Phi(a,u) = f(\zeta,u),
\end{aligned} \tag{8.89}$$

and the third order invariants $\delta X, \bar{\delta}X$ *and* ΔX.

Proof. See the proof to Proposition (8.5.8). □

Lemma 8.7.3. *The PP-wave metric belonging to the* $G_1 - I - 0$ *class will have the canonical form*

$$f(\zeta,u) = g_1(u)ln\zeta + g_2(u)\zeta$$

where g_1 *and* g_2 *are arbitrary complex functions. The functionally independent invariants are* α, *its conjugate and* $\mathbf{V}$ *where these invariants are expressed in the* $(a,\bar{a})$ *coordinates*

$$\alpha = -\frac{1}{2}e^{a+\bar{a}}\bar{g}_1^{-\frac{1}{2}}, \quad \mathbf{V} = e^{-a-3a}$$

The essential classifying functions are then

$$\begin{aligned}
\mu &= -\frac{1}{4}\frac{g_{1,u}}{g_1^{\frac{3}{2}}}\alpha^{-1}, \\
\bar{\delta}\alpha &= \alpha^2, \\
\nu &= \frac{1}{2}\alpha^{-1} + \frac{1}{2}\left(\frac{\mu^2}{\bar{\alpha}} + \frac{\Delta\mu}{\bar{\alpha}}\right) - g_2\mathbf{V},
\end{aligned} \tag{8.90}$$

and the third order invariants $\delta\mathbf{V}, \bar{\delta}\mathbf{V}$ *and* $\Delta\mathbf{V}$.

Proof. See the proof to Proposition (8.5.9). □

Lemma 8.7.4. *The PP-wave metric belonging to the* $G_1 - II - 1$ *class will have the canonical form*

$$f(\zeta,u) = F(u^{ik}\zeta)u^2 + g(u)\zeta$$

where g is an arbitrary complex function. The functionally independent invariants are α, its conjugate and any one of the second order invariants $\bar{\delta}\alpha$, μ, ν and their conjugates - we will work with $X = \mu\bar{\delta}\alpha\alpha^{-2} - \bar{\mu}$ from (8.50)*. These invariants are expressed in the $(a, \bar{a})$ coordinates,*

$$\alpha = -\tfrac{1}{2}\tfrac{e^{a-\bar{a}}}{Z_{,a}}, \quad X = e^{-a-\bar{a}}\left(\tfrac{L_{,u}+L_{,u}\bar{L}_{,\bar{a}}+\bar{L}_{,u}}{L_{,a}+\bar{L}_{,\bar{a}}+L_{,a}\bar{L}_{,\bar{a}}}\right)$$

where $\zeta = Z(a,u) = (F'')^{-1}(a - \frac{(1+ik)}{2}ln(u))u^{-2ik}$ and $L = ln(Z_{,a})$.The essential classifying functions are the remaining second order invariants

$$\begin{aligned}
\mu &= e^{-a-\bar{a}}log(Z_{,a})_{,u}\\
\bar{\delta}\alpha &= \bar{\alpha}\left(-\frac{1}{2}\frac{e^{a-\bar{a}}}{Z_{,a}}\right)_{,a}\\
\nu &= e^{-a-3\bar{a}}\left(Z_{,uu} + \frac{\bar{\Phi}_{\bar{a}}}{\bar{Z}_{\bar{a}}}\right), \quad \Phi(a,u) = f(\zeta, u).
\end{aligned} \tag{8.91}$$

At third order,

$$\delta X = \bar{\alpha}X, \quad \bar{\delta}X = \alpha X,$$

and $Re(\Delta X) = \Delta X_1$ has the particular form

$$\Delta X_1 = 2X_1^2. \tag{8.92}$$

Proof. See the proof to Proposition (8.5.10). □

Lemma 8.7.5. *The PP-wave metric belonging to the $G_1 - III - 1a$ class will have the canonical form*

$$f(\zeta, u) = F(e^{iu}\zeta) + g(u)\zeta$$

where g is an arbitrary complex function. The functionally independent invariants are α, its conjugate and any one of the second order invariants $\bar{\delta}\alpha$, μ, ν and their conjugates - we will work with $X = \mu\bar{\delta}\alpha\alpha^{-2} - \bar{\mu}$ from (8.50)*. These invariants are expressed in the $(a, \bar{a})$ coordinates,*

$$\alpha = -\tfrac{1}{2}\tfrac{e^{a-\bar{a}}}{Z_{,a}}, \quad X = e^{-a-\bar{a}}\left(\tfrac{L_{,u}+L_{,u}\bar{L}_{,\bar{a}}+\bar{L}_{,u}}{L_{,a}+\bar{L}_{,\bar{a}}+L_{,a}\bar{L}_{,\bar{a}}}\right)$$

where $\zeta = Z(a,u) = F(a - ig_1)e^{-2ig_1}$ *and* $L = ln(Z_{,a})$. *The essential classifying functions are the remaining second order invariants*

$$\begin{aligned}
\mu &= e^{-a-\bar{a}} log(Z_{,a})_{,u} \\
\bar{\delta}\alpha &= \bar{\alpha}\left(-\frac{1}{2}\frac{e^{a-\bar{a}}}{Z_{,a}}\right)_{,a} \\
\nu &= e^{-a-3\bar{a}}\left(Z_{,uu} + \frac{\bar{\Phi}_{\bar{a}}}{\bar{Z}_{\bar{a}}}\right), \quad \Phi(a,u) = f(\zeta, u)
\end{aligned} \tag{8.93}$$

At third order

$$\delta X = \bar{\alpha} X, \quad \bar{\delta} X = \alpha X,$$

and $Re(\Delta X) = \Delta X_1$ *has the particular form*

$$\Delta X_1 = 0. \tag{8.94}$$

Proof. See the proof to Proposition (8.5.11). □

8.7.2 $(0, 2, 3, 3)$; $\Delta\alpha = 0$ case:

Lemma 8.7.6. *The PP-wave metric belonging to the* $G_1 - III - 1b$ *class will have the canonical form*

$$f(\zeta, u) = F(\zeta) + g(u)\zeta$$

where g *is an arbitrary complex function. The functionally independent invariants are* α, *its conjugate and any one of the non-zero second order invariants* $\bar{\delta}\alpha$, ν *and their conjugates - we work with* $\bar{\delta}\alpha$. *These invariants are expressed in the* $(a, \bar{a})$ *coordinates,*

$$\alpha = -\tfrac{1}{2}\tfrac{e^{a-\bar{a}}}{Z_{,a}}, \quad \nu = e^{-a-3\bar{a}}\left(\tfrac{\bar{\Phi}_{\bar{a}}}{\bar{Z}_{\bar{a}}}\right), \quad \Phi(a,u) = f(\zeta, u)$$

where $\zeta = Z(a,u) = F(a - ig_1)e^{-2ig_1}$ *and* $L = ln(Z_{,a})$. *The essential classifying functions are the remaining second order invariants*

$$\begin{aligned}
\mu &= 0, \\
\bar{\delta}\alpha &= bar\alpha\left(-\frac{1}{2}\frac{e^{a-\bar{a}}}{Z_{,a}}\right)_{,a},
\end{aligned} \tag{8.95}$$

and the third order invariants $\delta\nu$, $\bar{\delta}\nu$ *and* $\Delta\nu$.

Proof. See the proof to Proposition (8.5.12). □

8.7.3 $(0,1,3,3)$; $\Delta\alpha \neq 0$ case:

Lemma 8.7.7. *The PP-wave metric belonging to the $G_1 - a - 0$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = \frac{1}{16e^{2C}} e^{4e^{C}e^{-iR'(u)}(\zeta - z(u))}, C \in \mathbb{R}.$$

The functions $R'(u)$ and $z(u)$ may be any function except those listed in (8.23) *and* (8.24) *respectively. Using the special coordinates in* (8.3)*, the functionally independent invariants are*

$$\alpha = e^{a-\bar{a}}e^{C}e^{iR'(u)}, \quad \boldsymbol{W'} = R'(u) - \frac{i}{2} ln\left(z_{,uu}/\bar{z}_{,uu}\right), \quad \boldsymbol{X} = e^{a+\bar{a}}.$$

The essential classifying functions are

$$\begin{aligned} \bar{\alpha} &= e^{2C}\alpha^{-1}, \\ \bar{\delta}\alpha &= -\alpha^2, \\ \mu &= iR'_{,u}(\boldsymbol{W})\boldsymbol{X}^{-1}, \\ \nu &= (4\alpha)^{-1} - a\left(\frac{\mu^2}{\bar{\alpha}} + \frac{\Delta\mu}{\bar{\alpha}}\right) - z_{,uu}(\boldsymbol{W})\alpha e^{-iR'(\boldsymbol{W})}e^{-C}\boldsymbol{X}^{-2}. \end{aligned} \tag{8.96}$$

Proof. In the $(a, \bar{a}, u, v)$ coordinate system used in the previous section, these solutions correspond to those metrics with $\zeta(a, u)$ of the form (8.10) with $C_1 = \pi/2$ or $3\pi/2$ mod 2π. By solving for a and integrating with respect to ζ in the coordinate system where the ζ-linear piece of $f(\zeta, u)$ has been set to zero we find the canonical form for the metric. The first order invariants are

$$\alpha = e^{a-\bar{a}}e^{C}e^{iR'(u)}, \quad \bar{\alpha} = e^{2C}\alpha^{-1}.$$

The second order invariants are then

$$\mu = iR'_{,u}e^{-a-\bar{a}}, \quad \mathbf{V} = z_{,uu}e^{-a-3\bar{a}},$$
$$\bar{\mu} = -\mu, \quad \nu = (4\alpha)^{-1} - a\left(\tfrac{\mu^2}{\bar{\alpha}} + \tfrac{\Delta\mu}{\bar{\alpha}}\right) - \mathbf{V}, \quad \bar{\delta}\alpha = -\alpha^2.$$

To continue we construct an appropriate pair of second order invariants, by first making a gauge transformation $R'(u) = R(u) + \Psi(u)$ such that $e^{i\Psi} = (z_{,uu}/\bar{z}_{,uu})^{\frac{1}{2}}$

$$\mathbf{W} = iln\left[\sqrt{\frac{V}{\bar{V}}}\frac{e^{C}}{\alpha}\right] = R(u).$$

Notice that $z_{,uu}(u)$ may locally be written in terms of $\mathbf{W}$ and hence is a functionally dependent invariant, which will be a classifying function for the spacetime. To determine the second invariant, we use this fact and eliminate this term in V,

$$\mathbf{X} = \left| \frac{\mathbf{V}}{z_{,uu}(\mathbf{W})} \right|^{-\frac{1}{2}} = e^{a+\bar{a}}.$$

The third order invariants are now

$$\delta\mathbf{W} = 0, \quad \Delta\mathbf{W} = R_{,u}(\mathbf{W})\mathbf{X}^{-1}, \quad \delta\mathbf{X} = \bar{\alpha}\mathbf{X}, \quad \Delta\mathbf{X} = 0.$$

We note that no new classifying functions are introduced at this stage. With the analysis complete, we may write down the non-trivial classifying functions

$$\begin{aligned} \bar{\alpha} &= e^{2C}\alpha^{-1} \\ \mu &= iR'_{,u}(\mathbf{W})\mathbf{X}^{-1}, \\ \nu &= (4\alpha)^{-1} - a\left(\frac{\mu^2}{\bar{\alpha}} + \frac{\Delta\mu}{\bar{\alpha}}\right) - z_{,uu}(\mathbf{W})\alpha e^{-iR'(\mathbf{W})}e^{-C}\mathbf{X}^{-2}. \end{aligned}$$

□

Lemma 8.7.8. *The PP-wave metric belonging to the* $G_1 - b - 0$ *class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$,

$$f(\zeta, u) = \frac{(e^{2iC_1}+1)^2}{2(e^{2iC_1}-3)(e^{2iC_1}-1)} \left(-\frac{(e^{iC_0}e^{R'(u)})^{e^{iC_1}}}{e^{2iC_1}+1} \right)^{\frac{4}{e^{2iC_1}+1}} (\zeta - z(u))^{-2itan(C_1)}, \quad C_0 \in \mathbb{R}.$$

If $C_1 \neq 0, \pi/2, \pi, 3\pi/2 \bmod 2\pi$ *the real-valued functions* $R'(u)$ *and* $z(u)$ *are arbitrary. If* $C_1 = 0, \pi \bmod 2\pi$, $R'(u)$ *and* $z(u)$ *with* $z_{,uu} \neq 0$ *may be any functions except those in* (8.29) *and* (8.30) *respectively. Using the special coordinates in* (8.3), *the functionally independent invariants are*

$$\alpha = \left(\frac{e^{e^{iC_1}a + e^{-iC_1}\bar{a}} e^{iC_0}}{e^{R'(u)}} \right)^{e^{-iC_1}}, \quad \boldsymbol{U} = \left(\frac{R'^2_{,u}}{|z_{,uu}|} \right)^2, \quad \boldsymbol{X} = e^{a+\bar{a}}.$$

The essential classifying functions are

$$\begin{aligned}
\bar{\alpha} &= \alpha^{e^{2iC_1}} e^{-2ie^{iC_1}C_0}, \\
\bar{\delta}\alpha &= e^{-2iC_1}\alpha^2, \\
\mu &= e^{iC_1} R'_{,u}(\boldsymbol{U})\boldsymbol{X}^{-1}, \\
\nu &= -\frac{1}{\alpha(e^{-2iC_1}-3)} - \frac{(\zeta - Z_1)}{\zeta_{,a}}\left(\frac{\mu^2}{\bar{\alpha}} + \frac{\Delta\mu}{\bar{\alpha}}\right) \\
&\quad -z_{,uu}(\boldsymbol{U})\left[e^{-iC_0 - R'(\boldsymbol{U})}\alpha^{e^{-2iC_1}}\boldsymbol{X}^{-1}\right]^{-2e^{2iC_1}} X^{-1}, \\
\Delta\boldsymbol{U} &= \boldsymbol{U}_{,u}\boldsymbol{X}^{-1}.
\end{aligned} \tag{8.97}$$

Proof. Opting for the $(a, \bar{a}, u, v)$ coordinates, these solutions correspond to those metrics with $\zeta(a, u)$ of the form (8.10) with $C_1 \neq \pi/2 or 3\pi/2 mod 2\pi$. By solving for a and integrating with respect to ζ in the coordinate system where the ζ-linear piece of $f(\zeta, u)$ has been set to zero we find the canonical form for the metric. The particular choice of R' and Z in the instances that $C_1 = 0, \pi/2, \pi$ or $3\pi/2 mod 2\pi$ are necessary to avoid those PP-wave spacetimes with invariant count $(0, 1, 2, 3, 3)$.

The first order invariant α and its conjugate become

$$\alpha = \left(\frac{e^{e^{iC_1}a + e^{-iC_1}\bar{a}} e^{iC_0}}{e^{R'(u)}}\right)^{e^{-iC_1}}, \quad \bar{\alpha} = \alpha^{e^{2iC_1}} e^{-2ie^{iC_1}C_0}.$$

The second order invariants are,

$$\mu = e^{iC_1} R'_{,u} e^{-a-\bar{a}}, \quad \mathbf{V} = z_{,uu} e^{-a-3\bar{a}},$$
$$\bar{\mu} = e^{-2iC_1}\mu, \quad \nu = -\tfrac{1}{\alpha(e^{-2iC_1}-3)} - \tfrac{(\zeta - Z_1)}{\zeta_{,a}}\left(\tfrac{\mu^2}{\bar{\alpha}} + \tfrac{\Delta\mu}{\bar{\alpha}}\right) - \mathbf{V}, \quad \bar{\delta}\alpha = e^{-2C_1}\alpha^2 \quad .$$

For ease of calculating the third order invariants, we will pick two new second order invariants. The first arises from a ratio

$$\mathbf{U} = \frac{|\mu|^2}{|\mathbf{V}|} = \left(\frac{R'^2_{,u}}{|z_{,uu}|}\right)^2.$$

As before we may express $z_{,uu}$ in terms of $\mathbf{U}$, and using this we find the second invariant at this iteration:

$$\mathbf{X} = \left(\frac{e^{iC_1}\sqrt{|z_{,uu}|}U^{\frac{1}{4}}}{\mu}\right) = e^{a+\bar{a}}.$$

Calculating the third order invariants,

$$\delta\mathbf{U} = 0, \quad \Delta\mathbf{U} = \mathbf{U}_{,u}\mathbf{X}^{-1}, \quad \delta\mathbf{X} = \bar{\alpha}\mathbf{X}, \quad \Delta\mathbf{X} = 0.$$

The remaining functionally dependent invariant arises at third order by expressing $\mathbf{U}_{,u}$ in terms of $\mathbf{U}$. The essential functionally dependent invariants are

$$\begin{aligned}
\bar{\alpha} &= \alpha^{e^{2iC_1}} e^{-2ie^{iC_1}C_0} \\
\mu &= e^{iC_1} R'_{,u}(\mathbf{U}')\mathbf{X}^{-1}, \\
\nu &= -\frac{1}{\alpha(e^{-2iC_1} - 3)} - \frac{(\zeta - Z_1)}{\zeta_{,a}} \left(\frac{\mu^2}{\bar{\alpha}} + \frac{\Delta\mu}{\bar{\alpha}} \right) \\
&\quad - z_{,uu}(\mathbf{U}) \left[e^{-iC_0 - R'(\mathbf{U})} \alpha^{e^{-2iC_1}} \mathbf{X}^{-1} \right]^{-2e^{2iC_1}} X^{-1}, \\
\Delta\mathbf{U} &= \mathbf{U}_{,u}\mathbf{X}^{-1}
\end{aligned}$$

□

8.7.4 $(0, 1, 3, 3)$; $\Delta\alpha = 0$ case:

Lemma 8.7.9. *The PP-wave metric belonging to the $G_1 - c - 0$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = \left(\frac{C}{4}\right)^2 e^{\frac{4}{C}(\zeta - z(u))}, \quad \bar{z} \neq z, \quad C \in \mathbb{R}.$$

The function $z(u)$ may be any complex-valued function with $\bar{z} \neq z$. Using the special coordinates in (8.3)*, the functionally independent invariants are*

$$\alpha = \frac{e^{a-\bar{a}}}{C}, \quad \boldsymbol{W}_0 = -iln\left[\frac{z_{,uu}}{\bar{z}_{,uu}}\right], \quad \boldsymbol{X} = e^{a+\bar{a}}.$$

while the essential classifying functions are

$$\begin{aligned}
\bar{\alpha} &= (C^2\alpha)^{-1}, \\
\bar{\delta}\alpha &= -\alpha^2, \\
\nu &= (4\alpha)^{-1} - z_{,uu}(\boldsymbol{W}_0)C\alpha\boldsymbol{X}^{-2}, \\
\Delta\boldsymbol{W}_0 &= \boldsymbol{W}_{1,u}(\boldsymbol{W}_0)\boldsymbol{X}^{-1}.
\end{aligned} \tag{8.98}$$

Proof. In the $(a, \bar{a}, u, v)$ coordinate system, these solutions correspond to those metrics with $\zeta(a, u)$ of the form (8.11) with $\epsilon = 0$. By solving for a and integrating with respect to ζ in the coordinate system where the ζ-linear piece of $f(\zeta, u)$ has been set to zero, we find the canonical form for the metric. The constant in (8.11) is c_4, which is complex, however by using the coordinate freedom this C may be set to be real. Equivalently if C is left as c_4 our invariants imply that the phase of c_4 is not a required constant, unlike its magnitude - to see this replace $C \to c_4$ in the following proof. The first order invariants are

$$\alpha = \frac{e^{a-\bar{a}}}{C}, \quad \bar{\alpha} = (C^2\alpha)^{-1}.$$

The second order invariants are then

$$\bar{\delta}\alpha = -\alpha^2, \quad \mu = 0, \quad \mathbf{V} = Z_{,uu}e^{-a-3\bar{a}}, \quad \nu = (4\alpha)^{-1} - \mathbf{V}, \quad \bar{\delta}\alpha = -\alpha^2.$$

To continue we construct an appropriate pair of second order invariants from $\mathbf{V}$ and $\bar{\mathbf{V}}$, the first will be a function of u only

$$\mathbf{W}_0 \quad = \quad -iln\left[(C\alpha)^{-2}\left(\frac{\mathbf{V}}{\bar{\mathbf{V}}}\right)\right] = -iln\left[\frac{z_{,uu}}{\bar{z}_{,uu}}\right].$$

With this invariant we see that locally $z_{,uu}$ may be written in terms of $\mathbf{W}_0$ and we may solve for the remaining invariant from $\mathbf{V}$,

$$\mathbf{X} \quad = \quad \left(\frac{V}{z_{,uu}(\mathbf{W}_0)C\alpha}\right)^{-\frac{1}{2}} = e^{a+\bar{a}}.$$

At third order the invariants are:

$$\delta\mathbf{W}_0 = 0, \quad \Delta\mathbf{W}_0 = \mathbf{W}_{1,u}(\mathbf{W}_0)\mathbf{X}^{-1}, \quad \delta\mathbf{X} = \bar{\alpha}\mathbf{X}, \quad \Delta\mathbf{Z} = 0.$$

The non-trivial classifying functions are then,

$$\begin{aligned}
\bar{\alpha} &= (C^2\alpha)^{-1}, \\
\bar{\delta}\alpha &= -\alpha^2, \\
\nu &= (4\alpha)^{-1} - z_{,uu}(\mathbf{W}_0)C\alpha\mathbf{X}^{-2}, \\
\Delta\mathbf{W}_0 &= \mathbf{W}_{1,u}(\mathbf{W}_0)\mathbf{X}^{-1}
\end{aligned}$$

□

Lemma 8.7.10. *The PP-wave metric belonging to the $G_1 - d - 0$ class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$,

$$f(\zeta, u) = \frac{C^{-4iC_0}(\zeta - z(u))^{4iC_0}}{(2 - 16C_0^2 - 12iC_0)}, \quad C, C_0 \in \mathbb{R}$$

with $C, C_0 \neq 0$ *and* $z_{,uu} \neq e^{iB(u)(1+iC_0)}$. *Using the special coordinates in* (8.3), *the functionally independent invariants are*

$$\alpha = \frac{e^{a+\left(\frac{1+2iC_0}{1-2iC_0}\right)\bar{a}}}{C}, \boldsymbol{R} = |z_{,uu}|e^{-2(a+\bar{a})}, \quad \boldsymbol{P} = \sqrt{\frac{z_{,uu}}{\bar{z}_{,uu}}}e^{(a-\bar{a})}.$$

The essential classifying functions are

$$\begin{aligned} \bar{\alpha} &= (\bar{C})^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}(C)^{-1}\alpha^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}, \\ \bar{\delta}\alpha &= \alpha^2\left(\frac{1-i2C_0}{1+i2C_0}\right), \\ \nu &= (4\alpha)^{-1} - \boldsymbol{RP}, \\ \Delta\boldsymbol{R} &= \boldsymbol{R}^{\frac{1}{2}}\frac{\boldsymbol{R}_u(\alpha, \boldsymbol{R}, \boldsymbol{P})}{\sqrt{|z_{,uu}(\alpha, \boldsymbol{R}, \boldsymbol{P})|}}, \\ \Delta\boldsymbol{P} &= \boldsymbol{R}^{\frac{1}{2}}\frac{\boldsymbol{P}_u(\alpha, \boldsymbol{R}, \boldsymbol{P})}{\sqrt{|z_{,uu}(\alpha, \boldsymbol{R}, \boldsymbol{P})|}} \end{aligned} \tag{8.99}$$

Proof. In the $(a, \bar{a}, u, v)$ coordinate system, these solutions correspond to those metrics with $\zeta(a, u)$ of the form (8.11) with $\epsilon \neq 0$ and $C_0 \neq 0$. By solving for a and integrating with respect to ζ in the coordinate system where the ζ-linear piece of $f(\zeta, u)$ has been set to zero, we find the canonical form for the metric. The constant in (8.11) is c_4, which is complex, however the form of $f(\zeta, u)$ allows for either the magnitude or phase of c_4 to be removed - in this case we opt for the magnitude. This arises from the fact that there are only two essential classifying functions for the three constants. To avoid the subclass of these spacetimes with an invariant count $(0, 1, 2, 3, 3)$ we require that $z(u) \neq e^{iB(u)(1+iC_0)}$, $\; C_0 \in \mathbb{R}$

$$\alpha = \frac{e^{a-\left(\frac{1-2iC_0}{1+2iC_0}\right)\bar{a}}}{\bar{C}},$$
$$\bar{\alpha} = (\bar{C})^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}(C)^{-1}\alpha^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}.$$

The second order invariants are then

$$\bar{\delta}\alpha = \alpha^2\left(\frac{1-i2C}{1+i2C}\right), \quad \mu = 0, \quad \mathbf{V} = z_{,uu}e^{-a-3\bar{a}}, \quad \nu = (4\alpha)^{-1} - \mathbf{V}.$$

To construct a set of invariants with the simplest third order functionally dependent invariants, we look at the polar representation of $\mathbf{V}$

$$\mathbf{R} = |z_{,uu}|e^{-2(a+\bar{a})}, \quad \mathbf{P} = \sqrt{\frac{z_{,uu}}{\bar{z}_{,uu}}} e^{(a-\bar{a})} \quad ;$$

At third order we find the following relations,

$$\delta\mathbf{R} = -2\mathbf{R}\bar{\alpha}, \quad \Delta\mathbf{R} = \mathbf{R}^{\frac{1}{2}} \frac{\mathbf{R}_u(\alpha,\mathbf{R},\mathbf{P})}{\sqrt{|z_{,uu}(\alpha,\mathbf{R},\mathbf{P})|}},$$
$$\delta\mathbf{P} = \mathbf{P}\bar{\alpha}, \quad \bar{\delta}\mathbf{R} = -\alpha\mathbf{R}, \quad \Delta\mathbf{P} = \mathbf{R}^{\frac{1}{2}} \frac{\mathbf{P}_u(\alpha,\mathbf{R},\mathbf{P})}{\sqrt{|z_{,uu}(\alpha,\mathbf{R},\mathbf{P})|}}$$

In summary the non-trivial classifying functions are,

$$\begin{aligned}
\bar{\alpha} &= (C)^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}(\bar{C})^{-1}\alpha^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}, \\
\bar{\delta}\alpha &= \alpha^2\left(\frac{1-i2C_0}{1+i2C_0}\right), \\
\nu &= (4\alpha)^{-1} - \mathbf{R}^{\frac{1}{2}}\mathbf{P}, \\
\Delta\mathbf{R} &= \mathbf{R}^{\frac{1}{2}}\frac{\mathbf{R}_u(\alpha,\mathbf{R},\mathbf{P})}{\sqrt{|z_{,uu}(\alpha,\mathbf{R},\mathbf{P})|}}, \\
\Delta\mathbf{P} &= \mathbf{R}^{\frac{1}{2}}\frac{\mathbf{P}_u(\alpha,\mathbf{R},\mathbf{P})}{\sqrt{|z_{,uu}(\alpha,\mathbf{R},\mathbf{P})|}}
\end{aligned}$$

□

Lemma 8.7.11. *The PP-wave metric belonging to the $G_1 - d_0 - 0$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = e^{-iC}ln(\zeta - z(u)), \quad C, \in \mathbb{R}$$

and $\bar{z} \neq z^{-1}$. Using the special coordinates in (8.3)*, the functionally independent invariants are*

$$\alpha = \frac{e^{a+\bar{a}}}{e^{-iC}}, \boldsymbol{R}_0 = |z_{,uu}|, \quad \boldsymbol{P} = \sqrt{\frac{z_{,uu}}{\bar{z}_{,uu}}}e^{(a-\bar{a})}.$$

The essential classifying functions are

$$\begin{aligned}
\bar{\alpha} &= e^{-2iC}\alpha, \\
\bar{\delta}\alpha &= \alpha^2, \\
\nu &= (4\alpha)^{-1} - \boldsymbol{R}\boldsymbol{P}, \qquad (8.100)\\
\Delta\boldsymbol{R}_0 &= (e^{iC}\alpha)^{-1}\boldsymbol{R}_u, \\
\Delta\boldsymbol{P} &= (e^{iC}\alpha)^{-1}\boldsymbol{P}_u
\end{aligned}$$

Proof. In the $(a, \bar{a}, u, v)$ coordinate system, these solutions correspond to those metrics with $\zeta(a, u)$ of the form (8.11) with $\epsilon \neq 0$ and $C_0 \neq 0$. By solving for a and integrating with respect to ζ in the coordinate system where the ζ-linear piece of $f(\zeta, u)$ has been set to zero, we find the canonical form for the metric. The constant in (8.11) is c_4, which is complex, however the form of $f(\zeta, u)$ allows for magnitude of c_4 to be removed, but not the phase. This arises from the fact that the term $c_4/\bar{c}_4$ is involved in the classifying functions. To avoid the subclass of these spacetimes with an invariant count $(0, 1, 2, 3, 3)$ we require that $z(u) \neq \bar{z}(u)$

$$\alpha = \frac{e^{a+\bar{a}}}{c_4},$$
$$\bar{\alpha} = (c_4)(\bar{C}_4)^{-1}\alpha.$$

The second order invariants are then

$$\bar{\delta}\alpha = \alpha^2, \quad \mu = 0, \quad \mathbf{V} = z_{,uu}e^{-a-3\bar{a}}, \quad \nu = (4\alpha)^{-1} - \mathbf{V}.$$

To construct a set of invariants with the simplest third order functionally dependent invariants, we construct an invariant dependent on u only,

$$\mathbf{W}_2 \quad = \quad \sqrt{\mathbf{V}\bar{\mathbf{V}}}(e^{-iC}\alpha)^{-2} = |z_{,uu}|.$$

To construct a set of invariants with the simplest third order functionally dependent invariants, we look at the polar representation of $\mathbf{V}$, however in this case R may be scaled to be a function of u only

$$\mathbf{R}_0 = |z_{,uu}|, \quad \mathbf{P} = \sqrt{\frac{z_{,uu}}{\bar{z}_{,uu}}e^{(a-\bar{a})}} \quad ;$$

At third order we find the following relations,

$$\delta\mathbf{R}_0 = 0, \quad \Delta\mathbf{R}_0 = (e^{iC}\alpha)^{-1}\mathbf{R}_u,$$
$$\delta\mathbf{P} = \mathbf{P}\bar{\alpha}, \quad \bar{\delta}\mathbf{R} = -\alpha\mathbf{R}, \quad \Delta\mathbf{P} = (e^{iC}\alpha)^{-1}\mathbf{P}_u$$

In summary the non-trivial classifying functions are,

$$\begin{aligned}
\bar{\alpha} &= c_4\bar{c}_4^{-1}\alpha, \\
\bar{\delta}\alpha &= \alpha^2, \\
\nu &= (4\alpha)^{-1} - \mathbf{RP}, \\
\Delta\mathbf{R}_0 &= (e^{iC}\alpha)^{-1}\mathbf{R}_u, \\
\Delta\mathbf{P} &= (e^{iC}\alpha)^{-1}\mathbf{P}_u
\end{aligned}$$

$\square$

8.7.5 $(0,1,2,3,3)$; $\Delta\alpha \neq 0$:

Lemma 8.7.12. *The PP-wave metric belonging to the $G_1 - b - 1$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$.*

$$f(\zeta,u) = \tfrac{C^{-2}e^{2iC_0}}{4u^2} ln\zeta + \tfrac{e^{C_1}e^{-iZ(u)}}{u^2}\zeta, \quad C, C_0, C_1 \in \mathbb{R}$$

where $Z \neq C_2 ln(u)$. Using the special coordinates in (8.3), *the functionally independent invariants are*

$$\alpha = Cue^{-iC_0}e^{a+\bar{a}}, \quad \boldsymbol{W}_1 = Z(u) - i(a-\bar{a}), \quad \Delta\boldsymbol{Y} = Z_{,u}.$$

The essential classifying functions are

$$\begin{aligned}
\bar{\alpha} &= e^{2iC_0}\alpha, \\
\mu &= (e^{iC_0}\alpha)^{-1}, \\
\nu &= \frac{1}{2\alpha} + \frac{1}{2}\left(\frac{\mu^2}{\bar{\alpha}} + \frac{\Delta\mu}{\bar{\alpha}}\right) \\
&\quad - \frac{e^{C_1}C^2e^{i\boldsymbol{W}_1}}{e^{2iC_0}\alpha^2}, \\
\Delta\boldsymbol{Y} &= Cu\,\boldsymbol{Y}_{,u}(e^{iC_1}\alpha)^{-1}\boldsymbol{X}^{-1}.
\end{aligned} \tag{8.101}$$

Proof. Opting for the $(a, \bar{a}, u, v)$ coordinates, these solutions correspond to those metrics with $\zeta(a,u)$ of the form (8.10) with $C_1 = 0$ or $\pi \mod 2\pi$ given in lemma (8.3.6), the coordinate transformation (8.19) was used to remove Z and set the ζ-linear term of $f(\zeta, u)$ to be $z_{,uu}$. The first order invariant α and its conjugate become

$$\alpha = Cue^{-iC_0}e^{a+\bar{a}}, \quad \bar{\alpha} = e^{2iC_0}\alpha.$$

The second order invariants are,

$$\begin{aligned}
&\mu = (e^{iC_0}\alpha)^{-1}, \quad \mathbf{V} = \tfrac{e^{C_2}e^{iZ(u)}}{u^2}e^{-a-3\bar{a}}, \\
&\nu = -\tfrac{1}{-2\alpha} - \tfrac{1}{2}\left(\tfrac{\mu^2}{\bar{\alpha}} + \tfrac{\Delta\mu}{\bar{\alpha}}\right) - \mathbf{V}, \quad \bar{\delta}\alpha = \alpha^2 \quad .
\end{aligned}$$

Instead of $\mathbf{V}$ we use the following second order invariant

$$\mathbf{W}_1 = -iln\left(\sqrt{\frac{\mathbf{V}}{\bar{\mathbf{V}}}}\right) = Z(u) - i(a-\bar{a}),$$

from which we have the following third order invariants

$$\delta \mathbf{W}_1 = i\bar{\alpha}, \quad \Delta \mathbf{W}_1 = CuZ_{,u}(e^{iC_0}\alpha)^{-1}.$$

Solving for $\mathbf{Y} = uZ_{,u}$ produces the last candidate for the third functionally independent invariant. In general this will be the case, except when the triple wedge product of α $\mathbf{W}_1$ and $\mathbf{Y}$ vanish. Calculating this condition we find a differential equation for Z:

$$Z = C_2 ln(u) + C_3,$$

implying that $\mathbf{Y} = C$ so that these are G_2 spacetimes, and so we must avoid this particular Z to have the desired invariant count. The remaining functionally dependent invariant arises at fourth order by expressing $\mathbf{Y}_{,u}$ in terms of $\mathbf{Y}$.The essential functionally dependent invariants are then

$$\begin{aligned}
\bar{\alpha} &= \alpha^{e^{2iC_0}}\alpha, \\
\mu &= (e^{iC_0}\alpha)^{-1}, \\
\nu &= \frac{1}{2\alpha} + \frac{1}{2}\left(\frac{\mu^2}{\bar{\alpha}} + \frac{\Delta\mu}{\bar{\alpha}}\right) \\
&\quad - \frac{e^{C_1}C^2 e^{i\mathbf{W}_1}}{e^{2iC_0}\alpha^2}, \\
\Delta \mathbf{Y} &= \mathbf{Y}_{,u}(e^{iC_0}\alpha)^{-1}\mathbf{X}^{-1}.
\end{aligned}$$

□

8.7.6 $(0, 1, 2, 3, 3)$; $\Delta\alpha = 0$:

Lemma 8.7.13. *The PP-wave metric, expressed in terms of a canonical coframe and belonging to the $G_1\!-\!c\!-\!1$ class in figure (8.1) will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = \frac{C^2}{16}e^{\frac{4}{C}(\zeta)} + e^{iC_0}e^{Z(u)}\zeta, \;\; C, C_0 \in \mathbb{R},$$

and $e^Z \neq C_0 ln(u)$, $\;C_0 \in \mathbb{R}$. Using the special coordinates in (8.3), the functionally independent invariants are

$$\alpha = e^{a-\bar{a}}C^{-1}, \quad \boldsymbol{U}_0 = e^{\frac{Z}{2}}e^{-a-\bar{a}}, \quad \boldsymbol{Y}_0 = (Z_{,u}e^{-\frac{Z}{2}})_{,u}.$$

The essential classifying functions are

$$\begin{aligned} \bar{\alpha} &= C^{-2}\alpha^{-1}, \\ \nu &= (4\alpha)^{-1} - e^{iC_0}e^{Z(\boldsymbol{Y}_0)}C\alpha\,\boldsymbol{U}_0^2, \\ \Delta\,\boldsymbol{Y}_0 &= (Z_{,u}e^{-\frac{Z}{2}})_{,u}e^{-\frac{Z}{2}}\,\boldsymbol{U}_0. \end{aligned} \tag{8.102}$$

Proof. This corresponds to $\Delta\alpha = 0$ case in Section (8.3) when $\epsilon = 0$ from the proof of Lemma (8.3.7) where the coordinate transformation (8.19) was used. By a rotation in the $(\zeta, \bar{\zeta})$-plane the complex constant in equation (8.11) may always set to be real. The first order invariants are

$$\alpha = e^{a-\bar{a}}C^{-1}, \bar{\alpha} = C^{-2}\alpha^{-1}.$$

The second order invariants will be,

$$\mu = 0, \quad \mathbf{V} = e^{Z}e^{-a-3\bar{a}}, \quad \nu = (4\alpha)^{-1} - e^{iC_0}\mathbf{V}, \quad \bar{\delta}\alpha = -\alpha^2,$$

from $\mathbf{V}$ we construct another real-valued invariant

$$\mathbf{U}_0 = (\bar{\mathbf{V}}\mathbf{V})^{\frac{1}{4}} = e^{\frac{Z}{2}}e^{-a-\bar{a}}.$$

The original invariant $\mathbf{V}$ must be expressed in terms of α, $\mathbf{U}_0$ and some third order invariant arising from the frame derivatives of $\mathbf{U}_0$:

$$\delta\mathbf{U}_0 = -\bar{\alpha}\mathbf{U}_0, \quad \Delta\mathbf{U}_0 = \frac{1}{2}Z_{,u}e^{-\frac{Z}{2}}\mathbf{U}_0^2;$$

so that at third order the last real-valued invariant will be,$\mathbf{Y}_0 = Z_{,u}e^{-\frac{Z}{2}}$. Taking the frame derivatives of this yields the final set of classification functions

$$\delta\mathbf{Y}_0 = 0, \Delta\mathbf{Y}_0 = (Z_{,u}e^{-\frac{Z}{2}})_{,u}e^{-\frac{Z}{2}}\mathbf{U}_0$$

The non-trivial classifying functions are:

$$\begin{aligned} \bar{\alpha} &= C^{-2}\alpha^{-1} \\ \nu &= (4\alpha)^{-1} - e^{iC_0}e^{Z(\mathbf{Y}_0)}C\alpha\mathbf{U}_0^2 \\ \Delta\mathbf{Y}_0 &= (Z_{,u}e^{-\frac{Z}{2}})_{,u}e^{-\frac{Z}{2}}\mathbf{U}_0. \end{aligned}$$

□

Lemma 8.7.14. *The PP-wave metric belonging to the* $G_1 - d - 1$ *class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$,

$$f(\zeta, u) = \frac{C^{-4iC_0}(\zeta)^{4iC_0}}{(2 - 16C_0^2 - 12iC_0)} + c_1 e^{(-4C_0 - I)Z(u)}\zeta, \quad C, C_0 \in \mathbb{R},\ c_1 \in \mathbb{R}$$

with $C, C_0 \neq 0$. *Using the special coordinates in* (8.3), *the functionally independent invariants are*

$$\alpha = \frac{e^{a + \left(\frac{1+2iC_0}{1-2iC_0}\right)\bar{a}}}{C}, \ \boldsymbol{V} = e^{(-4C_0 + I)Z(u)} e^{-a-3\bar{a}}, \quad \boldsymbol{W}_2 = Z_{,u} e^{-a-\bar{a}}.$$

The essential classifying functions are

$$\begin{aligned} \bar{\alpha} &= (C)^{\left(\frac{1+2iC_0}{1-2iC_0}\right)} (C)^{-1} \alpha^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}, \\ \bar{\delta}\alpha &= \alpha^2 \left(\frac{1 - i2C_0}{1 + i2C_0}\right), \\ \nu &= (4\alpha)^{-1} - c_1 \boldsymbol{V}, \qquad (8.103) \\ \Delta \boldsymbol{W}_2 &= \frac{z_{,uu}}{Z_{,u}^2} \boldsymbol{W}_2. \end{aligned}$$

$$(8.104)$$

Proof. In the $(a, \bar{a}, u, v)$ coordinate system, these solutions correspond to those metrics with $\zeta(a, u)$ of the form (8.11) with $\epsilon \neq 0$ and $C_0 \neq 0$ produced in the proof of Lemma (8.3.7) where the coordinate transformation (8.19) was used. It is always possible to make C real-valued, using a rotation in the $(\zeta, \bar{\zeta})$ plane.

$$\alpha = \frac{e^{a - \left(\frac{1-2iC_0}{1+2iC_0}\right)\bar{a}}}{\bar{C}},$$
$$\bar{\alpha} = (\bar{C})^{\left(\frac{1+2iC_0}{1-2iC_0}\right)} (C)^{-1} \alpha^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}.$$

The second order invariants are then

$$\bar{\delta}\alpha = \alpha^2 \left(\tfrac{1-i2C_0}{1+i2C_0}\right), \quad \mu = 0, \quad \mathbf{V} = e^{(-4C_0+I)Z(u)} e^{-a-3\bar{a}}, \quad \nu = (4\alpha)^{-1} - c_1 \mathbf{V}.$$

At third order the frame derivatives of $\mathbf{V}$ are

$$\delta \mathbf{V} = -\bar{\alpha}\mathbf{V}, \quad \bar{\delta}\mathbf{V} = -3\alpha\mathbf{V}, \quad \Delta\mathbf{V} = (-4C_0 + I)\mathbf{W}_2\mathbf{V}.$$

Solving for $\mathbf{W}_2$, its frame derivatives produce the fourth order invariants

$$\delta\mathbf{W}_2 = -\bar{\alpha}\mathbf{W}_2, \quad \Delta\mathbf{W}_2 \ = \ \frac{z_{,uu}}{Z_{,u}^2}\mathbf{W}_2.$$

Thus the essential classifying functions are

$$\begin{aligned}
\bar{\alpha} &= (\bar{C})^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}(C)^{-1}\alpha^{\left(\frac{1+2iC_0}{1-2iC_0}\right)}, \\
\bar{\delta}\alpha &= \alpha^2\left(\frac{1-i2C_0}{1+i2C_0}\right), \\
\nu &= (4\alpha)^{-1} - c_1\mathbf{V}, \\
\Delta\mathbf{W}_2 &= \frac{z_{,uu}}{Z_{,u}^2}\mathbf{W}_2.
\end{aligned} \tag{8.105}$$

$\square$

Lemma 8.7.15. *The PP-wave metric belonging to the $G_1 - d_0 - 1$ class in figure (8.1) will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = -\tfrac{e^{2iC}}{4}ln(\zeta) + e^{C_0}e^{iZ(u)}\zeta,\ C,\ C_0 \in \mathbb{R}$$

and $Z(u) \neq C_1 u,\ C_1 \in \mathbb{R}$. Using the special coordinates in (8.3), the functionally independent invariants are

$$\alpha = e^{a+\bar{a}}e^{iC}, \quad \boldsymbol{Z} = e^{iZ(u)}e^{-2\bar{a}}, \quad \boldsymbol{Y}_1 = Z_{,u} \quad .$$

The essential classifying functions are

$$\begin{aligned}
\bar{\alpha} &= e^{-2iC}\alpha \\
\nu &= (4\alpha)^{-1} - e^{C_0}\boldsymbol{Z}(C\alpha)^{-1}, \\
\Delta\boldsymbol{Y}_1 &= \boldsymbol{Y}_{1,u}(C\alpha)^{-1}.
\end{aligned} \tag{8.106}$$

Proof. This belongs to the final case in Section (8.3) with $\epsilon = 1$ and $C_3 = 0$, where we have used the coordinate transform (8.19). A rotation of $(\zeta, \bar{\zeta})$ the complex constant in equation (8.11) may always set to be real-valued and positive. The first order invariants are

$$\alpha = e^{a+\bar{a}}e^{iC}, \bar{\alpha} = e^{-2iC}\alpha.$$

The second order invariants will be,

$$\mu = 0, \quad \nu = (4\alpha)^{-1} - e^{C_0}\mathbf{V}, \quad \mathbf{V} = e^{iZ}e^{-a-3\bar{a}}, \quad \bar{\delta}\alpha = \alpha^2,$$

from $\mathbf{V}$ we construct another functionally independent invariant,

$$\mathbf{Z} = C\alpha\mathbf{V} = e^{iZ}e^{-2\bar{a}}.$$

The only relevant third order invariants arise from the frame derivatives of $\mathbf{X}$

$$\delta\mathbf{Z} = 0, \quad \bar{\delta}\mathbf{Z} = -2\alpha\mathbf{Z}, \quad \Delta\mathbf{Z} = iZ_{,u}(C\alpha)^{-1} \quad .$$

Finally, we produce the third functionally independent invariant as a real-valued function of u, $\mathbf{Y}_1 = Z_{,uu}$. Frame derivatives of $\mathbf{Y}$ yield the non-trivial fourth order invariants

$$\delta\mathbf{Y}_1 = 0, \quad \Delta\mathbf{Y}_1 = Z_{,uu}(C\alpha)^{-1}.$$

□

8.8 All G_2 Spacetimes

8.8.1 $(0, 2, 2)$; $\Delta\alpha \neq 0$:

For this class of spacetimes, an analysis of the vanishing of the necessary wedge products does not readily produce tractable equations. However if one considers the fact that a killing vector must annihilate all invariants, and that all invariants may be expressed in this subcase in terms of α and $\bar{\alpha}$: the normalization $\hat{\Delta}\alpha \to 0$ via a null rotation about ℓ will be a helpful choice as it will then be a linear combination of Killing vectors.

$$\mathbf{A} = \frac{\bar{\delta}\alpha}{\alpha^2}, \quad B = \mu A - \bar{\mu}, \tag{8.107}$$

$$X = B/(A\bar{A} - 1), \tag{8.108}$$

The X invariant will only be applicable in the case $A\bar{A} \neq 1$ $B \neq 0$, implying that $\mu \neq 0$. We study the case where $A = 1, \mu \neq 0$ first, and study the $\mu = 0$ case last.

Lemma 8.8.1. *The PP-wave metric belonging to the $G_2 - I$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = g(u)ln(\zeta),$$

where $g(u)$ is complex valued. Using the special coordinates in (8.3), the functionally independent invariants are α and its conjugate,

$$\alpha = -\frac{1}{2} e^{a+\bar{a}} \bar{g}^{-\frac{1}{2}}.$$

The essential classifying functions are

$$\begin{aligned} \mu &= -\frac{1}{4} \frac{g_{,u}}{g^{\frac{3}{2}}} \alpha^{-1}, \\ \bar{\delta}\alpha &= \alpha^2, \\ \nu &= \frac{1}{2}\alpha^{-1} + \frac{1}{2}\left(\frac{\mu^2}{\bar{\alpha}} + \frac{\Delta\mu}{\bar{\alpha}}\right). \end{aligned} \tag{8.109}$$

Proof. See the proof to Proposition (8.5.13). □

Lemma 8.8.2. *The PP-wave metric belonging to the $G_2 - II$ class in figure (8.1) will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = u^{-2} F(\zeta u^{ik}),$$

where F is a complex valued function of one variable and $k \in \mathbb{R}$. Using the special coordinates in (8.3), the two functionally independent invariants are α and its conjugate,

$$\alpha = -\tfrac{1}{2}\tfrac{e^{a-\bar{a}}}{Z_{,a}},$$

where $\zeta = Z(a,u) = (F'')^{-1}(a - \frac{(1+ik)}{2} ln(u))u^{-2ik}$ and $L = ln(Z_{,a})$. The essential classifying functions are the second and third order invariants

$$\begin{aligned} \mu &= e^{-a-\bar{a}} log(Z_{,a})_{,u} \\ \bar{\delta}\alpha &= \bar{\alpha}\left(-\frac{1}{2}\frac{e^{a-\bar{a}}}{Z_{,a}}\right)_{,a} \\ \nu &= e^{-a-3\bar{a}}\left(Z_{,uu} + \frac{\bar{\Phi}_{\bar{a}}}{\bar{Z}_{\bar{a}}}\right), \quad \Phi(a,u) = f(\zeta, u). \end{aligned} \tag{8.110}$$

At third order, there is another constraint aside from

$$\delta X = \bar{\alpha} X, \quad \bar{\delta} X = \alpha X, \quad Re(\Delta X) = \Delta X_1$$

that determines this class, given $\hat{\nu}$ given in (8.52) one requires that $\hat{\Upsilon}$ (8.53) must vanish:

$$\hat{\Upsilon} = \Delta(\hat{\nu}/\bar{X}) - 2\hat{\nu} + 1/\alpha - 4iXX_2/\bar{\alpha} = 0. \tag{8.111}$$

Proof. See the proof to Proposition (8.5.10). □

Lemma 8.8.3. *The PP-wave metric belonging to the $G_2 - IIIa$ class in figure (8.1) will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = F(\zeta e^{iku}),$$

where F is a complex-valued function of one variable and $k \in \mathbb{R}$. Using the special coordinates in (8.3), the two functionally independent invariants are α and its conjugate,

$$\alpha = -\tfrac{1}{2}\tfrac{e^{a-\bar{a}}}{Z_{,a}},$$

where $\zeta = Z(a, u) = F(a - ig_1)e^{-2ig_1}$ and $L = ln(Z_{,a})$. The essential classifying functions are the remaining second order invariants

$$\begin{aligned}
\mu &= e^{-a-\bar{a}} log(Z_{,a})_{,u} \\
\bar{\delta}\alpha &= \bar{\alpha}\left(-\frac{1}{2}\frac{e^{a-\bar{a}}}{Z_{,a}}\right)_{,a} \\
\nu &= e^{-a-3\bar{a}}\left(Z_{,uu} + \frac{\bar{\Phi}_{\bar{a}}}{\bar{Z}_{\bar{a}}}\right), \quad \Phi(a, u) = f(\zeta, u)
\end{aligned} \tag{8.112}$$

At third order, there is another constraint aside from

$$\delta X = \bar{\alpha}X, \quad \bar{\delta}X = \alpha X, \quad Re(\Delta X) = 0$$

that determines this class, given $\hat{\nu}$ given in (8.52) *one requires that $\hat{\Upsilon}$* (8.53) *must vanish:*

$$\hat{\Upsilon} = \Delta(\hat{\nu}/\bar{X}) - 2\hat{\nu} + 1/\alpha - 4iXX_2/\bar{\alpha} = 0. \tag{8.113}$$

Proof. See the proof to Proposition (8.5.11). □

8.8.2 $(0, 2, 2)$; $\Delta\alpha = 0$:

Lemma 8.8.4. *The PP-wave metric belonging to the $G_2 - III - 1b$ class will have the canonical form*

$$f(\zeta, u) = F(\zeta)$$

The functionally independent invariants are α, expressed in the $(a, \bar{a})$ coordinates,

$$\alpha = -\tfrac{1}{2}\tfrac{e^{a-\bar{a}}}{Z_{,a}}$$

where $\zeta = Z(a,u) = F(a - ig_1)e^{-2ig_1}$ and $L = ln(Z_{,a})$. The essential classifying functions are the remaining second order invariants

$$\begin{aligned} \mu &= 0, \\ \bar{\delta}\alpha &= \bar{\alpha}\left(-\frac{1}{2}\frac{e^{a-\bar{a}}}{Z_{,a}}\right)_{,a}, \\ \nu &= e^{-a-3\bar{a}}\left(\frac{\bar{\Phi}_{\bar{a}}}{\bar{Z}_{\bar{a}}}\right), \quad \Phi(a,u) = f(\zeta, u) \end{aligned} \tag{8.114}$$

At third order there is one more invariant that determines this class

$$\Delta\nu = 0. \tag{8.115}$$

Proof. See the proof to Proposition (8.5.16). □

8.8.3 $(0, 1, 2, 2)$:

Lemma 8.8.5. *The PP-wave metric belonging to the $G_2 - a - 0 - 0$ class in figure (8.1) will have the canonical form for $f(\zeta, u)$*

$$f(\zeta, u) = \tfrac{(e^{2iC_1}+1)^2}{(e^{2iC_1}-3)(e^{2iC_1}-1)}\left[\tfrac{-(e^{iC_0}e^{Cu})^{e^{iC_1}}}{e^{2iC_1}+1}\right]^{\frac{4}{e^{2iC_1}+1}}\zeta^{2itan(C_1)} + c_2 e^{\frac{-iCu}{sin(C_1)}}\zeta,$$

where $C, C_0, C_1 \in \mathbb{R}$ and $c_2 \in \mathbb{C}$. Using the special coordinates in (8.3), the two functionally independent invariants are α and μ

$$\alpha = e^{a+e^{-2iC_1}\bar{a}}\left(e^{-iC_0}e^{Cu}\right)^{-e^{-iC_1}}, \quad \mu = Ce^{-iC_1}e^{-a-\bar{a}}.$$

The essential classifying functions are

$$\begin{gathered} \bar{\alpha} = (e^{-2C_0}\alpha)^{e^{2iC_1}}, \\ \bar{\mu} = e^{2iC_1}\mu, \\ \nu = \tfrac{1}{4\alpha} + \tfrac{1}{e^{2iC_1}+1}\left(\tfrac{\mu^2}{\bar{\alpha}} + \tfrac{\Delta\mu}{\bar{\alpha}}\right) \\ -\left[((e^{-iC_0})^{e^{-iC_1}}\alpha e^{-iC_1}C^{-1}\mu)^3(e^{iC_0})^{e^{iC_1}}\bar{\alpha}e^{iC_1}C^{-1}\bar{\mu}\right]^{-\frac{1}{2sin(C_1)}}, \\ \Delta\mu = 0. \end{gathered} \tag{8.116}$$

Proof. This metric function arises in the analysis in section (8.3) from (8.10) where $R_{,uu} = 0$, $C_1 \neq 0, \pi/2, 3\pi/2, \pi \ mod \ 2\pi$ and $Z_1 = Z_{1a}$ in (8.27). By direct inspection we find that for this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = e^{a+e^{-2iC_1}\bar{a}}\left(e^{-iC_0}e^{Cu}\right)^{-e^{-iC_1}}, \quad \bar{\alpha} = (e^{-2C_0}\alpha)^{e^{2iC_1}} \quad .$$

while at second and third order we have,

$$\begin{gathered}
\mu = Ce^{-iC_1}e^{-a-\bar{a}}, \ \bar{\delta}\alpha = \alpha^2, \\
\nu = \tfrac{1}{4\alpha} + \tfrac{1}{e^{2iC_1}+1}\left(\tfrac{\mu^2}{\bar{\alpha}} + \tfrac{\Delta\mu}{\bar{\alpha}}\right) \\
-\left[((e^{-iC_0})^{e^{-iC_1}}\alpha e^{-iC_1}C^{-1}\mu)^3(e^{iC_0})^{e^{iC_1}}\bar{\alpha}e^{iC_1}C^{-1}\bar{\mu}\right]^{-\frac{1}{2sin(C_1)}}, \\
\delta\mu = -\bar{\alpha}\mu, \quad \Delta\mu = 0.
\end{gathered}$$

□

Lemma 8.8.6. *The PP-wave metric belonging to the $G_2 - a - 0 - 1$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$*

$$\begin{aligned}
f(\zeta, u) \quad &= \quad \frac{(e^{2iC_1}+1)^2}{(e^{2iC_1}-3)(e^{2iC_1}-1)}\left[\frac{-(e^{iC_0}C_3u^C)^{e^{iC_1}}}{e^{2iC_1}+1}\right]^{\frac{4}{e^{2iC_1}+1}}\zeta^{2itan(C_1)} \\
&+c_2u^{\frac{i(sin(C_1)+iln(C_3)+ie^{iC_1})}{sin(C_1)}},
\end{aligned}$$

where $C, C_0, C_1, C_3 \in \mathbb{R}$ and $c_2 \in \mathbb{C}$. Using the special coordinates in (8.3)*, the two functionally independent invariants are α and μ*

$$\alpha = e^{a+e^{-2iC_1}\bar{a}}\left(e^{-iC_0}e^{Cu}\right)^{-e^{-iC_1}}, \quad \mu = Cu^{-1}e^{iC_1}e^{-a-\bar{a}}.$$

The essential classifying functions are

$$\begin{gathered}
\bar{\alpha} = (e^{-2C_0}\alpha)^{e^{2iC_1}}, \\
\bar{\mu} = e^{2iC_1}\mu, \\
\nu = \tfrac{1}{4\alpha} + \tfrac{1}{e^{2iC_1}+1}\left(\tfrac{\mu^2}{\bar{\alpha}} + \tfrac{\Delta\mu}{\bar{\alpha}}\right) \\
-\left[((e^{-iC_0})^{e^{-iC_1}}\alpha e^{-iC_1}C^{-1}\mu)^3(e^{iC_0})^{e^{iC_1}}\bar{\alpha}e^{iC_1}C^{-1}\bar{\mu}\right]^{-\frac{1}{2sin(C_1)}}, \\
\Delta\mu = C^{-1}e^{iC_1}\mu^2.
\end{gathered} \tag{8.117}$$

Proof. This metric function arises in the analysis in section (8.3) from (8.10) where $R = Cln(u) + C_3$, $C_1 \neq 0, \pi/2, 3\pi/2, \pi \ mod \ 2\pi$ and $Z_1 = Z_{1b}$ in (8.28). By direct inspection we find that for this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = e^{a+e^{-2iC_1}\bar{a}}\left(e^{-iC_0}e^{Cu}\right)^{-e^{-iC_1}}, \quad \bar{\alpha} = (e^{-2C_0}\alpha)^{e^{2iC_1}} \quad .$$

while at second and third order we have,

$$\begin{gathered}\mu = Cu^{-1}e^{-iC_1}e^{-a-\bar{a}}, \ \bar{\delta}\alpha = \alpha^2, \\ \nu = \tfrac{1}{4\alpha} + \tfrac{1}{e^{2iC_1}+1}\left(\tfrac{\mu^2}{\bar{\alpha}} + \tfrac{\Delta\mu}{\bar{\alpha}}\right) \\ -\left[((e^{-iC_0})^{e^{-iC_1}}\alpha e^{-iC_1}C^{-1}\mu)^3(e^{iC_0})^{e^{iC_1}}\bar{\alpha}e^{iC_1}C^{-1}\bar{\mu}\right]^{-\frac{1}{2sin(C_1)}}, \\ \delta\mu = -\bar{\alpha}\mu, \quad \Delta\mu = C^{-1}e^{iC_1}\mu^2.\end{gathered}$$

□

Lemma 8.8.7. *The PP-wave metric belonging to the $G_2 - a - 0 - 2$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$*

$$f(\zeta, u) = \frac{1}{16e^{2C_0}}e^{4e^{C_0}(\zeta+C)e^{iC_1u}},$$

where $C, C_0, C_1 \in \mathbb{R}$. Using the special coordinates in (8.3)*, the two functionally independent invariants are α and μ*

$$\alpha = e^{a-\bar{a}}e^{-iC_1u+C_0}, \quad \mu = iC_1e^{-a-\bar{a}}.$$

The essential classifying functions are

$$\begin{gathered}\bar{\alpha} = e^{2C_0}\alpha^{-1}, \\ \nu = \tfrac{1}{4\alpha} + \tfrac{\mu^2}{2\bar{\alpha}}(ln(\tfrac{-i\mu\bar{\alpha}}{C_1}) - ln(16ce^{3C_0})), \\ \Delta\mu = 0.\end{gathered} \tag{8.118}$$

Notice that the constant found from ν is the ratio $\frac{C_1}{C}$, reflecting the coordinate freedom in u for fixing constants. This allows one to either scale C_1 or C to be equal to one.

Proof. This metric function arises in the analysis in section (8.3) from (8.10) where $R_{,uu} = 0$, $C_1 = \pi/2, 3\pi/2 \ mod \ 2\pi$ and $Z_1 = Z_{1a}$ in (8.24)- fixing these constants we relabel the remaining constants to have lower index numbering. By translating u we may set $c \in \mathbb{C}$ below to be real-valued. By direct inspection we find that for this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = e^{a-\bar{a}}e^{-iC_1u+C_0}, \quad \bar{\alpha} = e^{2C_0}\alpha^{-1}.$$

while at second and third order we have,

$$\mu = iC_1e^{-a-\bar{a}}, \; \bar{\delta}\alpha = -\alpha^2,$$
$$\nu = \tfrac{1}{4\alpha} + \tfrac{\mu^2}{2\bar{\alpha}}(ln(\tfrac{-i\mu\bar{\alpha}}{C_1}) - ln(16ce^{3C_0})),$$
$$\delta\mu = -\bar{\alpha}\mu, \quad \Delta\mu = 0.$$

□

Lemma 8.8.8. *The PP-wave metric belonging to the $G_2 - a - 0 - 3$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = \frac{1}{16e^{2C_0}u^2}e^{4e^{C_0}(\zeta+C)u^{iC_1}}.$$

Using the special coordinates in (8.3)*, the two functionally independent invariants are α and μ*

$$\alpha = e^{a-\bar{a}}e^{C_0}u^{-iC_1}, \quad \mu = \frac{iC_1e^{-a-\bar{a}}}{u}.$$

The essential classifying functions are

$$\begin{aligned} \bar{\alpha} &= e^{2C_0}\alpha^{-1}, \\ \nu &= \frac{1}{4\alpha} + \frac{\mu^2}{2\bar{\alpha}}\frac{C_1+i}{2C_1}(ln(\frac{-\mu\bar{\alpha}}{C_1}) - ln(16c) + 2C_0), \\ \Delta\mu &= \frac{i\mu^2}{C_1}. \end{aligned} \tag{8.119}$$

Proof. This metric function arises in the analysis in section (8.3) from (8.10) where $R_{,uu} \neq 0$, $C_1 = \pi/2, 3\pi/2 \; mod \; 2\pi$ and $Z_1 = Z_{1b}$ in (8.24) - fixing C_1 we relabel the remaining constants to have lower index numbering. By scaling u and translating z and v appropriately one may set the complex constant c in the following proof to be real-valued. By direct inspection we find that for this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = e^{a-\bar{a}}e^{C_0}u^{-iC_1}, \quad \bar{\alpha} = e^{2C_0}\alpha^{-1}.$$

while at second and third order we have,

$$\mu = \tfrac{iC_1 e^{-a-\bar{a}}}{u},\ \bar{\delta}\alpha = -\alpha^2,$$
$$\nu = \tfrac{1}{4\alpha} + \tfrac{\mu^2}{2\bar{\alpha}}\tfrac{C_1+i}{2C_1}(ln(\tfrac{-\mu\bar{\alpha}}{C_1}) - ln(16c) + 2C_0);$$
$$\delta\mu = -\bar{\alpha}\mu,\quad \Delta\mu = \tfrac{i\mu^2}{C_1}.$$

□

Lemma 8.8.9. *The PP-wave metric belonging to the* $G_2 - b - 0$ *class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$*,*

$$f(\zeta, u) = -e^{2iC}e^{-2R(u)}ln\zeta,$$

where $R(u)$ *is real-valued. Using the special coordinates in* (8.3)*, the two functionally independent invariants are* α *and* μ

$$\alpha = -e^{-iC}e^{a+\bar{a}}e^R,\quad \mu = -R_{,u}e^{-a-\bar{a}}.$$

The essential classifying functions are

$$\begin{aligned}\bar{\alpha} &= e^{2iC}\alpha\\ \nu &= \tfrac{1}{2\alpha} + \tfrac{1}{2}(\tfrac{\mu^2}{\bar{\alpha}} + \tfrac{\Delta\mu}{\bar{\alpha}}),\\ \Delta\mu &= -R_{,uu}e^{-2a-2\bar{a}}.\end{aligned} \tag{8.120}$$

Proof. This metric function arises in the analysis in section (8.3) from (8.10) where $Z_{1,uu} = 0$, $C_1 = 0, \pi\ mod\ 2\pi$ and R is arbitrary. For this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = -e^{-iC}e^{a+\bar{a}}e^R,\ \ \bar{\alpha} = e^{2iC}\alpha.$$

while at second and third order we have,

$$\mu = -R_{,u}e^{-a-\bar{a}},\ \bar{\delta}\alpha = \alpha^2$$
$$\nu = \tfrac{1}{2\alpha} + \tfrac{1}{2}(\tfrac{\mu^2}{\bar{\alpha}} + \tfrac{\Delta\mu}{\bar{\alpha}});$$
$$\Delta\mu = -R_{,uu}e^{-2a-2\bar{a}}.$$

□

Lemma 8.8.10. *The PP-wave metric belonging to the $G_2 - b - 1$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = u^{-2}\left(\frac{e^{2iC_0} ln\zeta}{4C^2} + C_1 u^{-2+iC_2} e^{iC_3}\right).$$

Using the special coordinates in (8.3)*, the two functionally independent invariants are α and $\boldsymbol{V}$*

$$\alpha = e^{a+\bar{a}} C e^{-iC_0} u, \quad \boldsymbol{V} = e^{-a-3\bar{a}} u^{-2-iC_2}.$$

The essential classifying functions, are

$$\begin{aligned} \bar{\alpha} &= e^{2iC_0}, \alpha \\ \mu &= \frac{C}{e^{iC_0}\alpha}, \\ \nu &= (2\alpha)^{-1} + C_1 e^{iC_3}\, \boldsymbol{V}, \\ \Delta \boldsymbol{V} &= -(2 + iC_3)\, \boldsymbol{V}\frac{C}{e^{iC_0}\alpha}. \end{aligned} \tag{8.121}$$

Proof. By direct inspection we find that for this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = e^{-iC} e^{a+\bar{a}} Cu, \quad \bar{\alpha} \;\; = \;\; e^{2iC_0}\alpha.$$

while at second and third order we have,

$$\mathbf{V} = e^{-a-3\bar{a}} u^{-2-iC_2}, \quad \mu = \tfrac{C}{e^{iC_0}\alpha}, \;\; \nu = (2\alpha)^{-1} + C_1 e^{iC_3}\mathbf{V}, \;\; \bar{\delta}\alpha = \alpha^2;$$
$$\delta\mathbf{V} = -\bar{\alpha}\mathbf{V}, \quad \bar{\delta}\mathbf{V} = -3\alpha\mathbf{V}, \quad \Delta\mathbf{V} = -(2 + iC_2)\mathbf{V}\tfrac{C}{e^{iC_0}\alpha}.$$

□

Lemma 8.8.11. *The PP-wave metric belonging to the $G_2 - c - 0$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = C^2 \frac{e^{\frac{4}{C}\zeta}}{16} + c_0\zeta.$$

Using the special coordinates in (8.3)*, the two functionally independent invariants are α and $\boldsymbol{V}$*

$$\alpha = \frac{e^{a-\bar{a}} u}{C}, \quad \boldsymbol{V} = e^{-a-3\bar{a}}.$$

The essential classifying functions, distinguishing one spacetime in this class from another, will be:

$$\bar{\alpha} = \tfrac{1}{C^2\alpha}, \quad \nu = (4\alpha)^{-1} - c_0\,\boldsymbol{V}. \tag{8.122}$$

Proof. This metric function arises in the analysis in section (8.3) from (8.11) where $\epsilon = 0$ and $Z_{1,uuu} = 0$, then by translating ζ one may absorb the phase of the complex constant c_4 and make it real-valued. By direct inspection we find that for this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = \frac{e^{a-\bar{a}}u}{C}, \quad \bar{\alpha} = \frac{1}{C^2\alpha}.$$

while at second and third order we have,

$$\mathbf{V} = e^{-a-3\bar{a}}, \ \bar{\delta}\alpha = -\alpha^2,$$
$$\mu = 0, \ \nu = (4\alpha)^{-1} - c_0\mathbf{V},$$
$$\delta\mathbf{V} = -\bar{\alpha}\mathbf{V}, \quad \bar{\delta}\mathbf{V} = -3\alpha\mathbf{V}, \quad \Delta\mathbf{V} = 0.$$

□

Lemma 8.8.12. *The PP-wave metric belonging to the* $G_2 - c - 1$ *class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$*,*

$$f(\zeta, u) = \frac{C^2 e^{\frac{4\zeta}{C}}}{16} + \frac{c_0\zeta}{u^2}.$$

Using the special coordinates in (8.3)*, the two functionally independent invariants are*

$$\alpha = \frac{e^{a-\bar{a}}}{C}, \ \boldsymbol{U}_1 = u^{-1}e^{-a-\bar{a}}.$$

The essential classifying functions are

$$\bar{\alpha} = \tfrac{1}{C^2\alpha}, \quad \nu = (4\alpha)^{-1} - c_0 C\,\boldsymbol{U}_1^2\alpha. \tag{8.123}$$

Proof. This metric function arises in the analysis in section (8.3) from (8.11) where $\epsilon = 0$ and $Z_{1,uu} = Cu^{-2}$, by rotating the ζ coordinates, we may shift the phase of c_4 onto the ζ-linear term. By direct inspection we find that for this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = \frac{e^{a-\bar{a}}}{C}, \quad \bar{\alpha} = \frac{1}{C^2\alpha}.$$

At second order we have the usual invariant $\mathbf{V} = u^{-2}e^{-a-3\bar{a}}$, however instead of $\mathbf{V}$ we work with the invariant $\mathbf{U}_1 = \sqrt{|\mathbf{V}|}$,

$$\mathbf{U}_1 = u^{-1}e^{-a-\bar{a}}.$$

We may write $\mathbf{V} = C\mathbf{U}_1^2\alpha$. The functionally dependent invariants are

$$\mu = 0,\ \nu = (4\alpha)^{-1} - c_0 C\mathbf{U}_1^2\alpha,\ \bar{\delta}\alpha = -\alpha^2,$$

while at third order we find,

$$\delta\mathbf{U}_1 = \bar{\alpha}\mathbf{U}_1, \quad \Delta\mathbf{U}_1 = -\mathbf{U}_1^2.$$

□

Lemma 8.8.13. *The PP-wave metric belonging to the G_2-d-0 class in figure (8.1) will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = \frac{c^2}{4C_1^2 + 2iC_1}\left(\frac{\zeta}{c}\right)^{2iC_1} + C_0\zeta,$$

where $C_0, C_1 \ \in \mathbb{R}$, $c \in \mathbb{C}$ and $C_1 \neq 0$. Using the special coordinates in (8.3), the two functionally independent invariants are

$$\alpha = \frac{e^a e^{(\frac{1+i2C_1}{1-2iC_1})\bar{a}}}{\bar{c}}, \quad \boldsymbol{W}_4 = e^{a-\bar{a}}.$$

The essential classifying functions are

$$\begin{aligned}
\bar{\alpha} &= \frac{(\bar{c}\alpha)^{\frac{1-2iC_1}{1+2iC_1}}}{c}, \\
\nu &= (\left(\frac{1+2iC_1}{1-2iC_1}\right)(4\alpha)^{-1} - C_0\,\boldsymbol{V}\alpha, \\
\bar{\delta}\alpha &= \left(\frac{1-2iC_1}{1+2iC_1}\right)\alpha^2. \\
\Delta\,\boldsymbol{W}_4 &= 0
\end{aligned} \tag{8.124}$$

Proof. This metric function arises in the analysis in section (8.3) from (8.11) where $\epsilon \neq 0$ and $Z_{1,uuu} = 0$. For this metric function $f(\zeta, u)$, at first order the invariants are

$$\alpha = \frac{e^a e^{(\frac{1+i2C_1}{1-2iC_1})\bar{a}}}{\bar{c}}, \quad \bar{\alpha} = \bar{\alpha} = \frac{(\bar{c}\alpha)^{\frac{1-2iC_1}{1+2iC_1}}}{c} \quad .$$

At second order we have the usual invariant $\mathbf{V} = e^{-a-3\bar{a}}$, however instead of $\mathbf{V}$ we work with the invariant $\mathbf{W}_4 = \sqrt{|\mathbf{V}|}$,

$$\mathbf{W}_4 = \sqrt{\frac{\mathbf{V}}{\bar{\mathbf{V}}}} = e^{a-\bar{a}}.$$

We may write $\mathbf{V} = (\alpha \mathbf{W}_4)^{\frac{-(1-2iC_1)}{2}} (\frac{\bar{\alpha}}{\mathbf{W}_4})^{\frac{-(1-2iC_1)}{2}}$. The functionally dependent second order invariants are

$$\mu = 0, \;\; \nu = \left(\tfrac{1+2iC_1}{1-2iC_1}\right)(4\alpha)^{-1} - C_0 \mathbf{V}\alpha, \;\; \bar{\delta}\alpha = \left(\tfrac{1-2iC_1}{1+2iC_1}\right)\alpha^2.$$

Taking the frame derivatives of $\mathbf{W}_4$, the third order invariants are

$$\delta \mathbf{W}_4 = \bar{\alpha}\mathbf{W}_4, \quad \Delta \mathbf{W}_4 = 0.$$

If $C_1 = 0$ in the above case, the form of $f(\zeta, u)$ changes slightly, $f(\zeta, u) = -C^2 ln(\frac{\zeta}{C}) + c_0\zeta$. However, by direct inspection, we see that this is a particular instance of the next subcase. $\square$

Lemma 8.8.14. *The PP-wave metric belonging to the $G_2 - d - 1$ class in figure* (8.1) *will have the canonical form for $f(\zeta, u)$,*

$$f(\zeta, u) = \frac{c^2}{4C_1^2 + 2iC_1}\left(\frac{\zeta}{c}\right)^{2iC_1} + c_0 u^{-2-\frac{i}{2C_1}}\zeta,$$

Using the special coordinates in (8.3), *the two functionally independent invariants are*

$$\alpha = \frac{e^a e^{(\frac{1+i2C_1}{1-2iC_1})\bar{a}}}{\bar{c}}, \;\; \boldsymbol{V} = u^{-2+\frac{i}{2C_1}} e^{-a-3\bar{a}}.$$

The essential classifying functions are

$$\begin{aligned}
\bar{\alpha} &= \frac{(\bar{c}\alpha)^{\frac{1-2iC_1}{1+2iC_1}}}{c}, \\
\nu &= (\left(\frac{1+2iC_1}{1-2iC_1}\right)(4\alpha)^{-1} - c_0\, \boldsymbol{V}\alpha, \\
\bar{\delta}\alpha &= \left(\frac{1-2iC_1}{1+2iC_1}\right)\alpha^2. \\
\Delta \boldsymbol{V} &= -\left(2 + \frac{i}{2C_1}\right)\boldsymbol{V}(\boldsymbol{V}\bar{\boldsymbol{V}})^{-\frac{1}{4}}
\end{aligned} \tag{8.125}$$

Proof. This metric function arises in the analysis in section (8.3) from (8.11) where $\epsilon \neq 0$ and $Z_{1,uu} = c_0 u^{-2-\frac{i}{2C_1}}$. For this metric function $f(\zeta, u)$, at first order the invariants are

$$\alpha = \frac{e^a e^{(\frac{1+i2C_1}{1-2iC_1})\bar{a}}}{\bar{c}}, \quad \bar{\alpha} = \bar{\alpha} = \frac{(\bar{c}\alpha)^{\frac{1-2iC_1}{1+2iC_1}}}{c} \quad .$$

At second order we have the usual invariant $\mathbf{V} = e^{-a-3\bar{a}}$. The functionally dependent second order invariants are

$$\mu = 0, \ \nu = \left(\tfrac{1+2iC_1}{1-2iC_1}\right)(4\alpha)^{-1} - C_0 \mathbf{V}\alpha, \ \bar{\delta}\alpha = \left(\tfrac{1-2iC_1}{1+2iC_1}\right)\alpha^2.$$

Taking the frame derivatives of $\mathbf{V}$, the third order invariants are

$$\delta\mathbf{V} = -\bar{\alpha}\mathbf{V}, \quad \bar{\delta}\mathbf{V} = -3\alpha\mathbf{V}, \quad \Delta\mathbf{V} = -(2 + \tfrac{i}{2C_1}\mathbf{V}(\mathbf{V}\bar{\mathbf{V}})^{-\frac{1}{4}}.$$

□

Lemma 8.8.15. *The PP-wave metric belonging to the* $G_2 - d_0 - 1$ *class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$,

$$f(\zeta, u) = e^{-2iC} ln\zeta + C_1 e^{iC_0 u}\zeta.$$

Using the special coordinates in (8.3), *the two functionally independent invariants are*

$$\alpha = e^{-iC} e^{a+\bar{a}}, \quad \boldsymbol{W}_5 = e^{iC_0 u} e^{a-\bar{a}};$$

the essential classifying functions are

$$\begin{aligned} \bar{\alpha} &= e^{2iC}\alpha, \\ \nu &= (4\alpha)^{-1} + C_1 \boldsymbol{V}, \\ \Delta \boldsymbol{W}_5 &= \frac{iC_0 \boldsymbol{W}_5}{\bar{c}\alpha} \end{aligned}$$

Proof. This metric function arises in the analysis in section (8.3) from (8.11) where $\epsilon \neq 0$, $C_3 = 0$ and $Z_{,uu} = c_0 e^{iC_0 u}$, a scaling of u sets the magnitude of c_4 equal to 1 and a rotation in the ζ-plane sets C_1 to be real-valued. For this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = e^{iC} e^{a+\bar{a}}, \quad \bar{\alpha} = e^{2iC}\alpha \quad .$$

At second order we have the usual invariant $\mathbf{V} = e^{iC_0u}e^{-a-3\bar{a}}$, however instead of $\mathbf{V}$ we work with the invariant,

$$\mathbf{W}_5 = \sqrt{\frac{\mathbf{V}}{\bar{\mathbf{V}}}} = e^{iC_0u}e^{a-\bar{a}}.$$

We may write $\mathbf{V} = \frac{\mathbf{W}_5}{e^{2iC}\alpha^2}$. The functionally dependent second order invariants are

$$\mu = 0,\ \nu = (4\alpha)^{-1} + C_1\mathbf{V},\ \bar{\delta}\alpha = \alpha^2,$$

Taking the frame derivatives of $\mathbf{W}_5$, the third order invariants are

$$\delta\mathbf{W}_5 = \bar{\alpha}\mathbf{W}_5, \quad \Delta\mathbf{W}_5 = \tfrac{iC_0\mathbf{W}_5}{e^{iC}\alpha}.$$

□

8.9 All G_3 Spacetimes with $\alpha \neq 0$

By imposing the condition that only that the invariant arising at second order is functionally dependent, a subset of the G_2 solutions produce G_3 solutions: G_3-c-0 arises from G_2-a-0-2, G_2-c-0 and G_2-c-1; G_2-b-0 and G_2-b-1 yield G_3-b-0; G_2-d-0, G_2-d-1 reduce to G_3-d-0; and G_2-d_0-0 gives G_3-d_0-0. Thus we have found the four subcases introduced by Kundt and Ehlers, although with different constants in two cases.

8.9.1 $(0,1,1)$:

Lemma 8.9.1. *The PP-wave metric belonging to the $G_3 - b - 0$ class in figure* (8.1) *will have the canonical form for $f(\zeta,u)$,*

$$f(\zeta,u) = u^{-2}\left(\frac{e^{2iC_0}ln\zeta}{4C^2}\right).$$

Using the special coordinates in (8.3), *the functionally independent invariant is*

$$\alpha = e^{a+\bar{a}}Ce^{-iC_0}u.$$

The essential classifying functions, are

$$\bar{\alpha} = e^{2iC_0}, \alpha, \quad \mu = \frac{C}{e^{iC_0}\alpha}. \tag{8.126}$$

Proof. By direct inspection we find that for this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = e^{-iC} e^{a+\bar{a}} Cu, \quad \bar{\alpha} \quad = \quad e^{2iC_0}\alpha.$$

while at second order we have,

$$\mu = \tfrac{C}{e^{iC_0}\alpha}, \;\; \nu = (2\alpha)^{-1}, \;\; \bar{\delta}\alpha = \alpha^2.$$

□

Lemma 8.9.2. *The PP-wave metric belonging to the* $G_3 - c - 0$ *class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$*,*

$$f(\zeta, u) = C^2 \frac{e^{\frac{4}{C}\zeta}}{16}.$$

Using the special coordinates in (8.3)*, the two functionally independent invariant is*

$$\alpha = \frac{e^{a-\bar{a}}u}{C}.$$

The essential classifying function, distinguishing one spacetime in this class from another, will be:

$$\bar{\alpha} = \tfrac{1}{C^2\alpha}. \tag{8.127}$$

Proof. By direct inspection we find that for this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = \frac{e^{a-\bar{a}}u}{C}, \quad \bar{\alpha} = \frac{1}{C^2\alpha}.$$

while at second and third order we have,

$$\bar{\delta}\alpha = -\alpha^2, \mu = 0, \;\; \nu = (4\alpha)^{-1}.$$

□

Lemma 8.9.3. *The PP-wave metric belonging to the* $G_2 - d - 0$ *class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$*,*

$$f(\zeta, u) = \frac{c^2}{4C_0^2 + 2iC_0}\left(\frac{\zeta}{c}\right)^{2iC_0},$$

where $C_0 \in \mathbb{R}$, $c \in \mathbb{C}$ *and* $C_0 \neq 0$. *Using the special coordinates in* (8.3), *the functionally independent invariant is*

$$\alpha = \frac{e^{a} e^{(\frac{1+i2C_1}{1-2iC_1})\bar{a}}}{\bar{c}}.$$

The essential classifying functions are

$$\bar{\alpha} = \frac{(\bar{c}\alpha)^{\frac{1-2iC_1}{1+2iC_1}}}{c}, \quad \bar{\delta}\alpha = \left(\frac{1-2iC_1}{1+2iC_1}\right)\alpha^2. \tag{8.128}$$

Proof. For this metric function $f(\zeta, u)$, at first order the invariants are

$$\alpha = \frac{e^{a} e^{(\frac{1+i2C_1}{1-2iC_1})\bar{a}}}{\bar{c}}, \quad \bar{\alpha} = \bar{\alpha} = \frac{(\bar{c}\alpha)^{\frac{1-2iC_1}{1+2iC_1}}}{c} \quad .$$

At second order we have the usual invariant

$$\mu = 0,\ \nu = \left(\frac{1+2iC_1}{1-2iC_1}\right)(4\alpha)^{-1},\ \bar{\delta}\alpha = \left(\frac{1-2iC_1}{1+2iC_1}\right)\alpha^2.$$

Notice that there are two invariant classifying constants, this is due to the fact that by bringing in the constant into the bracketed term one may absorb the phase of this constant by making a coordinate transformation □

Lemma 8.9.4. *The PP-wave metric, expressed in terms of a canonical coframe and belonging to the* $G_3 - d_0 - 0$ *class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$,

$$f(\zeta, u) = e^{-2iC} ln\zeta,$$

Using the special coordinates in (8.3), *the one functionally independent invariant is*

$$\alpha = \alpha = e^{-iC} e^{a+\bar{a}}.$$

The only classifying function is then

$$\bar{\alpha} = e^{2iC}\alpha \tag{8.129}$$

Proof. By direct computation, the first order and second order invariants are:

$$\alpha = e^{-iC} e^{a+\bar{a}}, \quad \bar{\alpha} = e^{2iC}\alpha,$$
$$\mu = 0,\ \nu = \frac{1}{4\alpha},\ \bar{\delta}\alpha = \alpha^2 \quad .$$

□

Lemma 8.9.5. *The PP-wave metric belonging to the* $G_2 - d_0 - 1$ *class in figure* (8.1) *will have the canonical form for* $f(\zeta, u)$,

$$f(\zeta, u) = e^{-2iC} ln\zeta.$$

Using the special coordinates in (8.3)*, the functionally independent invariant is*

$$\alpha = e^{-iC} e^{a+\bar{a}}.$$

The essential classifying functions is

$$\bar{\alpha} \quad = \quad e^{2iC}\alpha.$$

Proof. For this $f(\zeta, u)$ at first order the invariants are:

$$\alpha = e^{iC} e^{a+\bar{a}}, \quad \bar{\alpha} = e^{2iC}\alpha \quad .$$

At second order we have the usual invariants

$$\mu = 0, \ \nu = (4\alpha)^{-1}, \ \bar{\delta}\alpha = \alpha^2,$$

□

Chapter 9

Applications of Cartan Invariants to the Plane Waves

This chapter is based on:

A. Coley, R. Milson, D. McNutt (2012). Vacuum Plane Wave Spacetimes: Cartan Invariants and Physical Interpretation. *CQG* Vol 29, Issue 23, pp 235023-235034.

9.1 The Plane wave spacetimes and Cartan Invariants

The plane waves were introduced by Rosen [3] in 1937 to describe wave-like solutions to the Einstein equations. However, due to the choice of coordinates, Rosen concluded that these metrics were unphysical due to singularities in the metric components. Upon further analysis these singularities were shown to be coordinate dependent and easily eliminated by a change in coordinates, [6, 7]. In 1961 the plane waves were shown to belong to the class of PP-wave spacetimes[1] [9, 11] describing pure radiation far from an isolated source; these were originally studied by Brinkmann in 1925 as a special case of those Einstein spaces which are related by conformal transformations, with the Ricci scalar vanishing [2]. In fact the complete family of PP-waves were outlined in this paper for arbitrary dimension. Despite certain global problems, like closed null geodesic curves [14], these spacetimes have been studied in classical Relativity as well as its generalizations [50, 33, 54].

In a PP-wave spacetime, all polynomial scalar invariants vanish [55]; therefore, to classify such spacetimes we need to apply the Karlhede equivalence method [95, 94]. To calculate the Cartan invariants for a given spacetime one chooses a canonical null tetrad where the Weyl tensor component has been normalized (i.e., $\tilde{\Psi}_4 = 1$) by making a specific spin and boost. The first order Cartan invariants in the Karlhede

[1]These are Petrov type N spacetimes admitting a covariantly constant null vector, ℓ.

algorithm arise as components of the first and second order covariant derivatives of the Weyl tensor $\mathbf{\Psi}$. As Ψ_4 is constant, these additional invariants take the form of the spin-coefficient α, γ, and their conjugates $\bar{\alpha}, \bar{\gamma}$ which are introduced at first order as components of the covariant derivative of the Weyl tensor. To study the plane waves, we assume $\alpha = 0$. It should be noted that the canonical frame is not unique, as a null rotation about ℓ leaves $\Psi_4 = 1$ and γ invariant. We add a subscript c to indicate the fact that these are spin coefficients relative to this class of canonical coframes (in which $\Psi_4 = 1$), and hence Cartan invariants as well.

Following the analysis in the previous section, at second order the plane waves offer only one new Cartan invariant, $\Delta\gamma_c$ as all of the remaining second order invariants vanish, i.e. $\mu_c = \nu_c = \bar{\delta}\alpha_c = 0$. This is reflected in the Newman Penrose equations, where by setting these invariants to zero one may show that $\delta\gamma_c = D\gamma_c = 0$ implying that γ_c is a function of u only. From these facts we have a helpful proposition to classify the plane wave spacetimes

Proposition 9.1.1. *A plane wave spacetime may be locally described in an invariant manner, using the triplet of Cartan invariants* $\{\gamma_c, \bar{\gamma}_c, \Delta\gamma_c\}$*, where the remaining two invariants are expressed in terms of* γ_c *and hence do not change in any coordinate system.*

In the cases where γ_c is non-zero and constant,then relative to Brinkmann coordinates, one may show that these are the G_6 spacetimes given in table 9.1. Assuming γ_c is non-constant, we note that γ_c is a function of the null coordinate u only, so that instead of γ_c we may take the real or imaginary component of gamma as the functionally independent invariant and the classifying invariants $\bar{\gamma}_c$ and $\Delta\gamma_c$ are replaced with real valued functions.

For example if $\gamma_1 = Re(\gamma_c) \neq 0$ then the remaining invariants become $\gamma_2 = Im(\gamma_c)$ and $\Delta\gamma_1$. Locally we may take the inverse of this real-valued function and express u in terms of γ_1, of course if $\gamma_2 = Im(\gamma_c)$ is non-zero one could repeat this procedure. From this argument we have a helpful corollary:

Corollary 9.1.2. *If* $\gamma_1 = Re(\gamma_c)$ *(or* $\gamma_2 = Im(\gamma_c)$*) is non-constant and non-zero, one may write the null coordinate, u, in terms of* γ_1 *(or* γ_2*) , i.e.* $u = U(\gamma_1)$ *(or* $\tilde{U}(\gamma_2)$*).*

To see this, consider the Brinkmann form for the vacuum plane wave metric [50],

$$-2dudv - 2H(u)du^2 + 2d\zeta d\bar{\zeta}, \quad H(u,\zeta,\bar{\zeta}) = Re(f(\zeta,u)). \tag{9.1}$$

In this coordinate system the sole non-zero Weyl tensor component takes the form:

$$\Psi_4 = \frac{1}{2}H_{,\zeta\zeta} = 0. \tag{9.2}$$

Defining $a = \frac{1}{4}lnH_{,\zeta\zeta}$, we may write α as

$$\alpha = e^{a-\bar{a}}\bar{a}_{,\bar{\zeta}\bar{\zeta}} = 0, \tag{9.3}$$

clearly $\bar{a}_{,\bar{\zeta}} = 0$, so that $\bar{f}_{,\bar{\zeta}\bar{\zeta}\bar{\zeta}} = 0$, giving a solution of the form

$$f(\zeta,u) = A(u)\zeta^2. \tag{9.4}$$

Expressing γ_c in these coordinates:

$$\gamma_c = \frac{1}{4\sqrt{A\bar{A}}}ln(\bar{A})_{,u}. \tag{9.5}$$

While a particular coordinate system is being used, regardless of which coordinate system is used, we may calculate γ_c in the canonical coframe, as only Lorentz transformations are used. Supposing that $\gamma_c = \gamma_1 + i\gamma_2$ with $A = r(u)e^{i\theta(u)}$, one may solve for A in Brinkmann coordinates, in terms of the real-valued functions involved in γ, using the lemma:

Lemma 9.1.3. *For any pp-wave spacetime expressed in terms of a canonical coframe with $\alpha_c = 0$ and $\gamma_c = \gamma_1 + i\gamma_2, \quad \neq 0$ we may express the canonical form for $f(\zeta,u)$ as*

$$A = re^{i\theta}; \quad r(u) = [C_0 - \textstyle\int 4\gamma_1 du]^{-1}, \quad \theta(u) = -\textstyle\int 4r\gamma_2 du + C_1, \quad C_0, C_1 \in \mathbb{R} \tag{9.6}$$

Here, γ_c gives rise to the only functionally independent invariant and the essential classifying functions are $\bar{\gamma}_c(u)$ and $\Delta\gamma_c(u)$ expressed in terms of γ_c. If γ_c is constant, there are two possibilities for $A(u)$ depending on where γ_c lies in the complex plane these are given in table 9.1.

All of the remaining Cartan invariants involved depend on γ_c and hence are classifying functions invariantly describing the space.

	$f(\zeta,u)$	1	2
G_5	$\frac{A(u)}{2}\zeta^2$, (9.4)	γ_c; (9.6)	$\Delta\gamma_c$
G_6-a	$\frac{u^{\frac{iC_1}{C_0}-1}}{16C_0^2}\zeta^2$	$\gamma_c = C_0 + iC_1$	
G_6-b	$e^{iC_1 u}\zeta^2$	$\gamma_c = i\frac{C_1}{4}$	

Table 9.1: Summary of cases with $\alpha_c = 0$. Here, $C_0, C_1 \in \mathbb{R}$, and $A(u)$ is a complex valued function.

As an example we provide an invariant description for the class of spacetimes for which all timelike geodesic observers produce a linear polarization in terms of the equations of geodesic deviation [50], which will be discussed in the next section. For now we make the following definition in terms of Cartan invariants

Definition 9.1.4. A vacuum plane wave spacetime is *linearly polarized* when the Cartan invariant, γ_c, is real-valued.

Applying lemma (9.1.3) we find a particular form for the linearly polarized waves:

Corollary 9.1.5. *Given a vacuum plane wave spacetime, relative to the class of canonical coframes where $\Psi_4 = 1$ suppose $\bar{\gamma}_c = \gamma_c$. The metric expressed in Brinkmann coordinates has A real-valued and*

$$A = [C_0 - \textstyle\int 4\gamma_c du]^{-1}.$$

In the canonical coframe, we will say the plane wave is $'+'$ linearly polarized. Making a rotation in the spatial coordinates $\zeta' = e^{i\pi/4}\zeta$ (equivalently a spin in the transverse plane), the metric function is multiplied by i so that $\Psi_4 = i$, and we say this is $' \times$ ‘ linearly polarized. For more general polarization states the triplet of invariants $\{\gamma_c(u), \bar{\gamma}_c(u), \Delta\gamma_c(u)\}$ describes how the $'+'$ and $'\times'$ polarization states mix.

9.2 The Equations of Geodesic Deviation - Polarization Modes for all PP-wave Spacetimes

To study the polarization modes of gravitational waves in vacuum spacetimes with cosmological constant, a particular null tetrad is introduced relative to the Brinkmann coordinates . This null tetrad arises from the choice of an orthonormal frame in which the equations of geodesic deviation take a simpler form:

$$\frac{d^2 Z^\mu}{d\tau^2} = \ddot{Z}^\mu = -R^\mu{}_{\alpha\beta\gamma} u^\alpha Z^\beta u^\gamma. \tag{9.7}$$

Here $\dot{\mathbf{x}} = d\mathbf{x}/d\tau$, $|\mathbf{x}|^2 = -1$ is the four velocity of a timelike geodesic curve corresponding to a free test particle, τ is the proper time along this curve and $Z(\tau)$ is a displacement vector transverse to $\dot{\mathbf{x}}$. To construct the desired null tetrad, one first produces an orthogonal frame with $\dot{\mathbf{x}} = e_1$ and the remaining vectors $\{e_2, e_3, e_4\}$ from the local hypersurface orthogonal to e_1 (so that $< e_a, e_b >= g_\alpha \beta e_a^\alpha e_b^\beta = \eta_{ab}$). The dual basis will be $e^1 = -\dot{\mathbf{x}}$ and $e^i = e_i, i = 2, 3, 4$. This will hold at a point along the timelike geodesic $x^\mu(\tau)$. If we wish to have this hold on the entire curve the coframe must be parallely transported along the curve, yielding further conditions on the components of the metric and the four-velocity $\dot{\mathbf{x}}(\tau)$.

Choosing Kundt coordinates, the metric takes the form (9.1)

$$2d\zeta d\bar{\zeta} - 2dudv - 2Hdu^2, \quad H = Re(f(\zeta, u));$$

the plane waves are further constrained, as the analytic function must be of the form, $f(\zeta, u) = A(u)\zeta^2$. These solutions admit an isometry group of dimension five or six, which is reflected in the form of the complex function $A(u)$. Alternatively, in terms of Cartan invariants: the first order scalar, α, must vanish and the isometry group is six dimensional if and only if γ is constant, otherwise it is five dimensional.

Briefly, in Brinkmann coordinates the PP-wave spacetimes belong to the subclass of $KN(\Lambda)[\alpha', \beta']$ [33] with $\Lambda = 0$ and where the arbitrary functions α' and β' may be set to $\alpha' = 1$, $\beta' = 0$ via an appropriate coordinate transform preserving the metric form. We project the geodesic deviation equations onto this orthonormal frame in the case $\Lambda = 0$ and $\Psi_4 \neq 0$:

$$\ddot{Z}^1 = 0, \quad \ddot{Z}^2 = -A_+ Z^2 + A_\times Z^3, \quad \ddot{Z}^3 = A_+ Z^3 + A_\times Z^2, \quad \ddot{Z}^4 = 0 \tag{9.8}$$

Where the dot above a function denotes differentiation with respect to the proper time τ of the geodesic and

$$A_+ \equiv \frac{1}{4}(\Psi_4 + \bar{\Psi}_4), \quad A_\times \equiv \frac{i}{4}(\bar{\Psi}_4 - \Psi_4). \tag{9.9}$$

Using the null tetrad,

$$m_i = \frac{1}{\sqrt{2}}(e_2 + ie_3), \quad n_i = \frac{1}{\sqrt{2}}(e_1 - e_4), \quad \ell_i = \frac{1}{\sqrt{2}}(e_1 + e_4), \tag{9.10}$$

and denoting $Z = Z^0\ell_i + Z^1 n_i + z^2 m_i + z^3 \bar{m}_i$ with $\bar{z}^2 = z^3$, the equations of geodesic deviation become:

$$\begin{aligned} \ddot{Z}^0 = 0, \ddot{Z}^1 = 0, \\ \ddot{z}^2 = -(A_+ + iA_\times)z^3 = -\tfrac{1}{2}\Psi_4 z^3. \end{aligned} \tag{9.11}$$

To determine the form of the null tetrad (9.10) for the PP-wave spacetimes one must choose ℓ to be the preffered null direction along which the Weyl tensor has one non-vanishing component, Ψ_4. With this null direction, one may prove the following proposition

Proposition 9.2.1. *Let $\dot{\mathbf{x}}$ be the four velocity of a timelike geodesic, and ℓ some null vector. Then there exists a unit spacelike vector e_4 which is the projection of the null direction given by ℓ into the hypersurface orthogonal to $\dot{\mathbf{x}}$. This spatial vector is unique (up to reflections) and is given by $e_4 = -\dot{\mathbf{x}} + \sqrt{2}\ell$, where $< \ell, \dot{\mathbf{x}} >= -\frac{1}{\sqrt{2}}$. Another null vector n in the $(\dot{\mathbf{x}}, e_4)$ plane such that $< \ell, n >= -1$ is then given by $n = \sqrt{2}\dot{\mathbf{x}} - \ell$. The only remaining freedom are rotations in the (e_2, e_3) plane.*

In Kundt coordinates $(\zeta, \bar{\zeta}, u, v)$, we have two more propositions the first giving the form of the null tetrad:

Proposition 9.2.2. *In Kundt coordinates, the null tetrad tied to the 4-velocity of the geodesic, $\dot{\mathbf{x}} = (\dot{\zeta}, \dot{\bar{\zeta}}, \dot{u}, \dot{v})$, takes the simple form*

$$\begin{aligned} m_i^\mu = \left(-\tfrac{\dot{\zeta}}{\dot{u}}, 0, -1, 0\right), \ \bar{m}_i^\mu = \left(-\tfrac{\dot{\zeta}}{\dot{u}}, -1, 0, 0\right), \\ \ell_i^\mu = \left(\tfrac{1}{\sqrt{2}\dot{u}}, 0, 0, 0\right), \ n_i^\mu = \left(\sqrt{2}\dot{v} - \tfrac{1}{\sqrt{2}\dot{u}}, \sqrt{2}\dot{\zeta}, \sqrt{2}\dot{\bar{z}}, \sqrt{2}\dot{u}\right). \end{aligned} \tag{9.12}$$

where the function H in the metric (9.1) *is hidden away by the identity*

$$2\dot{\zeta}\dot{\bar{\zeta}} - 2\dot{u}\dot{v} - H\dot{u}^2 = -1.$$

Remark*: The null vector ℓ is no longer a covariantly constant null vector, as $\nabla_{\frac{\partial}{\partial x^\mu}}\ell_i = -\dot{u}\left(\frac{1}{\dot{u}}\right)_{,\mu}\ell_i$; however, it is a recurrent null vector and there is a covariant constant null vector proportional to the original vector ℓ.*

Of course, for an arbitrary unit timeike geodesic, $\dot{\mathbf{x}} = (\dot{\zeta}, \dot{\bar{\zeta}}, \dot{u}, \dot{v})$, one may reconstruct the usual metric coframe

$$m_n^\mu = (1,0,0,0), \quad \bar{m}_n^\mu = (0,1,0,0), \quad \ell_n^\mu = (0,0,0,1), \quad n_n^\mu = (0,0,1,-H) \tag{9.13}$$

from the interpretation tetrad $\{m_i, \bar{m}_i, \ell_i, n_i\}$ in (9.12) by applying the following Lorentz transformation

$$\begin{gathered} \ell_n = A\ell_i, \quad n_n = A^{-1}(\ell_i + Be^{iV}\bar{m}_i + \bar{B}e^{-i\mathfrak{V}}m_i + B\bar{B}n_i), \; m_n = e^{-i\mathfrak{V}}m_i + B\ell_i \\ A = \sqrt{2}\dot{u}, \quad B = -\sqrt{2}\dot{\zeta}, \quad \mathfrak{V} = \pi \end{gathered} \tag{9.14}$$

To relate this to a physical description, one must have a tetrad that will be defined on all points along the timelike geodesic curve, and not just one point. In the more general $KN(\Lambda)[\alpha', \beta']$ class this requirement imposes further differential constraints on the metric functions. Luckily in the case of the PP-waves, these constraints vanish and we have a final proposition.

Proposition 9.2.3. *For any timelike geodesic $x^\mu(\tau) = (\zeta, \bar{\zeta}, u, v)$ in a PP-wave spacetime, the null tetrad given by* (9.12) *is parallely transported along this geodesic.*

Proof. From Proposition 3 in [50], the tetrad arising from setting $\Lambda = 0$, $\alpha = 1$ and $\beta = 0$ [2] via a coordinate transform gives the following conditions for the null tetrad $A4$ in [50] to be parallel transported along the timelike geodesic:

$$\left(\frac{q}{p}\right)_{,\zeta} = \left(\frac{q}{p}\right)_{,\bar{\zeta}} = 0, \quad \dot{\mathfrak{V}}(\tau) = i\left(\frac{p_{,\zeta}}{p}\dot{\zeta} - \frac{p_{,\bar{\zeta}}}{p}\dot{\bar{\zeta}}\right). \tag{9.15}$$

In these coordiates, $p = q = 1$ and so the above vanishes, implying $\mathfrak{V}$ must be a constant as its dot derivative is zero. □

The interpretation tetrad (9.12) (up to constant spins and boosts) is the only tetrad which is parallel transported along the arbitrarily chosen timelike geodesic and provides the simplest form from which one can determine the polarization of a wave along that timelike geodesic. However, the geodesic deviation equations are frame dependent. As a simple example of this one may show that the magnitude of

[2]These are defined in [50] and are not to be confused with the spin-coefficients

the wave is dependent on the timelike observer. By applying a boost in an arbitrary direction with constant velocity (v_1, v_2, v_3), it is easily shown that

$$\Psi_4' = \frac{(1-v_3)^2}{1-v_1^2-v_2^2-v_3^2}\Psi_4 \tag{9.16}$$

the magnitude of the plane wave is dependent on the timelike observer as well. This can cause problems, as an observer travelling with a higher velocity relative to the original timelike geodesic will measure a smaller value for the magnitude of the wave. In fact, setting $v_1 = v_2 = 0$ and taking the limit $v_3 \to 1$ causes $\Psi_4' \to 0$.

A more elaborate example arises when we wish to express the geodesic deviation equations in terms of the canonical tetrad. To construct the canonical null tetrad from (9.12) one must apply a spin and boost to normalize $\tilde{\Psi}_4 = 1$:

$$\begin{aligned} \tilde{\ell} = e^X \ell_i, \quad \tilde{n} = e^{-X} n_i, \quad \tilde{m} = e^{iP} m_i \\ e^{2X} = |\Psi_4^i|, \quad e^{2IP} = \Psi_4^i / \bar{\Psi}_4^i. \end{aligned} \tag{9.17}$$

As long as we are concerned with points on the timelike geodesic the Weyl tensor component Ψ_4 will be a function of τ, because one may substitute $x^\mu = (\zeta(\tau), \bar{\zeta}(\tau), u(\tau), v(\tau))$ to make everything dependent on proper time. In this sense, the two frames may be related to each other along an arbitrary timelike geodesic,

$$\Psi_4^i = A^2 \Psi_4^n = \dot{u}^2 f_{,\zeta\zeta}, \tag{9.18}$$

where Ψ_4^i and Ψ_4^n are the Weyl tensor components relative to the interpretation tetrad (9.12) and natural tetrad (9.13) respectively.

The interpretation tetrad is best suited to give a physical description of a plane wave spacetime. If one is interested in the classification of the plane waves the canonical coframe and Cartan invariants provide a general classification that complements the study of the geodesic deviation equations. As an illustration of this, we will show that those spacetimes for which all timelike geodesics the equations of geodesic deviation are linearly polarized may be defined in an invariant fashion, and that the '+' and '×' linear polarization modes arise as a choice of coordinates.

Looking back at (9.17) and (9.18), $e^{i2P} = \Psi_4 / \bar{\Psi}_4$ is an invariant that is independent of the choice of timelike geodesic - as it lacks $\dot{u}$ and all of the other components of the timelike geodesic 4-velocity [3]. Any constant boost leaves this quantity invariant while a constant spin, $m' = e^{iC}m$, produces the net affect of adding a constant to Y, i.e. $P' = P + C$, C mod 2π. Rewriting e^{2iP} it is easily shown that there is only one real function involved, assuming $A_+ \neq 0$,

$$\Psi_4/\bar{\Psi}_4 = \frac{1 + i\frac{A_\times}{A_+}}{1 - i\frac{A_\times}{A_+}}.$$

While if $A_+ = 0$ the phase is already determined, i.e. $P = \pi/2$ mod 2π. Now if we apply lemma (9.1.3) and equation (9.6) from the previous section, we note that in the case of $'+'$ linear polariation, $A_+ = A = r(u)$ and $A_\times = 0$ while in the $'\times'$ linear polarization $A_+ = 0$, and $A_\times = ir(u)$. If the phase of Ψ_4 is constant in the complex plane this is called a linearly polarized wave ([50, 98, 22], etc).

Lemma 9.2.4. *Relative to the null tetrad* (9.12)*, if the phase P of Ψ_4, defined as*

$$P = \frac{1}{2} arctan \left(\frac{A_\times}{A_+} \right) \in (-\frac{\pi}{4}, \frac{\pi}{4}), \tag{9.19}$$

is constant or $A_+ = 0$ (implying $Y = \pm\frac{\pi}{4}$), then the vacuum plane wave spacetime is linearly polarized with constant phase $2P$ and γ_c must be real valued. In particular, if $P = 0$ the wave is in a $'+'$ linear polarization, and if $P = \pm\frac{\pi}{4}$ the wave is in the $'\times'$ linear polarization; each of these linear polarization modes are equivalent to each other via a spatial rotation.

Proof. To start we take the interpretation null tetrad (9.12) and apply a spin and boost to produce the canonical coframe in which $\Psi_4 = 1$. For any point along the arbitrarily chosen timelike geodesic, we may express the Cartan invariant γ_c (relative to the class of canonical frames) in terms of τ, as all of the coordinates may be replaced with functions of τ. Then by expressing Ψ_4 relative to the original interpretation coframe (9.12) and imposing the particular form for the plane waves: $f = A(u)\zeta^2$, the non-vanishing Weyl tensor component (9.18) along the timelike geodesic becomes:

$$\Psi_4 = \dot{u}A(u(\tau)).$$

[3] Glossing over the fact that the coordinates $(\zeta, \bar{\zeta}, u, v)$ may be written as function of τ for some timelike geodesic with proper time.

Imposing the condition that $P = \frac{1}{2}arctan\left(\frac{A_\times}{A_+}\right)$ is constant so that $\Psi_4 = S(\tau)e^{iP}$, S is a real valued function and

$$\frac{\Psi_4^i}{\bar{\Psi}_4^i} = \frac{\Psi_4^n}{\bar{\Psi}_4^n} = \frac{A}{\bar{A}}.$$

Since $\dot{u}$ is real valued already this implies that A must have constant phase P in the complex plane, by direct substitution into (9.5) one shows γ_c is real-valued, so that this is indeed a linearly polarized plane wave. Rotating the coordinates $(\zeta, \bar{\zeta})$ by $\theta/2$ [4] we may set $P = 0$, the plane wave is now '+' linearly polarized. Another rotation by $\pi/4$ will give the '$\times$' linear polarization, $P = \pi/2$. One can differentiate between these choices of coordinates by comparing Ψ_4 in the differing coordinate systems relative to (9.12). □

Thus, the two defining physical properties for the linearly polarized waves are the unchanging phase of the wave as $u(\tau)$ varies and the fact that the magnitude an observer measures depends on two functions: the value of $\dot{u}^2$ and the function,

$$\frac{|\Psi_4|}{\dot{u}^2} = A_+(u(\tau))$$

as $u(\tau)$ varies along the worldline of the observer.

9.3 The Vacuum Plane Waves and the Rosen Form

As another application of the classification we will consider a diffeomorphism that does not preserve Kundt form. We will use the transformation given in equation (24.49) in [22] to switch from Kundt form to Rosen form. In light of the results in [98, 99], where a general formalism was introduced for studying arbitrary polarization states of pp-wave spacetimes with $\alpha = 0$ in Rosen coordinates, we would like to apply Lemma (9.1.3) so that any novel solution found by this formalism may be expressed in Kundt coordinates.

In Rosen coordinates the metric is written in the simple form:

$$ds^2 = -2dudr + g_{AB}(u)dx^A dx^B, \quad A, B \in [1, 2]$$

[4]Equivalently applying a spin to the frame vectors m and $\bar{m}$.

where the three functions involved in the symmetric g_{AB} are tied together by the vanishing of the only non-zero Ricci tensor component arising from

$$R_{AB} = -\left(\frac{1}{4}g''_{AB} - \frac{1}{4}g'_{AC}g^{CD}g'_{DB}\right),$$

where differentiation with respect to u is denoted by primes. For a vacuum plane wave with arbitrary polarization we need two arbitrary functions of u. A particular form of the Rosen metric was introduced by Bondi, Pirani and Robinson [6, 7] to study gravitational plane waves:

$$\begin{aligned} ds^2 &= -e^{2Y}dudr + u^2 cosh2Z(dx^2 + dy^2) \\ &\quad + u^2 sinh2Z cos2W(dx^2 - dy^2) - u^2 sinh2Z sin2W(dxdy)]. \end{aligned} \tag{9.20}$$

where Y, Z, W are functions of u satisfying,

$$2Y' = u(Z^{'2} + W^{'2} sinh2W).$$

By examining the two independent components of the Riemann tensor: σ and ω defined in [7], the fixed plane polarization mode occurs if and only if $W = 0$. In this case the metric simplifies to be

$$ds^2 = -e^{2Y}dudr + u^2[e^{2Z}dx^2 + e^{-2Z}]dy^2.$$

Choosing a new null coordinate $\tilde{u} = \int e^{2Y(u)}du$ this becomes the usual Rosen metric for + linear polarization,

$$ds^2 = -d\tilde{u}dr + \tilde{Y}(\tilde{u})^2[e^{2Z}dx^2 + e^{-2Z}dy^2]$$

where $\tilde{Y}$ denotes the inverse function of e^{2Y}. One may apply a rotation of the $(\zeta, \bar{\zeta})$ coordinates to produce $\times$ linear polariation or any other linear polarization mode of fixed phase. With that observation we have proven a helpful lemma

Lemma 9.3.1. *In Rosen form, a vacuum plane wave is linearly polarized if and only if coordinates exist in which* $W' = 0$.

From which the results of the previous section imply:

Corollary 9.3.2. *Relative to the canonical coframe, if* γ_c *is real-valued, coordinates may be found in which* $W = 0$.

Even in the simpler form (9.20), the plane waves in Rosen form are much more complicated than their Kundt counterparts. For example, in the case of linear polarization modes, the equations connecting $\tilde{Y}$ and Z require considerably more analysis. In [98] and [99] this problem was studied. Using the metric form (9.20) along with the coordinate transformation $u' = \int e^{2Y} du$, the metric is now

$$\begin{gathered} ds^2 = -2dudr + S^2(u)[A'(u)dx^2 + 2B'(u)dxdy + C'(u)dy^2], \\ A' = cosh[X'(u)] + cos[\theta'(u)]sinh[X'(u)], \quad B' = sin[\theta'(u)]sinh[X'(u)], \\ C' = cosh[X'(u)] - cos[\theta'(u)]sinh[X'(u)]. \end{gathered} \tag{9.21}$$

If $\theta = 0$ this metric describes linearly $+$ polarized waves, while if it is constant one has a linear polarization of along the axes produced by rotating by an angle θ_0. For example, setting $\theta = \frac{\pi}{2}$ yields the linearly $\times$ polarized waves.

Noting that $A > 0$ for all values of u we may construct a null tetrad from this metric:

$$\ell = du, \quad n = dr, \quad m = S(u)\left[\sqrt{C - \frac{B^2}{A}}dy + i\sqrt{A}\left(dx + \frac{B}{A}dy\right)\right] \tag{9.22}$$

We boost and rotate this null tetrad to construct an invariant coframe for which $\Psi_4 = 1$ and γ_c is the only functionally independent invariant. In Kundt coordinates the metrics describing $+$ and $\times$ polarizations produce a real value and purely imaginary value for γ_c respectively; this fact holds true in the Rosen coordinates and so we may use lemma (9.1.3) to solve for the metric function A used to describe this spacetime in Kundt coordinates. An arbitrarily polarized wave will have $\bar{\gamma}_c \neq \pm\gamma_c$, however given γ_c in one coordinate system one may integrate to solve for A in the Kundt coordinates using lemma (9.1.3) .

9.4 An Example: The weak-field Circularly Polarized Waves

The circularly polarized waves were originally introduced as a weak-field solution, using the metric anzatz (9.21) and requiring that $X_{,u} = 0$, $\theta_{,u} \neq 0$ and $\theta_{,uu} = 0$. By imposing the final constraint that $X << 1$ we satisfy the weak-field vacuum condition. These metrics were generalized to a class of strong field solutions [98], [99] by requiring that $X = X_0$ and $\theta = \theta_0 u$ and the metric becomes

$$\begin{gathered} ds^2 = -2dudv + S^2(u)[A(u)dx^2 + 2B(u)dxdy + C(u)dy^2], \\ A = cosh[X_0] + cos[\theta_0 u]sinh[X_0], \quad B = sin[\theta_0 u]sinh[X_0], \\ C = cosh[X_0] - cos[\theta_0 u]sinh[X_0]. \end{gathered} \tag{9.23}$$

Remark 9.4.1. Notice that if $X_0 = 0$ this metric reduces to the Minkowski metric which cannot happen as we have assumed $\Psi_4 \neq 0$. Similarly by inspecting the Ricci and Weyl spinor components displayed below, we see that $\theta_0 \neq 0$ as well.

In these coordinates the sole non-vanishing Ricci tensor component is,

$$R_{00} = -\frac{1}{2}\left(4\frac{S_{,uu}}{S} + sinh^2(X_0)\theta_0^2\right) \tag{9.24}$$

Imposing vacuum conditions we find a form for S

$$S = S_0 cos\left(\frac{sinh(X_0)\theta_0(u-u_0)}{2}\right).$$

In the strong field regime the construction of the Cartan invariants is considerably more involved. To provide a simple application of our work, we examine the weak field conditions by imposing $X_0 << 1$ so that $X^2 = 0$.

Thus for an arbitrarily long interval of u the function S may be approximated to be a constant

$$S \approx S_0.$$

Without loss of generality we may always set $S_0 = 1$ and so the metric is approximately of the form

$$ds^2 = -2dudv + dx^2 + dy^2 + X_0[cos(\theta_0 u)(dx^2 - dy^2) + 2sin(\theta_0 u)dxdy].$$

Defining the following combinations of the functions A, B, C in (9.23),

$$\begin{gathered} D = C - \frac{B^2}{A} = -\frac{-1+X_0^2}{1+cos(\theta_0)X_0} = \frac{1}{1+cos(\theta_0)X_0} \\ E = \frac{B}{A} = \frac{sin(\theta_0 u)X_0}{1+cos(\theta_0)X_0}, \end{gathered}$$

we may construct a null tetrad from the metric

$$\ell = du,\ n = dv, \quad m = D^{\frac{1}{2}}dy + iA^{\frac{1}{2}}(dx + Edy).$$

To see that this is approximately a vacuum spacetime we calculate the sole component of the Ricci spinor which does not automatically vanish

$$\Phi_{22} = -\frac{1}{4}\frac{X_0^2\theta_0^2(2cos(\theta_0 u)X_0+X_0^2+1)}{1+2cos(\theta_0 u)X_0+cos(\theta_0 u)^2X_0^2},$$

imposing the weak-field condition it is clear that this does indeed vanish as $X_0^2 = 0$. For the remainder of this section we will assume this implicitely.

To produce the Cartan invariants for these spaces we must normalize Ψ_4. Relative to the natural coframe metric this component is:

$$\Psi_4 = \frac{(isin(\theta_0 u) - cos(\theta_0 u))\theta_0^2 X_0}{2(1+cos(\theta_0 u)X_0)^2}. \tag{9.25}$$

We introduce a new function, $a = \frac{1}{4}ln\Psi_4$, to produce the class of canoncial null tetrads for which $\Psi_4 = 1$,

$$\ell' = e^{a+\bar{a}}\ell,\ n' = e^{-a-\bar{a}}n,\quad m' = e^{a-\bar{a}}m$$

By direct calculation using the transformation laws for spin-coefficients we calculate γ relative to this frame. We add a subscript $''c''$ to indicate the fact that this is a Cartan invariant and a spin-coefficient as well:

$$\gamma_c = \frac{1}{2\sqrt{2}\sqrt{X_0}}\left(iX_0cos(\theta_0 u) + \frac{2X_0sin(\theta_0 u)}{(1+cos(\theta_0 u)X_0)^3} + (1+cos(\theta_0 u)X_0)\right) \tag{9.26}$$

Differentiating with respect to u

$$\gamma_{c,u} = \theta_0\left(iX_0sin(\theta_0 u) + \frac{2X_0sin(\theta_0 u)}{(1+cos(\theta_0 u)X_0)^3} + sin(\theta_0 u)X_0) + O(X_0^2)\right)$$

and multiplying by

$$e^{-a-\bar{a}} = \sqrt{\frac{X_0}{2}}\left(\frac{1+cos(\theta_0 u)X_0}{\theta_0}\right)$$

we produce the second order invariant needed to fully classify the space:

$$\Delta\gamma_c = \frac{1}{4}\left(iX_0sin(\theta_0 u) + \frac{2X_0sin(\theta_0 u)}{(1+cos(\theta_0 u)X_0)^3} + sin(\theta_0 u)X_0\right) + O(X_0^2). \tag{9.27}$$

By necessity X_0 and θ_0 must both be non-zero, and so the combination $Y = \sqrt{\frac{2}{X_0}}(\gamma_c - \bar{\gamma}_c)$ is a real-valued invariant with the simple form

$$Y = cos(\theta_0 u). \tag{9.28}$$

We may locally express γ_c and the second order invariant $\Delta\gamma_c$ in terms of Y. Treating the null coordiate u as an invariant in some open subset we write u in terms of Y, $u = \theta_0^{-1} arccos(Y)$. In this local region these will be classifying constants for the space. Substituting this into the original Cartan invariants at first and second order we find

$$\begin{aligned}\gamma_c &= \frac{1}{2\sqrt{2}\sqrt{X_0}}\left(iX_0Y + \frac{2X_0\sqrt{1-Y^2}}{(1+YX_0)^3} + (1+YX_0)\right),\\ \Delta\gamma_c &= \frac{1}{4}\left(iX_0\sqrt{1-Y^2} + \frac{2X_0\sqrt{1-Y^2}}{(1+YX_0)^3} + \sqrt{1-Y^2}X_0\right).\end{aligned}$$

We have expressed all of the original Cartan invariants in terms of the imaginary part of γ_c scaled by some real-valued constant,

$$2Im(\gamma_c) = \sqrt{\frac{X_0}{2}}Y.$$

These two non-vanishing constants uniquely determine the circularly polarized waves in the weak-field approximation.

To provide a physical description, we use the coordinate independent formalism developed in [50]. As these two spaces are equivalent, there is a diffeomorphism between the two coordinate systems. That is, the null tetrad built from the metric (9.13) in Kundt coordinates is mapped to the null tetrad (9.22). Therefore we may use (9.25) to express the equations of geodesic deviation. The real-valued functions A_+ and $A_\times$ are

$$A_+ = \frac{\theta_0^2 X_0 sin(\theta_0 u)}{4(1+cos(\theta_0 u)X_0)^2}, \quad A_\times = \frac{-\theta_0^2 X_0 cos(\theta_0 u)}{4(1+cos(\theta_0 u)X_0)^2}$$

Along an arbitrary timelike geodesic, the coordinates may be expressed in terms of the proper time, τ, then by applying a boost, spin and null rotation about ℓ one produces the coframe which is parallely transported along the curve [50]. Taking the spatial plane and using the null tetrad (9.10) with $\bar{z}^2 = z^3$, (9.11) becomes:

$$\ddot{z}^2 = \frac{-i\dot{u}^2 X_0 \theta_0^2 e^{-i\theta_0 u}}{4(1+cos(\theta_0 u)X_0)^2} z^3. \tag{9.29}$$

Here the $\dot{u}$ and u are (the only) functions of τ which identity the geodesic solution in the geodesic deviation equations, i.e. the particular timelike congruence. In this form we see that the phase of the gravitational wave, $e^{-iC_0 u}$, varies in a circular manner for any timelike geodesic, with θ_0 dictating how quickly the phase spins as $u(\tau)$ changes.

It is clear that the magnitude of the wave depends on the the timelike geodesic chosen. Given a particular timelike geodesic and corresponding interpretation frame, one may apply a boost in the positive direction of the wave to produce a new coframe in which the magnitude of Ψ_4 is changed [50]. However, along a particular timelike geodesic if the value of $\dot{u}^2$ were taken into account, the observer would notice the magnitude of the gravitational wave measured along the curve will additionally vary as $u(\tau)$ changes:

$$\frac{|\Psi_4|}{\dot{u}^2} = \frac{\theta_0^2 X_0}{4(1+cos(\theta_0 u)X_0)^2}.$$

These two properties: the change of phase moving in a circular motion and the additional change in magnitude as $u(\tau)$ varies along the worldline, determine the physical properties of the weak-field vacuum circularly polarized waves. As a comparison, recall the case of linear $''+''$ polarization modes where the phase of the wave is constant, and the magnitude of the wave depends on the value of $\dot{u}^2$ and the function,

$$\frac{|\Psi_4|}{\dot{u}^2} = A_+(u(\tau)).$$

9.5 One Last Application: The Plane Wave Spacetimes with $\bar{\gamma}_c = -\gamma_c$

In section (9.2) we saw that the class of plane waves with the invariant, γ_c, a real valued scalar, corresponds to those plane waves in which any timelike geodesic gives rise to a linear polarization mode in the form of the geodesic deviation equations along the geodesic. We now consider the plane waves with the classifying function, $\bar{\gamma}_c = -\gamma_c$, by expressing the metric in Kundt coordinates using Lemma (9.1.3). With a particular metric form we then examine the geodesic deviation equations relative to the complex null tetrad $\{\ell, n, m, \bar{m}\}$ in the form of (9.11).

Assuming $\gamma_c = ig(u)$ where g is real-valued, Lemma (9.1.3) implies $f(\zeta, u) = A(u)\zeta^2$ has the following form for $A = r(u)e^{i\theta(u)}$:

$$r(u) = \frac{1}{C_0^2}, \quad \theta(u) = \frac{4}{C_0^2}\int g du + C_1.$$

If we apply the transformation $u' = C_0 u$, $v' = \frac{v}{C_0}$, $\zeta' = e^{-\frac{C_1}{2}}\zeta$, these functions become,

$$r' = 1, \quad \theta' = \frac{4}{C_0}\int \tilde{g}(u')du'$$

As $A' = C_0^2 e^{-C_1} A$ we note that γ_c has the following transformation rule $\gamma_c' = \frac{\gamma_c}{C_0}$: equating the two we find that in the primed coordinate system,

$$g'(u') = \frac{\tilde{g}(u')}{C_0}.$$

Thus without loss of generality the form of the metric function for these plane waves is

$$A(u) = e^{4\int ig(u)du} = e^{4\int \gamma_c du}. \tag{9.30}$$

Relative to the metric coframe, the non-vanishing Weyl tensor component is

$$\Psi_4 = e^{4\int ig(u)du}$$

Applying a frame transformation of the form (9.14) to take the metric coframe to the interpretation coframe, the equations of geodesic deviation are now

$$\ddot{Z}^2 = -\frac{\dot{u}^2}{2}cos(4\int gdu)Z^2 + \frac{\dot{u}^2}{2}sin(4\int gdu)Z^3, \tag{9.31}$$

$$\ddot{Z}^3 = \frac{\dot{u}^2}{2}cos(4\int gdu)Z^3 + \frac{\dot{u}^2}{2}sin(4\int gdu)Z^2, \tag{9.32}$$

or relative to the complex null tetrad,

$$\ddot{z}^2 = \frac{\dot{u}^2}{2}e^{4\int ig(u)du}z^3. \tag{9.33}$$

The sole components of the geodesic curve $u, \dot{u}$ are involved in the above form, from which we see that the magnitude of the wave is directly related to $\dot{u}(\tau)^2$ as

$$\frac{|\Psi_4|}{\dot{u}^2} = 1.$$

How the wave polarization varies is directly related to P, defined by the equation (9.19), now a function of τ along the timelike geodesic:

$$P(\tau) = \frac{1}{2}arctan(tan(4\int gdu)) = 2\int g(u(\tau))\dot{u}d\tau$$

As $u(\tau)$ varies the polarization mode will vary as well. Thus the characteristic physical properties of these spaces consist of a magnitude entirely dependent on the compoe-nent $\dot{u}(\tau)$ of the timelike geodesic observer's 4-velocity, and that the phase, $e^{\int \gamma du}$ is determined by $\gamma(u)$ as $u(\tau)$ varies along the curve.

Imposing conditions on $\gamma_c = ig(u)$ can yield further conditions. As an example suppose consider the subcase where $\gamma = iC_0 u^n, \quad n \in \mathbb{Z}$ and $C_0 \in \mathbb{R}$, in this case the metric form is $A = e^{iC_0 u(\tau)^{n+1}}$. Tthe magnitude of the wave measured is influenced by the value of $\dot{u}$ alone, while the phase changes in a circular motion, with a certain orientation and rate of speed as determined by C_0 and n, as $u(\tau)$ varies along the worldline.

Chapter 10

The Karlhede Classification of the Vacuum Kundt Waves

This chapter is based on: D. McNutt, R. Milson, A. Coley (2013). Vacuum Kundt Waves. *CQG* Vol 30, Issue 5, 055010-055039.

10.1 Introduction

The Kundt waves, were originally defined by Kundt in 1961 [9], as a special subcase of the class of pure radiation solutions of Petrov type III or higher and Plebanski-Petrov (PP) type O or vacuum, admitting a non-twisting, non-expanding null congruence, ℓ, that is

$$\ell^a \ell_a = 0, \quad \ell^a{}_{;a} = 0, \quad \ell_{(a;b)}\ell^{a;b} = 0, \quad \ell_{[a;b]}\ell^{a;b} = 0.$$

These conditions restrict the Petrov type for the the plane-fronted waves to Petrov type N or O. The pp-waves are defined as the non-twisting plane-fronted waves, so that ℓ is a covariantly constant null vector $\ell_{a;b} = 0$; the Kundt waves are then the class of twisting plane-fronted waves. Choosing Kundt coordinates, the metric for the Kundt waves is

$$ds^2 = d\zeta d\bar{\zeta} - du\left(dv - \tfrac{2v}{\zeta+\bar{\zeta}}d\zeta - \tfrac{2v}{\zeta+\bar{\zeta}}d\bar{\zeta} + \left(4H(\zeta,\bar{\zeta},u)(\zeta+\bar{\zeta}) - \tfrac{v^2}{(\zeta+\bar{\zeta})^2}\right)du\right), \quad (10.1)$$

where u, v are null coordinates, $\zeta, \bar{\zeta}$ are complex coordinates for the transverse space[22].

All polynomial curvature invariants, built from contracting the Riemann tensor and covariant derivatives with each other, vanish for these spacetimes. Thus, the plane-fronted belong to the collection of VSI spacetimes where all polynomial curvature invariants vanish [55]; this is in turn a subclass of the CSI spacetimes in which all polynomial curvature invariants are constant [95]. These spaces have been explored in four dimensions and shown to belong to the class of degenerate Kundt metrics [92].

These are the Kundt metrics where the frame used to classify the Riemann tensor (i.e., Petrov or Riemann type [66]) and the kinematic frame are aligned, i.e., they are the same; it is expected that this is the case in higher dimensions as well [65, 92].

For a given spacetime in four dimensions, either a spacetime is uniquely determined by its polynomial scalar curvature invariants, a (locally) homogeneous space, or a degenerate Kundt spacetime [92]. For the degenerate Kundt spacetimes, the equivalence problem is particularly relevant, given that one cannot determine the inequivalence of two metrics of this class by comparing polynomial scalar curvature invariants [55, 95, 94]. To invariantly classify these spacetimes, one must use an alternative tool, the Karlhede algorithm, which utilizes the Cartan equivalence method [4] adapted to the case of Lorentzian manifolds [24] .

The first and second stages of the Karlhede algorithm were analyzed for all type N vacuum spacetimes with $\Lambda = 0$ by Collins [36], who produced a theoretical upper bound on the highest order, q, of the covariant derivatives of the curvature tensor required for each of the various subclasses of the type N spacetimes. Interestingly, this gives a hard upper bound for the VSI spacetimes [55, 95], as the pp-waves and vacuum Kundt waves make up the entirety of type N VSI spacetimes [33, 55, 50]. Collins has shown that the pp-waves require $q \leq 4$ while the vacuum Kundt waves need at most $q \leq 6$. We have shown that the pp-wave upper bound is sharp, and that the Kundt-wave's actual upper bound is five [47]. However, in 2000, Skea produced a *non-vacuum* Kundt wave in which $q = 5$, suggesting that there might be vacuum solutions for which $q = 5$ [52].

In this paper, we discuss the upper bound for the vacuum Kundt waves in the Karlhede algorithm or, equivalently, the highest order, q, covariant derivative of the curvature required to invariantly classify these spaces. We show that the upper bound may be lowered to be less than or equal to four by exploring all possible outcomes of the Karlhede algorithm (see figures (10.2), (10.3) and (10.4)). Out of all possible invariant counts only one actual vacuum Kundt wave may be integrated; namely, the class with invariant count $(0, 1, 3, 4, 4)$. Due to the exhaustive nature of this analysis we examine the remaining branches of possibilities in the algorithm to produce an invariant classification of all vacuum Kundt waves. This classification is summarized into two tables describing each of the non-diffeomorphic vacuum Kundt wave metrics arising by the choice of the metric function $f(\zeta, u)$. We present twelve propositions relating the form of the metric function $f(\zeta, u)$ to the essential Cartan invariants characterizing each spacetime. The final section contains all of the potential subcases of the Karlhede algorithm applied to the vacuum Kundt wave spacetimes prior to examining the geometric structure of these spacetimes.

10.2 Geometric Structure of the Vacuum Kundt Waves

If we wish to preserve the form of the metric, the permitted coordinate transformations will be [55]:

$$\zeta' = \zeta + i\tilde{C}, \quad u' = h(u), v' = \tfrac{v}{h_{,u}} - (\zeta + \bar{\zeta})^2 \tfrac{h_{,uu}}{2h_{,u}^2}, \qquad (10.2)$$
$$H' = \tfrac{H}{h_{,u}^2} + \tfrac{(\zeta+\bar{\zeta})}{4h_{,u}^4}(-3h_{,uu}^2 + 2h_{,u}h_{,uuu}),$$

where $\tilde{C}$ is a real constant and $h(u)$ is an arbitrary real function. Taking the metric (10.1), we work with the Newman-Penrose formalism [29] to calculate the non-vanishing curvature components of the Ricci (Φ) and Weyl (Ψ) spinors, respectively:

$$\Phi_{22} = xH_{,\zeta\bar{\zeta}}; \quad \Psi_4 = 2H_{,\bar{\zeta}\bar{\zeta}}.$$

To impose vacuum conditions, H must be harmonic and real-valued; as in the pp-waves, this will be the real part of an analytic function, $2H = f(\zeta, u) + \bar{f}(\bar{\zeta}, u)$. To examine the geometric structure of these spaces, we work with the class of coframes in which $\Psi_4 = 1$. These are found by applying an appropriate spin and boost to the natural metric coframe.

Without imposing the vacuum condition, the non-vanishing Bianchi identities imply the relationship between the spin-coefficients and the components of the Ricci and Weyl spinors [29] and their frame derivatives $D, \Delta, \delta, \bar{\delta}$:

$$\kappa = \sigma = \rho = 4\epsilon = 0, \; D\Phi_{22} = 0,$$
$$\bar{\delta}\Phi_{22} = (4\beta - \tau)\Psi_4 + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22}.$$

Imposing the vacuum conditions, we see that $\beta = \frac{\tau}{4}$. The non-vanishing Newman-Penrose field equations for the vacuum Kundt waves are

$$D\tau = 0, \quad D\alpha = 0 \tag{10.3}$$

$$D\gamma = \tfrac{5}{4}\tau\pi + \tau\alpha + \bar{\pi}\alpha + \tfrac{1}{4}\tau\bar{\tau}, \tag{10.4}$$

$$D\lambda - \bar{\delta}\pi = \pi^2 + \alpha\pi - \tfrac{1}{4}\bar{\tau}\pi, \tag{10.5}$$

$$D\mu - \delta\pi = \pi\bar{\pi} - \pi\bar{\alpha} + \tfrac{1}{4}\pi\tau, \tag{10.6}$$

$$D\nu - \Delta\pi = \pi\mu + \bar{\tau}\mu + \bar{\pi}\lambda + \tau\lambda + \gamma\pi - \bar{\gamma}\pi, \tag{10.7}$$

$$\Delta\lambda - \bar{\delta}\nu = -\mu\lambda - \bar{\mu}\lambda - 3\gamma\lambda + \bar{\gamma}\lambda + 3\alpha\nu + \pi\nu - \tfrac{3}{4}\bar{\tau}\nu - \Psi_4, \tag{10.8}$$

$$\delta\alpha - \tfrac{1}{4}\bar{\delta}\tau = \alpha\bar{\alpha} + \tfrac{1}{16}\tau\bar{\tau} - \tfrac{1}{2}\alpha\tau, \tag{10.9}$$

$$\delta\lambda - \bar{\delta}\mu = \mu\pi - \bar{\mu}\pi + \mu\alpha + \tfrac{1}{4}\mu\bar{\tau} + \lambda\bar{\alpha} - \tfrac{3}{4}\lambda\tau, \tag{10.10}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + \gamma\mu + \bar{\gamma}\mu - \bar{\nu}\pi + \tfrac{1}{4}\tau\nu - \bar{\alpha}\nu, \tag{10.11}$$

$$\delta\gamma - \tfrac{1}{4}\Delta\tau = \tfrac{1}{2}\tau\gamma - \bar{\alpha}\gamma + \tfrac{5}{4}\mu\tau + \tfrac{1}{4}\tau\bar{\gamma} + \alpha\bar{\lambda}, \tag{10.12}$$

$$\delta\tau = \tfrac{5}{4}\tau^2 - \tau\bar{\alpha}, \tag{10.13}$$

$$-\bar{\delta}\tau = -\tfrac{3}{4}\bar{\tau}\tau - \alpha\tau, \tag{10.14}$$

$$\Delta\alpha - \bar{\delta}\gamma = -\tfrac{5}{4}\tau\lambda + \bar{\gamma}\bar{\alpha} - \bar{\mu}\bar{\alpha} - \tfrac{3}{4}\tau\gamma, \tag{10.15}$$

while the commutator relations are

$$\begin{aligned}
(\Delta D - D\Delta)f &= [(\gamma + \bar{\gamma})D - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta]f, \\
(\delta D - D\delta)f &= [(\bar{\alpha} + \frac{\tau}{4} - \bar{\pi})D]f, \\
\delta\Delta - \Delta\delta)f &= [-\bar{\nu}D + (\frac{3\tau}{4} - \bar{\alpha})\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta]f, \\
(\bar{\delta}\delta - \delta\bar{\delta})f &= [(\bar{\mu} + \mu)D - (\bar{\alpha} - \frac{\tau}{4})\bar{\delta} - (\frac{\bar{\tau}}{4} - \alpha)\delta]f.
\end{aligned}$$

The benefit of working in the class of coframes for which $\Psi_4 = 1$ becomes apparent once one takes frame derivatives of the Weyl tensor, as only spin-coefficients and their derivatives appear as components of the Weyl tensor and its covariant derivatives. To illustrate, the first order derivatives of the Weyl tensor are

$$(D\Psi)_{50'} = 4\alpha, \quad (D\Psi)_{51'} = 4\gamma, \quad (D\Psi)_{40'} = 0,$$

$$(D\Psi)_{41'} = \tau, \quad (D\Psi)_{30'} = 0, \quad (D\Psi)_{31'} = 0.$$

At first order, one still has 2 degrees of frame freedom, using null rotations with complex parameter B, which affects the first order invariant γ and leaves α and τ unchanged:

$$\gamma' = \gamma + B\alpha + \frac{5}{4}\bar{B}\tau. \tag{10.16}$$

If $|\alpha| \neq \frac{5}{4}|\tau|$ it is always possible to set $\gamma' = 0$. However, if equality holds, only one degree of freedom can be fixed, and there are three subcases for the form of γ' [36]:

- $\bar{\alpha} = -\frac{5}{4}\tau$: $Im(\gamma') = 0$;
- $\bar{\alpha} = \frac{5}{4}\tau$: $Re(\gamma') = 0$;
- $\bar{\alpha} \neq \pm\frac{5}{4}\tau$: $Re(\gamma')$ or $Im(\gamma') = 0$, but not both.

Without fixing the frame freedom, the non-zero second order derivatives of the Weyl tensor are:

$$\begin{aligned}
(D^2\Psi)_{50';00'} &= 4D\alpha,\\
(D^2\Psi)_{50';01'} &= 4(\delta\alpha + 5\beta\alpha - \bar{\alpha}\alpha),\\
(D^2\Psi)_{50';10'} &= 4(\bar{\delta}\alpha + 5\alpha^2),\\
(D^2\Psi)_{50';11'} &= 4(\Delta\alpha + 5\gamma\alpha - \bar{\gamma}\alpha + \bar{\tau}\gamma),\\
(D^2\Psi)_{51';00'} &= 4(D\gamma - 5\pi\beta - \bar{\pi}\alpha),\\
(D^2\Psi)_{51';01'} &= 4(\delta\gamma - 5\mu\beta + 5\beta\gamma - \bar{\lambda}\alpha + \bar{\alpha}\gamma),\\
(D^2\Psi)_{51';10'} &= 4(\bar{\delta}\gamma - 5\lambda\beta + 5\alpha\gamma - \bar{\mu}\alpha + \bar{\beta}\gamma),\\
(D^2\Psi)_{51';11'} &= 4(\Delta\gamma - 5\nu\beta + 5\gamma^2 - \bar{\nu}\alpha + \bar{\gamma}\gamma),\\
(D^2\Psi)_{40';11'} &= 4(\tau\alpha + \bar{\tau}\beta),\\
(D^2\Psi)_{41';00'} &= 4D\beta,\\
(D^2\Psi)_{41';01'} &= 4(\delta\beta + 3\beta^2 + \bar{\alpha}\beta),\\
(D^2\Psi)_{41';10'} &= 4(\bar{\delta}\beta + 3\alpha\beta + \bar{\beta}\beta),\\
(D^2\Psi)_{41';11'} &= 4(\Delta\beta + 3\gamma\beta + \tau\gamma + \bar{\gamma}\beta),\\
(D^2\Psi)_{31';11'} &= 8\tau\beta.
\end{aligned}$$

If $|\alpha| = \frac{5}{4}|\tau|$, it is always possible to fix the last parameter of the frame freedom to fix $\Delta\tau$ so that $Re(\Delta\tau) = 0$. Manipulating the spin-coefficients and the remaining degrees of freedom, Collins produced a theoretical upper bound for these spaces [36], requiring at most six covariant derivatives. This bound was lowered to five covariant derivatives by Machados Ramos and Vickers [47] using the generalized GHP formalism. In both papers a particular choice of coordinates was avoided so that these bounds were not shown to be sharp.

10.3 An Alternative Proof That The Upper Bound for the Karlhede Algorithm is Less than Six

The Karlhede algorithm terminates if and only if the dimension of the isotropy group and number of functionally independent invariants do not change between two consecutive iterations using the invariant count notation, it is possible to map out all possibilities for the Karlhede algorithm. The case where the invariant count begins with $(0, 0, ...)$ is not permitted as the invariant τ must be non-constant at first order; if we assume τ is a constant we find from (10.13) and (10.14) that $\tau = 0$, which cannot be true since we are studying the vacuum Kundt waves. With this in mind, it is easily shown that there is only one scenario where $q = 6$ at most, (0,1,1,2,3,4,4)[1].

This invariant count would occur for the class of vacuum Kundt waves in which at first order only one functionally independent invariant appears and further that $|\alpha| = \frac{5|\tau|}{4}$. By choosing a particular coordinate system we may produce differential constraints on the metric function $H(\zeta, \bar{\zeta}, u) = Re(f(\zeta, u))$ by imposing the vanishing of the wedge products of the differentials of the spin-coefficients of α, τ and their conjugates. As the spins and boosts have been fixed to set $\Psi_4 = 1$, and since these two invariants α and τ are unchanged under the remainder of the isotropy group (null rotations about ℓ), these are already Cartan invariants. With a little effort and a change of coordinates we intend to prove the following theorem:

Theorem 10.3.1. *The vacuum Kundt waves require at most $q = 5$ iterations of the Karlhede algorithm to completely classify the spacetimes.*

To this end we introduce a new complex coordinate $a = \frac{1}{4}ln(f_{,\zeta\zeta})$. Relative to this new coordinate system, $\zeta = \zeta(a, u)$ and we find a differential constraint for the

[1]This notation is adopted in section D to summarize possible states of the Karlhede algorithm compactly.

metric function $\tilde{f}$,

$$\left(\frac{\tilde{f}_{,a}}{\zeta_{,a}}\right)_{,a} = e^{4a}\zeta_{,a}. \tag{10.17}$$

The metric coframe becomes:

$$\begin{aligned} m &= \zeta_{,a}da + \zeta_{,u}du, \\ \ell &= du, \\ n &= dv - \frac{2v}{\zeta+\bar{\zeta}}(\zeta_{,a}da + \zeta_{,u}du) - \frac{2v}{\zeta+\bar{\zeta}}\bar{\zeta}_{,\bar{a}}d\bar{a} + \bar{\zeta}_{,u}du) \\ &\quad + \left(2Re(\tilde{f}(a,u))(\zeta+\bar{\zeta}) - \frac{v^2}{(\zeta+\bar{\zeta})^2}\right)du. \end{aligned} \tag{10.18}$$

In these coordinates the non-zero component of the Weyl tensor is now

$$\Psi_4 = 2(\zeta+\bar{\zeta})e^{4b}.$$

Applying a spin and boost with $p = \frac{1}{4}ln(|\bar{\Psi}_4|) = a + \frac{1}{4}ln(2(\zeta+\bar{\zeta}))$ to the metric coframe (10.18), we produce a new coframe:

$$m' = e^{p-\bar{p}}m, \quad \ell' = e^{p+\bar{p}}\ell, \quad n' = e^{-p-\bar{p}}n. \tag{10.19}$$

Relative to this coframe, the non-vanishing Weyl tensor component has been normalized $\Psi_4' = 1$ and the spin-coefficients α and τ are already Cartan invariants as they are unaffected by the remaining isotropy.

By direct calculation we produce the following spin coefficients relative to this coframe:

Proposition 10.3.2. *The spin-coefficients relative to the class of coframes* (10.19)*, in which* $\Psi_4 = 1$*, may be expressed as*

$$\begin{gathered} \tau = 4\beta = -\bar{\pi} = -\tfrac{e^{\bar{a}-a}}{\zeta+\bar{\zeta}}, \\ \mu = \lambda = 0, \\ \alpha = \tfrac{\bar{\tau}}{4} + \sqrt{\tfrac{\bar{\tau}}{\tau}}(\bar{\zeta}_{,\bar{a}})^{-1}, \\ \gamma = -\tfrac{e^{-a-\bar{a}}|\tau|^{\frac{5}{2}}}{\sqrt{2}}\left(v + \tfrac{\bar{\zeta}_{,u}(\bar{\zeta}_{,a})^{-1}}{|\tau|^2}\right), \\ \nu = e^{-a-3\bar{a}}\left(\int \bar{\zeta}_{,\bar{a}}e^{4\bar{a}}d\bar{a} + f_1 - (f+\bar{f})|\tau|\right). \end{gathered} \tag{10.20}$$

Before we fix any more frame freedom to set all or a part of γ to zero, we may determine the explicit form of the metric function $f(\zeta, u)$ for the class of vacuum Kundt waves where *only one* functionally independent invariant arises in the set $\{\alpha, \tau, \bar{\alpha}, \bar{\tau}\}$:

Lemma 10.3.3. *Those spacetimes in which the spin-coefficients $\alpha, \bar{\alpha}$, τ and $\bar{\tau}$ are functionally dependent on one invariant will have the following form for the metric function $f(\zeta, u)$:*

$$f(\zeta, u) = \frac{C_0^2}{16} e^{\frac{-4i(\zeta - iG(u) + C_1)}{C_0}} + f_1(u)\zeta + f_2(u). \tag{10.21}$$

Relative to the coordinates $a = \frac{1}{4} ln(f_{,\zeta\zeta})$, the Cartan invariants α and τ are now

$$\tau = \frac{-e^{\bar{a}-a}}{iC_0(a-\bar{a}) + 2C_1}, \quad \alpha = \frac{\bar{\tau}}{4} + \frac{i}{C_0}\sqrt{\frac{\bar{\tau}}{\tau}}. \tag{10.22}$$

Proof. Taking τ in (10.20), we calculate the double wedge product of $d\tau$ and $d\bar{\tau}$ to get,

$$d\tau \wedge d\bar{\tau} = \frac{2}{(\zeta+\bar{\zeta})^3}\left[(\zeta_{,a} + \bar{\zeta}_{,\bar{a}})da \wedge d\bar{a} + (\zeta_{,u} + \bar{\zeta}_{,u})da \wedge du + (\zeta_{,u} + \bar{\zeta}_{,u})d\bar{a} \wedge du\right].$$

Requiring that this must vanish gives a set of equations:

$$\zeta_{,a} = -\bar{\zeta}_{,\bar{a}}, \quad \zeta_{,u} = -\bar{\zeta}_{,u}.$$

Thus $\zeta(a, u)$ is of the form

$$\zeta(a, u) = i(C_0 a + G(u)) + C_1. \tag{10.23}$$

Plugging this into the expressions for τ and α in (10.20) we recover (10.22), then solving for a and noting that $e^{4a} = f_{,\zeta\zeta}$ we may integrate twice to recover the function in the usual coordinate system. □

The vacuum Kundt wave spacetimes with this property will potentially contain at most two functionally independent invariants at first order: τ and γ which will simplify the search for those vacuum Kundt waves with only one functionally independent invariant at first order. Furthermore, as the necessary conditions for fixing the remaining isotropy is dependent on the Cartan invariants α and τ we may use the explicit form of these invariants from lemma 10.3.3 to show all isotropy may be fixed at first order i.e. $|\alpha| \neq \frac{5}{4}|\tau|$, and that no vacuum Kundt wave requires $q = 6$ in the algorithm.

Corollary 10.3.4. *The invariant count* $(0, 1, 1, 2, 3, 4, 4)$ *cannot occur in the Karlhede classification of the vacuum Kundt waves.*

Proof. From lemma 10.3.3 we calculate the equality $|\alpha| = \frac{5}{4}|\tau|$ using equation (10.22). We assume the equality holds and multiply both sides by $|\alpha|$, then $|\alpha|^2 = \frac{25}{16}|\tau|^2$. Expanding this we have:

$$\frac{25}{16}|\tau|^2 = \frac{1}{16}|\tau|^2 + \frac{1}{C_0^2}.$$

Using the $a, \bar{a}$ coordinates and simplifying the resulting equation yields

$$\frac{3}{2}C_0^2 = (iC_0(a - \bar{a}) + 2C_1)^2.$$

Differentiating with respect to a or $\bar{a}$ implies that $C_0 = 0$ which cannot happen as $\zeta_{,a} = C_0$ must be non-zero. This is a contradiction and so $|\alpha| \neq |\tau|$. □

As this is the only permitted state in the Karlhede algorithm for the vacuum Kundt waves with $q = 6$, and this case cannot occur, we conclude that the upper-bound for the vacuum Kundt waves may be lowered to less than or equal to five.

10.4 Reducing the Upper Bound to Less than Five

The goal of this section is to provide the necessary lemmas to prove the following theorem:

Theorem 10.4.1. *The vacuum Kundt wave spacetimes require, at most, four derivatives (i.e.,* $q = 4$*) to classify these spaces using the Karlhede algorithm.*

To study the sharpness of the upper bound, we examine the possible iteration scheme for the Karlhede algorithm applied to the vacuum Kundt waves as tree diagrams. This may be done exhaustively for the cases where there are at least one invariant at the first iteration of the algorithm.

Lemma 10.4.2. *The vacuum Kundt wave spacetimes for which the Karlhede algorithm requires five iterations have invariant counts*

$$(0, 1, 2, 3, 4, 4), \quad \textit{and} \quad (0, 2, 2, 3, 4, 4) \quad .$$

Proof. The trees for the various possibilities are included in section D. □

To prove theorem 10.4.1 we must examine the constraints on the vacuum Kundt waves to produce the invariant counts in lemma 10.4.2. To do so we break up the analysis into two subsections to examine the distinct subclasses of vacuum Kundt waves with either one or two functionally independent invariants appearing at first order.

10.4.1 Vacuum Kundt waves with $(0, 1, 2, ..)$

Applying the results of lemma 10.3.3 and corollary 10.3.4, we are able to say something about the upper bound in the first case, as the invariant coframe is produced from (10.19) by making a null rotation (10.16) to set $\gamma' = 0$. We must determine the form of the parameter B for the null rotation taking the coframe (10.19) to the invariant coframe required for the Karlhede algorithm:

$$\ell' = \ell, \quad n' = n + \bar{B}m + B\bar{m} + |B|^2\ell, \quad m' = m + B\ell. \tag{10.24}$$

To achieve this, we equate (10.16) to zero and solve for B,

$$B = -\sqrt{2}|\tau|^{\frac{5}{2}} e^{-a-\bar{a}} \sqrt{\tfrac{\tau}{\bar{\tau}}} \left(\tfrac{C_0^2|\tau| - iC_0}{3C_0^2|\tau|^2 - 2}\right) \left(v + \tfrac{G_{,u}}{C_0|\tau|^2}\right). \tag{10.25}$$

Using the dual of the invariant coframe, $\{\delta', \bar{\delta}', \Delta', D'\}$, we may compute the second order Cartan invariants as the frame derivatives of the first order Cartan invariants along with the following transformed spin coefficients:

$$\begin{aligned}
\pi' &= \pi + D\bar{B}, \\
\lambda' &= \frac{\bar{B}\bar{\tau}}{2} + \sqrt{\frac{\bar{\tau}}{\tau}} \frac{2\bar{B}}{\bar{\zeta}_{,\bar{a}}} + \bar{B}\pi + \bar{B}D\bar{B} + \bar{\delta}\bar{B}, \\
\mu' &= \frac{\bar{B}\tau}{2} + B\pi + BD\bar{B} + \delta\bar{B}, \\
\nu' &= \nu + 2\bar{B}\gamma + \frac{3}{2}\bar{B}^2\tau + B\bar{B}(\pi + 2\alpha) + \Delta\bar{B} + \bar{B}\delta\bar{B} + B\bar{\delta}B + B\bar{B}D\bar{B}.
\end{aligned} \tag{10.26}$$

These remaining invariants are expressed in terms of the coframe (10.19), the original spin coefficients (10.20), and the frame derivatives of B relative to the original coframe

(10.19) with $\Psi_4 = 1$:

$$\begin{aligned} D &= \sqrt{\frac{2}{|\tau|}} e^{a+\bar{a}} \frac{\partial}{\partial_v}, \\ \Delta &= \sqrt{\frac{|\tau|}{2}} e^{-a-\bar{a}} \left(\frac{\partial}{\partial_u} - \left(\frac{2(f+\bar{f})}{|\tau|} - v^2|\tau|^2 \right) \frac{\partial}{\partial_v} - \frac{\zeta_{,u}}{\zeta_{,a}} \frac{\partial}{\partial_a} - \frac{\bar{\zeta}_{,u}}{\bar{\zeta}_{,\bar{a}}} \frac{\partial}{\partial_{\bar{a}}} \right), \quad (10.27) \\ \delta &= \frac{e^{a-\bar{a}}}{\bar{\zeta}_{,\bar{a}}} \frac{\partial}{\partial_{\bar{a}}} - 2v\bar{\tau} \frac{\partial}{\partial_v}. \end{aligned}$$

Noting that $\bar{\pi} = -\tau$ in (10.20) we subtract $-\tau$ from $\bar{\pi}'$, it is clear that DB is an invariant; a quick calculation confirms that it is functionally dependent on τ

$$DB = -2|\tau|^2 \sqrt{\frac{\tau}{\bar{\tau}}} \left(\frac{C_0^2|\tau| - iC_0}{3C_0^2|\tau|^2 - 2} \right).$$

We now examine the second order invariant arising from the frame derivative of $|\tau|^{-1}$

$$\Delta'(|\tau|^{-1}) = e^{\bar{a}-a}B + e^{a-\bar{a}}\bar{B} = \sqrt{\frac{|\tau|}{2}} Re(e^{\bar{a}-a}DB) \left(v + \frac{G_{,u}}{C_0|\tau|^2} \right).$$

Removing all expressions involving τ leaves the helpful invariant:

$$\xi = e^{-a-\bar{a}} \left(v + \frac{G_{,u}}{C_0|\tau|^2} \right). \quad (10.28)$$

As $|\alpha| \neq \frac{5}{4}|\tau|$, the remaining invariants at second order may be simplified to the spin-coefficients μ', λ' and ν'. These spin-coefficients involve B and the remaining frame derivatives of this function:

$$\begin{aligned} \bar{\delta}B &= \tau e^{-a-\bar{a}} \sqrt{\frac{|\tau|}{2}} \left[\frac{1}{2} + B_0 \right] DB \left(v + \frac{G_{,u}}{C_0|\tau|^2} \right), \\ \delta B &= \bar{\tau} e^{-a-\bar{a}} \sqrt{\frac{|\tau|}{2}} \left[\frac{1}{2} + B_0 - \frac{2}{iC_0|\tau|} \right] DB \left(v + \frac{G_{,u}}{C_0|\tau|^2} \right) \\ \Delta B &= e^{-2a-2\bar{a}} \left[\frac{|\tau|G_{,u}}{C_0} \left(v + \frac{G_{,u}}{C_0|\tau|^2} \right) - (f+\bar{f}) + \frac{v^2|\tau|^3}{2} + \frac{G_{,uu}}{2C_0|\tau|} \right] DB, \end{aligned}$$

where B_0 is the following complex rational function of τ,

$$B_0 = \tfrac{2}{iC_0|\tau|} + \tfrac{C_0^2|\tau|}{C_0^2|\tau| - iC_0} - \tfrac{6C_0^2|\tau|^2}{3C_0^2|\tau|^2 - 2}.$$

Combining these functions, it is clear that both μ' and λ' are expressed entirely in terms of τ and ξ.

At this point we are able to prove that at second order, at least two functionally independent invariants are produced if we wish to produce a vacuum Kundt wave with $q \geq 4$ in the algorithm.

Lemma 10.4.3. *All vacuum Kundt waves with the metric function $f(\zeta, u)$ of the form* (10.21) *and an invariant count starting with* $(0, 1, 2, ...)$ *in the Karlhede algorithm must end at third order; i.e., with an invariant count* $(0, 1, 2, 2)$.

Proof. The last invariant given in $(D^2\Psi)_{51';11'}$ gives one new candidate for a functionally independent invariant: $\frac{5}{4}\tau\nu' + \bar{\nu}'\alpha$. Applying the transformation law for ν' it is seen that we may remove the majority of the terms in ν' and instead study the new invariant: $\frac{5}{4}\tau(\nu + \Delta\bar{B}) + \alpha(\bar{\nu} + \Delta B)$. As $|\alpha| \neq \frac{5}{4}|\tau|$, we may always combine this and the conjugate to produce a simpler invariant

$$\nu + \Delta\bar{B}. \tag{10.29}$$

To start on this calculation we denote $F_x = Re(f_1)$ and $F_y = Im(f_1)$ and use the special form on ν in (10.20). We first simplify $f + \bar{f}$ using the functions in (10.21) and (10.23) to produce

$$\begin{aligned} f + \bar{f} \quad &= \quad \frac{C_0^2}{16}(e^{4a} + e^{4\bar{a}}) + 2F_x C_1 - 2GF_y + Re(f_2), \\ &\quad\; -F_y C_0(a + \bar{a}) + iF_x C_0(a - \bar{a}). \end{aligned}$$

Integrating the remaining term and substituting this into ν in (10.20) to get

$$\begin{aligned} \sqrt{\frac{\tau}{\bar{\tau}}}\nu \quad &= \quad -\frac{iC_0}{4}\frac{\tau}{\bar{\tau}} + \bar{f}_1(u)e^{-2a-2\bar{a}} \\ &\quad\; -\left(\frac{C_0^2}{16}\frac{\bar{\tau}}{\tau} + \frac{C_0^2}{16}\frac{\tau}{\bar{\tau}} + (F_x(\zeta + \bar{\zeta}) + iF_y(\zeta - \bar{\zeta}) + 2Re(f_2))e^{-2a-2\bar{a}}\right)|\tau|. \end{aligned}$$

We may remove several terms in ν which are already functionally dependent on τ. Plugging $f + \bar{f}$ into $\Delta\bar{B}$ we find similar terms that may be removed from $\nu + \Delta\bar{B}$:

$$\begin{aligned} \frac{\Delta\bar{B}}{D\bar{B}} \quad &= \quad e^{-2a-2\bar{a}}\left[\frac{|\tau|G_{,u}}{C_0}\left(v + \frac{G_{,u}}{C_0|\tau|^2}\right) + \frac{v^2|\tau|^3}{2} + \frac{G_{,uu}}{2C_0|\tau|}\right] \\ &\quad\; -e^{-2a-2\bar{a}}\left[F_x(\zeta + \bar{\zeta}) + iF_y(\zeta - \bar{\zeta}) + 2Re(f_2)\right] - \frac{C_0^2}{16}\frac{\bar{\tau}}{\tau} - \frac{C_0^2}{16}\frac{\tau}{\bar{\tau}}. \end{aligned}$$

With this in mind we may remove even more terms from $\tilde{\nu}$, to produce a new invariant:

$$\begin{aligned} N &= \sqrt{\tfrac{\bar{\tau}}{\tau}}(\bar{f}_1|\tau|^{-1}e^{-2a-2\bar{a}} - N_0')|\tau| + D\bar{B}(N_1' - N_0') \\ N_0' &= \left(F_x(\zeta + \bar{\zeta}) + iF_y(\zeta - \bar{\zeta}) + 2Re(f_2)\right)e^{-2a-2\bar{a}}, \\ N_1' &= \left[\tfrac{|\tau|G_{,u}}{C_0}\left(v + \tfrac{G_{,u}}{C_0|\tau|^2}\right) + \tfrac{v^2|\tau|^3}{2} + \tfrac{G_{,uu}}{2C_0|\tau|}\right]e^{-2a-2\bar{a}}. \end{aligned} \tag{10.30}$$

Multiplying $\sqrt{\frac{\bar{\tau}}{\tau}} = e^{a-\bar{a}}$ to N and taking the difference of this new quantity with its conjugate,

$$-\frac{4iC_0|\tau|^2(N_1' - N_0')}{3C_0^2|\tau|^2 - 2} - 2iF_y e^{-2a-2\bar{a}} = e^{a-\bar{a}}N - e^{\bar{a}-a}\bar{N}.$$

Removing this term from N leaves

$$N_2' = (F_x|\tau|^{-1}e^{-2a-2\bar{a}} - N_0')|\tau| + C_0|\tau|F_y e^{-2a-2\bar{a}}. \tag{10.31}$$

Scaling N_2' by $|\tau|^{-1}$, we calculate the triple wedge product of this invariant with the previous invariants; the coefficients of the triple wedge product relative to the coordinate 3-form basis are extensive. However, only one is necessary if we wish that the triple wedge product vanishes: the $da \wedge d\bar{a} \wedge dv$ coefficient yields the equation

$$-e^{-3a-3\bar{a}}[4(-C_0F_y + iF_y(\zeta - \bar{\zeta}) + 2Re(f_2)) + 2C_0F_y] = 0.$$

As $\zeta - \bar{\zeta}$ is linear function in $a + \bar{a}$, F_y must vanish and hence $Re(f_2) = 0$ as well.

These constraints cause $(F_x|\tau|^{-1}e^{-2a-2\bar{a}} - N_0')$ to vanish and so we work with the remaining invariant $N' = N_1' - N_0' = (N_1' - F_x|\tau|^{-1}e^{-2a-2\bar{a}})|\tau|^{-3}$,

$$N' = \left[\frac{G_{,u}}{C_0|\tau|^2}\left(v + \frac{G_{,u}}{C_0|\tau|^2}\right) + \frac{v^2}{2} + \frac{G_{,uu} - 2C_0F_x}{2C_0|\tau|^4}\right]e^{-2a-2\bar{a}}.$$

Using the same procedure of equating the triple wedge product of $\bar{a} - a$, ξ and N', we produce a very large 3-form which will not be included. Instead we examine the $da \wedge du \wedge dv$-component:

$$e^{-3a-3\bar{a}}\frac{2G_{,uu}G_{,u} + C_0G_{,uuu} - 2C_0^2F_{x,u}}{2C_0^2|\tau|^4}.$$

Integrating we find that $F_x = \frac{G_{,uu}}{2C_0} + \frac{G_{,u}^2}{2C_0^2} + C_2$ and that

$$N' = \frac{\xi^2}{2} + \left[\frac{C_2}{|\tau|^4}\right]e^{-2a-2\bar{a}}.$$

To continue, we eliminate the parts of this invariant expressed in terms of previous invariants, by denoting $N'' = N' - \frac{\xi^2}{2}$. We take the triple wedge product of this invariant with $a - \bar{a}$ and ξ to produce the following equation in the $da \wedge d\bar{a} \wedge du$ component:

$$e^{-3a-3\bar{a}}\frac{G_{,uu}}{C_0^2|\tau|^4} = 0.$$

Denoting $G_{,u} = C_2$, the remaining invariant becomes $N'' = C_2 e^{-2a-2\bar{a}}(C_0^2|\tau|^4)^{-1}$. If we wish to have only two functionally independent invariants at second order, $C_2 = 0$. This is generically the case, since if $G_{,u} \neq 0$ we may always set $G = C_2 u + C_3$ to zero using the coordinate transformation, (10.2) of the form:

$$u' = h(u), \quad v' = \frac{v}{h_{,u}} + \frac{h_{,uu}}{2h_{,u}^2 \tau|^2}, \quad h_{,u} = e^{-\frac{2}{C_0}G}.$$

Applying this transformation, the analytic function $f(\zeta, u)$ becomes

$$f'(\zeta, u) \quad = \quad \frac{C_0^2}{16} e^{\frac{-4i(\zeta + C_1)}{C_0}}.$$

At third order there are no candidates for a third functionally independent invariant, $\Delta'\xi$, as the unprimed frame derivatives yield

$$D\xi = \sqrt{\tfrac{2}{|\tau|}}, \quad \delta\xi = \sqrt{\tfrac{\bar{\tau}}{\tau}} \left(\tfrac{\xi}{iC_0} + 2|\tau|\xi \right), \quad \Delta\xi = \sqrt{\tfrac{|\tau|}{2}} \xi^2 |\tau|^2.$$

The Karlhede algorithm terminates with an invariant count $(0, 1, 2, 2)$. □

Thus we have shown that vacuum Kundt waves with an invariant count of $(0, 1, 2, 3, 4, 4)$ in the Karlhede algorithm cannot occur as the metrics with invariant counts starting with $(0, 1, 2, ...)$ must have $(0, 1, 2, 2)$ at the next order.

10.4.2 Vacuum Kundt waves with $(0, 2, 2, ...)$

To begin this section we prove a more general result for the vacuum Kundt waves with invariant count $(0, n, ...)$ $1 \leq n \leq 4$ and $|\alpha| = \frac{5}{4}|\tau|$.

Lemma 10.4.4. *For those vacuum Kundt waves with at least one functionally independent invariant appearing at first order and such that $|\alpha| = \frac{5}{4}|\tau|$ then $\bar{\alpha} \neq e^{i\theta}\frac{5}{4}\tau$, $\theta \in \mathbb{R}$.*

Proof. Expanding the conjugate of α using (10.20) we find

$$\frac{\tau}{4} + e^{\bar{a}-a}\bar{\zeta}_{,\bar{a}}^{-1} = -\frac{5}{4}e^{i\theta}\tau.$$

Upon simplification this leads to the equation

$$\frac{1+5e^{i\theta}}{4}\bar{\zeta}_{,\bar{a}} = \zeta(a, u) + \bar{\zeta}(\bar{a}, u).$$

A contraction arises here, as we may differentiate with respect to a giving $\zeta_{,a} = 0$, which cannot be so. □

Recalling the comment after equation (10.16) there are three cases to consider depending on the phase of the conjugate of α. This lemma implies that the first two cases where $\bar{\alpha} = \pm\frac{5}{4}\tau$ cannot occur.

To start narrowing the possibilities for $f(\zeta, u)$ we consider the wedge products of invariants built out of α, τ and their conjugates. As $T = e^{\bar{a}-a} = \sqrt{\tau/\bar{\tau}}$, and $A = \zeta_{,a} = (e^{a-\bar{a}}(\bar{\alpha} - \tau))^{-1}$ are both invariants it will be helpful to consider the triple wedge product:

$$dA \wedge d\bar{A} \wedge dT = -T(\bar{\zeta}_{,\bar{a}\bar{a}}\zeta_{,au} - \bar{\zeta}_{,\bar{a}u}\zeta_{,aa})da \wedge d\bar{a} \wedge du. \tag{10.32}$$

Alternatively, using the invariant $M = |\tau|^{-1} = \zeta(a, u) + \bar{\zeta}(\bar{a}, u)$, we have another equation as the coefficient of the triple wedge product:

$$dT \wedge dM \wedge dA = -T(\bar{\zeta}_{,\bar{a}\bar{a}}\zeta_{,u} + \bar{\zeta}_{,\bar{a}\bar{a}}\bar{\zeta}_{,u} - \bar{\zeta}_{,\bar{a}u}\bar{\zeta}_{,\bar{a}} - \bar{\zeta}_{,\bar{a}u}\zeta_{,a})da \wedge d\bar{a} \wedge du. \tag{10.33}$$

Equating these two wedge products to zero, we have sufficient information to solve for $f(\zeta, u)$ in the vacuum Kundt wave metrics with invariant count $(0, 2, ...)$, and hence narrow down the possibilities for those spacetimes with invariant count $(0, 2, 2, ...)$.

Lemma 10.4.5. *The vacuum Kundt wave metrics for which the triple wedge product of α, τ and their conjugates vanish have the following form:*

$$\tilde{f}(\zeta, u) = -\frac{F(u)^2}{16} e^{\frac{4(\zeta - f_0(u))}{iF(u)}} + g(u)\zeta + g_0(u) \tag{10.34}$$

$$\tilde{f}(\zeta, u) = \frac{c^2}{16} e^{\frac{4(\zeta - f_1(u))}{c}} + g_1(u)\zeta + g_2(u), \quad Re(C) \neq 0 \tag{10.35}$$

$$\tilde{f}(\zeta, u) = f_2(\zeta - c_0 - iF_3(u)) + g_3(u)\zeta + g_4(u) \tag{10.36}$$

$$\tilde{f}(\zeta, u) = e^{-\int F_5(u)du} f_6\left(\zeta - i\int F_4(u)du\right) + g_5(u)\zeta + g_6(u). \tag{10.37}$$

Proof. Equating equations (10.32) and (10.33) to zero, we have two differential equations for $\zeta(a, u)$ and its conjugate. There will be four cases depending on whether $\zeta_{,aa}$ and $\zeta_{,ua}$ are zero or not.

Case 1 - $\zeta_{,aa} = 0,\ \zeta_{,ua} \neq 0$:

Equation (10.32) vanishes entirely while (10.33) implies $\zeta_{,a} = -\bar{\zeta}_{,\bar{a}}$, so that

$$\zeta = iF(u)a + f_0(u) \tag{10.38}$$

Solving for a and integrating $\tilde{f}_{,\zeta\zeta} = e^{4a}$:

$$\tilde{f}_{,\zeta\zeta} = e^{\frac{4(\zeta - f_0)}{iF}}$$

we find the form (10.34).

Case 2 - $\zeta_{,aa} = 0,\ \zeta_{,ua} = 0$:

Here the constraints immediately imply

$$\zeta = ca + f_1(u). \tag{10.39}$$

Solving for a and integrating $\tilde{f}_{,\zeta\zeta} = e^{4a}$:

$$\tilde{f}_{,\zeta\zeta} = e^{\frac{4(\zeta - f_1)}{c}},$$

which yields the analytic function (10.35).

Case 3 - $\zeta_{,aa} \neq 0,\ \zeta_{,ua} = 0$:

These assumptions cause (10.33) to become $\zeta_{,u} + \bar{\zeta}_{,u} = 0$, implying ζ takes the form:

$$\zeta = \dot{f}_2^{-1}(a) + iF_3(u) + C_0. \tag{10.40}$$

Solving for a and assuming $\dot{f}_2 = \frac{1}{4} ln \ddot{f}_2$, the expression $\tilde{f}_{,\zeta\zeta} = e^{4a}$ becomes,

$$\tilde{f}_{,\zeta\zeta} = \ddot{f}_2(\zeta - C_0 - iF_3).$$

As $\dot{f}_2$ and $\ddot{f}_2$ are arbitrary functions of u, we make one more assumption, $\ddot{f}_2 = f_{2,\zeta\zeta}$. Integrating twice yields the desired metric function (10.36).

Case 4 - $\zeta_{,aa} \neq 0,\ \zeta_{,ua} \neq 0$

Re-arranging the functions we find

$$\frac{\zeta_{,au}}{\zeta_{,aa}} = \frac{\bar{\zeta}_{,\bar{a}u}}{\bar{\zeta}_{,\bar{a}\bar{a}}}$$

which is equivalent to $\zeta_{,au} - F_5(u)\zeta_{,aa} = 0$. Integrating with respect to a yields

$$\zeta_{,u} - F_5(u)\zeta_{,a} = f_4(u).$$

Substituting this into (10.33) we find that $f_4 = iF_4$, so that ζ takes the form

$$\zeta = \dot{f}_6^{-1}\left(a + \int F_5 du\right) + i\int F_4 du. \tag{10.41}$$

Solving for a and assuming $\dot{f}_6 = \frac{1}{4} ln \ddot{f}_6$ we find

$$\tilde{f}_{,\zeta\zeta} = e^{-\int f_5 du} \ddot{f}_6 \left(\zeta - i \int F_4 du \right).$$

Assuming $\ddot{f}_6 = f_{6,\zeta\zeta}$ and integrating twice, we recover (10.37). □

These metrics do not yet belong to the $(0, 2, ...)$ class as we must determine whether γ may be set to zero or not. If γ is non-zero, the various triple wedge products involving γ with α, τ and their conjugates give further conditions on the metric function $f(\zeta, u)$. By lemma 10.4.4 we see that $\bar{\alpha} \neq \pm\tau$ and hence we may eliminate the real or imaginary part of γ but not both as the ratio of the real part to the imaginary part of the quantity,$\alpha B + \frac{5}{4}\bar{B}\tau$, is $tan[\frac{1}{2}(arg(\alpha) + arg(\tau))] = tan(arg(e^{iC})) = C \neq 0$ [36].

Opting to eliminate the real part of γ, we note that the purely imaginary invariant γ' is invariant under any null rotation preserving $Re(\gamma) = 0$ due to the proportionality of the real and imaginary part of $\alpha B + \frac{5}{4}\bar{B}\tau$. Thus, without fixing the frame any further, the transformed scalar γ' is a Cartan invariant:

$$\begin{aligned} \gamma' &= i(Im(\gamma) - CRe(\gamma)) \\ &= i\frac{\sqrt{|\tau|}}{2\sqrt{2}} \left[\frac{i\zeta_{,u}}{\zeta_{,a}} - \frac{i\bar{\zeta}_{,u}}{\bar{\zeta}_{,\bar{a}}} + C \left(|\tau|^2 v + \frac{\zeta_{,u}}{\zeta_{,a}} + \frac{i\bar{\zeta}_{,u}}{\bar{\zeta}_{,\bar{a}}} \right) \right] e^{-a-\bar{a}}, \end{aligned}$$

and so we may consider the triple wedge product of the differentials of three invariants constructed from γ', τ, α and their complex conjugates.

Lemma 10.4.6. *The class of vacuum Kundt waves with an invariant count beginning with (0,2,2,...) and* $|\alpha| = \frac{5}{4}|\tau|$ *cannot occur.*

Proof. Using the invariants $e^{\bar{a}-a}$, $\zeta_{,a}$ along with γ' the triple wedge product produced has a considerable number of terms in each coefficient of the coordinate basis for three-forms. Taking the coefficients of $da \wedge d\bar{a} \wedge dv$, $da \wedge du \wedge dv$ and $d\bar{a} \wedge du \wedge dv$ and equating these coefficients to zero we find two constraints:

$$\frac{iC|\tau|^{\frac{5}{2}}e^{-2a}}{2\sqrt{2}}\zeta_{,aa} = 0,$$
$$\frac{iC|\tau|^{\frac{5}{2}}e^{-2a}}{2\sqrt{2}}\zeta_{,au} = 0.$$

Immediately we see that the metric function must be of the form (10.35) with the corresponding form of $\zeta(a,u)$ given in (10.39). Expressing α and τ in terms of this function the required equality $|\alpha| = \frac{5}{4}|\tau|$ implies

$$\frac{24}{16}|\tau|^2 - \frac{|\tau|}{4}(c^{-1} + \bar{c}^{-1}) - |c|^{-2} = 0,$$

where $|\tau| = (C_0(a+\bar{a}) + iC_1(a-\bar{a}) + 2Re(f_0))^{-1}$ with either C_0 or C_1 non-zero. Multiplying by $|\tau|^{-2}|c|^2$ the above equation is now

$$\frac{24|c|^2}{16} - \frac{C_0|\tau|^{-1}}{2} - |\tau|^{-2} = 0,$$

then by expanding $|\tau|^{-1}$ and differentiating twice with respect to a we find a constant that must vanish:

$$C_0 + iC_1 = 0.$$

This produces a contradiction as we have assumed $\zeta_{,a} \neq 0$, thus there are no vacuum Kundt wave spacetimes with an invariant count $(0, 2, ...)$ where the first order Cartan invariants satisfy $|\alpha| = \frac{5}{4}|\tau|$. □

We have shown that the collection of vacuum Kundt waves must have either an invariant count $(0, 2, 2)$ with all isotropy fixed at first order, or an invariant count of $(0, 2, 3...)$ with $|\alpha| = \frac{5}{4}|\tau|$ implying all isotropy is fixed at second order. Regardless of either case, none of the potential spacetimes arising from these subclasses produce an invariant count with $q = 5$. This lemma completes the proof of theorem 10.4.1 as we have shown the two possibilities for the Karlhede algorithm requiring $q = 5$ cannot occur.

10.5 Sharpness of the $q \leq 4$ Upper Bound

In this section we will show that the new upper bound is indeed sharp by producing an explicit metric function $f(\zeta, u)$.

Theorem 10.5.1. *The vacuum Kundt waves with invariant count* $(0, 1, 3, 4, 4)$ *are of the form*

$$f(\zeta, u) = \frac{C_0^2}{16} e^{\frac{-4i(\zeta + C_1)}{C_0}} + f_1(u)\zeta + f_2(u)$$

where f_1 and f_2 are non-constant and satisfy either

$$
\begin{gathered}
f_1 = (C_2 + i)F_y, \quad Re(f_2) = C_3 + \tfrac{C_0}{2} ln(F_y)F_y, \quad F_y \neq Cu^{-2}, \\
or \\
f_1 = F_x, \quad Re(f_2) = C_3 F_x, \quad F_x \neq Cu^{-2}.
\end{gathered} \tag{10.42}
$$

Proof. To prove this fact we calculate the quadruple wedge product of the differentials of $a - \bar{a}$, ξ and two new invariants arising in N where the invariant N_2 in (10.31) is now denoted as N_0, and N_1 arises from the imaginary part of N,

$$
\begin{aligned}
N &= \sqrt{\tfrac{\bar{\tau}}{\tau}} \left(N_0|\tau| + DB(DB + D\bar{B})^{-1} \left(N_0 + \tfrac{|\tau|^2 \xi^2}{2} + N_1 \right) \right) \\
N_0 &= \left(-C_0 F_y + iF_y(\zeta - \bar{\zeta}) + 2Re(f_2) \right) e^{-2a-2\bar{a}}, \\
N_1 &= \left[\tfrac{(G_{,u}^2 + G_{,uu}C_0 - 2C_0^2 F_x)}{2C_0^2|\tau|} + \tfrac{[C_0|\tau|^2 - 1]F_y}{C_0|\tau|^2} \right] e^{-2a-2\bar{a}}.
\end{aligned} \tag{10.43}
$$

In these coordinates, the invariants are a bit complicated; one may make a coordinate transformation to remove $G(u)$ in the function $f(\zeta, u)$ in (10.21). Applying a coordinate transformation (10.2) of the form:

$$
u' = h(u), \quad v' = \frac{v}{h_{,u}} + \frac{h_{,uu}}{2h_{,u}^2 \tau|^2}, \quad h_{,u} = e^{-\frac{2}{C_0}G}, \tag{10.44}
$$

the analytic function $f(\zeta, u)$ becomes

$$
f'(\zeta, u) = \tfrac{C_0^2}{16} e^{\frac{-4i(\zeta + C_1)}{C_0}} + e^{\frac{4}{C_0}G} f_1(u)\zeta + e^{\frac{4}{C_0}G} f_2(u) + \frac{e^{\frac{4}{C_0}G}(G_{,u}^2 + C_0 G_{,uu})\zeta}{C_0^2}.
$$

Finally, noting that $f_1 = F_x + iF_y$ and f_2 are arbitrary functions of u, we may just relabel the quantities and write the function as:

$$
f'(\zeta, u) = \tfrac{C_0^2}{16} e^{\frac{-4i(\zeta + C_1)}{C_0}} + f_1'(u')\zeta + f_2'(u'). \tag{10.45}
$$

Dropping the primes and repeating the calculations in lemma 10.3.3 and subsection (10.4.1) with this new function, one finds that N_0 and N_1 are now

$$
\begin{aligned}
N_0 &= \left(-C_0 F_y + iF_y(\zeta - \bar{\zeta}) + 2Re(f_2) \right) e^{-2a-2\bar{a}}, \\
N_1 &= \left[\tfrac{-2C_0^2 F_x}{2C_0^2|\tau|} + \tfrac{[C_0^2|\tau|^2 - 1]F_y}{C_0|\tau|^2} \right] e^{-2a-2\bar{a}}.
\end{aligned} \tag{10.46}
$$

From the quadruple wedge product $d(a-\bar{a})\wedge d\xi\wedge dN_0\wedge dN_1$ we find the sole coefficient is:

$$(N_{0,a}+N_{0,\bar{a}})N_{1,u}-(N_{1,a}+N_{1,\bar{a}})N_{0,u}=0.$$

Expanding this equation we find three essential equations whose vanishing is necessary and sufficient for the 4-form to vanish

$$\begin{gathered}(Re(f_2)+\tfrac{C_0}{4}F_y)F_{x,u}-Re(f_2)_{,u}F_x=0\\ (Re(f_2)+\tfrac{C_0}{4}F_y)F_{y,u}-Re(f_2)_{,u}F_y=0\\ F_yF_{x,u}-F_{y,u}F_x=0.\end{gathered}$$

To solve these equations we must consider two cases depending on whether $F_y=0$ or not. In the case that F_y does vanish, we find that $Re(f_2)$ may be expressed in terms of derivatives F_x, an arbitrary function:

$$Re(f_2)=C_3F_x. \tag{10.47}$$

While if $F_y\neq 0$ and arbitrary, we find that

$$F_x=C_2F_y,\ \ Re(f_2)=[C_3+\frac{C_0}{4}ln(F_y)]F_y. \tag{10.48}$$

The choice of these functions is reflected in the structure of the invariants. Supposing that $F_y=0$, we may express N_1 in terms of $N_0=Re(f_2)e^{-2a-2\bar{a}}$,

$$N_1=\left[\tfrac{C_3}{|\tau|}\right]N_0.$$

In the case that $F_y\neq 0$ we find that N_0 and N_1 may be expressed in terms of N_2,

$$\begin{gathered}N_2=F_ye^{-2a-2\bar{a}},\\ N_0=N_2(C_0|\tau|^{-1}+2C_1+ln(N_2/2)),\\ N_1=\left[\tfrac{C_2}{|\tau|}+\tfrac{C_0^2|\tau|^2-2}{C_0|\tau|^2}\right]N_2.\end{gathered} \tag{10.49}$$

Regardless of whether $F_y\neq 0$ or not, the third second order invariant arising here is of the form

$$\tilde{N}=F_0(u)e^{-2a-2\bar{a}}.$$

The frame derivatives of this invariant produce only one new functionally independent invariant,

$$\sqrt{\frac{2}{|\tau|}}\Delta\tilde{N} = F_{0,u}e^{-3a-3\bar{a}}$$

To determine the full class of $(0,1,3,4,4)$ vacuum Kundt waves, we must avoid those functions F_0 which give the invariant count $(0,1,3,3)$, this can only happen if F_0 is constant or when it satisfies the following differential equation,

$$F_{0,u} = -2\sqrt{C_4^{-1}}F_0^{\frac{3}{2}}$$

by integrating one finds that $F_0 = C_4 u^{-2}$.

In the case that F_0 is constant, all of the metric functions in (10.45) are independent of u and hence this is a G_1 metric with no u-dependence. In the other case, we may make a coordinate transformation,

$$u' = h(u), \quad v' = \frac{v}{h_{,u}} + \frac{h_{,uu}}{2h_{,u}^2\tau|^2}, \quad h_{,u} = u^{-1}, . \tag{10.50}$$

Dropping the primes, in these new coordinates the $(0,1,3,3)$ metrics with $F_y = 0$ are now of the form

$$f(\zeta,u) = \frac{C_0^2}{16}e^{\frac{-4i(\zeta+\frac{iC_0u}{2}+C_1)}{C_0}} + C_2\zeta + C_3 \tag{10.51}$$

while those metrics with $F_y \neq 0$ are now

$$f(\zeta,u) = \frac{C_0^2}{16}e^{\frac{-4i(\zeta+\frac{iC_0u}{2}+C_1)}{C_0}} + C_2\zeta + iC_3\left(\zeta + \frac{iC_0u}{2}\right) + C_4. \tag{10.52}$$

From [27] we conclude these are all G_1 spacetimes. □

We conclude this section with the result that the sharpness of the upper bound has been confirmed.

10.6 Uniqueness of the Vacuum Kundt Waves with $q = 4$ in the Karlhede Algorithm

Applying lemma 10.7.2 it is easily proven that the vacuum Kundt waves with invariant count $(0,3,3,4,4)$ cannot occur. Thus we need only investigate the existence of the

$(0,2,3,4,4)$ vacuum Kundt waves to determine the uniqueness of the vacuum Kundt waves with $q=4$.

By examining the remaining admissible branches in figure (10.1), it is clear the only possibility of a vacuum Kundt waves attaining $q=4$ in the Karlhede algorithm lies in the cases with invariant count $(0,1,3,...)$ and $(0,2,3,...)$. In both of these cases γ may be set to zero and the invariant coframe is entirely fixed.

Instead of working with the spin-coefficients relative to the invariant coframe with $\Psi_4 = 1$ and $\gamma = 0$, we examine the second order invariants found by decomposing the invariants α and τ to construct the simpler invariants: $a - \bar{a} = \frac{1}{2} ln\left(\frac{\bar{\tau}}{\tau}\right)$, $\zeta_{,a} = \sqrt{\frac{\bar{\tau}}{\tau}}(\bar{\alpha} - 4\tau)^{-1}$ and $\zeta + \bar{\zeta} = |\tau|^{-1}$. To produce higher order invariants we apply the frame derivatives of the invariant coframe $\ell' = \ell$, $n' = n + \bar{B}m + B\bar{m} + |B|^2\ell$, $m' = m + B\ell$ to these invariants. By choosing coordinates, the frame derivatives take the form (10.27)

By direct calculation with this simpler set of invariants and the invariant coframe we may prove the following proposition.

Proposition 10.6.1. *For all vacuum Kundt waves with $|\alpha| \neq \frac{5}{4}|\tau|$, the second order Cartan invariants with no functional dependence on the previous invariants consist of the following frame-derivatives of the first order invariants: $\sqrt{\frac{|\tau|}{2}} Z_0 = |\zeta_{,a}|^2 \Delta(a - \bar{a})$, $\sqrt{\frac{|\tau|}{2}} Z_1 = \zeta_{,a}\Delta\zeta_{,a}$, $\sqrt{\frac{|\tau|}{2}} Z_2 = \Delta(\zeta + \bar{\zeta}) =$ and $\sqrt{\frac{\tau}{\bar{\tau}}} Z_3 = \zeta_{,a}\bar{\delta}\zeta_{,a}$, along with the spin-coefficients μ', λ', ν':*

$$\begin{aligned}
Z_0 &= -e^{-a-\bar{a}}(\zeta_{,u}\bar{\zeta}_{,\bar{a}} - \bar{\zeta}_{,u}\zeta_{,a}) + \frac{\tau}{\bar{\tau}} B'\bar{\zeta}_{,\bar{a}} - \frac{\bar{\tau}}{\tau}\bar{B}'\zeta_{,a}, \\
Z_1 &= e^{-a-\bar{a}}(\zeta_{,au}\zeta_{,a} - \zeta_{,u}\zeta_{,aa}) + \frac{\tau}{\bar{\tau}} B'\zeta_{,aa}, \\
Z_2 &= \frac{\tau}{\bar{\tau}} B' + \frac{\bar{\tau}}{\tau}\bar{B}', \\
Z_3 &= \zeta_{,aa}, \\
\lambda' &= \frac{\bar{B}\bar{\tau}}{2} + \sqrt{\frac{\bar{\tau}}{\tau}} \frac{2\bar{B}}{\bar{\zeta}_{,\bar{a}}} + \bar{B}\pi + \bar{B}D\bar{B} + \bar{\delta}\bar{B}, \\
\mu' &= \frac{\bar{B}\tau}{2} + B\pi + BD\bar{B} + \delta\bar{B}, \\
\nu' &= \nu + 2\bar{B}\gamma + \frac{3}{2}\bar{B}^2\tau + B\bar{B}(\pi + 2\alpha) + \Delta\bar{B} + \bar{B}\delta\bar{B} + B\bar{\delta}B + B\bar{B}D\bar{B}
\end{aligned}$$

where the unprimed spin-coefficients are defined in (10.20) *and $B' = \sqrt{\frac{\bar{\tau}}{\tau}}\sqrt{\frac{2}{|\tau|}} B$ is the* *complex-valued function*

$$B' = e^{-a-\bar{a}}\left[DB'v - \tfrac{5}{4}\left(\tfrac{\bar{\zeta}_{,u}}{\bar{\zeta}_{,\bar{a}}} - \tfrac{\zeta_{,u}}{\zeta_{,a}}\right) + \tfrac{\bar{\zeta}_{,u}}{\bar{\zeta}_{,\bar{a}}|\tau|^2}DB'\right],$$
$$DB' = \tfrac{16|\tau|^2}{25|\tau|^2-16|\alpha|^2}\left(|\tau| + \tfrac{1}{\bar{\zeta}_{,\bar{a}}}\right).$$

To inquire into the uniqueness of the $q=4$ vacuum Kundt waves, we classify the vacuum Kundt waves with invariant count beginning with $(0,2,3,...)$. This class of spacetimes is noteworthy as it contains the majority of metrics admitting one Killing vector, and will provide an invariant expression to differentiate those vacuum Kundt waves with invariant count $(0,2,3,4,4)$ from those with $(0,2,4,4)$. To find such an expression we consider the quadruple wedge products of

$$d(a-\bar{a}) \wedge d\zeta_{,a} \wedge dZ_i \wedge dZ_j, \quad \textit{and} \quad d(a-\bar{a}) \wedge d|\tau|^{-1} \wedge dZ_i \wedge dZ_j.$$

If there are only three functionally independent invariants at second order, all twelve quadruple wedge products must vanish, giving twelve equations:

$$Z_{i,u}Z_{j,v} - Z_{i,v}Z_{j,u} = \frac{\zeta_{,au}[Z_{j,v}(Z_{i,a}+Z_{i,\bar{a}}) - Z_{i,v}(Z_{j,a}+Z_{j,\bar{a}})]}{\zeta_{,aa}},$$
$$Z_{i,u}Z_{j,v} - Z_{i,v}Z_{j,u} = \frac{(|\tau|^{-1})_{,u}[Z_{j,v}(Z_{i,a}+Z_{i,\bar{a}}) - Z_{i,v}(Z_{j,a}+Z_{j,\bar{a}})]}{\zeta_{,a}+\bar{\zeta}_{,\bar{a}}}.$$

Fortunately the cases where $\zeta_{,au} \neq 0$ may be studied directly without resorting to wedge products:

Lemma 10.6.2. *The vacuum Kundt wave metrics with an analytic function of the form* (10.34)*,*

$$\tilde{f}(\zeta,u) = -\tfrac{F(u)^2}{16}e^{\frac{4(\zeta - F_0(u))}{iF(u)}} + g(u)\zeta + g_0(u)$$

have the invariant count $(0,2,4,4)$*.*

Proof. To start, we make a coordinate transformation to remove the imaginary part of f_0 in (10.34) using the coordinate transformation:

$$u' = h(u), \quad v' = \tfrac{v}{h_{,u}} - \tfrac{h_{,uu}}{2h_{,u}^2|\tau|^2}, \quad h_{,u} = e^{-\frac{2Im(f_0)}{F(u)}}.$$

Writing $\zeta(a,u) = iF(u)a + F_0(u)$, we find that the first order invariants arising from τ and α are

$$\frac{1}{2}ln(\bar{\tau}/\tau) = a - \bar{a}, \quad \zeta_{,a} = iF(u), \quad |\tau|^{-1} = iF(a-\bar{a}) + F_0. \tag{10.53}$$

As $F' \neq 0$, so as to avoid metrics of the form (10.35), we may take its inverse locally and express all other functions of u in terms of it.

$$F_0 = \mathfrak{F}_0(F).$$

Thus we are left with $a - \bar{a}$ and $\zeta_{,a}$ as invariants. Noting that $Z_1 = Z_1' + Z_2$, where Z_1 is

$$Z_1' = e^{-a-\bar{a}} F F_{,u} = e^{-a-\bar{a}} \mathfrak{F}(F).$$

Removing the u-dependent piece, we may solve for $a - \bar{a}$ as a third functionally independent invariant. Taking Z_2 in proposition 10.6.1 we eliminate all terms dependent on $a, \bar{a}$ and u leaving v as the last invariant at second order to complete the set $\{a - \bar{a}, a + \bar{a}, F(u), v\}$ with the spin-coefficients at first and second order acting as the classifying manifold along with the frame derivatives of v and $a + \bar{a}$. $\square$

Applying the lesson learned from section (10.5), we note that any Kundt wave metric with a function of the form (10.37) may be transformed into one of the form (10.36) using the coordinate transformation

$$u' = h(u), \quad v' = \tfrac{v}{h_{,u}} - \tfrac{h_{,uu}}{2h_{,u}^2|\tau|^2}, \quad h_{,u} = e^{-\frac{\int F_5 du}{2}}. \tag{10.54}$$

The division of the case with $\zeta_{,aa} \neq 0$ cannot be made by $\zeta_{,au}$ vanishing or not; it is a coordinate-dependent distinction. We may ignore the $\zeta_{,au} \neq 0$ in Lemma 10.4.5 case and study the simpler case.

With these two cases eliminated, we may set $\zeta_{,au} = 0$, giving only six equations

$$Z_{i,u}Z_{j,v} - Z_{i,v}Z_{j,u} = 0. \tag{10.55}$$

Lemma 10.6.3. *The vacuum Kundt wave metrics with analytic function of the form* (10.35)

$$\tilde{f}(\zeta,u) = \tfrac{c^2}{16} e^{4\left(\frac{\zeta}{c} - \frac{iF_1(u)}{|c|^2}\right)} + g_1(u)\zeta + g_2(u), \quad Re(c) \neq 0,$$

have the invariant count $(0,2,4,4)$ *except in the subclass of these metrics with*

$$\tilde{f}(\zeta,u) = \tfrac{c^2}{16}e^{\frac{4(\zeta-C_0-iC_1u)}{c}} + c_2\zeta + Im(c_2)C_1u + C_3$$

which have the invariant count $(0,2,3,3)$

Proof. We first examine the possibility of invariant counts of the form $(0,2,3,...)$ using the metric function (10.35). In this case the function is $\zeta(a,u) = ca + f_1(u)$, we find that the first order invariants arising from τ and α are

$$a-\bar{a}, \quad \zeta_{,a} = c, \quad |\tau|^{-1} = Re(c)(a+\bar{a}) + iIm(c)(a-\bar{a}) + Re(f_1). \tag{10.56}$$

At second order, $Z_3 = Z_1 = 0$, thus there is only one quadruple wedge product giving constraints on the metric functions. Mutiplying cZ_2 and adding it to Z_0 gives a useful invariant

$$\frac{Z_0'}{c+\bar{c}} = -\frac{e^{-a-\bar{a}}}{c+\bar{c}}Im(\zeta_{,u}\bar{c}) + \frac{\tau}{\bar{\tau}}B'.$$

To calculate the wedge product we scale Z_0' and Z_2 and use the following quantities:

$$Z_0'' = Z_0'\tfrac{25|\tau|^2-16|\alpha|^2}{16|\tau|^2(c+\bar{c})} \quad Z_2' = \tfrac{Z_2}{\frac{\tau}{\bar{\tau}}DB'+\frac{\bar{\tau}}{\tau}D\bar{B}'}.$$

Substituting into equation (10.55) and differentiating the whole expression by v to get

$$Z_{2,v}'Z_{0,uv}'' - Z_{0,v}''Z_{2,uv}' = |\tau|_{,u}.$$

Requiring this to vanish, we find that $Re(f_1') = 0$ implying that $a-\bar{a}$ and $a+\bar{a}$ are the only first order invariants.

Returning to the original invariants Z_0' and Z_2, substituting into equation (10.55) and denoting $Im(f_1) = F_1$ we find that this becomes,

$$iF_{1,uu}\left[\frac{\tau}{\bar{\tau}}DB' + \frac{\bar{\tau}}{\tau}D\bar{B}' + \frac{5(c+\bar{c})}{4|c|^2}(\frac{\tau}{\bar{\tau}}DB' + \frac{\bar{\tau}}{\tau}D\bar{B}') + \frac{c+\bar{c}|DB'|^2}{|c|^2|\tau|^2}\right].$$

As before, setting this equation to zero we find that $F_1 = Im(f_1) = C_1u$. Substituting the form of $f_1 = C_0 + iC_1u$ into B' in proposition 10.6.1 it is clear that one may peel away the terms and coefficients of the v-linear term in Z_2 to produce v as the last invariant.

It is clear that the only new functionally independent invariant arises in ν in (10.26), as this is the only function retaining u-dependence. Due to the formula for ν in (10.20) we may work with the simpler invariant,

$$V = |\tau|^{-1}g_1 + g_1\zeta + \bar{g}_1\bar{\zeta} + 2Re(g_2) \tag{10.57}$$

Denoting $g_1 = G_x + iG_y$. Taking the wedge product $da \wedge d\bar{a} \wedge dv \wedge dV$ and and equating this to zero we find that $g_1 = G_x + iIm(c_2)$, $Re(g_2) = Im(c_2)C_1u$ and $G_{x,u} = 0$ and so $g_1 = c_2 \in \mathbb{C}$. As all u-dependence has been removed from the invariants, it is clear this is a G_1 space; the classifying manifold consists of the first order and second order invariants in terms of $a, \bar{a}$ and v along with the frame derivatives of v.

In the $(0, 2, 4, 4)$ case, we may replace the complex-valued f_1 in (10.35) with a real-valued function of u. To do so, we apply the following coordinate transformation

$$u' = h(u), \quad v' = \frac{v}{h_{,u}} - \frac{h_{,uu}}{2h_{,u}^2|\tau|^2}, \quad h_{,u} = e^{-\frac{2}{|c|^2}(Re(f_1)Re(c)+Im(f_1)Im(c))}.$$

Then by making the gauge transformation, $F_1 = -Re(f_1)Im(c) + Im(f_1)Im(c)$, we recover the desired form. $\square$

Lemma 10.6.4. *The vacuum Kundt wave metrics with analytic function of the form* (10.36)

$$\tilde{f}(\zeta, u) = f_2(\zeta - C - iF_3(u)) + g_3(u)\zeta + g_4(u)$$

have the invariant count $(0, 2, 4, 4)$ *except in the subclass of these metrics with*

$$\tilde{f}(\zeta, u) = f_2(\zeta - C - iC_0u) + c_1\zeta + Im(c_1)C_0u) + C_2$$

which have the invariant count $(0, 1, 3, 3)$.

Proof. We first examine the possibility of invariant counts of the form $(0, 2, 3, ...)$. In this case the metric function is $\zeta(a, u) = Z(a) + C + iF_3(u)$, we find that the first order invariants arising from τ and α are

$$\zeta_{,a}, \quad |\tau|^{-1} = \zeta(a) + \bar{\zeta}(\bar{a}). \tag{10.58}$$

Locally we may take the inverse of $\zeta_{,a}$ to solve for a and use it as an invariant. Similarly we may do this for the conjugate, and hence at first order a and $\bar{a}$ may be treated as invariants. At second order, $Z_3 = \zeta_{,aa}$ gives no new information. If we define a new invariant $Z_1' = Z_1 Z_3^{-1}$, $Z_2 = Z_1 + \bar{Z}_1$ and $Z_0 = \bar{\zeta}_{,\bar{a}} Z_1' - \zeta_{,a} \bar{Z}_1'$, there is only one quadruple wedge product giving constraints on the metric functions.

Taking the quadruple wedge product with Z_1' and its conjugate and substituting into equation (10.55), we obtain

$$-iF_{3,uu}\left[\frac{\tau}{\bar{\tau}}DB' + \frac{\bar{\tau}}{\tau}D\bar{B}' + \frac{5(\zeta_{,a}+\bar{\zeta}_{,\bar{a}})}{4|\zeta_{,a}|^2}\frac{\tau}{\bar{\tau}}DB' + \frac{\bar{\tau}}{\tau}D\bar{B}' + \frac{\zeta_{,a}+\bar{\zeta}_{,\bar{a}}|DB'|^2}{|\zeta_{,a}|^2|\tau|^2}\right].$$

As before, setting this equation to zero we find that $F_3 = C_0 u$. Substituting $F_3(u)$ into B' in proposition 10.6.1 it is clear that one may peel away the terms and coefficients of the v-linear term in Z_2 to produce v as the last invariant.

From the remaining second order invariants, (10.26) it is clear that the only new functionally independent invariant arises in ν as this is the only function retaining u-dependence. Denoting $g_3 = g_x + ig_y$ we may work with the simpler invariant V in (10.57):

$$V \quad = \quad |\tau|^{-1} g_3 + g_3\zeta + \bar{g}_3\bar{\zeta} + 2Re(g_4)$$

Repeating the calculation of the wedge product $da \wedge d\bar{a} \wedge dv \wedge dV$ and equating this to zero we find that $g_3 = G_x + iIm(c_1)$, $Re(g_4) = Im(c_1)C_0 u$ and$g_3 = c_1 \in \mathbb{C}$. As all u-dependence has been removed from the invariants, it is clear this is a G_1 space, the classifying manifold consists of the first order and second order invariants in terms of $a, \bar{a}$ and v along with the frame derivatives of v. □

10.7 An Invariant Classification of Vacuum Kundt Waves

In proving the sharpness of the lowered upper bound we exhausted all of the branches of the invariant-count tree starting with $(0, 1, ...)$. Employing the first order Cartan invariants, α, τ and γ, we may eliminate several branches from the remaining invariant-count trees in (10.4).

Using the results in this section the possible scenarios for the invariant classification of the vacuum Kundt waves can be narrowed down further to the following diagrams in figure (10.1)

10.7.1 Vacuum Kundt waves with $|\alpha| = \frac{5}{4}|\tau|$

In the most general case, where a vacuum Kundt wave admits the following invariant counts, $(0, 4, ...)$, we may eliminate the scenario where $q = 2$ by counting coordinates involved in the first order invariants.

Lemma 10.7.1. *All vacuum Kundt waves with invariant count* $(0, 4, ...)$ *must have* $|\alpha| = \frac{5}{4}|\tau|$.

Proof. Choosing Kundt coordinates, we calculate the quadruple wedge product of the differentials of the first order Cartan invariants α, τ and their conjugates. As they are all functions of $a, \bar{a}$ and u relative to the special coordinate system, it is clear that

$$d\alpha \wedge d\bar{\alpha} \wedge d\tau \wedge d\bar{\tau} = 0.$$

If the magnitudes of α and τ were not proportional we would always be able to set $\gamma = 0$, contradicting our assumption that four invariants appear at first order. $\square$

With the upper bound $q \leq 4$ shown to be sharp in the subclass of the vacuum Kundt waves with invariant count $(0, 1, 3, 4, 4)$, we would like to determine whether vacuum Kundt waves with $q = 4$ exist in the other subclasses of the vacuum Kundt waves with invariant count begining with $(0, 2, ..)$ and $(0, 3, ...)$ respectively. Using the approach from the previous section, it is possible to show the $q = 4$ branch in (10.4) cannot occur in the $(0, 3, ...)$ case.

Lemma 10.7.2. *For all vacuum Kundt wave spacetimes with invariant count* $(0, 3, ...)$ *the magnitude of* α *is never proportional to that of* τ*; i.e.,* $|\alpha| \neq \frac{5}{4}|\tau|$. *All remaining frame freedom is exhausted at first order by setting* $\gamma = 0$.

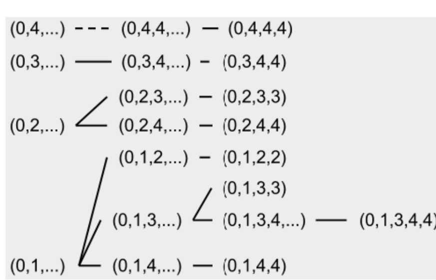

Figure 10.1: All permissible invariant-count trees for the vacuum Kundt waves

Proof. Let us assume that the two magnitudes are equal, then by lemma 10.4.4, $\alpha \neq \pm\tau$ and we may set either the real or imaginary part of γ to zero. As before, we eliminate the real part of γ. The purely imaginary invariant γ' is invariant under any null rotation preserving $Re(\gamma) = 0$ due to the proportionality of the real and imaginary part of $\alpha B + \frac{5}{4}\bar{B}\tau$. Thus, without fixing the frame any further, the transformed scalar γ' is a Cartan invariant:

$$\begin{aligned} \gamma' &= i(Im(\gamma) - C(a,\bar{a},u)Re(\gamma)) \\ &= i\frac{\sqrt{|\tau|}}{2\sqrt{2}}\left[\frac{i\zeta_{,u}}{\zeta_{,a}} - \frac{i\bar{\zeta}_{,u}}{\bar{\zeta}_{,\bar{a}}} + C\left(|\tau|v + \frac{\zeta_{,u}}{\zeta_{,a}} + \frac{i\bar{\zeta}_{,u}}{\bar{\zeta}_{,\bar{a}}}\right)\right]e^{-a-\bar{a}}, \end{aligned}$$

and so we may consider the quadruple wedge product of the differentials of three invariants constructed from γ', τ,α and their complex conjugates: $|\tau|^{-1}$, $e^{\bar{a}-a}$, $\zeta_{,a}$ and γ'. Doing so we find the sole coefficient of $da \wedge d\bar{a} \wedge du \wedge dv$ is:

$$\frac{ie^{2a}|\tau|^{\frac{3}{2}}C}{2\sqrt{2}}\left(\bar{\zeta}_{,\bar{a}\bar{a}}\zeta_{,u} + \bar{\zeta}_{,\bar{a}\bar{a}}\bar{\zeta}_{,u} - \bar{\zeta}_{,\bar{a}u}\bar{\zeta}_{,\bar{a}} - \bar{\zeta}_{,\bar{a}u}\zeta_{,a}\right).$$

If we wish to have three functionally independent invariants at first order, this wedge product must vanish; however, this is exactly equation (10.33) used to determine the class of vacuum Kundt wave metrics with invariant count $(0, 2, ...)$. This contradicts our assumption and so $|\alpha| \neq \frac{5}{4}|\tau|$. □

With this result we see that for all metrics with an invariant count $(0, n, ...)$, $n < 4$, we may always set $\gamma = 0$ as $\frac{5}{4}|\tau| \neq |\alpha|$.

10.7.2 Vacuum Kundt waves with $|\alpha| \neq \frac{5}{4}|\tau|$

To complete the classification of those spacetimes with invariant count $(0, 2, ...)$, we show that the class of vacuum Kundt waves with invariant count $(0, 2, 2)$ cannot occur.

Lemma 10.7.3. *If a vacuum Kundt wave spacetime admits a two-dimensional isometry group it must belong to the* $(0, 1, 2, 2)$ *class.*

Proof. Supposing that only two functionally independent invariants appear at first order, we require that the wedge products of $d(a - \bar{a}) \wedge d\zeta_{,a} \wedge dZ_i$ and $d(a - \bar{a}) \wedge$

$d|\tau|^{-1} \wedge dZ_i$ all vanish. Calculating the wedge products in a particular coordinate system gives

$$\begin{aligned} d(a-\bar{a}) \wedge d\zeta_{,a} \wedge dZ_i &= [\zeta_{,aa} Z_{i,u} - \zeta_{,au}(Z_{i,a} + Z_{i,\bar{a}})] da \wedge d\bar{a} \wedge du \\ &+ Z_{i,v} \zeta_{,aa} da \wedge d\bar{a} \wedge dv \\ &+ Z_{i,v} \zeta_{,au} (da \wedge du \wedge dv - d\bar{a} \wedge du \wedge dv), \end{aligned}$$

$$\begin{aligned} d(a-\bar{a}) \wedge d|\tau|^{-1} \wedge dZ_i &= [(\zeta_{,a} + \bar{\zeta}_{,\bar{a}}) Z_{i,u} - (|\tau|^{-1})_{,u} (Z_{i,a} + Z_{i,\bar{a}})] da \wedge d\bar{a} \wedge du \\ &+ Z_{i,v} (\zeta_{,a} + \bar{\zeta}_{,\bar{a}}) da \wedge d\bar{a} \wedge dv \\ &+ Z_{i,v} (\zeta_{,u} + \bar{\zeta}_{,u}) (da \wedge du \wedge dv - d\bar{a} \wedge du \wedge dv). \end{aligned}$$

If these wedge products are to vanish then either $\zeta_{,a} + \bar{\zeta}_{,\bar{a}} = \zeta_{,u} + \bar{\zeta}_{,u} = \zeta_{,au} = \zeta_{,aa} = 0$ or $Z_{i,v} = 0$. As in the proof of Lemma 10.7.4 we may use the same argument for metrics (10.34), (10.36) and (10.37) to show $Z_{i,v} \neq 0$, $i = 0, 1, 2$. In the case of the metric function (10.35) where $\zeta_{,aa} = 0$ and $\zeta_{,a} = -\bar{\zeta}_{,\bar{a}}$, $Z_{2,v} = 0$ occurs if and only if $\zeta_{,a} = 0$ which is not possible. If these wedge products do vanish, we must have $\zeta_{,a} + \bar{\zeta}_{,\bar{a}} = \zeta_{,u} + \bar{\zeta}_{,u} = \zeta_{,au} = \zeta_{,aa} = 0$, implying that this metric belongs to the $(0, 1, ...)$ class. □

To illustrate the utility of these invariants over the usual set of invariants arising from the spin coefficients relative to the invariant coframe, we prove that the class of vacuum Kundt waves with invariant count $(0, 3, 3)$ cannot occur.

Lemma 10.7.4. *If a vacuum Kundt wave spacetime admits three functionally independent invariants at first order of the Karlhede algorithm, it must belong to the* $(0, 3, 4, 4)$ *class.*

Proof. Supposing that we do have the invariant count $(0, 3, 3)$ we will show there is a contradiction. Denoting the triple wedge product $\Omega_3 = d(a - \bar{a}) \wedge d\zeta_{,a} \wedge d(\zeta + \bar{\zeta})$, we note

$$\Omega_3 = -(\bar{\zeta}_{,\bar{a}\bar{a}}(\zeta_{,u} + \bar{\zeta}_{,u}) - \bar{\zeta}_{,\bar{a}u}(\bar{\zeta}_{,\bar{a}} + \zeta_{,a})) da \wedge d\bar{a} \wedge du.$$

We recall from equation (10.33) that if this equation vanishes only two functionally independent invariants appear at first order of the algorithm; thus this must be non-zero if we wish to have three invariants at first order. To impose the condition that no new functionally independent invariants appear at second order, we require the vanishing of all quadruple wedge products with Z_I, $I = 1, 2, 3, 4$

$$\Omega_3 \wedge dZ_I = -Z_{I,v}(\bar{\zeta}_{,\bar{a}\bar{a}}(\zeta_{,u} + \bar{\zeta}_{,u}) - \bar{\zeta}_{,\bar{a}u}(\bar{\zeta}_{,\bar{a}} + \zeta_{,a}))da \wedge d\bar{a} \wedge du \wedge dv.$$

This can only occur if and only if $Z_{i,v} = 0$, $i = 1, 2, 3$. The v-coefficient of the first three Z_i yields two cases, depending on whether $\zeta_{,aa} = 0$ or not.

- If $\zeta_{,aa} \neq 0$, the vanishing of $\Omega_3 \wedge Z_1$ implies $Z_{1,v} = 0$; this can only occur if $DB = 0$which is not possible; otherwise one would have $|\tau| = -\zeta_{,a}^{-1}$. If one were to impose this constraint, it immediately implies, $\zeta_{,a} = 0$ which cannot be true.

- If $\zeta_{,aa} = 0$, the vanishing wedge products $\Omega_3 \wedge Z_0$ and $\Omega_3 \wedge Z_2$ give the following equations

$$\begin{aligned} \tfrac{\bar{\tau}}{\tau} D\bar{B}'\zeta_{,a} - \tfrac{\tau}{\bar{\tau}} DB'\bar{\zeta}_{,\bar{a}} &= 0, \\ \tfrac{\tau}{\bar{\tau}} DB' + \tfrac{\bar{\tau}}{\tau} D\bar{B}' &= 0. \end{aligned}$$

  As $DB \neq 0$, we may solve one equation and substitute into the other,

$$\left[\frac{\bar{\zeta}_{,\bar{a}}}{\zeta_{,a}} + 1\right] DB\frac{\tau}{\bar{\tau}} = 0.$$

  This will only vanish if $\zeta_{,a} = -\bar{\zeta}_{,\bar{a}}$; however, if this is the case, then (10.33) is satisfied and this spacetime belongs to the $(0, 2, ...)$ class, contradicting our assumption, and so it cannot occur.

□

Effectively we may differentiate those vacuum Kundt waves with invariant count $(0, 3, 4, 4)$ and $(0, 4, 4, 4)$ by the non-vanishing of the first order invariant $|\alpha| - \frac{5}{4}|\tau|$. The Newman-Penrose field equations provide further classifying functions.

10.8 Conclusions

In this paper we have invariantly classified all of the vacuum Kundt waves by exhaustively listing all invariant counts that appear as states in the Karlhede algorithm.

Using the invariants produced by this method, we examine each invariant count to determine if the spacetime is integrable. In many cases whole branches do not occur or are significantly simplified; the results of this analysis are summarized in table form in the following two tables (10.1) and (10.2).

This analysis was motivated by previous work on the upper bound of the Karlhede algorithm applied to type N spacetimes; it was conjectured that $q \leq 5$ [36, 47] for the vacuum Kundt waves; however, this upper bound was not shown to be sharp. We have lowered the upper bound to $q \leq 4$ and produced an example by integrating the class of vacuum Kundt waves with $(0, 1, 3, 4, 4)$ proving the sharpness of the bound. Furthermore, we proved this class is unique as it is the only class requiring the fourth derivative of the curvature to invariantly classify its members.

10.9 Vacuum Kundt Waves Admitting No Symmetry

In this section we collect all of necessary invariants required to sub-classify the vacuum Kundt waves admitting no Killing vectors, by identifying the functionally independent invariants and those functionally dependent invariants that are not generic to all vacuum Kundt waves. These functions constitute the essential classifying manifold, as all other curvature components to any order may be expressed in terms of these functions and their derivatives. In each list the use of a semi-colon indicates those elements arise from the next order covariant derivative of the curvature tensor than the predecessors in the list.

Proposition 10.9.1. *The metrics belonging to the* $(0, 4, 4, 4)$ *class may contain any analytic function,* $f(z, u)$*, not listed in the class of vacuum Kundt waves with invariant-counts beginning with* $(0, n, ...)$*,* $n < 3$.

Using the special coordinates $a = \frac{1}{4} ln(f_{,\zeta\zeta})$*, the four functionally independent invariants may be constructed from the spin-coefficients in* (10.20)*:*

$$i(a - \bar{a}), \quad \zeta_{,a}, \quad |\tau|^{-1}, \quad v.$$

The classifying functions at first and second order are:

$$a+\bar{a}=\tilde{Z}_0(i(a-\bar{a}),\zeta_{,a},|\tau|^{-1}),\quad \zeta_{,u}=\tilde{z}_1(i(a-\bar{a}),\zeta_{,a},|\tau|^{-1}),$$
$$\tfrac{24|\zeta_{,a}|^2}{16}+|\zeta_{,a}||\tau|^{-1}(\zeta_{,a}+\bar{\zeta}_{,\bar{a}})+|\tau|^{-2};$$
$$\zeta_{,aa}=\tilde{z}_2(i(a-\bar{a}),\zeta_{,a},|\tau|^{-1}),\quad \tilde{f}(a,u)=\tilde{z}_3(i(a-\bar{a}),\zeta_{,a},|\tau|^{-1}).$$

The invariant coframe arises from the coframe (10.19) *by using the null rotation parameters B' and B'' to satisfy the conditions at first and second order respectively:*

$$Im\left(\gamma+B'\alpha+\frac{5}{4}\bar{B}'\tau\right)=0;\quad i\Delta''(a-\bar{a})=0.$$

Proposition 10.9.2. *The metrics belonging to the* $(0,3,4,4)$ *class may contain any analytic function,$f(z,u)$, not listed in the class of vacuum Kundt waves with invariant-counts beginning with* $(0,n,...)$*,* $n<3$*.*

Using the special coordinates $a=\frac{1}{4}ln(f_{,\zeta\zeta})$, the four functionally independent invariants may be constructed from the spin-coefficients in (10.20) *even though the last invariant appears at second order:*

$$\zeta_{,a},\quad \bar{\zeta}_{,\bar{a}},\quad |\tau|^{-1};\quad v.$$

The classifying functions at first and second order are:

$$a=\tilde{z}_0(\zeta_{,a},\bar{\zeta}_{,\bar{a}},|\tau|^{-1});$$
$$\zeta_{,u}=\tilde{z}_2(\zeta_{,a},\bar{\zeta}_{,\bar{a}},|\tau|^{-1}),\quad \zeta_{,aa}=\tilde{z}_3(\zeta_{,a},\bar{\zeta}_{,\bar{a}},|\tau|^{-1}),\quad \tilde{f}(a,u)=\tilde{z}_4(\zeta_{,a},\bar{\zeta}_{,\bar{a}},|\tau|^{-1}).$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter B satisfying the conditions: $\gamma+B'\alpha+\frac{5}{4}\bar{B}'\tau=0$ which is explicitly given in proposition 10.6.1.*

Proposition 10.9.3. *The metric belonging to the* $(0,2,4,4)-0$ *class has the canonical form for $f(\zeta,u)$*

$$-\frac{F(u)^2}{16}e^{\frac{4(\zeta-F_0(u))}{iF(u)}}+g(u)\zeta+g_0(u),$$

where F, f_0, g and g_0 are arbitrary functions of u.

Using the special coordinates $a=\frac{1}{4}ln(f_{,\zeta\zeta})$, the four functionally independent invariants may be constructed from the spin-coefficients in (10.20) *even though the last two invariants arise at second order:*

$$a-\bar{a},\quad \zeta_{,a}=iF(u)\ with\ F_{,u}\neq 0;\quad a+\bar{a},\quad v.$$

The classifying functions at first and second order are:

$$|\tau|^{-1} = i\zeta_{,a}(a-\bar{a}) + F_0(u);$$
$$\zeta_{,aa} = 0, \quad F_{,u}(u), \quad g(u), \quad \bar{g}(u), \quad Re(g_0)(u).$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter B satisfying the conditions: $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ which is explicitly given in proposition 10.6.1.*

Proposition 10.9.4. *The metric belonging to the* $(0,2,4,4){-}1$ *class has the canonical form for $f(\zeta, u)$*

$$f(\zeta,u) = \frac{c^2}{16}e^{\frac{4(\zeta}{c} - \frac{iF(u)}{|c|^2}} + g_1(u)\zeta + g_2(u), \quad Re(c) \neq 0$$

where F_1, g_1, and g_2 are arbitrary functions of u.

Using the special coordinates $a = \frac{1}{4}ln(f_{,\zeta\zeta})$, the four functionally independent invariants may be constructed from the spin-coefficients in (10.20) *even though the last two invariants arise at second order:*

$$a-\bar{a}, \quad |\tau|^{-1}; \quad Z_0, \quad Z_2$$

where Z_0 and Z_2 are defined in proposition 10.6.1. The classifying functions at first, second and third order are:

$$\zeta_{,a} = c;$$
$$\zeta_{,aa} = 0, \quad a+\bar{a} = \tilde{Z}_0(a-\bar{a}, |\tau|^{-1}, -iZ_0, \quad Z_2), \quad v = \tilde{Z}_1(a-\bar{a}, |\tau|^{-1}, Z_0, Z_2);$$
$$\Delta Z_0 = i\tilde{Z}_2(a-\bar{a}, |\tau|^{-1}, Z_0, Z_2), \quad \Delta Z_3 = \tilde{Z}_4(a-\bar{a}, |\tau|^{-1}, Z_0, Z_2).$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter B satisfying the conditions: $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ which is explicitly given in proposition 10.6.1.*

Proposition 10.9.5. *The metric belonging to the* $(0,2,4,4){-}2$ *class has the canonical form for $f(\zeta, u)$*

$$f(\zeta,u) = f_2(\zeta - c_0 - iF_3(u)) + g_3(u)\zeta + g_4(u)$$

where F_3, g_3, and g_4 are arbitrary functions of u.

Using the special coordinates $a = \frac{1}{4}ln(f_{,\zeta\zeta})$*, the four functionally independent invariants may be constructed from the spin-coefficients in* (10.20) *even though the last two invariants arise at second order:*

$$a - \bar{a}, \quad |\tau|^{-1}; \quad Z_1', \quad \bar{Z}_1'$$

where $Z_1' = Z_1 Z_3^{-1}$ *as defined in proposition 10.6.1. The classifying functions at first, second and third order are:*

$$\zeta_{,a} = i\tilde{z}_0(a - \bar{a}, |\tau|^{-1});$$
$$a + \bar{a} = \tilde{Z}_1(a - \bar{a}, |\tau|^{-1}), \quad v = \tilde{Z}_2(a - \bar{a}, |\tau|^{-1}, Z_1', \bar{Z}_1');$$
$$\Delta Z_1' = i\tilde{z}_3(a - \bar{a}, |\tau|^{-1}, Z_1', \bar{Z}_1').$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter* B *satisfying the conditions:* $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ *which is explicitly given in proposition 10.6.1.*

Proposition 10.9.6. *The metric belonging to the* $(0, 1, 4, 4)$ *class has the canonical form for* $f(\zeta, u)$

$$f(\zeta, u) = \frac{C_0^2}{16} e^{\frac{-4i(\zeta + C_1)}{C_0}} + f_1(u)\zeta + f_2(u).$$

where f_1 *and* f_2 *may be any set of functions except those listed in the remaining invariant classes* $(0, 1, 3, 4, 4)$*,* $(0, 1, 3, 3)$ *and* $(0, 1, 2, 2)$*.*

Using the special coordinates $a = \frac{1}{4}ln(f_{,\zeta\zeta})$*, the four functionally independent invariants may be constructed from the spin-coefficients in* (10.20) *even though the last three invariants arise at second order:*

$$a - \bar{a}; \quad v, \quad N_0, \quad N_1$$

where N_0 *and* N_1 *are defined in equation* (10.46)*. The classifying functions at first, and second order are:*

$$\zeta_{,a} = iC_0, \quad |\tau|^{-1} = iC_0(a - \bar{a}) + 2C_1;$$
$$a + \bar{a} = \tilde{Z}_0(a - \bar{a}, N_0, N_1);$$
$$\Delta N_0 = \tilde{Z}_1(a - \bar{a}, v, N_0, N_1), \quad \Delta N_1 = \tilde{Z}_2(a - \bar{a}, v, N_0, N_1).$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter B satisfying the conditions: $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ which is explicitly given in proposition 10.6.1.*

Proposition 10.9.7. *The metric belonging to the* $(0,1,3,4,4){-}0$ *class has the canonical form for $f(\zeta, u)$*

$$f(\zeta,u) = \frac{C_0^2}{16}e^{\frac{-4i(\zeta+C_1)}{C_0}} + F_y[(C_2+i)\zeta + 2C_3 + ln(F_y^{\frac{C_0}{2}})]$$

where F_y may be any function except Cu^{-2}.

Using the special coordinates $a = \frac{1}{4}ln(f_{,\zeta\zeta})$, the four functionally independent invariants may be constructed from the spin-coefficients in (10.20) *even though the last three invariants arise at second order:*

$$a-\bar{a};\quad v,\quad N_2;\quad F(u) = \frac{F_{y,u}}{F_y{\frac{3}{2}}}$$

where N_2 is defined in equation (10.49)*. The classifying functions at first, and second order are:*

$$\zeta_{,a} = iC_0,\quad |\tau|^{-1} = iC_0(a-\bar{a}) + 2C_1;$$
$$N_1 = \left[\frac{C_2}{|\tau|} + \frac{C_0^2|\tau|^2-2}{C_0|\tau|^2}\right]N_2;$$
$$F_y = \tilde{Z}_0(F),\quad a+\bar{a} = \frac{1}{2}ln\left(\frac{N_2}{F_y}\right).$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter B satisfying the conditions: $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ which is explicitly given in proposition 10.6.1.*

Proposition 10.9.8. *The metric belonging to the* $(0,1,3,4,4){-}1$ *class has the canonical form for $f(\zeta, u)$*

$$f(\zeta,u) = \frac{C_0^2}{16}e^{\frac{-4i(\zeta+C_1)}{C_0}} + F_x(\zeta + C_2)$$

where F_x may be any function except Cu^{-2}.

Using the special coordinates $a = \frac{1}{4}ln(f_{,\zeta\zeta})$, the four functionally independent invariants may be constructed from the spin-coefficients in (10.20) *even though the last three invariants arise at second order:*

$$a-\bar{a};\quad v,\quad N_0;\quad F(u) = \frac{F_{x,u}}{F_x{\frac{3}{2}}}$$

where N_0 *is defined in equation* (10.49). *The classifying functions at first, and second order are:*

$$\zeta_{,a} = iC_0, \quad |\tau|^{-1} = iC_0(a - \bar{a}) + 2C_1;$$
$$N_1 = \frac{-N_2}{C_2|\tau|};$$
$$F_x = \tilde{Z}_0(F).$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter* B *satisfying the conditions:* $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ *which is explicitly given in proposition 10.6.1.*

10.10 Vacuum Kundt Waves Admitting a Symmetry

In this section we collect all of necessary invariants required to sub-classify the vacuum Kundt waves admitting one Killing vectors, by identifying the functionally independent invariants and those functionally dependent invariants that are not generic to all vacuum Kundt waves. These functions constitute the essential classifying manifold, as all other curvature components to any order may be expressed in terms of these functions and their derivatives. In each list the use of a semi-colon indicates those elements arise from the next order covariant derivative of the curvature tensor than the predecessors in the list

Proposition 10.10.1. *The metric belonging to the* $(0, 2, 3, 3) - 1$ *class has the canonical form for* $f(\zeta, u)$

$$\frac{c^2}{16} e^{\frac{4(\zeta - C_0 - iC_1 u)}{c}} + c_2\zeta + Im(c_2)C_1 u + C_3$$

where c, c_2 *and* C_0, C_1, C_3 *are arbitrary complex-valued and real-valued functions respectively.*

Using the special coordinates $a = \frac{1}{4}ln(f_{,\zeta\zeta})$, *the four functionally independent invariants may be constructed from the spin-coefficients in* (10.20) *even though the last two invariants arise at second order:*

$$a - \bar{a}, \quad a + \bar{a}; \quad v$$

where Z_0 and Z_2 are defined in proposition 10.6.1. The classifying functions at first, second and third order are:

$$\zeta_{,a} = c, \quad |\tau|^{-1} = Re(c)(a+\bar{a}) + Im(c)(a-\bar{a}) + C_0;$$
$$C_1, \quad c_2, \quad C_3.$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter B satisfying the conditions: $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ which is explicitly given in proposition 10.6.1.*

Proposition 10.10.2. *The metric belonging to the* $(0,2,3,3)-2$ *class has the canonical form for $f(\zeta, u)$*

$$f_2(\zeta - C - iC_0u) + c_1\zeta + Im(c_1)C_0u + C_2$$

where C, C_0, C_2, and c_1 are arbitrary real-valued and complex-valued constants. Using the special coordinates $a = \frac{1}{4}ln(f_{,\zeta\zeta})$, the four functionally independent invariants may be constructed from the spin-coefficients in (10.20) *even though the last two invariants arise at second order:*

$$\zeta_{,a}, \quad \bar{\zeta}_{,\bar{a}}; \quad v.$$

The classifying functions at first, second and third order are:

$$a - \bar{a} = i\tilde{Z}_0(\zeta_{,a}, \bar{\zeta}_{,ba}), \quad \zeta + \bar{\zeta} = \tilde{Z}_1(\zeta_{,a}, \bar{\zeta}_{,\bar{a}}), \quad C;$$
$$a + \bar{a} = \tilde{Z}_2(\zeta_{,a}, \bar{\zeta}_{,\bar{a}}), \quad c_1, \quad C_2.$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter B satisfying the conditions: $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ which is explicitly given in proposition 10.6.1.*

Proposition 10.10.3. *The metric belonging to the* $(0,1,3,3,3)$ *class has the canonical form for $f(\zeta, u)$*

$$\frac{C_0^2}{16} e^{\frac{-4i(\zeta - iC_0u + C_1)}{C_0}} + c_3\zeta + Im(c_3)C_2u + iC_2$$

where C_0, C_1, C_2, and c_3 are arbitrary real and complex valued constant, respectively.

Using the special coordinates $a = \frac{1}{4}ln(f_{,\zeta\zeta})$, *the four functionally independent invariants may be constructed from the spin-coefficients in* (10.20) *even though the last three invariants arise at second order:*

$$a - \bar{a}; \quad v, \quad u^{-2}e^{-2a-2\bar{a}}$$

The classifying functions at first, and second order are:

$$\zeta_{,a} = iC_0, \quad |\tau|^{-1} = iC_0(a - \bar{a}) + 2C_1;$$
$$C_2, \quad c_3.$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter* B *satisfying the conditions:* $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ *which is explicitly given in proposition 10.6.1.*

10.11 Vacuum Kundt Waves Admitting Two Symmetries

In this section we collect all of necessary invariants required to sub-classify the vacuum Kundt waves admitting two Killing vectors, by identifying the functionally independent invariants and those functionally dependent invariants that are not generic to all vacuum Kundt waves. These functions constitute the essential classifying manifold, as all other curvature components to any order may be expressed in terms of these functions and their derivatives. In each list the use of a semi-colon indicates those elements arise from the next order covariant derivative of the curvature tensor than the predecessors in the list

Proposition 10.11.1. *The metric belonging to the* $(0, 1, 2, 2)$ *class has the canonical form for* $f(\zeta, u)$

$$f(\zeta, u) = \frac{C_0^2}{16}e^{\frac{-4i(\zeta - iC_2 + C_1)}{C_0}}$$

where C_0 *and* C_1 *are arbitrary real-valued constants.*

Using the special coordinates $a = \frac{1}{4}ln(f_{,\zeta\zeta})$, *the four functionally independent invariants may be constructed from the spin-coefficients in* (10.20) *even though the last three invariants arise at second order:*

$$a - \bar{a}; \quad e^{-a-\bar{a}}v.$$

The classifying functions at first and second order are:

$$\zeta_{,a} = iC_0, \quad |\tau|^{-1} = iC_0(a - \bar{a}) + 2C_1.$$

The invariant coframe is found at first order by applying a null rotation to the coframe (10.27) *with parameter* B *satisfying the conditions:* $\gamma + B'\alpha + \frac{5}{4}\bar{B}'\tau = 0$ *which is explicitly given in proposition 10.6.1.*

10.12 All Potential Invariant Counts for the Vacuum Kundt Waves

To write down a potential case of the Karlhede algorithm up to a given iteration, p, we will use the following notation, $(t_1, t_2, ..., t_p, ...)$, where $t_i,\ i \in [1, p]$ denotes the number of functionally independent invariants at the i-th iteration of the Karlhede algorithm. We may map out all potential cases of the Karlhede algorithm, by using each potential invariant count as a node in a tree diagram where the existence of a non-trivial isotropy group from one iteration to the next will be denoted by a dashed line, while a solid line denotes a trivial isotropy group.

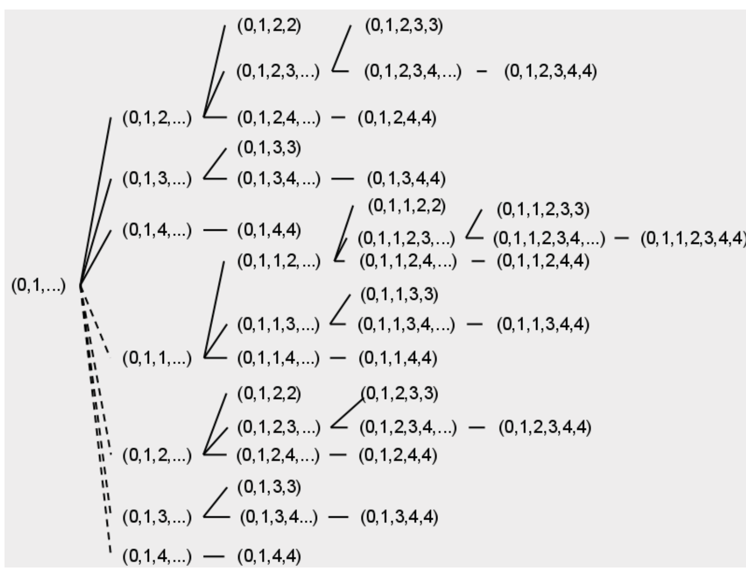

Figure 10.2: Potential invariant-count trees for the case where one functionally independent invariant appears at first order of the algorithm

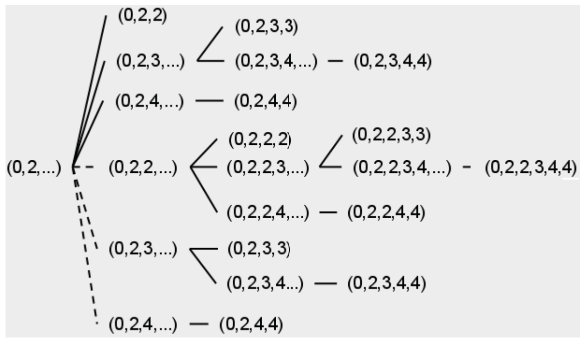

Figure 10.3: Potential invariant-count tree for the case where two functionally independent invariants appear at first order of the algorithm

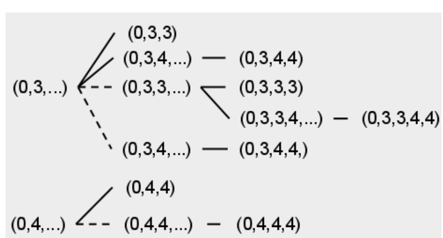

Figure 10.4: Potential invariant-count trees for the case where three or four functionally independent invariants appear at first order of the algorithm

Invariant Count	$f(\zeta, u)$
$(0,4,4,4)$	$f(\zeta,u), \quad \lvert\alpha\rvert - \frac{5}{4}\lvert\tau\rvert = 0$
$(0,3,4,4)$	$f(\zeta,u), \quad \lvert\alpha\rvert - \frac{5}{4}\lvert\tau\rvert \neq 0$
$(0,2,4,4)-0$	$-\frac{F(u)^2}{16}e^{\frac{4(\zeta - F_0(u))}{iF(u)}} + g(u)\zeta + g_0(u)$
$(0,2,4,4)-1$	$\frac{c^2}{16}e^{4\left(\frac{\zeta}{c} - \frac{iF_1(u)}{\lvert c\rvert^2}\right)} + g_1(u)\zeta + g_2(u)$
$(0,2,4,4)-2$	$f_2(\zeta - c_0 - iF_3(u)) + g_3(u)z + g_4(u)$
$(0,1,4,4)$	$\frac{C_0^2}{16}e^{\frac{-4i(\zeta+C_1)}{C_0}} + f_1(u)\zeta + f_2(u)$
$(0,1,3,4,4)$	$\frac{C_0^2}{16}e^{\frac{-4i(\zeta+C_1)}{C_0}} + f_y[(C_2+i)\zeta + 2C_3 + ln(f_y^{\frac{C_0}{2}})],$ $f_y(u) \neq Cu^{-2}$
$(0,1,3,4,4)$	$\frac{C_0^2}{16}e^{\frac{-4i(\zeta+C_1)}{C_0}} + f_x(\zeta + C_2),$ $f_x(u) \neq Cu^{-2}$

Table 10.1: All Vacuum Kundt waves admitting no symmetries

Invariant Count	$f(\zeta, u)$	Killing vector
$(0,2,3,3)$	$\frac{c^2}{16}e^{\frac{4(\zeta - C_0 - iC_1 u)}{c}} + c_2\zeta + Im(c_2)C_1u + C_3$	$U - C_1T$
$(0,2,3,3)$	$f_2(\zeta - C - iC_0u) + c_1\zeta + Im(c_1)C_0u + C_2$	$U - C_0T$
$(0,1,3,3)$	$\frac{C_0^2}{16}e^{\frac{-4i(\zeta - iC_0u + C_1)}{C_0}} + c_3\zeta + Im(c_3)C_0u + C_2$	$U - C_0T$
$(0,1,2,2)$	$\frac{C_0^2}{16}e^{\frac{-4i(\zeta+C_1)}{C_0}}$	U and $T + C_0^{-1}R$

Table 10.2: All Vacuum Kundt waves admitting symmetries; the Killing vectors are: $U = \frac{\partial}{\partial u}$,$R = i\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \bar{\zeta}}\right)$ and $T = \frac{u}{2}\frac{\partial}{\partial u}$.

Chapter 11

All Kundt Waves Admitting Isometries

11.1 An Invariant Coframe to Calculate Isometries

We are interested in the collection of Petrov Type N Kundt spacetimes with $\tau \neq 0$, the so called rotating plane fronted waves. In this paper our interest lies in the collection of these spacetimes admitting one or more isometries. Kundt had previously investigated these spacetimes [9], however we will use the notation in [22]. In the Newman-Penrose formalism, these spacetimes have $\sigma = \rho = 0$ and hence are Kundt spacetimes with $\tau \neq 0$; they admit vacuum, Null Einstein-Maxwell and Pure radiation as energy conditions.

Using the metric form (31.38) in [22], these spacetimes describe those vacuum Kundt spacetimes with $\tau \neq 0$. To restrict these to Petrov Type N, we set $W^0 = 0$. In these spacetimes, the vacuum condition implies H^0 is harmonic, if we relax this condition the spacetimes will have $\Phi_{22} \neq 0$ and hence allow for an energy tensor describing pure raditation or a null Einstein-Maxwell field. The metric will be

$$ds^2 = 2d\zeta d\bar{\zeta} - 2du(dv - \tfrac{4v}{\zeta+\zeta}Re(d\zeta) + Hdu, \quad H = H^0(\zeta, \bar{\zeta}, u) - \tfrac{v^2}{(\zeta+\bar{\zeta})^2}). \tag{11.1}$$

More conveniently this may be described by the coframe,

$$m = d\zeta, \quad \ell = du, \quad n = dv - \tfrac{4v}{\zeta+\bar{\zeta}}Re(d\zeta) + Hdu,$$
$$H = H^0(\zeta, \bar{\zeta}, u) - \tfrac{v^2}{(\zeta+\bar{\zeta})^2}.$$

Using this coframe and applying the appropriate Lorentz transformations to normalize particular Cartan invariants related to the Kundt waves, producing an invariant coframe well-suited to the geometry of these spaces.

The coframe approach to the metric provides another method to calculating the isometries of the spacetime using the Lie derivative, $\mathbb{L}$ by exploiting the fact that any coordinate transformation, Φ, maps each member of the coframe to its equivalent in the new coordinate system. That is for $\omega_i^A = \{m, \bar{m}, \ell v\}$ and $\omega_i^{A'} = \{m', \bar{m}', \ell' n'\}$ in the new coordinates:

$$\Phi^* \omega_i^{A'} = \omega_i^A. \tag{11.2}$$

In the typical situation where one is using a coframe which is not invariant, $\Phi^* \omega_i^{A'} = A_i^j(\zeta, \bar{\zeta}, u, v)\omega_j^A$ where A_i^j belongs to the Lorentz group.

Theorem 11.1.1. *Given a Lorentzian metric g_{ij}, $i, j \in [1,4]$ admitting an invariant coframe ω_i^A $A, B \in [1,4]$ such that $g_{ij} = \omega_i^A \omega_j^B \eta_{AB}$. Every vector-field X which annihilates the invariant coframe, $\mathbb{L}_X \omega_i^A = 0$ $\forall A \in [1,4]$, will be a Killing vector, $\mathbb{L}_X g_{ij} = 0$, and vice versa.*

Proof. Denoting $\Phi[X]_\epsilon$ as the flow of X through a neighborhood of a particular point in the manifold, the vanishing of the Lie Derivative implies $\Phi[X]_\epsilon^* \omega^A = \omega^A$. Vector fields belonging to this class give rise to the symmetries of a coframe. In the case of the invariant coframe these symmetries coincide with those of the metric.

Assuming such a vector field exists, we use the fact that $g_{ij} = \omega_i^A \omega_j^B \eta_{AB}$ and the Leibnitz property of the Lie derivative to find

$$\mathbb{L}_X g_{ij} = \mathbb{L}_X \omega_i^A \omega_j^B \eta_{AB} + \omega_i^A \mathbb{L}_X \omega_j^B \eta_{AB} = 0$$

Thus X is a Killing vector for the metric.

To prove the converse, we choose coordinates in which the Killing vector is part of the coordinate basis, i.e., $X = \frac{\partial}{\partial x^1}$. Relative to these coordinates, the coordinate transformation $\tilde{x}^1 = x^1 + C$, $\tilde{x}^i = x^i$, $i > 1$ is an isometry with the additional property $d\tilde{x}^a = dx^a$, $\quad a \in [1, i]$. Supposing $R(x^a)$ is an invariant function, for any coordinate transformation, $\tilde{R}(\tilde{x}^a) = R(x^a)$, in particular for an isometry this implies

$$R(x^1 + C, x^i) = R(x^1, x^i)$$

implying that $R_{,x^1} = 0$. Now if we consider an invariant coframe, $\omega^\alpha = \omega_a^\alpha dx^a$, the pullback of each member of the coframe is mapped to its equivalent in the $\tilde{x}$ system, i.e. $\tilde{\omega}_\alpha = \omega_\alpha$. As $d\tilde{x}^a = dx^a$ each component of the coframe member will be left invariant by the isometry:

$$\tilde{\omega}_a^\alpha = \omega_a^\alpha;$$

we conclude that in general $\omega^{\alpha}_{a,x^1} = 0$. Taking the Lie derivative of the invariant coframe ω^{α}_{a} in the direction of $X = \frac{\partial}{\partial x^1}$ we find:

$$\mathbb{L}_X \omega^{\alpha}_{a} = \omega^{\alpha}_{a,1} = 0,$$

X annihilates the invariant coframe.

□

11.2 Kundt Spacetimes with $\tau \neq 0$ Admitting a G_1 Isometry Group

Potentially this approach can lead to very complicated expressions involving the components of the Killing vector and coframe, and their first order coordinate derivatives. However by varying the isotropy to fix certain invariants we may produce a coframe for which these equations are much simpler.

To continue we make a simple gauge transformation $H^0 = 4xh^0(\zeta, \bar{\zeta}, u)$ and apply a boost to the coframe so that the Cartan invariant, $\Psi_4 = xh^0_{,\bar{\zeta}\bar{\zeta}}$, satisfies $|\Psi_4| = 1$:

$$m' = m, \quad \ell' = e^{\frac{a}{2}}\ell, \quad n' = e^{-\frac{a}{2}}n, \quad a = \frac{1}{2}\left(ln(x^2) + ln(|h^0_{,\bar{\zeta}\bar{\zeta}}|)\right). \tag{11.3}$$

Fixing the form of Ψ_4 reduces the dimension of the isotropy group by 1 so that $dim(\mathbb{H}_o) = 3$. In the context of the Karlhede algorithm, this is a bad choice of coframe as the algorithm would continue longer than necessary for classification. However we wish to produce a different invariant coframe to study the class of Kundt waves admitting symmetries.

To produce the necessary invariant coframe so we require that the remaining frame freedom be exhausted by setting the remaining parameters for a spin and null rotation to be set to zero, i.e., $\theta = B = \bar{B} = 0$. This choice is reflected in the components of the Weyl tensor and its derivatives where at zeroeth order $|\Psi_4| = 1$, at first order $\bar{\tau} = \tau$ and finally at second order the invariants $5\mu\beta - \bar{\lambda}\alpha$ and $5\lambda\beta - \bar{\mu}\alpha$ both vanish.

Thus the coframe stated in (11.3) is indeed an invariant coframe as it normalizes these invariant combinations of the Cartan invariants. It is in this coframe that we prove the following lemma.

Lemma 11.2.1. *If a Kundt spacetime with $\tau \neq 0$ admits an isometry it must be of the form,*

$$X = B\frac{\partial}{\partial y} + f(u)\frac{\partial}{\partial u} - [f'v + 2x^2 f'']\frac{\partial}{\partial v}, \tag{11.4}$$

and the metric function h^0 satisfies the differential equation,

$$2Bh^0_{,y} + 2fh^0_{,u} + 4f'h^0 - xf''' = 0. \tag{11.5}$$

Proof. Supposing the Killing vector field takes the form,

$$X = X^1(\zeta,\bar{\zeta},u,v)\tfrac{\partial}{\partial\zeta} + X^2(\zeta,\bar{\zeta},u,v)\tfrac{\partial}{\partial\bar{\zeta}} + f(\zeta,\bar{\zeta},u,v)\tfrac{\partial}{\partial u} + X^4(\zeta,\bar{\zeta},u,v)\tfrac{\partial}{\partial v},$$

where $\bar{X}^1 = X^2$, X^1 is $\mathbb{C}$-valued and X^3 and X^4 are $\mathbb{R}$-valued. We drop the primes in equation (11.3) and calculate the Lie derivative of m in the direction of X, using the Cartan formula $\mathbb{L}_X\omega = i_X d\omega + di_X\omega$:

$$dX^1 = 0.$$

The component X^1 and its conjugate X^2 are constant, $X^1 = C, C \in \mathbb{C}$. Applying the Lie derivative to ℓ in the direction of X yields,

$$\mathbb{L}_X\ell = \left[2Re(C(e^{\frac{a}{2}})_{,\zeta}) + (e^{\frac{a}{2}}X^3)_{,u}\right]du + e^{\frac{a}{2}}X^3{}_{,v}dv + 2Re(e^{\frac{a}{2}}X^3{}_{,\zeta}d\zeta)$$

This will vanish if $X^3 = f(u)$ and the quantity a satisfies the following partial differential equation

$$2Re(C(e^{\frac{a}{2}})_{,\zeta}) + (e^{\frac{a}{2}}f)_{,u} = 0. \tag{11.6}$$

To summarize the work so far, for arbitrary f and choice of $C \in \mathbb{C}$, if a in equation (11.3) satisfies the above differential equation (11.6) the local diffeomorphism related to the vector field

$$X = C\tfrac{\partial}{\partial\zeta} + \bar{C}\tfrac{\partial}{\partial\bar{\zeta}} + f(u)\tfrac{\partial}{\partial u} + X^4(\zeta,\bar{\zeta},u,v)\tfrac{\partial}{\partial v},$$

leaves the covectors $m, \bar{m}$ and ℓ invariant. We wish to have n vanish under the action of $\mathbb{L}_X$ as well, however to continue we will switch back to the real spatial coordinates involved in $\zeta = x + iy$ to simplify the process. Supposing $C = A + iB$, $A, B \in \mathbb{R}$, the vector field and differential equation become

$$X = A\tfrac{\partial}{\partial x} + B\tfrac{\partial}{\partial y} + f(u)\tfrac{\partial}{\partial u} + X^4(\zeta,\bar{\zeta},u,v)\tfrac{\partial}{\partial v}, \tag{11.7}$$
$$(e^{\frac{a}{2}}f)_{,u} + A(e^{\frac{a}{2}})_{,x} + B(e^{\frac{a}{2}})_{,y} = 0$$

The coframe transforms to be,

$$m = dx + idy, \quad \ell = e^{\frac{a}{2}} du, \quad n = e^{-\frac{a}{2}}(dv - \tfrac{2v}{x}dx + [x4h^0(x,y,u) - \tfrac{v^2}{4x^2}]du).$$

We note that $a = \frac{1}{2}\left(ln(x^2) + ln(|h^0_{,\bar{\zeta}\bar{\zeta}}|)\right)$ may be expressed in real coordinates, but that for now we will continue to express it in the complex coordinate system.

Calculating the lie derivative of n in the direction of X,

$$\begin{aligned}
\mathbb{L}_X n = \quad & \left[vAe^{-\frac{a}{2}}\left(\frac{a_{,x}}{x} + \frac{2}{x^2}\right) + \frac{e^{-\frac{a}{2}}}{x}(Ba_{,y} + fa_{,u}v - 2X^4) + e^{-\frac{a}{2}}X^4_{,x}\right] dx \\
& + \left[\left(fe^{-\frac{a}{2}}\left(2hx - \left(\frac{v}{2}\right)^2\right)\right)_{,u} + A\left(e^{-\frac{a}{2}}\left(2hx - \left(\frac{v}{2}\right)^2\right)\right)_{,x}\right] du \\
& + \left[B\left(e^{-\frac{a}{2}}\left(2hx - \left(\frac{v}{2}\right)^2\right)\right)_{,y} - \frac{X^4 e^{-\frac{a}{2}}v}{2x^2} + e^{-\frac{a}{2}}X^4_{,u}\right] du \\
& + \left[e^{-\frac{a}{2}}X^4_{,y}\right] dy + \left[e^{-\frac{a}{2}}X^4_{,v} - \frac{e^{-\frac{a}{2}}}{2}(Aa_{,x} + Ba_{,y} + fa_{,u})\right] dv.
\end{aligned} \tag{11.8}$$

It is easily seen that $X^4 = G(x,u,v)$ from the vanishing of the dy component. At this point it will be helpful to rework the differential equation (11.7)-B:

$$(e^{-\frac{a}{2}})_{,u}f + A(-e^{\frac{a}{2}})_{,x} + B(e^{\frac{a}{2}})_{,y} - e^{-\frac{a}{2}}f_{\ ,u} = 0.$$

Using this we find the dv component of $\mathbb{L}_X n$ simplifies to be $G_{,v} + f_{,u}$. Setting this to zero gives,

$$G = -f_{,u}v + g'(x,u).$$

Substitution of this form for $G(x,u,v)$ into the dx component along with another application of (11.7)-B gives further restrictions on G as a differential equation for the aribtrary function $g'_{,x} = 2g'x^{-1}$, the solution of which is now substituted into G

$$G = -f_{,u}v + x^2 g(u)$$

Finally, substituting (11.7)-B and the above G into the du component we find the following polynomial in terms of v that must vanish,

$$Av^2 + (g + 2f_{,uu})v + 4x(Ah^0_{,x} + Ah^0 + Bh^0_{,y} + fh^0_{,u} + 2f_{,u}h^0) - 2x^2 f_{,uuu}$$

Thus, (11.5) follows from the vanishing of the v^0 coefficient. It is easily shown that equation (11.6) with $A = 0$ may be derived from (11.5). □

We actually can do better than this, with a bit more work choosing the right coordinate system for X. Our goals will be to eliminate the arbitrary function f and provide a simple form for the metric function $h^0(x, y, u)$.

Theorem 11.2.2. *If a Kundt spacetime with $\tau \neq 0$ admits a Killing vector X. Then depending on the magnitude of X and the vanishing of $i_X du$ coordinates may be chosen in which the metric and vector field X takes the following forms:*

- $|X| > 0$ *and* $i_X du \neq 0$*:* $X = \frac{\partial}{\partial y} + \frac{\partial}{\partial u}$

$$ds^2 = dx^2 + dy^2 - du\left(dv - \frac{2v}{x}dx + \left(4xh^0(x, y-u) - \frac{v^2}{4x^2}\right)du\right). \quad (11.9)$$

- $|X| > 0$ *and* $i_X du = 0$*:* $X = \frac{\partial}{\partial y}$

$$ds^2 = dx^2 + dy^2 - du\left(dv - \frac{2v}{x}dx + \left(4xh^0(x, u) - \frac{v^2}{4x^2}\right)du\right). \quad (11.10)$$

We note these spacetimes cannot describe vacuum solutions as

$$\Phi_{22} = x(h^0_{,xx}) = 0$$

implies $h^0_{,xx} = 0$; this causes a contradiction as $\Psi_4 = h^0_{,xx} \neq 0$.

- $|X| = 0$ *and* $i_X du \neq 0$*:* $X = \frac{\partial}{\partial u}$

$$ds^2 = dx^2 + dy^2 - du\left(dv - \frac{2v}{x}dx + \left(4xh^0(x, y) - \frac{v^2}{4x^2}\right)du\right). \quad (11.11)$$

Proof. There are two cases to consider, depending on the vanishing of f. If $f \equiv 0$, we find immediately that $X = \frac{\partial}{\partial y}$ and the metric function h^0 must satisfy the differential equation, $Bh^0_{,y} = 0$, and the metric takes the form

$$ds^2 = dx^2 + dy^2 - du\left(dv - \frac{2v}{x}dx + \left(4xh^0(x, u) - \frac{v^2}{4x^2}\right)du\right).$$

Next we assume that $f \neq 0$. We may choose coordinates so that $f(u) \equiv 1$ in (11.4),

$$\tilde{u} = \int \frac{1}{f}du = F(u), \quad \tilde{v} = f(u)v - \frac{x^2 f'}{2f}, \quad \tilde{x} = x, \quad \tilde{y} = y, \quad (11.12)$$

so that the metric becomes (dropping the tildes)

$$dx^2 + dy^2 - du\left(dv + \left[f^2 4h^0 x - \left(\frac{x^2 f'}{f}\right)_{,u} - \frac{x^2 f'^2}{4f^2} - \frac{v^2}{4x^2}\right]du - \frac{2v}{x}dx\right).$$

As $h^0(x,y,u)$ was arbitrary we may make a gauge transformation to absorb the extra v^0-terms in the du component in n, $\tilde{h}^0 = f^2 4h^0 x - \left(\frac{x^2 f'}{f}\right)_{,u} - \frac{x^2 f'^2}{4f^2}$. Dropping the tilde yields a Kundt metric for which the killing vector

$$\tilde{X} = B\frac{\partial}{\partial y} + \frac{\partial}{\partial u},$$

and the metric function h^0 satisfies the differential equation,

$$2Bh^0_{,y} + 2h^0_{,u} = 0.$$

We again have two subcases, depending on whether or not $B = 0$. If $B = 0$ it is clear that the metric must be independent of the retarded time coordinate u, and that the metric takes the form

$$ds^2 = dx^2 + dy^2 - du\left(dv - \frac{2v}{x}dx + \left(4xh^0(x,y) - \frac{v^2}{4x^2}\right)du\right).$$

If $B \neq 0$, without lost of generality we scale the y coordinate so that $B = 1$ and so $X = \frac{\partial}{\partial y} + \frac{\partial}{\partial u}$. The differential constraint on h^0 in (11.5) is now

$$h_{,y} + h_{,u} = 0.$$

This is easily solved, giving the final form for the metric

$$ds^2 = dx^2 + dy^2 - du\left(dv - \frac{2v}{x}dx + \left(4xh^0(x,y-u) - \frac{v^2}{4x^2}\right)du\right).$$

□

11.3 Kundt Spacetimes with $\tau \neq 0$ Admitting a G_2 Isometry Group

Taking the metric for each case in theorem (11.2.2), we analyze the Killing equations for a new Killing vector $Y \neq X$. In each case the form of X will determine the nature of Y and (11.5) with Y switched with X determines h^0 up to constant.

Theorem 11.3.1. *If a Kundt spacetime admits a two dimensional group of isometries, then for some $C_0 \in \mathbb{R}$ the metric will belong to one of two classes. Writing $V = C_1X + C_2Y,\ C_1, C_2 \in \mathbb{R}$, depending on whether $i_V\ell \neq 0$ or not, the metrics will be*

$$ds^2 = dx^2 + dy^2 - du(dv - \tfrac{2v}{x}dx + \left(4xh_A(x)e^{C_0(y-u)} + \tfrac{C_0^2x^2}{4} + \tfrac{v^2}{4x^2}\right)du), \quad (11.13)$$

$$ds^2 = dx^2 + dy^2 - du\left(dv - \tfrac{2v}{x}dx + \left(4xh_B(x)e^{-C_0y} - \tfrac{v^2}{4x^2}\right)du\right), \quad (11.14)$$

admitting the following Killing vectors with commutation relations $[X_A, Y_A] = \frac{C_0}{2} Y_A$ *and* $[X_B, Y_B] = \frac{C_0}{2} X_B$ *respectively:*

$$X_A = \tfrac{\partial}{\partial y} + \tfrac{\partial}{\partial u}, \quad Y_A = e^{\frac{C_0}{2}u}\left[\tfrac{\partial}{\partial u} - \left(\tfrac{C_0 v}{2} + \tfrac{C_0^2}{4}\right)\tfrac{\partial}{\partial v}\right], \tag{11.15}$$

$$X_B = \tfrac{\partial}{\partial u}, \quad Y_B = \tfrac{\partial}{\partial y} + \tfrac{C_0 u}{2}\tfrac{\partial}{\partial u} \qquad . \tag{11.16}$$

Proof. We will go through each case and analyze the Killing equations:

- If $X = \frac{\partial}{\partial y} + \frac{\partial}{\partial u}$ the second Killing vector, Y, must have $B = 0$, as it may be removed by adding an appropriate scaling of X. Setting $B = 0$ and $h^0(x, y-u)$ in (11.4) and (11.5)

$$Y = f(u)\frac{\partial}{\partial u} - [f'v + 2x^2 f'']\frac{\partial}{\partial v},$$

  and $h^0(x, y-u)$ must satisfy the additional differential equation

$$2fh^0_{,u} + 4f'h^0 - xf''' = 0.$$

  Differentiating the above identity with respect to y, yields a seperable equation

$$\frac{2f'}{f} = -\frac{h^0_{,uy}}{h^0_{,y}} \equiv C_0, \quad C_0 \in \mathbb{R},$$

  we find a simple differential equation for $f(u)$ for which the solution is $f = C_1 e^{\frac{C_0}{2}u}$,$C_1 \in \mathbb{R}$. Substituting this into the original equation (11.5) with $B = 0$ and $h^0(x, y-u)$ another differential equation is found

$$\frac{C_0^3 x}{16} - h^0_{,u} - C_0 h^0 = 0$$

  Denoting $w = y - u$, we note that $h_{,u} = -h_{,w}$ and so we may write the above equation as

$$(e^{C_0 w}h^0)_{,w} = \frac{C_0^3 x}{16}e^{C_0 w}$$

  from which $h^0(x, y-u)$ is easily found by integrating and introducing one arbitrary function $h(x)$. We conclude that the metric

$$ds^2 = dx^2 + dy^2 - du(dv - \tfrac{2v}{x}dx + \left(4x\left(h(x)e^{C_0(y-u)} + \tfrac{C_0^2 x}{16}\right) - \tfrac{v^2}{4x^2}\right)du).$$

admits the following Killing vectors with commutator, $[X, Y] = \frac{C_0}{2} Y$,

$$X = \frac{\partial}{\partial y} + \frac{\partial}{\partial u}, \quad Y = e^{\frac{C_0}{2}u}\left[\frac{\partial}{\partial u} - \left(\frac{C_0 v}{2} + \frac{C_0^2}{4}\right)\frac{\partial}{\partial v}\right].$$

In these coordinates $\Phi_{22} = x(h^0_{,xx} + h^0_{,yy}) \neq 0$ and so in general these spacetimes will describe pure radiation or null Einstein-Maxwell fields. In the special case that Φ_{22} we find $h_{,xx} + C_0^2 h = 0$, the most general solution of which is

$$h(x) = C_1 cos(C_0 x) + C_2 sin(C_0 x), \quad C_1, C_2 \in \mathbb{R}.$$

- If $X = \frac{\partial}{\partial y}$ we may always scale X and add this to the second Killing vector Y to set $B = 0$, thus $f \neq 0$, and we may fix coordinates to set $f(u)$ equal to some fixed constant. Using the same coordinate transformation as in the proof of (11.2.2) we find that (11.4) and (11.5) become

$$Y = \frac{\partial}{\partial u}, \quad h_{,u} = 0$$

and hence the metric takes the form

$$ds^2 = dx^2 + dy^2 - du\left(dv - \frac{2v}{x}dx + \left(4xh^0(x) - \frac{v^2}{4x^2}\right)du\right). \qquad (11.17)$$

This admits two commuting Killing vectors $X = \frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial u}$. We note that this metric can be found from the first metric (11.13) by setting $C_0 = 0$.

- If $X = \frac{\partial}{\partial u}$ we find that

$$Y = B\frac{\partial}{\partial y} + f(u)\frac{\partial}{\partial u} - [f'v + 2x^2 f'']\frac{\partial}{\partial v}$$

where $h^0(x, y)$ satisfies

$$Bh^0_{,y} + 2fh^0 = \frac{xf'''}{2}$$

There are two cases here, depending on whether $B = 0$ or not. Let us suppose $B = 0$ $f' \neq 0$ otherwise $X = Y$. Assuming this we may solve for h^0 algebraically from (11.5) with $B = 0$ and $h^0(x, y)$ and so the metric is of the form

$$h^0 = \frac{-xf'''}{4f'}.$$

However this implies $\Psi_4 = 0$ and so the constant B cannot vanish. Taking the original identity and differentiating with respect to y a separable equation is found

$$-\frac{h^0_{,yy}}{h^0_{,y}} = \frac{2f'}{B} \equiv C_0 \in \mathbb{R}.$$

Thus $f(u) = \frac{BC_0}{2}u$ and a rescaling of y sets $B = 1$, substituting this into the differential equation simplifies to be

$$h_{,y} + C_0 h = 0.$$

Solving this we find that the metric

$$ds^2 = dx^2 + dy^2 - du\left(dv - \frac{2v}{x}dx + \left(4xh(x)e^{-C_0 y} - \frac{v^2}{4x^2}\right)du\right).$$

admits the Killing vectors satisfying the commutator relation $[X, Y] = \frac{C_0}{2}X$

$$X = \frac{\partial}{\partial u}, \quad Y = \frac{\partial}{\partial y} + \frac{C_0 u}{2}\frac{\partial}{\partial u}$$

Requiring these to be vacuum forces $h_{,xx} + C_0^2 h = 0$, the most general solutions of which are $h = C_2 cos(C_0 x) + C_3 sin(C_0 x)$ $C_2, C_3 \in \mathbb{R}$. Notice that for $C_0 = 0$ we recover the second metric (11.17).

□

To show that the class of Kundt spacetimes with $\tau \neq 0$ cannot admit any symmetry group larger than a G_2, there are two approaches. Either we may construct our standard fixed coframe and show that all invariants are written in terms of x and v, implying that there are two functionally independent invariants. We may use the result from [22], Chapter 9, relating then number of functionally independent invariants to the dimension of the symmetry group to show that $dim(I) - dim(H_o) = 0$ where H_o denotes the isotropy group. Alternatively we may suppose $Z \neq X, Y$ and analyze the Killing equations once more. In either case we find that Z takes the same form as Y except $C_0' \neq C_0$

$$h_A(x)e^{C_0(y-u)} + \frac{C_0^2 x^2}{16} = h_A'(x)e^{C_0'(y-u)} + \frac{C_0'^2 x^2}{16}$$
$$or \quad h_b(x)e^{C_0 y} = h_b'(x)e^{C_0' y};$$

this immediately causes a contradiction as both of these can give a seperable equation for which $C_0 = C_0'$ must hold.

Proposition 11.3.2. *The class of Kundt spacetimes with $\tau \neq 0$ admit at most a two-dimensional group of isometries.*

Chapter 12

Concluding Remarks

In this thesis we have examined several aspects of the equivalence problem for the degenerate Kundt spacetimes in arbitrary dimension. This is a particularly relevant question, as the straightforward approach of computing polynomial scalar curvature invariants to classify such spacetimes is no longer applicable, even in principle. In fact, in four dimensions the degenerate Kundt spacetimes constitute all of the spacetimes with this property, while in higher dimensions the degenerate Kundt class may not make up the entirety of the spacetimes whose polynomial scalar curvature invariants do not uniquely determine the spacetime. This is in contrast with the case of Riemannian manifolds where the polynomial scalar curvature invariants for a given metric uniquely characterize the space uniquely, independent of dimension.

In order to solve the equivalence problem for Lorentzian manifolds of any dimension, we must employ Cartan's equivalence algorithm in order to construct a set of scalar curvature invariants, the Cartan invariants, that uniquely characterize the spacetimes. In theory, this algorithm would provide a complete answer to the equivalence problem; in practice, this algorithm is not easily implemented. For example, in four dimensions Cartan's algorithm was adapted to the formalisms of General Relativity by Karlhede, and significant effort has gone into applying the Karlhede algorithm to a large subset of all four dimensional spacetimes, although there are still many spacetimes for which the Karlhede algorithm is not feasible in practice. In higher dimensions there has been no work towards applying Cartan's algorithm.

With the higher dimensional equivalence problem for spacetimes in mind, we consider necessary conditions for two degenerate Kundt metrics to be diffeomorphic. As the ultimate goal is to implement Cartan's algorithm in arbitrary dimension, we will look at invariant conditions that are related to the properties of the set of Cartan invariants. The existence of isometries in a spacetime influence the dimension of the isotropy group and the number of functionally independent invariants; for example, in an N dimensional space where all isotropy is exhausted in Cartan's algorithm the existence of an isometry decreases the number of functionally independent invariants by one.

In analogy with Kundt and Ehlers' classification of the symmetry groups of the vacuum PP-waves in four dimensions, we study two well-known subclasses of the degenerate Kundt spacetimes: the CSI spacetimes, those metrics for which all polynomial scalar curvature invariants are constant, and the $CCNV$ spacetimes, where the metrics admit a covariantly constant null vector ℓ. Both of these spacetimes are generalizations of the PP-wave spacetimes which belong to the class of VSI spacetimes in which all polynomial scalar curvature invariants vanish, and which constitute all of the four dimensional $CCNV$ spacetimes. For both classes of spacetimes, we present the possible forms for the non-spacelike Killing vector and the constraints on the metric functions for each case. By fixing the dimension and making a choice of locally transverse space we can use these conditions to determine the maximal symmetry group on a case by case basis. To illustrate this, in the eleventh chapter of this thesis we use the conditions for the vacuum Kundt waves to admit an isometry to prove the maximal symmetry group is two dimensional.

As an illustration of the importance of the degenerate Kundt spacetimes in higher dimensions and the utility of the study of symmetry groups, we take an aside into an alternative gravity theory: supergravity. If we wish to preserve some fraction of symmetry, one requires the existence of a Killing spinor which give rise to non-spacelike Killing vectors. For this reason we examine the $CCNV$ spacetimes and produce two specific examples that preserve a non-minimal fraction of supersymmetry. More generally, it has been shown that CSI spacetimes, when provided with the appropriate sources, are solutions to other gravity theories as well. In fact, in four dimensions there are classical solutions in general relativity, the *universal* spacetimes, for which all quantum corrections are multiples of the metric and hence are solutions to all quantum gravity theories; it has been shown that these spacetimes belong to the CSI subclass.

Although all four dimensional universal spacetimes have been shown to be belong to the CSI subclass, they have yet to be fully identified; e.g., it is still unknown if every CSI spacetime is universal. Originally this property was discovered in the class of the PP-waves, where the vanishing of the polynomial scalar curvature invariants ensured that all quantum corrections vanished; generalizing this idea one can show that the VSI spacetimes are universal as well. To push this further we considered the class of spacetimes in which all polynomial invariants vanish or are expressed as polynomials in terms of the cosmological constant Λ, which we have named the CSI_Λ spacetimes, which are universal also. To describe this broad class of metrics we provide conditions on the Newman Penrose curvature scalars, which provides an invariant characterization for the whole class.

Using these conditions we integrated the metric functions and related these metrics to the standard metric form in Kundt coordinates. Although the CSI_Λ spacetimes are of Petrov type III, N or O, we focus on the Petrov type N metrics for two reasons: physically they describe all plane-fronted gravitational waves in spacetimes with cosmological constant, and they provide an alternative classification in terms of the sign of two invariants, Λ and the sole component of the second Lie derivative of the metric in the direction of the null coframe vector ℓ, denoted by τ. This classification divides the plane-fronted gravitational waves into five distinct subclasses; however, it cannot prove the equivalence of two Type N CSI_Λ metrics with the same values for Λ and τ. Applying the Karlhede algorithm to the entire collection of Petrov type N CSI_Λ spacetimes is still a significant task, one that is outside the scope of this PhD thesis. In order to present an example of the Karlhede algorithm in use, we focus on the collection of all vacuum Petrov Type N $CSI_{\Lambda=0} = VSI$ spacetimes, consisting of the vacuum PP-waves and vacuum Kundt waves.

In each class the Karlhede algorithm allows us to classify each distinct subclass first by discrete quantities like the invariant count at each order and the dimension of the isotropy group. Within each broad subclass determined by these discrete quantities, the functionally independent and dependent Cartan invariants produce a classifying manifold which allows for a complete classification for each subclass. This analysis answers two important questions relating to the upper-bound on the covariant derivatives of curvature required for the Karlhede algorithm to invariantly classify the vacuum PP-waves and Kundt waves, proving that the fourth covariant derivative is the maximum required. Furthermore, a collection of PP-waves was found that was missing from Kundt and Ehler's original classification using symmetry groups.

In the context of General Relativity, the measurements of physical properties are independent of the coordinates used and thus are invariants. It is a reasonable question to ask if the Cartan invariants of a spacetime are connected to physical properties in it, since the Cartan invariants give a complete characterization of the spacetime locally. The actual relationship between the classifying manifold and physical properties in the spacetime is unknown.

To this end we examine a particularly simple and well-known subcase of the PP-wave spacetimes, the vacuum plane wave spacetimes; these spacetimes are ideal due to their simple form in Brinkmann coordinates, the small number of Cartan invariants, $\{\gamma(u), \bar{\gamma}(u), \Delta\gamma(u)\}$ where γ is a spin coefficient relative to the coframe with $\Psi_4 = 1$ and Δ belongs to the dual of this coframe, and the direct relationship between the choice of γ and the sole metric function $A(u)$ in terms of an integral. We showed that the magnitude and phase of a plane wave spacetime are related to the form of the classifying manifold; imposing the condition that $\bar{\gamma} = \pm\gamma$ we produce two classes of spacetimes in which an arbitrary time-like observer would measure: i) a fixed value for the magnitude of Ψ_4 and a varying phase, ii) a fixed value for the phase with the magnitude of Ψ_4 varying. As a final example we examine the weak-field vacuum circularly polarized plane waves and determine the form of the classifying functions in terms of the simplest Cartan invariants.

In the future, I hope to continue my work on the equivalence problem for Lorentzian manifolds in four dimensions and in higher dimensions. In four dimensions there is still a considerable amount of work required to apply the Karlhede algorithm to the remaining degenerate Kundt spacetimes; in particular, the CSI spacetimes are largely unexplored in four dimensions. By applying the Karlhede algorithm to these four dimensional spacetimes we will gain insight into implementing Cartan's algorithm for the higher dimensional CSI and $CCNV$ spacetimes and the four dimensional degenerate Kundt spacetimes as well.

Independent of the analysis of the four dimensional CSI and degenerate Kundt spacetimes, the class of VSI $CCNV$ spacetimes in N dimensions offer a chance to apply the equivalence algorithm to these higher dimensional spacetimes. Furthermore since there is now a formalism for the physical interpretation of higher dimensional spacetimes, it is hoped that this formalism along with the invariant classification of these spacetimes will provide a generalization of the work done relating the Cartan invariants of the vacuum gravitational plane waves and their physical properties.

Appendix A

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A.1 Classical and Quantum Gravity

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A.2 International Journal of Geometric Methods in Modern Physics

A.2.1 Response

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A.3 International Journal of Modern Physics: D

A.3.1 Response

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A.5 Mathematical Physics, Analysis and Geometry

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