EXACTLY SOLVABLE DIFFUSION EQUATIONS AND PRICING MODELS BASED ON EXCEPTIONAL HERMITE POLYNOMIALS

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Submitted in partial fulfillment of the requirements
for the degree of Master of Science

at

Dalhousie University
Halifax, Nova Scotia
May 2023

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Abstract

In 1973, Black-Scholes and Merton developed a partial differential equation that models the price evolution of a European call option, now referred to as the Black-Scholes equation. Because of its importance in options pricing, there has been a lot of research put into developing solvable derivative models. Through a gauge transformation, the classical Black-Scholes equation can be transformed into a Schrodinger equation. From there, we apply supersymmetric methods to construct a family of orthogonal solutions in terms of exceptional Hermite polynomials. We use these techniques to generalize the classical Black-Scholes equation and obtain solvable derivative models.

Chapter 1

Introduction

In 1973, Black-Scholes [1] and Merton [2] developed a partial differential equation that models the price evolution of a European call option. Because of the importance of the Black-Scholes in options pricing, there has been a lot of research put into developing solvable derivative models. The Black-Scholes equation is a diffusion equation that can be transformed into a Schrödinger equation by a change of variables and by gauge transformations. Once in Schrödinger form, we can use the method of supersymmetric quantum mechanics (SUSYQM) applied to the quantum harmonic oscillator to generate a new family of exact orthogonal solutions. These solutions will be generated by the exceptional Hermite polynomials, orthogonal polynomials that generalize the classical Hermite polynomials. This will generate solvable models for derivative pricing.

We begin with a brief introduction to stochastic differential equations. We introduce some stochastic calculus and show that stochastic differential equations can be modeled as backward Kolmogorov equations. We then review some Strum-Liouville theory, since the Black-Scholes equations and its generalizations are all second order equations, and separation of variables.

In chapters 3 and 4, we develop the transformation techniques needed to transform backwards Kolmogorov equations into Schrödinger form and introduce SUSYQM. What's important to note is that every second order differential equation can be put in Schrödinger form using a change of variables and a gauge transformation. These transformations are critical for us to derive other solvable derivative models. We also introduce the Darboux transformation and its generalization, the Darboux-Crum transformations. These are important transformations for the SUSYQM method.

Next, we introduce the classical Hermite polynomials and their properties. This gives us the tools we need to derive exact orthogonal solutions from the Schrödinger equation with potential. A brief introduction to hypergeometric functions is given and we show that the classic Hermite polynomials satisfy a hypergeometric equation. We then use SUSY to construct the translationally invariant shape potential. We use them to derive the exceptional Hermite polynomials and discuss some important properties that will later allow us to construct solvable derivative models.

We then introduce the classical Black-Scholes equation and discuss some of its generalizations. We begin with the derivation of the classical Black-Scholes equation and its solutions. Then we use the techniques developed in chapters 3 and 4 to trasnform the Black-Scholes equation into Schrödinger form. This gives us the Black-Scholes Hamiltonian.

Finally, we apply the SUSYQM method and exceptional Hermite polynomials to construct solvable derivative models. We use these to examine the double knockout barrier model discussed by Jana and Roy [10], as well as the solvable model with variable drift and the solvable model with varrying carrying cost.

Chapter 2

Differential Equations

Differential equations appear throughout math, science, and engineering to model various physical processes. This chapter is an introduction to stochastic differential equations, differential equations with both a deterministic component and a random, stochastic component, and stochastic calculus. These topics will reappear in chapter 6 when we will use them to derive the Black-Scholes equation and Black-Scholes formula. We also discuss Sturm-Liouville problems, which are the main class of differential equations we will be manipulating. We finish with a review of separation of variables on the class of differential equations that will be of interest to us.

2.1 Stochastic Differential Equations

For functions $b(x)$ and $\sigma(x)$, a time-homogeneous Ito-process is a solution to the stochastic differential equation

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x. \tag{2.1}$$

This is a purely formal notation. It means that the process X_t satisfies the integral equation

$$X_t = X_0 + \int_0^t \mu(X_u)du + \int_0^t \sigma(X_u)dW_u \tag{2.2}$$

The change in X_t consists of a deterministic part and a stochastic part. We represent the deterministic part as $\mu\,dt$, where μ is the deterministic growth rate. The stochastic component in is denoted by $\sigma\,dW_t$, where σ is called the volatility because it measures the standard deviation of X_t. The measure dW_t is a random variable taking values in a continuous analogue of the normal distribution, which is known as a Wiener process, or Brownian motion. We normalize this distribution so that dW_t has the mean of zero and variance equal to dt.

The integral with respect to du in (2.2) is an ordinary integral, while the integral with respect to dW_t is a Ito integral. An Itô integral

$$\int_a^b f(\tau)dW_\tau$$

is defined as the limit $n \to \infty$ of

$$\sum_{i=1}^{n} f(t_{i-1})[W_{t_i} - W_{t_{i-1}}] \tag{2.3}$$

where t_i is a partition of $[a, b]$ into n equal intervals.

A helpful interpretation of the stochastic differential equation (2.1) is that in a small time interval of length Δt the stochastic process X_t changes its value by an amount that is normally distributed with expectation $\mu(X_t)\Delta t$ and variance $\sigma(X_t)^2\Delta t$, and is independent of the past behavior of the process.

This follows from the fundamental properties of the Wiener process W_t. The independence property is that for every $0 \leq s_1 < t_1 \leq s_2 < t_2$ the random variables $W_{t_2} - W_{s_2}$ and $W_{t_1} - W_{s_1}$ are independent. The other key property is that $W_{t+\Delta t} - W_t$ is normally distributed with with zero mean and variance equal to Δt.

For a function $F(\phi)$ of a normally distributed random variable ϕ the expectation operator is

$$\mathbb{E}[F(\phi)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)e^{-x^2/2}\, dx,$$

To make sure that W_t has zero mean and variance of dt, we can write

$$dW_t = \phi\sqrt{dt}$$

To check this, firstly,

$$\mathbb{E}[\phi] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-x^2/2}\, dx = 0.$$

Then,

$$\mathbb{E}[dW_t] = \mathbb{E}[\phi\sqrt{dt}] = \sqrt{dt}\,\mathbb{E}[\phi] = 0.$$

Secondly, using integration by parts,

$$\int_{-\infty}^{\infty} x^2e^{-x^2/2}\, dx = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

Then

$$\mathbb{E}[\phi^2] = 1, \quad \mathbb{E}[dW_t^2] = \mathbb{E}[\phi^2\, dt] = dt$$

Hence,

$$Var[dW_t] = \mathbb{E}[dW_t^2] - \mathbb{E}[dW_t]^2 = \mathbb{E}[dW_t^2] = dt.$$

Another important property of W_t is that the expected value of an Itô integral is zero,

$$\mathbb{E}\left[\int_a^b f(\tau)dW_\tau\right] = 0 \tag{2.4}$$

where $f(t)$ is a non-anticipating process. The integral has zero expected value because each term in the approximating sum (2.3) has zero expected value:

$$\mathbb{E}\left[f(t_i)(W_{t_{i+1}} - W_{t_i})\right] = \mathbb{E}\left[f(t_i)\right]\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})\right] = 0.$$

Equivalently, the process

$$m(t) = \int_a^t f(\tau)dW_\tau$$

is a Martingale.

2.2 The Feynman-Kac-Formula

A backward Kolmogorov equation is a partial differential equation

$$u_t + \frac{1}{2}\sigma(x)^2 u_{xx} + \mu(x)u_x - \nu(x)u = 0 \tag{2.5}$$

with a terminal condition $u(x, T) = g(x)$.

Proposition 2.2.1. Let X_t be an Itô-process defined by the stochastic differential equation (2.1). Then, a solution of (2.5) is the function u defined by the conditional expectation

$$u(x,t) = \mathbb{E}\left[e^{-\int_t^T \nu(X_s)\,ds} g(X_T) \mid X_t = x\right],$$

Proof. For $0 \leq t_0 < T$ let Y_t be the Itô process

$$Y_t = e^{-\int_{t_0}^t \nu(X_\tau)\,d\tau} u(X_t, t)$$

By Itô's lemma,

$$d(f(X_t, t)) = \frac{1}{2} f_{xx} dX_t^2 + f_x dX_t + f_t dt.$$

From (2.1),

$$dX_t^2 = \sigma(X_t)dW_t^2 = \sigma(X_t)dt.$$

Then,

$$\begin{aligned} d(f(X_t,t)) &= f_x dX_t + \left(\frac{1}{2}\sigma(X_t)^2 f_{xx} + f_t\right) dt \\ &= \left(\frac{1}{2}\sigma(X_t)^2 f_{xx} + \mu(X_t) f_x + f_t\right) dt + \sigma(X_t) f_x dW_t \end{aligned}$$

Now we have

$$\begin{aligned} dY_t &= \partial_t \left(e^{-\int_{t_0}^t \nu(X_\tau)\,d\tau}\right) u(X_t,t)dt + e^{-\int_{t_0}^t \nu(X_\tau)\,d\tau}\, d(u(X_t,t)) \\ &= e^{-\int_{t_0}^t \nu(X_\tau)\,d\tau} \left(-\nu(X_t)u(X_t,t)dt + d(u(X_t,t))\right) \\ &= e^{-\int_{t_0}^t \nu(X_\tau)\,d\tau} \left(\left(u_t + \frac{1}{2}\sigma(X_t)^2 u_{xx} + \mu(X_t)u_x - \nu(X_t)u\right) dt + \sigma(X_t)u_x\, dW_t\right) \end{aligned}$$

The dt coefficient vanishes by (2.5). Then

$$dY_t = e^{-\int_{t_0}^t \nu(X_\tau)\,d\tau} \sigma(X_t) u_x(X_t,t)\, dW_t$$

Integrating this equation from t to T,

$$Y_T - Y_t = \int_t^T e^{-\int_t^s \nu(X_\tau)\,d\tau} \sigma(X_s)\, u_x(X_s,s)\, dW_s$$

Taking expectations, the right side is zero by (2.4). Then,

$$\mathbb{E}[Y_T - Y_t \mid X_t = x] = 0$$

We have

$$\mathbb{E}[Y_t \mid X_t = x] = \mathbb{E}\left[e^{-\int_t^t \nu(X_\tau)\,d\tau}\, u(X_t,t) \mid X_t = x\right] = u(x,t)$$

And,

$$\begin{aligned} \mathbb{E}[Y_T \mid X_t = x] &= \mathbb{E}\left[e^{-\int_t^T \nu(X_\tau)\,d\tau}\, u(X_T,t) \mid X_t = x\right] \\ &= \mathbb{E}\left[e^{-\int_t^T \nu(X_\tau)\,d\tau}\, g(X_T) \mid X_t = x\right] \end{aligned}$$

Hence,

$$u(x,t) = \mathbb{E}\left[e^{-\int_t^T \nu(X_\tau)\,d\tau}\, g(X_T) \mid X_t = x\right]$$

□

2.3 Sturm-Liouville Problems

We are interested in eigenvalue problems posed as a second-order Sturm-Liouville equation

$$(P(x)y'(x))' + R(x)y(x) = \lambda W(x)y(x), \quad X(a) = X(b) = 0 \tag{2.6}$$

where $-\infty \leq a < b \leq \infty$. Dividing by $W(x)$ and setting

$$\begin{aligned} \frac{1}{2}\sigma(x)^2 &= \frac{P(x)}{W(x)} \\ \mu(x) &= \frac{P'(x)}{W(x)} \\ \nu(x) &= -\frac{R(x)}{W(x)} \end{aligned}$$

transforms (2.6) into the second-order eigenvalue problem

$$\frac{1}{2}\sigma(x)^2 y'' + \mu(x)y' - \nu(x)y = \lambda y,$$

which we encounter in dealing with diffusion equations and stochastic differential equations. Another type of Sturm-Liouville problem that will be of interest is the Schrodinger equation

$$y'' - U(x)y = \lambda y.$$

By integration by parts, we can derive Green's identity.

$$\int_a^b ((Py_1')'y_2 - (Py_2')'y_1)\, dx \;=\; P(x)(y_1'(x)y_2(x) - y_2'(x)y_1(x))\Big|_a^b$$

Suppose that $y_1(x), y_2(x)$ are solutions of (2.6) with eigenvalues $\lambda_1 \neq \lambda_2$. Then, Green's identity implies

$$(\lambda_1 - \lambda_2)\int_a^b y_1(x)y_2(x)W(x)dx = P(x)(y_1'(x)y_2(x) - y_2'(x)y_1(x))\Big|_a^b,$$

If we impose boundary conditions so that the right-side is zero, then we get that the eigenfunctions of (2.6) are orthogonal. We also assume that $W(x) > 0$ so that

$$\int_a^b y_1(x)y_2(x)W(x)dx$$

is a positive-definite inner product. We will also assume that there exists a complete basis of solutions $y_n(x),\ n = 1, 2, \ldots$ corresponding to eigenvalues λ_n. Without loss of generality, we assume that the eigenfunctions are orthonormal:

$$\int_a^b y_n(x)^2 W(x) dx = K$$

for all n. This means that for a function $\phi(x)$ that is square-integrable on $a \leq x \leq b$ we have

$$\phi(x) = \sum_{n=1}^{\infty} A_n y_n(x),$$

where the A_n are generalized Fourier coefficients

$$A_n = \frac{1}{K} \int_a^b \phi(x) y_n(x) W(x) dx.$$

2.4 Method of Separation of Variables

Consider the partial differential equation in form of

$$\frac{1}{2}\sigma(x)^2 u_{xx} + \mu(x) u_x + u_t = \nu(x) u \tag{2.7}$$

where $u = u(x, t)$. The separated solution to it is of the form

$$u(x, t) = X(x)\, T(t) \tag{2.8}$$

Plugging the form (2.8) into the PDE (2.7), we get

$$\frac{1}{2}\sigma(x)^2 X''(x)T(t) + \mu(x)X'(x)T(t) + X(x)T'(t) = \nu(x)X(x)T(t)$$

Dividing by $X(x)T(t)$,

$$\frac{1}{2}\sigma^2(x)\,\frac{X''(x)}{X(x)} + \mu(x)\,\frac{X'(x)}{X(x)} - \nu(x) = -\frac{T'(t)}{T(t)}$$

Since the left side is a function of x and the right side a function of t, both sides must be equal to a constant.

Then the PDE is equivalent to a pair of separated ordinary differential equations for $X(x)$ and $T(t)$:

$$-\frac{1}{2}\sigma^2\, X'' - \mu\, X' + \nu X = \lambda X$$

$$T' = \lambda\, T$$

Hence the solution of the diffusion equation (2.7) can be given as

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) e^{\lambda_n t}, \tag{2.9}$$

where $X = X_n(x)$ is a solution of the second-order boundary-value problem

$$-\frac{1}{2}\sigma^2 X'' - \mu X' + \nu X = \lambda_n X, \quad X(a) = X(b) = 0. \tag{2.10}$$

In particular if all the eigenvalues $\lambda_n > 0$ are positive and the $\{X_n\}_n$ are a complete orthonormal basis of solutions to (2.10), then the diffusion equation (2.7) with terminal condition $u(x,T) = \phi(x)$ can be solved as

$$u(x,t) = \sum_{i=1}^{\infty} a_i e^{\lambda_i t} X_i(x)$$

where the a_i are the generalized Fourier coefficients of $\phi(x)$ relative to the basis $X_i(x)$.

If we are solving a diffusion equation with an initial condition, then we make the transformation $t = T - \tau$ to obtain

$$u_\tau = \frac{1}{2}\sigma(x)^2 u_{xx} + \mu(x) u_x + \nu(x) u$$

In this case the form of the solution is

$$u(x,\tau) = \sum_{n=1}^{\infty} X_n(x) e^{-\lambda_n \tau} \tag{2.11}$$

Chapter 3

Transformation Techniques

A second-order eigenvalue equation is exactly solvable if it can be changed its solution can be expressed in terms of previously known functions. We will later express solutions in terms of hypergeometric functions, especially Hermite polynomials. We will need the following transformations: a change of variables and gauge transformations. Using separation of variables to reduce diffusion equations to second-order equations in one variable, we can obtain a large class of solvable diffusion equations by applying such transformations to the hypergeometric equation, specifically the Hermite differential equation.

We are interested in second-order differential equations of the form

$$\frac{1}{2}\sigma(x)^2\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x} = \nu(x)f \tag{3.1}$$

which we express in operator form as

$$\mathcal{L}f = 0, \tag{3.2}$$

where $f = f(x)$ is the unknown function and

$$\mathcal{L} = \frac{1}{2}\sigma(x)^2\partial_x^2 + b(x)\partial_x - \nu(x). \tag{3.3}$$

3.1 Change of Variables

A change of variables is an invertible transformation $y = Y(x)$. Let $x = X(y)$ be the inverse transformation. Hence we have $X(Y(x)) = x$. The corresponding transformation of the unknown function $f \mapsto g$ is

$$g(y) = f(X(y)). \tag{3.4}$$

A change of variables transforms the second-order operator equation (3.2) into

$$\hat{\mathcal{L}}g = 0,$$

where the transformed operator

$$\hat{\mathcal{L}} = \frac{1}{2}\hat{\sigma}(y)^2\partial_{yy} + \hat{b}(y)\partial_y - \hat{\nu}(y) \tag{3.5}$$

is given by

$$\begin{aligned}\hat{\sigma}(y) &= \frac{\sigma(X(y))}{X'(y)} \\ \hat{b}(y) &= \frac{b(X(y))}{X'(y)} - \frac{\sigma(X(y))^2 X''(y)}{2X'(y)^3} \\ \hat{\nu}(y) &= \nu(X(y))\end{aligned} \tag{3.6}$$

Here is the derivation of this transformation law. We want

$$(\hat{\mathcal{L}}g)(y) = (\mathcal{L}f)(X(y)).$$

Using $f(x) = g(Y(x))$ and (3.3) gives

$$\begin{aligned}(\mathcal{L}f)(x) &= \frac{1}{2}\sigma(x)^2\,\partial_x\left[g'(Y(x))Y'(x)\right] + b(x)\,g'(Y(x))Y'(x) - \nu(x)g(Y(x)) \\ &= \frac{1}{2}\sigma(x)^2(Y'(x))^2\,g''(y) + \left(\frac{1}{2}\sigma(x)^2Y''(x) + b(x)Y'(x)\right)\,g'(y) - \nu(x)g(y)\end{aligned}$$

From there, taking derivative of

$$X(Y(x)) = x,$$

we have

$$\begin{aligned}Y'(x) &= \frac{1}{X'(Y(x))} = \frac{1}{X'(y)} \\ Y''(x) &= -\frac{1}{X'(Y(x))^2}X''(Y(x))Y'(x) = -\frac{X''(y)}{X'(y)^3}\end{aligned}$$

using which yields the transformation law (3.6).

3.2 Gauge Transformation

A gauge transformation is a transformation of the dependent variable by a multiplication operator:

$$g(x) = h(x)\,f(x), \tag{3.7}$$

where $h(x)$ is some invertible function. The corresponding operator transformation is

$$\begin{aligned}\hat{\mathcal{L}} &= h\mathcal{L}h^{-1} \\ &= \frac{1}{2}\hat{\sigma}(x)^2\partial_x^2 + \hat{b}(x)\partial_x - \hat{\nu}(x)\end{aligned}$$

The transformation law for the components is

$$\begin{aligned}\hat{\sigma}(x) &= \sigma(x) \\ \hat{b}(x) &= b(x) - \sigma(x)^2\frac{h'(x)}{h(x)} \\ \hat{\nu}(x) &= \nu(x) + \frac{1}{2}\sigma(x)^2\left(\frac{h''(x)}{h(x)} - 2\left(\frac{h'(x)}{h(x)}\right)^2\right) + b(x)\frac{h'(x)}{h(x)}\end{aligned} \tag{3.8}$$

To derive this transformation law note that

$$\begin{aligned}\partial_x\left(\frac{g(x)}{h(x)}\right) &= \frac{g'(x)}{h(x)} - \frac{h'(x)g(x)}{h(x)^2} \\ \partial_x^2\left(\frac{g(x)}{h(x)}\right) &= \frac{g''(x)}{h(x)} - \frac{2h'(x)g'(x)}{h(x)^2} + \left(\frac{2h'(x)^2}{h(x)^3} - \frac{h''(x)}{h(x)^2}\right)g(x)\end{aligned}$$

and hence that

$$\begin{aligned}\mathcal{L}\left[\frac{g(x)}{h(x)}\right] &= h^{-1}\hat{\mathcal{L}}g \\ \hat{\mathcal{L}}g &= h(x)\mathcal{L}\left[\frac{g(x)}{h(x)}\right] \\ &= \frac{1}{2}\sigma(x)^2g''(x) + \left(b(x) - \sigma(x)^2\frac{h'(x)}{h(x)}\right)g'(x) + \\ &\quad - \left[\nu(x) - \frac{1}{2}\sigma(x)^2\left(\frac{2h'(x)^2}{h(x)^2} - \frac{h''(x)}{h(x)}\right) + b(x)\frac{h'(x)}{h(x)}\right]g(x)\end{aligned}$$

A Schrodinger operator is a special class of second-order differential operators of the form

$$\mathcal{L} = \partial_{zz} - U(z),$$

where the function $U(z)$ is called the potential.

The Schrodinger form is a canonical form for second-order operators because every second-order operator can be put into Schrodinger form by a scaling and a gauge transformation.

Consider a second-order operator

$$T = p(x)\partial_{xx} + q(x)\partial_x + r(x)$$

Apply the change of variables $x = \zeta(z)$ that satisfies

$$\zeta'(z)^2 = p(\zeta(z)). \tag{3.9}$$

Explicitly,

$$z = \int^{x=\zeta(z)} \frac{dx}{\sqrt{p(x)}}.$$

In this way

$$\partial_{zz} = p(x)\partial_{xx} + \frac{1}{2}p'(x)\partial_x.$$

Set

$$\mu(x) = \exp\left(\frac{1}{2}\int \frac{q(x) - \frac{1}{2}p'(x)}{p(x)}dx\right).$$

A direct calculation shows that

$$\mu T \mu^{-1} = p(x)\partial_{xx} + \frac{1}{2}p'(x)\partial_x + V(x),$$

where

$$V(x) = \frac{p''(z)}{4} - \frac{q'(x)}{2} - \frac{\big(q(x) - \frac{1}{2}p'(x)\big)\big(q(x) - \frac{3}{2}p'(x)\big)}{4p(x)} + r(x).$$

Set

$$H = -\partial_{zz} - V\big(\zeta(z)\big),$$

so that $T[y] = \lambda y$ if and only if $H[\psi] = -\lambda\psi$, where

$$\psi(z) = \mu\big(\zeta(z)\big)y\big(\zeta(z)\big).$$

Let us apply this method to the isotonic oscillator.

Example 1. For the generalized Laguerre equation, the operator T above is given by

$$T = x\partial_{xx} + (\alpha + 1 - x)\partial_x + n.$$

We can calculate the potential for the Schrodinger form of T using the formula for V above

$$\begin{aligned} V(x) &= \frac{p''(z)}{4} - \frac{q'(x)}{2} - \frac{\big(q(x) - \frac{1}{2}p'(x)\big)\big(q(x) - \frac{3}{2}p'(x)\big)}{4p(x)} + r(x). \\ &= \frac{1}{2} - \frac{(\alpha + 1 - x - \frac{1}{2})(\alpha + 1 - x - \frac{3}{2})}{4x} + n \\ &= -\frac{x}{4} + \left(n + \frac{1}{2} + \frac{\alpha}{2}\right) + \frac{1 - 4\alpha^2}{4x} \end{aligned}$$

The change of variables to transform T into Schrodinger form is given by

$$z = \int^{x=\zeta(z)} \frac{1}{\sqrt{x}} dx = 2\sqrt{\zeta(x)}$$

Hence the desired transformation is

$$x = \zeta(z) = \frac{z^2}{4}.$$

The Hamiltonian for the isotonic oscillator can now be written in Schrodinger form

$$H = -\partial_{zz} - V(z),$$

where the potential is given by

$$V(z) = -\frac{z^2}{16} + \left(n + \frac{1}{2} + \frac{\alpha}{2}\right) + \frac{1 - 4\alpha^2}{z^2}.$$

Chapter 4

The Supersymmetric Method

In 1882, G.Darboux [13] studied the eigenvalue problem of Schrodinger equation of the form

$$-\phi'' + U\phi = \lambda\phi.$$

If ϕ_0 is a solution for $\lambda = 0$, then $\hat{\phi}_0 = \phi - \frac{\phi_0'}{\phi_0}\phi$ is a solution to

$$-\hat{\phi}'' + \hat{U}\hat{\phi} = \lambda\hat{\phi}$$

where

$$\hat{U} = U - 2\partial_x^2 \log \phi_0$$

. The transformation transforms $U \mapsto \hat{U}$ that satisfies the same equation is called the Darboux transformation. We will see below that there is a way to relate the eigenfunctions of the two potentials. This means that if U is exactly solvable, then so is $\hat{U}$.

The concept of "Supersymmetry" was discovered by Ramond [14] in 1971. He proposed a wave equation for free fermions based on the structure of the dual model for bosons. In 1982, Witten [15] studied SUSY in the simplest case of SUSY quantum mechanics (SUSYQM). It involves pairs of operators that have a particular relation, which are called partner operators with the corresponding partner potentials[16].

4.1 Darboux Transformation(Factorization Method)

Let $H = -\partial_x^2 + U$ be a Schrodinger operator where ∂_x is the first-order derivative with respect to x, and $U = U(x)$ is the potential. Consider an eigenfunction $\Phi_0 = \Phi_0(x)$ with eigenvalue λ_0, that is $H\Phi_0 = \lambda_0\Phi_0$, or more explicitly,

$$-\Phi_0'' + U\Phi_0 = \lambda_0\Phi_0 \tag{4.1}$$

Set

$$W = \frac{\Phi_0'}{\Phi_0} = \partial_x \log \Phi_0, \tag{4.2}$$

and observe that

$$W' + W^2 = \frac{\Phi_0''}{\Phi_0}. \tag{4.3}$$

Hence, dividing the linear equation (4.1) by Φ_0 we obtain the Ricatti equation

$$W' + W^2 = U - \lambda_0 \tag{4.4}$$

Set

$$A = -\partial_x + W \tag{4.5}$$

$$A^\dagger = \partial_x + W \tag{4.6}$$

and use (4.4) to factorize the second-order Schrodinger operator as

$$\begin{aligned} H - \lambda_0 &= A^\dagger A \\ &= (\partial_x + W)(-\partial_x + W) \\ &= -\partial_x^2 - W\partial_x + W\partial_x + W' + W^2 \\ &= -\partial_x^2 + U - \lambda_0 \end{aligned} \tag{4.7}$$

By construction $A\Phi_0 = 0$, and therefore the factorization (4.7) determines the eigenfunction Φ_0, up to a constant multiple, as the kernel of A. For this reason we will call Φ_0 the factorizing eigenfunction and λ_0 the factorizing eigenvalue.

Exchange the order of the factors in (4.7), and introduce

$$\hat{H} = AA^\dagger + \lambda_0 \tag{4.8}$$

which is called the partner operator. By an explicit calculation,

$$\begin{aligned} \hat{H} - \lambda_0 &= (-\partial_x + W)(\partial_x + W) \\ &= -\partial_x^2 + W^2 - W' \\ &= -\partial_x^2 + U - 2W', \end{aligned}$$

and hence $\hat{H}$ is also a Schrodinger operator with potential

$$\hat{U} = U - 2\partial_x^2 \log \Phi_0. \tag{4.9}$$

The transformation from $H \mapsto \hat{H}$, or equivalently $U \mapsto \hat{U}$ is called a Darboux transformation. A given operator H has infinitely many possible Darboux transformations. Each such transformation is governed by a choice of factorizing eigenfunction.

Now introduce the function $\hat{\Phi}_0 = 1/\Phi_0$ and observe that $B\hat{\Phi}_0 = 0$, and that

$$\frac{\hat{\Phi}_0'}{\hat{\Phi}_0} = -W.$$

Therefore, $\hat{H}\hat{\Phi}_0 = \lambda_0\hat{\Phi}_0$, and so we can use $\hat{\Phi}_0$ as the factorizing function on $\hat{H}$ to obtain the factorization (4.8). Therefore, every Darboux transformation can has an inverse transformation $H \to \hat{H}$.

4.2 Intertwining relations

Let $H = A^\dagger A + \lambda_0$ and $\hat{H} = AA^\dagger + \lambda_0$ be partner operators, as defined above. Multiplying H by $A^\dagger$ from right and $\hat{H}$ by $A^\dagger$ from left, we get

$$HA^\dagger = A^\dagger AA^\dagger + \lambda A^\dagger$$

and

$$A^\dagger \hat{H} = A^\dagger AA^\dagger + \lambda A^\dagger.$$

Thus, we obtain the intertwining relation

$$HA^\dagger = A^\dagger \hat{H}.$$

Similarly, we have the dual relation

$$AH = \hat{H}A. \tag{4.10}$$

Now suppose that Φ is an eigenfunction of H with eigenvalue $\lambda \neq \lambda_0$. Explicitly,

$$H\Phi = \lambda\Phi, \quad \lambda \neq \lambda_0.$$

Set

$$\hat{\Phi} = A\Phi,$$

and check that by the intertwining relation above,

$$\hat{H}\hat{\Phi} = (\hat{H}A)\Phi = (AH)\Phi = A(H\Phi) = \lambda A\Phi = \lambda\hat{\Phi} \tag{4.11}$$

Since $\lambda \neq \lambda_0$, the function $\hat{\Phi} \neq 0$. Therefore $\hat{\Phi}$ is and eigenfunction of $\hat{H}$ with eigenvalue λ.

Conversely, suppose that

$$\hat{H}\hat{\Phi} = \lambda\hat{\Phi}, \quad \lambda \neq \lambda_0.$$

Set

$$\Phi = \frac{1}{\lambda - \lambda_0} A^\dagger \hat{\Phi}.$$

By the intertwining relations,

$$H\Phi = \frac{1}{\lambda - \lambda_0} A^\dagger \hat{H}\hat{\Phi} = \frac{\lambda}{\lambda - \lambda_0} A^\dagger \hat{\Phi} = \lambda\Phi.$$

In other words, Φ is an eigenfunction of H with eigenvalue λ. Also,

$$\begin{aligned} A\Phi &= \frac{1}{\lambda - \lambda_0} AA^\dagger \hat{\Phi} \\ &= \frac{1}{\lambda - \lambda_0} (\hat{H} - \lambda_0)\hat{\Phi} \\ &= \hat{\Phi} \end{aligned}$$

Therefore, if $\lambda \neq \lambda_0$ the the eigenfunctions of H and $\hat{H}$ are in one-to-one correspondence.

For the λ_0 eigenvalue, the correspondence between eigenfunctions is different. We already saw that $A\Phi_0 = 0$. Therefore we cannot construct an eigenfunction of $\hat{H}$ in this way. In this case, set $\hat{\Phi}_0 = 1/\Phi_0$ and hence by (4.2) and (4.3)

$$\begin{aligned} \hat{H}\hat{\Phi}_0 &= -\partial_x^2(1/\Phi_0) + (W^2 - W')/\Phi_0 + \lambda_0/\Phi_0 \\ &= \Phi_0''/\Phi_0^2 - 2(\Phi_0')^2/\Phi_0^3 + ((\Phi_0'/\Phi_0)^2 - \Phi_0''/\Phi_0 + (\Phi_0'/\Phi_0)^2)/\Phi_0 + \lambda_0/\Phi_0 \\ &= \lambda_0/\Phi_0 = \lambda_0\hat{\Phi}_0. \end{aligned}$$

4.3 Darboux-Crum Transformations

In 1955, Crum[17] generalized Darboux transformation with the regular eigenvalue problem by n successive transformations that are similar to Darboux transformation. A Darboux-Crum transformation is just an iterated Darboux transformation that uses $n \geq 1$ factorization functions to construct a partner operator. The original operator and the partner are then related by an n-th order intertwining relation.

Consider a Schrodinger operator $H = -D^2+U$, and let $\Phi_1, \Phi_2, ..., \Phi_n$ be eigenfunctions with the corresponding eigenvalues $\lambda_1...., \lambda_n$. We assume that these eigenvalues are distinct.

Set $H_0 = H,\ U_0 = U$ and let $H_n = -\partial_x^2 + U_n$ be a Schrodinger operator where

$$U_n = U_0 - 2\partial_x^2 \log Wr(\Phi_1, ..., \Phi_n). \tag{4.12}$$

Note that in the case $n = 1$, we define $Wr(\Phi_1) = \Phi_1$. Suppose that Φ is an eigenfunction of H_0 with eigenvalue λ such that $\lambda \notin \{\lambda_1...., \lambda_n\}$.

Theorem 4.3.1. Set

$$\Phi_{1...n} = \frac{Wr(\Phi_1, ..., \Phi_n, \Phi)}{Wr(\Phi_1, ..., \Phi_n)}$$

We claim that $\Phi_{1...n}$ is an eigenfunction of H_n with eigenvalue λ.

Proof. We proceed by induction on n. The case $n = 1$ was proved in the previous section. For the inductive step, suppose the result holds for $n-1$ and prove it for n. Let

$$\hat{\Phi}_n = \frac{Wr(\Phi_1, ..., \Phi_n)}{Wr(\Phi_1, ..., \Phi_{n-1})}.$$

By the inductive hypothesis, this is and eigenfunction of H_{n-1} with eigenvalue λ_n. Hence, we can use $\hat{\Phi}_n$ as the factorizing eigenfunction to factorize H_{n-1} as follows:

$$H_{n-1} = A_n^\dagger A_n + \lambda_n$$

where

$$A_n = -\partial_x + \hat{W}_n, \quad A_n^\dagger = \partial_x + \hat{W}_n,$$

where

$$\begin{aligned}\hat{W}_n &= \partial_x \log \hat{\Phi}_n \\ &= \partial_x \log Wr(\Phi_1, ..., \Phi_n) - \partial_x \log Wr(\Phi_1, ..., \Phi_{n-1}).\end{aligned}$$

Observe that by (4.12),

$$U_n = U_{n-1} - 2\partial_x \hat{W}_n,$$

Hence, by (4.9), H_n is the partner operator for H_{n-1} with the above factorizing function.

By the inductive hypothesis, the function

$$\Phi_{1\dots n-1} = \frac{Wr(\Phi_1, ..., \Phi_n, \Phi)}{Wr(\Phi_1, ..., \Phi_{n-1})}$$

is an eigenfunction of H_{n-1} with eigenfunction λ. Hence, by (4.11), the function $A_n\Phi_{1\dots n-1}$ is an eigenfunction of H_n with eigenvalue λ also. To conclude our claim, we have to show that

$$A_n\Phi_{1\dots n-1} = -\Phi_{1\dots n}.$$

This is accomplished by the following calculation

$$\begin{aligned} A_n\Phi_{1\dots n-1} &= -\partial_x\Phi_{1\dots n-1} + \hat{W}_n\Phi_{1\dots n-1} \\ &= \frac{Wr(\Phi_{1\dots n-1}, \hat{\Phi}_n)}{\hat{\Phi}_n} \\ &= \frac{Wr\left(\frac{Wr(\Phi_1,\dots,\Phi_{n-1},\Phi)}{Wr(\Phi_1,\dots,\Phi_{n-1})}, \frac{Wr(\Phi_1,\dots,\Phi_{n-1},\Phi_n)}{Wr(\Phi_1,\dots,\Phi_{n-1})}\right)}{\frac{Wr(\Phi_1,\dots,\Phi_{n-1},\Phi_n)}{Wr(\Phi_1,\dots,\Phi_{n-1})}} \end{aligned}$$

We now use the Wronskian identities. Applying the identity which is proved in [24]

$$Wr(Wr(f_1, \dots, f_n, g), Wr(f_1, \dots, f_n, h)) = Wr(f_1, \dots, f_n,)Wr(f_1, \dots, f_n, g, h).$$

gives

$$A_n\Phi_{1\dots n-1} = \frac{Wr(Wr(\Phi_1, ..., \Phi_{n-1}, \Phi), Wr(\Phi_1, ..., \Phi_{n-1}, \Phi_n))}{Wr(\Phi_1, ..., \Phi_{n-1})Wr(\Phi_1, ..., \Phi_{n-1}, \Phi_n)}$$

Alternativity gives us

$$\begin{aligned} A_n\Phi_{1\dots n-1} &= \frac{Wr(\Phi_1, ..., \Phi_{n-1}, \Phi, \Phi_n)}{Wr(\Phi_1, ..., \Phi_{n-1}, \Phi_n)} \\ &= -\frac{Wr(\Phi_1, ..., \Phi_{n-1}, \Phi_n, \Phi)}{Wr(\Phi_1, ..., \Phi_{n-1}, \Phi_n)} \end{aligned}$$

as was to be shown. Next, define the n-th order operator

$$\hat{A}_n f = \frac{Wr(\Phi_1, ..., \Phi_n, f)}{Wr(\Phi_1, ..., \Phi_n)}, \tag{4.13}$$

where $f = f(x)$ is a smooth function. We claim that the following higher-order intertwining relation holds,

$$H_n\hat{A}_n = \hat{A}_n H_0. \tag{4.14}$$

First, we claim that

$$\hat{A}_n = (-1)^n A_n \cdots A_1.$$

The proof is by induction. The case $n = 1$ was proved in the previous section. Suppose that the assumption is true for $n-1$, and prove it for n. So we have to show that

$$\hat{A}_n = -A_n \hat{A}_{n-1}.$$

Set

$$f_{1 \dots n} = \hat{A}_n f.$$

In other words, we have to prove that

$$f_{1 \dots n} = -A_n f_{1 \dots n-1}.$$

This is the same proof as above, but with Φ replaced by f. So, now we have to show that

$$H_n A_n \cdots A_1 = A_n \cdots A_1 H_0.$$

Again, the proof is by induction. The $n = 1$ case is just (4.10). Suppose that the result holds for $n-1$ and prove it for n. So assume that

$$H_{n-1} A_{n-1} \cdots A_1 = A_{n-1} \cdots A_1 H_0.$$

Again, by (4.10),

$$H_n A_n = A_n H_{n-1}.$$

Hence,

$$H_n A_n A_{n-1} \cdots A_1 = A_n H_{n-1} A_{n-1} \cdots A_1 = A_n A_{n-1} \cdots A_1 H_0$$

as was to be shown. □

Chapter 5

Classical and Exceptional Hermite Polynomials

We begin this chapter by reviewing the classical Hermite polynomials, which are the solutions to the Hermite differential equation. We review some important properties of the classical Hermite polynomials that we would like to remain once we generalize them to the exceptional Hermite polynomials. Classical Hermite polynomials are examples of hypergeometric functions, a generalized class of functions we briefly introduce. The classical Hermite polynomials can be generalized using SUSY methods. These are used in order to derive the exceptional Hermite polynomials. We spend the remainder of this section studying the exceptional Hermite polynomials and showing that they satisfy many of the same important properties that the classical Hermite polynomials do. Later, we will use the properties in order to derive families of orthogonal solutions that will become new families of derivative models.

5.1 Classical Hermite Polynomials

The classical orthogonal polynomials give the exact solutions of a variety of quantum mechanical potentials. One of them, Hermite polynomials, named after the French mathematician Charles Hermite, were invented in the 19th century but found use in quantum mechanics. The harmonic oscillator is a solvable potential precisely because the eigenfunctions are given in terms of Hermite polynomials. We outline the key properties of Hermite polynomials here.

5.2 Key Properties

Recurrence Relations The Hermite polynomials are orthogonal on the interval $(-\infty, \infty)$ with a weight function $w(x) = e^{-x^2}$. They may be defined by a 3-term recurrence relation.

Definition 5.2.1. For $n \geq 1,\ n \in \mathbb{Z}$, Hermite polynomials $h_n(x)$ are polynomials of degree n that satisfy the recurrence relation

$$h_{n+1}(x) = 2xh_n(x) - 2nh_{n-1}(x), \tag{5.1}$$

with $h_0(x) = 1$ and $h_1(x) = 2x$.

Applying an induction argument gives the another recurrence relation, which relates the derivative of $h_n(x)$ with $h_{n-1}(x)$,

$$h_n'(x) = 2nh_{n-1}(x). \tag{5.2}$$

Relation (5.2) is true by inspection for $n = 1, 2$. Suppose that this relation is true for some $n \geq 3$. Taking derivatives of (5.1) gives

$$\begin{aligned} h_{n+1}'(x) &= 2h_n(x) + 2n\Big(2xh_{n-1}(x) - 2(n-1)h_{n-2}(x)\Big) \\ &= 2h_n(x) + 2nh_n \\ &= 2(n+1)h_n(x), \end{aligned}$$

which proves that the relation also holds for $n + 1$.

Hermite Differential Equation The function $h_n(x)$ satisfies the Hermite differential equation which has the form

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \quad n \in \{0, 1, 2, ...\} \tag{5.3}$$

To derive this equation, we make use of (5.2). We have

$$h_n''(x) = 4n(n-1)h_{n-2}(x)$$

Hence,

$$h_n''(x) - 2xh_n'(x) + 2nh_n(x) = 4n(n-1)h_{n-2}(x) - 4xnh_{n-1}(x) + 2nh_n(x).$$

The right-side vanishes by (5.1), which proves (5.3)

Orthogonality Relation The Hermite polynomials satisfy the orthogonality relation

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} h_m(x)h_n(x)e^{-x^2}\, dx = 2^n\, n!\, \delta_{mn}, \quad m, n \in \{0, 1, 2, ...\} \tag{5.4}$$

This implies that the family of Hermite polynomials is orthogonal with respect to the weight e^{-x^2}.

Completeness To write a function in terms of Hermite polynomials, we state the completeness property of Hermite polynomials. The Hermite polynomials form an orthogonal basis of the Hilbert space of functions satisfying

$$\int_{-\infty}^{\infty} |f(x)|^2 \, w(x) \, dx < \infty, \tag{5.5}$$

in which the inner product is given by the integral including the Gaussian weight function $w(x)$ defined as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \, \overline{g(x)} \, w(x) \, dx. \tag{5.6}$$

Thus Hermite polynomials form a complete basis for $L^2(\mathbb{R},\, w(x)dx)$, the L^2 space on $\mathbb{R}$ with weight $e^{-x^2}dx$. A proof of the completeness property can be found in[22].

Rodrigues' Formula The Rodrigues' formula for the Hermite polynomials is given by

$$h_n(x) = (-1)^n \, e^{x^2} \, \frac{d^n}{dx^n}(e^{-x^2}). \tag{5.7}$$

We will prove (5.7) by establishing a relation between the Hermite polynomials and a quantum Hamiltonian.

5.3 Hypergeometric Functions

Hypergeometric functions are solutions of the hypergeometric differential equation. Hypergeometric functions have been studied by Euler, Gauss, Riemann, and Kummer. They have a wide range of applications, including differential equations, number theory, combinatorics, and representation theory. We will briefly introduce hypergeometric functions and show that they can be used to express classes of orthogonal polynomials, namely the Laguerre and Hermite polynomials.

Hypergeometric Equation The hypergeometric equation is a differential equation of the form

$$x(1-x)y'' + [c - (a+b+1)x] \, y' - ab \, y = 0, \tag{5.8}$$

where $a, b, c \in \mathbb{R}$ are constants. If $c \geq 0$, then the hypergeometric equation has the solution

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!}. \tag{5.9}$$

We call $F(a, b, c; x)$ the *hypergeometric function*. The general solution to the hypergeometric equation is

$$y(x) = AF(a, n, c; x) + Bx^{1-c}F(a - c + 1, b - c + 1, 2 - c; x) \tag{5.10}$$

where A and B are arbitrary constants.

Confluent Hypergeometric Equation The confluent hypergeometric equation has the form

$$xy'' + (c - x)y' - ay = 0 \tag{5.11}$$

which is also known as Kummer's equation. It has a regular singularity at $x = 0$ and an essential singularity at $x = \infty$.

The two linearly independent solutions are

$$y_1(x) = M(a, c; x) = \sum_{n=0}^{\infty} \frac{a^{(n)}x^n}{c^{(n)}n!} \tag{5.12}$$

$$y_2(x) = x^{1-c}M(a - c + 1, 2 - c; x) \tag{5.13}$$

where

$$a^{(n)} = \prod_{k=0}^{n-1}(a + k)$$

is the rising factorial and $M(a, c; x)$ is called the *confluent hypergeometric function*. Then the second solution to (5.11) can be given by

$$U(a, c; x) = \frac{\Gamma(1 - c)}{\Gamma(a - c + 1)}M(a, c; x) + \frac{\Gamma(c - 1)}{\Gamma(a)}x^{1-c}M(a - c + 1, 2 - c; x) \tag{5.14}$$

Laguerre and Hermite Polynomials Observe that (1) is a special case of the confluent hypergeometric equation.

$$L_n^{(\alpha)}(x) = \binom{n + \alpha}{n} M(-n, \alpha + 1; x) \tag{5.15}$$

or equivalently

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} U(-n, \alpha + 1; x) \tag{5.16}$$

Classical Hermite polynomials can be given in terms of Lagerre polynomials

$$h_{2n}(x) = (-4)^n \, n! \, L_n^{(-\frac{1}{2})}(x^2) = 4^n \, n! \sum_{i=0}^{n}(-1)^{n-i}\binom{n - \frac{1}{2}}{n - i}\frac{x^{2i}}{i!} \tag{5.17}$$

$$h_{2n+1}(x) = 2(-4)^n \, n! \, x \, L_n^{(\frac{1}{2})}(x^2) = 2 \cdot 4^n \, n! \sum_{i=0}^{n}(-1)^{n-i}\binom{n + \frac{1}{2}}{n - i}\frac{x^{2i+1}}{i!} \tag{5.18}$$

and thus can also be regarded as solutions of the confluent hypergeometric equation.

$$h_n(x) = 2^n\, U\left(-\frac{n}{2}, \frac{1}{2}; x^2\right). \tag{5.19}$$

5.4 The Quantum Harmonic Oscillator

In quantum mechanics, the Hamiltonian for the harmonic oscillator is the Schrodinger operator

$$\mathcal{H} = -\partial_x^2 + x^2. \tag{5.20}$$

For physical reasons, we want the eigenfunctions to vanish at $\pm\infty$. To obtain the eigenfunctions, we perform a gauge transformation and a shift

$$\mathcal{T} = -e^{x^2/2}\mathcal{H}e^{-x^2/2} + 1.$$

Using the transformation law (3.8) this gives

$$\mathcal{T} = \partial_x^2 - 2x\partial_x,$$

which is the Hermite differential operator. From the Hermite differential equation

$$\mathcal{T}h_n = -2nh_n,$$

we therefore obtain the eigenfunctions of the Harmonic oscillator as

$$\psi_n = e^{-x^2/2}h_n. \tag{5.21}$$

Because of the change of sign and the +1 shift between $\mathcal{T}$ and $\mathcal{H}$, the eigenvalues of the harmonic oscillator operator are

$$\mathcal{H}\psi_n = (2n+1)\psi_n, \quad n = 0, 1, 2, \ldots$$

5.5 Ladder Operators

In this section, we use SUSY methods to introduce the translationally shape invariant potentials with showing the ladder operators A and $A^\dagger$, which respectively lowers and raises the quantum number of state of the system. Then we derive the Rodrigues' formula from ladder operators.

Apply the Darboux transformation to the harmonic oscillator operator in (5.20). As the factorizing eigenfunction we take $\psi_0 = e^{-x^2/2}$ and $\lambda = 1$ as the factorizing eigenvalue.

Hence,

$$\begin{aligned} W &= \frac{\psi_0'}{\psi_0} = -x, \\ A &= -\partial_x - x \\ A^\dagger &= \partial_x - x, \end{aligned}$$

which gives the factorization

$$\mathcal{H} = A^\dagger A + \lambda_0 = (\partial_x - x)(-\partial_x - x) + 1$$

and the Darboux transformation

$$\begin{aligned} \hat{\mathcal{H}} &= AA^\dagger + \lambda_0 \\ &= -\partial_x^2 + x^2 - 2\partial_x W \\ &= -\partial_x^2 + x^2 + 2 \\ &= \mathcal{H} + 2 \end{aligned}$$

When a Darboux transformation gives back a the same potential up to a constant shift, we call the potential translationally shape invariant.

Translationally shape invariant potentials have closed-form expressions for the superpotentials and the eigenfunctions. Grandati and Berard propose an alternate way to determine the eigenfunctions of a translationally shape invariant potential in [18]. In [19], they generate an infinite set of solvable rational extensions for some translationally shape invariant potentials. Translationally shape invariant potentials is also used to extend Krein-Adler theorem and thereby to establish novel bilinear Wronskian for classical orthogonal polynomials[20].

Observe that

$$\begin{aligned} A &= -\partial_x - x \\ &= -e^{-x^2/2} \circ \partial_x \circ e^{x^2/2} \end{aligned} \tag{5.22}$$

$$\begin{aligned} A^\dagger &= \partial_x - x \\ &= -e^{-x^2/2} \circ (\partial_x - 2x) \circ e^{x^2/2} \end{aligned} \tag{5.23}$$

Hence, using identities (5.21) and (5.2) we obtain

$$\begin{aligned} -A\psi_n &= e^{-x^2/2}\partial_x h_n \\ &= 2ne^{-x^2/2}h_{n-1} \\ &= 2n\psi_{n-1} \end{aligned}$$

Similarly,

$$\begin{aligned} -A^\dagger\psi_n &= -e^{-x^2/2}(\partial_x - x)h_n \\ &= = e^{-x^2/2}h_{n+1} \\ &= \psi_{n+1} \end{aligned}$$

We therefore refer to A as the lowering operator and to $A^\dagger$ as the raising operator, and call both of these ladder operators. In particular, applying the raising operator n times to the ground eigenstate ψ_0 gives the nth eigenstate:

$$(-A^\dagger)^n\psi_0 = \psi_n$$

Also, observe that

$$-A^\dagger = \partial_x - x = e^{x^2/2} \circ \partial_x \circ e^{-x^2/2}$$

Hence,

$$(-A^\dagger)^n = e^{x^2/2} \circ \partial_x^n \circ e^{-x^2/2}.$$

Hence, by (5.21),

$$h_n = e^{x^2/2}\psi_n = (-1)^n e^{x^2}\partial_x^n e^{-x^2}.$$

which gives a proof of the classical Rodrigues formula using ladder operators.

5.6 Exceptional Hermite polynomials

Let $K = \{k_1, \dots, k_l\}$ be a finite subset of positive integers arranged in ascending order $k_1 < k_2 < \cdots < k_l$. Let $h_n(x)$ be the classical Hermite polynomial of degree n. Define the polynomials

$$h_K := Wr[h_{k_1}, \dots, h_{k_l}] \tag{5.24}$$

$$h_{K,n} := Wr[h_{k_1}, \dots, h_{k_l}, h_n], \quad n \notin K, \tag{5.25}$$

where Wr is the Wronskian determinant. We assume that $n \notin K$ because otherwise, by properties of the Wronskian, the polynomial $h_{K,n}$ would be zero.

Proposition 5.6.1. The degree of h_K and $h_{K,n}$ is equal to

$$\deg h_K = \sum_{j=1}^{l} k_j - \frac{1}{2}(l-1)l \tag{5.26}$$

$$\deg h_{K,n} = \sum_{j=1}^{l} k_j + n - \frac{1}{2}l(l+1) = deg h_K + n - l. \tag{5.27}$$

Proof. The leading degree term of h_K is a constant times

$$Wr[x^{k_1}, x^{k_2}, \ldots, x^{k_l}] = \begin{vmatrix} x^{k_1} & x^{k_2} & \ldots & x^{k_l} \\ k_1 x^{k_1-1} & k_2 x^{k_2-1} & \ldots & k_l x^{k_l-1} \\ k_1(k_1-1)x^{k_1-2} & k_2(k_2-1)x^{k_2-2} & \ldots & k_l(k_l-1)x^{k_l-2} \\ \vdots & \vdots & \ddots & \vdots \\ (k_1)_{l-1}x^{k_1-l+1} & (k_2)_{l-1}x^{k_2-l+1} & \ldots & (k_l)_{l-1}x^{k_l-l+1} \end{vmatrix}.$$

where

$$(k)_n = k(k-1)(k-2)\cdots(k-n+1)$$

is the Pocchammer symbol. Each of the terms in the determinant expansion is a constant times x raised to the power

$$p = \sum_{j=1}^{l}(k_j - j) = \left(\sum_j k_j\right) - \frac{1}{2}l(l+1).$$

Therefore the Wronskian above is equal to Cx^p where

$$C = V(k_1, \ldots, k_l)$$

is a polynomial in $k_1, \ldots, k_l$. By inspection, this polynomial has total degree

$$0 + 1 + \cdots + l - 1 = \frac{1}{2}(l-1)l.$$

Also if $k_i = k_j$ for two indices $i \neq j$, then $C = 0$. This implies that the constant C must be the Vandermonde determinant

$$V(k_1, \ldots, k_l) = \prod_{1 \le i < j \le l}(k_i - k_j),$$

and that it isn't zero because we have assumed that all the k_i are distinct. This proves equation (5.24). Equation (5.25) is proved in a similar way. □

The following Theorem was originally proved in [8].

Theorem 5.6.1. Fix a set of natural numbers $K \subset \mathbb{N}$. The polynomial $y(x) = h_{K,n}(x)$, $n \notin I_K$ is a solutions of the differential equation

$$y''(x) - 2\left(x + \frac{h'_K(x)}{h_K(x)}\right) y'(x) + \left(\frac{h''_K(x)}{h_K(x)} + 2x\frac{h'_K(x)}{h_K(x)}\right) y(x) = 2(l-n)y(x) \quad (5.28)$$

Proof. We introduce the Schrodinger operator

$$\mathcal{H}_\lambda := -\partial_x^2 + U_\lambda(x), \quad (5.29)$$

where

$$\begin{aligned} U_K(x) &:= x^2 - 2\partial_x^2 \log h_K + 2l \\ &= x^2 + 2\left(\frac{h'_K}{h_K}\right)^2 - \frac{2h''_K}{h_K} + 2l \end{aligned}$$

is a rational extension of the harmonic oscillator.

Set

$$\sigma_K(x) = \frac{x^2}{2} + \log h_K(x)$$

and define the operator

$$\mathcal{T}_K := -e^{\sigma_K(x)} \circ \mathcal{H}_K \circ e^{-\sigma_K(x)} + 2l + 1. \quad (5.30)$$

A special case of the gauge transformation law (3.8) is

$$e^\sigma \circ \partial_x^2 \circ e^{-\sigma} = \partial_x^2 - 2\sigma'(x)\partial_x - \sigma''(x) + \sigma'(x)^2$$

Observe that

$$\begin{aligned} \sigma'_K &= x + \frac{h'_K}{h_K}, \\ \sigma''_K &= 1 + \frac{h''_K}{h_K} - \left(\frac{h'_K}{h_K}\right)^2 \\ -\sigma''_K(x) + \sigma'_K(x)^2 &= -1 - \frac{h''_K}{h_K} + 2\left(\frac{h'_K}{h_K}\right)^2 + x^2 + 2x\frac{h'_K}{h_K} \\ &= U_K(x) + \frac{h''_K}{h_K} + 2x\frac{h'_K}{h_K} - 2l - 1 \end{aligned}$$

Hence,

$$\mathcal{T}_K = \partial_x^2 - 2\left(x + \frac{h''_K}{h_K}\right)\partial_x - \sigma''_K(x) + \sigma'_K(x)^2 - U_K(x) + 2l + 1$$

is the exceptional Hermite operator in the left side of equation (5.28).

Let $\psi_n(x)$ be the eigenfunction of the classical Harmonic oscillator as defined in (5.21). Applying the Darboux-Crum transformation (4.12) with $\psi_{k_1}, \ldots, \psi_{k_l}$ as factorization functions gives the potential

$$\hat{U}_K(x) := x^2 - 2\partial_x^2 \log Wr[\psi_{k_1}, ..., \psi_{k_l}]$$

By properties of the Wronskian,

$$Wr[\psi_{k_1}, ..., \psi_{k_l}] = e^{-lx^2/2} Wr[h_{k_1}, \ldots, h_{k_l}]$$

Hence

$$\hat{U}_K = x^2 - 2\partial_x^2 \log h_K + 2l \tag{5.31}$$

is equal to U_K, the potential of the operator $\mathcal{H}_K$ introduced above.

By the Darboux-Crum Theorem 4.3.1, the function

$$\psi_{K,j} := \frac{Wr[\psi_{k_1}, \ldots, \psi_{k_l}, \psi_j]}{Wr[\psi_{k_1}, \ldots, \psi_{k_l}]}$$

is an eigenfunction of $\mathcal{H}_K$, so that

$$\mathcal{H}_K \psi_{K,j} = (2j+1)\psi_{K,j}.$$

Again, by the properties of Wronskian,

$$\begin{aligned} \psi_{K,j} &= \frac{e^{-(l+1)x^2/2} Wr[h_{k_1}, \ldots, h_{k_l}, h_n]}{e^{-lx^2/2} Wr[h_{k_1}, \ldots, h_{k_l}]} \\ &= \frac{e^{-x^2/2}}{h_K(x)} h_{K,j}(x). \end{aligned} \tag{5.32}$$

Therefore, by (5.30),

$$\mathcal{T}_K, h_{K,j} = \Big(-(2j+1) + 2l + 1 \Big) h_{K,j},$$

which proves the theorem. □

5.7 Orthogonality

Define the weight function

$$W_K(x) := \frac{e^{-x^2}}{h_K(x)^2}, \tag{5.33}$$

and observe that the weight is non-singular as long as $h_K(x) \neq 0$ for real x.

Definition 5.7.1. We say that $K = \{k_1, \ldots, k_l\}$ where $0 < k_1 < \cdots < k_l$ is a Krein-Adler sequence provided $l = 2j$ is even and $k_{2i-1} = k_{2i} + 1$ for all $i = 1, \ldots, j$. Here are some examples of Krein-Adler sequences:

$$\{1, 2, 4, 5\}, \quad \{2, 3, 4, 5, 9, 10\}, \quad \{2, 3, 6, 7, 9, 10\}.$$

The following Theorem was proved in [8] using earlier results by Krein and Adler.

Theorem 5.7.1. The Wronskian

$$h_K = Wr[h_{k_1}, \ldots, h_{k_l}]$$

has no real zeros if and only if K is a Krein-Adler sequence.

Hence, if K is a Krein-Adler sequence we have a well-defined inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, W_K(x)\, dx. \tag{5.34}$$

Note that in general, $\deg h_{K,n} \neq n$. Therefore it's convenient to define the exceptional Hermite polynomials as follows.

Definition 5.7.2. Suppose that $K = \{k_1, \ldots, k_l\} \subset \mathbb{N}$ is a Krein-Adler sequence and let

$$N_K = \deg h_K = \sum_{j=1}^{l} k_j - \frac{1}{2} l(l-1).$$

The family of polynomials $\{h_{K,j} : j \notin K\}$ is missing degrees

$$J_K = \{0, 1, \ldots, N_K - l - 1\} \cup \{k + N_K - l : k \in K\}$$

These are the exceptional degrees. We define the index set for the remaining degrees as the complement

$$I_K = \mathbb{N}_0 \setminus J_K$$

and define the exceptional Hermite polynomials as

$$\hat{h}_{K,n}(x) = \frac{h_{K,j}(x)}{(j-k_1)\cdots(j-k_l)} \tag{5.35}$$

where $j = n + l - N_K$. It then follows by (5.26) that $\deg \hat{h}_{K,n} = n$ provided $n \in I_K$.

We now show that the Exceptional Hermite polynomials are orthogonal with respect to W_K.

Proposition 5.7.1. We have

$$\int_{-\infty}^{\infty} \hat{h}_{K,n}\hat{h}_{K,m} W_K \, dx = \delta_{mn} \frac{\sqrt{\pi}\, 2^{j+l}\, j!}{\prod_{i=1}^{l}(j-k_i)}, \tag{5.36}$$

where $j = n + l - N_K$ and where $N_K = \deg h_K$ by (5.26).

Proof. In the proof of Theorem 5.6.1 we showed that the exceptional Hermite operator $\mathcal{T}_K$ is gauge-equivalent to the Schrodinger operator in (5.29) with a gauge factor of $\frac{e^{-x^2/2}}{H_K(x)}$. The eigenfunctions of a Schrodinger operator at different energies are orthogonal with respect to dx. This means that if $i \neq j$ then

$$\int_{-\infty}^{+\infty} \psi_{K,i}(x)\psi_{K,j}(x)dx = 0.$$

Substitute (5.32) into above to get

$$\int_{-\infty}^{+\infty} h_{K,i}(x)h_{K,j}(x)\frac{e^{-x^2}}{h_K(x)^2}dx = 0, \quad i \neq j.$$

The proof for the case of $i = j$ can be found in [21]. □

5.8 Example

Consider the case where $K = \{1, 2\}$ and

$$h_K = Wr[h_1, h_2] = 4(1 + 2x^2).$$

For this case, the exceptional Hermite differential equation is

$$y'' - \left(2x + \frac{8x}{1+2x^2}\right)y' + 2ny = 0,$$

where

$$y = \hat{h}_{K,n} = \frac{Wr[h_1, h_2, h_n]}{(n-2)(n-1)}$$

is the corresponding Exceptional Hermite polynomial of degree n. Note that, by construction $n \notin \{1, 2\}$. These are the excluded "exceptional" degrees. The degree set is therefore

$$I_K = \{0, 3, 4, 5, \ldots\}.$$

Proposition 5.8.1. We have

$$\hat{h}_{K,n} = 8(h_n + 4nh_{n-2} + 4n(n-3)h_{n-4}), \quad n \in I_K \tag{5.37}$$

Proof. Recall that $h_1 = 2x, h_2 = 4x^2 - 2$. Hence

$$\frac{1}{8}Wr[h_1, h_2, h_3] = \frac{1}{2}(4h_n(x) - 4xh_n'(x) + (1 + 2x^2)h_n''(x))$$

Using the identities

$$2xh_n = h_{n+1} + 2nh_{n-1}$$
$$h_n' = 2nh_{n-1}$$

we have

$$\begin{aligned}
\frac{1}{2}(4h_n(x) - 4xh_n'(x) + (1 + 2x^2)h_n''(x)) &= 2h_n(x) - 2xh_n'(x) + \frac{1}{2}h_n''(x) + x^2h_n''(x) \\
&= 2h_n(x) - 4nxh_{n-1}(x) + nh_{n-1}'(x) + 2nx^2h_{n-1}'(x) \\
&= -2(n-1)h_n(x)2(n-1)n(2x^2-1)h_{n-2}(x) \\
&= -2(n-1)h_n(x) + 2(n-1)nxh_{n-1}(x) + -2(n-1)nh_{n-2}(x) \\
&\quad +4(n-2)(n-1)nxh_{n-3}(x) \\
&= (n-2)(n-1)h_n(x) + 4(n-2)(n-1)nh_{n-2}(x) + 4(n-3)(n-2)(n-1)nh_{n-4}(x) \\
&= (n-2)(n-1)[h_n(x) + 4nh_{n-2}(x) + 4(n-3)nh_{n-4}(x)]
\end{aligned}$$

. Dividing by $(n-2)(n-1)$ yields the desired result. □

Note that equation (5.8.1) is another indication that $1, 2$ are exceptional degrees. The formula for $\hat{h}_{K,n}$ is simply not valid for these values of n.

Also observe that $K = \{1, 2\}$ is a Krein-Adler sequence. Consequently, $h_K(x)$ does not vanish for real x and the corresponding weight

$$W_K = \frac{e^{-x^2}}{16(1 + 2x^2)^2}$$

is non-singular.

Next, let's consider the example of $K = \{1, 3\}$. An explicit calculation shows that

$$h_K = Wr[h_1, h_3] = Wr[2x, 8x^3 - 12x] = 32x^3.$$

In this case, K is not a Krein-Adler sequence, and the corresponding weight W_K has a singularity at $x = 0$. Thus the corresponding polynomials $\hat{h}_{K,n}$ are eigenfunctions of a 2nd order differential equation, but they cannot be considered to be orthogonal polynomials.

Chapter 6

The Black Scholes Equation and its Generalizations

In this chapter we introduce introduce the general interpretation of stochastic differential equations as diffusion equations. We then show how use the general solutions of the diffusion equations to construct pricing models of derivative assets. We then apply these results to derive the classical Black-Scholes equation and formula for a call option.

The generalized Black-Scholes partial differential equation plays an important and fundamental role in valuing all derivative securities. We begin by considering the general form of a second order linear PDE with all the variables coefficients being functions of S. This approach can be generalized to time-dependence, but we limit ourselves to time-homogeneous, backward Kolmogorov equations.

$$\frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 V}{\partial S^2} + \mu(S)S\,\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = \nu(S)\,V \tag{6.1}$$

$$t \in (0,T),\ V(T,S) = g(S). \tag{6.2}$$

In the model, the diffusion coefficient σ is the volatility function, and the coefficient μ is the (absolute) risk-neutral drift. The coefficient ν is the cost of carrying the claim once a particular asset is chosen to finance the premium.

We change of variable $S = S(x), x = x(S)$ and set

$$dx = \frac{dS}{S\sigma(S)}$$

or

$$x = \int_{S_0}^{S} \frac{dz}{z\sigma(z)} \tag{6.3}$$

Using 3.8, set $U(x,t) = V(S(x),t)$. Hence the original PDE becomes

$$\frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \beta(x)\,\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = \gamma(x)U$$

where by (3.6)

$$\beta(x) = \frac{\mu(S)}{\sigma(S)} - \frac{1}{2}\sigma(S) - \frac{1}{2}S\sigma'(S),$$
$$\gamma(x) = \nu(S)$$

We can't interpret γ^c as a carrying cost because y is the log of an asset price. Since everything is time independent we should suppress dependence on t for the equation coefficients.

Now consider a gauge transformation

$$U^c(x,t) = \exp\left[\int_0^x \beta(y)\,dy\right] U(x,t) \tag{6.4}$$

By (3.8) the transformed diffusion equation is the canonical Schrodinger operator form:

$$\frac{\partial U^c}{\partial t} = -\frac{1}{2}\frac{\partial^2 U^c}{\partial x^2} + \gamma^c(x)\,U^c$$

where

$$\gamma^c = \gamma + \frac{1}{2}\,\frac{\partial \beta}{\partial x} + \frac{1}{2}\,\beta^2 \tag{6.5}$$

and $U^c(x,t)$ is the dependent variable and $\gamma^c(x)$ is the potential function.

6.1 Differential Equations as Pricing Models

Let S_t be a stochastic process that represents the price of an asset. Then dS_t/S_t is the rate of return. Suppose that the evolution of the asset price is modeled by the stochastic differential equation,

$$\frac{dS_t}{S_t} = \mu(S)\,dt + \sigma(S)\,dW_t, \tag{6.6}$$

where W_t is a normalized Wiener process, σ is the volatility function, and μ is the (absolute) risk-neutral drift. Here for simplicity we assume that these parameters are time-independent and are purely functions of the price of the underlying asset.

Consider a derivative security whose price is a function $f = f(S,t)$ of the underlying asset price and time. The Taylor expansion of

$$f(S+dS, t+dt) = f(S,t) + \Delta f$$

is

$$\Delta f = f(S,t) + \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 + higher\ order\ terms$$

Since dS is a small amount and $dW \sim O(\sqrt{dt})$,

$$\begin{aligned} (dS)^2 &= (\mu S dt + \sigma S dW)^2 \\ &= \mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 dt\, dX + \sigma^2 S^2 (dW)^2 \end{aligned}$$

then the leading term of $(dS)^2$ is $\sigma^2 S^2 (dW)^2$, which approaches $\sigma^2 S^2 dt$ as $dt \to 0$. This result is called Itô's lemma.

Therefore the evolution of the derivative price is governed by the equation

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 \\ &= \sigma S \frac{\partial f}{\partial S}dW + \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial s^2}\right) dt \end{aligned} \tag{6.7}$$

Now we construct a risk-less portfolio $g = f - \Delta S$, which is long one unit of the derivative and short Δ units of underlying asset. We want to balance the portfolio so that during the small time interval dt, the quantity Δ remains constant and dg is totally deterministic. Using (6.6) and (6.7)

$$\begin{aligned} dg &= df - \Delta dS \\ &= \sigma S \left(\frac{\partial f}{\partial S} - \Delta\right) dW + \left(\mu S \left(\frac{\partial f}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right) dt \end{aligned}$$

Therefore, we can eliminate the randomness dW by choosing $\Delta = \frac{\partial f}{\partial S}$. Hence,

$$dg = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right) dt$$

The return of the risk-less portfolio would see a growth during a time dt as

$$\frac{dg}{g} = r dt,$$

where r is the risk-free interest rate. Hence,

$$\begin{aligned} rg dt &= r\left(f - \frac{\partial f}{\partial S}S\right) dt \\ &= \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right) dt \end{aligned}$$

From there, we have

$$rf - r\frac{\partial f}{\partial S}S = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$$

i.e.

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf.$$

which is the Black-Scholes partial differential equation.

6.2 Black-Scholes Hamiltonian

The Black-Scholes equation for option pricing with constant volatility is given by

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC$$

where C, S, σ and r denotes the price of the option, the stock price(random variable), the constant volatility of the stock price and the constant risk-free sport interest rate, respectively.

Using separation of variables and a substitution,

$$C(S,t) = e^{\lambda t}\psi(x)$$

where $S = e^x, \ -\infty < x < \infty$

We have

$$\begin{aligned} \frac{\partial C}{\partial t} &= \lambda e^{\lambda t}\psi \\ S\frac{\partial C}{\partial S} &= e^{\lambda t}\psi_x \\ S^2\frac{\partial^2 C}{\partial S^2} &= e^{\lambda t}\left(\psi_{xx} - \psi_x\right) \end{aligned}$$

Plugging in the Black-Scholes equation yields,

$$-\frac{\sigma^2}{2}\frac{d^2\psi}{dx^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{d\psi}{dx} + r\psi = \lambda\psi$$

For the right-hand side, we define the Black-Scholes Hamiltonian as

$$H_{BS} = -\frac{\sigma^2}{2}\frac{d^2}{dx^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{d}{dx} + r,$$

and we rewrite the above eigenvalue problem as

$$H_{BS}\psi = \lambda\psi.$$

We will derive the solution to the Black-Scholes equation

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0, \tag{6.8}$$

by transforming it into the diffusion equation, which is solvable.

First, let

$$x = \log\left(\frac{S}{K}\right) \Rightarrow S = Ke^x$$

$$\tau = \frac{\sigma^2}{2}(T-t) \Rightarrow t = T - \frac{2\tau}{\sigma^2}$$

$$U(x,\tau) = \frac{1}{K}V(S,t)$$

Taking partial derivatives, we get

$$\frac{\partial V}{\partial t} = -\frac{K\sigma^2}{2}\frac{\partial U}{\partial \tau}$$
$$\frac{\partial V}{\partial S} = e^{-x}\frac{\partial U}{\partial x}$$
$$\frac{\partial^2 V}{\partial S^2} = \frac{e^{-2x}}{K}\left(\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x}\right)$$

Substituting these back into (6.8), we have

$$-\frac{\sigma^2}{2}\frac{\partial u}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial U}{\partial x} - rU = 0. \tag{6.9}$$

Next let $k = \frac{2r}{\sigma^2}$. Then (6.9) becomes

$$-\frac{\partial U}{\partial \tau} + \frac{\partial^2 U}{\partial x^2} + (k-1)\frac{\partial U}{\partial x} - kU = 0. \tag{6.10}$$

Now let

$$W(x,\tau) = e^{\alpha x + \beta^2 \tau}U(x,\tau), \quad \alpha = \frac{1}{2}(k-1), \beta = \frac{1}{2}(k+1).$$

The partial derivatives become

$$\frac{\partial U}{\partial \tau} = e^{-\alpha x - \beta^2 \tau}\left(\frac{\partial W}{\partial \tau} - \beta^2 W\right)$$
$$\frac{\partial U}{\partial x} = e^{-\alpha x - \beta^2 \tau}\left(\frac{\partial W}{\partial x} - \alpha W\right)$$
$$\frac{\partial^2 U}{\partial x^2} = e^{-\alpha x - \beta^2 \tau}\left(\frac{\partial^2 W}{\partial x^2} - 2\alpha\frac{\partial W}{\partial x} + \alpha^2 W\right)$$

Substituting these into (6.9) and simplifying, we arrive at the diffusion equation

$$\beta^2 W - \frac{\partial W}{\partial \tau} + \frac{\partial^2 W}{\partial x^2} - 2\alpha\frac{\partial W}{\partial x} + \alpha^2 W + (k-1)\frac{\partial W}{\partial x}$$
$$-\alpha(k-1)W - kW = 0$$
$$\frac{1}{4}(k+1)^2 W - \frac{\partial W}{\partial \tau} + \frac{\partial^2 W}{\partial x^2} - (k-1)\frac{\partial W}{\partial x} + \frac{1}{4}(k-1)^2 W$$
$$+(k-1)\frac{\partial W}{\partial x} - \frac{1}{2}(k-1)^2 W - kW = 0.$$
$$\frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial \tau} = 0$$
$$\frac{\partial^2 W}{\partial x^2} = \frac{\partial W}{\partial \tau}$$

The payoff option for a European style call option is given by

$$U_0(x_T) = \frac{1}{K}V(S_T - K)^+ = (e^{x_T} - 1)^+,$$

Where the + in the superscript indicates the postive part of the function. Since we transformed the Black-Scholes diffusion equation into the heat equation into the heat equation, we can describe the solution with the given boundary conditions using convolution wih the fundamental solution

$$G(x,\xi) = e^{-\frac{(x-\xi)^2}{4\tau}}.$$

The resulting solution is

$$W(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4\tau}} W_0(\xi)d\xi$$
$$= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4\tau}} \exp\left((\beta x_T - \alpha x_T)^+\right) d\xi$$

Let $z = \frac{\xi - x}{\sqrt{2\tau}}$ so that $\xi = x + z\sqrt{2\tau}, d\xi = \sqrt{2\tau}dz$. Then we have

$$W(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \exp\left(\left(\beta[\sqrt{2\tau}z + x] - \alpha[\sqrt{2\tau}z + x]\right)^+\right) dz.$$

This expression is nonzero if and only if

$$\beta[\sqrt{2\tau}z + x] > \alpha[\sqrt{2\tau}z + x] \Rightarrow z > -\frac{x}{\sqrt{2\tau}}.$$

Hence we have

$$
\begin{aligned}
W(x,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}}\exp(\beta[\sqrt{2\tau}z+x])dz\\
&- \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}}\exp(\alpha[\sqrt{2\tau}z+x])dz\\
&= I_1 - I_2
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \exp\left(-\frac{1}{2}z^2+\beta\sqrt{2\tau}z+\beta x\right)dz\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \exp\left(-\frac{1}{2}(z-\beta\sqrt{2\tau})^2+\beta x+\beta^2\tau\right)dz
\end{aligned}
$$

Let $y = z-\beta\sqrt{2\tau}, dy = dz$, so that

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}}e^{\beta x+\beta^2\tau}\int_{-\frac{x}{\sqrt{2\tau}}-\beta\sqrt{2\tau}}^{\infty} e^{-\frac{y^2}{2}}dy\\
&= e^{\beta x+\beta^2\tau}\left(1-\Phi\left(-\frac{x}{\sqrt{2\tau}}-\beta\sqrt{2\tau}\right)\right)\\
&= e^{\beta x+\beta^2\tau}\Phi\left(\frac{x}{\sqrt{2\tau}}+\beta\sqrt{2\tau}\right),
\end{aligned}
$$

where $\Phi(x)$ is the cumulative normal distribution function. I_2 can be solved similarly to get

$$
I_2 = e^{\alpha x+\alpha^2\tau}\Phi\left(\frac{x}{\sqrt{2\tau}}+\alpha\sqrt{2\tau}\right).
$$

Set

$$
\begin{aligned}
d_1 &= \frac{x}{\sqrt{2\tau}}+\beta\sqrt{2\tau} = \frac{x+\beta(2\tau)}{\sqrt{2\tau}} = \frac{\log\left(\frac{S}{K}\right)+\frac{1}{2}(k+1)(T-t)}{\sigma\sqrt{T-t}}\\
&= \frac{\log\left(\frac{S}{K}\right)+\frac{1}{2}(\frac{\sigma^2}{2}+r)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

$$
\begin{aligned}
d_2 &= \frac{x}{\sqrt{2\tau}}+\alpha\sqrt{2\tau} = \frac{x+\alpha(2\tau)}{\sqrt{2\tau}} = \frac{\log\left(\frac{S}{K}\right)+\frac{1}{2}(k-1)(T-t)}{\sigma\sqrt{T-t}}\\
&= d_1-\sigma\sqrt{T-t}
\end{aligned}
$$

Substituting the values for I_1 and I_2 in for $W(x,\tau)$, we have

$$W(x,\tau) = e^{\beta x+\beta^2\tau}\Phi(d_1) - e^{\alpha x+\alpha^2\tau}\Phi(d_2).$$

Back-substitution yields,

$$\begin{aligned} V(S,t) &= Ke^{-\alpha x-\beta^2\tau}\left(e^{\beta x+\beta^2\tau}\Phi(d_1) - e^{\alpha x+\alpha^2\tau}\Phi(d_2)\right) \\ &= Ke^{(\beta-\alpha)x}\Phi(d_1) - Ke^{(\alpha^2-\beta^2)\tau}\Phi(d_2) \end{aligned}$$

Finally, substituting in for α and β gives us the Black-Scholes formula

$$V(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2). \tag{6.11}$$

Chapter 7

Application of Supersymmetric Methods to Derivative Pricing

In this chapter, we give some examples of application of supersymmetric methods and exceptional Hermite polynomials to construct solvable derivative models. We will discuss knockout models, models with a variable drift and models with a variable carrying cost.

First of all, it is not possible to reduce a diffusion equation with a general potential to the heat equation. In the absence of a fundamental solution, we replace the convolution described above with a representation in terms of generalized Fourier coefficients.

Consider a generalized diffusion equation

$$\frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \beta(x)\,\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = \gamma(x)U$$

If β vanishes, we obtain a generalized heat equation driven by a Schrödinger operator with potential $\gamma(x)$. In finance models it is useful to retain the general form because the first-order part of the operator can be used to model a stochastic process with drift. However, to obtain a generalized Fourier expansion we first perform a gauge transformation get the operator into Schrödinger form.

$$\frac{1}{2}\frac{\partial^2 U^c}{\partial x^2} + \frac{\partial U^c}{\partial t} = \gamma^c(x)U^c$$

where

$$U^c(x,t) = \exp\left[-\int_a^x \beta(y)\,dy\right] U(x,t) \tag{7.1}$$

$$\gamma^c = \gamma + \frac{1}{2}\,\frac{\partial \beta}{\partial x} + \frac{1}{2}\,\beta^2 \tag{7.2}$$

We therefore assume that the potential $\gamma^c(x)$ is positive and exactly solvable, meaning that we there are explicit eigenvalues $\lambda_n > 0,\ n = 0, 1, 2, \ldots$, eigenfunctions

$\phi_n(x),\ n = 0, 1, 2, \ldots$ and normalization constants $\nu_n > 0$. Explicitly, this means that

$$-\frac{1}{2}\frac{\partial^2 \phi_n}{\partial x^2} + \gamma^c(x)\phi_n = \lambda_n \phi_n, \quad n = 0, 1, 2, \ldots$$

and that

$$\int_a^b \phi_n(x)^2 dx = \nu_n$$

where the (a, b) is the natural domain dictated by the form of the potential $\gamma^c(x)$. In other words, we apply boundary conditions to $U^c(x)$ at $x = a$ and $x = b$.

Given an initial profile $U_0(x),\ a < x < b$, we let

$$U_0^c = \exp\left[-\int_a^x \beta(y)\, dy\right] U_0(x)$$

and construct the generalized Fourier expansion

$$U_0^c(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

where

$$a_n = \frac{1}{\nu_n}\int_a^b U_0^c(x)\phi_n(x) dx$$

The general solution of the diffusion equation can now be given using separation of variables as

$$U(x, t) = \exp\left[\int_a^x \beta(y)\, dy\right] \sum_{n=0}^{\infty} a_n e^{-\lambda_n t}\phi_n(x).$$

By truncating the above sum to a finite number of terms we can then obtain numerical solutions of a given diffusion model.

7.1 Supersymmetric double knockout model

A double barrier option is a path dependent option which restricts the value of the stock to be within two barriers, which we denote by $S = e^a$ and $S = e^b$. In other words, a Brownian motion of the underlying asset price e^x is immediately worthless as the asset price hits the barrier from below or above. We modify the Black-Scholes model by imposing an infinite carrying cost to the left and to the right of the barrier prices.

To get the knockout barrier model take

$$U(x) = \begin{cases} 0 & a < x < b \\ \infty & \text{otherwise} \end{cases}$$

The physical picture of this is the problem of a particle in an infinitely deep quantum well. Then the Schrödinger equation we obtained becomes

$$-\phi''(x) = \lambda\phi(x) \tag{7.3}$$

with boundary conditions $\phi(a) = \phi(b) = 0$. The general solution of this second-order equation is

$$\phi(x) = A \sin[\sqrt{\lambda}(x - C)]$$

Using the boundary conditions, $C = a$, and

$$(b - a)\sqrt{\lambda} = (n + 1)\pi, \quad n = 0, 1, 2, \ldots$$

Thus, the eigenvalues are

$$\lambda_n = \left[\frac{(n + 1)\pi}{b - a}\right]^2, \quad n = 0, 1, 2, \ldots$$

Normalize the norm of the eigenfunction to have

$$\int_a^b \phi_n^2 \; dx = 1$$

This gives

$$1 = \int_a^b A^2 \sin^2\left[\frac{\pi(x - a)}{b - a}\right] \; dx = \frac{b - a}{2} A^2$$

Therefore,

$$A = \sqrt{\frac{2}{b - a}}$$

$$\phi_n(x) = \sqrt{\frac{2}{b - a}} \sin\left[\sqrt{\lambda_n}(x - a)\right]$$

For the double barrier Hamiltonian $H_{BS} + V$ we have

$$\epsilon_n = \frac{\sigma^2}{2}\lambda_n + \frac{1}{2\sigma^2}\left(r + \frac{\sigma^2}{2}\right)^2$$

and

$$\psi_n = \sqrt{\frac{2}{b - a}} \exp\left[\left(\frac{1}{2} - \frac{r}{\sigma^2}\right) x\right] \sin\left[\sqrt{\lambda_n}(x - a)\right], \; n = 0, 1, 2, \ldots$$

We will now use the supersymmetric method discussed in Section 4 to find another solution of (7.3). Define the operator $A = -\frac{d}{dx} + tanh(x)$ and its adjoint $A^\dagger = \frac{d}{dx} + tanh(x)$. Then we have

$$A^\dagger A\phi = -\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + sech^2(x)\phi + tanh^2(x)\phi = -\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \phi$$
$$= (1+\lambda)\phi$$

The isospectral partner to this is

$$AA^\dagger\phi = (1+\lambda)\phi$$

Setting $\phi = \rho^{-1}\psi$, where $\rho = e^{\left(\frac{1}{2}-\frac{r}{\sigma^2}\right)}$, we can rewrite these equations as

$$BB^\dagger\psi = (1+\lambda)\psi$$
$$B^\dagger B\psi = (1+\lambda)\psi,$$

where

$$B = -\frac{\mathrm{d}}{\mathrm{d}x} + tanh(x) + \left(\frac{1}{2} + \frac{r}{\sigma^2}\right)$$
$$B^\dagger = \frac{\mathrm{d}}{\mathrm{d}x} + tanh(x) - \left(\frac{1}{2} - \frac{r}{\sigma^2}\right)$$

Expanding these expressions yields

$$BB^\dagger\psi = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + tanh(x) + \left(\frac{1}{2} + \frac{r}{\sigma^2}\right)\right)\left(\frac{\mathrm{d}\psi}{\mathrm{d}x} + tanh(x)\psi - \left(\frac{1}{2} - \frac{r}{\sigma^2}\right)\psi\right)$$
$$= -\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \psi - 2sech^2(x)\psi + 2\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)\frac{\mathrm{d}\psi}{\mathrm{d}x} - \left(\frac{1}{2} - \frac{r}{\sigma^2}\right)^2\psi$$
$$= (1+\lambda)\psi$$

$$B^\dagger B\psi = \left(\frac{\mathrm{d}}{\mathrm{d}x} + tanh(x) - \left(\frac{1}{2} - \frac{r}{\sigma^2}\right)\right)\left(-\frac{\mathrm{d}\psi}{\mathrm{d}x} + tanh(x)\psi + \left(\frac{1}{2} + \frac{r}{\sigma^2}\right)\psi\right)$$
$$= -\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \psi + 2\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)\frac{\mathrm{d}\psi}{\mathrm{d}x} - \left(\frac{1}{2} - \frac{r}{\sigma^2}\right)^2\psi$$
$$= (1+\lambda)\psi$$

Simplifying the above expressions, we have for $BB^\dagger$ that

$$H_{BB^\dagger} = -\frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\mathrm{d}}{\mathrm{d}x} - \sigma^2 sech^2(x) + r, \tag{7.4}$$

and for $B^\dagger B$

$$H_{B^\dagger B} = -\frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\mathrm{d}}{\mathrm{d}x} + r.$$

With the new Hamiltonian $H_{BB^\dagger}$ we can solve $H_{BB^\dagger}\psi = \lambda\psi$ to get

$$\psi(x) = \frac{1}{\sqrt{2}} e^{\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)x} sech(x). \tag{7.5}$$

We will find the potential for $H_{BB^\dagger}$ and it's partner potential. To find the potential of $H_{BB^\dagger}$, we first need to convert it into Schrödinger form. We can use the change of variables outlined in chapter 3.

$$z = \int^{x=\zeta(z)} \frac{dx}{\sqrt{p(x)}} = \int^{x=\zeta(z)} \frac{dx}{\sqrt{-\sigma^2}} = -\frac{i}{\sigma}\zeta(z).$$

$$x = \zeta(z) = i\sigma z.$$

Then we have

$$V(x) = \left(\frac{\sigma}{4} - \frac{r}{2\sigma}\right)^2 + r - \sigma^2 sech^2(x).$$

Then the new Hamiltonian becomes

$$H_{BB^\dagger} = \partial_{zz} - V(\zeta(z))$$

$$H_{BB^\dagger} = -\partial_{zz} + \sigma^2 sec^2(\sigma z) - r - \left(\frac{\sigma}{4} - \frac{r}{2\sigma}\right)^2$$

Hence the potential function is

$$U(z) = \sigma^2 sec^2(\sigma z) - r - \left(\frac{\sigma}{4} - \frac{r}{2\sigma}\right)^2. \tag{7.6}$$

Applying the transformation $x = \zeta(z)$ to the eigenfunction of the Hamiltonian, (7.5) gives

$$\psi(z) = \frac{1}{\sqrt{2}} e^{i\left(\frac{\sigma}{2} - \frac{r}{\sigma}\right)z} sec(\sigma z).$$

We can now apply the Darboux transformation as outlined in chapter 4 to find the partner potential. From (4.9) we have

$$\hat{U} = U - 2\partial_{xx}\log(\psi(z))$$

$$\hat{U} = -\frac{1}{16\sigma^2}(\sigma^4 + 12r\sigma^2 + 4r^2 - 16\sigma^4 sec(\sigma z)^2). \tag{7.7}$$

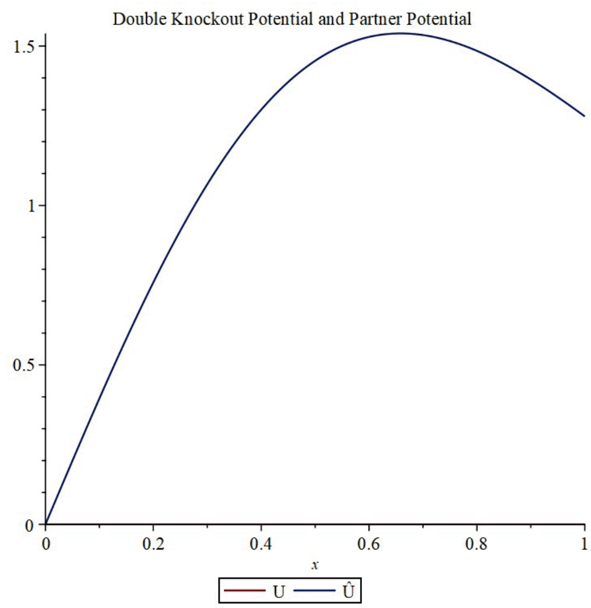

Figure 7.1: Potential $U = 0, 0 < x < 1$ for the double knockout model and the partner potential $\hat{U} = 4tanh(x)sech^2(x)$ obtained from SUSY.

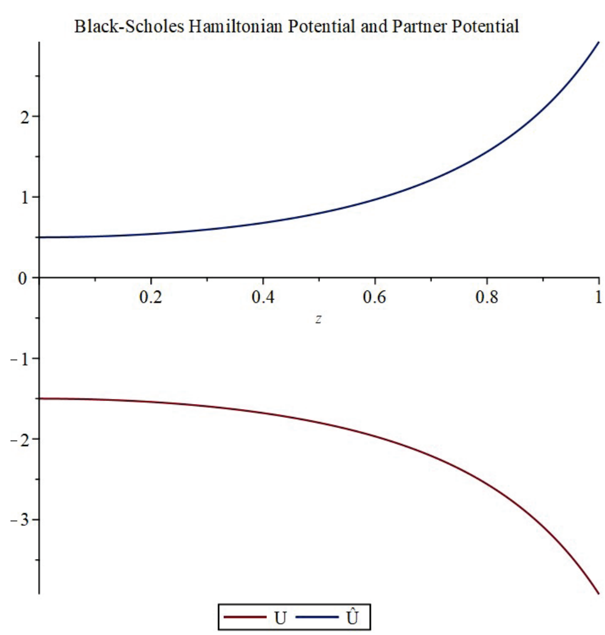

Figure 7.2: Potentials U and $\hat{U}$ defined by (7.6) and (7.7) for $0 < z < 1$, $\sigma = 1, r = \frac{1}{2}$

7.2 Solvable model with a Variable Drift

We consider a genralized Black-Scholes equation with constant diffusion $\sigma = \sigma_0$ and carrying cost $r = \gamma(S)$ that depends on the cost of the underlying asset. This modified Black-Scholes equation becomes

$$\frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \gamma(S)S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = \gamma(S)V.$$

Our goal is to transform the above equation into a diffusion equation with a variable drift function. Using the change of variables outlined in chapter 3:

$$S = e^x, \quad U(x) = S\,V(S), \quad \beta(x) = \gamma(S),$$

yields the diffusion equation

$$\frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \left(\beta(x) + \frac{1}{2}\right)\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = 0. \tag{7.8}$$

We now suppose that

$$\gamma(S) = \sigma_0\left(-\log S - \frac{\sigma_0}{2}\right)$$

so that

$$\beta(x) = -x - \frac{1}{2}$$

and we obtain

$$\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial x^2} - x\frac{\partial U}{\partial x} = 0,$$

a diffusion equation which can be solved using Classical Hermite polynomials. By (5.3), the general solution is

$$U(x,t) = \sum_{n=0}^{\infty} k_n e^{nt} h_n(x),$$

where the coefficients k_n are determined by the payoff function. Explicitly, the k_n's are given by

$$k_n = \nu_n^{-1}\int_{-\infty}^{\infty} U(x,0)h_n(x)e^{-x^2}\,dx.$$

where the norming constants ν_n are given in (5.4).

To construct supersymmetric generalizations of the above model we consider an exceptional Hermite DE (5.28) where the zero order coefficient term is a constant. This happens for example when $K = \{1, 2\}$, in which case

$$h_K = Wr[h_1, h_2] = 8x^2 + 4$$

$$\frac{h''_K(x)}{h_K(x)} + 2x\frac{h'_K(x)}{h_K(x)} = 4$$

In this case, the exceptional Hermite DE is

$$y'' - \left(2x + \frac{8x}{1+2x^2}\right) y' = -2ny$$

with solutions

$$y = \hat{h}_n = \frac{Wr[h_1, h_2, h_n]}{16(n-1)(n-2)}, \quad n \in \{0, 3, 4, 5, \dots\}$$

We therefore consider the diffusion equation

$$\frac{1}{2}\frac{\partial^2 U}{\partial x^2} - \left(x + \frac{4x}{1+2x^2}\right)\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = 0.$$

From 5.8, we know that

$$\hat{h}_n = h_n + 4nh_{n-1} + 4n(n-3)h_{n-2}, \tag{7.9}$$

and the general solution is

$$\sum_{n=0,3,4,5,\dots}^{\infty} k_n\, e^{nt}\left(h_n + 4nh_{n-1} + 4n(n-3)h_{n-2}\right)$$

where the coefficients k_n depend on the payoff function. Since

$$\beta(x) = -\left(x + \frac{4x}{1+2x^2}\right) = -x\left(\frac{x^2 + \frac{5}{2}}{x^2 + \frac{1}{2}}\right)$$

we obtain

$$\mu(S) = \sigma_0\left(\beta(x) + \frac{\sigma_0}{2}\right) = -\sigma_0 \log S\left(\frac{\log S^2 + \frac{5}{2}}{\log S^2 + \frac{1}{2}}\right) + \frac{\sigma_0}{2}$$

Changing $x = \log S$ gives the price $V(S,t)$ of an option for a security with the variable drift as above.

7.3 Solvable model with a Variable Carrying Cost

In this chapter, we consider a generalized Black-Scholes model with constant diffusion $\sigma = \sigma_0$, but a price dependent carrying cost. We modify the diffusion equation (3.1) to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_0^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - \nu(S)C = 0$$

to include a variable carrying cost, which may be due to a varying interest rate or dividends paid by the underlying security.

The transformations described in Section 3 are

$$S = e^x, \quad \gamma(x) = \nu(S), \quad V(S,t) = e^{-x}U^c(x,t), \quad K = r - \frac{1}{2}.$$

. The transformed equation is

$$\frac{\partial U^c}{\partial t} = -\frac{1}{2}\frac{\partial^2 U^c}{\partial x^2} + \left(\gamma(x) - \frac{1}{2}K^2\right)U^c$$

We now take

$$\nu(S) = r - (A - \frac{1}{2}\log(S)^2)$$

so that

$$\gamma(x) = r - A + \frac{1}{2}x^2$$

thereby obtaining the solvable harmonic oscillator. This model represents a security that pays a variable dividend with A being the value of the proportional dividend when the price of the security is $S = 1$ [4]. The general solution of this diffusion equation is

$$U^c(x,t) = \sum_{n=0}^{\infty} k_n e^{(n+\frac{1}{2}-K^2)t-\frac{1}{2}x^2} h_n(x)$$

where k_n are determined by the payoff function. Explicitly, the k_n's are given by

$$k_n = \frac{1}{\nu_n}\int_{-\infty}^{\infty} U^c(x,0)e^{-\frac{1}{2}x^2}h_n(x)dx,$$

where the normalizing constants ν_n are given in (5.4).

7.4 Example

We now generalize the above model by using Supersymmetric methods. Take $K = \{1,2\}$ so that

$$h_K = Wr[h_1, h_2] = 8x^2 + 4$$

Using formula (5.31) gives the potential

$$\hat{U}(x) = x^2 + 2\left(\frac{2x^2 - 1}{(2x^2+1)^2}\right)$$

which was already discussed in Section 5.7. With this potential, the carrying cost becomes

$$\nu(S) = r - \left(A - \left(\frac{1}{2}\log S^2 + \frac{2\log S^2 - 1}{(2\log S^2 + 1)^2}\right)\right)$$

The general solution of the diffusion equation becomes

$$U^c(x,t) = \sum_{n=0,3,4,\ldots}^{\infty} k_n e^{(n+\frac{3}{2}-K^2)t - \frac{1}{2}x^2} \frac{\hat{h}_n(x)}{2x^2+1}$$

where the exceptional Hermite polynomial is given by formula (7.9) and where

$$k_n = \frac{1}{\hat{\nu}_n} \int_{-\infty}^{\infty} U^c(x,0) e^{-\frac{1}{2}x^2} \hat{h}_n(x) dx.$$

The normalizing constants $\hat{\nu}_n$ are no longer the classical Hermite norms, but rather are given by the modified formula (5.36).

Chapter 8

Conclusion

We have shown how to use SUSYQM and exceptional Hermite polynomials to derive solvable derivative models. We introduced a change of variables and gauge transformation to convert a second order PDE into Schrödinger form. We then introduced supersymmetric methods and Darboux transformations and applied them to the classical Hermite polynomials to derive the exceptional Hermite polynomials and derive important properties of them analogous to the classical case. We then derived the Black-Scholes Hamiltonian which allowed us to apply SUSYQM methods. We finally applied these techniques to several examples to obtain solvable derivative models.

Recall that the classical Black-Scholes equation (6.8) has a concise solution in the form of the Black-Scholes formula (6.11). Notice however that the models presented in the previous chapter do not yield such concise solution analogous to the classical Black-Scholes formula with the methods we have presented so far. Finding a concise formula analogous to the classical Black-Scholes formula for the generalized Black-Scholes equation is a subject for future research.

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