LIGHT PROPAGATION IN NONLINEAR MEDIA

by

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Abstract

The study of the light propagation in some complex media and its out-of-equilibrium dynamics is reported. Our research is based on the light-matter interaction hydrodynamic model under the multiscale expansion framework. At the proper scale of our mathematical analysis, the optical Benney-Luke equation is obtained. Also, we have proposed and demonstrated analytically, a novel and simpler variational approach to describe nonlinear open systems. Our methodology has been extended to the polarization instability effect on the beat length of propagating optical fields in a nonlinear birefringent Kerr medium. We have described the generation and dynamics of shock waves, specifically in disordered colloids. Finally, a preliminary theory based on reaction-diffusion dynamical system theory has been developed as a new goal for this project.

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Chapter 1

Introduction

This chapter introduces the main theme of the thesis. The state-of-the-art is reviewed. Thesis motivations and objectives are presented.

1.1 Overview

The interest in nonlinear waves in open physical, chemical, and biological systems, has been in the forefront of contemporary nonlinear science. A laser beam propagating in a nonlinear media [1, 2, 3, 4, 5, 6, 7, 8, 9] produces a richness of physical phenomenons due to the interplay between nonlinearity and dispersion which brings defocussing (self-focusing) effects to be present [10, 11, 12].

Nonlocality, which is related to the direct interaction of physical objects that are not in proximity, plays an important role in real experiments. Thus, it has shown that the state of a system is strongly dependent on disorder-induced scattering [13]. This could be related to nonlocality (σ), limiting the wave evolution and propagation length [14]. Therefore the nonlocality effects on wave dynamics [15, 16] are still a subject of an intense research interest [7, 17].

Let us begin with the study of a phenomenological model for a nonlinear Kerr media where nonlocal effects are considered. The laser beam intensity $I(x, z)$ propagating in the media produces a change Δn in the index of refraction n. This can be represented through a phenomenological equation as

$$\Delta n(I) = \pm \int_{-\infty}^{\infty} R(x' - x) I(x', z) dx', \tag{1.1}$$

where $R(x)$ is the response function of the nonlocal medium. The sign $\pm$ refers to the type of nonlinearity, focusing or defocusing. Also, the x and z denote transverse and propagation coordinates, respectively.

From here, it is straightforward to see that the local case is obtained from the above equation when $R(x) = \delta(x)$. This means that the refractive index value at a given point is solely determined by the light intensity at that very point.

$$\Delta n(I) = \pm \int_{-\infty}^{\infty} \delta(x' - x) I(x', z) dx' \equiv \pm I(x, z). \tag{1.2}$$

Nonlocality can be modeled mathematically by increasing the width of $R(x)$. This corresponds to the case when the light intensity in the vicinity of the point x also contributes to the index change at that point. Notice that in this thesis we are not interested in the case of highly nonlocal response[18, 19].

When the nonlocality is weak, i.e. when the response function $R(x)$ is narrow compared to the extent of the beam, we can expand $I(x', z)$ around the point $x' = x$ to obtain a partial differential equation (1.4) (a diffusion-like). This equation together with the NLSE forms the following system that in the paraxial wave equation approximation describes the propagation of a 1D beam propagating in a weakly nonlocal nonlinear defocusing medium with a dissipative term

$$i\,\epsilon \frac{\partial \psi}{\partial z} + \frac{\epsilon^2}{2} \frac{\partial^2 \psi}{\partial x^2} - \theta\, \psi = -i\,\epsilon \frac{\alpha}{2} \psi\,, \tag{1.3}$$

where the dimensionless z and x are the spatial evolutionary variable and the transverse coordinate, respectively. Also, $\psi = A/\sqrt{I_0}$, A is the complex electric field envelop, I_0 is the peak intensity, $\theta = k_0 L_{nl} \Delta n$ is a real function that denotes the nonlinear nonlocal change of the refractive index depending on the intensity $I = |\psi|^2$. $k = k_0 n \equiv (\omega/c) n$ $L_{nl} = (k_0 |n_2| I_0)^{-1}$, n_2 $(\Delta n = n_2 |A|^2)$ in the local Kerr coefficient and Δn is the refractive index change of nonlinear origin. Also $L_d = kw_0^2$, $L = \sqrt{L_{nl}L_d}$ and w_0 is the gaussian waist. $\alpha = \bar{\alpha}L$, $\bar{\alpha}$ is the intensity loss rate [20, 21], $\epsilon = L_{nl}/L \equiv \sqrt{L_{nl}/L_d}$ is a small quantity that deal with the weakly diffracting regime[22].

It is important to notice that the scalar NLSE is only valid if we assume in our physical model the response of the material and the electric field are transverse and the polarization is linear.

For example, in the reference system x, y, z, we assume that the wave vector is referred to along the z-axis and so the polarization occurs in the plane x, y. Then, a mode excited with its polarization in the x-direction would not couple to the mode with the orthogonal y-polarization states. Conventional pulse-shaping technology involves modulating the spectral amplitudes and phases of linearly polarized light pulses. In such methods, the optical electric field is treated as a scalar field.

The above expression is complemented by the diffusion-like equation

$$-\sigma^2 \frac{\partial^2\theta}{\partial x^2} + \theta = |\psi|^2, \tag{1.4}$$

where the parameter σ is a spatial scale (setting the diffusion length) that measures the degree of nonlocality [22].

This system of equations characterizes the behavior of a beam when it propagates through a spatially nonlocal defocusing nonlinear media, i.e., defocusing liquid solutions exhibiting thermal nonlinearities [17] or nematic liquid crystals [23, 24].

This phenomenological model can be solved by means of either the Mandelung or WKB transformations which permit casting the NLSE equation into a hydrodynamic formulation (i.e., Euler's equations) [14, 22], similarly to a classical fluid (i.e., undular bores, shock waves) [5, 8].

In general, hydrodynamics describes the low-frequency, long-wavelength response of a system that is disturbed from equilibrium. These conditions are mimicked by the propagation of a laser beam in complex media where we rely on a nonlocal nonlinear Schrödinger equation (NLSE) coupled through nonlocality with a diffusion-like expression [22, 25].

Specifically, in the case of the defocusing regime, the asymptotic behavior of the hydrodynamical model is analyzed using a version of the reductive perturbation theory, based on a multiscale asymptotic expansion. This allows the authors to derive a dynamical regime described by a Korteweg-de Vries equation (KdV) [5, 8, 25].

In [25] Horikis and Frantzeskakis reported an analogy between nonlocality in optics and surface tension in weakly nonlinear shallow water waves. Thus, some similar structures can appear in both media, i.e., X-, H-, or Y-shaped waves. These results suggest other equivalent phenomena observed in water waves may also occur during light propagation.

In a weakly nonlocal defocusing nonlinear media, as in the shallow water case, when the wave amplitude is assumed to be a small and slowly-varying modulation of the steady state, and at proper scales of the asymptotic analysis – intermediate step of the multiscale expansion – an optical version of the Benney-Luke equation (BLE) type is obtained [25, 26, 27].

In [25], the authors have proven how the different scenarios for solitary waves, specifically their amplitude and speed, depend on the nonlocal parameter σ through the optical surface tension strength γ [25]. The latter is a function of the degree of nonlocality, which changes the sign of dispersion, similar to what surface tension does in the shallow water wave problem. The study includes not only the case for strong nonlocality (or $\gamma < 0$) but also when weak nonlocality (or $\gamma > 0$) is present. The solutions in the form of solitary waves above and below the continuous wave (CW) background are shown.

Although a variational method is widely used to study the interaction between dark solitons in weakly nonlocal medium, it has not been applied to the system of equations above mentioned. The main focus is on the relative center mass position of two dark-soliton solutions corresponding to the time variation of a relative distance parameter x_0 and to obtain the equation of motion for it by employing the Lagrangian approach [28, 29, 30, 31].

We propose a new Lagrange density such that its Euler-Lagrange equations are identical to the hydrodynamic ones obtained from applying the Mandelung transformation in the NLS-diffusion original system. It is based on the one pioneered by Infeld, in the context for water waves, nonlinear equations [32] and also superconductivity [33].

In addition, it is worth noticing that conversely to the work from Infeld where the author uses the Euler equations in classical fluids like water, in our proposed Lagrange density we need to consider the quantum pressure term given its important contribution to the boundaries. Also, this term allows, in our results, to get the optical analog to the surface tension contribution in the dispersion term of the KdV [25].

Our interest is also extended to complex media and its out-of-equilibrium dynamics, through nonlinear optics experiments and theoretical models which have enjoyed an intense period of activity over the last years [11, 21] after the pioneering work of A. Ashkin in Ref. [10], and other authors [11, 12, 34]. The general problem of beam propagation and scattering in a turbid medium is important to fields such as biology and medical imaging as a diagnostic tool [35, 36]. Specially due to the potential to acquire information non-invasively through the sample's optical properties, especially the generation and detection of traveling waves.

Recent theoretical and experimental investigations have described the generation and propagation of shock waves in non-local and disordered media in response to an incident laser beam [14, 37, 38]. An important contribution to this subject has been made by the authors in Ref. [39] where some experiments and their respective theory on optical manipulation of the local properties of dense, particulate-loaded, highly-scattering(opaque) suspensions of dielectric nanoparticles in a liquid were introduced. The experiment, which was done for the self-focusing case, has proven that multiple-scattered light can give rise to concentration shock fronts propagating deep inside the opaque suspensions.

These particle density shock waves are primarily the result of the interplay between the two components of the applied radiation force, namely the scattering and the optical gradient forces respectively. The first one represents the momentum transfer from the external radiation field to the nanoparticle by scattering and absorption, and is pointing along the axis of the energy flux of the light beam, whereas the gradient force is directed along the intensity gradient of the beam. A partial differential equation (PDE) is well known to support shock-like waves which apply for both focusing and defocusing cases.

Finally, polarization instability in a medium arises when the nonlinear change of the refractive index is comparable with the linear birefringence. This phenomenon manifests when the nonlinear birefringence cancels completely the linear birefringence and the beat length escalates to infinity. Physically, the beat length (L_B^{eff}) is the length at which the optical power is transferred from one polarization to another. In a nonlinear medium, such as the Kerr medium, the L_B^{eff} length becomes infinite at a critical input power for a propagating light that is polarized along the fast axis [20, 40, 41]. It then follows that a substantial change in the output polarization state is observed when the input power (or its polarization state) is slightly differing.

To optimize the operation of some photonic devices [42], extensive efforts have been done theoretically and experimentally in order to control the polarization dynamics [43, 44, 45].

Interestingly, for propagating optical fields in a non-resonant Kerr nonlinear medium, a biasing electric field induces birefringence even if the medium is optically isotropic [46]. In [47], the authors have studied the impact of applying a DC electric field (i.e., E_{ext}), to a third-order nonlinear medium, on the evolution of propagating optical waves. They found that the polarization evolution can be controlled by the applied E_{ext} field. As a matter of fact, the E_{ext} field turns the third-order nonlinearity into a second-order-like as if one deals with an electro-optic-like effect.

1.2 Thesis Motivations and Objectives

From the theoretical point of view, there are still several open problems concerning the physics of optical fluid-like description (hydrodynamic model). One of them is to find a different kind of solution for the Optical Benney-Luke equation with appropriate boundary conditions in a nonlocal nonlinear defocusing media. This equation has been usually neglected for its far-field limit expansion, the Korteweg-de Vries (KdV), or its 2D variant, the Kadomtsev-Petviashvilli (KPI/KPII) equations due to the KdV (or KP) integrability and solitonic solutions. We intend to study their analytical dependency in terms of optical surface tension and nonlocality. The interplay between these two effects has not yet been studied systematically. We would like to elucidate the behavior of a wide class of open wave systems displaying OBL-type equations and the evolution of certain properties of the system, which are valid in an intermediate propagation regime. Thus, we would provide important information that will help to detail the transition from the original system of equations to the long wavelength limit where the KdV occurs. Furthermore, an open problem is to demonstrate the evolution of the generalized NLSE model in the intermediate asymptotics regime that satisfies the reported homeomorphism or parallelism between optics and shallow water waves.

Another goal is to propose and demonstrate analytically a novel and simpler variational approach to study the asymptotic behavior of a continuous wave laser beam propagating in a nonlinear nonlocal medium (i.e., a variational multiscale asymptotics mathematical method) and its extension to (2+1)-dim.

The idea is to propose a new Lagrange density such that its Euler-Lagrange equations are identical to the hydrodynamic ones obtained from applying the Mandelung transformation in the NLS-diffusion original system.

The study of the evolution of small-amplitude waves generated by the interaction of a laser beam with nanoparticles dispersed in a liquid medium was proven experimentally but the theoretical explanation was not well developed. Here the goal is to derive a dynamical system where a partial differential equation describes in the first approximation a wave propagation in a form of a kink shock wave.

More theoretical research on the polarization beat length of propagating optical fields in a nonlinear birefringent Kerr medium is investigated in the presence of an externally applied DC electric field. Here, we have focused our attention on the possibility to control when the polarization beat length becomes infinite by means of adjusting the externally applied electric field.

1.3 Thesis Outline

This thesis is based on papers published publications by the author and others. Each chapter is dedicated to one paper while the introduction to each and every chapter identifies and summarizes the author's original contribution.

In Chapter 2, a detailed theoretical treatment is developed, and a thorough analysis performed for the Optical Benney-Luke equation (OBLE). The analytical solutions in the form of weakly localized cnoidal waves (CnWs) are introduced. The study has been done in a nonlocal nonlinear defocusing media. The OBLE solutions lead to periodic waves when appropriated boundary conditions are taken into account. It is found that the wave frequency and wavelengths depend on the nonlocality and the optical surface tension parameter. The results are extended in Chapter 3 to obtain the exact solitary wave (SW) profiles, for the light intensity and its phase chirp in terms of the optical surface tension, a function of the degree of nonlocality. The solution dynamics have been demonstrated numerically. Our results show that the OBLE satisfies the reported ”homeomorphism” between optics and shallow-water waves and gives an insight into the nonlocal nonlinear Schrodinger equation (NLSE) evolution in the intermediate asymptotics regime.

In Chapter 4, it is proposed the asymptotic variational multiscale approach which is based on a new Lagrange density such that its Euler-Lagrange equations are identical to the hydrodynamic ones obtained from applying the Mandelung transformation in the NLSE system of equations.

The results of Chapter 4 are generalized to higher dimensions in Chapter 5. Here, the Kadomtsev-Petviashvili (KP) type equation is obtained. For the first time, to the best of our knowledge, the variational multiscale asymptotics method is used to describe nonlinear open systems.

In Chapter 6, we consider the evolution of small-amplitude waves generated by the interaction of a laser beam with nanoparticles dispersed in a liquid medium. Under the asymptotic multiscale expansion framework and assuming a low concentration of beads, we have derived a dynamical system where a partial differential equation describes in the first approximation a wave propagation in a form of a kink shock wave. This front forms a depletion region with a vanishing concentration of beads which consequently allows light propagation through the medium. The possible presence of absorption in the system could be shown through the complex expressions for the phase and group velocities in the case of linear propagation of waves.

In Chapter 7, the polarization beat length of propagating optical fields in a nonlinear birefringent Kerr medium is investigated in the presence of an externally applied DC electric field. We show that the critical power, at which the effective polarization beat length becomes infinite, can be controlled by adjusting the externally applied electric field. The principle of operation is based on modifying the polarization instability by electronically adjusting the effective birefringence through an external electrical bias. The presented analytical expressions describe the beat length and the polarization instability as a function of the applied electric field for an arbitrary optical input state.

Chapter 8 presents a brief discussion and concluding remarks.

Chapter 2

Cnoidal wave solutions for the Optical Benney-Luke equation

This chapter's work is published in the paper entitled "Cnoidal wave solutions for the Optical Benney-Luke equation", Journal of Optics, vol. 22, pp. 105401, 2020. The authors are Artorix de la Cruz and Michael Cada. The paper proposes by means of an Optical Benney-Luke equation (OBLE), with appropriate boundary conditions, a nonlinear periodic wave solution. The corresponding analytical expressions and numerical plots of the cnoidal waves in the phase plane are presented using the Sagdeev pseudo-potential approach. Our results are written in terms of the media nonlocality through the optical surface tension parameter.

My contribution to this work includes generating the fundamental ideas, performing the analytical description and the necessary analysis, and carrying out numerical simulations. I also wrote the first version of the paper.

2.1 Introduction

The study of a continuous wave (CW) laser beam in a nonlocal nonlinear defocusing media has gained importance due to its application in understanding light propagating in liquid solutions exhibiting thermal nonlinearities [14, 15, 17, 21, 37, 48, 49, 50, 51, 52] or nematic liquid crystals [8, 23, 24, 53, 54]. Mathematically, a generalized nonlinear Schrödinger equation (NLSE) coupled with a diffusion-like expression govern this behavior.

In similitude with the shallow water case, when the wave amplitude is assumed to be a small and slowly-varying modulation of the steady-state, and at proper scales of the asymptotic analysis – intermediate step of the multiscale expansion – an optical version of the Benney-Luke equation (BLE) type is obtained [26, 25, 27]. Furthermore, Horikis and Frantzeskakis reported an analogy between nonlocality in optics and surface tension in weakly nonlinear shallow water waves [25].

The Benney-Luke equation has been intensely studied in different scenarios such as mathematical analysis and hydrodynamic physics [55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65]. In optics has been usually neglected for its far-field limit expansion, the Korteweg-de Vries (KdV), or its 2D variant, the Kadomtsev-Petviashvilli (KPI/KPII) equations [25]. This is due to the KdV (or KP) integrability and solitonic solutions.

The present paper aims at a further analytical and numerical study of light propagating in a weakly absorbing nonlocal nonlinear media. The main distinctions of the situation considered here compared to other relevant researches are as follows:

(i) The analytical solutions for the Optical Benney-Luke equation (OBLE) in the form of weakly localized cnoidal waves (CnWs) are introduced. We show that by solving the OBLE, the system of equations (NLSE-diffusion) possess – in the intermediate asymptotics – cnoidal wave solutions for the intensity profile. Although the OBLE admits waves in two opposing directions, we restrict its applicability to uni-directional wave motion. This 1D case illustrates basic physics with the least complexity.

(ii) We prove how the different scenarios for CnWs, specifically their amplitude and speed, depend on the nonlocal parameter σ through the optical surface tension strength γ [25]. The latter is a function of the degree of nonlocality, which changes the sign of dispersion, similar to what surface tension does in the shallow water wave problem. Our study specifically includes the case for strong nonlocality (or $\gamma < 0$) but it can be trivially generalized also when we are in the presence of weak nonlocality (or $\gamma > 0$).

2.2 Basic Model and Derivation of Optical Benney-Luke equation

For the sake of clarity and completeness, we briefly review the mathematical model for the OBLE derivation following the same method published in [25]. The start point in the analysis is the light propagation in a weakly nonlocal nonlinear defocusing media described by a normalized NLSE (8.1) [54] and diffusion-like (2.1b) equations [21, 25],

$$i\,\frac{\partial\psi}{\partial z}+\frac{1}{2}\,\frac{\partial^2\psi}{\partial x^2}-n\,\psi=0\,, \tag{2.1a}$$

$$-\sigma^2\,\frac{\partial^2 n}{\partial x^2}+n=|\psi|^2. \tag{2.1b}$$

The z is the spatial evolutionary variable and x is a transverse coordinate. ψ is the complex electric field envelop and σ is a spatial scale (setting the diffusion length) that measures the degree of nonlocality. The real function n denotes the nonlinear nonlocal change of the refractive index depending on the intensity $I=|\psi|^2$ and obeys the diffusion-like equation (2.1b).

To derive the full nonlinear version of the optical BL equation, the wave amplitude is assumed to be a small and slowly-varying modulation of the steady state. The solutions of (2.1) can be proposed in the form

$$\psi=\psi_0\sqrt{\rho}\,\exp(-i|\psi_0|^2 z+i\,\epsilon^{1/2}\Phi), \tag{2.2}$$

together with the asymptotic expansions $\rho=1+\sum_{j=1}^{\infty}\epsilon^j\rho_j$ and $n=|\psi_0|^2+\sum_{j=1}^{\infty}\epsilon^j n_j$. Here ϵ is a formal parameter ($0<\epsilon\ll 1$), while the phase Φ and amplitudes ρ_j and n_j are unknown real functions of the slow variables $X=\epsilon^{1/2}x$ and $Z=\epsilon^{1/2}\,z$, (for more details on the model see [25]).

Substituting the expression (2.2) into (2.1), the following results are valid for the leading-order equation at $\mathcal{O}(\epsilon)$,

$$n_1 = |\psi_0|^2 \rho_1, \quad n_1 = -\frac{\partial \Phi}{\partial Z} \equiv -\epsilon^{-1/2} \frac{\partial \Phi}{\partial z}. \tag{2.3}$$

A scaled Benney-Luke equation is obtained as

$$\frac{\partial^2 \Phi}{\partial Z^2} - C^2 \frac{\partial^2 \Phi}{\partial X^2} + \epsilon \left[\frac{\gamma}{4} \frac{\partial^4 \Phi}{\partial X^4} + \frac{1}{2} \frac{\partial}{\partial Z} \left(\frac{\partial \Phi}{\partial X} \right)^2 + \frac{\partial}{\partial X} \left(\frac{\partial \Phi}{\partial Z} \frac{\partial \Phi}{\partial X} \right) \right] = 0, \tag{2.4}$$

for an approximation up to $\mathcal{O}(\epsilon^2)$.

Our goal is to solve Eq. (8.6) which is a partial differential equation (PDE) in terms of phase Φ. This PDE has travelling wave solutions. The phase behaves like a wave as the laser propagates in the media.

Notice that when $\epsilon = 0$, the above expression has the form of a well-known wave equation. Then, it is possible to identify the wave velocity as $C^2 = |\psi_0|^2$. The effective surface tension is given by $\gamma = 1 - 4\,\sigma^2\,|\psi_0|^2$.

The equation (8.6) has some similarity with a model of small amplitude long water waves with finite depth originally derived by Benney and Luke [26] as a description for bidirectional shallow water waves.

2.3 Cnoidal wave solutions

In this section the effects of nonlocality (σ) on the propagation of cnoidal waves are investigated. Let us assume travelling wave in the form $\Phi(X, Z) = V(X - Z) \equiv V(\eta)$, and substituting it into Equation (8.6), it yields

$$(1 - C^2) \frac{d^2 V}{d\eta^2} + \frac{\epsilon \gamma}{4} \frac{d^4 V}{d\eta^4} - \frac{3\epsilon}{2} \frac{d}{d\eta} \left(\frac{dV}{d\eta} \right)^2 = 0. \tag{2.5}$$

By integrating once and changing variable as $\varphi = dV/d\eta$, we get the following expression

$$(1 - C^2)\,\varphi + \frac{\epsilon\,\gamma}{4}\frac{d^2\varphi}{d\eta^2} - \frac{3\epsilon}{2}\varphi^2 = k_0, \tag{2.6}$$

where k_0 is an integration constant to be determined later.

Now, multiplying by $d\varphi/d\eta$,

$$(1 - C^2)\varphi\,\frac{d\varphi}{d\eta} + \frac{\epsilon\gamma}{4}\frac{d^2\varphi}{d\eta^2}\frac{d\varphi}{d\eta} - \frac{3\epsilon}{2}\varphi^2\,\frac{d\varphi}{d\eta} = k_0\,\frac{d\varphi}{d\eta}. \tag{2.7}$$

Finally, we multiply the above equation by φ and integrate once more to get

$$-\frac{\gamma}{4}\left(\frac{d\varphi}{d\eta}\right)^2 = -\varphi^3 + H\varphi^2 + \frac{2k_0}{\epsilon}\varphi + \frac{2}{\epsilon}k_1, \tag{2.8}$$

where we have defined

$$H = (\,1 - C^2\,)/\,\epsilon. \tag{2.9}$$

and k_1 is another integration constant to be determined as well. In Eq. (2.8) when the two constants k_0 and k_1 become zero, it is possible to obtain solitary wave solutions. For the case when $k_0 \neq 0$, and $k_1 \neq 0$, spatially periodic traveling wave solutions are derived. To compute them, let us first rewrite Eq. (2.8) as a energy conservation law as follows:

$$\frac{1}{2}\left(\frac{\partial\,\varphi}{\partial\,\eta}\right)^2 + V(\varphi) = 0\ , \tag{2.10}$$

where the Sagdeev-like potential $V(\varphi)$ is given by

$$V(\varphi) = \kappa\,\varphi^3 - \kappa\,H\,\varphi^2 - \frac{2\kappa}{\epsilon}k_0\,\varphi - \frac{2\kappa}{\epsilon}k_1\,, \tag{2.11}$$

where $\kappa = 2/(-\gamma)$.

On can compute the integration constant k_0 from the differential equation (2.6) using the following initial conditions for $\varphi(\eta)$ as $\varphi(0) = \varphi_0$ and $d\,\varphi(0)/d\eta = 0$. Then,

$$k_0 = \epsilon\,\varphi_0 \left(H - \frac{3}{2}\,\varphi_0 \right) . \tag{2.12}$$

Furthermore, the integration constant k_1 can be determined by taking into account the above initial conditions in Eq. (2.11) as well. Therefore,

$$k_1 = \frac{\epsilon}{2}\,\varphi_0^2\,(\,4\,\varphi_0 - 3\,H\,) . \tag{2.13}$$

In Fig. 2.1 we have depicted the variation in the Sagdeev potential $V(\varphi)$ with respect to

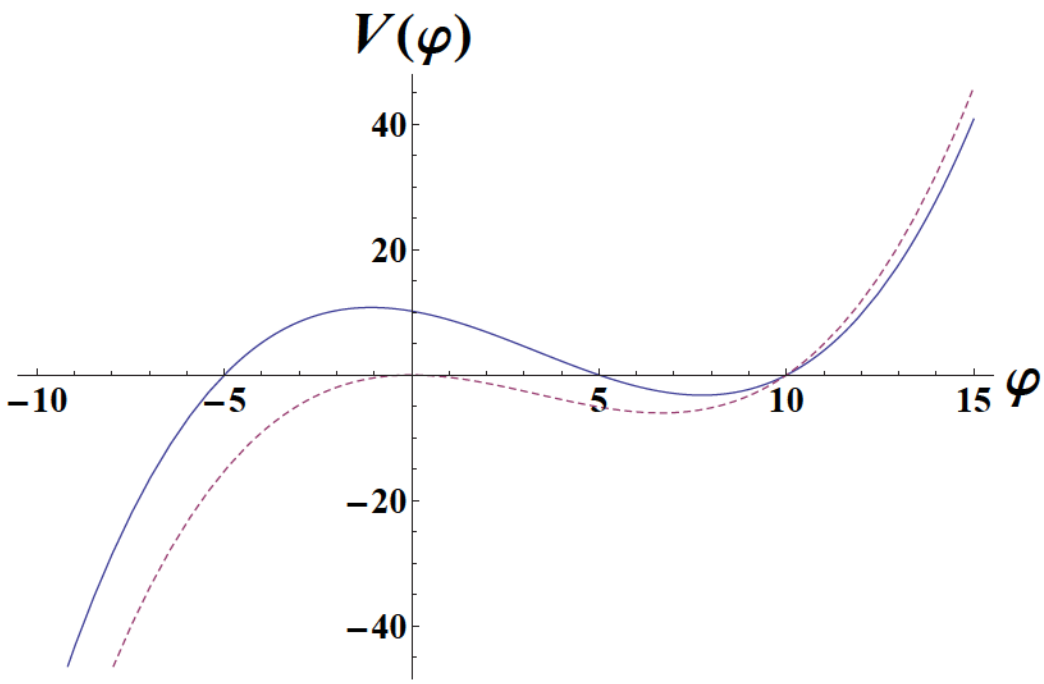

Figure 2.1: Variation in Sagdeev potential $V(\varphi)$ with respect to values of φ using Eq. (2.11). The parameters used to obtain the figure are $\sigma = 5.0$, $\epsilon = 0.05$, $\varphi_0 = 5.0$ and $C = 0.70$.

φ using Eq. (2.11). The parameters chosen are representative of the nonlocal nonlinear defocusing media found in a thermal one. The solid (the dashed) curve demonstrates the

Sagdeev potential corresponding to the cnoidal wave (solitary wave).

In the case of a cnoidal wave (i.e., $k_0 \neq 0$ and $k_1 \neq 0$), the three real zeros of $V(\varphi)$ are φ_0, φ_1, and φ_2. Furthermore, this potential does not become zero at $\varphi = 0$. Accordingly, the potential structure is repeated and the distance between repetitions of the wave shape is equal to one wavelength. On the other hand, in the case of a soliton (i.e., $k_0 = 0$ and $k_1 = 0$), the potential $V(\varphi)$ becomes zero at $\varphi = 0$.

Now, let us substitute Eqs. (2.12) and (2.13) in Eq. (2.10). Then,

$$\left(\frac{\partial\varphi}{\partial\eta}\right)^2 = -2\,\kappa\,V(\varphi)\ , \tag{2.14}$$

where

$$V = \varphi^3 - H\varphi^2 - 2\varphi_0\left(H - \frac{3}{2}\varphi_0\right)\varphi - \varphi_0^2\,(4\varphi_0 - 3H)\,, \tag{2.15}$$

The above expression (2.15) has the form of a third-degree polynomial equation. Then, by using the well-known cubic formula and after some algebraic manipulations, the factorized expression is

$$\left(\frac{\partial\varphi}{\partial\eta}\right)^2 = \kappa\,(\varphi_0 - \varphi)(\varphi - \varphi_1)(\varphi - \varphi_2), \tag{2.16}$$

where

$$\varphi_{1,2} = \frac{(H - \varphi_0)}{2} \pm \frac{1}{2}\left[(\Psi_1 - \varphi_0)\,(\varphi_0 - \Psi_2)\right]^{1/2}\ , \tag{2.17}$$

and

$$\Psi_{1,2} = \frac{H}{3} \pm \frac{2}{3}\sqrt{H^2 + 6\,k_0/\epsilon}. \tag{2.18}$$

The last relations indicate that the inequality $\Psi_2 \leq \varphi_0 \leq \Psi_1$, or $\Psi_1 \leq \varphi_0 \leq \Psi_2$, should be satisfied. Also, φ must lie between the two-zeros φ_0 and φ_1 which corresponds to the higher and lowe values of the wave amplitude, respectively. Their difference is the total dimensionless wave height

$$h = \varphi_0 - \varphi_1 \, . \tag{2.19}$$

The periodic wave solution of Eq. (2.10) in original coordinates yields

$$\varphi = \varphi_1 + h \;\; \mathrm{Cn}^2 \left[a \; (x - z) \, , \, m \right] , \tag{2.20}$$

where

$$a \;\; = \;\; \frac{1}{2} \sqrt{\epsilon \, \kappa \, (\varphi_0 - \varphi_2)} \, , \tag{2.21}$$

$$m^2 \;\; = \;\; \frac{\varphi_0 - \varphi_1}{\varphi_0 - \varphi_2} \, , \tag{2.22}$$

and $\mathrm{Cn}(u, k)$ is a Jacobian elliptic function with argument u and elliptic modulus $k(0 \leq k \leq 1) = \sqrt{m}$. Physically, the elliptic parameter m (the modulus) may be viewed as a fair indicator of the nonlinearity with the linear limit being $m \to 0$ and the extreme nonlinear limit being $m \to 1$. The conditions for the existence of a cnoidal solution (2.20) require that $\varphi_0 > \varphi_1 \geq \varphi_2$ and $\varphi_1 \leq \varphi \leq \varphi_0$ and therefore the dimensionless wavelength λ_{Cn} of a cnoidal wave is given by

$$\lambda_{\mathrm{Cn}} = 4 \sqrt{\frac{m}{\epsilon \, \kappa \, h}} \; K(m) \, , \tag{2.23}$$

where $K(m)$ is the standard symbol for the complete elliptic integral of the first kind. The wave speed is found by adding φ_1, φ_2 and φ_3 from Eqs. (2.10) and (2.16):

$$C^2 = 1 + \frac{2\,\epsilon}{\gamma}\left(\varphi_0 + \varphi_1 + \varphi_2\right). \tag{2.24}$$

In Fig. 2.2 we have plotted the phase plane using Eqs. (2.10)-(2.16). From the numerical results displayed one can note that the phase curve is repeated on the same path and one complete cycle corresponds to a wavelength in the physical space. Therefore, the closed curve in the phase plane implies that the trajectory is a periodic orbit.

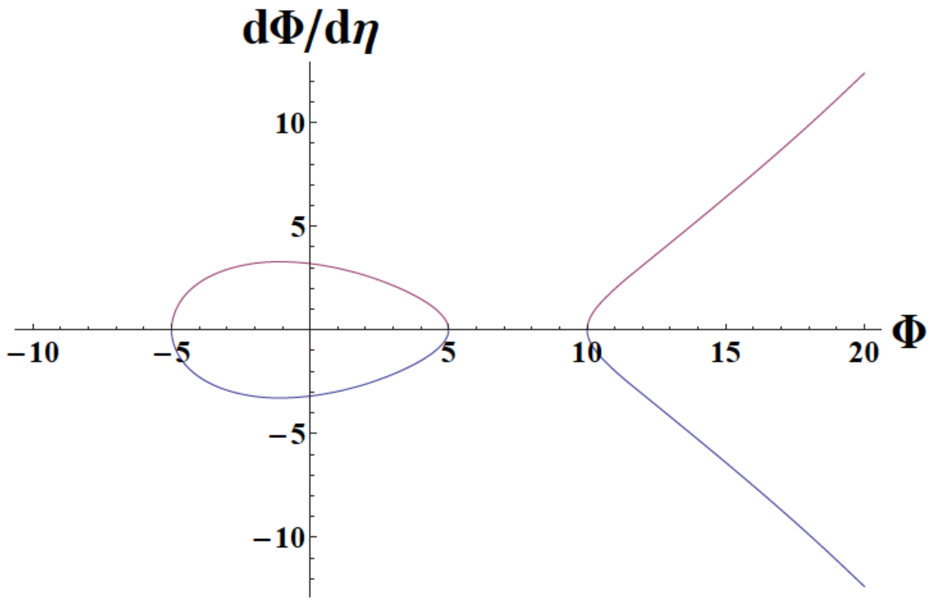

Figure 2.2: Phase curves using Eqs. (2.10) and (2.16), with $\sigma = 5.0$, $\epsilon = 0.05$, $\varphi_0 = 5.0$ and $C = 0.70$.

2.4 Limit solution: From Cnoidal to Soliton

Now, to describe the soliton solution as a limit case for (2.20), we assume $m \to 1$ and $k_0 = k_1 = 0$, which can be realized at $\varphi_1 = \varphi_2 = 0$. Therefore, $h = \varphi_0$ and $a = \sqrt{\epsilon\,\kappa\,\varphi_0}/2 \equiv 1/W$, and $\mathrm{Cn}(\cdot) \to \mathrm{sech}(\cdot)$ [66], then Eq. (2.20) becomes

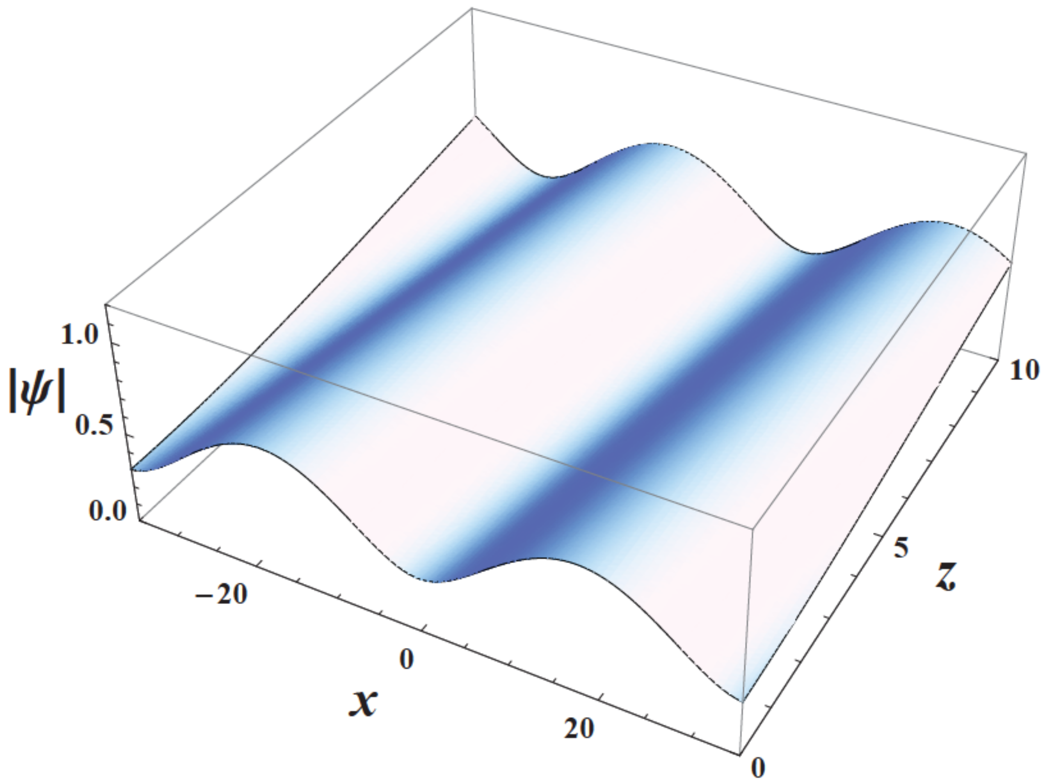

Figure 2.3: Evolution of the cnoidal waves in the case of strong nonlocality. The parameters used to obtain the figure are $\sigma = 5.0$, $\epsilon = 0.05$, $\varphi_0 = 5.0$ and $C = 0.70$.

$$\varphi(x,z) = \varphi_0 \ \text{sech}^2 \left[(x-z)/W \right] . \tag{2.25}$$

where φ_0 and W are the amplitude and the width of the soliton wave, respectively.

2.5 Final general solution

Until now we have obtained the solutions for the OBLE Eq. (8.6). In this section, our goal is to finally find the solution of the original problem given by the system (2.1). If we recall that $\varphi = dV/d\eta$, then the integration of φ in (2.20) yields -expressed in original

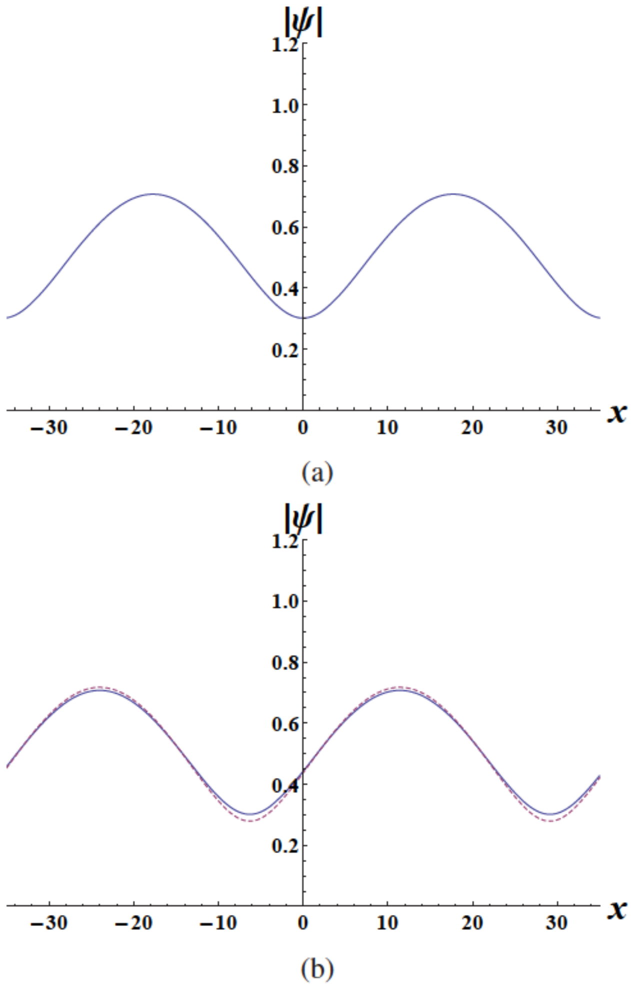

Figure 2.4: Evolution of the solitary waves in the case of strong nonlocality $\sigma = 5.0$ (weak optical surface tension $\gamma < 0$), $\varphi_0 = 5.0$, $C = 0.70$ and $\epsilon = 0.05$. In both panels the blue (black) solid lines represent the analytical (numerical) wavefunction's modulus $\psi(x, z)$. The panel (a) shows $|\psi(x = 0, z = 0)|$. The panel (b) is for $|\psi(x = 30, z = 100)|$.

coordinates-,

$$\Phi(x, z) = \frac{b\,h}{a\,c} + \varphi_1\,x + h\,\left[1 - \frac{1}{m}\right]\,(x - z), \tag{2.26}$$

where

$$\begin{aligned} b &\equiv E\left[\mathrm{am}[a(x-z), m], m\right]\left(\mathrm{Cn}^2[a(x-z), m] + \frac{1}{m} - 1\right), \\ c &\equiv \mathrm{dn}\left[a(x-z), m\right]\left(1 - m\,\mathrm{Sn}^2\left[a(x-z), m\right]\right), \end{aligned} \tag{2.27}$$

and $\mathrm{am}(u,k)$ is a Jacobi amplitude function. Also the $\mathrm{Sn}(u,k)$ and $\mathrm{dn}(u,k)$ are the Jacobian elliptic functions while $\mathrm{E}(u,k)$ is the Elliptic integral of the second kind [66].

Transforming back using Eqs. (2.2)- (2.3) leads to the following approximate [up to order $\mathcal{O}(\epsilon)$] solution for the macroscopic wavefunction $\psi \sim \psi_0\,(\rho_0 + \epsilon\,\rho_1)^{1/2}$ and $n = n_0 + \epsilon\, n_1$ as

$$\psi \;=\; \psi_0\sqrt{1-\frac{\epsilon^{1/2}}{|\psi_0|^2}\frac{\partial\Phi}{\partial z}}\;\;\exp(-i|\psi_0|^2 z + i\epsilon^{1/2}\Phi)\;, \tag{2.28}$$

$$n \;=\; |\psi_0|^2 - \epsilon^{1/2}\,\frac{\partial\Phi}{\partial z}\;, \tag{2.29}$$

where $\partial\Phi/\partial z$ can be obtained readily from (2.26). These results can be seen in Fig. 2.3 where we have plotted the profile of the cnoidal wave $|\psi|$ versus x and z using Eq. 4.17 .

Finally, the analytical results presented above have been confirmed by numerical simulation with the use of the appropriated scaling for the original system (2.1) and experimental values on nonlocal spatial media [21] . This task was done with the help of a split-step Fourier method [67, 68]. For the initial conditions we used corresponding analytical forms given by (2.26).

By taking into consideration strong nonlocal media, Fig. 2.4 shows snapshots of the evolution of $|\psi|$ versus x in two different values of dimensionless z. When $x = 0$ at $z = 0$ in Fig. 2.4a. The 2.4b is for $x = 30$ at $z = 100$. In general, the numerically obtained cnoidal wave profiles coincide, to a good accuracy, with the analytically determined ones. The relative maximum error in our numerical computations to estimate the solitary waves' minimum or maximum is relatively small with values around 2.7 %.

2.6 Conclusion

The analytical solutions for the Optical Benney-Luke equation (OBLE) in the form of weakly localized cnoidal waves (CnWs) are introduced. The study has been done in a nonlocal nonlinear defocusing media. The OBLE solutions lead to periodic waves when appropriated boundary conditions are taken into account. It is found that the wave frequency and wavelengths depend on the nonlocality and the optical surface tension parameter. The new results are of importance both for the mathematical theory of the OBLE waves, and also for their physical significance since they shine light on the intermediate asymptotic scenario for the NLSE. The results of the paper are, to our knowledge, original and they could be of significant interest, in particular in the context of research on optical spatial waves in liquid crystals.

2.7 Acknowledgment

Artorix de la Cruz thanks the financial support from Killam Trust Predoctoral and Nova Scotia Research scholarships. M. Cada acknowledges support from the Natural Sciences and Engineering Research Council (NSERC) of Canada.

Chapter 3

Optical Benney-Luke equation

This chapter's work is published in the paper entitled ”Optical Benney-Luke equation”, Analytical and Numerical Methods in Differential Equations. 100th birthday of the prominent Russian mathematician and Academician Nikolai Nikolaevich Yanenko. Conference Proceedings: http://math.sut.ac.th/conference (July 2021). The authors are T. Diaz-Chang, A. de la Cruz, Ch. Liang, J. Pistora and M. Cada. The paper analyzes, within the framework of an Optical Benney-Luke equation (OBLE), the light propagation in a nonlocal nonlinear defocusing media. The exact solitary wave (SW) profiles, for the light intensity and its phase chirp, have been obtained analytically in terms of the optical surface tension, which depends on the degree of nonlocality. The solutions dynamics have been demonstrated numerically. Our results show that the OBLE satisfies the reported ”homeomorphism” between optics and shallow-water waves and gives an insight into the nonlocal nonlinear Schrodinger equation (NLSE) evolution in the intermediate asymptotics regime.

My contribution to this work includes generating the fundamental ideas, performing the analytical description and the necessary analysis, and carrying out numerical simulations. I also wrote the first version of the paper.

3.1 Theory

The starting point in the analysis is the light propagation in a weakly nonlocal nonlinear defocusing medium described by [21] normalized NLSE with a dissipative term and a diffusion-like equation for the response of the nonlocal medium

$$i\,\epsilon\,\frac{\partial\Psi}{\partial z}+\frac{\epsilon^2}{2}\,\frac{\partial^2\,\Psi}{\partial x^2}-\Theta\,\Psi=0 \quad \text{and} \quad -\,\sigma^2\,\nabla^2\,\Theta+\Theta=|\Psi|^2, \tag{3.1}$$

The z and x are the spatial evolutionary variable and the transverse coordinates, respectively. Ψ is the complex electric field envelop peak intensity, Θ is a real function that denotes the nonlinear nonlocal change of the refractive index depending on the intensity. The $\epsilon \ll 1$ is a small quantity that deal with the weakly diffracting regime. The parameter σ is a spatial scale (setting the diffusion length) that measures the degree of nonlocality.

3.2 Our results

The solutions can be proposed in the form $\psi = \psi_0\sqrt{\rho}\,\exp(-i|\psi_0|^2 z + i\,\epsilon^{1/2}\Phi)$. A scaled Benney-Luke equation [25] is obtained as

$$\frac{\partial^2\Phi}{\partial Z^2}-C^2\frac{\partial^2\Phi}{\partial X^2}+\epsilon\left[\frac{\sigma}{4}\,\frac{\partial^4\Phi}{\partial X^4}+\frac{1}{2}\,\frac{\partial}{\partial Z}\left(\frac{\partial\Phi}{\partial X}\right)^2+\frac{\partial}{\partial X}\left(\frac{\partial\Phi}{\partial Z}\frac{\partial\Phi}{\partial X}\right)\right]=0, \tag{3.2}$$

Let us propose traveling wave solutions in the form $\Phi(X,Z) = V(X-Z) \equiv V(\eta)$. Integrating once and changing the variable as $\varphi = dV/d\eta$, we obtain the following relation

Integrating (8.6) one obtains the phase Φ associated with this solution, which in terms of the original (dimensionless) coordinates, x and z reads:

$$\Phi(x,z)=\sqrt{\frac{-\,\gamma\,H}{\epsilon}}\;\tanh\left[\frac{1}{W}\,x-v_g\;z\right]\;, \tag{3.3}$$

where $v_g = \sqrt{-(\epsilon\,H)/\gamma}$ and $W = -(\gamma/H)\sqrt{-H/\gamma}, H = (1-C^2)/\epsilon$.

As a function of the original (dimensionless) z and x, one may write down an approximate [up to order $\mathcal{O}(\epsilon)$] solution for the macroscopic wavefunction $\psi \sim \psi_0\,(\rho_0+\epsilon\,\rho_1)^{1/2}$

and $n = n_0 + \epsilon\, n_1$ as follows:

$$\psi = \psi_0 \sqrt{1 - \frac{\epsilon^{1/2}}{|\psi_0|^2}\frac{\partial \Phi}{\partial z}}\ \exp(-i|\psi_0|^2 z + i\epsilon^{1/2}\Phi), \quad \text{and} \quad n = |\psi_0|^2 - \epsilon^{1/2}\frac{\partial \Phi}{\partial z}\ , \quad (3.4)$$

3.3 Conclusion

In conclusion, we have explored theoretically and numerically the light propagation in a nonlocal nonlinear defocusing media through the Optical Benney-Luke equation. We have proven that solutions of this equation exhibit sech-type solitary waves. We also discussed the regimes in which the form of the solitary waves can be quantitatively described depending on the optical surface tension. The insights gained enabled us to interpret our findings in terms of the nonlocality of the media. The new results are of importance both for the mathematical theory of the OBLE solitary waves, and also for their physical significance since they shine light on the intermediate asymptotic scenario for the NLSE.

3.4 Acknowledgments

Artorix de la Cruz thanks the financial support from Killam Trust Predoctoral and Nova Scotia Research scholarships. This work was supported by NSERC of Canada and by the IT4Innovations National Supercomputing Center - Path to exascale project (EF16-013/0001791).

Chapter 4

Asymptotic variational approach to study light propagation in a nonlocal nonlinear medium

This chapter's work is published in the paper entitled ”Asymptotic variational approach to study light propagation in a nonlocal nonlinear medium”, Results in Physics (27), 104536 (2021). The authors are Artorix de la Cruz, Michael Cada, Jaromir Pistora and Tamara Diaz-Chang.

We propose and demonstrate analytically, within the framework of a hydrodynamic model, a novel and simpler variational approach to study the asymptotic behavior of a continuous wave (cw) laser beam propagating in a nonlinear nonlocal medium.

My contribution to this work includes generating the fundamental ideas, performing the analytical description and the necessary analysis, and carrying out numerical simulations. I also wrote the first version of the paper.

4.1 Theory

The starting point in the analysis is the light propagation in a weakly nonlocal nonlinear defocusing medium described by normalized NLSE

$$i\,\psi_z + \frac{1}{2}\,\psi_{x,x} - \varphi\,\psi = 0\,, \tag{4.1}$$

where the dimensionless z and x are the spatial evolutionary variable and the transverse coordinates, respectively. Also, ψ is the complex electric field envelop, φ is a real function that denotes the nonlinear nonlocal change of the refractive index depending on the intensity $I = |\psi|^2$. Finally $0 < \epsilon \ll 1$ is a small quantity that deal with the weakly diffracting regime (see [25] for more details). Other examples of light propagating in different media are [69, 70, 71, 72, 73, 74, 75, 76].

The above expression is coupled to a diffusion-like equation for the response of the nonlocal medium

$$-\sigma^2 \varphi_{x,x} + \varphi = |\psi|^2, \tag{4.2}$$

where the parameter σ is a spatial scale (setting the diffusion length) that measures the degree of nonlocality.

4.2 Proposed mathematical framework

From now on we proceed to develop our method. To start, we consider small amplitude slowly varying modulations of the steady state given by a continuous wave $\psi = \psi_0 \exp(-i\,|\psi_0|^2\,z)$, where ψ_0 is an arbitrary complex constant, $|\psi_0|^2 = 1$ and the constant $\varphi = |\psi_0|^2$.

Applying the Mandelung transformation $\psi(z,x) = \rho^{1/2}(z,x)\exp[\,i\,h(z,x)\,]$ and retaining leading orders in ϵ, it is possible to obtain the following equations

$$\rho_z + (\rho\,h_x)_x = 0, \tag{4.3a}$$

$$h_z + \frac{1}{2}\,h_x^2 + \frac{1}{2}\,\rho^{-1/2}\,\rho^{1/2}_{x,x} + \varphi = 0, \tag{4.3b}$$

$$-\sigma^2\,\varphi_{x,x} + \varphi = \rho, \tag{4.3c}$$

The above system of equations can be derived from the appropriate Lagrangian density

$$L = \rho\left[\frac{h_x^2}{2} + h_z + \varphi - 1\right] + \frac{1}{2}(\rho_x^{1/2})^2 - \frac{1}{2}\left[\varphi^2 + (\sigma\,\varphi_x)^2 - 1\right]. \tag{4.4}$$

Euler-Lagrange variation with respect to h yields (4.3a) whereas the ρ and φ variations yield (4.3b) and (4.3c), respectively.

To discuss the wave envelop dynamics in this long-wavelength limit due to weak non-linear and weak dispersive effects, we introduce the stretched variables

$$\xi = \epsilon^{1/2}\,(x - z) \quad \text{and} \quad \tau = \epsilon^{3/2}\,z,$$

where ξ allows us to study the system on different, slowly, moving frames and by τ, longer propagation distance z. Also, ϵ is a measure of the deviation from the background ψ_0. Using the perturbation expansions

$$\rho(\xi,\tau) = \rho_0 + \sum_{j=1}^{\infty} \epsilon^j\,\rho^{(j)}(\xi,\tau)\,, \tag{4.5a}$$

$$\varphi(\xi,\tau) = \varphi_0 + \sum_{j=1}^{\infty} \epsilon^j\,\varphi^{(j)}(\xi,\tau)\,, \tag{4.5b}$$

$$h(\xi,\tau) = \sum_{j=0}^{\infty} \epsilon^{j+1/2}\;h^{(j+1)}(\xi,\tau)\,. \tag{4.5c}$$

where $\rho_0 = 1$, $\varphi_0 = |\psi_0|^2$.

Therefore we can expand the Lagrangian density for small amplitudes following the method in [32, 33].

$$L = \epsilon\,L^{(1)} + \epsilon^2\,L^{(2)} + \epsilon^3\,L^{(3)} + \mathcal{O}(\epsilon^4). \tag{4.6}$$

For ϵ:

$$L^{(1)} = -h^{(1)}_{\xi},$$

from where no relevant information is obtained.

For ϵ^2 :

$$L^{(2)}=\frac{1}{2}h^{(1)^2}_{\xi}-2\rho^{(1)}h^{(1)}_{\xi}+\rho^{(1)}\varphi^{(1)}+h^{(1)}_{\tau}-h^{(2)}_{\xi}-\frac{1}{2}\varphi^{(1)^2} \tag{4.7}$$

from where we have obtained the following expression as Euler-Lagrange equations

$$\delta\rho^{(1)}\ :\quad h^{(1)}_{\xi} = \varphi^{(1)}, \tag{4.8a}$$

$$\delta\varphi^{(1)}\ :\quad \rho^{(1)} = \varphi^{(1)}, \tag{4.8b}$$

$$\delta h^{(1)}\ :\quad \rho^{(1)} = h^{(1)}_{\xi}, \tag{4.8c}$$

and the relation

$$h^{(2)}_{\xi} = -\rho^{(1)}\, h^{(1)}_{\xi}. \tag{4.9}$$

The ϵ^3 final Lagrangian is obtained with help of (4.8) and (4.9) as

$$L^{(3)}=\frac{1}{2}h^{(1)^3}_{\xi} - \rho^{(2)}h^{(1)}_{\xi} + 2h^{(1)}_{\tau}h^{(1)}_{\xi}+\frac{\gamma}{8}h^{(1)^2}_{\xi\xi}+h^{(2)}_{\tau} \tag{4.10}$$

providing the condition

$$h^{(2)}_{\tau} = -\rho^{(2)}\, h^{(1)}_{\xi}. \tag{4.11}$$

where $\gamma = (1 - 4\,\sigma^2)$ is the optical analogue to surface tension [25]. Second-approximation terms $h^{(2)}$ and $\rho^{(2)}$ could be obtained and studied [77] using the expressions (4.9)-(4.11).

Assuming $u = h_x$, the preceding equation yields, as its Euler-Lagrange equation, a

KdV type [25, 78]

$$u_\tau + \frac{3}{2}\, u\, u_\xi - \frac{\gamma}{8}\, u_{\xi\xi\xi} = 0. \tag{4.12}$$

The solution of (6.24) is given by

$$\rho(\xi,\tau) \equiv u = N\, sech^2 \left[\sqrt{\frac{N}{2\,\gamma}} \left(\xi - \frac{N}{4}\,\tau \right) \right]. \tag{4.13}$$

where N is the soliton amplitude. In original coordinates

$$u(z,x) = N\, sech^2 \left\{ \frac{1}{4} \sqrt{\frac{\epsilon\, N}{\gamma}} \left[x - \left(1 + \frac{\epsilon N}{8} \right) z \right] \right\}, \tag{4.14}$$

and $h(z,x)$ can be obtained readily from (4.8c),

$$h = -\frac{4\gamma}{\epsilon} \sqrt{\frac{\epsilon N}{\gamma}} \tanh \left\{ \frac{1}{4} \sqrt{\frac{\epsilon N}{\gamma}} \left[x - \left(1 + \frac{\epsilon N}{8} \right) z \right] \right\}. \tag{4.15}$$

In the original (dimensionless) x and z, one may write down an approximate [up to order $\mathcal{O}(\epsilon)$] solution for the macroscopic wavefunction ψ

$$\psi = \psi_0 \sqrt{\rho_0 + \epsilon\, \rho_1} \;\; \exp\left[-i\, |\psi_0|^2 z + i\, h(z,x) \right], \tag{4.16}$$

$$\varphi = |\psi_0|^2 + \epsilon\, \varphi_1\,, \tag{4.17}$$

where φ_1 is written as (4.8b) and (4.14).

4.3 Conclusions

We have explored theoretically light propagation in a nonlocal nonlinear defocusing media through a proposed alternative simpler method, the asymptotic variational multiscale approach. The obtained KdV equation is similar to the one derived using reductive multiscale technique. Our results advance the understanding of nonlinear phenomena.

4.4 Acknowledgments

Artorix de la Cruz thanks the financial support from Killam Trust Predoctoral and Nova Scotia Research scholarships. This work was supported by NSERC of Canada and by the IT4Innovations National Supercomputing Center - Path to exascale project (EF16-013/0001791).

Chapter 5

(2+1)-dim Asymptotic variational theory for light propagating in a nonlocal nonlinear dissipative medium

This chapter's work is published in the paper entitled "(2+1)-dim Asymptotic variational theory for light propagating in a nonlocal nonlinear dissipative medium", Analytical and Numerical Methods in Differential Equations. 100th birthday of the prominent Russian mathematician and Academician Nikolai Nikolaevich Yanenko. Conference Proceedings: http://math.sut.ac.th/conference (July 2021). The authors are A.de la Cruz, T. Diaz-Chang, Ch. Liang, J. Pistora and M. Cada.

We propose and demonstrate analytically, within the framework of a hydrodynamic model, a novel and simpler variational approach to study the asymptotic behavior of a continuous wave (cw) laser beam propagating in a weakly absorbing defocusing nonlinear nonlocal media. The Kadomtsev-Petviashvili (KP) type equation is obtained. For the first time, to the best of our knowledge, the variational multiscale asymptotics method is used to describe nonlinear open systems.

My contribution to this work includes generating the fundamental ideas, performing the analytical description and the necessary analysis, and carrying out numerical simulations. I also wrote the first version of the paper.

5.1 Introduction

The starting point in the analysis is the light propagation in a weakly nonlocal nonlinear defocusing medium described by [22] normalized NLSE with a dissipative term and a diffusion-like equation for the response of the nonlocal medium

$$i\,\epsilon\,\frac{\partial\Psi}{\partial z}+\frac{\epsilon^2}{2}\nabla^2\,\Psi-\Theta\,\Psi=-i\,\epsilon\,\frac{\alpha}{2}\Psi\quad\text{and}\quad-\,\sigma^2\,\nabla^2\,\Theta+\Theta=|\Psi|^2,\tag{5.1}$$

where $\nabla^2=\partial_x^2+\partial_y^2$. The z and $\mathbf{r}=(x,y)$ are the spatial evolutionary variable and the transverse coordinates, respectively. Ψ is the complex electric field envelop peak intensity, Θ is a real function that denotes the nonlinear nonlocal change of the refractive index depending on the intensity. α is the intensity loss rate and $\epsilon \ll 1$ is a small quantity that deal with the weakly diffracting regime. The parameter σ is a spatial scale (setting the diffusion length) that measures the degree of nonlocality. A hydrodynamic model with the help of the Mandelung transformation $\Psi(z,\mathbf{r})=\rho^{1/2}(z,\mathbf{r})\,\exp[\,i\,h(z,\mathbf{r})\,]$. Both functions Ψ and Θ are assumed to be non-zero at the boundaries (infinities). Using $\Psi(z,\mathbf{r})=\psi_b(z)\,\psi(z,\mathbf{r})$ and $\Theta(z,\mathbf{r})=\theta_b(z)\,\varphi(z,\mathbf{r})$ in the above system of equations, the background equations $\psi_b(z)$ and $\theta_b(z)$ are to be determined as well.

5.2 Proposed theory and results

We propose that above system of equations can be derived from the appropriate Lagrangian density

$$L=\rho\left[\frac{(\nabla h)^2}{2}+\frac{\partial h}{\partial z}+\varphi-1\right]-\frac{1}{2}\left[\varphi^2+(\sigma\nabla\varphi)^2-1\right]+\frac{\left(\nabla\sqrt{\rho}\right)^2}{2}.\tag{5.2}$$

By means of the stretched variables $\xi=\epsilon^{1/2}\,(x-z)$, $\eta=\epsilon\,y$, $\tau=\epsilon^{3/2}\,z$, it is possible to write the Lagrangian as

$$L=\epsilon\,L^{(1)}+\epsilon^2\,L^{(2)}+\epsilon^3\,L^{(3)}+\mathcal{O}(\epsilon^4).\tag{5.3}$$

Introducing the variable transformations $\tau \to -(8\gamma)\,\tau$, $\eta \to (\sqrt{3|\gamma|/2})\,\eta$ and $u = -(\gamma/2)\,U$, we arrive to a KP equation [25]

$$\partial_\xi \left(U_\tau + 6\,U_\xi^2 + U_{\xi\xi\xi} + 3\,\varsigma^2\,U_{\eta\eta}\right) = 0. \tag{5.4}$$

where $\varsigma^2 = -\mathrm{sgn}\gamma$.

5.3 Conclusion

We have explored theoretically light propagation in a nonlocal nonlinear defocusing media through a proposed alternative simpler method, the asymptotic variational multiscale approach. The obtained Kadomtsev-Petviashvili equation is similar to the one derived using reductive multiscale technique.

Chapter 6

Small-Amplitude front due to laser radiation force in opaque colloid media

This chapter's work is published in the paper entitled "Small-Amplitude front due to laser radiation force in opaque colloid media", Physical Review A 96, 033850 (2017). I am the sole author.

We consider the evolution of small-amplitude waves generated by the interaction of a laser beam with nanoparticles dispersed in a liquid medium. Under the asymptotic multi-scale expansion framework and assuming a low concentration of beads, we have derived a dynamical system where a partial differential equation describes in the first approximation a wave propagation in a form of a kink shock wave. This front forms a depletion region with vanishing concentration of beads which consequently allows the light propagation through the medium. The possible presence of absorption in the system could be shown through the complex expressions for the phase and group velocities in the case of a linear propagation of waves.

My contribution to this work includes generating the fundamental ideas, performing the analytical description and the necessary analysis, and carrying out numerical simulations. I also wrote the paper.

6.1 Introduction

The study of complex media and its out-of-equilibrium dynamics, through nonlinear optics experiments and theoretical models, has enjoyed an intense period of activity over the last years [21, 79, 80] after the pioneering work of A. Ashkin in Ref. [10], and other authors [11, 12, 34]. The general problem of a beam propagation and scattering in a turbid medium is important to fields such as biology and medical imaging as a diagnostic tool [35, 36]. Specially due to the potential to acquire information non-invasively through the sample's optical properties, especially the generation and detection of travelling waves.

Recent theoretical and experimental investigations have described the generation and propagation of shock waves in non-local and disorded media in response to an incident laser beam [14, 37, 38]. An important contribution to this subject has been made by the authors in Ref. [39] where some experiments and their respective theory on optical manipulation of the local properties of dense, particulate-loaded, highly-scattering(opaque) suspensions of dielectric nanoparticles in a liquid were introduced. The study, which was done for the self-focusing case, has proven that multiple-scattered light can give rise to concentration shock fronts propagating deep inside the opaque suspensions.

These particle density ρ shock waves are primarily the result of the interplay between the two components of the applied radiation force, namely the scattering and the optical gradient forces respectively. The first one represents the momentum transfer from the external radiation field to the nanoparticle by scattering and absorption, and is pointing along the axis of the energy flux of the light beam, whereas the gradient force is directed along the intensity gradient of the beam. To characterize this kind of behavior observed experimentally, a mathematical description was developed, resulting in a nonlinear Burger's-type equation. This kind of partial differential equation(PDE) is well known to support shock-like waves which applies for both focusing and defocusing cases.

Some research groups [81, 82, 83] have proposed a new class of synthetic colloidal suspensions capable of exhibiting negative polarizabilities (defocusing case). They obtained robust propagation and enhanced transmission of self-trapped light over long distances that would have been otherwise impossible in conventional suspensions with positive polarizabilities. However, no asymptotic analysis to study the long-evolution in time and space of the concentration wave related disturbance has been reported under the conditions introduced in Refs.[39, 81, 82, 83].

In Ref. [22] the research work have put emphasis on nonlinear optical propagation of laser beam in a dye-doped strongly absorbing nanoscale colloid. The authors predicted and studied the absorption and temperature effects on the formation and propagation of a matter-shock wave. They have used optically induced thermodiffusion to explain the formation of the ρ shock and have arrived also to a Burger's-like wave equation. This temperature gradient induces a drift velocity on the nanoparticle density ρ since near the optical field the density travels faster than the surrounding regions.

Conversely to the assertion done in [22], this paper has assumed a very low or non-absorbing colloids. This condition allows to rule out any thermal effects in our computation. An example could be the Ref. [39] experimental setup but in our case with nanoparticle beads with negative polarisability. Attention is given to a particular circumstance where dispersive and nonlinear effects are present during the wave propagation as well as their role in its long-term evolution. We study a kink wave formation in the out-of-equilibrium nanoscale particles self-defocusing medium under the influence of a laser excitation. We have proved that Burger's-like equations do not describe the asymptotic behaviour of the colloid nanoparticle dynamics. The analytic expressions for the wave envelope and its corresponding propagation speed that support our claim are obtained. We show that at low background concentration of particles(that is, concentration in absence of laser light) a front shock could exist asymptotically and in the absence of a temperature gradient based diffusion. It is also shown that this light-induced disturbance produces a density depletion in the material, which correspondingly becomes transparent and allows light propagation.

As mentioned before, the necessary condition for the existence and propagation of these kind of waves is found to be a low nanoparticles density ($0 \sim \rho << 1$). Experimentally this regime corresponds to the defocusing one.

To develop our theoretical model we have stated a system of equations from first physical principles and studied it in the mathematical framework of the asymptotic analysis, specifically the method of multiscale expansions [84, 85]. This leads to the identification and derivation of a dynamical regime where a partial differential equation governs the motion of the first-order wave velocity and particle density corrections.

6.2 Physical Model

6.2.1 Governing system of equations

In our problem we have assumed a dielectric nanosized colloidal beads concentration ρ per unit of volume immersed in a liquid solution such that the dielectric particles and the surrounding liquid refractive indices $n_{particle}$ and n respectively, satisfy the relation $(n_{particle}/\, n) < 1$. The real part of the particle response to a radiation field(polarisability) is $\alpha' < 0$ which corresponds to the self-defocusing case where the particles expelling them from the large light energy density region. As observed and reported, the scattered light induces localized, directional spearhead shaped, concentration shock-fronts propagating distances of several photon transport MFPs into the fluid. We have studied the radiation force acting on the nanoparticles and the wave propagation in the z-direction. Consequently, we have considered a one-dimensional geometry which can be justified by assuming the wave propagation in a thin microfluidic channel experimental setup [39].

The system is illuminated by a laser with intensity $I = \vec{E} \cdot \vec{E}^{*}$, where $\vec{E}$ accounts for the incident electric field. The nanoparticles are under the influence of the two types of the radiation forces: scattering ($\vec{F}_{scat}$) and gradient($\vec{F}_{grad}$) ones. In the case of the scattering one, its magnitude is proportional to the light intensity, and its direction is along the time-average Poynting vector $< \vec{S} >$. For simplicity, $< \vec{S} >$ will be in the z-direction. Then

$\vec{F}_{scat}$ on one particle is expressed as

$$\vec{F}_{scat} = \hbar\omega\,\sigma_s(1-g)\,\vec{J}_{ph}/\tilde{c}, \tag{6.1}$$

where $\hbar$ is the Plank constant divided by 2π and $\sigma_s(1-g)$ is the particle scattering cross-section with g being the anisotropy factor, which is a measure of the isotropy of the scattering profile. The quantities $\tilde{c}$ and ω are the speed of light in the medium and the frequency of light respectively.

In Eq.(6.1) $\vec{J}_{ph} = -\tilde{c}\,\vec{\nabla}\tilde{\varphi}/[3\rho\,\sigma_s(1-g)]$ accounts for the photon flux under the assumption that the wave transport in random media acquires diffusive behaviour due to multiple scattering and satisfies the so-called first Flick's law $\partial\tilde{\varphi}/\partial t + \vec{\nabla}\cdot\vec{J}_{ph} = 0$. The expression that relates the light intensity I with the photon density $\tilde{\varphi}$ is given by $\varepsilon_0\,n^2\,I/2 = \hbar\omega\,\tilde{\varphi}$.

As mentioned above, $\vec{F}_{grad}$ is the other component of the radiation force acting on a dielectric nanoparticle. This gradient force is due to the Lorentz force influence on the dipole induced by non-uniform electromagnetic fields on the particles. The direction of $\vec{F}_{grad}$ is along the gradient of the light intensity I. Therefore, $\vec{F}_{grad}$ is expressed as

$$\vec{F}_{grad} = \frac{1}{4}\,\alpha'\,\vec{\nabla}E^2 \equiv \frac{\alpha'\,\hbar\omega}{2\,\varepsilon_0\,n^2}\,\vec{\nabla}\tilde{\varphi}, \tag{6.2}$$

Finally, in Ref. [39] the authors have obtained an expression, which is also valid in our case, that relates the photon density with the nanoparticles concentration ρ. Based on the assumption that the $\tilde{\varphi}$ always reaches a steady state much faster than ρ, we can write

$$\frac{\partial\tilde{\varphi}}{\partial z} = \frac{3\,v_{drag}}{\hbar\omega\,\mu_m}\,\rho\,, \tag{6.3}$$

where v_{drag} is the drag velocity and μ_m is the particle mobility. To introduce our governing system of equations in this theoretical model we first take α' negative, although we get both forces($\vec{F}_{scat}, \vec{F}_{grad}$) acting in the same direction. Therefore, we can state the equation of motion for a nanoparticle of mass m and velocity $\vec{v}$ as

$$m\,\frac{d\vec{v}}{dt} \;=\; \vec{F}_{scat} + \vec{F}_{grad}\,, \tag{6.4}$$

or due to equations (6.1) and (6.2) as

$$\frac{dv}{dt} = \frac{\hbar\omega}{3\,m}\;\frac{1}{\rho}\;\frac{\partial\tilde{\varphi}}{\partial z} + \frac{1}{2}\;\frac{\alpha'\,\hbar\omega}{m\,\varepsilon_0 n^2}\,\frac{\partial\tilde{\varphi}}{\partial z}\,. \tag{6.5}$$

Also the continuity equation for ρ in the colloid is given by $\partial\rho/\partial t + \vec{\nabla}\cdot(\rho\vec{v}\,) = 0$, and together with (6.3), the normalized and dimensionless equations read

$$\frac{\partial\tilde{\varphi}}{\partial z} \;=\; \rho\;, \tag{6.6}$$

$$\frac{\partial\,\rho}{\partial t} \;=\; -\frac{\partial}{\partial\,z}\,(\rho\,v), \tag{6.7}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} \;=\; \frac{1}{\rho}\,\frac{\partial\tilde{\varphi}}{\partial z} + \frac{\partial\tilde{\varphi}}{\partial z}\;, \tag{6.8}$$

where we have used the convective derivative $d/dt = \partial/\,\partial t + v\,\partial/\partial z$ for dv/dt and have performed the following scalings $v = v_{drag}\,v'$, $\tilde{\varphi} = \varphi_0\,\tilde{\varphi}'$, $z = z_0\,z'$, $z_0 = k\,w_0^2$, $k = 2\pi n/\lambda$ where λ is the wavelength, k is the wave number. $t = t_0\,t'$, $t_0 = z_0/v_{drag}$, and $\rho = \rho_0\,\rho'$. Also we have defined the relations $m = \hbar\omega\,\rho_0\,\varphi_0\,t_0/(3\,z_0\,v_{drag})$, $\alpha = (2/3)\,\rho_0\,\varepsilon_0\,n^2$ and $\mu_m = 3\,v_{drag}\,z_0\,\rho_0/(\hbar\,\omega\,\varphi_0)$.

Assuming polarization in the x-direction and propagation in the $+z$ direction for an applied Gaussian beam (TEM_{00}) with an incident intensity $I_0 = P_0/\pi\,w_0^2$ where P_0 and w_0 are the incident laser power and the input gaussian waist, respectively. From now on, paying attention only to the field on the $z-$axis, we can rewrite

$$\frac{\varepsilon_0\, n^2}{2}\, I(0,z) = \hbar\,\omega\, \frac{\tilde{\varphi}}{\varphi_0}, \tag{6.9}$$

where the light $z-$dependents intensity and spot-size are

$$\begin{aligned} I(r,z) &= I_0\, \frac{w_0}{w(z)}\, e^{-2\, r^2/w^2(z)}\,, && (6.10)\\ w(z) &= w_0\left[1 + (z/z_R)^2\right]^{1/2}. && (6.11)\end{aligned}$$

The $z_R = \pi\, w_0^2/\lambda$ is called the Rayleigh range. $\varphi_0 = \varepsilon_0\, n^2\, P_0/(2\pi\,\omega\,\hbar\, w_0^2)$. The smallness parameter $\epsilon = [w_0^2/(2^{1/2}\, z_R)]^{1/2}$ will allows to study the v and ρ evolutions in the weakly diffracting regime by the multiscale asymptotic analysis. To complete our mathematical description, we have imposed the boundary conditions that for $z \to \pm\infty$, $\rho \sim \rho_0$ and $v = 0$ since the particles are only affected around the localized propagating disturbance. Finally, we have dropped the tilde for the sake of clarity.

6.2.2 Dispersion relation

The dispersion relation for linear wave propagation can be obtained if we restrict the system of equations(6.6-6.8) to solutions which are close to the stationary equilibrium state. To linearize we substitute in the equations the expressions $v(z,t) = v_0 + \delta v(z,t)$, $\rho(z,t) = \bar{\rho} + \delta\rho(z,t)$ and $\tilde{\varphi}(z,t) = \tilde{\varphi}_0 + \delta\tilde{\varphi}(z,t)$. If the nonlinear terms are neglected, it yields the expression

$$\frac{\partial^2\, \delta v}{\partial t^2} = \varrho\, \frac{\partial\, \delta v}{\partial z}\,, \tag{6.12}$$

where $\varrho = -(\bar{\rho} + 1)$. Assuming the perturbed values δv with dependency of the form $e^{i(kz-\omega t)}$ and $\bar{\rho} = 1$, the condition on the allowed values k and ω is

$$\omega^2 = i\,2\,k\ , \tag{6.13}$$

Using the relation $\sqrt{i} = \pm(i+1)/\sqrt{2}$, we rewrite Eq.(6.13) as $\omega = \pm\,(1+i)\,k^{1/2}$ or $\omega = \omega_R + i\,\omega_I$, where $\omega_R \equiv \omega_I = \pm k^{1/2}$. From the above expressions it is possible to observe that the linearized system allows the propagation of a sinusoidal wave with frequency ω_R. Notice that at the same time the wave is getting damped with coefficient ω_I. The phase velocity is

$$c_p \ \equiv\ \omega/k = (1+i)\ k^{-1/2}, \tag{6.14}$$

or $(1+i)\,(\lambda/2\,\pi)^{1/2}$. The dependency of the wave velocity on the square root of λ implies a very important variation over the range of wavelengths of interest.

The group velocity is a complex number $c_g \equiv d\omega/dk = v_{gr} + i\,v_{gi}$ given by $c_g = c_p/2$. Since $v_{gi} \equiv \mathrm{Im}(d\,\omega/\,dk) \neq 0$ the medium is not dissipation free. Complex group velocity is common in absorbing and active media in general, yet its precise physical meaning is unclear. Conversely to the case of a nondissipative medium the group velocity of the propagating waves is exactly equal to the observable energy velocity defined as the ratio between the energy flux and the total energy density. In a dissipative medium v_g cannot be recognized as the velocity of energy transport, and it may contain information about the wave energy absorption in the medium [86].

The dispersion relation $k \propto \omega^2$ is obtained from a similar expression to Eq.(6.12) in the case of gravity waves on deep water or as a limit case for surface water waves in [87] and [88].

6.3 Small-Amplitude Waves Propagation

6.3.1 Nanoparticle dynamics

To account for the slow variation of the wave-form due to nonlinear and dispersive(or dissipative) effects we have applied a multiscale asymptotic analysis by introducing in Eqs(6.6-6.8), after [89], a long time and space scale transformations for the independent variables; i.e., $\xi = \epsilon^{1/2}(z-t)$ and $\eta = \epsilon^{3/2}t$, where the respective derivatives take the form $\partial/\partial z = \epsilon^{1/2}\,\partial/\partial\xi$ and $\partial/\partial t = -\,\epsilon^{1/2}\,\partial/\partial\xi + \epsilon^{3/2}\,\partial/\partial\eta$. Thus, for a finite ϵ with $0 < \epsilon \lesssim 1$, one can observe slow variations of the wave amplitude in the moving frame of reference. Then the dimensionless system gets the form,

$$\epsilon^{1/2}\,\frac{\partial\tilde{\varphi}}{\partial\xi} = \rho\ , \tag{6.15}$$

$$\frac{\partial(\rho v)}{\partial\,\xi} + \epsilon\,\frac{\partial\rho}{\partial\eta} = \frac{\partial\rho}{\partial\xi}\ , \tag{6.16}$$

$$-\,\rho\left(\frac{\partial v}{\partial\xi} + \frac{\partial\tilde{\varphi}}{\partial\xi}\right) + \rho\, v\,\frac{\partial v}{\partial\xi} + \rho\,\epsilon\,\frac{\partial v}{\partial\eta} = \frac{\partial\tilde{\varphi}}{\partial\xi}. \tag{6.17}$$

The Eq.(6.17) exhibits different scaling behaviours depending on the balance between its different terms. These are the inertia and the scattering forces on the left-hand side and the gradient force on the right-hand side. With the goal to analyse the most general case, we are going to choose the scaling in a way that takes into account all the forces introduced here to describe the dynamical system.

We continue our analysis by assuming the deviation from a distribution of particles is small. This allows to expand the dependent variables as $\rho = \sum_{j=0}^{\infty}\,\epsilon^j\,\rho^{(j)},\ \ v = \sum_{j=1}^{\infty}\,\epsilon^j\,v^{(j)},\ \ \tilde{\varphi} = \sum_{j=1}^{\infty}\,\epsilon^j\,\tilde{\varphi}^{(j)}$, where $\rho^{(0)} = \epsilon^{1/2}$, which is consistent with our assumption of low particle density.

Then, from Eqs.(6.15-6.17) we get the relations for the first order in ϵ as

$$\frac{\partial\,\rho^{(1)}}{\partial\xi}=\frac{\partial\,v^{(1)}}{\partial\xi} \;=\; -\frac{\partial\,\tilde{\varphi}^{(1)}}{\partial\xi}, \tag{6.18}$$

$$\frac{\partial\tilde{\varphi}^{(1)}}{\partial\xi}-\rho^{(1)} \;=\; 0, \tag{6.19}$$

where by taking into account the boundary conditions we have $\rho^{(1)}=v^{(1)}=-\,\tilde{\varphi}^{(1)}$.

For the term of order $\epsilon^{3/2}$

$$v^{(1)}\frac{\partial v^{(1)}}{\partial\xi}-\rho^{(1)}\frac{\partial v^{(1)}}{\partial\xi}-\rho^{(1)}\frac{\partial\tilde{\varphi}^{(1)}}{\partial\xi}=\frac{\partial\tilde{\varphi}^{(2)}}{\partial\xi}. \tag{6.20}$$

Another result is the system of equations from terms of order ϵ^2,

$$\frac{\partial v^{(2)}}{\partial\xi} \;=\; \frac{\partial\rho^{(2)}}{\partial\xi}-\frac{\partial\rho^{(1)}}{\partial\eta}, \tag{6.21}$$

$$\rho^{(1)}\,v^{(1)}\frac{\partial v^{(1)}}{\partial\xi}-\frac{\partial v^{(2)}}{\partial\xi} \;=\; \frac{\partial\tilde{\varphi}^{(2)}}{\partial\xi}-\frac{\partial v^{(1)}}{\partial\eta}, \tag{6.22}$$

$$\frac{\partial\tilde{\varphi}^{(2)}}{\partial\xi} \;=\; \rho^{(2)}. \tag{6.23}$$

Combining the equations from ϵ, $\epsilon^{3/2}$ and ϵ^2 we have arrived to a nonlinear PDE for $v^{(1)}$

$$2\,\frac{\partial v}{\partial\eta}+v\frac{\partial^2 v}{\partial\xi^2}+v\,(1+v)\,\frac{\partial v}{\partial\xi}+\left(\frac{\partial v}{\partial\xi}\right)^2=0, \tag{6.24}$$

where $v\equiv v^{(1)}$.

This expression can be rearranged into

$$2\,\frac{\partial v}{\partial\eta}+v\,\frac{\partial v}{\partial\xi}+v^2\frac{\partial v}{\partial\xi}+\frac{\partial}{\partial\xi}\left(v\,\frac{\partial v}{\partial\xi}\right)=0. \tag{6.25}$$

The nonlinearity of Eq.(6.25) is an inherent property of the interaction of light with light-scattering suspensions. In detail it derives from the optical gradient frame acting on the particles which at the same time scatter the optical field diffusively.

6.3.2 Kink-shape front velocity

Let us look for travelling wave solutions by assuming $v(\xi, \eta) = v(\zeta)$, where $\zeta = \xi - \beta\,\eta$ and β is a constant arbitrary velocity of propagation. Integrating the equation in the co-moving frame Eq.(6.25) reads

$$-2\,\beta\,v + \frac{v^2}{2} + \frac{v^3}{3} + v\,\frac{dv}{d\zeta} = B, \tag{6.26}$$

where the integration constant becomes $B = 0$ under the assumption that we are looking for bounded solutions of Eq.(6.25).

$$\begin{aligned} \frac{dv}{d\zeta} &= 2\beta - \frac{v}{2} - \frac{v^2}{3} \\ &= -\frac{1}{3}\left[\left(v + \frac{3}{4}\right)^2 - \frac{1}{2}\left(12\,\beta + \frac{9}{8}\right)\right]. \end{aligned} \tag{6.27}$$

Now we can rewrite the last equation as

$$\frac{dv}{(v-a)^2 - b^2} = -\frac{1}{3}\,d\,\zeta, \tag{6.28}$$

where $a = -3/4$ and $b = [(1/2)(12\,\beta + 9/8)]^{1/2}$.

Integrating both parts and using

$$\int \frac{dv}{(v-a)^2 - b^2} = \frac{1}{2b}\ln\left|\frac{v-a-b}{v-a+b}\right|\ , \tag{6.29}$$

it yields

$$v = \frac{v_2 + v_1\, e^{-K\,\zeta}}{(1 + e^{-K\,\zeta})}, \tag{6.30}$$

where the leading edge of the wavefront velocity $v_2 = a+b$ and the trailing edge $v_1 = a-b$, $K = 2\,b/3$ and have taken into account that $v_2 > v > v_1$. Note that $v(\infty) = v_2$ and $v(-\infty) = v_1 \equiv 0$. This last expression is based on the relation $v^{(1)} = \rho^{(1)} \geq 0$ since the particle density can never be negative.

The function (6.30) is called a kink or rarefaction wave, a similar structure can be seen in the solution to the Sine-Gordon equation. It clearly represents a kink wavefront moving with velocity $\beta = (v_2 - v_1)/2$ in the $(\xi, \eta)-$coordinates reference. Then using the identity

$$\frac{2\,e^{-z/2}}{e^{z/2} + e^{-z/2}} = 1 - \tanh\left(z/2\right), \tag{6.31}$$

allows us to write the solution as

$$v(\xi, \eta) \;\; = \;\; \beta\,\left(1 \,+\, \tanh\left[\beta\,\left(\xi - \beta\,\eta\right)\right]\right), \tag{6.32}$$

from where by the assumption that the disturbance moves with speed equal to the average of its asymptotic values, $16\,\beta\,(4\,\beta - 3) = 0$, which gives the speed velocity $\beta = 3/4$. Equation (6.32) represents a kink, which is physically a truly localized excitation since all the energy and momentum associated with this wave are centered around the disturbance location. This kind of waves travels to the right without loss of shape or speed and its presence is supported by the continuous-wave background applied laser beam.

The kink-shape front in the material forms a depletion region with vanishing concentration of colloids beads as shown by the curve's tail. This creates a zone that allows the light propagation through the otherwise forbidden in the linear regime by an absorbing medium. Similar behaviour can be observed in the defocusing early shock formation experiments

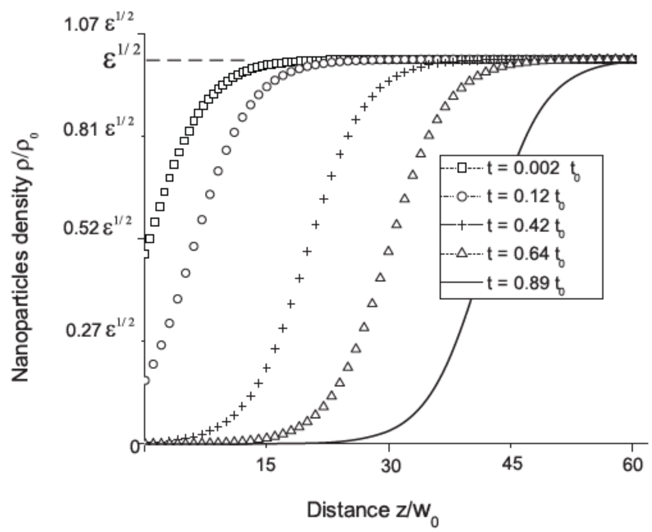

Figure 6.1: Dimensionless particle density versus z, normalized by w_0. The envelope is a travelling wave driven by the scattering light. Correspondingly, the material becomes transparent to the optical field, whose propagation is forbidden in the linear regime by strong absorption. The values used are $w_0 = 10\,\mu m$, $v_{drag} = 400\,\mu m/s$, $t_0 = 47ms$ and $\lambda = 532\,nm$.

and simulations [22]. Consequently, as the matter front advances, the beam enters the material that becomes progressively transparent.

Fig.1 shows the asymptotic behaviour of the density profile at several different propagation time. The scattering effects produce a depletion region (where $\rho << 1$), which induces an early front wave formation.As the density reaches the value $\rho = 0$, the shock front starts moving in the material. Figure 2 represents the nanoparticle density time variation at an specific distance z/w_0. The matter-shock front advances inside the colloid it becomes progressively transparent.

Consistent with the assumption of small amplitude waves in our analysis, the kink-shape shock does not present that sudden increment [22] in concentration but a monotonically decrease of the function. Now, taking into account the scaling relations from ξ and η

given above we can write

$$v = \beta\, v_{drag} \left(1 + \tanh \left[\frac{\beta\, \epsilon^{1/2}}{z_0} \{ z - v_{drag}\, t\, (1 + \epsilon\beta) \} \right] \right), \tag{6.33}$$

$A = \beta\, v_{drag}$ is a measure of the wave amplitude and $\beta\, \epsilon^{1/2}/z_0$ is a measure of the steepens of the wave.

Thus the wave profile takes the form

$$v \;=\; A \left(1 + \tanh \left[\frac{\epsilon^{1/2}\, \beta^2}{A\, t_0}\, (z - c_{wave}\, t) \right] \right), \tag{6.34}$$

where the wave velocity is

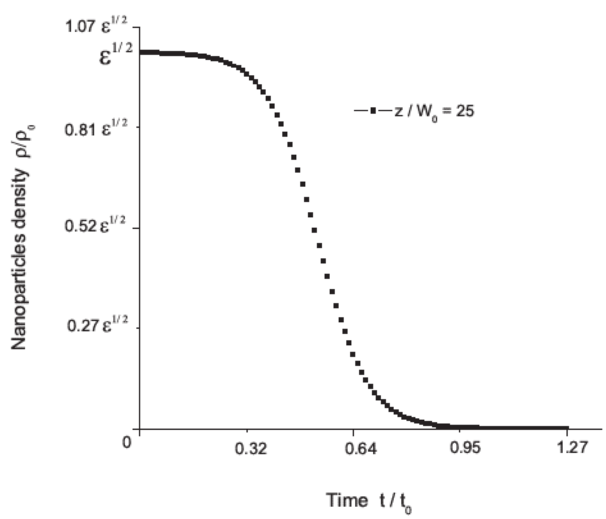

Figure 6.2: Density kink wave propagation versus time at fixed distance z/w_0.

$$c_{wave} \;=\; v_{drag} + \epsilon\, A. \tag{6.35}$$

It follows from Eqs.(6.33) and (6.34) that the amplitude and steepening of the newly discovered kink depend on the wave velocity. Thus larger amplitude waves move with a higher speed than smaller amplitude ones. The wave steepen depends on the inverse of the amplitude. The lower the speed is, the more $v(z,t)$ tends to a step function and the transition layer between the two asymptotic values of the solution becomes sharper. Therefore, taller shock waves are narrower in steepens and travel faster than shorter ones.

In the idealized limit of no dissipative energy loss, these waves propagate without degradation of shape; they are indeed the natural stable, localized modes for propagation in the colloid. In real conditions where dissipative mechanisms cause waves to lose energy, the individual kinks therefore broaden.

Finally, it is worth to mention that light experiences various dynamic phases during propagation in a nonlocal nonlinear medium. The asymptotic evolution problem of a wave is dependent on its initial conditions such as the beam waist, power, etc.

As the nonlinear wave propagates, its profile evolution ca be studied in terms of the initial critical power level. In other words, by properly choosing the experimental conditions, we can establish the occurrence of different kinds of small-amplitude nonlinear waves on top of the background by using the corresponding initial conditions. Multiscale expansion is an important mathematical tool to study the continuous transition from shock waves, Benny-Luke waves and Solitons.

6.4 Conclusion

In conclusion, we have investigated the evolution and nonlinear properties of small-amplitude kink-like shock waves in colloids under the asymptotic multiscale expansion framework and assuming a low concentration of beads. The system behavior changes to a regime that allows beam propagation. We have derived the dispersion relation for linear propagation of waves in the colloid. We consider the evolution of small-amplitude waves generated by the interaction of a laser beam with nanoparticles dispersed in a liquid medium.

Chapter 7

Electronically Controlled Polarization Beat Length in Kerr Nonlinear Media

This chapter's work is published in the paper entitled "Electronically Controlled Polarization Beat Length in Kerr Nonlinear Media", Results in Physics, Vol 25, pp. 104232, 2021. The authors are Artorix de la Cruz, Montasir Qasymeh, Jaromir Pistora, and Michael Cada.

The polarization beat length of propagating optical fields in nonlinear birefringent Kerr medium is investigated in the presence of an externally applied DC electric field. We show that the critical power, at which the effective polarization beat length becomes infinite, can be controlled through adjusting the externally applied electric field. The principle of operation is based on modifying the polarization instability by electronically adjusting the effective birefringence through an external electrical bias. The presented analytical expressions describe the beat length and the polarization instability as a function of the applied electric field for an arbitrary optical input state.

My contribution to this work includes generating the fundamental ideas, performing the analytical description and the necessary analysis, and carrying out numerical simulations. I also wrote a first draft of the paper.

7.1 Introduction

Polarization instability in a medium arises when the nonlinear change of the refractive index is comparable with the linear birefringence. This phenomenon manifests when the nonlinear birefringence cancels completely the linear birefringence and the beat length escalates to infinity. Physically, the beat length (L_B^{eff}) is the length at which the optical power is transferred from one polarization to another. In a nonlinear medium, such as the Kerr medium, the L_B^{eff} length becomes infinite at a critical input power for a propagating light that is polarized along the fast axis [20, 40, 41]. It then follows that a substantial change in the output polarization state is observed when the input power (or its polarization state) is slightly differing.

Controlling the polarization dynamics and obtaining non-trivial polarization evolution is vital [43, 44, 45, 90, 91, 92, 93, 94, 95, 96, 97, 98] to optimize the operation of several photonic devices [42]. These include the birefringent optical fibers (BOFs), [99, 100, 101, 102, 103], the multimode interference (MMI) couplers [104], the Y-branches [105] and also, the integrated photonic circuits [106], specially the electric-field-induced second harmonic generation (EFISHG) could be considered as a practical possibility, in integrated photonics, due to the fact that the nonlinear susceptibility χ^3 in silicon is two order of magnitude larger than in silicon oxide, and that in integrated photonics the non-linear modal area is reduced by a large factor when compared to typical optical fibers [107]. This keeps the electric fields required below the silicon breakdown, although not too far from it. Another favourable condition of integrated devices is that the required field may be produced across a small distance (few microns), thus avoiding the requirement of high voltage components [108]. Interestingly, for propagating optical fields in non-resonant Kerr nonlinear medium, a biasing electric field induces birefringence even if the medium is optically isotropic [46]. In [47], the authors have studied the impact of applying a DC electric field (i.e., E_{ext}), to a third-order nonlinear medium, on the evolution of propagating optical waves. They found that the polarization evolution can be controlled by the applied E_{ext} field. As a matter of fact, the E_{ext} field turns the third-order nonlinearity into a second-order-like as if one deals with an electro-optic-like effect.

While these effects in a nonlinear and birefringent medium have been known for long, and examined in details [47], the polarization instability in nonlinear medium with the presence of externally applied DC electric field has received little attention. Both the L_B, which is the beat length when nonlinear optical effects are neglected, and the L_B^{eff} are important quantities that must be characterized in fibers and waveguides. For instance, reducing these lengths can improve the stability of the optical system and enhance the communication capacity significantly [109]. However, in practice, these lengths are fixed once the geometry, the materials, and the input power are selected. Thus, a limited capability to adaptively designing/monitoring the performance of the pertinent optical systems is experienced.

In this letter, we present a theoretical description for electronically controlled polarization instability [47]. The considered scheme implies adjusting the critical power (at which the polarization instability takes place) through modifying the effective birefringence by applying an external electric field E_{ext}. We have carried out analytical expressions that relate L_B and L_B^{eff} with the DC applied field. The derived expression shows that both the critical power and the effective beat length can be arbitrarily shifted by adjusting the applied DC electric field.

7.2 Mathematical framework

The theoretical analysis begins by deriving the governing coupled differential equations of the evolution of the optical field in Kerr nonlinear medium while an external electric field is applied. On assuming a slow-varying-envelope approximation (whereby the second derivatives are neglected), considering harmonic fields, and omitting the transverse variations, a well-known first-order differential equation relating the optical electric field and the polarization is obtained, given by [47]:

$$\frac{\partial \vec{E}}{\partial z} = -i\,\frac{k}{2\,\varepsilon}\,\vec{P}_{NL}, \tag{7.1}$$

where $\vec{E}$ is the optical electric vector field, k is the propagation constant, ε is the material permittivity, and $\vec{P}_{NL}$ is the nonlinear polarization vector.

In the following, without losing any aspect of generality, we analyze the $x-$polarization component while it is coupled to the $y-$polarization component. It then follows that the nonlinear polarization $P_{x_{NL}}$ is given by:

$$P_{x_{NL}} = \epsilon_0 \chi \left[\frac{A_x}{4} |A_{x,y}|^2 + \frac{A_x^*}{8} A_{x,y}^2 + \frac{3}{8} A_x E_{ext}^2 \right], \tag{7.2}$$

where A_x and A_y are the complex amplitudes of two orthogonal modes (in case of an optical fiber) or TE modes (in case of planar waveguides), ε_0 is the vacuum permittivity, and $\chi \equiv \chi^{(3)}$ is the nonlinear susceptibility.

Substituting (7.2) into (7.1) yields the spatial evolution of the polarization state, given by:

$$\frac{1}{A_x} \frac{\partial A_x}{\partial z} = i\gamma \left[|A_x|^2 + |A_y|^2 + 4E_{ext}^2 + \frac{1}{3} |A_y|^2 \left(\frac{A_x^* A_y}{A_x A_y^*} - 1 \right) \right], \tag{7.3}$$

Here, A and E_{ext} are normalized such that $|A|^2$ and E_{ext}^2 are in power unit (i.e., W). The parameter $\gamma = 3\chi k_0/(8n_{_L})$ and $n_L = \sqrt{\varepsilon_0(1+\chi^{(1)})}$, where $\chi^{(1)}$ is the linear susceptibility.

At this point, we propose to re-write (7.3) in the following form:

$$\frac{\partial A_x}{\partial z} = i\,\gamma \left[\left(|A_x|^2 + \frac{2}{3} |A_y|^2 \right) A_x + 4\, E_{ext}^2\, A_y + \frac{1}{3} \frac{A_x^*\, A_y}{A_x\, A_y^*} |A_y|^2\, A_x \right], \tag{7.4}$$

here we have $(A_x^* A_y / A_x A_y^*)|A_y|^2 A_x = A_y^2\, A_x^*$, yielding:

$$\begin{aligned} \frac{\partial A_x}{\partial z} &= i\gamma \left[\left(|A_x|^2 + \frac{2}{3} |A_y|^2 \right) A_x + \frac{1}{3} A_y^2\, A_x^* + 4\, E_{ext}^2 A_x \right], \\ \frac{\partial A_y}{\partial z} &= i\gamma \left[\left(|A_y|^2 + \frac{2}{3} |A_x|^2 \right) A_y + \frac{1}{3} A_x^2\, A_y^* + \frac{4}{3}\, E_{ext}^2 A_y \right]. \end{aligned} \tag{7.5}$$

The above two equations can be expressed in the circular polarization bases using the following transformation:

$$A_1 = \frac{A_x + i A_y}{\sqrt{2}} \exp\left(-i\frac{8}{3}\kappa_{ext}\, z\right),$$

$$A_2 = \frac{A_x - i A_y}{\sqrt{2}} \exp\left(-i\frac{8}{3}\kappa_{ext}\, z\right). \tag{7.6}$$

where $\kappa_{ext} = \gamma\, E_{ext}^2$. It then follows that the evolution equations are given by:

$$i\,\frac{\partial A_s(z)}{\partial z} = \kappa_{ext}\, A_{3-s} + \frac{2\,\gamma}{3}\left(|A_s|^2 + \frac{2}{3}|A_{3-s}|^2\right) A_s\,, \tag{7.7}$$

where $s = 1, 2$ pertains to the right 1 and left 2 circular polarization. This equation governs the self-induced polarization rotation- and the polarization instability, resulting from a subtle balance between linear birefringence and self- as well as cross-phase modulation. This is a significant result, which is a generalization of [20, 110, 111, 112]. Hereby, the polarization dynamics in this regime are controlled by a static electric field. Here, κ_{ext} (which is in m^{-1} unit) is a controlled parameter that is a function of the applied DC electric field. For $E_{ext} = 0$, the equations in (7.7) are identical to those in [110, 111, 112] which governs nondispersive cross-phase modulation (XPM) in birefringent fibers. The solutions in [110, 111, 112] are also applicable for short pulses (i.e., 100 ps) given that the fiber length is adequately shorter than the dispersion length and the walk-off length [20].

The polarization state is determined by the complex ratio $\xi = A_1/A_2$. The azimuth of the polarization ellipse is $\theta = (1/2)\arg(\xi)$. We consider an input beam linearly polarized at angle θ_0 with respect to the slow axis. Thus, the slow axis is represented by $\theta_0 = 0^o$ and the fast axis is represented by $\theta_0 = 90^o$.

The solutions of (7.7) can be sought in the form of $A_s = (P_{cr}\, p_s)^{1/2}\ \exp(i\, 2\theta_0)$ [40, 20], where for convenience we have defined the normalization parameter by $P_{cr} = 3\, E_{ext}^2$. Here, p_s are the normalized power in the s mode satisfying $p \equiv P_0/P_{cr} = p_s + p_{(3-s)}$, where P_0

is the total power launched into the medium.

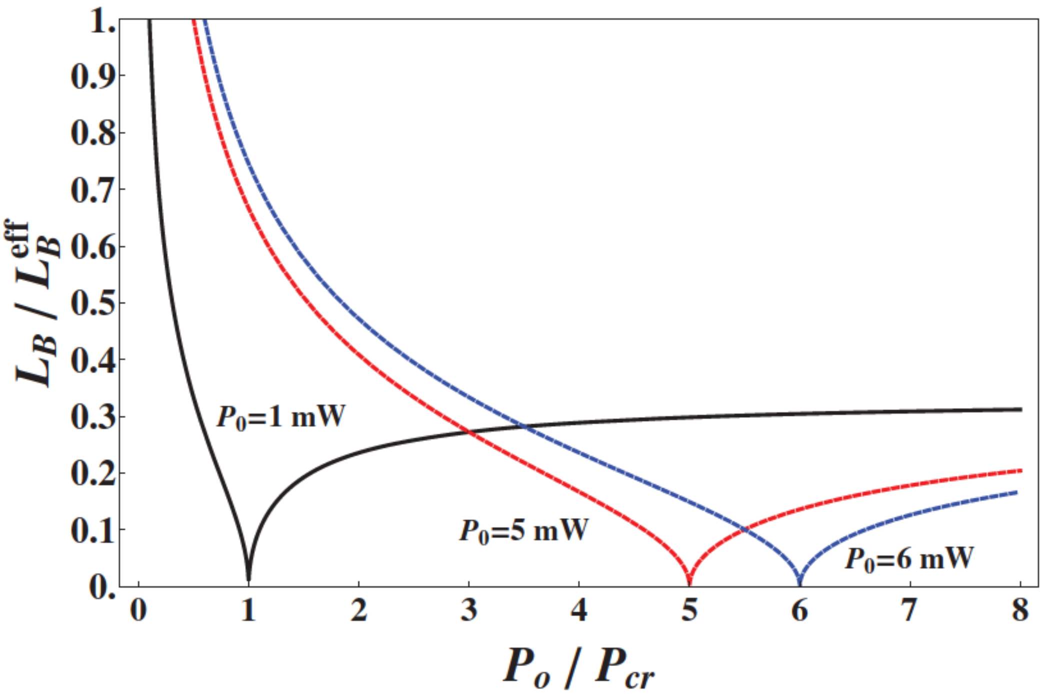

Figure 7.1: Evolution of the inverse effective beat length versus P_{cr}. Different P_0 are considered.

It follows from (7.7) that when optical nonlinear effects are neglected, the medium shows only linear birefringence. Considering this scenario, the propagating light beams along the principal axes preserve their polarization state and the instability is not taking place. The governing equation in this case can be written as:

$$i\,\frac{\partial A_s(z)}{\partial z} = \gamma\, E_{ext}^2\, A_{3-s}. \tag{7.8}$$

From (7.8), one can infer the low-power polarization beat length, given by: $L_B = \pi/(\gamma\, E_{ext}^2)$. It is straight forward to reduce the above expressions in (7.8) as a system of uncoupled linear ordinary differential equations which behaves like a harmonic oscillator. See [40] for more details. For an isotropic medium, as the external electric field $E_{ext} \to 0$ the beat length also $L_B \to \infty$ [20]. However, for birefingent medium, the right hand side of (7.8) is given by $(\Delta\beta/2 + \gamma E_{ext}^2)A_{3-s}$, where $\Delta\beta = \beta_{0x} - \beta_{0y}$. Here, β_{0x} and β_{0y} are the propagation constants of slow and fast polarization modes, respectively. Thus, L_B approaches $(2\pi/\Delta\beta)$ as $E_{ext} \to 0$. In this work, we consider the case of isotropic optical medium. We also remark that in a case of pulse propagation, a similar system of equations (7.7) can be obtained in the quasi-CW regime [113].

On the other hand, for intensive optical input power, the polarization evolution can be described in term of Jacobian elliptic function as detailed in [40, 20]. By following the same approach, one can obtain the effective beat length from (7.7), given by:

$$\begin{aligned} L_B^{eff}(P_0; P_{cr}) &= \frac{2\,\mathrm{K}(m)}{\gamma\sqrt{|q|}}\,\frac{1}{E_{ext}^2}, \\ &= \frac{2\,\mathrm{K}(m)}{\pi\sqrt{|q|}}\,L_B, \end{aligned} \tag{7.9}$$

where $\mathrm{K}(m)$ is the quarter-period argument of the Jacobian elliptic function. For completeness, we also present the solution of the power evolution p_s , given by:

$$p_s(z) = \frac{P_0}{2\,P_{cr}} - \sqrt{m|q|}\;\mathrm{Cn}\left[\sqrt{|q|}\,2\gamma\,E_{ext}^2\,z + \mathrm{K}(m)\right], \tag{7.10}$$

where $\mathrm{Cn}(.|m)$ is the Jacobian elliptic function, $m = [1 - Re(q)/|q|]/2$, and $q = 1 + p\,e^{i\,2\theta_0}$. In Fig. (7.1), the normalized inverse effective beat length is calculated against the normalized input power p. Here, the propagating beam is polarized along the fast axes $(\theta = 90^0)$. We have also computed 3 examples with different P_0. The first one is the black continuous curve (one on the left) for which we have assumed $P_0 = 1\,mW$, with the instability present at $p = 1$ when P_{cr} reaches the value of P_0. As can be seen, as P_{cr} varies , the critical power shifts as governed by (7.9). This scenario can be utilized for electronically controlling the optical switching. Also, the instability broadens in terms of the power p while having smaller interval of L_B^{eff} affected by the instability for larger P_{cr} (i.e., larger DC electrical field). While similar what was observed previously in [40], the

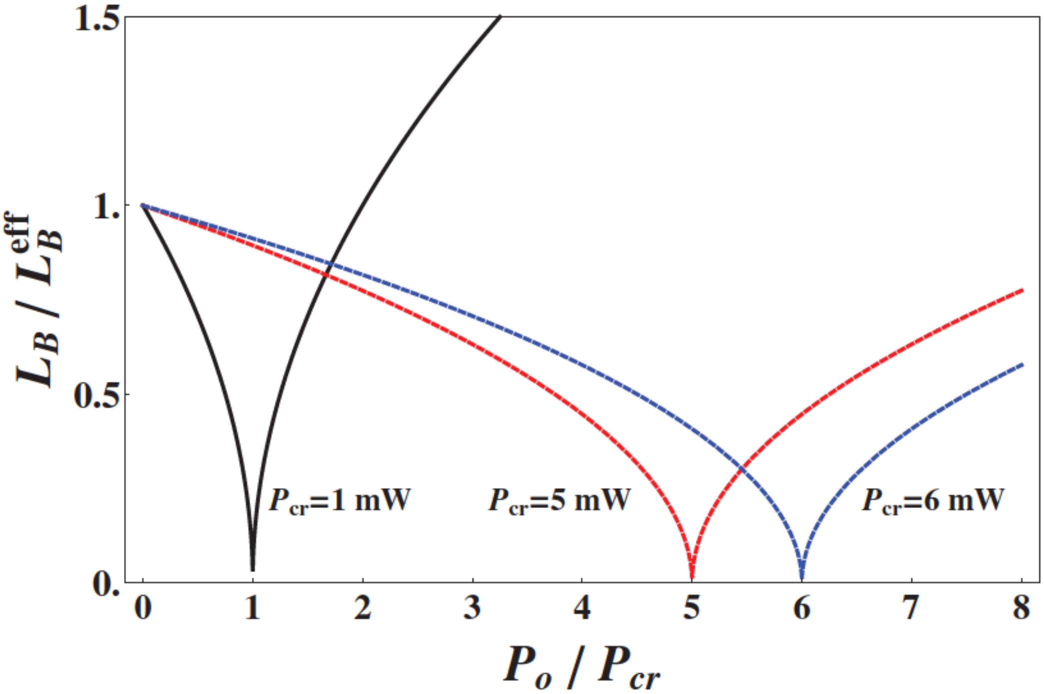

Figure 7.2: Evolution of the inverse effective beat length versus P_0. Different P_{cr} are considered.

beat length monotonically decreases for power values increased beyond the critical power.

Fig. (7.2) is devoted to show the inverse beat length versus P. This is illustrated by computing some examples of very distinctive regions for constant P_{cr} while P_0 is increased. Similar to Fig. (7.1), the effective beat length becomes infinite as input power becomes identical to the critical power. Further increment in the input power turns the fiber birefringent again but with reversed slow and fast axes. Once the condition for the instability is passed (the input power is increased beyond the critical power), L_B^{eff} decreases monotonically in a similar behaviour to the case of slow axis oriented beam.

Finally, we present an illustrative example using real experimental parameters. We consider an optical fiber with $6.6\,\mu m$ effective mode radius, $n_L = 1.46$ refractive index, and $\gamma = 0.0043\,W^{-1}m^{-1}$ nonlinear coefficient. The corresponding normalized effective beat length for these values is presented in Fig. (7.3) as function of P_{cr}. As an example, if one considers $P_0 = 1\,mW$ and $E'_{ext} = 80\,V/\mu m$, the beat lengths are $L_B = 3.11\,m$ and

$L_B^{eff} = 9.35\,m$. We note that the quantity $E_{ext} = (2\,n_L\,A_{eff}^2\,\epsilon_0\,c)^{1/2}\,E'_{ext}$ in our theoretical description is normalized so that E_{ext}^2 is in $[W]$ unit, while E'_{ext} is the physical applied DC electric field in $[V/m]$ and c is the light speed in free space.

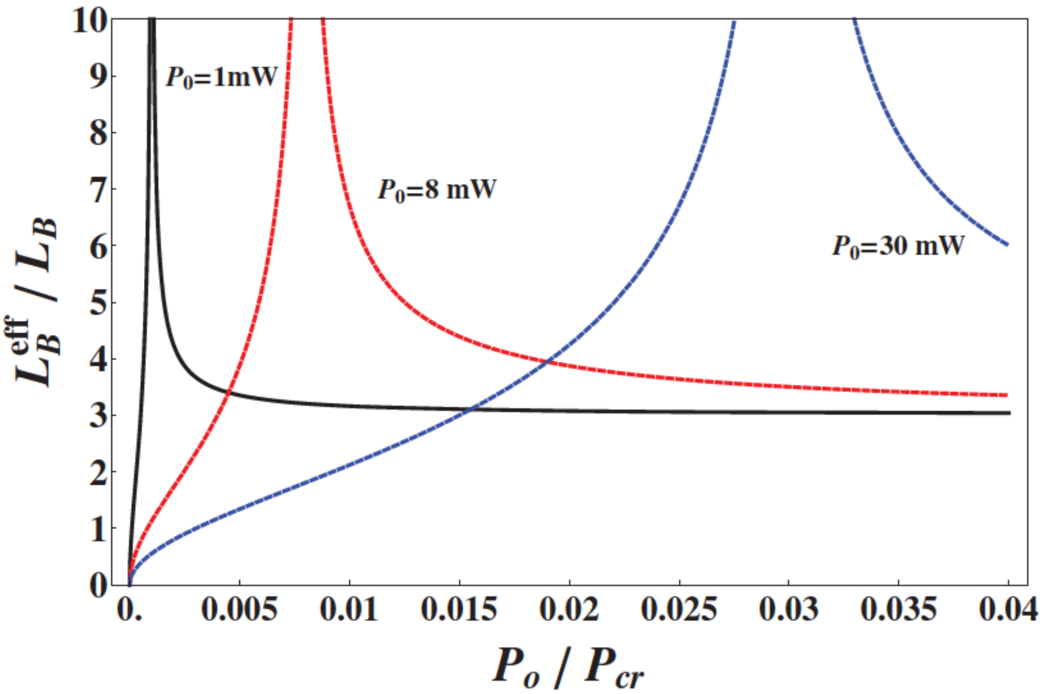

Figure 7.3: Normalized effective beat length evolution versus P_{cr}. Different P_0 are considered.

The presented scheme in this work is also applicable for micro-photonic devices including planar waveguides and photonic integrated circuits. Several interesting devices/systems are natural candidates to benefit from electronically controlled polarization instability. These include fiber laser devices that incorporate birefringent cavities [114], supercontinuum photonic crystal fibers that use polarization dynamics of Raman solitons [115], vector cavity solitons in birefringent resonators [116], semiconductor lasers that utilize vertical cavity resonators, and vertical cavity surface-emitting lasers, just to mention few examples. Alternatively, the proposed modality can be devised as a sensitive polarizer (utilizing the polarization instability) that is directly integrable with a specific device (e.g., a semiconductor vertical-cavity surface-emitting laser) to monitor its extreme operation [117].

7.3 Conclusions

We have theoretically demonstrated the possibility of controlling the polarization instability of optical fields propagating in Kerr nonlinear medium through applying an external electric field. The proposed scheme implies electronically modifying the effective birefringence of the medium and thus varying the critical power required for the polarization instability.

7.4 Acknowledgments

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Chapter 8

Conclusion

This chapter discusses the thesis work from an overall prospective, draws conclusions and suggests future work.

8.1 Discussion and Conclusion Remarks

In Chapter 2, the analytical solutions for the Optical Benney-Luke equation (OBLE) in the form of weakly localized cnoidal waves (CnWs) are introduced. The study has been done in a nonlocal nonlinear defocusing media. The OBLE solutions lead to periodic waves when appropriated boundary conditions are taken into account. It is found that the wave frequency and wavelengths depend on the nonlocality and the optical surface tension parameter. The new results are of importance both for the mathematical theory of the OBLE waves and also for their physical significance since they shine a light on the intermediate asymptotic scenario for the NLSE. The results of the paper are, to our knowledge, original and they could be of significant interest, in particular in the context of research on optical spatial waves in liquid crystals.

The corresponding results in Chapter 2 have motivated the author to further investigate the possibility of a different nonlinear wave. In Chapter 3, we have explored theoretically and numerically the light propagation in a nonlocal nonlinear defocusing media through the Optical Benney-Luke equation. We have proven that solutions of this equation exhibit sech-type solitary waves. We also discussed the regimes in which the form of the solitary waves can be quantitatively described depending on the optical surface tension. The insights gained enabled us to interpret our findings in terms of the nonlocality of the media.

The new results are of importance both for the mathematical theory of the OBLE solitary waves and also for their physical significance since they shine a light on the intermediate asymptotic scenario for the NLSE.

In Chapter 4, we have explored theoretical light propagation in a nonlocal nonlinear defocusing media through a proposed alternative simpler method, the asymptotic variational multiscale approach. The obtained KdV equation is similar to the one derived using the reductive multiscale technique. Our results advance the understanding of nonlinear phenomena. These results were extended to higher dimensions in Chapter 5.

In Chapter 6, we have investigated the evolution and nonlinear properties of small-amplitude kink-like shock waves in colloids under the asymptotic multiscale expansion framework and assuming a low concentration of beads. The system behavior changes to a regime that allows beam propagation. We have derived the dispersion relation for linear propagation of waves in the colloid. We consider the evolution of small-amplitude waves generated by the interaction of a laser beam with nanoparticles dispersed in a liquid medium.

Finally, in Chapter 7, we have theoretically demonstrated the possibility of controlling the polarization instability of optical fields propagating in Kerr nonlinear medium by applying an external electric field. The proposed scheme implies electronically modifying the effective birefringence of the medium and thus varying the critical power required for the polarization instability.

Finally, it is important to know that besides controlling the light polarization via an external DC electric field, there exists in optical wave engineering the possibility to control the polarization by coherent light. An example is polarization switching induced by optical feedback which has attracted considerable interest [118, 119, 120, 121, 122], including polarization control by changing the anisotropic values of the external feedback cavity.

The polarization dynamics of the laser light subjected to weak optical feedback from the birefringence external cavity have been explained theoretically. Experimentally, polarization flipping with hysteresis was induced by birefringence feedback, and the intensities of two eigenstates were both modulated by the external cavity length [123].

Artificial birefringence has been extensively employed in technology applications [124], e.g., in photoelasticity. Birefringence has been used for analyzing stress distribution in solids, while in flat panel displays, electrically induced birefringence can modulate the intensity of light by using a polarizer inside a light modulator. Nowadays, the ever-growing security and environmental needs in a modern society have made remote sensing one of the most important technical approaches in explosives detection and pollution control.

8.2 Future work

Inspired by the findings of this work, on the other hand, we recommend exploring further following topics.

It would be interesting to keep investigating the interaction of a laser beam with nanoparticles dispersed in a liquid medium under the asymptotic multiscale expansion framework and assuming a low concentration of beads. The possibility of deriving a KdV-type equation. Its soliton solution could allow deeper penetration of light propagation through the medium. If we assume the presence of absorption in the system, it will lead to a KdV-Burgers equation. Also, the variational approach developed in this thesis can be applied here to study the asymptotic behavior and see if the Kadomtsev-Petviashvili (KP) type equation is obtained.

Another possibility is to study the propagation of light in a particular nonlinear medium as a reaction-diffusion problem. Specifically the description of dark solitons where Raman self-pumping is present [125]. It is possible to show that the NLSE in the asymptotics can be expressed as a reaction-diffusion equation. A detailed derivation follows:

Introduction. The experimentally perturbation-induced dynamics observed temporal self-shift of dark solitons produced by the Raman self-pumping is given by the perturbed NLS equation. Because the Raman response function of fused silica is extremely short, an approximate response function has been successfully used to model the Raman contribution in the NLS equation by a local term.

$$i\,\frac{\partial\Psi}{\partial x}-\frac{\partial^2\Psi}{\partial t^2}+2\,|\Psi|^2\,\Psi=\alpha\,\Psi\,\frac{\partial}{\partial t}\left(|\Psi|^2\right)=0, \tag{8.1}$$

where $\Psi(x,t)$ is the complex electric-field amplitude envelope with the general solution

$$\Psi(x,t)=[u_0+a(x,t)]\exp\left[2\,i\,u_0^2+\,i\,\phi(x,t)\right], \tag{8.2}$$

and $\phi(x,t)$ is the phase, u_0 is the CW solution Ψ=u_0 = 1. The right-hand side term represents the Raman contribution to the nonlinear refractive index. α being in proportion to the Raman gain parameter α' (see, e.g., Ref. [125] for more details on the model). Assuming

$$\tau=\epsilon\,(t-C\,x)\ ;\ \xi=\epsilon^3\,x\,, \tag{8.3}$$

and expanding in terms if small-parameter ϵ as

$$\begin{aligned} a(x,t) &= \epsilon^2\psi_0+\epsilon^4\psi_1+\ldots\,, \\ \phi(x,t) &= \epsilon\phi_0+\epsilon^3\phi_1+\ldots\,, \end{aligned} \tag{8.4}$$

the resulting first-order equation for the amplitude has the form

$$\frac{\partial^3\psi}{\partial\tau^3}-24\,\psi\,\frac{\partial\psi}{\partial\tau}-2\,C\,\frac{\partial\psi}{\partial\xi}+\eta\,\frac{\partial^2\psi}{\partial\tau^2}=0, \tag{8.5}$$

where for simplicity we write $\psi_0\equiv\psi$. Also, $C^2=4$ is the wave velocity of the CW background linear excitation and $\eta=2\,\alpha/\epsilon$.

The Equation (8.5) is the Korteweg-de Vries-Burgers equation (KdVB) arising in different branches of physics, mostly in hydrodynamics.

Front propagation. Let us propose traveling wave solutions in the form

$$\psi(\tau,\xi)=\varphi(s), \quad \text{with} \quad s=\tau-v\,\xi. \tag{8.6}$$

If we suppose $d^2\varphi/ds^2$ and $d\varphi/ds \to 0$ as $|s|\to\infty$ then $\varphi(s)$ tends to a constant value as the variable s tends to infinity. However, $\varphi(s)$ need not tend to the same constant value in both directions. Suppose, then that

$$\varphi\to 1 \qquad \text{as} \qquad s\to-\infty \tag{8.7}$$

$$\varphi\to 0 \qquad \text{as} \qquad s\to\infty \tag{8.8}$$

Then the equation is given by

$$\frac{d^2\varphi}{ds^2}+\eta\,\frac{d\varphi}{ds}+f(\varphi)=0, \tag{8.9}$$

where

$$f(\varphi)=6\,C\,\left(\frac{v}{3}-\varphi\right)\varphi. \tag{8.10}$$

Notice that we must scale (8.10) given it does not satisfy the boundary condition (8.7). Therefore, let us make a change a variable as $\varphi=K\,\theta$, where K is to be determined. Then the function θ satisfies

$$\frac{d^2\theta}{ds^2}+\eta\,\frac{d\theta}{ds}+2\,C\,v\,\theta-6\,C\,K\,\theta^2=0, \tag{8.11}$$

Then Eq.(8.11) can be finally written as

$$\frac{d^2\theta}{ds^2} + \eta^* \frac{d\theta}{ds} + (1-\theta)\,\theta = 0, \tag{8.12}$$

for $K = v/3$. Also, we have determined that $\eta^* = \eta/(\sqrt{2\,v\,C})$ and for the sake of simplicity and clarity we have define the scaled variable $s \equiv\ s\,\sqrt{(2\,v\,C)}$.

Thus, the the original problem can be expressed as a classic reaction-diffusion problem,

$$\frac{d^2\theta}{ds^2} + \eta^* \frac{d\theta}{ds} + (1-\theta)\,\theta = 0, \tag{8.13}$$

with new boundary conditions,

$$\theta \to 1 \qquad \text{as} \qquad s \to -\infty \tag{8.14}$$

$$\theta \to 0 \qquad \text{as} \qquad s \to \infty. \tag{8.15}$$

The form of η^* expresses the dependency of α on the background velocity C and wave velocity v.

Finally, it is worth noticing that conversely to the standard reaction-diffusion theory where the quantity η^* represents the front propagation speed, in the Eq.(8.12), η^* is the minimum value at which the front exists.

Several methods can be used to solve the differential equation with a plethora of new physics to be analyzed.

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