# AN EXPLORATION OF ORTHOGONAL COLOURINGS

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## Abstract

Two colourings of a graph are *orthogonal* if when two elements are coloured with the same colour in one of the colourings, then those elements receive distinct colours in the other colouring. First, we study the orthogonal chromatic number of Cayley graphs and bipartite graphs. In particular, we determine which cycle graphs, Paley graphs, circulant graphs, and tree graphs have an optimal orthogonal colouring.

Orthogonal colourings of graphs that are constructed by graph products are then explored. We show that if one component has an optimal orthogonal colouring, then the resulting Cartesian, tensor, and strong product graph has an optimal orthogonal colouring under certain conditions. In addition, we determine which hypercube graphs and Hamming graphs have an optimal orthogonal colouring.

Next, orthogonal colourings of graphs that are randomly generated are considered. In particular, we study the random geometric model and the Erdős-Rényi model. We show which random geometric graphs have an optimal orthogonal colouring with high probability. Additionally, we obtain an upper bound on the orthogonal chromatic number in terms of the chromatic number with high probability for both models.

Lastly, a variation of orthogonal colourings, called $(k, t)$-orthogonal colourings, is discussed. We establish a categorization of graphs having an optimal $(k, t)$-orthogonal colouring. Next, we generalize the results for orthogonal colourings of graph products to $(k, t)$-orthogonal colourings of graph products. Also, we show which cycle graphs have an optimal $(2, t)$-orthogonal colouring.

## List of Symbols Used

$V(G)$ denotes the vertex set of a graph.

$E(G)$ denotes the edge set of a graph.

$N(v)$ denotes the open neighbourhood of a vertex.

$N[v]$ denotes the closed neighbourhood of a vertex.

$\alpha(G)$ denotes the independence number of a graph.

$\Delta(G)$ denotes the maximum degree of a graph.

$\chi(G)$ denotes the chromatic number of a graph.

$O\chi(G)$ denotes the orthogonal chromatic number of a graph.

$O\chi_k(G)$ denotes the $k$-orthogonal chromatic number of a graph.

$O\chi_{(k,t)}(G)$ denotes the $(k,t)$-orthogonal chromatic number of a graph.

$C_n$ denotes the cycle graph with $n$ vertices.

$D_n$ denotes the double star graph with $n$ vertices.

$\Gamma(G,S)$ denotes the Cayley graph of a group $G$ with $S$ as its generating set.

$H(n,m)$ denotes the hamming graph with $nm$ vertices.

$K_n$ denotes the complete graph with $n$ vertices.

$\bar{K}_n$ denotes the empty graph with $n$ vertices.

$Q_n$ denotes the hypercube graph with $n$ vertices.

$QR(n)$ denotes the Paley graph with $n$ vertices.

$T_n$ denotes the tree graph with $n$ vertices.

$U_n$ denotes the universal graph with $n$ vertices.

$\mathbb{F}_n$ denotes the finite field with $n$ elements.

$\mathbb{Z}_n$ denotes the group of integers modulo with $n$ elements.

$G \times H$ denotes the tensor graph product of two graphs.

$G \square H$ denotes the Cartesian graph product of two graphs.

$G \boxtimes H$ denotes the strong graph product of two graphs.

$G \circ H$ denotes the lexicographic graph product of two graphs.

$G(n,r)$ is the random geometric graph with threshold function $r$ and $n$ vertices.

$G(n,p)$ is the Erdős-Rényi graph with probability function $p$ and $n$ vertices.

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# Chapter 1

# Introduction

## 1.1 Preliminary Definitions and Notations

The reader is referred to [13] for any graph theory definitions omitted in this section. Throughout this thesis, a *graph* $G$ is the pair $(V(G), E(G))$, where $V(G)$ is the *vertex set* and $E(G)$ is the *edge set*. The *order* of $G$ is $|V(G)|$ and the *size* of $G$ is $|E(G)|$. If $u$ and $v$ are vertices, then $u$ is *adjacent* to $v$ if $\{u, v\} \in E(G)$. In such a case, we denote $\{u, v\}$ by $uv$. All graphs considered in this thesis are finite, undirected, and simple, meaning that there are no loops or parallel edges.

For a vertex $v \in V(G)$, the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ is the *open neighbourhood* of $v$. The set $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of $v$. The *degree* of a vertex $v$, denoted $\deg(v)$, is defined by $\deg(v) = |N(v)|$. A vertex of degree 1 is called a *pendant vertex*. The *maximum degree* of a graph $G$, denoted $\Delta(G)$, is the maximum degree of its vertices.

A graph $H = (V(H), E(H))$ is a *subgraph* of a graph $G = (V(G), E(G))$, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph $H$ is an *induced subgraph* of a graph $G$ if it is required that for all $u, v \in V(H)$, $uv \in E(H)$ if and only if $uv \in E(G)$. An induced subgraph $H$ with $m$ vertices and all $\binom{m}{2}$ possible edges is called a *clique*. An induced subgraph $H$ with $m$ vertices and no edges is called an *independent set*.

A graph is *bipartite* if its vertex set can be partitioned into one or two independent sets. A graph is *connected* if there is a path between every pair of vertices in its vertex set and *disconnected* otherwise. The *complement* of a graph $G$, denoted $\bar{G}$, is the graph on the same vertex set with $u$ and $v$ adjacent in $G$ if and only if they are not adjacent in $\bar{G}$.

A *$k$ colouring* of a graph $G$, is a labelling $f : V(G) \to S$, where $|S| = k$. The labels are called *colours*. The set of all the vertices that receive the same colour are called a *colour class*. A $k$ colouring is *proper* if adjacent vertices receive different colours. A graph is called *$k$ colourable* if it has a proper $k$ colouring. A *colour conflict* occurs if two adjacent vertices receive the same colour. The *chromatic number* of a graph $G$, denoted $\chi(G)$, is the least $k$ such that $G$ is $k$ colourable.

Common graphs that we will use in this thesis include the *path graph* on $n$ vertices, denoted $P_n$, the *cycle graph* on $n$ vertices, denoted $C_n$, and the *complete graph* on $n$ vertices, denoted $K_n$. An example of each of these graphs is given in Figure 1.1.1.

Figure 1.1.1: Example of $P_4, C_4$, and $K_4$

The *complete bipartite graph* with partitions of size $n$ and $m$ is denoted by $K_{n,m}$. The *empty graph* on $n$ vertices, denoted $\bar{K}_n$, is the graph with no edges. An acyclic connected graph is called a *tree graph.*

A graph is *$d$-degenerate graph* if there exists an ordering of the vertices, in which each vertex has $d$ or fewer neighbours that are earlier in the ordering. Such an ordering of the vertices is called a *degenerate ordering.* The *degeneracy* of a graph is the smallest value of $d$ for which it is $d$-degenerate.

Given two positive numbers $a$ and $n$, *$a$ modulo $n$*, abbreviated as $a(\text{mod } n)$, is the remainder of the Euclidean division of $a$ by $n$, where $a$ is the dividend and $n$ is the divisor. Two integers $a$ and $b$ are said to be *congruent modulo* n, denoted by $a \equiv b(\text{mod } n)$, if $n$ is a divisor of their difference. The reader is referred to [48] for any number theory definitions and terms omitted in this section.

For two positive real valued functions $f(n)$ and $g(n)$, we say that $f(n) = o(g(n))$, read $f(n)$ is *little-o* of $g(n)$, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. In particular, for a real valued function $f(n)$, we say $f(n) = o(1)$ if $\lim_{n\to\infty} f(n) = 0$. For two positive real value functions $f(n)$ and $g(n)$, we say that $f(n) = \Omega(g(n))$, read $f(n)$ is *big omega* of $g(n)$, if $\limsup_{n\to\infty} \frac{f(x)}{g(x)} > 0$.

An event $E$ occurs *with high probability* if as $n$ tends to infinity, the probability that $E$ occurs tends to one. An event $E$ occurs *with exponential probability* if the probability that the event does not happen is $e^{-f(n)}$, where $f(n) = \Omega(\ln^2(n))$. The reader is referred to [25] for any probability theory definitions and terms omitted in this section.

## 1.2 History of Colourings

Graph colourings are a special type of graph labellings, where elements of a graph are assigned labels, called colours, subject to certain constraints. The convention of using colours originated from colouring the countries of a map. Guthrie [28] posed the Four Colour Theorem in 1852, which said that four colours are sufficient to colour a map so that regions sharing a border do not receive the same colour.

In 1879, Kempe [36] published a paper that claimed to prove the Four Colour Theorem. However in 1890, Heawood [33] noticed that Kempe's proof was incorrect. The Four Colour Theorem was correctly proved in 1976 by Appel and Haken [2]. This proof is noteworthy for being the first major computer-aided proof. The pursuit of proving the Four Colour Theorem has given rise to the field of graph colourings.

Graph colourings have applications other than colouring maps, such as scheduling problems [43] and register allocation in compilers [12, 55]. To study graph colourings and their applications, various approaches have been implemented over the years. For instance the chromatic polynomial, introduced in 1912 by Birkhoff [6], was originally created to study the Four Colour Theorem.

Another method is to study different variations of graph colourings. For instance list colourings, defined in 1979 by Vizing and independently by Erdős, Rubin, and Taylor [19], have applications to frequency assignments [54]. Another example, equitable colourings, defined in 1973 by Meyer [46], have applications to more involved scheduling problems, like assigning final exam time slots [23]. In this thesis, we study orthogonal colourings, another graph colouring variation.

Two proper colourings of a graph are *orthogonal* if when two elements are coloured with the same colour in one of the colourings, then those elements receive distinct colours in the other colouring. This variation was proposed in 1985 by Archdeacon, Dinitz, and Harary [3]. They studied orthogonal colourings in the context of edge colourings. In this thesis, the vertex variation of orthogonal colouring is considered.

The vertex variation of orthogonal colouring was originally studied in 1999 by Caro and Yuster [11]. In 2013, Ballif [4] then explored upper bounds on the number of orthogonal colourings that a graph can have. Lastly in 2019, Andres et al. [1] studied a game version of orthogonal colouring. Applications of orthogonal colourings are now discussed.

## 1.3 Applications of Orthogonal Colourings

In this section, we discuss two potential applications of orthogonal colourings. First, we mention how orthogonal colourings can create independent tranversals. We then show that orthogonal colourings can generalize independent coverings. Lastly, we discuss how orthogonal colourings can extend the applications of transversal designs by utilizing graph structure.

### 1.3.1 Applications to Independent Coverings

An *independent transversal* of a graph $G$, with respect to a vertex partition $P$, is an independent set in $G$ that has exactly one vertex from each vertex class of $P$. Finding sufficient conditions for the existence of independent transversals is an active area of research. This is because many problems can be answered by finding an independent transversal of a graph. List colouring [40] is one example of this.

This problem of finding a sufficient condition for the existence of an independent traversal was originally studied in 1975 by Bollobás, Erdős, and Szemerédi [7]. They conjectured that if each vertex class of a graph $G$ in a partition $P$ has size at least $2\Delta(G)$, then an independent transversal with respect to $P$ exists. This conjecture was later proved in 2001 by Haxell [31]. In 2006, this bound was then proved to be best possible by Szabó and Tardos [51].

An *independent covering* of a graph $G$, with respect to a vertex partition $P$, is a collection of disjoint independent transversals with respect to $P$ that span all of the vertices. Independent coverings are important to study because they extend independent transversal applications. For instance, independent coverings have applications to forming multiple committees and also have applications to forming replacement committees [32].

Currently, no general degree condition giving the existence of an independent covering exists. Hence, the main area of research for independent coverings is in finding sufficient conditions for specific families of graphs. For example, Yuster [56] found conditions for $[n, k, r]$-partite graphs to have independent coverings. A graph is $[n, k, r]$*-partite* if the vertices can be partitioned into $n$ independent sets of size $k$, where all of the edges between any two independent sets is a matching of size $r$.

Additionally, independent transversals of $[n, k, r]$-partite graphs have also been studied. In 1994, Erdős, Gyárfás, and Łuczak [17] studied what the maximum $n$ is such that every $[n, k, 1]$-partite graph has an independent transversal. They provided an asymptotic bound that $n$ is at most $(1 + o(1))k^2$. In 2020, Glock and Sudakov [24] proved that this result is actually the best possible.

Note that an independent covering provides an orthogonal colouring. The vertex classes of the partition $P$ are taken as the colour classes for the first colouring. The disjoint independent transversals with respect to $P$ are then taken as the colour classes for the second colouring.

Since the vertex classes and independent transversals are independent sets, the first and second colourings are proper. Also, each independent transversal contains exactly one vertex from each vertex class, so each colour pair assigned is unique. Therefore, the two colourings constructed are orthogonal colourings.

On the other hand, an orthogonal colouring provides an independent covering if the sizes of the colour classes in the first colouring are the same and the sizes of the colour classes in the second colouring are the number of colours used in the first colouring. The colour classes in either the first or second colouring can be taken as the partition. The colour classes in the other colouring can then be taken as the independent transversals.

Since the colour classes have the same size and the two colourings are orthogonal, each independent transversal will contain exactly one vertex from each vertex class. Therefore, orthogonal colourings where all colour classes have the same size can be applied to create independent coverings of graphs with a square number of vertices. The benefit of using such an orthogonal colouring is that they are not restricted to a particular vertex partition. Instead, this type of orthogonal colouring provides a partition that has an independent covering.

Also, the colour classes in an orthogonal colourings can be viewed as a relaxation of independent transversals. The colour classes have at most one vertex from each vertex class, rather than exactly one vertex. Similarly, an orthogonal colouring can be viewed as a generalization of independent coverings. All of the vertices are still covered, except now by the colour classes, rather than by the independent transversals.

### 1.3.2 Applications to Transversal Designs

A $(n, k, \lambda)$-*transversal design* is a triple, $(V, G, B)$, where $V$ is a set of $kn$ points, $G$ is a partition of these points into $k$ disjoint sets (called *groups*) each containing $n$ points, and $B$ is a set of $n^2$ $k$-tuples (called *blocks*), satisfying the following properties. Each pair of points from different groups appears in precisely $\lambda$ blocks, and no block contains more than one point from each group. Transversal designs are important to study due to their applications to fields such as experimental designs, error-correcting codes, and cryptography [5, 15].

Note that an orthogonal colouring of a graph with $n^2$ vertices where all colour classes are of size $n$ corresponds to a $(n, 2, 1)$-transversal design. To see this, let the colours used in the first and second colouring be denoted by $C = \{1, 2, \ldots, n\}$. Next, construct the sets $C_1 = \{1, 2, \ldots, n\}$ and $C_2 = \{n + 1, n + 2, \ldots, 2n\}$.

Now, let $V = \{1, 2, \ldots, 2n\}$, $G = \{C_1, C_2\}$, and $B$ be the $n^2$ pairs obtained by taking the colour pairs and adding $n$ to the second colour. Since the colourings are orthogonal, each pair appears in 1 block. Also, each block in $B$ contains only one element from each group. Therefore, an $(n, 2, 1)$-transversal design was created.

The same argument shows that $k$ mutually orthogonal colourings of a graph with $n^2$ vertices where all colour classes are of size $n$ provides a $(n, k, 1)$-transversal design. Notice that the transversal designs created this way did not depend on the graph. Hence, one application of $k$ mutually orthogonal colourings where all colour classes have the same size is to utilize the graph structure.

For example, suppose that an experiment to investigate the effects of pesticide treatment levels on crop production is conducted on $n^2$ plots. Each week, $n$ different treatments are going to be assigned to $n$ different plots in such a way that each plot receives exactly one treatment. Additionally, it is required that adjacent plots receive different treatments and that adjacent plots receive treatments on different weeks. Let $G$ be the grid graph modelling the adjacencies of the plots.

An orthogonal colouring of $G$ would then model this experiment. Let the colours in the first colouring be the weeks and let the colours in the second colouring be the treatments. The orthogonality of the colourings gives that each plot receives exactly one treatment. The properness of the colourings gives that adjacent plots receive different treatments and adjacent plots receive treatments on different weeks.

## 1.4 Summary of Known Orthogonal Vertex Colouring Results

The following definitions are motivated by the applications. A $k$ *orthogonal colouring* of a graph $G$ is a collection of $k$ mutually orthogonal proper vertex colourings of $G$. The $k$ *orthogonal chromatic number* of a graph $G$, denoted by $O\chi_k(G)$, is the minimum number of colours required for a $k$-orthogonal colouring of $G$.

For brevity, a 2 orthogonal colouring is called an *orthogonal colouring*. Similarly, the 2 orthogonal chromatic number is denoted by $O\chi(G)$ and called the *orthogonal chromatic number*. For example, an orthogonal colouring of $C_6$ using 3 colours is provided in Figure 1.4.1.

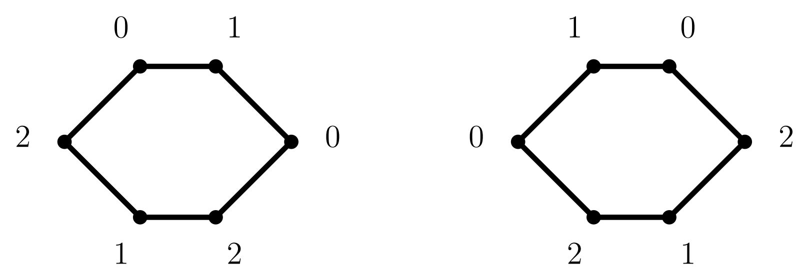


Figure 1.4.1: Orthogonal Colouring of $C_6$

Rather than drawing the graph twice with two separate colourings, one copy of the graph is drawn, and the two colourings are superimposed. That is, displayed next to each vertex are the colours assigned in both the first and the second colouring. For example, see Figure 1.4.2. The pairs of colours that are assigned to each vertex are called *colour pairs*. Similarly, for a $k$ orthogonal colouring, a $k$-tuple of the colours assigned by each of the $k$ colourings are displayed next to each vertex. The $k$-tuples of colours that are assigned to each vertex are analogously called *colour $k$-tuples*.

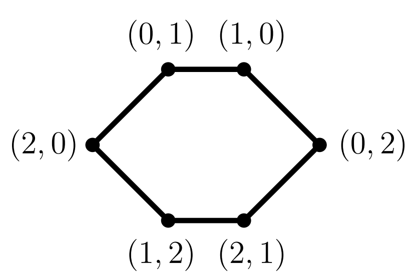


Figure 1.4.2: Orthogonal Colouring of $C_6$

An assignment of colour pairs to the vertices of a graph is called an *orthogonal assignment* if no colour pair is assigned more than once. If some colour pair occurs more than once in an assignment of colour pairs to the vertices, this is called an *orthogonal conflict*. Therefore, an orthogonal colouring has no colour conflicts and no orthogonal conflicts.

For a graph $G$ with $n$ vertices, $O\chi_k(G) \geq \lceil \sqrt{n}\, \rceil$. Otherwise, there are less than $n$ colour pairs, so in each pair of colourings, some colour pair is assigned twice, resulting in an orthogonal conflict. Also, $O\chi_k(G) \geq \chi(G)$. This is because each of the colourings need to be proper vertex colourings. On the other hand, $O\chi_k(G) \leq n$. This is because assigning each vertex $v_i$ the colour $k$-tuple $(i, i, \ldots, i)$ is a $k$-orthogonal colouring. Therefore, combining these bounds gives the following chain of inequalities on the $k$-orthogonal chromatic number.

$$\max\left\{\chi(G), \left\lceil \sqrt{n}\, \right\rceil\right\} \leq O\chi_k(G) \leq n$$

Based on this inequality, there are three directions of research. Firstly, one could try to find improved upper bounds on the $k$-orthogonal chromatic number. This was originally studied by Caro and Yuster [11]. Secondly, one could focus on graphs having the same chromatic number and orthogonal chromatic number. Due to a lack of known applications, this is not studied here. Lastly, one could focus on graphs having orthogonal chromatic number of $\lceil \sqrt{n}\, \rceil$. Due to the previously discussed applications of orthogonal colourings, this is the focus of our research in this thesis.

If $O\chi_k(G) = \lceil \sqrt{n}\, \rceil$, then $G$ is said to have an *optimal $k$-orthogonal colouring*. An optimal 2-orthogonal colouring is simply called an *optimal orthogonal colouring*. In the previous section, we showed that optimal orthogonal colourings of graphs with a square number of vertices have applications to independent coverings and transversal designs. Therefore, determining which graphs have optimal orthogonal colourings is an important area of research.

This question was originally answered by Caro and Yuster [11] by using orthogonal Latin squares. A *Latin square* of order $n$ is an $n \times n$ array, filled with $n$ different symbols, each occurring exactly once in each row and column. Two Latin squares are *orthogonal*, if when superimposed, each ordered pair occurs exactly once. A collection of Latin squares is *mutually orthogonal* if all pairs of Latin squares are orthogonal. For example, two orthogonal Latin squares of order 3 are given in Figure 1.4.3.

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 2 | 0 |
| 2 | 0 | 1 |

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 2 | 0 | 1 |
| 1 | 2 | 0 |

Figure 1.4.3: Orthogonal Latin Squares

Let $n \geq 2$ be an integer and let $n = p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$ be the factorization of $n$ into distinct prime numbers. It is known that there are at least $\min\{p_i^{e_i} - 1 : i = 1, 2, \ldots, k\}$ orthogonal Latin squares of size $n$. This implies for all odd $n$ and all $n \equiv 0 \pmod 4$, that there exists orthogonal Latin squares of size $n$. The remaining cases left to consider are when $n \equiv 2 \pmod 4$.

It was originally conjectured in 1782 by Euler [21] that there were no orthogonal Latin squares of this size. One can quickly verify, that for $n = 2$, this is indeed correct. For $n = 6$, it was shown in 1900 by Tarry [52] that no orthogonal Latin squares of this size exist. This suggested that Euler's conjecture might be true for all $n \equiv 2 \pmod 4$. However in 1960, Euler's conjecture was disproved by the combined efforts of Bose, Shrikhande, and Parker [9]. Therefore, there exists a pair of orthogonal Latin squares of size $n$ if and only if $n \neq 2, 6$.

Caro and Yuster [11] constructed a family of graphs having an optimal $k$-orthogonal colouring by using $k$ mutually orthogonal Latin squares. Let $C = \{L_1, L_2, \ldots, L_k\}$ be a collection of $k$ mutually orthogonal Latin squares. For $1 \leq r \leq k$ and $1 \leq i, j \leq n$, let $L_r(i, j)$ denote the entry of $L_r$ in the $i$-th row and $j$-th column. The graph $U(C)$ has $n^2$ vertices, which are denoted by the ordered pairs $(i, j)$. The vertex $(i_1, j_1)$ is adjacent to the vertex $(i_2, j_2)$ if for every $r$, $L_r(i_1, j_1) \neq L_r(i_2, j_2)$.

For $U(C)$, consider the colourings $c_r((i, j)) = L_r(i, j)$. If $(i_1, j_1)$ is adjacent to $(i_2, j_2)$, then by construction, $L_r(i_1, j_1) \neq L_r(i_2, j_2)$. Therefore, each $c_r$ is a proper colouring. Since the Latin squares are mutually orthogonal, it follows that the colourings are mutually orthogonal as well.

For example, the orthogonal colouring of the graph constructed by using the Latin squares $L_1$ and $L_2$ in Figure 1.4.3 is given in Figure 1.4.4. The entries in $L_1$ provide the colours assigned in the first colouring and the entries in $L_2$ provide the colours assigned in the second colouring. That is, the assigned colour pairs are precisely the superimposed Latin squares positions. Then, vertices that do not have a colour conflict in either colouring are connected by an edge.

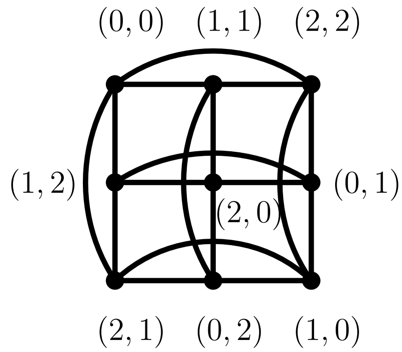


Figure 1.4.4: Orthogonal Colouring of $U(L_1, L_2)$

Interestingly, the graph obtained through Caro and Yuster's process does not depend on the choice of Latin squares when only two Latin squares are used. Thus, the graph can be denoted by $U_n$ in this case. Notice that $U_n$ is the largest graph with orthogonal chromatic number $n$ in the sense that no more edges can be added without breaking the properness of the colourings. This lead to the following result.

**Lemma 1.4.1.** *(Caro and Yuster [11]) For any graph $G$, $O\chi(G) \leq n$ if and only if $G \subseteq U_n$.*

Lemma 1.4.1 gives a way to view orthogonal colourings as a graph homomorphism problem. It also provides a categorization of graphs having an optimal orthogonal colouring. If $G$ is a graph with $n$ vertices and $N = \lceil \sqrt{n}\, \rceil$, then $G$ will have an optimal orthogonal colouring if and only if $G \subseteq U_N$. We give an equivalent categorization in terms of tensor graph products later in this thesis.

If $C$ is a collection of $k > 2$ mutually orthogonal Latin squares, then $U(C)$ is a graph with an optimal $k$-orthogonal colouring. However, it can be shown that the graph $U(C)$ changes based on the chosen Latin squares. Thus, a categorization like Lemma 1.4.1 in terms of a single graph does not exist. Instead, some other categorization would need to be considered. For instance, the following result provides a partial categorization in terms of the maximum degree.

**Theorem 1.4.2.** *(Caro and Yuster [11]) Let $L(k)$ be an integer such that if $r \geq L(k)$, then there exists a collection of $k$-orthogonal Latin squares of size $r$. If $G$ is a graph with $n$ vertices where $n \geq (L(k-2))^2$ and $\Delta(G) \leq \frac{\sqrt{n}-1}{2k}$, then*

$$O\chi_k(G) = \lceil \sqrt{n}\, \rceil$$

.

The proof of Theorem 1.4.2 follows from the following colouring algorithm, which will be improved later in this thesis. First, consider the graph $G$ as a graph with no edges and $k$ orthogonally colour the graph. At each step of the algorithm, an edge of $G$ is returned to the graph. If this results in a colour conflict, then the colourings are modified so that they remain proper and mutually orthogonal.

The condition that $n \geq (L(k-2))^2$ is to guarantee that an optimal $k$-orthogonal colouring of an independent set of size $n$ exists in the first step. The condition that $\Delta(G) \leq \frac{\sqrt{n}-1}{2k}$ is to guarantee that at each step of the algorithm, the colouring modification process is possible. In our improvement of this algorithm, a particular ordering of the edges is used, rather than an arbitrary one.

### 1.4.1 Orthogonal Chromatic Number of Graphs

Recall that $L(k)$ is an integer such that if $r \geq L(k)$, then there exists a collection of $k$-orthogonal Latin squares of size $r$. The following result by Caro and Yuster determines the orthogonal chromatic number of empty graphs by using mutually orthogonal Latin squares.

**Theorem 1.4.3.** *(Caro and Yuster [11]) For an empty graph $\bar{K}_n$,*

$$O\chi_k(\bar{K}_n) \leq \max\left\{\left\lceil\sqrt{n}\,\right\rceil, L(k-2)\right\}.$$

The proof of Theorem 1.4.3 is generalized in Chapter 5. Thus, a summary of the proof is given here. Suppose that $\lceil\sqrt{n}\,\rceil \geq L(k-2)$, so $k-2$ orthogonal Latin squares of size $\lceil\sqrt{n}\,\rceil$ exist. Let $L_1, \ldots, L_{k-2}$ be $k-2$ orthogonal Latin squares of order $\lceil\sqrt{n}\,\rceil$. For each $v_{i,j}$, where $1 \leq i \leq \lceil\sqrt{n}\,\rceil$ and $1 \leq j \leq \lceil\sqrt{n}\,\rceil$, define $c_1(v_{i,j}) = i, c_2(v_{i,j}) = j$, and for $3 \leq s \leq k$, define $c_s(v_{i,j}) = L_{s-2}(i,j)$. It can be shown that $c_1, \ldots, c_k$ are all mutually orthogonal.

Note that in the case where $k = 2$ and $k = 3$, that $L(k-2) = 1$. Thus, no Latin squares are required to construct the orthogonal colouring. Therefore, for any $n$, $O\chi(\bar{K}_n) = O\chi_3(\bar{K}_n) = \lceil\sqrt{n}\,\rceil$. The following result by Caro and Yuster applied Theorem 1.4.3 to find the orthogonal chromatic number of complete $t$-partite graphs.

**Theorem 1.4.4.** *(Caro and Yuster [11]) If $G$ is a complete $t$-partite graph with vertex classes of sizes $s_1, s_2, \ldots, s_t$ and if $m$ is the number of vertex classes whose size $s_i$ satisfies $\lceil \sqrt{s_i} \rceil \lfloor \sqrt{s_i} \rfloor \geq s_i$ but is not an integer square, then*

$$O\chi(G) = \sum_{i=1}^{t} \sqrt{s_i} - \left\lfloor \frac{m}{2} \right\rfloor.$$

The proof of Theorem 1.4.4 is generalized in Chapter 5. Thus, a summary of the proof is given here. Let $S_1, \ldots, S_t$ denote the vertex classes of $G$, where $|S_i| = s_i$. Suppose that the sizes of the first $m$ classes have the property that $s_i$ is not an integer square and $\lceil \sqrt{s_i} \rceil \lfloor \sqrt{s_i} \rfloor \geq s_i$. For $i = m+1, \ldots, t$, apply Theorem 1.4.3 to orthogonally colour the vertices of $S_i$ with $\lceil \sqrt{s_i} \rceil$ distinct colours.

Next, for the $m$ other classes, apply Theorem 1.4.3 to orthogonally colour $S_i$ with $\lceil \sqrt{s_i} \rceil$ distinct colours. It can be shown that for even $m$, by pairing the $S_i$'s together, one colour can be removed from each pair. For odd $m$, there is one class left unpaired. Lastly, a minimization argument is used to show that fewer colours cannot be used.

### 1.4.2 Orthogonal Chromatic Number Upper Bounds

Here, known upper bounds on the $k$-orthogonal chromatic number are discussed. The first bound relies on an equitable vertex colouring result. An *equitable vertex colouring* of a graph is a proper vertex colouring such that the number of vertices in any two colour classes differ by at most one. The following result by Hajnal and Szemerédi gives an equitable colouring of any graph in terms of its maximum degree.

**Theorem 1.4.5.** *(Hajnal and Szemerédi [29]) Every graph $G$ has an equitable vertex colouring with $\Delta(G) + 1$ colours.*

Theorem 1.4.5 was originally conjectured in 1964 by Erdős and was later proved in 1970 by Hajnal and Szemerédi . Their original proof was quite long, and a simpler proof was given in 2008 by Kierstead and Kostochka [37]. The following theorem uses this result iteratively to obtain an upper bound on the $k$-orthogonal chromatic number in terms of the maximum degree.

**Theorem 1.4.6.** *(Caro and Yuster [11]) If $G$ is a graph with $n$ vertices, then*

$$O\chi_k(G) \leq (k-1) \left\lceil \frac{n}{\Delta(G)+1} \right\rceil + \Delta(G).$$

The proof of Theorem 1.4.6 is as follows. First, consider an equitable colouring $c_1$ of $G$ using $\Delta(G)+1$ colours, which exists by Theorem 1.4.5. Next, let $G_1$ be the graph obtained by adding the edges between vertices that have received the same colour in $c_1$. Notice that $G_1$ has maximum degree at most $\Delta(G) + \left\lceil \frac{n}{\Delta(G)+1} \right\rceil - 1$ because each vertex is adjacent to at most $\Delta(G)$ vertices in $G$, and there are at most $\left\lceil \frac{n}{\Delta(G)+1} \right\rceil - 1$ other vertices that have received the same colour in $c_1$.

Now, apply Theorem 1.4.5 again, this time to $G_1$ to obtain an equitable colouring $c_2$ of $G_1$ using $\Delta(G_1)+1$ colours. Next, define $G_2$ by adding to $G_1$ the edges between vertices that have received the same colour in $c_2$. After multiple iterations of this process, eventually the graph $G_{k-1}$ with $\Delta(G_{k-1}) \leq \left\lceil \frac{n}{\Delta(G_{k-2})+1} \right\rceil + \Delta(G_{k-2}) - 1$ is obtained. Notice that $\Delta(G_{k-1}) \leq (k-1) \left\lceil \frac{n}{\Delta(G)+1} \right\rceil + \Delta(G) - 1$ by substitution.

Lastly, let $c_k$ denote a greedy colouring of $G_{k-1}$ with $\Delta(G_{k-1})+1$ colours. By the construction of $G_{k-1}$, $\{c_1, \ldots, c_k\}$ will all be mutually orthogonal. Therefore, since the $k$-orthogonal colouring of $G_{k-1}$ is also a $k$-orthogonal colouring for $G$, it follows that $O\chi_k(G) \leq O\chi_k(G_{k-1}) \leq \Delta(G_{k-1}) + 1 \leq (k-1) \left\lceil \frac{n}{\Delta(G)+1} \right\rceil + \Delta(G)$.

If the chromatic number of a graph is large, and close to the maximum degree, then the bound in Theorem 1.4.6 is very good. However, if the maximum degree is small, then the bound becomes worse. The following result determines an upper bound on the $k$-orthogonal chromatic number of a graph in terms of the chromatic number.

**Theorem 1.4.7.** *(Caro and Yuster [11]) If $G$ is a graph with $n$ vertices, then*

$$O\chi_k(G) \leq \chi(G)L(k-2) + \chi(G) + \sqrt{\chi(G)}\sqrt{n}.$$

The proof of Theorem 1.4.7 is as follows. Let $f$ be a proper $\chi(G)$ vertex colouring. Consider the colour classes of $f$ as independent sets $I_1, I_2, \ldots I_{\chi(G)}$. By Theorem 1.4.3, each independent set $I_i$ can be orthogonally coloured with $\max\{L(k-2), \left\lceil \sqrt{|I_i|} \right\rceil\}$ colours. The argument is then to orthogonally colour each of these independent sets with a disjoint number of colours.

Lastly, an upper bound using the degeneracy of a graph was determined. A graph is called *d-degenerate* if there exists an ordering of the vertices in which each vertex has $d$ or fewer neighbours that are earlier in the ordering. The following result gives an upper bound on the $k$-orthogonal chromatic number is terms of the degeneracy.

**Theorem 1.4.8.** *(Caro and Yuster [11]) Let $G$ be a $d$-degenerate graph with $n$ vertices. If $(t-d)^k > \binom{k}{2}(n-d-1)t^{k-2}$, then*

$$O\chi_k(G) \le t.$$

The proof of Theorem 1.4.8 is as follows. Let $\{v_1, v_2, \ldots, v_n\}$ be a degenerate ordering of the vertices and for $i = 1, \ldots, d$ and $j = 1, \ldots, k$, colour $v_i$ with $c_j(v_i) = i$. Suppose that for $l \ge d$, the vertices $v_1, \ldots, v_l$, have been properly $k$-orthogonally coloured by using no more than $t$ colours. If $(t-d)^k > \binom{k}{2}(n-d-1)t^{k-2}$, then one can greedily find an unused colour $k$-tuple to assign to $v_{l+1}$ that is proper.

This summarizes all of the known techniques for creating orthogonal colourings. In the next section, we provide an overview of the results that we obtain in thesis as well as the new approaches that we develop for constructing orthogonal colourings.

## 1.5 Summary of Thesis

First, we study the $k$-orthogonal chromatic number of some classic graph families. In particular, we explore cycle graphs, circulant graphs, Cayley graphs, and bipartite graphs. One reason for the study of these graphs is to help expand the catalogue of graphs with known $k$-orthogonal chromatic number. Additionally, by studying these classic graph families, we find different methods for constructing orthogonal colourings. These results were published in [34, 42].

Secondly, we consider the $k$-orthogonal chromatic number of product graphs. In particular, we study the tensor, Cartesian, and strong product graphs. One reason for the study of these product graphs is to take existing graphs with known $k$-orthogonal colourings and create new families with known $k$-orthogonal colourings. Also, we show that product graphs give a categorization of optimal orthogonal colourings. These results were published in [34] and led to the paper [41].

Thirdly, we explore the orthogonal chromatic number of randomly generated graphs as well as randomly generated orthogonal colourings. Specifically, we study the random geometric model, the Erdős-Rényi model, and the entropy compression method. We give asymptotic bounds on the orthogonal chromatic number for graphs sampled from these models.

Lastly, we discuss a variation of $k$-orthogonal colouring. Two colourings of a graph $G$ are *$t$-orthogonal* if they have the property that when $t+1$ vertices are coloured with the same colour in one of the colourings, then at least one of these $t+1$ vertices must have a distinct colour in the other colouring. A *$(k,t)$-orthogonal colouring* of $G$ is a collection of $k$ mutually $t$-orthogonal colourings. The *$(k,t)$-orthogonal chromatic number*, denoted $O\chi_{(k,t)}(G)$, is the minimum number of colours required so that $G$ has a $(k,t)$-orthogonal colouring.

If $O\chi_{(k,t)}(G) = \lceil \sqrt{\lceil \frac{n}{t} \rceil} \rceil$, then $G$ is said to have an *optimal $(k,t)$-orthogonal colouring*. We show that optimal $(k,t)$-orthogonal colourings have applications to $(n,k,\lambda)$-transversal designs, mentioned in the introduction. We give a categorization of graphs that have an optimal $(k,t)$-orthogonal colourings. Additionally, we study the $(k,t)$-orthogonal chromatic number of product graphs, complete $r$-partite graphs, and cycle graphs.

# Chapter 2

## Orthogonal Colourings of Classic Graphs

To start, orthogonal colourings of cycle graphs, denoted $C_n$, are studied. We show that $C_n$ has an optimal orthogonal colouring if and only if $n > 4$. Additionally, we show that if $\lceil \sqrt{n} \rceil$ is prime, then $C_n$ has an optimal $(p-2)$-orthogonal colouring. Orthogonal colourings of Paley graphs, denoted $QR(n)$, are then considered. We show that $QR(p^{2r})$ has an optimal $\left(\frac{p^r+1}{2}\right)$-orthogonal colouring.

The results obtained for cycle graphs and Paley graphs are then generalized to circulant graphs of prime square order. We show that if $|S| < \frac{p-1}{2}$, then an optimal orthogonal colouring of the circulant graph $\Gamma(\mathbb{Z}_{p^2}, S)$ exists. This is accomplished by constructing a family of orthogonal assignments and then proving that at least one of them is also a proper colouring.

Also, we show that if there are no multiples of $p$ in the generating set and $|S| < p$, then an optimal orthogonal colouring of the circulant graph $\Gamma(\mathbb{Z}_{p^2}, S)$ exists. This is similarly accomplished by constructing a different family of orthogonal assignments and then proving that at least one of them is also a proper colouring. Additionally, we discuss other constraints on the generating set.

Next, we disprove an open conjecture stating that all tree graphs with $n$ vertices and maximum degree less than $\frac{n}{2}$ have an optimal orthogonal colouring. We show that all tree graphs with $n$ vertices and maximum degree less than $\frac{\sqrt{n}-3}{2}$ do have an optimal orthogonal colouring. Also, we show that for even $m$, the double star, denoted $D_m$, has an optimal orthogonal colouring if and only if $m < \lceil \sqrt{m} \rceil^2 - 1$.

To conclude, we disprove an open conjecture stating that all $[n, k, r]$-partite graphs have an independent covering with respect to the $[n, k, r]$-partition. We show that all $[3, 3, 3]$-partite graphs have an optimal orthogonal colouring. Lastly, we show that all $[\frac{k}{2}, k, k]$-partite graphs have an independent covering with respect to the $[n, k, r]$-partition. This is achieved by constructing an orthogonal colouring via Hall's Condition.

## 2.1 Orthogonal Colourings of Cayley Graphs

A *generating set* of a group is a subset of the group set not containing the identity element such that every element of the group can be expressed as a combination under the group operation of finitely many elements of the subset and their inverses. Let $(G, \circ)$ denote a group $G$ with operation $\circ$ and let $S$ be a generating set of $(G, \circ)$. The associated Cayley graph, denoted $\Gamma(G, S)$, has a vertex for each element of $G$, and there is a directed edge from $i$ to $j$, if and only if $i \circ j^{-1} \in S$.

As an example, consider the group $\mathbb{Z}_9$ with group operation $+_9$. Let a generating set of $(\mathbb{Z}_9, +_9)$ be the set $S = \{1, 3\}$. There is a directed edge from $i$ to $j$ if and only if $(i - j)(\text{mod } 9) = 1$ or $(i - j)(\text{mod } 9) = 3$. The Cayley digraph $\Gamma(\mathbb{Z}_9, S)$ illustrating these adjacencies is shown in Figure 2.1.1.

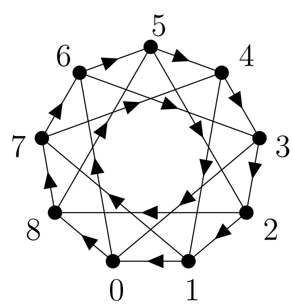


Figure 2.1.1: Digraph $\Gamma(\mathbb{Z}_9, \{1, 3\})$

A set $S$ is *closed under taking inverses* if it has the following property: If $i \in S$, then $i^{-1} \in S$. Throughout the remainder of this chapter, generating sets that are closed under taking inverses are considered. The reason for this condition is that it allows directed edges to be replaced with undirected edges. To see this, suppose $i$ is adjacent to $j$. By the definition of $S$, we get that $i \circ j^{-1} \in S$. Now, by the self inverse property, $(i \circ j^{-1})^{-1} = j \circ i^{-1} \in S$. Hence, $j$ is adjacent to $i$. Therefore, if $i$ is adjacent to $j$, then $j$ is adjacent to $i$. Thus, we can replace the directed edges with undirected edges.

Furthermore, it is assumed that the generating sets do not contain the identity element. The reason for this second condition is that it results in the Cayley graphs not having any loops. Therefore, by imposing these two conditions, the Cayley graphs considered in this chapter will be simple and undirected.

### 2.1.1 Orthogonal Colourings of Cycle Graphs

To start, orthogonal colourings of the cycle graph $C_n$ are explored. To see that $C_n$ is a Cayley graph, consider $\mathbb{Z}_n$ with group operation $+_n$, addition modulo $n$. It follows that $\Gamma(\mathbb{Z}_n, \{1, n-1\}) \cong C_n$. The following lemma shows that in most cases, $C_n$ has an optimal orthogonal colouring.

**Lemma 2.1.1.** *For $n > 4$, if $\lceil \sqrt{n} \rceil \nmid (n-1)$ and $\lceil \sqrt{n} \rceil \nmid \left(n - 1 + \left\lfloor \frac{n-1}{\lceil \sqrt{n} \rceil} \right\rfloor\right)$, then $O\chi(C_n) = \lceil \sqrt{n} \rceil$.*

**Proof:** Let $v_i \in \mathbb{Z}_n$ where $0 \leq i < n$ and let $N = \lceil \sqrt{n} \rceil$. Define the two colourings as $c_1(v_i) = i (\text{mod } N)$ and $c_2(v_i) = \left(i + \left\lfloor \frac{i}{N} \right\rfloor\right) (\text{mod } N)$. For illustration, these two colourings are applied to $C_9$ in Figure 2.1.2.

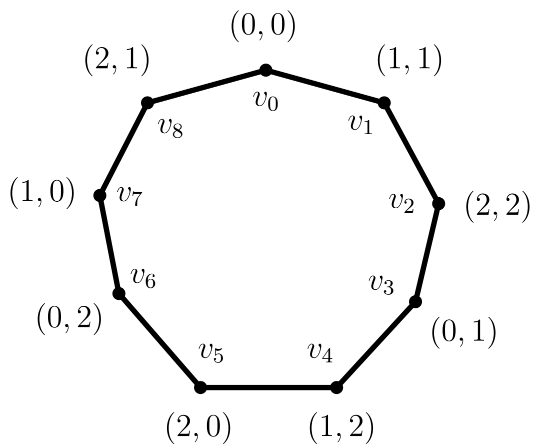


Figure 2.1.2: Orthogonal Colouring of $C_9$

We now show that $c_1$ and $c_2$ are both proper colourings. For $0 \leq i \leq n-2$, $c_1(v_i) = i(\text{mod } N) \neq (i+1)(\text{mod } N) = c_1(v_{i+1})$. Since $N \nmid (n-1)$ by assumption, it follows that $c_1(v_{n-1}) = (n-1)(\text{mod } N) \neq 0 = c_1(v_0)$. Therefore, $c_1$ is a proper colouring.

To show that $c_2$ is a proper colouring, notice that $\left\lfloor \frac{i}{N} \right\rfloor \leq \left\lfloor \frac{i+1}{N} \right\rfloor \leq \left\lfloor \frac{i}{N} \right\rfloor + 1$. Therefore, for $0 \leq i \leq n-2$, it follows that $1 \leq c_2(v_{i+1}) - c_2(v_i) \leq 2$. Since $N > 2$, it follows that $c_2(v_i) \neq c_2(v_{i+1})$. Now, since $N \nmid \left(n - 1 + \left\lfloor \frac{n-1}{N} \right\rfloor\right)$ by assumption, it follows that $c_2(v_{n-1}) = \left(n - 1 + \left\lfloor \frac{n-1}{N} \right\rfloor\right) (\text{mod } N) \neq 0 = c_2(v_0)$. Therefore, $c_2$ is a proper colouring.

We now show that $c_1$ and $c_2$ are orthogonal colourings. Suppose otherwise, that is, $c_1(v_i) = c_1(v_j)$ and $c_2(v_i) = c_2(v_j)$ where $i \neq j$. Since $c_1(v_i) = c_1(v_j)$, this implies that $i = j + mN$, where $0 < m < N$. Therefore:

$$\begin{aligned} c_2(v_i) &\equiv \left(j + \left\lfloor \frac{j+mN}{N} \right\rfloor\right) (\text{mod } N) \\ &\equiv \left(j + m + \left\lfloor \frac{j}{N} \right\rfloor\right) (\text{mod } N) \\ &\equiv (c_2(v_j) + m)(\text{mod } N). \end{aligned}$$

Since $c_2(v_i) = c_2(v_j)$, this gives that $m \equiv 0(\text{mod } N)$, contradicting $0 < m < N$. Therefore, $c_1$ and $c_2$ are orthogonal colourings of $C_n$. Since $c_1$ and $c_2$ both used $N$ colours, the fewest possible, $O\chi(C_n) = N$. $\square$

Notice that the orthogonality property of $c_1$ and $c_2$ in Lemma 2.1.1 did not depend on the assumed divisibility conditions. Therefore, the problem with using $c_1$ and $c_2$ in the cases where $N \mid (n-1)$ and $N \mid \left(n - 1 + \left\lfloor \frac{n-1}{N} \right\rfloor\right)$, is that there is some sort of colour conflict. We show with the following lemma that this conflict can be resolved by assigning $v_{n-1}$ a different colour pair.

**Lemma 2.1.2.** *For $n > 16$, if $\lceil \sqrt{n} \rceil \mid (n-1)$ or $\lceil \sqrt{n} \rceil \mid \left(n - 1 + \left\lfloor \frac{n-1}{\lceil \sqrt{n} \rceil} \right\rfloor\right)$, then $O\chi(C_n) = \lceil \sqrt{n} \rceil$.*

**Proof:** Let $v_i \in \mathbb{Z}_n$ where $0 \leq i < n$ and let $N = \lceil \sqrt{n} \rceil$. Let $c_1(v_i) = i(\text{mod } N)$ and $c_2(v_i) = \left(i + \left\lfloor \frac{i}{N} \right\rfloor\right) (\text{mod } N)$ be the two colourings from Lemma 2.1.1. There are four different cases to consider here, which are **Case 1**: $N \mid (n-1)$ and $N \nmid \left(n + \left\lfloor \frac{n}{N} \right\rfloor\right)$, **Case 2**: $N \nmid n$ and $N \mid \left(n - 1 + \left\lfloor \frac{n-1}{N} \right\rfloor\right)$, **Case 3**: $N \mid (n-1)$ and $N \mid \left(n + \left\lfloor \frac{n}{N} \right\rfloor\right)$, and **Case 4**: $N \mid n$ and $N \mid \left(n - 1 + \left\lfloor \frac{n-1}{N} \right\rfloor\right)$. For Case 1 and Case 2, the orthogonal colouring $(\bar{c}_1, \bar{c}_2)$ is used, where $\bar{c}_1$ and $\bar{c}_2$ are defined as follows:

$$\bar{c}_1(v_i) = \begin{cases} c_1(v_i) & 0 \leq i \leq n-2 \\ n(\text{mod } N) & i = n-1 \end{cases}$$

$$\bar{c}_2(v_i) = \begin{cases} c_2(v_i) & 0 \leq i \leq n-2 \\ \left(n + \left\lfloor \frac{n}{N} \right\rfloor\right) (\text{mod } N) & i = n-1 \end{cases}$$

In words, the pair of colourings $(\bar{c}_1, \bar{c}_2)$ replaces the colour pair that would have been assigned to $v_{n-1}$ by $(c_1, c_2)$ with the next available colour pair in the sequence. For example, an orthogonal colouring of $C_{18}$ using $(\bar{c}_1, \bar{c}_2)$ is given in Figure 2.1.3. Originally, the colour pair $(2, 0)$ would have been assigned by $(c_1, c_2)$. Therefore, the next available colour pair, $(3, 1)$, is assigned instead.

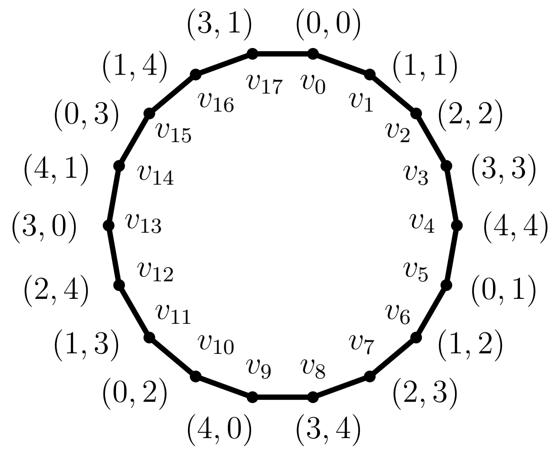


Figure 2.1.3: Orthogonal Colouring of $C_{18}$

For $0 \leq i \leq n-2$, $\bar{c}_1(v_i) = c_1(v_i)$ and $\bar{c}_2(v_i) = c_2(v_i)$. Therefore, by the proof of Lemma 2.1.1, there are no colour conflicts between these vertices. Note that in both Case 1 and Case 2, $n(\text{mod } N) \neq 0$ and $\left(n + \left\lfloor \frac{n}{N} \right\rfloor\right)(\text{mod } N) \neq 0$. Therefore, $\bar{c}_1(v_{n-1}) \neq 0 = \bar{c}_1(v_0)$, and $\bar{c}_2(v_{n-1}) \neq 0 = \bar{c}_2(v_0)$. Thus, there are no colour conflicts between the vertices $v_0$ and $v_{n-1}$.

Notice that $\left\lfloor \frac{n-2}{N} \right\rfloor \leq \left\lfloor \frac{n}{N} \right\rfloor \leq \left\lfloor \frac{n-2}{N} \right\rfloor + 1$. Therefore, $2 \leq \bar{c}_2(v_{n-1}) - \bar{c}_2(v_{n-2}) \leq 3$. Since $N > 4$ by assumption, this implies that $\bar{c}_2(v_{n-1}) \neq \bar{c}_2(v_{n-2})$. Also, since $N > 4$, $\bar{c}_1(v_{n-2}) = (n-2)(\text{mod } N) \not\equiv n(\text{mod } N) = \bar{c}_1(v_{n-1})$. Thus, there are no colour conflicts between the vertices $v_{n-2}$ and $v_{n-1}$. Therefore, $\bar{c}_1$ and $\bar{c}_2$ are proper colourings of $C_n$. We now show that $\bar{c}_1$ and $\bar{c}_2$ are orthogonal colourings.

For $0 \leq i \leq n-2$, there are no orthogonal conflicts between the vertices $v_i$ by the proof of Lemma 2.1.1. In Case 1, since $n \equiv 1(\text{mod } N)$, the colour pair $(1, \left(1 + \left\lfloor \frac{n}{N} \right\rfloor\right)(\text{mod } N))$ is assigned to $v_{n-1}$. Let $i \equiv 1(\text{mod } N)$ and $i < n$. Let $m_1$ and $m_2$ be integers so that $i = m_1 N + 1$ and $n = m_2 N + 1$. Since $i < n$, this gives that $m_1 < m_2$. Therefore, $\bar{c}_2(v_i) = \left(1 + \left\lfloor \frac{m_1 N + 1}{N} \right\rfloor\right)(\text{mod } N)) \equiv (1 + m_1)(\text{mod } N) \not\equiv (1 + m_2)(\text{mod } N) = \bar{c}_2(v_{n-1})$. Hence, there are no orthogonal conflicts.

In Case 2, since $N \mid \left(n-1+\lfloor\frac{n-1}{N}\rfloor\right)$, the colour pair $(n(\text{mod } N), 1)$ is assigned to $v_{n-1}$. A similar argument as in Case 1 shows that there are no orthogonal conflicts. Therefore, in both Case 1 and Case 2, $\bar{c}_1$ and $\bar{c}_2$ are proper orthogonal colourings. For Case 3 and Case 4, a different orthogonal colouring $(\hat{c}_1, \hat{c}_2)$ is used, where $\hat{c}_1$ and $\hat{c}_2$ are defined as follows:

$$\hat{c}_1(v_i) = \begin{cases} c_1(v_i) & 0 \le i \le n-2 \\ (n+1)(\text{mod } N) & i = n-1 \end{cases}$$

$$\hat{c}_2(v_i) = \begin{cases} c_2(v_i) & 0 \le i \le n-2 \\ \left(n+1+\lfloor\frac{n+1}{N}\rfloor\right)(\text{mod } N) & i = n-1 \end{cases}$$

In words, the pair of colourings $(\hat{c}_1, \hat{c}_2)$ replaces the colour pair that would have been assigned to $v_{n-1}$ by $(c_1, c_2)$ with the second available colour pair in the sequence. For example, an orthogonal colouring of $C_{18}$ using $(\hat{c}_1, \hat{c}_2)$ is given in Figure 2.1.4. Originally, the colour pair $(2, 0)$ would have been assigned by $(c_1, c_2)$. Therefore, the next available colour pair $(4, 2)$ is assigned instead.

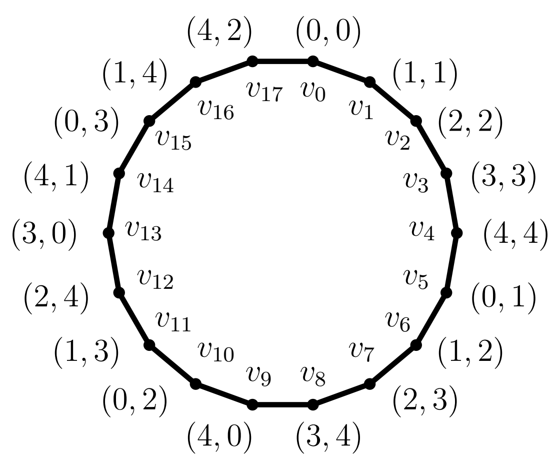


Figure 2.1.4: Orthogonal Colouring of $C_{18}$

We now show that $\hat{c}_1$ and $\hat{c}_2$ are proper colourings in both cases. For $0 \le i \le n-2$, $\hat{c}_1(v_i) = c_1(v_i)$ and $\hat{c}_2(v_i) = c_2(v_i)$. Therefore, by the proof of Lemma 2.1.1, there are no colour conflicts between these vertices. Note that in both Case 3 and Case 4, $(n+1)(\text{mod } N) \ne 0$ and $\left(n+1+\lfloor\frac{n+1}{N}\rfloor\right)(\text{mod } N) \ne 0$. Thus, $\hat{c}_1(v_{n-1}) \ne 0 = \hat{c}_1(v_0)$, and $\hat{c}_2(v_{n-1}) \ne 0 = \hat{c}_2(v_0)$.

Notice that $\lfloor \frac{n-2}{N} \rfloor \le \lfloor \frac{n+1}{N} \rfloor \le \lfloor \frac{n-2}{N} \rfloor + 1$. Therefore, $3 \le \hat{c}_2(v_{n-1}) - \hat{c}_2(v_{n-2}) \le 4$. Since $N > 4$ by assumption, this implies that $\hat{c}_2(v_{n-2}) \ne \hat{c}_2(v_{n-1})$. Also, since $N > 4$, $\hat{c}_1(v_{n-2}) = (n-2)(\text{mod } N) \not\equiv (n+1)(\text{mod } N) = \hat{c}_1(v_{n-1})$. Therefore, $\hat{c}_1$ and $\hat{c}_2$ are proper colourings of $C_n$. We now show that $\hat{c}_1$ and $\hat{c}_2$ are orthogonal colourings.

For $0 \le i \le n-2$, there are no orthogonal conflicts on the vertices $v_i$ by the proof of Lemma 2.1.1. In Case 3, since $n \equiv 1(\text{mod } N)$, the colour pair $(2, \left(2 + \lfloor \frac{n+2}{N} \rfloor\right)(\text{mod } N))$ is assigned to $v_{n-1}$. Let $i \equiv 2(\text{mod } N)$ and $i < n$. Let $m_1$ and $m_2$ be integers so that $i = m_1N + 2$ and $n = m_2N + 1$. Since $i < n$, this gives that $m_1 < m_2$. Therefore, $\hat{c}_2(v_i) = \left(2 + \lfloor \frac{m_1N+1}{N} \rfloor\right)(\text{mod } N) \equiv (2+m_1)(\text{mod } N) \not\equiv (2+m_2)(\text{mod } N) = \hat{c}_2(v_{n-1})$.

In Case 4, since $N \mid \left(n - 1 + \lfloor \frac{n-1}{N} \rfloor\right)$, the colour pair $((n+2)(\text{mod } N), 2)$ is assigned to $v_{n-1}$. A similar argument as in Case 3 shows that there are no orthogonal conflicts. Thus, in both Case 3 and Case 4, $\hat{c}_1$ and $\hat{c}_2$ are orthogonal colourings. $\square$

For $5 \le n \le 16$, the remaining orders of $n$ that are not covered by Lemma 2.1.1 are $n = 6, 7, 8, 11, 13, 14$. These can all be orthogonally coloured with $\lceil \sqrt{n} \rceil$ colours, as we show in Figure 2.1.5 and Figure 2.1.6.

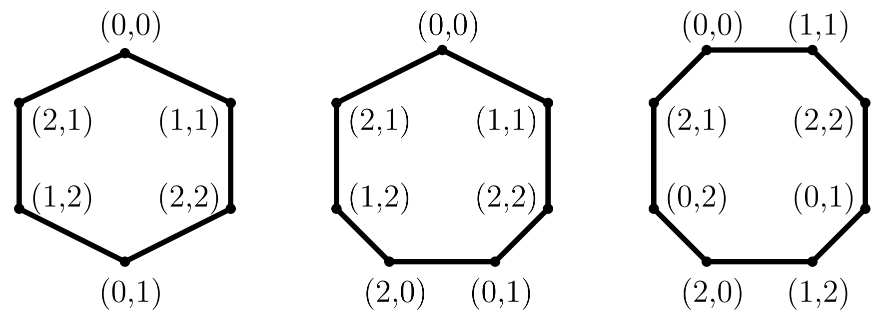


Figure 2.1.5: Orthogonal Colourings of $C_6, C_7$, and $C_8$.

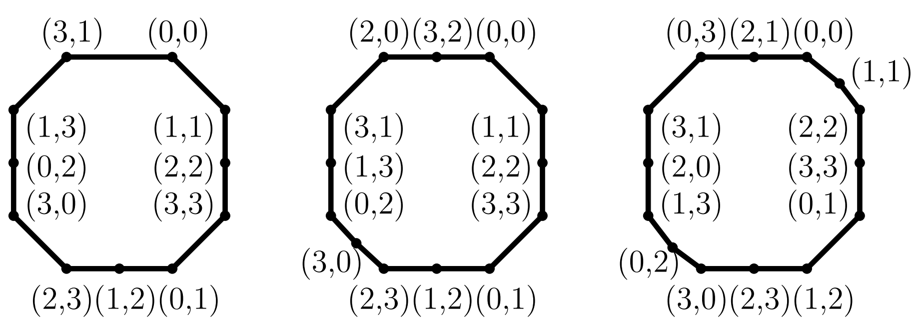


Figure 2.1.6: Orthogonal Colouring of $C_{11}, C_{13}$, and $C_{14}$.

**Theorem 2.1.3.** *$O\chi(C_n) = \lceil \sqrt{n} \rceil$ if and only if $n > 4$.*

**Proof:** If $n > 4$, then $O\chi(C_n) = \lceil \sqrt{n} \rceil$ from the combined results of Lemma 2.1.1, Lemma 2.1.2, Figure 2.1.5, and Figure 2.1.6. Thus, we now show that $C_3$ and $C_4$ cannot be optimally orthogonally coloured. Note that a vertex colouring of a connected graph with 2 colours is unique, up to relabelling of the colours.

Therefore, an orthogonal colouring with 2 colours does not exist, unless the graph is $K_2$. This is because the first and second colouring are the same, and thus if any colour is used twice in the first colouring, then that colour pair occurs twice. Hence, an orthogonal colouring of $C_3$ and $C_4$ with 2 colours does not exist. However, an orthogonal colouring with 3 colours does exist, as we show in Figure 2.1.7. $\square$

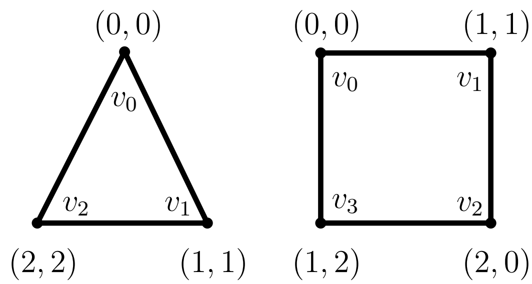


Figure 2.1.7: Orthogonal Colourings of $C_3$ and $C_4$.

Next, we construct multiple mutual orthogonal colourings of $C_n$. The following theorem gives a method to construct multiple orthogonal colourings if $\lceil \sqrt{n} \rceil$ is a prime number.

**Theorem 2.1.4.** *If $\lceil \sqrt{n} \rceil = p$ is a prime number, then $O\chi_{p-2}(C_n) = p$.*

**Proof:** Let $v_i \in \mathbb{Z}_n$ where $0 \leq i < n$. We now show that there are $p - 1$ orthogonal assignments. For $0 \leq k < p$, consider the following $p - 1$ colourings:

$$c_k(v_i) = \left(i + k \left\lfloor \frac{i}{p} \right\rfloor \right) \pmod{p}$$

We now show that any two are mutually orthogonal. Suppose otherwise, that is, $c_t(v_i) = c_t(v_j)$ and $c_s(v_i) = c_s(v_j)$, where $0 \leq t < s < p$ and $0 \leq i < j < n$. Then

$$(i - j) \equiv t \left( \left\lfloor \frac{j}{p} \right\rfloor - \left\lfloor \frac{i}{p} \right\rfloor \right) \pmod{p} \tag{2.1}$$

$$(i - j) \equiv s \left( \left\lfloor \frac{j}{p} \right\rfloor - \left\lfloor \frac{i}{p} \right\rfloor \right) \pmod{p} \tag{2.2}$$

Note that $t, s \in \mathbb{Z}_p$, which is a field because $p$ is a prime. Thus if $t \neq 0$, then $t^{-1}$ and $s^{-1}$ exist. Equation 2.1 and Equation 2.2 can then be rearranged to give that $(i-j)(s^{-1}-t^{-1}) \equiv 0 (\mathrm{mod}\ p)$. However, $0 \leq t < s < p$, so $t^{-1} \not\equiv s^{-1} (\mathrm{mod}\ p)$. Therefore, it must be that $i \equiv j (\mathrm{mod}\ p)$, or equivalently, $j = i + mp$ where $0 < m < p$. Since $s, m \neq 0$, $sm \not\equiv 0 (\mathrm{mod}\ p)$ because $\mathbb{Z}_p$ has no zero divisors. This implies that:

$$\begin{aligned}\left(j + s\left\lfloor \frac{j}{p} \right\rfloor\right) (\mathrm{mod}\ p) &\equiv \left(i + s\left\lfloor \frac{i+mp}{p} \right\rfloor\right) (\mathrm{mod}\ p)\\ &\equiv \left(i + sm + s\left\lfloor \frac{i}{p} \right\rfloor\right) (\mathrm{mod}\ p)\\ &\not\equiv \left(i + s\left\lfloor \frac{i}{p} \right\rfloor\right) (\mathrm{mod}\ p).\end{aligned}$$

This contradicts Equation 2.2 however, so $t = 0$. Putting $t = 0$ into Equation 2.1 gives that $i \equiv j (\mathrm{mod}\ p)$, and the same contradiction arises. Thus, the colourings are all mutually orthogonal. We now show that $p-2$ of the colourings are proper. Notice that $k\left\lfloor \frac{i}{p} \right\rfloor \leq k\left\lfloor \frac{i+1}{p} \right\rfloor \leq k\left\lfloor \frac{i}{p} \right\rfloor + k$.

Therefore, $1 \leq c_k(v_{i+1}) - c_k(v_i) \leq k$, where $1 \leq i \leq n-2$. Since $k < p$, this gives that $c_k(v_{i+1}) \neq c_k(v_i)$. Now, notice that the colour 0 is assigned to $v_0$ in all of the colourings. Therefore, by the mutual orthogonality of the colourings, at most one colouring has 0 assigned to the vertex $v_{n-1}$. Therefore, choosing the $k-1 = p-2$ colourings that don't assign the colour 0 to the vertex $v_{n-1}$ gives a $(p-2)$-orthogonal colouring. $\square$

For example, we give a 3-orthogonal colouring of $C_{18}$ in Figure 2.1.8.

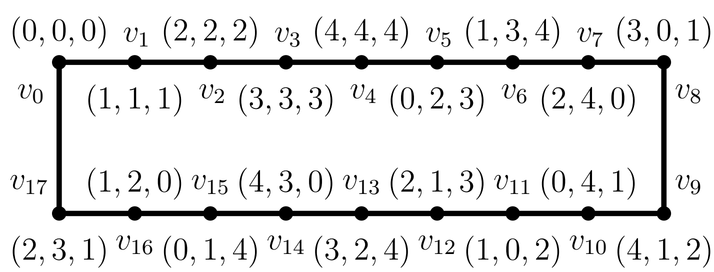


Figure 2.1.8: 3-Orthogonal Colouring of $C_{18}$

**Corollary 2.1.5.** *If* $\lceil \sqrt{n} \rceil = p$ *is a prime number, then* $O\chi_{p-1}(P_n) = p$.

**Proof:** Since the edge between $v_{n-1}$ and $v_0$ is not present in $P_n$, all of the colourings in the proof of Theorem 2.1.4 on $P_n$ are proper and orthogonal. $\square$

### 2.1.2 Orthogonal Colourings of Paley Graphs

The reader is referred to [22] for any group theory definitions or material omitted in this section. Some relevant properties of finite fields are summarized. Finite fields of order $q$ exist if and only if $q = p^k$ where $p$ is a prime and $k \in \mathbb{Z}^+$. These fields are unique up to isomorphism, so they are denoted $\mathbb{F}_q$. The multiplicative group, denoted $\mathbb{F}_q^*$, is cyclic, so all non-zero elements can be expressed as powers of a single element, called a *primitive element* of the field.

Finite fields can be explicitly constructed as such. If $q = p^k$, then $\mathbb{F}_q \cong \mathbb{Z}_p[X]/(P)$ where $(P)$ is the ideal generated by an irreducible polynomial $P$ of degree $k$ in $\mathbb{Z}_p[X]$. Such a polynomial always exists. That is, the elements of $\mathbb{F}_q$ are polynomials over $\mathbb{Z}_p$ whose degree is strictly less than $k$. Addition and subtraction in $\mathbb{F}_q$ is defined as addition and subtraction over $\mathbb{Z}_p$. Multiplication in $\mathbb{F}_q$ is defined as the remainder of Euclidean division by $P$ in $\mathbb{Z}_p[X]$.

The *Paley graph*, denoted $QR(q)$, can be constructed as a Cayley graph of $\mathbb{F}_q$. Let $\alpha$ be a primitive element of $\mathbb{F}_q$ and let $S = \{\alpha^{2m} : 1 \leq m \leq \frac{q-1}{2}\}$. Notice that $S$ is the set of quadratic residues in $\mathbb{F}_q$. The Paley graph is $QR(q) = \Gamma(\mathbb{F}_q, S)$, where the group operation is addition. For example, $QR(9)$ is shown in Figure 2.1.9. In this case, $P = x^2 + 1$ is an irreducible polynomial, $\alpha = x + 1$ is a primitive element, and $S = \{1, 2, x, 2x\}$ is the set of quadratic residues.

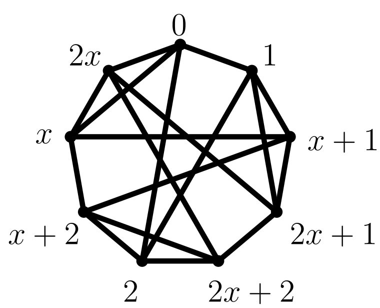


Figure 2.1.9: $QR(9)$

A graph is *self complementary* if it is isomorphic to its graph complement. Paley graphs have a variety of interesting properties, but the one that we use is that they are self-complementary [35]. This allows the cliques in the Paley graphs to be considered as colour classes, since they can be turned into independent sets by taking the graph complement. The following lemma describes a relation between cosets.

**Lemma 2.1.6** (Fraleigh [22])**.** *Let $G$ be a group, $H \leq G$, and $K \leq G$. For any $a, b, c \in G$, either $(a + H) \cap (b + K) = \emptyset$ or $(a + H) \cap (b + K) = c + (H \cap K)$.*

The following theorem gives a method for constructing orthogonal colourings of Paley graphs by utilizing the structure of the finite field. The idea is to find particular subgroups that share at most one element. Lemma 2.1.6 then gives that their cosets share at most one element. These cosets can then be used as the colour classes. The following theorem formalizes this argument.

**Theorem 2.1.7.** *For $p > 2$ a prime and an integer $r \geq 1$, $O\chi_{\frac{p^r+1}{2}}(QR(p^{2r})) = p^r$.*

**Proof:** Let $G = \mathbb{F}_{p^{2r}}$. Since $p^r \mid p^{2r}$, there exists a subfield $H \subset G$, where $H \cong \mathbb{F}_{p^r}$. Also, notice that $H^*$ is a multiplicative subgroup of $G^*$ with index $\frac{p^{2r}-1}{p^r-1} = p^r + 1$. Thus, $H = \{0\} \bigcup \{\alpha^{m(p^r+1)} | 1 \leq m < p^r - 1\}$, where $\alpha$ is a primitive element of $G$.

Next, for $0 \leq i < \frac{p^r+1}{2}$, consider the sets $H_i = \{0\} \bigcup \{\alpha^{m(p^r+1)+2i} | 1 \leq m < p^r - 1\}$. The sets $H_i$ are additive subgroups of $G$. To see this, it suffices to show for $x, y \in H_i$ that $x - y \in H_i$. Without loss of generality, suppose that $m > n$.

$$\begin{aligned}
x - y &= \alpha^{m(p^r+1)+2i} - \alpha^{n(p^r+1)+2i} && \text{by the definition of } H_i \\
&= \alpha^{2i}(\alpha^{m(p^r+1)} - \alpha^{n(p^r+1)}) \\
&= \alpha^{2i}(\alpha^{t(p^r+1)}) && \text{because } \mathbb{F}_{p^{2r}} \text{ is a field} \\
&= \alpha^{t(p^r+1)+2i} \\
&\in H_i && \text{by the definition of } H_i.
\end{aligned}$$

Therefore, the sets $H_i$ are additive subgroups of $G$. Since $\alpha^{m(p^r+1)+2i}$ is an even power of $\alpha$, $H_i \subset S$ and so $H_i$ are cliques in $QR(p^{2r})$. Similarly, the $p^r - 1$ cosets of $H_i$ are cliques in $QR(p^{2r})$. Therefore, $H_i$ and its cosets are independent sets in the complement graph. For $1 \leq j \leq p^r - 1$, let $H_i + k_j$ be the $j$th coset of $H_i$. Then, define the colourings of the complement as $c_i(x) = j$ for $x \in H_i + k_j$. Since these are independent sets, the $c_i$ are proper colourings.

Since $0 \leq 2i, 2j < p^r + 1$ and $1 \leq m < p^r - 1$, it follows that $\alpha^{m(p^r+1)+2i} \neq \alpha^{m(p^r+1)+2j}$. Therefore, it follows that $H_i \bigcap H_j = \{0\}$. Thus, by Lemma 2.1.6, any coset of $H_i$ and any coset of $H_j$ will intersect in at most 1 element. Therefore, the colour pair $(i, j)$ is only assigned to at most one vertex. Hence, there is no orthogonal conflict. For example, an orthogonal colouring of $QR(9)$ is given in Figure 2.1.10 $\square$

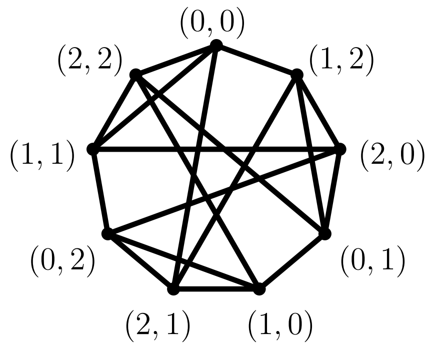


Figure 2.1.10: Orthogonal Colouring of $QR(9)$

### 2.1.3 Orthogonal Colourings of Circulant Graphs

A *circulant graph* is a Cayley graph of a cyclic group. For instance, the cycle graph and Paley graph are circulant graphs. In this section, circulant graphs on the additive group $\mathbb{Z}_{p^2}$ are now considered, where $p$ is a prime. We focus on the group $\mathbb{Z}_{p^2}$ so that the Cayley graphs have a square number of vertices and so that every element in $\mathbb{Z}_p$ has a multiplicative inverse.

Also, the size of the generating set $S$ is varied throughout this section. To start, a method for orthogonally colouring $\Gamma(\mathbb{Z}_{p^2}, S)$ is constructed when $|S|$ is sufficiently small. The following function was obtained by trying to generalize the orthogonal assignment used in the proof of Lemma 2.1.1. Let $\alpha \in \mathbb{Z}_p\backslash\{0\}$ and consider the function $\hat{F}_{\alpha,p} : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_{p^2}$ defined by

$$\hat{F}_{\alpha,p}(i, j) = ((\alpha(j - i)(\text{mod } p)) + p(2i - j))(\text{mod } p^2).$$

This function assigns colour pairs from $\mathbb{Z}_p \times \mathbb{Z}_p$ to the vertices of $\Gamma(\mathbb{Z}_{p^2}, S)$. Thus, the goal is to show that the inverse of this function is an orthogonal colouring. The following lemma shows that this assignment is bijective.

**Lemma 2.1.8.** *If $\alpha \in \mathbb{Z}_p\backslash\{0\}$, then $\hat{F}_{\alpha,p}(i, j)$ is a bijection.*

**Proof:** Since $|\mathbb{Z}_p \times \mathbb{Z}_p| = |\mathbb{Z}_{p^2}| = p^2$, it is sufficient to show that $\hat{F}_{\alpha,p}$ is an injective function. Suppose $\hat{F}_{\alpha,p}(i, j) = \hat{F}_{\alpha,p}(r, s)$. For this equality to be true, the two modular components of $\hat{F}_{\alpha,p}$ must be equal. That is:

$$\alpha(j - i)(\text{mod } p) = \alpha(s - r)(\text{mod } p). \tag{2.3}$$

$$p(2i - j)(\text{mod } p^2) = p(2r - s)(\text{mod } p^2). \tag{2.4}$$

Note that $\alpha$ has a multiplicative inverse in $\mathbb{Z}_p$, which is denoted by $\alpha^{-1}$. Therefore, Equation (2.3) can be rewritten as $(i-j)(\text{mod } p) = (r-s)(\text{mod } p)$. Multiplying by $p$ gives that $p(i-j)(\text{mod } p^2) = p(r-s)(\text{mod } p^2)$. Substituting this into Equation (2.4) gives that $pi(\text{mod } p^2) = pr(\text{mod } p^2)$. Note that this implies that $i(\text{mod } p) = r(\text{mod } p)$. Since $i$ and $r$ are elements of $\mathbb{Z}_p$, it follows that $i = r$. Lastly, substituting $i = r$ into $(i-j)(\text{mod } p) = (r-s)(\text{mod } p)$ gives that $j(\text{mod } p) = s(\text{mod } p)$. Similarly, since $j$ and $s$ are elements of $\mathbb{Z}_p$, it follows that $j = s$. Therefore, $(i,j) = (r,s)$. $\square$

In particular, $\hat{F}_{\alpha,p}^{-1} : \mathbb{Z}_{p^2} \to \mathbb{Z}_p \times \mathbb{Z}_p$ is an injective function. Since $\hat{F}_{\alpha,p}^{-1}$ is injective into $\mathbb{Z}_p \times \mathbb{Z}_p$, $\hat{F}_{\alpha,p}^{-1}$ will orthogonally assign the vertices to colour pairs in $\mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, if we can show that the components of $\hat{F}_{\alpha,p}^{-1}$ are both proper colourings, then $\hat{F}_{\alpha,p}^{-1}$ will be an orthogonal colouring. For a general generating set $S$ and $\alpha$, this is not always the case. However, the following theorem shows that if the size of the generating set $S$ is sufficiently small, then there is an $\alpha$ for which $\hat{F}_{\alpha,p}^{-1}$ is a proper colouring of $\Gamma(\mathbb{Z}_{p^2}, S)$.

**Theorem 2.1.9.** *If $|S| < \frac{p-1}{2}$, then $O\chi(\Gamma(\mathbb{Z}_{p^2}, S)) = p$.*

**Proof:** The goal is to show that $\hat{F}_{\alpha,p}^{-1}$ is an orthogonal colouring for some $\alpha \in \mathbb{Z}_p \backslash \{0\}$. By Lemma 2.1.8, $\hat{F}_{\alpha,p}^{-1}$ is an orthogonal assignment to the vertices. We now show that there is an $\alpha$ such that $\hat{F}_{\alpha,p}^{-1}$ is a proper colouring. Suppose two vertices $k$ and $l$ receive the same colour in the first colouring. That is, $\hat{F}_{\alpha,p}^{-1}(k) = (i, (j+x)(\text{mod } p))$ and $\hat{F}_{\alpha,p}^{-1}(l) = (i,j)$ for some $i,j \in \mathbb{Z}_p$ and $x \in \mathbb{Z}_p \backslash \{0\}$. Then:

$$k - l = \hat{F}_{\alpha,p}(i, (j+x)(\text{mod } p)) - \hat{F}_{\alpha,p}(i,j) = (((\alpha x)(\text{mod } p)) - px)(\text{mod } p^2).$$

Similarly, suppose two vertices $k$ and $l$ receive the same colour in the second colouring. That is, $\hat{F}_{\alpha,p}^{-1}(k) = ((i+x)(\text{mod } p), j)$ and $\hat{F}_{\alpha,p}^{-1}(l) = (i,j)$ for some $i,j \in \mathbb{Z}_p$ and $x \in \mathbb{Z}_p \backslash \{0\}$. Then:

$$k - l = \hat{F}_{\alpha,p}((i+x)(\text{mod } p), j) - \hat{F}_{\alpha,p}(i,j) = (((-\alpha x)(\text{mod } p)) + 2px)(\text{mod } p^2).$$

Let $A_\alpha$ be the set of differences in the first colouring and $B_\alpha$ be the set of differences in the second colouring. That is, $A_\alpha = \{(((\alpha x)(\text{mod } p)) - px)(\text{mod } p^2) | x \in \mathbb{Z}_p \backslash \{0\}\}$ and $B_\alpha = \{(((-\alpha x)(\text{mod } p)) + 2px)(\text{mod } p^2) | x \in \mathbb{Z}_p \backslash \{0\}\}$. Therefore, there is a colour conflict in the first colouring if and only if $S \cap A_\alpha \neq \emptyset$. Similarly, there is a colour conflict in the second colouring in and only if $S \cap B_\alpha \neq \emptyset$.

Properties of $A_\alpha$ and $B_\alpha$ are now discussed. The first is that $A_\alpha = B_{2\alpha (\text{mod } p)}$. Since $\mathbb{Z}_p$ is a field, 2 has a multiplicative inverse in $\mathbb{Z}_p$, which is denoted $2^{-1}$. Since $\mathbb{Z}_p$ has no zero divisors, $\mathbb{Z}_p \backslash \{0\} = -2\mathbb{Z}_p \backslash \{0\}$. Therefore, it follows that

$$\begin{aligned} B_{2\alpha(\text{mod } p)} &= \{(((-2\alpha x)(\text{mod } p)) + 2px)(\text{mod } p^2) | x \in \mathbb{Z}_p \backslash \{0\}\} \\ &= \{(((-2\alpha(-2^{-1}x))(\text{mod } p)) + 2p(-2^{-1}x))(\text{mod } p^2) | x \in -2\mathbb{Z}_p \backslash \{0\}\} \\ &= \{(((\alpha x)(\text{mod } p)) - px)(\text{mod } p^2) | x \in -2\mathbb{Z}_p \backslash \{0\}\} \\ &= \{(((\alpha x)(\text{mod } p)) - px)(\text{mod } p^2) | x \in \mathbb{Z}_p \backslash \{0\}\} \\ &= A_\alpha. \end{aligned}$$

A second property of $A_\alpha$ is the following. The $A_\alpha$s together with $\{m | m \in \mathbb{Z}_p \backslash \{0\}\}$ and $\{mp | m \in \mathbb{Z}_p \backslash \{0\}\}$ are disjoint. First, notice that $\alpha x (\text{mod } p) \neq 0$ for any $x \in \mathbb{Z}_p \backslash \{0\}$. Therefore, it follows that $A_\alpha \cap \{mp | m \in \mathbb{Z}_p \backslash \{0\}\} = \emptyset$. Next, notice that $-px (\text{mod } p^2) \neq 0$ for any $x \in \mathbb{Z}_p \backslash \{0\}$. Therefore, $A_\alpha \cap \{m | m \in \mathbb{Z}_p \backslash \{0\}\} = \emptyset$. It is also the case that $\{mp | m \in \mathbb{Z}_p \backslash \{0\}\} \cap \{m | m \in \mathbb{Z}_p \backslash \{0\}\} = \emptyset$. This is because the first set consists of multiples of $p$ and the second set does not. We now show that the $A_\alpha$s are all mutually disjoint.

Let $\alpha_1, \alpha_2 \in \mathbb{Z}_p \backslash \{0\}$ where $\alpha_1 \neq \alpha_2$. Suppose that $A_{\alpha_1} \cap A_{\alpha_2} \neq \emptyset$. There then exists some $c = ((\alpha_1 x (\text{mod } p)) - px)(\text{mod } p^2)$ and $c = ((\alpha_2 y (\text{mod } p)) - py)(\text{mod } p^2)$ for some $x, y \in \mathbb{Z}_p \backslash \{0\}$. Note that $c$ can be uniquely written as $r - px$, where $r \in \mathbb{Z}_p \backslash \{0\}$. This means that $-px (\text{mod } p^2) = -py (\text{mod } p^2)$, so $x (\text{mod } p) = y (\text{mod } p)$. Since $x$ and $y$ are elements of $\mathbb{Z}_p$, its follows that $x = y$.

However, since $c$ can be uniquely written as $r - px$, it follows that $\alpha_1 x (\text{mod } p) = \alpha_2 y (\text{mod } p)$. Substituting $x = y$ into this gives that $\alpha_1 (\text{mod } p) = \alpha_2 (\text{mod } p)$. Since $\alpha_1$ and $\alpha_2$ are elements of $\mathbb{Z}_p \backslash \{0\}$, it then follows that $\alpha_1 = \alpha_2$. Therefore, it follows that the $A_\alpha$s along with $\{mp | m \in \mathbb{Z}_p \backslash \{0\}\}$ and $\{m | m \in \mathbb{Z}_p \backslash \{0\}\}$ are all mutually disjoint.

We now show that there is a choice of $\alpha$ for which $\hat{F}^{-1}_{\alpha,p}$ is a proper colouring. Recall that there is a colour conflict in the first colouring if and only if $S \cap A_\alpha \neq \emptyset$ and there is a colour conflict in the second colouring if and only if $S \cap B_\alpha \neq \emptyset$. Let $c \in S$ and suppose that $c \in \{mp | m \in \mathbb{Z}_p \backslash \{0\}\}$ or $c \in \{m | m \in \mathbb{Z}_p \backslash \{0\}\}$. Since the $A_\alpha$s, along with $\{mp | m \in \mathbb{Z}_p \backslash \{0\}\}$ and $\{m | m \in \mathbb{Z}_p \backslash \{0\}\}$ are all mutually disjoint, $c \notin A_\alpha$ for any $\alpha \in \mathbb{Z}_p \backslash \{0\}$. Since $B_{2\alpha(\text{mod } p)} = A_\alpha$, $c \notin B_\alpha$ for any $\alpha$ as well.

Suppose now that $c \in A_{\alpha_1}$ for some $\alpha_1 \in \mathbb{Z}_p \backslash \{0\}$. Since the $A_\alpha$'s are disjoint, for any $\alpha_2 \neq \alpha_1$, $c \notin A_{\alpha_2}$. Thus, since $B_{2\alpha (\text{mod } p)} = A_\alpha$, it follows that $c \in B_{2\alpha_1}$ but $c \notin B_{\alpha_3}$ for any $\alpha_3 \neq 2\alpha_1$. Hence, if $\alpha_1$ is chosen, then $c \in A_{\alpha_1}$, and there is a colour conflict in the first colouring. Similarly, if $2\alpha_1$ is chosen, then $c \in B_{2\alpha_1}$, and there is a colour conflict in the second colouring.

Any other choice of $\alpha$ will result in $c \notin A_\alpha$ and $c \notin B_\alpha$. Thus, each $c \in S$ will result in at most 2 restrictions on the choice of $\alpha$. Since there are at most $|S| < \frac{p-1}{2}$ elements in $S$, there are fewer than $p-1$ restrictions on the choice of $\alpha$. Since $\alpha \in \mathbb{Z}_p \backslash \{0\}$, there are $p-1$ choices for $\alpha$. Therefore, there are more choices than restrictions. $\square$

Theorem 2.1.9 says that if the size of the generating set is sufficiently small, then an orthogonal colouring can be constructed using only $p$ colours. This leads to the following question.

**Question 2.1.10.** *What is the largest value of $m$ such that, if $|S| < m$, then $O\chi(\Gamma(\mathbb{Z}_{p^2}, S)) = p$?*

Theorem 2.1.9 gives a lower bound of $m \geq \frac{p-1}{2}$. We showed with the Paley graph that there exists circulant graphs with $|S| = \frac{p^2-1}{2}$ that have an optimal orthogonal colouring. However, on the other hand, the following theorem by Klotz and Sander gives an upper bound on the value of $m$.

**Theorem 2.1.11** (Klotz and Sander [38])**.** *If $S = \{\pm 1, \ldots, \pm(p-1)\}$, then $\Gamma(\mathbb{Z}_{p^2}, S)$ is uniquely $p$-colourable.*

Recall that uniquely $p$-colourable graphs cannot be orthogonally coloured with $p$ colours, unless the graph is $K_p$. Therefore, since $|S|$ in Theorem 2.1.11 is of size $2p-2$, this gives an upper bound of $2p-2$.

We now show that if no multiples of $p$ are in $S$, then the size of the generating set can be increased. Consider the following function: $F_{\alpha,p} : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_{p^2}$ defined by

$$F_{\alpha,p}(i, j) = (ip + j\alpha)(\text{mod } p^2).$$

This function will be used to assign colour pairs to $\Gamma(\mathbb{Z}_{p^2}, S)$, when there are no multiples of $p$ in $S$. Thus, the goal is to show that the inverse is an orthogonal colouring. The following lemma shows that this assignment of colour pairs is injective.

**Lemma 2.1.12.** *If $\alpha \in \mathbb{Z}_{p^2} \backslash \{xp | 0 \leq x < p\}$, then $F_{\alpha,p}(i,j)$ is a bijection.*

**Proof:** Since $|\mathbb{Z}_p \times \mathbb{Z}_p| = |\mathbb{Z}_{p^2}| = p^2$, it is sufficient to show that $F_{\alpha,p}$ is surjective. Let $x \in \mathbb{Z}_{p^2}$. Since $\gcd(p^2, \alpha) = 1$, $\alpha$ has a multiplicative inverse $\alpha^{-1} \in \mathbb{Z}_{p^2}$. Now, by the Division Algorithm, there exist unique integers $q$ and $r$ such that $x = qp + r$ where $0 \leq q, r < p$. Let $i = q$ and $j = r\alpha^{-1}(\text{mod } p^2)$. Substituting gives that $F_{\alpha,p}(i,j) = (qp + r)(\text{mod } p^2) = x(\text{mod } p^2)$. Therefore, $F_{\alpha,p}$ is bijective. □

Note $F_{\alpha,p}^{-1}$ will orthogonally assign the vertices to colour pairs in $\mathbb{Z}_p \times \mathbb{Z}_p$. The following lemma shows that if the size of the generating set $S$ is sufficiently small and $S$ contains no multiples of $p$, then there is an $\alpha$ for which $F_{\alpha,p}^{-1}$ is a proper colouring.

**Theorem 2.1.13.** *For $p$ a prime, if $|S| < p$ and $xp \notin S$ for all $x$ where $x \in \mathbb{Z}_p \backslash \{0\}$, then $O\chi(\Gamma(\mathbb{Z}_{p^2}, S)) = p$.*

**Proof:** We will show that $F_{\alpha,p}^{-1}$ is an orthogonal colouring for some $\alpha \in \mathbb{Z}_{p^2}$ where $\gcd(p^2, \alpha) = 1$. By Lemma 2.1.12, $F_{\alpha,p}^{-1}$ is an orthogonal assignment of the vertices. We now show there is a choice for $\alpha$ such that $F_{\alpha,p}^{-1}$ is a proper colouring.

Suppose two vertices $k$ and $l$ receive the same colour in the first colouring. That is, $F_{\alpha,p}^{-1}(k) = (i, (j+x)(\text{mod } p))$ and $F_{\alpha,p}^{-1}(l) = (i, j)$ for some $i, j \in \mathbb{Z}_p$ and $x \in \mathbb{Z}_p \backslash \{0\}$. Note that $F_{\alpha,p}(i, (j + x)(\text{mod } p)) = ip + (j + x)\alpha = k$ and $F_{\alpha,p}(i, j) = ip + j\alpha = l$. Therefore, there is a colour conflict in the first colouring if and only if $k$ and $l$ are adjacent, which occurs when $k - l = x\alpha \in S$.

Suppose two vertices $k$ and $l$ receive the same colour in the second colouring. That is, $F_{\alpha,p}^{-1}(k) = (i, j)$ and $F_{\alpha,p}^{-1}(l) = ((i+x)(\text{mod } p), j)$ for some $i, j \in \mathbb{Z}_p$ and $x \in \mathbb{Z}_p \backslash \{0\}$. Note that $F_{\alpha,p}(i, j) = ip + j\alpha = k$ and $F_{\alpha,p}((i + x)(\text{mod } p), j) = (i + x)p + j\alpha = l$. Therefore, there is a colour conflict in the second colouring if and only if $k$ and $l$ are adjacent, so when $k - l = xp \in S$. By assumption, $xp \notin S$ for all $x$, so there are no colour conflicts in the second colouring.

So the only conflict that can occur is when $x\alpha \in S$. Since $x \in \mathbb{Z}_p \backslash \{0\}$, which is a field, $x^{-1}$ exists. Therefore, if $\alpha \notin \bigcup_x \{x^{-1}S\}$, then there will be no colour conflicts. Note that since $p$ is a prime, there are $p(p - 1)$ choices for $\alpha$ so that $\gcd(p^2, \alpha) = 1$. Since there are $p - 1$ choices for $x$, there are at most $(p - 1)|S| < (p - 1)p$ elements in $\bigcup_x \{x^{-1}S\}$. Therefore, there is an $\alpha$ such that $\alpha \notin \bigcup_x \{x^{-1}S\}$. □

## 2.2 Orthogonal Colourings of r-partite Graphs

An *$r$-partite graph* is a graph whose vertices can be partitioned into $r$ independent sets. That is, it is a graph that can be vertex coloured with $r$ colours. In the case where $r = 2$, these graphs are called *bipartite graphs.* If all of the edges exist between the different independent sets, then the graph is called a *complete $r$-partite graph.*

As mentioned in the introduction, Caro and Yuster [11] determined the orthogonal chromatic number of complete $r$-partite graphs. In the previous section, we constructed orthogonal colourings of even cycles, which are an example of bipartite graphs. Also, in the next chapter, we investigate another family of bipartite graphs, the hypercube graph.

In this section, we explore two more types of r-partite graphs. First, we investigate tree graphs, which are bipartite graphs containing no cycles. Next, we study a family of $k$-regular $r$-partite graphs, called $[n, k, r]$-partite graphs. By studying these two graphs, we disprove two open conjectures. Additionally, we establish an improved condition for optimal orthogonal colourings and we obtain an improved bound for $[n, k, r]$-partite graphs.

### 2.2.1 Orthogonal Colourings of Tree Graphs

Orthogonal colourings of tree graphs are interesting because there are only two values that the orthogonal chromatic number can be. Caro and Yuster [11] showed that if $T$ is a tree graph with $n$ vertices, then $O\chi(T) = \lceil \sqrt{n}\, \rceil$ or $O\chi(T) = \lceil \sqrt{n}\, \rceil + 1$. They proposed the following conjecture because of an incorrect categorization of the orthogonal chromatic number of double star tree graphs.

**Conjecture 2.2.1** (Caro and Yuster [11]). *If $T$ is a tree graph with $n$ vertices and $\Delta(T) < \frac{n}{2}$, then $O\chi(T) = \lceil \sqrt{n}\, \rceil$.*

In this section, we show that Conjecture 2.2.1 is false. We then establish a degree condition that guarantees the existence of an orthogonal colouring using $\lceil \sqrt{n}\, \rceil$ colours. This result gives a degree condition for independent coverings in the special case where the graph has $n^2$ vertices. Also, this result provides a partial categorization of the orthogonal chromatic number of tree graphs.

Conjecture 2.2.1 was posed due to a false categorization of double star graphs, which are now defined and the correct orthogonal chromatic number is determined. For even $m$, let $D_m$ denote the *double star graph*, obtained by joining the roots of two $K_{1,\frac{m}{2}-1}$ graphs. Caro and Yuster assert in [11] that $O\chi(D_m) = \lceil\sqrt{m}\,\rceil + 1$ if $m$ is even and satisfies $\lceil\sqrt{m}\,\rceil\lceil\sqrt{m} - 1\rceil < m$.

The flaw in their proof was that they assumed that no colour could appear $c$ times on leaves, where $c$ is the total number of colours used. This assumption is incorrect, as we show in the following example. For $m = 14$, the condition $\lceil\sqrt{m}\,\rceil\lceil\sqrt{m}-1\rceil < m$ holds, but $O\chi(D_{14}) = 4$ as shown in Figure 2.2.1. Also, the colour 4 is used on 4 leaves. The following theorem correctly establishes the orthogonal chromatic number of $D_m$.

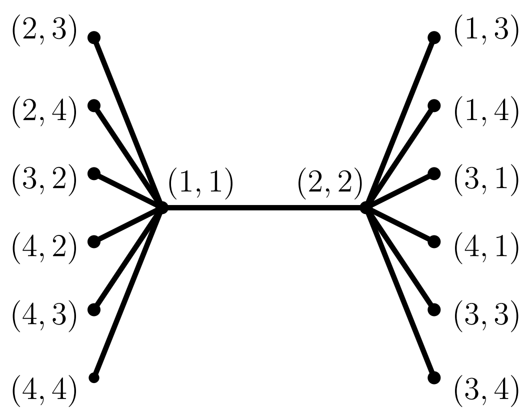


Figure 2.2.1: Orthogonal Colouring of $D_{14}$

**Theorem 2.2.2.** *For even $m$, $O\chi(D_m) = \lceil\sqrt{m}\,\rceil = N$ if and only if $m < N^2 - 1$.*

**Proof:** In the following, let $x_0$ and $y_0$ denote the root vertices of $D_m$. Next, let $x_1, x_2, \ldots, x_{\frac{m}{2}-1}$ and $y_1, y_2, \ldots, y_{\frac{m}{2}-1}$ be the leaves adjacent to $x_0$ and $y_0$ respectively. Suppose $m = N^2 - 2$ in the case where $N$ is even and suppose $m = N^2 - 3$ in the case where $N$ is odd.

In both cases, assign the colour pair $(1, 1)$ to $x_0$ and the colour pair $(2, 2)$ to $y_0$. We now extend this assignment to the leaves. For $1 \leq i \leq N - 2$, assign the colour pair $(2, i + 2)$ to $x_i$ and the colour pair $(1, i + 2)$ to $y_i$. For $N - 1 \leq j \leq 2N - 4$, assign the colour pair $(j - N + 4, 2)$ to $x_j$ and the colour pair $(j - N + 4, 1)$ to $y_j$. Note that $i + 2$ and $j - N + 4$ are greater than 2, thus the assigned colour pairs will not cause any colour conflict.

For $3 \leq r, s \leq N$, the colour pairs $(r, s)$ can be assigned to the remaining leaves in any order. This is because these colour pairs do not conflict with the roots $x_0$ and $y_0$. Therefore, in the case where $N$ is even, by arbitrarily assigning all of these colour pairs to the remaining leaves, an orthogonal colouring of $D_m$ using $N$ colours has been constructed. Similarly, in the case where $N$ is odd, by arbitrarily assigning all but one of these colour pairs to the remaining leaves, an orthogonal colouring of $D_m$ using $N$ colours has been constructed.

Note that if $m < N^2 - 2$ and $N$ is even, then the orthogonal colouring of $D_{N^2-2}$ constructed in this proof can be restricted to $D_m$ to give an orthogonal colouring using $N$ colours. Similarly, if $m < N^2 - 3$ and $N$ is odd, then the orthogonal colouring of $D_{N^2-3}$ constructed in this proof can be restricted to $D_m$ to give an orthogonal colouring using $N$ colours.

We now show that for $m = N^2$ when $N$ is even, and $m = N^2 - 1$ when $N$ is odd, that there are no orthogonal colourings using $N$ colours. Let $c_1$ and $c_2$ be two colourings of $D_m$. In $c_1$ (similarly in $c_2$), $x_0$ and $y_0$ must receive different colours. Give $x_0$ the colour pair $(a, b)$ and give $y_0$ the colour $(c, d)$, then the colour pair $(c, b)$ (similarly, the colour pair $(a, d)$) cannot be assigned to any leaf.

However, in the case where $N$ is even, every colour pair must be used since there are $N^2$ vertices. Thus, no orthogonal colouring using $N$ colours exists. Similarly, in the case where $N$ is odd, all but one colour pair must be used since there are $N^2 - 1$ vertices, Thus, no orthogonal colouring using $N$ colours exists in this case either. $\square$

Theorem 2.2.2 shows that for some even $m$, there are trees with maximum degree $\frac{m}{2}$ that require $\lceil \sqrt{m} \rceil$ colours and also trees that require $\lceil \sqrt{m} \rceil + 1$ colours. This shows that the maximum degree cannot be used to completely classify the orthogonal chromatic number of tree graphs. However, Conjecture 2.2.1 says that if a tree graph with $n$ vertices has maximum degree less than $\frac{n}{2}$, then an orthogonal colouring using $\lceil \sqrt{n} \rceil$ colours exists. This conjecture is false, as shown with the following proposition.

**Proposition 2.2.3.** *For each odd $n$, there exists a tree graph $T$ with $n^2$ vertices such that $\Delta(T) < \frac{n^2}{2}$ and $O\chi(T) = n + 1$.*

**Proof:** Let $T$ be the tree graph obtained by taking the double star graph $D_{n^2-1}$ and adding a vertex on the edge between the two root vertices. For example, the tree graph obtained for $n = 3$ is shown in Figure 2.2.2.

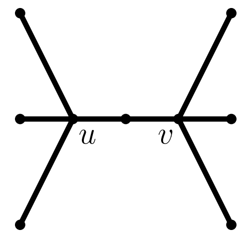


Figure 2.2.2: Counterexample Tree

Suppose that the vertex $u$ has the colour pair $(a, b)$ and the vertex $v$ has the colour pair $(c, d)$ where $(a, b) \neq (c, d)$. Since every other vertex is adjacent to either $u$ or $v$, the colour pair $(a, d)$ and the colour pair $(c, b)$ cannot be assigned to a vertex without resulting in a colour conflict. Since the tree graph has $n^2$ vertices, an orthogonal colouring with $n$ colours requires that every colour pair is used. Thus, an orthogonal colouring of Figure 2.2.2 with $n$ colours does not exist and $O\chi(T) = n + 1$. $\square$

Since the maximum degree of the graph in Figure 2.2.3 is $\lfloor \frac{n}{2} \rfloor < \frac{n}{2}$ for odd $n$, it follows that Conjecture 2.2.1 is false. This raises the question, if the maximum degree is sufficiently small, does an orthogonal colouring using $\lceil \sqrt{n}\, \rceil$ colours exist? Caro and Yuster [11] showed in Theorem 1.4.2 that for any graph $G$, if $\Delta(G) \leq \frac{\sqrt{n}-1}{4}$, then $O\chi(G) = \lceil \sqrt{n}\, \rceil$. The following theorem improves upon this result for tree graphs by using degenerate orderings.

Recall that a graph is *d-degenerate graph* if there exists an ordering of the vertices, in which each vertex has $d$ or fewer neighbours that are earlier in the ordering. Such an ordering of the vertices is called a *degenerate ordering.* The *degeneracy* of a graph is the smallest value of $d$ for which it is $d$-degenerate. In particular, tree graphs are 1-degenerate. We generalize Theorem 1.4.2 in the following theorem. The same argument is used, except now the degeneracy is applied to give a better bound on the orthogonal chromatic number.

**Theorem 2.2.4.** *If $G$ is $d$-degenerate with $\Delta(G) < \frac{\sqrt{n}-2d-1}{2}$, then $O\chi(G) = \lceil \sqrt{n}\, \rceil$.*

**Proof:** Consider a $d$-degenerate ordering of the vertices $\{v_1, v_2, \ldots, v_n\}$. Let $G_t$ be the graph where all the edges incident with the vertices $v_{t+1}, \ldots, v_n$ are removed. Our strategy is to inductively colour $G_t$ with $\lceil \sqrt{n}\, \rceil$ colours, correcting the colourings if necessary. For $t = 1$, $G_1 = \bar{K}_n$. This graph can be orthogonally coloured with $\lceil \sqrt{n}\, \rceil$ colours by Theorem 1.4.3.

Now, suppose for $t \geq 1$ that $G_{t-1}$ has an orthogonal colouring using $\lceil \sqrt{n} \, \rceil$ colours. Assign the same colouring to $G_t$. This colouring may not be proper however, because the edges incident to $v_t$ from $\{v_1, v_2, \ldots, v_{t-1}\}$ are now present. So we show that if this is the case, then the colouring can be corrected while maintaining orthogonality.

Let $N_t(v_t)$ be the neighbourhood of $v_t$ in $G_t$. Notice that $|N_t(v_t)| \leq d$ because $v_t$ is adjacent to at most $d$ vertices in $\{v_1, v_2, \ldots v_{t-1}\}$. Next, let $W$ be the set of vertices $w \in V(G)$, having the property that for some vertex $v \in N_t(v_t)$, $c_1(v) = c_1(w)$ or $c_2(v) = c_2(w)$. Notice that $|W| \leq 2d\lceil \sqrt{n} \, \rceil$ because there are at most $d$ neighbours of $v_t$ in $G_t$ and each colour appears at most $\lceil \sqrt{n} \, \rceil$ times in each colouring.

Next, let $Y_t$ denote the set of vertices $y \in V(G) \backslash \{v_t\}$, such that $c_1(y) = c_1(v_t)$ or $c_2(y) = c_2(v_t)$. Notice that $|Y_t| \leq 2(\lceil \sqrt{n} \, \rceil - 1)$ because there are two colourings and there are at most $\lceil \sqrt{n} \, \rceil - 1$ available colours in each colouring. Now, let $N(Y_t)$ be the union of open neighbourhoods of these vertices in $G$. Thus, $|N(Y_t)| \leq |Y_t|\Delta(G)$. Lastly, let $X = V(G) \backslash (W \cup N(Y_t))$. So $X$ is the set of vertices that do not conflict with the colour assigned to vertices in $N_t(v_t)$ and are not adjacent to vertices that have the same colour as $v_t$.

$$
\begin{aligned}
\Delta(G) &< \frac{\sqrt{n} - 2d - 1}{2} && \text{by assumption of the theorem.} \\
&< \frac{\frac{n}{\lceil \sqrt{n} \, \rceil} - 2d}{2} && \text{because } n > \lceil \sqrt{n} \, \rceil \sqrt{n} - \lceil \sqrt{n} \, \rceil. \\
&= \frac{n - 2d\lceil \sqrt{n} \, \rceil}{2\lceil \sqrt{n} \, \rceil} && \text{by factoring } 1/\lceil \sqrt{n} \, \rceil. \\
&< \frac{n - 2d\lceil \sqrt{n} \, \rceil}{2\lceil \sqrt{n} \, \rceil - 2} && \text{because } 2\lceil \sqrt{n} \, \rceil > 2\lceil \sqrt{n} \, \rceil - 2.
\end{aligned}
$$

Therefore, the following chain of inequalities is obtained:

$$
\begin{aligned}
|X| &\geq n - |W| - |N(Y_t)| \\
&> n - 2d\lceil \sqrt{n} \, \rceil - (2\lceil \sqrt{n} \, \rceil - 2) \left( \frac{n - 2d\lceil \sqrt{n} \, \rceil}{2\lceil \sqrt{n} \, \rceil - 2} \right) \\
&= 0
\end{aligned}
$$

Therefore, the set $X$ is non-empty, so let $x \in X$. Since $x \notin W$, the colour pair assigned to $x$ does not conflict with the neighbours of $v_t$. Also, since $x \notin N(Y_t)$, the colour pair assigned to $v_t$ does not conflict with the neighbours of $x$. Thus, interchanging the colours assigned to $x$ and $v_t$ results in an orthogonal colouring. $\square$

In particular, since tree graphs are 1-degenerate, Theorem 2.2.4 gives that if $T$ is a tree graph with $n$ vertices and if $\Delta(T) < \frac{\sqrt{n}-3}{2}$, then $O\chi(T) = \lceil \sqrt{n} \rceil$. Note that in the case where the degeneracy of the graph is the maximum degree, Theorem 2.2.4 agrees with Theorem 1.4.2.

### 2.2.2 Orthogonal Colourings of $[n, k, r]$-Partite Graphs

Recall that an *independent transversal* of a graph, with respect to a vertex partition $P$, is an independent set that contains exactly one vertex from each vertex class. An *independent covering* of a graph, with respect to a vertex partition $P$, is a collection of disjoint independent transversals with respect to $P$ that spans all of the vertices. A graph is $[n, k, r]$-*partite* if the vertices can be partitioned into $n$ independent sets of size $k$, where the edges between any two independent sets is a matching of size $r$.

Let $c(k, r)$ denote the maximal $n$ such that all $[n, k, r]$-partite graphs have an independent covering with respect to the given $[n, k, r]$-partition. This maximum value was shown to exist by Yuster [56] when he proved $k \geq c(k, r) \geq \min\{k, k-r+2\}$. Note that when $r = 1, 2$, the upper bound and lower bound coincide, giving that $c(k, 1) = c(k, 2) = k$. This led to the following conjecture:

**Conjecture 2.2.5** (Yuster [56])**.** *For all $r \leq k$, $c(k, r) = k$.*

In this section, we disprove Conjecture 2.2.5 by showing that there is a specific $[3, 3, 3]$-partite graph that does not have an independent covering with respect to the given $[3, 3, 3]$-partition. We then obtain a lower bound of $c(k, r) \geq \lceil \frac{k}{2} \rceil$ by using orthogonal colourings. This gives an improved lower bound on $c(k, r)$ for $r > \frac{k}{2} + 2$. The following theorem shows that $c(3, 3) \neq 3$.

**Theorem 2.2.6.** *There exists a $[3, 3, 3]$-partite graph that does not have an independent covering with respect to the $[3, 3, 3]$-partition. That is, $c(3, 3) \neq 3$.*

**Proof:** We will show that the $[3, 3, 3]$-partite graph $G$ in Figure 2.2.3 does not have an independent covering with respect to $P = \{\{x_0, x_1, x_2\}, \{y_0, y_1, y_2\}, \{z_0, z_1, z_2\}\}$. The three transversals are defined as $T_0$, $T_1$, $T_2$ and $T_0$, $T_1$, $T_2$ are populated later. Suppose for the sake of contradiction that $G$ does have an independent covering with respect to $P$ with the independent transversals $T_0, T_1$, and $T_2$. Without loss of generality, suppose that $x_0 \in T_0$, $x_1 \in T_1$, and $x_2 \in T_2$.

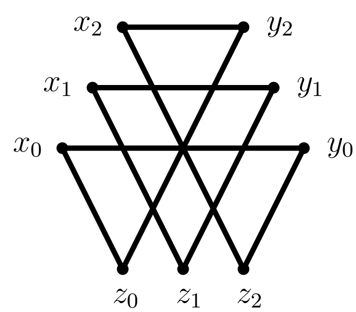


Figure 2.2.3: Counterexample Graph

There are two properties used to show how the independent covering must be formed when $T_0$ is given. The first property is that each vertex in $\{x_0, x_1, x_2\}$, $\{y_0, y_1, y_2\}$, and $\{z_0, z_1, z_2\}$ must be in a different independent transversal. This is because an independent transversal can only have one vertex from each vertex class. The second property is that if two vertices are adjacent, then they must be in different transversals. This is because an independent transversal is an independent set.

**Case 1:** $T_0 = \{x_0, y_1, z_2\}$**.** Since $x_2y_2 \in E(G)$ and $x_2 \in T_2$, $y_2 \notin T_2$ by the second property. Also, since $y_1 \in T_0$, $y_2 \notin T_0$ by the first property. Therefore, $y_2 \in T_1$ is the only option available. Now, if $z_1 \in T_1$, then $x_1z_1 \in E(G)$ and $x_1, z_1 \in T_1$, which contradicts that $T_1$ is independent. If $z_0 \in T_1$, then $y_2z_0 \in E(G)$ and $y_2, z_0 \in T_1$, which contradicts that $T_1$ is independent.

**Case 2:** $T_0 = \{x_0, y_2, z_1\}$**.** Since $x_1y_1 \in E(G)$ and $x_1 \in T_1$, $y_1 \notin T_1$ by the second property. Also, since $y_2 \in T_0$, $y_1 \notin T_0$ by the first property. Therefore, $y_1 \in T_2$ is the only option available. Thus, $y_0 \in T_1$ is the only option available. Now, if $z_2 \in T_2$, then $x_2z_2 \in E(G)$ and $x_2, z_2 \in T_2$, which contradicts that $T_2$ is independent. If $z_2 \in T_1$, then $y_0z_2 \in E(G)$ and $y_0, z_2 \in T_1$, which contradicts that $T_1$ is independent.

**Case 3:** $T_0 = \{x_0, y_2, z_2\}$**.** Since $x_1y_1 \in E(G)$ and $x_1 \in T_1$, $y_1 \notin T_1$ by the second property. Also, since $y_2 \in T_0$, $y_1 \notin T_0$ by the first property. Therefore, $y_1 \in T_2$ is the only option available. Now, if $z_1 \in T_1$, then $x_1z_1 \in E(G)$ and $x_1, z_1 \in T_1$, which contradicts that $T_1$ is independent. If $z_1 \in T_2$, then $y_1z_1 \in E(G)$ and $y_1, z_1 \in T_2$, which contradicts that $T_2$ is independent. $\square$

Theorem 2.2.6 illustrates that not every $[3, 3, 3]$-partite graphs has an independent covering with respect to the $[3, 3, 3]$-partition. The following theorem shows that all $[3, 3, 3]$-partite graphs have an independent covering with respect to some partition.

**Theorem 2.2.7.** *Let $G$ be a $[3,3,3]$-partite graph, then $O\chi(G) = 3$.*

**Proof:** It can be shown that with some work that there are three $[3,3,3]$-partite graphs: $G_1 = K_3 \cup K_3 \cup K_3$, $G_2 = C_9$, and $G_3 = K_3 \cup C_6$. Orthogonal colourings, where each colour class has the same size, of $G_1, G_2$, and $G_3$ are given in Figure 2.2.4, Figure 2.2.5, and Figure 2.2.6 respectively. Therefore, every $[3,3,3]$-partite graph has an independent covering with respect to the partition provided by the orthogonal colouring. □

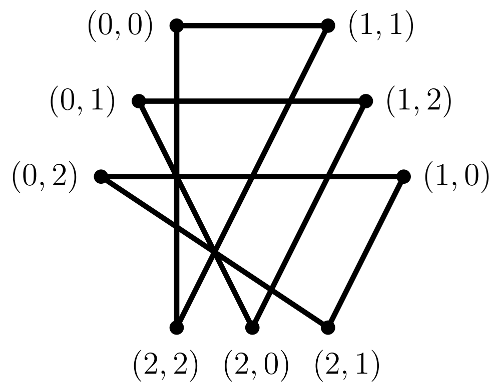


Figure 2.2.4: Orthogonal Colouring of $G_1$

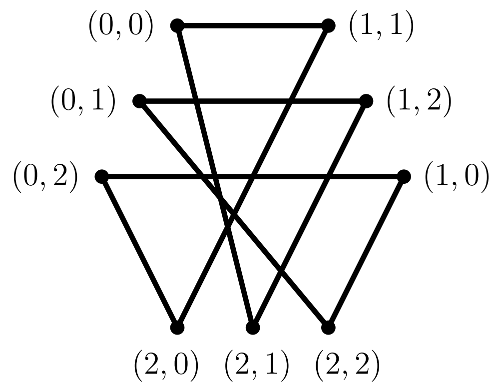


Figure 2.2.5: Orthogonal Colouring of $G_2$

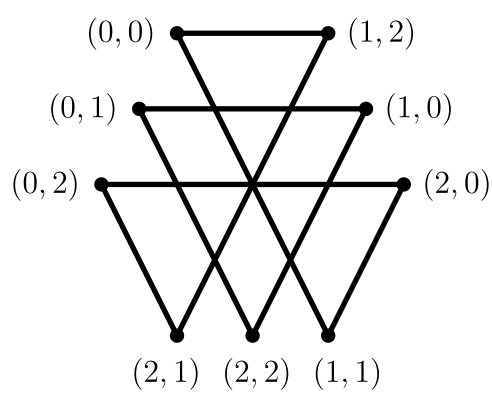


Figure 2.2.6: Orthogonal Colouring of $G_3$

Theorem 2.2.6 suggests that if $n > \lceil \frac{k}{2} \rceil$, then an independent covering with respect to the $[n, k, k]$-partition may not exist, but an independent covering with respect to some vertex partition might. However, if $n \leq \lceil \frac{k}{2} \rceil$, then an independent covering with respect to the original $[n, k, k]$-partition does exist. We will illustrate this in Theorem 2.2.10 by applying Hall's condition.

To state Hall's condition, some definitions are required. Let $S$ be a possibly infinite family of finite subsets of $X$, where the members of $S$ are counted with multiplicity. A *system of distinct representatives* for $S$ is the image of an injective function $f$ that selects one representative from each set in $S$ in such a way that no two of these representatives are equal.

**Theorem 2.2.8** (Hall's Condition [30]). *Let $S$ be a possibly infinite family of finite subsets of $X$, where the members of $S$ are counted with multiplicity. The collection $S$ has a system of distinct representatives if for each subfamily $W \subseteq S$,*

$$|W| \leq | \cup_{A \in W} A|$$

In particular, we apply the following corollary of Hall's Condition. This result will be used iteratively to construct an independent covering of $[\frac{k}{2}, k, k]$-partite graphs.

**Corollary 2.2.9** (Van [53]). *Let $X$ have $2n$ elements and let $S$ be a family of $2n$ subsets of $X$. If each subset contains at least $n$ elements and each element appears in at least $n$ of the subsets, then a system of distinct representatives for $S$ exists.*

**Theorem 2.2.10.** *All $[\frac{k}{2}, k, k]$-partite graphs have an independent covering with respect to the $[\frac{k}{2}, k, k]$-partition. That is, $c(k, k) \geq \lceil \frac{k}{2} \rceil$.*

**Proof:** Let $n = \lceil \frac{k}{2} \rceil$ and let $G$ be an $[n, k, k]$-partite graph. To prove the statement, an independent covering with respect to the $[n, k, k]$-partition is required. This is done by constructing an orthogonal colouring such that the colour classes in the first colouring are from the $[n, k, r]$-partition and the sizes of each colour class in the second colouring is equal to the number of colours used in the first colouring. This way, the orthogonal colouring constructed will provide an independent covering with respect to the $[n, k, r]$-partition.

For $1 \leq i \leq n$ and $1 \leq j \leq k$, let $A_i$ denote the $i$-th vertex class and let $v_{i,j}$ denote the $j$-th vertex in $A_i$. For the first colouring, define $f_1(v_{i,j}) = i$. Since each $A_i$ is an independent set, there is no colour conflict. For $1 \leq m \leq n$, an independent covering with respect to the $[n, k, k]$-partition is established by constructing a second colouring $f_2$ using induction on $m$. We now show that for each $i$, that each vertex of $A_i$ can be assigned a unique colour while not causing any colour conflicts.

For the base case $m = 1$, define the second colouring on $A_1$ as $f_2(v_{1,j}) = j$. Since each $v_{1,j}$ receives a different colour, there is no colour conflict. Suppose now that for $1 < m \leq n$, that the second colouring is properly defined on the sets $A_1, \ldots, A_{m-1}$ where each colour appears exactly once on each $A_i$. We now show that $f_2$ can be defined on $A_m$.

Let $C_j$ denote the set of available colours for the vertex $v_{k,j}$ in the second colouring. By definition, each vertex in $A_m$ has exactly one neighbour in each of $A_1, \ldots, A_{m-1}$. Therefore, at most $m-1$ colours are appearing on the neighbours of each $v_{k,j}$. Since $f_2$ uses $k$ colours, this means there are at least $k-(m-1) \geq n$ colours available for $v_{k,j}$. Therefore, it follows that $|C_j| \geq n$. By the induction hypothesis, all colours appear exactly once on each $A_i$. Therefore, each colour appears in at least $k - (m-1) \geq n$ of the $C_j$.

Let $X = \{1, 2, \ldots, k\}$ and $S = \{C_1, C_2, \ldots, C_k\}$. Since each subset of $X$ contains at least $n$ elements and each element appears in at least $n$ of the subsets, a system of distinct representatives for $S$ exists by Corollary 2.2.9. Assigning the distinct representative of $C_j$ to $v_{k,j}$ will complete the orthogonal colouring. Since each colour class in the second colouring is equal to the number of colours used in the first colouring, this orthogonal colouring corresponds to an independent covering. $\square$

# Chapter 3

## Orthogonal Colourings of Product Graphs

To start, we study orthogonal colourings of tensor product graphs, denoted $G \times H$. We show that a graph with $n$ vertices has an optimal orthogonal colouring if and only if it is a subgraph of $K_N \times K_N$, where $N = \lceil \sqrt{n} \rceil$. Then, we show that if both tensor components have a square number of vertices and if one tensor component has an optimal orthogonal colouring, then the tensor product graph will have an optimal orthogonal colouring.

We then combine these two results to show that the orthogonal chromatic number of a tensor product graph is at most the product of the orthogonal chromatic number of its tensor components. Lastly, we show that if $G$ has $n^2$ vertices and a $k$-optimal orthogonal colouring, and if $H$ has $p^2$ vertices where $k \leq p$ and $p$ is prime, then $G \times H$ has a $k$-optimal orthogonal colouring.

We then explore orthogonal colourings of Cartesian product graphs, denoted $G \square H$. We show that if both Cartesian components have a square number of vertices and if the larger Cartesian component has an optimal orthogonal colouring, then the Cartesian product graph will have an optimal orthogonal colouring. We then show that the orthogonal chromatic number of a Cartesian product graph is at most the product of the orthogonal chromatic number of the Cartesian components.

Next, we found the orthogonal chromatic number of hypercube graphs, denoted $Q_n$. We show that if $n \neq 2, 3, 9$, then $Q_n$ has an optimal orthogonal colouring. Lastly, we discuss orthogonal colourings of Hamming graphs, denoted $H(d, q)$. We show that if $q \neq 2, 6$, then $H(2d, q)$ has an optimal orthogonal colouring.

To conclude, we study orthogonal colourings of strong product graphs, denoted $G \boxtimes H$. We show that if both components have a $k$-optimal orthogonal colouring and a square number of vertices, then the strong product graph will have a $k$-optimal orthogonal colouring. We also establish the analogous result for tensor product graphs and Cartesian product graphs.

## 3.1 Orthogonal Colourings of Tensor Graphs

The *tensor product* of two graphs $G$ and $H$, denoted by $G \times H$, has vertex set $V(G) \times V(H)$, and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ in $G \times H$ are adjacent if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. If $G$ is a graph that is created by the tensor product of two graphs, then $G$ is called a *tensor graph.* The two graphs in a tensor graph product are called *tensor components.*

To start, we show that a graph $G$ has $O\chi(G) \leq n$ if and only if is a subgraph of $K_n \times K_n$. In this thesis, this is denoted by $G \subseteq K_n \times K_n$. Therefore in particular, the following result gives that a graph $G$ with $n$ vertices has an optimal orthogonal colouring if and only if $G \subseteq K_N \times K_N$ where $N = \lceil \sqrt{n}\, \rceil$.

**Theorem 3.1.1.** *For every graph $G$, $O\chi(G) \leq n$ if and only if $G \subseteq K_n \times K_n$.*

**Proof:** For $1 \leq i, j \leq n$, let $(i, j)$ denote the vertices of the graph $K_n \times K_n$. First, suppose that $G \subseteq K_n \times K_n$. We will show that $K_n \times K_n$ has an optimal orthogonal colouring. If this is the case, then the orthogonal colouring of $K_n \times K_n$ restricted to $G$ is an orthogonal colouring of $G$ using $n$ colours, giving $O\chi(G) \leq n$. Assign the vertex $(i, j)$ the colour $i$ in the first colouring and the colour $j$ in the second colouring. For example, this orthogonal colouring is applied to $K_3 \times K_3$ in Figure 3.1.1.

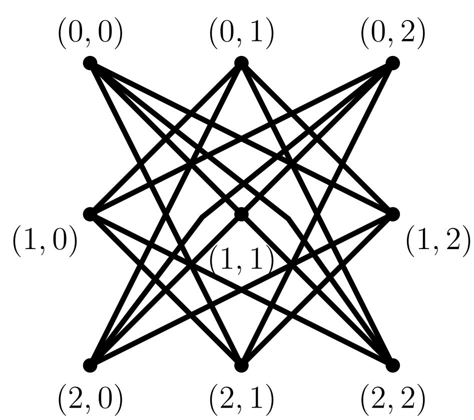


Figure 3.1.1: Orthogonal Colouring of $K_3 \times K_3$

Notice that this assignment of colours has no orthogonal conflicts. We now check that there are no colour conflicts. Now, by the definition of the tensor product, for $1 \leq i_1, i_2, j_1, j_2 \leq n$, two vertices $(i_1, j_1)$ and $(i_2, j_2)$ in $K_n \times K_n$ are adjacent if and only if $i_1 \neq i_2$ and $j_1 \neq j_2$. Therefore, there are also no colour conflicts.

Now, suppose that $O\chi(G) \leq n$ and that $(g_1, g_2)$ is an orthogonal colouring of $G$ using the colours $\{1, 2, \ldots, n\}$. To show that $G \subseteq K_n \times K_n$, an injective map that preserves edges is required. Let $F : G \to K_n \times K_n$ by $F(v) = (g_1(v), g_2(v))$. We now show that $F$ is injective and preserves edges.

Since $(g_1, g_2)$ is an orthogonal colouring of $G$, each colour pair is only assigned once. Thus, $F$ is injective. Now, if $v_1v_2 \in E(G)$, then $g_1(v_1) \neq g_1(v_2)$ and $g_2(v_1) \neq g_2(v_2)$ because $g_1$ and $g_2$ are proper. Therefore, $(g_1(v_1), g_2(v_1))(g_1(v_2), g_2(v_2)) \in E(K_n \times K_n)$ by the definition of the edges in $K_n \times K_n$. Thus, $F$ preserves edges. Since $F$ is injective and preserves edges, $G \subseteq K_n \times K_n$. □

Theorem 3.1.1 gives a way to reformulate the problem of determining if a graph has an optimal orthogonal colouring. This will be used later with the following theorem to obtain an upper bound on the orthogonal chromatic number of general tensor graphs and other product graphs. The general idea of the proof is to take the orthogonal colouring of one tensor component and then extend it to the other copies of that tensor component in the tensor graph.

**Theorem 3.1.2.** *If $G$ has $n^2$ vertices, $H$ has $m^2$ vertices, and $O\chi(G) = n$, then $O\chi(G \times H) = nm$.*

**Proof:** Label $V(G) = \{v_k : 0 \leq k < n^2\}$ and $V(H) = \{(u_i, u_j) : 0 \leq i, j < m\}$. Let $f = (f_1, f_2)$ be a proper orthogonal colouring of $G$ where $f_1$ and $f_2$ use the colours $\{0, 1, \ldots, n-1\}$. We will show that $g = (g_1, g_2)$ is an orthogonal colouring of $G \times H$ using $nm$ colours, where $g_1$ and $g_2$ are defined as:

$$g_1((v_k, (u_i, u_j))) = f_1(v_k) + in$$

and

$$g_2((v_k, (u_i, u_j))) = f_2(v_k) + jn.$$

First, we will show that $g$ has no orthogonal conflicts. Let $v_{k_1}, v_{k_2} \in V(G)$ and let $(u_{i_1}, u_{j_1}), (u_{i_2}, u_{j_2}) \in V(H)$. If $g((v_{k_1}, (u_{i_1}, u_{j_1}))) = g((v_{k_2}, (u_{i_2}, u_{j_2})))$, then:

$$f_1(v_{k_1}) + i_1 n = f_1(v_{k_2}) + i_2 n \tag{3.1}$$

and

$$f_2(v_{k_1}) + j_1 n = f_2(v_{k_2}) + j_2 n. \tag{3.2}$$

Without loss of generality, suppose that $i_1 < i_2$. Then, since the largest colour used by $f$ is $n-1$, it follows that:

$$\begin{aligned} f_1(v_{k_1}) + i_1 n &< n + i_1 n \\ &\leq i_2 n \\ &\leq f_1(v_{k_2}) + i_2 n. \end{aligned}$$

Therefore, $f_1(v_{k_1}) + i_1 n < f_1(v_{k_2}) + i_2 n$, which contradicts Equation (3.1), thus $i_1 = i_2$. A similar argument shows that $j_1 = j_2$. Substituting $i_1 = i_2$ and $j_1 = j_2$ into Equations (3.1) and (3.2), gives $f_1(v_{k_1}) = f_1(v_{k_2})$ and $f_2(v_{k_1}) = f_2(v_{k_2})$. Hence, it follows that $v_{k_1} = v_{k_2}$ because $f$ is an orthogonal colouring of $G$. Thus, it follows that $(v_{k_1}, (u_{i_1}, u_{j_1})) = (v_{k_2}, (u_{i_2}, u_{j_2}))$.

We will now show that $g_1$ and $g_2$ are proper colourings of $G \times H$. Suppose that $v_{k_1}v_{k_2} \in E(G)$ and $(u_{i_1}, u_{j_1})(u_{i_2}, u_{j_2}) \in E(H)$. If $i_1 = i_2 = i$, then since $f_1$ is a proper colouring of $G$, $g_1((v_{k_1}, (u_{i_1}, u_{j_1}))) = f_1(v_{k_1}) + in \neq f_1(v_{k_2}) + in = g_1((v_{k_2}, (u_{i_2}, u_{j_2})))$. Thus, there are no colour conflicts between these vertices.

Now, without loss of generality, suppose that $i_1 < i_2$. Then, it follows that $g_1((v_{k_1}, (u_{i_1}, u_{j_1}))) = f_1(v_{k_1}) + i_1 n < n + i_1 n \leq i_2 n \leq f_1(v_{k_2}) + i_2 n = g_1((v_{k_1}, (u_{i_2}, u_{j_2})))$. Hence, $g_1((v_{k_1}, (u_{i_1}, u_{j_1}))) < g_1((v_{k_1}, (u_{i_2}, u_{j_2})))$. Therefore, $g_1$ is a proper colouring. A similar argument shows that $g_2$ is a proper colouring. Thus, $g$ is an orthogonal colouring of $G \times H$ using $nm$ colours. Since $G \times H$ has $n^2m^2$ vertices, this gives that $O\chi(G \times H) = nm$. □

Theorem 3.1.2 provides a method for constructing optimal orthogonal colourings out of graphs that have optimal orthogonal colourings. Theorem 3.1.1 shows that $K_n \times K_n$ is the maximum graph with $n$ as its orthogonal chromatic number. Combining these two results provides the following upper bound for tensor graphs.

**Corollary 3.1.3.** *For all graphs $G$ and $H$, if $O\chi(G) = n$ and $O\chi(H) = m$, then $O\chi(G \times H) \leq nm$.*

**Proof:** Since $O\chi(G) = n$ and $O\chi(H) = m$, $G \subseteq K_n \times K_n$ and $H \subseteq K_m \times K_m$ by Theorem 3.1.1. Therefore, $G \times H \subseteq (K_n \times K_n) \times (K_m \times K_m)$. Since $|V(K_n \times K_n)| = n^2$, $|V(K_m \times K_m)| = m^2$, and $O\chi(K_n \times K_n) = n$, $O\chi((K_n \times K_n) \times (K_m \times K_m)) = nm$ by Theorem 3.1.2. Therefore, $(K_n \times K_n) \times (K_m \times K_m) \subseteq K_{nm} \times K_{nm}$ by Theorem 3.1.1. Then, $G \times H \subseteq K_{nm} \times K_{nm}$, and so by Theorem 3.1.1, $O\chi(G \times H) \leq nm$. □

The last result creates optimal $k$-orthogonal colourings of tensor graphs. In this case, it is required that one tensor component has an optimal $k$-orthogonal colouring and the other tensor component has a prime square number of vertices.

**Theorem 3.1.4.** *If $G$ has $n^2$ vertices with $O\chi_k(G) = n$, $H$ has $p^2$ vertices where $p$ is a prime, and $k \leq p$, then $O\chi_k(G \times H) = np$.*

**Proof:** Label $V(H) = \{(u_i, u_j) : 0 \leq i, j < p\}$. For $0 \leq r < k$ and $0 \leq s < n$, let $I_{r,s}$ be the $s$-th colour class in the $r$-th colouring of $G$. Then, for $0 \leq j < p$, let $\hat{I}_{r,s,j} = \{(v, (u_i, u_{(ir+j)(\text{mod } p)})) \mid v \in I_{r,s}, 0 \leq i < p\}$. The goal is to show that $C_r = \{\hat{I}_{r,s,j} \mid 0 \leq s < n, 0 \leq j < p\}$ is a partition of $G \times H$ into $np$ independent sets. That is, $C_r$ is a proper colouring of $G \times H$ using $np$ colours.

First, we will show that each $\hat{I}_{r,s,j}$ is an independent set. Since each $I_{r,s}$ is an independent set in $G$, for each $v_1, v_2 \in I_{r,s}$, $v_1v_2 \notin E(G)$. Thus, by the definition of the tensor graph product, $(v_1, (u_i, u_{(ir+j)(\text{mod } p)}))(v_2, (u_i, u_{(ir+j)(\text{mod } p)})) \notin E(G \times H)$. Therefore, each $\hat{I}_{r,s,j}$ is an independent set. Next, it is shown that $C_r$ is a partition of $G \times H$ into independent sets.

Consider a vertex $(v, (u_x, u_y))$ in $G \times H$. Since $\{I_{r,s} \mid 0 \leq s < n\}$ is a partition of $G$, $v \in I_{r,s}$ for some $s$. Now, notice that for $0 \leq j < p$, $\{(u_i, u_{(ir+j)(\text{mod } p)}) \mid 0 \leq i < p\}$ is a partition of $H$. Therefore, $(u_x, u_y)$ is in one of these sets. In particular, this occurs for $i = x$ and $j = y - r$. Therefore, there is in a unique set $\hat{I}_{r,s,j}$ that contains $(v, (u_x, u_y))$. Thus, $C_r$ is a partition of $G \times H$ into independent sets.

Now it remains to show that each of the colourings are mutually orthogonal. That is, it remains to show for $r_1 \neq r_2$, $s_1, s_2$ and $j_1, j_2$ fixed, that $|\hat{I}_{r_1,s_1,j_1} \cap \hat{I}_{r_2,s_2,j_2}| = 1$. Since $|I_{r_1,s_1} \cap I_{r_2,s_2}| = 1$, let $v$ be this vertex. Therefore, if it can be shown that $\{(u_i, u_{ir_1+j_1}) | 0 \leq i\} \cap \{(u_i, u_{ir_2+j_2}) | 0 \leq i\} = 1$, then we are done. The only way $(u_i, u_{ir_1+j_1}) = (u_i, u_{ir_2+j_2})$ is if $i_1 = i_2$. Thus, call this $i$. Now, need to show $(ir_1 + j_1)(\text{mod } p) = (ir_2 + j_2)(\text{mod } p)$ which can be rewritten in an equivalent form as $i(r_1 - r_2)(\text{mod } p) = (j_2 - j_1)(\text{mod } p)$.

Since $k \leq p$, and $r_1 \neq r_2$, $r_1 - r_2 \not\equiv 0$. Thus, $r_1 - r_2 = r$ and $j_2 - j_1 = j$. Since $p$ is a prime, $\mathbb{Z}_p$ has no zero divisors. Therefore, $ir(\text{mod } p) = j(\text{mod } p)$ has a unique solution, call this unique solution $(i, j)$. Thus, $(v, (i, j))$ is the unique element in the $\hat{I}_{r_1,s_1,j_1} \cap \hat{I}_{r_2,s_2,j_2}$. Hence, the colourings are all mutually orthogonal. Since each of these colourings using $np$ colours, and $G \times H$ has $n^2p^2$ vertices, $O\chi(G \times H) = np$. $\square$

### 3.2 Orthogonal Colourings of Cartesian Graphs

The *Cartesian graph product* of two graphs $G$ and $H$, denoted by $G\Box H$, is a graph with the following properties. The vertex set is $V(G)\times V(H)$ and the edge set is $E(G\Box H)=\{(u_1,v_1)(u_2,v_2)\mid u_1=u_2 \text{ and } v_1v_2\in E(H) \text{ or } v_1=v_2 \text{ and } u_1u_2\in E(G)\}$. A graph constructed by the Cartesian graph product of two graphs is referred to as a *Cartesian product graph.* The two graphs in a Cartesian graph product are called *Cartesian components.* We first prove the following result.

**Theorem 3.2.1.** *If $G$ has $n^2$ vertices, $H$ has $m^2$ vertices, and $O\chi(G)=n\geq m$, then $O\chi(G\Box H)=nm$.*

**Proof:** Label $V(G)=\{v_k : 0\leq k<n^2\}$ and $V(H)=\{(u_i,u_j): 0\leq i,j<m\}$. Let $f=(f_1,f_2)$ be an orthogonal colouring of $G$ using the set of colours $\{0,1,\ldots,n-1\}$. We show that $g=(g_1,g_2)$ is an orthogonal colouring of $G\Box H$ where:

$$\begin{aligned}g_1((v_k,(u_i,u_j)))&=(f_1(v_k)+j)(\mathrm{mod}\ n)+in\\ g_2((v_k,(u_i,u_j)))&=(f_2(v_k)+i)(\mathrm{mod}\ n)+jn.\end{aligned}$$

Firstly, we will show that $g$ has no orthogonal conflicts. Let $v_{k_1},v_{k_2}\in V(G)$ and let $(u_{i_1},u_{j_1}),(u_{i_2},u_{j_2})\in V(H)$. If $g((v_{k_1},(u_{i_1},u_{j_1})))=g((v_{k_2},(u_{i_2},u_{j_2})))$, then:

$$(f_1(v_{k_1})+j_1)(\mathrm{mod}\ n)+i_1n=(f_1(v_{k_2})+j_2)(\mathrm{mod}\ n)+i_2n. \tag{3.3}$$

$$(f_2(v_{k_1})+i_1)(\mathrm{mod}\ n)+j_1n=(f_2(v_{k_2})+i_2)(\mathrm{mod}\ n)+j_2n. \tag{3.4}$$

Without loss of generality, suppose that $i_1<i_2$. Then, since the largest colour used by $f$ is $n-1$, it follows that

$$\begin{aligned}(f_1(v_{k_1})+j_1)(\mathrm{mod}\ n)+i_1n&<n+i_1n\\ &\leq i_2n\\ &\leq (f_1(v_{k_2})+j_2)(\mathrm{mod}\ n)+i_2n.\end{aligned}$$

Therefore, by transitivity, $(f_1(v_{k_1})+j_1)(\mathrm{mod}\ n)+i_1n<(f_1(v_{k_2})+j_2)(\mathrm{mod}\ n)+i_2n$, which contradicts Equation (3.3), thus $i_1=i_2$. A similar argument shows that $j_1=j_2$. Substituting $i_1=i_2$ and $j_1=j_2$ into Equations (3.3) and (3.4), gives $f_1(v_{k_1})=f_1(v_{k_2})$ and $f_2(v_{k_1})=f_2(v_{k_2})$. Hence, $v_{k_1}=v_{k_2}$ because $f$ is an orthogonal colouring. Thus, $(v_{k_1},(u_{i_1},u_{j_1}))=(v_{k_2},(u_{i_2},u_{j_2}))$, and there are no orthogonal conflicts.

We will now show that $g_1$ and $g_2$ are proper colourings of $G\Box H$. First, consider adjacencies of the form $(v_{k_1},(u_i,u_j)) \sim (v_{k_2},(u_i,u_j))$. Since $v_{k_1}v_{k_2} \in E(G)$ and $f_1$ is a proper colouring of $G$, $f_1(v_{k_1}) \neq f_1(v_{k_2})$. Thus:

$$\begin{aligned} g_1((v_{k_1},(u_i,u_j))) &= (f_1(v_{k_1})+j)(\text{mod } n)+in \\ &\neq (f_1(v_{k_2})+j)(\text{mod } n)+in \\ &= g_1((v_{k_2},(u_i,u_j)). \end{aligned}$$

Next, consider adjacencies of the form $(v_k,(u_{i_1},u_{j_1})) \sim (v_k,(u_{i_2},u_{j_2}))$. Suppose that $i_1 = i_2$ and $j_1 \neq j_2$. Since $n \geq m$, it follows that:

$$\begin{aligned} g_1((v_k,(u_{i_1},u_{j_1}))) &= (f_1(v_k)+j_1)(\text{mod } n)+i_1n \\ &\neq (f_1(v_k)+j_2)(\text{mod } n)+i_1n \\ &= (f_1(v_k)+j_2)(\text{mod } n)+i_2n \\ &= g_1((v_k,(u_{i_2},u_{j_2}))). \end{aligned}$$

Thus, there are no colour conflicts in this case. If $i_1 \neq i_2$, then the argument used to prove the orthogonality of $g$ shows that $g_1((v_k,(u_{i_1},u_{j_1}))) \neq g_1((v_k,(u_{i_2},u_{j_2})))$. The same argument can be applied to show that $g_2$ has no colour conflicts. Therefore, $g$ is an orthogonal colouring of $G\Box H$ using $nm$ colours. Since $G\Box H$ has $n^2m^2$ vertices, this gives that $O\chi(G\Box H) = nm$. □

Theorem 3.2.1 provides a method for constructing optimal orthogonal colourings in the case where both Cartesian components have a square number of vertices and the larger Cartesian component has an optimal orthogonal colouring. This also provides the following upper bound.

**Corollary 3.2.2.** *If $O\chi(G) = n$ and $O\chi(H) = m$, then $O\chi(G\Box H) \leq nm$.*

**Proof:** Recall that Theorem 3.1.1 gives that $O\chi(G) \leq n$ if and only if $G \subseteq K_n \times K_n$. Since $O\chi(G) = n$ and $O\chi(H) = m$, it follows that $G \subseteq K_n \times K_n$ and $H \subseteq K_m \times K_m$. Therefore, $G\Box H \subseteq (K_n \times K_n)\Box(K_m\Box K_m)$. Now, suppose that $n \geq m$.

Since $|V(K_n \times K_n)| = n^2$, $|V(K_m \times K_m)| = m^2$, and $O\chi(K_n \times K_n) = n \geq m$, $O\chi((K_n \times K_n)\Box(K_m \times K_m)) = nm$ by Theorem 3.2.1. Therefore, it follows that $(K_n \times K_n)\Box(K_m \times K_m) \subseteq K_{nm} \times K_{nm}$. Then, $G\Box H \subseteq K_{nm} \times K_{nm}$ by transitivity, and thus by Theorem 3.1.1, it follows that $O\chi(G\Box H) \leq nm$. □

### 3.2.1 Orthogonal Colourings of Hamming Graphs

Hamming graphs are a special class of graphs named after Richard Hamming and they can be constructed as follows. Let $S$ be a set of $q$ elements and let $d$ be a positive integer. The *Hamming graph*, denoted $H(d,q)$, has vertex set $S^d$, that is, the set of ordered $d$-tuples of elements of S. Two vertices are adjacent if they differ in precisely one coordinate; that is, if their Hamming distance is one.

The graph $H(d,q)$ is also the Cartesian product of $d$ complete graphs $K_q$. To start our study of Hamming graphs, the orthogonal chromatic number of *hypercube graphs*, denoted $Q_n = H(n,2) = K_2 \square K_2 \square \cdots \square K_2$, is established. Corollary 3.2.2 gives an upper bound of $O\chi(Q_n) \leq 2O\chi(Q_{n-1})$. However, an improved upper bound can be achieved with the following result. Additionally, this result will provide the exact orthogonal chromatic number for small hypercube graphs.

**Lemma 3.2.3.** *If $O\chi(G) = n$, where $n \equiv 0 (mod\ 4)$, then $O\chi(G \square K_2) \leq \frac{3n}{2}$.*

**Proof:** Let $G$ be a graph and suppose that $O\chi(G) = n$ and $n \equiv 0(\text{mod } 4)$. By Theorem 3.1.1, $G \subseteq K_n \times K_n$. Thus, $G \square K_2 \subseteq (K_n \times K_n) \square K_2$. Therefore, if an orthogonal colouring of $(K_n \times K_n) \square K_2$ with $\frac{3n}{2}$ colours can be constructed, then by applying Theorem 3.1.1 again, an orthogonal colouring of $G \square K_2$ with $\frac{3n}{2}$ colours is obtained. We show $g = (g_1, g_2)$ is an orthogonal colouring of $(K_n \times K_n) \square K_2$, where:

$$g(((u_i,u_j),v_0)) = \begin{cases} (i,j) & 0 \leq i < \frac{n}{2}. \\ (i+\frac{n}{2}, j+n) & j < \frac{n}{2} \leq i < n. \\ (i+\frac{n}{2}, j) & \frac{n}{2} \leq i,j < n. \end{cases}$$

$$g(((u_i,u_j),v_1)) = \begin{cases} (\frac{3n}{2}-i-1, \frac{n}{2}-j-1) & 0 \leq j < \frac{n}{2}. \\ (\frac{n}{2}-i-1, j+\frac{n}{2}) & i < \frac{n}{2} \leq j < n. \\ (\frac{3n}{2}-i-1, j+\frac{n}{2}) & \frac{n}{2} \leq i,j < n. \end{cases}$$

For example, the grid form of this orthogonal colouring is applied to $(K_4 \times K_4) \square K_2$ in Figure 3.2.1. The rows represent the colours assigned in the first colouring and the columns represent the colours assigned in the second colouring. For simplicity, $x_{i,j}$ denotes the vertices of the form $((u_i,u_j),v_0)$ and $y_{i,j}$ denotes the vertices of the form $((u_i,u_j),v_1)$. For example, $((u_2,u_2),v_1) = y_{2,2}$ receives the colour pair $(3,4)$.

| $\backslash$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | $x_{0,0}$ | $x_{0,1}$ | $x_{0,2}$ | $x_{0,3}$ | $y_{1,2}$ | $y_{1,3}$ |
| 1 | $x_{1,0}$ | $x_{1,1}$ | $x_{1,2}$ | $x_{1,3}$ | $y_{0,2}$ | $y_{0,3}$ |
| 2 | $y_{3,1}$ | $y_{3,0}$ | | | $y_{3,2}$ | $y_{3,3}$ |
| 3 | $y_{2,1}$ | $y_{2,0}$ | | | $y_{2,2}$ | $y_{2,3}$ |
| 4 | $y_{1,1}$ | $y_{1,0}$ | $x_{2,2}$ | $x_{2,3}$ | $x_{2,0}$ | $x_{2,1}$ |
| 5 | $y_{0,1}$ | $y_{0,0}$ | $x_{3,2}$ | $x_{3,3}$ | $x_{3,0}$ | $x_{3,1}$ |

Figure 3.2.1: Grid Orthogonal Colouring of $(K_4 \times K_4)\Box K_2$

Suppose there is an orthogonal conflict. Then, there are three cases to consider. **Case 1**: $g(((u_{i_1}, u_{j_1}), v_0)) = g(((u_{i_2}, u_{j_2}), v_0))$. If $i_1 < \frac{n}{2}$ and $i_2 \geq \frac{n}{2}$, then $i_1 < i_2 + \frac{n}{2}$. Thus, $g_1(((u_{i_1}, u_{j_1}), v_0)) \neq g_1(((u_{i_2}, u_{j_2}), v_0))$, a contradiction. The same argument applies if $i_1 \geq \frac{n}{2}$ and $i_2 < \frac{n}{2}$. Thus, it must be the case that $i_1, i_2 < \frac{n}{2}$ or $i_1, i_2 \geq \frac{n}{2}$. In both cases, since $g_1(((u_{i_1}, u_{j_1}), v_0)) = g_1(((u_{i_2}, u_{j_2}), v_0))$, it follows that $i_1 = i_2$.

Now, if $i_1 = i_2 < \frac{n}{2}$, then $j_1 = j_2$ since $g_2(((u_{i_1}, u_{j_1}), v_0)) = g_2(((u_{i_2}, u_{j_2}), v_0))$. If $i_1 = i_2 \geq \frac{n}{2}$, where $j_1 < \frac{n}{2}$ and $j_2 \geq \frac{n}{2}$, then $j_1 + n \geq n > j_2$. Therefore, $g_2(((u_{i_1}, u_{j_1}), v_0)) \neq g_2(((u_{i_2}, u_{j_2}), v_0))$, a contradiction. Thus, it must be the case that $j_1, j_2 < \frac{n}{2}$ or $j_1, j_2 \geq \frac{n}{2}$. In both cases, since $g_2(((u_{i_1}, u_{j_1}), v_0)) = g_2(((u_{i_2}, u_{j_2}), v_0))$, it follows that $j_1 = j_2$. Therefore, $(i_1, j_1) = (i_2, j_2)$, and there is no orthogonal conflict.

**Case 2**: $g(((u_{i_1}, u_{j_1}), v_1)) = g(((u_{i_2}, u_{j_2}), v_1))$. If $j_1 < \frac{n}{2}$ and $j_2 \geq \frac{n}{2}$, then it follows that $\frac{n}{2} - j_1 - 1 < n \leq j_2 + \frac{n}{2}$. Therefore, $g_2(((u_{i_1}, u_{j_1}), v_1)) \neq g_2(((u_{i_2}, u_{j_2}), v_1))$, a contradiction. The same argument applies if $j_1 \geq \frac{n}{2}$ and $j_2 < \frac{n}{2}$. Thus, $j_1, j_2 < \frac{n}{2}$ or $j_1, j_2 \geq \frac{n}{2}$. In both cases, since $g_2(((u_{i_1}, u_{j_1}), v_1)) = g_2(((u_{i_2}, u_{j_2}), v_1))$, it follows that $j_1 = j_2$.

Now, if $j_1 = j_2 < \frac{n}{2}$, then $i_1 = i_2$ since $g_1(((u_{i_1}, u_{j_1}), v_1)) = g_1(((u_{i_2}, u_{j_2}), v_1))$. If $j_1 = j_2 \geq \frac{n}{2}$ where $i_1 < \frac{n}{2}$ and $i_2 \geq \frac{n}{2}$, then $\frac{n}{2} - i_1 - 1 < \frac{n}{2} \leq \frac{3n}{2} - i_2 - 1$. Therefore, $g_1(((u_{i_1}, u_{j_1}), v_1)) \neq g_1(((u_{i_2}, u_{j_2}), v_1))$, a contradiction. Thus, it must be the case that $i_1, i_2 < \frac{n}{2}$ or $i_1, i_2 \geq \frac{n}{2}$. In both cases, since $g_1(((u_{i_1}, u_{j_1}), v_1)) = g_1(((u_{i_2}, u_{j_2}), v_1))$, it follows that $i_1 = i_2$. Therefore, $(i_1, j_1) = (i_2, j_2)$, and there is no orthogonal conflict.

**Case 3**: $g(((u_{i_1}, u_{j_1}), v_0)) = g(((u_{i_2}, u_{j_2}), v_1))$. It will be shown that in all four subcases, a contradiction arises. **Subcase 1**: $i_1, j_2 < \frac{n}{2}$. Then, $\frac{3n}{2} - i_2 - 1 \geq \frac{n}{2} > i_1$. Thus, $g_1(((u_{i_1}, u_{j_1}), v_0)) \neq g_1(((u_{i_2}, u_{j_2}), v_1))$, a contradiction. **Subcase 2**: $i_1, j_2 \geq \frac{n}{2}$. If $i_2 \geq \frac{n}{2}$, then $i_1 + \frac{n}{2} \geq n > \frac{3n}{2} - i_2 - 1$. If $i_2 < \frac{n}{2}$, then $i_1 + \frac{n}{2} \geq \frac{n}{2} \geq \frac{n}{2} - i_2 - 1$. Thus, for all values of $i_2$, it follows that $g_1(((u_{i_1}, u_{j_1}), v_0)) \neq g_1(((u_{i_2}, u_{j_2}), v_1))$, a contradiction.

**Subcase 3**: Suppose $i_1 < \frac{n}{2}$ and $j_2 \geq \frac{n}{2}$. Then, it follows that $j_1 < n \leq j_2 + \frac{n}{2}$. Thus, $g_2(((u_{i_1}, u_{j_1}), v_0)) \neq g_2(((u_{i_2}, u_{j_2}), v_1))$, a contradiction. **Subcase 4**: $i_1 \geq \frac{n}{2}$ and $j_2 < \frac{n}{2}$. If $j_1 \geq \frac{n}{2}$, then $j_1 \geq \frac{n}{2} > \frac{n}{2} - j_2 - 1$. If $j_1 < \frac{n}{2}$, then $j_1 + n > \frac{n}{2} > \frac{n}{2} - j_2 - 1$. Thus, for all values of $j_1$, it follows that, $g_2(((u_{i_1}, u_{j_1}), v_0)) \neq g_2(((u_{i_2}, u_{j_2}), v_1))$, a contradiction. Therefore, a contradiction arises in all cases. Thus, $(i_1, j_1) = (i_2, j_2)$, and there is no orthogonal conflict. Hence, $g = (g_1, g_2)$ has no orthogonal conflicts.

We now show that $g_1$ and $g_2$ are proper colourings of $(K_n \times K_n)\Box K_2$. First, consider adjacent vertices $((u_{i_1}, u_{j_1}), v_0)$ and $((u_{i_2}, u_{j_2}), v_0)$ where $i_1 \neq i_2$ and $j_1 \neq j_2$. If $i_1 < \frac{n}{2}$ and $i_2 \geq \frac{n}{2}$, then $i_1 \neq i_2 + \frac{n}{2}$. Thus, $g_1(((u_{i_1}, u_{j_1}), v_0)) \neq g_1(((u_{i_2}, u_{j_2}), v_0))$. If $i_1 \geq \frac{n}{2}$ and $i_2 < \frac{n}{2}$, then $i_2 \neq i_1 + \frac{n}{2}$. Thus, $g_1(((u_{i_1}, u_{j_1}), v_0)) \neq g_1(((u_{i_2}, u_{j_2}), v_0))$. Then, since $i_1 \neq i_2$, it follows that $i_1 + \frac{n}{2} \neq i_2 + \frac{n}{2}$. Thus, if $i_1, i_2 < \frac{n}{2}$ or $i_1, i_2 \geq \frac{n}{2}$, then $g_1(((u_{i_1}, u_{j_1}), v_0)) \neq g_1(((u_{i_2}, u_{j_2}), v_0))$. Therefore, there are no colour conflicts in the first colouring.

Now, suppose that $j_1 < \frac{n}{2}$ and $i_1, i_2, j_2 \geq \frac{n}{2}$. Then, it follows that $j_1 + n \neq j_2$. Thus, $g_2(((u_{i_1}, u_{j_1}), v_0)) \neq g_2(((u_{i_2}, u_{j_2}), v_0))$. If $j_1, i_1, i_2 \geq \frac{n}{2}$ and $j_2 \leq \frac{n}{2}$, then it follows that $j_2 + n \neq j_1$. Thus, $g_2(((u_{i_1}, u_{j_1}), v_0)) \neq g_2(((u_{i_2}, u_{j_2}), v_0))$. Lastly, since $j_1 \neq j_2$, it follows that $j_1 + n \neq j_2 + n$. Thus, if $i_1, i_2 < \frac{n}{2}$ or $i_1, i_2, j_1, j_2 \geq \frac{n}{2}$ or $i_1, i_2 \geq \frac{n}{2}$ and $j_1, j_2 < \frac{n}{2}$, then $g_2(((u_{i_1}, u_{j_1}), v_0)) \neq g_2(((u_{i_2}, u_{j_2}), v_0))$. Therefore, in both colourings, there are no colour conflicts between $((u_{i_1}, u_{j_1}), v_0)$ and $((u_{i_2}, u_{j_2}), v_0)$.

Secondly, consider adjacent vertices $((u_{i_1}, u_{j_1}), v_1)$ and $((u_{i_2}, u_{j_2}), v_1)$ where $i_1 \neq i_2$ and $j_1 \neq j_2$. First, suppose that $j_1 < \frac{n}{2}$ and $j_2 \geq \frac{n}{2}$, then $\frac{n}{2} - j_1 - 1 \neq j_2 + \frac{n}{2}$. Thus, $g_2(((u_{i_1}, u_{j_1}), v_1)) \neq g_2(((u_{i_2}, u_{j_2}), v_1))$. If $j_2 \geq \frac{n}{2}$ and $j_2 < \frac{n}{2}$, then $\frac{n}{2} - j_2 - 1 \neq j_1 + \frac{n}{2}$. Thus, $g_2(((u_{i_1}, u_{j_1}), v_1)) \neq g_2(((u_{i_2}, u_{j_2}), v_1))$. Then, since $j_1 \neq j_2$, it follows that $\frac{n}{2} - j_1 - 1 \neq \frac{n}{2} - j_2 - 1$ and $j_1 + \frac{n}{2} \neq j_2 + \frac{n}{2}$. Thus, if $j_1, j_2 < \frac{n}{2}$ or $j_1, j_2 \geq \frac{n}{2}$, then it follows that $g_2(((u_{i_1}, u_{j_1}), v_1)) \neq g_2(((u_{i_2}, u_{j_2}), v_1))$. Therefore, there are no colour conflicts in the second colouring.

Now, suppose that $i_1 < \frac{n}{2}$ and $i_2, j_1, j_2 \geq \frac{n}{2}$. Then, $\frac{n}{2} - i_1 - 1 \neq \frac{3n}{2} - i_2 - 1$. Thus, $g_1(((u_{i_1}, u_{j_1}), v_1)) \neq g_1(((u_{i_2}, u_{j_2}), v_1))$. If $i_2 < \frac{n}{2}$ and $i_1, j_1, j_2 \geq \frac{n}{2}$. Then, $\frac{n}{2} - i_2 - 1 \neq \frac{3n}{2} - i_1 - 1$. Thus, $g_1(((u_{i_1}, u_{j_1}), v_1)) \neq g_1(((u_{i_2}, u_{j_2}), v_1))$. Lastly, since $i_1 \neq i_2$, it follows that $\frac{3n}{2} - i_1 - 1 \neq \frac{3n}{2} - i_2 - 1$ and $\frac{n}{2} - i_1 - 1 \neq \frac{n}{2} - i_1 - 1$. Thus, if $j_1, j_2 < \frac{n}{2}$ or $i_1, i_2, j_1, j_2 \geq \frac{n}{2}$ or $j_1, j_2 \geq \frac{n}{2}$ and $i_1, i_2 < \frac{n}{2}$, then it follows that $g_1(((u_{i_1}, u_{j_1}), v_1)) \neq g_1(((u_{i_2}, u_{j_2}), v_1))$. Therefore, in both colourings, there are no colour conflicts between $((u_{i_1}, u_{j_1}), v_1)$ and $((u_{i_2}, u_{j_2}), v_1)$.

Lastly, consider adjacencies of the form $((u_i, u_j), v_0)$ and $((u_i, u_j), v_1)$. If $i < \frac{n}{2}$ and $j \geq \frac{n}{2}$, then since $n \equiv 0 (\text{mod } 4)$, it follows that $i \neq \frac{n}{2} - i - 1$ and $j \neq j + \frac{n}{2}$. Thus, $g_1(((u_i, u_j), v_0)) \neq g_1(((u_i, u_j), v_1))$ and $g_2(((u_i, u_j), v_0)) \neq g_2(((u_i, u_j), v_1))$. If $i \geq \frac{n}{2}$ and $j < \frac{n}{2}$, then it follows that $i + \frac{n}{2} \neq \frac{3n}{2} - i - 1$ and $j + n \neq \frac{n}{2} - j - 1$. Thus, $g_1(((u_i, u_j), v_0)) \neq g_1(((u_i, u_j), v_1))$ and $g_2(((u_i, u_j), v_0)) \neq g_2(((u_i, u_j), v_1))$.

If $i, j < \frac{n}{2}$, then since $n \equiv 0 (\text{mod } 4)$, it follows that $i \neq \frac{3n}{2} - i - 1$ and $j \neq \frac{n}{2} - j - 1$. Thus, $g_1(((u_i, u_j), v_0)) \neq g_1(((u_i, u_j), v_1))$ and $g_2(((u_i, u_j), v_0)) \neq g_2(((u_i, u_j), v_1))$. Lastly, if $i, j \geq \frac{n}{2}$, then it follows that $i + \frac{n}{2} \neq \frac{3n}{2} - i - 1$ and $j \neq j + \frac{n}{2}$. Thus, $g_1(((u_i, u_j), v_0)) \neq g_1(((u_i, u_j), v_1))$ and $g_2(((u_i, u_j), v_0)) \neq g_2(((u_i, u_j), v_1))$. Hence, in both colourings, there are no colour conflicts between $((u_i, u_j), v_0)$ and $((u_i, u_j), v_1)$.

Therefore, $g_1$ and $g_2$ have no colour conflicts across all of the vertices. Thus, $g = (g_1, g_2)$ is a proper orthogonal colouring of $(K_n \times K_n) \Box K_2$ using $\frac{3n}{2} = M$ colours. Hence, by Theorem 3.1.1, $(K_n \times K_n) \Box K_2 \subseteq K_M \times K_M$. Thus, $G \Box K_2 \subseteq K_M \times K_M$. Therefore, by Theorem 3.1.1, $O\chi(G \Box K_2) \leq M = \frac{3n}{2}$. $\square$

Now that Lemma 3.2.3 has been proved, orthogonal colourings of hypercube graphs can be considered. The proof will be broken into two cases based on the parity of $n$. For $n \equiv 0 (\text{mod } 4)$, Theorem 3.2.1 will be used inductively. For odd $n$, Lemma 3.2.3 and Theorem 1.4.2 will be applied.

**Theorem 3.2.4.** *If $n \neq 2, 3, 9$ then $O\chi(Q_n) = \lceil \sqrt{2^n} \rceil$.*

**Proof:** Note that $Q_2 = C_4$, which by Theorem 2.1.3 can not be orthogonally coloured with 2 colours. Also, it was shown [4] that $Q_3$ cannot be orthogonally coloured with 3 colours. Lastly, for $n = 9$, we suspect that an optimal orthogonal colouring does exist. However, due to the number of vertices and number of colours required, we do not explicitly construct one.

Therefore, we will first show that for even $n > 2$, that $O\chi(Q_n) = 2^{\frac{n}{2}}$. This will be done by proceeding with induction on the number of vertices. To start, consider the base case, $Q_4$. This graph has an orthogonal colouring using 4 colours, as shown in Figure 3.2.2.

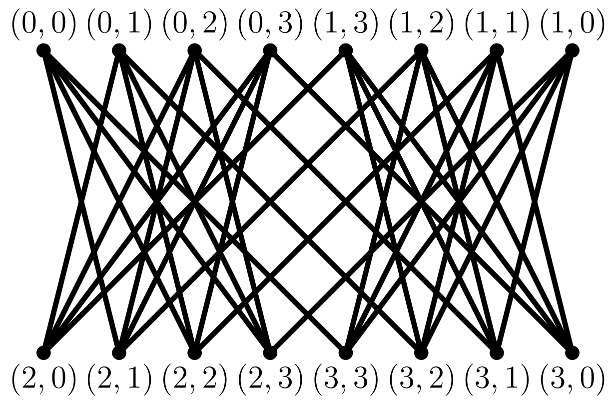


Figure 3.2.2: Orthogonal Colouring of $Q_4$

For the induction step, assume for $k$ even and $k \geq 4$, that $O\chi(Q_k) = 2^{\frac{k}{2}}$. Then, notice that $Q_{k+2} \cong Q_k \square (K_2 \square K_2)$. Now, since the graph $Q_k$ has $(2^{\frac{k}{2}})^2$ vertices, the graph $K_2 \square K_2$ has $2^2$ vertices, and $O\chi(Q_k) = 2^{\frac{k}{2}}$. By Theorem 3.2.1, it follows that $O\chi(Q_{k+2}) = O\chi(Q_k \square (K_n \square K_n)) = 2(2^{\frac{k}{2}}) = 2^{\frac{k+2}{2}}$. Therefore, if $n \neq 2$ is even, then $O\chi(Q_n) = 2^{\frac{n}{2}}$ by induction. We now show that $O\chi(Q_n) = \lceil \sqrt{2^n} \rceil$ for odd $n$.

Now, suppose that $n \geq 5$ is odd. Then, $Q_n = Q_{n-1} \square K_2$ and $Q_{n-1}$ has $(n-1)^2$ vertices. Since $n$ is odd, it follows that $(n-1)^2 \equiv 0 (\text{mod } 4)$. Thus, by Lemma 3.2.3 and the even case, $O\chi(Q_n) = O\chi(Q_{n-1} \square K_2) \leq \frac{3}{2} O\chi(Q_{n-1}) = 3(2^{\frac{n-3}{2}})$. Now, notice than for $n = 5$ and $n = 7$ that $\lceil \sqrt{2^n} \rceil = 3(2^{\frac{n-3}{2}})$. However, for $n = 9$, $\lceil \sqrt{2^n} \rceil + 1 = 3(2^{\frac{n-3}{2}})$.

Thus, Lemma 3.2.3 gives one more colour than necessary for an optimal orthogonal colouring. However, for $n \geq 11$, notice that $\frac{\lceil \sqrt{2^n} \rceil - 1}{4} > \frac{\sqrt{2^n}}{2^2} = 2^{\frac{n-4}{2}}$. Also, for $n \geq 11$, it follows that $n < 2^{\frac{n-4}{2}}$. Therefore, since $Q_n$ has $2^n$ vertices and $\Delta(Q_n) = n$, Theorem 1.4.2 can be applied to give that $O\chi(Q_n) = \lceil \sqrt{2^n} \rceil$ for $n \geq 11$ in this case. $\square$

Now, for a general Hamming graph, $H(d+2, q) = H(d, q) \square H(2, q)$. An orthogonal colouring of $H(2, q) = K_q \square K_q$ is equivalent to finding a pair of orthogonal Latin squares of size $q$. As long as $q \neq 2, 6$, orthogonal Latin squares exist, as shown in the introduction. Thus, an orthogonal colouring of $H(2, q)$ using $q$ colours also exists.

The case $q = 2$ is completed in Theorem 3.2.4. For the case $q = 6$, an orthogonal colouring of $H(4, 6)$ would need to be constructed. However, due to the number of vertices and number of colours required for $H(4, 6)$, we do not explicitly construct one.

**Theorem 3.2.5.** *If $q \neq 2, 6$, then $O\chi(H(2d, q)) = q^d$.*

**Proof:** The orthogonal colourings of $H(2d, q)$ will be found by induction. For $d = 1$, an orthogonal colouring of $H(2, q)$ is equivalent to a pair of orthogonal Latin squares, which exists. Suppose for $k \geq 1$ that $O\chi(H(2k, q)) = q^k$. Consider $H(2(k + 1), q) = H(2k, q)\Box H(2, q)$. Then, by Theorem 3.2.1, $O\chi(H(2(k + 1), q)) = q(q^k) = q^{k+1}$. $\Box$

## 3.3 Orthogonal Colourings of Strong Graphs

The *strong graph product* of two graphs $G$ and $H$, denoted by $G \boxtimes H$, is a graph with the following properties. The vertex set is $V(G) \times V(H)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ in $G \boxtimes H$ are adjacent if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. A graph constructed by the strong graph product of two graphs is referred to as a *strong product graph*. The two graphs in a strong graph product are called *strong components*. We first prove the following result for optimal $k$-orthogonal colourings of strong product graphs.

**Theorem 3.3.1.** *If $G$ has $n^2$ vertices with $O\chi_k(G) = n$ and $H$ has $m^2$ vertices with $O\chi_k(H) = m$, then $O\chi_k(G \boxtimes H) = nm$.*

**Proof:** For $0 \leq r < k$ and $0 \leq i < n$, let $G_{r,i}$ be the $i$-th colour class in the $r$-th colouring of $G$. Then, for $0 \leq r < k$ and $0 \leq j < m$, let $H_{r,j}$ be the $j$-th colour class in the $r$-th colouring of $H$. Next, let $I_{r,i,j} = \{(u, v) \mid u \in G_{r,i}, v \in H_{r,j}\}$. We will show that $C_r = \{I_{r,i,j} \mid 0 \leq i < n, 0 \leq j < m\}$ is a collection of disjoint spanning independent sets of $G \boxtimes H$. That is, $C_r$ is a proper colouring of $G \boxtimes H$ using $nm$ colours.

First, we show that each $I_{r,i,j}$ is an independent set. Let $(u_1, v_1), (u_2, v_2) \in I_{r,i,j}$. Then $u_1, u_2 \in G_{r,i}$ and $v_1, v_2 \in H_{r,j}$. However, $G_{r,i}$ and $H_{r,j}$ are independent sets, thus $u_1u_2 \notin E(G)$ and $v_1v_2 \notin E(H)$. Therefore, $(u_1, v_1)(u_2, v_2) \notin E(G \times H)$. Now, let $(u,v) \in V(G \times H)$. Since $\{G_{r,i} \mid 0 \leq i < n\}$ is a spanning set of $G$, $u \in G_{r,i}$ for some $i$. Similarly, $v \in H_{r,j}$ for some $j$. Therefore, $(u,v) \in I_{r,i,j}$.

Now, suppose that $(u,v) \in I_{r,i_1,j_1}$ and $(u,v) \in I_{r,i_2,j_2}$. If $i_1 \neq i_2$ then $u \in G_{r,i_1}$ and $u \in G_{r,i_2}$. However, this contradicts that $\{G_{r,i} \mid 0 \leq i < n\}$ is a colouring of $G$. Similarly, if $j_1 \neq j_2$, then $v \in H_{r,j_1}$ and $v \in H_{r,j_2}$. However, this contradicts that $\{H_{r,j_1} \mid |0 \leq j < m\}$ is a colouring of $H$. Therefore, there is a unique set $I_{r,i,j}$ that contains $(u,v)$. Thus, $C_r$ is a proper colouring of $G \boxtimes H$ using $nm$ colours.

We now show that each of the colourings are mutually orthogonal. Consider $I_{r_1,i_1,j_1}$ and $I_{r_2,i_2,j_2}$, where $r_1 \neq r_2$. If $(u,v) \in I_{r_1,i_1,j_1}$ and $(u,v) \in I_{r_2,i_2,j_2}$, then $u \in G_{r_1,i_1}$ and $u \in G_{r_2,i_2}$. However, $G_{r_1,i_1} \cap G_{r_2,i_2} = 1$, so let $u$ be this unique vertex. Similarly, $v \in H_{r_1,j_1}$ and $v \in H_{r_2,j_2}$. However, $H_{r_1,j_1} \cap H_{r_2,j_2} = 1$, so let $v$ be this unique vertex. Therefore, there is a unique vertex $(u,v)$ that resides in both $I_{r_1,i_1,j_1}$ and $I_{r_2,i_2,j_2}$.

Hence, each of the $C_r$ are mutually orthogonal. Thus, this provides an orthogonal colouring of $G \boxtimes H$ using $nm$ colours. Since $G \boxtimes H$ has $n^2m^2$ vertices, this gives that $O\chi(G \boxtimes H) = nm$. $\square$

Note that the strong product of two graphs has both the edges of the tensor graph product and the Cartesian graph product. Thus, an orthogonal colouring of a strong product graph also yields orthogonal colourings for these product graphs. Therefore, the following corollary is obtained.

**Corollary 3.3.2.** *If $G$ has $n^2$ vertices with $O\chi_k(G) = n$ and $H$ has $m^2$ vertices with $O\chi_k(H) = m$, then $O\chi_k(G \square H) = nm$ and $O\chi_k(G \times H) = nm$.*

**Proof:** Let $I_{r,i,j}$ be the same set as in Theorem 3.3.1. Then, note that $I_{r,i,j}$ is an independent set in $G \square H$ and $G \times H$. Therefore, this results follows by applying the proof of Theorem 3.3.1. $\square$

Theorem 3.3.1 gives a method to construct optimal $k$-orthogonal colourings when both strong components have an optimal $k$-orthogonal colouring. We use this to find an upper bound on the $k$-orthogonal chromatic number of strong product graphs. Recall Theorem 3.1.1, which gives a way to reformulate the problem as a subgraph question. Unlike optimal orthogonal colourings, for optimal $k$-orthogonal colourings, there are multiple graphs required to reformulate the problem.

Caro and Yuster [11] showed that a graph $G$ with $n$ vertices has an optimal $k$-orthogonal colouring if and only if it is a subgraph of a graph obtained by removing $k$ edge disjoint $K_N$-covers from $K_{N^2}$, where $N = \lceil \sqrt{n}\, \rceil$. Let $K_{N^2}[k]$ denote this family of graphs. Thus, for $k = 2$, Theorem 3.1.1 gives that $K_{N^2}[2] = K_N \times K_N$. Therefore, using the same argumentation as Corollary 3.1.3, but using this family of graphs, we obtain the following upper bound.

**Corollary 3.3.3.** *If* $O\chi_k(G) = n$ *and* $O\chi_k(H) = m$. *then* $O\chi_k(G \boxtimes H) \leq nm$.

**Proof:** Suppose that $O\chi_k(G) = n$. Then $G \subseteq \bar{G}$ and $H \subseteq \bar{H}$ for some $\bar{G} \in K_{n^2}[k]$ and $\bar{H} \in K_{m^2}[k]$. Then, since $\bar{G}$ has $n^2$ vertices with $O\chi_k(G) = n$ and $\bar{H}$ has $m^2$ vertices with $O\chi_k(\bar{H}) = m$, $O\chi_k(\bar{G} \boxtimes \bar{H}) = nm$ by Theorem 3.3.1. Therefore, since $G \boxtimes H \subseteq \bar{G} \boxtimes \bar{H}$, $O\chi_k(G \boxtimes H) \leq nm$ by restricting the $k$-orthogonal colouring. $\square$

# Chapter 4

# Orthogonal Colourings of Random Graphs

In this chapter, we explore orthogonal colourings of random graphs. To start, graphs sampled from the random geometric graph model, denoted $G \sim RG(n, r)$, are studied. We show that if $\frac{r^2 n}{\ln n} \to 0$ as $n \to \infty$, then $G \sim RG(n, r)$ has an optimal orthogonal colouring with high probability. Orthogonal colourings of a graph required later, denoted $H(m, d, t)$, are then explored. We show that if $d \geq \frac{\sqrt{m}}{\sqrt{t}}$, then $H(m, d, t)$ has an orthogonal colouring with $t(d + 1) + 1$ colours.

Orthogonal colourings of clique grids, denoted $L(m, d, t)$, are also studied. We show that if $d \geq \frac{\sqrt{m}}{\sqrt{t}}$, then $L(m^2, d, t^2)$ has an orthogonal colouring using $(t(d+1)+1)^2$ colours. The results obtained for clique grids are then applied to dense random geometric graphs. We show that if $0 \leq \alpha \leq \frac{1}{4}$ and $r = n^{-\alpha}$, then $G \sim RG(n, r)$ has an orthogonal colouring with $n^{1-2\alpha}(1 + o(1))$ colours with high probability.

We also investigate graphs sampled from the Erdős-Rényi model, denoted $G \sim G(n, p)$, are then studied. In particular, the Erdős-Rényi model having probability function $p = \frac{1}{2}$ is considered. We show that if $G \sim G(n, \frac{1}{2})$, then $G$ has an orthogonal colouring using $(4 + o(1))\chi(G)$ colours with high probability.

## 4.1 Orthogonal Colourings of Random Geometric Graphs

The *random geometric graph model*, denoted $RG(n, r)$, is defined as follows. In this model, $n$ points are placed in the unit square, $[0, 1]^2$, uniformly at random. Two vertices are then connected by an edge if and only if the Euclidean distance between the two vertices is less than $r$. If $G$ is a graph sampled from the random geometric graph model, then this is denoted by $G \sim RG(n, r)$. Random geometric graphs are interesting to study since they can be used to model real world networks [39].

An important property for random geometric graphs to have is connectedness. Penrose [49] showed that if $\frac{r^2 n}{\ln n} \to 0$ as $n \to \infty$, then $G \sim RG(n, r)$ is disconnected with high probability, and if $\frac{r^2 n}{\ln n} \to \infty$ as $n \to \infty$, then $G$ is connected with high probability. This led McDiarmid [44] to define the following three classes. If $\frac{r^2 n}{\ln n} \to 0$ as $n \to \infty$, then $G$ is called a *sparse random geometric graph*. Secondly, if $\frac{r^2 n}{\ln n} \to \infty$ as $n \to \infty$, then $G$ is called a *dense random geometric graph*. Lastly, if $\frac{r^2 n}{\ln n} \to c$ as $n \to \infty$, where $0 < c < \infty$, then $G$ is called an *intermediate random geometric graph*.

McDiarmid [44] first studied vertex colourings of sparse random geometric graphs. McDiarmid and Müller [45] then studied the chromatic number of intermediate random geometric graphs. In both cases, it was shown that for $G \sim RG(n, r)$, that $\chi(G) \leq \frac{\ln n}{\ln\left(\frac{\ln n}{r^2 n}\right)}$ with high probability. Lastly, McDiarmid [44] showed that $\chi(G) \leq \frac{\sqrt{3}}{2} r^2 n$ for dense random geometric graphs with high probability.

### 4.1.1 Orthogonal Colourings of Sparse Random Geometric Graphs

To show that sparse random geometric graphs have an optimal orthogonal colourings with high probability, the following lemma by McDiarmid [44] that provides an upper bound on the maximum degree is used.

**Lemma 4.1.1** (McDiarmid [44])**.** *If* $\frac{r^2 n}{\ln n} \to 0$ *and* $G \sim RG(n, r)$*, then* $\Delta(G) \leq \frac{\ln n}{\ln\left(\frac{\ln n}{r^2 n}\right)}$ *with high probability.*

Recall Theorem 1.4.2, which states that for every graph $G$ with $n$ vertices, if $\Delta(G) < \frac{\sqrt{n}-1}{4}$, then $G$ has an optimal orthogonal colouring. The following theorem determines the orthogonal chromatic number of sparse random geometric graphs by combining Lemma 4.1.1 and Theorem 1.4.2.

**Theorem 4.1.2.** *If* $\frac{r^2 n}{\ln n} \to 0$ *and* $G \sim RG(n, r)$*,* $O\chi(G) = \lceil \sqrt{n}\, \rceil$ *with high probability.*

**Proof:** To start, Lemma 4.1.1 gives that $\Delta(G(n, r)) \leq \frac{\ln n}{\ln\left(\frac{\ln n}{r^2 n}\right)}$ with high probability. Since $\frac{r^2 n}{\ln n} \to 0$ by assumption, it follows that $\frac{\ln n}{r^2 n} \to \infty$. So in particular, as $n \to \infty$, $\frac{\ln n}{r^2 n} \geq e$. Therefore, as $n \to \infty$, we get the following chain of inequalities:

$$\Delta(G) \leq \frac{\ln n}{\ln\left(\frac{\ln n}{r^2 n}\right)} \leq \frac{\ln n}{\ln e} < \frac{\sqrt{n}-1}{4}$$

Therefore, $\Delta(G) < \frac{\sqrt{n}-1}{4}$ with high probability. Thus, by Theorem 1.4.2, it follows that $O\chi(G) = \lceil \sqrt{n}\, \rceil$ with high probability. □

By using the maximum degree, Theorem 4.1.2 gives that sparse random geometric graphs have optimal orthogonal colourings with high probability. For dense random geometric graphs, a different approach is required.

### 4.1.2 Orthogonal Colourings of Dense Random Geometric Graphs

The following graph will be used to construct another graph called a clique grid. This clique grid will then be used to construct an orthogonal colouring of dense random geometric graphs. Let $H(m,d,t)$ be the graph with vertices labelled $v_i^j$ for $0 \leq j < t$ and $0 \leq i < m$, where two vertices $v_{i_1}^{j_1}$ and $v_{i_2}^{j_2}$ are adjacent if and only if $|i_1 - i_2| \leq d$. That is, $H(m,d,t)$ is the graph obtained by taking $m$ cliques of size $t$, denoted $C_i$, where all of the vertices in $C_{i_1}$ and $C_{i_2}$ are adjacent to one another if and only if $|i_1 - i_2| \leq d$. For example, $H(9,1,2)$ is given in Figure 4.1.1. In this case, there are nine cliques of size two, and two cliques are adjacent if their indices differ by one.

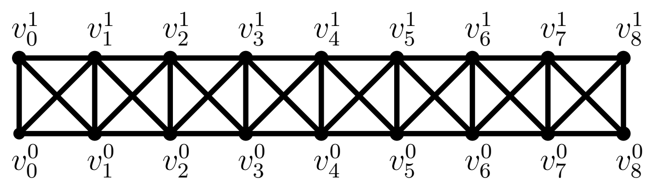


Figure 4.1.1: $H(9,1,2)$

Notice that by greedily colouring the vertices in the order of $C_0, C_1, \ldots, C_{m-1}$, a vertex colouring with $t(d+1)$ colours is obtained. Also, since $C_0, C_1, \ldots, C_d$ forms a clique of size $t(d+1)$, a vertex colouring with less that $t(d+1)$ colours does not exist. Therefore, it follows that $\chi(H(m,d,t)) = t(d+1)$. The following lemma determines that for the right choice of $d$, the orthogonal chromatic number of $H(m,d,t)$ is at most one more than its chromatic number.

**Lemma 4.1.3.** *If $d \geq \frac{\sqrt{m}}{\sqrt{t}}$, then $O\chi(H(m,d,t)) \leq t(d+1)+1$.*

**Proof:** Let $N = t(d+1)+1$ and consider the following two colourings of $H(m,d,t)$: $c_1(v_i^j) = (j+it)(\text{mod } N)$ and $c_2(v_i^j) = \left(j + it + \lfloor \frac{j+it}{N} \rfloor\right)(\text{mod } N)$. First, we will show that these two colourings are proper. Suppose that the vertices $v_{i_1}^{j_1}$ and $v_{i_2}^{j_2}$ are adjacent, that is, $|i_1 - i_2| \leq d$. Therefore, since $|j_1 - j_2| < t$, it follows that $1 \leq |c_1(v_{i_1}^{j_1}) - c_1(v_{i_2}^{j_2})| < t(d+1)$. Since $N > t(d+1)$, it follows that $c_1(v_{i_1}^{j_1}) \neq c_1(v_{i_1}^{j_1})$.

Without loss of generality, suppose $i_1t+j_1 > i_2t+j_2$. Since $|i_1t+j_1-i_2t-j_2| < N$, it follows that $\lfloor \frac{i_1t+j_1}{N} \rfloor \leq \lfloor \frac{i_2t+j_2}{N} \rfloor + 1$. Therefore, by the definition of $c_2$, it follows that $1 \leq |c_2(v_{i_1}^{j_1}) - c_2(v_{i_2}^{j_2})| \leq t(d+1)$. Since $N > t(d+1)$, it follows that $c_2(v_{i_1}^{j_1}) \neq c_2(v_{i_2}^{j_2})$. Hence, $c_1$ and $c_2$ are proper colourings of $H(m,d,t)$.

We will now show that $c_1$ and $c_2$ are orthogonal colourings. First, since $d \geq \frac{\sqrt{m}}{\sqrt{t}}$, it follows that $N \geq t\left(\frac{\sqrt{m}}{\sqrt{t}} + 1\right) + 1 > \sqrt{mt}$. Therefore, there are more colour pairs than vertices. Now, suppose that $c_1(u_{i_1}^{j_1}) = c_1(u_{i_2}^{j_2})$ and $c_2(u_{i_1}^{j_1}) = c_2(u_{i_2}^{j_2})$ where $i_1 \neq i_2$ or $j_1 \neq j_2$. Since $c_1(u_{i_1}^{j_1}) = c_1(u_{i_2}^{j_2})$, this implies that $i_1 t + j_1 = i_2 t + j_2 + cN$ where $0 < c < N$. Hence, it follows that:

$$\begin{aligned} c_2(u_{i_1}^{j_1}) &= \left(i_2 t + j_2 + \left\lfloor \frac{i_2 t + j_2 + cN}{N} \right\rfloor\right) (\text{mod } N) \\ &= \left(i_2 t + j_2 + c + \left\lfloor \frac{i_2 t + j_2}{N} \right\rfloor\right) (\text{mod } N) \\ &= (c_2(v_{i_2}^{j_2}) + c) (\text{mod } N). \end{aligned}$$

Since $c_2(u_{i_1}^{j_1}) = c_2(u_{i_2}^{j_2})$, this gives that $c \equiv 0 (\text{mod } N)$, contradicting $0 < c < N$. Therefore, $c_1$ and $c_2$ are orthogonal colourings using $t(d+1)+1$ colours. $\square$

It still remains an open problem to determine whether an orthogonal colouring with $t(d+1)$ colours exists. By taking the strong product of $H(m,d,t)$ with itself, the clique grid is obtained. The graph $L(m^2,d,t^2) = H(m,d,t) \boxtimes H(m,d,t)$ is called the *clique grid*. Alternatively, $L(m^2,d,t^2)$ can be viewed as $m^2$ cliques of size $t^2$, denoted $C_{i,j}$, where all of the vertices in $C_{i_1,j_1}$ and $C_{i_2,j_2}$ are adjacent to one another if and only if $|i_1 - i_2| \leq d$ and $|j_1 - j_2| \leq d$. For example, the clique grid $L(25,1,1)$ is given in Figure 4.1.2.

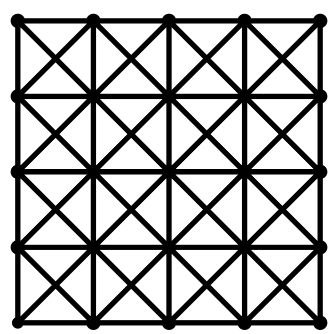

Figure 4.1.2: $L(25,1,1)$

We will show that a dense random geometric graph is a subgraph of a clique grid with high probability. Therefore, by constructing an orthogonal colouring of $L(m^2,d,t^2)$, an upper bound on the orthogonal chromatic number of dense random geometric graphs is obtained. We obtain the following by combining Corollary 3.3.3 and Lemma 4.1.3.

**Lemma 4.1.4.** *If* $d \geq \frac{\sqrt{m}}{\sqrt{t}}$, *then* $(t(d+1))^2 \leq O\chi(L(m^2,d,t^2)) \leq (t(d+1)+1)^2$.

**Proof:** By Corollary 3.3.3, the orthogonal chromatic number of the strong product of two graphs is at most the product of the orthogonal chromatic number of the components. Now, by Lemma 4.1.3, it follows that $O\chi(H(m,d,t)) \leq t(d+1)+1$. Therefore, $O\chi(L(m^2,d,t^2)) \leq O\chi(H(m,d,t))^2 \leq (t(d+1)+1)^2$.

On the other hand, $H(m,d,t)$ has a clique of size $t(d+1)$. Therefore, in $L(m^2,d,t^2)$, there is a clique of size $(t(d+1))^2$. Thus, since the orthogonal chromatic number is at least the size of the chromatic number, $(t(d+1))^2 \leq O\chi(L(m^2,d,t^2))$. $\square$

We now provide a general sketch of the proof for orthogonal colourings of dense random geometric graphs. We will shown that with high probability and for the appropriate choice of parameters, that $G \sim RG(n,r)$ is isomorphic to a subgraph of $L(m^2,d,t^2)$. To obtain this subgraph isomorphism, the unit square is divided into $m \times m$ equal size squares. In particular, for $l = \frac{1}{m}$, the set $S_{ij}$ gives the vertices of $G$ in the square with the following dimensions $((i-1)l, il) \times ((j-1)l, jl)$.

To show that $G$ is isomorphic to a subgraph of $L(m^2,d,t^2)$, all of the vertices in $S_{ij}$ are mapped to vertices in the cliques $C_{ij}$ in $L(m^2,d,t^2)$. To show that this is a subgraph isomorphism with $H \subseteq L(m^2,d,t^2)$, we show that with high probability, for all $i,j$, that $|S_{ij}| \leq |C_{ij}| = t^2$. Additionally, we show that if two vertices are adjacent in $G$, then their images in $H$ are adjacent.

Now, notice that two vertices $u \in C_{i_1,j_1}$ and $v \in C_{i_2,j_2}$ are adjacent if and only if $|i_1 - i_2| \leq d$ and $|j_1 - j_2| \leq d$. On the other hand, two vertices $u \in S_{i_1,j_1}$ and $v \in S_{i_2,j_2}$ are adjacent in $G$ if and only if there Euclidean distance is less than $r$. To distinguish between Euclidean distance and the absolute value, the *Euclidean distance* between two points $u$ and $v$ is denoted by $||u-v||$.

We will show in Lemma 4.1.8 that $||u-v|| < r$ implies that $|i_1 - i_2| < \frac{r}{l} + 1$ and $|j_1 - j_2| < \frac{r}{l} + 1$. Therefore, we define $r = n^{-\alpha}, l = \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil^{-1}$, and $d = \left\lceil \frac{n^{\frac{1}{2}-\alpha}}{\ln n} \right\rceil + 2$ so

$$\begin{aligned}
\frac{r}{l} + 1 &= n^{-\alpha} \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil + 1 \\
&\leq \frac{n^{1/2-\alpha}}{\ln n} + n^{-\alpha} + 1 \\
&< \left\lceil \frac{n^{1/2-\alpha}}{\ln n} \right\rceil + 2 = d
\end{aligned}$$

Therefore, Lemma 4.1.8 will give that if $||u - v|| < r$ then $|i_1 - i_2| < \frac{r}{l} + 1 < d$. Hence, the subgraph isomorphism described will preserve the edges. Additionally, we define two other parameters, $t = \lceil \ln n \rceil$ and $l = \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil^{-1}$. These two parameters are defined in this way to satisfy the probabilistic lemmas proved later. For reference, the follow parameters are used throughout this section.

$$t = \lceil \ln n \rceil \tag{4.1}$$

$$m = \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil \tag{4.2}$$

$$l = \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil^{-1} \tag{4.3}$$

$$d = \left\lceil \frac{n^{\frac{1}{2}-\alpha}}{\ln n} \right\rceil + 2 \tag{4.4}$$

$$r = n^{-\alpha} \tag{4.5}$$

Recall that an event $E$ occurs *with high probability* if as $n$ tends to infinity, the probability that $E$ occurs tends to one. We will show that with high probability and for all $i, j$, that $|S_{ij}| \leq |C_{ij}| = t^2$. We prove this result with Chernoff's bound, which is now stated.

**Lemma 4.1.5** (Chernoff's Bound [14])**.** *Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables taking values in $\{0, 1\}$. Let $X$ denote their sum and let $\mu = \mathbb{E}[X]$. For any $\delta \geq 0$, it follows that*

$$\mathbb{P}(X > (1 + \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}$$

Chernoff's bound gives exponentially decreasing bounds and can be applied to bound the probability that $|S_{ij}| \leq |C_{ij}|$ for a single square. However, it is required that this inequality holds for all squares in the partition. To extend this result to all squares, the Union Bound is required. For the events $E_1, \ldots, E_n$, let $\cup_i E_i$ denote the event that at least one of the events occurs. The Union Bound is then as follows.

**Lemma 4.1.6** (Union Bound [8])**.** *For the events $E_1, \ldots, E_n$, it follows that*

$$\mathbb{P}(\cup_i E_i) \leq \sum_i^n \mathbb{P}(E_i)$$

**Lemma 4.1.7.** *Let $t, m, l, r$ be the parameters in Equations 4.1, 4.2, 4.3, and 4.5. For $G \sim RG(n,r)$ and for all $1 \leq i, j \leq m$, let $S_{ij}$ denotes the vertices of $G$ in the square $((i-1)l, il) \times ((j-1)l, jl)$. With high probability and for all $i, j$, $|S_{i,j}| \leq t^2$.*

**Proof:** First, fix the indices $i$ and $j$. For all vertices $v \in V(G)$, define the random variable $X_v$ as $X_v = 0$ if $v \notin S_{i,j}$ and $X_v = 1$ if $v \in S_{i,j}$. Let $X_{i,j}$ denote the sum of the random variables. That is,

$$X_{i,j} = \sum_v X_v = |S_{i,j}|$$

Recall that the points of $G$ are placed uniformly at random in the unit square, which has an area of 1. Also, the area of each of the $m^2$ squares in the partition of the unit square is $\left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil^{-2}$. Therefore, the probability that $X_v = 1$ is given by

$$\left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil^{-2}$$

Since there are $n$ points and the probability that $X_v = 1$ is $\left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil^{-2}$, the expected number of points in the fixed square is given by $\mu = \mathbb{E}(X_{ij}) = n \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil^{-2} \leq t^2$. Lastly, let $\delta = \frac{\sqrt{3 \ln n}}{\sqrt{n}} \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil > 0$. By applying Chernoff's bound, it follows that

$$\mathbb{P}(X_{i,j} > (1+\delta)\mu) \leq e^{-\frac{\left(\frac{\sqrt{3\ln n}}{\sqrt{n}} \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil\right)^2 n \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil^{-2}}{3}} = e^{-\ln n} = n^{-1}$$

Thus with high probability and for a fixed $i, j$, $|S_{i,j}| = \mathbb{E}(X_{ij}) \leq t^2$. However, to obtain our result, it is required that with high probability and for all $i, j$, that $|S_{i,j}| \leq t^2$. We obtain this by applying the Union Bound:

$$\begin{aligned}
\mathbb{P}(\cup_{i,j}\{X_{i,j} > (1+\delta)\mu\}) &\leq \sum_{i=1,j=1}^{m} \mathbb{P}(X_{i,j} > (1+\delta)\mu) \\
&\leq mn^{-1} \\
&= \left\lceil \frac{\sqrt{n}}{\ln n} \right\rceil^{2} n^{-1} \\
&\leq (\ln n)^{-2} + 2(\sqrt{n}\ln n)^{-1} + n^{-1}
\end{aligned}$$

Note that each of terms in this expression tends to zero as $n$ tends to infinity. Therefore with high probability and for all $i, j$, $|S_{i,j}| \leq t^2$. $\square$

Lemma 4.1.7 gives that the vertices in $S_{i,j}$ in the random geometric graph can be mapped into the cliques $C_{i,j}$ of $L(m^2, d, t^2)$. Thus, one required property of the subgraph isomorphism is obtained. It remains to show that the edges are preserved under this mapping. The following lemma provides this property.

**Lemma 4.1.8.** *Let $t, m, l, d$ be the parameters in Equations 4.1, 4.2, 4.3, and 4.4, and 4.5. For $G \sim RG(n, r)$ and for all $1 \leq i, j \leq m$, let $S_{ij}$ denote the vertices of $G$ in the square $((i-1)l, il) \times ((j-1)l, jl)$. For $u, v \in V(G)$, suppose that $u \in S_{i_1,j_1}$ and $v \in S_{i_2,j_2}$. If $uv \in E(G)$, then $|i_1 - i_2| < d$ and $|j_1 - j_2| < d$.*

**Proof:** Recall that $d$ is the parameter that provides which cliques in $L(m^2, d, t^2)$ are adjacent. Suppose that $u \in S_{i_1,j_1}$, $v \in S_{i_2,j_2}$, and $uv \in E(G)$. It then follows that $|i_1 - i_2| < \frac{r}{l} + 1$ and $|j_1 - j_2| < \frac{r}{l} + 1$. This is because there are at most $\frac{r}{l} + 1$ inclusive squares between $S_{i_1,j}$ and $S_{i_1,j}$. Similarly, there are at most $\frac{r}{l} + 1$ inclusive squares between $S_{i,j_1}$ and $S_{i,j_2}$. However, by the specific choice of parameters, $\frac{r}{l} + 1 < d$, as shown earlier. Therefore, $|i_1 - i_2| < \frac{r}{l} + 1 < d$ and $|j_1 - j_2| < \frac{r}{l} + 1 < d$. □

Lemma 4.1.8 provides that the previously described mapping will preserve all of the edges. Therefore, by combining Lemma 4.1.7 and Lemma 4.1.8, we obtain the following result.

**Theorem 4.1.9.** *Let $t, m, l, d$ be the parameters in Equations 4.1, 4.2, 4.3, and 4.4. For $G \sim RG(n, r)$ where $r = n^{-\alpha}$, if $0 \leq \alpha \leq \frac{1}{4}$, then with high probability*

$$\frac{\sqrt{3}}{2} n^{1-2\alpha} \leq O\chi(G) \leq n^{1-2\alpha}(1 + o(1))$$

*In particular, if $\alpha = \frac{1}{4}$, then with high probability*

$$O\chi(G) = \sqrt{n}(1 + o(1))$$

**Proof:** Consider partitioning the unit square into $m \times m$ equal size squares. Let $S_{ij}$ denote the vertices of $G$ in the square with dimensions $((i-1)l, il) \times ((j-1)l, jl)$. Consider the mapping that takes all of the vertices in $S_{i,j}$ and maps them to vertices in $C_{i,j}$ in $L(m^2, d, t^2)$. Lemma 4.1.7 and Lemma 4.1.8 then gives that with high probability, $G$ is a subgraph of $L(m^2, d, t^2)$ through this mapping. To apply Lemma 4.1.4 to find an orthogonal colouring of $L(m^2, d, t^2)$, it is required that $d \geq \frac{\sqrt{m}}{\sqrt{t}}$. By the choice of parameters, it follows that

$$\begin{aligned}
d &\geq \left\lceil \frac{n^{1/4}}{\ln n} \right\rceil + 1 && \text{Since } 0 \leq \alpha \leq \frac{1}{4} \\
&\geq \frac{n^{1/4}}{\ln n} + \frac{1}{\sqrt{\ln n}} \\
&\geq \frac{\sqrt{\frac{\sqrt{n}}{\ln n} + 1}}{\sqrt{\ln n}} \\
&\geq \frac{\sqrt{m}}{\sqrt{t}}.
\end{aligned}$$

Therefore, since $d \geq \frac{\sqrt{m}}{\sqrt{t}}$, Lemma 4.1.4 can be applied to find an orthogonal colouring of $L(m^2, d, t^2)$. By substituting the parameters into Lemma 4.1.4, it follows that

$$\begin{aligned}
O\chi(L(m^2, d, t^2)) &\leq (t(d+1)+1)^2 \\
&\leq \left( (\ln n + 1) \left( \frac{n^{\frac{1}{2}-\alpha}}{\ln n} + 2 \right) + 1 \right)^2 \\
&\leq n^{1-2\alpha}(1 + o(1))
\end{aligned}$$

In particular, for $\alpha = \frac{1}{4}$, this gives that $O\chi(G) \leq \sqrt{n}(1 + o(1))$. On the other hand, it is known [44] that with high probability, $\chi(G) = \frac{\sqrt{3}}{2} n^{1-2\alpha}$. Therefore, since the orthogonal chromatic number is at least the chromatic number, this gives that $O\chi(G) \geq \frac{\sqrt{3}}{2} n^{1-2\alpha}$ with high probability. □

Theorem 4.1.9 shows for $\alpha < \frac{1}{4}$, the orthogonal chromatic number of dense random geometric graphs is close to the chromatic number. For $\frac{1}{4} \leq \alpha$, Theorem 4.1.9 gives that the orthogonal chromatic number is asymptotically close to $\lceil \sqrt{n} \rceil$. This gives the following corollary.

**Corollary 4.1.10.** *For $G \sim RG(n, r)$, if $r < n^{-1/4}$, then with high probability,*

$$O\chi(G) = (1 + o(1))\lceil \sqrt{n} \rceil$$

It remains an open problem to show that the asymptotic bound for dense random geometric graphs when $r < n^{-1/4}$ can be reduced to an optimal orthogonal colouring with high probability. Also, it remains an open problem to show that the asymptotic bound for dense random geometric graphs when $r > n^{-1/4}$ can be reduced to an orthogonal colouring with just $\frac{\sqrt{3}}{2} n^{1-2\alpha}$ many colours.

## 4.2 Orthogonal Colourings of Erdős-Rényi Random Graphs

The *Erdős-Rényi random graph model*, denoted $G(n,p)$, is a probability distribution over all graphs. In this model, each edge is included in the graph with probability $p$, independently from every other edge. If a graph $G$ is sampled from the distribution, the notation $G \sim G(n,p)$ is used.

The parameter $p$ in this model can be thought of as a weighting function. As $p$ increases from 0 to 1, the model becomes more and more likely to include graphs with more edges and less and less likely to include graphs with fewer edges. In particular, the case $p = \frac{1}{2}$ corresponds to the case where all $2^{\binom{n}{2}}$ graphs on $n$ vertices are chosen with equal probability.

As with the random geometric graph, the weighting function $p$ categorizes when $G \sim G(n,p)$ is connected. Erdős and Rényi [18] showed that if $p < \frac{(1-\epsilon)\ln n}{n}$, then $G$ will be disconnected with high probability. Conversely, if $p > \frac{(1-\epsilon)\ln n}{n}$, then $G$ will be connected with high probability. Therefore, $\frac{\ln n}{n}$ is a sharp threshold for connectedness in this model.

The chromatic number of a graph sampled from $G(n, \frac{1}{2})$ was originally determined by Grimmett and McDiarmid [27]. They showed that $\chi(G) = (1+o(1))\frac{n}{2\log_2(n)}$ with high probability. However, to construct an orthogonal colouring of $G \sim G(n, \frac{1}{2})$, we use the following result of Bukh.

The result applied the following definition. Recall that an event occurs *with exponential probability* if the probability that the event does not happen is exponentially small, that is, $e^{-f(n)}$, where $f(n) \in \Omega(\log^2(n))$. The following result gives a method to construct an orthogonal colouring of $G \sim G(n,p)$ with high probability.

**Lemma 4.2.1** (Bukh [10])**.** *For $m = \left\lfloor \frac{n}{\log_2^2(n)} \right\rfloor$ and $k = 2\log_2(n) + \log_2(\log_2(n))$, with exponential probability, for $G \sim G(n, \frac{1}{2})$, every set of size $m$ contains an independent set of size $k$. In particular, a vertex colouring of $G$ having $\frac{n}{k}$ colour classes of size $k$ and $m$ colour classes of size 1 exists.*

The fact that this occurs for every set of size $m$ is important, since it will allow us to apply Lemma 4.2.1 linearly many times. For the first colouring, the same colouring used by Bukh [10] in his alternate proof of the chromatic number of $G(n, \frac{1}{2})$ is used. By then using Lemma 4.2.1, a second orthogonal colouring is constructed.

**Theorem 4.2.2.** *If* $G \sim G(n, \frac{1}{2})$*,* $O\chi(G) \leq (4 + o(1))\frac{n}{2\log_2(n)}$ *with high probability.*

**Proof:** Let $m = \left\lfloor \frac{n}{\log_2^2(n)} \right\rfloor$ and let $k = 2\log_2(n) + \log_2(\log_2(n))$. By Lemma 4.2.1, a colouring $c_1$ of $G \sim G(n, p)$ with $\frac{n}{k}$ colour classes of size $k$ and $m$ colour classes of size 1 exists. Next, notice that

$$\lim_{n\to\infty} \frac{\log_2^2(n)}{2\log_2(n) + \log_2(\log_2(n))} = \infty.$$

Hence as $n \to \infty$, $\frac{n}{k} > m$. Thus, there are at least $m$ independent sets of size $k$ used in the first colouring. Assume now that every set of size $m$ contains an independent set of size $k$. By Lemma 4.2.1, this happens with exponential probability. It is now shown that under this assumption, a second orthogonal colouring can be constructed.

Define $c_2$ as follows: Take one vertex from each of the colour classes of the first colouring. There are $\frac{n}{k} + m > m$ vertices in this set, so this set will contain an independent set of size $k$ by the previous assumption. Remove these $k$ vertices from the graph. If there are at least $m$ colour classes remaining, then repeat this process.

Let $t$ be the number of independent sets formed this way. Since there are $k$ vertices in each of these independent sets, there are at most $\frac{n}{k}$ independenet sets formed this way. Therefore, it follows that $t \leq \frac{n}{k}$. Now, assign these independent sets the colours $1, 2, \ldots, t$. Since each vertex in these independent sets has a different colour in $c_1$ by construction, there are no orthogonal conflicts.

The remaining vertices will each receive a new colour. After the first step, there are less than $m$ colour classes of $c_1$ remaining. Also, each of these colour classes contain at most $k$ vertices. Hence, there are at most $km$ remaining vertices to receive a distinct colour. Therefore, the total number of colours used is

$$\begin{aligned}
\frac{n}{k} + km &= \frac{n}{2\log_2(n) + \log_2(\log_2(n))} + (2\log_2(n) + \log_2(\log_2(n)))\frac{n}{\log_2^2(n)} \\
&= \frac{n}{2\log_2(n)}\left(\frac{1}{1 + \frac{\log_2(\log_2(n))}{2\log_2(n)}} + \frac{4\log_2(n) + 2\log_2(\log_2(n))}{\log_2(n)}\right) \\
&= \frac{n}{2\log_2(n)}\left(\frac{1}{1 + \frac{\log_2(\log_2(n))}{2\log_2(n)}} + 4 + \frac{2\log_2(\log_2(n))}{\log_2(n)}\right) \\
&= \frac{n}{2\log_2(n)}(4 + \mathrm{o}(1))
\end{aligned}$$

Therefore, since $\chi(G) = \frac{n}{2\log_2(n)}$ with high probability, $O\chi(G) \leq (4 + o(1))\chi(G)$ with high probability. $\square$

## 4.3 Entropy Compression Method

In probability theory, if a group of events are mutually independent, and each event has probability less than 1, then there is a chance none of the events will occur. The Lovász Local Lemma lets this mutual independence condition be relaxed. Informally, the Lovász Local Lemma states that if the events are mostly mutually independent and individually not too likely, then there is a chance that none of the events will occur. Formally, the Lovász Local Lemma is stated below.

**Lemma 4.3.1** (Lovász Local Lemma [50]). *Let $A_1, A_2, \ldots, A_k$ be a sequence of events such that each event occurs with probability at most $p$ and is dependent with at most $d$ of the events. If $ep(d+1) \leq 1$, then there is a nonzero probability that none of the events occurs.*

In terms of graph colourings, the Lovász Local Lemma is used to determine upper bounds of various chromatic numbers [26]. The drawback of the Lovász Local Lemma is that it only proves existence. In 2010 however, Moser and Tardos [47] created an algorithmic version of Lovász Local Lemma. The general idea of their algorithm is to resample the events until none of the events occur.

To prove that this algorithm terminates, a compact record of what the algorithm does at each step is kept. The record needs to have enough information such that when combined with the partial assignment at step $t$, each step of the algorithm up to step $t$ can be determined. This process of using a compact record to prove that an algorithm terminates has been coined as the entropy compression method.

Entropy compression has also been applied to graph colouring problems. For example, Dujmović et al [16] applied entropy compression to non-repetitive colourings. Esperet and Parreau [20] obtained an upper bound on the acyclic chromatic number that is better than the upper bound obtained by the Lovász Local Lemma by using the entropy compression method.

A colour configuration is a graph together with a specific vertex colouring. Esperet and Parreau [20] developed a framework that can be applied to any colouring where some colour configurations are forbidden. For instance, in the case of star colouring, there are two configurations, a single edge with both ends having the same colour and a $P_4$ that is properly 2-coloured.

In this section, orthogonal colourings are created using the entropy compression method. Orthogonal colourings cannot be described as a colouring with forbidden colour configurations. Therefore, the general framework by Esperet and Parreau [20] cannot be used. The basic principles of entropy compression are used instead.

The goal is to use entropy compression to improve upon the best known upper bounds on the orthogonal chromatic number. For a graph $G$, write the maximum degree in terms of the size of $G$ as $\Delta(G) = \alpha\sqrt{n}$. By substituting this into Theorem 1.4.2, Theorem 1.4.8, and Theorem 1.4.6, the following corollaries are obtained.

**Corollary 4.3.2.** *If $\alpha \leq \frac{1}{4}$ and $\Delta(G) = \alpha(\sqrt{n} - 1)$, then $O\chi(G) = \lceil \sqrt{n} \rceil$.*

**Proof:** If $\alpha \leq \frac{1}{4}$ and $\Delta(G) = \alpha(\sqrt{n} - 1)$, then $\Delta \leq \frac{\sqrt{n}-1}{4}$. Therefore, by Theorem 1.4.2, it follows that $O\chi(G) = \lceil \sqrt{n} \rceil$. □

**Corollary 4.3.3.** *If $\Delta(G) = \alpha\sqrt{n}$, then $O\chi(G) \leq (\alpha + 1)\lceil \sqrt{n} \rceil$*

**Proof:** By Theorem 1.4.8, $O\chi(G) \leq \Delta(G) + \sqrt{n - \Delta(G)}$. Therefore, by substituting $\Delta(G) = \alpha\sqrt{n}$, it follows that $O\chi(G) \leq \alpha\sqrt{n} + \sqrt{n - \alpha\sqrt{n}} \leq (\alpha + 1)\sqrt{n}$. □

**Corollary 4.3.4.** *If $\Delta(G) = \alpha\sqrt{n}$, then $O\chi(G) \leq (\alpha + \frac{1}{\alpha} + \frac{2}{\sqrt{n}})\lceil \sqrt{n} \rceil$*

**Proof:** By Theorem 1.4.6, $O\chi(G) \leq \left\lceil \frac{n}{\Delta(G)} \right\rceil + \Delta(G) + 1$. Therefore, by substituting $\Delta(G) = \alpha\sqrt{n}$, it follows that $O\chi(G) \leq \left\lceil \frac{n}{\alpha\sqrt{n}} \right\rceil + \alpha\sqrt{n} + 1 \leq (\alpha + \frac{1}{\alpha} + \frac{2}{\sqrt{n}})\lceil \sqrt{n} \rceil$. □

Recall that Theorem 1.4.6 was obtained from applying Theorem 1.4.5 and then using a greedy vertex colouring. On the other hand, Theorem 1.4.8 was obtained by greedily assigning colour pairs to the vertices. Here, we obtain a greedy orthogonal colouring by randomly assigning colour pairs and then using the entropy compression method.

This might explain why our result, Theorem 4.3.9, tends towards the result of Theorem 1.4.8. The general approach developed here may be improved if restricted to particular classes of graphs, rather than arbitrary graphs. For $\Delta(G) = \alpha\sqrt{n}$, the graph in Figure 4.3.1 illustrates which of Corollary 4.3.2, Corollary 4.3.3, and Corollary 4.3.4 give the best upper bound on the orthogonal chromatic number in terms of the maximum degree. The red line illustrates the bound to be achieved in this section.

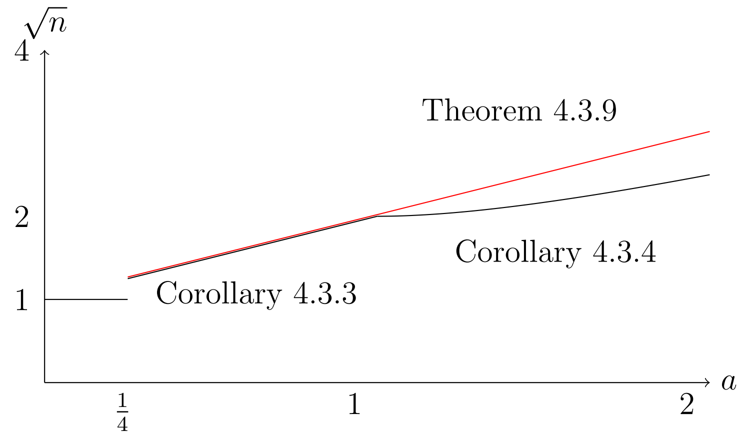


Figure 4.3.1: Best Upper Bounds on Orthogonal Chromatic Number

### 4.3.1 Entropy Compression Using Maximum Degree

An algorithm that maintains a proper, partial orthogonal colouring at each step is considered. The algorithm only terminates early if an orthogonal colouring of the entire graph is found. For this algorithm, an arbitrary input vector is encoded as an orthogonal colouring. If it can be shown that for some time step, that the number of vectors that cause the algorithm not to terminate is less than the total number of input vectors, then there will exist an input vector that does terminate the algorithm. By then using this input vector, an orthogonal colouring of the entire graph is obtained.

This is where the entropy compression technique is implemented. Suppose that the steps of the algorithm can be recorded in a compact form, such that the partial colouring of the algorithm at any past time step can be recovered from the current partial colouring and this record. If the amount of additional new information that is recorded at each step of the process is less than the total amount of information generated at each step, then the algorithm eventually terminates. This is because the difference in total information cannot exceed the fixed amount of information.

This is the general process for using entropy compression. To state the algorithm applied here, some definitions are required. Let $G$ be a graph and let $v_1, v_2, \dots, v_n$ be a labelling $L$ of the vertices. Let $\kappa = \Delta(G) + \sqrt{n+2}$ denote the number of colours used. At step $t$, let $A_{t,i}$ and $B_{t,i}$ denote the sets of colours not used on $v_i$'s neighbours, in the first and second colouring respectively. After step $t$, let $P_t$ denote the partial colouring of the graph.

For $t \in \mathbb{Z}$, consider two random vectors $X, Y \in \{1, 2, \ldots, \sqrt{n+2}\}^t$. The $i$-th element of $X$ and $Y$ are denoted by $X_i$ and $Y_i$ respectively. The reason the entries of $X$ and $Y$ range over $\{1, 2, \ldots, \sqrt{n+2}\}$ is so that $|A_{t,i}| \geq \kappa - \Delta(G) \geq \sqrt{n+2}$ and $|B_{t,i}| \geq \kappa - \Delta(G) \geq \sqrt{n+2}$. Therefore, the $X_i$-th smallest element of $A_{t,i}$ and the $Y_i$-th smallest element of $B_{t,i}$ will exist, which is used in the algorithm.

A record $\{R_i : i \leq t\}$ of the process up to step $t$ is also maintained. In each step, one uncoloured vertex will be coloured. If the colour pair is already present in the current colouring at another vertex, then that vertex will be uncoloured. The index of that uncoloured vertex will be recorded. If there is no conflict, then the record will be set to zero. The orthogonal colouring algorithm using entropy compression and maximum degree is now stated.

**Algorithm** Entropy Compression Using Maximum Degree

**Input:** $(G, L, t, X, Y)$.

**Output:** $(P_t, \{R_i \mid i \leq t\})$.

1: Order the vertices of $G$ according to $L$.

2: For $i = 1, \ldots, t$, follow steps 3 through 7.

3: Find $v_j$, the uncoloured vertex with the smallest index.

4: If all vertices coloured, then stop.

5: Compute $A_{t,j}$ and $B_{t,j}$. Let $c_1$ be the $X_i$-th smallest element of $A_{t,j}$ and $c_2$ be the $Y_i$-th smallest element of $B_{t,j}$.

6: Assign the pair $(c_1, c_2)$ to $v_j$.

7: If $(c_1, c_2)$ already occurs on another vertex, $v_k$, then uncolour $v_k$ and set $R_i = k$. Otherwise, set $R_i = 0$.

Note that since $A_{t,i}$ and $B_{t,i}$ are the sets of unused colours, the partial colouring will not have any colour conflicts. Also, since duplicated colour pairs get removed, there will be no orthogonal conflicts. Therefore, the following lemma holds.

**Lemma 4.3.5.** *At each step $t$ of the algorithm, $P_t$ remains proper and orthogonal.*

The goal is to show that at step $i$, the set of records until step $i$, $\{R_j \mid j \leq i\}$, along with the partial colouring at step $i$, are enough to determine $\{(X_j, Y_j) \mid j \leq i\}$. That is, the colour pairs assigned until step $i$ of the algorithm can all be determined. Let $U_i$ be the set of uncoloured vertices after step $i$ of the algorithm.

**Lemma 4.3.6.** *At each step $i$, $1 \leq i \leq t$, $U_i$ is uniquely determined by $\{R_j \mid j \leq i\}$.*

**Proof:** This result is proved by induction on $i$. All vertices start uncoloured, and at each step the smallest uncoloured vertex is coloured. Therefore, $v_1$ is coloured after step 1, and $U_1 = V(G)\backslash\{v_1\}$. Thus, $U_1$ is uniquely determined.

Assume for some $k \geq 1$, that $U_k$ is uniquely determined by $\{R_j \mid j \leq k\}$. Now, consider $\{R_j \mid j \leq k+1\}$. By the induction step, $U_k$ is uniquely determined. Let $r$ be the smallest index in $U_k$. Then, $v_r$ is the smallest uncoloured vertex in step $k+1$. If $R_{k+1} = 0$, then $v_r$ was coloured and there was no conflict created, so $U_{k+1} = U_k\backslash\{v_r\}$. If $R_{k+1} \neq 0$, suppose $R_{k+1} = s$, then $v_s$ was uncoloured and $v_r$ was coloured. Thus, $U_{k+1} = (U_k\backslash\{v_r\}) \cup \{v_s\}$. Therefore, in both cases, $U_{k+1}$ is uniquely determined. □

Let $(X,Y)(t)$ be the set of vectors $(X,Y)$ such that at step $t$ of the algorithm, the graph $G$ has not been completely coloured. Then, $|(X,Y)(t)| \leq |(X,Y)| = (\sqrt{n+2})^{2t}$. If this inequality is strict, then there is an input vector $(X,Y)$ such that the graph does get completely coloured. Therefore, the algorithm applied to this $(X,Y)$ results in an orthogonal colouring of the graph using $\kappa$ colours. To prove that the inequality is strict, let $F$ be a function that takes input $(X,Y)$, and returns $(\{R_j \mid j \leq k\}, P_i)$. The following lemma shows that $F$ is an injective map.

**Lemma 4.3.7.** *At each step $i$, $1 \leq i \leq t$, the function $F$ is injective.*

**Proof:** This result is proved by induction on $i$. It is shown that $\{R_j \mid j \leq i\}$ and $P_i$ uniquely determine $\{(X_j, Y_j) \mid j \leq i\}$. After the first step, the colour of $v_1$ in $P_1$ is $(X_1, Y_1)$. Assume that for some $k \geq 1$, $\{(X_j, Y_j) \mid j \leq k\}$ is uniquely determined by $\{R_j \mid j \leq k\}$ and $P_k$. Consider $\{R_j \mid j \leq k+1\}$ and $P_{k+1}$. By Lemma 4.3.6, it follows that $U_{k+1}$ and $U_k$ are uniquely determined. Therefore, the one element in $U_k\backslash U_{k+1}$, call this $v_r$, is the vertex coloured at step $k+1$.

Assume first that $R_i = 0$. Then $P_k$ is obtained from $P_{k+1}$ by uncolouring $v_r$. By the induction step, it follows that $\{(X_j, Y_j) \mid j \leq k\}$ is uniquely determined. Let $(c_1, c_2)$ be the colour pair assigned to $v_r$ in $P_{k+1}$. Then, let $a$ be the number of different colours, smaller than $c_1$, that appear on vertices adjacent to $v_r$ in $P_k$ in the first colouring. Similarly, define $b$ on the number of colours in the second colouring. Then, $(X_{k+1}, Y_{k+1}) = (c_1 - a, c_2 - b)$ and $\{(X_j, Y_j) \mid j \leq k+1\}$ is uniquely determined.

Now, assume that $R_i = s$. The colouring $P_k$ is obtained from $P_{k+1}$ by colouring $v_s$ with the colour on $v_r$ in $P_{k+1}$ and uncolouring $v_r$. By the induction step, it follows that $\{(X_j, Y_j) \mid j \leq k\}$ is uniquely determined. The pair $(X_{k+1}, Y_{k+1})$ is obtained the same way as in the previous case. Therefore, in both cases, $\{(X_j, Y_j) \mid j \leq k+1\}$ is uniquely determined. $\square$

**Lemma 4.3.8.** $|(X, Y)(t)| \leq (\kappa^2 + 1)^n (n+1)^t$.

**Proof:** Each vertex can be assigned a colour pair or not, so there are at most $(\kappa^2 + 1)^n$ partial colourings of $G$. Each entry of $R$ has $n + 1$ options, so at step $t$, there are at most $(n + 1)^t$ possible records. Therefore, applying Lemma 4.3.7 with these bounds proves the result. $\square$

**Theorem 4.3.9.** *For a graph $G$ with $n$ vertices, $O\chi(G) \leq \Delta(G) + \sqrt{n+2}$.*

**Proof:** By Lemma 4.3.8, it suffices to show $(\kappa^2 + 1)^n (n+1)^t < \sqrt{n+2}^{2t}$ for some step $t$. This is shown by taking the limit as $t$ goes to infinity. By rearranging the equation and taking the limit, it suffices to show that

$$\lim_{t \to \infty} \frac{(\kappa^2 + 1)^n (n+1)^t}{(\sqrt{n+2})^{2t}} < 1$$

This is a geometric sequence, so this limit hold if $n+1 < (\sqrt{n+2})^2 = n+2$. Therefore, this limit holds, and thus there is some step $t$ where the inequality is strict. Therefore, $G$ has an orthogonal colouring with $\Delta(G) + \sqrt{n+2}$ colours. $\square$

# Chapter 5

# Variations of Orthogonal Colourings

In this chapter we study a variation of orthogonal colourings called $(k,t)$-orthogonal colourings. For this variation, a colour pair can be assigned at most $t$ times. We show that this variation has applications to $[n,k,t]$-transversal designs. We also study the concept of an optimal $(k,t)$-orthogonal colouring. We show that a graph with $n$ vertices has an optimal $(2,t)$-orthogonal colouring if and only if for $N = \lceil \sqrt{n}\, \rceil$, $G$ is a subgraph of $(K_N \times K_N) \circ \bar{K}_t$, where $G \circ H$ is the lexicographic product graph.

We then generalize the results that were obtained for orthogonal colourings of tensor product graphs to $(2,t)$-orthogonal colourings. First, we show that if $G$ has $n^2 t$ vertices, $H$ has $m^2$ vertices, and $O\chi_{(2,t)}(G) = n$, then $O\chi_{(2,t)}(G \times H) = nm$. This then provides the corollary that if $O\chi_{(2,t)}(G) = n$ and $O\chi_{(2,t^2)}(H) = m$, then $O\chi_{(2,t)}(G \times H) \leq tnm$.

Next, we generalize the results that were obtained for orthogonal colourings of Cartesian product graphs to $(2,t)$-orthogonal colourings. First, we show that if $G$ has $n^2 t$ vertices, $H$ has $m^2$ vertices, and $O\chi_{(2,t)}(G) = n \geq m$, then $O\chi_{(2,t)}(G \square H) = nm$. This then provides the corollary that if $O\chi_{(2,t)}(G) = n$ and $O\chi_{(2,t^2)}(H) = m$, and $n \geq m$, then $O\chi_{(2,t)}(G \square H) \leq tnm$.

We then generalize the results that were obtained for orthogonal colouring of strong product graphs to $(k,t)$-orthogonal colourings. First, we show that if $G$ has $n^2 t$ vertices with $O\chi_{(k,t)} = n$ and $H$ has $m^2 r$ vertices with $O\chi_{(k,r)} = m$, then $O\chi_{(k,tr)}(G \boxtimes H) = nm$. This then provides the corollary that if $O\chi_{(k,t)}(G) = n$ and $O\chi_{(k,r)}(H) = m$, then $O\chi_{(k,tr)}(G \boxtimes H) \leq nm$.

Lastly, we find the $(k,t)$-orthogonal chromatic number of some families of graphs. First, we determine an upper bound on the $(2,t)$-orthogonal chromatic number of complete $r$-partite graphs. We then show that the independent sets and cycle graphs have an optimal $(2,t)$-orthogonal colouring under some restrictions. Both results are obtained by extending the known optimal orthogonal colourings.

## 5.1 $(k,t)$-Orthogonal Colourings

In Chapter 1, the concept of $k$-orthogonal colourings were introduced. A $k$-orthogonal colouring is a is a set of $k$ colourings so that each pair of colourings is mutually orthogonal. This means that, for each pair of colourings, each colour pair occurs at most once. Here, a $(k,t)$-orthogonal colouring relaxes this condition by allowing for each pair of colourings, each colour pair occurs at most $t$ times. A formal definition of $(k,t)$-orthogonal colouring is now given.

Two colourings of a graph $G$ are *$t$-orthogonal* if they have the property that when $t+1$ vertices are coloured with the same colour in one of the colourings, then at least one of these $t+1$ vertices must have a distinct colour in the other colouring. A *$(k,t)$-orthogonal colouring* of $G$ is a collection of $k$ mutually $t$-orthogonal colourings. The *$(k,t)$-orthogonal chromatic number*, denoted $O\chi_{(k,t)}(G)$, is the minimum number of colours required so that $G$ has a $(k,t)$-orthogonal colouring. For example, a $(2,3)$-orthogonal colouring of $C_6$ using 2 colours is given in Figure 5.1.1.

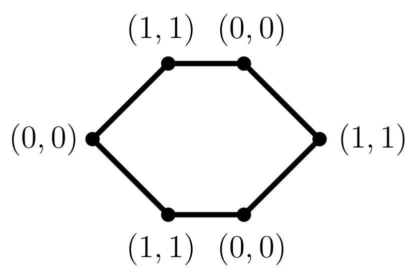


Figure 5.1.1: $(2,3)$-Orthogonal Colouring of $C_6$

Displayed next to each vertex are the colours assigned in both the first and the second colouring. The pairs of colours that are assigned to each vertex are called *colour pairs*. Similarly, for a general $(k,t)$-orthogonal colouring, a $k$-tuple of the colours assigned by each of the $k$ colourings are displayed next to each vertex. The $k$-tuples of colours that are assigned to each vertex are called *colour $k$-tuples*.

Therefore, a $(k,t)$-orthogonal colouring can be equivalently stated as each colour pair can occur at most $t$ times for each pair of colouring. Notice that in a normal orthogonal colouring of $C_6$, three colours are required. Here, in the $(2,3)$-orthogonal colouring of Figure 5.1.1, only 2 colours were required. In fact, this is the best possible, since the chromatic number of $C_6$ is 2. This is a general case of the following theorem.

**Theorem 5.1.1.** *If $t \geq \alpha(G)$, then $O\chi_{(k,t)} = \chi(G)$.*

**Proof:** Recall that $\alpha(G)$ is the size of largest independent set in a graph. Suppose that $t \geq \alpha(G)$. Then, a colour pair can be used at most $\alpha(G)$ times. Consider a vertex colouring $f$ of $G$ that uses $\chi(G)$ colours. All colour classes in $f$ have size at most $\alpha(G)$. Therefore, for all $k$ and each vertex $v \in G$, let $c_k = (f(v), f(v))$. Since each colour class has at most $\alpha(G)$ vertices, each colour pair will occur at most $\alpha(G) \leq t$ times in each pair of colourings. Thus, $O\chi_{(k,t)} = \chi(G)$. □

Notice that for a graph $G$ with $n$ vertices, $O\chi_{(k,t)}(G) \geq \left\lceil \sqrt{\lceil \frac{n}{t} \rceil} \right\rceil$. Otherwise, there are less than $\lceil \frac{n}{t} \rceil$ total colour pairs, and then by the pigeonhole principle, some colour pair would be assigned $t+1$ times, contradicting the $(k,t)$-orthogonality condition. Also, $O\chi_{(k,t)}(G) \geq \chi(G)$, since each of the colourings are proper vertex colourings. Also, since each pair of colourings in a $k$-orthogonal colouring has that each colour pair occurs at most once, $O\chi_{(k,t)}(G) \leq O_k\chi(G)$. Thus, the following is obtained.

$$\max\left\{\chi(G), \left\lceil \sqrt{\left\lceil \frac{n}{t} \right\rceil} \right\rceil\right\} \leq O\chi_{(k,t)}(G) \leq O\chi_k(G)$$

If $O\chi_{(k,t)}(G) = \lceil \sqrt{\lceil \frac{n}{t} \rceil} \rceil$, then $G$ is said to have an *optimal $(k,t)$-orthogonal colouring*. Therefore, optimal $(k,t)$-orthogonal colourings of graphs that have a square number of vertices have applications to $[n,k,t]$-transversal designs, as discussed in the introduction. Thus, determining which graphs have optimal $(k,t)$-orthogonal colourings is of interest.

Optimal $(2,t)$-orthogonal colourings can be categorized with the following graph product. The *lexicographic product* $G \circ H$ of two graphs $G$ and $H$ has vertex set $V(G) \times V(H)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent whenever $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. A graph constructed by the lexicographic product of two graphs is referred to as a *lexicographic product graph*. For example, the lexicographic product graph $K_3 \circ \bar{K}_3$ is given in Figure 5.1.2.

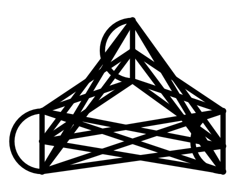

Figure 5.1.2: $K_3 \circ \bar{K}_3$

**Theorem 5.1.2.** *For a graph $G$, $O\chi_{(2,t)}(G) \leq n$ if and only if $G \subseteq (K_n \times K_n) \circ \bar{K}_t$.*

**Proof:** First, we show that $O\chi_{(2,t)}((K_n \times K_n) \circ \bar{K}_t) = n$. Therefore, for any subgraph $H$ of $(K_n \times K_n) \circ \bar{K}_t$, $O\chi_{(2,t)}(H) \leq n$. For $1 \leq i, j \leq n$ and $1 \leq k \leq t$, let $(i, j, k)$ denote the vertices of the graph $(K_n \times K_n) \circ \bar{K}_t$. Thus, $(i_1, j_1, k_1)$ is adjacent to $(i_2, j_2, k_2)$ if and only if $i_1 \neq i_2$ and $j_1 \neq j_2$. Assign the vertex $(i, j, k)$ the colour $i$ in the first colouring and the colour $j$ in the second colouring. For example, this colouring is applied to $(K_2 \times K_2) \circ \bar{K}_2$ in Figure 5.1.3.

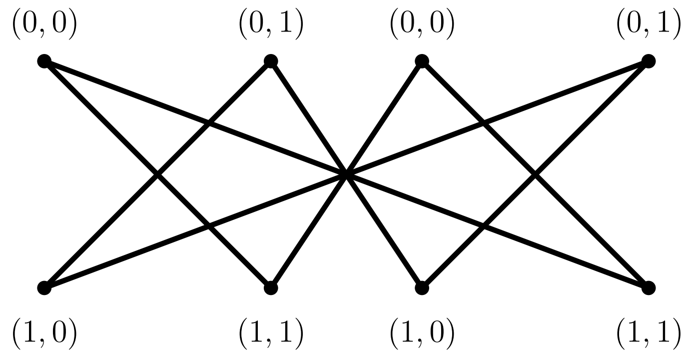


Figure 5.1.3: Orthogonal Colouring of $(K_2 \times K_2) \circ \bar{K}_2$

Note that this assignment of colours has no $(2, t)$-orthogonal conflicts. This is because on each copy of $K_n \times K_n$, each colour pair is used once. Thus, since there are $t$ copies, each colour pair is used $t$ times. Next, since two vertices $(i_1, j_1, k_1)$ and $(i_2, j_2, k_2)$ are adjacent if and only if $i_1 \neq i_2$ and $j_1 \neq j_2$ and they receive the colour pairs $(i_1, j_1)$ and $(i_2, j_2)$ respectively, there are no colour conflicts.

Next, the converse is proved, namely if $O\chi_{(2,t)}(G) \leq n$, then $G \subseteq (K_n \times K_n) \circ \bar{K}_t$. To show that $G \subseteq (K_n \times K_n) \circ \bar{K}_t$, an injective map that preserves edges is required. Let $F : G \to (K_n \times K_n) \circ \bar{K}_t$ by mapping the set of vertices of $G$ with colour pair $(i, j)$ to the vertices $(i, j, 1), \ldots, (i, j, s)$, where $s$ is the number of vertices in $G$ that received this colour pair. We now show that $F$ is injective and preserves edges.

Note that, since the colouring is $(2, t)$-orthogonal, $s \leq t$. Thus, $F$ is injective. Let $(g_1, g_2)$ be a $(2, t)$-orthogonal colouring of $G$ using $n$ colours. If $v_1 v_2 \in E(G)$, then $g_1(v_1) \neq g_1(v_2)$ and $g_2(v_1) \neq g_2(v_2)$ because $g_1$ and $g_2$ are proper. Therefore, it follows that the edge $(g_1(v_1), g_2(v_1), k_1)(g_1(v_2), g_2(v_2), k_2) \in E((K_n \times K_n) \circ \bar{K}_t)$. Thus, $F$ preserves edges and $G \subseteq (K_n \times K_n) \circ \bar{K}_t$. $\square$

Theorem 5.1.2 gives a way to reformulate the problem of determining if a graph has an optimal $(2,t)$-orthogonal colouring. It can now be restated as a subgraph problem involving the tensor graph $K_n \times K_n$ for original orthogonal colourings, and the lexicographic product for $(k,t)$-orthogonal colourings. In particular, it provides the following corollary.

**Corollary 5.1.3.** *A graph $G$ with $n$ vertices has an optimal $(2,t)$-orthogonal colouring if and only if $G \subseteq (K_N \times K_N) \circ \bar{K}_t$ where $N = \left\lceil \sqrt{\lceil \frac{n}{t} \rceil} \right\rceil$.*

Recall that $K_{n^2}[k]$ denotes the family of graphs obtained by removing $k$ edge disjoint $K_n$ covers from $K_{n^2}$. This family of graphs categorizes graphs having optimal $k$-orthogonal colourings. Thus, let $K_{n^2}[k,t]$ denote the family of graphs $H \circ \bar{K}_t$ where $H \in K_{n^2}[k]$. In particular, $K_{n^2}[2,t] = (K_n \times K_n) \circ \bar{K}_t$. Therefore, by applying the same argument used in Theorem 5.1.2, the following theorem is obtained.

**Theorem 5.1.4.** *$O\chi_{(k,t)}(G) \leq n$ if and only if $G \subseteq H$, where $H \in K_{n^2}[k,t]$.*

Next, notice that the lexicographic product of $G$ with $\bar{K}_t$ is $t$ copies of $G$, with edges between $G$ that resemble the edges of a tensor graph product. In the graph $G \times K_t$, there are $t$ edgeless copies of $G$, with the edges between these copies. Therefore, the tensor graph $G \times K_t$ is a subgraph of $G \circ \bar{K}_t$. By applying this observation, the following result for $(k,t)$-orthogonal colourings of tensor graphs is obtained.

**Corollary 5.1.5.** *If $O\chi_k(G) = n$ and $H$ has $t$ vertices, then $O\chi_{(k,t)}(G \times H) = n$.*

**Proof:** Suppose that $O\chi_k(G) = n$ and $H$ has $t$ vertices. Then, $G \subseteq R$, where $R \in K_{n^2}[k]$. Therefore, $G \times H$ is a subgraph of $R \times H$. Thus, by the observation above, $R \times H \subseteq R \circ \bar{K}_t$. Therefore, by transitivity, $G \times H \subseteq R \circ \bar{K}_t$. Thus, it follows by Theorem 5.1.4 that $O\chi_{(k,t)}(G \times H) = n$. □

Corollary 5.1.5 provides a way of constructing graphs with optimal $(k,t)$-orthogonal colouring by using graphs with an optimal $k$-orthogonal colouring. Therefore, the optimal $k$-orthogonal colourings of the graphs collected in this thesis can be applied to give a collection of graphs having $(k,t)$-optimal orthogonal colourings.

In the next section, one of the product components will have an optimal $(k,t)$-orthogonal colouring. Then, by using the Cartesian, tensor, and strong graph products, a new graph with an optimal $(k,t)$-orthogonal colouring is constructed. This provides an alternate method of constructing optimal $(k,t)$-orthogonal colourings.

### 5.1.1 Optimal $(k,t)$-Orthogonal Colourings Using Graph Products

To start, we consider the tensor graph product. The following result gives a method to construct tensor graphs having optimal $(2,t)$-orthogonal colourings. The same general argument as Theorem 3.1.2 is used, except now an optimal $(2,t)$-orthogonal colouring is considered.

**Theorem 5.1.6.** *If $G$ has $n^2t$ vertices, $H$ has $m^2$ vertices, and $O\chi_{(2,t)}(G) = n$, then $O\chi_{(2,t)}(G \times H) = nm$.*

**Proof:** Label $V(G) = \{v_k : 0 \leq k < n^2t\}$ and $V(H) = \{(u_i, u_j) : 0 \leq i, j < m\}$. Let $f = (f_1, f_2)$ be a proper $(2,t)$-orthogonal colouring of $G$ where $f_1$ and $f_2$ use the colours $\{0, 1, \ldots, n-1\}$. It is shown that $g = (g_1, g_2)$ is a $(2,t)$-orthogonal colouring of $G \times H$, where

$$g_1((v_k, (u_i, u_j))) = f_1(v_k) + in$$

and

$$g_2((v_k, (u_i, u_j))) = f_2(v_k) + jn.$$

First, it is shown that $g$ has no $(2,t)$-orthogonal conflicts. Let $v_{k_1}, v_{k_2} \in V(G)$ and let $(u_{i_1}, u_{j_1}), (u_{i_2}, u_{j_2}) \in V(H)$. If $g((v_{k_1}, (u_{i_1}, u_{j_1}))) = g((v_{k_2}, (u_{i_2}, u_{j_2})))$, then:

$$f_1(v_{k_1}) + i_1 n = f_1(v_{k_2}) + i_2 n \tag{5.1}$$

and

$$f_2(v_{k_1}) + j_1 n = f_2(v_{k_2}) + j_2 n. \tag{5.2}$$

Without loss of generality, suppose that $i_1 < i_2$. Then it follows that:

$$\begin{aligned} f_1(v_{k_1}) + i_1 n &< n + i_1 n \\ &\leq i_2 n \\ &\leq f_1(v_{k_2}) + i_2 n. \end{aligned}$$

Therefore, $f_1(v_{k_1}) + i_1 n < f_1(v_{k_2}) + i_2 n$, which contradicts Equation (5.1), thus $i_1 = i_2$. A similar argument shows that $j_1 = j_2$. Substituting $i_1 = i_2$ and $j_1 = j_2$ into Equations (5.1) and (5.2), gives $f_1(v_{k_1}) = f_1(v_{k_2})$ and $f_2(v_{k_1}) = f_2(v_{k_2})$. Since $f$ is a $(2,t)$-orthogonal colouring, there are $t$ options for $v_{k_2}$. Hence, there are $t$ options for $(v_{k_2}, (u_{i_2}, u_{j_2}))$. Therefore, each colour pair is only assigned $t$ times.

It remains to show that $g_1$ and $g_2$ are proper colourings of $G \times H$. Suppose that $v_{k_1}v_{k_2} \in E(G)$ and $(u_{i_1}, u_{j_1})(u_{i_2}, u_{j_2}) \in E(H)$. If $i_1 = i_2 = i$, then since $f_1$ is a proper colouring of $G$, $g_1((v_{k_1}, (u_{i_1}, u_{j_1}))) = f_1(v_{k_1}) + in \neq f_1(v_{k_2}) + in = g_1((v_{k_2}, (u_{i_2}, u_{j_2})))$. Thus, there are no colour conflicts between these vertices.

Now, without loss of generality, suppose that $i_1 < i_2$. Then, it follows that $g_1((v_{k_1}, (u_{i_1}, u_{j_1}))) = f_1(v_{k_1}) + i_1 n < n + i_1 n \leq i_2 n \leq f_1(v_{k_2}) + i_2 n = g_1((v_{k_1}, (u_{i_2}, u_{j_2})))$. Hence, $g_1((v_{k_1}, (u_{i_1}, u_{j_1}))) < g_1((v_{k_1}, (u_{i_2}, u_{j_2})))$. Thus, $g_1$ is a proper colouring. A similar argument shows $g_2$ is proper. Thus, $g$ is a proper $(2,t)$-orthogonal colouring. Since $G \times H$ has $tn^2m^2$ vertices and $g$ uses $nm$ colours, $O\chi_{(2,t)}(G \times H) = nm$. □

Theorem 5.1.6 provides a way for constructing optimal $(2,t)$-orthogonal colourings out of graphs that have an optimal $(2,t)$-orthogonal colouring. Corollary 5.1.3 gives that $(K_n \times K_n) \circ \bar{K}_t$ is the maximum graph with $n$ as its $(2,t)$-orthogonal chromatic number. By combining these two results, an upper bound on the $(2,t)$-orthogonal chromatic number of certain tensor graphs is obtained.

**Corollary 5.1.7.** *If $O\chi_{(2,t)}(G) = n$ and $O\chi_{(2,t^2)}(H) = m$, then $O\chi_{(2,t)}(G \times H) \leq tnm$.*

**Proof:** Since $O\chi_{(2,t)}(G) = n$ and $O\chi_{(2,t^2)}(H) = m$, it follows that $G \subseteq (K_n \times K_n) \circ \bar{K}_t$ and $H \subseteq (K_m \times K_m) \circ \bar{K}_{t^2}$ by Corollary 5.1.3. Therefore,

$$G \times H \subseteq ((K_n \times K_n) \circ \bar{K}_t) \times ((K_m \times K_m) \circ \bar{K}_{t^2})$$

Now, notice that $|V((K_n \times K_n) \circ \bar{K}_t)| = tn^2$, $|V((K_n \times K_m) \circ \bar{K}_{t^2})| = (tm)^2$, and $O\chi_{(2,t)}((K_n \times K_n) \circ \bar{K}_t) = n$. Therefore, by Theorem 5.1.6, it follows that

$$O\chi(((K_n \times K_n) \circ \bar{K}_t) \times ((K_m \times K_m) \circ \bar{K}_{t^2})) = tnm$$

Thus, $((K_n \times K_n) \circ \bar{K}_t) \times ((K_m \times K_m) \circ \bar{K}_{t^2}) \subseteq (K_{nmt} \times K_{nmt}) \circ \bar{K}_t$ by Theorem 5.1.2. Then, $G \times H \subseteq (K_{nmt} \times K_{nmt}) \circ \bar{K}_t$. Therefore, by Theorem 5.1.2, it follows that $O\chi_{(2,t)}(G \times H) \leq nmt$. □

Corollary 5.1.7 gives an upper bound on the $(2,t)$-orthogonal chromatic number of $G \times H$ in the case where the $(2,t)$-orthogonal chromatic number of $G$ and the $(2,t^2)$-orthogonal chromatic number of $H$ is known. It is interesting to see if this could be reduced to just knowing the $(2,t)$-orthogonal chromatic numbers of $G$ and $H$.

Next, the Cartesian graph product is considered. The following result gives a method to construct Cartesian graphs having optimal $(2,t)$-orthogonal colourings. The same argument as Theorem 3.2.1 is used, except now an optimal $(2,t)$-orthogonal colouring is applied.

**Theorem 5.1.8.** *If $G$ has $n^2t$ vertices, $H$ has $m^2$ vertices, and $O\chi_{(2,t)}(G) = n \geq m$, then $O\chi_{(2,t)}(G \square H) = nm$.*

**Proof:** Label $V(G) = \{v_k : 0 \leq k < tn^2\}$ and $V(H) = \{(u_i, u_j) : 0 \leq i, j < m\}$. Let $f = (f_1, f_2)$ be a proper $(2,t)$-orthogonal colouring of $G$ using the set of colours $\{0, 1, \ldots, n-1\}$. It is shown that $g = (g_1, g_2)$ is a $(2,t)$-orthogonal colouring of $G \square H$

$$g_1((v_k, (u_i, u_j))) = (f_1(v_k) + j)(\text{mod } n) + in$$
$$g_2((v_k, (u_i, u_j))) = (f_2(v_k) + i)(\text{mod } n) + jn.$$

Firstly, it is shown that $g$ has no $(2,t)$-orthogonal conflicts. Let $v_{k_1}, v_{k_2} \in V(G)$ and let $(u_{i_1}, u_{j_1}), (u_{i_2}, u_{j_2}) \in V(H)$. If $g((v_{k_1}, (u_{i_1}, u_{j_1}))) = g((v_{k_2}, (u_{i_2}, u_{j_2})))$, then:

$$(f_1(v_{k_1}) + j_1)(\text{mod } n) + i_1 n = (f_1(v_{k_2}) + j_2)(\text{mod } n) + i_2 n. \tag{5.3}$$
$$(f_2(v_{k_1}) + i_1)(\text{mod } n) + j_1 n = (f_2(v_{k_2}) + i_2)(\text{mod } n) + j_2 n. \tag{5.4}$$

Without loss of generality, suppose that $i_1 < i_2$. Then:

$$\begin{aligned}(f_1(v_{k_1}) + j_1)(\text{mod } n) + i_1 n &< n + i_1 n \\ &= n(1 + i_1) \\ &\leq i_2 n \\ &\leq (f_1(v_{k_2}) + j_2)(\text{mod } n) + i_2 n.\end{aligned}$$

Therefore, by transitivity, $(f_1(v_{k_1}) + j_1)(\text{mod } n) + i_1 n < (f_1(v_{k_2}) + j_2)(\text{mod } n) + i_2 n$, which contradicts Equation (5.3), thus $i_1 = i_2$. A similar argument shows that $j_1 = j_2$. Substituting $i_1 = i_2$ and $j_1 = j_2$ into Equations (5.3) and (5.4), gives $f_1(v_{k_1}) = f_1(v_{k_2})$ and $f_2(v_{k_1}) = f_2(v_{k_2})$. Since $f$ is a $(2,t)$-orthogonal colouring, there are $t$ options for $v_{k_2}$. Hence, there are $t$ options for $(v_{k_2}, (u_{i_2}, u_{j_2}))$. Therefore, each colour pair is only assigned $t$ times. Thus, there are no $(2,t)$-orthogonal conflicts.

It remains to show that $g_1$ and $g_2$ are proper colourings of $G \square H$. First, consider adjacencies of the form $(v_{k_1}, (u_i, u_j)) \sim (v_{k_2}, (u_i, u_j))$. Since $v_{k_1} \sim v_{k_2}$ in $G$ and $f_1$ is a proper colouring of $G$, $f_1(v_{k_1}) \neq f_1(v_{k_2})$. Thus:

$$\begin{aligned} g_1((v_{k_1},(u_i,u_j))) &= (f_1(v_{k_1})+j)(\text{mod } n)+in \\ &\neq (f_1(v_{k_2})+j)(\text{mod } n)+in \\ &= g_1((v_{k_2},(u_i,u_j)). \end{aligned}$$

Next, consider adjacencies of the form $(v_k,(u_{i_1},u_{j_1})) \sim (v_k,(u_{i_2},u_{j_2}))$. Suppose that $i_1 = i_2$ and $j_1 \neq j_2$. Then, because $n \geq m$, it follows that:

$$\begin{aligned} g_1((v_k,(u_{i_1},u_{j_1}))) &= (f_1(v_k)+j_1)(\text{mod } n)+i_1 n \\ &\neq (f_1(v_k)+j_2)(\text{mod } n)+i_1 n \\ &= (f_1(v_k)+j_2)(\text{mod } n)+i_2 n \\ &= g_1((v_k,(u_{i_2},u_{j_2}))). \end{aligned}$$

If $i_1 \neq i_2$, then the argument used to prove the orthogonality of $g$ shows that $g_1((v_k,(u_{i_1},u_{j_1}))) \neq g_1((v_k,(u_{i_2},u_{j_2})))$. The same argument can be applied to show that $g_2$ has no colour conflicts. Therefore, $g$ is a $(2,t)$-orthogonal colouring of $G\Box H$. Since $G\Box H$ has $tn^2m^2$ vertices and $g$ uses $nm$ colours, $O\chi_{(2,t)}(G\Box H) = nm$. □

Theorem 3.2.1 provides a way to construct optimal $(2,t)$-orthogonal colourings out of graphs that have an optimal $(2,t)$-orthogonal colouring. This results in the following corollary.

**Corollary 5.1.9.** *If $O\chi_{(2,t)}(G) = n$, $O\chi_{(2,t^2)}(H) = m$, and $n \geq m$, then it follows that $O\chi_{(2,t)}(G\Box H) \leq tnm$.*

**Proof:** Since $O\chi_{(2,t)}(G) = n$ and $O\chi_{(2,t^2)}(H) = m$, it follows that $G \subseteq (K_n \times K_n) \circ \bar{K}_t$ and $H \subseteq (K_m \times K_m) \circ \bar{K}_{t^2}$ by Corollary 5.1.3. Therefore,

$$G\Box H \subseteq ((K_n \times K_n) \circ \bar{K}_t)\Box((K_m \times K_m) \circ \bar{K}_{t^2})$$

Now, notice that $|V((K_n \times K_n) \circ \bar{K}_t)| = tn^2$, $|V((K_n \times K_m) \circ \bar{K}_{t^2})| = (tm)^2$, and $O\chi_{(2,t)}((K_n \times K_n) \circ \bar{K}_t) = n \geq m$. Therefore, by Theorem 5.1.8, it follows that

$$O\chi(((K_n \times K_n) \circ \bar{K}_t)\Box((K_m \times K_m) \circ \bar{K}_{t^2})) = tnm$$

Thus, $((K_n \times K_n) \circ \bar{K}_t)\Box((K_m \times K_m) \circ \bar{K}_{t^2}) \subseteq (K_{nmt} \times K_{nmt}) \circ \bar{K}_t$ by Theorem 5.1.2. Then, $G\Box H \subseteq (K_{nmt} \times K_{nmt}) \circ \bar{K}_t$, and by Theorem 5.1.2, $O\chi_{(2,t)}(G\Box H) \leq nmt$. □

Next, the strong graph product is considered. The following theorem gives a method to construct optimal $(k, tr)$-optimal orthogonal colourings of strong graphs when one component has an optimal $(k, t)$-orthogonal colouring and the other has an optimal $(k, r)$-orthogonal colouring.

**Theorem 5.1.10.** *If $G$ has $n^2t$ vertices with $O\chi_{(k,t)}(G) = n$ and $H$ has $m^2r$ vertices with $O\chi_{(k,r)}(H) = m$, then $O\chi_{(k,tr)}(G \boxtimes H) = nm$.*

**Proof:** For $0 \leq s < k$ and $0 \leq i < n$, let $G_{s,i}$ be the $i$-th colour class in the $s$-th colouring of $G$. Then, for $0 \leq s < k$ and $0 \leq j < m$, let $H_{s,j}$ be the $j$-th colour class in the $s$-th colouring of $H$. Next, let $I_{s,i,j} = \{(u, v) \mid u \in G_{s,i}, v \in H_{s,j}\}$. It will be shown that $C_s = \{I_{s,i,j} \mid 0 \leq i < n, 0 \leq j < m\}$ is a collection of disjoint spanning independent sets. That is, $C_s$ is a proper colouring of $G \boxtimes H$ using $nm$ colours.

First, it is shown that each $I_{s,i,j}$ is an independent set. Let $(u_1, v_1), (u_2, v_2) \in I_{s,i,j}$. Then, $u_1, u_2 \in G_{s,i}$ and $v_1, v_2 \in H_{s,j}$. However, $G_{s,i}$ and $H_{s,j}$ are independent sets. Thus, $u_1u_2 \notin E(G)$ and $v_1v_2 \notin E(H)$. Therefore, $(u_1, v_1)(u_2, v_2) \notin E(G \times H)$. Now, let $(u, v) \in V(G \times H)$. Since $\{G_{s,i} \mid 0 \leq i < n\}$ is a spanning set of $G$, $u \in G_{s,i}$ for some $i$. Similarly, $v \in H_{s,j}$ for some $j$. Therefore, $(u, v) \in I_{s,i,j}$.

Now, suppose that $(u, v) \in I_{s,i_1,j_1}$ and $(u, v) \in I_{s,i_1,j_2}$. If $i_1 \neq i_2$, then $u \in G_{s,i_1}$ and $u \in G_{s,i_2}$. However, this contradicts that $\{G_{s,i} \mid 0 \leq i < n\}$ is a colouring of $G$. Similarly, if $j_1 \neq j_2$, then $v \in H_{s,j_1}$ and $v \in H_{s,j_2}$. However, this contradicts that $\{H_{s,j} \mid 0 \leq j < m\}$ is a colouring of $H$. Therefore, there is a unique set $I_{s,i,j}$ that contains $(u, v)$. Thus, $C_r$ is a proper colouring of $G \boxtimes H$ using $nm$ colours.

It remains to show that the colourings are mutually $tr$-orthogonal. Consider $I_{s_1,i_1,j_1}$ and $I_{s_2,i_2,j_2}$, where $s_1 \neq s_2$. If $(u, v) \in I_{s_1,i_1,j_1}$ and $(u, v) \in I_{s_2,i_2,j_2}$, then $u \in G_{s_1,i_1}$ and $u \in G_{s_2,i_2}$. However, $G_{s_1,i_1} \cap G_{s_2,i_2} = t$, so let $u$ be one of these $t$ vertices. Similarly, $v \in H_{s_1,j_1}$ and $v \in H_{s_2,j_2}$. However, $H_{s_1,j_1} \cap H_{s_2,j_2} = r$, so let $v$ be one of these $r$ vertices. Thus, there are $tr$ vertices $(u, v)$ in both $I_{s_1,i_1,j_1}$ and $I_{s_2,i_2,j_2}$.

Hence, each of the $C_s$ are mutually $tr$-orthogonal. Thus, this provides an $(k, tr)$-orthogonal colouring of $G \boxtimes H$ using $nm$ colours. Since $G \boxtimes H$ has $trn^2m^2$ vertices, this gives that $O\chi_{(k,tr)}(G \boxtimes H) = nm$. $\square$

Note that the strong product of two graphs has both the edges of the tensor graph product and the Cartesian graph product. Therefore, a $(k, t)$-orthogonal colouring of a strong product graph also yields $(k, t)$-orthogonal colourings for these graphs.

**Corollary 5.1.11.** *If $G$ has $n^2$ vertices with $O\chi_{(k,t)}(G) = n$ and $H$ has $m^2$ vertices with $O\chi_{(k,r)}(H) = m$, then $O\chi_{(k,tr)}(G \Box H) = nm$ and $O\chi_{(k,tr)}(G \times H) = nm$.*

**Proof:** Let $I_{s,i,j}$ be the same set as in Theorem 3.3.1. Then, note that $I_{s,i,j}$ is an independent set in $G \Box H$ and $G \times H$. Therefore, this results follows by applying the proof of Theorem 5.1.10. $\square$

Theorem 3.3.1 gives a method to construct optimal $(k, t^2)$-orthogonal colourings when both components have an optimal $(k, t)$-orthogonal colouring. This is now used to find an upper bound on the $(k, t^2)$-orthogonal chromatic number of general strong product graphs. This is done in the following corollary.

**Corollary 5.1.12.** *If $O\chi_{(k,t)}(G) = n$, $O\chi_{(k,r)}(H) = m$, then $O\chi_{(k,tr)}(G \boxtimes H) \leq nm$.*

**Proof:** Since $O\chi_{(k,t)}(G) = n$ and $O\chi_{(k,r)}(H) = m$, it follows that $G \subseteq R$ and $H \subseteq S$ where $R \in K_{n^2}[k, t]$ and $S \in K_{m^2}[k, r]$ by Theorem 5.1.4. Therefore,

$$G \boxtimes H \subseteq R \boxtimes S$$

Now, notice that $|V(R)| = tn^2$, $|V(S)| = rm^2$, and $O\chi_{(k,t)}(R) = n$ and $O\chi_{(k,t)}(S) = m$. Therefore, by Theorem 5.1.10, it follows that

$$O\chi_{(k,tr)}(R \boxtimes S) = nm$$

Thus, $R \boxtimes S \subseteq T$, where $T \in K_{n^2m^2}[k, tr]$ by Theorem 5.1.4. Then, $G \boxtimes H \subseteq T$, and by Theorem 5.1.4, $O\chi_{(k,tr)}(G \boxtimes H) \leq nm$. $\square$

Again, since the strong product of two graphs has both the edges of the tensor graph product and the Cartesian graph product, the following Corollary is obtained.

**Corollary 5.1.13.** *If $O\chi_{(k,t)}(G) = n$, $O\chi_{(k,r)}(H) = m$, then $O\chi_{(k,tr)}(G \times H) \leq nm$ and $O\chi_{(k,tr)}(G \Box H) \leq nm$.*

**Proof:** By Corollary 5.1.11, a $(k, tr)$-orthogonal colouring of $G \boxtimes H$ exists. Therefore, restrict the $(k, tr)$-orthogonal colouring to $G \Box H$ and $G \times H$. $\square$

This concludes our exploration of graph products on $(k, t)$-orthogonal colourings. We showed that graph products can be used to take existing optimal $(2, t)$-orthogonal colourings to construct new optimal $(2, t)$-orthogonal colourings. Therefore, it is beneficial to have a collection of graphs with optimal $(2, t)$-orthogonal colourings. This is done in the next section.

### 5.1.2 Graphs With Optimal $(2,t)$-Orthogonal Colourings

In this section, $(2,t)$-orthogonal colourings of graphs are constructed. This is done by taking the orthogonal colouring considered in this thesis, and extending them to a $(2,t)$-orthogonal colouring. Firstly, for use later, $(k,t)$-orthogonal colourings of independent sets are considered. Recall that that $L(k)$ is an integer such that if $r \geq L(k)$, then there exists a collection of $k$-orthogonal Latin squares of size $r$.

**Lemma 5.1.14.** *For $1 \leq t \leq n$, $O\chi_{(k,t)}(\bar{K}_n) = \max\{\lceil \sqrt{\lceil \frac{n}{t} \rceil}\, \rceil, L(k-2)\}$. In particular, $O\chi_{(2,t)}(\bar{K}_n) = O\chi_{(3,t)}(\bar{K}_n) = \lceil \sqrt{\lceil \frac{n}{t} \rceil}\, \rceil$.*

**Proof:** First, partition $\bar{K}_n$ into $t$ disjoint independent sets, denoted $I_1, I_2, \ldots, I_t$, having size at most $\lceil \frac{n}{t} \rceil$. By Theorem 1.4.3, each set can be orthogonally coloured with $\max\{\lceil \sqrt{\lceil \frac{n}{t} \rceil}\, \rceil, L(k-2)\}$ colours. Therefore, by using the same set of colours on each independent set, any given colour pair occurs at most $t$ times. Thus, this gives a $(k,t)$-orthogonal colouring of $\bar{K}_n$ using $\max\{\lceil \sqrt{\lceil \frac{n}{t} \rceil}\, \rceil, L(k-2)\}$ colours. □

Lemma 5.1.14 is applied to get an upper bound on the $(2,t)$-orthogonal chromatic number of complete $r$-partite graphs. This argument is similar to the one used to prove Theorem 1.4.4.

**Theorem 5.1.15.** *Let $G$ be a complete $r$-partite graph with vertex classes of sizes $s_1, s_2, \ldots, s_r$. Then*

$$O\chi_{(2,t)}(G) = \sum_{i=1}^{r} \left\lceil \sqrt{\left\lceil \frac{s_i}{t} \right\rceil} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor$$

*where $m$ is the number of vertex classes having a non-integer square number of vertices that also satisfy the relation $\left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil \left\lfloor \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rfloor \geq \frac{s_i}{t}$.*

**Proof:** First, it is shown that there exists a $(2,t)$-orthogonal colouring of $G$ with the required number of colours. For $1 \leq i \leq r$, let $S_i$ denote the vertex class of size $s_i$. Without loss of generality, suppose that the first $m$ classes have the property that $s_i$ is not an integer square and that $\left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil \left\lfloor \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rfloor \geq \frac{s_i}{t}$.

Each vertex class is an independent set. Therefore, by Lemma 5.1.14, $S_i$ can be $(2,t)$-orthogonally coloured with $\left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil$ colours. Summing over the vertex classes, this results in a $(2,t)$-orthogonal colouring using $\sum_{i=1}^{r} \left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil$ colours, which is $\lfloor \frac{m}{2} \rfloor$ more colours than required. The following method gives a way to remove these unnecessary colours.

Consider the $\lfloor \frac{m}{2} \rfloor$ pairs of vertex classes $\{S_1, S_2\}, \{S_3, S_4\}, \ldots, \{S_{2\lfloor \frac{m}{2} \rfloor - 1}, S_{2\lfloor \frac{m}{2} \rfloor}\}$. Let $C_j$ be a set of $z$ distinct colours where $z = \left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil + \left\lceil \sqrt{\lceil \frac{s_{i+1}}{t} \rceil} \right\rceil - 1$. Consider all the ordered pairs of the form $(w_p, w_q)$ where $1 \leq p \leq \left\lfloor \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rfloor$ and $1 \leq q \leq \left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil$. There are at least $\left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil \left\lfloor \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rfloor \geq \frac{s_i}{t}$ pairs by assumption. These colour pairs are used $t$ times to colour the first independent set in the pairs.

Now for the other independent set, consider all the ordered pairs of the form $(w_p, w_q)$ where $\left\lfloor \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rfloor + 1 = \left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil \leq p \leq z$ and $\left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil + 1 \leq q \leq z$. There are at least $\left\lceil \sqrt{\lceil \frac{s_{i+1}}{t} \rceil} \right\rceil \left\lfloor \sqrt{\lceil \frac{s_{i+1}}{t} \rceil} \right\rfloor \geq \frac{s_{i+1}}{t}$ pairs by assumption. These colour pairs are used $t$ times to colour the second independent set in the pairs.

Note that the colour pairs assigned to the first independent set in a pair are disjoint from the colour pairs assigned to the second independent set in a pair. Thus, there are no colour conflicts. Then, since each colour pair is used at most $t$ times, there are no $t$-orthogonal conflicts.

Therefore, each of these $\lfloor \frac{m}{2} \rfloor$ pairs of independent sets can be $t$-orthogonally coloured with 1 less colour than by $t$-orthogonally colouring them separately. Thus, a $(2,t)$-orthogonal colouring using $\sum_{i=1}^{r} \left\lceil \sqrt{\lceil \frac{s_i}{t} \rceil} \right\rceil - \lfloor \frac{m}{2} \rfloor$ colours is obtained. □

Lastly, optimal $(k,t)$-orthogonal colourings of cycle graphs are constructed. In Chapter 2, it was shown that every $C_n$ where $n > 4$ has an optimal orthogonal colouring. In particular, these orthogonal colourings all had the property that $v_0$ received the colour pair $(0,0)$. Therefore, by using this fact, the following result can be proved.

**Theorem 5.1.16.** *If $1 \leq t \leq \frac{n}{2}$ and $\lfloor \frac{n}{t} \rfloor > 4$, then $O\chi_{(2,t)}(C_n) = \left\lceil \sqrt{\lceil \frac{n}{t} \rceil} \right\rceil$.*

**Proof:** First, partition $C_n$ into $t$ path graphs $P_1, \ldots, P_t$ where each path graph has at most $\lceil \frac{n}{t} \rceil$ vertices and at least $\lfloor \frac{n}{t} \rfloor$ vertices. Since $\lfloor \frac{n}{t} \rfloor > 4$, there exists an orthogonal colouring of $C_{\lfloor \frac{n}{t} \rfloor}$ using $\left\lceil \sqrt{\lfloor \frac{n}{t} \rfloor} \right\rceil$ colours such that the first vertex receives the colour pair $(0,0)$ and the last vertex does not conflict with this vertex.

Therefore, restrict these orthogonal colourings of the cycles to the path graphs. Then, since each starting vertex receives the colour pair $(0,0)$ and the last vertex does not conflict with this colour pair, there will be no colour conflicts on the graph $C_n$. Thus, since there are $t$ path graphs, each colour pair is used at most $t$ times. □

For example, an optimal $(2,2)$-orthogonal colouring of $C_{10}$ is given in Figure 5.1.4.

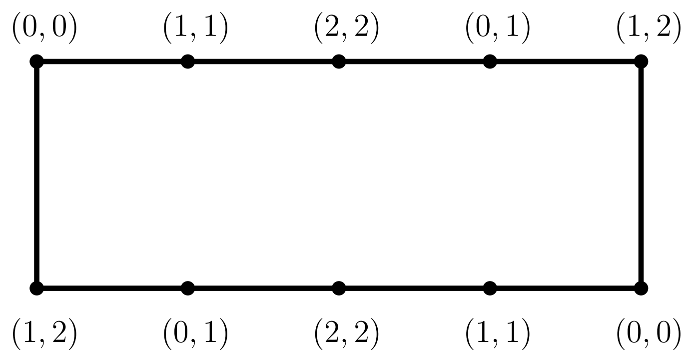


Figure 5.1.4: Optimal $(2, 2)$-Orthogonal Colouring of $C_{10}$

Theorem 5.1.16 used the optimal orthogonal colourings of cycles to construct an optimal $(2, t)$-orthogonal colourings of cycles. Similarly, the following theorem uses the optimal $k$-orthogonal colourings of cycles to construct optimal $(k, t)$-orthogonal colourings of cycles. The same general argument is used.

**Theorem 5.1.17.** *If* $\left\lceil \sqrt{\left\lceil \frac{n}{t} \right\rceil} \right\rceil = p$ *and* $\left\lceil \sqrt{\left\lfloor \frac{n}{t} \right\rfloor} \right\rceil = p$ *is a prime number, then* $O\chi_{(p-2,t)}(C_n) = p$.

**Proof:** First, partition $C_n$ into $t$ path graphs $P_1, \ldots, P_t$ where each path graph has at most $\left\lceil \frac{n}{t} \right\rceil$ vertices and at least $\left\lfloor \frac{n}{t} \right\rfloor$ vertices. Since $\left\lceil \sqrt{\left\lceil \frac{n}{t} \right\rceil} \right\rceil = p$ and $\left\lceil \sqrt{\left\lfloor \frac{n}{t} \right\rfloor} \right\rceil = p$, there exists a $(p-2)$-orthogonal colouring of $C_{\left\lfloor \frac{n}{t} \right\rfloor}$ and $C_{\left\lceil \frac{n}{t} \right\rceil}$ using $p$ colours such that the first vertex receives the colour pair $(0, 0)$ and the last vertex does not conflict with this vertex.

Therefore, restrict these orthogonal colourings of the cycles to the $t$ path graphs. Then, since each starting vertex receives the colour pair $(0, 0)$ and the last vertex does not conflict with this colour pair, there will be no colour conflicts on the graph $C_n$. Thus, since there are $t$ path graphs, each colour pair is used at most $t$ times. $\square$

# Chapter 6

# Conclusions and Future Work

This thesis explored many different methods for constructing orthogonal colourings. Optimal orthogonal colourings of cycle graphs, Paley graphs, circulant graphs, tree graphs, and $[n, k, r]$-partite graphs were found. Then, optimal orthogonal colourings were created out of existing ones by using graph products. Also, optimal orthogonal colourings of randomly generated graphs were constructed with high probability. Lastly, a brief study of $(k, t)$-orthogonal colourings was given.

In this section, some open of the problems not studied and some open conjectures are discussed. Recall that in the second chapter, orthogonal colouring of Cayley graphs were considered. It was shown that if the group was $\mathbb{Z}_{p^2}$, and if the generating set had size $|S| < \frac{p-1}{2}$, then $\Gamma(\mathbb{Z}_{p^2}, S)$ has an optimal orthogonal colouring. The following is conjectured for $k$-optimal orthogonal colourings of Cayley graphs.

**Conjecture 6.0.1.** *If $|S| < \frac{p-1}{2k}$, then the Cayley graph $\Gamma(\mathbb{Z}_{p^2}, S)$ has an optimal $k$-orthogonal colouring.*

The case where $k = 2$ was proved by taking a collection of orthogonal assignments, and showing that at least one of them is proper. Hence, one possible method of studying Conjecture 6.0.1 is to similarly find a collection of $k$-orthogonal assignments and show that at least one of them is proper. Another area of research is studying Cayley graphs of more general groups.

Now, recall that in the third chapter, orthogonal colourings of product graphs were studied. In particular, it was proved for the tensor and Cartesian graph product, that if one component had an optimal orthogonal colouring and the other component had some size restriction, then the resulting product graph has an optimal orthogonal colouring. This leads to the following question.

**Question 6.0.2.** *If $G$ has an optimal $k$-orthogonal colouring, what size restrictions on $H$ result in $G \times H$ and $G \square H$ having an optimal $k$-orthogonal colouring.*

The orthogonal colourings constructed for $G \times H$ and $G \Box H$ both incorporated an $in$ and a $jn$ term in their colourings. This term was present due to the fact that both $G$ and $H$ had a square number of vertices. Hence, if $H$ had $n^k$ vertices, then an optimal $k$-orthogonal colouring of $G \times H$ and $G \Box H$ may exist. In this situation, there could be $k$ of the $in$ terms.

Now, recall in the fourth chapter, that orthogonal colourings of random graphs were explored. In particular, it was proved that with high probability, an intermediate random geometric graph has an orthogonal colouring with $(1 + o(1))\lceil \sqrt{n} \rceil$ colours. This was also proved for dense random geometric graphs having threshold function $r(n) < n^{-\frac{1}{4}}$. Therefore, this leads us to conjecture the following.

**Conjecture 6.0.3.** *If $\frac{nr^2}{\ln n} \to c$, where $0 < c < \infty$, and $G \sim RG(n, r)$, then with high probability, $O\chi(G) = \lceil \sqrt{n} \rceil$.*

Conjecture 6.0.3 says that the asymptotic $o(1)$ term can be removed. Recall that in the original orthogonal colouring, under the appropriate choices of parameters, $G \sim RG(n, r)$ was mapped into a subgraph of the clique grid. Instead, if it can be shown that $G$ is a subgraph of $K_n \times K_n$ with high probability, then the desired result will be obtained.

In this thesis, the unit square, the Euclidean metric, and the uniform probability distribution were considered. Another possible area of research is random geometric graphs where these identifiers are generalized. For instance, suppose that instead of the unit square, a $k$-dimensional unit hypercube was used to define the random geometric graph model. This leads to the following question.

**Question 6.0.4.** *Suppose that the $k$-dimensional unit hypercube was used to define the random geometric graph model. How can orthogonal colourings of $G \sim RG(n, r)$ be constructed in this case?*

Recall that for original dense random geometric graphs, the method used was showing that $G \sim RG(n, r)$ was a subgraph of the clique graph, $H(m, d, t) \boxtimes H(m, d, t)$. Hence, one possible method of constructing orthogonal colourings in the case where the $k$-dimensional unit hypercube is used is to take the $k$ fold product of $H(m, d, t)$ with itself, to get the $k$-dimensional analogue of the clique grid. Other generalizations of the random geometric graph could also be considered.

Lastly, a brief introduction to $(k,t)$-orthogonal colourings and some preliminary results were obtained. In particular, it was shown that the $(k,\alpha)$-orthogonal chromatic number is the chromatic number and that the $(2,1)$-orthogonal chromatic number is the orthogonal chromatic number. As $t$ increases from 1 to $\alpha(G)$, the $(2,t)$-orthogonal chromatic number decreases from $O\chi(G)$ to $\chi(G)$. This leads us to ask the following question.

**Question 6.0.5.** *Is there a positive function $f(t)$ such that $O\chi_{(2,t)}(G) \leq \frac{O\chi(G)}{f(t)}$?*

In the $(2,t)$-orthogonal colouring constructed in this thesis, the general approach was to take the existing orthogonal colouring and extend it to the larger graph. However, this was possible because copies of graph structure lay within the graphs studied. It may still be possible to take the existing orthogonal colouring and use it to construct a $(2,t)$-orthogonal colouring however. Additionally, other orthogonal colouring variants could be created and studied.

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