SOLVING THE DIRICHLET PROBLEM FOR VIBRATION OF PARALLELOGRAM-SHAPED MEMBRANE USING METHOD OF PARTIAL DOMAINS

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ABSTRACT

In this study, the Dirichlet boundary problem for vibration of a parallelogram-shaped membrane is solved. The simplicity and transparency of the proposed procedures allow one to clarify the specific features of some state-of-the-art approaches to solve similar problems of mathematical physics. For many types of domains, including a wide range of non-canonical ones, the use of the concept of a general solution of the boundary value problem makes it possible to construct a numerical-analytical solution to the problem. In this case, sets of partial solutions for the basic equations of mathematical physics are used. The main idea is to indicate effective ways to determine arbitrary coefficients and functions that are part of a general solution. The conventional approach for deriving numerical-analytical solutions is used based on the mean square deviation minimization and collocation methods.

KEYWORDS: *vibration; membrane; parallelogram; Dirichlet boundary value problem; mean square deviation minimization; collocation method.*

1. INTRODUCTION

Consider the problem of harmonic vibration of a parallelogram-shaped membrane, as shown in Fig. 1.

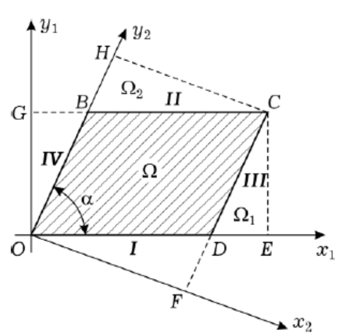


Figure 1. Parallelogram-shaped vibrating membrane

In the absence of a mechanical load within the area occupied by the membrane, the oscillatory process is reduced to the Dirichlet problem and the Helmholtz equation [1,2], which has the following form

$$\begin{cases} \nabla^2\phi + k^2\phi = 0, & (x,y) \in \Omega, \\ \phi(x,y) = \Phi(x,y), & (x,y) \in \Gamma. \end{cases} \qquad (1)$$

Here, ϕ is lateral displacements of the membrane, Ω is the area it occupies, $\Gamma = \bigcup_{i=I}^{IV} \Gamma_i$ is the boundary of this region, while Γ_I, Γ_{II}, Γ_{III}, and Γ_{IV} are the parallelogram sides OD, BC, DC, and OB, respectively. From a mechanical point of view, this problem describes the kinematic excitation of the membrane. To solve it, the method of partial domains can be successfully applied [3].

The vibration problem of a membrane with fixed edges, excited by a mechanical load with a density $F(x,y)$ distributed along its surface (dynamic excitation), can also be solved by the method used for solving the expression (1). This problem has the form

$$\begin{cases} \nabla^2\phi + k^2\phi = F(x,y), & (x,y) \in \Omega, \\ \phi(x,y) = 0, & (x,y) \in \Gamma. \end{cases} \qquad (2)$$

To obtain the solution to this problem, we use the mathematical system of fundamental solutions satisfying the following equation

$$\nabla^2\phi^* + k^2\phi^* = \delta(x,y) \qquad (3)$$

From a mechanical point of view, the fundamental solution is a function that satisfies the equation of oscillations of the membrane (i.e., the Helmholtz equation) under the action of a bulk oscillating force.

In the case under study, the fundamental solution requires no implementation of specific boundary conditions, and we can use the fundamental solution of the form

$$\varphi^*(x,y) = -\frac{1}{4} Y_0(\sqrt{x^2 + y^2}). \qquad (4)$$

In this case, the amplitude of displacements at an arbitrary point of the membrane caused by the action of a load with density $F(x,y)$ distributed along its surface Ω is calculated in the form of its convolution with the fundamental solution (4):

$$\hat{\varphi}(x,y) = -\frac{1}{4}\int_\Omega F(\xi,\eta) Y_0(\sqrt{(x-\xi)^2 + (y-\eta)^2})\, d\xi d\eta. \qquad (5)$$

We now assign a function with a reversed sign as a function in the boundary conditions of problem (1):

$$\Phi(x,y)=\frac{1}{4}\int_{\Omega}F(\xi,\eta)Y_0(\sqrt{(x-\xi)^2+(y-\eta)^2})d\xi d\eta\Big|_{\Gamma}. \quad (6)$$

Having solved the latter problem for the given boundary conditions and taking the sum of the solution obtained, which we denote by $\tilde{\varphi}$, with the forced (nonuniform) solution $\hat{\varphi}$ to (5), we get the desired solution to the problem of forced oscillations of the membrane with a fixed edge:

$$\varphi(x,y)=\tilde{\varphi}(x,y)+\hat{\varphi}(x,y) \quad (7)$$

2. The problem solution by the method of partial areas

The main objective of the mathematical modeling of physical oscillations is to obtain their quantitative characteristics, depending on the geometric parameters of their area of existence and boundary conditions. Typically, this goal is achieved by some iterative processes. This may be, for example, the process of increasing the number of members in the Fourier series, which is used to represent the desired function. For the effective implementation of such iterative processes, it is crucial to guarantee that the choice of characteristics of the process ensures its stability and reliable quantitative estimates.

In this respect, the method of partial domains, which leads to the construction of general solutions of boundary problems, has a substantial advantage over other computation techniques. The stability and convergence of computational procedures in its implementation are provided by the fundamental properties of the completeness of the systems of functions used to represent solutions. Consider two coordinate systems $(x_1,y_1)\equiv(x,y)$ and (x_2,y_2), which refer to the scheme shown in Fig. 1 and are linked by the following relations:

$$\begin{cases} x_1 = x_2\sin\alpha + y_2\cos\alpha, \\ y_1 = -x_2\cos\alpha + y_2\sin\alpha; \end{cases} \quad (8)$$

$$\begin{cases} x_2 = x_1\sin\alpha - y_1\cos\alpha, \\ y_2 = x_1\cos\alpha + y_1\sin\alpha. \end{cases} \quad (9)$$

Next, we introduce the following notations (Fig. 1): Ω1 and Ω2 are the areas bounded by OGCE and OHCF rectangles, respectively. Thus, $\Omega=\Omega_1 \cap \Omega_2$. Let a, b, a', b', h, and L be lengths of OF, OH, OD, OB, OG, and OE segments, respectively. It is obvious that $a=a'\sin\alpha$, $b=b'+DF=b'+a'\cos\alpha$, $h=b'\sin\alpha$, while the set of points of the boundary Γ can be described as follows:

Γ_I : $y_1 = 0$; $x_1 \in [0; a']$; (10)

Γ_{II} : $y_1 = h$; $x_1 \in [b'\cos\alpha; a' + b'\cos\alpha]$; (11)

Γ_{III} : $x_2 = a; y_2 \in [a'\cos\alpha; b' + a'\cos\alpha]$; (12)

Γ_{IV} : $x_2 = 0; y_2 \in [0; b']$. (13)

The boundary conditions for equation (1) can be reduced to the following form:

$$\varphi_{\Gamma_I}(x_1) = -\frac{\cosh \dfrac{\pi\left(x_1 - \dfrac{a'}{2}\right)}{a'}}{\cosh \dfrac{\pi}{2}} + 1, \tag{14}$$

$$\varphi_{\Gamma_{II}}(x_1) = -\frac{\cosh \dfrac{\pi\left(-\dfrac{a'}{2} - b'\cos(\alpha) + x_1\right)}{a'}}{\cosh \dfrac{\pi}{2}} + 1, \tag{15}$$

$$\varphi_{\Gamma_{III}}(y_2) = \cos \frac{2\pi\left(-a'\cos\alpha - \dfrac{b'}{2} + y_2\right)}{b'} + 1, \tag{16}$$

$$\varphi_{\Gamma_{IV}}(y_2) = \cos \frac{2\pi\left(y_2 - \dfrac{b'}{2}\right)}{b'} + 1, \tag{17}$$

According to the method of partial regions, the desired function $\varphi(x, y)$ can be represented by the sum of solutions to the Dirichlet problem $\varphi_1(x_1, y_1)$ and $\varphi_2(x_2, y_2)$ in the rectangular areas Ω_1 and Ω_2, respectively:

$$\varphi(x, y) = \varphi_1(x_1, y_1) + \varphi_2(x_2, y_2), \tag{18}$$

where

$$\begin{cases} x_1 = x, \\ y_1 = y; \end{cases} \qquad \begin{cases} x_2 = x\sin\alpha - y\cos\alpha, \\ y_2 = x\cos\alpha + y\sin\alpha. \end{cases} \tag{19}$$

In this case, we can make an arbitrary choice of boundary conditions for functions φ_1 and φ_2 within separate sections of the region boundaries Ω_1 and Ω_2. This method is also known as the Schwarz-Neumann method [4]. Next, functions φ_1 and φ_2 should have zero values within segments DE, GB, BH, and DF.

Using the following designation

$$\bar{\varphi}_1(x_2, y_2) = \varphi_1(x_1(x_2, y_2), y_1(x_2, y_2)), \quad (20)$$

$$\bar{\varphi}_2(x_1, y_1) = \varphi_2(x_2(x_1, y_1), y_2(x_1, y_1)) \quad (21)$$

we can formulate boundary conditions for the sum of functions (18)

$$\varphi_1(x_1, 0) + \bar{\varphi}_2(x_1, 0) = \begin{cases} \varphi_I(x_1), & 0 < x_1 < a', \\ \bar{\psi}_I(x_1), & a' < x_1 < L; \end{cases} \quad (22)$$

$$\varphi_1(x_1, h) + \bar{\varphi}_2(x_1, h) = \begin{cases} \varphi_{II}(x_1), & b'\cos\alpha < x_1 < a' + b'\cos\alpha, \\ \bar{\psi}_{II}(x_1), & 0 < x_1 < b'\cos\alpha; \end{cases} \quad (23)$$

$$\bar{\varphi}_1(a, y_2) + \varphi_2(a, y_2) = \begin{cases} \varphi_{III}(y_2), & a'\cos\alpha < y_2 < b' + a'\cos\alpha, \\ \bar{\psi}_{III}(y_2), & 0 < y_2 < a'\cos\alpha; \end{cases} \quad (24)$$

$$\bar{\varphi}_1(0, y_2) + \varphi_2(0, y_2) = \begin{cases} \varphi_{IV}(y_2), & 0 < y_2 < b', \\ \bar{\psi}_{IV}(y_2), & b' < y_2 < b. \end{cases} \quad (25)$$

Arbitrary functions can be used as $\bar{\psi}_I(x_1)$, $\bar{\psi}_{II}(x_1)$, $\bar{\psi}_{III}(y_2)$, $\bar{\psi}_{IV}(y_2)$. Without loss of universality, they can later be set to zero.
The type of regions Ω_1 and Ω_2 allows one to represent their solutions by infinite series:

$$\varphi_1(x_1, y_1) = \sum_{n=1}^{\infty} A_n(y_1) \sin\left(\frac{n\pi}{L} x_1\right), \quad (26)$$

$$\varphi_2(x_2, y_2) = \sum_{p=1}^{\infty} D_p(x_2) \sin\left(\frac{p\pi}{b} y_2\right) \quad (27)$$

Insofar as $\varphi_1(x_1, y_1)$ has to be the Helmholtz equation solution, this yields the following system of equations concerning functions $A_n(y_1)$:

$$\sum_{n=1}^{\infty}\left(\frac{d^2 A_n(y_1)}{dy_1^{\,2}} - \left(\frac{n\pi}{L}\right)^2 A_n(y_1) + k^2 A_n(y_1)\right)\sin\left(\frac{n\pi}{L}x_1\right) = 0 \quad . \qquad (28)$$

Since $\sin\left(\frac{n\pi}{L}x_1\right) \neq 0$, to determine $A_n(y_1)$ in (28), we get the following system of ordinary differential equations:

$$\frac{d^2 A_n(y_1)}{dy_1^{\,2}} - \left(\left(\frac{n\pi}{L}\right)^2 - k^2\right) A_n(y_1) = 0, \quad n = 1,2,... \qquad (29)$$

Using $\lambda_n = \frac{n\pi}{L}, \quad \mu_p = \frac{p\pi}{b},$ we can write the solution to system (29) as follows:

$$A_n(y_1) = A_n^{(1)}\begin{cases} e^{-\sqrt{\lambda_n^2 - k^2}\, y_1}, & \lambda_n^2 \geq k^2 \\ \cos\sqrt{k^2 - \lambda_n^2}\, y_1, & \lambda_n^2 < k^2 \end{cases} + A_n^{(2)}\begin{cases} e^{\sqrt{\lambda_n^2 - k^2}\,(y_1 - h)}, & \lambda_n^2 \geq k^2 \\ \cos\sqrt{k^2 - \lambda_n^2}\,(y_1 - h), & \lambda_n^2 < k^2 \end{cases} \qquad (30)$$

We designate

$$F_n^{(1)}(y_1) = \begin{cases} e^{-\sqrt{\lambda_n^2 - k^2}\, y_1}, & \lambda_n^2 \geq k^2, \\ \cos\sqrt{k^2 - \lambda_n^2}\, y_1, & \lambda_n^2 < k^2, \end{cases}$$

$$F_n^{(2)}(y_1) = \begin{cases} e^{\sqrt{\lambda_n^2 - k^2}\,(y_1 - h)}, & \lambda_n^2 \geq k^2, \\ \cos\sqrt{k^2 - \lambda_n^2}\,(y_1 - h), & \lambda_n^2 < k^2, \end{cases}$$

This allows one to reduce expansion (26) to

$$\varphi_1(x_1, y_1) = \sum_{n=1}^{\infty} A_n^{(1)} \sin(\lambda_n x_1) F_n^{(1)}(y_1) + \sum_{n=1}^{\infty} A_n^{(2)} \sin(\lambda_n x_1) F_n^{(2)}(y_1). \qquad (31)$$

Similarly, we derive $\varphi_2(x_2, y_2)$:

$$D_p(x_2) = D_p^{(1)}\begin{cases} e^{-\sqrt{\mu_p^2 - k^2}\, x_2}, & \mu_p^2 \geq k^2 \\ \cos\sqrt{k^2 - \mu_p^2}\, x_2, & \mu_p^2 < k^2 \end{cases} + D_p^{(2)}\begin{cases} e^{\sqrt{\mu_p^2 - k^2}\,(x_2 - a)}, & \mu_p^2 \geq k^2 \\ \cos\sqrt{k^2 - \mu_p^2}\,(x_2 - a), & \mu_p^2 < k^2 \end{cases} \qquad (32)$$

Next, we apply the following designations:

$$G_p^{(1)}(x_2)=\begin{cases} e^{-\sqrt{\mu_p^2-k^2}x_2}, & \mu_p^2\ge k^2,\\ \cos\sqrt{k^2-\mu_p^2}x_2, & \mu_p^2<k^2,\end{cases}$$

$$G_p^{(2)}(x_2)=\begin{cases} e^{\sqrt{\mu_p^2-k^2}(x_2-a)}, & \mu_p^2\ge k^2,\\ \cos\sqrt{k^2-\mu_p^2}(x_2-a), & \mu_p^2<k^2,\end{cases}$$

This allows one to reduce series (27) to the following form

$$\varphi_2(x_2,y_2)=\sum_{p=1}^{\infty}D_p^{(1)}\sin(\mu_p y_2)G_p^{(1)}(x_2)+\sum_{p=1}^{\infty}D_p^{(2)}\sin(\mu_p y_2)G_p^{(2)}(x_2). \tag{33}$$

Using (31) and (33), we can derive the general solution to (18)

$$\varphi(x,y)=\sum_{n=1}^{\infty}A_n^{(1)}\sin(\lambda_n x_1)F_n^{(1)}(y_1)+\sum_{n=1}^{\infty}A_n^{(2)}\sin(\lambda_n x_1)F_n^{(2)}(y_1)+$$
$$+\sum_{p=1}^{\infty}D_p^{(1)}\sin(\mu_p y_2)G_p^{(1)}(x_2)+\sum_{p=1}^{\infty}D_p^{(2)}\sin(\mu_p y_2)G_p^{(2)}(x_2), \tag{34}$$

which coordinates are derived via Eq.(19).

3. The mean square deviation minimization (MSDM) method

Definition of constants $A_n^{(1)}$, $A_n^{(2)}$, $D_p^{(1)}$ and $D_p^{(2)}$ in (34) is carried out by the mean square deviation minimization method (MSDM) [5-20]. As projection systems of functions , we choose the systems $\left\{\sin(\frac{\tilde{n}\pi}{L}x_1)\right\}$, $\tilde{n}=1,2,...$ for Γ_I and Γ_{II} boundaries and $\left\{\sin(\frac{\tilde{p}\pi}{b}y_2)\right\}$, $\tilde{p}=1,2,...$ for Γ_{III} and Γ_{IV} boundaries.

We define the boundary conditions for the boundaries of regions Ω_1 and Ω_2, which include the sides of the parallelogram. Taking into account the reduction of infinite series in the representation of the general solution (4.34) to the finite ones with the number of their members N and P, respectively, as well as the corresponding choice of the number of projective functions, we obtain a system of $2N+2P$ linear algebraic equations relative to $A_n^{(1)}$, $A_n^{(2)}$, $n=\overline{1,N}$ and $D_p^{(1)}$, $D_p^{(2)}$, $p=\overline{1,P}$:

$$\begin{aligned}
&\sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{n}}^{I} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{n}}^{I} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{n}}^{I} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{n}}^{I} = \varphi_{\tilde{n}}^{I},\\
&\sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{n}}^{II} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{n}}^{II} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{n}}^{II} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{n}}^{II} = \varphi_{\tilde{n}}^{II},\\
&\sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{p}}^{III} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{p}}^{III} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{p}}^{III} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{p}}^{III} = \varphi_{\tilde{p}}^{III},\\
&\sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{p}}^{IV} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{p}}^{IV} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{p}}^{IV} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{p}}^{IV} = \varphi_{\tilde{p}}^{IV},
\end{aligned} \tag{35}$$

Here, the following designations are used

$$a_{1n\tilde{n}}^{I} = \int_0^L \sin(\lambda_n x_1)\sin(\lambda_{\tilde{n}} x_1)dx_1 = \frac{1}{2}L\delta_{n,\tilde{n}},$$

$$a_{2n\tilde{n}}^{I} = \int_0^L \sin(\lambda_n x_1)\sin(\lambda_{\tilde{n}} x_1)\begin{cases} e^{-h\sqrt{\lambda_n^2-k^2}}, & \lambda_n^2 \ge k^2 \\ \cos(-h\sqrt{k^2-\lambda_n^2}), & \lambda_n^2 < k^2 \end{cases} dx_1 =$$

$$= \begin{cases} \frac{1}{2}L\delta_{n,\tilde{n}} e^{-h\sqrt{\lambda_n^2-k^2}}, & \lambda_n^2 \ge k^2, \\ \frac{1}{2}L\delta_{n,\tilde{n}}\cos(-h\sqrt{k^2-\lambda_n^2}), & \lambda_n^2 < k^2, \end{cases}$$

$$x_2 = x_1 \sin\alpha,$$

$$y_2 = x_1 \cos\alpha,$$

$$d_{1p\tilde{n}}^{I} = \begin{cases} \int_0^{a'} \sin(\lambda_{\tilde{n}} x_1)\sin(\mu_p x_1\cos\alpha) e^{-x_1\sin\alpha\sqrt{\mu_p^2-k^2}} dx_1, & \mu_p^2 \ge k^2, \\ \int_0^{a'} \sin(\lambda_{\tilde{n}} x_1)\sin(\mu_p x_1\cos\alpha)\cos(x_1\sin\alpha\sqrt{k^2-\mu_p^2})dx_1, & \mu_p^2 < k^2, \end{cases}$$

$$d_{2p\tilde{n}}^{I} = \begin{cases} \int_0^{a'} \sin(\lambda_{\tilde{n}} x_1)\sin(\mu_p x_1 \cos\alpha) e^{-(x_1\sin\alpha - a)\sqrt{\mu_p^2 - k^2}} dx_1, & \mu_p^2 \ge k^2, \\ \int_0^{a'} \sin(\lambda_{\tilde{n}} x_1)\sin(\mu_p x_1 \cos\alpha)\cos((x_1\sin\alpha - a)\sqrt{k^2-\mu_p^2}) dx_1, & \mu_p^2 < k^2, \end{cases}$$

$$\varphi_{\tilde{n}}^{I} = \int_0^{a'} \varphi_{\Gamma_I}(x_1)\sin(\lambda_{\tilde{n}} x_1) dx_1 + \int_{a'}^{L} \overline{\psi}_{\Gamma_I}(x_1)\sin(\lambda_{\tilde{n}} x_1) dx_1 =$$

$$= \int_0^{a'} \left(1 - \frac{\cosh \frac{\pi\left(x_1 - \frac{a'}{2}\right)}{a'}}{\cosh \frac{\pi}{2}} \right) \sin(\lambda_{\tilde{n}} x_1) dx_1 + 0, \tag{36}$$

$$a_{1n\tilde{n}}^{II} = \begin{cases} \frac{1}{2} L \delta_{n,\tilde{n}} e^{-h\sqrt{\lambda_n^2 - k^2}}, & \lambda_n^2 \ge k^2, \\ \frac{1}{2} L \delta_{n,\tilde{n}} \cos(h\sqrt{k^2 - \lambda_n^2}), & \lambda_n^2 < k^2, \end{cases}$$

$$a_{2n\tilde{n}}^{II} = \frac{1}{2} L \delta_{n,\tilde{n}},$$

$$d_{2p\tilde{n}}^{II} = \begin{cases} \int_{b'\cos\alpha}^{L} \sin(\lambda_{\tilde{n}} x_1)\sin(\mu_p(h\sin\alpha + x_1\cos\alpha)) e^{(-h\cos\alpha + x_1\sin\alpha - a)\sqrt{\mu_p^2 - k^2}} dx_1, & \mu_p^2 \ge k^2, \\ \int_{b'\cos\alpha}^{L} \sin(\lambda_{\tilde{n}} x_1)\sin(\mu_p(h\sin\alpha + x_1\cos\alpha))\cos((-h\cos\alpha + x_1\sin\alpha - a)\sqrt{k^2 - \mu_p^2})\, dx_1, & \mu_p^2 < k^2, \end{cases}$$

$$d_{1p\tilde{n}}^{II} = \begin{cases} \int_{b'\cos\alpha}^{L} \sin(\lambda_{\tilde{n}} x_1)\sin(\mu_p(h\sin\alpha + x_1\cos\alpha)) e^{-(-h\cos\alpha + x_1\sin\alpha)\sqrt{\mu_p^2 - k^2}} dx_1, & \mu_p^2 \ge k^2, \\ \int_{b'\cos\alpha}^{L} \sin(\lambda_{\tilde{n}} x_1)\sin(\mu_p(h\sin\alpha + x_1\cos\alpha))\cos((-h\cos\alpha + x_1\sin\alpha)\sqrt{k^2 - \mu_p^2})\, dx_1, & \mu_p^2 < k^2, \end{cases}$$

$$\varphi_{\tilde{n}}^{II} = \int_{b'\cos\alpha}^{L} \varphi_{\Gamma_{II}}(x_1)\sin(\lambda_{\tilde{n}}x_1)dx_1 + \int_{0}^{b'\cos\alpha} \overline{\psi}_{\Gamma_{II}}(x_1)\sin(\lambda_{\tilde{n}}x_1)dx_1 =$$

$$= \int_{b'\cos\alpha}^{L} \left(1 - \frac{\cosh \dfrac{\pi\left(x_1 - \dfrac{a'}{2} - b'\cos\alpha\right)}{a'}}{\cosh \dfrac{\pi}{2}} \right) \sin(\lambda_{\tilde{n}}x_1)dx_1 + 0,$$

(37)

$$a_{1n\tilde{p}}^{III} = \begin{cases} \displaystyle\int_{a'\cos\alpha}^{b} \sin(\mu_{\tilde{p}}y_2)\sin(\lambda_n(a\sin\alpha + y_2\cos\alpha))e^{-(y_2\sin\alpha - a\cos\alpha)\sqrt{\lambda_n^2 - k^2}}dy_2, & \lambda_n^2 \ge k^2, \\ \displaystyle\int_{a'\cos\alpha}^{b} \sin(\mu_{\tilde{p}}y_2)\sin(\lambda_n(a\sin\alpha + y_2\cos\alpha))\cos((y_2\sin\alpha - a\cos\alpha)\sqrt{k^2 - \lambda_n^2})dy_2, & \lambda_n^2 < k^2, \end{cases}$$

$$a_{2n\tilde{p}}^{III} = \begin{cases} \displaystyle\int_{a'\cos\alpha}^{b} \sin(\mu_{\tilde{p}}y_2)\sin(\lambda_n(a\sin\alpha + y_2\cos\alpha))e^{(y_2\sin\alpha - a\cos\alpha - h)\sqrt{\lambda_n^2 - k^2}}dy_2, & \lambda_n^2 \ge k^2, \\ \displaystyle\int_{a'\cos\alpha}^{b} \sin(\mu_{\tilde{p}}y_2)\sin(\lambda_n(a\sin\alpha + y_2\cos\alpha))\cos((y_2\sin\alpha - a\cos\alpha - h)\sqrt{k^2 - \lambda_n^2})\,dy_2, & \lambda_n^2 < k^2, \end{cases}$$

$$d_{1p\tilde{p}}^{III} = \begin{cases} \dfrac{1}{2}b\delta_{p,\tilde{p}}e^{-a\sqrt{\mu_p^2 - k^2}}, & \mu_p^2 \ge k^2, \\ \dfrac{1}{2}b\delta_{p,\tilde{p}}\cos(a\sqrt{k^2 - \mu_p^2}), & \mu_p^2 < k^2, \end{cases}$$

$$d_{2p\tilde{p}}^{III} = \frac{1}{2}b\delta_{p,\tilde{p}}\ ,$$

$$\varphi_{\tilde{p}}^{III} = \int_{a'\cos\alpha}^{b} \varphi_{\Gamma_{III}}(y_2)\sin(\mu_{\tilde{p}}y_2)dy_2 + \int_{0}^{a'\cos\alpha} \overline{\psi}_{\Gamma_{III}}(y_2)\sin(\mu_{\tilde{p}}y_2)dy_2 =$$

$$= \int_{a'\cos\alpha}^{b} \left(\cos \frac{2\pi\left(y_2 - a'\cos\alpha - \dfrac{b'}{2}\right)}{b'} + 1 \right) \sin(\mu_{\tilde{p}}y_2)dy_2 + 0,$$

(38)

$$a_{1n\tilde{p}}^{IV}=\begin{cases}\int_0^{b'}\sin(\mu_{\tilde{p}}y_2)\sin(\lambda_n y_2\cos\alpha)e^{-y_2\sin\alpha\sqrt{\lambda_n^2-k^2}}dy_2, & \lambda_n^2\ge k^2,\\ \int_0^{b'}\sin(\mu_{\tilde{p}}y_2)\sin(\lambda_n y_2\cos\alpha)\cos(y_2\sin\alpha\sqrt{k^2-\lambda_n^2})\,dy_2, & \lambda_n^2<k^2,\end{cases}$$

$$a_{2n\tilde{p}}^{IV}=\begin{cases}\int_{a'\cos\alpha}^{b}\sin(\mu_{\tilde{p}}y_2)\sin(\lambda_n y_2\cos\alpha)e^{(y_2\sin\alpha-h)\sqrt{\lambda_n^2-k^2}}dy_2, & \lambda_n^2\ge k^2,\\ \int_{a'\cos\alpha}^{b}\sin(\mu_{\tilde{p}}y_2)\sin(\lambda_n y_2\cos\alpha)\cos((y_2\sin\alpha-h)\sqrt{k^2-\lambda_n^2})\,dy_2, & \lambda_n^2<k^2,\end{cases}$$

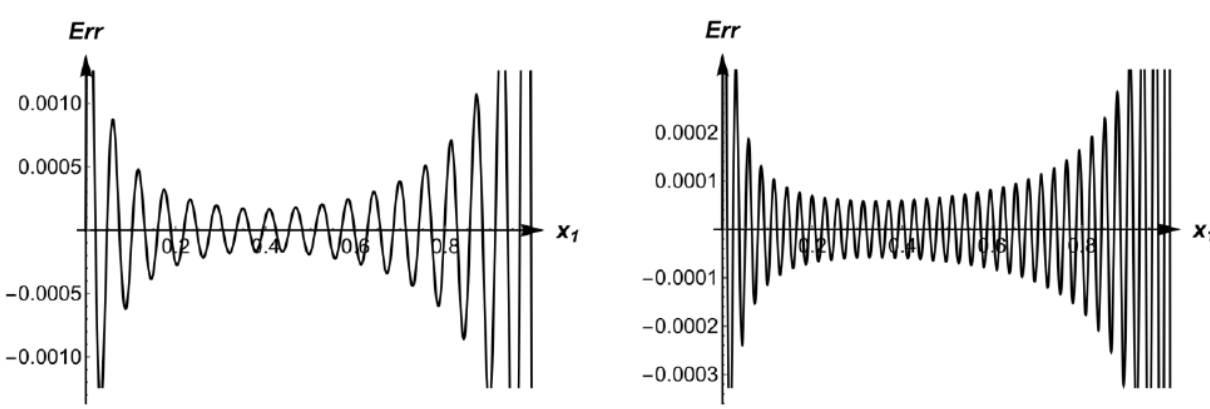
$$d_{1p\tilde{p}}^{IV}=\frac{1}{2}b\delta_{p,\tilde{p}},$$

$$d_{2p\tilde{p}}^{IV}=\begin{cases}\frac{1}{2}b\delta_{p,\tilde{p}}e^{-a\sqrt{\mu_p^2-k^2}}, & \mu_p^2\ge k^2,\\ \frac{1}{2}b\delta_{p,\tilde{p}}\cos(-a\sqrt{k^2-\mu_p^2}), & \mu_p^2<k^2,\end{cases}$$

$$\varphi_{\tilde{p}}^{IV}=\int_0^{b'}\varphi_{\Gamma_{IV}}(y_2)\sin(\mu_{\tilde{p}}y_2)dy_2+\int_{b'}^{b}\overline{\psi}_{\Gamma_{IV}}(y_2)\sin(\mu_{\tilde{p}}y_2)dy_2=$$

$$=\int_0^{b'}\left(\cos\frac{2\pi\left(y_2-\frac{b'}{2}\right)}{b'}+1\right)\sin(\mu_{\tilde{p}}y_2)dy_2+0, \tag{39}$$

The numerical realization of the proposed method was carried out using the Wolfram Mathematica system. The calculations were conducted at N=P=60 and N=P=120. The method mismatch with the boundary solutions was assessed, as shown in Fig. 2. It can be seen that the obtained solutions have the largest oscillation mismatch/error in the vicinity of the parallelogram angular points.



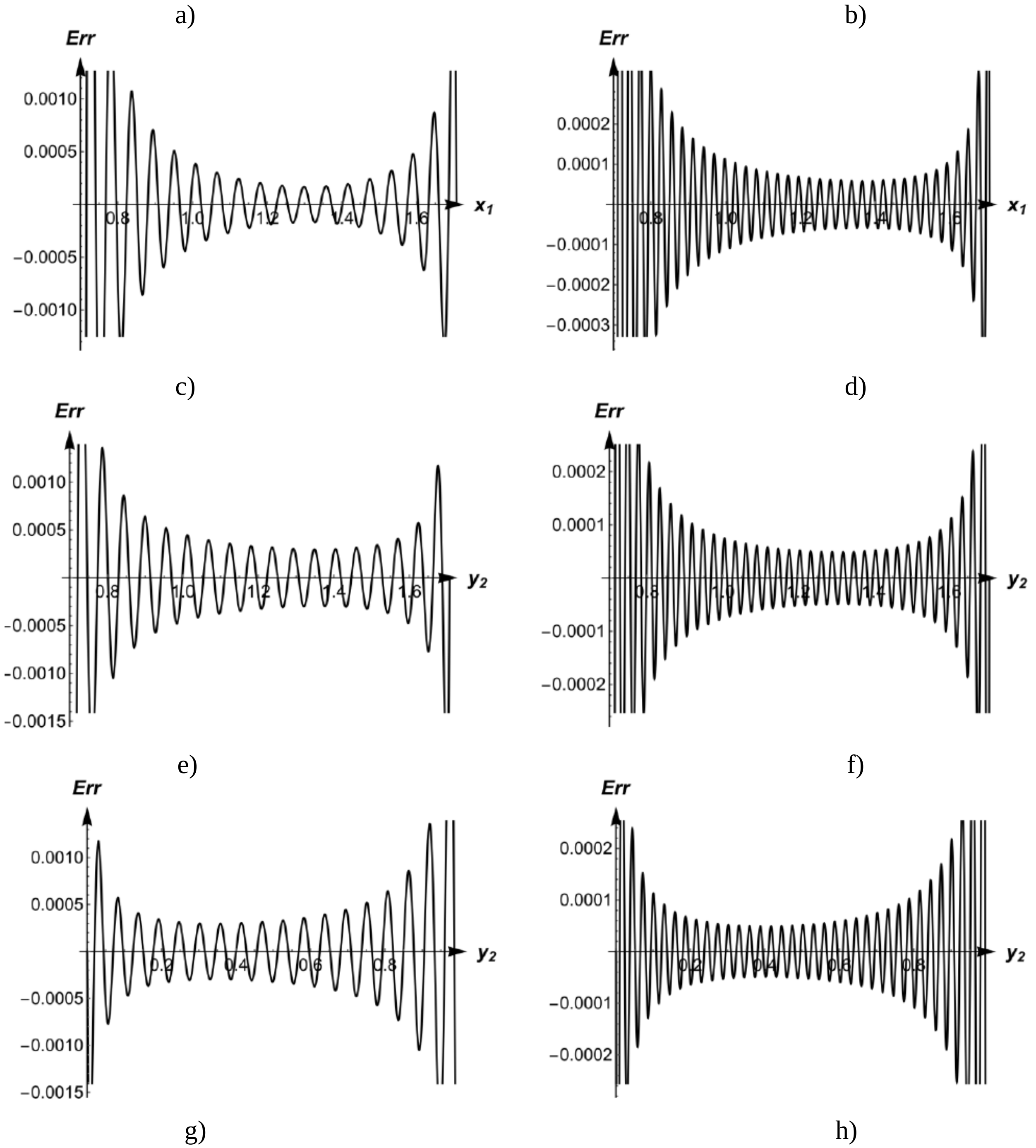


Fig. 2. Mismatch with boundary conditions obtained by the method of the mean square minimization method:

a) at boundary Γ_I for N=P=60 at k=8; b) at boundary Γ_I for N=P=120 at k=8;

c) at boundary Γ_{II} for N=P=60 at k=8; d) at boundary Γ_{II} for N=P=120 at k=8;

e) at boundary Γ_{III} for N=P=60 at k=8; f) at boundary Γ_{III} for N=P=120 at k=8;

g) at boundary Γ_{IV} for N=P=60 at k=8; h) at boundary Γ_{IV} for N=P=120 at k=8;

4. The collocation method

Alternatively, constants $A_n^{(1)}$, $A_n^{(2)}$, $D_p^{(1)}$, $D_p^{(2)}$ in (34) can be derived via the collocation method, which can be reduced to point-by-point implementation of boundary conditions in particular boundary points (also referred to as collocation points). In this case, the system of linear algebraic equations takes the following form:

$$\begin{aligned}
&\Gamma_I^{main}: \sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{n}}^{I} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{n}}^{I} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{n}}^{I} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{n}}^{I} = \varphi_{\tilde{n}}^{I},\\
&\Gamma_I^{add}: \sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{n}}^{Iadd} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{n}}^{Iadd} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{n}}^{Iadd} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{n}}^{Iadd} = \varphi_{\tilde{n}}^{Iadd},\\
&\Gamma_{II}^{main}: \sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{n}}^{II} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{n}}^{II} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{n}}^{II} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{n}}^{II} = \varphi_{\tilde{n}}^{II},\\
&\Gamma_{II}^{add}: \sum_{n=1}^{N} A_n^{1} a_{1n\tilde{n}}^{IIadd} + \sum_{n=1}^{N} A_n^{2} a_{2n\tilde{n}}^{IIadd} + \sum_{p=1}^{P} D_p^{1} d_{1p\tilde{n}}^{IIadd} + \sum_{p=1}^{P} D_p^{2} d_{2p\tilde{n}}^{IIadd} = \varphi_{\tilde{n}}^{IIadd},\\
&\Gamma_{III}^{main}: \sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{p}}^{III} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{p}}^{III} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{p}}^{III} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{p}}^{III} = \varphi_{\tilde{p}}^{III},\\
&\Gamma_{III}^{add}: \sum_{n=1}^{N} A_n^{1} a_{1n\tilde{p}}^{IIIadd} + \sum_{n=1}^{N} A_n^{2} a_{2n\tilde{p}}^{IIIadd} + \sum_{p=1}^{P} D_p^{1} d_{1p\tilde{p}}^{IIIadd} + \sum_{p=1}^{P} D_p^{2} d_{2p\tilde{p}}^{IIIadd} = \varphi_{\tilde{p}}^{IIIadd},\\
&\Gamma_{IV}^{main}: \sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{p}}^{IV} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{p}}^{IV} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{p}}^{IV} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{p}}^{IV} = \varphi_{\tilde{p}}^{IV},\\
&\Gamma_{IV}^{add}: \sum_{n=1}^{N} A_n^{(1)} a_{1n\tilde{p}}^{IVadd} + \sum_{n=1}^{N} A_n^{(2)} a_{2n\tilde{p}}^{IVadd} + \sum_{p=1}^{P} D_p^{(1)} d_{1p\tilde{p}}^{IVadd} + \sum_{p=1}^{P} D_p^{(2)} d_{2p\tilde{p}}^{IVadd} = \varphi_{\tilde{p}}^{IVadd},
\end{aligned} \tag{40}$$

Here $N_{main}^{(I/II)}$ and $P_{main}^{(III/IV)}$ have to meet the requirements of $\{x_{1n},(0,h)\}\in(I/II)$ and $\{(a,0),y_{2n}\}\in(III/IV)$, respectively, while $N_{add}^{(I/II)}$ and $P_{add}^{(III/IV)}$ are supplements to $N_{main}^{(I/II)}$ and $P_{main}^{(III/IV)}$, respectively.

This yields:

$$\begin{cases} N_{main}^{(I/II)} + N_{add}^{(I/II)} = N \\ P_{main}^{(III/IV)} + P_{add}^{(III/IV)} = P \end{cases}$$

The system of 2N+2P equations (34) allows one to derive 2N+2P unknown expansion coefficients.

$$a_{1n\tilde{n}}^{I} = \sin(\lambda_n x_{1\tilde{n}}),$$

$$a_{2n\tilde{n}}^{I} = \begin{cases} \sin(\lambda_n x_{1\tilde{n}}) e^{-h\sqrt{\lambda_n^2 - k^2}}, & \lambda_n^2 \ge k^2, \\ \sin(\lambda_n x_{1\tilde{n}}) \cos(-h\sqrt{\lambda_n^2 - k^2}), & \lambda_n^2 < k^2, \end{cases}$$

$$d_{1p\tilde{n}}^{\mathrm{I}} = \begin{cases} \sin(\mu_p x_{1\tilde{n}} \cos\alpha) e^{-x_{1\tilde{n}} \sin\alpha \sqrt{\mu_p^2 - k^2}}, & \mu_p^2 \ge k^2, \\ \sin(\mu_p x_{1\tilde{n}} \cos\alpha) \cos(x_{1\tilde{n}} \sin\alpha \sqrt{k^2 - \mu_p^2}), & \mu_p^2 < k^2, \end{cases}$$

$$d_{2p\tilde{n}}^{\mathrm{I}} = \begin{cases} \sin(\mu_p x_{1\tilde{n}} \cos\alpha) e^{-(x_{1\tilde{n}} \sin\alpha - a)\sqrt{\mu_p^2 - k^2}}, & \mu_p^2 \ge k^2, \\ \sin(\mu_p x_{1\tilde{n}} \cos\alpha) \cos((x_{1\tilde{n}} \sin\alpha - a)\sqrt{k^2 - \mu_p^2}), & \mu_p^2 < k^2, \end{cases}$$

$$\varphi_{\tilde{n}}^{I} = \varphi_{\Gamma_I}(x_{1\tilde{n}}), x_{1\tilde{n}} = 0 + \tilde{n}\frac{L}{(N+1)}, \tilde{n} = \overline{1, N_{main}^{I}}; N = N_{main}^{I} + N_{add}^{I},$$

$$a_{1n\tilde{n}}^{Iadd} = a_{1n\tilde{n}}^{I}, a_{2n\tilde{n}}^{Iadd} = a_{2n\tilde{n}}^{I}, d_{1p\tilde{n}}^{Iadd} = 0, d_{2p\tilde{n}}^{Iadd} = 0, \varphi_{\tilde{n}}^{Iadd} = \overline{\psi}_{\Gamma_I}(x_{1\tilde{n}}), \tilde{n} = \overline{N_{main}^{I} + 1, N},$$

(41)

$$a_{1n\tilde{n}}^{II} = \begin{cases} \sin(\lambda_n x_{1\tilde{n}}) e^{-h\sqrt{\lambda_n^2 - k^2}}, & \lambda_n^2 \ge k^2, \\ \sin(\lambda_n x_{1\tilde{n}}) \cos(h\sqrt{k^2 - \lambda_n^2}), & \lambda_n^2 < k^2, \end{cases}$$

$$a_{2n\tilde{n}}^{II} = \sin(\lambda_n x_{1\tilde{n}}),$$

$$d_{1p\tilde{n}}^{II} = \begin{cases} \sin(\mu_p (h\sin\alpha + x_{1\tilde{n}}\cos\alpha)) e^{-(-h\cos\alpha + x_{1\tilde{n}}\sin\alpha)\sqrt{\mu_p^2 - k^2}}, & \mu_p^2 \ge k^2, \\ \sin(\mu_p (h\sin\alpha + x_{1\tilde{n}}\cos\alpha)) \cos((-h\cos\alpha + x_{1\tilde{n}}\sin\alpha)\sqrt{k^2 - \mu_p^2}), & \mu_p^2 < k^2, \end{cases}$$

$$d_{2p\tilde{n}}^{II} = \begin{cases} \sin(\mu_p (h\sin\alpha + x_{1\tilde{n}}\cos\alpha)) e^{(-h\cos\alpha + x_{1\tilde{n}}\sin\alpha - a)\sqrt{\mu_p^2 - k^2}}, & \mu_p^2 \ge k^2, \\ \sin(\mu_p (h\sin\alpha + x_{1\tilde{n}}\cos\alpha)) \cos((-h\cos\alpha + x_{1\tilde{n}}\sin\alpha - a)\sqrt{k^2 - \mu_p^2}), & \mu_p^2 < k^2, \end{cases}$$

$$\varphi_{\tilde{n}}^{II} = \varphi_{\Gamma_{II}}(x_{1\tilde{n}}), \quad \tilde{n} = \overline{N_{add}^{II} + 1, N},$$

$$a_{1n\tilde{n}}^{IIadd} = a_{1n\tilde{n}}^{II}, a_{2n\tilde{n}}^{IIadd} = a_{2n\tilde{n}}^{II}, d_{1p\tilde{n}}^{IIadd} = 0, d_{2p\tilde{n}}^{IIadd} = 0, \varphi_{\tilde{n}}^{IIadd} = \overline{\psi}_{\Gamma_{II}}(x_{1\tilde{n}}), \tilde{n} = \overline{1, N_{add}^{II}},$$

(42)

$$a_{1n\tilde{p}}^{III}=\begin{cases}\sin(\lambda_n(a\sin\alpha+y_{2\tilde{p}}\cos\alpha))e^{-(y_{2\tilde{p}}\sin\alpha-a\cos\alpha)\sqrt{\lambda_n^2-k^2}}, & \lambda_n^2\geq k^2,\\ \sin(\lambda_n(a\sin\alpha+y_{2\tilde{p}}\cos\alpha))\cos((y_{2\tilde{p}}\sin\alpha-a\cos\alpha)\sqrt{k^2-\lambda_n^2}), & \lambda_n^2<k^2,\end{cases}$$

$$a_{2n\tilde{p}}^{III}=\begin{cases}\sin(\lambda_n(a\sin\alpha+y_{2\tilde{p}}\cos\alpha))e^{(y_{2\tilde{p}}\sin\alpha-a\cos\alpha-h)\sqrt{\lambda_n^2-k^2}}, & \lambda_n^2\geq k^2,\\ \sin(\lambda_n(a\sin\alpha+y_{2\tilde{p}}\cos\alpha))\cos((y_{2\tilde{p}}\sin\alpha-a\cos\alpha-h)\sqrt{k^2-\lambda_n^2}), & \lambda_n^2<k^2,\end{cases}$$

$$d_{1p\tilde{p}}^{III}=\begin{cases}\sin(\mu_p y_{2\tilde{p}})e^{-a\sqrt{\mu_p^2-k^2}}, & \mu_p^2\geq k^2,\\ \sin(\mu_p y_{2\tilde{p}})\cos(a\sqrt{k^2-\mu_p^2}), & \mu_p^2<k^2,\end{cases}$$

$$d_{2p\tilde{p}}^{III}=\sin(\mu_p y_{2\tilde{p}}),$$

$$\varphi_{\tilde{p}}^{III}=\varphi_{\Gamma_{III}}(y_{2\tilde{p}}),\quad \tilde{p}=\overline{P_{add}^{III}+1,P},$$

$$a_{1n\tilde{p}}^{IIIadd}=0,\quad a_{2n\tilde{p}}^{IIIadd}=0,\quad d_{1p\tilde{p}}^{IIIadd}=d_{1p\tilde{p}}^{III},\quad d_{2p\tilde{n}}^{IIIadd}=d_{2p\tilde{p}}^{III},$$

$$\varphi_{\tilde{p}}^{IIIadd}=\overline{\psi}_{\Gamma_{III}}(y_{2\tilde{p}}),\quad \tilde{p}=\overline{1,P_{add}^{III}},\tag{43}$$

$$a_{1n\tilde{p}}^{IV}=\begin{cases}\sin(\lambda_n y_{2\tilde{p}}\cos\alpha)e^{-y_{2\tilde{p}}\sin\alpha\sqrt{\lambda_n^2-k^2}}, & \lambda_n^2\geq k^2,\\ \sin(\lambda_n y_{2\tilde{p}}\cos\alpha)\cos(y_{2\tilde{p}}\sin\alpha\sqrt{k^2-\lambda_n^2}), & \lambda_n^2<k^2,\end{cases}$$

$$a_{2n\tilde{p}}^{IV}=\begin{cases}\sin(\lambda_n y_{2\tilde{p}}\cos\alpha)e^{(y_{2\tilde{p}}\sin\alpha-h)\sqrt{\lambda_n^2-k^2}}, & \lambda_n^2\geq k^2,\\ \sin(\lambda_n y_{2\tilde{p}}\cos\alpha)\cos((y_{2\tilde{p}}\sin\alpha-h)\sqrt{k^2-\lambda_n^2}), & \lambda_n^2<k^2,\end{cases}$$

$$d_{1p\tilde{p}}^{IV}=\sin(\mu_p y_{2\tilde{p}}),$$

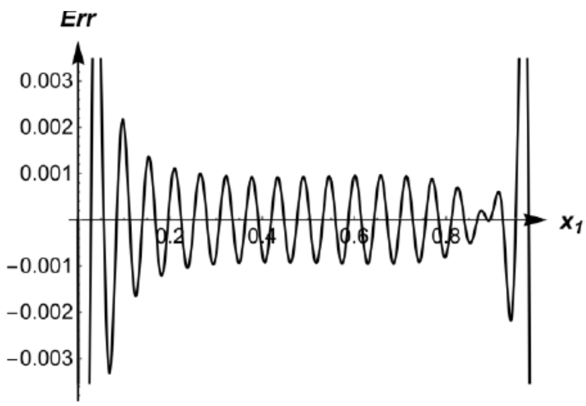
$$d_{2p\tilde{p}}^{IV}=\begin{cases}\sin(\mu_{\tilde{p}} y_{2\tilde{p}})e^{-a\sqrt{\mu_p^2-k^2}}, & \mu_p^2\geq k^2,\\ \sin(\mu_{\tilde{p}} y_{2\tilde{p}})\cos(-a\sqrt{k^2-\mu_p^2}), & \mu_p^2<k^2,\end{cases}$$

$$\varphi_{\tilde{p}}^{IV}=\varphi_{\Gamma_{IV}}(y_{2\tilde{p}}),\tilde{p}=\overline{1,P_{main}^{IV}},$$

$$a_{1n\tilde{p}}^{IVadd}=0,\quad a_{2n\tilde{p}}^{IVadd}=0,\quad d_{1p\tilde{p}}^{IVadd}=d_{1p\tilde{p}}^{IV},\quad d_{2p\tilde{p}}^{IVadd}=d_{2p\tilde{p}}^{IV},$$

$$\varphi_{\tilde{p}}^{IVadd}=\overline{\psi}_{\Gamma_{IV}}(y_{2\tilde{p}}),\quad \tilde{p}=\overline{P_{main}^{IV}+1,P},\tag{44}$$

The mismatch with boundary conditions by the collocation method was estimated for different cases at N=P=60 and N=P=120, as shown in Fig.3.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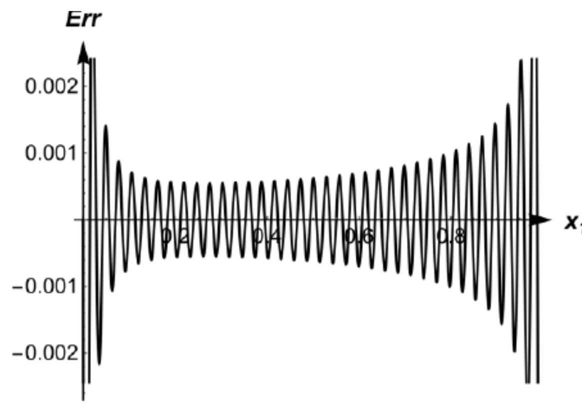
0.001
−0.001
−0.002
−0.003
0.2
0.4
0.6
0.8
x1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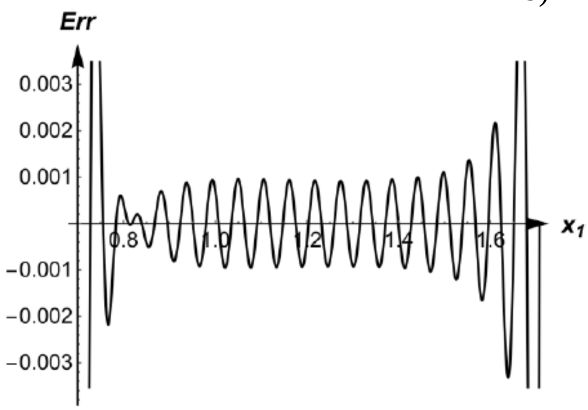
−0.001
−0.002
0.2
0.4
0.6
0.8
x1

b)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0.8
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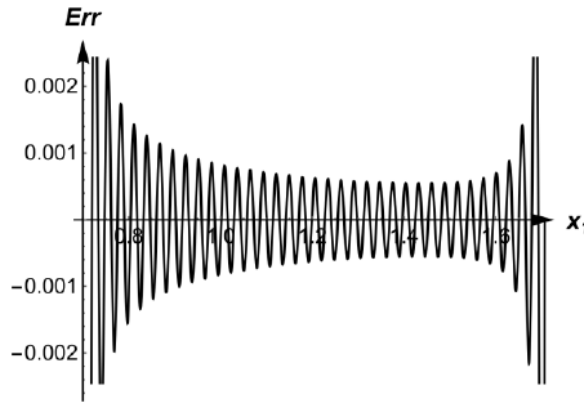
1.2
1.4
1.6
x1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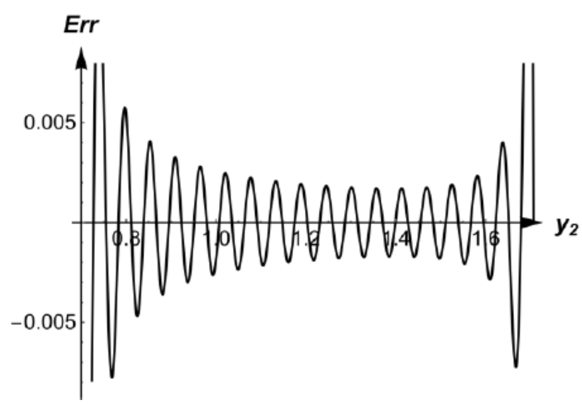
1.2
1.4
1.6
x1

c) d)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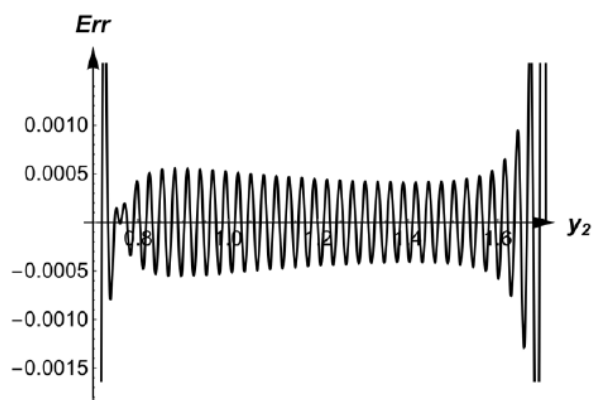
1.6
y2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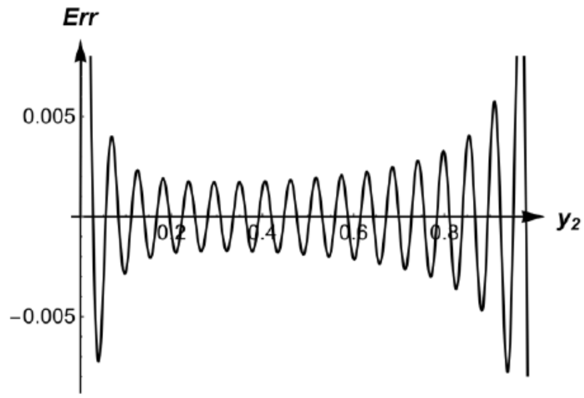
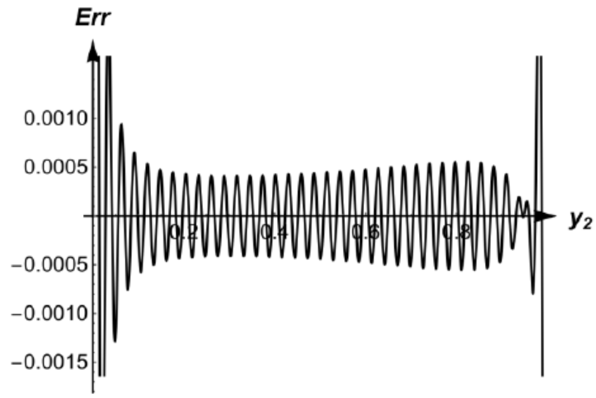
1.0
1.2
1.4
1.6
y2

e) f)





g) h)

Fig. 2. Mismatch with boundary conditions obtained by the collocation method:
a) at boundary Γ_I for N=P=60 at k=8; b) at boundary Γ_I for N=P=120 at k=8;
c) at boundary Γ_{II} for N=P=60 at k=8; d) at boundary Γ_{II} for N=P=120 at k=8;
e) at boundary Γ_{III} for N=P=60 at k=8; f) at boundary Γ_{III} for N=P=120 at k=8;
g) at boundary Γ_{IV} for N=P=60 at k=8; h) at boundary Γ_{IV} for N=P=120 at k=8;

The comparison of the results obtained for the two systems of collocation points (containing 60 and 120 points, respectively) strongly indicates that an increase in the number of collocation points improves the quality of boundary conditions' implementation. Noteworthy is the presence of a relatively high error in very narrow areas near the ends of the boundary segments. This can be treated as a specific feature of the collocation method, which can be omitted by using the alternative technique, e.g., the mean square deviation minimization method. However, local deviations in boundary conditions have no significant effect when the spectrum of eigenfrequencies is assessed by the collocation method. Moreover, the comparative analysis of data for 60 and 120 points of collocation indicates that an increase in the number of collocation points improves the accuracy and the number of such points can be significantly increased, if required.

5. Oscillation of the membrane with a dynamic excitation; The mean square deviation minimization (MSDM) method

Consider a parallelogram-shaped membrane subjected to a distributed load of density

$$F(x,y)=\sin\left(\frac{\pi y}{h}\right)\sin\left(\frac{\pi(x-y\cot\alpha)}{a'}\right). \quad (45)$$

The nonuniform solution (5) corresponding to such a loading case is

$$\hat{\varphi}(x_1,y_1)=-\frac{1}{4}\int_{\Omega}\sin\left(\frac{\pi\eta}{h}\right)\sin\left(\frac{\pi(\xi-\eta\cot\alpha)}{a'}\right)Y_0(\sqrt{(x_1-\xi)^2+(y_1-\eta)^2})d\xi d\eta. \quad (46)$$

In this case, coefficients of the linear algebraic system (35) remain unchanged, as compared to those in the problem of kinematically excited membrane oscillations, while variations occur only in the right parts of equations representing the problem boundary conditions. When the MSDM is used, functions $\overline{\psi}_I(x_1)$, $\overline{\psi}_{II}(x_1)$, $\overline{\psi}_{III}(y_2)$ and $\overline{\psi}_{IV}(y_2)$ are considered to be equal to zero, which reduces the right parts of the system equations to the following form:

$$\varphi_{\tilde{n}}^{I} = \frac{1}{4}\int_0^{a'} \sin\left(\lambda_{\tilde{n}} x_1\right)\left(\int_\Omega \sin\left(\frac{\pi\eta}{h}\right)\sin\left(\frac{\pi(\xi-\eta\cot\alpha)}{a'}\right)Y_0(k\sqrt{\eta^2+(x_1-\xi)^2})d\eta d\xi\right)dx_1,$$

$$\varphi_{\tilde{n}}^{II} = \frac{1}{4}\int_{b'\cos\alpha}^{L} \sin(\lambda_{\tilde{n}} x_1)\left(\int_\Omega \sin\left(\frac{\pi\eta}{h}\right)\sin\left(\frac{\pi(\xi-\eta\cot\alpha)}{a'}\right)Y_0\left(k\sqrt{(x_1-\xi)^2+(h-\eta)^2}\right)d\eta d\xi\right)dx_1,$$

$$\varphi_{\tilde{p}}^{III} = \frac{1}{4}\int_{a'\cos\alpha}^{b} \sin(\mu_{\tilde{p}} y_2)\left(\int_\Omega \sin\left(\frac{\pi\eta}{h}\right)\sin\left(\frac{\pi(\xi-\eta\cot\alpha)}{a'}\right)\times\right.$$
$$\left.\times Y_0\left(k\sqrt{((-a\cos\alpha+y_2\sin\alpha)-\eta)^2+((a\sin\alpha+y_2\cos\alpha)-\xi)^2}\right)d\eta d\xi\right)dy_2,$$

$$\varphi_{\tilde{p}}^{IV} = \frac{1}{4}\int_0^{b'} \sin(\mu_{\tilde{p}} y_2)\left(\int_\Omega \sin\left(\frac{\pi\eta}{h}\right)\sin\left(\frac{\pi(\xi-\eta\cot\alpha)}{a'}\right)\times\right.$$
$$\left.\times Y_0\left(k\sqrt{(y_2\sin\alpha-\eta)^2+(y_2\cos\alpha-\xi)^2}\right)d\eta d\xi\right)dy_2, \tag{47}$$

By solving the obtained linear algebraic system, we can derive the reduced series coefficients. Next, to assess the displacement amplitude at an arbitrary point of the membrane, the uniform solution should be derived for the nonuniform one in (46).

6. Oscillation of the membrane with a dynamic excitation; The collocation method

The coefficients of the linear algebraic system (40) remain unchanged, as compared to those in in the problem of kinematically excited membrane oscillations solved by the collocation method, while the right parts of the system equations take the following form:

$$\varphi_{\Gamma_I}(x_{1\tilde{n}}) = \frac{1}{4}\int_\Omega \sin\left(\frac{\pi\eta}{h}\right)\sin\left(\frac{\pi(\xi-\eta\cot\alpha)}{\tilde{a}}\right)Y_0\left(k\sqrt{\eta^2+(x_{1\tilde{n}}-\xi)^2}\right)d\eta d\xi,$$

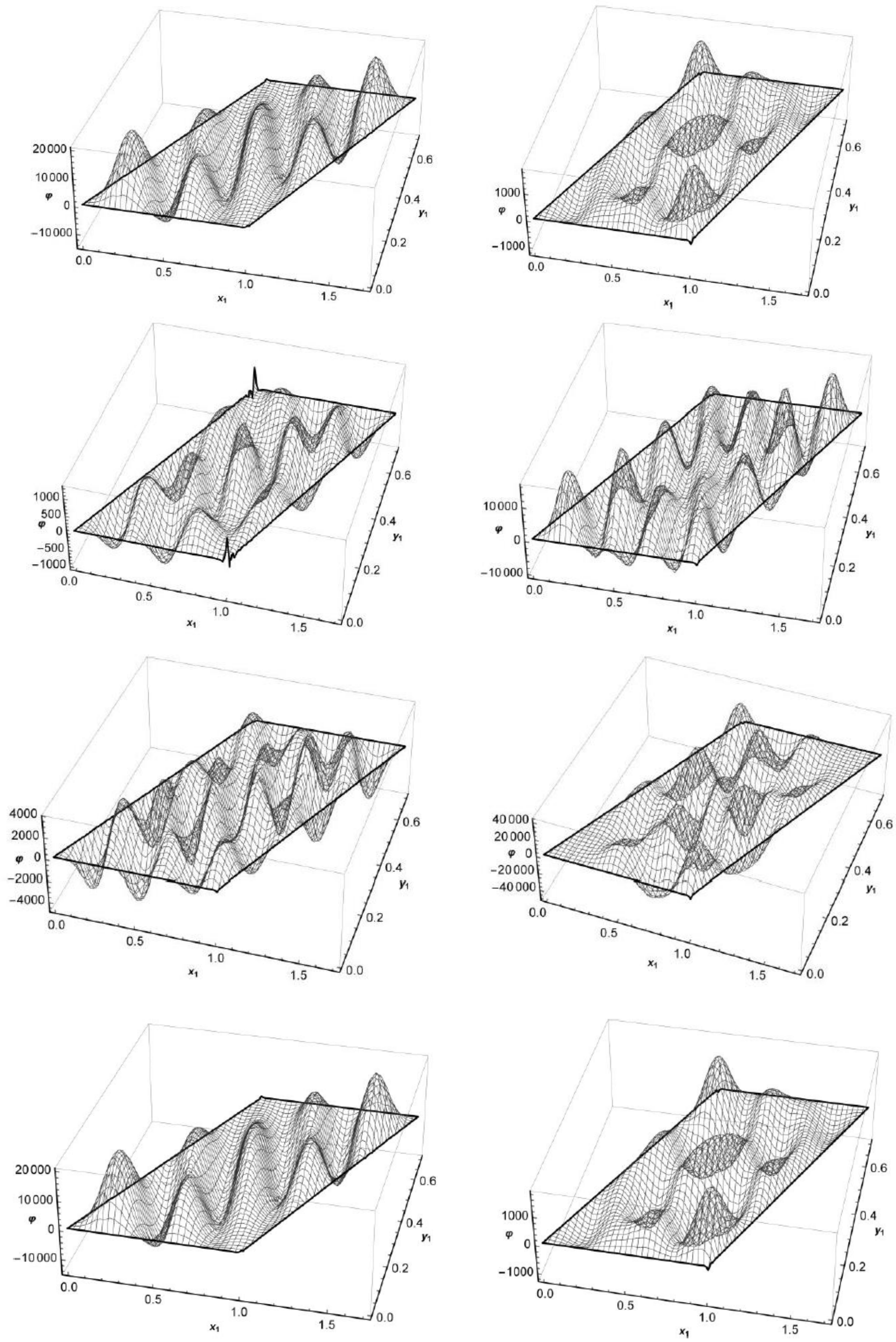
$$\varphi_{\Gamma_{II}}(x_{1\tilde{n}}) = \frac{1}{4}\int_\Omega \sin\left(\frac{\pi\eta}{h}\right)\sin\left(\frac{\pi(\xi-\eta\cot\alpha)}{\tilde{a}}\right)Y_0\left(k\sqrt{(x_{1\tilde{n}}-\xi)^2+(h-\eta)^2}\right)d\eta d\xi,$$

$$\varphi_{\Gamma_{III}}(y_{2\tilde{p}}) = \frac{1}{4}\int_\Omega \sin\left(\frac{\pi\eta}{h}\right)\sin\left(\frac{\pi(\xi-\eta\cot\alpha}{\tilde{a}}\right)\times$$
$$\times Y_0\left(k\sqrt{((-a\cos\alpha+y_{2\tilde{p}}\sin\alpha)-\eta)^2+((a\sin\alpha+y_{2\tilde{p}}\cos\alpha)-\xi)^2}\right)d\eta d\xi,$$

$$\varphi_{\Gamma_{IV}}(y_{2\tilde{p}}) = \frac{1}{4}\int_\Omega \sin\left(\frac{\pi\eta}{h}\right)\sin\left(\frac{\pi(\xi-\eta\cot\alpha}{\tilde{a}}\right)\times$$
$$\times Y_0\left(k\sqrt{(y_{2\tilde{p}}\sin\alpha-\eta)^2+(y_{2\tilde{p}}\cos\alpha-\xi)^2}\right)d\eta d\xi, \tag{48}$$

The behavior of solution errors (mismatches) provided by the MSDM and collocation methods, in case of dynamically excited membrane oscillations, is similar to those revealed in the kinematic excitation cases.

Forms of oscillation of the membrane with a kinematic excitation are depicted in Fig.3.



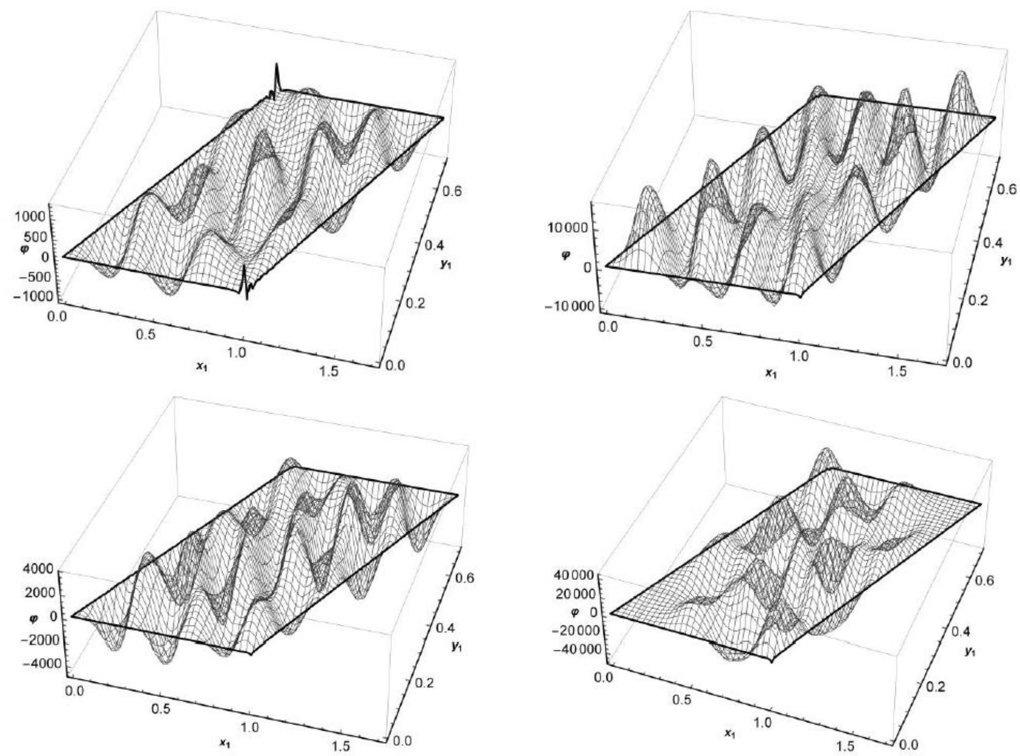
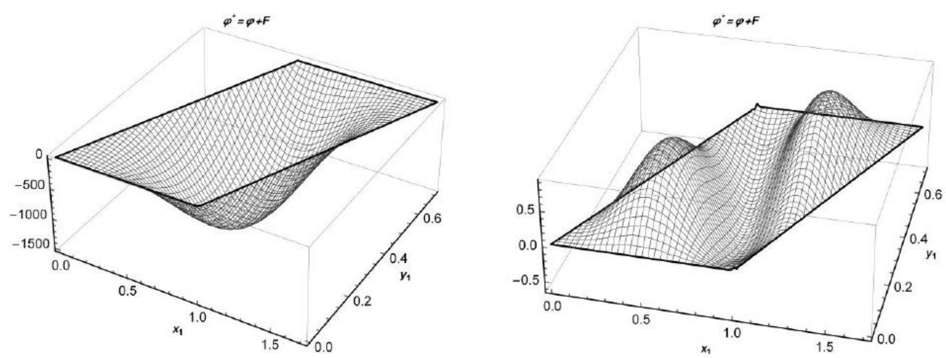


Fig.3. Forms of oscillation of the membrane with a kinematic excitation for different values of k: k=5.895; b) k=10.013; c) k=13.6; d) k=14.77; e) k=17.15; f) k=19.75; g) k=20.76; h) k=23.17; j) k=23.354; k) k=27.224; l) k=27.877; m) k=29.22

Forms of oscillation of the membrane with a dynamic excitation are presented in Fig.4, whereas the calculated amplitude-frequency characteristic (AFC) of the membrane is plotted in Fig.5.



a) b)

Fig.4. Forms of oscillation of the membrane with a dynamic excitation for two values of k: a) k=5.895; b) k=10.013.

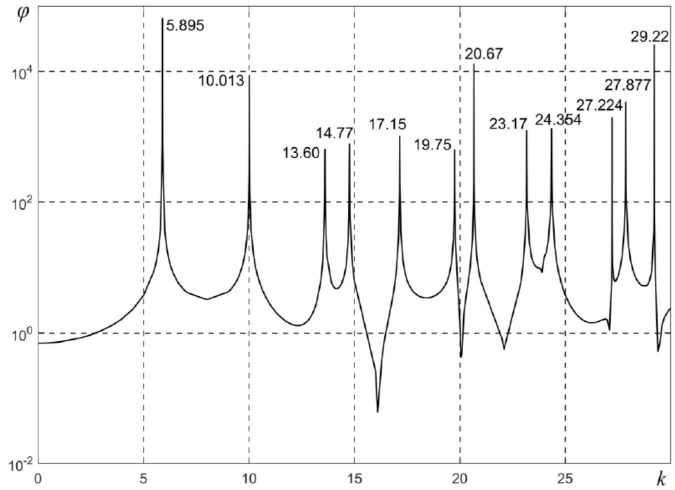


Fig.5. The calculated amplitude-frequency characteristic of the dynamically excited membrane.

7. Conclusions

This study attempted to provide solutions to the oscillation problems of complex-shaped membranes, including the case of dynamically excited finite regions. The main feature of the dynamic processes in such regions is the presence of resonance phenomena. The results obtained strongly indicate that the analysis of forced oscillations in a wide range of frequencies of interfering factors makes it possible to determine precisely their eigenfrequencies and eigenforms of oscillations. The amplitude-frequency characteristic of the membrane central point was derived, which indicates the effect of the complexity of the membrane shape on the spectrum of its eigenfrequencies. There is no distribution regularity, in contrast to rectangular membranes. Some discrepancies in the magnitudes of oscillation amplitudes for frequencies close to resonant ones is due to a slight difference in the values of different frequencies from the membrane eigenfrequencies. However, a significant increase in the membrane deviation from the static equilibrium state (by 2-3 orders of magnitude) strongly indicates the sufficient accuracy of the obtained estimates. All calculations made using 120 collocation points fully satisfied the boundary conditions. The additional proof of the sufficient accuracy of predictions is the fact that the magnitude of oscillation amplitudes of kinematic perturbations of exciting circuits approaches zero. Several characteristic intrinsic forms of oscillation were derived for various wavenumbers and the number of used collocation points. It is shown that an increase in the wavenumber leads to a rise in the number of nodal lines in its oscillation eigenfrequencies. The performed analysis of oscillation forms confirms that the chosen number of collocation points is sufficient to determine the dynamic characteristics of the system with high eigenfrequencies containing about ten local maxima in the membrane area.

8. Acknowledgements

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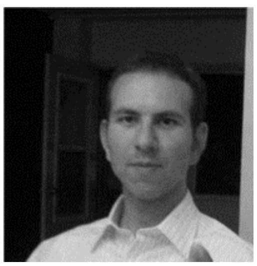

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