Procurement Models for Multiple Consumable Spare Parts with Application to Canadian Military Self-Contained Units and Similar Organizations

by

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LIST OF SYMBOLS AND ABBREVIATIONS

t = fixed cycle period (15 days, 500 hours, 1 year,..)

U = expected usage (miles, hours,..) of each identical equipment during period t

M = number of identical equipments operating at the location

B = fixed available budget (\$) during period t

R = constant and common interest rate for inventory holding costs H (\$/\$/time period)

j = index of part (item) types ($j=1,2,..,J$)

f_j = time-to-failure density (pdf) of item j

τ_j = item or component failure rate (fr/time)

p_j = Poisson probability mass function (pmf)

δ_j = Poisson rate parameter for a fixed time period

F_j = Cumulative distribution function (cdf)

P_j = Complementary cdf = $1-F_j$

h_j = c_jR = inventory holding costs for item j

H = Σh_j = total inventory holding costs (\$)

c_j = purchase cost for item j (\$/item)

C_S = Σc_jS_j = total purchase costs (\$)

S_j = up-to inventory level for item j at the beginning of the period

$\{S\}=\{S_j, j=1,2,..,J\}$ = inventory quantity vector of all items at the beginning of the period

BO_j = expected number of backorders for item j

BO = total expected system backorders = ΣBO_j

A_j = availability for item j

A_S = system availability = πA_j

AA_S = expected proportion of equipments still operational at the end of the period

i = index for number of locations ($i=1,2,..,I$)

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ABSTRACT

This thesis investigates the current inventory model used in the Canadian Armed Forces (Army) to field spare parts to various self-contained organization (units) to sustain operational equipments for fixed cycle periods. Since the current model determines inventory levels for each spare j, j=1,..,J up to a specified fixed availability service measure which is the same for each item, we propose and analyze two distinct models in which the objective sought is to determine the optimal number of spares S_j, j=1,..,J (considered consumables or throw-away modules subject to Poisson demands) required at the beginning of the period in order to optimize system performance by either 1) maximizing system availability A_S and/or minimizing total expected system backorders BO, constrained to a specified budget. Both models result in optimized stock levels $\{S_j, j=1,..,J\}$ that are approximately the same. Various solution methods for solving this non-linear integer optimization type of problem such as dynamic programming, marginal and Lagrange analysis are investigated and compare both models vs the current military model; simple heuristics are developed to improve near optimal solutions. We also analyze the link between both performance measures and develop a more appropriate measure of system performance: the expected number (and proportion) of equipments still operational at the end of the period or AA_S, with and without part failure dependencies. Variants and extensions to multiple location and indentured types of systems are also discussed. We include randomly generated numerical test problems, whose results significantly underscore the usefulness of the proposed procedures across all measures of system performance A_S, BO and AA_S.

CHAPTER 1: INTRODUCTION

1.1 GENERAL.

In the past, several traditional types of inventory systems have been studied and account for hundreds of papers and articles that have been published in the literature. Various policies were analyzed such as continuous and periodic review models, with and without reorder point levels, deterministic and variable delivery lead time, constant and stochastic demands, backorders vs lost sales, consumables vs repairables,.. A class of inventory systems that have been the focus of further study in the past 20 years are those associated with various combinations of multiple item, multiple indentures, multiple location and multi-echelon types of inventory systems for both consumable and repairable items.

In this thesis, we restrict our attention to the current model used by the Canadian Armed Forces (Land element) to field spares to self-contained units and we investigate and compare two alternative multiple item inventory models, each subjected to a one-period, stationary Poisson failure process for consumable or throw-away repairable items.

It is motivated by the case where a set of identical operating equipments or machines, subject to random and independent failures, need to be supported by an inventory of spare assemblies or parts that will maximize or minimize a defined system performance measure; several important variants of the basic models are investigated.

In particular, the objective is to determine how many of each type of assemblies or items to purchase and stock in inventory, or the order-up-to quantity vector $\{S\} = \{S_j, j=1,2,..,J\}$ at the beginning of each cycle period either to 1) maximize system availability A_S and/or 2) minimize total expected system backorders E(BO) which will be denoted BO, and subject to a fixed single investment budgetary constraint consisting of purchasing costs. These two models will be formulated in chapter 2 and be known as model P1 and P2 respectively.

1.2 ORGANIZATION

The remainder of chapter 1 will describe the current model used by the Canadian Armed Forces (Land element), and the system performance measures used to determine the number of spares $\{S_j, j=1,..,J\}$ in order to keep a set of M identical equipments operating during fixed cycle periods.

In chapter 2, the necessary notation, terminology and definitions used throughout the thesis are introduced and includes the description, formulation and assumptions for the two different single location, multiple item inventory models P1 and P2 for consumables or repairable (throw-away type), the basic building block from which more complex extensions can be analyzed in later chapters. It also discusses various solution methods and model variants that yield the same solutions but lead to different managerial interpretation. A literature review is included.

Chapters 3 to 6 describe the various methods and important factors used to obtain the near or optimal solution vector $\{S_j, j=1,..,J\}$ for each model P1 and P2 and their related variants, so that they may be compared with the current military model and considered for eventual implementation.

In chapter 3, dynamic programming methodology is described to solve for the optimal solution to the problem; undominated solution vectors $\{S_j\}$ can be generated for various budget values and are guaranteed to be optimal only if all possible budget values are enumerated at every stage; associated serious computational difficulties are discussed which lead to a DP approximation methodology with increments (or discretized values) at every stage and a lower bound on the total cost solution C_S is derived. Equivalent FULL and GAP network structures illustrating both models are presented. The requirement to develop faster and more efficient methods is also discussed.

Chapter 4 analyzes the marginal analysis procedure used to obtain near or optimal solutions for both models P1 and P2, based on the

Poisson distribution, by generating successive undominated allocations (but not necessarily all of them) until the maximum available budget has been reached; error bounds on the system availability performance measure A_S for model P1 and for expected system backorders BO of model P2 as well as error bounds on the total costs solutions C_S are developed. We shall extend the analysis by deriving a simple heuristic, which will be called the "top-up" marginal analysis procedure, that can significantly improve the performance measures, particularly for low budget values, and reduce the total cost solution C_S to within the least expensive of all J items or an error less than the min $\{c_j, j=1,..,J\}$.

Switching the objective function with the constraint in each of the model results in two variants of the models when minimizing costs, called models P1a and P2a, which give the same sequence of undominated solution vectors $\{S_j, j=1,..,J\}$ but lead to a different managerial interpretation.

Chapter 5 derives Lagrange multipliers for each model and provides accurate and useful lower and upper bounds to help us calculate (as opposed to guessing) the first estimated value of the multiplier, from which the efficient bisection search technique can then be used to obtain the near or optimal solution to the problem even faster than the marginal analysis procedure; the Lagrange multipliers for model variants P1a and P2a are also included.

Chapter 6 uses simulation to analyze the end of cycle effect and its impact on system availability A_S due to part failure dependencies; an equivalent system availability measure EA_S is defined and is compared to an alternate and more appropriate measure of system performance AA_S = average proportion of equipments still operational at the end of the cycle, by varying the number of operational equipments available at a location. Theoretical derivations of AA_S and practical results are compared with and without part failure dependencies.

Chapters 7 discusses various extensions to models P1 and P2. An analysis of a system with multiple indenture levels of assemblies and their components, a derivation of the conditions under which it becomes worthwhile to disaggregate the Poisson process for an assembly into smaller ones (for its components) once additional information on their failure rates and costs become more accurate, and how a marginal analysis technique can be used for our models to solve for the near or optimal solution vector $\{S_j\}$, even though the sum of the costs of individual components may be more than the cost of the whole assembly while still providing higher system availability A_S.

It also provides a further extension from the one location system to the multiple location type of systems by formulating two new models: model (P1b) to maximize A_S and model (P2b) to minimize BO; specifically, the computational difficulties associated with the DP methodology to allocate the total available budget B among the various locations i, i=1,..,I are examined; an I=3 location and J=5 item example is shown to be equivalent to a problem of I=1 x J=15 items in which the same parameters of the 5 items are appended 3 times at the same location. Marginal analysis and Lagrange multiplier techniques can then be used to obtain a near or optimal solution much faster than DP, and both methods provide a fast and convenient procedure to optimally determine and allocate budget levels to each location.

Network analysis, presented earlier in chapter 3, is also shown to be applicable to the multiple location models; the FULL network structure is equivalent to the optimal DP methodology with enumeration of all possible budget values at each stage. The GAP network, however, drastically reduces network size and we show that an error bound on C_S is exactly the same as the single location model if items with equal costs c_{ij} are listed adjacently, thus reducing the error on C_S by a factor of I (number of locations). The concept of "reverse" marginal analysis is used to demonstrate how to optimally re-distribute stocked items from a central location (warehouse or base) among various operational locations.

Chapter 8 provides a set of 40 randomly (correlated) generated, and realistic larger scale and practical problems for the single location case, covering an appropriate range of values for the number of items J and the number of equipments M to study the effects on the average proportion of equipments operational at the end of the cycle or AA_S, an important alternate and more appropriate measure of system performance. We also develop a simple heuristic to estimate AA_S within ± 1% when compared to simulated values for AA_S while taking into account part failure dependencies.

Concluding remarks are in chapter 9 and include recommendations for possible implementation and further areas of study for similar organizations.

1.3 CURRENT MILITARY MODEL

1.3.1 General. When the Canadian Armed Forces (Land element) decide to purchase a fleet of M identical vehicles or weapon systems, a series of procedures are set in motion which ultimately lead to the selection of a contractor and the signature of a contract to acquire these capital assets, which will eventually be distributed to several self-contained field units, along with the appropriate technical documentation support for operating and maintaining the equipments with replacement, repair and overhaul of spares.

A portion of the total budget, under the responsibility of a project manager, is allocated to the purchase and distribution of selected spare assemblies and related components to all the units and bases receiving the equipments. Since each unit has its own role and organization, each one may receive a different number of equipments, thus requiring different levels of spares to support the equipments.

All first line units are required to hold in inventory a minimum of 15 days supply of spares as defined in Canadian Forces Administration Orders, in order to complete their assigned mission, which may vary depending on whether it is an infantry, armoured, artillery unit, or engineering squadron,... Periodically, some units are deployed for various periods other than 15 days, which can be up to 6 months or longer when they are assigned as part of a United Nations force, or when they deploy for simulated training operations for various lengths of time and budgets for fuel, spares, rations,.. must be planned and budgeted accordingly.

Second line field units such as service batallions must also carry a minimum of 15 days spares to support their own equipments and 30 days spares to support first line field units when they deploy together, which is not always the case; all first and second line spares are calculated based on a modified Poisson formula up to 99.8% confidence level for each item, as described further in the next section. Bases provide further support by holding various amounts of spares based on 4 months mean demands and depot stocks based on 23 months mean demands and constitute "pipeline" stocks, used for replenishment during normal peacetime operations. Overhaul quantity applicable for some types of equipments only, may also be calculated based on 10% of the total number of equipments. Wartime spares are based on the AQ system as well and are handled by simply multiplying Poisson mean failure demands in peacetime by a factor to be determined as the situation may dictate.

Immediately following the signature of the major contract, a procedure known as the initial provisioning sequence is initiated, whereby spare assemblies and/or components are earmarked for eventual purchase, and distribution to all the first and second line field units, bases and depots. The total amount of money required for spares for all first line units (and other pipeline stocks) depends on the number of equipments each one holds, individual item costs and failure rates, and is not known until the calculations for each spare and each unit is executed and then tabulated during the initial provisioning.

The number of equipments each unit will receive is known (fixed), and costs and failure rates are estimated based on prior experience from similar equipments, preliminary cost and test data provided by the manufacturer or independent reliability tests performed on prototypes, and the procedure to determine stock levels for all spares j, j=1,..,J is based on the "AQ" or assessed quantity system and the related scaling model, both of which will be descrided in more details in the next section.

Once the calculations of the stock level S for item j have been done based on the scaling model, the same type of calculations are then executed for all other spares j=1,..J for that unit and the same process is repeated for each of the remaining units (including pipeline stocks such as second line field units, base and depot stocks.

The total costs of all spares C_S to be distributed at all first line units (and other pipeline stocks) are then tabulated; if there is not enough budget available to buy all the spares, the scaling quantities or S_j values j=1,..,J are reduced either proportionately, most expensive items first, pipeline stocks first, or any other selection method or combination devised by the project manager or his representative, a process that is not standardized and the consequences of which remain unknown.

Once the quantities have been finalized and eventually meet the budget requirements, all spares are purchased and distributed to all the appropriate locations, along with a scaling document listing all applicable spare items within the equipment. The published document includes the AQ quantity assigned for each item and the same standard look-up tables of the scaling quantities S for fixed time equivalent usage period of U=15 and U=30 days for various values of AQ (from 1 to 100 in increments of 5) and for various M values (from 5 to 120, also in increments of 5). The inventory level or S_j values can be found at the intersection of any combination of the AQ and M values applicable to the unit.

Since the computer program to determine stock levels for all first and second line field units, bases and depots processes one item at a time, irrespective of its cost, there is no mechanism set in place to control the total costs of all spares and, as a result, require adhoc procedures to lower stock levels when the total budget is exceeded. Furthermore, there is no aggregate system performance against which we can measure the effectiveness of the calculation of spares for first line units.

Therefore, the objective will be two-fold:

1. to present alternative models to calculate spare levels for all first line units during the initial provisioning phase to ensure a fixed available budget is not exceeded and;

2. to enable each first line unit to optimally calculate appropriate system performance measure quickly and efficiently for any specified cycle period.

Note that the thesis does not analyze the effect of pipeline stocks, as they are currently not used as a measure of system performance and first line units are required to hold at least 15 days worth of spares, without the possible impact of such pipeline stocks. Furthermore, that requirement becomes evident when a first line unit deploys and conducts simulated training exercises, often by themselves, and are without the benefit of periodic replenishment during the period, which may also vary in length other than 15 days.

We now describe the AQ system and the related military scaling model to achieve the 99.8% availability measure for each item, discuss its relevancy and appropriate system performance measures.

1.3.2 The AQ system. The AQ system was originally documented by [Gibson 1976] and intended to calculate all the necessary individual scaled quantities S_j, $j=1,..,J$ for every first and second line units, bases and depot stocks and determine the total costs required to support the equipments; it was also designed to provide standard look-up tables published in documents from which individual units can determine the scaled quantity of each spare S_j it should hold in stock during a specified (fixed) period of either 15 or 30 days, and based on the number of equipments it will receive.

The AQ system assigns for each earmarked spare, a required average quantity to support M=100 equipments operating for U=1 year's usage at mid-life; it assumes an exponential distribution for the mean time between failures or MTBF of all items, and considers each item equally vital to the operation of the equipment, whether it is an engine or a component, and whether that component may be an "expendable" type of item such as a nut, bolt or screw. The Provisional Parts Breakdown or PPB sequence of parts provided by the manufacturer may be used (but is often unreliable) in such cases to annotate indenture levels to parts that relate them to their next higher assembly, in a similar manner than the Bill of Material is used to make up a Material Requirement Planning or MRP.

As a result of the assumption of exponential failure times, each AQ is a Poisson process with mean parameter based on M=100 equipments, and U=1 year's usage; for example an item j whose estimated failure rate is τ_j = 1 failure/20,000 kilometers (or conversely, an $MTBF_j$ = 20,000 kms), which is required to operate for U=10,000 km/year will yield an AQ = $M \times U \times \tau$ = 100 x 10,000 km x 1/20,000 = 50, and becomes the Poisson mean parameter δ_j for that part. The AQ system is thus a system that assigns a mean Poisson parameter for M=100 equipments. From this AQ value, the scaling quantity S is obtained by finding the number of spares S required to achieve an "availability" of at least .998 or 99.8%, irrespective of its cost, and using the modified Poisson model described below:

$$A(S) = \sum_{x=0}^{S} \delta^{x} \frac{\exp(-\delta x)}{x!} + \sum_{x=S+1}^{\infty} \left(\frac{M-1}{M}\right)^{x-S} \delta^{x} \frac{\exp(-\delta x)}{x!} \qquad (1.1)$$

$$\text{where } \delta = \frac{AQ \times M}{100 \times 24}$$

For example, if the AQ value for an item is 50 (which is for 100 equipments and U=1 year's expected usage), a first line (U=15 days or 1 year/24) unit having M=36 equipments will result in a mean δ = (50 x 36) / (100 x 24) = 0.75; then the required scaled quantity S is computed from equation (1.1) above until its availability exceeds 99.8% or S = 2 in this case.

1.3.3 Comments. We first note that the model optimizes an availability measure set arbitrarily high at 99.8%, is based for an individual item whose importance to the successful operation of the equipment is the same as the next, and does not take into account its cost. As originally documented by [Gibson 1976], the availability measure was set very high so that the probability of any equipment being unoperational due to a lack of spares, considering that there may be several items in series, would be reasonable, but does not include further analysis. The model is equivalent to equalizing service levels across all items regardless of their costs; thus, it is possible to achieve much higher system availability (depending on how it is measured) for the same budget.

As to the model itself, the first summation in the above equation is the straight cumulative Poisson and the second summation is the probability that the (S+1)st failure does not occur on the k th equipment which is (M-1)/M nor that the (S+2)nd failure occur on the k th equipment which is $((M-1)/M)^2$ and so on; thus the modified Poisson model is the probability that the k th equipment does not fail during the period due to a lack of spares.

[Vincent 1982] has later published a paper commenting on the scaling model by Gibson; he first points out the error in the second summation by replacing the term $((M-1)/M)^{x-S}$ in equation (1.1) above with another expression for successive failures exceeding S, which take into account the decrease in the number of operational equipments, and is given by (M-1)/M for the (S+1)st failure, (M-1)/M x (M-2)/(M-1) for the (S+2)nd failure, and so on... In general, if there are x > S failures, the number of ways the x-S failures do not affect the k th equipment is $_{(M-1)}C_{(x-S)}$ and the total number of ways they can occur is $_{M}C_{(x-S)}$; therefore the probability that the x-S failures cause equipments to be DOWN (due to lack of spares) other than on the k th equipment will be $_{(M-1)}C_{(x-S)} \div {_{M}C_{(x-S)}}$, which reduces to (M+S-x/M) and is valid only until the (M+S)th failure since at this failure, the k th equipment will definitely be DOWN due to lack of spares. The second summation term in equation (1.1) above should thus be revised and the model becomes as follows:

$$A(S) = \sum_{x=0}^{S} \delta^x \frac{\exp(-\delta x)}{x!} + \sum_{x=S+1}^{M+S-1} \frac{(M+S-x)}{M} \delta^x \frac{\exp(-\delta x)}{x!} \qquad (1.2)$$

The author correctly points out that this correction factor makes little difference at such a high level of assurance of 99.8%. More importantly, he questions the usefulness of the modified Poisson model in its current form and suggests dropping the second summation term in either (1.1) or (1.2) since the performance measure is related to the probability of a specific equipment, say the k th equipment failing before the end of the period due to a lack of spares, which is not meaningful to unit commanders and unnecessarily provides higher S values, as compared to the straight cumulative Poisson given by the first summation in either equation.

A more useful expression for the probability of a subset of the M equipments not being in a failed state at the end of the period due to a lack of spares has also been included but the expression derived assumes there is only one type of spare. We will extend the analysis about this important measure of performance and discuss it later in chapter 6.

More recently, [Hebert 1995] has pointed out that for parameter values of interest (AQ and M), a specified 99.8% availability level used to calculate S values would yield corresponding cumulative Poisson availability values ranging approximately from 86% to 99%. For example, if $\delta = 5$ and $M = 10$, then using equation (1.1) to a .998 availability yields $S = 11$, and would result in an availability of .994 or 99.4% if the cumulative Poisson is used; but for higher reliability parts, such as $\delta = .75$ and $M = 36$, the modified Poisson up to .998 availability yields $S = 2$ and with $S = 2$ corresponds to a cumulative Poisson availability of .959 or 95.9%.

In his document, Hebert also provides a comparative analysis with another model [Shapelavey and Mickel 1989] to calculate spares required at the unit level, for possible various usage periods and easy implementation by units on microcomputers. The model uses an approximation to the Poisson, however, and the performance measure used remains at the individual item level, and therefore will not be pursued any further.

1.3.4 Applicability of the model. Today, the current model still uses the original modified Poisson model as given by equation (1.1) to determine all item stockage levels for all first line units in the

Canadian Armed Forces (Land element). Since the performance measure used is calculated for individual items, system performance and total costs are unknown at the time of procurement until all calculations have been done, creating additional problems if the budget has been exceeded. Furthermore, procedures and modeling at unit level have not yet been devised to link all items together and provide for meaningful models that can be implemented for various usage periods and budgeted or costed accordingly. Furthermore, the level of system performance measures that can be compared with other models is very restricted, unless the second summation is dropped, as described earlier.

We intend to develop and demonstrate the usefulness of other such models which could also be widely applicable to any similar organization that operates a set of identical equipments and do not use an aggregate performance measure. For example, municipalities that operate a fleet of buses, airlines that fly regional jets, manufacturers with numerically controlled machines; we shall demonstrate that it is possible to achieve significantly higher system availability, higher average number (and proportion) of equipments operational at the end of the period, and lower total expected system backorders for any specified available budget level, when compared to the current military model, which we will refer to as the equal A_j model.

1.4 MEASURES OF SYSTEM PERFORMANCE

In particular, the objective is to determine how many of each type of assemblies or items to purchase or the order-up-to quantity vector $\{S\} = \{S_j, j=1,2,..,J\}$ at the beginning of each cycle period either to 1) maximize system availability A_S and/or 2) minimize total expected system backorders E(BO) which will be denoted BO, and subject to a fixed single investment budgetary constraint consisting of purchasing costs.

The models can thus be considered a trade-off between product operational requirements and the total purchasing costs. The thesis will provide a comprehensive analysis of solution methods to optimally (or near optimally) solve this problem, point out the computational difficulties associated with each one as well as provide appropriate error bounds for near optimal solution methods.

Maximizing system availability A_S links all items together through the multiplication of individual item availabilities A_j's, j=1,..,J and provides a convenient and practical performance measure for managers to optimally allocate a limited budget during a fixed time period among various assemblies (or to components of an assembly) so as to maximize the probability of not running out of any one of them or the probability that all equipments will not be "down" due to a lack of any spare. It is equivalent to completing the mission with all its equipments still functional at the end of the period, assuming mission reliability is only related to spares availability.

Maximizing the expected number (and proportion) of equipments operational at the end of the period AA_S would be another valid and more appropriate measure of system performance than A_S; as we shall demonstrate, however, its distribution is complex and mathematically untractable for even moderate combination values for the number of items J and the total number of equipments M that typical organizational units have ($10 \leq J \leq 50$ and $1 \leq M \leq 20$), except for the special case M=1 (and regardless of J), where an exact value can be obtained. Given a stock level vector $\{S_j, j=1,..,J\}$, we can use simulation methodology to obtain accurate values of this important system performance measure, and will be treated in chapter 6.

As we have already pointed out, several different organizations such as in the military, industrial or service sectors are continually faced with that type of decision on a seasonal, quarterly or annual basis; budget allocations for spares of identical equipments (vehicles, airplanes, buses, equipment machinery,..) from one period to the next may either substantially decrease while the same number of equipments are kept operational, or increase significantly due to new capital acquisitions programs, or widely fluctuate due to the number of replacements or overhauls of older equipments.

In the Canadian Armed Forces (Land element), the importance of providing solution methods that link items together with an aggregate performance measure and are quick, easily implemented (specially on microcomputers, see [DesRochers 1984], [Shepelavey and Mickel 1989], [Hebert 1995]) and interpreted correctly at individual locations or in multiple locations types of situations where transshipments may not be possible or allowed, cannot be overemphasized. However, true optimal solutions can still be elusive for this type of non-linear integer optimization problem as we shall soon demonstrate, and the need to develop, calculate and compare error bounds for near optimal solutions will form an important part of this thesis.

Similarly, minimizing the total expected system backorders BO links all items together through the sum of individual backorders for each type of assembly BO_j's, j=1,..,J and provides another convenient and practical performance measure to optimally allocate a limited budget during a fixed time period among several different types of assemblies. The total expected system backorders measures the average number of stockouts expected during the period, whether they are due to a starter or a radiator is irrelevant; for example, if on average, we expect to run out of 1.2 starters and 1.8 radiators during the period, then the total expected BO = 3 (on average) during the period, which also means that if I had M=10 equipments at the beginning of the period, I can expect to have 7 equipments still operational at the end of the period and 3 equipments in a failed state due to lack of spares (either starters or radiators). It also provides an alternate performance measure whose optimal (or near optimal) stockage level solution vector $\{S_j,\ j=1,..,J\}$ is strikingly similar and practically the same (not always equal but slightly different) as the solution vector $\{S_j,\ j=1,..,J\}$ obtained when maximizing A_S. Since all stockouts will occur towards the end of the cycle period, the impact of measuring system BO as a function of time is considered negligible and will not be analyzed here.

Although both performance measures A_S and BO practically yield the same results, managerial interpretation can be quite different. For example, the same optimal stockage level solution vector $\{S_j\}$ resulting in A_S=.70 may appear quite low when maximizing A_S but if the total number of equipments is M=10, it may yield an average of 9 out of 10 equipments (.90 proportion) still operational at the end of the period, which can be more than acceptable, and may also yield correspondingly very low values for total expected system backorders (say BO = .10 for example).

We shall thus develop alternative models with system performance measures A_S and/or BO to be optimized for any specified budget level, and simulate the corresponding AA_S values. We will then be able to compare their values with A_S and corresponding simulated AA_S values obtained as a result of solving the Equal item availability at the same specified budget level. Since all models are solved starting from $0, the response curves $\{A_S,C_S\}$ also gives valuable information to managers by quickly showing the improvement in system performance.

CHAPTER 2: PROBLEM FORMULATION

2.1 GENERAL

Consider first a single location system having several identical equipments (vehicles, airplanes,..) each one of which operates for a fixed period of time t. Each identical equipment consists of several replaceable assemblies (simply referred to as components or items) such as engine, transmission, battery,.. and that failure of any one of the j items (j=1 to J) within the equipment occurs independently and will result in a failed or unavailable equipment until the end of the cycle period of fixed length t if no spare of the failed item is available.

Therefore, it is assumed that each item life has exponential density $f_j(x)$ with constant failure rate τ_j and thus the failure rate process for multiple identical items is a Poisson process. When any of the possible J items within an equipment fails, it is immediately replaced by a spare item, if one is available, from the inventory, otherwise the equipment remains DOWN or in a failed state until the end of the period.

With this type of inventory system, the objective is to determine the optimal number of spare items S_j of each type j=1,..,J to stock at the beginning of each cycle time period of fixed duration t (or equivalently for a usage period of U days, months, hours,..) in order to ensure that an adequate system service level is provided during the whole time period subject to a fixed investment budget constraint made up of purchasing costs.

As a result of formulating and analyzing related models whose objective functions to be optimized will link all items together through relevant system performance measures and subject to a fixed available amount of money (budget), organizations such as the military that implement these models will be able to effectively allocate restricted resources in an optimal way at the time of procurement while providing user sub-units (or clients) with increased awareness and flexibility in planning inventory levels.

2.2 NOTATION

The index j (j=1 to J) represents the number of items making up an equipment and the index i (i=1 to I) represents the number of different locations where identical equipments are operational; the index i is deferred until it is introduced in a later chapter when the multiple location case will be analyzed; so, for the single location models, the following notation will be used:

- t = fixed cycle period (say 1 year)
- U = expected usage (miles, hours,..) of each identical equipment during period t
- M = number of identical equipments operating at the location
- B = fixed available budget ($) during period t
- R = constant and common interest rate for inventory holding costs H ($/$/time period)
- j = index of part (item) types (j=1,2,..,J)
- f_j = time-to-failure density (pdf) of item j
- τ_j = item or component failure rate (fr/time)
- p_j = Poisson probability mass function (pmf)
- δ_j = Poisson rate parameter for a fixed time period
- F_j = Cumulative distribution function (cdf)
- P_j = Complementary cdf = $1-F_j$
- $h_j = c_jR$ = inventory holding costs for item j
- $H = \Sigma h_j$ = total inventory holding costs ($)
- c_j = purchase cost for item j ($/item)
- $C = \Sigma c_jS_j$ = total purchase costs ($)
- C_s = total systems costs = (C+H)
- S_j = up-to inventory level for item j at the beginning of the period
- $\{S\}=\{S_j, j=1,2,..,J\}$ = inventory quantity vector of all items at the beginning of the period
- BO_j= expected number of backorders for item j

BO = total expected system backorders = $\Sigma\ BO_j$

A_j = availability for item j

A_S = system availability = $\pi\ A_j$

i = index for number of locations (i=1,2,..,I)

As an example, consider an item (within an equipment) whose failure rate τ_j is assumed constant and τ_j = 1 failure/100,000 kilometers. Within a specific location, M=15 identical equipments are each expected to operate for U=10,000 kilometers during t=1 year; then the number of failures for that item during the period will be Poisson distributed with expected parameter value δ_j = 15 x 10,000 x 1/100,000 = 1.5 failures or $\delta_j = MU\tau_j$. We will refer from now on to the failure (demand) process for each item j as Poisson with mean rate $\{\delta\} = \{\delta_j, j=1,2,..,J\}$.

2.3 MODEL FORMULATION

2.3.1 Model P1 to maximize A_S. If S_j items of type j are available (held) in inventory at the beginning of the period, then a measure of the service level provided for any item j is its availability or the probability of not running out of stock which is defined by its cumulative Poisson probability density function as:

$$A_j = \sum_{x=0}^{S_j} p_j(x,\delta_j) = F_j \tag{2.1}$$

where $p_j(x,\delta_j)$ is Poisson (rate δ_j)

or $p_j = \delta_j^x.\exp(-\delta_j)/x!$

$$A_j^C = 1-A_j = \sum_{x=S_j+1}^{\infty} p_j(x,\delta_j) \quad \text{(Unavailability)} \tag{2.2}$$

The aggregate performance measure that links all items together by their equal contribution to the equipment's operational effectiveness is the system availability or the joint probability of not running out of stock of any type of item given by:

$$A_S = \prod_{j=1}^{J} A_j = \prod_{j=1}^{J} \left(\sum_{x=0}^{S_j} p_j(x) \right) \tag{2.3}$$

Since this measures the probability of not running out of any type of item j, j=1,..,J, it can also be interpreted as the mission reliability or the probability that all M equipments will remain operational (not "down") until the end of the cycle period, if it is critical that we do not run out of any type of spare j during the period, regardless of the number of equipments at the beginning of the cycle. This measure of system performance will make up the objective function to be optimized (maximized) in model P1, formulated below.

If we only have M=1 equipment available at the beginning of the period and we do run out of any one type of spares, then it will remain in a failed state until the end of the cycle. If multiple equipments are involved, then the expected proportion of the M equipments still operational at the end of the cycle, denoted AA_S and defined as the expected number of equipments still operational ÷ M, may be a more meaningful and appropriate measure of system performance; its distribution is complex and will be analyzed in a later chapter.

The objective sought by the inventory policy adopted is to order at the beginning of the cycle period t enough of each type of item or order-up-to quantity vector $\{S\} = \{S_j, j=1,2,..,J\}$ in order to maximize system availability A_S while not exceeding a fixed available investment budget B. Model P1 can thus be formulated as follows:

$$\text{Max} \quad A_S = \prod_{j=1}^{J} A_j = \prod_{j=1}^{J} \left(\sum_{x=0}^{S_j} p_j(x) \right) \tag{P1}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \leq B \quad \text{(Budget)} \tag{2.4}$$

$$S_j \geq 0 \qquad j=1,2,..,J \tag{2.5}$$

Switching the objective function and the constraint yields the following equivalent model variant P1a below:

$$\text{Min} \qquad C_S = \sum_{j=1}^{J} c_j S_j \qquad \text{(P1a)}$$

$$\text{s.t.} \qquad A_S = \prod_{j=1}^{J} A_j = \prod_{j=1}^{J} \left(\sum_{x=0}^{S_j} p_j(x) \right) \geq \alpha \qquad (2.6)$$

$$S_j \geq 0 \qquad j=1,2,..,J \qquad (2.7)$$

Marginal analysis will be used in a later chapter to demonstrate that the model variant P1a i.e. minimizing total costs $C_S(.)$ and subject to a minimum specified system availability $A_S \geq \alpha$, will yield the same optimal stockage level vectors $\{S_j,\ j=1,..,J\}$ and the same response curve $\{A_S, C_S(.)\}$ as model P1, from which managers can select the optimal stockage level vector $\{S_j,\ j=1,..,J\}$ corresponding to a desired value on the response curve.

The managerial interpretation between models P1 and P1a are different: model P1 seeks to maximize A_S subject to total available costs and is usually encountered in organizations where budgets allocated are limited and/or difficult to predict, such as large military and public organizations whereby service levels are not the main concern; managers are thus faced with the decision to obtain maximum efficiency from the allocated budget and is practically impossible to secure additional funds to meet a service level, and sometimes are faced with downsizing or reduced budgets.

Model P1a, however shifts the emphasis on meeting a minimum service level such as system availability A_S here, regardless of the costs and would usually be of primary concern to organizations facing tough competition within the same industry: the industrial and service sectors are prime examples of the applications of this type of model, along with model P2a discussed below. Managers are forced to meet the service level objective at minimum costs.

2.3.2 Model P2 to minimize BO. A service level frequently used in the performance of an inventory system is called 'part-fill' criterion or item shortages, often called the expected number of backorders = BO_j for any item j, given by equation (2.8) below, and is not dependent on time; their duration is ignored since for all practical purposes, they will only be incurred at the end of the period as described and shown in figure 2.1 below:

$$BO_j = \sum_{x=S_j+1}^{\infty} (x-S_j) . p_j(x) \tag{2.8}$$

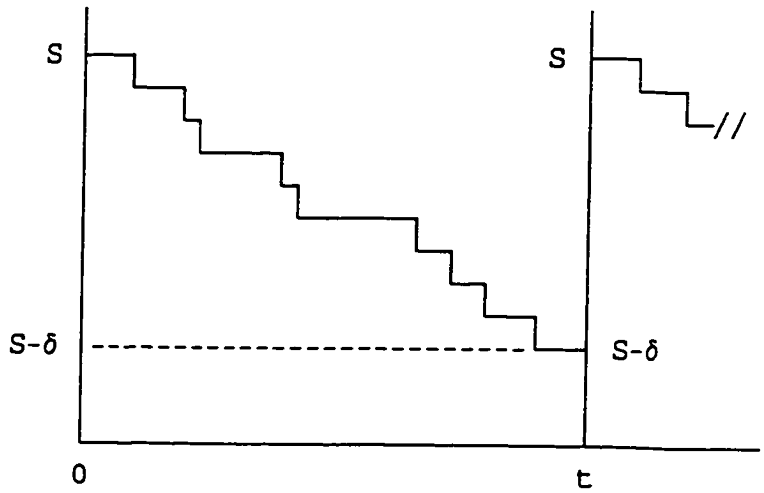


Figure 2.1: Inventory level vs time

Again, the aggregate service level linking all items together can be measured by the total expected system backorders BO, by summing all individual item backorders as:

$$BO = \sum_{j=1}^{J} BO_j = \sum_{j=1}^{J} \sum_{x=S_j+1}^{\infty} (x-S_j) . p_j(x) \tag{2.9}$$

and will make up the the objective function to be optimized (minimized) in model P2, formulated as follows:

$$\text{Min} \quad BO = \sum_{j=1}^{J} (BO_j) = \sum_{j=1}^{J} \sum_{x=S_j+1}^{\infty} (x-S_j).p_j(x) \qquad (P2)$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \leq B \text{ (Budget)} \qquad (2.10)$$

$$S_j \geq 0 \qquad j=1,2,..,J \qquad (2.11)$$

Switching the objective function and the constraint yields the following equivalent model variant P2a below:

$$\text{Min} \quad CS = \sum_{j=1}^{J} c_j S_j \qquad (P2a)$$

$$\text{s.t.} \quad BO = \sum_{j=1}^{J} (BO_j) = \sum_{j=1}^{J} \sum_{x=S_j+1}^{\infty} (x-S_j).p_j(x) \leq ß \qquad (2.12)$$

$$S_j \geq 0 \qquad j=1,2,..,J \qquad (2.13)$$

Marginal analysis will be used in a later chapter to demonstrate that the model variant P2a i.e. minimizing total costs $C_S(.)$ and subject to a maximum specified expected number of backorders $BO \geq ß$, will yield the same optimal stockage level vectors $\{S_j, j=1,..,J\}$ and the same response curve $\{BO, C_S(.)\}$ as model P2, from which managers can select the optimal stockage level vector $\{S_j, j=1,..,J\}$ corresponding to a desired value on the response curve.

The managerial interpretation between both models P2 and P2a is essentially the same as the one discussed for models P1 and P1a; the system performance measure used here is the total expected system backorders BO obtained as a result of summing all individual item backorders BO_j, $j=1,..,J$, whereby each item is equally crucial to the operation of the equipment.

2.4 ASSUMPTIONS FOR MODELS P1 AND P2.

2.4.1 <u>Model assumptions</u>. Assumptions for both models P1 and P2 are as follows:

A1. All equipments at a location are identical and operate independently.

A2. Each equipment consists of J single types of items each one of which is required to operate for the equipment to remain operational.

A3. Each item j (j=1,2,..,J) is subject to random failure (constant failure rate τ_j).

A4. Any item that fails is replaced immediately by an identical item if one is available from the inventory; otherwise, the equipment remains in a failed state until the end of the period t.

A5. At the beginning of the period, a stockage level vector $\{S\}=\{S_j\ j=1,2,..,J\}$ is purchased with no possibility of resupply until the end of the period.

A6. A fixed budget B is available to cover the purchasing costs for all items j until the end of the period.

A7. There is no ordering cost for any item, or if there is, the order cost/item is fixed and can be included in the purchase cost c_j.

A8. Failure of an equipment is caused by only one item.

A9. Instantaneous delivery.

Because failures of each item occur independently and at a constant rate, the times between failures are assumed to be exponential and the number of failures in a fixed interval of time [0,t] are Poisson distributed with rate δ_j. The consequence of assumption (A8) is that each failure results in a single demand for that item and demands occur independently of one another.

This one-period review model of length t is concerned with the determination of an order-up-to quantity vector $\{S\}=\{S_j\ j=1,2,..,J\}$ to last the whole period without possibility of resupply during the period. Instantaneous delivery ensures that the quantity ordered prior to the end of the period is delivered so that each period starts with a constant inventory level vector $\{S\} = \{S_j\ j=1,2,..,J\}$ for each type of item, therefore stochastic lead times are not considered here. As shown in figure 2.1 earlier, the inventory level for one specific item as a function of time starting with S_j identical items, and is equivalent to a periodic review inventory system where the cycle repeats itself after each period t.

2.4.2 Equivalent system availability EA_S. Since we assume that the number of failures for each part type $j \epsilon J$ in a cycle of fixed length t as illustrated in figure 2.1 is Poisson distributed with parameter $\delta_j = \tau_j t$, then the times between failures is exponential with parameter $(1/\tau_j)$. Thus, if S_j items type j are stocked at the beginning of the period, the distribution of the time at which the $(S_j+1)^{th}$ failure occurs is the sum of exponential variates $(1/\tau_j)$ and known to be Gamma distributed with parameters $(S_j+1, 1/\tau_j)$, also called the m-Erlang distribution since S_j+1 is an integer.

From the definition of system availability A_S defined earlier by the product of individual item availability A_j, given S_j parts type j are available at the beginning of the period, then an equivalent definition of system availability performance measure, denoted EAs, can be defined as:

$$EAs|\{S\} = \text{Prob}\ \{\underset{j \epsilon J}{\text{Min}}\ \text{Gamma}_j\ (S_j+1, 1/\tau_j) > t\} \qquad (2.14)$$

In other words, if we were given an initial inventory quantity vector $\{S\} = \{S_j,\ j=1,2,..,J\}$ for each type of part at the beginning of the period, we would start running out of parts at the time the first of the $(S_j+1)^{th}$ failure occurs. This can be equated to a reliability system of J sub-systems in series, each one of which consists of S_j available redundant parts failing at an exponential rate of τ_j.

The distribution of the minimum of the Gamma_j variates approaches asymptotically a Weibull distribution but its parameters can not be derived analytically (See [Hahn and Shapiro 1967] for a discussion on the distribution of various minimums). We could solve this problem, however, by simulating the system for N cycles for various inventory stockage level vectors {S}, estimate EAs with equation (2.15), that is the proportion of the N cycles that the system survived past t without running out of any parts. Unfortunately, repeating this methodology for every stockage level vector {S} in order to construct the entire response curve $\{A_S, B\}$ to find the best stockage level vector {S} that will maximize P1 for a given budget B falls in the category of combinatorial analysis, would be analytically time consuming and therefore not pursued.

Nevertheless, a program was developed to simulate N cycles of this type of inventory system (see chapter 9) to compare and discuss the empirical results with those obtained with exact and more efficient methods to be presented in subsequent chapters. Furthermore, the simulation methodology provides us with an opportunity to analyze the effects of part failure dependencies, discussed in the next section, and to introduce another more appropriate measure of performance, denoted AA_S or alternate system availability.

2.4.3 <u>Alternate system availability AA_S</u>. Noting that both performance measures used in models P1 and P2 put equal emphasis on each type of part, the optimal stockage level vector $\{S\} = \{S_j, j=1,2,..,J\}$ for each model will usually be similar. Furthermore, the product of individual availability service measure is a conservative estimate and therefore a lower bound on the exact availability achieved (system performance service level) since a part that runs out of inventory before the end of the cycle will have the effect of decreasing the expected number of failures for other parts as the number of equipments operating towards the end of the period decrease; this end of cycle phenomenon is illustrated in figure 2.2a and 2.2b below for two different parts. [Ernst and Pyke 1992] have discussed the end of cycle effects in more detail and verified with simulation results that increasing inventory for one part will benefit product service level only up to a certain extent and tend to equalize inventories across all items individual service level.

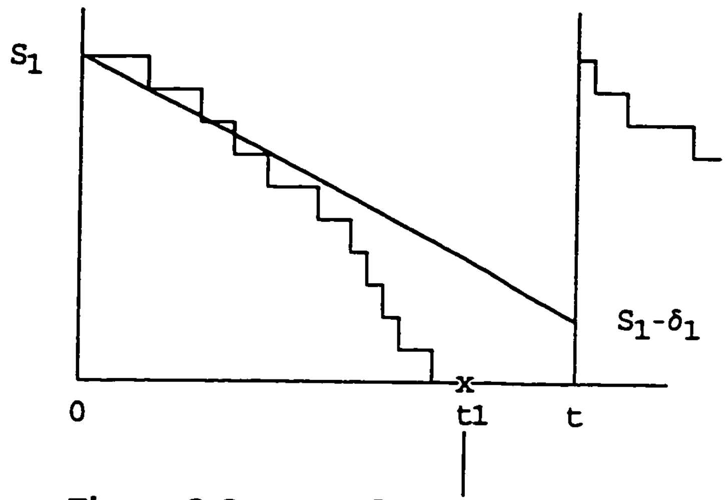


Figure 2.2a: Actual Inventory - Part 1

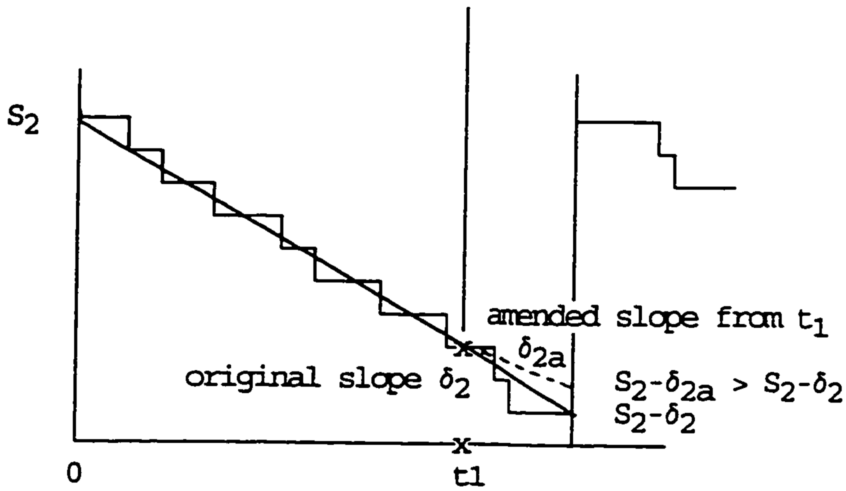


Figure 2.2b: Actual Inventory - Part 2

The system availability aggregate performance measure A_S and its equivalent counterpart EAs defined in the earlier section with the Gamma distribution, is the probability of not running out of any item during the time period. The measures of performance A_S or EA_S then provide us with a lower bound LB (and therefore conservative measures) on the true proportion (or number) of equipments still operational at the end of the cycle, denoted AA_S, since part failure dependencies cause the expected number of failures (Poisson parameters $\{\delta_j, j=1,..,J\}$) for parts to actually decrease towards the end of the cycle.

This alternative measure of system availability to model part dependencies or AA_S, and defined as the expected proportion of equipments that are still operational at the end of the cycle period when $\{S\} = \{S_j \; j=1,2,..,J\}$ spare items are available at the beginning of the cycle, will be shown in a later chapter with simulation methodology, to be a more appropriate measure of actual system performance when multiple equipments are involved.

2.5 LITERATURE REVIEW

The constrained multi-item inventory models P1 and P2 introduced above have been treated in the literature in various forms and is model dependent. Surprisingly, the system availability performance measure has seldom been analyzed and has been discussed mainly in the context of least cost allocation of redundant units in parallel to improve the system's "availability" during a specific time period. Also, the budget or investment constraint consisting of purchasing costs has not been extensively used other than in multiple-item "Newsboy" types of problems where single objective functions trading off costs or profits vs shortage costs are analyzed.

Several practical applications in solving this type of problem can be implemented effectively in different types of organizations. For example, military organizations and public institutions traditionally include purchasing costs only and do not include inventory holding costs in their formulations although they are presumably "accounted for" in some other logistic support organizations (supply, warehousing,.). Due to the on-going major restructuring and downsizing by the defense industry and government organizations, both types of costs should be included in the analysis and provide for better accountability as to the real costs incurred, an extension to both models P1 and P2 that include inventory holding costs, will be analyzed in a later chapter.

Another practical application is in the retail industry where inventory holding costs are a significant factor when planning for the purchase of a range of similar items on a seasonal or annual basis. From a managerial point of view, if only a fixed investment budget is available for a period, optimizing a service level objective such as maximizing A_S in model P1 subject to a fixed budget is clearly not the same as trying to minimize costs subject to a minimum A_S or simply trying to maximize profits as commonly found in the literature. Therefore, the objective function of model P1 or P2 may be much more appropriate than minimizing total costs subject to a specified minimum (maximum) service level.

The reason for describing the different systems below is because most of them focus on solution procedures that are similar to the ones that will be used throughout this analysis. Because they are also model dependent, necessary derivations are carried out for the 2 models of interest, as related to the Poisson distribution, and major considerations and key factors to consider for selection of various specific solution procedures will be studied and compared.

If the function to be optimized is linear and the single constraint is also linear, the problem is commonly referred to as a distribution of effort or knapsack problem. It attempts to determine the optimal number (0-1 variables or multiple items) of each type of items to include in a knapsack if a benefit w_j is obtained for each item j; it is formulated in [Wagner 1975a] as follows:

$$\text{Max } Z = \sum_{j=1}^{J} w_j x_j \tag{P3}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j x_j \le B \quad \text{(budget or volume)} \tag{2.15}$$

$$x_j = 0,1 \text{ or } \ge 0 \text{ and integer} \tag{2.16}$$

where w_j = return/benefit of item j

and x_j = number of items type j to be included in the knapsack (usually 0-1 variables).

Other practical applications often encountered include the fly-away or tool-kit problem discussed by [Geisler and Karr 1956], [Hadley and Whitin 1963] and the submarine spare parts provisioning problem also discussed in [Hadley and Whitin 1963] and [Silver and Peterson 1985]. Solution procedures usually involve branch and bound ([Wagner 1975a]), dynamic programming ([Winston 1994] and [Hadley and Whitin 1963]) and network analysis ([Winston 1994], [Lawler 1976] and Wagner 1975b]). For both models P1 and P2 treated here, multiple values other than 0,1 for the variables are required and will be further discussed below.

Variants of this problem usually attempt to optimize a cost function or the expected number of part shortages (i.e. backorders BO) when subjected to a total investment constraint or some service level measure such as "Fill rate". A typical formulation from [Hadley and Whitin 1963 pp 304-307] which seeks to minimize weighted backorders for all items is as follows:

$$\text{Min} \quad \sum_{x=S_j+1}^{\infty} L_j . (x-S_j) p_j(x) \tag{P4}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \leq B \quad \text{(budget or volume)} \tag{2.17}$$

$$S_j \geq 0 \text{ and non-negative integers} \tag{2.18}$$

$$L_j = \text{penalty cost for a backorder } j$$

Model (P4) is similar to our model of interest P2 except that all part shortages (i.e backorders) are equally crucial to the successful operation of equipments, i.e. all parts have equal penalties $L_j=1$ in (P4). Although the model to be developed is mostly concerned with the availability aggregate performance measure (model P1) which is non-linear, it will include comparison with the backorder objective function for model P2 previously described and similar to model (P4) above.

[Black and Proschan 1959] have developed a similar model to maximize system availability of a spare parts kit subject to a fixed budget but the solution procedure presented is based on an approximation of Poisson's exponential binomial limits, published in

Molina's tables for high reliability parts/ components, to determine an initial allocation S_j for part j from which the stock levels for all other parts are calculated. The result is a short-cut equivalent to the marginal analysis analyzed here in that the sequence of stock level vectors $\{S_j, j=1,..J\}$ generated are all undominated allocations but skips over several iterations of the marginal analysis procedure and yields much wider gaps in the budget, which may cause large errors from the exact solution.

The authors briefly mention but do not describe in detail the more efficient marginal analysis procedure to generate successive undominated allocations. This thesis extends the analysis by deriving and describing the procedure for all models included here, including the necessary conditions for its application, by developing error bounds and comparing the maximum possible number of iterations (order of magnitude) with DP, approximate DP and Lagrange relaxation methods. We also provide two simple and useful heuristics to further improve the solution.

[Kettelle 1962] has analyzed the system's availability objective in least cost allocation of redundancy units (in parallel) subject to an investment constraint. The system's availability measure is defined by $A_S = \pi\ [1-(1-a_j)^{nj}]$ where a_j is a fixed availability for item j and only item purchasing costs are considered. Dynamic programming and marginal improvements in availability per dollar invested are used to solve the problem.

Fox [1966] has published a widely quoted paper on the general application of the marginal analysis, stating the conditions under which it is justified and an important proof that the sequence of points generated are undominated (or efficient) when the objective function is separable by item, each one of which is concave and strictly increasing when maximizing, which is the case for model P1 here, or convex and strictly decreasing, which is also the case for model P2. It also formally established the close relationship of the marginal analysis with the Lagrange relaxation method.

The paper by Fox further states that the solution obtained depends on the spacings of the successive allocations generated until the budget B is exceeded and should be "sufficiently near" optimal for practical purposes, otherwise the exact solution must be found by DP at the expense of much more computational effort. The marginal analysis applicable to the models treated here, will be analyzed in further details in chapter 4 by describing its characteristics applicable to the Poisson distribution, including error bounds and introducing two simple heuristics to further improve the near optimal solutions.

Due to the generic nature of the procedure, several authors (briefly described below) have successfully used its results in a wide range of applications, including all the models presented here. We explore in more details the computational difficulties associated with DP and the errors that can be generated by a DP approximation procedure, derive the procedure and the error bounds (or "spacings" as described by Fox) based on cost for several other solution methods, including reducing the size of the spacings, and provide extensive numerical results on randomly generated test problems.

Another performance measure similar to the availability criterion is the "job completion" or "job fill" rate criterion which has been used in a variety of models mostly concerned with the optimal "repair kit" type of problem where solution procedures developed for our models P1 and P2 are similar.

The [Smith, Chambers and Schlifer 1980] model minimizes the cost of a repair parts kit (0-1 variables) based on the fraction of jobs completed without stockout, inventory holding costs and incurring penalty costs for each job non-completion; solution procedure involves separable programming for which marginal analysis applies.

[Graves 1982] selects a spare parts kit (0-1 variables) that has the minimum inventory costs for a specified job completion criterion. He eliminated the need to specify shortage costs which are difficult to evaluate in the Smith, Chambers and Schlifer model above which essentially results in a binary knapsack problem that can be solved with the specialized optimization procedures developed by [Balas and Zemel 1980] for large scale problems of this type.

[Mamer and Smith 1982] has extended the Smith, Chambers, Shlifer model by allowing for parts demands that are not necessarily independent and shows how it can be solved by a maximum flow/ minimum cut network algorithm. A communication by [Hausman 1982] has commented on the set of efficient points for the three (3) models described above and discussed mixed strategy solutions but may be difficult or impractical to implement.

[Brumelle and Granot 1993] examined the properties of some of the models above and incorporated the theory of lattice programming and the structure of the Pareto set of the convex hull of the 0-1 variables included in the repair kit models, resulting in some computational simplifications.

[Schaefer 1983] has developed another model based on 3 alternative job completion rate criteria (different obective functions) for the selection of repair parts in the context of equipment overhaul at periodic time intervals and allow for multiple units of each part to be stocked; solution procedures for the 3 models include a brief comparison between dynamic programming (assuming costs are integer) and marginal analysis. A key assumption in the application of the model is that Poisson mean rates are so low, i.e much smaller than 1, so that the probability of 2 or more failures for any part j during the cycle is negligible, and requires a job non-completion penalty (difficult to assess), both of which are not required here. Similar procedures to derive error bounds from the marginal analysis are developed here for all our models but we extend the analysis further by reducing the error bounds with simple heuristics.

[Cohen, Kleindorfer and Lee 1989] analyzed a similar modified version of P1 by switching the objective function with the constraint of model P1 and thus seek to minimize costs subject to a minimum system availability service level, which lead to a different managerial interpretation. They developed a general Lagrangian relaxation procedure to obtain the solution and demonstrated that it is closely related to marginal analysis (or greedy algorithm), as did [Fox 1966] earlier. Small scale test problems (with J=3,6 and 9 items) are included but are restricted to the binomial distribution and Lagrange analysis.

We extend the analysis to the Poisson distribution and comparative results for randomly generated test problems up to J=99 items for other models and include a more appropriate measure of system performance when multiple equipments are involved, with and without part failure dependencies. The top-up marginal analysis procedure is also shown to further improve the solution.

Consider a multi-indenture equipment made up of assemblies, sub-assemblies and components as depicted in figure 2.3 below. Random failure of such an equipment is caused by random failures of specific components. In this context, the amount of spare items held in inventory will affect the availability or some other suitable aggregate performance service level (SL) of complete equipments depending on its defined structure. [Audet 1986, 1984], [Bitran and Hax 1981], [Demmy 1970], [Muckstadt 1973] and [Svoronos and Zipkin 1988].

These types of systems were initially developed in the context of reliability studies where an equipment is broken down into individual repairable or throw-away type modules. If it is assumed that failure of an equipment is caused by at most 1 failure of a module (or component) and that failures occur independently, then the possibility of disaggregation of the resulting Poisson failure process becomes most important. We develop an extension to both models in a later chapter in which a 2-phase approach can effectively be applied at the assembly level first to optimize system performance, followed by optimization at component level, when more reliable information is known and it can be worthwile to disaggregate the Poisson process into smaller ones and demonstrate how this method can thus be used to increase system performance measure. This procedure has the net effect of reducing the size of the original problem to a more manageable size by successively optimizing a number of assemblies, then its sub-assemblies, and so on down to component level, similar to calculating requirement levels for components using an MRP system (Material Requirement Planning).

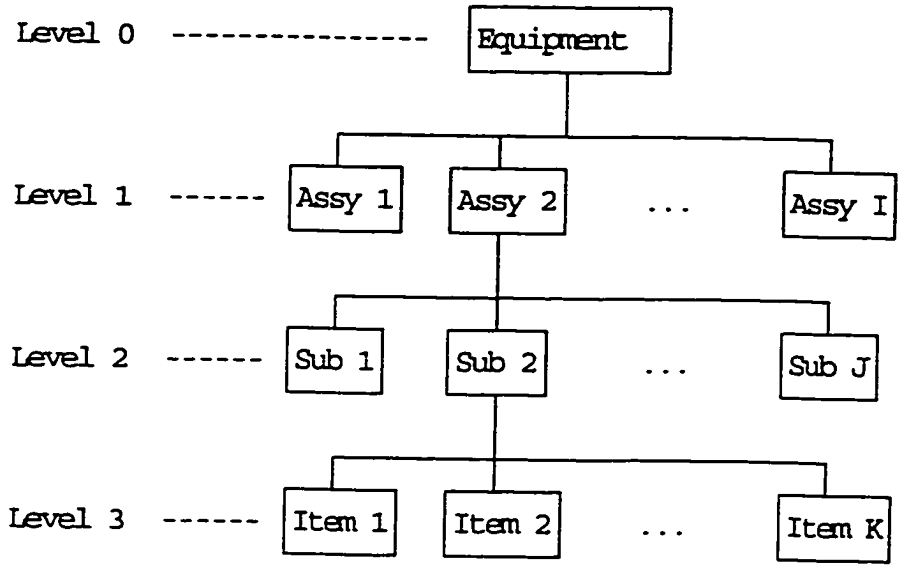


Figure 2.3: Multiple indentured equipment

Multi-location inventory systems involve failures of items which result in demands at different rates depending on the location (figure 2.4 below). Models developed in this context often refer to consumable items and retail outlets and cover a wide variety of assumptions and solution procedures [Badinelli and Schwarz 1988], [Deuermeyer and Schwarz 1981], [Eppen and Schrage 1981]. An extension to models P1 and P2 studied here will demonstrate that a problem with I locations x J items is equivalent to a single location with I x J items, under some of the assumptions of the models treated here.

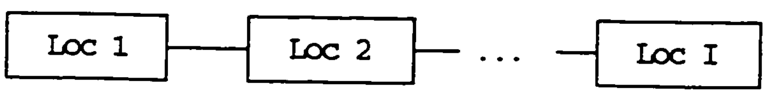


Figure 2.4: Multiple location system

Consider now a multi-location, multi-echelon organization made up of several locations each operating several identical equipments as shown in figure 2.5. When a failure occurs, the component which has caused the failure is identified and a decision as to where the repair should be done is made in accordance with the maintenance policy.

If a spare item is available locally, it is replaced immediately and the equipment is restored back to an operating condition; the failed item is then repaired locally or sent to a higher echelon and stored back on the shelf. If a spare is not available locally, then a stockout occurs and a request is sent to the central location; a spare item is shipped back if available otherwise the stockout condition remains in effect until the failed item has been repaired either locally or centrally.

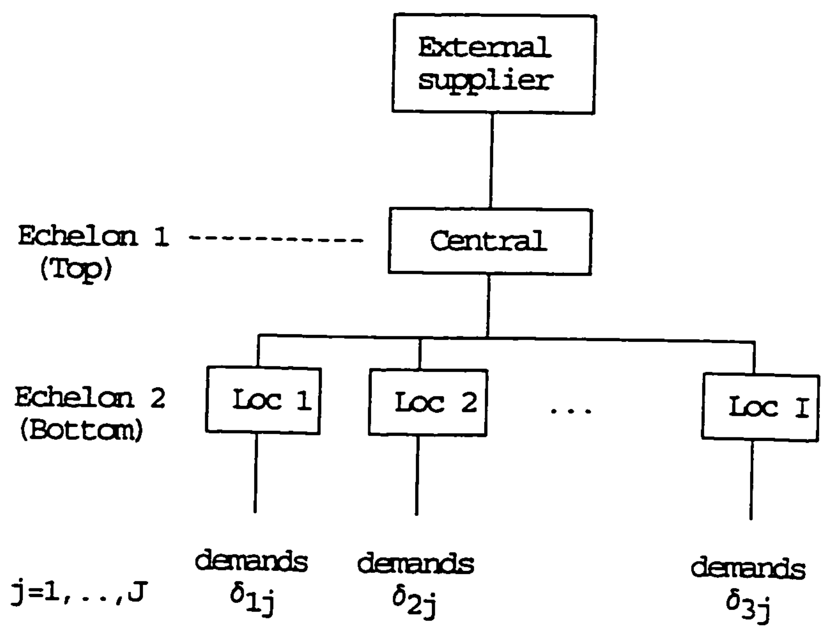


Figure 2.5: Multiple echelon system

The situation described above has become common in practice for several types of large organizations. The development of various combinations of multi-item, multi-indenture, multi-location, multi-echelon inventory models has received considerable attention in the past several years. The development of repairable items maintenance policies has been mostly concentrated in military organizations, such as the US Air Force, where operating equipments in several bases are supported by a central depot in the case of repairables and in the retail industry where several retailers are supplied by a central warehouse in the case of consumables.(See [Clark and Scarf 1960] for early multi-echelon models).

In most of the models developed in this type of environment, the decisions involve the determination of the optimal number of spare items (repairable or consumable items) quantity vector $\{S\} = \{S_j$ j=1,2,..,J$\}$ to be stocked at each location and at each echelon in order to achieve a desired performance service level subject to a specified available budget. Most formulations attempt to minimize the sum of system (depot and locations) backorders expressed as a function of time, which are not applicable here; some of the common solution procedures, however, will be commented throughout the thesis.

Complex and time-consuming Multi Echelon Techniques for Recoverable Items Control or METRIC based programs originally discussed, developed for various military applications and reported by [Sherbrooke 1968] have been extensively modified and improved upon over the past 20 years. The most important models were derived by [Simon 1971], [Muckstadt 1973, 1976a and 1976b] for the MOD-METRIC model, [Muckstadt and Thomas 1980], [Haber and Sitgreaves 1975], [Hillestadt 1982] for the DYNA-METRIC model and [Graves 1985], [Sherbooke 1986] and [Slay 1984] for the VARI-METRIC model.

Other authors like [Jackson 1988], [Johnson and Silver 1987], [Cohen, Kleindorfer and Lee 1986], [Nordin and Maier 1989], [Svoronos and Zipkin 1991] and [Nahmias and Smith 1994] derived results under various assumptions.

A comprehensive review up to 1980 on this subject can be found in chapters of [Schwarz 1981], written by [Nahmias 1981], [Demmy and Pressuti 1981] and [Clark 1981] and more recently by [Hausman and Erkip 1994] who compare multi-echelon sub-optimization with single-echelon inventory control policies.

The vast majority of these models involve repairable items and requires Palm's assumption [Palm 1938], [Little 1961], of "ample service" or infinite repair capacity and hence assumes independent repair times; the extensive use of the well known Palm's theorem from queuing theory can lead to serious errors as reported by [Gross 1982], and [Hausman and Scudder 1982] improved system performance by up to 20% reduction in spares inventory levels when simulating improved priority scheduling rules for repair facilities (See also [Pyke 1990]).

[Gross and Ince 1981 and 1978] modeled the classic machine repair problem as cyclic queues or closed queuing networks originally developed by [Mirasol 1964]; later, [Gross, Miller and Solland 1983] and [Gross and Harris 1985] have modeled a two-echelon system for repairable items also as a closed queuing network to determine optimal spares level and repair capacities.

[Ebeling 1991] used the machine repair M/M/s finite population queueing model results and dynamic programming to allocate a fixed budget among various components in order to max A_S subject to an investment constraint consisting of the sum of the purchase cost of each type of item and the individual cost of repair facilities for each type of item; his model structure for repairables is similar to the one used here for consumables but presents serious computational difficulties due to the integer requirements of budget allocation amounts at various stages and the cost of repair facilities for each type of item may prove difficult to obtain just like shortage costs are to the multi-item "Newsboy" class of problems; we will show how his model, can be solved using the FULL and GAP network structures to be discussed in the next chapter.

CHAPTER 3: DYNAMIC PROGRAMMING SOLUTION PROCEDURE

3.1 EXAMPLE 1

In this chapter, we present the dynamic programming method to solve both models P1 and P2 with a sample problem numerically using both models P1 and P2 and demonstrate the computational difficulties associated with this procedure, particularly for moderate to larger and more realistic problems, including multiple location problems to be discussed in a later chapter. This chapter also includes an analysis of an approximate DP (or incremental, mesh or grid) procedure where an error bound on the total cost solution $C_S(.)$ is developed. Finally, illustration with equivalent FULL (and GAP network approximation) network structures are presented.

Before we develop the various solution methods with an example, both models P1 and P2 are repeated below for convenience:

model P1:

$$\text{Max} \quad A_S = \prod_{j=1}^{J} A_j = \prod_{j=1}^{J} \left(\sum_{x=0}^{S_j} p_j(x) \right) \tag{P1}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \le B \quad \text{(Budget)} \tag{3.1}$$

$$S_j \ge 0 \qquad j=1,2,..,J \tag{3.2}$$

model P2:

$$\text{Min} \quad BO = \sum_{j=1}^{J} (BO_j) = \sum_{j=1}^{J} \sum_{x=S_j+1}^{\infty} (x-S_j).p_j(x) \tag{P2}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \le B \quad \text{(Budget)} \tag{3.1a}$$

$$S_j \ge 0 \qquad j=1,2,..,J \tag{3.2a}$$

The following numerical example (to be referred to as example 1 throughout) will be used to illustrate the application of various solution procedures for both models P1 and P2. With a budget B = \$50, J=3 items (and M=1 equipment, the parameters are: $\{\delta_j\} = \{1, 1.5, 2\}$ and $\{c_j\}=\{\$5, \$3, \$2\}$. Note that the order of magnitude of the budget and cost parameters is irrelevant in the sense that the solution vector $\{S_j, j=1,..,J\}$ obtained would be the same as the equivalent example with B=\$5000 and the following costs: $\{c_j\} = \{\$500., \$300. \text{ and } \$200.\}$.

The exponential failure rate parameters τ_j and the corresponding expected number of failures Poisson parameters δ_j during a fixed period of length t are similar to results of reliability studies of various Canadian Armed Forces military equipments such as the Canadian Leopard Main Battle Tank as reported by [Turmel and Wright 1986 and 1985] or such as the Jeep Iltis Canadian reported by [Parsons 1985] and [Dufour and Parsons 1984]. Only a subset of assemblies/parts were taken and the costs c_j and the budget B have deliberately been scaled down for simplicity and convenience in the presentation of results.

The objective is to determine the optimal number of each type of spares or vector quantity $\{S\} = \{S_j, j=1,2,..,J=3\} = \{S_1, S_2, S_3\}$ at the beginning of the period in order to maximize the system availability A_S (model P1) subject to a fixed budget B consisting of purchasing costs.

3.2 FORWARD AND BACKWARD RECURSION FOR MODEL P1

3.2.1 <u>Procedure</u>. A forward or backward recursion formula for the case of a multiplicative return function, can easily be found to solve model P1 using the following notation:

<u>Stage</u> j=1,2,3 = item j

<u>State</u> b_j = amount available for allocation up to the end of stage j

<u>Decision variables</u>

S_j = number of items allocated at stage j

y_j = amount allocated for S_j items at stage j = c_jS_j

<u>Return function</u>

$A_j(S_j)$ = availability for item j

$$= \sum_{x=0}^{S_j} p_j(x, \delta_j) \quad = \text{Poisson cdf} = F_j(S_j)$$

Forward recursive formula

$$f_0(b_j) \quad = 1$$

$$f_j(b_j) \quad = A_j(b_j) . f_{j-1}(b_j - y_j) \quad \text{for } j=1,2,3$$

Optimal value $f_j^*(b_j)$ at any stage

$$f_j^*(b_j) \quad = \max_{\min y_j \le y_j \le \max y_j} \{ A_j(b_j) . f_{j-1}(b_j - y_j) \}$$

$$= \max_{\min S_j \le S_j \le \max S_j} \{A_j(S_j) . f_{j-1}(b_j - c_j S_j)\}$$

where b_j= all possible available budget values

y_j= all possible values to be allocated
= $[\min y_j, \max y_j]$

$\min y_j = [c_j(\min S_j)]$

$$\max y_j = [B - \sum_{\substack{i=1 \\ i<>j}}^{J} c_i(\max S_i)]$$

where "<>" means "not equal to"

The solution procedure is to start at stage j=1 (item 1) and to determine sequentially the optimal allocation S_j (or y_j=c_jS_j) for all its possible values, when b_j is available up to that stage; once the optimal allocation for a given budget B has been calculated at the last stage J, the optimal allocation S_j at each stage is determined by working backwards from J to stage 1.

Table 3.1 (Stages 1 and 2 on page 1 and stage 3 on page 2) displays the complete DP solution for example 1 with J=3 types of items.

Table 3.1: DP stages 1,2 with B=$50

Stage j=1	Lamda =		1	c1,min,in		$5	$0	$5				
b1\S1	0	1	2	3	4	5	6	7	8	9	10	rowmax
$0	0,36788	0	0	0	0	0	0	0	0	0	0	0,36788
$5	0	0,73576	0	0	0	0	0	0	0	0	0	0,73576
$10	0	0	0,91970	0	0	0	0	0	0	0	0	0,91970
$15	0	0	0	0,98101	0	0	0	0	0	0	0	0,98101
$20	0	0	0	0	0,99634	0	0	0	0	0	0	0,99634
$25	0	0	0	0	0	0,99941	0	0	0	0	0	0,99941
$30	0	0	0	0	0	0	0,99992	0	0	0	0	0,99992
$35	0	0	0	0	0	0	0	0,99999	0	0	0	0,99999
$40	0	0	0	0	0	0	0	0	1,00000	0	0	1,00000
$45	0	0	0	0	0	0	0	0	0	1	0	1,00000
$50	0	0	0	0	0	0	0	0	0	0	1	1,00000
Stage j=2	Lamda =		1,5	c2,min,in		$3	$0	$1				
b2\S2	0	1	2	3	4	5	6	7	8	9	10	rowmax
$0	0,08208	0	0	0	0	0	0	0	0	0	0	0,08208
$1	0,08208	0	0	0	0	0	0	0	0	0	0	0,08208
$2	0,08208	0	0	0	0	0	0	0	0	0	0	0,08208
$3	0,08208	0,20521	0	0	0	0	0	0	0	0	0	0,20521
$4	0,08208	0,20521	0	0	0	0	0	0	0	0	0	0,20521
$5	0,16417	0,20521	0	0	0	0	0	0	0	0	0	0,20521
$6	0,16417	0,20521	0,29756	0	0	0	0	0	0	0	0	0,29756
$7	0,16417	0,20521	0,29756	0	0	0	0	0	0	0	0	0,29756
$8	0,16417	0,41042	0,29756	0	0	0	0	0	0	0	0	0,41042
$9	0,16417	0,41042	0,29756	0,34373	0	0	0	0	0	0	0	0,41042
$10	0,20521	0,41042	0,29756	0,34373	0	0	0	0	0	0	0	0,41042
$11	0,20521	0,41042	0,59512	0,34373	0	0	0	0	0	0	0	0,59512
$12	0,20521	0,41042	0,59512	0,34373	0,36105	0	0	0	0	0	0	0,59512
$13	0,20521	0,51303	0,59512	0,34373	0,36105	0	0	0	0	0	0	0,59512
$14	0,20521	0,51303	0,59512	0,68746	0,36105	0	0	0	0	0	0	0,68746
$15	0,21889	0,51303	0,59512	0,68746	0,36105	0,36624	0	0	0	0	0	0,68746
$16	0,21889	0,51303	0,74390	0,68746	0,36105	0,36624	0	0	0	0	0	0,74390
$17	0,21889	0,51303	0,74390	0,68746	0,72209	0,36624	0	0	0	0	0	0,74390
$18	0,21889	0,54723	0,74390	0,68746	0,72209	0,36624	0,36754	0	0	0	0	0,74390
$19	0,21889	0,54723	0,74390	0,85933	0,72209	0,36624	0,36754	0	0	0	0	0,85933
$20	0,22231	0,54723	0,74390	0,85933	0,72209	0,73248	0,36754	0	0	0	0	0,85933
$21	0,22231	0,54723	0,79349	0,85933	0,72209	0,73248	0,36754	0,36782	0	0	0	0,85933
$22	0,22231	0,54723	0,79349	0,85933	0,90261	0,73248	0,36754	0,36782	0	0	0	0,90261
$23	0,22231	0,55578	0,79349	0,85933	0,90261	0,73248	0,73508	0,36782	0	0	0	0,90261
$24	0,22231	0,55578	0,79349	0,91662	0,90261	0,73248	0,73508	0,36782	0,36787	0	0	0,91662
$25	0,22300	0,55578	0,79349	0,91662	0,90261	0,91560	0,73508	0,36782	0,36787	0	0	0,91662
$26	0,22300	0,55578	0,80589	0,91662	0,90261	0,91560	0,73508	0,73563	0,36787	0	0	0,91662
$27	0,22300	0,55578	0,80589	0,91662	0,96279	0,91560	0,73508	0,73563	0,36787	0,36788	0	0,96279
$28	0,22300	0,55749	0,80589	0,91662	0,96279	0,91560	0,91885	0,73563	0,36787	0,36788	0	0,96279
$29	0,22300	0,55749	0,80589	0,93094	0,96279	0,91560	0,91885	0,73563	0,73574	0,36788	0	0,96279
$30	0,22311	0,55749	0,80589	0,93094	0,96279	0,97664	0,91885	0,73563	0,73574	0,36788	0,36788	0,97664
$31	0,22311	0,55749	0,80837	0,93094	0,96279	0,97664	0,91885	0,91954	0,73574	0,36788	0,36788	0,97664
$32	0,22311	0,55749	0,80837	0,93094	0,97783	0,97664	0,91885	0,91954	0,73574	0,73576	0,36788	0,97783
$33	0,22311	0,55778	0,80837	0,93094	0,97783	0,97664	0,98010	0,91954	0,73574	0,73576	0,36788	0,98010
$34	0,22311	0,55778	0,80837	0,93380	0,97783	0,97664	0,98010	0,91954	0,91967	0,73576	0,36788	0,98010
$35	0,22313	0,55778	0,80837	0,93380	0,97783	0,99190	0,98010	0,91954	0,91967	0,73576	0,73576	0,99190
$36	0,22313	0,55778	0,80878	0,93380	0,97783	0,99190	0,98010	0,98085	0,91967	0,73576	0,73576	0,99190
$37	0,22313	0,55778	0,80878	0,93380	0,98084	0,99190	0,98010	0,98085	0,91967	0,91969	0,73576	0,99190
$38	0,22313	0,55782	0,80878	0,93380	0,98084	0,99190	0,99542	0,98085	0,91967	0,91969	0,73576	0,99542
$39	0,22313	0,55782	0,80878	0,93428	0,98084	0,99190	0,99542	0,98085	0,98098	0,91969	0,73576	0,99542
$40	0,22313	0,55782	0,80878	0,93428	0,98084	0,99495	0,99542	0,98085	0,98098	0,91969	0,91970	0,99542
$41	0,22313	0,55782	0,80884	0,93428	0,98084	0,99495	0,99542	0,99617	0,98098	0,91969	0,91970	0,99617
$42	0,22313	0,55782	0,80884	0,93428	0,98134	0,99495	0,99542	0,99617	0,98098	0,98101	0,91970	0,99617
$43	0,22313	0,55782	0,80884	0,93428	0,98134	0,99495	0,99848	0,99617	0,98098	0,98101	0,91970	0,99848
$44	0,22313	0,55782	0,80884	0,93435	0,98134	0,99495	0,99848	0,99617	0,99631	0,98101	0,91970	0,99848
$45	0,22313	0,55782	0,80884	0,93435	0,98134	0,99546	0,99848	0,99617	0,99631	0,98101	0,98101	0,99848
$46	0,22313	0,55782	0,80885	0,93435	0,98134	0,99546	0,99848	0,99924	0,99631	0,98101	0,98101	0,99924
$47	0,22313	0,55782	0,80885	0,93435	0,98141	0,99546	0,99848	0,99924	0,99631	0,99634	0,98101	0,99924
$48	0,22313	0,55783	0,80885	0,93435	0,98141	0,99546	0,99899	0,99924	0,99631	0,99634	0,98101	0,99924
$49	0,22313	0,55783	0,80885	0,93436	0,98141	0,99546	0,99899	0,99924	0,99938	0,99634	0,98101	0,99938
$50	0,22313	0,55783	0,80885	0,93436	0,98141	0,99553	0,99899	0,99924	0,99938	0,99634	0,99634	0,99938

Stage j = 3	Lamda =		2	c3,min,incr		#2	#0	#1				
b3\S3	0	1	2	3	4	5	6	7	8	9	10	rowmax
#0	0,01111	0	0	0	0	0	0	0	0	0	0	0,01111
#1	0,01111	0	0	0	0	0	0	0	0	0	0	0,01111
#2	0,01111	0,03333	0	0	0	0	0	0	0	0	0	0,03333
#3	0,02777	0,03333	0	0	0	0	0	0	0	0	0	0,03333
#4	0,02777	0,03333	0,05554	0	0	0	0	0	0	0	0	0,05554
#5	0,02777	0,08332	0,05554	0	0	0	0	0	0	0	0	0,08332
#6	0,04027	0,08332	0,05554	0,07036	0	0	0	0	0	0	0	0,08332
#7	0,04027	0,08332	0,13886	0,07036	0	0	0	0	0	0	0	0,13886
#8	0,05554	0,12081	0,13886	0,07036	0,07776	0	0	0	0	0	0	0,13886
#9	0,05554	0,12081	0,13886	0,17589	0,07776	0	0	0	0	0	0	0,17589
#10	0,05554	0,16663	0,20135	0,17589	0,07776	0,08073	0	0	0	0	0	0,20135
#11	0,08054	0,16663	0,20135	0,17589	0,19441	0,08073	0	0	0	0	0	0,20135
#12	0,08054	0,16663	0,27772	0,25504	0,19441	0,08073	0,08171	0	0	0	0	0,27772
#13	0,08054	0,24162	0,27772	0,25504	0,19441	0,20181	0,08171	0	0	0	0	0,27772
#14	0,09304	0,24162	0,27772	0,35178	0,28189	0,20181	0,08171	0,08199	0	0	0	0,35178
#15	0,09304	0,24162	0,40270	0,35178	0,28189	0,20181	0,20428	0,08199	0	0	0	0,40270
#16	0,10068	0,27911	0,40270	0,35178	0,38881	0,29263	0,20428	0,08199	0,08207	0	0	0,40270
#17	0,10068	0,27911	0,40270	0,51009	0,38881	0,29263	0,20428	0,20499	0,08207	0	0	0,51009
#18	0,10068	0,30203	0,46519	0,51009	0,38881	0,40363	0,29621	0,20499	0,08207	0,08208	0	0,51009
#19	0,11630	0,30203	0,46519	0,51009	0,56378	0,40363	0,29621	0,20499	0,20516	0,08208	0	0,56378
#20	0,11630	0,30203	0,50338	0,58924	0,56378	0,40363	0,40856	0,29723	0,20516	0,08208	0,08208	0,58924
#21	0,11630	0,34889	0,50338	0,58924	0,56378	0,58526	0,40856	0,29723	0,20516	0,20520	0,08208	0,58924
#22	0,12216	0,34889	0,50338	0,63761	0,65126	0,58526	0,40856	0,40997	0,29749	0,20520	0,08208	0,65126
#23	0,12216	0,34889	0,58149	0,63761	0,65126	0,58526	0,59242	0,40997	0,29749	0,20520	0,20521	0,65126
#24	0,12405	0,36647	0,58149	0,63761	0,70473	0,67608	0,59242	0,40997	0,41033	0,29754	0,20521	0,70473
#25	0,12405	0,36647	0,58149	0,73655	0,70473	0,67608	0,59242	0,59446	0,41033	0,29754	0,20521	0,73655
#26	0,12405	0,37215	0,61078	0,73655	0,70473	0,73157	0,68435	0,59446	0,41033	0,41041	0,29756	0,73655
#27	0,13030	0,37215	0,61078	0,73655	0,81408	0,73157	0,68435	0,59446	0,59497	0,41041	0,29756	0,81408
#28	0,13030	0,37215	0,62025	0,77365	0,81408	0,73157	0,74052	0,68671	0,59497	0,41041	0,41042	0,81408
#29	0,13030	0,39090	0,62025	0,77365	0,81408	0,84509	0,74052	0,68671	0,59497	0,59509	0,41042	0,84509
#30	0,13217	0,39090	0,62025	0,78565	0,85509	0,84509	0,74052	0,74308	0,68730	0,59509	0,41042	0,85509
#31	0,13217	0,39090	0,65150	0,78565	0,85509	0,84509	0,85543	0,74308	0,68730	0,59509	0,59511	0,85543
#32	0,13234	0,39652	0,65150	0,78565	0,86835	0,88766	0,85543	0,74308	0,74372	0,68743	0,59511	0,88766
#33	0,13264	0,39652	0,65150	0,82523	0,86835	0,88766	0,85543	0,85838	0,74372	0,68743	0,59511	0,88766
#34	0,13264	0,39701	0,66087	0,82523	0,86835	0,90143	0,89852	0,85838	0,74372	0,74386	0,68746	0,90143
#35	0,13424	0,39793	0,66087	0,82523	0,91209	0,90143	0,89852	0,85838	0,85912	0,74386	0,68746	0,91209
#36	0,13424	0,39793	0,66168	0,83710	0,91209	0,90143	0,91246	0,90162	0,85912	0,74386	0,74389	0,91246
#37	0,13424	0,40272	0,66321	0,83710	0,91209	0,94684	0,91246	0,90162	0,85912	0,85929	0,74389	0,94684
#38	0,13472	0,40272	0,66321	0,83812	0,92522	0,94684	0,91246	0,91561	0,90240	0,85929	0,74389	0,94684
#39	0,13472	0,40272	0,67120	0,84007	0,92522	0,94684	0,95842	0,91561	0,90240	0,85929	0,85932	0,95842
#40	0,13472	0,40415	0,67120	0,84007	0,92635	0,96046	0,95842	0,91561	0,91640	0,90257	0,85932	0,96046
#41	0,13482	0,40415	0,67120	0,85018	0,92850	0,96046	0,95842	0,96173	0,91640	0,90257	0,85932	0,96173
#42	0,13482	0,40415	0,67358	0,85018	0,92850	0,96164	0,97221	0,96173	0,91640	0,91657	0,90261	0,97221
#43	0,13513	0,40445	0,67358	0,85018	0,93967	0,96387	0,97221	0,96173	0,96256	0,91657	0,90261	0,97221
#44	0,13513	0,40445	0,67358	0,85320	0,93967	0,96387	0,97340	0,97557	0,96256	0,91657	0,91661	0,97557
#45	0,13513	0,40539	0,67409	0,85320	0,93967	0,97547	0,97566	0,97557	0,96256	0,96274	0,91661	0,97566
#46	0,13523	0,40539	0,67409	0,85320	0,94301	0,97547	0,97566	0,97676	0,97641	0,96274	0,91661	0,97676
#47	0,13523	0,40539	0,67565	0,85384	0,94301	0,97547	0,98740	0,97903	0,97641	0,96274	0,96278	0,98740
#48	0,13523	0,40570	0,67565	0,85384	0,94301	0,97893	0,98740	0,97903	0,97760	0,97660	0,96278	0,98740
#49	0,13525	0,40570	0,67565	0,85582	0,94372	0,97893	0,98740	0,99081	0,97987	0,97660	0,96278	0,99081
#50	0,13525	0,40570	0,67616	0,85582	0,94372	0,97893	0,99090	0,99081	0,97987	0,97779	0,97663	0,99090

Row increments of \$1 up to B=\$50 have been included to reflect the entire range of possible allocation amounts remaining at each stage. Although row increments in \$1 increments are not really necessary for the first and the last stage, it will become of crucial importance for all intermediate stages from j=2 to j=J-1 (j=2 only for example 1) in larger scale problems.

3.2.2 Optimal solution. The corresponding numerical values of the objective function measuring the system availability performance measure A_S at each stage j=1,2 and 3 and the optimal solution can easily be obtained from table 3.1 by working backwards from stage j=3 with any specified available budget B up to \$50 for this example.

Thus, with an available budget of say B=\$20 at stage 3, the best system availability from stage j=3 is found to be A_S = .5892 by allocating S_3 = 3 items type j=3 for a total cost of 3 x \$2 = \$6 and, therefore, \$20-\$6 = \$14 remains to be allocated at stage j=2 from which the highest cell in row \$14 is to allocate S_2=3 items type j=2 for a total cost of 3 x \$3 = \$9 and \$14-\$9 = \$5 remains for stage j=1 from which we allocate S_1=1 item type j=1. The optimal solution vector thus becomes $\{S_j\}$ = {1,3,3} and yields A_S = .5892 at a total cost C_S = \$20 which is exactly the same as the specified available budget B.

There is no other allocation S' yielding a higher A_S at a lower cost C_S. The major advantage of DP is that it guarantees an optimal solution but only if all possible budget values are considered, and in general, solving model P1 for a given budget B results in a whole set of allocations S_B which also solve P1 for smaller B values. As shown in table 3.1, the set of allocations $\{S_B\}$ and the corresponding response curve $\{A_S,B\}$ can easily be determined for the whole range of B from \$0 to \$50 in \$1 increments.

As a result of applying the dynamic programming procedure, all of the allocations in S_B that maximize A_S are undominated as described below:

- If $S' \epsilon S_B$, then $A_S(S) > A_S(S')$ ==> $C_S(S) > C_S(S')$ for all other allocations S, hence S' is the optimal solution for P1 with $C_S(S')$ as the budget.

- If $S'\epsilon S_B$, then $C_S(S)>C_S(S')$ ==> $A_S(S)>A_S(S')$ for all other allocations S.

In other words, the sequence of allocations generated for any possible budget values (from \$0 to \$50 in example 1) gives us the entire trade-off curve between the system availability A_S and the system costs C_S for $C_S \le B$ that is: $\{A_S(S), C_S(S)\}$; or, similarly, one allocation is said to dominate another if it has either (1) more availability at no more cost, or (2) no less availability at less cost (See [Kettelle 1963]). We also note that A_S is nondecreasing as C_S increases and approaches 1 for an arbitrary large budget value.

3.2.3 Advantages and disadvantages. The major advantage of DP methodology is that the solution vectors $\{S_j, j=1,..,J\}$ and its corresponding total costs C_S obtained from this procedure is guaranteed to be undominated and therefore optimal, but only if all possible budget values are examined for all intermediate stages j=1,..,J-1, and there lies its major problem: the computational difficulties associated with this approach; otherwise, the solution resulting from the use of the procedure no longer guarantees that it will be the true optimal solution.

For example 1, table 3.1 shows increments of \$1 (the lowest common denominator of all items costs c_j's) for stage j=2 which include all possible combinations of numerical budget values that can be allocated between \$0 and the total available budget B = \$50. However, the problem can best be described as follows: suppose we had used incremental amounts of c_2=\$3 from \$0, \$3, \$6, .., \$48 at stage j=2, and that we had \$17 remaining to allocate at stage 2 (from stage 3), then this "approximate" DP procedure would have resulted in an allocation of S_2=3 items from the best row element at \$15 (row above \$18) resulting in a system availability of 0.68746, which is clearly non-optimal since increments of \$1 in table 3.1 shows that a higher system availability of 0.74389 can be achieved by allocating S_2=2 items, had we included row \$17. This problem can occur for several available budget values when optimal allocation vectors $\{S^*\}$ are determined by working backwards from stage 3.

Furthermore, the solution obtained is also a function of the order or the sequence in which the items are listed. Thus, keeping the $3 item at stage 2 in increments of $3 and listing the $5 item first at stage 1 and the $1 item at stage 3 vs the reverse ($1 item first and $5 item last), can lead to different solution vectors $\{S_j\}$ and therefore different values for A_S.

To illustrate these effects, table 3.1 has been redrawn with c_2=$3 increments at stage 2 listing the $5 item first and listing the $5 item last. A comparative summary of the performance measure A_S and the possible significant errors for some budget values, when selecting equally sized increments of c_2=$3 for item type j=2 and the decreasing (vs increasing) order in which the items are listed ($5 item first vs last), were compared with the true optimal DP solutions using all possible incremental values of $1 at stage 2 and are presented in table 3.2 below.

Note that no particular order ($5 item first vs last) dominates the other one and therefore no conclusions can be drawn about which one is the best sequence. Figure 3.1 shows the optimal response curve $\{A_S,B\}$ from B = $20 to $50 in increments of $1 as compared with increments of $3 at stage 2 to highlight those errors and clearly show the crucial importance of enumerating all possible budget amounts that could be available (remaining) at each stage, otherwise optimal solutions from the approximate DP procedure are no longer guaranteed, and can be unpredictable. Although the approximate DP procedure may reduce the combinatorial nature of the problem to a more manageable size, the selection of incremental values of budget amounts to be allocated at each stage j is subjective and lead to unpredictable errors for some available budget values.

The important conclusion is that any selection of incremental values (mesh or grid) for the amount y_i allocated for S_j at each stage j can no longer guarantee that the true optimal solutions will be found unless the size of the increments is sufficiently small so that the process results in the enumeration of all possible budget values, at the expense of increasing the state space to possibly unmanageable proportions for more realistic complex problems, as the number of items J increase.

Short of enumerating all possible allocation vectors $\{S_j\}$ and calculating A_S for each one, in order to guarantee that the true optimal solution is found, we would have to find all possible budget combinations which quickly become unmanageable as the number of items J increase; for example, if each one of J = 20 different items can take an average of 10 possible values, then 10^{20} possible combinations (rows at the later stages) would be required, if the amount allocated at each stage is assumed to be continuous. If the amount to be allocated at each stage can be assumed to be an integer, then $1 increments could be selected for each intermediate stage and if a total budget of say $1,000,000 or 10^6 is available, then the total combimations would be reduced to an order of magnitude of 20 x 10^6, and the solution obtained would no longer be guaranteed to be optimal, which lead to our second major problem, the appropriate interval or incremental values to choose for each intermediate stage j=1,..,J-1.

One such logical size increment for each row is c_j or simply the item cost c_2 = $3 we have chosen for our example. Regardless of the increment, the optimal solution can no longer be guaranteed and can lead to significant errors on A_S from the optimal solution, unless the increments become sufficiently small but increase the state space to unacceptable levels, as discussed previously.

Table 3.2: Optimal As vs c2=$3 increment ($5 item first, last)

$Budget	Optimal As($1 incr)	$5 item first As($3 incr)	% Error	$5 item last As($3 incr)	% Error
$0	0,01111	0,01111	0,00	0,01111	0,00
$1	0,01111	0,01111	0,00	0,01111	0,00
$2	0,03333	0,03333	0,00	0,01111	66,67
$3	0,03333	0,03333	0,00	0,03333	0,00
$4	0,05554	0,05554	0,00	0,03333	40,00
$5	0,08332	0,08332	0,00	0,03333	60,00
$6	0,08332	0,08332	0,00	0,08332	0,00
$7	0,13886	0,13886	0,00	0,08332	40,00
$8	0,13886	0,13886	0,00	0,08332	40,00
$9	0,17589	0,17589	0,00	0,17589	0,00
$10	0,20135	0,20135	0,00	0,17589	12,64
$11	0,20135	0,20135	0,00	0,17589	12,64
$12	0,27772	0,25504	8,17	0,25504	8,17
$13	0,27772	0,27772	0,00	0,25504	8,17
$14	0,35178	0,28189	19,87	0,35178	0,00
$15	0,40270	0,35178	12,64	0,35178	12,64
$16	0,40270	0,40270	0,00	0,35178	12,64
$17	0,51009	0,40270	21,05	0,51009	0,00
$18	0,51009	0,51009	0,00	0,51009	0,00
$19	0,56378	0,51009	9,52	0,51009	9,52
$20	0,58924	0,56378	4,32	0,58924	0,00
$21	0,58924	0,58924	0,00	0,58924	0,00
$22	0,65126	0,58924	9,52	0,63761	2,10
$23	0,65126	0,65126	0,00	0,65126	0,00
$24	0,70473	0,65126	7,59	0,65126	7,59
$25	0,73655	0,67608	8,21	0,73655	0,00
$26	0,73655	0,70473	4,32	0,73655	0,00
$27	0,81408	0,73655	9,52	0,73655	9,52
$28	0,81408	0,73655	9,52	0,81408	0,00
$29	0,84509	0,81408	3,67	0,81408	3,67
$30	0,85509	0,81408	4,80	0,81408	4,80
$31	0,85543	0,84509	1,21	0,85543	0,00
$32	0,88766	0,86835	2,18	0,85543	3,63
$33	0,88766	0,86835	2,18	0,86835	2,18
$34	0,90143	0,90143	0,00	0,89852	0,32
$35	0,91209	0,91209	0,00	0,89852	1,49
$36	0,91246	0,91246	0,00	0,91246	0,00
$37	0,94684	0,94684	0,00	0,91246	3,63
$38	0,94684	0,94684	0,00	0,91246	3,63
$39	0,95842	0,95842	0,00	0,95842	0,00
$40	0,96046	0,96046	0,00	0,95842	0,21
$41	0,96173	0,96173	0,00	0,95842	0,34
$42	0,97221	0,97221	0,00	0,97221	0,00
$43	0,97221	0,97221	0,00	0,97221	0,00
$44	0,97557	0,97557	0,00	0,97340	0,22
$45	0,97566	0,97566	0,00	0,97566	0,00
$46	0,97676	0,97641	0,04	0,97566	0,11
$47	0,98740	0,97903	0,85	0,98740	0,00
$48	0,98740	0,98740	0,00	0,98740	0,00
$49	0,99081	0,98740	0,34	0,98740	0,34
$50	0,99090	0,99081	0,01	0,99090	0,00

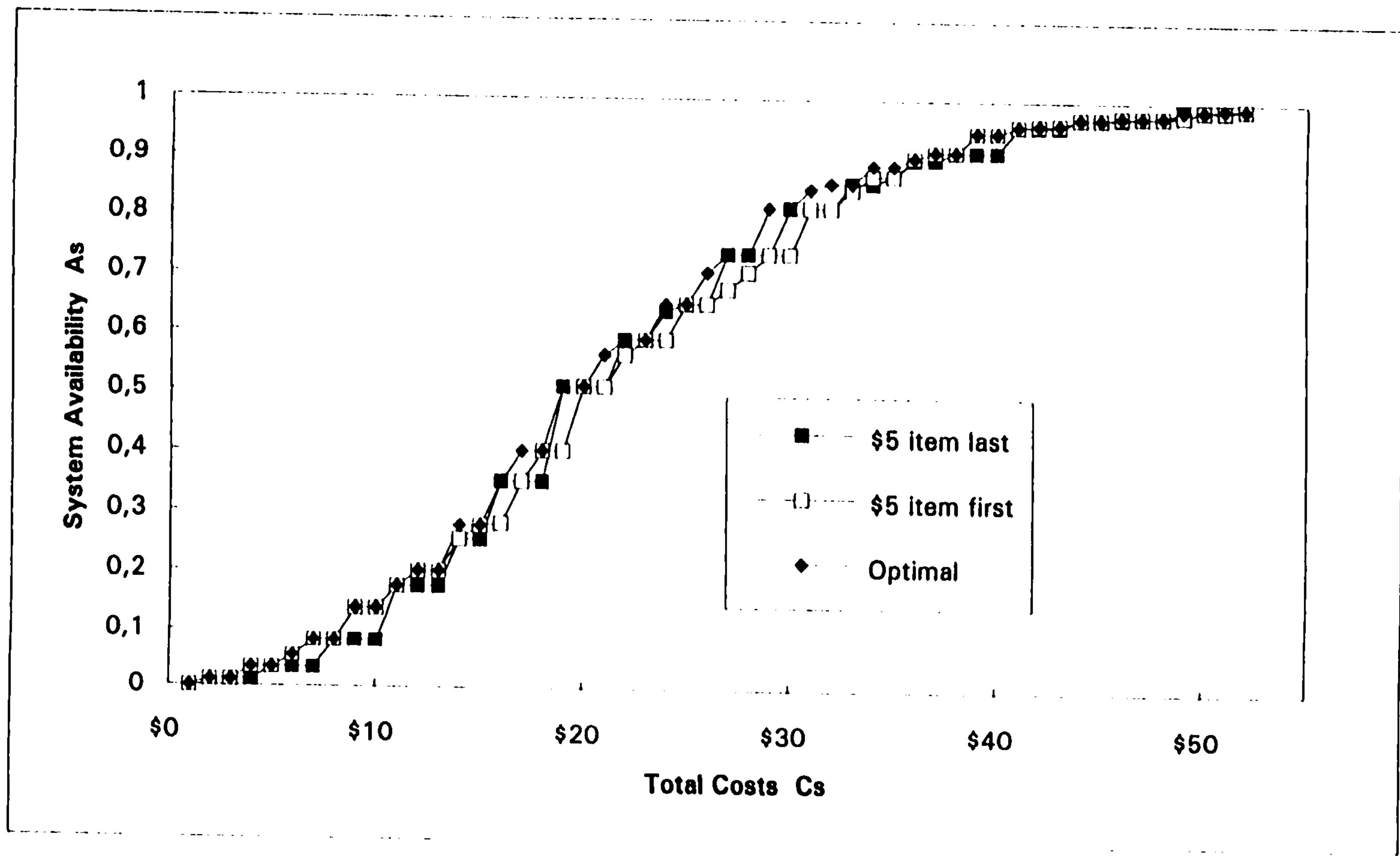


Figure 3.1: DP solution As optimal vs $cj incr ($5 item first, last)

As we have shown, any approximate DP procedure (incremental values at intermediate stages) could lead to significant errors in the measures of system performance such as A_S, and is unpredictable; the problem is further compounded since all item individual availabilities A_j's, j=1,..,J are multiplied together, some of which may be high reliability parts and the difference of 1 item can have significant impact on the performance measure A_S. This problem becomes progressively worse in the multiple location case and therefore, other methods that yield near (or optimal) solutions faster and within reasonable and acceptable error bounds are required.

3.2.4 Maximum and Minimum S_j's. It is possible to reduce the state space shown in table 3.1 to a smaller size since there is obviously an upper bound for each item $UB(S_j$, j=1,..,J) that can be derived from constraint (3.1) and is given by $UB(S_j)$ = Integer $[B/c_j]$; thus, the upper bound vector $\{UB(S_j)\}$ cannot be expanded beyond the numerical values $\{UB(Sj)\}=\{4,6,10\}$ for each type of item in example 1.

For more realistic examples, the state space can become quite large but it is possible to further reduce it even after taking into account the $\{UB(S_j)\}$ discussed above, by simply replacing constraint (3.2) $S_j \geq 0$ with the constraint $S_j \geq [\delta_j]$ j=1,2,..,J which constitutes a lower bound $LB(S_j)$ for each item j, j=1,..J. This procedure essentially means that the stockage level S_j for each item must be at least as large as its mean integer Poisson failure demand parameter which is not overly restrictive since it ensures that each item j will have a minimum $A_j(S_j)$ of approximately .55 to .60 for the Poisson distribution.

When all A_j's are multiplied together, we ensure a minimum $A_S > \pi A_j = (.55 \text{ to } .60)^J$; for example, a minimum specified A_j = 60% for an equipment consisting of J=10 items will result in an approximate $A_S \approx .60^{10}$ or .006 (less than 1%); since it is hardly conceivable to obtain such an unacceptably low value for A_S, even higher minimum values for each A_j, such as $A_j \geq .70$ or .80 can be specified as long as the overall cost of this initial starting allocation does not exceed the budget or $\Sigma c_j S_j \le B$, otherwise an unfeasible solution will result.

The number of equipments M operating during the period become a more important consideration since lower values for A_S when J increases may be perfectly acceptable if the expected number (proportion) of M equipments remain high. For example, if J=50 items and each $A_j \approx .99$, then $A_S = .60$ which might be unacceptably low, but if the average number of the M = 20 equipments still operational at the end of the period is 19, then a 95% proportion (defined earlier as AA_S) might be perfectly acceptable to managers, as we shall demonstrate later.

In our example 1, given a maximum available budget B=\$20, the lower bound vector $\{LB(S_j)\}$ for each item j results in the minimum stockage level $\{LB(S_j)\}=\{1,2,2\}$. Specifying the lower bound $LB(Sj) = [\delta_j]$ for each item's stockage level S_j, j=1,..,J automatically implies that a tighter or better $UB(S_j)$ can easily be found by applying constraint (3.1) again and defined as:

$$UB(S_j) = (B - \sum_{\substack{i=1 \\ i<>j}}^{J} (c_i(\text{Min } S_i)) \; / \; c_j \qquad (3.3)$$

For example 1, the result of applying (3.3) to item 1 when $\{LB(S_j)\}=\{1,2,2\}$ means that $UB(S_1)=$ [(\$20-\$10)/\$5] = 2. Similarly, we can derive the following complete upper bound vector of stockage levels for each item as $\{UB(Sj)\}=\{2,3,4\}$. Therefore, we need to be concerned with only the following range of possible values for each item S_j: $S_1=[1,2]$, $S_2=[2,3]$ and $S_3=[2,3,4]$ which has been considerably reduced in size as compared to the original problem of table 3.1 earlier.

Even though the $\{LB(S_j)\}$ and $\{UB(S_j)\}$ vectors provide a realistic and most convenient way to reduce the number of possible combinations to a more manageable size, for more practical and real-world problems of larger sizes, this procedure compounds the problems defined earlier (incremental values at intermediate stages and sequence in which the items are listed) and creates an additional problem: selecting the minimum and the maximum allocation amount (range of possible values) for each intermediate stage in the DP table. For example, selecting incremental values of c_2=\$3 at stage 2 from \$0, \$3,... vs from \$1, \$4,... vs \$2, \$5,.. will likely result in different solution vectors $\{S_j\}$, which is also unpredictable.

Another more useful upper bound vector $\{UB_j, j=1,..,J\}$ would be based on the magnitude of the mean Poisson parameter vector $\{\delta_j, j=1,..,J\}$. For example 1, if the budget B=\$50, the UB_3 would become = 25 since its cost c_3=\$2, but with a Poisson mean parameter of $\delta_3=2$, the item availability A_3 would become extremely high; we could thus specify a max $A_j = 0.999999$ (or any other reasonable number such as 5 times its standard deviation) for each item, which would further refine the respective {LB} and {UB} vectors. This procedure shall be the one adopted for the remaining examples, including the numerical examples presented in a later chapter. However, we must ensure that the optimal solution vector $\{S_j\}$ using this procedure does not contain any S_j, j=1,..,J that is at its minimum or maximum value, otherwise, the solution is not guaranteed to be the true optimal one; thus a simple check on $\{S_j\}$ optimal values compared to its minimum and maximum value is required.

3.2.5 <u>Lower bound on C_S</u>. We can establish a lower bound $LB(C_S)$ on the total cost solution $C_S(.)$ obtained when increments are used from stage J=J-1,..,1; suppose we are given a budget of \$50 and we use increments of c_1=\$5 and c_2=\$3 as opposed to \$1 increments in table 3.1, then we can lose no more than \$3 at stage 2 since rows are separated by at most \$3 and by at most \$5 at stage 1 for a total of stricly less than \$8. In general the total loss, denoted ß, will be $< \Sigma c_j$, j=J-1,..,1. So, given a starting available budget of B=\$50 at stage j=3, the total cost solution vector $C_S(.) \geq B - ß$ or \$50 - \$8 = \$42.

The size of the error ß on how far away can C_S be from the starting budget B can also be expressed as % of the budget or ßx100%/B and depends on the size of increments used at each stage j=1,..,J-1 as well as the sequence the items are listed, and the budget size B. Comparing the 2 different sequences (\$5,\$3,\$2 vs \$2,\$3,\$5) for a given available budget of B=\$50 at stage 3 gives us a maximum a-priori error of (\$5+\$3)/\$50 or 16% vs (\$2+\$3)/\$50 or 10% respectively and a corresponding $LB(C_S)$ of \$42 and \$45., which can be used as a guide for acceptability of results by an analyst.

Thus, if working backward from stage 3, listing the items in increasing order from j=1,..,J-1 would yield the lowest possible theoretical error. Unfortunately, no conclusion can be drawn about the effect on the performance measure A_S (or the expected system backorders BO) as can be seen from the summary table 3.2 above, but the % error on the system performance measure A_S can be unacceptable for lower available budget values, and progressively improve for higher budget values, as would be expected.

As shown in table 3.2 (column with \$5 item first) with B=\$50 from stage j=3, we obtain the following near optimal solution vector {S} = {4,5,7} and A_S = .99081 at a total C_S = 4x\$5 + 5x\$3 + 7x\$2 = \$49 which is greater than the a-priori $LB(C_S)$ of \$42 as expected and the error achieved ß = \$1 (or 2% of B=\$50), compared with the true optimal solution vector {S} = {4,6,6} and A_S = .99090 at a total cost C_S = \$50 which is greater than $LB(C_S)$ = \$45, and the error achieved ß = 0 (or 0% of B=\$50). The solution vector {S} = {4,6,6} with A_S = .99090 is also guaranteed to be the optimal solution as confirmed by table 3.1 since all possible budget values were enumerated.

3.2.6 <u>Summary</u>. The first major problem of DP methodology is to ensure all possible budget values are enumerated at intermediate stages j=2,..,J if we are to guarantee that true optimal solution is found. We could solve the problem by enumerating at each stage j=2,..,J all possible budget allocations (combinations) resulting from every budget value examined at the previous stage j-1 and successively adding budget increments of $\$c_j$ (item's cost from the previous stage), up to the maximum possible amount that can be allocated at the current stage.

Or, alternatively, we could accomplish the same objective starting from the last stage J and working backward for each stage from j=J-1,..,1 to stage 1 by successively subtracting from every budget value examined at the previous stage j+1 an amount equal to c_j (item's cost from the current stage), up to the maximum possible amount that can be deducted as a result of increasing S at stage j+1.

This latter approach will be shown next to be equivalent to a FULL network representation of the DP problem and guarantee the true optimal solution will be found; furthermore, the total number of rows in the DP tables as a result of the application of this technique will be equal to the total number of nodes N in the network minus 1 (destination node). This process, however, could also become quickly unmanageable as the number of items J increases (number of stages) since the possible budget values expand exponentially, particularly when non-integer (real) budget allocation values are introduced whenever c_j's for any item type j are real valued (dollars and cents) or of different orders of magnitude when compared with one another, which is very likely in real-world applications.

We are thus still faced with an enumeration type of problem to solve. Another solution would be to find the lowest common denominator of all cost terms c_j, j=1,..,J and proceed as described in the previous paragraph. Since item costs may have fractional values, this can lead to very small increments and increase significantly the number of possible allocation amounts to be examined at each stage and would also remain an enumeration type of problem for which the total number of possible combinations is unacceptable.

The second major problem with DP (other than enumerating all possible budget allocation values for all stages) is associated with the possible selection of equally sized increments to build the DP tables at each stage, creating gaps between the rows listed in the DP table for each intermediate stage j=2,..,J-1, and thus possibly losing part of the budget remaining for each subsequent stage. However, this procedure no longer guarantees that the true optimal solution will be found and therefore, lower system availability A_S may occur as a result; furthermore, it is subjective and can lead to different solution vectors $\{S_j, j=1,..,J\}$ depending on the size of the increments and also on the sequence in which the items are listed in the DP tables, and the starting/ending values at each stage, as table

3.2 has clearly shown.

The subjective and unpredictable nature of the approximate DP procedure leads us to a search for faster, more efficient and practical solution methods such as the marginal analysis and/or Lagrange relaxation methods, to be presented in the next 2 chapters respectively. As a quick overview of what lies ahead, table 3.3 below provides a comparative summary of the optimal (or near optimal) solutions for example 1, given an available budget B from \$17 to \$22, obtained from various other solution methods to be studied in the next chapters.

<u>Table 3.3: Optimal solutions for P1 with B=[17,22]</u>

B	S^*	$A_S(S^*)$	$C_S(S^*)$	Method used
17	(1,2,3)	.5101	17	D M L F
18	(1,2,3)	.5101	17	D F G
19	(1,2,4)	.5638	19	D M L F G
20	(1,3,3)	.5892	20	D F
21	(1,3,3)	.5892	20	D F G
22	(1,3,4)	.6513	22	D M L F

<u>Notes</u>: D = Dynamic programming
M = Marginal analysis
L = Lagrange multiplier
F = Full network
G = Gap network (gaps in the budget)

3.3 FORWARD AND BACKWARD RECURSION FOR MODEL P2

3.3.1 Procedure. A forward/backward recursion formula for the case of an additive return function, can also be found to solve model P2 using the following notation:

Stage $j=1,2,3$ = item j

State b_j = amount available for allocation up to the end of stage j

Decision variable

S_j = number of items from the allocation of y_j at stage j

y_j = amount allocated for S_j items at stage $j = \Sigma c_j S_j$

Return function

$BO_j(S_j)$ = Expected backorders for item j

$$= \sum_{x=S_j}^{\infty} (x-S_j) . p_j(x, \delta_j)$$

Forward recursive formula

$$f_0(b_j=0) = 0$$

$$f_j(b_j) = BO_j(y_j) + f_{j-1}(b_j-y_j) \quad \text{for } j=1,2,3$$

Optimal value $f_j^*(b_j)$ at any stage

$$f_j^*(b_j) = \min_{\min y_j \le y_j \le \max y_j} \{ BO_j(y_j) + f_{j-1}(b_j-y_j) \}$$

$$= \min_{\min S_j \le S_j \le \max S_j} \{ BO_j(S_j) + f_{j-1}(b_j-c_jS_j) \}$$

where b_j = all possible available budget values

y_j = all possible values to be allocated

$= [\min y_j, \max y_j]$

$$\min y_j = [c_j(\min S_j)]$$

$$\max y_j = [B - \sum_{\substack{i=1 \\ i<>j}}^{J} c_i(\max S_i)]$$

The solution procedure is to start at stage j=1 (item 1) and determine the optimal allocation y_j (or S_j) for all its possible values, when b_j is available up to that stage; once the optimal allocation has been calculated for a budget B at the last stage J has been calculated, the optimal allocation $\{S_j\}$ is determined by working backwards from J to stage 1.

3.3.2 Optimal solution. The sequence of optimal allocations for example 1 in the range B=[17,22] for model P2 to minimize total expected system backorders BO is shown in the summary table 3.4 below (no DP tables are included) along with other possible solution methods, to be discussed later. All of the allocations generated by the DP procedure are guaranteed to be undominated, if all possible budget values that can be allocated at each intermediate stage similar to model P1, and are described as follows:

- If $S' \epsilon S_B$, then $BO(S) < BO(S') \implies C_S(S) > C_S(S')$ for all other allocations S, hence S' is the optimal solution for P1 with $C_S(S')$ as the budget
- If $S' \epsilon S_B$, then $C_S(S) > C_S(S') \implies BO(S) < BO(S')$ for all other allocations S

In other words, the sequence of allocations generated for any specified budgetary values (from \$17 to \$22 shown in this example) gives us the entire trade-off curve between the total system backorders BO and the system costs C_S for $C_S \leq B$ that is: $\{BO(S), C_S(S)\}$; or, similarly, one allocation is said to dominate another if it has either (1) less backorders at no more cost, or (2) no more backorders at less cost. We also note that BO is nonincreasing as C_S increases and approaches 0 for an arbitrary large budget value.

Table 3.4: Optimal solutions for P2 with B=[17,22]

B	S^*	$BO_S(S^*)$	$C_S(S^*)$	Method used
17	(1,2,3)	.8669	17	D M L
18	(1,2,3)	.8669	17	D
19	(1,2,4)	.7240	19	D M L
20	(1,3,3)	.6760	20	D
21	(1,2,5)	.6702	21	D
22	(1,3,4)	.5328	22	D M L

Notes: D = Dynamic programming
M = Marginal analysis
L = Lagrange multiplier

3.3.3 Summary. The same comments apply for model P2 to minimize BO as for model P1 to maximize A_S previously described. The procedure guarantees that all the allocations in $\{S_B\}$ that minimize BO are undominated, as long as all possible budget values remaining are enumerated for all intermediate stages j=J-1,..,1, which lead to the approximate DP procedure with the problems of selecting incremental values, the sequence items are listed and the beginning and end range of possible allocation budget values remaining at each stage.

In the next section, we develop a FULL network structure (equivalent to the exact DP method) of the same problem and therefore guarantees that the true optimal solution will be found, while a GAP network structure (with GAPS in the budget) is equivalent to choosing increments of c_j at various stages; although, for the GAP network, the error bound on C_S from the budget B is identical to the one calculated previously, i.e. $\Sigma\ c_j$, j=J-1,..,1, the GAP network procedure can be shown to be slightly more accurate and therefore likely to give better results than the approximate DP procedure with increments.

Both types of network structures (FULL and GAP) will also be applicable to all other models studied here in this thesis (models P1, P1a, P1b, P2, P2a, P2b, P2' and PE), as well as for Ebeling's multiple machine repair model, denoted model (PE) [see Ebeling 1991], which uses similar DP solution methodology, but uses an impractical or unrealistic numerical example and does not discuss the computational difficulties associated with it.

3.4 FULL NETWORK STRUCTURE

3.4.1 Concept. The contents of this section describe the network structures and the solution procedures to solve problems related to all our models. The main reasons for discussing the network structures are as follows: 1) the lack of practical models represented as networks in the literature, 2) drawing parallels with DP methodology and its equivalent structure and 3) comparing the GAP network with the approximate DP methodology.

The optimal solution can be found with a node labeling procedure such as Dijkstra's algorithm for the shortest path in a network and the techniques outlined here are purely academic and for small scale problems only; for larger scale problems, applying DP at each successive stage or alternate methods discussed in the next chapters would be far more productive.

The concept is to construct a network in which each stage (j=1,2,..,J) represents an item type and whereby part of the budget is allocated for S_j items. Nodes represent the exact budget remaining to be allocated for the subsequent stages after purchasing S_j items at stage j and each arc length represents the cumulative Poisson availability A_{jk} resulting from the allocation (purchase) of k items at stage j. The same concept is applied in the case of model P2 to minimize BO except that the length of each arc represents the number of backorders BO_{jk} resulting from the allocation of k items at stage j.

For both models P1 and P2, the dynamic programming method is used to develop each stage of the network and the marginal analysis to create each node and each arc within each stage. While a stage is developed, the sequence of node numbering becomes a critical factor since we must keep track of the exact budget label for each node created (hence FULL network) which can be discarded once we proceed to the next stage.

The result is a directed acyclic network for which the optimal solution for model P1 is obtained by finding the most reliable path in the network through a matrix multiplication algorithm or equivalently, finding the shortest path in a network when the transformation to $-\ln(A_{jk})$ for each arc is performed.

For model P2 to minimize BO, the shortest path can be directly applied to the network to obtain the optimal solution, without any arc transformation, by calculating for each arc the expected backorders BO_{jk} as a result of allocating k items type j at each stage.

In the literature, there are very few inventory systems that are solved using network analysis but the method described in this chapter offers a better and faster alternative other than DP for finding exact optimal solutions to models P1 and P2.

The network for model P1 is constructed similar to the resource allocation problem described in [Winston 1994] which is set up as a linear knapsack problem; the Turnpike theorem discussed by Winston, based on sorting the benefit to cost ratio of each item by decreasing order, is valid only when the objective function is linear and is equivalent to the marginal analysis procedure derived in the next chapter, except that the benefit to cost ratio presented here is non-linear i.e. the increase in availability as a result of adding one more item type j is based on the cumulative Poisson distribution and is therefore not constant.

[Wagner 1975b] has also included a few more examples of non-linear objective functions subject to a single constraint (see his chapters 8 and 10) such as distribution management effort and financial allocation to projects whereby the DP problem structure can be represented as networks and solved with a shortest (or longest) path algorithm.

Other examples of network analysis can be found in [Lawler 1976 ch 2 p.64] for the 0-1 spare parts profit maximization problem in knapsack form where the solution is to find the longest path in the network. [Mamer and Smith 1982] has analyzed a 0-1 type of "tool kit" problem which is solved using a maximum flow/ minimum cut algorithm as described in [Lawler 1976 ch 4 p.125] for a spares provisioning type of problem.

3.4.2 FULL network for model P1 (Max As). The same example 1 previously described for model P1 with the dynamic programming, will now be solved using the FULL network analysis with B=$20 to keep the network structure to a reasonable size. The parameters are repeated here for convenience: (B=$20, $\delta_1=1$, $\delta_2=1.5$, $\delta_3=2$, $c_1=\$5$, $c_2=\$3$ and $c_3=\$2$, min $A_j=.001$ and max $A_j=.999$). The procedure results in an acyclic network and is illustrated in figure 3.2 below. Each arc length thus represents the cumulative Poisson probability A_{jk} if k items are allocated at that stage; stage 1 for item type 1 with k=[min=0, max=(B/c_1)] or k=[0,4]; we must keep track of the exact budget left after allocating k items at stage j=1.

Note that at the last stage j=3, in our example, the "sink" or "destination" node number 24 is created and the reverse procedure is used to determine the possible allocations (number of items type 3 and A_{3k}, k=[0,8]) from the preceding nodes to the destination node. The final result is an acyclic directed network consisting of N=24 nodes and A=42 arcs for which the optimal solution can be found by applying Dijkstra's shortest route node labelling procedure or some other method.

Within any stage j=2 to j=J-1, nodes are created only if no other node with the same budget label (called a matching label) has been previously created; this prevents multiple nodes with the same budget labels from being unnecessarily created and ensures that multiple arcs emanating from different nodes at the previous stage will in fact be directed to the same numbered node with the same exact budget remaining at the current stage. The number of times an arc is directed to an already existing node or sharing the same budget labels are called matching labels and are cumulated with the value M which is 3 for our example.

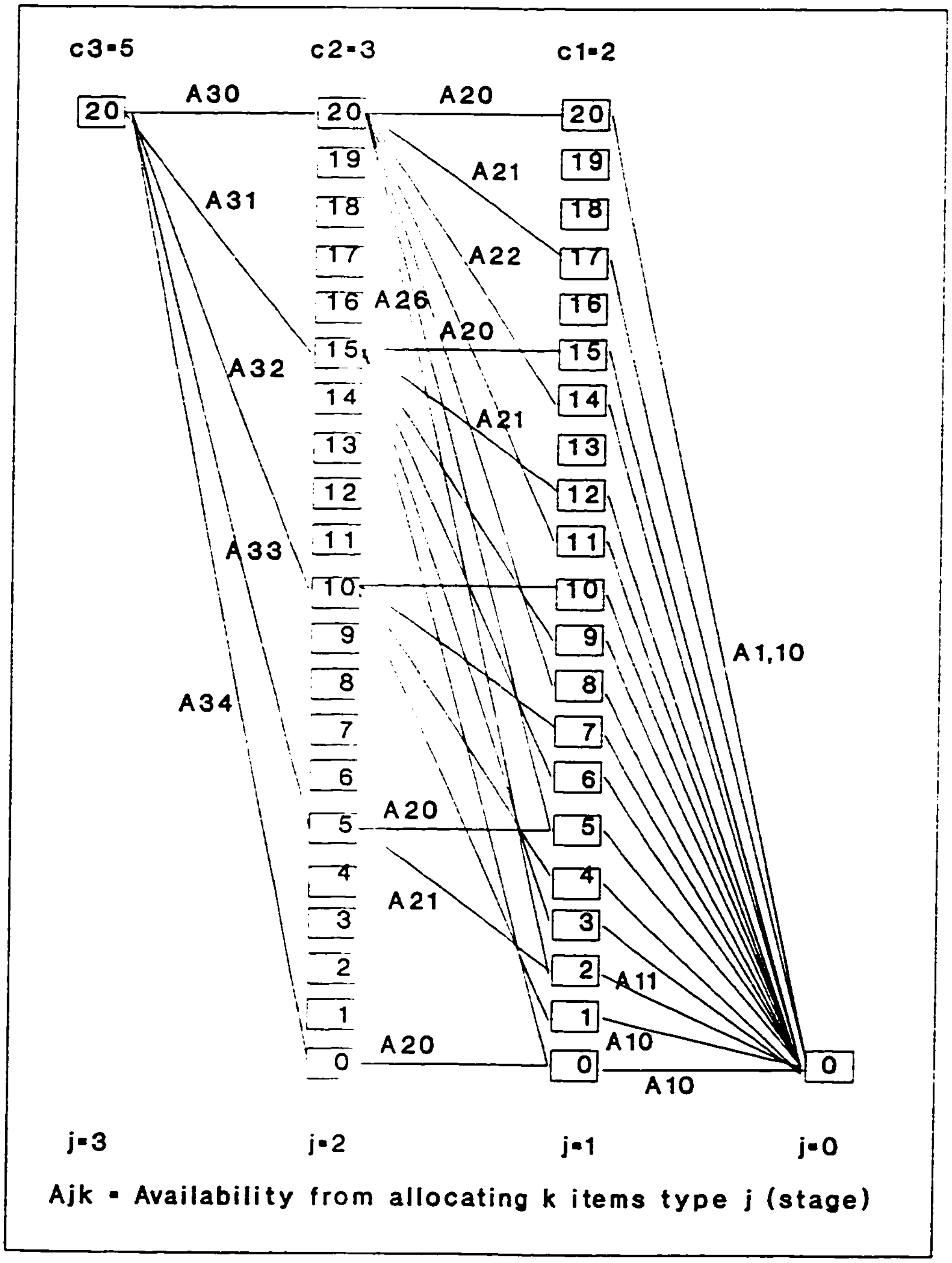


Figure 3.2: Full network for example 1 with B=$20

The solution using Dijkstra's method for this small example yields the optimal allocation vector $\{S^*\}=\{1,3,3\}$ by selecting the shortest path through nodes 1-3-17-24 in the network for a total path length or distance of $\Sigma -\ln(A_{jk}) = .5289223$ and a corresponding system availability A_S of $\exp(-.5289223) = .5892397$ which is the same optimal solution obtained from exact DP earlier.

3.4.3 Comments. One of the critical computational problem associated with this approach (equivalent to establishing rows in DP tables) is that the budget remaining when creating nodes and its associated budget node labels, is a real (fractional) value as opposed to an integer value, caused by individual item costs as is usually the case in practice. This will result in a large increase in the number of nodes created as the number of items (and therefore number of stages) increases and will quickly become unmanageable, as was the case with the exact DP method; when faced with the decision to select "suitable" incremental row values of budget allocations for approximate DP tables described earlier, the results can become unpredictable. The reduction of network size with the various available techniques just mentioned in the various sections above become of critical importance, just as it was for DP, but leads to additional problems, as discussed earlier and reiterated below.

We can significantly decrease the number of possible allocations by specifying lower $\{LB_j\}$ and upper bounds $\{UB_j\}$ quantity vectors on the number of k possible items of each type $j=1,..,J$ to consider, but could lead to possible infeasibility for small budgets or that one of any item j is at its lower or upper bound, in which case the solution vector $\{S_j\}$ obtained is no longer guaranteed to be optimal.

The selection of how individual items should be considered before the network is created, which is equivalent to the sequence the items are listed, becomes an important factor. Although the number of possible "matching labels" M cannot be accurately predicted, regardless of the order in which the items are considered (and therefore the number of nodes and arcs), then listing the most expensive items first (in decreasing order of costs $c_{(1)} > c_{(2)} > .. > c_{(J)}$) should ensure that less nodes and arcs are created in later stages, and equivalently that the number of corresponding rows in DP tables is less. That is why item j=1 with c_1=\$5 in figure 3.2 has been listed first followed by c_2=\$3 and c_3=\$2 as the last item.

Had we listed the least expensive item first, i.e. c_1=\$2 followed by c_2=\$3 and c_3=\$5, the network created by this procedure using the same original parameters of example 1 would have resulted in N=24 nodes, A=42 arcs and M=3 matching labels vs N=19, A=59 and M=34 for the original network, with the most expensive item listed first (c_1=\$5).

The important conclusion here is that the total number of rows in the DP table = N-1 nodes created as a result of the FULL network setup procedure and that the total number of DP cell evaluations = total number of arcs A in the network, as shown in the equivalent DP table 3.5 below. The total number of rows of the DP table is exactly equal to the total number of nodes N of the equivalent network less 1 (destination node) or N-1 = 24-1 = 23 in figure 3.2 and that the total number of entries to be evaluated in the DP table is equal to the number of arcs A = 42; the procedure developed here gives us a convenient way to quickly estimate the size of the problem should DP be used to solve the problem; the number of matching labels M simply mean the number of duplicate rows that would have the same budget value within a particular stage.

Table 3.5a: DP FULL network equivalent (Min Aj=.001)

Stage j=1	Lamda =	2.00		c1.min.in		$2.00	$0.00	$1.00				
b1\S1	0	1	2	3	4	5	6	7	8	9	10	rowmax
$0	0.13534	0	0	0	0	0	0	0	0	0	0	0.13534
$1	0.13534	0	0	0	0	0					0	0.13534
$2	0	0.40601	0	0	0	0					0	0.40601
$3	0	0.40601	0	0	0	0					0	0.40601
$4	0	0	0.67668	0	0	0					0	0.67668
$5	0	0	0.67668	0	0	0	0	0	0	0	0	0.67668
$6	0	0	0	0.85712	0	0	0	0	0	0	0	0.85712
$7	0	0	0	0.85712	0	0	0	0	0	0	0	0.85712
$8	0	0	0	0	0.94735	0	0	0	0	0	0	0.94735
$9	0	0	0	0	0.94735	0	0	0	0	0	0	0.94735
$10	0	0	0	0	0	0.98344	0	0	0	0	0	0.98344
$11	0	0	0	0	0	0.98344	0	0	0	0	0	0.98344
$12	0	0	0	0	0	0	0.99547	0	0	0	0	0.99547
$14	0	0	0	0	0	0	0	0.99890	0	0	0	0.99890
$15	0	0	0	0	0	0	0	0.99890	0	0	0	0.99890
$17	0	0	0	0	0	0	0	0	0.99976	0	0	0.99976
$20	0	0	0	0	0	0	0	0	0	0	0.99999	0.99999

DP rows = Nodes - 1 = 23
DP entries = Arcs = 42

Stage j=2	Lamda =	1.50		c2.min.in		$3.00	$0.00	$1.00				
b2\S2	0	1	2	3	4	5	6	7	8	9	10	rowmax
$0	0.03020	0	0	0	0	0	0	0	0	0	0	0.03020
$5	0.15099	0.22648	0	0	0	0	0	0	0	0	0	0.22648
$10	0.21943	0.47813	0.54733	0.12645	0	0	0	0	0	0	0	0.54733
$15	0.22289	0.55530	0.76626	0.80086	0.39846	0.13473	0	0	0	0	0	0.80086
$20	0.22313	0.55769	0.80796	0.91888	0.92975	0.67366	0.40563	0	0	0	0	0.92975

Stage j=3	Lamda =	1.00		c3.min.in		$5.00	$0.00	$1.00				
b3\S3	0	1	2	3	4	5	6	7	8	9	10	rowmax
$20	0.34204	0.58924	0.50338	0.22218	0.03009	0	0	0	0	0	0	0.58924

Table 3.5b: DP FULL network equivalent (Min Aj=.6)

Stage j=1	Lamda =	2.00		c1.min.in		$2.00	$0.00	$1.00				
b1\S1	0	1	2	3	4	5	6	7	8	9	10	rowmax
$0	0	0	0	0	0	0	0	0	0	0	0	0
$1	0	0	0	0	0	0					0	0
$3	0	0	0	0	0	0					0	0
$4	0	0	0.67668	0	0	0					0	0.67668
$6	0	0	0	0.85712	0	0					0	0.85712
$9	0	0	0	0	0.94735	0	0	0	0	0	0	0.94735

DP rows = Nodes - 1 = 11
DP entries = Arcs = 13

Stage j=2	Lamda =	1.50		c2.min.in		$3.00	$0.00	$1.00				
b2\S2	0	1	2	3	4	5	6	7	8	9	10	rowmax
$0	0	0	0	0	0	0	0	0	0	0	0	0
$5	0	0	0	0	0	0	0	0	0	0	0	0
$10	0	0	0.54733	0.12645	0	0	0	0	0	0	0	0.54733
$15	0	0	0.76626	0.80086	0.39846	0.13473	0	0	0	0	0	0.80086

Stage j=3	Lamda =	1.00		c3.min.in		$5.00	$0.00	$1.00				
b3\S3	0	1	2	3	4	5	6	7	8	9	10	rowmax
$20	0	0.58924	0.50338	0.22218	0.03009	0	0	0	0	0	0	0.58924

Starting from stage J=3 with B=\$20 (first row of the DP table 3.5a and therefore the first or origin node of the network), we have four entries for row \$20, creating rows \$20, \$15, \$10, \$5 and \$0 at stage J-1=2 as a result of the possible allocation of S_3 = 0,1,2,3 or 4 items type j=3 costing c_3=\$5 each. The procedure is then repeated from stage j=2 to create all rows at stage j=1 as a result of the possible allocation of S_2 = 2,3 or 4 items type j=2 costing \$3 each. Note that rows in the DP table (or nodes in the network) need to be created before the actual optimal calculations of A_S are executed, just as was the requirement to create the nodes and arcs of the network before the shortest route optimal algorithm in the network is executed.

The following 5-item example illustrates how a network (and the number of rows in exact DP method) increases in size very quickly: with an initial available budget of \$500, min A_j=.70 and max A_j=.999 for 5 different types of items whose cost vector $\{\$c_j\}$ = {19.99, 17.67, 15.00, 11.11, 9.99} and Poisson parameter $\{\delta_j\}$ = {1,2,8,3,5} results in a FULL network structure consisting of N = 2012 nodes, A=3109 arcs and M=41 matching labels. The equivalent exact DP method with enumeration of all possible budget allocation values at each intermediate stage would thus require a total of N-1 or 2011 rows and A = 3109 availability calculations within all of the 2011 rows, without counting the comparisons for each row to determine the row maximum; this compares with N = 9245, A = 19454 and M = 1990 had we specified a minimum A_j = .001 for each item j, j=1,..,5.

The GAP network discussed in a later section will deal specifically with a powerful technique to reduce the network to more manageable sizes, and is equivalent to the approximate DP method (with incremental budget values at each intermediate stage), and although it cannot guarantee that the solution found is the true optimal one, an error bound on the total costs C_S similar to the approximate DP is shown, on average, to be slightly superior.

3.4.4 Full network for model P2 (Min BO). The modifications required (if minimizing total system backorders BO in model P2) to utilize the full network concept and the solution procedures just described for model P1 can easily be accommodated because its objective function and its constraint(s) are also separable by item.

As for model P1, the network structure for model P2 is set up as per dynamic programming, that is, each item represents a stage where nodes represent the budget left as a result of adding 1 more item j from a specified minimum k value to a specified maximum, and arc lengths represent the expected number of backorders BO_j from the possible allocation of S_j items.

Successive arcs from each node then use the same marginal benefit concept that was utilized in the marginal analysis solution procedure for model P2 earlier, in that, adding 1 more item type j from S_j-1 to S_j results in a decrease of backorders by its complimentary distribution function $P_j(S_j)$.

Since the length of each arc does not require any logarithmic transformation unlike model P1, the optimal solution is found by directly applying a shortest route algorithm such as Dijkstra's from the origin to the destination. Therefore, the network structure for model P2 is the same as the one shown in figure 3.1 for model P1 except that the arc lengths represent the number of $BO_{jk}(S_j)$ instead of the cumulative Poisson probabilities A_{jk}'s.

The same comments apply for model P2 as for model P1 to reduce the network to a more manageable size: sorting and listing the most expensive items first and a judicious selection of $\{LB(S_j)\}$ and $\{UB(S_j)\}$ vectors for $\{S_j\}$.

Because of the similarities between max A_S in model P1 and min BO in model P2, the $\{LB(S_j)\}$ or minimum S_j's can be specified as $\{S_j\} > \{[\delta_j]\}$ or min A_j's=A_S such that $\Sigma\, p_j(x)$ from 0 to S_j is $\geq$ min A_S and appropriate maximum S_j's or $\{UB(S_j)\}$ selected such that $\Sigma\, p_j(x)$ from 0 to S_j is $\leq$ max A_j or applying equations (3.3) as before.

The permanent labelling process is also the same as for model P1, since the nodes with the lowest number of backorders (bottom arcs of each node) will be favored and as a result, will improve the performance of the shortest route algorithm. Although example 1 was not solved for model P2 using the full network analysis method, it can be programmed along the same logic as for model P1.

3.4.5 Machine repair model. The FULL network analysis techniques (and the GAP network next) can be applied to other models such as the classic machine repair problem for repairable items described by [Ebeling 1991]. His model, denoted (PE) below, seeks to determine the optimal allocation of repair channels (or crews) K_j and spares levels S_j for each item type j {j=1,..,J} in order to maximize the total operational system availability A_S subject to an investment constraint made up of total purchasing costs $c_{1j}S_j$ and total repair channel costs $c_{2j}K_j$ which must be ≤ the available budget B. The model is briefly summarized as follows:

$$\text{Max } A_S = \prod_{j=1}^{J} A_j(S_j,K_j) = \prod_{j=1}^{J} \sum_{n=0}^{S_j} A_{n,j}(S_j,K_j) \qquad \text{(PE)}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} (c_{1j}S_j + c_{2j}K_j) \le B \qquad (3.4)$$

$$S_j = 0,1,..,[(B-c_{2j})/c_{1j}] \quad \text{(integer)} \qquad (3.5a)$$

$$K_j = 1,2,..,[B/c_{2j}] \quad (\text{"}) \qquad (3.5b)$$

where:

A_S = joint probability that the number of failed items do not exceed their stock levels S_j or
= probability that all L systems are operating (from queueing theory)

S_j = number of items type j held as spares

K_j = number of repair channels for item type j

c_{1j} = cost per item type j

c_{2j} = cost per repair channel type j

L = number of operating systems (items)

$$A_j(S_j,K_j) = \sum_{n=0}^{S_j} A_{n,j}(S_j,K_j)$$

= steady state probability of $\leq S_j$ items in repair, based on the $M/M/K_j$ queueing system with $(L + S_j)$ items where at least L items are operating.

As described in his paper, Ebeling optimizes A_S in two steps. The first determines the optimal stock levels S_j and the number of repair channels K_j for each item j, j=1,2,..,J, using a direct search technique that must include a comparison of the optimal $A_{n,j}(S_j,K_j)$ for each possible budget allocation, unlike the network technique we will be using here. The second step allocates the total budget B among the J types of items using dynamic programming, similar to the procedure described earlier.

Besides the problem of obtaining accurate estimates for the repair channel costs c_{2j}'s, the model would become much more complex if any of the cost elements c_{1j} or c_{2j} was real valued, as is usually the case in practice. The numerical example provided by Ebeling consists of a J=4 item example with a maximum available budget B of $300., L=10 operating equipments and the following failure rate parameters: $\{\delta_j\}$ = {.50,1.00,.25,.50}, the repair rates $\{\mu_j\}$ = {4,6,3,5} and the item and repair channel costs: $\{c_{1j}\}$ = {15,5,10,5} and $\{c_{2j}\}$ = {20,10,20,15}.

This example can be illustrated as a FULL network structure as shown in figure 3.3 below. The procedure used to create the nodes and arcs is almost the same as for model P1 for consumables seen earlier in the chapter.

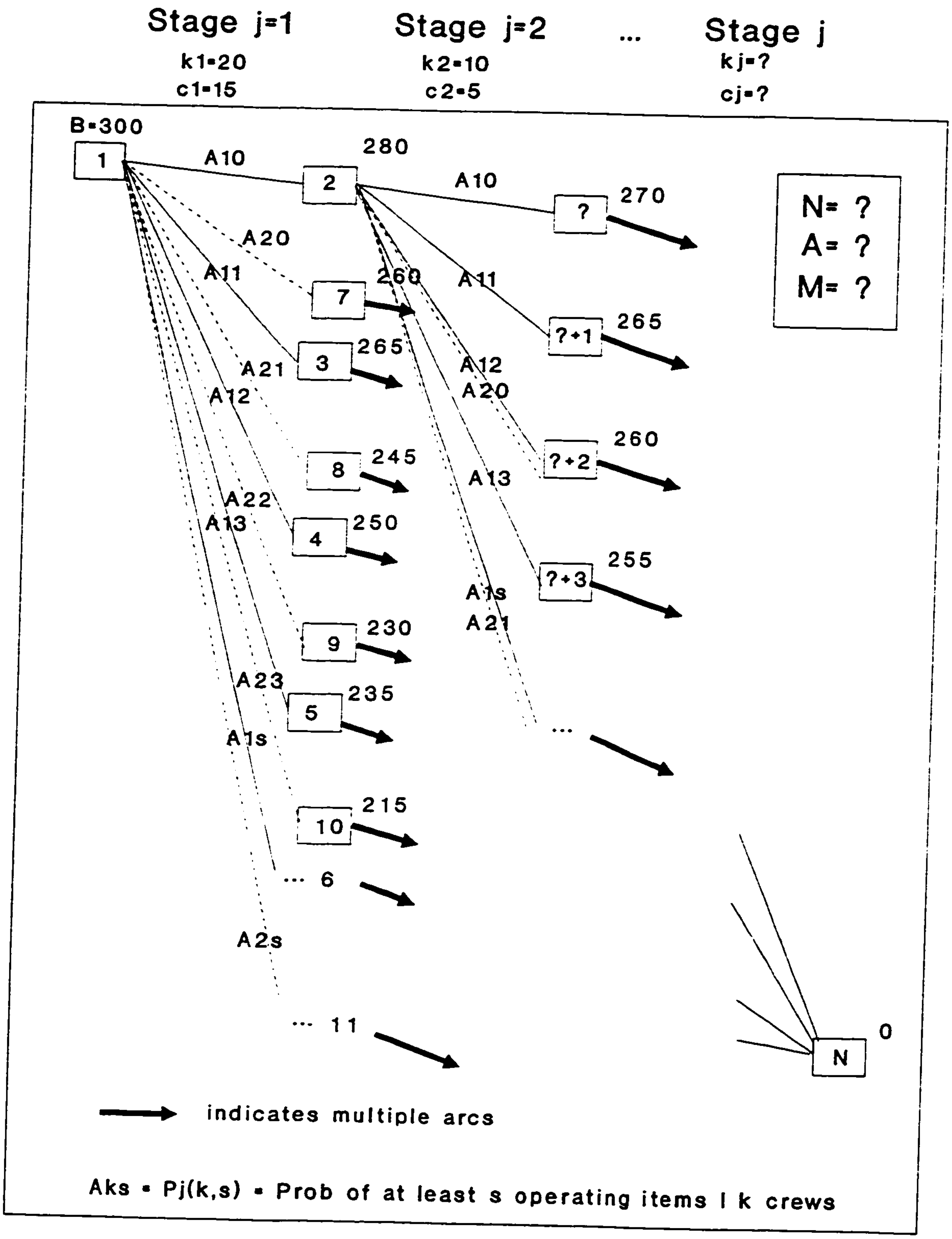


Figure 3.3: Network for Ebeling example

Dynamic programming is used in stages, one stage for each item type j; within each stage, arcs emanating from each of the nodes created at the previous stage (starting from the top node having the highest budget remaining) are directed to previously created nodes if they currently exist and have the same matching budget label, or are directed into new nodes numbered in sequence. Each arc represents the steady state probability $A_{n,j}(S_j,K_j)$ or A_{SK} shown in figure 3.3, that the number of failed items do not exceed S_j given K_j repair channels for item type j.

From each node, repair channels K_j are successively added starting from a minimum value of $K_j=1$ (at least 1 repair channel to ensure items can be returned to an operational state) to a specified maximum value of $[B/\Sigma c_{2j}]$; then, for each K_j value, spare items S_j are also successively added from its minimum of 0 to its maximum specified value of $[(B-c_{2j})/c_{1j}]$, thus creating an arc emanating from a node with a budget value label of say b_j and directed to a node with a budget label value reduced to $b_j - (c_{1j}S_j+c_{2j}K_j)$.

The procedure is repeated until stage J-1 and the reverse procedure is then applied for the last stage J, as was done for model P1. The result is an acyclic network whose optimal solution can be obtained by applying a shortest path algorithm once the length of each arc has been transformed to $-\ln(A_{n,j}(S_j,K_j))$.

The network procedural setup described above for Ebeling's model differs from model P1 in that there is no requirement to check the optimal combination of the two variables S_j and K_j within each stage since the shortest path algorithm and its corresponding temporary/permanent labeling procedure automatically determines the best alternativeas, as already described previously.

The same network analysis technique can be used if the objective function in Ebeling's model was changed to minimize the total expected backorders function BO; in this case, we could use standard queueing results and seek to minimize $L_S = \Sigma\, n.A_{n,j}(S_j,K_j)$ from $n=S_j+1$ to infinity; the length of each arc would then represent the expected number of failed items times the steady state probability which is solved as a result of having S_j items held as spares and K_j repair channels/crews for each item type j, j=1,..,J.

The network reduction techniques studied earlier can also be easily implemented; for example, lower and upper bounds $\{LB_j\}$ and $\{UB_j\}$ for S_j and K_j can also be specified as a function of steady state availability $A_{n,j}(S_j,K_j)$ which can be much tighter restrictions than only the $\{UB_j\}$'s specified by equations (3.5a) and (3.5b) in Ebeling's model above.

Finally, the powerful reduction technique described next as a GAP network will considerably help reduce the network of Ebeling's model to a more reasonable size as the number of items J increases.

3.5 GAP NETWORK STRUCTURE

3.5.1 Concept. A powerful alternative to the full network analysis is a network structure that can quickly approximate the optimal solution for both models P1 and P2 by reducing the network size significantly. The concept is nearly identical to the FULL network structure and will be called a GAP network, one in which gaps in budget node labels are occurring while arcs and nodes are created, and is equivalent to the approximate DP method with increments of c_j within each intermediate stage.

The procedure is again represented by a network whereby each item type j represents a stage just like dynamic programming and budget node labels indicate the budget left for subsequent stages after allocating S_j items at stage j, j=1,..,J. At the beginning of each stage and from the top node of that stage, new arcs and nodes with their budget labels are successively created as a result of adding k items of type j from its specified lower bound until its specified upper bound has been reached or until there is no more budget available.

The only critical difference from the full network structure is that from the remaining nodes within each stage, arcs are again successively created and the budget remaining as a result of adding one more item type j is calculated, then the arc is directed into a node already created at the current stage if it matches its budget label; otherwise, if the budget remaining falls between two budget node labels already created i.e. no match is found, then the arc is directed into the lower of the two budget node labels instead of creating a new node as was the case for the FULL network procedure. Furthermore, the arc is directed into a newly created node (if no match is found or it doesn't fall between two already existing nodes) with the new node having an assigned budget label (remaining for subsequent stages) that is exact i.e. no budget loss is incurred.

We note that since all nodes created within a stage are spread by at most the amount c_j as a result of successively adding one more item type j, a maximum possible budget loss of c_j may occur within each stage by forcing the arc to be directed into a lower budget node and therefore can incur a total possible maximum loss of $\Sigma\ c_j$, $j=2,..,J$.

The result is a directed acyclic network for which the optimal solution is obtained by applying a shortest path algorithm just like the full network analysis. Since this procedure causes a possible loss in the budget, the solution vector {S} may not be the true optimal solution but it constitutes a conservative lower bound LB on the system availability A_S. It is also a true optimal solution for the corresponding lower budget if a budget loss has occurred.

However, the major advantage over the full network structure is that it considerably reduces the network size (number of nodes and arcs in the network) by increasing the number of matching labels and therefore dramatically improves the performance of the shortest path algorithm used to determine the near or optimal solution. Before we formalize the results described above, we shall demonstrate the application of the GAP procedure to example 1 and compare its results with the FULL network.

3.5.2 GAP network for model P1 (Max As). The parameters for example 1 with B=\$20 are conveniently repeated here: (B=20, $\delta_1=1$, $\delta_2=1.5$, $\delta_3=2$, $c_1=5$, $c_2=3$ and $c_3=2$, min $A_j=.001$). The network structure as a result of the GAP network procedure is presented in figure 3.4 below.

Once the first stage is completed (from node 1 to nodes 2 to 6 in figure 3.4), at each subsequent stage j=2,..,N, nodes and arcs are created as before from the top node of each stage to the bottom node (emanating from nodes 2 to 6). From each node, a series of arcs is created, emanating from that node representing the cumulative availability A_{jk} to nodes being created and numbered in sequence with the identifying budget remaining label after the possible allocation of k items.

For our example, from node 2, arcs with length $-\ln(A_{20})$ to $-\ln(A_{26})$ directed to nodes 7 to 13 are created (A_{jk} where j=2 and k=[0,6]), since with a possible budget remaining of 20, only from 0 to 6 items type 2 costing c_2=\$3 each can be allocated or up to the specified maximum of A_j=.999 or until no more budget remains to be allocated. Thus, nodes 7 to 13 are created exactly c_2 = \$3 apart with budget node labels of \$20, \$17, \$14, \$11, \$8, \$5 and \$2 budget remaining to be allocated for subsequent stages.

The <u>only critical difference</u> between the FULL and GAP network structures occurs when the process is repeated for nodes 3 to 6 to complete stage j=2 before proceeding to the next stage. From node 3 (budget of \$15), an arc $A_{jk} = A_{20}$ resulting from the allocation of k=0 items type j=2 should be directed to a new node to be created (node 14) with a budget label of \$15 remaining; however, nodes 8 and 9 with respective budgets of \$17 and \$14 remaining (the cost spread between these 2 nodes can be at most a maximum of c_2 = \$3 only) have already been created from an earlier node (node 2) within the same stage j=2, therefore, node 14 is NOT created and arc A_{20} is instead directed to the lower budget node number 9, incuring a budget loss of \$15-\$14=\$1 in the process.

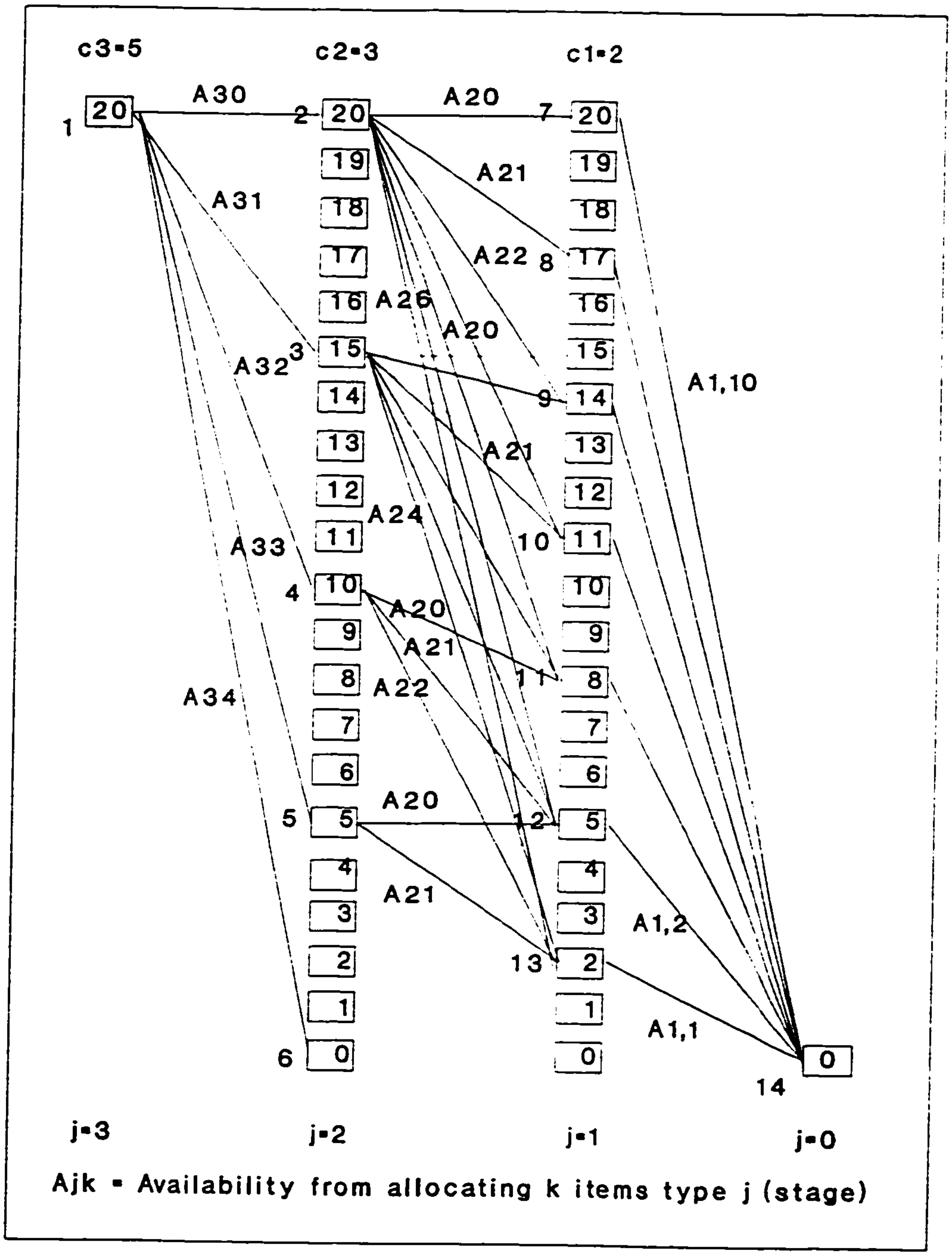


Figure 3.4: Gap network for example 1 with B=$20

The top node (node 2) of the current stage j=2 thus determines the starting budget node labels to be used for the next stage (j=3) and $c_j = c_2$ = \$3 determines the interval between budget node labels for all nodes created at the current stage j=2 and from which new arcs and nodes will be created for the next stage j=3.

The same procedure is then successively applied for the next arc $A_{jk} = A_{21}$ directed into node 10 until the maximum k=5 value for arc A_{25}, at which time node 14 is created at the bottom with a budget remaining of exactly \$0. The process is repeated for all remaining nodes 4 to 6 to complete the network structure at stage j=2. Budget labels for each of nodes 2 to 6 can then be discarded, just like the FULL network, as they are no longer required for the next stage.

Finally at the last stage j=3, the "sink" or "destination" node assigned number 15 is created and the reverse procedure identical to the FULL network structure is used to determine the possible allocations A_{3k} from the preceding nodes 7 to 14 to the destination node 15. The final result is an acyclic directed network consisting of a total number of N=15 nodes, A=33 arcs and M=12 matching labels during execution, (vs N=24, A=41 and M=3 for the FULL network structure in figure 3.1).

The solution using Dijkstra's method yields the near optimal solution $\{S^*\}=\{1,2,4\}$ at a total cost C_S=\$19, by selecting the shortest path through nodes 1-3-11-15 in the network. The total path length or distance is = $\Sigma \ln(-A_{jk})$ = .5730885 and a corresponding system availability $A_S = \exp(-.5730885) = .5637815$ which compares with the exact solution of $\{S^*\}=\{1,3,3\}$ and A_S=.5892397 obtained from the FULL network structure and dynamic programming, or 4.32% error from the true optimal solution.

3.5.3 <u>Comments</u>. In addition to the comments that applied for the FULL network to reduce the size of the network (most expensive item first, judicious selection of lower $\{LB_j\}$ and upper bound $\{UB_j\}$ vectors for S_j's, the concept of using GAPS in the budget to construct the network constitutes a most powerful technique to reduce its size and thus provides a convenient way to determine the total number of rows and cell evaluations within each row if the approximate DP procedure with c_j increments at each intermediate stage.

Other than considerably reducing the network size, the other major advantage is that it can easily accommodate real budget node values (or fractional) without modifications, which is a major problem for the full network (and the equivalent exact DP method) since it causes the number of nodes and arcs to increase significantly as the number of items also increase and quickly become unmanageable. Finally, the GAP network technique can also be applied to the classic machine repair model and Ebeling's model (PE) as was the FULL network technique.

The GAP network procedure constructs networks that have interesting properties and closely match those of the FULL network procedure described earlier. Since the procedure specified increments of c_j in our example (increments of any other size can be accomodated), we know that the solution obtained is not guaranteed to be the optimal one, as was the case for the approximate DP method discussed earlier.

It can be easily shown (proof omitted) that the maximum possible budget loss that may result from the application of the GAP network procedure is $\Sigma\ c_j$, $j=2,..,J$. For example 1, the maximum total possible loss in budget is therefore $\Sigma\ c_j$ $j=2,..,3 = c_2 + c_3$ = \$3+\$2 = \$5 and whose total costs $C_S(.) \geq B - \Sigma\ c_j$, $j=2,..,J$ or $C_S(.) \geq \$20$ - (\$3+\$2=\$5) = \$15.

We can also state that we cannot obtain a system availability A_S lower than if we had started with a budget of $C_S(.) = B - \Sigma\ c_j$, $j=2,..J$ or \$20-\$5 = \$15 for example 1; in other words, the solution vector $\{S_j, j=1,..,J\}$ generated from the GAP network structure provides us with a lower bound $LB(A_S) = .5730$ on the true optimal system availability of $A_S = .5892$ obtained from DP earlier.

Also as a direct consequence, the maximum (a-priori) error ß of the total costs $C_S(.)$ of the solution vector $\{S_j\}$ obtained as a result of the application of the GAP network procedure and a shortest path algorithm, denoted $B - C_S(.)$, has to be smaller or equal to $\Sigma\ c_j$,

$j=2,..,J$ or, expressed as a percentage (%) of the total available budget B, $[(B-C_S(.))/B] \times 100\% \leq [\Sigma c_j/B] \times 100\% = ß(\%)$, $j=2,..,J$. (Note that it is the same as saying that a lower bound on $C_S(.)$ is $B - \Sigma c_j(1+R)$, $j=2,..,J$).

For our small example, the a-priori error ß(%) = ($3+$2)/$20 x 100% = 25% (or a lower bound for $C_S(.)$ will be $20-($3+$2) = $15); since the GAP network solution vector obtained was $\{S_j\}=\{1,2,4\}$ at a total cost $C_S(S)$=$19, the error ß(%) achieved was thus only [($20-$19)/$20] x 100% = 5% of the total available budget B.

The GAP network procedure described in this chapter is <u>almost</u> equivalent to the approximate DP method (with increments of c_j at each intermediate stage $j=J-1,..,1$). Although the error bound on $C_S(.)$ is the same as the one developed for the DP structure with equal sized increments of $\$c_j$, the GAP network increments is governed not only by the top node at each stage but also the second top node etc.. so it is possible for any given stage $j=2,..J$ to create nodes with exact budget labels remaining from the second top node or the third top node etc.., thus giving solution vectors whose total costs $C_S(.)$ are likely to be better (but cannot be worse).

3.5.4 <u>Gap network for model P2 (Min BO)</u>. Finally, the same type of GAP network structure can be setup for model P2 to minimize the expected system backorders BO as was the case here for model P1 to maximize A_S, the only difference being the arc lengths which would represent the expected number of backorders BO_{jk} as a result of allocating k items type j instead of the Poisson cumulative availability A_{jk} as in model P1. The lower bound on C_S and the corresponding error ß(%) of the budget B also remain the same as for model P1.

3.5.5 <u>Comparison Full vs Gap networks (J=5 item example)</u>. We can now demonstrate the powerful effect of the GAP network structure compared with the FULL network with another example consisting of only J=5 items which involves decimals: B=$500., min A_j=.7, max A_j=.999, item cost vector $\{c_j\}$ = {19.99, 17.67, 15.00, 11.11 and 9.99} and Poisson parameter $\{\delta_j\} = \{1,2,8,3,5\}$. The results for the FULL and GAP network structures in table 3.6 below indicate savings of more than 1 order of magnitude in the number of nodes N and arcs A created and savings of more than 2 orders of magnitude in setup running time. The more realistic cost parameters are such that budget node matching labels in the resulting networks (and possible duplicate rows in DP tables) tend to be low and conversely, the number of nodes and arcs grow rapidly.

Table 3.6: FULL and GAP networks summary (J=5 items)

Type of network	FULL	GAP
Number of nodes N =	2012	86
Number of arcs A =	3109	331
Number of matches M =	41	224
Setup time (seconds) T =	17.3	0.7
Optimal solution		
Total costs $C_S(S)$ =	\$499.89	\$478.33
System availability A_S =	0.8719	0.8323

We note that DP methodology would require N-1 or 2011 rows and A = 3109 cell calculations (A_j x the highest availability at the previous stage) and comparisons and selection of the highest of all A_j's for each of the 2011 rows to obtain the true optimal solution by enumerating all possible budget combinations, without counting the comparisons of A_j's required for each row.

The approximate DP methodology, if using increments of size $\$c_j$, resulted in the following near optimal solution vector: $\{S_j\}$ = $\{2,5,12,7,9\}$ with A_S = .81005 and C_S = \$476.01, a loss of \$500 - \$476.01 = \$23.99 or 4.8% of B.

We also note that the a-priori maximum error $\beta(\%)$ for $C_S(.)$ from the original budget B=\$500., which is applicable to the approximate DP procedure with $\$c_j$ increments at each stage and the GAP network procedure, are both identical and ≤ [(\$17.67 + \$15.00 + \$11.11 +\$9.99)/\$500] x 100% = [\$53.77/\$500] x 100% = 10.75% from the original budget B=\$500 when listing the items in increasing order of $\$c_j$ starting at stage 1. Thus, the lower bound on $C_S(.)$ will be \$500-\$53.77 = \$446.23; the actual error achieved was [(\$500-\$478.33)/\$500] x 100% = [\$21.67/\$500] x 100% = 4.33% of the budget for the GAP network (vs 4.8% for DP with $\$c_j$ increments).

The optimal allocation vector for the J=5 item example above, was $\{S^*\} = \{3,5,13,6,9\}$ at a total cost C_S=\$499.89 and A_S=.8719; this solution compares with the GAP network near optimal solution vector {S} = {3,4,12,7,9} and A_S = 0.8323 for a total cost of C_S=\$478.33, which is different and slightly better than the approximate DP procedure with the same $\$c_j$ increments. Furthermore, a total of 21 nodes were never examined during the execution of Dijkstra's shortest path algorithm for the GAP network.

3.6 CHAPTER SUMMARY

This chapter described how to solve models P1 and P2 with the dynamic programming method, given an available budget value; the results yield the optimal solution vector $\{S_j, j=1,..,J\}$ only if all budget allocation amounts are enumerated at every stage j=J-1,..,1, which significantly increase the computational efforts as the budget B and/or the number of items J increases. Both models can be represented as FULL network structures, from which the optimal solution can be found and are equivalent to the DP method with a total number of rows equal to N-1 nodes of the network and cell evaluations equal to the total number of arcs in the corresponding network structures. Both types of networks can thus be effectively used to setup the DP tables and determine an a-priori error bound on C_S, if using the GAP network.

The approximate DP strategy of enumerating possible budget allocation values in increments at every stage no longer guarantees that the optimal solution will be found; the near optimal and unpredictable solutions obtained with this method are a function of the size of the possible allocation increments used at every stage, which has a cumulative effect, the sequence in which items are listed or make up the DP stages, and the starting possible budget allocation values at every stage. This method is equivalent to GAP network structures for which a LB on the total cost C_S can be developed.

As a result of these problems, we will analyze alternative solution methods for our models such as the marginal analysis in chapter 4 next, followed by the Lagrange relaxation method in the following chapter 5, both of which are faster and more practical to implement and will be shown to yield near or optimal solution vectors and corresponding useful response curves $\{A_S, C_S\}$ for model P1 and/or $\{BO, C_S\}$ for model P2 from which error bounds on A_S, BO and C_S can be readily obtained. We shall then compare the results with the military model discussed earlier when these methods are applied to larger scale and more representative problems.

CHAPTER 4: MARGINAL ANALYSIS SOLUTION PROCEDURE

4.1 INTRODUCTION

Marginal analysis, also commonly referred to as incremental analysis or the greedy algorithm, can be used effectively to solve both P1 and P2. The analysis can be applied due to the structure of the objective function and the constraint. When the objective function and the constraint are both separable by item and the constraint is linear, the iterative procedure consists of calculating the benefit to cost ratio for every item, selecting the one with the best ratio and adding it to the current solution; the procedure is then repeated until the budget B is exceeded by the addition of the last item at the k+1 st iteration.

A widely quoted paper on the subject is by [Fox 1966] who relied on the generalized Lagrangian multiplier method developed earlier by [Everett 1963] and discussed in the next chapter; Fox proves that the sequence of allocations generated by the marginal analysis are undominated and also proves optimality for the special case of identical unit costs. Variants and the conditions for which it can be applied, including the case of a non-linear constraint and various applications are also discussed.

[Rolfe 1971] has derived an application of the marginal allocation in multiple-server service systems by seeking to minimize expected queueing time subject to a fixed number of available servers and is similar to the machine repair problem analyzed by [Ebeling 1991], which was discussed earlier. [Shih 1974] has demonstrated this alternative procedure for a special class of resource allocation problems which is much simpler and faster than dynamic programming. He also has provided a heuristic proof of its optimality in two simple cases. [Mjelde 1975] has provided a general proof for the distribution of effort problem with one linear constraint. [Kao 1976] has provided an alternative proof to Fox and has shown that the fundamental issue of several applications in [Fox 1966], [Barlow and Proschan 1965] and

[Rolfe 1971] is to show that the return functions are concave.

4.2 GENERAL PROCEDURE

The general sequential marginal allocation procedure can be applied [see Fox 1966] if each term in the objective function is separable by variable and if the function is concave when maximizing or convex when minimizing. Adopting a similar notation given in [Fox 1966] for the general procedure:

$$\text{Max} \quad \Phi(x) = \sum_{j=1}^{J} \Phi_j(x_j) \tag{P5}$$

$$\text{s.t.} \quad C(x) = \sum_{j=1}^{J} c_j x_j \leq M \tag{4.1}$$

$$x_j \geq 0 \text{ and integer} \tag{4.2}$$

<u>Steps</u>:
1. Start with the allocation vector $\{x^0\}=\{0,..,0\}$;
2. Set k=1;
3. Calculate $x^k=x^{k-1}+\epsilon_j$ where ϵ_i is any index for which $[\Phi_j(x_j^{k-1}+1)-\Phi_j(x_j^{k-1})]/c_j$ is maximum;
4. if $C(x^k)>M$, terminate; otherwise set k=k+1 and go to step 3.

The procedure described above selects the most profitable item to be added at each iteration until the total costs $C(x^k) > M$ and generates a sequence of undominated allocations <u>but</u> not necessarily all of them. The procedure does not guarantee an optimal solution unless the total costs $C(x^k)$ is exactly equal to M.

4.3 PROCEDURE FOR MODEL P1

4.3.1 <u>Derivation</u>. For convenience, we repeat our model P1 below:

$$\text{Max} \quad A_S = \prod_{j=1}^{J} A_j = \prod_{j=1}^{J} \left(\sum_{x=0}^{S_j} p_j(x) \right) \tag{P1}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \leq B \tag{4.3}$$

$$S_j \geq 0 \qquad j=1,2,..,J \tag{4.4}$$

For our model, we can easily obtain separability by variable S_j, which is the first requirement of the marginal analysis procedure, by transforming the objective function from a product form into a sum of separate functions (and variables S_j). Since Max $A_S = \pi A_j$ is equivalent to Max $\ln(A_S)$, we obtain the following equivalent model (P1'):

$$\text{Max} \ln A_S = \sum_{j=1}^{J} \ln(A_j) = \sum_{j=1}^{J} (\ln \sum_{x=0}^{S_j} p_j(x)) \tag{P1'}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \leq B \tag{4.3}$$

$$S_j \geq 0 \qquad j=1,2,..,J \tag{4.4}$$

The second requirement is that the objective function be concave. [Fox 1966] has supplied a general proof and [Black and Proschan 1959] have proved it for the exponential/Poisson case for all values of S_j defined over the integer set $S_j=0,1,..$ Since one can show that if each A_j is concave (non-decreasing) in S_j, the ln (A_j) is also concave and the sum of concave functions is also concave; therefore, marginal analysis procedure as described by Fox can be applied to model P1 with $B=M$, $A_j(S_j)=\Phi_j(x_j)$ and $S_j=x_j$. Once P1 has been transformed to model (P1'), we can define the function:

$$d[\ln(A_j(S_j+1))] = \ln(A_j(S_j+1))-\ln(A_j(S_j)) \tag{4.5}$$

which represents the increase in the objective function ln A_S by adding one item type j in stock at a cost of c_j per item, then the benefit to cost ratio as a result of adding item j to the current solution would become:

$$d[\ln(A_j(S_j+1))] = \frac{[\ln(A_j(S_j+1)) - \ln(A_j(S_j))]}{c_j} \tag{4.6}$$

The procedure can be applied starting with any undominated initial allocation vector; since $\{S^{k=0}\}=\{0,..,0\}$ is obviously undominated, the procedure is valid and can thus be formally defined as follows:

- Start with $S^{k=0}=(0,0,..,0)$;
- Set k=1;
- Calculate the benefit to cost ratio for each item: $d[\ln(A_j(S_j+1))]/ c_j$ and select the item which has the maximum ratio;
- If $C_S(S^k)>B$, stop, otherwise set k=k+1 and go to step 3 (repeat until B is exceeded).

4.3.2 Solution. The solution for example 1 given earlier with B=$20 and the 3 item parameters $\{\delta_j\} = \{1, 1.5, 2\}$ and $\{c_j\} = \{5, 3, 2\}$ yield the following sequence of undominated allocations given that we start from the initial allocation vector $\{S^{k=0}\}=\{0,0,0\}$:

Table 4.1: Marginal analysis allocations for P1

k	C_S	S	A_S
0	0	(0,0,0)	.011
1	2	(0,0,1)	.033
2	5	(0,1,1)	.083
3	7	(0,1,2)	.139
4	12	(1,1,2)	.278
5	15	(1,2,2)	.403
6	17	(1,2,3)	.510
7	19	(1,2,4)	.564
8	22	(1,3,4)	.651

4.3.3 Comments on the solution. The sequence of allocations generated by the marginal analysis procedure for model P1 are undominated. In our example, given a budget B=$20, the solution vector $\{S_j\}=\{1,2,4\}$ with a total cost C_S = $19 and A_S = .564 is an undominated allocation vector if B had been $19, and thus yields a near optimal solution for the given budget \$20. The optimal solution obtained with dynamic programming earlier was: $\{S^*\}=\{1,3,3\}$ at a total cost $C_S(S^*)=\$20$ and $A_S(S^*)=.589$. If we had specified an available budget of \$19 then the solution obtained from the application of marginal analysis procedure, resulting in $\{S\}=\{1,2,4\}$ at a total cost $C_S(S)=\$19$ would be optimal and would have also been obtained with the DP method as shown in table 3.1 earlier.

So, the sequence of allocations obtained from marginal analysis and shown in table 4.1 above does not necessarily gives all undominated allocation vectors $\{S\}$ for all possible budget values but yields all optimal solutions for A_S for all intermediate budget values B = 0,2,5,7,12,15,17,19,22 in our example.

The major advantages of marginal analysis over dynamic programming are that real valued total costs do not, in any way, hinder the process, and that its execution time is much faster since the maximum total number of iterations (items added in the sequence) is simply the sum of the possible range (max S_j - min S_j) for each item or Σ Max S_j, j=1,..,J when starting from the initial allocation vector {min S_j, j=1,..,J} = $\{S\}=\{0,0,..,0\}$. For example 1 with a specified budget B=\$20, the UB vector is simply {4,6,10} and the procedure will require at most a total of { (4-0)+(6-0)+(10-0) } = 20 iterations.

A better initial allocation vector quantity $S=\{S^k\}$ with k total parts would be to specify a minimum target for system availability A_S such as, say $A_S > .60$ or any other appropriate value; since the objective function for P1 is to multiply individual item availabilities A_j, we know that each $A_j > A_S$ and a starting allocation will be S_j= smallest integer that satisfies:

$$A_j = \sum_{x=0}^{S_j} p_j(x) > A_S \quad (j=1,2,..,J) \tag{4.7}$$

$$S^k = (S_1, S_2, .., S_J) \text{ with } k = \sum_{j=1}^{J} S_j \tag{4.8}$$

and, as a result, determine the corresponding upper bound vector {UB} as described in the previous chapter. If a $\{LB_j, j=1,..,J\}$ is specified, then it may yield an infeasible solution since the total costs C_S may have already exceeded the specified available budget value B, but more importantly, the marginal analysis procedure will guarantee a sequence of undominated allocations as soon as all items j=1,..,J have been increased by at least one, thus requiring a simple check of the starting and final allocation vector to validate the procedure; therefore, the maximum number of possible iterations simply become $\Sigma\{UB-LB\}_j$, j=1,..,J.

4.3.4 Error bound on A_S. Using the marginal (or incremental) allocation sequencing procedure, we can derive an error bound α on the system availability A_S as shown by the following proposition:

Proposition 4.1: Error $\alpha = A_S(S^*) - A_S(S^k) < A_S(S^{k+1}) - A_S(S^k)$, the right-hand side obtained from marginal analysis after (k+1) parts just exceeded the budget.

Proof. Let $\{S^*\}$ be the optimal solution vector to P1 consisting of a total of k parts of all types; as $C_S(S)$ increases with the incremental addition of a part of any type at each iteration, then the following inequality must hold:

$$C_S(S^k) \le C_S(S^*) \le B < C_S(S^{k+1}) \tag{4.9}$$

since at some point during the allocation sequence of parts, the $(k+1)^{st}$ part will eventually cause the budget B to be exceeded. From the total costs $C_S(.)$ in (4.9) above, we therefore obtain the following corresponding system availabilities $A_S(.)$:

$$A_S(S^k) \le A_S(S^*) < A_S(S^{k+1}) \tag{4.10}$$

the last inequality must also hold since $A_S(S)$ increases as $C_S(S)$ increases as a result of the incremental addition of a part of any type j whose total increases from k to k+1; we then obtain the error α on A_S after subtracting $A_S(S^k)$ from all three sides of (4.10):

$$0 \le A_S(S^*) - A_S(S^k) < A_S(S^{k+1}) - A_S(S^k), \text{ or} \tag{4.11}$$
$$0 \le \alpha < A_S(S^{k+1}) - A_S(S^k)$$

the right-hand side obtained from marginal analysis after (k+1) parts just exceeded the budget and this completes the proof. Then the solution with k parts not exceeding the budget is our near or optimal solution.

For example 1, from the sequence of allocations shown in table 4.1 after the k=8th part (costing \$3) was added and resulted in the total costs C_S to jump from \$19 to \$22, thereby exceeding the specified available budget B=\$20; the marginal allocation sequence procedure was therefore terminated after that iteration and the solution with k=7 parts is the near optimal solution. We can then determine the theoretical error α on A_S to be at most = $A_S(S^{k=8}) - A_S(S^{k=7})$ = $A_S(S=1,3,4) - A_S(S=1,2,4)$ = .651-.564 = .087

In other words, we conclude that the true optimal solution A_S for our target budget B=\$20 lies between A_S=.564 at a cost of C_S=\$19 and A_S=.651 at a cost of C_S=\$22. The actual error achieved can be compared with the optimal DP solution (table 3.1) and is $A_S(S^*) - A_S(S^k)$ = $A_S(S^*=1,3,3) - A_S(S=1,2,4)$ = .589 -.564 = .025 or less than one-third of the theoretical error α. We also note that a convenient lower bound LB on the exact system availability A_S is readily obtained from the marginal analysis sequence of allocations: given B=\$20, the lower bound LB(A_S{S=1,2,4})= .564 at a total cost of C_S=\$19 and, in general:

$$LB(A_S) = LB(S^k) \qquad (4.12)$$

such that $A_S(S^k)$ with k parts is the last allocation (before B is exceeded at the next iteration k+1) obtained from marginal analysis with $C_S(S^k) \le B$.

4.3.5 <u>Error bound on C_S</u>. Before the marginal analysis procedure is carried out, we can also determine how close the total solution cost $C_S(S^k)$ will be from the specified budget value B, the corresponding maximum (a-priori) error ß.

<u>Proposition 4.2</u>: The maximum (a-priori) error ß(%) for $C_S(S^k)$ from the budget B as a result of the application of the marginal analysis procedure will be $\le$ max $\{c_j, j=1,..,J\}$ *100 /B. The proof is similar to proposition 4.1 earlier.

Proof. Let $\{S^*\}$ be the optimal solution vector to P1 consisting of a total of k parts of all types; as $C_S(S)$ increases with the incremental addition of a part of any type j at each iteration, then the following inequality must hold:

$$C_S(S^k) \le C_S(S^*) \le B < C_S(S^{k+1})$$

since at some point during the allocation sequence of parts, the $(k+1)^{st}$ part will eventually cause the budget B to be exceeded. Subtracting $C_S(S^k)$ on all three sides of the inequality, we obtain: $0 \le C_S(S^*)-C_S(S^k) \le B-C_S(S^k) < C_S(S^{k+1})-C_S(S^k)$, but we know that as a result of the incremental addition of a part of any type j at each iteration of the marginal analysis, the total cost increase from the current solution $C_S(S^k)$ with k total parts to $C_S(S^{k+1})$ with k+1 total parts or simply $C_S(S^{k+1})-C_S(S^k)$ must be $\le$ max $\{c_j, j=1,..,J\}$ as the increase cannot exceed the incremental cost of the most expensive of all items j, j=1,..,J}. Therefore, the above inequality becomes: $0 \le C_S(S^*)-C_S(S^k) \le B-C_S(S^k) < C_S(S^{k+1})-C_S(S^k) \le \max \{c_j, j=1,..,J\}$.

Dropping the terms $C_S(S^{k+1})-C_S(S^k)$ no longer needed and dividing all sides of the above inequality by B to express the error as a relative proportion to the budget, we obtain:

$$0 \le \underset{\text{(exact)}}{[C_S(S^*)-C_S(S^k)]\ /B} \le \underset{\text{(achieved)}}{[B-C_S(S^k)]\ /B} < \underset{\text{(a-priori)}}{\max \{c_j,\ j\epsilon J\}/B}$$

and multiplying by 100% to express the error as a percentage of the budget B, we finally obtain:

$$\underset{\text{(a-priori)}}{\text{Error } \beta(\%)} \le \max \{c_j, j=1,..,J\} \times 100\% \ /B \qquad (4.13)$$

which completes the proof.

In a later section, we shall describe the top-up marginal analysis procedure that will further reduce the error on C_S from the most expensive item or max $\{c_j, j=1,..,J\}$ down to the least expensive item or min $\{c_j, j=1,..,J\}$. So, for example 1, since the most expensive item costs $5, we know before we apply the procedure that the solution vector $\{S_j\}$ will yield a total cost solution C_S whose a-priori error $\beta(\%)$ will be:

$$\text{Error } \beta(\%) \text{ (a-priori)} = \$5 \times 100\% / \$20 = 25\%$$

which turns out to be fairly high for this small example with a limited budget; we also know that the error achieved once the procedure has been carried out, will be:

$$\text{Error } \beta(\%) \text{ (achieved)} = [B - C_S(S^k)] \times 100\% / B \tag{4.14}$$

which, for our example 1, turned out to be:

$$\text{Error } \beta(\%) \text{ (achieved)} = [20-19] \times 100\% / 20 = 5\%$$

The exact solution obtained from DP earlier resulted in C_S exactly equal to the budget B=$20, the error = 0% while the error achieved from the application of marginal analysis above was reduced from a possible theoretical 25% (a-priori) to an actual 5% error from the budget. Since the error size is based on the most expensive item as compared to the budget B, increasing the available budget will automatically reduce the error size proportionally.

4.3.6 <u>Example 2 (J=10 items)</u>. We now illustrate the marginal analysis procedure with a larger scale J=10 item example 2 (file 10_10 in chapter 8). With a specified available budget B = $15,000., item costs ranging from the least expensive = $152 to the most expensive = $860. and mean Poisson parameters ranging from 0.293 to 1.398; thus, the a-priori maximum error ß of C_S from B will be ≤ $860/$15000 or ≤ 5.73% of the budget. By specifying a min A_j = .000001 and max A_j = 0.999999 for each item j=1,..,J=10, the maximum number of possible iterations of the procedure will be $\Sigma\ \{\max - \min\}_j$, j=1,..,10 which turn out to be 87.

Starting from the initial alocation vector $\{S^{k=0}\}=\{0,..,0\}$, figure 4.1 below shows the response curve $\{A_S \text{ vs } C_S\}$ as a result of the sequence of iterations until the budget has been exceeded at iteration k+1 = 33 rd; the initial iterations resulting in $A_S \le .01$ are not shown for any of the more complex problem in order not to downgrade the visual appearance of the figures due to scale.

The solution vector $\{S_j\}$ = {2,2,5,3,3,3,2,3,4,5} at a total cost

C_S = \$14,851 (achieved error on $C_S \leq$ \$149/\$15,000 or less than 0.99 % from B) at the k=32 nd iteration resulting in A_S = 0.93010 which is a lower bound LB(A_S) while the k=33 rd iteration caused the budget B=\$15,000 to be exceeded by the addition of an item costing \$860 (coincidence that it happens to be the most expensive) for a total cost C_S = \$15,711 and A_S = 0.94853 which becomes the UB(A_S); thus, the true optimal solution lies between LB = 0.93010 $\leq A^* <$ UB = 0.94853 or less than 0.01843 or 1.98% possible increase over the LB.

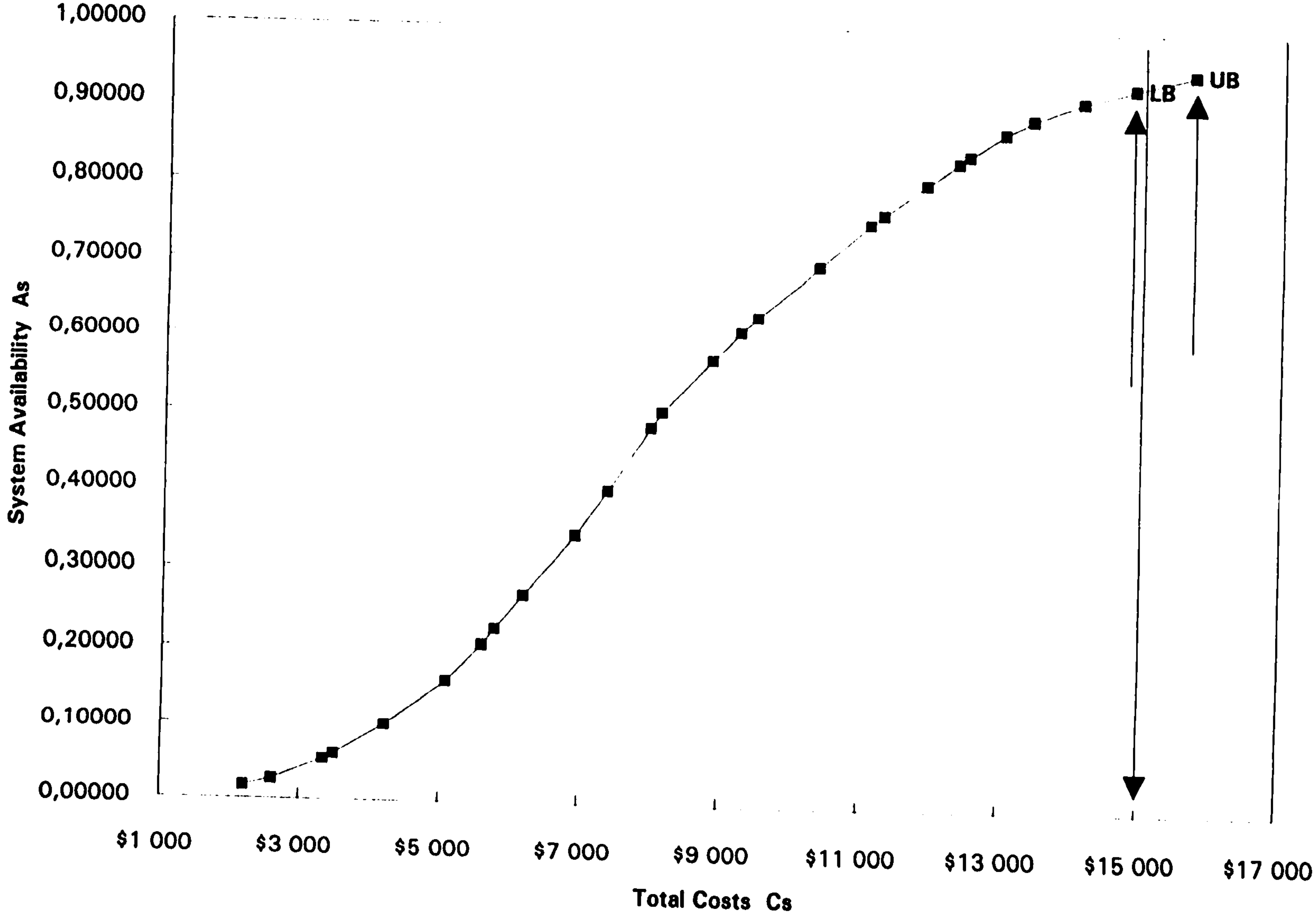


Figure 4.1: Marginal analysis for model P1 to max As (J = 10 items)

The marginal analysis procedure proves to be a very fast and efficient method to use repeatedly for several equipment types, and the response curve $\{A_S \text{ vs } C_S\}$ gives managers a very versatile tool from which knowledgeable decisions can be made, if the aggregate system availability $A_S = \pi A_j$ $j=1,..,J$ is the performance measure to be optimized. The next section discusses model P2 to minimize BO subject to the same constraint and whose solution vector $\{S_j, j=1,..,10\}$ practically yields the same results, i.e. the response curve $\{BO \text{ vs } C_S\}$ obtained by minimizing BO also gives a corresponding response curve $\{A_S \text{ vs } C_S\}$ nearly identical to the one just described above and thus also gives near identical solution vectors $\{S_j \; j=1,..,J\}$. This conclusion is valid for all larger scale numerical problems presented in chapter 8.

4.3.7 <u>Variant of P1 (model P1a)</u>. Solving P1 for an arbitrary high budget B with the marginal analysis procedure described earlier, will also solve the equivalent model P1a shown below:

$$\text{Min} \qquad C_S = \sum_{j=1}^{J} c_j S_j \tag{P1a}$$

$$\text{s.t.} \qquad A_S = \pi_{j=1}^{J} A_j = \pi_{j=1}^{J} \left(\sum_{x=0}^{S_j} p_j(x) \right) \geq \alpha \tag{4.15}$$

$$S_j \geq 0 \qquad j=1,2,...,J \tag{4.16}$$

where α is the minimum system availability specified by the user. Example 1 could have been solved using this procedure for any α (as given by discrete integer steps of S_j's) only by specifying an arbitrarily high budget value (say B=$100 vs B=$20) and omitting to specify maximum $\{S_j\}$ values.

The reason is that the natural log transform of (4.15) above and the marginal analysis procedure can be applied to the model: Minimizing C_S s.t. $\Sigma \ln (A_j) \geq \ln \alpha$. The model is separable by item and the objective function is convex, which is equivalent to Maximizing - C_S, which is concave. The marginal analysis procedure would select the item with the minimum cost to benefit ratio or simply $c_j / [\ln(A_j(S_j+1)) - \ln(A_j(S_j))]$, which is the reciprocal of equation (4.6) seen earlier for model P1. The sequence of iterations, as a result of adding items one at a time until the budget is exceeded, will thus be exactly the same.

The managerial interpretations of both models P1 and P1a are clearly different from an operational standpoint; model P1 seeks to max A_S subject to a maximum cost constraint (allowable budget) while P1a attempts to minimize total costs subject to a minimum system availability performance measure. Since model P1a is just the inverse of model P1, its objective function and constraint are both separable by item (variable S_j) and its cost function is clearly convex, marginal analysis can thus be applied to both models and will result in the same sequence of undominated stockage level allocations $\{S_j\}$.

[Cohen, Kleindorfer and Lee 1989] analyzed a similar model, without purchasing costs but including transportation and ordering costs. The solution procedure presented is the general Lagrange relaxation method for this type of model and proved (as Fox did earlier in 1966) the close relationship between the two procedures in that the solution obtained from marginal analysis when maximizing A_S is the same as the solution obtained from Lagrangian analysis.

4.4 PROCEDURE FOR MODEL P2

4.4.1 <u>Derivation</u>. As for model P1, we need an objective function which is separable by item; model P2, however, is already separable by item as the backorder function is the sum of individual item backorder functions, each one of which is convex in S_j as will be demonstrated shortly.

Therefore, marginal analysis is also applicable for model P2 since we could simply multiply the objective function by -1 and maximize the sum of concave functions instead of minimizing as described in [Fox 1966] and [Kao 1976].

$$\text{Min} \quad BO = \sum_{j=1}^{J} BO_j = \sum_{j=1}^{J} \sum_{x=S_j+1}^{\infty} (x-S_j).p_j(x) \qquad \text{(P2)}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \leq B \qquad (4.17)$$

$$S_j \geq 0 \qquad j=1,2,..,J \qquad (4.18)$$

<u>Theorem 4.2</u>: The $BO_j(x)$ is convex for all values of x defined over the integer set $\{I\}=\{0,1,..\}$. A proof is supplied below even though it may not necessarily be original.

<u>Proof</u>: We need to prove that the function BO(x) is increasing in x or that the first difference of the function $h(x) = h(x+1)-h(x)$ is increasing in x. We first define the first successive difference functions:

$h_1(x) = BO(x+1) - BO(x)$ and

$h_2(x) = BO(x+2) - BO(x+1)$

and require to show that $h_2(x) > h_1(x)$. First, we know that $BO(x) - BO(x+1) = 1 - F(x+1)$ and that $BO(x+1) - BO(x+2) = 1 - F(x+2)$ by expansion of each backorder function, an example of which is given below. Then:

$h_2(x) = BO(x+2) - BO(x+1) = F(x+2) - 1$ and

$h_1(x) = BO(x+1) - BO(x) \quad = F(x+1) - 1$; therefore,

$h_2(x) = F(x+2) - 1 > h_1(x) = F(x+1) - 1$ and the strict inequality holds since $F(x+2) > F(x+1)$ for all values of $x \in \{I\}$ which completes the proof.

To apply the procedure, we need then to determine the benefit to cost ratio for each item and select the best one; the marginal benefit of adding one (1) item type j from S_j to S_j+1 will result in a reduction of backorder by its complementary cdf $P_j(S_j+1) = 1 - F_j(S_j)$ or simply $1 - A_j(S_j)$ as shown below with a simple example:

$$\text{from } S_j = 3 \implies BO_j = \sum_{x=S_j}^{\infty} (x-S_j)p_j(x) = \sum_{x=3}^{\infty} (x-3)p_j(x)$$

$$= 0.p(3)+1.p(4)+2.p(5)+...$$

$$\text{to} \quad S_j+1 = 4 \implies BO_j = \sum_{x=S_j}^{\infty} (x-S_j)p_j(x) = \sum_{x=4}^{\infty} (x-4)p_j(x)$$

$$= 0.p(4)+1.p(5)+2.p(6)+\ldots$$

results in a net reduction in BO for item j of:

$$BO(S_j=3)-BO(S_j=4) = 1.p(4)+1.p(5)+1.p(6)+\ldots$$

$$BO(S_j)-BO(S_j+1) = \sum_{x=S_j+1}^{\infty} p_j(x) = P_j(S_j+1) = 1-F_j(S_j) = 1-A_j(S_j) \quad (4.19)$$

The cost of purchasing and adding the item to inventory from the current solution will be c_j, the same as model P1 when maximizing A_S. We note that if we start with an initial allocation of k=0 parts or $\{S^{k=0}\}=\{0,0,..,0\}$, we can immediately calculate the system expected backorder function $BO = \Sigma\, BO_j(S_j=0)$ which is simply the sum of the mean expected number of failures or $\Sigma\, \delta_j$, $j=1,..,J$ and the marginal analysis procedure can thus be formalized as follows:

- Start with $\{S^{k=0}\}=\{0,..,0\}$ and calculate $BO = \Sigma\delta_j$ given the starting allocation vector S^0;
- Set k=1;
- Calculate the benefit to cost ratio for each item: $P_j(S_j)/\ c_j$ and select the item which has the maximum ratio i.e. provides the max <u>net reduction</u> of BO from all J items;
- If $C_S(S^k)>B$, stop, otherwise set k=k+1 and go to step 3 (repeat until B is exceeded).

4.4.2 <u>Solution</u>. The solution for example 1 given earlier with B=$20 and the 3 item parameters $\{\delta_j\} = \{1,\ 1.5,\ 2\}$ and $\{c_j\} = \{5,3,2\}$ yields the following sequence of allocations given that we start from the initial allocation vector $\{S^{k=0}\}=\{0,0,0\}$:

Table 4.2: Marginal analysis allocations for P2

k	C_S	S	BO
0	0	(0,0,0)	4.500
1	2	(0,0,1)	3.635
2	4	(0,0,2)	3.041
3	7	(0,1,2)	2.264
4	9	(0,1,3)	1.941
5	12	(0,2,3)	1.499
6	17	(1,2,3)	0.867
7	19	(1,2,4)	0.724
8	22	(1,3,4)	0.533

$= \sum_{j=1}^{3} \delta_j$

4.4.3 Comments on the solution. The sequence of allocations generated by the marginal analysis procedure for model P2 are undominated for BO as was the case for A_S in model P1. For example 1, given a budget of B=\$20, the solution resulted in {S}={1,2,4} with a cost C_S(S)=\$19 and BO(S) = 0.724.

The optimal solution obtained with the dynamic programming solution procedure described in the previous chapter for model P2 yielded the optimal solution vector {S}={1,3,3} while the solution obtained from the application of marginal analysis resulted in the near optimal solution {S}={1,2,4} with a C_S=\$19; if we had started with a budget of B=\$19, then {S}={1,2,4} with C_S(S)=\$19 would have been optimal and the same solution would heve been obtained with DP.

So, the sequence of allocations obtained from marginal analysis also yields optimal solutions for BO for all intermediate budget values B=0,2,4,7,9,12,17,19,22 in our example, whereby a total of k=7 iterations or items were added, but does not necessarily produce all the undominated possible allocations, as was the case for model P1.

We also note that the sequence of allocations generated by marginal analysis for model P2 is not necessarily the same sequence generated for model P1 but is highly correlated and, for all practical purposes, nearly the same since Max A_S is similar to Min BO. All examples, including larger scale problems with up to J=99 items with different parameter values presented in a later chapter confirmed this finding and execution time proved to be approximately twice as fast as model P1.

4.4.4 Error bounds on BO and C_S. As with model P1, we can derive an error bound α for the total expected system backorders BO (and the total costs C_S) when the marginal analysis procedure is applied to model P2.

Proposition 4.3: Error $\alpha = BO(S^*) - BO(S^k) < BO(S^{k+1}) - BO(S^k)$, the right-hand side obtained from marginal analysis after (k+1) parts just exceeded the budget.

Proof: Let $\{S^*\}$ be the optimal solution vector to P2 consisting of a total of k parts of all types; as $C_S(S)$ increases with the incremental addition of a part of any type at each iteration, then the following inequality must hold: $C_S(S^k) \le C_S(S^*) \le B < C_S(S^{k+1})$ since at some point during the allocation sequence of parts, the $(k+1)^{st}$ part will eventually cause the budget B to be exceeded. We therefore have:

$$BO(S^k) \le BO(S^*) < BO(S^{k+1}) \tag{4.20}$$

the last inequality must also hold since BO is non-increasing (convex) as $C_S(S)$ increases; we can then obtain the following error α on BO after subtracting $BO(S^k)$ from all sides:

$$0 \le \alpha = BO(S^*) - BO(S^k) < BO(S^{k+1}) - BO(S^k) \tag{4.21}$$

the right-hand side obtained from marginal analysis after (k+1) parts just exceeded the budget and this completes the proof. The solution with k parts not exceeding the budget is our near or optimal solution.

For our example 1, from the sequence of allocations shown in table 4.2 after the k=8th part (costing \$3) was added and resulted in the specified available budget B=\$20 to jump from \$19 to \$22 and therefore the marginal allocation procedure terminated after that particular iteration. We can determine the theoretical error α on BO to be at most $= BO(S^{k=8}) - BO(S^{k=7}) = BO(S=1,3,4) - BO(S=1,2,4) = .724 - .533 = .191$

In other words, we can conclude that the true optimal solution BO for our target budget of $20 lies between BO=.533 at a cost of B=$19 and BO=.724 at a cost of B=$22. The actual error achieved can be compared with the optimal DP solution shown in table 3.6 is BO(S=1,2,4) - BO(S^*=1,2,5) = .670-.533 = .137. We also note that a convenient upper bound UB on BO is readily obtained from the marginal analysis sequence of allocations as follows: given B=$20, the upper bound UB(BO{S=1,2,4})= .724 and, in general:

$$UB(BO) = BO(S^k) \tag{4.22}$$

such that BO(S^k) with k parts is the last allocation (before B is exceeded) obtained from marginal analysis with $C_S(S^k) \leq B$.

Finally, the error bound on C_S is the same as for model P1, namely that C_S will be at least as close to B as the most expensive item or max $\{c_j, j=1,..,J\}$, thus yielding an a-priori error ß ≤ max $\{c_j, j=1,..,J\}$ * 100 / B, and the error achieved as a result of the procedure will be = (B - C_S) * 100/B.

4.4.5 <u>Example 2 (J=10 items)</u>. We now illustrate the marginal analysis procedure for model P2 with the larger J=10 item example 2 described earlier for P1. With a specified available budget B = $15,000., item costs ranging from the least expensive = $152 to the most expensive = $860. and mean Poisson parameters ranging from 0.293 to 1.398; thus, the a-priori maximum error ß of C_S from B will be ≤ $860/$15000 or ≤ 5.73% of the budget. By specifying a min A_j = .000001 and max A_j = 0.999999 for each item j=1,..,J=10, the maximum number of possible iterations of the procedure will be Σ $\{max - min\}_j$, j=1,..,10 which turn out to be 87. All of the above remain the same as model P1 to max A_S earlier.

Starting from the initial alocation vector $\{S^{k=0}\}=\{0,..,0\}$, figure 4.2 below shows the response curve {BO vs C_S} as a result of the sequence of iterations until the budget has been exceeded at iteration k+1 = 33 rd. The solution vector $\{S_j\}$ = {2,2,5,3,3,3,2,3,4,5} at a total cost C_S = $14,851 (achieved error on C_S ≤ $149/$15,000 or less than 0.99 % from B) at the k=32 nd iteration resulting in BO = 0.08472 which is an upper bound UB(BO) while the k=33 rd iteration caused the budget B=\$15,000 to be exceeded by the addition of an item costing \$860 (coincidence that it happens to be the most expensive) for a total cost C_S = \$15,711 and BO = 0.06210 which becomes the LB(BO); the true optimal solution lies between LB = 0.06210 < BO^* ≤ UB = 0.08472 or less than 0.02262 or 26.7% possible decrease over the UB. The % errors can become fairly high as BO tends towards 0.

The same comments apply for model P2 as for model P1 when using the marginal analysis procedure; it is much faster than DP and very efficient in providing a response curve {BO vs C_S} that is extremely useful for managers to plan and make decisions, if the aggregate system performance measure to be used is to minimize BO.

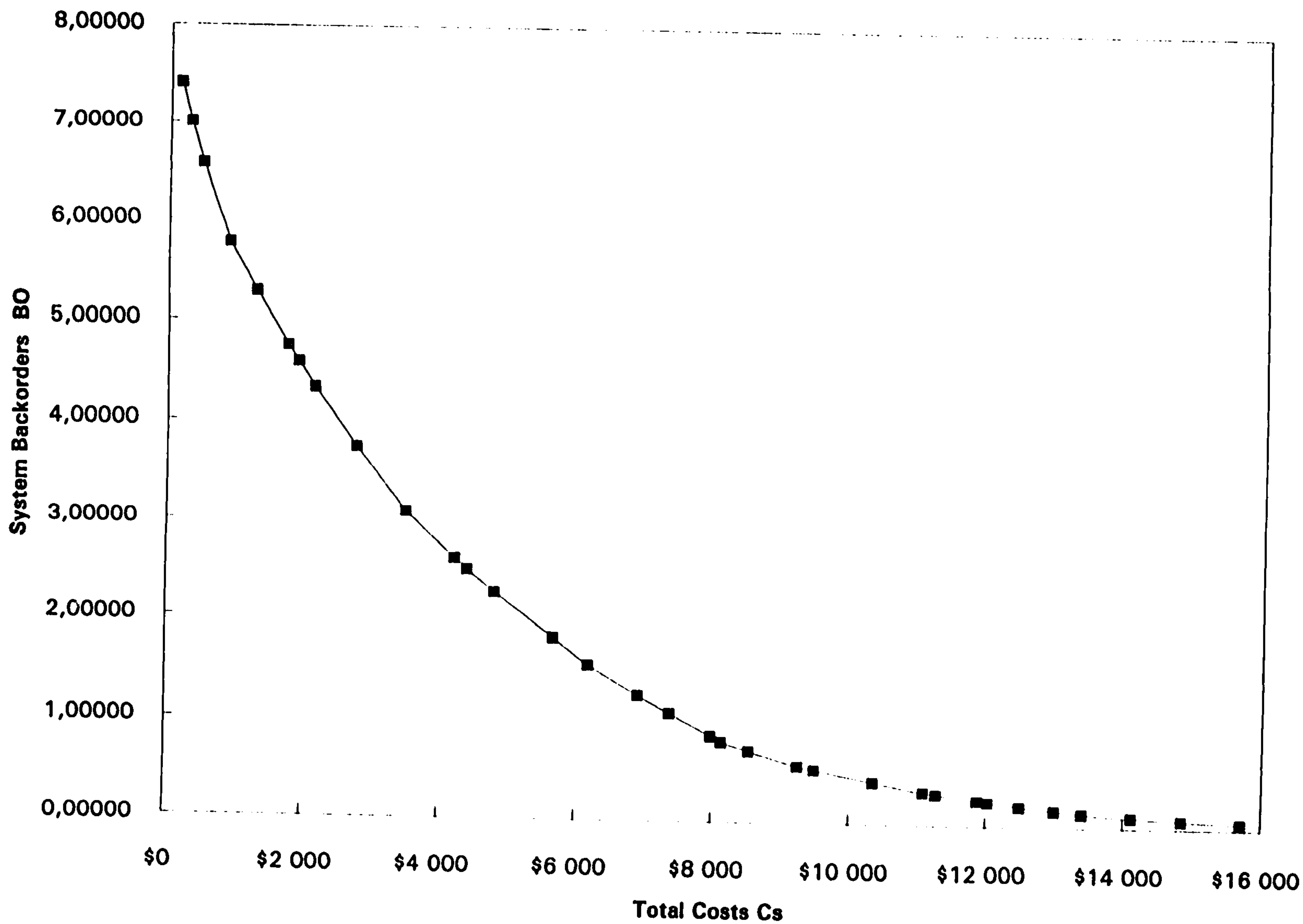


Figure 4.2: Marginal analysis for model P2 to min BO (J=10 items)

4.4.6 <u>Comparison model P1 vs P2</u>. We now compare the 2 models P1 and P2 and find out that solving both models practically give the same results; for example 1 with J=3 items and example 2 with J=10 items, the results are exactly the same stock level vectors $\{S_j,\ j=1,..,J\}$, given the specified budget values, therefore the corresponding A_S and BO values are also the same.

Figure 4.3 below gives the overlaid response curves $\{A_S$ vs $C_S\}$ when solving P1 to maximize A_S (same as figure 4.1) and the corresponding response curve $\{A_S$ vs $C_S\}$ when solving P2 to minimize BO (same as figure 4.2). As the figure clearly shows, most of the undominated points generated with the marginal analysis procedure are identical for both models and do not differ by much when some iterations are not the same. For all larger scale problems presented in chapter 8, the results follow the same pattern and can be considered practically the same.

The reason for this striking similarity can be analyzed using the marginal analysis derivation procedure; at each iteration k, model P1 seeks to max A_S by selecting the best (maximum) ratio $\{\ln A(S_j+1) - \ln A(S_j)\}/c_j$, $j=1,..,J$ which is concave for all values of $S_j=0,1,..$ but the resulting increase in A_j per dollar invested is not concave for values of S_j below the mean δ_j for the Poisson distribution, but do become concave for values $S_j > \delta_j$; individual A_j's provide the greatest impact on A_S when S_j's are close to δ_j. For model P2 the best (maximum) ratio $\{BO(S_j) - BO(S_j+1)\}/c_j = (1-A_j(S_j))/c_j$, $j=1,..,J$ or the item providing the most net reduction in BO per dollar invested and is convex for all values of S_j. Thus, for higher values of A_S, all individual item availabilities A_j's have progressively higher S_j values and once all of them have S_j values $> \delta_j$, the tails of the Poisson distribution become well behaved exponential functions and the ratios for both models are closely related: model P1 behaves like increasing the $A_j(S_j+1) - A_j(S_j) = p_j(S_j+1)$ while model P2 is $(1-A_j) = \Sigma\ p_j(x_j)$, from $x_j=S_j+1$ to infinity, which tends towards $p_j(S_j+1)$ as S_j becomes increasingly larger and tends towards infinity.

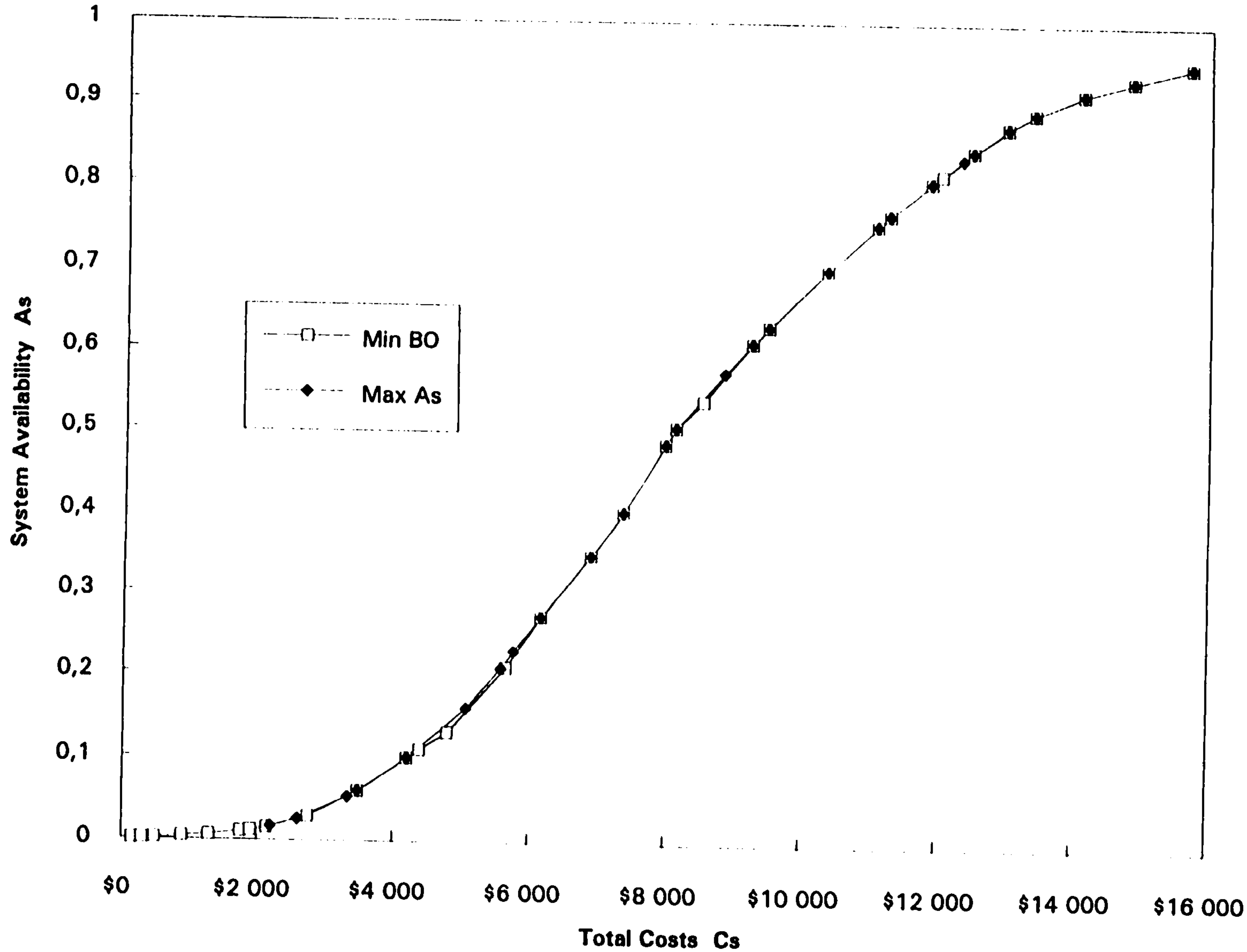


Figure 4.3: As values for model P1 vs P2 (J = 10 items)

This explains why the response curve $\{A_S$ vs $C_S\}$ between the 2 models have several points that differ for low values of A_S, when A_j's are such that stock levels $\{S_j$'s$\}$ in the current solution are below and/or close to their Poisson mean parameter values $\{\delta_j$'s$\}$ where the increase in A_j values impacts A_S the most, and progressively share more undominated common points as A_S increases past their means, and eventually become equal by equalizing inventories across all items j, j=1,..,J, since the ratios become less affected by their individual costs c_j.

4.4.7 Variant of P2 (model P2a). For the identical reasons described earlier for the variant of model P1, solving P2 for an arbitrary high budget value B with the marginal analysis procedure for backorders will also solve the following equivalent model P2a:

$$\text{Min} \quad C_S = \sum_{j=1}^{J} c_j S_j \tag{P2a}$$

$$\text{s.t.} \quad BO = \sum_{j=1}^{J} BO_j = \sum_{j=1}^{J} \sum_{x=S_j+1}^{\infty} (x-S_j).p_j(x) \le \text{ß} \tag{4.23}$$

$$S_j \ge 0 \qquad j=1,2,\ldots,J \tag{4.24}$$

where ß is the maximum allowable total expected number of backorders for an arbitrarily high budget value B and given that we start at some stockage vector quantity $\{S\}$. Example 1 could have easily been solved by the marginal analysis procedure by setting a budget of say B=$100. and the initial allocation vector quantity $\{S^0\}=\{0,0,0\}$.

With this initial allocation, the starting number of total expected system backorders is easily determined to be $= \Sigma\ \delta_j$ and the procedure will allocate one item at a time selecting the maximum benefit to cost ratio from all items at each iteration which is the maximum net reduction in backorders per dollar invested and stop when the total system backorders has decreased to the threshold value ß set by the analyst.

Again, the managerial interpretations of both models P2 and P2a are also clearly different as were models P1 and P1a, but marginal analysis applied to both models will result in the same sequence of undominated stockage levels allocation vectors $\{S_j\}$ and therefore can provide managers with an alternative solution methodology to choose from in order to select the optimal stockage level vector $\{S_j, j=1,..,J\}$ based on the objective function they wish to optimize.

4.5 TOP-UP MARGINAL ANALYSIS PROCEDURE

We have seen that the regular marginal analysis procedure for both models P1 and P2 (and others) results in a sequence of undominated allocation vectors $\{S_j, j=1,..,J\}$ and a corresponding response curve $\{A_S, C_S\}$ or $\{BO, C_S\}$ that should prove extremely useful to managers in deciding the best appropriate service level to choose from an arbitrarily high specified budget value. The procedure is much faster than the dynamic programming procedure since the number of iterations required is much smaller; before the procedure can be carried out, the total possible number of iterations can easily be determined as the sum of the range of possible values for each item or simply $\Sigma \{UB - LB\}_j$, $j=1,..,J$.

Furthermore, we know that the procedure increases one item at a time until the last iteration k at a total cost $C^k \leq B$ when the next item to be added caused the budget to be exceeded at iteration k+1 for a total cost $\{C^{k+1}\}$ strictly greater than B; since each successive undominated point $\{C^{k+1} - C^k\}$ can be spaced by at most the most expensive cost item, an a-priori error of at most $\{C^{k+1} - C^k\} = \{\max c_j, j=1,..,J\}$ can easily be calculated. We now turn our attention to the top-up marginal analysis procedure which will reduce the spacing between iteration k and k+1 from at most the most expensive to the least expensive item, thus reducing the error to $\{\min c_j, j=1,..,J\}$.

Once the last iteration k of the marginal analysis procedure has given us the undominated point $\{A_S, C_S(k)\}$ and therefore a lower bound on A_S or $LB(A_S)$, the next iteration k+1 that added the next most profitable item j whose cost c_j caused the budget to be exceeded, also gave us the undominated point $\{A_S, C_S(k+1)\}$ and therefore an upper bound $UB(A_S)$. It is possible to further improve the LB on A_S by simply topping up the current optimized stock level vector $\{S_j\}$, obtained at iteration k, by using marginal analysis and adding lower cost items than the item cost at iteration k+1.

This procedure essentially starts by first eliminating all items whose cost $c_j \geq$ cost of the item added at the (k+1)st iteration since none of them can be added to the current stock levels without exceeding the budget; then, from the remaining items whose costs are stricly smaller, we select the most profitable one still using the marginal analysis procedure. If the item selected has a cost $c_j \leq B - \{C_S(k)\}$, it is added to the current solution, otherwise, all remaining items whose cost exceed the selected item are eliminated. The procedure is repeated until no more items can be added or, conversely, until all items have successively been eliminated, the last point obtained on the response curve $\{A_S, C_S\}$ constitutes an improved LB on A_S or $LB^*(A_S)$, but does not guarantee that it is undominated.

Adding an item using this method will automatically increases A_S the most profitable way and thus improves its lower bound from $LB(A_S)$ to $LB^*(A_S)$ and will bring the total cost C_S closer to B. Since the process of eliminating items whose costs c_j causes the budget to be exceeded, we know that the total cost solution C_S will be at least as close to the budget B as the least expensive of all cost item or min $\{c_j, j=1,..,J\}$, or an a-priori error $ß(\%) = (B - \min\{c_j, j=1,..,J\})*100/B)$, which is substantially better than the one derived earlier, which was based on the most expensive item or max $\{c_j, j=1,..,J\}$.

The top-up procedure is also applicable to model P2 and all other models for which the regular marginal analysis applies, including [Schaefer 1983] and [Ebeling 1991] models discussed earlier. In fact, Schaefer used integer inventory costs {1,2,1} for J=3 items and the top-up procedure would have resulted in the complete optimal response curve for the model described in the paper, while incorporating more realistic cost parameters in Ebeling's model would have resulted in a much more computationally difficult problem to solve using DP and marginal analysis.

We now illustrate the topup marginal analysis procedure with the same larger scale J=10 item example 2 introduced earlier but at a lower budget B=$10,000 vs $15,000. With a specified available budget B = $10,000., item costs ranging from the least expensive = $152 to the most expensive = $860. and mean Poisson parameters ranging from 0.293 to 1.398, as before; thus, the a-priori maximum error ß of C_S from B based on the least expensive item will be ≤ $152/$10000 or ≤ 1.52% of the budget, which compares with 8.60% based on the most expensive item for the regular procedure. By specifying a min A_j = .000001 and max A_j = 0.999999 for each item j=1,..,J=10, the maximum number of possible iterations of the procedure will be Σ $\{max - min\}_j$, j=1,..,10 which turn out to be 87.

Starting from the initial alocation vector $\{S^{k=0}\}=\{0,..,0\}$, figure 4.4 below shows the sequence of iterations until the budget has been exceeded at iteration k+1 = 23 rd, using the regular marginal analysis procedure. The solution vector $\{S_j\}$ = {1,1,4,2,2,2,2,2,2,4} at a total cost C_S = $9484 (actual error on C_S ≤ $516/$10,000 or less than 5.16 % from B) at the k=22 nd iteration resulting in $LB(A_S)$ = 0.62961 while the k=23 rd iteration caused the budget B=$10,000 to be exeeded by the addition of an item costing $860 (coincidence that it happens to be the most expensive) for a total cost C_S = $10,344 and $UB(A_S)$ = 0.69948; the true optimal solution lies between 0.62961 ≤ A^* ≤ 0.69948.

The topup marginal analysis procedure then reverted back to the solution vector obtained at the 22 nd iteration and proceeded to eliminate all items whose cost exceeded c_j=$860 and added the next most profitable item from the remaining ones, whose cost c_j was ≤ $10,344 - $10,000 = $344; as a result of the topup procedure, figure 4.4 shows that three more items (2 type j=3 and 1 more type j=4) were subsequently added to yield $\{S_j\}$ = {1,1,6,3,2,2,2,2,2,4} for a total cost C_S = $9,974 (actual error on C_S ≤ $26/$10,000 or less than .26% from B) and improved the lower bound from $LB(A_S)$ = 0.62961 to $LB^*(A_S)$ = 0.64895 or a 3.07% increase over the regular procedure. Every successive point ensures the function is strictly increasing and dominates the previous one in the sequence, but cannot guarantee that it is the true optimal solution. The true value of A_S is likely to be slightly ≥ the one found by the top-up procedure.

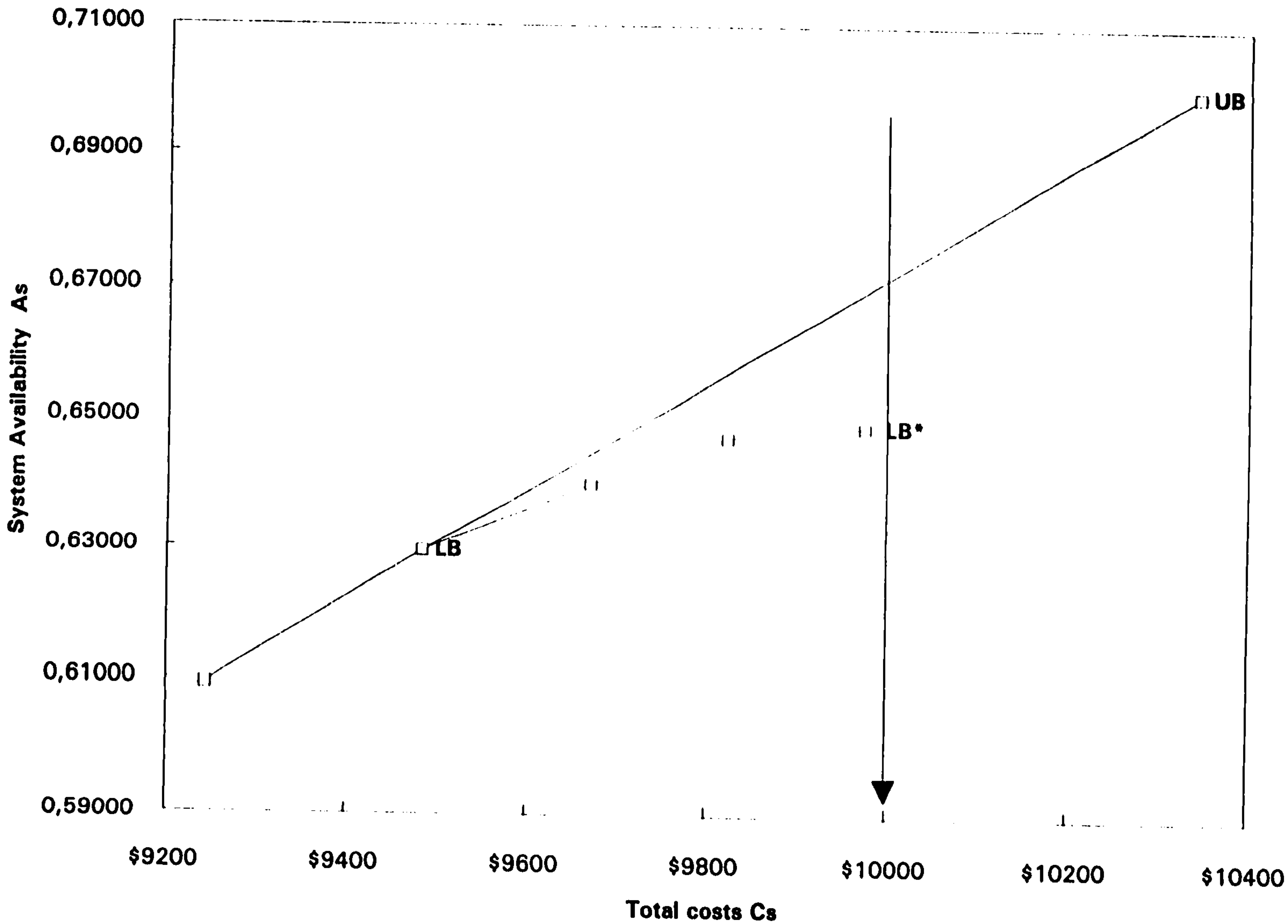


Figure 4.4: Topup procedure P1 (J = 10 items)

4.6 INTERPOLATION PROCEDURE

Since the topup procedure is likely to slightly underestimate the true value of A_S, another heuristic procedure is to use a simple linear interpolation at the budget value B between the two undominated points LB (lower than B) and UB (higher than B) found earlier with the marginal analysis procedure. Since this technique is equivalent to drawing a straight line between the two points as opposed to the increasing concave function drawn as a result of the top-up procedure, the estimate for A_S is likely to be slightly higher than the true value of A_S.

For the same J=10 item problem discussed above with a budget B=$10,000., the linear interpolation between the two points LB and UB, evaluated at B=$10,000 is illustrated in figure 4.5 below and found as follows:

$$\text{Interp } A_S = LB(A_S) + \frac{(B-C_S(LB))*(UB(A_S)-LB(A_S))}{(C_S(UB)-C_S(LB))} \quad (4.25)$$

$$= .62961 + \frac{(10000-9484)*(.69948-.62961)}{(10344-9484)}$$

$$= .62961 + .04192 = .67153$$

which compares with the .64895 value or a 3.48% increase over the topup procedure described in the previous section. This method is shown in figure 4.5 below at the intersection of the straight line drawn from LB to UB values for A_S and on the vertical line at B = $10,000. Even though the estimate found using the interpolation procedure is, on average, likely to be higher than the estimate found with the top-up procedure, it cannot guarantee that it is an undominated point, nor that it will always be higher.

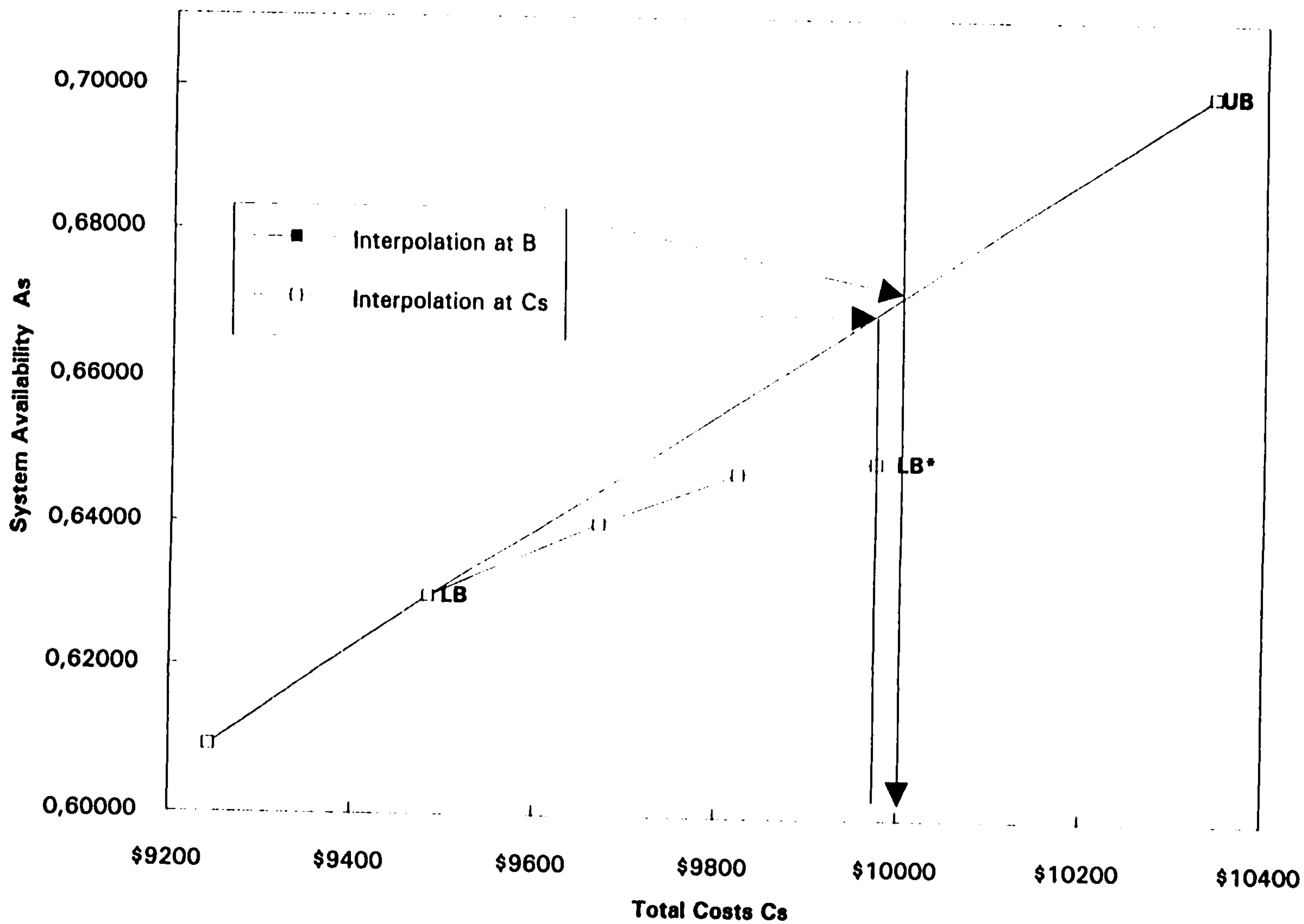


Figure 4.5: Interpolation procedure for P1 (J = 10 items)

In order to compare the two methods more equally, the interpolation of A_S should also be carried out at the C_S value obtained as a result of the top-up procedure as opposed to the budget value B; therefore, replacing B by C_S obtained at the $LB^*(A_S)$ yields the following alternate formula to (4.25):

$$\text{Interp } A_S = LB(A_S) + \frac{(C_S(LB^*) - C_S(LB)) * (UB(A_S) - LB(A_S))}{(C_S(UB) - C_S(LB))} \qquad (4.26)$$

$$= .62961 + \frac{(9974-9484)*(.69948-.62961)}{(10344-9484)}$$

$$= .62961 + .03981 = .66942$$

which is close to the A_S value of .67153 when interpolated at B. Comparisons of the interpolation procedure with the top-up marginal analysis procedure for model P1 when maximizing A_S is further discussed for several larger scale problems in chapter 8, which will also include summary results vs the current military model when optimizing items individually. The same procedure can be carried out for model P2 to minimize BO. Since the end result of this procedure may give us a better estimate for A_S, it does not give us the resulting stock level vector $\{S_j,\ j=1,..,J\}$ and will therefore not be analyzed any further.

4.7 COMPARISON MODEL P1 vs EQUAL Aj's

We are now in a position to compare the response curves $\{A_S \text{ vs } C_S\}$ between the current military model whereby equal A_j's are computed vs the marginal analysis procedure, whereby the most cost benefit items are added one at a time until the budget B is exceeded. For the equal A_j's model (military), the response curve $\{A_S \text{ vs } C_S\}$ was derived as follows: a starting low value for A_j was specified and for each item $j=1,..,J$, the stock level S_j was obtained from the cumulative Poisson distribution, from which the system availability $A_S = \pi\ A_j,\ j=1,..,J$ and the resulting C_S was calculated; then A_j was successively increased to $A_j + (1-A_j)/1000$ and the procedure repeated until C_S exceeded the budget B.

If the aggregate system performance measure A_S is the criterion to be optimized, then the solution vector $\{S_j,\ j=1,..,J\}$ and the resulting points $\{A_S \text{ vs } C_S\}$ obtained from marginal analysis will always dominate the solution obtained by equalizing A_j's across all items j=1,..,J. For a given specific available budget B, the solution vector and the last undominated point $\{A_S \text{ vs } C_S(k)\}$ obtained from marginal analysis at iteration k and the next undominated point $\{A_S \text{ vs } C_S(k+1)\}$ at iteration k+1 may be dominated by equalizing A_j's across all items when $C_S(k) \le B \le C_S(k+1)$, but is highly unlikely. The reason would be mainly due to the jump in costs from $C_S(k)$ to $C_S(k+1)$, where $C_S \le B \le C_S(k+1)$, as a result of one of the most expensive of all J items, which is impossible to predict and can be considered a highly unlikely event, as figure 4.6 for example 1 (J=3 items) below clearly shows.

The response curves $\{A_S \text{ vs } C_S\}$ for the 3 models: P1, P2 and Equal A_j's for a specified available budget value B=\$50 in figure 4.6 do not differ much and thus, result in stock levels $\{S_j,\ j=1,2,3\}$ that are identical (or nearly) for practically all possible intermediate budget values from B=\$0 to B=\$50, except for budget values that could fall in between undominated points. For example, suppose we had specified B=\$26, marginal analysis for P1 causes both points {0.65126, \$22} and {0.81408, \$27} to be undominated at these costs but the equal A_j (military) model would give the better solution {0.73655, \$25} if B=\$25 or \$26 and would also give the same solution {0.81408, \$27} for B=\$27; this event could theoretically occur for any problem but is increasingly unlikely as the number of items J increases.

The immediate consequence of this conclusion is that significant savings can be achieved by optimizing an aggregate system performance measure such as maximizing A_S and/or minimizing BO. Since models P1 and P2 result in identical (or nearly) stock level vectors $\{S_j,\ j=1,..,J\}$, we shall mostly restrict our analysis to the comparison between the current military model or Equal Aj's and model P1 to maximize A_S.

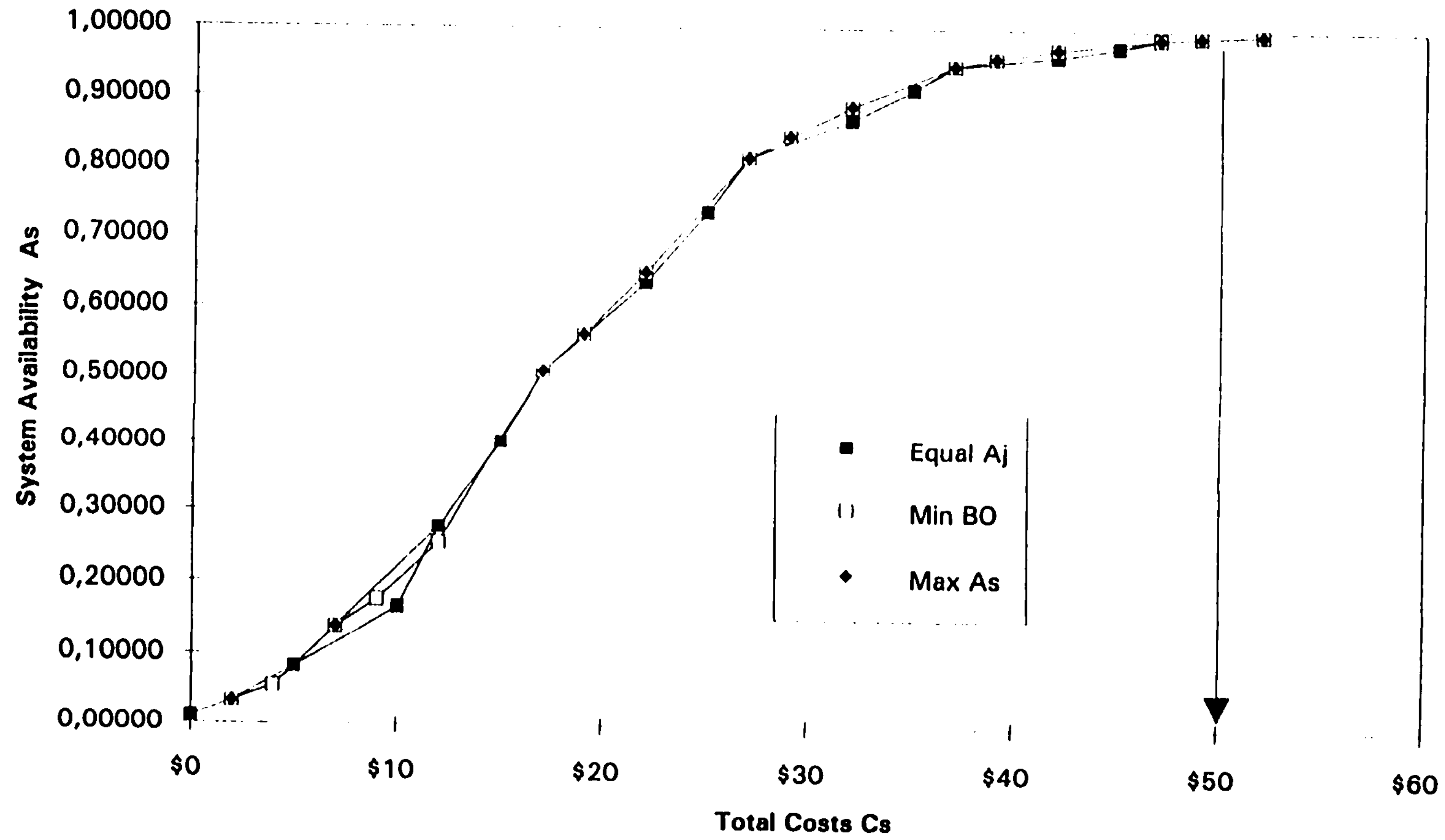


Figure 4.6: As values for models P1 vs P2 vs Equal Aj's (J=3 items)

Figure 4.7 below illustrates for the J=10 item example 2 discussed earlier, the differences between the response curves {A_S vs C_S} between all 3 models: P1, P2 and Equal A_j's (military model) for 2 different possible available budget values at B=$10,000 and B=$15,000., or at any other possible budget value, for which the following comparative results were obtained:

B=$10,000	A_S	C_S
Model P1 (Max A_S, no topup):	0.62961	$9,484
Model P1 (topup, not shown):	0.64895	$9,974
Model P2 (Min BO, no topup):	0.62961	$9,484
Model P2 (topup, not shown):	0.64895	$9,974
Model Equal Aj's :	0.61040	$9,550

B=$15,000	A_S	C_S
Model P1 (Max A_S, no topup):	0.93010	$14,851
Model P1 (topup, not shown):	0.93010	$14,851
Model P2 (Min BO, no topup):	0.93010	$14,851
Model P2 (topup, not shown):	0.93010	$14,851
Model Equal Aj's :	0.90469	$14,513

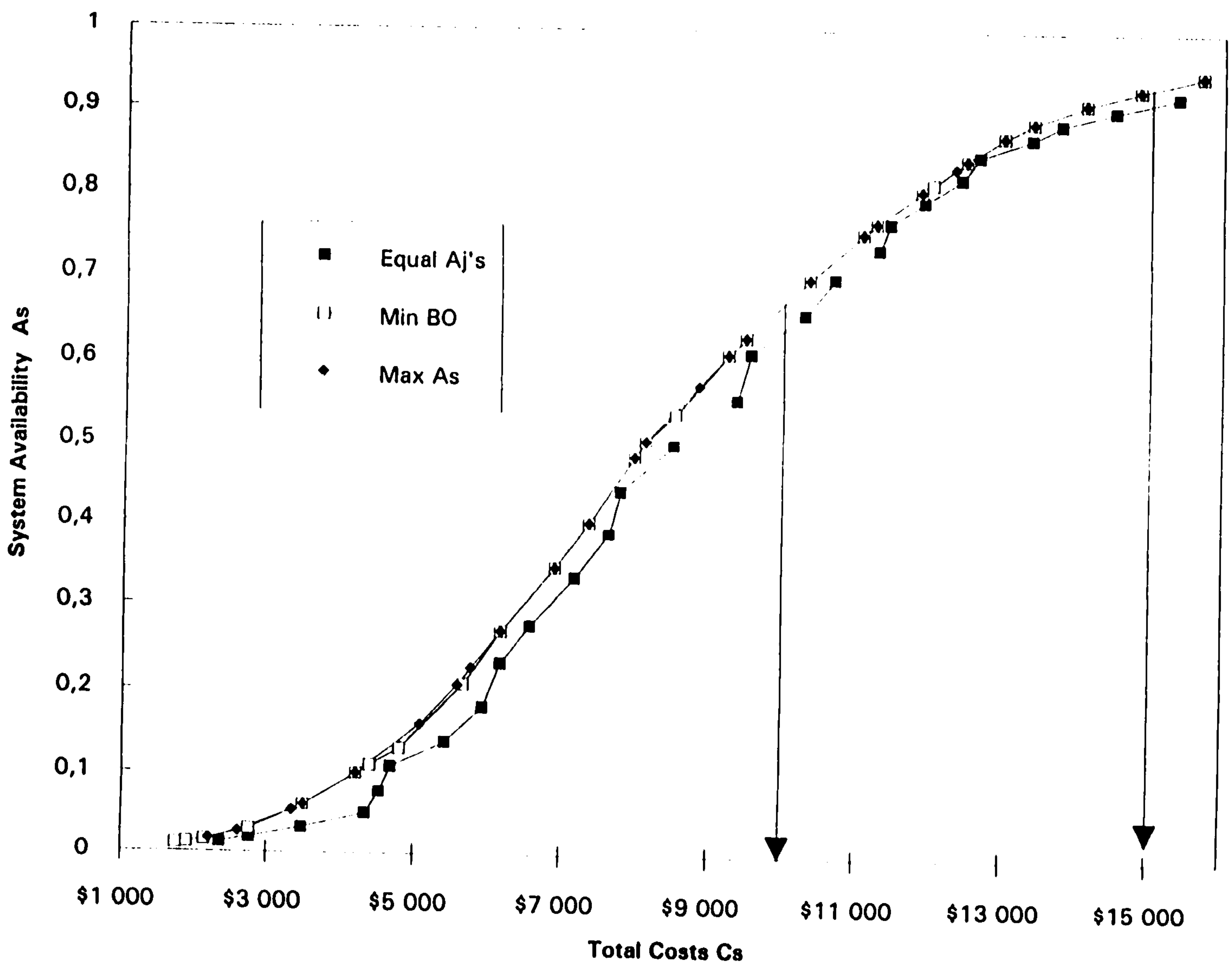


Figure 4.7: As values for models P1 vs P2 vs Equal Aj's (J=10 items)

As can be seen from figure 4.7, the equal A_j's model equalizes all individual item availabilities A_j's regardless of their cost and will underperform the marginal analysis procedure; we note that the response curves $\{A_S \text{ vs } C_S\}$ between model P1 and the model with equal A_j's is the same from the starting allocation vector of $\{S\} = \{0,..,0\}$ whose total cost is C_S=\$0, and progressively widens in the middle portion of the A_S curve as stock levels S_j's increase around their Poisson means and cause the greatest increase in item availabilities A_j's; finally, it tends to become the same again for higher A_S values since individual A_j's become high enough that equalizing stock levels across all items at a given A_j, j=1,..,J, regardless of their costs, become optimal. For example 2 with J=10 items and A_S =.90 in figure 4.7 means that all A_j's > $.90^{(1/10)}$ = 0.99; at their current S_j levels, the increase in A_S as a result of adding any item will become less and less significant and less dependent on their cost parameters.

So, we can conclude from the above data that if only \$10,000 was available for the period, the stock level vector $\{S_j\}$ obtained from the topup marginal analysis shown earlier in figure 4.4 would cost C_S = \$9,974. and result in A_S = 0.64895 while the equal A_j's model (military) stock level vector $\{S_j\}$ would cost \$9,950. and A_S = 0.61040, a 5.94% decrease in A_S at approximately the same costs.

Conversely, as the model P1a variant would show, in order to achieve a minimum system availability A_S = .64, the vector $\{S_j\}$ from marginal analysis would cost \$9,974. compared to approximately \$10,284. for the military model with equal A_j's, or a 3.0% savings (The exact values at each iteration have not been shown here but are available on request).

If the analysis is projected over several locations, each having a different number of equipments M which proportionately increase /decrease the Poisson mean parameters $\{\delta_j$, j=1,..,J$\}$, then the increase in A_S at each location, given a budget level or, conversely, the savings in costs to achieve a minimum A_S at each location can become significant when a system performance measure such as A_S is maximized (or when BO is minimized for model P2) with the marginal analysis procedure as opposed to simply equalizing inventory levels across all items at a given availability level.

In figure 4.8 below, we analyze the differences in stock level vectors $\{S_j\}$ between the 2 models of the J=10 item example 2 by plotting the difference between $\{S_j,\ j=1,..,10\}$ obtained with the marginal analysis and $\{S_j,\ j=1,..,10)$ obtained from the equal A_j's model, denoted $\{$Marg S_j - Equal $S_j\}$ on the Y-axis vs the cost to MTBF ratio for each item, denoted $c_j/MTBF_j$ on the X-axis. Thus a positive (+) difference for an item j indicates the number of those items that the marginal analysis stocks more of that type than the equal A_j model and vice versa for a negative (-) difference.

As expected, items with (+) difference have lower cost to MTBF ratio, meaning that the marginal analysis procedure will stock more of an item that has either (or both) lower cost or/and high MTBF, i.e. reliable items at low costs, as compared with the indiscriminate process of the equal A_j model, which tends to stock more of an item with high cost or/and low MTBF or less reliable and expensive parts, as indicated by the (-) differences on the right side of the X-axis. This finding becomes more accentuated as the Poisson mean parameters $\{\delta_j\}$ increase; in our J=10 item example 2, parameters vary only from 0.293 to 1.398 but if the number of equipments M operating at a location become larger, then the Poisson means become proportionately higher and the same pattern becomes accentuated as will be shown in chapter 8 for larger scale problems.

The size of the gap between the 2 models can easily vary up to 10% in the middle portion of the curve as the range between cost to MTBF ratios for items within an equipment become progressively larger, as would be expected, since the marginal analysis procedure will always select the most profitable item to add at each iteration while the equal A_j model disregards costs entirely and selects the number of items to achieve a specified A_j across all items $j=1,..,J$; as discussed earlier, the differences will be largest when stock levels S_j are close to their means δ_j and cause the greatest impact on A_S.

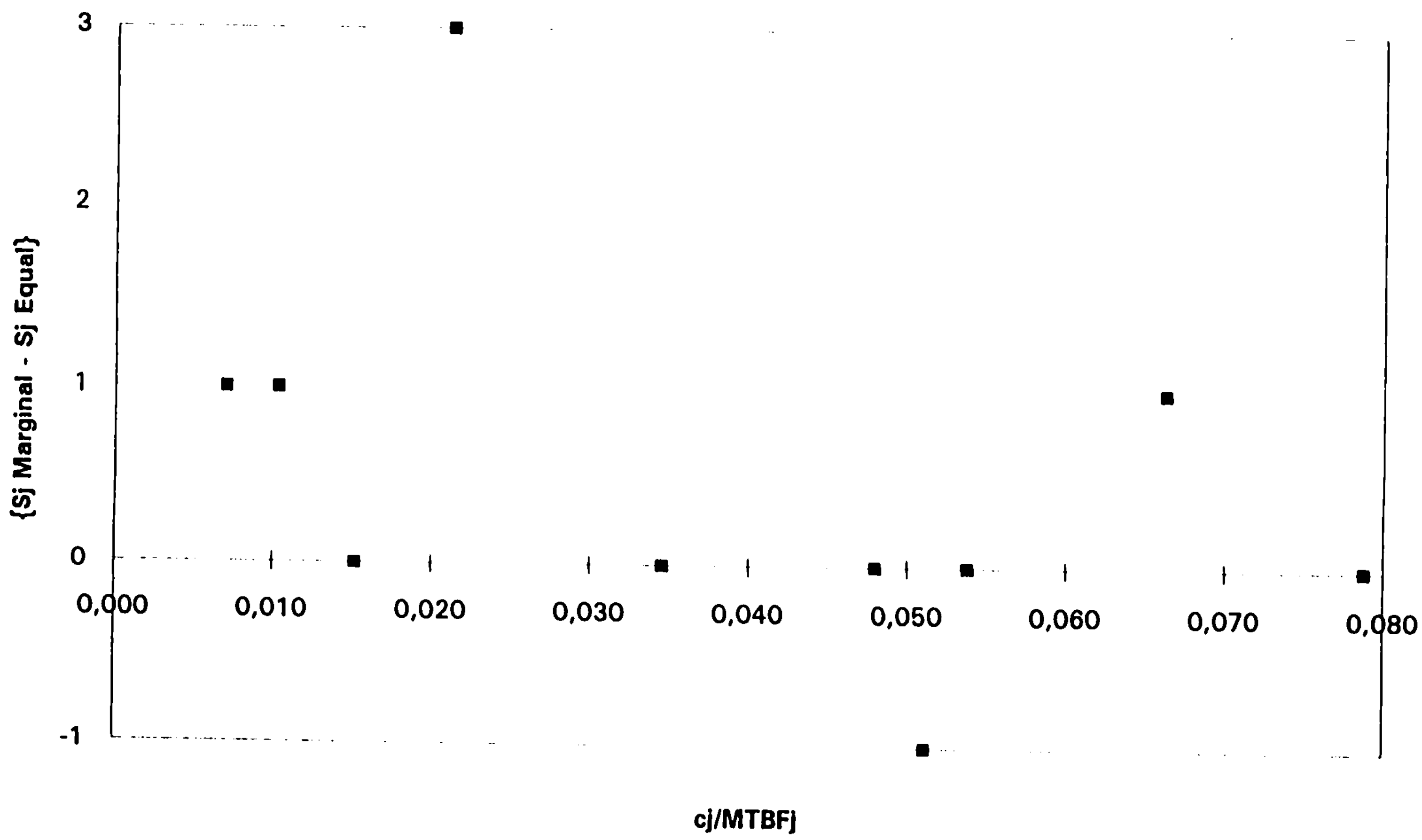


Figure 4.8: Differences in {Sj} between P1 and Equal Aj's (J=10 items)

CHAPTER 5: LAGRANGE MULTIPLIER SOLUTION PROCEDURE

5.1 GENERAL

The Lagrangian multiplier method has also been used to solve models similar to P1 and P2. Although the procedure is general in nature, the derivation to find the best and hopefully optimal value for the multiplier is model dependent. We therefore have to manipulate the models before we can begin a systematic iterative search procedure.

Two widely quoted papers on Lagrange multipliers, which are important in our context, have been written by [Everett 1963] and [Fox and Landi 1970] and applied in a variety of models. Everett has developed a generalized Lagrange multiplier method for optimizing functions that are not necessarily differentiable, which is well suited for resource allocation models. The method does not guarantee a feasible solution but, if one is found, then the solution obtained from the application of the method is undominated.

For discrete types of models (like P1 and P2), Everett suggests maximizing analytically the Lagrangian function assuming it is continuous, and then testing the integer on each side, selecting the one that maximizes the Lagrangian function. The parallel components example used by [Kettelle 1962] to determine the system reliability, is solved via the Lagrange multiplier method by Everett to illustrate the procedure and the source of possible "gaps" caused by a discrete function.

Fox and Landi compared different search methods to obtain the multiplier(s) and showed that the minimax sequential search is the bisection method. The technique can be applied in multi-echelon type of inventory systems developed by [Sherbrooke 1968], [Muckstadt 1973 and 1978], [Muckstadt and Thomas 1980], [Cohen et al 1992] as examples.

The general procedures have also been described in textbooks such as [Hadley and Whitin 1963], [Eiselt, Pederzoli and Sandblom 1987], [Bertsekas 1982], [Hillier and Lieberman 1990], [Winston 1994], and others.

[Nahmias and Schmidt 1984] have studied a special case where the multi-item "Newsboy" type of problem when demands are from a continuous function (normal distribution) and the Lagrangian method causes difficulties when optimal stockage levels for S_j's are set close to their mean number of failures.

[Cohen, Kleindorfer and Lee 1989] have generalized the procedure for the reverse of model P1 treated here, which is equivalent to the variant model P1a, already discussed, and lead to a different managerial interpretation of the results. In this chapter, we derive the procedure for the Poisson distribution for both models P1 and P2 (which will also be extended to the multiple location case models P1b and P2b, and develop the bounds on the multipliers, its initial estimate and error bounds on the solutions.

The exploratory comparative test results for model P1a provided by Cohen et all, were restricted to n=3,6 and 9 items and to the Lagrange relaxation and the binomial distribution, which will be extended to 16 randomly generated test problems for the Poisson distribution with J=10 up to J=99 items and the number of equipments M=1 up to 20; results will be compared and summarized in a later chapter. Although no formal comparison in execution time between the two procedures are presented, experimental empirical results indicate that the Lagrange relaxation method is approximately 5 to 10 times faster than the marginal analysis for models P1 and P2, P1b and P2b (multiple location models), as successive iterations of the Lagrange relaxation skip over several iterations of the marginal analysis procedure.

5.2 LAGRANGE MULTIPLIER METHOD FOR P1

5.2.1 Procedure derivation. Model P1 and its modified version (P1') are repeated here to illustrate the procedure:

$$\text{Max} \quad A_S = \prod_{j=1}^{J} A_j = \prod_{j=1}^{J} \left(\sum_{x=0}^{S_j} p_j(x) \right) \qquad \text{(P1)}$$

$$\text{s.t. } \sum_{j=1}^{J} c_j S_j \quad \leq B \tag{5.1}$$

$$S_j \geq 0 \qquad j=1,2,..,J \tag{5.2}$$

$$\text{Max } \ln A_S = \sum_{j=1}^{J} \ln(A_j) = \sum_{j=1}^{J} (\ln \sum_{x=0}^{S_j} p_j(x) \ ) \tag{P1'}$$

$$\text{s.t. } \sum_{j=1}^{J} c_j S_j \quad \leq B \tag{5.1}$$

$$S_j \geq 0 \qquad j=1,2,..,J \tag{5.2}$$

We form the following Lagrangian function P1'' with the Lagrange multiplier θ based on model (P1') which is separable by item and use the general notation for continuous functions f_j for $p_j(x)$ and F_j the cdf, instead of A_j:

$$\text{Max } L = \sum_{j=1}^{J} \ln(F_j) + \theta \ [B - \sum_{j=1}^{J} c_j S_j \ ] \tag{P1''}$$

The solution to P1'' is to find the set of S_j that will maximize L, by setting the partial derivatives to 0, i.e. $dL/dS_j = 0 = f_j/F_j - \theta.c_j$ which yields:

$$\theta^* = f_j/[F_j c_j] \qquad j=1,2,..,J \tag{5.3}$$

The procedure is then to:

- select an initial value for θ, say θ_1;
- determine all S_j's from (5.3);
- calculate $\Sigma\, c_j S_j = B_1$;
- if $B_1 < B$, select $\theta_2 < \theta_1$ or if $B_1 > B$, select $\theta_2 > \theta_1$ with a search procedure such as bisection and repeat until $B_n = B$ or $|\theta_n - \theta_{n-1}| <$ error α;
- round UP or DOWN all S_j values.

The problems with the general procedure applied to discrete functions are as follows:

- calculating upper and lower bounds for θ;

- guessing the initial estimate θ_1 for θ;
- can lead to serious problems when S_j's are small, say ≤ 5, which is not unusual in practice with expensive/low demand items since small changes in θ may cause large variations in the solution;
- the continuous assumption for $p_j(x)$ for our model, which is discrete (Poisson demand distribution) and lead to integrality problems.

Although the procedure is well defined, the first two problems described above become a significant factor as discussed by [Muckstadt 1978] for larger size problems in the U.S. Air Force; since the determination of upper and lower bounds and initial estimates for θ can be impossible to obtain, it can result in a few unsuccessful trial runs before the systematic search procedure is undertaken. For model P1, however, we can solve these problems as shown in the next sections. [Nahmias and Schmidt 1984] discusses the third problem while [Everett 1963] and [Fox and Landi 1970] analyzed the integrality problems caused by discrete functions. Everett recommends to test the integer solution obtained from the procedure on either side to maximize the Lagrangian function.

5.2.2 Upper and lower bounds for θ. Based on (5.3), note that as F_j tends to 0, the ratio f_j/F_j tends to 1 for the Poisson distribution since $p(x=0)/P[x\leq 0] = 1$; furthermore, the ratio f_j/F_j decreases to 0 as F_j tends to 1, since $p(x=n)/P[x\leq n] = 0$ for n in the extreme right tail of the Poisson distribution. Thus the lower bound for θ becomes:

$$LB(\theta) = 0 \quad \text{as } F_j \text{ tends to } 1 \tag{5.4}$$

and, for the upper bound UB, as F_j tends to 0:

$$UB(\theta) = 1/\min\,[c_j,\ j=1,..,J]; \tag{5.5}$$

in other words, UB(θ) is 1/ least expensive of the J items.

5.2.3 Initial estimate for θ. Although any initial estimate θ_1 for θ between its UB and LB can be successful, it is desirable to reduce the number of iterations to as few as possible; the procedure outlined next will ensure an excellent starting value for θ_1. The initial estimate θ_1 for θ can be determined as follows: for a reasonably high budget B (usually the case in practice), we wish F_j to tend towards 1 resulting in θ^* closer to its LB of 0 since the ratio f_j/F_j tends to 0, and we also want $\theta_1 \geq \theta^*$ so that the corresponding value of B_1 is $\leq B$ for the simple reason that the next value θ_2 using the bisection search method will halve the interval between θ_1 and LB=0 as illustrated in figure 5.1 below. Successive iterations will quickly tend towards θ^*.

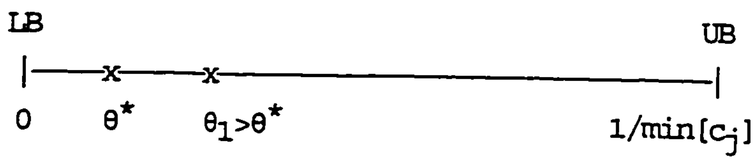


Figure 5.1: Bounds and estimate of θ for P1

Since all item availabilities F_j (or A_j) must be at least as large as the system availability A_S, that is, $F_j \geq A_S$ $(j=1,2,..,J)$, we can use $(1-A_S)$ as a numerator in (5.3) for the ratio f_j/F_j which tend towards 0, and use $\max[c_j]$ in the denominator of (5.3) as well to compensate for the fact that all A_j's will in fact be $> A_S$ and therefore $1-A_j \geq 1-A_S$; $(1-A_S)$ is a conservative estimate for $(1-A_j)$. Several examples suggest that the initial estimate works very well for reasonable parameter values:

Initial estimate: $$\theta_1 = \frac{(1-A_S)}{\max\,[c_j]} \qquad j=1,2,..,J \qquad (5.6)$$

5.2.4 Solution. The solution example 1 given earlier with B=\$20 and the 3 item parameters $\{\delta_j\}=\{1, 1.5, 2\}$ and $\{c_j\}=\{\$5, \$3 \text{ and } \$2\}$ and a minimum A_S = .001 and .600 displayed in tables 5.1 and 5.2 below for comparative purposes. The solution yielded the following Lagrange multiplier sequence of allocations which stopped after 14 and 12 iterations respectively when a specified error was reached:

Table 5.1: Lagrange iterations for P1, Min As=.001 (J=3 items)

θ	S	$C_S(S)$	A_S
.1998	0,1,2	7	.139
.0999	1,2,3	17	.510
.05	1,2,3	17	.510
.025	2,3,4	27	.814
.037	2,3,4	27	.814
.044	1,3,4	22	.651
.047	1,2,4	19	.564
.045	1,2,4	19	.564
.044	1,3,4	22	.651
.045	1,2,4	19	.564
...	...	..	...

Table 5.2: Lagrange iterations for P1, Min As=.600 (J=3 items)

θ	S	$C_S(S)$	A_S
.08	1,2,3	17	.510
.04	2,3,4	27	.814
.06	1,2,3	17	.510
.05	1,2,3	17	.510
.045	1,2,4	19	.564
.0425	1,3,4	22	.651
.044	1,3,4	22	.651
.044	1,3,4	22	.651
.045	1,2,4	19	.564
...	...	..	...

5.2.5 Comments. First we note that the method does not guarantee an optimal solution for B=\$20 which was $\{S^*\}=\{1,3,3\}$ with A_S=.589 and C_S=\$20 obtained earlier with DP because of the discrete nature of the Poisson distribution and the derivation of the procedure that assumed a continuous function. As recommended by [Everett 1963], we could test all possible combinations of S_j's on either side of the integer solution which would have resulted in the optimal allocation vector

$\{S^*\}=\{1,3,3\}$. Second, this method is much faster than the marginal analysis described earlier, which is itself faster than dynamic programming since the procedure skips several iterations of the marginal analysis described earlier. Third, we notice the validity of the UB, LB and the initial estimate θ_1. For our example 1:

A_S=.001, LB=0, UB=1/2=0.5, θ_1=(1-.001)/5 = 0.1998, B_1=7

A_S=.600, LB=0, UB=1/2=0.5, θ_1=(1-.600)/5 = 0.0800, B_1=17.

Fourth, the interpretation of the multiplier value as the shadow price for the original constraint is more difficult to interpret for our model. Since the modification from P1 to (P1') uses the log transformation of model P1, we can say that small unit increases in the budget B will result in an increase of the objective function $\ln(A_S)$, by approximately θ^*. Since this is valid for small increases in B coupled with the integrality problem due to the discrete nature of the model, we must be cautious about its significance.

Finally, [Cohen, Kleidorfer and Lee 1992] have reported some experimental results of using this procedure with a model very similar to model P1a, described earlier; although a restricted number of items J=3,6 and 9 items and the Binomial distribution instead of the Poisson distribution have been used, no comparison with the marginal analysis has been made and the authors have quoted a referee that Lagrange relaxation may do fewer iterations than marginal analysis but each Lagrange iteration may require more calculations than an iteration done with marginal analysis.

The experimental results of numerous problems (more than 100 generated problems varying J=1 up to 99) indicate that Lagrange is approximately 5 to 10 times faster than the marginal analysis, and all of them giving the optimal solution in less than 30 iterations. Thus the speed of execution is much faster and although it can be a valid criterion for selecting this method over marginal analysis, the response curve $\{A_S, C_S\}$ can be made up of only a few points (generally less than 20), which do not give valuable information in between the total initial costs C_S of the initial starting vector $\{S_j, j=1,..,J\}$ and the available budget B.

If more points on the response curve $\{A_S, C_S\}$ provided by Lagrange relaxation is desired, then one could conceivably use a different search procedure than the binary search procedure described by Fox 1966 and used here; for example, since we have derived a lower and an upper bound for the multiplier θ, we could divide the range (UB - LB) into say 100 equally divided values for θ and solve for each one; the most important drawback of this procedure is that most of the points on the curve may be above the available budget B.

The authors have also proved (as Fox did earlier in 1966) that the optimal solution vector using the Lagrange relaxation method for model P1a (and consequently model P1) will be the same as the optimal solution obtained from marginal analysis. The numerical experiments performed for the Poisson distribution involving up to J=99 items and budgets up to $5 million dollar budgets confirmed this important theorem.

5.2.6 Lagrange multiplier method for P1a. As with the marginal procedure, solving P1 for an arbitrarily high budget value B will also solve model P1a to minimize total cost C_S subject to a minimum specified service level as described earlier in chapter 4. The reader can easily verify that the Lagrange multiplier found for P1a will become:

$$\theta^* = [F_j c_j] / f_j \qquad (5.7)$$

which is just the inverse (reciprocal of θ^*) derived above for P1. The natural logarithm tranformation must first be performed on the availability constraint and the derivation is along the same steps and the continuous assumption used previously for model P1. Bounds and initial estimate also remain the same as above but will be for $1/\theta^*$.

5.3 LAGRANGE MULTIPLIER METHOD FOR P2

5.3.1 Procedure derivation. Model P2 is repeated here for convenience:

$$\text{Min} \quad BO = \sum_{j=1}^{J} E(BO_j) = \sum_{j=1}^{J} \sum_{x=S_j+1}^{\infty} (x-S_j).p_j(x) \tag{P2}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \leq B \tag{5.8}$$

$$S_j \geq 0 \qquad j=1,2,..,J \tag{5.9}$$

We form the Lagrangian function L, assuming that $p_j(x)$ is a continuous function f_j and F_j, its pdf and cdf respectively:

$$\text{Min } L = \sum_{j=1}^{J} \int_{x=S_j+1}^{\infty} (x-S_j).f_j(x)dx + \theta\ [\ B - \Sigma\ c_j S_j\ ] \tag{P2'}$$

The solution to P2' is to find the set of S_j that will minimize L, by setting the partial derivatives to 0, i.e. $dL/dS_j = 0 = -(1-F_j) - \theta c_j$ which yields:

$$\theta^* = -\ (1-F_j)/c_j \qquad j=1,2,..,J \tag{5.10}$$

The value of θ^* is negative and can be interpreted as the decrease in BO resulting from a budget unit increase for the investment constraint. The procedure described below, the problems about bounds for θ, its initial estimate θ_1 and the integrality requirements are the same as for model P1 except the reverse sign of θ; we therefore assume it is positive and adopt the same procedure as for P1:

- select an initial value for θ, say θ_1;
- determine all S_j's from (5.7);
- calculate $\Sigma\ c_j S_j = B_1$;
- If $B_1 < B$, select $\theta_2 < \theta_1$ or if $B_1 > B$, select $\theta_2 > \theta_1$ with a search procedure such as bisection and repeat until $B_n = B$ or $|\theta_n - \theta_{n-1}| <$ error α;
- round UP or DOWN all S_j values.

5.3.2 Upper and lower bounds for θ. We can also derive UB and LB for θ as follows: as F_j tends toward 1, θ tends towards 0, therefore:

$$LB(\theta)=0; \text{ as } F_j \text{ tends towards } 0 \tag{5.11}$$

and the ratio $(1-F_j)/[c_j]$ will tend towards $1/[c_j]$, leading to the upper bound:

$$UB(\theta)=1/\min[c_j,\ j=1,..,J] \tag{5.12}$$

or the least expensive of all J items.

5.3.3 Initial estimate for θ. The initial estimate θ_1 for θ can be determined as follows: for a reasonably high budget B (usually the case in practice), we wish F_j to tend towards 1 resulting in θ^* closer to its LB of 0 since the numerator $(1-F_j)$ in (5.10) tends to 0, and we also want $\theta_1 \geq \theta^*$ so that the corresponding value of B_1 is $\leq B$ for the simple reason that the next value θ_2 using the bisection search method will halve the interval between θ_1 and LB=0 which is illustrated in figure 5.2 below. Successive iterations will quickly tend towards the optimal multiplier value θ^*.

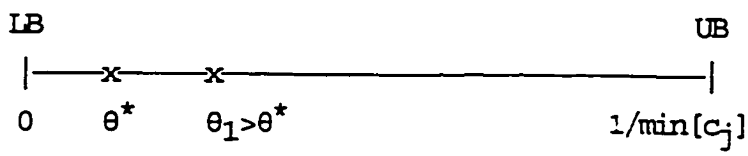


Figure 5.2: Bounds and estimate of θ for P2

Since all item availabilities F_j (or A_j) must be at least as large as the system availability A_S, that is, $F_j \geq A_S$ $(j=1,2,..,J)$, we can substitute A_S for F_j and use $(1-A_S)$ as a numerator in (5.10) and use $\max[c_j]$ in the denominator of (5.10) to compensate for the fact that all A_j's will be $> A_S$ and $(1-A_S)$ is a conservative estimate for $(1-A_j)$. Several examples also suggest as was the case for model P1 that the initial estimate works very well for these examples.

$$\textbf{Initial estimate:}\quad \theta_1 = \frac{(1-A_S)}{\max\ [c_j]} \quad j=1,2,..,J \tag{5.13}$$

5.3.4 Solution. The solutions for example 1 with B=\$20 and the 3 item parameters given earlier $\{\delta_j\}=\{1,\ 1.5,\ 2\}$ and $\{c_j\}=\{\$5,\ \$3,\ \$2\}$ and a minimum A_S = .001 and .600 are displayed in tables 5.3 and 5.4 below for comparative purposes. The solutions yielded the following Lagrange multiplier sequence of allocations which stopped after 14 and 12 iterations respectively (same as P1 but not the exact same sequence for lower A_S values) when a specified error was reached:

Table 5.3: Lagrange iterations for P2, Min As=.001 (J=3 items)

θ<0	S	C_S(S)	BO
.1998	0,0,1	2	3.635
.0999	0,1,2	7	2.264
.05	1,2,3	17	.867
.025	1,2,4	19	.724
.0125	2,3,4	27	.269
.0187	1,3,4	22	.533
.0219	1,3,4	22	.533
.0234	1,2,4	19	.724
.0226	1,2,4	19	.724
.0224	1,2,4	19	.724
...	...	..	...

Table 5.4: Lagrange iterations for P2, Min As=.600 (J=3 items)

θ<0	S	C_S(S)	BO
.08	0,1,2	7	2.264
.04	1,2,3	17	.867
.02	1,3,4	22	.533
.03	1,2,3	17	.867
.025	1,2,4	19	.724
.0225	1,2,4	19	.724
.0213	1,3,4	22	.533
.0219	1,3,4	22	.533
.0222	1,2,4	19	.724
...	...	..	...

5.3.5 Comments. The same comments apply for model P2 as for model P1 described earlier. We notice the validity of the UB, LB and initial estimate θ_1 which turn out to be the same as model P1 except in reversed sign and help to explain why the optimal solution tends to be the same for both models P1 and P2 particularly for high F_j values. The initial allocations, as shown in tables 5.1 and 5.3 (or 5.2 and 5.4), are not the same however, since the starting allocations have low F_j values. For our example:

A_S=.001, LB=0, UB=1/2=0.5, θ_1=(1-.001)/5 = 0.1998 and B_1=2

A_S=.600, LB=0, UB=1/2=0.5, θ_1=(1-.600)/5 = 0.0800 and B_1=7.

The interpretation of the multiplier value as the shadow price for the original constraint indicates that small unit increases in the budget B will improve the objection function by approximately θ^*, which means a reduction of BO since $\theta^*<0$. Again, this is valid for small increases in B coupled with the integrality problem due to the discrete nature of the model.

5.3.6 Lagrange multiplier method for P2a. As with the marginal procedure, solving P2 for an arbitrarily high budget value B will also solve model variant P2a to minimize C_S subject to a specified upper bound on BO as described earlier in chapter 4. The reader can easily verify that the Lagrange multiplier found for P2a will become:

$$\theta^* = -\, c_j/(1-F_j); \qquad (5.14)$$

which is just the inverse (reciprocal of θ^*) derived above for model P2. The derivation is along the same steps and the continuous assumption used previously for model P2. Bounds and initial estimate also remain the same as for P2 above but will be for $1/\theta^*$.

5.4 COMPARISON OF MODELS P1 vs P2

Taking up the same J=10 item example 2 earlier with a specified available budget of B=$15,000., figure 5.3 below illustrates the sequence of iterations for both models P1 and P2 and their effects on the response curve {A_S, C_S}. As discussed in the previous chapter with marginal analysis, the sequence between both models P1 and P2 is closely related and results in identical stock level vectors {S_j,

$j=1,..,10\}$. They do not always give the same results but are (nearly or) identical and can be considered the same for practical purposes, as shown in chapter 8 later, unless the level of precision required for a specific problem is warranted.

The reason both sequences are nearly identical and do not differ by much when they do differ can also be analyzed with the Lagrange multipliers. For model P1, the optimal $\theta^* = f_j/(F_j c_j)$ and $\theta^* = (1-F_j)$ for model P2. As was the case with the marginal analysis procedure, the close relationship between the multipliers can be analyzed as follows: for P1 the ratio f_j/F_j or f_j/A_j tends towards $p_j(S_j+1)$ as S_j becomes increasingly larger while for model P2, the term $1-F_j = 1-A_j = \Sigma\, p_j(x_j)$ from $x_j = S_j+1$ to infinity becomes $p_j(S_j+1) + p_j(S_j+2) + \ldots$ and the term $p_j(S_j+1)$ becomes the largest term in value as S_j becomes increasingly larger and tends towards infinity, the remaining terms becoming less and less significant.

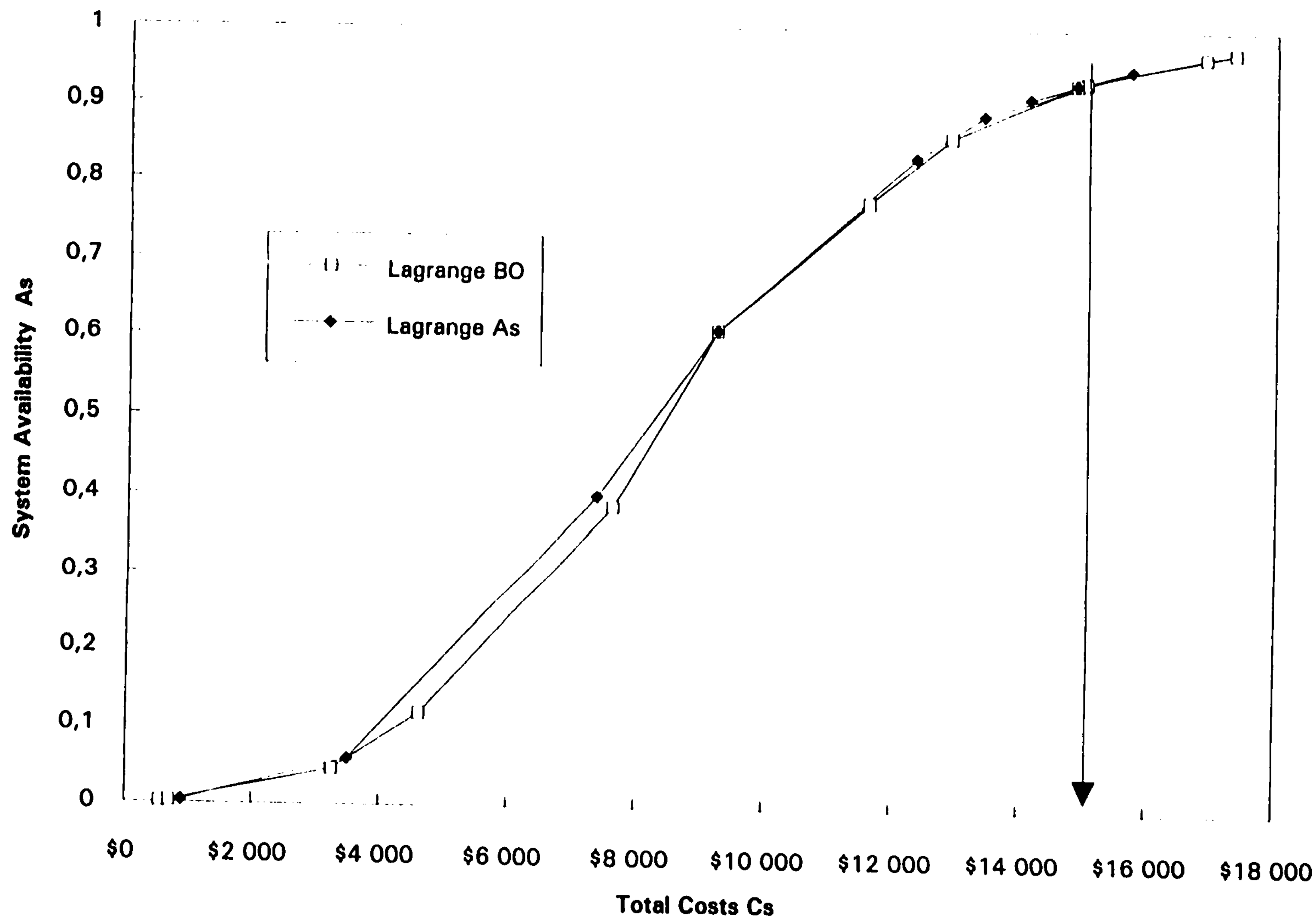


Figure 5.3: Lagrange iterations for models P1 vs P2 (J=10 items)

CHAPTER 6: END OF CYCLE EFFECT AND PART FAILURE DEPENDENCIES

6.1 PERFORMANCE MEASURES: REVISITED

So far, several solution procedures have been presented to maximize the system availability A_S performance measure specified in model P1 or to minimize expected system backorders BO in model P2 subject to a budget investment constraint consisting of purchasing costs.

$$\text{Max} \quad A_S = \prod_{j=1}^{J} A_j = \prod_{j=1}^{J} \left(\sum_{x=0}^{S_j} p_j(x) \right) \tag{P1}$$

$$\text{Min} \quad BO = \sum_{j=1}^{J} E(BO_j) = \sum_{j=1}^{J} \sum_{x=S_j+1}^{\infty} (x-S_j) . p_j(x) \tag{P2}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j \leq B \tag{6.1}$$

We noted that the DP (and equivalent network) procedure presented in chapter 3 guaranteed an optimal solution solution vector $\{S_j, j=1,..,J\}$ only if all possible budget values are enumerated at every stage; the faster and more efficient marginal analysis procedure in chapter 4 and the even faster Lagrange multiplier method in chapter 5 provided more practical and near or optimal solutions to both models, within an acceptable margin of error. We also proved with the marginal analysis procedure that maximizing the system performance measure E as the proportion of item demands that are satisfied immediately from inventory or:

$$E = 1 - BO/(BO|\{S\}=0\}) = 1 - BO/ \sum_{j=1}^{J} \delta_j \tag{6.3}$$

is equivalent to minimizing total expected system backorders BO performance measure of model P2. We also stated that simulation, although impractical to use as a search to the optimal solution vector

$\{S\}$, can be used to determine the system availability A_S, given an initial stockage vector quantity $\{S\}$; A_S can thus be re-defined and the equivalent performance measure EA_S for system availability is given by:

$$EA_S | \{S_j\} = \text{Prob } \{\underset{j \in J}{\text{Min}} \text{ Gamma}_j \ (S_j+1, 1/\tau_j)\} > t \qquad (6.4)$$

This chapter introduces a third numerical example (known as example 3 throughout with J=4 items and B=$50) which will be solved with the various solution methods presented thus far in previous chapters to obtain the exact opimal solution and/or near optimal solutions with their appropriate error bounds. We then demonstrate the equivalence of both performance measures A_S and EA_S (equation 6.4 above) with the use of simulation and introduce a new performance measure, denoted AA_S and defined as the average proportion of equipments still operational at the end of the cycle, taking into account part failure dependencies. Thus, AA_S = average number of equipments still operational (UP) at the end of the cycle and divided by the number of equipments M available at the beginning of the cycle or simply average number UP/M; equivalently, AA_S = [(M - the average number DOWN) /M] or simply [1 - average number DOWN/M].

Since AA_S may be a more appropriate and useful measure of system performance, we shall demonstrate that the distribution of the average number of equipments still operational at the end of the cycle can be analytically solved for very small combination values of J items and M equipments and without taking into account part failure dependencies), but it quickly becomes too complex or mathematically untractable to solve. The simulation methodology developed here will be critical in obtaining solutions to estimate AA_S with and without part failure dependencies as referred to in [Ernst and Pyke 1992] and discussed earlier.

6.2 EXAMPLE 3 (J=4 ITEMS)

6.2.1 Parameters. Example 3 consists of a one location, J=4 types of items with the Poisson mean parameters $\{\delta_j\} = \{1,2,3,5\}$ and cost $\{\$c_j\}$

= {\$7,\$5,\$2,\$1}. With a budget B=\$50 and min A_j = 0.700 and max A_j = 0.999, the range of possible stockage levels for each item become $\{LB(S_j)\}=\{1,3,4,6\}$ and $\{UB(S_j)\}=\{5,8,10,13\}$ from the Poisson cumulative probabilities. A minimum available budget of B=\$36 at stage j=4 is required to ensure the minimum allocation vector $\{S_j\} = \{1,3,4,6\}$ at a total cost C_S = \$36 is met.

6.2.2 Optimal solution for P1 (J=4). Using DP with an available budget B=\$50 and \$1 increments at each stage to ensure optimality, we can show that the true optimal solution vector is $\{S_j\} = \{2,3,6,9\}$ and a corresponding system availability A_S = .7376 at a total cost C_S = \$50. The order in which the items are listed is of no consequence if all possible allocation values at intermediate stages j=2,..,J-1 are enumerated, and will yield the same response curve $\{A_S, C_S\}$.

For comparative purposes, when items at stages 2 and 3 are incremented by \$c2=\$5 and \$c3=\$2 respectively, the solution vectors $\{S_j$, j=1,..,4} obtained result in near optimal solutions at some budget values. As an example, for an available budget B=\$50, we obtain the following solution vector $\{S_j\} = \{2,3,5,8\}$ and A_S = .67296 at a total cost of C_S = \$47. We also remember that the approximate DP methodology developed in chapter 3 earlier in which increments of $\$c_j$ are used for all intermediate stages will yield a solution vector $C_S(.)$ whose lower bound cannot be lower than B - Σ c_j for j=J-1,..,1 or \$50 - (\$2+\$5+\$7) = \$36.; thus, $C_S(.)$ can be as much as \$14 lower than B = \$50 (or 28% from B). Listing the items in reverse order, however, would give an improved lower bound on $C_S(.)$ of \$5+\$2+\$1 = \$8 (or 16% of B).

Table 6.1 summarizes and compares the effects on system availability A_S when dynamic programming is used without enumerating all possible budget allocation values for intermediate stages j=2,..,J-1 (j=2 and j=3 for example 3); errors are as high as 10% for upper budget values and illustrate the unpredictability of the results and the computational difficulties of the methodology as discussed earlier in chapter 3. We also note that another response curve $\{A_S, B\}$ would have been obtained (just as was the case for example 1 earlier with J=3 items) if items had been listed in reverse order i.e. least expensive items first, when increments used at intermediate stages do not include all possible budget values that can be allocated at these stages.

Table 6.1: DP solution A_S optimal vs \$cj increments (J=4)

\$Budget	As(*)	As(incr)	(%)error
36	.39186	.39186	0.0
37	.44556	.44556	0.0
38	.47912	.47912	0.0
39	.50066	.50066	0.0
40	.53837	.53837	0.0
41	.55932	.55932	0.0
42	.56980	.56980	0.0
43	.59010	.59010	0.0
44	.60115	.60115	0.0
45	.60618	.60618	0.0
46	.62582	.61459	1.8
47	.67296	.61973	7.9
48	.69915	.62978	9.9
49	.71225	.65222	8.4
50	.73763	.67296	8.8

The corresponding FULL and GAP network structures for this example could easily be constructed from which we can compare the total number of nodes N, arcs A and matching labels M for both of them: for the FULL network, N=43 nodes, A=78 arcs and M=24 matching labels while for the GAP network, N=27, A=54 and M=21. The error ß(%) as a percentage of the budget B described in chapter 3 earlier for the total costs $C_S(.)$ solution vector $\{S_j\}$ of the GAP network is $\Sigma\ c_j$, j=2,..,4 and is equal to $(c_2+c_3+c_4)$ = (\$5+\$2+\$1) = \$8 or (\$8/\$50) x 100% = 16% from which we can easily calculate the lower bound for the total costs $C_S(.)$ solution vector $\{S_j\}$ as equal to B - $\Sigma\ c_j$, j=2,..,4 or \$50-\$8 = \$42 or 84% of the total available budget B. In other words, $C_S(.)$ of the solution vector $\{S_j\}$ obtained with the GAP network will be at least \$42 or 84% of the budget. FULL network analysis corresponding to the DP approach also gave the same optimal solution, while network analysis with the GAP network resulted in the solution vector quantity $\{S_j\}=\{2,3,5,8\}$ for a total cost $C_S=\$47$ or 94% of B=\$50 (which compares with a lower bound of \$42 calculated earlier) and a system availability of $A_S=.67297$, an 8.7% error from the exact solution $A_S=.7376$ for this small problem.

The solution obtained from marginal analysis for model P1 described in chapter 4 took only 6 iterations from the starting lower bound vector $\{S_j\}=\{1,3,4,6\}$. The sequence of incremental cost allocations by adding the most "cost effective" item, one at a time, is shown in table 6.2 below and yielded the optimal solution vector $\{S_j\}=\{2,3,6,9\}$ for a total cost C_S of \$50., $A_S=.7376$, BO=.4264 and individual item availabilities $\{A_j\} = \{.9197, .8571, .9665, .9682\}$.

Table 6.2: Marginal allocation sequence for P1 (J=4 items)

{S}				C_S	A_S	BO
1	3	4	6	36	.3918	1.3986
1	3	4	7	37	.4456	1.1607
1	3	4	8	38	.4791	1.0274
1	3	5	8	40	.5384	0.8426
1	3	5	9	41	.5593	0.7745
2	3	5	9	48	.6991	0.5103
2	3	6	9	50	.7376	0.4264

Since the total cost C_S is exactly the specified budget B of \$50. and none of the items has reached its specified maximum availability of $A_j=0.999$, the solution is the true optimal solution as confirmed by dynamic programming. We note that the lower bound for $C_S(.)$ developed in chapter 4 for marginal analysis is better than the one developed for the approximate DP methodology with $\$c_j$ increments; the $LB(C_S)$ for marginal analysis is B - max c_j, j=1,..,J or \$50 - \$7 = \$43 (vs \$14 or \$8 depending on the sequence used for DP). The Lagrange solution method described in chapter 5 yielded the same optimal allocation vector

$\{S\}=\{2,3,6,9\}$ in only 2 iterations.

6.2.3 Optimal solution for P2. Marginal analysis for minimizing total system backorders BO in model P2 resulted in the same optimal allocation vector $\{S\}=\{2,3,6,9\}$ in 6 iterations as shown in table 6.3 below, one of which was not in the same sequence as for maximizing A_S in model P1 and the Lagrange relaxation method resulted in a total cost C_S=\$44 with $\{S\}=\{1,3,6,10\}$ after 6 iterations.

Table 6.3: Marginal allocation sequence for P2 (J=4 items)

{S}				C_S	A_S	BO
1	3	4	6	36	.3918	1.3986
1	3	4	7	37	.4456	1.1607
1	3	4	8	38	.4791	1.0274
1	3	5	8	40	.5384	0.8426
1	3	5	9	41	.5593	0.7745
1	3	6	9	43	.5901	0.6906
2	3	6	9	50	.7376	0.4264

6.3 END OF CYCLE EFFECTS

6.3.1 Introduction. Example 3 described above would be valid for M=1 equipment (say a vehicle) consisting of J=4 major assemblies, having the following exponential failure rates $\{\tau_j\}$ = {1/10,000 km, 1/5,000 km, 1/3333.333 km and 1/2,000 km}. If it is scheduled to operate for 10,000 kilometers during the planning period, then the expected number of failures is Poisson distributed with parameters $\delta_j = M \times U \times \tau_j$. For assembly number one (j=1), then δ_1 = 1 equipment x 10,000 km x 1 failure/10,000 km = 1; applying the same formula yields the following Poisson parameter vector $\{\delta_j\} = \{1,2,3,5\}$.

However, the system performance measures A_S and/or BO would be exactly the same if we had M=2 identical equipments each operating for 5,000 km during the period instead of M=1 equipment operating for

10,000 km. Both situations clearly result in the same Poisson parameters $\{\delta_j\}$ and therefore, would also yield the same optimal solution vector $\{S_j\} = \{2,3,6,9\}$, regardless of the method used to optimize $\{S_j\}$.

This leads to the interpretation of the performance measure A_S = 0.7376 calculated for example 3 above: it means that the probability of not running out of any spare is 0.7376, regardless of how many equipments M were used to calculate the Poisson parameters. Conversely, it also means that I have (1 - 0.7376) = .2624 or 26.24% chance of running out of at least 1 type of spare, but it does not answer the question: what is the probability of running out of 2,3,.. spares ?.

So, A_S should be used if we want to measure the mission reliability and its importance would be crucial when all M equipments (whether M=1 or M>1) must remain operational till the end of the cycle without requiring any spare, in order to complete the mission successfully. The latter has several military applications such as an armoured squadron (of tanks, helicopters, fighter aircrafts, ...) and its effectiveness is seriously hampered if one or more equipments fail due to lack of any spares during a mission and it is deemed a mission failure when it happens. The same implications can be found in the retail industry when retailers do not want to run out of any stocked items in a specialty group (say lawnmowers, patio sets,..) advertised in catalogs or ads during a specified time period; the same situation can apply in the manufacturing sector when a group of numerically controlled machines are required to operate successfully for an extended period of time in order to complete one or more orders on time.

If the proportion of equipments still operational at the end of the cycle (time period) or AA_S is a more appropriate measure of performance of the system, then A_S is the same as AA_S when M=1 and constitutes a conservative lower bound on AA_S when more than M=1 equipment is involved. The reason is that if an equipment (M=1) is running out of any type of spares as a result of more failures than expected before the end of the cycle, the equipment will remain in a failed state until the end of the period (no re-supply permitted). If multiple equipments are involved, i.e $M > 1$, then running out of any type of spare causes one equipment to remain in a failed state and the remaining M-1 equipments (and related assemblies) to keep operating but the respective expected number of failures are reduced by 1/M, giving rise to the end of cycle effect.

Thus, the performance measure $A_S = 0.7376$ calculated earlier is exact if the number of equipments M=1, and is a lower bound (albeit a progressively worse one as M increases) on the true proportion of equipments still operational at the end of the cycle or AA_S, when multiple equipments are involved. Similarly, the total expected system backorders $BO = \Sigma BO_j$ j=1,..,J has an upper bound of one when M=1 since we cannot run out of more than 1 spare of any type j; only one more failure than the initial stockage vector $\{S_j, j=1,..J\}$ is required to cause the equipment to remain in a failed state (DOWN) until the end of the cycle; and, in general, $UB(BO) = M$ whenever $M \geq 1$. The next sections introduce simulation methodology to determine the effect of part failure dependencies on remaining equipments.

6.3.2 <u>Simulating EA_S</u>. In order to determine the effect of multiple operating equipments and part failure dependencies towards the end of cycle, a simulation program based on the GAMMA variates as the sum of exponential variates and another based on sequential EXPONENTIAL variates, were first developed using GWBASIC.EXE.

The system was simulated n=5,000 times (cycles), calculating EA_S with equation (6.4) as the proportion p of the n cycles that the equipment survived past t=10,000 km when the available stockage vector quantity at the beginning of the period is set to $\{S_j\}=\{2,3,6,9\}$. In other words, finding the time at which the minimum of the 3rd, 4th, 7th and 9th failure occured during each of the n cycles. The estimate p obtained with simulation was 0.7330 and is consistent with the exact system availability A_S of .7376 calculated in section 6.2 for example 3 when M=1.

We also can establish a 95% confidence interval for π = the true proportion of cycles that the time at which we ran out of any type of spares was less (greater) than 10,000 km based on this estimate.

$$CI(1-\alpha) \text{ for } \pi = p \pm z(\alpha/2) \times \sqrt{[p*(1-p)/n]} \quad (6.5)$$
$$= .733 \pm 1.96 \times \sqrt{[.733*.267/5000]}$$
$$= .733 \pm .0123$$
$$= [\ .7207,\ .7453]$$

the interval containing the exact value A_S=.7376 found earlier with other methods. We could also conduct the usual hypothesis testing procedure to determine the p_value obtained from the simulation as follows:

H_0: π = .7376 (hypothesized π_0 value)

H_A: π <> .7376

and calculate the p_value for this test

$$p\_value = 2*P(Z<z_c) \quad (6.6)$$
$$= 2*P(Z<(p-\pi_0)/\sqrt{(\pi_0*(1-\pi_0))})$$
$$= 2*P(Z<-.735) = 2*.2312 = .4624$$

and we cannot reject the null hypothesis that π=.7376

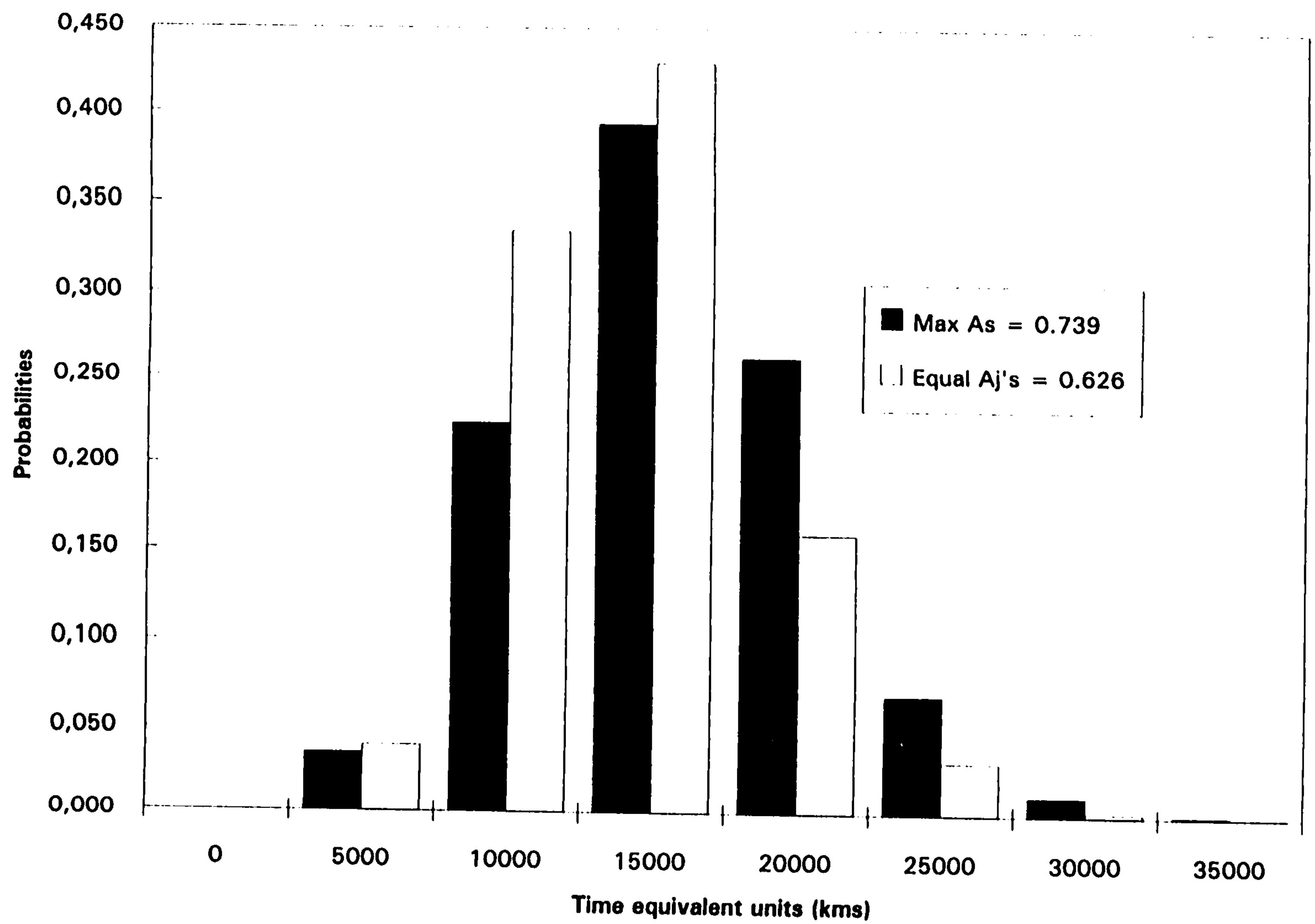
The time distribution of the minimum of the $Gamma_j$ variates was also analyzed using the distribution fitting software package UNIFIT2, version 2.0, by [Law and Vincent 1991] and BestFit, version 2.0a, by [Palisade Corporation, NJ, 1995], both of which use slightly different techniques to determine maximum likelihood estimators of the distribution parameters.

Both distribution fitting software find the Weibull distribution (See [Hahn and Shapiro 1967]) as the best one to describe the data as evidenced by the usual equal-probability Chi-Square, Kolmogorov-Smirnov and Anderson-Darling test statistics based on 40 intervals at the α=.10 significance level and shown in table 6.4, as well as other comparison measures described in [Law and Kelton 1991] such as the P-P and Q-Q plots, model moments, etc.., which all indicate a better fit for the Weibull.

Table 6.4: Test statistics of fitted Weibull vs Normal

	Weibull	Normal
Chi-Square	48.768	80.568 *
K-S	0.012	0.020 *
Anderson-Darling	1.084	4.437 *
Conclusion	Do not reject	* Reject

Figure 6.1 below illustrates a typical histogram of the sampled data for example 3 (J=4 items).



<u>**Figure 6.1: Distribution 1st stockout (J=4 items)**</u>

6.3.3 Alternate performance measure AA_S. In order to study the end of cycle effect (part failure dependencies) when multiple equipments are involved during the cycle (when $M > 1$), another simulation program S_INVL.BAS using GWBASIC.EXE was then developed to calculate the more appropriate measure of system performance AA_S or the average proportion of equipments that are still operational (or UP) at the end of the cycle, denoted AA_S, and defined as the average number of equipments NOT (or DOWN) operational at the end of the cycle divided by M, and when subtracted from 1, becomes the average proportion of equipments operational (or UP) at the end of the cycle; thus $AA_S = 1$ - (average number DOWN /M)

As we shall demonstrate next, the distribution of the number of equipments still operational at the end of the cycle and without taking into account part failure dependencies, given a stockage level vector $\{S_j\}$, depends on both M and J (number of item types) and quickly become analytically untractable as M and/or J increases and we must therefore resort to simulation to estimate AA_S.

6.3.4 AA_S for special case M=1. When M=1, the number of equipments UP at the end of the cycle will either be 1 or 0; it will be 1 x UP (or 0 DOWN) with $P(X_1 \le S_1).P(X_2 \le S_2)...P(X_J \le S_J) = \pi A_j$, that is if we do not run out of any type of spare item $j=1,..,J$; and it will be 0 x UP (or 1 DOWN) if we run out of exactly 1 type of spare $j=1,..,J$ with $P(X_1 = S_1+1).P(X_2 \le S_2)...P(X_J \le S_J)$ or(+) $P(X_1 \le S_1).P(X_2 \le S_2+1)...P(X_J \le S_J)$ ... or(+) $P(X_1 \le S_1).P(X_2 \le S_2)...P(X_J = S_J+1)$; applying the expected value operator $E(.) = \Sigma\, x.p(x)$ to the expression above simply reduces to πA_j, $j=1,..,J$ and, thus the proportion of equipments still operational at the end of the cycle $AA_S = \pi A_j/M=1 = \pi A_j$. The most important and useful conclusion for the special case M=1 is that maximizing $A_S = \pi A_j$, $j=1,..,J$ in model P1 is also equivalent to maximizing AA_S.

6.3.5 AA_S for the general case M>1. For larger and more realistic values of J and M, the distribution of the number of equipments operational at the end of the cycle become far more complex and πA_j is no longer a reliable (and progressively worse) estimate for AA_S as M increases; we show the derivation for AA_S without part failure dependencies for the case J=3 and M=2 which will enable us to develop a simple heuristic based on total expected system BO later.

For J=3 and M=2, we can have either 2, 1 or 0 equipments UP or operational at the end of the cycle (2 UP is equivalent to not running out of any spare, 1 UP is equivalent to running out of exactly 1 spare,...), each one occuring with the following probabilities:

2 UP: $P(X_1 \le S_1).P(X_2 \le S_2).P(X_3 \le S_3) = A_1.A_2.A_3 = \pi A_j \; j=1,2,3$

1 UP: $P(X_1 = S_1+1).P(X_2 \le S_2).P(X_3 \le S_3)$ or(+)
$P(X_1 \le S_1).P(X_2 = S_2+1).P(X_3 \le S_3)$ or(+)
$P(X_1 \le S_1).P(X_2 \le S_2).P(X_3 = S_3+1)$

0 UP: $P(X_1 = S_1+1).P(X_2 = S_2+1).P(X_3 \le S_3)$ or(+)
$P(X_1 = S_1+1).P(X_2 \le S_2).P(X_3 = S_3+1)$ or(+)
$P(X_1 \le S_1).P(X_2 = S_2+1).P(X_3 = S_3+1)$ or(+)
$P(X_1 = S_1+2).P(X_2 \le S_2).P(X_3 \le S_3)$ or(+)
$P(X_1 \le S_1).P(X_2 = S_2+2).P(X_3 \le S_3)$ or(+)
$P(X_1 \le S_1).P(X_2 \le S_2).P(X_3 = S_3+2)$

The expression above can easily be reduced and calculated analytically, although the combinatorial nature of these expressions as M and/or J increase (specially M, assuming J>10 which is usually the case in practice) become far too complex and mathematically untractable; they could not be readily calculated for say M=10 and J=20 as we would have to enumerate all possible combinations of failures when 10,9,..,0 equipments are UP at the end of the cycle; within each one (say 2 UP which is equivalent to running out of at most 8 spares of any type exactly), all possible probability combinations of 8 failures out of J=20 items would have to be evaluated, only one of which would be: $P(X_1 = S_1+5).P(X_2 \le S_2)...P(X_6 = S_6+1)...P(X_8 = S_8+2)...P(X_{20} \le S_{20})$.

We also note that the probabilities in the expressions above do not even take into account failure dependencies; for example, if we run out of a particular type of spare, one equipment will be DOWN until the end of the cycle and only M-1 equipments are left operating from then on, causing a reduction of (1/M) in expected number of failures for each part j=1,..,J, until the end of the cycle.

For this reason, AA_S calculated above will always be smaller and thus always underestimates the true value of AA_S with part failure dependencies; for high system availability A_S > say .90, the optimized stockage levels $\{S_j\}$ will be high compared to the Poisson mean rates $\{\delta_j\}$ calculated based on the original M value at the beginning of the cycle and the error between the two will be small since running out of spares (if any) will occur close to the end of the cycle only.

As we demonstrated above, estimating AA_S with A_S for the special case when M=1 is exact (with or without part failure dependencies since they are both equal), but cannot be relied upon when M > 1; we shall therefore develop better estimates for AA_S with and without part failure dependencies, based on modified system backorders BO in the last chapter where numerical experiments with larger scale problems are analyzed. The next two sections illustrate the differences using simulation methodology, first for example 3 (J=4 items, B=$50) followed by example 1 (J=3 items, B=$20) for which we also expand and calculate the probability expressions developed above for comparative purposes.

6.3.6 Simulating AA_S for example 3 (J=4). Given the same original failure rate parameters $\{\tau_j\}$ and the same optimal stockage vector quantity $\{S_j=2,3,6,9\}$ for example 3 when B=$50, we know that A_S = .7376 and is exact when M=1, as calculated in the earlier sections, and resulted in a simulated EA_S value = .7330 with the GAMMA distribution.

In this section, we present the results for AA_S with and without dependencies, obtained from the simulation program S_INVL.BAS, by varying the number of equipments M=1,2,3,4,5,10 and 20, and the corresponding length of the period t adjusted accordingly in order to keep the same Poisson parameters $\{\delta_j\}$.

Two key characteristics diffentiate both programs for simulating EA_S vs AA_S (whether taking failure dependencies or not): the first one is that we can no longer use the Gamma distribution as the means to simulate the system if we want to analyze the impact of multiple equipments and calculate the average number of equipments still operational at the end of the cycle.

The reason is that we must obtain the time at which we first run out of parts of any type, therefore we must use the exponential distribution sequentially, otherwise we would not be able to assess the impact on the overall failure rates of items if the minimum of all $Gamma_j$ variates exceed t; in other words, we would not know how many failures for each type of item occured before the end of the simulation time period t.

The second reason is that for estimating AA_S when taking into account failure dependencies, we want to keep track of the time at which we start running out of spare items if it is < t in order to be able to reduce the failure rates by 1/M from that time onwards, until the end of the cycle, since 1 less equipment is operating, thereby affecting all other types of items. This methodology enables us to determine the effect on the overall number of failures for each item type j=1,..,J for the whole simulation period t and is measured as a proportion of the original $\{\delta_j\}$ values.

Table 6.5 below demonstrates the effect on AA_S, denoted AA_S(not) and AA_S(mod) respectively, for increasing values of M when NOT modifying (without dependencies) the Poisson parameters until the end of the cycle when we run out of spares during the cycle or AA_S(not), and compared with AA_S(mod) when we DO modify (with dependencies) Poisson parameters by reducing each one by (1/M) every time we run out of spares during a cycle or AA_S(mod). Since simulating AA_S(not) always underestimates the true value of AA_S(mod) to indicate dependencies, the percentage error was calculated as (AA_S(mod) - AA_S(not))*100/ AA_S(mod).

The results clearly show that simulating M=1 equipment for t=10,000 kms does not have the same effect on AA_S (whether modified or not due to part failure dependencies) as simulating M=2 equipments each operating for t=5,000 kms; for example the average proportion of equipments still operational at the end of the cycle was AA_S(mod) = 0.7318 when M=1 vs 0.8418 when M=2. The same interpretation can be made for higher values of M.

The second important result is the confirmation that NOT modifying

$\{\delta_j\}$ values when we start running out of spares at the end of the cycle always underestimates the true values of AA_S, as indicated by the error(%) values; a lower stockage level vector $\{S_j\}$ would also result in smaller A_S values.

Table 6.5: AA_S vs M for example 3 (J=4 items)

M	t	AA_S (Not)	AA_S (Mod)	Error (%)
1	10,000	.7318	.7318	0.00
2	5,000	.8213	.8418	2.43
3	3,333	.8638	.8843	2.32
4	2,500	.8955	.9093	1.52
5	2,000	.9148	.9224	0.82
10	1,000	.9581	.9596	0.16
20	500	.9788	.9794	0.06
25	400	.9835	.9837	0.02

6.3.7 Simulating AA_S for example 1 (J=3). This section presents the simulation results for example 1 originally derived in earlier chapters but with the additional calculation of the probability expressions for AA_S(not), which enables us to validate the simulation program S_INVL.BAS; the last chapter on numerical experimentations for randomly generated larger scale problems will rely heavily on simulation results obtained from this program to develop accurate estimates of AA_S(mod) when taking into account the impact caused by part failure dependencies.

Even though the probability expressions for the distribution of the number of equipments at the end of the cycle developed earlier do not take into account part failure dependencies, table 6.6 below indicates the results of those calculations for example 1 (J=3, M=1,2 and 3) and varying the budget B from \$50 to \$20 in intervals of \$5 and optimizing stockage levels $\{S_j\}$ for each one, using marginal analysis and corresponding exact A_S values in columns 1,2 and 3 respectively, which are the same for M=1,2 or 3.

Given the vectors $\{S_j\}$ for each budget level, simulation based on N=10,000 cycles each of AA_S(mod) thus taking into account part failure dependencies are shown in column 4 followed by simulated estimated values for AA_S(not) without dependence in column 5 and AA_S(not) values calculated using the expansion of probability expressions in the last column.

The interpretation of the results exhibit the same patterns and support the same conclusions drawn for example 1 (J=3 items) earlier: the average proportion of equipments AA_S(mod) still operational at the end of each cycle (column 4) increases as M increases for each given stockage level $\{S_j\}$; since varying the available budget B downwards from \$50 to \$20 yields lower stock levels {Sj} and correspondingly lower system availability A_S, it is evident that with or without part dependencies (columns 4,5 and 6) will also yield progressively lower estimates of AA_S, the percentage errors between AA_S(mod) and AA_S(not) becoming more significant as B (and therefore $\{S_j\}$ and A_S as well) decreases. This will be crucial for larger scale problems analyzed in the last chapter.

What is also interesting is to compare the values in the last two columns (5 and 6): the simulated estimated AA_S(not) value in column 5 with the AA_S(not) value, shown in column 6, using the probability expansion terms for the exact distribution of AA_S(not), which also does not take part failure dependencies into account; the values in columns 5 exhibit very small percentage errors as compared to values in column 6 across each row, regardless of M, B, $\{S_j\}$ or A_S.

Table 6.6: AA_S vs M for example 1 (J=3 items)

M=1 B	{S_j}	A_S Marginal	Sim AAs (Mod)	Sim AAs (Not)	AAs expand
50	4,5,7	.99081	.99160	na	na
45	3,5,7	.97557	.97330	na	na
40	3,4,6	.95842	.95600	na	na
35	2,4,6	.89852	.90090	na	na
30	2,3,5	.84509	.84120	na	na
25	1,3,5	.67608	.67340	na	na
20	1,2,4	.56378	.56330	na	na
M=2 B	**{S_j}**	**A_S Marginal**	**Sim AAs (Mod)**	**Sim AAs (Not)**	**AAs expand**
50	4,5,7	.99081	.99535	.99535	.99445
45	3,5,7	.97557	.98655	.98535	.98533
40	3,4,6	.95842	.97585	.97575	.97446
35	2,4,6	.89852	.93980	.93700	.93649
30	2,3,5	.84509	.90650	.89825	.89972
25	1,3,5	.67608	.80350	.78720	.78103
20	1,2,4	.56378	.72175	.68370	.68742
M=3 B	**{S_j}**	**A_S Marginal**	**Sim AAs (Mod)**	**Sim AAs (Not)**	**AAs expand**
50	4,5,7	.99081	.99700	.99680	.99624
45	3,5,7	.97557	.99037	.98953	.98993
40	3,4,6	.95842	.98377	.98073	.98234
35	2,4,6	.89852	.95967	.95737	.95580
30	2,3,5	.84509	.93680	.92863	*
25	1,3,5	.67608	.85903	.83993	*
20	1,2,4	.56378	.79673	.77250	*

* not calculated

The last chapter consists of numerical experiments with larger scale problems and will focus on developing reliable estimates for AA_S(mod) using simple heuristics. Before we do so, however, the next chapter considers further important considerations and extensions to the models.

Figure 6.2 below illustrates the typical distribution of the number of equipments operational at the end of the cycle for a randomly generated problem referred to as example 2 earlier (J=10 items x M=10 equipments), and simulated for N=5,000 cycles whereby the budget has been purposely set very low to B= $6,000. in order to have an even lower stock level vector $\{S_j\}$ than with the original B=$15,000; the differences in measuring AA_S due to end of cycle effects (with vs without dependencies) resulted in an average number of equipments operational at the end of the period AA_S(mod) = 0.84966 vs an average AA_S(not) = .80804, or a (.84966-.80804)*100/.84966 = 4.9% difference.

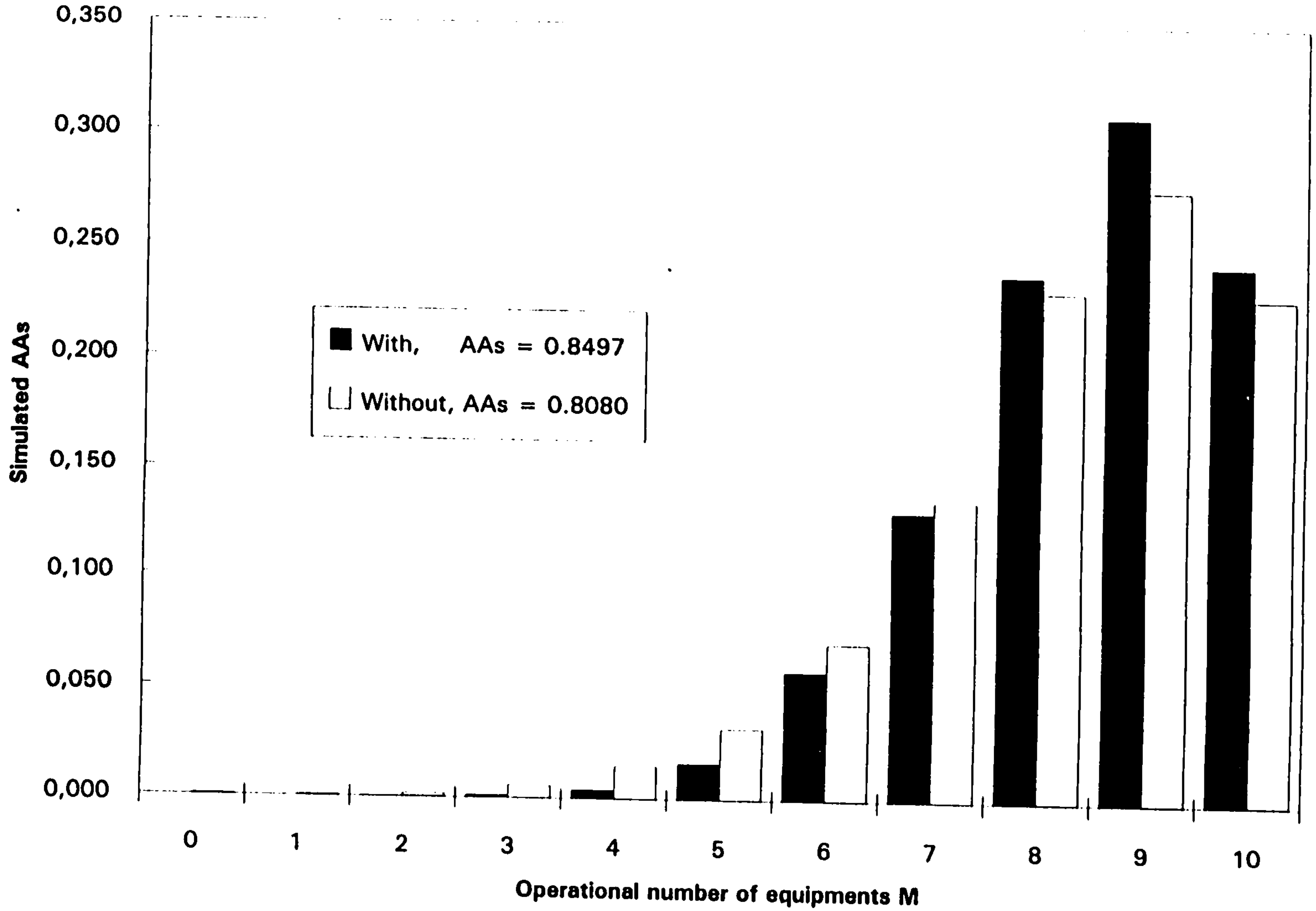


Figure 6.2: End of cycle effects (J = 10 item)

CHAPTER 7: MODEL EXTENSIONS

7.1 MULTIPLE INDENTURE SYSTEMS

7.1.1 Introduction. As discussed in chapter 2, multi-indenture systems consist of incorporating different component levels within the equipment. They then become an integral part of the analysis of optimal stockage levels for both the equipments, its assemblies and components. It appears that no papers on a priori analysis of the disaggregation process have been published. Most studies published in the literature regarding this topic simply lump the different component levels together without prior analysis as to the impact of individual parts' costs and are usually based on a single operating equipment, a single level, deterministic costs and/or availability during a fixed time period. Relevant papers include [Black and Proschan 1959], [Kettelle 1962], [Goodwin and Geese 1965] who studied a single level of indenture and the optimal level of redundancy for each repairable part.

Since information about assemblies and their component levels are usually based on best-parameter distribution "fit" and underlying assumptions, the relationship between assemblies and their components become difficult to analyze. The objective is to determine the conditions under which we can decide whether it is worthwhile to include lower level indenture stockage levels for components (i.e. purchase and stock components independently from its assembly) in a larger scale one (1) period inventory systems when additional information concerning failure (demand) data about these components becomes available to the analyst. The solution procedure will use the marginal analysis discussed earlier and is applied to a simple example to illustrate the concept.

7.1.2 Example. Suppose we had the following 1 location, 3 item system with the following parameters: $\{\delta_j\} = \{1,5,10\}$ and $\{c_j\} = \{\$1000, \$300, \$200\}$. Specifying min A_j=0.80 and max A_j = .99999 for j=1,2,3 and with a budget of $9900., the last few iterations and the optimal solution for this example obtained via marginal analysis are shown in table 7.1:

Table 7.1: Marginal analysis for 3 assemblies

S	C_S	A_S
...	...	...
2,9,17	8100	.8777
3,9,17	9100	.9362
3,10,17	9400	.9538
3,10,18	9600	.9606
3,11,18	9900	.9687

Since the total costs C_S achieved is exactly the available budget B=$9900 and we know that the allocations obtained from marginal analysis are undominated, then the solution vector $\{S^*\}=\{3,11,18\}$ is optimal with $A_S=.9687 = A_1A_2A_3 = .981x.994x.9928 = .9687$ at a total cost of $C_S=(3x1000)+(11x300)+(18x200)=\$9900$.

Suppose we had additional information concerning assembly 3 ($\delta_3=10$) that lead us to believe that 20% of its failures during the period were caused by a component within assembly 3 (denoted as component 31) and the remaining 80% by various other remaining components within the assembly (denoted as components 32). We can therefore estimate $\delta_{31}=2$ and $\delta_{32}=8$ since the failure process is also Poisson (disaggregation of Poisson process into individual Poisson processes described in [Ross 1989 ch 5]).

We now wish to know whether it is worthwhile to purchase (or produce) and stock component 31 independently of assembly 3 for the next period; specifically, we want to determine an upper bound on the cost c_{31} for which A_S will remain ≥ .9687 for the same or lower budget B ≤ $9900 or correspondingly a budget of (18 x $200) = $3600 for component 31 and assembly 3. We could estimate c_{31} (say c_{31}=$40) and since we already know other parts failures will be replaced by complete assemblies costing c_{32}=$200 each, we can solve the entire problem with the following modified parameters:

Table 7.2: Modified parameters for example

B=\$9900		Min A_j=.80	Max A_j=.99999
δ_1=1	δ_2=5	δ_{31}=2	δ_{32}=8
c_1=1000	c_2=300	c_{31}=40	c_{32}=200

for which the solution obtained via marginal analysis is as follows (only the last few iterations shown):

Table 7.3: Marginal analysis for 4 assemblies

S	B	A_S	
...	...	...	
3,10,7,14	9080	.9498	
3,10,7,15	9280	.9586	
3,11,7,15	9580	.9666	
3,11,7,16	9780	.9710	
3,11,8,16	9820	.9718	*
4,11,8,16	10820	.9870	

and the solution is near optimal with {S}={3,11,8,16}, B=\$9820 but A_S = .981x.994x.9998x.9963 = .9718 is greater than A_S = .9687 obtained earlier with the 3 original assemblies and that A_{31}xA_{32} = .9998x.9963 = .9960 is greater than A_3 = .9928 also obtained from the same solution earlier. We note from table 7.3 that the previous iteration with {S}={3,11,7,16} also yields a system availability of A_S = .981x.994x.9989x.9963 = .9710 which is still greater than .9687 at a lower cost C_S of \$9780 and that A_{31}xA_{32} = .9989x.9963 =.9952 > A_S = .9928.

There are 2 conditions that must be met in order to be able to achieve a better A_S at equal or lower C_S and therefore worthwhile to disaggregate the Poisson process into smaller components. These are:

1. $c_{31}S_{31}+c_{32}S_{32} \le c_3S_3$ = (\$200x18) = \$3600 (7.1)
2. $A_{31}xA_{32} \ge A_3$ = .9928 (7.2)

which are satisfied for the iteration yielding C_S=\$9820 as well as the previous iteration with C_S=\$9780. We can now describe a convenient way to solve the upper bound UB for c_{31} using the solution above by simply going back to the iteration in table 7.4 yielding the lowest A_S>.9687 which is .9710 and calculating the upper bound for c_{31} as follows:

$$UB(c_{31}) = (18-16)x\$200/7 = 400/7 = \$57.143 \qquad (7.3)$$

which is simply to calculate the savings achieved for assembly 3 i.e. the difference between the original (S_3=18) and the new one (S_{32}=16) times its cost and dividing by the new stockage level (S_{31}=7) obtained from the solution; this guarantees that both conditions will be satisfied.

There are 2 problems associated with this procedure: first, the stock levels for other parts (S_1=3 and S_2=11) are not guaranteed to stay the same which will invalidate the upper bound should they change and second, the entire problem would have to be solved again with the new parameters without knowing the results beforehand (or whether it was worthwhile). If we had several tens or hundreds of items, this procedure would prove to be very inefficient and practically impossible to work with.

7.1.3 <u>Efficient solution procedure</u>. Fortunately, the problems described above can easily be handled as follows: due to the separability by item of model P1, we could have simply solve the problem using marginal analysis separately for assembly 3 with \$3600 as a budget and the following parameters: B=(18x\$200)=\$3600 as per the original problem, c_{31}=\$40 (initial guess), c_{32}=\$200, δ_{31}=2 and δ_{32}=8 and obtain the following solution (only the last few iterations shown):

Table 7.4: Marginal analysis for 2 sub-assemblies

S	C_S	A_S
...	...	...
7,14	3080	.9817
7,15	3280	.9910
7,16	3480	.9952
8,16	3520	.9960 *
8,17	3720	.9982

This procedure guarantees that both conditions will be met when an upper bound for c_{31} is calculated; first, the budget will not exceed \$3600 (marginal analysis stops when B is exceeded) and by picking the lowest entry for which $A_S = A_{31} x A_{32} > A_3 = .9928$ in the original solution. In table 7.4, this entry is .9952 and the upper bound $UB(c_{31}) = (18-16) x \$200/7 = \$57.143$ as before. Solving the entire original problem with the additional information becomes much more mathematically tractable, and is very efficient with marginal analysis. The steps for computing an upper bound on the cost of a component are summarized below:

- solve the original problem at the assembly level with a given budget using marginal analysis;
- with additional information about an assembly, solve the problem separately with a best guess about its estimated cost and a budget equal to the stock level S_j obtained from the original solution;
- compute the UB for the part using the separate solution where the lowest entry is such that A_S (components) is greater than A_S (original assy) and the stock levels $\{S_{jk}$ (k=1,..)$\}$ given by the solution.

7.1.4 Comments. The procedure described above using marginal analysis is a very efficient way to include the analysis for multi-indenture systems and separable items like model P1. It is thus possible to quickly determine an upper bound on a component's cost when additional information becomes available and disaggregation of Poisson processes is considered without having to solve the entire problem and discover afterwards that either the total resulting costs C_S would exceed the budget or that we achieve a lower system availability A_S.

It has also been shown that even though the sum of the components' individual costs may exceed the cost of an assembly, $\Sigma\ c_{jk} > c_j$ (k=1,..), it may be possible under some conditions, to achieve a higher system availability A_S at a lower overall cost C_S. For our example, even if the cost of component c_{31} is set at its upper bound $57.14, the total costs of disaggregation $c_{31}+c_{32}$ = $57.14 + $200 = $257.14 is greater than the original cost of the whole assembly c_3=$200, but a higher system availability was achieved (A_S=.9710 vs .9687) as a result of the additional information which enabled us to disaggregate the original Poisson process into two smaller ones.

It is therefore possible to use a 2-phase (or multi-level phase) approach to optimize P1 or P2; the first phase consists of maximizing A_S subject to the budget constraint for an equipment made up of J types of major assemblies, using any method described in earlier chapters, which will give us S_j, A_j, and the total purchasing costs for each item, say C_j, j=1,..,J; phase 2 can then be apllied by using C_j as the available budget for assembly j along with the additional information about its sub-assemblies (c_{jk}'s and failure rates) to determine whether it is worthwhile to disaggregate the Poisson process into smaller processes and still increase A_S for the same cost.

The end result is the ability to reduce a large scale problem into separate smaller problems whereby we optimize J major assemblies first and use the total costs obtained for an assembly to optimize its k major components while still achieving a better aggregate system performance measure, as demonstrated above. The current military model could well handle this type of situation since the list of items within an equipment submitted by a manufacturer is supposed to follow strict guidelines as to the breakdown of the equipment into components and must contain a code for each item identifying its relationship to the next higher assembly, similar to how a bill of materials is used in an MRP system (Material Requirement Planning).

7.2 MULTIPLE LOCATION MODELS

7.2.1 Introduction. The multiple location model extension to problems P1 (Max A_S) and/or P2 (Min BO) is of practical importance for organizations having several identical systems (vehicles, machines,..) each operating in different locations and becomes essential in the military environment for the initial purchase (and subsequent periodic budgetary process) of spares for several locations when a fixed budget is available. The objective is to decide how much of the overall available budget B should be allocated to each location in a decentralized operational environment to purchase and stock spare assemblies/parts while optimizing a system performance measure. The budget allocation vector $\{BA_i\}$ to each location is then used for the determination of the optimal stock levels $\{S_{ij},\ i=1,..,I\ j=1,..,J\}$ for each part j at every location i.

A numerical example will be introduced following the methodology discussed below. We shall present the derivation for solving model P1b (maximizing A_S) only, as the derivation for solving model P2b (minimizing BO) follows essentially the same methodology.

Probably the most extensive research incorporating several levels of indentures has been in the context of multi-echelon inventory systems for repairable parts described in chapter 2 earlier; most optimization models are not mathematically tractable but nevertheless employ several techniques to obtain stockage levels close to the "true" optimal levels usually obtained from simulation of the system under study for long periods of time and/or under several assumptions and/or restrictions. Valuable papers on the subject include [Sherbrooke 1971], [Muckstadt 1973], and others as discussed in the literature review of chapter 2.

7.2.2 Terminology and notation. Adding subscript i (i=1,..I) to the notation introduced earlier, $\{\delta_{ij}\}$ i=1,..I locations and j=1,..,J items represents Poisson failure rates based on annual expected common usage U and the number of identical equipments $\{M_i\}$ operating at each of I locations.

For example, if each of M_i=10 identical equipments at location i are expected to operate U=10,000 kilometers during the period and item j is expected to fail at a constant (exponential) rate of $\tau_j = 1$ fr/20,000 kilometers, then the distribution of the number of failures of part j is Poisson with rate $\delta_{ij} = M_i \times U \times \tau_j = 10 \times 10{,}000 \times 1/20{,}000 = 5$. A second location having 20 equipments will thus double all Poisson rates of the former location. The aggregate Poisson process can then be represented as:

$$\delta_{.j} = \sum_{i=1}^{I} \delta_{ij} = \text{Sum of all Poisson rates for item j (across all locations)}$$

$$\delta_{i.} = \sum_{j=1}^{J} \delta_{ij} = \text{Sum of all Poisson rates at loc i (across all items)}$$

7.2.3 Formulation model P1b (Max A_S). The multiple item problem P1 to maximize A_S, described in earlier chapters and solved using various solution methods, can be similarly extended to include multiple locations and is formulated as model P1b as follows:

$$\text{Max } A_S = \prod_{i=1}^{I} \prod_{j=1}^{J} A_{ij}(S_{ij}) = \prod_{i=1}^{I} \prod_{j=1}^{J} \left(\sum_{x=0}^{S_{ij}} p_{ij}(x) \right) \qquad \text{(P1b)}$$

s.t.

$$\sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} S_{ij} \le B \qquad (7.4)$$

$$S_{ij} \ge 0 \quad i=1,..,I \quad j=1,..,J \qquad (7.5)$$

As before, the analyst could specify practical and meaningful lower and upper bounds vectors {LB(Sij)} and {UB(Sij)} i=1,..,I and j=1,..J by simply specifying a common minimum availability for each item, say {min A_{ij}} ≥ 0.60 or higher for each part j at each location which will automatically reduce the state space required to obtain the exact solution vector of stockage levels {S_{ij}^*}. Caution must be exercised however, since specifying too high {LB(S_{ij}'s)} can lead to an infeasible solution as the initial allocation may already exceed the specified budget value B, as discussed earlier.

7.2.4 <u>DP solution procedure</u>. The forward/backward recursion used for the DP solution procedure of chapter 3 can also be applied to solve model P1b to maximize A_S but a 2-dimensional state space requires first the specification of various amounts of budget to be allocated at each location {BA_i, i=1,..,I}, followed by optimizing stock level vector {S_{ij}, j=1,..,J} within each location i=1,..,I; thus, it involves a stage for each of the I locations, and within each location (or stage), a number of possible budget amounts y_i to be allocated to optimize the selected performance measure for each of the J items. Furthermore, budget allocations of significant amounts, possibly involving several hundreds or thousands of dollars for items at all locations, require us to choose appropriate increments z_i of say $10, $50, $100,.. in order to reach a solution. The recursive DP formulation procedure is as follows:

<u>Stage</u> i=1,..,I = location i

<u>State</u> b_i = amount available for allocation with i stages remaining

<u>Decision variables</u>

y_i = amount allocated for {S_{ij}} items at stage i

{S_{ij}}= allocation stockage level vector for all j items at location i

<u>Return function</u>

$A_i(y_i)$ = availability from allocating \$$y_i$ stage i

$A_i(S_{ij})$ = availability from {S_{ij}} items at stage i

$$= \prod_{j=1}^{J} A_{ij}(S_{ij}) = \prod_{j=1}^{J} (\Sigma\, p_{ij}(x,\delta ij))$$

$\{S_{ij}\}$ = allocation vector of stockage levels at location i = $\{S_{i1}, S_{i2}, .., S_{iJ}\}$

Forward recursive formula

$$f_0(b_i) = 1$$

$$f_i(b_i, y_i) = A_i(y_i) . f_{i-1}(b_i - y_i) \quad i=1,..,I$$

Optimal value $f_i^*(b_i)$ at any stage

$$f_i^*(b_i) = \max_{\min y_i \le y_i \le \max y_i} \{A_i\{y_i\} . f_{i-1}(b_i - y_i)\}$$

$$= \max_{\min \le \{S_{ij}\} \le \max} \{A_i\{S_{ij}\} . f_{i-1}(b_i - \sum_{j=1}^{J} c_{ij} S_{ij})$$

where b_i = all possible available budget values in discrete increments of z_i (intervals)

y_i = all possible values to be allocated in discrete increments of z_i (intervals)

$$\min y_i = \sum_{j=1}^{J} c_{ij}(\min S_{ij})$$

$$\max y_i = B - \sum_{\substack{k=1 \\ k<>i}}^{I} \min y_k$$

The procedure is to start at stage i=1 (location 1) and determine the optimal allocation y_i from its corresponding stockage levels $\{S_{ij}\}$ from all its possible values in the range [min y_i to max y_i] in increments of z_i, when b_i is available with i stages remaining; once the last stage I optimal allocation for a given budget B has been calculated, work backward from stage I to stage 1 to determine the optimal allocation vector $\{S_{ij}\}$ at each stage.

7.2.5 Comments. The major problem associated with this method is that since the possible budget allocations y_i at each stage is considered a continuous variable, we must choose "suitable" incremental values z_i for y_i and therefore, the solution obtained can no longer guaranteed to be optimal.

These incremental z_i values at each stage for the eventual allocation of an amount y_i have a similar interpretation to the network analysis with budget gaps whereby part of the budget up to the incremental value may be lost from stage to stage. These values can be specified in increments of c_j values to minimize possible budget losses at each stage.

Second, for each y_i considered at each stage, we must find an optimal solution involving multiple items for which the DP solution methodology of chapter 3 for a single location may be applied, leading to a time consuming procedure, rapidly increasing in complexity and exploding the state space to unacceptable levels. This multi-dimensional DP procedure for an I=3 locations problem can be represented as shown in table 7.5 below; within each cell, we must find the optimal allocation vector {Sij} given each possible allocation y_i={BA_i} at stage i.

Table 7.5: DP structure I=3 locations (stages)

Stage I=1	LB(y1) = min y1					UB(y1) = max y1		
b1\y1	min y1	BA1-z1	mid y1	BA1+z1		...y1 max	y1(*)	A1(*)
min b1	A1(y1) |----							
..		A1(y1) |----						
BA1			A1(y1) |----					
..				A1(y1) |----				
max b1					...			
Stage I=2	LB(y2) = min y2					UB(y2) = max y2		
b2\y2	min y2 ..	BA2-z2	mid y2	BA2+z2		...y2 max	y2(*)	A2(*)
min b1 + min b2	|----							
		|----						
			|----					
				|----				
				...	|----			
	----|		A2(y2)xA1*(b2-y2)			|----		
-z2		----|	A2(y2)xA1*(b2-y2)					
BA1+BA2			----| A2(y2)xA1*(b2-y2)					
+z2			| A2(y2)xA1*(b2-y2)					
			| A2(y2)xA1*(b2-y2)					
			| ...					
				----|			
					----|		
					|		
						----|	
max b1 + max b2						|	
						----|	
Stage I=3	LB(y3) = min y3					UB(y3) = max y3		
b3\y3	min y3	BA3-z3	mid y3	BA3+z3		...y3max	y3(*)	A3(*)
Budget B		...	A3(B)xA2*(b3-y3)	...				

zi = interval or increment size

mid b1 = BA1 = (La1./La..)*B
mid b2 = BA2 = (La2./La..)*B
mid b3 = BA3 = (La3./La..)*B

min bi = B - Sum {min bk} k=1,...,I k<>i

min b1 = B - Sum {min bk} k=2,3
= B - {min b2 + min b3}

7.2.6 Example (I=3 x J=5 items). The following numerical example has been solved for an I=3 location, J=5 items problem, with B=\$30,000., and $\{M\}=\{2,3,5\}$ = number of identical operating systems at each of 3 locations, each one expected to accumulate an annual usage of U=10,000 miles and the given costs parameters c_j and failure rates τ_j for each item which are obviously the same regardless of the location. From these parameters, the expected number of failures (Poisson parameters) are easily derived as before using the equation $\delta_{ij} = M_i.U.\tau_j = M_i.U.1/MTBF_j$ and are indicated in table 7.6 below.

Table 7.6: Example Parameters (I=3 x J=5)

	Loc i = / M =		1 / 2	2 / 3	3 / 5	10
j	c_{ij}	$MTBF_j$	δ_{1j}	δ_{2j}	δ_{3j}	$\delta_{.j}$
1	1000	20000	1	1.5	2.5	5
2	400	10000	2	3	5	10
3	200	5000	4	6	10	20
4	200	4000	5	7.5	12.5	25
5	100	2500	8	12	20	40
			---	---	---	---
		$\delta_{i.}$ =	20	30	50	100

From the data shown above, it is pointed out that the Poisson parameters $\{\delta_{ij}\}$ at each location is simply a multiple of another location parameter which is proportional to their number of equipments M. For instance, for item j=4 at location i=1, the parameter $\delta_{14}=5$ is based on $M_1U\tau_4$ = 2x10000x1/4000 = 5 while the same item j=4 at location i=3 is δ_{34} = 12.5 since it has 2 1/2 times more equipments ($M_3=5$ vs $M_1=2$) than at location i=1.

To alleviate the problem of finding the possible range of allocation amounts y_i to various locations, a heuristic initial allocation solution methodology would be to assign the budget B in proportion to the number of equipments held at each location such that

$\{BA_i\}=\{(M_i/M) \times B\}$, which is the same as $\{BA_i\} = \{(\delta_{i.}/\delta_{..}) \times B\}$ for $i=1,2,..,I$ resulting in the following mid-range y_i allocation vector $\{BA_i\} = \{BA_1=\$6000, BA_2=\$9000$ and $BA_3=\$15000\}$ for our example, and then solve for the optimal stockage level vector independently for each location, using either the dynamic programming methodology of chapter 3, or other much faster solution methods such as marginal analysis or Lagrange relaxation.

7.2.7 DP solution (I=3 x J=5). The example above has been solved using dynamic programming methodology where each stage represents a location i, and within each cell, marginal analysis was used to determine the near or optimal allocation for each possible budget allocation amount y_i from its minimum $LB(y_i)$ up to its maximum $UB(y_i)$ values, whereby the maximum values were set arbitrarily but sufficiently higher than the proportional budget allocation $\{BA_i\} = \{\$6000, \$9000, \$15000\}$ calculated earlier to ensure undominated or optimal solutions at each stage are found. The incremental values z_i were therefore not constant since they were chosen from the results of marginal analysis performed independently for each location. The results are indicated in table 7.7 below and are summarized as follows:

$\{S_{ij}\} = \{i=1: 1,3,7,8,13;$ $\quad C_{1S} = 6500$

$\quad i=2: 2,4,9,11,17;$ $\quad C_{2S} = 9300$

$\quad i=3: 3,7,13,16,26\};$ $\quad C_{3S} = 14200$

$\{S_{.j}\} = \{6,14,29,35,56\};$ $\quad C_{.S} = 30000$

$\{BA_i\} = \{6500,9300,14200\};$

$A_S = .1277$ and $BO = 3.4702$

Due to the increasing computational difficulties as the number of locations and/or items are added and for the same reasons given earlier in chapter 3 for the single location case, the DP solution procedure for the multiple location models become impractical for several more locations (say $I \geq 10$) and budgets involving tens or hundreds of thousands of dollars. It is desirable to develop better and more efficient solution methods to solve this problem.

Table 7.7: DP solution (I=3 x J=5)

Stage i=1		LB(y1)=	$6 000		UB(y1)=	$7 600		
b1\y1	$6 000	$6 200	$6 400	$6 500	$6 900	$7 900	y1(*)	A1(*)
$6 000	0,4550 |—						$6 000	0,4550
$6 200		0,4893 |—					$6 200	0,4893
$6 400			0,5221 |—				$6 400	0,5221
$6 500				0,5386 |—			$6 500	0,5386
$6 900					0,5953		$6 900	0,5953

Stage i=2		LB(y2)=	$9 000		UB(y2)=	$10 500				
b2\y2	$8 900	$9 100	$9 300	$9 700	$9 800	$10 000	$10 200	$10 300	y2(*)	A2(*)
$14 900	0,2054 |—								$8 900	0,2054
$15 100	0,2209	0,2221 |—							$9 100	0,2221
$15 300	0,2357	0,2388	0,2372 |—						$9 100	0,2388
$15 700	0,2431	0,2629	0,2721	0,2665 |—					$9 300	0,2721
$15 800	0,2687	0,2629	0,2807	0,2665	0,2737 |—				$9 300	0,2807
$16 000	—|	0,2906	0,2807	0,2865	0,2944	0,2861 |—			$9 800	0,2944
$16 200		—|	0,3103	0,3154	0,3141	0,3076	0,2974 |—		$9 700	0,3154
$16 300			—|	0,3154	0,3240	0,3076	0,2974	0,3024	$9 800	0,3240
$16 400				0,3154	0,3240	0,3282	0,3199	0,3024	$10 000	0,3282
$16 500				0,3154	0,3240	0,3386	0,3199	0,3252	$10 000	0,3386
$16 600				0,3486	0,3240	0,3386	0,3413	0,3252	$9 700	0,3486
$16 700				—|	0,3581	0,3386	0,3521	0,3470	$9 800	0,3581
$16 800					—|	0,3386	0,3521	0,3580	$10 300	0,3580
$16 900						0,3743	0,3521	0,3580	$10 000	0,3743
$17 000						—|	0,3521	0,3580	$10 300	0,3580
$17 100							0,3891	0,3580	$10 200	0,3891
$17 200							—|	0,3957	$10 300	0,3957

Stage i=3		LB(y3)=	$13 800		UB(y3)=	$15 000				
b3\y3	$13 500	$13 600	$13 800	$14 200	$14 400	$14 500	$14 700	$15 000	y3(*)	A3(*)
$30 000	0,1210	0,1218	0,1262	0,1277	0,1152	0,1101	0,1073	0,1134	$14 200	0,1277

Solution	y3= $14 200	A3= 0,45495
Budget allocation	y2= $9 300	A2= 0,52119
(DP + Marginal)	y1= $6 500	A1= 0,53858
	B= $30 000	As= 0,12771
Solution	y3= $15 000	A3= 0,55234
Budget allocation	y2= $9 000	A2= 0,45139
(Prop + Marginal)	y1= $6 000	A1= 0,45503
	B= $30 000	As= 0,11345

For the initial procurement and distribution of spares in the military environment, when a new capital acquisition program is activated, the budget allocation at each location cannot be pre-assigned since spares are individually calculated up to the same availability level (99.8%) using the current model described in chapter 1 earlier and does not discrimate between more expensive and/or cheaper items.

7.2.8 Marginal analysis equivalent (I=1 x J=15). Recognizing that the multiple location multiple item model P1b is also separable by item and by location, then marginal analysis with the natural log transformation of the objective function terms can be used by combining all 3 locations into a single one and would result in the same optimal solution obtained with DP methodology. Therefore, model P1b below, subject to (7.4) and (7.5) earlier and repeated here becomes:

$$\text{Max } A_S = \prod_{i=1}^{I} \prod_{j=1}^{J} A_{ij}(S_{ij}) = \prod_{i=1}^{I} \prod_{j=1}^{J} \left(\sum_{x=0}^{S_{ij}} p_{ij}(x) \right) \qquad \text{(P1b)}$$

s.t.

$$\sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} S_{ij} \leq B \qquad (7.4)$$

$$\{S_{ij}\} \geq 0 \quad i=1,..,I \quad j=1,..,J \qquad (7.5)$$

upon the log transform of the objective function:

$$\text{Max } \ln(A_S) = \sum_{i=1}^{I} \sum_{j=1}^{J} \ln\{A_{ij}(S_{ij})\} \qquad \text{(P1b')}$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{J} \ln \left(\sum_{x=0}^{S_{ij}} p_{ij}(x) \right)$$

subject to (7.4) and (7.5), where each function $\ln A_{ij}$ is concave as proven earlier. The marginal increase in the objective function value of model (P1b)' as a result of adding 1 more part j at location i from S_{ij} to $S_{ij}+1$ therefore becomes similar to the single location case as derived earlier:

$$d[\ln\{A_{ij}(S_{ij}+1)] = \ln(A_{ij}(S_{ij}+1)) - \ln(A_{ij}(S_{ij})) \qquad (7.6)$$

at an additional cost of c_{ij} and the procedure is therefore equivalent to a model with 1 location having J=15 items that is much faster and simpler to solve (a few seconds only) than DP. The only modification to be made to validate the procedure is to ensure $\{\delta_{ij}\}$ are proportionally adjusted to compensate for the different number of equipments at each location as indicated in table 7.8 below.

Table 7.8: Equivalent parameters (I=1 x J=15 items)

B=\$30,000		I=1 J=15 items
Min A_j=.001		Max A_j=.99999
Item	δ_j	c_j
1	1.0	1000
2	2.0	400
3	4.0	200
4	5.0	200
5	8.0	100
6	1.5	1000
7	3.0	400
8	6.0	200
9	7.5	200
10	12.0	100
11	2.5	1000
12	5.0	400
13	10.0	200
14	12.5	200
15	20.0	100

The optimal solution using the marginal analysis procedure for the equivalent single location model I=1 location x J=15 items resulted, after dropping the location index i, in exactly the same optimal allocation vector:

$\{S_j{}^*\} = \{1,3,7,8,13,2,4,9,11,17,3,7,13,16,26\}$; and tallying the results for each location:

$\{S_{ij}\} = \{$i=1: 1,3,7,8,13; $C_{1S} = 6500$

i=2: 2,4,9,11,17; $C_{2S} = 9300$

i=3: 3,7,13,16,26; $C_{3S} = 14200$

$\{S_{.j}\} = \{6,14,29,35,56\}$; $C_{.S} = 30000$

$\{BA_i\} = \{6500,9300,14200\}$

$A_S = .1277$ and BO = 3.4702

and is the true optimal solution vector $\{S_j,\ j=1,..,15\}$ since the total costs C_S is exactly equal to B. Each individual item availability corresponding to the optimal allocation vector $\{S_j{}^*\}$ is shown below for information:

$\{A_j\} = \{.73, .86, .95, .93, .97,$

$.81, .82, .92, .92, .94,$

$.76, .87, .86, .87, .92\}.$

7.2.9 <u>FULL network analysis</u>. As was the case for model P1 to max A_S and model P2 to min BO analyzed earlier for single location problems, the FULL network analysis can also be implemented for the multiple location models P1b and P2b. The FULL network structures can be setup using the same techniques and will yield the same true optimal solution as DP methodology (as long as all possible budget allocation values are enumerated).

First, all items j=1,..,J at the first location i=1 are considered, one at a time and thus make up the first J stages of the network; then, the pattern is again repeated for the same item types j=1,..J for each of the remaining location i=2,..,I. The final network will therefore consist of a total of IxJ stages whose true optimal solution can be found by applying DP sequentially in stages or a shortest path algorithm in the network, from its origin to the destination node N. The network structure can be used effectively to determine the number of rows = (N-1) total nodes and cell evaluations within all rows = total number of arcs A in the network.

The properties of the single location networks for the one location problems are also applicable to the multiple location networks built using the same procedure, namely: the networks will be acyclic and any path from the origin node 1 to the destination node N will be made up of exactly one arc from each stage for a total of IxJ arcs.

Furthermore, the same techniques used to help reduce the network size (total number of nodes and arcs) can be applied here as well by processing higher cost items first within each stage and specifying appropriate lower $\{LB(S_{ij})\}$ and upper bounds $\{UB(S_{ij})\}$. As a result of applying the procedure to our (I=3 locations x J=5 items) example with the same parameters described in the previous sections, the FULL network structure consists of a total of N=1425 nodes, A=10087 arcs and M=8626 matches by processing the higher cost items within each location first and repeating the process for each subsequent location, thus I=3 groups of J=5 items for a total of IxJ = 15 stages. (We note that the lowest common denominator is $100 which avoids the creation of a substantially and unacceptably high number of nodes, explained by the number of matching node labels M = 8626).

It is also possible to reduce the size of the network even more, by exploiting an interesting property of the multiple location models. Since all items j=1,..J are considered at each of the location i=1,..I, it is reasonable to expect that the cost structure $\{c_{ij}\}$ is the same for any item type j, regardless of its location i.e. $c_{1j}=c_{2j}=\ldots=c_{Ij}$, j=1,..,J; therefore the number of nodes and arcs can still be reduced by listing the higher cost item type j in I successive (adjacent) stages eventhough they may have different Poisson parameters.

The reason is that nodes created within any current stage j are created apart by an amount = $\$c_j$, as a result of adding one more item type j; if the next item type j to be listed is the same type of item with equal costs but from a different location, then all nodes to be created will also be created exactly by the same amount $\$c_j$ apart and all arcs incident to them will come from nodes which were also created $\$c_j$ apart at the previous stage j-1. Therefore, the net result will be to direct arcs into nodes already created at the current stage with budget remaining labels that "match" them and thus prevent the creation of additional nodes or avoid possible budget loss by directing arcs into a lower budget node, as described in the next section for the GAP networks.

For our example, by processing the higher cost items sequentially and irrespective of their location i.e. the first 3 items costing $c_{11} = c_{21} = c_{31}$ = \$1000 each, the network reduces to a total of N=831 nodes (41.7% reduction), A=5765 arcs (42.8% reduction) and M=4898 matches which is rather significant as a simple but highly efficient technique to achieve more manageable network structures (or setting up DP tables) while guatanteeing the true optimal solution to the models.

We can still achieve even further reduction by listing the higher cost items first, irrespective of their locations <u>and</u> sorting the items having equal costs in decreasing order of their Poisson parameters i.e. $\delta_{33} = 2.5 > \delta_{23} = 1.5 > \delta_{13} = 1.0$; the net result is a network structure consisting of N=787 nodes, A=5407 arcs and M=4584 matches. Although the network size cannot be predicted, this simple technique appears to be effective for all (albeit few) examples analyzed and should be the subject of further analysis.

7.2.10 <u>GAP network analysis</u>. The GAP network analysis procedure along with all its properties discussed in chapter 3 earlier for the single location models P1 and P2 can also be implemented for the multiple location models P1b and P2b treated in this chapter; thus, the networks constructed by using the procedure are acyclic, any path from the origin node 1 to the destination node N is made up of exactly IxJ arcs, one from each stage.

For our example, listing the higher cost items c_{ij} first within each location and constructing the GAP network by processing the 15 items in I=3 groups (locations) of J=5 items (higher cost items listed first within each location) resulted in a network with a total of N=1055 nodes, A=7162 arcs and M=6090 matches.

We can still apply the error bound for the near optimal solution obtained as a result as was done in chapter 3; thus it follows that the total cost solution $C_S(.)$ will be at least B - $\Sigma\Sigma c_{ij}$, i=1,..,I, j=2,..,J since only the first item j=1 at the first location i=1 will guarantee that no budget loss occurs (the first item to be listed and making up the first of the IxJ=15 stages). For our example, the total cost solution $C_S(.)$ from the network (or DP tables) will be at least \$30,000 - \$4,400 = \$25,600 or a 14.7% maximum error from B=\$30,000 by processing the items in I=3 groups of J=5 items.

The network reduction techniques are also applicable here but the order in which the items make up the IxJ stages become of much greater significance in improving the lower bound on $C_S(.)$; as shown earlier, if the stages are made up by sorting the higher cost items first and irrespective of their locations, the net result will be to improve the lower bound $C_S(.)$ to at least B - Σc_j j=2,..J, for the single location model, since no budget loss can occur when equal cost items are listed in adjacent stages. For the multiple location models however, the result for $C_S(.)$ is much closer to the total available budget B as shown by the following important proposition:

<u>Proposition 7.1</u>: The total cost solution for $C_S(.)$ as a result of applying the GAP network procedure and processing the higher cost items first, regardless of their location (and assumed to have equal purchasing costs) will have the same lower bound as the single location models i.e. B - Σc_j, j=2,..,J.

<u>Proof</u>: We know from the single location model that $C_S(.)$ will have a lower bound of at least B - Σc_j when listing higher cost items first and applying the GAP network procedure since nodes created as a result of stage j=1 have exact budget node labels and a possible budget loss of c_j can occur for each of the remaining stages j=2,..J.

Assuming the same items have identical costs regardless of their location, listing all of them in adjacent stages will result in exact budget node labels at stage i=1,..,I for the first j items (the first I stages) since nodes (or DP rows) at the current stage are created exactly c_j apart and have arcs incident to them only from nodes created at the preceding stage, also c_j apart; it then follows that if two items have equal costs, no budget loss for nodes created will occur.

As a result of grouping equal cost items in adjacent stages, only J-1 stages will have a preceding stage with a different cost item and the total maximum budget loss can only be $\Sigma\ c_j$ $j=2,..,J$, exactly the same as the single location model and this completes the proof.

The immediate consequence, however, is to significantly lower the maximum relative error ß(%) of the shortest path solution $C_S(.)$ which then becomes much closer to the total available budget B. For our example, sorting the items using this simple technique (listing all 3 items costing $1000. each as the first 3 stages) reduces the network size to a total of N=728 nodes, A=5036 arcs and M=4272 matches, while the total cost solution $C_S(.)$ will be at least B - $\Sigma\ c_j$ which becomes $30,000 - ($400+$200+$200+$100) = $29,100 or less than $900/$30,000 or 3% maximal error which compares with $25,600 or a 14.7% error above.

Furthermore, it appears that the net result of sorting equal cost items in decreasing order of their Poisson parameters can also have a positive effect on reducing network size. Thus, by listing higher cost items first, irrespective of their locations and by decreasing order of their respective Poisson parameters δ_{ij} in the example, resulted in a network total of N=680 nodes, A=4648 arcs and M=3932 matches; other examples seem to confirm this finding but cannot be formally proven due to the combinatorial nature of "matching" labels which cannot be accurately predicted.

Table 7.9 below summarizes and compares the results of FULL and GAP networks for our multiple location example (I=3 x J=5 items); each network built using the appropriate sorting techniques presented in the two previous sections is compared in terms of its size i.e. the total number of N=nodes, A=arcs and M=matches to highlight related savings and the last column states the maximal error ß(%) for the total cost solution $C_S(.)$ from the total available budget B obtained by applying a shortest algorithm to GAP networks. We also note that the savings achieved from the FULL network to the GAP network structure is significant but would be much more had we used cost data more representative of real world problems, unlike the $100 common denominator used for all items here.

Table 7.9: FULL vs GAP network solutions (I=1 x J=15)

FULL network	N	A	M	ß(%)
1. Higher cost items first within each location (I=3 x J=5 items)	1425	10087	8626	NA
2. Higher cost (and equal) items first regardless of its location	831	5765	4898	NA
3. Higher cost (and equal) items first regardless of their location and in decreasing order of their Poisson parameters	787	5407	4584	NA
GAP network				
1. Higher cost items first within each location (I=3 x J=5 items)	1055	7162	6090	14.7
2. Higher cost (and equal) items first regardless of its location	728	5036	4272	3.0
3. Higher cost (and equal) items first regardless of their location and in decreasing order of their Poisson parameters	680	4648	3932	3.0

7.2.11 Formulation model P2b (Min BO). Similar to model P1b to maximize A_S, model P2 to minimize BO can also be extended to the multiple location model P2b shown below, due to item separability and convexity of each individual BO_{ij} function:

$$\text{Min } BO = \sum_{i=1}^{I} \sum_{j=1}^{J} BO_{ij} \qquad \text{(P2b)}$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{x=S_{ij}+1}^{\infty} (x-S_{ij}) . p_{ij}(x)$$

s.t.

$$\sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} S_{ij} \leq B \tag{7.7}$$

$$S_{ij} \geq 0 \qquad i=1,..,I \quad j=1,..,J \tag{7.8}$$

The dynamic programming method of chapter 3 earlier can also be applied to the multiple location model P2b as well, in order to minimize BO with a similar dynamic programming recursion of model (P2a) described earlier except that the return function is simply the sum of BO functions instead of the multiplicative recursive formula involving system availability. The complete solution methodology, including the budget allocation problem $\{BA_i\}$ to each location, will therefore be omitted since we have already shown much more efficient and quicker solution methodologies, described below.

Since model P2b is already separable by item and by location, the marginal analysis procedure can also be applied directly. As described for the backorder case, the marginal benefit of adding 1 more item type ij from S_{ij} to $S_{ij}+1$ will result in a net reduction of the number of expected backorders by its complimentary cdf $P_{ij}(S_{ij}) = 1-F_{ij}(S_{ij})$ at an additional cost of c_{ij}. The results obtained from the marginal analysis procedure (dropping the index location i), indicate that C_S=\$30,000. was exactly the specified available budget B, and therefore is an undominated and the optimal solution vector:

$\{S_j\}$ = {1,3,6,8,13,2,4,9,11,18,3,6,14,17,29}; and tallying the results for each location:

$\{S_{ij}\}$ = {i=1: 1,3,6,8,13; C_{1S} = 6300
i=2: 2,4,9,11,18; C_{2S} = 9400
i=3: 3,6,14,17,29; C_{3S} = 14300
$\{S_{.j}\}$ = {4,15,32,38,60}; $C_{.S}$ = 30000
$\{BA_i\}$ = {6300,9400,14300}
A_S = .1241 and BO = 3.41196

which is slightly different than the solution obtained when maximizing A_S, with the same procedure. Since the total costs C_S obtained as a result of the procedure is exactly the specified budget B=\$30,000, then the corresponding stockage level vector $\{S_j^*\}$ is guaranteed to be the optimal solution for minimizing BO; that is, no other stockage levels would have resulted in a lower expected total system backorders of BO = 3.41196 for a budget B = \$30,000. The solution obtained with marginal analysis when maximizing A_S was slightly different (.1277 vs .1241 here) and the corresponding BO = 3.4706 which is, as expected, slightly greater than the value BO = 3.41196 obtained here when minimizing BO.

The FULL and GAP network analysis procedures for the single location model P2 to minimize BO also apply for the multiple location model P2b. The procedure to setup the network remains the same as for model P1b discussed above except that the branches are represented by the cumulative Poisson number of backorders $\Sigma\ (x-S_j)p(x)$, $x=S_j+1$ to ∞ which is decreasing as the number of spares S_j increases. Thus, DP or the shortest path in the network can be direcly applied to the network to solve for the optimal spare parts stockage level vector $\{S_j, j=1,..,J\}$. We note that the significance of the system performance A_S becomes less meaningful as I and J increase; the multiplication of all item availabilities A_{ij} practically 0 which means that we are practically certain to run out of at least 1 type of spare but do not say much else; on the other hand, the total expected system backorders gradually become more meaningful since we can relate it directly to another more appropriate system performance measure AA_S = the average number (proportion) of equipments expected to be operational at the end of the period, by approximating AA_S value with the expression 1 - BO/M, and will be discussed when comparing several larger scale problems in chapter 8. For our example, BO = 3.41, thus 1 - BO/M = 1 - 3.41/10 = 0.659 or 65.9% approximates the simulated AA_S = 75% and as we shall demonstrate, constitute a lower bound on AA_S.

7.3 MODEL COMPARISON P1 and P2 vs EQUAL Aj's

The comparison of the 3 models for the I=3 x J=5 example just described when an available budget of \$30,000 is specified was measured by the system availability performance measure tracked for all 3 models and are shown in figure 7.1 below, as indicated by an optimal A_S = 0.1277 obtained from DP and marginal analysis, A_S = .1241 when min BO and A_S = .07823 with the equal A_j's model used in the military. Even the proportional budget allocation method $\{BA_i\}$ is superior to the military model with A_S = .1134. The measure of performance A_S used to optimize stock levels $\{S_{ij}\}$ still means the probability of not running out of any spares j at any location i and, although it may be nice to know, becomes less meaningful in the multiple location models, since a different number of equipments may be operational for each location and therefore, the impact on AA_S may be more significant for locations having a lesser number than at others. The problem of allocating budget levels at each location also remains.

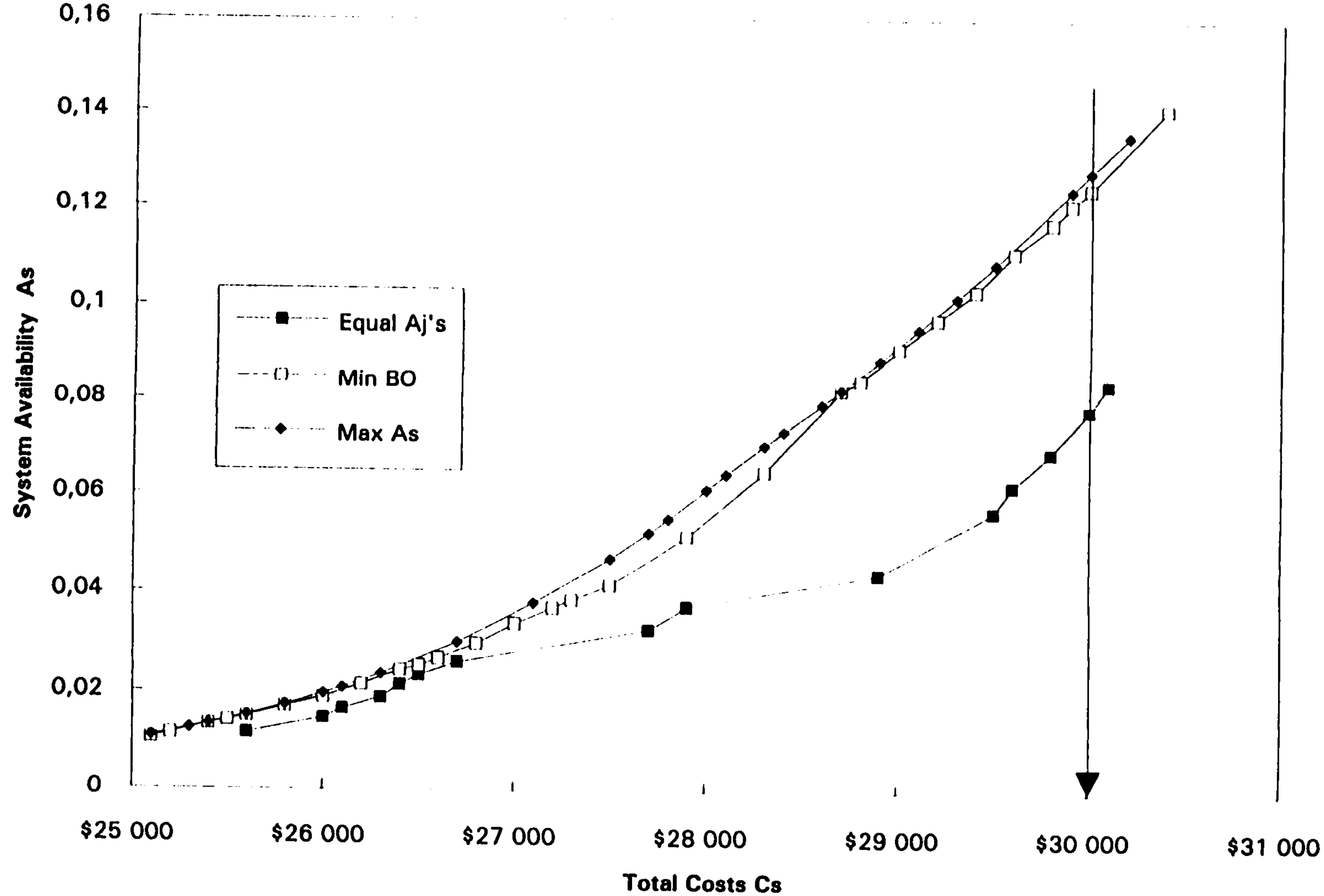


Figure 7.1: Models P1 and P2 vs Equal Aj's (I=3 x J=5)

The effect on the more appropriate performance measure AA_S or the average proportion of equipments still operational at the end of the period can also be significant. The stock level vectors as a result of solving P1, P2 and the Equal Aj's model were simulated for each location and N=5,000 cycles by allocating the optimal budget $\{BA_i*, i=1,2,3\}$ = {$6500,$9300 and $14200} at each location obtained from marginal analysis (I=1 x J=15), and by allocating the budget proportionally $\{BA_i\}$ = {$6000, $9000 and $15000} or in proportion to the number of equipments M=2,3 and 5 held respectively at each location, and are presented in table 7.10 below.

Table 7.10: Simulated AAs for P1b and Equal Aj's (I=3 x J=5)

Budget\AAs(i) \Loc i	AAs1 M=2	AAs2 M=3	AAs3 M=5	AAs M=10
- {BAi*}	$6500	$9300	$14200	$30000
- P1b (or P2b)	1.4102	2.2672	3.8900	7.5674
% operational	70.5%	75.6%	77.8%	75.7%
- Equal Aj's	1.2092	1.9896	3.5678	6.7666
% operational	60.5%	66.3%	71.4%	67.6%
- {BAi} prop	$6000	$9000	$15000	$30000
- P1b (or P2b)	1.2726	2.1548	4.1820	7.6094
% operational	63.6%	71.8%	83.6%	76.1%
- Equal Aj's	0.9216	1.8832	3.8392	6.6440
% operational	46.1%	62.8%	76.8%	66.4%

There are 4 important conclusions to draw from the data: first, using marginal analysis to solve model P1b (or P2b) resulted in stock level vectors $\{S_{ij}\}$ that yield significant higher AA_S values when compared to the equal A_j's model (current military model) regardless of the method the budget is allocated; this is not surprising since we have just shown that model P1b is equivalent to a single location case P1 with the same number of items appended I times, and marginal analysis to optimize A_S will always be superior to the equal A_j's model.

Second, there does not seem to be a significant difference between the same model when comparing both budget allocation methods; for P1b to maximize A_S in the example above, AA_S = 75.95% using $\{BA_i{*}\}$ and 76.09% using the proportional $\{BA_i\}$ and is also evidenced by the equal Aj's model (67.54% vs 66.44%). This makes sense and can be explained as follows: given a specified available budget, if the same procedure is used at all locations to allocate items, regardless of the budget allocated, then AA_S will simply decrease at one location if less money is available but will be compensated at another location by having more money, as long as the total available budget B for all locations remain the same, and the same method to obtain stock level vectors $\{S\}$ is used across all locations. The important consequence of this is that we can use the proportional budget allocation method easily by simply assigning $\{BA_i\}$ = $\{M_i/M\ i=1,..,I\}$ as an available budget at each location, from which the aggregate system performance measure to be optimized can be implemented efficiently. The disadvantage of this budget allocation method is that we know AA_S may not be optimal for some locations with fewer equipments, but will be compensated by a higher AA_S at others, but the total AA_S across all locations remaining the same.

In our example, AA_S = 1.4114 + 2.2810 + 3.9028 = 7.5952 out of 10 or 75.95% equipments operational for model P1b (optimal budget of \$6500, \$9300 and \$14200) vs AA_S = 1.2726 + 2.1548 + 4.1820 = 7.6094 out of 10 or 76.094% for P1b (proportional budget of \$6000, \$9000 and \$15000); so, the decrease in AA_S at location i=1 from 1.4114 (out of M=2 equipments) to 1.2726 due to less budget available (\$6500 vs \$6000) and a slight decrease at location i=2 from 2.2810 (out of M=3 equipments) to 2.1548 (\$9300 vs \$9000) has been compensated by an increase in AA_S at location i=3 from 3.9028 (out of M=5 equipments) to 4.1820 (\$14200 vs \$15000) to give approximately the same overall AA_S of 7.5952 vs 7.6094 (out of M=10 equipments) at all 3 locations. All values were the result of the same total available budget B = \$30000 and using the same marginal analysis procedure in both types of budget allocation methods. The same analysis holds true for the equal A_j's model.

Third, if the total budget available for all locations ($30,000 for our example) and using the same optimal procedure (DP or marginal), it seems that the optimal solution vector will favor higher stock levels to the location having less equipments rather than one having more, due to the proportionally lower values of the Poisson mean parameters $\{\delta_j\}$, which makes sense since the impact of running out of spares at a location with less equipments will be greatest. Another way to look at it is to use the expression 1 - BO/M as an approximation to the more appropriate performance measure AA_S, and will be discussed in chapter 8.

Fourth, and probably the most important, is that it gives us a method to allocate the budget to each location, enabling us to solve large scale problems quickly. Thus, we could choose to allocate the budget in proportion to the number of equipments held at each location, then use topup marginal analysis at each location to (near or) optimally determine stock levels $\{S_{ij}\}$, ensuring we meet the specified budget, unlike the current military model. The result should also be close to the AA_S value for all equipments across all locations, but will tend to give higher AA_{Si} values for locations having more equipments than others. We can equalize those AA_{Si} values across all locations by adopting a budget allocation method by solving P1b with the topup marginal analysis by appending J items I times but at the expense of possibly increasing the equivalent model to a point where system A_S value become practically meaningless to interpret, and also requiring further manipulations, such as tallying results and computing A_S for each location, eventhough the marginal analysis can easily handle such large problems.

We conclude this section by solving a second numerical example I=3 locations x J=10 items, to further support the findings discussed above. With a total budget of $16,000 and $\{M_i\}$ = {20, 30 and 50} respectively, cost parameters $\{c_{ij}\}$ = {$100.,..,$650} and $\{\delta_{ij}\}$ = {0.445 to 8.333}, we solved P1b using marginal analysis at each of the

I=3 locations by allocating the budget proportionally $\{BA_i\} = \{M_i/M\}$ vs the optimal budget obtained with marginal analysis for the equivalent model (I=1 x J=30 items), stopping at B=$16,000 and tallying the resulting $\{BA_i\}$ at each location. We then repeated the same procedure for the equal A_j's model, and the simulated AAs values (N=5,000 cycles each) obtained as a result of this methodology are presented in table 7.11 below:

<u>Table 7.11: Simulated AAs for P1b and Equal Aj's (I=3 x J=10)</u>

Budget\AAs(i) \Loc i	AAs1 M=20	AAs2 M=30	AAs3 M=50	AAs M=100
- {BAi*}	$3540	$5325	$7135	$16000
- P1b (or P2b)	18.202	27.996	46.476	92.674
% operational	91.0%	93.3%	93.0%	92.7%
- Equal Aj's	17.173	26.810	44.767	88.750
% operational	85.9%	89.4%	89.5%	88.7%
- {BAi} prop	$3200	$4800	$8000	$16000
- P1b (or P2b)	17.972	27.558	47.122	92.652
% operational	89.9%	91.9%	94.2%	92.7%
- Equal Aj's	16.609	26.557	45.712	88.878
% operational	83.0%	88.5%	91.4%	88.9%

7.4 REVERSE MARGINAL ANALYSIS.

An interesting concept resulting from the multiple location models is the re-distribution of items either at the beginning of a period as a result of purchasing $\{S_{ij}\}$ items or re-distribution at any time during the period if the inventory position is known, in order to optimize our selected performance measures. This concept is particularly important even for an organization that does not hold some quantity of items at a central warehouse, otherwise, it would become a multi-echelon type of system that has been extensively studied under various assumptions of lead-time distributions, repairable items,.. as stated in the literature review of chapter 3.

The marginal analysis procedure described for the single or multiple location models always involves the selection at each iteration of an item j which provides the greatest increase of the objective function per dollar invested when maximizing A_S or the greatest decrease in the number of backorders per dollar invested. We can thus apply the procedure in "reverse" in order to allocate in a near or optimal way to each location i, the stock recently purchased or to re-allocate any time during the period, given that we know the current inventory position for item j at every location i.

This can be accomplished by simply combining all quantities held for each item j, say $q_j = \Sigma\ inv_j$ summed over the location index i=1,..,I and applying the marginal analysis by optimizing A_S in reverse, starting from this initial allocation quantity $\{S_j=q_j,$ $i=1,..,I\}$ and reducing the "pooled" quantitie q_{ij} for each item to the different locations until it reaches 0. The procedure will guarantee that the allocation quantity vector {Sij} obtained is the optimal solution since at every iteration, the solution obtained is undominated (see [Fox 1966]). Once stock levels of each item j for all locations have been determined, a simple redistribution of items between locations can take place, assuming that transportation (transshipment) costs are negligible, otherwise, a new model would be required to minimize transportation costs.

7.5 INVENTORY HOLDING COSTS

In most organizations, the inventory holding costs are usually an integral part of the accounting systems and financial statements of any business, and are measured using a cost of capital, denoted R (either as \$/\$/time unit or as a % of the individual item value), and eventhough its accuracy may be doubtful, they can be a significant part of total costs in addition to the purchasing costs of the items. For a public institution, such as the military, the cost of capital and the inventory holding costs cannot be measured against alternative projects yielding a say 20% return on capital investment. Therefore, this section will specifically deal with the additional costs incurred as a result of carrying the items in inventory during the time period, and is applicable mostly for organizations that include these costs with appropriate accounting practices. Model P1 can thus be re-formulated as model P1c defined below:

$$\text{Max} \quad A_S = \prod_{j=1}^{J} A_j = \prod_{j=1}^{J} \left(\sum_{x=0}^{S_j} p_j(x) \right) \tag{P1c}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} c_j S_j + \sum_{j=1}^{J} (S_j + S_j - \delta_j) c_j R/2 \le B \tag{7.9}$$

or, upon rearranging terms for constraint (7.9) as explained below:

$$\text{s.t.} \quad \sum_{j=1}^{J} (c_j S_j (1+R) - \delta_j c_j R/2) \le B \tag{7.9a}$$

$$S_j \ge [\delta_j] \qquad j=1,2,..,J \tag{7.10}$$

The reason constraint (7.10) must be added is to start with an inventory level vector $\{S_j,\ j=1,..,J\}$ that is the smallest integer greater or equal to the expected number of failures during the time period (Poisson rate parameter) for each item j=1,2,..,J, and will ensure that costs will always be positive because of the second term on the left side of the inequality in constraint (7.9a).

The second aggregate performance measure for such an inventory system is to minimize the total expected number of backorders BO and model P2 can be re-formulated as follows:

$$\text{Min} \quad BO = \sum_{j=1}^{J} (BO_j) = \sum_{j=1}^{J} \sum_{x=S_j+1}^{\infty} (x-S_j).p_j(x) \tag{P2c}$$

subject to equations (7.9a) and (7.10) as for model P1c. Constraint (7.9) or (7.9a) in both models P1c and P2c consists of the total purchasing costs $C = \Sigma\, c_j S_j$ when S_j items of type j are purchased at the beginning of the cycle and the total inventory costs $H = \Sigma (S_j+S_j-\delta_j)c_jR/2$ incurred during the period given by the second term on the left side of constraint (7.9a) which is the average inventory times the item's cost multiplied by the interest rate factor for the period.

The inventory costs can be explained as follows: at the beginning of the period, I have S_j items of type j and I expect a demand of δ_j items during the period, therefore the Expected Net Stock E(NS) for the period = $S_j-\delta_j$ items and the average inventory will be $(S_j+S_j-\delta_j)/2$ at a cost of c_jR for each one (see [Silver and Peterson 1985]). The holding costs are thus based on average inventory as opposed to ending inventory.

In this context, it is assumed that the expected number of backorders BO is small compared to the Expected On-Hand inventory E(OH) so that the expected net stock E(NS) ≈ E(OH). The exact expressions for these quantities are as follows:

$$E(OH) = \sum_{x=0}^{S} (S-x).p(x) \qquad (7.11)$$

$$BO = \sum_{x=S+1}^{\infty} (x-S).p(x) \qquad (7.12)$$

$$E(NS) = E(OH) - BO \qquad (7.13)$$

$$\approx E(OH) \quad \text{when } BO/E(OH) \approx 0$$

The above approximation is valid when sufficiently high availability A_S is specified since it will result in higher A_j's for individual item types, or the number of items J is high enough such that individual Aj's ≥ .90; for example, if J=20 item types and A_S = 0.10 after the optimization, then individual A_j's will average $.20^{(1/20)}=0.922$. Then, suppose S=10, δ=6, then $A_j(S_j=10|\delta_j=6) = 0.96$ and exact values for E(OH) and BO are 4.077 and .077 respectively and E(NS) = 4. Therefore, BO approaches 0 as S increases and the ratio BO/E(OH) = .077/4.077 = 0.0108 also approaches 0; thus, E(NS) = 10-6 = 4 is a close approximation of E(OH) = 4.077 or E(NS) = 4 ≈ E(OH) or less than 2% error.

Since E(NS) can be negative for S_j smaller than the average demand δ_j and thereby resulting in negative total costs when R>0 in constraint (7.9a), the allocation vector $\{S_j, j=1,..,J\}$ is restricted to be smallest integer ≥ the average demand δ_j during the period by adding constraint (7.10); this requirement will ensure that this situation will not arise and if R=0, then this requirement can simply be dropped since no inventory costs would be incurred.

The marginal increase in the cost of an additional item as a result of applying the marginal analysis procedure for both models P1c or P2c simply become $c_j(1+R)$ or $c_j + c_jR$ which is the sum of purchasing 1 additional item type j at the cost c_j and a holding cost of c_jR; thus, we only need to replace the term c_j by $c_j(1+R)$ in all the models P1, P1a, P1b or P2, P2a, P2b to implement the marginal analysis procedure (or others), including the topup procedure. The marginal increase in total costs as a result of adding an additional item or $c_j(1+R)$ also become a very effective and efficient way to incorporate inventory holding costs to setup the FULL and/or GAP network structures for all models, including the Ebeling model, since all of them use the cumulative distribution.

Although the comparison with a public organization is irrelevant here, we nevertheless can illustrate the possible impact of adding inventory holding costs to the models for those organizations having appropriate accounting practices. The end result is that for sufficiently high A_S (depending on the number of items J), if individual item availabilities A_j's, j=1,..,J are such that they are > .90, then the small errors in approximating E(OH) or On-Hand inventory by E(NS) or Net Stock will be relatively accurate, when compared with R values traditionally in the order of 20% on an annual basis.

Example 2 (J=10 items) illustrates the possible impact of adding inventory holding costs during a period, figure 7.2 below shows the marginal analysis sequence of iterations with B=$15,000 for models P1c and Equal Aj's each with R=0 and R=20%.

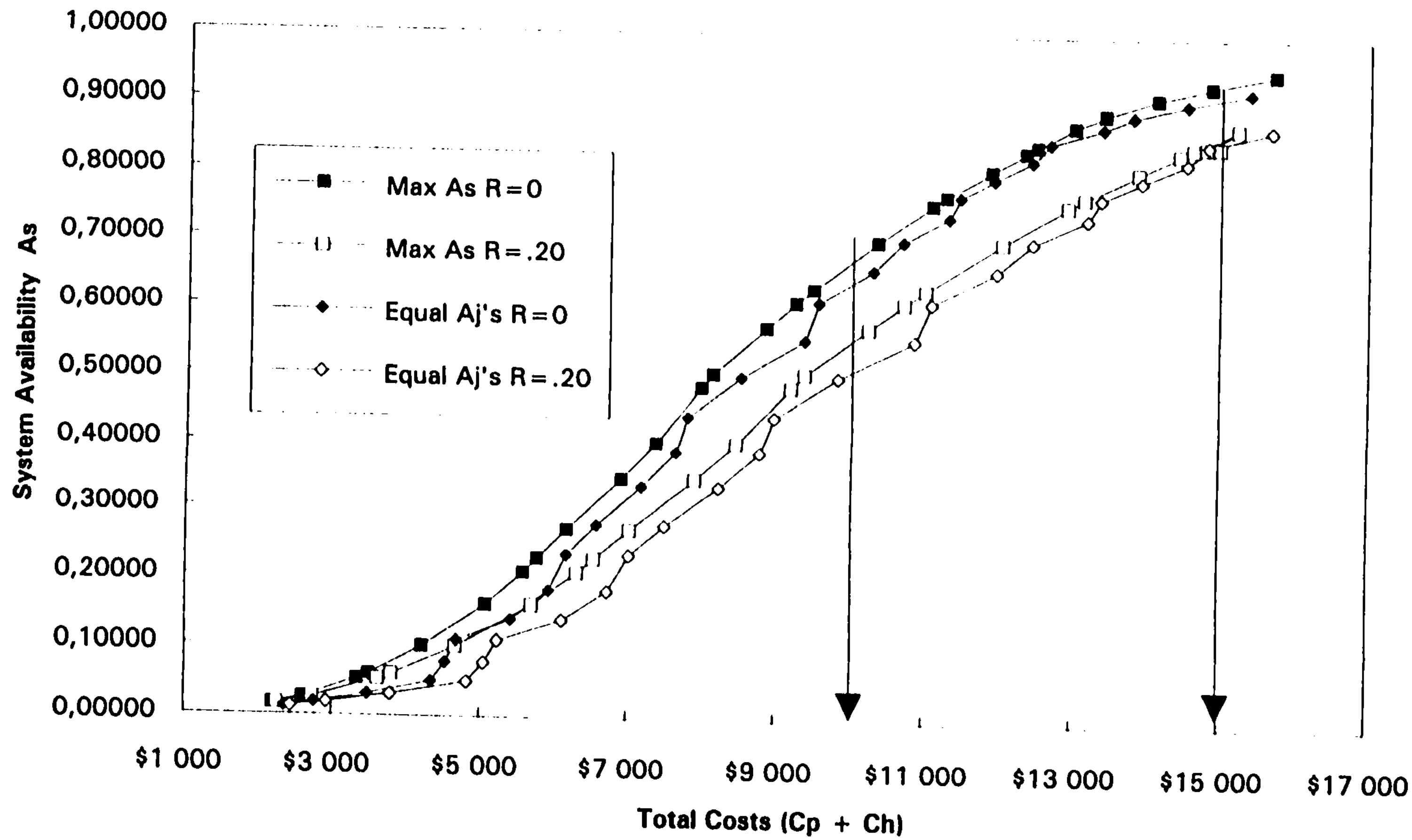


Figure 7.2: Comparison P1c vs Equal Aj's, R=0 vs 20% (J=10 items)

CHAPTER 8: NUMERICAL EXPERIMENTS

8.1 TEST PROBLEMS

This chapter deals with larger scale problems by applying the various solution methods developed in earlier chapters, in order to confirm or validate some of the more important theoretical and practical results concerning the performance measures of the operating system. To this end, we created a series of sixteen (16) randomly generated test problems most likely to be found in practice in the military as well as other organizations that wish to implement an aggregate system performance measure such as min A_S and/or min BO, covering an appropriate range of values for $J = 10,20,50$ and 99 items and $M = 1,5,10$ and 20 number of equipments.

For each problem, a trial run was necessary to find a suitable available budget value that ensured a high system availability or $A_S >$ .90 for both model P1 to Maximize A_S and for model P2 to Minimize BO applied to the same data, and mostly concentrating on the marginal analysis procedure, including the topup procedure and simulation techniques of chapter 6 for the measure of performance AA_S (with part failure dependencies incorporated). In addition, each of the 16 problems was solved at a lower value of $A_S \approx 0.50$ to 0.60 and for those problems with $M = 10$ or 20 equipments, also solved for very low values of $A_S \approx 0.20$ to 0.30 to obtain more useful results for the corresponding AA_S values, as well as enabling us to better interpret the averages. Thus, a total of forty (40) test problems were used to solve models P1, P2 and Equal Aj's and simulated AA_S values for P1 vs Equal Aj's model were also compared.

For each (J x M) problem, the cost c_j and failure rate τ_j (or conversely its $MTBF_j = 1/\tau_j$) for $j=1,..,J$ within each problem were randomly generated as follows: first, MTBFj's (measured in kms) were obtained from a truncated exponential distribution with mean 15000 kms with an acceptance region of between 5000 and 50000 kms, otherwise, it was rejected. This method ensured a proper range of MTBFs likely to be encountered in practice for vehicle systems, not only in military applications but in other organizations as well; the secondary realistic effect of using the exponential is to generate most items in the lower range of possible extremes.

Second, the cost parameters for each item $\{c_j, j=1,..,J\}$ was generated from a truncated exponential distribution with mean $250 with an acceptance criterion of between $50 and $5000; for each c_j, a procedure to incorporate a correlation factor was implemented to vary c_j in relation to its MTBF; this ensured that most items with low MTBFs also had low costs, and items with high MTBFs either had high or low costs, an example of which is illustrated in figure 8.1 below for 1 of the 40 problems (J=99, M=10). The average correlation factor was approximately 0.6 across all problems.

For example, if an item j generated parameter $MTBF_j$ = 36,000 kms and the annual usage rate U for an equipment (M=1) was expected to be 12,000 kms, then M = 10 equipments each operating on average of 12,000 kms/yr would yield an estimated mean number of failures = M x U x $(1/MTBF_j)$ = 10 x 12,000 x 1/36,000 kms = 3.33 failures (POISSON distributed) as seen earlier in chapter 2. For all the problems presented (40 test problems) here, the annual usage was assumed to be 12,000 kms and all Poisson means converted to an equivalent of 15 to 30 days, reflecting the military requirement of carrying enough spares without resupply for that period.

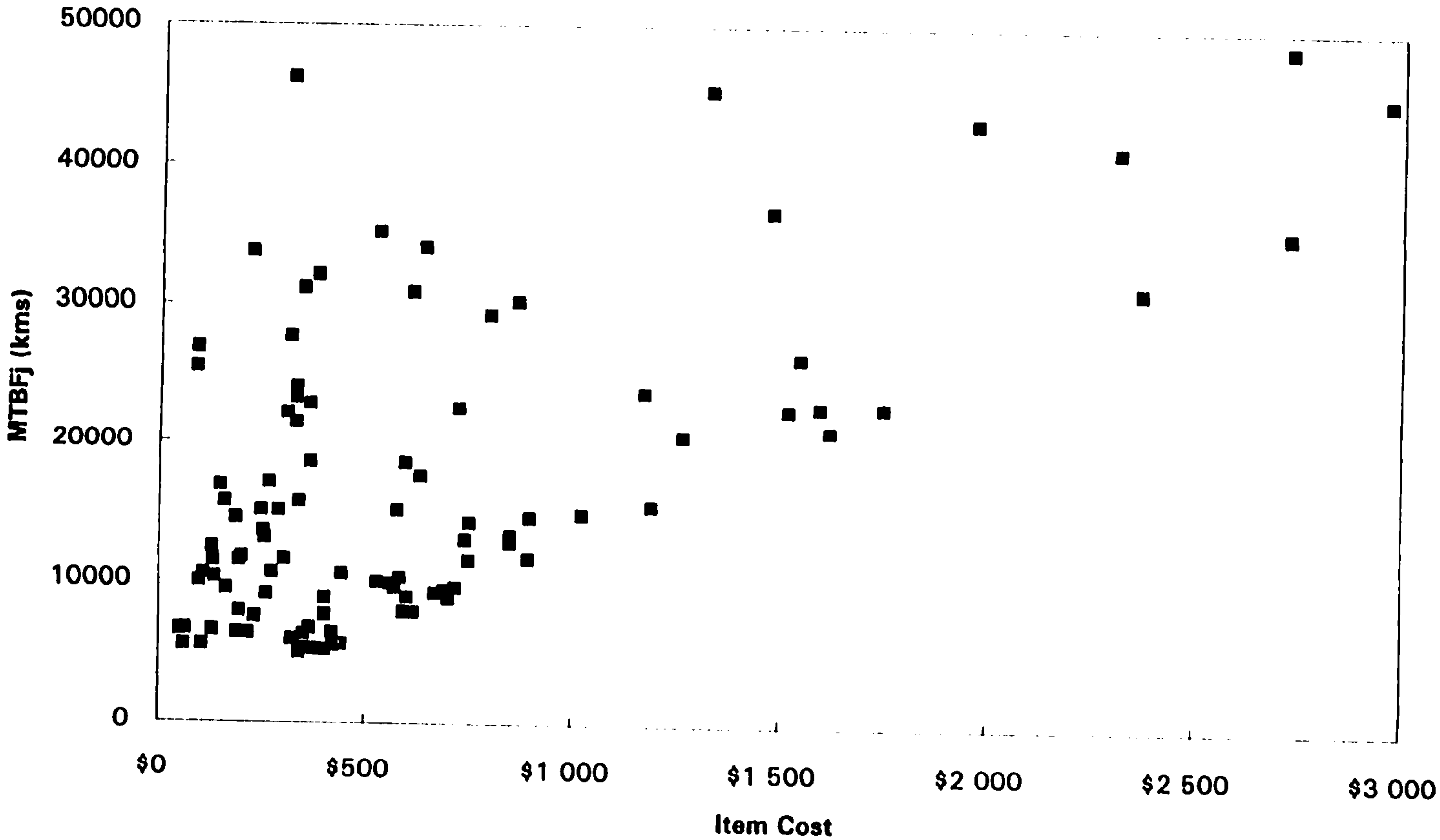


Figure 8.1: MTBF vs Item Costs (J = 99 x M = 10)

All the results are presented into separate tables as described below and start with table 8.1 giving the general parameters data for each of the 40 problems, such as J, M and B, the minimum and maximum c_j, from which the error on the total costs C_S from B can be readily calculated before the marginal analysis procedure is applied; tables 8.2a presents comparative results obtained from marginal analysis vs topup when solving P1 to maximize A_S and table 8.2b compares model P1 with the Equal Aj's model (military model) against simulated AA_S values. Table 8.3a and 8.3b repeats the same comparative data when solving P2 to minimize BO.

Because of the amount of data involved, table 8.1 general data was not repeated and incorporated into other tables, but can easily be referred to when reading tables 8.2a, 8.2b, 8.3a or 8.3b as they simply extend table 8.1 into more columns.

<u>table 8.1 :</u> Random test problem data varying J, M and B
<u>table 8.2a</u>: Marginal analysis models P1 vs Topup
<u>table 8.2b</u>: Simulated AAs vs P1 and Equal Aj's
<u>table 8.3a</u>: Marginal analysis models P2 vs Topup
<u>table 8.3b</u>: Simulated AAs vs P2 and Equal Aj's

Relevant averages are shown at the bottom of each appropriate column for convenience; since error size decreases as problems become more complex, careful interpretation of each appropriate column average must be made. Their individual values may not be as useful or as indicative as the increase (or decrease) between 2 column averages.

From table 8.1 below varying J, M and B, we note the filename used to describe the J x M data, annotated with an appropriate symbol L = Low A_S or V = very low A_S; so the file 1010L referred to in earlier chapters as example 2 indicate J=10 items x M=10 equipments results in a L = low A_S value with a budget B = $10,000. The number of items J and the number of equipments M become progressively higher.

Minimum and Maximum cost (least and most expensive of J=10 items for 1010L) items are $152 and $860 respectively in columns (5) and (7), giving us the margin of error or a lower bound on C_S from B (as a proportion of B) in columns (6) and (8) when topup and marginal analysis are applied to solve P1 or P2, as derived in chapter 4.

For file 1010, 1010L or 1010V, the total costs C_S of the solution obtained from marginal analysis will be within \$860/\$10,000 or -.086 of B or -8.6%, meaning C_S will be > (\$10,000-\$860) or at least \$9140, while the topup procedure will result in C_S > (\$10,000-\$152) or at least \$9848. The average LB on C_S for all 40 problems indicated at the bottom of each column (5) and (7) indicate a possible improvement of C_S from an average of -12.7% down to less than 0.7% by applying the topup procedure following the regular marginal analysis procedure.

Finally, the last column (9) in table 8.1 indicates the maximum number of iterations as a result of implementing the marginal analysis procedure; all 40 problems specified a minimum A_j = 0.00001 and maximum A_j = 0.999999 for each of the J items. As J (and M) increases, so does the possible maximum number of iterations (average of 356 iterations) from applying the regular marginal analysis; this can be considered the most efficient and convenient of all methods analyzed in earlier chapter to give us the response curve {A_S vs C_S} for P1 and/or {BO vs C_S} for P2.

Table 8.1: Random test problems varying J,M,B

(1) filename	(2) No. Items J	(3) No. Eqpts M	(4) Budget B	(5) Min Cj	(6) Prop vs B	(7) Max Cj	(8) Prop vs B	(9) Max Iter
1001	10	1	$3000	$66	-0,022	$811	-0,270	44
1001L	10	1	$1000	$66	-0,066	$811	-0,811	44
1005	10	5	$12000	$54	-0,005	$1673	-0,139	71
1005L	10	5	$6000	$54	-0,009	$1673	-0,279	71
1010	10	10	$15000	$152	-0,010	$860	-0,057	87
1010L	10	10	$10000	$152	-0,015	$860	-0,086	87
1010V	10	10	$6000	$152	-0,025	$860	-0,143	87
1020	10	20	$25000	$237	-0,009	$1368	-0,055	98
1020L	10	20	$15000	$237	-0,016	$1368	-0,091	98
1020V	10	20	$11000	$237	-0,022	$1368	-0,124	98
2001	20	1	$12000	$54	-0,005	$2469	-0,206	91
2001L	20	1	$2000	$54	-0,027	$2469	-1,235	91
2005	20	5	$25000	$51	-0,002	$2946	-0,118	130
2005L	20	5	$19000	$51	-0,003	$2946	-0,155	130
2010	20	10	$25000	$75	-0,003	$1357	-0,054	186
2010L	20	10	$18000	$75	-0,004	$1357	-0,075	186
2010V	20	10	$12000	$75	-0,006	$1357	-0,113	186
2020	20	20	$70000	$120	-0,002	$3053	-0,044	227
2020L	20	20	$50000	$120	-0,002	$3053	-0,061	227
2020V	20	20	$40000	$120	-0,003	$3053	-0,076	227
5001	50	1	$35000	$54	-0,002	$3397	-0,097	226
5001L	50	1	$16000	$54	-0,003	$3397	-0,212	226
5005	50	5	$90000	$86	-0,001	$3253	-0,036	326
5005L	50	5	$60000	$86	-0,001	$3253	-0,054	326
5010	50	10	$110000	$73	-0,001	$3040	-0,028	432
5010L	50	10	$80000	$73	-0,001	$3040	-0,038	432
5010V	50	10	$65000	$73	-0,001	$3040	-0,047	432
5020	50	20	$150000	$51	0,000	$2450	-0,016	593
5020L	50	20	$120000	$51	0,000	$2450	-0,020	593
5020V	50	20	$100000	$51	-0,001	$2450	-0,025	593
9901	99	1	$80000	$52	-0,001	$3418	-0,043	448
9901L	99	1	$50000	$52	-0,001	$3418	-0,068	448
9905	99	5	$180000	$58	0,000	$2963	-0,016	664
9905L	99	5	$120000	$58	0,000	$2963	-0,025	664
9910	99	10	$220000	$51	0,000	$2968	-0,013	863
9910L	99	10	$170000	$51	0,000	$2968	-0,017	863
9910V	99	10	$140000	$51	0,000	$2968	-0,021	863
9920	99	20	$330000	$53	0,000	$3243	-0,010	923
9920L	99	20	$250000	$53	0,000	$3243	-0,013	923
9920V	99	20	$200000	$53	0,000	$3243	-0,016	923
Total	40	10,2	Averages	$85	-0,007	$2422	-0,125	356

8.2 TABLES 8.2a,b FOR MODEL P1

Table 8.2a below mostly represents data related to model P1 to maximize A_S and compares the results of solving P1 by applying the regular marginal analysis and the improvement as a result of applying the topup procedure. The following gives a brief explanation of each column for convenience and easy reference purposes.

Table 8.2a: Marginal analysis for P1 vs Topup

columns (10) to (14): results of marginal analysis for P1

(10) Iter: actual number of iterations k to max A_S (marginal analysis)

(11) LB C_S: total costs of the optimal solution vector $\{S_j, j=1,..,J\}$ (which is a lower bound LB)

(12) Prop vs B: actual difference of C_S vs B as a proportion of B.

(13) LB A_S: lower bound on A_S (last iteration k)

(14) UB A_S: upper bound on A_S (as a result of iteration k+1)

columns (15) to (19): results of topup marginal analysis for P1

(15) Iter: actual number of iterations as a result of topping up the solution vector obtained from column (10).

(16) LB*C_S: total cost of the improved solution vector $\{S_j, j=1,..,J\}$ as a result of topping up (which is the improved lower bound).

(17) Prop vs B: actual difference of C_S vs B as a proportion of B.

(18) LB*A_S: improved LB on A_S as a result of topping up.

(19) Prop vs LB A_S: relative improvement (increase) in A_S as a result of topping up from iteration k.

Table 8.2a: Marginal analysis P1 vs Topup

(10) Marg P1 Iter	(11) LB Cs	(12) Prop vs B	(13) LB As	(14) UB As	(15) Topup P1 Iter	(16) LB* Cs	(17) Prop vs B	(18) LB* As	(19) Prop vs LB_As
8	$2845	-0,052	0,89181	0,94264	14	$2977	-0,008	0,89367	0,002
4	$838	-0,162	0,67685	0,72017	9	$970	-0,030	0,67827	0,002
25	$11343	-0,055	0,92281	0,94756	39	$11994	-0,001	0,92961	0,007
16	$5105	-0,149	0,52364	0,65979	26	$5946	-0,009	0,55067	0,052
32	$14851	-0,010	0,93010	0,94852	35	$14851	-0,010	0,93010	0,000
22	$9484	-0,052	0,62961	0,69948	33	$9974	-0,003	0,64895	0,031
14	$5765	-0,039	0,22800	0,27056	19	$5917	-0,014	0,23748	0,042
40	$23767	-0,049	0,92737	0,94959	47	$24972	-0,001	0,94076	0,014
25	$14184	-0,054	0,52146	0,57969	32	$14963	-0,002	0,55510	0,065
19	$10387	-0,056	0,25121	0,33174	23	$10769	-0,021	0,27648	0,101
23	$11711	-0,024	0,89552	0,92059	30	$11974	-0,002	0,89800	0,003
9	$1809	-0,096	0,53947	0,57832	17	$1983	-0,009	0,54807	0,016
41	$24984	-0,001	0,86442	0,88874	46	$24984	-0,001	0,86442	0,000
36	$16556	-0,129	0,64610	0,72041	47	$18955	-0,002	0,68910	0,067
77	$24463	-0,021	0,88530	0,90789	82	$24949	-0,002	0,89897	0,015
60	$17772	-0,013	0,62010	0,64318	69	$17950	-0,003	0,62910	0,015
45	$11740	-0,022	0,18596	0,20829	53	$11927	-0,006	0,19376	0,042
114	$69284	-0,010	0,92072	0,93620	122	$69919	-0,001	0,92425	0,004
88	$48558	-0,029	0,58238	0,64041	96	$49909	-0,002	0,60765	0,043
73	$39389	-0,015	0,31994	0,35672	79	$39880	-0,003	0,33357	0,043
66	$33287	-0,049	0,89402	0,92263	79	$34996	0,000	0,90928	0,017
46	$15037	-0,060	0,56661	0,59234	56	$15982	-0,001	0,58166	0,027
131	$89160	-0,009	0,91002	0,91860	136	$89982	0,000	0,91350	0,004
93	$59377	-0,010	0,59347	0,60410	102	$59938	-0,001	0,60095	0,013
199	$109875	-0,001	0,90742	0,91513	204	$109998	0,000	0,90800	0,001
157	$79773	-0,003	0,57704	0,60335	165	$79954	-0,001	0,57941	0,004
135	$64684	-0,005	0,31872	0,34743	143	$64933	-0,001	0,32205	0,010
320	$149894	-0,001	0,90929	0,92059	328	$149996	0,000	0,90952	0,000
260	$119835	-0,001	0,63770	0,64942	271	$119970	0,000	0,63965	0,003
232	$99618	-0,004	0,30985	0,31916	240	$99983	0,000	0,31471	0,016
147	$79623	-0,005	0,91512	0,91824	154	$79963	0,000	0,91669	0,002
108	$49560	-0,009	0,57208	0,59668	119	$49977	0,000	0,57729	0,009
315	$179426	-0,003	0,93710	0,94361	325	$179972	0,000	0,93832	0,001
228	$118828	-0,010	0,59459	0,61109	238	$119961	0,000	0,60668	0,020
435	$219852	-0,001	0,91027	0,91593	441	$219959	0,000	0,91063	0,000
360	$169740	-0,002	0,61920	0,62180	368	$169996	0,000	0,62106	0,003
301	$139752	-0,002	0,28095	0,29068	306	$139984	0,000	0,28355	0,009
443	$329942	0,000	0,90087	0,90120	449	$329998	0,000	0,90093	0,000
354	$249806	-0,001	0,48901	0,49226	365	$249996	0,000	0,49028	0,003
285	$199588	-0,002	0,11669	0,12475	292	$199981	0,000	0,11894	0,019
135		-0,030	0,64807		142		-0,003	0,65678	0,018

From the data in table 8.2a above, we can easily compare the results of solving model P1 with marginal analysis vs the topup procedure.

First, we note the validity of the lower bound on C_S derived in chapter 4 earlier for the marginal analysis procedure, based on the maximum cost item; we knew that the LB on CS will be $\geq B - \max\{c_j, j=1,..,J\}$ as indicated in table 8.1 earlier, that is based on the most expensive item; on average the theoretical LB on C_S from B was -12.5% from the budget and the actual error was less than -3.0% (column 12) and improved further with the topup procedure to less than 0.3%.

Second, and probably the most important is the average 1.8% (column 19) relative increase in A_S to 65.678% as a result of the topup procedure, a significant improvement in system performance, by simply adding items after the k th iteration with lesser cost than the one that caused the budget to be exceeded at iteration k+1, thus confirming the validity of the procedure.

Third, the very small price to pay for this significant improvement in system performance is an average increase of only 7 iterations (from 135 to 142), eventhough the marginal analysis could easily handle several thousand iterations quickly and efficiently. The program counts one comparison as an iteration eventhough no item may be added, so in fact less than 7 items were further added as a result of the procedure.

Although it does not guarantee the solution to be the true optimal one, the topup procedure shall therefore be the method closest to the true optimal value of the system performance A_S in table 8.2 and BO in table 8.3; therefore, all other methods shall always be compared against that one, denoted LB^*A_S.

The following table 8.2b compares the results of the topup marginal analysis procedure for P1 with the Equal Aj's model and gives the simulated comparative AA_S values obtained as a result of solving both models.

Table 8.2b: Simulated AAs vs P1 and Equal Aj's

(20) Sim AAs P1: simulated AAs value (N=5,000 cycles) obtained from the solution vector $\{S_j\}$ from the topup marginal analysis for P1 to max A_S
(21) Prop vs LB*As: relative (proportional) increase of AAs vs LB*A_S (valid when M=1 only)
(22) Corresp BO: corresponding BO value when solving P1 to max A_S
(23) Estimate AAs = 1 - BO/M: the value of 1-BO/M as an estimate of AA_S when compared to its simulated value
(24) Prop vs Sim AAs: relative (proportional) difference of the estimate 1-BO/M vs AAs

<u>columns (25) to (28)</u>: comparison Equal Aj's vs P1 and AA_S.
(25) Equal Aj's A_S: the A_S value obtained by solving the current military model with equal Aj's
(26) Prop LB*A_S vs Equal: relative (proportional) increase in A_S values of LB*A_S (topup) vs Equal Aj's model
(27) Sim AA_S Equal Aj's: simulated AAs value (N=5,000 cycles) obtained from the solution vector $\{S_j\}$ from the Equal Aj's model.
(28) Prop Sim AA_S* vs Sim AA_S: relative (proportional) difference in simulated AA_S values of the Topup marginal analysis vs Equal Aj's models.

Table 8.2b: AAs for P1 vs Equal Aj's

(20) Sim P1 AAs*	(21) Prop vs LB*As	(22) Corresp BO	(23) Estimate AAs 1-BO/M	(24) Prop vs Sim AAs*	(25) Equal Aj's As	(26) Prop LB*As Equal	(27) Sim Equal AAs	(28) Prop Sim AAs* Sim Aj's
0,89280	-0,001	0,11323	0,88677	-0,007	0,89181	0,002	0,88800	0,005
0,67140	-0,010	0,38877	0,61123	-0,090	0,58198	0,165	0,58320	0,151
0,98516	0,060	0,07820	0,98436	-0,001	0,90612	0,026	0,97944	0,006
0,88800	0,613	0,60601	0,87880	-0,010	0,33685	0,635	0,79532	0,117
0,99200	0,067	0,08472	0,99153	0,000	0,90469	0,028	0,98804	0,004
0,95004	0,464	0,51436	0,94856	-0,002	0,61040	0,063	0,94416	0,006
0,84758	2,569	1,73279	0,82672	-0,025	0,18064	0,315	0,82558	0,027
0,99589	0,059	0,07312	0,99634	0,000	0,93784	0,003	0,99588	0,000
0,96575	0,740	0,72035	0,96398	-0,002	0,51410	0,080	0,95861	0,007
0,92511	2,346	1,58266	0,92087	-0,005	0,26271	0,052	0,92264	0,003
0,89780	0,000	0,10824	0,89176	-0,007	0,85204	0,054	0,85280	0,053
0,56120	0,024	0,60316	0,39684	-0,293	0,37482	0,462	0,38080	0,474
0,97132	0,124	0,15757	0,96849	-0,003	0,79626	0,086	0,94676	0,026
0,92764	0,346	0,38464	0,92307	-0,005	0,51780	0,331	0,85832	0,081
0,98786	0,099	0,12605	0,98740	0,000	0,88211	0,019	0,98364	0,004
0,94418	0,501	0,55658	0,94434	0,000	0,49616	0,268	0,91946	0,027
0,83060	3,287	1,92353	0,80765	-0,028	0,08570	1,261	0,75068	0,106
0,99529	0,077	0,09245	0,99538	0,000	0,87804	0,053	0,99271	0,003
0,97167	0,599	0,60028	0,96999	-0,002	0,48702	0,248	0,95832	0,014
0,93524	1,804	1,38869	0,93057	-0,005	0,26912	0,239	0,92387	0,012
0,90600	-0,004	0,09600	0,90400	-0,002	0,80974	0,123	0,82500	0,098
0,57540	-0,011	0,54405	0,45595	-0,208	0,46156	0,260	0,51060	0,127
0,98132	0,074	0,09600	0,98080	-0,001	0,87398	0,045	0,97296	0,009
0,89884	0,496	0,54792	0,89042	-0,009	0,51384	0,170	0,87144	0,031
0,98918	0,089	0,10830	0,98917	0,000	0,87495	0,038	0,98660	0,003
0,94130	0,625	0,61665	0,93834	-0,003	0,40287	0,438	0,90548	0,040
0,88406	1,745	1,30090	0,86991	-0,016	0,20214	0,593	0,84448	0,047
0,99441	0,093	0,11505	0,99425	0,000	0,88471	0,028	0,99287	0,002
0,97276	0,521	0,56864	0,97157	-0,001	0,57269	0,117	0,96692	0,006
0,93309	1,965	1,46546	0,92673	-0,007	0,18471	0,704	0,89885	0,038
0,91840	0,002	0,08889	0,91111	-0,008	0,90360	0,014	0,90440	0,015
0,59060	0,023	0,55506	0,44494	-0,247	0,44960	0,284	0,49680	0,189
0,98640	0,051	0,06770	0,98646	0,000	0,91890	0,021	0,98204	0,004
0,90116	0,485	0,53713	0,89257	-0,010	0,48879	0,241	0,86524	0,042
0,98966	0,087	0,10550	0,98945	0,000	0,88600	0,028	0,98574	0,004
0,94914	0,528	0,54155	0,94585	-0,003	0,46606	0,333	0,91714	0,035
0,86994	2,068	1,46967	0,85303	-0,019	0,20046	0,414	0,83888	0,037
0,99318	0,102	0,12314	0,99384	0,001	0,86454	0,042	0,99180	0,001
0,95974	0,958	0,86901	0,95655	-0,003	0,35996	0,362	0,93930	0,022
0,88509	6,441	2,60862	0,86957	-0,018	0,02924	3,068	0,80817	0,095
0,90641	0,753	0,60652	0,88973	-0,026	0,57786	0,293	0,87382	0,049

The results presented in table 8.2b above compares the A_S values obtained from solving model P1 (topup) and the equal Aj's model (current military model).

First, the most important conclusion to be drawn as indicated in column 26, the A_S values obtained as a result of the topup marginal analysis procedure show a 29.3% average proportional increase over the Equal Aj's model, by comparing A_S values in column 25 and column 18 of the topup procedure in table 8.2a above; the absolute difference in average A_S values is 0.65678 - 0.57786 = 0.07892, or a 13.7% improvement when averaging these absolute values before computing and comparing the averages.

We knew that the equal Aj's model can never outperform model P1 (regular and topup marginal analysis) as expected, but the data also shows that the average difference is seriously affected by low available budget values and for low or very low A_S values, two conditions likely to be encountered in practical problems and not only restricted to the military. Such would be the case when downsizing or restructuring occurs, when a manager decides (either by choice or otherwise) to allocate small budget values for maintenance, or simply because a large number of items J is included in the analysis resulting in lower A_S values when multiplied together; as discussed earlier, the response curve is likely to be wider at such values. Thus, the relative increase in system performance can be significant by implementing a model that can discriminate between cost items such as model P1 and/or P2.

Second, the effect on AA_S (average number/proportion of equipments operational at the end of the cycle period) can also be significant, as indicated in table 8.2b; as a result of solving both models P1 and Equal Aj's, the AA_S value was simulated for each stock level vector $\{S_j\}$ for N=5,000 cycles each, taking into account failure dependencies, and are presented in columns 20 and 27 respectively; column 28 indicates the relative difference between the two models and shows that model P1 outperforms the equal Aj's model by an average of 4.9% across all 40 problems, even though the number of equipments M goes as high as

20 and the differences for such problems tend to be smaller. The average absolute difference in simulated AA_S values across all 40 problems (columns 20 and 27) is 0.90641 - 0.87382 = 0.03259. For organizations having M ≤ 5 equipments, the difference in AA_S can be far more serious, specially for low (or very low) values of A_S, if the Equal Aj's model is used as opposed to model P1.

Third, because of the similarities between models P1 and P2, which will be commented on shortly in tables 8.3a and 8.3b next, we also reported in table 8.2b here the corresponding BO value obtained as a result of solving P1 in column 22 and 1 - BO/M, an estimate of AA_S shown in column 23, as a proportional increase when compared to the AA_S simulated value, in column 24. The reason is as follows: the ratio BO divided by M is the sum of all BO_j, j=1,..,J and when divided by M, constitutes an estimate of the average proportion of equipments NOT operational at the end of the cycle; when subtracted from 1, the expression 1 - BO/M becomes an estimate of the average proportion of equipments operational (or UP) at the end of the cycle. As the data clearly shows, the estimate 1-BO/M is a close approximation of AA_S for most 40 problems except those with M=1 (lower values of M) and underestimates AA_S by an average of less than 2.6%.

Except for problems with M=1, the average estimate would be very close to the simulated AA_S value, even for low A_S values. Since Max A_S is equivalent to Max AA_S when M=1, we should compare the improved LB^*A_S obtained from the topup procedure with the simulated values of AA_S for only those 8 problems where M=1 in columns 20 and 21.

8.3 TABLES 8.3a,b FOR MODEL P2

Table 8.3a below mostly represents data related to model P2 to minimize BO and compares the results of solving P2 by applying the regular marginal analysis and the improvement as a result of applying the topup procedure. The following gives a brief explanation of each column for convenience and easy reference purposes.

Table 8.3a: Marginal analysis vs Topup for P2

<u>columns (10) to (14): results of marginal analysis for P2</u>

(10) Iter = actual number of iterations k to min BO (marginal analysis)

(11) LB C_S = total costs of the optimal solution vector $\{S_j,\ j=1,..,J\}$ (which is a lower bound LB)

(12) Prop vs B = actual difference of C_S vs B as a proportion of B.

(13) UB BO = upper bound on BO (last iteration k)

(14) LB BO = lower bound on BO (as a result of iteration k+1)

<u>columns (15) to (19): results of topup marginal analysis for P2</u>

(15) Iter = actual number of iterations as a result of topping up the solution vector obtained from column (10).

(16) UB*C_S = total cost of the improved solution vector $\{S_j,\ j=1,..,J\}$ as a result of topping up (which is the improved upper bound).

(17) Prop vs B = actual difference of C_S vs B as a proportion of B.

(18) UB*BO = improved UB on BO as a result of topping up.

(19) Prop vs UB BO = relative improvement (decrease) in BO as a result of topping up from iteration k.

Table 8.3a: Marginal analysis P2 vs Topup

(10) Marg P2 Iter	(11) LB Cs	(12) Prop vs B	(13) UB BO	(14) LB BO	(15) Topup P2 Iter	(16) UB* Cs	(17) Prop vs B	(18) UB* BO	(19) Prop vs UB_BO
8	$2845	-0,052	0,11536	0,05995	14	$2977	-0,008	0,11323	-0,018
4	$838	-0,162	0,39090	0,32891	9	$970	-0,030	0,38877	-0,005
26	$11644	-0,030	0,08023	0,05175	39	$11994	-0,001	0,07820	-0,025
16	$5105	-0,149	0,66488	0,58594	26	$5946	-0,009	0,60601	-0,089
32	$14851	-0,010	0,08472	0,06209	34	$14851	-0,010	0,08472	0,000
22	$9484	-0,052	0,55071	0,43045	33	$9974	-0,003	0,51436	-0,066
14	$5657	-0,057	1,81509	1,55665	21	$5995	-0,001	1,74260	-0,040
40	$23767	-0,049	0,09120	0,06362	47	$24972	-0,001	0,07311	-0,198
26	$14489	-0,034	0,75533	0,63441	29	$14871	-0,009	0,70595	-0,065
20	$10769	-0,021	1,58266	1,25144	24	$10769	-0,021	1,58266	0,000
23	$11711	-0,024	0,11106	0,08345	30	$11974	-0,002	0,10824	-0,025
9	$1809	-0,096	0,61965	0,55018	17	$1983	-0,009	0,60316	-0,027
41	$24984	-0,001	0,15757	0,12767	46	$24984	-0,001	0,15757	0,000
36	$16556	-0,129	0,45465	0,34601	46	$18955	-0,002	0,38464	-0,154
78	$24764	-0,009	0,13232	0,10287	86	$24998	0,000	0,12717	-0,039
60	$17772	-0,013	0,57451	0,52690	68	$17950	-0,003	0,55658	-0,031
46	$11274	-0,061	2,10102	1,79176	57	$11984	-0,001	1,90800	-0,092
116	$69919	-0,001	0,09245	0,08496	121	$69919	-0,001	0,09245	0,000
90	$49080	-0,018	0,63057	0,52262	98	$49991	0,000	0,59909	-0,050
73	$37346	-0,066	1,70723	1,31134	79	$39903	-0,002	1,40265	-0,178
66	$33287	-0,049	0,11345	0,08196	79	$34996	0,000	0,09601	-0,154
46	$15037	-0,060	0,57138	0,52546	56	$15982	-0,001	0,54405	-0,048
131	$89160	-0,009	0,10010	0,09029	140	$89952	-0,001	0,09607	-0,040
94	$59690	-0,005	0,55302	0,53417	101	$59938	-0,001	0,54792	-0,009
199	$109875	-0,001	0,10910	0,10697	205	$109998	0,000	0,10830	-0,007
157	$78403	-0,020	0,66232	0,61340	165	$79935	-0,001	0,61673	-0,069
135	$64684	-0,005	1,31417	1,28253	143	$64933	-0,001	1,30089	-0,010
322	$149978	0,000	0,11438	0,10981	325	$149978	0,000	0,11438	0,000
266	$119781	-0,002	0,57022	0,49735	272	$119998	0,000	0,56501	-0,009
235	$99498	-0,005	1,48677	1,43236	243	$99968	0,000	1,45933	-0,018
147	$79623	-0,005	0,09064	0,08715	154	$79963	0,000	0,08889	-0,019
108	$49560	-0,009	0,56438	0,52229	119	$49977	0,000	0,55506	-0,017
316	$179972	0,000	0,06765	0,06046	327	$179972	0,000	0,06765	0,000
228	$119588	-0,003	0,54327	0,53258	236	$119944	0,000	0,53739	-0,011
435	$219852	-0,001	0,10598	0,10481	440	$219959	0,000	0,10552	-0,004
362	$169647	-0,002	0,54625	0,52515	375	$169971	0,000	0,54181	-0,008
304	$139370	-0,005	1,49313	1,45262	318	$139999	0,000	1,46602	-0,018
444	$329530	-0,001	0,12461	0,12231	456	$329986	0,000	0,12325	-0,011
361	$248810	-0,005	0,88825	0,85977	375	$249987	0,000	0,86476	-0,026
292	$199413	-0,003	2,62388	2,54061	303	$199950	0,000	2,59578	-0,011
136		-0,031	0,63388	0,56388	144		-0,003	0,60560	-0,040

The results of solving P2 to minimize BO presented in table 8.3a above are entirely consistent with those presented earlier for model P1; in fact, the next table (table 8.3b) will show that 23 out of the 40 problems yielded the exact same solution vectors $\{S_j\}$ as for P1.

First, because of the similarities between both models P1 and P2, it is hardly surprising that the actual number of iterations were practically the same for both models and for both the regular and the topup marginal analysis procedure. The theoretical error on C_S based on the most expensive item for the regular procedure and based on the least expensive item for the topup procedure were the same as for model P1. The actual relative differences (as a proportion of B) were also practically the same as for model P1, as shown in columns 11 and 12 (regular) and columns 16 and 17 (topup), showing an error from an average of less than 3.1% (regular) to less than .3% (topup), which is quite an improvement when using the topup procedure, eventhough the procedure does not guarantee that the last point on the response curve $\{BO, C_S\}$ is undominated.

Second, we also note a significant average 4.0% further decrease in BO (last column 19) by following up the regular procedure with the topup procedure, which will also help us in obtaining estimates for AA_S with the expression 1 - BO/M and 1 - TBO/M, to be discussed in the next table 8.3b.

The following table 8.3b compares the results of the topup marginal analysis procedure for P2 with the Equal Aj's model and gives the simulated comparative AA_S values obtained as a result of solving both models.

Table 8.3b: Simulated AAs vs P1 and Equal Aj's

(20) Sim AAs P2: simulated AAs value (N=5,000 cycles) obtained from the solution vector $\{S_j\}$ from the topup marginal analysis for P2 to min BO
(21) Prop vs Sim AA_S*: relative difference vs Sim AA_S* value obtained from topup marginal analysis when solving P1 to max A_S
(22) Corresp A_S: corresponding A_S value when solving P2 to min BO
(23) Prop vs LB*As: relative difference of A_S vs LB*A_S
(24) Estimate AAs = 1 - BO/M: the value of 1-BO/M as an estimate of AA_S when compared to the simulated AA_S value when solving P1 to max A_S
(25) Prop vs Sim AA_S*: relative difference of 1-BO/M vs AAs

<u>columns (26) to (29)</u>: comparison heuristic TBO vs simulated AA_S.
(26) TBO : "Better" estimate for actual BO when summing up to S_j+M
(27) Estimate 1-TBO/M: estimate of AA_S with 1 - TBO/M
(28) Prop vs sim AA_S*: relative difference between the estimate and the simulated AA_S* value from topup marginal analysis when solving P1
(29) TBO vs Sim AA_S*: relative difference when 1-TBO/M is compared to the AA_S estimated values from simulation, except when M=1, since Max A_S is equivalent to Max AA_S and is exact for those problems.

Table 8.3b: AAs for P2 vs Equal Aj's

(20) Sim P2 AAs	(21) Prop vs Sim AAs*	(22) Corresp As	(23) Prop vs LB*As	(24) Estimate AAs 1-BO/M	(25) Prop vs Sim AAs*	(26) TBO	(27) Estimate AAs 1-TBO/M	(28) Prop vs Sim AAs*	(29) TBO vs Sim AAs*
same	0,000	0,89367	0,000	0,88677	-0,007	0,05995	0,94005	0,053	
same	0,000	0,67827	0,000	0,61123	-0,090	0,32890	0,67110	0,000	
same	0,000	0,92961	0,000	0,98436	-0,001	0,05818	0,98836	0,003	0,003
same	0,000	0,55067	0,000	0,87880	-0,010	0,43593	0,91281	0,028	0,028
same	0,000	0,93010	0,000	0,99153	0,000	0,06209	0,99379	0,002	0,002
same	0,000	0,64895	0,000	0,94856	-0,002	0,43048	0,95695	0,007	0,007
0,84522	-0,003	0,22057	-0,071	0,82574	-0,023	1,55664	0,84434	-0,001	-0,001
same	0,000	0,94076	0,000	0,99634	0,000	0,06362	0,99682	0,001	0,001
0,96526	-0,001	0,55804	0,005	0,96470	-0,001	0,67641	0,96618	0,001	0,001
same	0,000	0,27648	0,000	0,92087	-0,005	1,38769	0,93062	0,006	0,006
same	0,000	0,89800	0,000	0,89176	-0,007	0,08345	0,91655	0,021	
same	0,000	0,54807	0,000	0,39684	-0,293	0,55018	0,44982	-0,198	
same	0,000	0,86442	0,000	0,96849	-0,003	0,15757	0,96849	-0,003	-0,003
same	0,000	0,68910	0,000	0,92307	-0,005	0,38464	0,92307	-0,005	-0,005
0,98776	0,000	0,89771	-0,001	0,98728	0,000	0,11502	0,98850	0,001	0,001
same	0,000	0,62910	0,000	0,94434	0,000	0,52689	0,94731	0,003	0,003
0,83558	0,006	0,18399	-0,050	0,80920	-0,032	1,82763	0,81724	-0,022	-0,022
same	0,000	0,92425	0,000	0,99538	0,000	0,07875	0,99606	0,001	0,001
0,97149	0,000	0,60582	-0,003	0,97005	-0,001	0,54670	0,97267	0,001	0,001
0,93365	-0,002	0,29855	-0,105	0,92987	-0,004	1,31132	0,93443	0,001	0,001
same	0,000	0,90928	0,000	0,90399	-0,002	0,08196	0,91804	0,013	
same	0,000	0,58166	0,000	0,45595	-0,208	0,52546	0,47454	-0,175	
0,98100	0,000	0,91334	0,000	0,98079	0,000	0,09029	0,98194	0,001	0,001
same	0,000	0,60095	0,000	0,89042	-0,009	0,54347	0,89131	-0,008	-0,008
same	0,000	0,90800	0,000	0,98917	0,000	0,09971	0,99003	0,001	0,001
0,94164	0,000	0,57828	-0,002	0,93833	-0,004	0,57242	0,94276	0,001	0,001
same	0,000	0,32205	0,000	0,86991	-0,016	1,21659	0,87834	-0,006	-0,006
0,99422	0,000	0,90936	0,000	0,99428	0,000	0,10141	0,99493	0,001	0,001
0,97267	0,000	0,63494	-0,007	0,97175	-0,001	0,54996	0,97250	0,000	0,000
0,93267	0,000	0,31058	-0,013	0,92703	-0,006	1,45051	0,92747	-0,006	-0,006
same	0,000	0,91669	0,000	0,91111	-0,008	0,08715	0,91285	-0,006	
same	0,000	0,57729	0,000	0,44494	-0,247	0,52228	0,47772	-0,191	
same	0,000	0,93832	0,000	0,98647	0,000	0,06189	0,98762	0,001	0,001
0,89808	-0,003	0,60663	0,000	0,89252	-0,006	0,53006	0,89399	-0,005	-0,005
same	0,000	0,91063	0,000	0,98945	0,000	0,09915	0,99009	0,000	0,000
0,94822	-0,001	0,61936	-0,003	0,94582	-0,003	0,54032	0,94597	-0,002	-0,002
0,86980	0,000	0,28177	-0,006	0,85340	-0,019	1,44075	0,85593	-0,016	-0,016
0,99349	0,000	0,90066	0,000	0,99384	0,000	0,12273	0,99386	0,000	0,000
0,95837	-0,001	0,48607	-0,009	0,95676	-0,002	0,86237	0,95688	-0,002	-0,002
0,88192	-0,004	0,11710	-0,015	0,87021	-0,013	2,55210	0,87240	-0,011	-0,011
23/40	0,000	0,65473	-0,007	0,88978	-0,026	0,56732	0,89936	-0,013	-0,001

First, and probably the most important conclusion from the results presented in table 8.3b above when solving P2 to min BO is that 23 out of 40 problems yielded the exact same solution vector $\{S_j\}$, even for larger scale problems up to J=99 items and even after the topup marginal analysis procedure is applied, where differences are more likely to occur; for these 23 problems, we used the same simulated AA_S* values obtained for P1.

For problems that gave different solution vectors, the differences were very small when compared to model P1 system performance measures A_S, corresponding BO and simulated AA_S values. For example, the corresponding A_S value obtained from solving P2 to min BO shown in column 22 were an average of less than -.7% (column 23) from the A_S value obtained from solving P1 to max A_S. The most differences occured at lower A_S values where just an item or two difference will have the most impact, as previously discussed, when stock level vectors are not yet past their mean Poisson mean parameter values $\{\delta_j\}$.

Second, a most interesting and important conclusion to be drawn here is that we can use the UB*BO (column 18 in previous table 8.3a) as an accurate and much more reliable estimate of the proportion of equipments still operational at the end of the cycle or AA_S, than using A_S when M>1 and when A_S is high or > 0.90, even when taking into account part failure dependencies!. The estimate 1-BO/M is shown in column 24 and compared with the simulated AA_S value of column 20 and the proportional differences in column 25 shown an average of less than 2.4% difference and most differences occur when M=1. If M≥5, then all estimates 1-BO/M underestimate AA_S by less than 3.2% (highest difference).

The reason is as follows: the ratio BO divided by M is the sum of all BO_j, j=1,..,J and when divided by M, constitutes an estimate of the average proportion of equipments NOT operational at the end of the cycle; when subtracted from 1, the expression 1 - BO/M becomes an estimate of the average proportion of equipments operational (or UP) at the end of the cycle. For example, if we expect to run out of a total of say 2.5 items (of all types, as calculated by the backorders for all items) and we have M=10 equipments at the beginning of the cycle, we would expect to have 2.5/10 or 25% of equipments NOT operational (and conversely AA_S = 75% operational) at the end of the cycle.

Third, columns 26 and 27 indicate the estimate of AA_S with the expression 1-TBO/M when compared to the simulated AA_S value in column 20; the relative differences of less than an average of 1.3% are shown in column 28 and clearly establish the validity of using TBO as a better estimate for calculating UB*BO by summing the terms from S_j+1 to S_j+M instead of up to infinity; we therefore count less BO, as we should, and most differences occur for low values of M, specially when M=1 in column 27 and 28.

We have seen in chapter 6 that the proportion of equipments still operational at the end of the cycle without taking into account part failure dependencies, was essentially a combinatorial type of problem where the probability expressions become far too complex as the values of J and/or M (specially M) increase; taking into account part failure dependencies become even more complex and simulation methodology must be used. Using the expression 1-BO/M or better still 1-TBO/M thus provides us with a valuable estimate of AA_S when M>1. When M=1, we have already shown earlier that model P1 should be used instead since Max A_S is equivalent to Max AA_S and thus gives the exact value of AA_S with or without part failure dependencies.

However, the estimate of 1 - BO/M becomes a progressively worse estimate of AA_S as the system availability A_S decreases (having lower budgets yield lower $\{S_j\}$ and therefore lower A_S), as we shall see in table 8.4 next; the reason is that the estimate can theoretically become negative since we count too many backorders.

As an example, suppose J=3 and M=2; then the probability expression for each of the

$$BO_j = \sum_{x=S_j+1}^{\infty} (x-S_j).p(x)$$

$$= 1.p(x=S_j+1) + 2.p(x=S_j+2) + \ldots$$

and goes to infinity; however, it is impossible to run out of more than 2 spares of any type (or any combination) since we have only M=2 equipments; the first time we run out of any type of spare, the first equipment remains DOWN (failed state) and once we run out of a second spare, the second equipment remains DOWN until the end of the cycle.

Therefore, all the probability expressions involving more than M=2 spares cannot occur; the terms $3.p(x=S_j+3)$ + ... cannot be counted in the BO functions, j=1,..,J when estimating AA_S. Thus, using the estimate 1 - BO/M will badly underestimate the true value of AA_S since it counts too many BO; it is an acceptable estimate for AA_S for high values of A_S since high stock levels $\{S_j, j=1,..,J\}$ will yield very small probabilities when calculating BO_j's beyond the value S_j + M in the summation terms.

In order to estimate all the probability expressions expanded for AA_S earlier in chapter 6 which dealt with end of cycle effects, we can use the following estimate 1 - TBO/M for AA_S (instead of 1 - BO/M), where TBO = "truer" measure of system BO which will count BO_j only up to S_j + M, instead of up to infinity, and will be further analyzed in table 8.4 for problems with small values of M.

In summary, when $A_S > 0.90$ (HIGH), AA_S can be estimated (exactly) by A_S when M = 1 and estimated by 1 - BO/M when M > 1; in fact, for lower values of A_S or < 0.90 and M = 1, A_S should be used to estimate AA_S with or without failure part dependencies since $A_S = AA_S$ when M = 1. When M > 1, however, then 1 - BO/M can only be relied upon to accurately estimate AA_S when $A_S > 0.90$ or 1 - TBO/M for progressively lower A_S values as M increases; for lower values of A_S or < 0.70, it becomes progressively unacceptable and significantly underestimates the true value of AA_S, and simulation should be used for these cases. This constitutes the most important conclusions for the AA_S measure of system performance.

8.4 TABLE 8.4 FOR AAs

8.4.1 Introduction. Table 8.4 below (7 pages) deals mostly with a selection of some randomly generated test problems to analyze the performance measure AA_S. Since the estimate 1 - BO/M performed very well for high system availability A_S (A_S > .90) and became progressively worse for lower A_S values, the budget B was purposely lowered for each test problem in decremental steps to yield lower optimal stockage levels $\{S_j\}$ and therefore lower A_S values. We then applied the marginal analysis with the top-up procedure by minimizing BO as opposed to max A_S since AA_S becomes more a function of system backorders as defined earlier, rather than using A_S which yields very poor estimates of AA_S when M > 1.

We shall present the results of using 1 - TBO/M to estimate AA_S taking into account part failure dependencies and compare its accuracy with simulation results. The following briefly explains the different column headings used in table 8.4; only page 1 is needed as subsequent pages repeat the same data but is extended to include additional problems. The first 3 problems j3_1, j3_2 and j3_3 summarize earlier results for example 1 (J=3 items) and example 3 (J=4 items) referred to in chapter 6 of the thesis.

Table 8.4: Estmate 1 - TBO/M vs AAs

columns (1) to (4): general parameters

(1) File = filename used (J x M).

(2) b1, b2,.. = budget decreased to yield lower optimal $\{S_j\}$ as a result of min BO using the marginal analysis top-up procedure.

(3) B = available budget.

(4) A_S = corresponding system availability as a result of min BO for each file for the specified budget.

<u>columns (5) to (7): related to BO</u>

(5) BO = system backorders obtained as a result of the top-up marginal analysis.

(6) 1-BO/M = estimate of AA_S

(7) %diff vs sim = % difference between 1-BO/M vs simulated AA_S value shown in column (11).

<u>columns (8) to (10): related to TBO</u>

(8) TBO up to M = "Truer" method of calculating BO, where TBO only sums backorder terms from S_j+1 up to M instead of up to infinity.

(9) 1-TBO/M = more accurate estimate of AA_S taking account part failure dependencies.

(10) %diff 1-TBO/M vs sim = % difference between 1-TBO/M vs simulated AA_S value shown in column (11).

<u>columns (11) to (14): related to simulated AA_S value</u>

(11) AA_S = simulated AA_S value or the proportion of equipments still operational at the end of the cycle, obtained from the program S_INVL.BAS which takes into account part failure dependencies. All problems were simulated N=5,000 cycles for $M \leq 5$ and N=1,000 cycles for $M > 5$.

(12) Sim again 10,000 = a second simulation with N=10,000 cycles to confirm only those initial simulation results close to ± 1%.

(13) %diff = recalculated % differences between the estimate 1-TBO/M and the second simulated AA_S value with N=10,000 cycles if applicable.

(14) A_S = category used for A_S, either High when $A_S > .90$, Medium when $.60 < A_S < .90$ or Low when $A_S < .60$).

<u>Note:</u> All values of A_S lower than .60 in column (4) and % differences in columns (7), (10) and (13) are boldfaced to highlight unsatisfactory results, i.e the % differences that do not satisfy the criterion of ± 1% between the estimates used for AA_S and its simulated value.

Table 8.4: Estimates As and 1 - TBO/M vs sim AAs

(1) Simul AAs File	(2)	(3) B	(4) As	(5) BO	(6) 1-bo/m	(7) %diff 1-bo/m vs sim	(8) TBO up to M	(9) 1-tbo/m	(10) %diff 1-tbo/m vs sim	(11) AAs sim 1;5000	(12) Sim again 10000	(13) %diff High = Med = Low =	(14) As H>.9 M>.6 L<.6
Ex 1	b1	50	0,99081	0,01132	0,98868	-0,29%	na	na	na	0,99160			H
j3_1	b2	45	0,97557	0,03031	0,96969	-0,37%	na	na	na	0,97330			H
	b3	40	0,95842	0,05342	0,94658	-0,99%	na	na	na	0,95600			H
	b4	35	0,89852	0,13372	0,86628	-3,84%	na	na	na	0,90090			M
	b5	30	0,84509	0,21593	0,78407	-6,79%	na	na	na	0,84120			M
	b6	25	0,67608	0,48017	0,51983	-22,81%	na	na	na	0,67340	no dep	no dep	M
	b7	20	0,56378	0,72398	0,27602	-51,00%	na	na	na	0,56330	sim	prob	L
Ex 1	b1	50	0,99081	0,01132	0,99434	-0,10%	0,01037	0,99482	-0,05%	0,99535	0,99535	0,99445	H
j3_2	b2	45	0,97557	0,03031	0,98485	-0,17%	0,02774	0,98613	-0,04%	0,98655	0,98535	0,98533	H
	b3	40	0,95842	0,05342	0,97329	-0,26%	0,04780	0,97610	0,03%	0,97585	0,97575	0,97446	H
	b4	35	0,89852	0,13372	0,93314	-0,71%	0,11831	0,94085	0,11%	0,93980	0,93700	0,93649	M
	b5	30	0,84509	0,21593	0,89204	-1,60%	0,18618	0,90691	0,05%	0,90650	0,89825	0,89972	M
	b6	25	0,67608	0,48017	0,75992	-5,42%	0,40078	0,79961	-0,48%	0,80350	0,78720	0,78103	M
	b7	20	0,56378	0,72398	0,63801	-11,60%	0,58636	0,70682	-2,07%	0,72175	0,68370	0,68742	L
Ex 1	b1	50	0,99081	0,01132	0,99623	-0,08%	0,01113	0,99629	-0,07%	0,99700	0,99680	0,99624	H
j3_3	b2	45	0,97557	0,03031	0,98990	-0,05%	0,02982	0,99006	-0,03%	0,99037	0,98953	0,98993	H
	b3	40	0,95842	0,05342	0,98219	-0,16%	0,05217	0,98261	-0,12%	0,98377	0,98073	0,98234	H
	b4	35	0,89852	0,13372	0,95543	-0,44%	0,13034	0,95655	-0,32%	0,95967	0,95737	0,95580	M
	b5	30	0,84509	0,21593	0,92802	-0,94%	0,20854	0,93049	-0,67%	0,93680	0,92863		M
	b6	25	0,67608	0,48017	0,83994	-2,22%	0,45993	0,84669	-1,44%	0,85903	0,83993		M
	b7	20	0,56378	0,72398	0,75867	-4,78%	0,68502	0,77166	-3,15%	0,79673	0,77250		L
Ex 2	b1	50	0,73763	0,42637	0,57363	-21,61%	na	na	na	0,73180			M
j4_1	b2	48	0,61933	0,61158	0,38842	-37,15%	na	na	na	0,61800			M
	b3	46	0,60618	0,64509	0,35491	-41,89%	na	na	na	0,61080			M
	b4	44	0,59011	0,69062	0,30938	-47,49%	na	na	na	0,58920			L
	b5	42	0,55933	0,77453	0,22547	-60,36%	na	na	na	0,56880			L

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(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
j10_2a	b1	20000	0,98050	0,02690	0,98655	-0,30%	0,02231	0,98885	-0,07%	0,98950		-0,07%	H
Min BO	b2	19000	0,96609	0,04898	0,97551	-0,41%	0,03890	0,98055	0,11%	0,97950		0,11%	H
top-up	b3	18000	0,94884	0,07482	0,96259	-0,30%	0,05920	0,97040	0,51%	0,96550		0,51%	H
	b4	17000	0,91833	0,12057	0,93972	-0,56%	0,09589	0,95206	0,75%	0,94500		0,75%	H
	b5	16000	0,87774	0,18967	0,90517	-1,24%	0,14568	0,92716	1,16%	0,91650	0,92430	0,31%	M
	b6	15000	0,80759	0,31435	0,84283	-2,73%	0,23630	0,88185	1,77%	0,86650	0,87345	0,96%	M
j10_2b	b1	15000	0,88792	0,17142	0,91429	-2,20%	0,13234	0,93383	-0,11%	0,93490		-0,11%	M
	b2	14000	0,82846	0,28636	0,85682	-3,91%	0,20458	0,89771	0,67%	0,89170		0,67%	M
	b3	13000	0,74945	0,43003	0,78499	-7,16%	0,31259	0,84371	-0,21%	0,84550		-0,21%	M
	b4	12000	0,64846	0,66199	0,66901	-12,43%	0,45537	0,77232	1,09%	0,76400	0,76670	0,73%	M
	b5	11000	0,51165	1,00859	0,49571	-25,20%	0,69150	0,65425	-1,28%	0,66270	0,66580	-1,73%	L
	b6	10000	0,38275	1,48148	0,25926	-52,88%	0,94195	0,52903	-3,85%	0,55020		-3,85%	L
j10_2c	b1	15000	0,89848	0,14114	0,92943	-1,19%	0,11987	0,94007	-0,06%	0,94060		-0,06%	M
	b2	14000	0,86275	0,20316	0,89842	-2,24%	0,16425	0,91788	-0,12%	0,91900		-0,12%	M
	b3	13000	0,76349	0,36238	0,81881	-2,86%	0,29277	0,85362	1,27%	0,84290	0,85625	-0,31%	M
	b4	12000	0,67936	0,51630	0,74185	-6,72%	0,41884	0,79058	-0,59%	0,79530		-0,59%	M
	b5	11000	0,58598	0,73608	0,63196	-13,00%	0,57869	0,71066	-2,17%	0,72640		-2,17%	L
	b6	10000	0,44683	1,20723	0,39639	-35,17%	0,83082	0,58459	-4,39%	0,61140		-4,39%	L
j10_2d	b1	20000	0,88751	0,18282	0,90859	-2,09%	0,13197	0,93402	0,65%	0,92800		0,65%	M
	b2	19000	0,83492	0,27975	0,86013	-3,42%	0,19747	0,90127	1,20%	0,89060	0,89835	0,32%	M
	b3	18000	0,76373	0,41989	0,79006	-5,70%	0,29319	0,85341	1,86%	0,83780	0,85170	0,20%	M
	b4	17000	0,68421	0,61549	0,69226	-12,33%	0,40387	0,79807	1,07%	0,78960	0,78860	1,20%	M
	b5	16000	0,56476	0,90761	0,54620	-22,44%	0,60607	0,69697	-1,03%	0,70420	0,69490	0,30%	L
	b6	15000	0,45314	1,29720	0,35140	-41,97%	0,80920	0,59540	-1,68%	0,60560	0,60525	-1,63%	L
j10_2e	b1	14000	0,90167	0,13663	0,93169	-1,14%	0,11405	0,94298	0,06%	0,94240		0,06%	H
	b2	13000	0,84985	0,22519	0,88741	-2,78%	0,17956	0,91022	-0,28%	0,91280		-0,28%	M
	b3	12000	0,78673	0,34733	0,82634	-4,65%	0,26314	0,86843	0,21%	0,86660		0,21%	M
	b4	11000	0,65617	0,60495	0,69753	-10,33%	0,45358	0,77321	-0,60%	0,77790		-0,60%	M
	b5	10000	0,52643	0,89211	0,55395	-18,51%	0,67832	0,66084	-2,79%	0,67980		-2,79%	L

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(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
j20_2a	b1	40000	0,95768	0,06041	0,96980	-0,74%	0,04858	0,97571	-0,13%	0,97700		-0,13%	H
	b2	39000	0,94560	0,07894	0,96053	-0,51%	0,06274	0,96863	0,32%	0,96550		0,32%	H
	b3	38000	0,92903	0,10302	0,94849	-0,84%	0,08272	0,95864	0,22%	0,95650		0,22%	H
	b4	37000	0,91052	0,13331	0,93335	-0,97%	0,10505	0,94748	0,53%	0,94250		0,53%	H
	b5	36000	0,88599	0,17251	0,91375	-1,46%	0,13552	0,93224	0,53%	0,92730		0,53%	M
	b6	35000	0,85840	0,21750	0,89125	-2,23%	0,17083	0,91459	0,33%	0,91160		0,33%	M
	b7	33000	0,78297	0,35561	0,82220	-5,06%	0,27167	0,86417	-0,21%	0,86600		-0,21%	M
	b8	30000	0,62611	0,70221	0,64890	-13,60%	0,51359	0,74321	-1,04%	0,75105	0,75430	-1,47%	M
j20_2b	b1	60000	0,88024	0,18690	0,90655	-1,64%	0,14064	0,92968	0,87%	0,92170		0,87%	M
	b2	58000	0,84560	0,24891	0,87555	-2,76%	0,18495	0,90753	0,79%	0,90040		0,79%	M
	b3	56000	0,79565	0,34233	0,82884	-5,06%	0,24848	0,87576	0,32%	0,87300		0,32%	M
	b4	54000	0,73470	0,44327	0,77837	-6,53%	0,33684	0,83158	-0,13%	0,83270		-0,13%	M
	b5	52000	0,67957	0,55801	0,72100	-10,13%	0,42257	0,78872	-1,69%	0,80230	0,80100	-1,53%	M
	b6	50000	0,62511	0,71192	0,64404	-13,69%	0,50443	0,74779	0,21%	0,74620		0,21%	M
j20_2c	b1	30000	0,93576	0,08893	0,95554	-0,81%	0,07430	0,96285	-0,05%	0,96330		-0,05%	H
	b2	28000	0,89178	0,15791	0,92105	-1,51%	0,12771	0,93615	0,10%	0,93520		0,10%	M
	b3	26000	0,80712	0,29749	0,85126	-2,88%	0,23686	0,88157	0,58%	0,87650		0,58%	M
	b4	24000	0,70845	0,48054	0,75973	-7,28%	0,38129	0,80936	-1,23%	0,81940	0,81710	-0,95%	M
	b5	22000	0,56019	0,82330	0,58835	-17,64%	0,63330	0,68335	-4,35%	0,71440	0,71145	-3,95%	L
	b6	20000	0,36571	1,38967	0,30517	-45,11%	1,07440	0,46280	-16,76%	0,55600	0,54685	-15,37%	L
j50_2a	b1	170000	0,97397	0,03604	0,98198	0,05%	0,02967	0,98517	0,37%	0,98150		0,37%	H
	b2	165000	0,96313	0,05154	0,97423	-0,39%	0,04228	0,97886	0,09%	0,97800		0,09%	H
	b3	160000	0,94719	0,07617	0,96192	-0,73%	0,06094	0,96953	0,05%	0,96900		0,05%	H
	b4	155000	0,92612	0,10838	0,94581	-0,34%	0,08615	0,95693	0,84%	0,94900		0,84%	H
	b5	150000	0,89749	0,15307	0,92347	-2,07%	0,12152	0,93924	-0,40%	0,94300		-0,40%	M
	b6	140000	0,81458	0,29706	0,85147	-2,07%	0,22888	0,88556	1,85%	0,86950	0,88510	0,05%	M
	b7	130000	0,67510	0,57471	0,71265	-9,67%	0,43572	0,78214	-0,86%	0,78890		-0,86%	M
	b8	120000	0,49186	1,06398	0,46801	-26,96%	0,77834	0,61083	-4,68%	0,64080		-4,68%	L

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(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
j10_3a	b1	30000	0,95700	0,07045	0,97652	-0,27%	0,06021	0,97993	0,07%	0,97920		0,07%	H
	b2	28000	0,91635	0,14152	0,95283	-0,97%	0,11993	0,96002	-0,23%	0,96220		-0,23%	H
	b3	26000	0,85073	0,26902	0,91033	-2,19%	0,22119	0,92627	-0,47%	0,93067		-0,47%	M
	b4	24000	0,74380	0,49124	0,83625	-3,84%	0,40199	0,86600	-0,42%	0,86967		-0,42%	M
	b5	22000	0,59843	0,88060	0,70647	-9,66%	0,69208	0,76931	-1,62%	0,78200		-1,62%	L
	b6	20000	0,42567	1,53396	0,48868	-26,62%	1,11867	0,62711	-5,84%	0,66600		-5,84%	L
j10_3b	b1	15000	0,84727	0,26562	0,91146	-1,85%	0,22266	0,92578	-0,30%	0,92860		-0,30%	M
	b2	14500	0,80026	0,35464	0,88179	-2,81%	0,29928	0,90024	-0,77%	0,90727		-0,77%	M
	b3	14000	0,74269	0,46488	0,84504	-3,62%	0,39327	0,86891	-0,90%	0,87680		-0,90%	M
	b4	13500	0,68709	0,61385	0,79538	-5,65%	0,50061	0,83313	-1,17%	0,84300	0,84183	-1,03%	M
	b5	13000	0,61256	0,79573	0,73476	-9,22%	0,65014	0,78329	-3,23%	0,80940		-3,23%	M
j10_3c	b1	22000	0,77365	0,38707	0,87098	-2,50%	0,34102	0,88633	-0,78%	0,89327		-0,78%	M
	b2	21500	0,74983	0,43708	0,85431	-3,60%	0,38386	0,87205	-1,60%	0,88620	0,87923	-0,82%	M
	b3	21000	0,72156	0,49192	0,83603	-3,96%	0,43318	0,85561	-1,71%	0,87047		-1,71%	M
	b4	20500	0,67193	0,62360	0,79213	-4,98%	0,52290	0,82570	-0,96%	0,83367		-0,96%	M
	b5	20000	0,64000	0,68515	0,77162	-5,66%	0,58291	0,80570	-1,49%	0,81787	0,82060	-1,82%	M
j10_3d	b1	42000	0,78239	0,46830	0,84390	-4,31%	0,36822	0,87726	-0,52%	0,88187		-0,52%	M
	b2	41000	0,73860	0,55839	0,81387	-5,69%	0,40073	0,86642	0,40%	0,86293		0,40%	M
	b3	40000	0,68870	0,65857	0,78048	-6,98%	0,49014	0,83662	-0,28%	0,83900		-0,28%	M
	b4	39000	0,65224	0,81032	0,72989	-11,01%	0,54162	0,81946	-0,09%	0,82020		-0,09%	M
	b5	38000	0,61071	0,92459	0,69180	-11,41%	0,62735	0,79088	1,28%	0,78087	0,78697	0,50%	M
j10_3e	b1	33000	0,79543	0,40762	0,86413	-2,85%	0,31019	0,89660	0,80%	0,88947		0,80%	M
	b2	32000	0,75502	0,50308	0,83231	-4,89%	0,38103	0,87299	-0,24%	0,87513		-0,24%	M
	b3	31000	0,69832	0,63463	0,78846	-6,24%	0,48715	0,83762	-0,39%	0,84093		-0,39%	M
	b4	30000	0,63673	0,81783	0,72739	-9,66%	0,59462	0,80179	-0,41%	0,80513		-0,41%	M
	b5	29000	0,58231	1,00403	0,66532	-12,79%	0,71235	0,76255	-0,04%	0,76287		-0,04%	L
j20_3a	b1	40000	0,88373	0,17539	0,94154	-0,45%	0,16251	0,94583	0,00%	0,94580		0,00%	M
	b2	38000	0,82865	0,27217	0,90928	-1,71%	0,24914	0,91695	-0,88%	0,92507		-0,88%	M
	b3	36000	0,75910	0,40212	0,86596	-2,15%	0,36647	0,87784	-0,81%	0,88500		-0,81%	M
	b4	34000	0,66126	0,60984	0,79672	-4,13%	0,54963	0,81679	-1,71%	0,83100		-1,71%	M
	b5	32000	0,55425	0,88215	0,70595	-9,11%	0,78556	0,73815	-4,96%	0,77667		-4,96%	L
	b6	30000	0,43108	1,28604	0,57132	-17,03%	1,12103	0,62632	-9,04%	0,68860		-9,04%	L
j60_3a	b1	230000	0,95240	0,06882	0,97706	-0,33%	0,06317	0,97894	-0,14%	0,98030		-0,14%	H
	b2	225000	0,93920	0,09054	0,96982	0,19%	0,08208	0,97265	0,48%	0,96800		0,48%	H
	b3	220000	0,92137	0,12088	0,95971	-0,65%	0,10810	0,96397	-0,21%	0,96600		-0,21%	H
	b4	215000	0,89544	0,16210	0,94597	-1,08%	0,14579	0,95140	-0,51%	0,95630		-0,51%	M
	b5	210000	0,86421	0,21678	0,92774	-1,96%	0,19340	0,93553	-1,14%	0,94630	0,94190	-0,68%	M
	b6	205000	0,82739	0,28241	0,90586	-0,75%	0,25096	0,91635	0,40%	0,91270		0,40%	M
	b7	200000	0,78300	0,36428	0,87857	-1,87%	0,32387	0,89204	-0,36%	0,89530		-0,36%	M
	b8	190000	0,67291	0,60803	0,79732	-4,97%	0,52965	0,82345	-1,85%	0,83900	0,83497	-1,38%	M
	b9	180000	0,52179	1,01307	0,66231	-12,00%	0,87233	0,70922	-5,76%	0,75260		-5,76%	L

Table 8.4: page 4

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
j10_4a	b1	25000	0,78857	0,47457	0,88136	-3,28%	0,38179	0,90455	-0,73%	0,91120		-0,73%	M
	b2	24000	0,72155	0,67193	0,83202	-4,78%	0,51457	0,87136	-0,28%	0,87380		-0,28%	M
	b3	23000	0,64413	0,95819	0,76045	-8,72%	0,68158	0,82961	-0,42%	0,83310		-0,42%	M
	b4	22000	0,55304	1,25433	0,68642	-13,19%	0,92414	0,76897	-2,75%	0,79075		-2,75%	L
j10_4b	b1	24000	0,82388	0,30202	0,92450	-1,29%	0,28638	0,92841	-0,87%	0,93655		-0,87%	M
	b2	23000	0,78408	0,39544	0,90114	-1,58%	0,36712	0,90822	-0,81%	0,91560		-0,81%	M
	b3	22000	0,69721	0,57964	0,85509	-3,05%	0,53937	0,86516	-1,90%	0,88195	0,87915	-1,59%	M
	b4	21000	0,62512	0,77368	0,80658	-4,14%	0,70960	0,82260	-2,24%	0,84145		-2,24%	M
j10_4c	b1	30000	0,83801	0,32129	0,91968	-1,14%	0,28254	0,92937	-0,10%	0,93030		-0,10%	M
	b2	29000	0,78222	0,43975	0,89006	-2,76%	0,38937	0,90266	-1,38%	0,91530	0,91208	-1,03%	M
	b3	28000	0,72752	0,57982	0,85505	-3,17%	0,50597	0,87351	-1,08%	0,88305	0,88647	-1,46%	M
	b4	27000	0,65736	0,78962	0,80260	-5,54%	0,67315	0,83171	-2,12%	0,84970		-2,12%	M
j10_5a	b1	30000	0,97488	0,03935	0,99213	0,09%	0,03822	0,99236	0,12%	0,99120		0,12%	H
	b2	29000	0,96130	0,06098	0,98780	0,06%	0,05946	0,98811	0,09%	0,98720		0,09%	H
	b3	28000	0,94258	0,09149	0,98170	-0,46%	0,08927	0,98215	-0,41%	0,98620		-0,41%	H
	b4	27000	0,92151	0,13306	0,97339	-0,29%	0,12767	0,97447	-0,18%	0,97620		-0,18%	H
	b5	26000	0,88516	0,19641	0,96072	-0,46%	0,18990	0,96202	-0,33%	0,96520		-0,33%	M
	b6	25000	0,83680	0,28612	0,94278	-1,01%	0,27657	0,94469	-0,81%	0,95240		-0,81%	M
j20_5a	b1	90000	0,93624	0,11180	0,97764	-0,27%	0,10680	0,97864	-0,17%	0,98032		-0,17%	H
	b2	85000	0,87206	0,23424	0,95315	-0,44%	0,22294	0,95541	-0,21%	0,95740		-0,21%	M
	b3	80000	0,77113	0,46323	0,90735	-1,72%	0,43602	0,91280	-1,13%	0,92320	0,92296	-1,10%	M
	b4	75000	0,62859	0,85880	0,82824	-4,60%	0,79173	0,84165	-3,05%	0,86816	0,86248	-2,41%	M
	b5	70000	0,44555	1,57265	0,68647	-10,27%	1,41126	0,71775	-6,05%	0,76396		-6,05%	L
	b6	65000	0,25052	2,73092	0,45382	-28,11%	2,40893	0,51821	-17,91%	0,63128		-17,91%	L
j50_5a	b1	250000	0,97168	0,04754	0,99049	0,13%	0,04541	0,99092	0,17%	0,98920		0,17%	H
	b2	245000	0,96396	0,06645	0,98671	-0,13%	0,06300	0,98740	-0,06%	0,98800		-0,06%	H
	b3	240000	0,94608	0,09282	0,98144	-0,20%	0,08833	0,98233	-0,11%	0,98340		-0,11%	H
	b4	235000	0,92846	0,12728	0,97454	-0,39%	0,11986	0,97603	-0,24%	0,97840		-0,24%	H
	b5	230000	0,90498	0,17574	0,96485	-0,41%	0,16831	0,96634	-0,25%	0,96880		-0,25%	H
	b6	225000	0,87328	0,23755	0,95249	-0,33%	0,22254	0,95549	-0,01%	0,95560		-0,01%	M
	b7	220000	0,83790	0,32288	0,93542	-1,89%	0,29709	0,94058	-1,34%	0,95340	0,94648	-0,62%	M
	b8	200000	0,59336	1,01635	0,79673	-6,13%	0,89710	0,82058	-3,32%	0,84880		-3,32%	L
j99_5a	b1	580000	0,95396	0,07672	0,98466	-0,36%	0,07356	0,98529	-0,29%	0,98820		-0,29%	H
	b2	570000	0,93917	0,10329	0,97934	-0,07%	0,09912	0,98018	0,02%	0,98000		0,02%	H
	b3	560000	0,91928	0,13786	0,97243	-0,41%	0,13241	0,97352	-0,30%	0,97640		-0,30%	H
	b4	550000	0,89649	0,18016	0,96397	-0,72%	0,17266	0,96547	-0,57%	0,97100		-0,57%	M
	b5	540000	0,86674	0,24068	0,95186	0,22%	0,22940	0,95412	0,45%	0,94980		0,45%	M
	b6	530000	0,82870	0,31741	0,93652	-1,21%	0,30244	0,93951	-0,90%	0,94800		-0,90%	M

Table 8.4: page 5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
j10_10a	b1	70000	0,96670	0,06459	0,99354	0,17%	0,06451	0,99355	0,17%	0,99190		0,17%	H
	b2	65000	0,92049	0,16368	0,98363	-0,03%	0,16325	0,98368	-0,02%	0,98390		-0,02%	H
	b3	60000	0,82011	0,41573	0,95843	-0,14%	0,41431	0,95857	-0,13%	0,95980		-0,13%	M
	b4	55000	0,65652	0,90491	0,90951	-1,89%	0,89953	0,91005	-1,83%	0,92700	0,91992	-1,07%	M
	b5	53000	0,57134	1,22418	0,87758	-0,93%	1,21527	0,87847	-0,83%	0,88580		-0,83%	L
	b6	50000	0,41799	1,93978	0,80602	-5,43%	1,92138	0,80786	-5,21%	0,85230		-5,21%	L
j20_10a	b1	180000	0,96410	0,07175	0,99283	-0,10%	0,07131	0,99287	-0,09%	0,99380		-0,09%	H
	b2	170000	0,90954	0,19541	0,98046	0,04%	0,19370	0,98063	0,05%	0,98010		0,05%	H
	b3	160000	0,79432	0,49969	0,95003	-0,60%	0,49365	0,95064	-0,54%	0,95580		-0,54%	M
	b4	155000	0,70949	0,77728	0,92227	-0,69%	0,76598	0,92340	-0,57%	0,92870		-0,57%	M
	b5	150000	0,59260	1,17748	0,88225	-2,47%	1,15819	0,88418	-2,26%	0,90460		-2,26%	L
	b6	145000	0,46943	1,75782	0,82422	-3,56%	1,72379	0,82762	-3,16%	0,85460		-3,16%	L
j50_10a	b1	400000	0,96818	0,06048	0,99395	0,02%	0,06048	0,99395	0,02%	0,99380		0,02%	H
	b2	395000	0,95880	0,07784	0,99222	-0,03%	0,07084	0,99292	0,04%	0,99250		0,04%	H
	b3	390000	0,94815	0,10030	0,98997	-0,16%	0,10029	0,98997	-0,16%	0,99160		-0,16%	H
	b4	385000	0,93521	0,12718	0,98728	0,08%	0,12717	0,98728	0,08%	0,98650		0,08%	H
	b5	380000	0,91859	0,16373	0,98363	0,00%	0,16347	0,98365	0,01%	0,98360		0,01%	H
	b6	375000	0,89790	0,20902	0,97910	0,03%	0,22848	0,97715	-0,17%	0,97880		-0,17%	M
	b7	360000	0,81346	0,41524	0,95848	-0,02%	0,41291	0,95871	0,00%	0,95870		0,00%	M
	b8	350000	0,72793	0,65304	0,93470	-0,52%	0,64829	0,93517	-0,47%	0,93960		-0,47%	M
	b9	340000	0,62009	1,00226	0,89977	-0,79%	0,99330	0,90067	-0,69%	0,90690		-0,69%	M

Table 8.4: page 6

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
j10_20a	b1	130000	0,96336	0,09625	0,99519	0,06%	0,09624	0,99519	0,06%	0,99460		0,06%	H
	b2	125000	0,92718	0,22391	0,98880	0,07%	0,22391	0,98880	0,07%	0,98810		0,07%	H
	b3	120000	0,84954	0,49070	0,97547	-0,17%	0,49060	0,97547	-0,17%	0,97715		-0,17%	M
	b4	115000	0,71222	1,01576	0,94921	-0,67%	1,01513	0,94924	-0,67%	0,95565		-0,67%	M
	b5	110000	0,53831	1,96380	0,90181	-0,03%	1,96167	0,90192	-0,01%	0,90205		-0,01%	L
	b6	105000	0,33763	3,59440	0,82028	-4,57%	3,58823	0,82059	-4,54%	0,85960		-4,54%	L
j10_20b	b1	110000	0,94428	0,13651	0,99317	0,00%	0,13650	0,99318	0,00%	0,99320		0,00%	H
	b2	108000	0,92440	0,18877	0,99056	0,03%	0,18877	0,99056	0,03%	0,99030		0,03%	H
	b3	106000	0,89661	0,26680	0,98666	-0,21%	0,26680	0,98666	-0,21%	0,98875		-0,21%	M
	b4	104000	0,86883	0,35728	0,98214	-0,07%	0,35728	0,98214	-0,07%	0,98285		-0,07%	M
	b5	102000	0,82364	0,49012	0,97549	-0,27%	0,49012	0,97549	-0,27%	0,97815		-0,27%	M
	b6	100000	0,77477	0,66323	0,96684	-0,31%	0,66323	0,96684	-0,31%	0,96980		-0,31%	M
j10_20c	b1	90000	0,96976	0,07899	0,99605	0,01%	0,07882	0,99606	0,01%	0,99595		0,01%	H
	b2	88000	0,95063	0,13627	0,99319	-0,04%	0,13601	0,99320	-0,04%	0,99355		-0,04%	H
	b3	86000	0,91779	0,23089	0,98846	0,02%	0,23053	0,98847	0,02%	0,98825		0,02%	H
	b4	84000	0,87150	0,38791	0,98060	0,03%	0,38719	0,98064	0,03%	0,98030		0,03%	M
	b5	82000	0,80215	0,62929	0,96854	-0,45%	0,62787	0,96861	-0,45%	0,97295		-0,45%	M
	b6	80000	0,72859	0,97939	0,95103	-0,49%	0,97616	0,95119	-0,48%	0,95575		-0,48%	M
	b7	75000	0,42181	2,82150	0,85893	-2,51%	2,80295	0,85985	-2,41%	0,88108		-2,41%	L
j20_20a	b1	220000	0,93595	0,15035	0,99248	0,07%	0,14996	0,99250	0,08%	0,99175		0,08%	H
	b2	215000	0,90121	0,24286	0,98786	0,25%	0,24247	0,98788	0,26%	0,98535		0,26%	H
	b3	210000	0,85588	0,37799	0,98110	0,01%	0,37757	0,98112	0,01%	0,98100		0,01%	M
	b4	205000	0,78727	0,59120	0,97044	-0,28%	0,59073	0,97046	-0,28%	0,97320		-0,28%	M
	b5	200000	0,69592	0,90599	0,95470	-0,32%	0,90544	0,95473	-0,32%	0,95780		-0,32%	M
j50_20a	b2	590000	0,94555	0,13427	0,99329	0,18%	0,13427	0,99329	0,18%	0,99150		0,18%	H
	b3	580000	0,91841	0,20897	0,98955	-0,02%	0,20897	0,98955	-0,02%	0,98975		-0,02%	H
	b4	570000	0,88058	0,31860	0,98407	0,18%	0,31860	0,98407	0,18%	0,98230		0,18%	M
	b5	560000	0,82662	0,48276	0,97586	-0,47%	0,48276	0,97586	-0,47%	0,98050		-0,47%	M
	b6	550000	0,75892	0,72422	0,96379	-0,49%	0,72422	0,96379	-0,49%	0,96850		-0,49%	M
	b7	525000	0,51794	1,84998	0,90750	-1,20%	1,84866	0,90757	-1,20%	0,91855		-1,20%	L
	b8	500000	0,23025	4,37644	0,78118	-5,84%	4,36877	0,78156	-5,80%	0,82965		-5,80%	L

Table 8.4: page 7

8.4.2 Estimates of AA_S using A_S (M=1). As we have already seen, the system availability A_S should be the performance measure used when estimating the proportion of equipments still operational at the end of the cycle AA_S, and will be exact with or without part failure dependencies since they are the same when M=1. This will be exact for any value of the optimized value A_S in model P1, therefore irrespective of the available budget B and the stock level $\{S_j, j=1,..,J\}$ obtained as a result.

We have also proved earlier that maximizing A_S is equivalent to maximizing AA_S when M=1. Since table 8.1 for model P1 has clearly demonstrated the accuracy and reliability of this measure of performance, the special case M=1 has been solved and will not be discussed further.

8.4.3 Estimates of AA_S using BO (M>1). Table 8.3b has also previously demonstrated that the estimate 1 - BO/M can also be used as an accurate estimate of AA_S, with or without part failure dependencies, when M > 1 but for high A_S values or > 0.90 only, even if the total expected system BO = Σ BO_j, j=1,..,J counts too many backorders; the reason is that the terms in the summation quickly become negligible for the extreme right tails of the Poisson distribution and, therefore, do not significantly affect the accuracy of the estimate 1 - BO/M.

Table 8.4 clearly shows the progressively worsening results for lower A_S values indicated by the % differences shown in column (7) as stock levels $\{S_j\}$ are deliberately set lower. One important conclusion though, is that, as J and M increase, 1 - BO/M progressively becomes a better estimate and for J ≥ 10 and M ≥ 5, a high proportion of all those problems indicate an error of less than 1% from AA_S even for medium to some low A_S values!. For example, the last five 5 problems with M=20 shows that all estimates 1 - BO/M vs AA_S are within 1% when A_S is > 0.60 (Medium) and could be considered acceptable even for some low values of A_S < 0.60.

8.4.4 Estimates of AA_S using TBO (M>1). As explained in earlier sections, the expression 1 - TBO/M can be used instead of 1 - BO/M to estimate AA_S with part failure dependencies, for lower values of A_S. Numerical experiments have shown that this simple heuristics significantly improves the estimate of AA_S to within 1% for a fairly wide range of J x M value combinations and for medium values of A_S.

Since 1 - TBO/M is most sensitive for small values of M and large values of J, table 8.4 was constructed using problems with parameters in appropriate range of interest, specially with M=2,3,4. The results clearly show the dramatic improvement of the estimate for AA_S for small $M \leq 5$ and essentially show lesser improvement as $M \geq 10$; the reason is that for smaller values of M, the number of backorders BO counted is considerably reduced, as it should, and improves the estimate for AA_S by reducing the % error by half; as M increases, the summation terms for BO beyond the value of S_j + M quickly become negligible and TBO tends towards BO.

The estimate 1 - TBO/M also shows values that are consistently lower than the simulated AA_S values; underestimating AA_S is an important property since it errs on the conservative side of the true value for AA_S, in that it is better to underestimate than to overestimate this important measure of system performance.

As an example, the % differences between 1 - BO/M and 1 - TBO/M vs AA_S(mod) simulated values improves by several orders of magnitude for low A_S values (ex: j10_2b,c and d) and improves to a lesser degree as M increases to M=3 (ex: j10_3b,c and d) and are about the same for higher M values (ex: problems at the end with M=20). Further study would be required to analyze and improve the reliability or accuracy of other estimates for AA_S possibly based on the ratio J/M.

From table 8.4, it is clear that the estimate 1 - TBO/M significantly improves the 1 - BO/M estimate for AA_S across a whole range of A_S values, even taking into account the part failure dependency problem defined earlier. Other than being within ± 1% for all J x M combinations for all A_S values > 0.90, it also met that criterion for all problems whose A_S > 0.70 except for only j20_5a with A_S = 0.77113 and underestiamted AA_S by at most 3.05% for all those cases where $A_S > 0.60$.

CHAPTER 9: CONCLUDING REMARKS

This thesis has analyzed the sparing model currently used by the military Land Forces for the initial provisioning and determination of inventory levels for first line unit organizations required to carry the equivalent of 15 days worth of spare items, which is based on a modified Poisson formula to sequentially spare every individual item j, j=1,..,J up to a specified 99.8% confidence level at every location i, i=1,..,I, irrespective of their costs. The problem is compounded further when only a fixed budget is available, with the ensuing adhoc procedures to lower stock levels when it is exceeded. Recommendations to change the current model and adopt the straight Poisson cumulative distribution or Equal Aj's model, specially for microcomputer use, has been made for the past 25 years (see Vincent 1982, DesRochers 1984a and 1984b and Hebert 1995) but have not yet been acted upon.

In the thesis, it has been shown that linking items together with practical aggregate system performance measures to be optimized for either of two models: model P1 to maximize system availability A_S and/or model P2 to minimize total expected system backorders BO, both subject to a specified available fixed budget B consisting of total purchasing costs, would significantly improve the performance when compared with the equal Aj's model, or conversely, that significant dollar savings would be achieved to attain a specified system performance measure. Empirical results for 40 randomly generated test problems with parameters in the range of interest, indicate an average relative increase in A_S of 29.3% over the current model, and even more for low A_S values.

It has also been shown that adopting either model P1 and/or P2 would also significantly increase the average number (and proportion) of equipments AA_S still operational at the end of the period, which may be considered a more appropriate measure of system performance when multiple identical equipments (M > 1) are operating during the period. Simulation of stock levels obtained as a result of optimizing P1 and/or P2 across the same 40 test problems show an average relative increase in AA_S of 4.9% over the current model, which can be considered significant since the average A_S value is 65.7% and M averages 10.2 equipments across the 40 problems.

The measure of system performance AA_S is affected by the attrition of equipments M and the number of items J involved; the derivation of its exact theoretical distribution is shown to be essentially a combinatorial type of problem and mathematically intractable even for moderate values of J x M. For the special case M=1, however, it has been proved that solving model P1 when maximizing A_S also maximizes AA_S, regardless of the available budget, resulting in stock levels and A_S values that can be quite low.

For M > 1, the estimate 1 - BO/M proves to be a particularly effective and reliable (all less than 1.0% from AA_S) for all problems where $A_S \geq 0.90$, including more than 80 other test problems not reported on here. For lower A_S values ≤ 0.90, the expression 1 - TBO/M significantly improves the estimate 1 - BO/M for the true value of AA_S, by summing the BO function up to S_j+M instead of up to infinity, since with M equipments, no backorders can occur past S_j+M. For the 32 test problems where $M \geq 5$, 1 - TBO/M obtained as a result of solving P1 or P2 differed by an average of less than 0.1% when compared to AA_S simulated values (2.8% the highest); most differences tend to occur for problems where the number of equipments is small ($M \leq 5$) and lower A_S values ($A_S \leq 0.30$).

From the solution methods studied to determine optimal inventory stock level vectors $\{S_j, j=1,..,J\}$, and possible implementation on (personal) micro-computers, the dynamic programming (DP) approach presents serious computational difficulties and the approximate DP strategy with incremental budget values at each stage, is considered impractical and unpredictable in calculating error size. We have also shown that this non-linear integer optimization problem can be represented by equivalent FULL (optimal) and GAP (near optimal) network structures and how they can be effectively used to determine the size of the DP problem by its total number of nodes N equals the total DP rows - 1 and the total number of arcs A equals the total availability calculations in DP rows.

The highly efficient, much more versatile and faster marginal analysis procedure gives a sequence of undominated points spaced by not more than the cost of the most expensive of all J items, and a response curve $\{A_S \text{ vs } C_S\}$ for model P1 and/or the response curve $\{BO \text{ vs } C_S\}$ for model P2 which will always dominate the response curves generated by the equal Aj's model. Furthermore, the optimal (or near optimal) solution generated by this procedure can be extremely useful for managers to decide the level of spares required as well as a key factor for its implementation on microcomputers, even on a large scale basis involving hundreds of items or more, from which organizational units can also budget spares for different time periods as required. Finally, the solution vector $\{S_j\}$ obtained from this method, demonstrates that it allocates items optimally by stocking more of the most reliable and least expensive items and less of least reliable and expensive items, as would be expected when adopting a system performance measure such as A_S or BO, as opposed to the current military model indiscriminate method.

In cases where near optimal solutions are obtained from the regular marginal analysis procedure, we introduced the topup marginal analysis procedure, which does not seem to be part of the literature, and has also been shown to further improve related performance measures such as increasing A_S by an average of 1.8% and decreasing BO by an average 4.0% across the 40 test problems. The relative percentage difference can easily become more than 5% in cases where limited budgets are available (causing lower A_S values) as a result of restructuring, downsizing or implementing cost reduction measures not only applicable to the military but to other organizations as well. Although the last and closest solution point to the available budget obtained as a result of topping up is not guaranteed to be undominated, it guarantees to dominate the last iteration point obtained from the regular marginal analysis, and its total costs C_S will be within the least expensive of all J items.

The Lagrange relaxation method applied to both models P1 and P2 and an accurate initial estimate for the optimal multiplier derived here, can also be used to obtain the optimal (or near) solution vectors $\{S_j\}$ even faster than the regular marginal analysis by skipping over several iterations at a time, when a bisection search procedure is implemented; tests problems conducted for model P1 indicated an approximate 5 to 10 times faster execution time to obtain the exact same solution vectors.

The faster execution speed of this procedure would be a definitive advantage over the marginal analysis procedure for calculating optimal stock levels on large scale problems and/or when multiple locations are involved, but gives fewer iterative points on the response curve of interest. It can also be used to quickly evaluate total costs for these fewer points and provide valuable information to program managers, if an arbitrarily large budget value is assigned. This advantage will quickly disappear if more points on the curve are desired and a different search technique is used to give more undominated points, since it will tend towards an order of magnitude similar to the marginal analysis procedure.

The same solution methods described for the single location model have been extended to multiple indentured and multiple location models P1b to maximize A_S and/or model P2b to minimize BO, the aggregate performance measures calculated for these models. For multiple indentured systems, it has been shown that, under certain conditions, it may be possible to achieve a higher system availability A_S at a lower overall cost C_S when additional failure information about individual components of an assembly is available, even though the sum of the components' costs exceed the cost of the whole assembly. A method to derive upper bounds for components' costs and a multi-phase approach to optimize system performance measures is included.

The multiple location models P1b to maximize A_S and P2b to minimize BO with I locations and J items is shown to be equivalent to a single location model with (I x J) items, which means that optimal stock level vectors $\{S_{ij},\ i=1,..,I\ j=1,..,J\}$, can also be solved using the procedures described for the single location model.

Due to increased computational difficulties, the additional problem of allocating budget levels at each location i, i=1,..,I cumulates or compounds the calculation of all measures of system performance achieved as a result of implementing model P1b, P2b and the equal Aj's model. Comparisons between the models show a marked improvement in AA_S across all locations by adopting P1 and/or P2 over the equal Aj's model; it also appears that the proportional budget allocation method to allocate budget amounts in proportion to the number of equipments held at each location yields the same overall results for AA_S as if they were optimally allocated, and locations with higher number of equipments tend to have higher AA_S values.

Summarizing the above discussion, we conclude:

1. a system performance measure linking items together such as models P1 (maximize A_S) or model P2 (minimize BO) is clearly superior to the current model;

2. the marginal analysis solution procedure complemented by the topup additional procedure to either maximize A_S and/or minimize BO, could benefitially be considered for implementation on a system wide basis, both for initial calculation of first (and possibly second line units) provisioning of parts given a maximum available budget to be allocated; it would ensure that stockage levels at every location would be optimized and budget levels not exceeded, thus avoiding the otherwise necessary manipulation of end results;

3. the standard lookup tables to determine entitlements of stock levels on an individual item basis (up to 99.8% confidence level currently) and based on the number of equipments M held at each unit, could be replaced by a standard microcomputer program (compiled BASIC version or other Windows based program) to be distributed to every unit receiving the equipments and related spares; this would ensure that each location could determine its own optimal stock levels for any cycle period as required by the budget planning process, based on the

same cost and failure rate parameter data established through the AQ system.

The end result would be a far more effective and efficient system for determinig optimal spares levels on a national basis for budgeting initial provisioning of spares to every first line unit and a most efficient way for individual units to determine their own stockage levels for any usage period as required.

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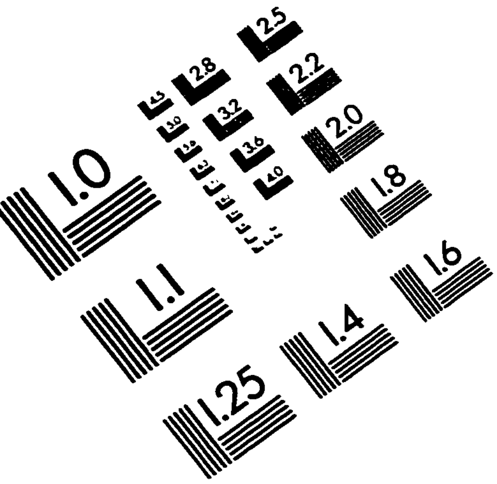
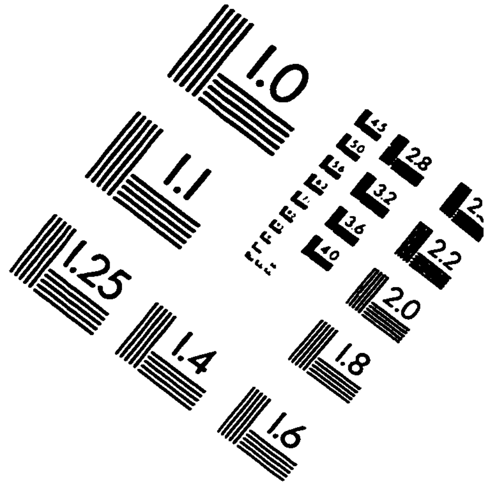
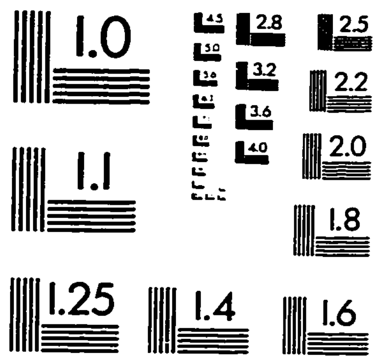
IMAGE EVALUATION TEST TARGET (QA-3)







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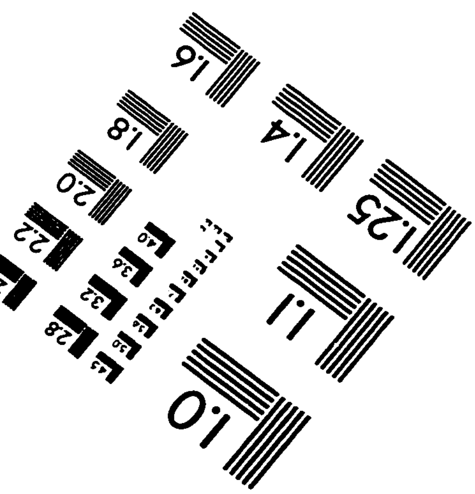

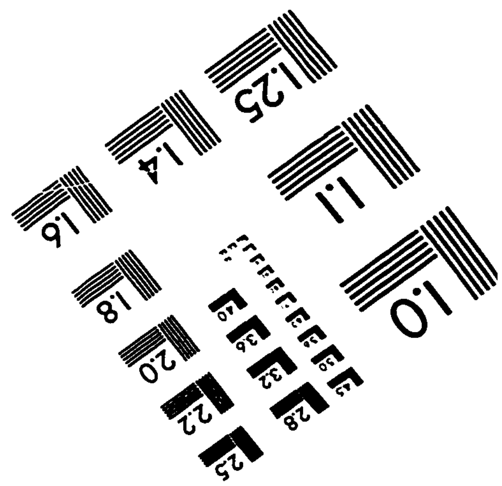

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