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SOME RESULTS ON QUASI-MONOTONE AND
PSEUDO-MONOTONE OPERATORS AND APPLICATIONS

By
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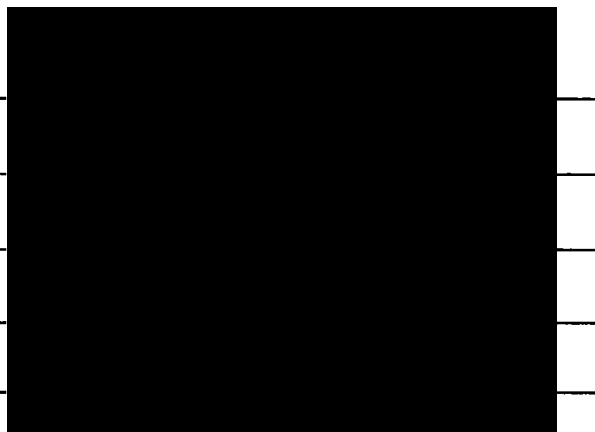
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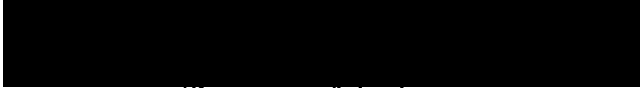
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*In the name of 'Allah'— Lord of the Worlds,
Who is the Most Merciful and the Most
Benevolent!*

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Abstract

In this thesis, we shall introduce the concepts of h -quasi-monotone, quasi-monotone, bi-quasi-monotone, h -quasi-semi-monotone, quasi-semi-monotone, quasi-nonexpansive, semi-nonexpansive, lower hemi-continuous, upper hemi-continuous, weakly lower (respectively, upper) demi-continuous, strongly lower (respectively, upper) demi-continuous, strong h -pseudo-monotone, strong pseudo-monotone, h -pseudo-monotone, pseudo-monotone, h -demi-monotone, and demi-monotone operators. We shall first obtain some generalizations of Ky Fan's minimax inequality. As applications, we shall obtain results on fixed point theorems, generalized variational, quasi-variational and bi-quasi-variational inequalities, and complementarity and bi-complementarity problems.

In Chapter 2, we shall first obtain a minimax inequality which generalizes Ky Fan's minimax inequality in several respects. Then, we shall obtain a Knaster-Kuratowski-Mazurkiewicz (in short KKM) type lemma which will be more general than KKM Lemma in all of its practical applications. By applying our KKM type lemma, we shall obtain a generalization of Brézis-Nirenberg-Stampacchia's generalization of Ky Fan's minimax inequality.

In Chapter 3, as applications of the minimax inequalities of Chapter 2, and as applications of most of the above mentioned operators, we shall obtain several existence theorems for compact and non-compact generalized variational inequalities and for non-compact generalized complementarity problems in topological vector spaces.

Finally, in Chapter 4, as applications of the generalized variational inequalities of Chapter 3, we shall first obtain some fixed point theorems in Hilbert spaces for some of the operators introduced in this thesis. Applying the minimax inequalities of Chapter 2 or some minimax inequalities in the literature and/or some of the operators introduced in this thesis, we shall obtain several existence theorems for non-compact generalized quasi-variational inequalities as well as for both compact and non-compact generalized bi-quasi-variational inequalities, and non-compact bi-complementarity problems in locally convex Hausdorff topological vector spaces.

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brought up their children through a great hardship. Their love always accompany me throughout my study in Canada.

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Table of Symbols and Abbreviations

Φ	either the real field \mathbf{R} or the complex field \mathbf{C} .
\mathbf{C}	the set all complex numbers.
\mathbf{R}	the real line.
\mathbf{N}	the set of all natural numbers.
2^X	the family of all non-empty subsets of X .
$\mathcal{F}(X)$	the family of all non-empty finite subsets of X .
\emptyset	the empty set.
$\text{co}(X)$	the convex hull of X .
$\text{cl}_X(A)$	the closure of A in X .
$\partial_H(X)$	the topological boundary of X in the Hilbert space H .
$I_X(y)$	the inward set of X at y .
E^*	the dual space of E .
$G(T)$	the graph of the mapping T .
$bc(X)$	the family of all non-empty bounded and closed subsets of X .
D	the Hausdorff metric on $bc(X)$ induced by the metric d or norm $\ \cdot\ $.
KKM theorem	Knaster-Kuratowski-Mazurkiewicz theorem.
G - KKM map	generalized KKM map.

Chapter 1

Introduction

Ky Fan [48] obtained a minimax inequality in 1972 which was celebrated at that time. Let us go back to the history behind obtaining this minimax inequality. We shall start with the classical theorem of Knaster-Kuratowski-Mazurkiewicz (in short KKM) [72]. This classical theorem is often called the KKM Theorem or KKM Lemma. The KKM Theorem has numerous applications in various fields of pure and applied mathematics. Today, the studies and applications of the KKM Theorem are called the KKM Theorey.

Ky Fan [44] obtained a generalization of the classical KKM Theorem [72] in 1961 to infinite dimensional Hausdorff topological vector spaces and established an elementary but very basic 'Geometric Lemma' for set-valued mappings. Later, Browder [21] obtained a fixed point form of Fan's Geometric Lemma in 1968 which is called Fan-Browder Fixed Point Theorem today. Since then there have been numerous generalizations of Fan-Browder Fixed Point Theorem with applications in coincidence and fixed point theory, minimax inequalities, variational inequalities, nonlinear analysis, convex analysis, game theory and mathematical economics.

Ky Fan [48] applied the above Geometric Lemma in obtaining his 1972 celebrated minimax inequality. Ky Fan's minimax inequality plays a fundamental role in nonlinear analysis and mathematical economics and has been applied to potential theory, partial differential equations, monotone operators, variational inequalities, optimization, game theory, linear and nonlinear programming, operator theory, topological groups and linear

algebra. By using Ky Fan's minimax inequality, a more general form of the Fan-Glicksberg Fixed Point Theorem was derived for set-valued operators which were inward (or outward) as defined by Ky Fan in 1969. Ky Fan's definitions of inward (or outward) mappings were more general than Halpern's [56] definitions for inward (or outward) mappings of 1965.

In Chapter 2 of this thesis, we shall obtain two new minimax inequalities in topological vector spaces which generalize Ky Fan's minimax inequality [48, Theorem 1] in several respects.

To obtain the second minimax inequality, we shall first establish a KKM type lemma. This lemma will be more general than KKM Lemma in all of its practical applications. By applying our KKM type lemma, we shall generalize and extend Brézis-Nirenberg-Stampacchia's generalization [16] of Ky Fan's minimax inequality [48].

Let X be a non-empty subset of a topological vector space E and E^* be the continuous dual of E . Let $T : X \rightarrow 2^{E^*}$ be a map, then the *generalized variational inequality problem* associated with X and T is to find $\hat{y} \in X$ such that the *generalized variational inequality* $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq 0$ for all $x \in X$ holds, or to find $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in X$ holds. When T is single-valued, a generalized variational inequality is called a *variational inequality*.

The topic of variational inequalities has only been studied systematically since 1960s (e.g., see Fichera [51] and Stampacchia [102] and others). The variational inequality theory is related to the simple fact that the minimum of the differentiable convex functional I on a convex set D in a real Hilbert space can be characterized by an inequality of the type $\langle I'(u), v - u \rangle \leq 0$ for all $v \in D$, where $I'(u)$ is the derivative of the functional $I(u)$. However, it is remarkable that the variational inequality theory has many diversified applications. During the last three decades which have elapsed since its discovery, the important developments in variational theory are formulations that variational inequalities can be used to study problems of fluid flow through porous media (e.g., see Baiocchi and Capelo [7]), contact problems in elasticity (e.g., see Kikuchi and Oden [69]), transportation problems (see Bertsekas and Tsitsiklis [14] and Harker [57]) and economic equilibria (see Dafermos [34]). An additional main area of applications for variational inequalities arises

in control problems with a quadratic objective functional, where the control equations are partial differential equations. A detailed discussion of this can be found in Lions [75]. The connection between control problems and quasi-variational inequalities is presented in Aubin [3] and Zeidler [114]. There also exist intimate interconnections between variational inequalities, stochastic differential equations, and stochastic optimization. We can find these in Friedman [52]-[53], Bensoussan, Goursat and Lions [12] and Bensoussan [11]. Browder [19] and Hartman and Stampacchia [59] first introduced variational inequalities. Since then, there have been many generalizations, e.g., see [1], [6], [7], [16], [21], [42], [71], [91], [98], [99], [104], and [112], etc.

The area of mathematical programming which is known as complementarity theory, is equally important. In 1965, Lemke [74] first introduced and studied complementarity theory. Cottle and Dantzig [30] defined the complementarity problem and called it the fundamental problem. For recent results and applications, we refer to Harker and Pang [58], Noor, Noor and Rassias [80] and references therein. However, it was Karamardian [68], who proved that if the set involved in a variational inequality and complementarity problem is a convex cone, then both problems are equivalent. After that, many generalizations have been given by Shih and Tan [93], Ding [36], Isac [66]-[67], Chang and Huang [24] and references therein. For more detail on the discussion between the variational inequalities and complementarity problems, we refer to Cottle, Giannessi and Lions' book [31] and references therein.

The purpose of Chapter 3 is to present existence theorems for generalized variational inequalities with applications to existence theorems for generalized complementarity problems. To this end, we shall first introduce the notions of lower hemi-continuous, upper hemi-continuous, h -quasi-monotone, quasi-monotone, h -quasi-semi-monotone, quasi-semi-monotone, quasi-nonexpansive and semi-nonexpansive operators. As applications of the above operators and the minimax inequalities in Section 2 of Chapter 2, we shall present some existence theorems for generalized variational inequalities and generalized complementarity problems in topological vector spaces. These results will extend or improve the corresponding results in the literature, e.g., see [6], [27] and [91]. Surjectivity

of monotone or semi-monotone operators will also be discussed. Moreover, we shall introduce the notions of weakly lower (respectively, upper) demi-continuous, strongly lower (respectively, upper) demi-continuous operators and the notions of quasi-monotone and quasi-semi-monotone operators in more general settings. As applications of these operators, we shall obtain some existence theorems on generalized variational inequalities in topological vector spaces and in non-reflexive Banach spaces. Using the concept of escaping sequences introduced by Border in [15], we shall also obtain some existence theorems for the above operators on generalized variational inequalities in non-compact settings.

Finally, in this chapter we shall introduce the notions of h -pseudo-monotone, pseudo-monotone, h -demi-monotone and demi-monotone operators. Our definition of pseudo-monotone operators is a generalization of the single-valued pseudo-monotone operators defined by Brézis-Nirenberg-Stampacchia in [16]. As applications, we shall present some existence theorems for generalized variational inequalities and generalized complementarity problems for pseudo-monotone and demi-monotone operators. Our results for demi-monotone operators will extend the corresponding results in [6], [16], [27] and [91]. The results for pseudo-monotone operators will generalize the corresponding results in [16] and extend those in [6], [27] and [91]. Surjectivity of demi-monotone operators will also be discussed.

We remark here that the development of variational inequalities can be viewed as the simultaneous pursuit of two different lines of research: On the one side, it reveals the fundamental facts on the qualitative behaviour of solutions (such as its existence, uniqueness and regularity) to important classes of problems. On the other side, it enables us to develop highly efficient and powerful new numerical methods to solve, for example, free and moving boundary value problems and the general equilibrium problems. A comprehensive investigation of numerical methods of variational inequalities is contained in Glowinski, Lions and Tremolieres's book [55]. For more details, we refer to Cottle, Giannessi and Lions [31], Crank [32], Harker and Pang [58], Noor [78]-[79], Noor, Noor and Rassias [80], Rodrigues [88] and Shi [90] etc. Among the most effective numerical

techniques are projection methods and its variant forms, linear approximation method, relaxation method, auxiliary principle and penalty function techniques. In addition to these methods, the finite element technique which is also being applied for the approximate solution of variational inequalities, have been obtained by many mathematicians including Falk [43], Mosco and Strang [76] and Noor, Noor and Rassias [80] and references therein.

It is well known that fixed point theory is very important in mathematics. The close relationship between fixed point theory and mathematical economics can be illustrated in many ways. In order to explore this relationship, one can study, for example, the topics in [17], [77] and [35]. The usefulness of Brouwer's fixed point theorem in [17] was recognized by John von Neumann [77] when he developed the foundations of game theory in 1928.

In Chapter 4 of this thesis, we shall give several applications of the generalized variational inequalities of Chapter 3 and the minimax inequalities of Chapter 2. We shall mainly apply the generalized variational inequalities of Chapter 3 in obtaining fixed point theorems in Hilbert spaces. Applying the minimax inequalities of Chapter 2 or some minimax inequalities in the literature and/or some of the operators introduced in this thesis, we shall obtain several existence theorems for non-compact generalized quasi-variational inequalities as well as several existence theorems for both compact and non-compact generalized bi-quasi-variational inequalities, and non-compact bi-complementarity problems in locally convex Hausdorff topological vector spaces.

We shall first investigate fixed point theorems in Hilbert spaces for lower or upper hemi-continuous operators T such that $I - T$ is either quasi-monotone or a quasi-semimonotone operator. Our results will extend or improve the corresponding fixed point theorems in the literature, e.g., see [6], [18], [27] and [91]. As special cases of these fixed point theorems, we shall also obtain fixed point theorems for quasi-nonexpansive or semi-nonexpansive operators.

Next, we shall investigate some fixed point theorems in Hilbert spaces H for set-valued operators T which have some kind of upper semicontinuity and such that $I - T$ is pseudo-monotone or demi-monotone. These fixed point theorems will extend or improve the corresponding fixed point theorems in the literature, e.g., see [6], [27] and [91].

In recent years, various extensions and generalizations of variational inequalities have been considered and studied. It is clear that in a variational inequalities formulation, the convex set involved does not depend on solutions. If the convex set does depend on solutions, then variational inequalities are called quasi-variational inequalities.

Let X be a non-empty subset of a topological vector space E and E^* be the continuous dual of E . Given the maps $S : X \rightarrow 2^X$ and $T : X \rightarrow E^*$, the quasi-variational inequality (QVI) problem is to find a point $\hat{y} \in S(\hat{y})$ such that $Re\langle T(\hat{y}), \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. The QVI was introduced by Bensoussan and Lions in 1973 (see, e.g., [13]) in connection with impulse control. Applications of quasi-variational inequalities can be found in Aubin [3], Aubin and Cellina [4] and Zeidler [114]. Again, if we consider a set-valued map $T : X \rightarrow 2^{E^*}$, then the generalized quasi-variational inequality (GQVI) problem is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$.

In 1982, for the study of operations research, mathematical programming and optimization theory, Chan and Pang [23] first introduced the so-called generalized quasi-variational inequalities in finite dimensional Euclidean spaces. The existence theorem of Chan and Pang [23] is illustrated as follows:

Theorem 1.0.1 *Let X be a non-empty compact convex subset of \mathbf{R}^n and $S : X \rightarrow 2^X$ and $T : X \rightarrow 2^{\mathbf{R}^n}$ be such that $S(x)$ is compact convex and $T(x)$ is contractible and compact for each $x \in X$. Moreover assume that S is continuous and T is upper semicontinuous. Then GQVI has at least one solution.*

In 1985, Shih and Tan [92] were the first to study the GQVI in infinite dimensional locally convex Hausdorff topological vector spaces. The following result illustrates an existence theorem of Shih and Tan [92] on GQVI:

Theorem 1.0.2 *Let E be a normed space, E^* be the continuous dual space of E and X be a non-empty compact convex subset of E . Let $S : X \rightarrow 2^X$ be continuous such that for each $x \in X$, $S(x)$ is a non-empty closed convex subset of X , and $T : X \rightarrow 2^{E^*}$ be upper semicontinuous from the relative topology of X to the strong topology of E^**

such that for each $x \in X$, $T(x)$ is a non-empty strongly compact subset of E^* . Then there exists a point $\hat{y} \in X$ such that

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) $\sup_{x \in S(\hat{y})} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq 0$.

Since then, there have been a number of generalizations of the existence theorems about GQVI, e.g., Cubiotti [33], Ding and Tan [39], Harker and Pang [58], Kim [70], Shih and Tan [100] and Tian and Zhou [111] and references therein. These results have wide applications to problems in game theory and economics, mathematical programming (e.g., see Aubin [3], Aubin and Ekeland [5], Chan and Pang [23], Harker and Pang [58] and references therein). Most existence theorems mentioned above, however, are obtained on compact sets in finite dimensional spaces or infinite dimensional locally convex Hausdorff topological vector spaces, and both S and T are either continuous or upper (or lower) semicontinuous.

On the other hand, in economic and game applications, it is known that the choice space (or the space of feasible allocations) generally is not compact in any topology (even though it is closed and bounded), a key situation in infinite dimensional topological vector spaces. Moreover, we note that there is practically no existence theorem for solutions of generalized quasi-variational inequalities on non-compact sets in infinite dimensional spaces. Motivated by this observation, we shall obtain some results on existence theorems for generalized quasi-variational inequalities on paracompact sets for operators which are either monotone and lower hemi-continuous along line segments or semi-monotone and upper hemi-continuous along line segments or upper semicontinuous. Our results generalize the corresponding results in [70] and [92].

Moreover, we shall obtain some results on existence theorems for generalized quasi-variational inequalities on paracompact sets for operators which are either strong h -pseudo-monotone or h -pseudo-monotone and which have some kind of upper semi-continuity. Our results will extend the corresponding results in [70] and [92].

In 1989, Shih and Tan [100] first introduced the generalized bi-quasi-variational inequalities in Hausdorff topological vector spaces. Shih and Tan's generalized bi-quasi-variational inequality can be described as follows:

Let E and F be Hausdorff topological vector spaces over the field Φ (which is either the real field or the complex field), let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional, and let X be a non-empty subset of E . Given a set-valued map $S : X \rightarrow 2^X$ and two set-valued maps $M, T : X \rightarrow 2^F$, the *generalized bi-quasi-variational inequality* (GBQVI) problem is to find a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

Motivated by Shih and Tan's GBQVI in [100], we shall obtain some results on existence theorems of generalized bi-quasi-variational inequalities in locally convex Hausdorff topological vector spaces on compact sets. Then as applications of our results and the results in [100], using the concept of escaping sequences introduced by Border in [15], we shall obtain some existence theorems on non-compact generalized bi-quasi-variational inequalities and generalized bi-complementarity problems for semi-monotone operators. Our results will extend the corresponding results in [100].

Furthermore, in this thesis, we shall introduce the concept of bi-quasi-monotone operators. As applications of bi-quasi-monotone operators, we shall obtain some results on existence theorems for generalized bi-quasi-variational inequalities in locally convex Hausdorff topological vector spaces.

Finally, even though we have some results for demi-operators, generalized quasi-monotone, generalized quasi-semi-monotone, bi-quasi-semi-monotone, and hemi-continuous operators on generalized variational inequalities or generalized quasi-variational inequalities, they have not been included here. We have completed some work on these topics and wish to continue to work on these topics soon. Moreover, we do not cover the topics on generalized KKM (in short G-KKM) maps, minimax inequalities and existence theorems of equilibria for \mathcal{G}_C^L -majorized correspondences in generalized convex (or G-convex) spaces for which we refer to M. S. R. Chowdhury [26], M. S. R. Chowdhury and K.-K. Tan [28]-[29] and some references therein.

Chapter 2

Generalizations of Ky Fan's Minimax Inequality

2.1 Introduction

In Chapter 1 of this thesis, we have given a brief history of Ky Fan's minimax inequality [48]. Indeed, Ky Fan's minimax inequality has become a versatile tool in nonlinear functional analysis [48], convex analysis, game theory and economic theory [3]. There have been numerous generalizations of Ky Fan's minimax inequality by weakening the compactness assumption or the convexity assumption; e.g., due to Allen [1], Bae-Kim-Tan [6], Brézis-Nirenberg-Stampacchia [16], Ding and Tan [39], Shih and Tan [91], Tan [104], Tan and Yuan [109], Yen [112] and Fan himself [49].

Horvath obtained some generalizations of Fan's Geometric Lemma and his minimax inequality in [60] and [61] in 1983 and 1987 respectively, by replacing convexity assumption with topological properties: pseudo-convexity and contractibility. In 1988, Bardaro and Ceppitelli [8] further generalized Ky Fan's minimax inequality into H -spaces, which are topological spaces, by utilizing Horvath's concept.

Following this line, a number of generalizations of Ky Fan's minimax inequality were given by Horvath [62], Bardaro and Ceppitelli [10], Ding and Tan [40], Ding, Kim and Tan [37]-[38], Chang and Ma [25], Park [81], Tarafdar [110] and Tan, Yu and Yuan [108]

in H -spaces which do not have a linear structure.

Recently, Park and Kim introduced the concept of a generalized convex (or G -convex) space in [86] which is a generalization of convexity in vector spaces, Horvath's pseudo-convex spaces [60], c -structure [62] and H -spaces [8]-[10]. G -convex spaces are adequate to establish theories on fixed points, coincidence points, KKM maps, G -KKM maps, minimax inequalities, equilibrium existence theorems, intersection theorems, variational inequalities, best approximations and many others. For details, see [28], [29], [82], [83], [84], [85], [86], [107].

On the other hand, for applications, various generalizations of the classical KKM Theorem and Sperner's Lemma [101] have been given by Fan [45], [46], [47] and [50], Ding and Tan [40], Gale [54], Idzik and Tan [65], Shapley [89], Shih and Tan [95], [96], [97], Ichiishi [63], and Ichiishi and Idzik [64].

In Section 2 of this chapter, we shall obtain a new minimax inequality in topological vector spaces which generalizes the celebrated 1972 Ky Fan's minimax inequality [48, Theorem 1] in several respects. We shall establish that this minimax inequality is equivalent to all the minimax inequalities in [109].

In Section 3, we shall obtain a KKM type lemma. This lemma will not be a direct generalization of the classical KKM Theorem since convexity of the underlying subset is needed but which is not required in the KKM Theorem. But we shall observe that in all practical applications of the classical KKM Theorem the underlying subset is always convex. Hence for all practical applications, our KKM type lemma of Section 3 will be more general. Moreover, we shall apply this lemma to generalize and extend Brézis-Nirenberg-Stampacchia's generalization [16] of Ky Fan's minimax inequality [48]. This minimax inequality will also generalize the corresponding minimax inequalities in [27] and [91]. As a special case of this minimax inequality, a third new minimax inequality will be obtained. Four fixed point theorems and four equivalent formulations of the third minimax inequality will also be obtained.

2.2 Generalization of Ky Fan's Minimax Inequality in topological vector spaces

First we introduce and recall some notations and definitions. Throughout this thesis \mathbf{R} denotes the set of all real numbers and $\mathbf{R}^+ = \{r \in \mathbf{R} : r > 0\}$. If A is a set, we shall denote by 2^A the family of all non-empty subsets of A and by $\mathcal{F}(A)$ the family of all non-empty finite subsets of A . If A is a subset of a topological space X , we shall denote by $\text{int}_X(A)$ the interior of A in X and by $\text{cl}_X(A)$ the closure of A in X . If A is a subset of a vector space, we shall denote by $\text{co}(A)$ the convex hull of A . If A is a non-empty subset of a topological vector space E and $G : A \rightarrow 2^E$ is a correspondence, then $\text{co}G : A \rightarrow 2^E$ is a correspondence defined by $(\text{co}G)(x) = \text{co}(G(x))$ for each $x \in A$.

Let X and Y be subsets of a vector space E such that $\text{co}(X) \subset Y$. Then $F : X \rightarrow 2^Y$ is called a *KKM-map* if for each $A \in \mathcal{F}(X)$, $\text{co}(A) \subset \bigcup_{x \in A} F(x)$. Note that if F is a KKM-map, then $x \in F(x)$ for all $x \in X$. In this thesis, topological vector spaces are not assumed to be Hausdorff unless it is explicitly stated.

We shall need Ky Fan's following infinite dimensional generalization [44, Lemma 1] of the classical Knaster-Kuratowski-Mazurkiewicz Theorem [72]:

Theorem 2.2.1 *Let E be a topological vector space, X and Y be non-empty subsets of E such that $X \subset Y$ and Y is convex. Suppose $F : X \rightarrow 2^Y$ is such that*

- (a) *F is a KKM-map;*
- (b) *for each $x \in X$, $F(x)$ is closed in Y ;*
- (c) *there exists $x_0 \in X$ such that $F(x_0)$ is compact.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Theorem 2.2.1 as stated above is slightly more general than Ky Fan originally stated in [44]. This was observed by Ding and Tan in [39, p.234].

We now state Ky Fan's minimax inequality [48, Theorem 1].

Theorem 2.2.2 *Let E be a topological vector space and X a non-empty compact convex subset of E . Let f be a real-valued function defined on $X \times X$ such that*

- (a) *for each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X ;*
- (b) *for each fixed $y \in X$, $f(x, y)$ is a quasi-concave function of x on X .*

Then the minimax inequality

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x)$$

holds.

The following result is another formulation of Theorem 2.2.2:

Theorem 2.2.3 *Let E be a topological vector space and X be a non-empty compact convex subset of E . Let f be a real-valued function defined on $X \times X$ such that*

- (a) *for each $x \in X$, $f(x, x) \leq 0$;*
- (b) *for each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X ;*
- (c) *for each fixed $y \in X$, $f(x, y)$ is a quasi-concave function of x on X .*

Then there exists $\hat{y} \in X$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

We remark here that in stating Theorems 2.2.1, 2.2.2 and 2.2.3 above, the space E is not required to be Hausdorff as Ky Fan originally stated. This fact was observed in Ding and Tan's paper [39].

In this section we shall obtain a new minimax inequality as a generalization of Ky Fan's minimax inequality to non-compact sets.

We shall begin with the following result:

Theorem 2.2.4 *Let E be a topological vector space, X be a non-empty convex subset of E and $f, g : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ be such that*

- (a₁) *for each $x, y \in X$, $f(x, y) > 0$ implies $g(x, y) > 0$;*
- (b₁) *for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semicontinuous on non-empty compact subsets of X ;*
- (c₁) *for each $A \in \mathcal{F}(X)$ and for each $y \in \text{co}(A)$, $\min_{x \in A} g(x, y) \leq 0$;*

(d₁) there exist a non-empty closed and compact subset K of X and $x_0 \in K$ such that $g(x_0, y) > 0$ for all $y \in X \setminus K$.

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

Proof: Define $F : X \rightarrow 2^K$ by

$$F(x) = \{y \in K : f(x, y) \leq 0\} \text{ for all } x \in X.$$

Note that by (b₁), each $F(x)$ is closed in K . We shall first show that the family $\{F(x) : x \in X\}$ has the finite intersection property. Indeed, let $\{x_1, \dots, x_n\}$ be any finite subset of X . Set $C = \text{co}(\{x_0, x_1, \dots, x_n\})$, then C is non-empty compact convex. Note that by (c₁), $g(x, x) \leq 0$ for all $x \in X$. Define $G : C \rightarrow 2^C$ by $G(x) = \{y \in C : g(x, y) \leq 0\}$ for all $x \in C$. We observe that: (i) if A is any finite subset of C , then $\text{co}(A) \subset \cup_{x \in A} G(x)$; for if this were false, then there exist a finite subset $\{z_1, \dots, z_m\}$ of C and $z \in \text{co}(\{z_1, \dots, z_m\})$ with $z \notin \cup_{j=1}^m G(z_j)$ so that $g(z_j, z) > 0$ for all $j = 1, \dots, m$ which contradicts (c₁); (ii) for each $x \in C$, $\text{cl}_C(G(x))$ is closed in C and is therefore also compact. By Theorem 2.2.1, $\cap_{x \in C} \text{cl}_C(G(x)) \neq \emptyset$. Take any $\bar{y} \in \cap_{x \in C} \text{cl}_C(G(x))$. Note that $x_0 \in C$ and $G(x_0) \subset K$ by (d₁); thus $\bar{y} \in \text{cl}_C(G(x_0)) \subset \text{cl}_X(G(x_0)) = \text{cl}_K(G(x_0)) \subset K$. Since we also have $\bar{y} \in \cap_{j=1}^n \text{cl}_C(G(x_j))$ and for each $j = 1, \dots, n$, $\text{cl}_C(G(x_j)) = \text{cl}_C(\{y \in C : g(x_j, y) \leq 0\}) \subset \text{cl}_C(\{y \in C : f(x_j, y) \leq 0\}) = \{y \in C : f(x_j, y) \leq 0\}$ by (a₁) and (b₁), we have $f(x_j, \bar{y}) \leq 0$ for all $j = 1, \dots, n$ and hence $\bar{y} \in \cap_{j=1}^n F(x_j)$. Therefore $\{F(x) : x \in X\}$ has the finite intersection property.

By compactness of K , $\cap_{x \in X} F(x) \neq \emptyset$. Take any $\hat{y} \in \cap_{x \in X} F(x)$, then $\hat{y} \in K$ and $f(x, \hat{y}) \leq 0$ for all $x \in X$. \square

For another generalization of Theorem 2.2.3 (and also generalizations of the corresponding minimax inequalities in [16] and [91]), we refer to Section 2.3.

The following fixed point theorem is equivalent to Theorem 2.2.4:

Theorem 2.2.5 *Let E be a topological vector space, X be a non-empty convex subset of E and $F, G : X \rightarrow 2^X \cup \{\emptyset\}$ be such that*

(a₂) *for each $x \in X$, $F(x) \subset G(x)$;*

(b₂) for each $x \in X$, $F^{-1}(x)$ is compactly open (i.e., $F^{-1}(x) \cap C$ is open in C for each non-empty compact subset C of X);

(c₂) there exist a non-empty closed and compact subset K of X and $x_0 \in K$ such that $X \setminus K \subset G^{-1}(x_0)$;

(d₂) for each $x \in K$, $F(x) \neq \emptyset$,

(e₂) for each $x \in X$, $G(x)$ is convex.

Then there exists $\bar{y} \in X$ such that $\bar{y} \in G(\bar{y})$.

To show Theorem 2.2.4 implies Theorem 2.2.5:

Define $f, g : X \times X \rightarrow \mathbf{R}$ by

$$f(x, y) = \begin{cases} 1 & \text{if } x \in F(y), \\ 0 & \text{if } x \notin F(y), \end{cases}$$

$$g(x, y) = \begin{cases} 1 & \text{if } x \in G(y), \\ 0 & \text{if } x \notin G(y) \end{cases}$$

for all $x, y \in X$. It is easy to see that the conditions (a₁), (b₁) and (d₁) of Theorem 2.2.4 are satisfied. If the hypothesis (c₁) of Theorem 2.2.4 is also satisfied, then by Theorem 2.2.4, there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$. It follows that $F(\hat{y}) = \emptyset$ which is impossible. Thus the hypothesis (c₁) of Theorem 2.2.4 does not hold. Hence there exist $A \in \mathcal{F}(X)$ and $\bar{y} \in \text{co}(A)$ such that $\min_{x \in A} g(x, \bar{y}) > 0$ so that $x \in G(\bar{y})$ for all $x \in A$. Therefore $\bar{y} \in \text{co}(A) \subset G(\bar{y})$ by (e₂). \square

To show Theorem 2.2.5 implies Theorem 2.2.4:

Define $F, G : X \rightarrow 2^X \cup \{\emptyset\}$ by $F(y) = \{x \in X : f(x, y) > 0\}$ and $G(y) = \text{co}(\{x \in X : g(x, y) > 0\})$ for all $y \in X$. It is easy to see that the conditions (a₂), (b₂), (c₂) and (e₂) of Theorem 2.2.5 are satisfied. If the hypothesis (d₂) of Theorem 2.2.5 is also satisfied, then by Theorem 2.2.5, there exists $\bar{y} \in X$ such that $\bar{y} \in G(\bar{y})$. But then there exist $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in [0, 1]$ such that $g(x_i, \bar{y}) > 0$ for all $i = 1, \dots, n$, $\sum_{i=1}^n \lambda_i = 1$ and $\bar{y} = \sum_{i=1}^n \lambda_i x_i$. This contradicts (c₁) because $\bar{y} \in \text{co}(A)$, where $A = \{x_1, \dots, x_n\}$. Hence the hypothesis (d₂) of Theorem 2.2.5 does not hold.

Thus there exists $\hat{y} \in K$ such that $F(\hat{y}) = \emptyset$. It follows that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

□

Note that Theorem 2.2.5 is Theorem 2.4' in [109].

Clearly, Theorem 2.2.4 implies the following result which is Theorem 2.2 in [109]:

Theorem 2.2.6 *Let X be a non-empty convex subset of a topological vector space and $\phi, \psi : X \times X \rightarrow \mathbf{R} \cup \{-\infty, \infty\}$ be such that*

(a) $\phi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times X$;

(b) for each fixed $x \in X$, $y \mapsto \phi(x, y)$ is lower semi-continuous on non-empty compact subsets of X ;

(c) for each $A \in \mathcal{F}(X)$ and for each $y \in \text{co}(A)$, $\min_{x \in A} \psi(x, y) \leq 0$;

(d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \setminus K$.

Then there exists $y \in K$ such that $\phi(x, y) \leq 0$ for all $x \in X$.

It is shown in [109] that Theorem 2.2.6 implies the following result which is Theorem 2.4 in [109]:

Theorem 2.2.7 *Let X be a non-empty convex subset of a topological vector space and $\phi, \psi : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ be such that*

(a) $\phi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times X$ and $\psi(x, x) \leq 0$ for each $x \in X$;

(b) for each fixed $x \in X$, $y \mapsto \phi(x, y)$ is lower semicontinuous on non-empty compact subsets of X ;

(c) for each fixed $y \in X$, the set $\{x \in X : \psi(x, y) > 0\}$ is convex;

(d) there exist a non-empty closed and compact subset K of X and a point $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \setminus K$.

Then there exists $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$.

It is shown in [109] that Theorem 2.2.7 is equivalent to Theorem 2.2.5. Thus Theorems 2.2.4, 2.2.5 (Theorem 2.4' in [109]), 2.2.6 (Theorem 2.2 in [109]) and 2.2.7 (Theorem 2.4 in [109]) above are all equivalent and are also equivalent to Theorems 2.2', 2.2'', 2.3''',

2.4'', 2.4''', 2.4'''' and 2.4''''' in [109]. Note however that the equivalence of Theorem 2.2 in [109] and Theorem 2.4 in [109] was not established in [109]. Note also that Theorem 2.2.6 does not imply Theorem 2.2.4 directly and Theorem 2.2.7 does not imply Theorems 2.2.4 and 2.2.6 directly. For applications to existence of equilibrium points of generalized games, we refer to [109].

2.3 Further generalization of Ky Fan's Minimax Inequality in topological vector spaces

The purpose of this section is to present a further generalization of Ky Fan's minimax inequality to non-compact sets.

We shall begin with the following result which is Lemma 1 of Brézis, Nirenberg and Stampacchia in [16, p.294]:

Lemma 2.3.1 *Let E be a Hausdorff topological vector space, X, Y be non-empty subsets of E and $F : X \rightarrow 2^Y$ be a KKM-map such that*

- (a) $cl_Y F(x_0)$ is compact for some $x_0 \in X$;
- (b) for each $x \in X$, the intersection of $F(x)$ with any finite dimensional subspace L of E is closed in L ;
- (c) for each convex subset D of E ,

$$(cl_Y(\bigcap_{x \in X \cap D} F(x))) \cap D = (\bigcap_{x \in X \cap D} F(x)) \cap D.$$

Then

$$\bigcap_{x \in X} F(x) \neq \emptyset.$$

Lemma 2.3.1 as stated above is slightly more general than Brézis, Nirenberg and Stampacchia originally stated in [16, p.294]. This was observed by K.-K. Tan in [106].

Now, we shall establish the following result:

Lemma 2.3.2 *Let E be a topological vector space, X be a non-empty convex subset of E . Let $F : X \rightarrow 2^X$ be a KKM-map such that*

- (a) $cl_X F(x_0)$ is compact for some $x_0 \in X$;
- (b) for each $A \in \mathcal{F}(X)$ with $x_0 \in A$ and each $x \in co(A)$, $F(x) \cap co(A)$ is closed in $co(A)$ and
- (c) for each $A \in \mathcal{F}(X)$ with $x_0 \in A$,

$$(cl_X(\bigcap_{x \in co(A)} F(x))) \cap co(A) = (\bigcap_{x \in co(A)} F(x)) \cap co(A).$$

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Proof: Fix any $A \in \mathcal{F}(X)$ with $x_0 \in A$. Define $G_A : \text{co}(A) \rightarrow 2^{\text{co}(A)}$ by $G_A(x) = F(x) \cap \text{co}(A)$ for each $x \in \text{co}(A)$.

Now, for each $x \in \text{co}(A)$, $G_A(x)$ is non-empty since F is a KKM map and closed in $\text{co}(A)$ by (b). Note that $\text{co}(A)$ is compact. Thus each $G_A(x)$ is also compact. For each $B \in \mathcal{F}(\text{co}(A))$ we have $B \in \mathcal{F}(X)$ as $\text{co}(A) \subset X$ and so $\text{co}(B) \subset \bigcup_{x \in B} F(x)$. But $\text{co}(B) \subset \text{co}(A)$; it follows that

$$\text{co}(B) \subset \left(\bigcup_{x \in B} F(x) \right) \cap \text{co}(A) = \bigcup_{x \in B} (F(x) \cap \text{co}(A)) = \bigcup_{x \in B} G_A(x).$$

Thus G_A is a KKM-map on $\text{co}(A)$. Hence by Theorem 2.2.1, we have

$$\bigcap_{x \in \text{co}(A)} G_A(x) \neq \emptyset, \quad \text{i.e.,} \quad \bigcap_{x \in \text{co}(A)} F(x) \cap \text{co}(A) \neq \emptyset. \quad (2.1)$$

Let $\{E_i\}_{i \in I}$ be the family of all convex hulls of finite subsets of X containing the point x_0 , partially ordered by \subset .

Now, for each $i \in I$, let $E_i = \text{co}(A_i)$, where $A_i \in \mathcal{F}_0(X) =$ the family of all non-empty finite subsets of X containing the point x_0 .

By (2.1), for each $i \in I$, $\bigcap_{x \in E_i} F(x) \cap E_i \neq \emptyset$. Fix any $u_i \in \bigcap_{x \in E_i} F(x) \cap E_i$. For each $i \in I$, let

$$\Phi_i = \{u_j \mid j \geq i, j \in I\}.$$

Clearly, (i) $\{\Phi_i \mid i \in I\}$ has the finite intersection property and (ii) $\Phi_i \subset F(x_0)$ for all $i \in I$. Then $\text{cl}_X \Phi_i \subset \text{cl}_X F(x_0)$ for all $i \in I$. By compactness of $\text{cl}_X F(x_0)$, $\bigcap_{i \in I} \text{cl}_X \Phi_i \neq \emptyset$. Choose any $\hat{x} \in \bigcap_{i \in I} \text{cl}_X \Phi_i$. Note that for any $i \in I$ and for all $j \in I$ with $j \geq i$,

$$u_j \in \bigcap_{x \in E_j} F(x) \cap E_j \subset \left(\bigcap_{x \in E_i} F(x) \cap E_j \right) \subset \bigcap_{x \in E_i} F(x).$$

Therefore

$$\Phi_i \subset \bigcap_{x \in E_i} F(x). \quad (2.2)$$

Now, for any $x \in X$, there exists $i_0 \in I$ such that $x, \hat{x} \in E_{i_0}$. Therefore for all $i \geq i_0$,

we have $x, \hat{x} \in E_{i_0} \subset E_i$. It follows that for all $i \geq i_0$,

$$\begin{aligned} \hat{x} \in E_i \cap \text{cl}_X \Phi_i &\subset E_i \cap (\text{cl}_X \bigcap_{z \in E_i} F(z)) \\ &= (\bigcap_{z \in E_i} F(z)) \cap E_i \quad (\text{by (c)}) \\ &= (\bigcap_{z \in E_i} F(z) \cap E_i) \subset F(x). \end{aligned}$$

Thus $\hat{x} \in F(x)$ for all $x \in X$. Hence $\bigcap_{x \in X} F(x) \neq \emptyset$. \square

Under the hypotheses of Lemma 2.3.2, we see that if for each $x \in X$ and each finite dimensional subspace L of E , $F(x) \cap L$ is closed in L , then for each $A \in \mathcal{F}(X)$ with $x_0 \in A$ and each $x \in \text{co}(A)$, $F(x) \cap \text{co}(A)$ is also closed in $\text{co}(A)$. The following example shows that the converse is not true in general.

Example 2.3.3 Let $E = \mathbf{R}^2$. Consider the following non-empty convex subset X of E :

$$X = \{(u, v) \in \mathbf{R}^2 \mid 0 < u < 1 \text{ and } 0 < v \leq 1 - u\} \cup \{(u, v) \in \mathbf{R}^2 \mid u = 0 \text{ and } 0 \leq v \leq 1\}.$$

Fix $x_0 = (\frac{1}{2}, \frac{1}{2}) \in X$. For each $x \in X$ with $x \neq (0, 0)$ and $x \neq x_0$, let A_x denote the following set:

$A_x =$ the closed region in X bounded by the line $v = 1 - u$ and the line passing through the point x and parallel to the line $v = 1 - u$.

Now, we define $F : X \rightarrow 2^X$ by

$$F(x) = \begin{cases} A_x \cup \{(0, 0)\} \cup \{(\frac{1}{n+2}, \frac{1}{n+2}) : n = 1, 2, 3, \dots\}, & \text{if } x \neq (0, 0) \text{ and } x \neq x_0; \\ X, & \text{if } x = (0, 0); \\ \{(0, 0)\} \cup \{(\frac{1}{n+1}, \frac{1}{n+1}) : n = 1, 2, \dots\}, & \text{if } x = x_0. \end{cases}$$

Then for each $A \in \mathcal{F}(X)$ with $x_0 \in A$ and for each $x \in \text{co}(A)$, $F(x) \cap \text{co}(A)$ is closed in $\text{co}(A)$. However, consider $L = \mathbf{R}^2$ and $x = (0, 0)$; then $F(x) \cap L = F((0, 0)) \cap \mathbf{R}^2 = X$ is not closed in L . Note that F is a KKM-map such that $\text{cl}_X F(x_0) = F(x_0)$ is compact and the condition (c) of Lemma 2.3.2 is also satisfied. Thus our Lemma 2.3.2 is applicable but Lemma 2.3.1 is not.

We remark here that Lemma 2.3.1 and Lemma 2.3.2 are not comparable. Note that when $X = Y$ and is convex then Lemma 2.3.2 improves Lemma 2.3.1. However, in all

applications of Lemma 1 in [44] or Lemma 2.3.1, the sets X and Y are equal and X is always assumed to be convex.

We shall now establish the following minimax inequality:

Theorem 2.3.4 *Let E be a topological vector space, X be a non-empty convex subset of E , $h : X \rightarrow \mathbf{R}$ be lower semicontinuous on $co(A)$ for each $A \in \mathcal{F}(X)$ and $f : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ be such that*

(a) *for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, $y \mapsto f(x, y)$ is lower semicontinuous on $co(A)$;*

(b) *for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, $\min_{x \in A} [f(x, y) + h(y) - h(x)] \leq 0$;*

(c) *for each $A \in \mathcal{F}(X)$ and each $x, y \in co(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y with*

$$f(tx + (1-t)y, y_\alpha) + h(y_\alpha) - h(tx + (1-t)y) \leq 0 \text{ for all } \alpha \in \Gamma \text{ and all } t \in [0, 1],$$

we have $f(x, y) + h(y) - h(x) \leq 0$;

(d) *there exist a non-empty closed and compact subset K of X and $x_0 \in K$ such that $f(x_0, y) + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq h(x) - h(\hat{y})$ for all $x \in X$.

Proof: Define $F : X \rightarrow 2^X$ by

$$F(x) = \{y \in X : f(x, y) + h(y) - h(x) \leq 0\} \text{ for each } x \in X.$$

If F is not a KKM -map, then for some finite subset $\{x_1, \dots, x_n\}$ of X and $\alpha_i \geq 0$ for $i = 1, \dots, n$ with $\sum_{i=1}^n \alpha_i = 1$, we have $\bar{y} = \sum_{i=1}^n \alpha_i x_i \notin \bigcup_{i=1}^n F(x_i)$. Thus $f(x_i, \bar{y}) + h(\bar{y}) - h(x_i) > 0$ for $i = 1, \dots, n$ so that

$$\min_{1 \leq i \leq n} [f(x_i, \bar{y}) + h(\bar{y}) - h(x_i)] > 0,$$

which contradicts the assumption (b). Hence $F : X \rightarrow 2^X$ is a KKM -map. Moreover we have,

(i) $F(x_0) \subset K$ by (d), so that $cl_X F(x_0) \subset cl_X K = K$ and hence $cl_X F(x_0)$ is compact in X ;

(ii) for each $A \in \mathcal{F}(X)$ with $x_0 \in A$ and each $x \in \text{co}(A)$,

$$\begin{aligned} F(x) \cap \text{co}(A) &= \{y \in \text{co}(A) : f(x, y) + h(y) - h(x) \leq 0\} \\ &= \{y \in \text{co}(A) : f(x, y) + h(y) \leq h(x)\} \end{aligned}$$

is closed in $\text{co}(A)$ by (a) and the fact that h is lower semicontinuous on $\text{co}(A)$;

(iii) for each $A \in \mathcal{F}(X)$ with $x_0 \in A$, if $y \in (\text{cl}_X(\bigcap_{x \in \text{co}(A)} F(x))) \cap \text{co}(A)$, then $y \in \text{co}(A)$ and there is a net $\{y_\alpha\}_{\alpha \in \Gamma}$ in $\bigcap_{x \in \text{co}(A)} F(x)$ such that $y_\alpha \rightarrow y$. For each $x \in \text{co}(A)$, since $tx + (1-t)y \in \text{co}(A)$ for all $t \in [0, 1]$, we have $y_\alpha \in F(tx + (1-t)y)$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$. This implies that $f(tx + (1-t)y, y_\alpha) + h(y_\alpha) - h(tx + (1-t)y) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$ so that by (c), $f(x, y) + h(y) - h(x) \leq 0$; it follows that $y \in (\bigcap_{x \in \text{co}(A)} F(x)) \cap \text{co}(A)$. Hence, $(\text{cl}_X(\bigcap_{x \in \text{co}(A)} F(x))) \cap \text{co}(A) = (\bigcap_{x \in \text{co}(A)} F(x)) \cap \text{co}(A)$.

Hence by Lemma 2.3.2 we have $\bigcap_{x \in X} F(x) \neq \emptyset$. Then there exists $\hat{y} \in \bigcap_{x \in X} F(x)$, so that $f(x, \hat{y}) + h(\hat{y}) - h(x) \leq 0$ for all $x \in X$. By (d), \hat{y} necessarily belongs to K . \square

When $h \equiv 0$, Theorem 2.3.4 reduces to the following:

Theorem 2.3.5 *Let E be a topological vector space, X be a non-empty convex subset of E and $f : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ be such that*

(a) *for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $y \mapsto f(x, y)$ is lower semicontinuous on $\text{co}(A)$;*

(b) *for each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, $\min_{x \in A} f(x, y) \leq 0$;*

(c) *for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y with $f(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$, we have $f(x, y) \leq 0$;*

(d) *there exist a non-empty closed and compact subset K of X and $x_0 \in K$ such that $f(x_0, y) > 0$ for all $y \in X \setminus K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

Next we show that Theorem 2.3.5 implies the following minimax inequality:

Theorem 2.3.6 *Let E be a topological vector space, X be a non-empty convex subset of E and $f : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ be such that*

(a) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $y \mapsto f(x, y)$ is lower semicontinuous on $\text{co}(A)$;

(b) for each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, $\min_{x \in A} f(x, y) \leq 0$;

(c) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y with $f(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$, we have $f(x, y) \leq 0$;

(d) there exist a non-empty closed and compact subset K of X and $x_0 \in K$ such that whenever $\sup_{x \in X} f(x, x) < \infty$, $f(x_0, y) > \sup_{x \in X} f(x, x)$ for all $y \in X \setminus K$.

Then the minimax inequality

$$\min_{y \in K} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x)$$

holds.

Proof: Let $t = \sup_{x \in X} f(x, x)$. Clearly, we may assume that $t < +\infty$. Define for any $x, y \in X$, $g(x, y) = f(x, y) - t$. Then g satisfies all the hypotheses of Theorem 2.3.5 when f is replaced by g . Hence by Theorem 2.3.5, there exists an $\hat{y} \in K$ such that $g(x, \hat{y}) \leq 0$ for all $x \in X$. This implies $f(x, \hat{y}) \leq t$ for all $x \in X$, so that $\sup_{x \in X} f(x, \hat{y}) \leq t$ and therefore

$$\min_{y \in K} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, \hat{y}) \leq t = \sup_{x \in X} f(x, x),$$

i.e.,

$$\min_{y \in K} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x). \square$$

Theorem 2.3.6 generalizes Theorem 2.2.2 in several ways.

Theorem 2.3.7 Let E be a topological vector space, X be a non-empty convex subset of E . Let $f, g : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ be such that

(a) $f(x, y) \leq g(x, y)$ for all $x, y \in X$ and $g(x, x) \leq 0$ for all $x \in X$;

(b) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $y \mapsto f(x, y)$ is lower semicontinuous on $\text{co}(A)$;

(c) for each $y \in X$, the set $\{x \in X : g(x, y) > 0\}$ is convex;

(d) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y with $f(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$, we have $f(x, y) \leq 0$;

(e) there exist a non-empty closed and compact subset K of X and $x_0 \in K$ such that $f(x_0, y) > 0$ for all $y \in X \setminus K$.

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

Proof: It is easy to see that the conditions (a) and (c) here imply the condition (b) of Theorem 2.3.5 so that the conclusion follows. \square

Note that Theorem 2.3.7 generalizes Theorem 1 of Shih and Tan in [91, pp.280-282].

Theorem 2.3.8 Let E be topological vector space, C be a non-empty closed convex subset of E and $f : C \times C \rightarrow \mathbf{R}$ be such that

(a) $f(x, x) \leq 0$ for all $x \in C$;

(b) for each $A \in \mathcal{F}(C)$ and each fixed $x \in \text{co}(A)$, $y \mapsto f(x, y)$ is lower semicontinuous on $\text{co}(A)$;

(c) for each $y \in C$, the set $\{x \in C : f(x, y) > 0\}$ is convex;

(d) for each $A \in \mathcal{F}(C)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in C converging to y with $f(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$, we have $f(x, y) \leq 0$;

(e) there exist a non-empty closed and compact subset L of E and $x_0 \in C \cap L$ such that $f(x_0, y) > 0$ for all $y \in C \setminus L$.

Then there exists $\hat{y} \in C \cap L$ such that $f(x, \hat{y}) \leq 0$ for all $x \in C$.

Proof: Let $f = g$, $K = C \cap L$ and $X = C$ in Theorem 2.3.7, the conclusion follows.

\square

Theorem 2.3.8 improves Theorem 1 of Brézis-Nirenberg-Stampacchia in [16]. Note that if the compact set L is a subset of C , C is not required to be closed in E in Theorem 2.3.8. Note also that in Theorem 1 of [16], the set C was not assumed to be closed in E . However this is false in general as is observed by the following example in [106, Example 1.3.14].

Example 2.3.9 Let $E = \mathbf{R}^2$, $C = \{(u, v) \in \mathbf{R}^2 : u^2 + v^2 \leq 1, u, v > 0\}$, $L = \{(u, v) \in \mathbf{R}^2 : u^2 + v^2 \leq \frac{1}{4}\}$, $x_0 = (\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$, and $f : C \times C \rightarrow \mathbf{R}$ be defined by $f(x, y) = \|y\| - \|x\|$ for all $x, y \in C$. Then all the hypotheses of Theorem 1 in [16] are satisfied. However there does not exist $\hat{y} \in C \cap L$ such that $f(x, \hat{y}) \leq 0$ for all $x \in C$.

Following the ideas of Ky Fan [48, pp.104-106], Ding and Tan [39] and Tan and Yuan [109, pp.486-489], we shall obtain several equivalent formulations of Theorem 2.3.5 and fixed point theorems:

Theorem 2.3.5-A. (First Geometric Form) Let X be a non-empty convex subset of a topological vector space E and $N \subset X \times X$ be such that

(a₁) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, the set $\{y \in \text{co}(A) : (x, y) \in N\}$ is open in $\text{co}(A)$;

(b₁) for each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, there exists $x \in A$ such that $(x, y) \notin N$;

(c₁) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $(tx + (1-t)y, y_\alpha) \notin N$ for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $(x, y) \notin N$;

(d₁) there exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $(x_0, y) \in N$, for all $y \in X \setminus K$.

Then there exists $\hat{y} \in K$ such that the set $\{x \in X : (x, \hat{y}) \in N\} = \emptyset$.

Theorem 2.3.5-B. (Second Geometric Form) Let X be a non-empty convex subset of a topological vector space E and let $M \subset X \times X$ be such that

(a₂) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, the set $\{y \in \text{co}(A) : (x, y) \in M\}$ is closed in $\text{co}(A)$;

(b₂) for each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, there exists $x \in A$ such that $(x, y) \in M$;

(c₂) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $(tx + (1-t)y, y_\alpha) \in M$ for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $(x, y) \in M$;

(d₂) there exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $(x_0, y) \notin M$, for all $y \in X \setminus K$.

Then there exists a point $\hat{y} \in K$ such that $X \times \{\hat{y}\} \subset M$.

Theorem 2.3.5-C. (Maximal Element Version) Let X be a non-empty convex subset of a topological vector space E and let $G : X \rightarrow 2^X \cup \{\emptyset\}$ be a set-valued map such that

(a₃) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $G^{-1}(x) \cap \text{co}(A) = \{y \in \text{co}(A) : x \in G(y)\}$ is open in $\text{co}(A)$;

(b₃) for each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, there exists $x \in A$ such that $x \notin G(y)$;

(c₃) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $tx + (1-t)y \notin G(y_\alpha)$, for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $x \notin G(y)$;

(d₃) there exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $x_0 \in G(y)$, for all $y \in X \setminus K$.

Then there exists a point $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$.

Theorem 2.3.5-D. (Fixed Point Version) Let X be a non-empty convex subset of a topological vector space E and let $G : X \rightarrow 2^X \cup \{\emptyset\}$ be a set-valued map such that

(a₄) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $G^{-1}(x) \cap \text{co}(A)$ is open in $\text{co}(A)$;

(b₄) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $tx + (1-t)y \notin G(y_\alpha)$, for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $x \notin G(y)$;

(c₄) there exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $x_0 \in G(y)$, for all $y \in X \setminus K$;

(d₄) for each $y \in K$, $G(y) \neq \emptyset$.

Then there exists $y_0 \in X$ such that $y_0 \in \text{co}(G(y_0))$.

Theorem 2.3.5-D implies the following fixed point theorem:

Theorem 2.3.10-A. Let X be a non-empty convex subset of a topological vector space E and let $G : X \rightarrow 2^X \cup \{\emptyset\}$ be a set-valued map such that

(a₅) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $G^{-1}(x) \cap \text{co}(A)$ is open in $\text{co}(A)$;

(b₅) for each $y \in X$, $G(y)$ is convex;

(c₅) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $tx + (1-t)y \notin G(y_\alpha)$, for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $x \notin G(y)$;

(d₅) there exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $x_0 \in G(y)$, for all $y \in X \setminus K$;

(e₅) for each $y \in K$, $G(y) \neq \emptyset$.

Then there exists a point $y_0 \in X$ such that $y_0 \in G(y_0)$.

The following fixed point theorem is equivalent to Theorem 2.3.10-A.

Theorem 2.3.10-B Let X be a non-empty convex subset of a topological vector space E and let $Q : X \rightarrow 2^X \cup \{\emptyset\}$ be a set-valued map such that

(a₆) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $Q(x) \cap \text{co}(A)$ is open in $\text{co}(A)$;

(b₆) for each $y \in X$, $Q^{-1}(y)$ is convex;

(c₆) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $tx + (1-t)y \notin Q^{-1}(y_\alpha)$, for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $x \notin Q^{-1}(y)$;

(d₆) there exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $x_0 \in Q^{-1}(y)$, for all $y \in X \setminus K$;

(e₆) for each $y \in K$, $Q^{-1}(y) \neq \emptyset$.

Then there exists a point $y_0 \in X$ such that $y_0 \in Q(y_0)$.

The following fixed point theorem follows from Theorem 2.3.10-A:

Theorem 2.3.11. Let X be a non-empty convex subset of a topological vector space E and let $G : X \rightarrow 2^X \cup \{\emptyset\}$ be a set-valued map such that

(a₇) for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $G^{-1}(x) \cap \text{co}(A)$ is open in $\text{co}(A)$;

(b₇) for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $tx + (1-t)y \notin \text{co}G(y_\alpha)$, for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$,

we have $x \notin \text{co}G(y)$;

(c_7) there exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $x_0 \in \text{co}(G(y))$, for all $y \in X \setminus K$;

(d_7) for each $y \in K$, $G(y) \neq \emptyset$.

Then there exists $y_0 \in X$ such that $y_0 \in \text{co}(G(y_0))$.

Proof: Theorem 2.3.5 \Rightarrow Theorem 2.3.5-A:

Let $f : X \times X \rightarrow \mathbf{R}$ be such that

$$f(x, y) = \begin{cases} 1, & \text{if } (x, y) \in N; \\ 0, & \text{if } (x, y) \notin N. \end{cases}$$

Then we have the following.

(a) For each $\lambda \in \mathbf{R}$, each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, the set

$$\{y \in \text{co}(A) : f(x, y) \leq \lambda\} = \begin{cases} \emptyset, & \text{if } \lambda < 0; \\ \{y \in \text{co}(A) : (x, y) \notin N\}, & \text{if } 0 \leq \lambda < 1; \\ \text{co}(A), & \text{if } \lambda \geq 1. \end{cases}$$

is closed in $\text{co}(A)$. Thus for each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $y \mapsto f(x, y)$ is lower semicontinuous on $\text{co}(A)$.

(b) For each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, there exists $x \in A$ such that $(x, y) \notin N$. Thus $f(x, y) = 0$. Hence $\min_{x \in A} f(x, y) \leq 0$.

(c) By hypothesis (c_1), for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $f(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $f(x, y) \leq 0$.

(d) There exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $(x_0, y) \in N$, i.e., $f(x_0, y) > 0$ for all $y \in X \setminus K$.

Hence, all the hypotheses of Theorem 2.3.5 are satisfied. Therefore, by Theorem 2.3.5, there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$; i.e., there exists $\hat{y} \in K$ such that $\{x \in X : (x, \hat{y}) \in N\} = \emptyset$. \square

Proof: Theorem 2.3.5-A \Rightarrow Theorem 2.3.5:

Let $N = \{(x, y) \in X \times X : f(x, y) > 0\}$. Then we have the following.

(a₁) For each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, the set $\{y \in \text{co}(A) : f(x, y) > 0\}$ is open in $\text{co}(A)$. Hence the set $\{y \in \text{co}(A) : (x, y) \in N\}$ is open in $\text{co}(A)$.

(b₁) For each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, $\min_{x \in A} f(x, y) \leq 0$. Thus there exists $x \in A$ such that $f(x, y) \leq 0$, i.e., $(x, y) \notin N$;

(c₁) By condition (c), for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $(tx + (1-t)y, y_\alpha) \notin N$ for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $(x, y) \notin N$.

(d₁) By condition (d), there exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $(x_0, y) \in N$ for all $y \in X \setminus K$.

Hence, all the hypotheses of Theorem 2.3.5-A are satisfied. Therefore, by Theorem 2.3.5-A, there exists $\hat{y} \in K$ such that $\{x \in X : (x, \hat{y}) \in N\} = \emptyset$. Thus $(x, \hat{y}) \notin N$ for all $x \in X$. Hence $f(x, \hat{y}) \leq 0$ for all $x \in X$. \square

Proof: Theorem 2.3.5-A \Rightarrow Theorem 2.3.5-B:

Let $N = X \times X \setminus M$. Then we have the following.

(a₁) For each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, the set $\{y \in \text{co}(A) : (x, y) \in N\}$ is open in $\text{co}(A)$.

(b₁) For each $A \in \mathcal{F}(x)$ and each $y \in \text{co}(A)$, there exists $x \in A$ such that $(x, y) \notin N$.

(c₁) For each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $(tx + (1-t)y, y_\alpha) \notin N$ for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $(x, y) \notin N$.

(d₁) There exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $(x_0, y) \in N$, for all $y \in X \setminus K$.

Hence, all the hypotheses of Theorem 2.3.5-A are satisfied. Therefore, by Theorem 2.3.5-A, there exists $\hat{y} \in K$ such that $(x, \hat{y}) \notin N$ for all $x \in X$. Thus $(x, \hat{y}) \in M$ for all $x \in X$. Hence $X \times \{\hat{y}\} \subset M$. \square

Proof: Theorem 2.3.5-B \Rightarrow Theorem 2.3.5-A:

Let $M = X \times X \setminus N$. Then the proof is similar to the above proof and therefore, by Theorem 2.3.5-B, there exists $\hat{y} \in K$ such that $X \times \{\hat{y}\} \subset M$ and hence $(x, \hat{y}) \notin N$ for all $x \in X$, i.e., the set $\{x \in X : (x, \hat{y}) \in N\} = \emptyset$. \square

Proof: Theorem 2.3.5-B \Rightarrow Theorem 2.3.5-C:

Let $M = \{(x, y) \in X \times X : x \notin G(y)\}$. Then we have the following.

(a₂) For each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, the set $\{y \in \text{co}(A) : x \notin G(y)\} = \{y \in \text{co}(A) : (x, y) \in M\}$ is closed in $\text{co}(A)$.

(b₂) For each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, there exists $x \in A$ such that $x \notin G(y)$ so that $(x, y) \in M$.

(c₂) By condition (c₃), for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $(tx + (1-t)y, y_\alpha) \in M$ for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $(x, y) \in M$.

(d₂) By condition (d₃), there exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $(x_0, y) \notin M$ for all $y \in X \setminus K$.

Hence, all the hypotheses of Theorem 2.3.5-B are satisfied. Therefore, by Theorem 2.3.5-B, there exists $\hat{y} \in K$ such that $X \times \{\hat{y}\} \subset M$. Thus $x \notin G(\hat{y})$ for all $x \in X$. Hence $G(\hat{y}) = \emptyset$. \square

Proof: Theorem 2.3.5-C \Rightarrow Theorem 2.3.5-B:

Let $G : X \rightarrow 2^X$ be defined by $G(y) = \{x \in X : (x, y) \notin M\}$ for all $y \in X$. Then we have the following.

(a₃) For each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, the set $\{y \in \text{co}(A) : (x, y) \notin M\} = \{y \in \text{co}(A) : x \in G(y)\}$ is open in $\text{co}(A)$.

(b₃) For each $A \in \mathcal{F}(X)$ and each $y \in \text{co}(A)$, there exists $x \in A$ such that $(x, y) \in M$ so that $x \notin G(y)$.

(c₃) By condition (c₂), for each $A \in \mathcal{F}(X)$ and each $x, y \in \text{co}(A)$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y such that $(tx + (1-t)y) \notin G(y_\alpha)$ for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$, we have $x \notin G(y)$.

(d₃) There exists a non-empty closed and compact subset K of X and $x_0 \in K$ such that $(x_0, y) \notin M$ for all $y \in X \setminus K$ so that $x_0 \in G(y)$ for all $y \in X \setminus K$.

Hence, all the hypotheses of Theorem 2.3.5-C are satisfied. Therefore, by Theorem 2.3.5-C, there exists a point $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$. Thus $x \notin G(\hat{y})$ for all $x \in X$. Hence $(x, \hat{y}) \in M$ for all $x \in X$, i.e., $X \times \{\hat{y}\} \subset M$. \square

Proof: Theorem 2.3.5-C \Rightarrow Theorem 2.3.5-D:

By Theorem 2.3.5-C, there exist $A \in \mathcal{F}(X)$ and $y_0 \in \text{co}(A)$ such that $x \in G(y_0)$ for all $x \in A$. Thus $y_0 \in \text{co}(A) \subset \text{co}(G(y_0))$. \square

Proof: Theorem 2.3.5-D \Rightarrow Theorem 2.3.5-C:

From the hypotheses of Theorem 2.3.5-C we see that the conditions $(a_4), (b_4)$ and (c_4) follow from the conditions $(a_3), (c_3)$ and (d_3) respectively. Suppose for each $y \in K$, $G(y) \neq \emptyset$. Then condition (d_4) of Theorem 2.3.5-D is satisfied. Hence all the hypotheses of Theorem 2.3.5-D are satisfied. Therefore by Theorem 2.3.5-D, there exists $y_0 \in X$ such that $y_0 \in \text{co}G(y_0)$. Thus there exist $x_1, \dots, x_n \in G(y_0)$ and $\lambda_1, \dots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that $y_0 = \sum_{i=1}^n \lambda_i x_i \in \text{co}G(y_0)$. Let $A = \{x_1, \dots, x_n\}$. Then $A \in \mathcal{F}(X)$ and $y_0 \in \text{co}(A)$. Hence by condition (b_3) of Theorem 2.3.5-C, there exists $x_i \in A$ such that $x_i \notin G(y_0)$, which is a contradiction. Hence there exists $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$. \square

Proof: Theorem 2.3.5-D \Rightarrow Theorem 2.3.10-A:

This is obvious, because by Theorem 2.3.5-D there exists $y_0 \in X$ such that $y_0 \in \text{co}G(y_0)$. But by (b_5) of Theorem 2.3.10-A, $G(y_0)$ is convex. Hence $y_0 \in G(y_0)$. \square

Proof: Theorem 2.3.10-A \Leftrightarrow Theorem 2.3.10-B:

(\Rightarrow) Let $G = Q^{-1}$, then $G^{-1} = Q$. Therefore by Theorem 2.3.10-A, there exists $y_0 \in X$ such that $y_0 \in G(y_0) = Q^{-1}(y_0)$. Hence $y_0 \in Q(y_0)$.

(\Leftarrow) Let $Q = G^{-1}$, then $Q^{-1} = G$. Therefore by Theorem 2.3.10-B, there exists $y_0 \in X$ such that $y_0 \in Q(y_0) = G^{-1}(y_0)$. Hence $y_0 \in G(y_0)$. \square

Proof: Theorem 2.3.10-A \Rightarrow Theorem 2.3.11:

Let $F(y) = \text{co}G(y)$. Then we have the following.

(a_5) For each $A \in \mathcal{F}(X)$ and each fixed $x \in \text{co}(A)$, $F^{-1}(x) \cap \text{co}(A)$ is open in $\text{co}(A)$. For, let $y \in (\text{co}G)^{-1}(x) \cap \text{co}(A)$, then $y \in \text{co}(A)$ and $x \in \text{co}(G(y))$. Let $y_1, \dots, y_n \in G(y)$ and $\lambda_1, \dots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that $x = \sum_{i=1}^n \lambda_i y_i$. Now for each $i = 1, \dots, n$ $G^{-1}(y_i) \cap \text{co}(A)$ is open in $\text{co}(A)$ and $y \in G^{-1}(y_i) \cap \text{co}(A)$ for all i . Let $U = \bigcap_{i=1}^n (G^{-1}(y_i) \cap \text{co}(A))$. Then U is an open neighbourhood of y in $\text{co}(A)$. If $z \in U$ then $z \in \text{co}(A)$ and $y_i \in G(z)$ for all $i = 1, \dots, n$ so that $x = \sum_{i=1}^n \lambda_i y_i \in \text{co}(G(z))$ and

hence $z \in (\text{co}G)^{-1}(x) \cap \text{co}(A)$ for all $z \in U$.

(b_5) For each $y \in X$, $F(y) = \text{co}(G(y))$ is convex;

Conditions (c_5) and (d_5) are obvious. But for each $y \in K$, $G(y) \neq \emptyset$ implies $F(y) \neq \emptyset$. Thus (e_5) holds. Therefore by Theorem 2.3.10-A, there exists $y_0 \in X$ such that $y_0 \in F(y_0) = \text{co}(G(y_0))$. \square

Chapter 3

Generalized Variational Inequalities

3.1 Introduction

If X is a non-empty subset of a topological vector space E and $T : X \rightarrow 2^{E^*}$, then the *generalized variational inequality problem* associated with X and T is to find $\hat{y} \in X$ such that the *generalized variational inequality* $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq 0$ for all $x \in X$ holds, or to find $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in X$ holds. When T is single-valued, a generalized variational inequality is called a *variational inequality*. Browder [19] and Hartman and Stampacchia [59] first introduced variational inequalities. Since then, there have been many generalizations, e.g., see [1], [6], [7], [16], [21], [42], [71], [91], [98], [99], [104], and [112], etc.

The purpose of this chapter is to present existence theorems for generalized variational inequalities with applications to existence theorems for generalized complementarity problems. Our main results are listed as Theorems 3.2.23, 3.2.28, 3.2.33, 3.3.12, 3.3.15, 3.4.4, 3.4.7, 3.5.1, 3.5.3, 3.6.4 and 3.6.9. This chapter is organized as follows:

In Section 2 of this chapter, we shall introduce the notions of lower hemi-continuous, upper hemi-continuous, h -quasi-monotone, quasi-monotone, h -quasi-semi-monotone, quasi-semi-monotone, quasi-nonexpansive and semi-nonexpansive operators. Some basic results

as well as some examples are also given.

Next, as applications of minimax inequalities of Section 2.2, we present some existence theorems for generalized variational inequalities and existence theorems for generalized complementarity problems. Our results extend or improve the corresponding results in the literature, e.g., see [6], [27] and [91]. Some results will also be obtained on surjectivity of monotone or semi-monotone operators.

In Section 3, we shall introduce the notions of weakly lower (respectively, upper) demi-continuous, strongly lower (respectively, upper) demi-continuous operators and the notion of quasi-monotone operators in more general settings. Some basic results and examples will also be given. As applications of these operators, we shall obtain some existence theorems on generalized variational inequalities in topological vector spaces and in non-reflexive Banach spaces. A result on surjectivity will also be obtained.

In Section 4, we shall introduce the notion of quasi-semi-monotone operators in more general settings and obtain some existence theorems for quasi-semi-monotone and upper demi-continuous operators on generalized variational inequalities in topological vector spaces and in non-reflexive Banach spaces. A result on surjectivity will also be obtained.

In Section 5, we shall obtain some existence theorems for the operators introduced in Section 3.3 and 3.4 on generalized variational inequalities in non-compact settings using escaping sequences introduced by Border in [15]. As applications, some results are obtained in non-reflexive Banach spaces.

In Section 6, the notions of h -pseudo-monotone, pseudo-monotone, h -demi-monotone and demi-monotone operators will be first introduced. Then, applying the minimax inequalities of Section 2.3, we shall present some existence theorems for generalized variational inequalities and existence theorems for generalized complementarity problems for pseudo-monotone and demi-monotone operators. Our results for demi-monotone operators extend the corresponding results in [6], [16], [27] and [91]. The results for pseudo-monotone operators generalize the corresponding results in [16] and extend those in [6], [27] and [91]. Some results will also be obtained on surjectivity of demi-monotone operators.

3.2 Generalized Variational Inequalities for Quasi-Monotone and Quasi-Semi-Monotone Operators

In this section, as applications of the minimax inequalities of Section 2.2, existence theorems for generalized variational inequalities and generalized complementarity problems are obtained in topological vector spaces for quasi-monotone and quasi-semi-monotone operators. Our results extend or improve the corresponding results in the literature, e.g., see [6] and [91]. Our results also improve and generalize the corresponding results in [27].

Let E be a topological vector space. We shall denote by E^* the continuous dual of E , by $\langle w, x \rangle$ the pairing between E^* and E for $w \in E^*$ and $x \in E$ and by $Re\langle w, x \rangle$ the real part of $\langle w, x \rangle$.

For each $x_0 \in E$, each non-empty subset A of E and each $\epsilon > 0$, let $W(x_0; \epsilon) := \{y \in E^* : |\langle y, x_0 \rangle| < \epsilon\}$ and $U(A; \epsilon) := \{y \in E^* : \sup_{x \in A} |\langle y, x \rangle| < \epsilon\}$. Let $\sigma\langle E^*, E \rangle$ be the topology on E^* generated by the family $\{W(x; \epsilon) : x \in E \text{ and } \epsilon > 0\}$ as a subbase for the neighbourhood system at 0 and $\delta\langle E^*, E \rangle$ be the topology on E^* generated by the family $\{U(A; \epsilon) : A \text{ is a non-empty bounded subset of } E \text{ and } \epsilon > 0\}$ as a base for the neighbourhood system at 0. We note that E^* , when equipped with the topology $\sigma\langle E^*, E \rangle$ or the topology $\delta\langle E^*, E \rangle$, becomes a locally convex Hausdorff topological vector space. Furthermore, for a net $\{y_\alpha\}_{\alpha \in \Gamma}$ in E^* and for $y \in E^*$, (i) $y_\alpha \rightarrow y$ in $\sigma\langle E^*, E \rangle$ if and only if $\langle y_\alpha, x \rangle \rightarrow \langle y, x \rangle$ for each $x \in E$ and (ii) $y_\alpha \rightarrow y$ in $\delta\langle E^*, E \rangle$ if and only if $\langle y_\alpha, x \rangle \rightarrow \langle y, x \rangle$ uniformly for $x \in A$ for each non-empty bounded subset A of E . The topology $\sigma\langle E^*, E \rangle$ (respectively, $\delta\langle E^*, E \rangle$) is called the *weak* topology* (respectively, the *strong topology*) on E^* . If $p \in E$, \tilde{p} is the linear functional on E^* defined by $\tilde{p}(f) = f(p)$ for each $f \in E^*$.

Let X be a non-empty subset of E . Then X is a *cone* in E if X is convex and $\lambda X \subset X$ for all $\lambda \geq 0$. If X is a cone in E , then $\widehat{X} = \{w \in E^* : Re\langle w, x \rangle \geq 0 \text{ for all } x \in X\}$ is also a cone in E^* , called the *dual cone* of X .

Let X be a non-empty subset of E and $T : X \rightarrow 2^{E^*}$. Then T is said to be

(i) *monotone* (on X) if for each $x, y \in X$, each $u \in T(x)$ and each $w \in T(y)$,

$Re\langle w - u, y - x \rangle \geq 0$;

(ii) *semi-monotone* [6, pp.236-237] (on X) if for each $x, y \in X$, $\inf_{u \in T(x)} Re\langle u, y - x \rangle \leq \inf_{w \in T(y)} Re\langle w, y - x \rangle$.

Let $y \in E$. Then the *inward set* of y with respect to X [56] is the set $I_X(y) = \{x \in E : x = y + r(u - y) \text{ for some } u \in X \text{ and } r > 0\}$. We shall denote by $\overline{I_X(y)}$ the closure of $I_X(y)$ in E .

Let X be a non-empty convex subset of E . Then for each $x, y \in X$, the line segment in X joining x and y is the set $\{z \in X : z = \lambda x + (1 - \lambda)y \text{ for all } \lambda \in [0, 1]\}$.

It is clear that if T is monotone, then T is semi-monotone. The converse is in general false, see Example 2 in [6]. Clearly, these two notions coincide for single-valued operators.

Let X and Y be topological spaces and $T : X \rightarrow 2^Y$. Then T is said to be

(i) *upper* (respectively, *lower*) *semicontinuous* at $x_0 \in X$ if for each open set G in Y with $T(x_0) \subset G$ (respectively, $T(x_0) \cap G \neq \emptyset$), there exists an open neighbourhood U of x_0 in X such that $T(x) \subset G$ (respectively, $T(x) \cap G \neq \emptyset$) for all $x \in U$;

(ii) *upper* (respectively, *lower*) *semicontinuous* on X if T is upper (respectively, lower) semicontinuous at each point of X ;

(iii) *continuous* on X if T is both lower and upper semicontinuous on X .

We shall need the following Kneser's minimax theorem [73] (see also Aubin [2, pp.40-41]):

Theorem 3.2.1 *Let X be a non-empty convex subset of a vector space and Y be a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, $f(x, y)$ is lower semicontinuous and convex on Y and for each fixed $y \in Y$, $f(x, y)$ is concave on X . Then*

$$\min_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \min_{y \in Y} f(x, y).$$

The following result is essentially Lemma 2 in [94] (see also [106, Lemma 2.4.1]) which was first given by Karamardian for single-valued operators in [68, Lemma 3.1]:

Lemma 3.2.2 *Let X be a cone in a topological vector space E and $T : X \rightarrow \mathfrak{2}^{E^*}$. Then the following statements are equivalent:*

- (a) *There exists $\hat{y} \in X$ such that $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq 0$ for all $x \in X$.*
- (b) *There exists $\hat{y} \in X$ such that $\operatorname{Re}\langle w, \hat{y} \rangle = 0$ for all $w \in T(\hat{y})$ and $T(\hat{y}) \subset \widehat{X}$.*

The following is essentially a result of S. C. Fang (e.g. see [23] and [94, p.59]) (see also [106, Lemma 2.4.2]):

Lemma 3.2.3 *Let X be a cone in a topological vector space E and $T : X \rightarrow \mathfrak{2}^{E^*}$. Then the following statements are equivalent:*

- (a) *There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in X$.*
- (b) *There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} \rangle = 0$ and $\hat{w} \in \widehat{X}$.*

The following simple result is Lemma 2.1.6 in [106]:

Lemma 3.2.4 *Let E be a topological vector space and A be a non-empty bounded subset of E . Let C be a non-empty strongly compact subset of E^* . Define $f : A \rightarrow \mathbf{R}$ by $f(x) = \min_{u \in C} \operatorname{Re}\langle u, x \rangle$ for all $x \in A$. Then f is weakly continuous on A .*

We shall begin with the following:

Definition 3.2.5 *Let E be a topological vector space, X be a non-empty subset of E and $T : X \rightarrow \mathfrak{2}^{E^*}$. Then T is said to be*

(a) *lower hemi-continuous on X if and only if for each $p \in E$, the function $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$, defined by*

$$f_p(z) = \sup_{u \in T(z)} \operatorname{Re}\langle u, p \rangle \text{ for each } z \in X,$$

is lower semicontinuous on X (if and only if for each $p \in E$, the function $g_p : X \rightarrow \mathbf{R} \cup \{-\infty\}$, defined by

$$g_p(z) = \inf_{u \in T(z)} \operatorname{Re}\langle u, p \rangle \text{ for each } z \in X,$$

is upper semicontinuous on X);

(b) upper hemi-continuous on X if and only if for each $p \in E$, the function $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$, defined by

$$f_p(z) = \sup_{u \in T(z)} \operatorname{Re}\langle u, p \rangle \text{ for each } z \in X,$$

is upper semicontinuous on X (if and only if for each $p \in E$, the function $g_p : X \rightarrow \mathbf{R} \cup \{-\infty\}$, defined by

$$g_p(z) = \inf_{u \in T(z)} \operatorname{Re}\langle u, p \rangle \text{ for each } z \in X,$$

is lower semicontinuous on X).

Note that if X is convex, then the notions of lower hemi-continuity along line segments in X and upper hemi-continuity along line segments in X are independent of the vector topology τ on E as long as τ is Hausdorff and the continuous dual E^* remains unchanged. Note also that if $T, S : X \rightarrow 2^{E^*}$ are lower (respectively, upper) hemi-continuous and $\alpha \in \mathbf{R}$, then $T + S$ and αT are also lower (respectively, upper) hemi-continuous.

Proposition 3.2.6 *Let E be a topological vector space and X be a non-empty subset of E . Let $T : X \rightarrow 2^{E^*}$ be lower semicontinuous from the relative topology on X to the weak* topology $\sigma(E^*, E)$ on E^* . Then T is lower hemi-continuous on X .*

Proof: For each fixed $p \in E$, define $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$f_p(z) = \sup_{u \in T(z)} \operatorname{Re}\langle u, p \rangle \text{ for each } z \in X.$$

Fix any $p \in E$. Let $\lambda \in \mathbf{R}$ be given and let $A = \{z \in X : f_p(z) > \lambda\}$. Take any $z_0 \in A$. Then $f_p(z_0) = \sup_{u \in T(z_0)} \operatorname{Re}\langle u, p \rangle = \sup_{u \in T(z_0)} \operatorname{Re} \tilde{p}(u) > \lambda$. Choose any $u_0 \in T(z_0)$ such that $\operatorname{Re} \tilde{p}(u_0) > \lambda$. Thus $(\operatorname{Re} \tilde{p})^{-1}(\lambda, \infty) \cap T(z_0) \neq \emptyset$ where $(\operatorname{Re} \tilde{p})^{-1}(\lambda, \infty)$ is a weak* open set in E^* . Since T is lower semicontinuous at z_0 , there exists an open neighbourhood N_{z_0} of z_0 in X such that $T(z) \cap (\operatorname{Re} \tilde{p})^{-1}(\lambda, \infty) \neq \emptyset$ for all $z \in N_{z_0}$. Hence $f_p(z) = \sup_{u \in T(z)} \operatorname{Re} \tilde{p}(u) = \sup_{u \in T(z)} \operatorname{Re}\langle u, p \rangle > \lambda$ for all $z \in N_{z_0}$. Thus $N_{z_0} \subset A$. Consequently, f_p is lower semicontinuous on X . Hence T is lower hemi-continuous on X .

□

The converse of Proposition 3.2.6 is not true in general as can be seen in the following example.

Example 3.2.7 Let $X = [0, 1]$ and $E = \mathbf{R}$. Then $E^* = \mathbf{R}$. Let $T : X \rightarrow \mathcal{P}E^*$ be defined by

$$T(x) = \begin{cases} \{1, 3\}, & \text{if } x < 1, \\ \{1, 2, 3\}, & \text{if } x = 1. \end{cases}$$

If $p \in E$ and $p \geq 0$, the function $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$, defined by $f_p(z) = \sup_{u \in T(z)} \operatorname{Re}\langle u, p \rangle = 3p$ for each $z \in X$, is continuous. If $p \in E$ and $p < 0$, the function $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$, defined by $f_p(z) = \sup_{u \in T(z)} \operatorname{Re}\langle u, p \rangle = p$ for each $z \in X$, is also continuous. Thus T is lower (and upper) hemi-continuous on X .

But T is not lower semicontinuous (along line segments) in X . Indeed, let $x_0 = 1$ and $U = (\frac{3}{2}, \frac{5}{2})$, then U is an open set in \mathbf{R} such that $U \cap T(x_0) = \{2\} \neq \emptyset$. But for any open neighbourhood V of x_0 in X and for any $x \in V$ with $x \neq x_0$, $U \cap T(x) = \emptyset$. This shows that T is not lower semicontinuous (along line segments) in X .

Proposition 3.2.8 Let E be a topological vector space and X be a non-empty subset of E . Let $T : X \rightarrow \mathcal{P}E^*$ be upper semicontinuous from the relative topology on X to the weak* topology $\sigma(E^*, E)$ on E^* . Then T is upper hemi-continuous on X .

Proof: For each fixed $p \in E$, define $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$f_p(z) = \sup_{u \in T(z)} \operatorname{Re}\langle u, p \rangle, \text{ for each } z \in X.$$

Fix any $p \in E$. Let $\lambda \in \mathbf{R}$ be given and let $A = \{z \in X : f_p(z) < \lambda\}$. Take any $z_0 \in A$. Then $f_p(z_0) = \sup_{u \in T(z_0)} \operatorname{Re}\langle u, p \rangle = \sup_{u \in T(z_0)} \operatorname{Re} \tilde{p}(u) < \lambda$. Thus there exists $\epsilon > 0$ such that $f_p(z_0) < \lambda - \epsilon < \lambda$. Therefore $\operatorname{Re} \tilde{p}(u) < \lambda - \epsilon < \lambda$ for all $u \in T(z_0)$. Hence $T(z_0) \subset (\operatorname{Re} \tilde{p})^{-1}(-\infty, \lambda - \epsilon)$ which is weak* open in E^* . Since T is upper semicontinuous at z_0 , there exists an open neighbourhood N_{z_0} of z_0 in X such that $T(z) \subset (\operatorname{Re} \tilde{p})^{-1}(-\infty, \lambda - \epsilon)$ for all $z \in N_{z_0}$. Thus $\operatorname{Re} \tilde{p}(u) < \lambda - \epsilon$ for all $u \in T(z)$ and for all $z \in N_{z_0}$. Hence $\sup_{u \in T(z)} \operatorname{Re} \tilde{p}(u) \leq \lambda - \epsilon < \lambda$ for all $z \in N_{z_0}$; i.e., $f_p(z) = \sup_{u \in T(z)} \operatorname{Re}\langle u, p \rangle \leq \lambda - \epsilon < \lambda$ for all $z \in N_{z_0}$. Therefore $N_{z_0} \subset A$ so

that A is open in X . Consequently, f_p is upper semicontinuous on X . Hence T is upper hemi-continuous on X . \square

The converse of Proposition 3.2.8 is not true in general as can be seen in the following example which is Example 2.3 in [105, p.392]:

Example 3.2.9 Let $E = \mathbf{R}^2$ and $X = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1 \text{ and } x, y > 0\}$. Define $f, g : X \rightarrow 2^{E^*}$ by

$$f(rcos\theta, rsin\theta) = \{(tcos\theta, tsin\theta) : r \leq t \leq 2\} \text{ for all } r \in (0, 1), \theta \in (0, \frac{\pi}{2}),$$

and

$$g(x, y) = \{(z, 0) : z \geq x\} \text{ for all } (x, y) \in X.$$

Then f and g are upper semicontinuous on X so that f and g are upper hemi-continuous on X by Proposition 3.2.8 and hence $f + g$ is also upper hemi-continuous. However it is easy to see that $f + g$ is not upper semicontinuous (along line segments) in X .

Definition 3.2.10 Let E be a topological vector space, X be a non-empty subset of E and $T : X \rightarrow 2^{E^*}$. If $h : X \rightarrow \mathbf{R}$, then T is said to be

- (1) h -quasi-monotone if for each $x, y \in X$, $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) > 0$ whenever $\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$;
- (2) quasi-monotone if T is h -quasi-monotone with $h \equiv 0$;
- (3) h -quasi-semi-monotone if for each $x, y \in X$, $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) > 0$ whenever $\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$;
- (4) quasi-semi-monotone if T is h -quasi-semi-monotone with $h \equiv 0$.

Clearly, monotonicity implies quasi-monotonicity, but the converse is not true as can be seen in the following simple example:

Example 3.2.11 Define $T : \mathbf{R}^+ \rightarrow 2^{\mathbf{R}}$ by $T(x) = [x, 2x]$ for all $x \in \mathbf{R}^+$.

Suppose $x, y \in \mathbf{R}^+$ such that $x < y < 2x$. Choose $u = 2x \in T(x)$ and $w = y \in T(y)$. Then $\langle w - u, y - x \rangle = \langle y - 2x, y - x \rangle = (y - 2x)(y - x) < 0$, which shows that T is not monotone.

But T is quasi-monotone. Indeed, let $x, y \in \mathbf{R}^+$. Clearly, if $\sup_{u \in T(x)} \langle u, y - x \rangle = 2x(y - x) > 0$ then $y > x$. Thus $\inf_{w \in T(y)} \langle w, y - x \rangle = y(y - x) > 0$. Hence T is quasi-monotone.

The following example shows that quasi-monotonicity does not imply semi-monotonicity.

Example 3.2.12 Define $T : \mathbf{R}^+ \rightarrow 2^{\mathbf{R}}$ by

$$T(x) = \begin{cases} [x, \frac{1}{x}], & \text{if } 0 < x < 1; \\ [\frac{1}{x}, x], & \text{if } x \geq 1. \end{cases}$$

Let $x, y \in \mathbf{R}^+$. If

$$\sup_{u \in T(x)} \langle u, y - x \rangle = \begin{cases} \frac{1}{x}(y - x), & \text{for } 0 < x < 1; \\ x(y - x), & \text{for } x \geq 1; \end{cases} > 0,$$

then $y > x$. Thus

$$\inf_{w \in T(y)} \langle w, y - x \rangle = \begin{cases} y(y - x), & \text{if } 0 < y < 1; \\ \frac{1}{y}(y - x), & \text{if } y \geq 1, \end{cases} > 0.$$

Hence T is quasi-monotone.

But T is not semi-monotone. Indeed, let $x = \frac{1}{2}$ and $y = 3$. Then $T(\frac{1}{2}) = [\frac{1}{2}, 2]$ and $T(3) = [\frac{1}{3}, 3]$. Hence $\inf_{u \in T(\frac{1}{2})} \langle u, y - x \rangle = \inf_{u \in [\frac{1}{2}, 2]} u(y - x) = \inf_{u \in [\frac{1}{2}, 2]} u(3 - \frac{1}{2}) = \inf_{u \in [\frac{1}{2}, 2]} \frac{5u}{2} = \frac{5}{4}$; and $\inf_{w \in T(3)} \langle w, y - x \rangle = \inf_{w \in [\frac{1}{3}, 3]} w(y - x) = \inf_{w \in [\frac{1}{3}, 3]} w(3 - \frac{1}{2}) = \inf_{w \in [\frac{1}{3}, 3]} \frac{5w}{2} = \frac{5}{6}$. Thus the inequality $\inf_{u \in T(x)} \text{Re} \langle u, y - x \rangle \leq \inf_{w \in T(y)} \text{Re} \langle w, y - x \rangle$ does not hold.

The following is Example 2 of Bae-Kim-Tan in [6, pp.241-242] which shows that semi-monotonicity does not imply quasi-monotonicity.

Example 3.2.13 Define $T : \mathbf{R} \rightarrow 2^{\mathbf{R}}$ by

$$T(x) = \begin{cases} [0, 2x], & \text{if } x \geq 0; \\ [2x, 0], & \text{if } x < 0. \end{cases}$$

It is shown in [6, p.241] that T is semi-monotone. Taking $x = 1$ and $y = 2$, we see that

$$\sup_{u \in T(x)} \langle u, y - x \rangle = \sup_{u \in [0,2]} u(y - x) = 2 > 0.$$

and

$$\inf_{w \in T(y)} \langle w, y - x \rangle = \inf_{w \in [0,4]} w(y - x) = 0,$$

so that T is not quasi-monotone. Note that T is also not monotone.

Clearly, semi-monotonicity implies quasi-semi-monotonicity and quasi-monotonicity implies quasi-semi-monotonicity; but the converses are not true in general as shown in Examples 3.2.12 and 3.2.13 respectively.

Definition 3.2.14 Let $(E, \|\cdot\|)$ be a normed space and X be a non-empty subset of E . Then $T : X \rightarrow 2^{E^*}$ is quasi-nonexpansive if for each $x, y \in X$, each $u \in T(x)$ and each $w \in T(y)$, $\operatorname{Re}\langle w - u, y - x \rangle \leq \|y - x\|^2$.

It is clear that if T is single-valued and nonexpansive (i.e., $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in X$), then T is quasi-nonexpansive. The converse is false in general as can be seen in the following example.

Example 3.2.15 Let $X = [0, 1]$ and define $T : X \rightarrow \mathbf{R}$ by $T(x) = -x^2$ for all $x \in X$. Then for each $x, y \in X$, $\langle T(y) - T(x), y - x \rangle = -(y^2 - x^2)(y - x) = -(x + y)(y - x)^2 \leq |y - x|^2$ so that T is quasi-nonexpansive. On the other hand, since

$$|T(x) - T(y)| = |y^2 - x^2| = |y + x||y - x| > |y - x|$$

whenever $y \neq x$ with $x + y > 1$, T is not nonexpansive.

We shall denote by I the identity operator on a Hilbert space H ; i.e., $I(x) = x$ for all $x \in H$.

Proposition 3.2.16 If X is a non-empty subset of a Hilbert space H and $T : X \rightarrow 2^H$, then T is quasi-nonexpansive if and only if $I - T$ is monotone.

Proof: Suppose T is quasi-nonexpansive. Let $x, y \in X$ be given and choose any $u_0 \in T(x)$. Then for each $w \in T(y)$,

$$\begin{aligned} \operatorname{Re}\langle y - w, y - x \rangle &= \operatorname{Re}\langle y - x + x - u_0 + u_0 - w, y - x \rangle \\ &= \|y - x\|^2 + \operatorname{Re}\langle x - u_0, y - x \rangle + \operatorname{Re}\langle u_0 - w, y - x \rangle \\ &\geq \operatorname{Re}\langle x - u_0, y - x \rangle \end{aligned}$$

since T is quasi-nonexpansive. Thus $\inf_{w \in T(y)} \operatorname{Re}\langle y - w, y - x \rangle \geq \operatorname{Re}\langle x - u_0, y - x \rangle$. As $u_0 \in T(x)$ is arbitrary, $\inf_{w \in T(y)} \operatorname{Re}\langle y - w, y - x \rangle \geq \sup_{u \in T(x)} \operatorname{Re}\langle x - u, y - x \rangle$; i.e.,

$$\inf_{w \in (I-T)(y)} \operatorname{Re}\langle w, y - x \rangle \geq \sup_{u \in (I-T)(x)} \operatorname{Re}\langle u, y - x \rangle.$$

Thus $I - T$ is monotone.

Conversely, suppose $I - T$ is monotone. Then for each $x, y \in X, u \in T(x)$ and $w \in T(y)$, $\operatorname{Re}\langle (y - w) - (x - u), y - x \rangle \geq 0$ so that $\operatorname{Re}\langle w - u, y - x \rangle = \operatorname{Re}\langle w - y + y - x + x - u, y - x \rangle = \operatorname{Re}\langle w - y, y - x \rangle + \|y - x\|^2 + \operatorname{Re}\langle x - u, y - x \rangle \leq \|y - x\|^2$. Thus T is quasi-nonexpansive. \square

Proposition 3.2.16 is a generalization of Proposition 1 in [20].

Definition 3.2.17 Let $(E, \|\cdot\|)$ be a normed space and X be a non-empty subset of E . Then $T : X \rightarrow 2^{E^*}$ is semi-nonexpansive if for each $x, y \in X$,

$$\inf_{u \in T(x)} \sup_{w \in T(y)} \operatorname{Re}\langle w - u, y - x \rangle \leq \|y - x\|^2.$$

Proposition 3.2.18 If X is a non-empty subset of a Hilbert space H and $T : X \rightarrow 2^H$, then T is semi-nonexpansive if and only if $I - T$ is semi-monotone.

Proof: Suppose T is semi-nonexpansive. Let $x, y \in X$ be given. Then $\inf_{u \in T(x)} \sup_{w \in T(y)} \operatorname{Re}\langle w - u, y - x \rangle \leq \|y - x\|^2$. Let $\epsilon > 0$ be arbitrarily fixed. Then there exists $u_0 \in T(x)$ with $\sup_{w \in T(y)} \operatorname{Re}\langle w - u_0, y - x \rangle < \|y - x\|^2 + \epsilon$. It follows that for each $w_0 \in T(y)$,

$$\begin{aligned} \operatorname{Re}\langle x - u_0, y - x \rangle &= \operatorname{Re}\langle x - y + y - w_0 + w_0 - u_0, y - x \rangle \\ &= -\|y - x\|^2 + \operatorname{Re}\langle y - w_0, y - x \rangle + \operatorname{Re}\langle w_0 - u_0, y - x \rangle \\ &< \epsilon + \operatorname{Re}\langle y - w_0, y - x \rangle \end{aligned}$$

so that $Re\langle x - u_0, y - x \rangle \leq \inf_{w \in T(y)} Re\langle y - w, y - x \rangle + \epsilon$ as $w_0 \in T(y)$ is arbitrary. Since $\epsilon > 0$ is also arbitrary, $Re\langle x - u_0, y - x \rangle \leq \inf_{w \in T(y)} Re\langle y - w, y - x \rangle$. Therefore $\inf_{u \in T(x)} Re\langle x - u, y - x \rangle \leq \inf_{w \in T(y)} Re\langle y - w, y - x \rangle$. Thus $I - T$ is semi-monotone.

Conversely, suppose $I - T$ is semi-monotone. Then for each $x, y \in X$, $\inf_{u \in T(x)} Re\langle x - u, y - x \rangle \leq \inf_{w \in T(y)} Re\langle y - w, y - x \rangle$. Let $\epsilon > 0$ be arbitrarily fixed. There is $u_0 \in T(x)$ with $Re\langle x - u_0, y - x \rangle < \inf_{w \in T(y)} Re\langle y - w, y - x \rangle + \epsilon$. It follows that for any $w_0 \in T(y)$, $Re\langle w_0 - u_0, y - x \rangle = Re\langle w_0 - y + y - x + x - u_0, y - x \rangle = Re\langle w_0 - y, y - x \rangle + \|y - x\|^2 + Re\langle x - u_0, y - x \rangle < -Re\langle y - w_0, y - x \rangle + \|y - x\|^2 + \inf_{w \in T(y)} Re\langle y - w, y - x \rangle + \epsilon \leq \|y - x\|^2 + \epsilon$. As $w_0 \in T(y)$ is arbitrary, $\sup_{w \in T(y)} Re\langle w - u_0, y - x \rangle \leq \|y - x\|^2 + \epsilon$. Thus $\inf_{u \in T(x)} \sup_{w \in T(y)} Re\langle w - u, y - x \rangle \leq \|y - x\|^2 + \epsilon$. As $\epsilon > 0$ is also arbitrary, $\inf_{u \in T(x)} \sup_{w \in T(y)} Re\langle w - u, y - x \rangle \leq \|y - x\|^2$. Therefore T is semi-nonexpansive. \square

It is clear from the definitions that a quasi-nonexpansive operator is semi-nonexpansive. The converse does not hold in general: The operator T defined in Example 3.2.13 is semi-monotone but not monotone. Then by Propositions 3.2.16 and 3.2.18, the operator $S = I - T$ is semi-nonexpansive but not quasi-nonexpansive.

In this section, we shall apply Theorem 2.2.4 to obtain existence theorems for generalized variational inequalities together with applications to existence theorems for generalized complementarity problems. Some results on maximality of monotone operators and surjectivity of monotone or semi-monotone operators will also be given. We shall begin with the following:

Lemma 3.2.19 *Let E be a topological vector space, X be a non-empty convex subset of E , $h : X \rightarrow \mathbf{R}$ be convex and $T : X \rightarrow 2^{E^*}$ be lower hemi-continuous along line segments in X . Suppose $\hat{y} \in X$ is such that $\sup_{u \in T(x)} Re\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Then*

$$\sup_{w \in T(\hat{y})} Re\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in X.$$

Proof: Suppose that $\sup_{u \in T(x)} Re\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Let $x \in X$ be arbitrarily fixed. Let $z_t = tx + (1 - t)\hat{y} = \hat{y} - t(\hat{y} - x)$ for all $t \in [0, 1]$. Then $z_t \in X$

as X is convex. Let $L = \{z_t : t \in [0, 1]\}$. Thus $\sup_{u \in T(z_t)} \operatorname{Re}\langle u, \hat{y} - z_t \rangle \leq h(z_t) - h(\hat{y})$ for all $t \in [0, 1]$. Therefore $\sup_{u \in T(z_t)} \operatorname{Re}\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $t \in (0, 1]$.

Since T is lower hemi-continuous on L , the function $f_{\hat{y}-x} : L \rightarrow \mathbf{R} \cup \{+\infty\}$, defined by

$$f_{\hat{y}-x}(z_t) = \sup_{u \in T(z_t)} \operatorname{Re}\langle u, \hat{y} - x \rangle \text{ for each } z_t \in L,$$

is lower semicontinuous on L . Thus the set $A = \{z_t \in L : f_{\hat{y}-x}(z_t) \leq h(x) - h(\hat{y})\}$ is closed in L . Now $z_t \rightarrow \hat{y}$ in L as $t \rightarrow 0^+$. Since $z_t \in A$ for all $t \in (0, 1]$ we have $\hat{y} \in A$. Hence $f_{\hat{y}-x}(\hat{y}) = \sup_{u \in T(\hat{y})} \operatorname{Re}\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$. Since $x \in X$ is arbitrary, we have $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. \square

By modifying the above proof, we have the following result whose proof is omitted:

Lemma 3.2.20 *Let E be a topological vector space, X be a non-empty convex subset of E , $h : X \rightarrow \mathbf{R}$ be convex and $T : X \rightarrow 2^{E^*}$ be upper hemi-continuous along line segments in X . Suppose $\hat{y} \in X$ is such that $\inf_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Then*

$$\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in X.$$

Note that if E is a locally convex space, X is a non-empty convex subset of E and $h : X \rightarrow \mathbf{R}$ is convex, then h is lower semicontinuous on X if and only if h is weakly lower semicontinuous on X .

Lemma 3.2.21 *Let E be a topological vector space, X be a non-empty convex subset of E and $h : E \rightarrow \mathbf{R}$ be convex. Suppose $\hat{y} \in X$ and $\hat{w} \in E^*$ are such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$, then $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.*

Proof: Let $x \in I_X(\hat{y})$ be arbitrarily fixed; then $x = \hat{y} + r(u - \hat{y})$ for some $u \in X$ and $r > 0$.

Case 1. Suppose $0 < r \leq 1$, then $x = ru + (1 - r)\hat{y} \in X$ as X is convex and $u, \hat{y} \in X$. By assumption, we have $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$.

Case 2. Suppose $r > 1$, then $u = (1 - \frac{1}{r})\hat{y} + \frac{1}{r}x$. Since $u \in X$, by assumption again, we have

$$\begin{aligned} \frac{1}{r}Re\langle \hat{w}, \hat{y} - x \rangle &= Re\langle \hat{w}, \hat{y} - u \rangle \\ &\leq h(u) - h(\hat{y}) \leq (1 - \frac{1}{r})h(\hat{y}) + \frac{1}{r}h(x) - h(\hat{y}) \\ &= \frac{1}{r}(h(x) - h(\hat{y})) \end{aligned}$$

so that $Re\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$.

Thus in either case, $Re\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$. \square

Theorem 3.2.22 *Let E be a topological vector space, X be a non-empty convex subset of E , $h : X \rightarrow \mathbf{R}$ be convex and weakly lower semicontinuous on weakly compact subsets of X and $T : X \rightarrow 2^{E^*}$ be h -quasi-monotone. Suppose there exist a non-empty weakly closed and weakly compact subset K of X and $x_0 \in K$ such that $\inf_{w \in T(y)} Re\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that $\sup_{u \in T(x)} Re\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$.*

Proof: Define $f, g : X \times X \rightarrow \mathbf{R}$ by

$$f(x, y) = \sup_{u \in T(x)} Re\langle u, y - x \rangle + h(y) - h(x),$$

$$g(x, y) = \inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)$$

for all $x, y \in X$. Then we have the following:

- (1) For each $x, y \in X$, since T is h -quasi-monotone, $f(x, y) > 0$ implies $g(x, y) > 0$.
- (2) For each fixed $x \in X$, $y \mapsto f(x, y)$ is weakly lower semicontinuous on non-empty weakly compact subsets of X .

(3) For each $A \in \mathcal{F}(X)$ and $y \in co(A)$, $\min_{x \in A} g(x, y) \leq 0$. Indeed, if this were false, then for some $A = \{x_1, \dots, x_n\} \in \mathcal{F}(X)$ and some $y \in co(A)$, say $y = \sum_{i=1}^n \lambda_i x_i$ where $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, such that $\min_{1 \leq i \leq n} g(x_i, y) > 0$. Then for each $i = 1, \dots, n$, $\inf_{w \in T(y)} Re\langle w, y - x_i \rangle + h(y) - h(x_i) > 0$ so that $0 = g(y, y) = \inf_{w \in T(y)} Re\langle w, y - \sum_{i=1}^n \lambda_i x_i \rangle + h(y) - h(\sum_{i=1}^n \lambda_i x_i) \geq \sum_{i=1}^n \lambda_i (\inf_{w \in T(y)} Re\langle w, y - x_i \rangle + h(y) - h(x_i)) > 0$, which is a contradiction.

(4) K is a weakly closed and weakly compact subset of X and $x_0 \in K$ such that for all $y \in X \setminus K$, $g(x_0, y) > 0$.

Equip E with the weak topology. Then f and g satisfy all the hypotheses of Theorem 2.2.4 so that by Theorem 2.2.4, there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$; i.e., $\sup_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in X$. \square

Theorem 3.2.23 *Let E be a topological vector space, X be a non-empty convex subset of E , $h : X \rightarrow \mathbf{R}$ be convex and weakly lower semicontinuous on weakly compact subsets of X and $T : X \rightarrow 2^{E^*}$ be h -quasi-monotone and lower hemi-continuous along line segments in X to the weak*-topology on E^* . Suppose there exist a non-empty weakly closed and weakly compact subset K of X and $x_0 \in K$ such that $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Moreover, if h is defined on all of E and is convex, then $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.*

Proof: By Theorem 3.2.22, there exists $\hat{y} \in K$ such that $\sup_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$.

Since h is convex and T is lower hemi-continuous along line segments in X , by Lemma 3.2.19, we have

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in X. \quad (3.1)$$

Now if h is defined on all of E and is convex, then by (3.1) and Lemma 3.2.21, we have

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in I_X(\hat{y}). \square$$

Note that Theorem 2.2.6 (i.e., Theorem 2.2 in [109]) can not be applied directly to prove Theorem 3.2.23.

Remark 3.2.24 *Theorem 3.2.23 improves Theorem 3 of Shih and Tan in [91, pp.283-285] in the following ways:*

- (1) f is h -quasi-monotone instead of monotone;

(2) f is lower hemi-continuous along line segments instead of lower semicontinuous along line segments in X .

Note that there are typos in the original statement of Theorem 3 in [91].

Theorem 3.2.25 Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E , $h : X \rightarrow \mathbf{R}$ be convex and lower semicontinuous on weakly compact subsets of X and $T : X \rightarrow 2^{E^*}$ be h -quasi-monotone and lower hemi-continuous along line segments in X to the weak topology on E^* . Suppose there is $x_0 \in X$ such that

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0. \quad (3.2)$$

Then there exists $\hat{y} \in X$ such that $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Moreover, if h is defined on all of E and is convex, then $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.

Proof: Let $\alpha = \lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0)$. Then by (3.2), $\alpha > 0$. Let $M > 0$ be such that $\|x_0\| \leq M$ and $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > \frac{\alpha}{2}$ for all $y \in X$ with $\|y\| > M$. Let $K = \{x \in X : \|x\| \leq M\}$; then K is a non-empty weakly compact subset of X . Note that for any $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > \frac{\alpha}{2} > 0$. The conclusion now follows from Theorem 3.2.23. \square

By taking $h \equiv 0$ in Theorem 3.2.23 and applying Lemma 3.2.2, we have the following existence theorem for a generalized complementarity problem:

Theorem 3.2.26 Let X be a cone in a topological vector space E . Let $T : X \rightarrow 2^{E^*}$ be quasi-monotone and lower hemi-continuous along line segments in X to the weak*-topology on E^* . Suppose there exist a non-empty weakly closed and weakly compact subset K of X and $x_0 \in K$ such that $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that $\operatorname{Re}\langle w, \hat{y} \rangle = 0$ for all $w \in T(\hat{y})$ and $T(\hat{y}) \subset \widehat{X}$.

By taking $h \equiv 0$ in Theorem 3.2.25 and applying Lemma 3.2.2 (or by the same argument as in the proof of Theorem 3.2.25 and by Theorem 3.2.26), we have the following existence theorem for a generalized complementarity problem:

Theorem 3.2.27 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a closed cone in E and $T : X \rightarrow 2^{E^*}$ be quasi-monotone and lower hemi-continuous along line segments in X to the weak topology on E^* . Suppose there is $x_0 \in X$ such that*

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle > 0.$$

Then there exists $\hat{y} \in X$ such that $\operatorname{Re}\langle w, \hat{y} \rangle = 0$ for all $w \in T(\hat{y})$ and $T(\hat{y}) \subset \widehat{X}$.

Theorem 3.2.28 *Let E be a Hausdorff topological vector space, X be a non-empty convex subset of E , $h : X \rightarrow \mathbf{R}$ be convex and weakly lower semicontinuous on weakly compact subsets of X and $T : X \rightarrow 2^{E^*}$ be h -quasi-monotone and upper hemi-continuous along line segments in X to the weak* topology on E^* such that each $T(x)$ is weak* compact convex. Suppose there exist a non-empty weakly closed and weakly compact subset K of X and $x_0 \in K$ such that $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Moreover, if h is defined on all of E and is convex, then $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.*

Proof: By Theorem 3.2.22, there exists $\hat{y} \in K$ such that $\sup_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in X$. It follows that $\inf_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in X$. Since h is convex and T is upper hemi-continuous, by Lemma 3.2.20, $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in X$.

Define $\phi : X \times T(\hat{y}) \rightarrow \mathbf{R}$ by $\phi(x, w) = \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ for all $(x, w) \in X \times T(\hat{y})$. Then for each fixed $x \in X$, $w \mapsto \phi(x, w)$ is weak* lower semicontinuous and convex and for each fixed $w \in T(\hat{y})$, $x \mapsto \phi(x, w)$ is concave. By Theorem 3.2.1,

$$\min_{w \in T(\hat{y})} \sup_{x \in X} \phi(x, w) = \sup_{x \in X} \min_{w \in T(\hat{y})} \phi(x, w) \leq 0.$$

Since $T(\hat{y})$ is weak*-compact, there exists $\hat{w} \in T(\hat{y})$ such that

$$\sup_{x \in X} \phi(x, \hat{w}) = \min_{w \in T(\hat{y})} \sup_{x \in X} \phi(x, w) \leq 0.$$

Therefore

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in X. \quad (3.3)$$

Now suppose h is defined on all of E and is convex. Then by (3.3) and Lemma 3.2.21,

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in I_X(\hat{y}). \square$$

Remark 3.2.29 *Theorem 3.2.28 extends Theorem 5 of Bae-Kim-Tan in [6, pp.238-240] in the following ways:*

- (1) E^* is not equipped with the strong topology,
- (2) T is h -quasi-monotone instead of semi-monotone,
- (3) Each $T(x)$ is weak*-compact instead of strongly compact,
- (4) T is upper hemi-continuous along line segments instead of upper semicontinuous along line segments in X .

Note however that the coercive conditions in our Theorem 3.2.28 here and in Theorem 5 of [6] are not comparable.

Theorem 3.2.30 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E , $h : X \rightarrow \mathbf{R}$ be convex and lower semicontinuous on weakly compact subsets of X and $T : X \rightarrow 2^{E^*}$ be h -quasi-monotone and upper hemi-continuous along line segments in X to the weak topology on E^* such that each $T(x)$ is weakly compact convex. Suppose there is $x_0 \in X$ such that*

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0.$$

Then there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Moreover, if h is defined on all of E and is convex, then $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.

Proof: By using the same argument in the proof of Theorem 3.2.25 and by Theorem 3.2.28, the conclusion follows. \square

By taking $h \equiv 0$ in Theorem 3.2.28 and applying Lemma 3.2.3 we have the following existence theorem for a generalized complementarity problem:

Theorem 3.2.31 *Let X be a cone in a Hausdorff topological vector space E . Let $T : X \rightarrow \mathcal{P}^{E^*}$ be quasi-monotone and upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is weak*-compact convex. Suppose there exist a non-empty weakly closed and weakly compact subset K of X and $x_0 \in K$ such that $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle > 0$ for all $y \in X \setminus K$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} \rangle = 0$ and $\hat{w} \in \widehat{X}$.*

By taking $h \equiv 0$ in Theorem 3.2.30 and applying Lemma 3.2.3 (or by a similar argument in proving Theorem 3.2.25 and by Theorem 3.2.31), we have the following existence theorem for a generalized complementarity problem:

Theorem 3.2.32 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a closed cone in E and $T : X \rightarrow \mathcal{P}^{E^*}$ be quasi-monotone and upper hemi-continuous along line segments in X to the weak topology on E^* such that each $T(x)$ is weakly compact convex. Suppose there is $x_0 \in X$ such that*

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle > 0.$$

Then there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} \rangle = 0$ and $\hat{w} \in \widehat{X}$.

Theorem 3.2.33 *Let E be a Hausdorff locally convex topological vector space, X be a non-empty convex subset of E , $h : X \rightarrow \mathbf{R}$ be convex and weakly lower semicontinuous on weakly compact subsets of X and $T : X \rightarrow \mathcal{P}^{E^*}$ be h -quasi-semi-monotone and upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is strongly compact convex. Suppose there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\min_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Moreover, if h is defined on all of E and is convex, then $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.*

Proof: Define $f, g : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ by

$$f(x, y) = \min_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x),$$

$$g(x, y) = \min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)$$

for all $x, y \in X$. Then we have the following:

(1) For each $x, y \in X$, since T is h -quasi-semi-monotone, $f(x, y) > 0$ implies $g(x, y) > 0$.

(2) For each fixed $x \in X$, since $T(x)$ is strongly compact, by Lemma 3.2.4, $y \mapsto f(x, y)$ is weakly lower semicontinuous on non-empty bounded subsets of X and hence also weakly lower semicontinuous on weakly compact subsets of X .

(3) For each $A \in \mathcal{F}(X)$ and $y \in \operatorname{co}(A)$, $\min_{x \in A} g(x, y) \leq 0$ by using the same argument as for (3) in the proof of Theorem 3.2.22.

(4) By assumption, K is a weakly closed and weakly compact subset of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\min_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$, i.e., $g(x_0, y) > 0$.

Equip E with the weak topology. Then f and g satisfy all the hypotheses of Theorem 2.2.4 so that by Theorem 2.2.4, there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$; i.e., $\min_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) \leq 0$ for all $x \in X$. Since h is convex and T is upper hemi-continuous, by Lemma 3.2.20, we have

$$\min_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in X.$$

By following the same argument as in proving Theorem 3.2.28, the conclusion follows. \square

Remark 3.2.34 *Theorem 3.2.33 extends Theorem 5 of Bae-Kim-Tan in [6, pp.238-240] in the following ways:*

(1) *T is upper hemi-continuous along line segments instead of upper semicontinuous along line segments in X ,*

(2) *T is h -quasi-semi-monotone instead of semi-monotone.*

Note however that the coercive conditions in our Theorem 3.2.33 here and in Theorem 5 of [6] are not comparable.

Theorem 3.2.35 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E , $h : X \rightarrow \mathbf{R}$ be convex and lower semicontinuous on weakly*

compact subsets of X and $T : X \rightarrow \mathfrak{2}^{E^*}$ be h -quasi-semi-monotone and upper hemi-continuous along line segments in X to the weak topology on E^* such that each $T(x)$ is compact convex. Suppose there is $x_0 \in X$ such that

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0.$$

Then there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Moreover, if h is defined on all of E and is convex, then $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.

Proof: By using the same argument in the proof of Theorem 3.2.25 and by Theorem 3.2.33, the conclusion follows. \square

By taking $h \equiv 0$ in Theorem 3.2.33 and applying Lemma 3.2.3, we have the following existence theorem for a generalized complementarity problem:

Theorem 3.2.36 *Let X be a cone in a Hausdorff locally convex topological vector space E . Let $T : X \rightarrow \mathfrak{2}^{E^*}$ be quasi-semi-monotone and upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is strongly compact convex. Suppose there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\min_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle > 0$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} \rangle = 0$ and $\hat{w} \in \widehat{X}$.*

Theorem 3.2.37 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a closed cone in E and $T : X \rightarrow \mathfrak{2}^{E^*}$ be quasi-semi-monotone and upper hemi-continuous along line segments in X to the weak topology on E^* such that each $T(x)$ is compact convex. Suppose there is $x_0 \in X$ such that*

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle > 0.$$

Then there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} \rangle = 0$ and $\hat{w} \in \widehat{X}$.

We observe that in all the generalized variational inequalities and generalized complementarity problems stated above, (1) when T is lower hemi-continuous along line segments, T is only required to have non-empty values, (2) when T is upper hemi-continuous along line segments and quasi-monotone, T is required to have weak*-compact-convex values and (3) when T is upper hemi-continuous along line segments and quasi-semi-monotone, T is required to have strongly-compact-convex values.

Next we shall discuss maximality of monotone operators.

Let X be a non-empty subset of a topological vector space E ; then $T : X \rightarrow \mathcal{2}^{E^*}$ is *maximal monotone* if T is monotone and if $T^* : X \rightarrow \mathcal{2}^{E^*}$ is monotone such that $T(x) \subset T^*(x)$ for all $x \in X$, then $T = T^*$.

Theorem 3.2.38 *Let E be a topological vector space and $T : E \rightarrow \mathcal{2}^{E^*}$ be monotone and lower (respectively, upper) hemi-continuous along line segments in E such that each $T(x)$ is weak* compact convex. Then T is maximal monotone.*

Proof: Let $T^* : E \rightarrow \mathcal{2}^{E^*}$ be monotone such that $T(x) \subset T^*(x)$ for all $x \in X$. Let $y_0 \in E$ be arbitrarily fixed and let $w_0 \in T^*(y_0)$. Since T^* is monotone, for each $x \in E$ and each $u \in T^*(x)$, $\text{Re}\langle u - w_0, y_0 - x \rangle \leq 0$. It follows that $\sup_{u \in T^*(x)} \text{Re}\langle u - w_0, y_0 - x \rangle \leq 0$ for all $x \in E$. By Lemma 3.2.19 (respectively, Lemma 3.2.20), $\sup_{w \in T(y_0)} \text{Re}\langle w - w_0, y_0 - x \rangle \leq 0$ (respectively, $\inf_{w \in T(y_0)} \text{Re}\langle w - w_0, y_0 - x \rangle \leq 0$) for all $x \in E$. Thus $\sup_{x \in E} \inf_{w \in T(y_0)} \text{Re}\langle w - w_0, y_0 - x \rangle \leq 0$. By Theorem 3.2.1, $\inf_{w \in T(y_0)} \sup_{x \in E} \text{Re}\langle w - w_0, y_0 - x \rangle \leq 0$. Since $T(y_0)$ is weak* compact, there exists $\hat{w} \in T(y_0)$ such that $\sup_{x \in E} \text{Re}\langle \hat{w} - w_0, y_0 - x \rangle = \inf_{w \in T(y_0)} \sup_{x \in E} \text{Re}\langle w - w_0, y_0 - x \rangle \leq 0$. Therefore $w_0 = \hat{w} \in T(y_0)$. Since $w_0 \in T^*(y_0)$ is arbitrary, $T(y_0) = T^*(y_0)$. Since $y_0 \in E$ is also arbitrary, we conclude that $T = T^*$. Hence T is maximal monotone. \square

Theorem 3.2.38 improves Lemma 3 of [98] in several respects.

Finally in this section we shall prove some results on the surjectivity of monotone or semi-monotone operators.

Theorem 3.2.39 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E and $T : X \rightarrow \mathcal{2}^{E^*}$ be monotone and lower hemi-continuous along*

line segments in X to the weak topology on E^* . Suppose there is $x_0 \in X$ such that

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle / \|y\| = \infty.$$

Then for each given $w_0 \in E^*$, there exist $\hat{y} \in X$ such that $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w - w_0, \hat{y} - x \rangle \leq 0$ for all $x \in X$. In particular, when $X = E$, then T is surjective; in fact, for each $w \in E^*$, there is $y \in E$ such that $T(y) = \{w\}$.

Proof: Let $w_0 \in E^*$ be given. Then

$$\begin{aligned} & \lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} (\inf_{w \in T(y)} \operatorname{Re}\langle w - w_0, y - x_0 \rangle / \|y\|) \\ &= \lim_{\|y\| \rightarrow \infty} ((\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle / \|y\|) - \|w_0\|) = \infty. \end{aligned}$$

Define $T^* : X \rightarrow 2^{E^*}$ by $T^*(x) = T(x) - w_0$ for all $x \in X$. Then T^* is monotone and lower hemi-continuous along line segments in X to the weak topology on E^* and

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T^*(y)} \operatorname{Re}\langle w, y - x_0 \rangle / \|y\| = \infty.$$

Therefore by Theorem 3.2.25, there exist $\hat{y} \in X$ such that $\sup_{w \in T^*(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq 0$ for all $x \in X$. That is, $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w - w_0, \hat{y} - x \rangle \leq 0$ for all $x \in X$. Now if $X = E$, then $w - w_0 = 0$ so that $w_0 = w$ for all $w \in T(\hat{y})$ and hence $T(\hat{y}) = \{w_0\}$. This shows that T is surjective such that for each $w \in E^*$, there is $y \in E$ with $T(y) = \{w\}$. \square

Theorem 3.2.40 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E and $T : X \rightarrow 2^{E^*}$ be monotone and upper hemi-continuous along line segments in X to the weak topology on E^* such that each $T(x)$ is weakly compact convex. Suppose there is $x_0 \in X$ such that*

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle / \|y\| = \infty.$$

Then for each given $w_0 \in E^$, there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w} - w_0, \hat{y} - x \rangle \leq 0$ for all $x \in X$. In particular, if $X = E$, then T is surjective.*

Proof: Let $w_0 \in E^*$ be given. Then

$$\begin{aligned} & \lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} (\inf_{w \in T(y)} \operatorname{Re} \langle w - w_0, y - x_0 \rangle / \|y\|) \\ &= \lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} ((\inf_{w \in T(y)} \operatorname{Re} \langle w, y - x_0 \rangle / \|y\|) - \|w_0\|) = \infty. \end{aligned}$$

Define $T^* : X \rightarrow 2^{E^*}$ by $T^*(x) = T(x) - w_0$ for all $x \in X$. Then T^* is monotone and upper hemi-continuous along line segments in X to the weak topology on E^* such that each $T^*(x)$ is weakly compact convex and

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T^*(y)} \operatorname{Re} \langle w, y - x_0 \rangle / \|y\| = \infty.$$

Therefore by Theorem 3.2.30, there exist $\hat{y} \in X$ and $\bar{w} \in T^*(\hat{y})$ such that $\operatorname{Re} \langle \bar{w}, \hat{y} - x \rangle \leq 0$ for all $x \in X$. But then there exists $\hat{w} \in T(\hat{y})$ with $\bar{w} = \hat{w} - w_0$ so that $\operatorname{Re} \langle \hat{w} - w_0, \hat{y} - x \rangle \leq 0$ for all $x \in X$. Now if $X = E$, then $\hat{w} - w_0 = 0$ so that $w_0 = \hat{w} \in T(\hat{y})$. This shows that T is surjective. \square

By using an argument similar to the proof of Theorem 3.2.40 and by applying Theorem 3.2.35 (instead of Theorem 3.2.30), we have the following surjectivity of semi-monotone operators:

Theorem 3.2.41 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E and $T : X \rightarrow 2^{E^*}$ be semi-monotone and upper hemi-continuous along line segments in X to the weak topology on E^* such that each $T(x)$ is compact convex. Suppose there is $x_0 \in X$ such that*

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re} \langle w, y - x_0 \rangle / \|y\| = \infty.$$

Then for each given $w_0 \in E^$, there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re} \langle \hat{w} - w_0, \hat{y} - x \rangle \leq 0$ for all $x \in X$. In particular, if $X = E$, then T is surjective.*

We remark here that the proofs of Theorems 3.2.39 and 3.2.40 are slight modification of the proof of Theorem 2 in [98] and improve Theorem 2 in [98] from upper semicontinuous along line segments to lower or upper hemi-continuous along line segments.

3.3 Generalized Variational Inequalities for Quasi-Monotone and Lower Demi-Continuous Operators

In this section we shall obtain some results in topological vector spaces for the existence of solutions for some generalized variational inequalities with quasi-monotone and lower demi-continuous operators. Some applications are given in non-reflexive Banach spaces for these solutions for generalized variational inequalities with quasi-monotone and lower demi-continuous operators. A result on surjectivity will also be obtained.

We shall denote by Φ either the real field \mathbf{R} or the complex field \mathbf{C} throughout Sections 3.3 to 3.5. If X and Y are topological spaces and $T : X \rightarrow 2^Y$, then the graph of T is the set $G(T) := \{(x, y) \in X \times Y : y \in T(x)\}$.

Let X be a topological space such that $X = \bigcup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact subsets of X . Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ [15, p.34] if for each $n \in \mathbf{N}$, there exists $m \in \mathbf{N}$ such that $x_k \notin C_n$ for all $k \geq m$.

Let E be a topological vector space over Φ , F be a vector space over Φ and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional. For each $x_0 \in E$, for each non-empty subset A of E and for $\epsilon > 0$, let $W(x_0; \epsilon) := \{y \in F : |\langle y, x_0 \rangle| < \epsilon\}$ and $U(A; \epsilon) := \{y \in F : \sup_{x \in A} |\langle y, x \rangle| < \epsilon\}$. Let $\sigma\langle F, E \rangle$ be the (weak) topology on F generated by the family $\{W(x; \epsilon) : x \in E \text{ and } \epsilon > 0\}$ as a subbase for the neighbourhood system at 0 and $\delta\langle F, E \rangle$ be the (strong) topology on F generated by the family $\{U(A; \epsilon) : A \text{ is a non-empty bounded subset of } E \text{ and } \epsilon > 0\}$ as a base for the neighbourhood system at 0. We note then that F , when equipped with the (weak) topology $\sigma\langle F, E \rangle$ or the (strong) topology $\delta\langle F, E \rangle$, becomes a locally convex topological vector space which is not necessarily Hausdorff. But if the bilinear functional $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ separates points in F , i.e., for each $y \in F$ with $y \neq 0$, there exists $x \in E$ such that $\langle y, x \rangle \neq 0$, then F becomes Hausdorff. Furthermore, for a net $\{y_\alpha\}_{\alpha \in \Gamma}$ in F and for $y \in F$, (i) $y_\alpha \rightarrow y$ in $\sigma\langle F, E \rangle$ if and only if $\langle y_\alpha, x \rangle \rightarrow \langle y, x \rangle$ for each $x \in E$ and (ii) $y_\alpha \rightarrow y$ in $\delta\langle F, E \rangle$ if and only if $\langle y_\alpha, x \rangle \rightarrow \langle y, x \rangle$ uniformly for $x \in A$ for each non-empty bounded subset A of E .

Now if X is a non-empty subset of E , then a map $T : X \rightarrow 2^F$ is called (i) monotone (with respect to the bilinear functional $\langle \cdot, \cdot \rangle$) if for each $x, y \in X$, each $u \in T(x)$ and each $w \in T(y)$, $Re\langle w - u, y - x \rangle \geq 0$ and (ii) semi-monotone (with respect to the bilinear functional $\langle \cdot, \cdot \rangle$) if for each $x, y \in X$, $\inf_{u \in T(x)} Re\langle u, y - x \rangle \leq \inf_{w \in T(y)} Re\langle w, y - x \rangle$. Note that when $F = E^*$, the vector space of all continuous linear functionals on E and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E , the monotonicity and semi-monotonicity notions coincide with the usual definitions (see, e.g., Browder [22, p.79] and Bae-Kim-Tan [6, p.237] respectively). Note also that $T : X \rightarrow 2^F$ is monotone if and only if its graph $G(T)$ is a monotone subset of $X \times F$; i.e., for all $(x_1, y_1), (x_2, y_2) \in G(T)$, $Re\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0$.

We now state the following result which follows from Theorem 2.2.4 and is a generalization of Ky Fan's minimax inequality in [48, Theorem 1]:

Theorem 3.3.1 *Let E be a topological vector space, and X be a non-empty compact convex subset of E . Suppose that $f, g : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ are two mappings satisfying the following conditions:*

(i) *for each $x \in X$, $g(x, x) \leq 0$ and for each $x, y \in X$, $f(x, y) > 0$ implies $g(x, y) > 0$;*

(ii) *for each fixed $x \in X$, the map $y \mapsto f(x, y)$ is lower semicontinuous on X ;*

(iii) *for each fixed $y \in X$, the set $\{x \in X : g(x, y) > 0\}$ is convex.*

Then there exists a point $\hat{y} \in X$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

We shall begin with the following result:

Lemma 3.3.2 *Let E be a Hausdorff topological vector space over Φ , F be a vector space over Φ and C be a non-empty compact subset of E . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that for each fixed $y \in F$, the map $x \mapsto Re\langle y, x \rangle$ is continuous on E . Equip F with the strong topology $\delta\langle F, E \rangle$ and let A be a non-empty (strongly) bounded subset of F . Define $f : A \rightarrow \mathbf{R}$ by $f(y) = \min_{x \in C} Re\langle y, x \rangle$ for each $y \in A$. Then f is lower semicontinuous on A from its relative $\delta\langle F, E \rangle$ topology to the*

usual topology of \mathbf{R} . If in addition, for each fixed $x \in E$, the map $y \mapsto \operatorname{Re}\langle y, x \rangle$ is continuous on A , then f becomes continuous on A from its relative $\delta\langle F, E \rangle$ topology to \mathbf{R} .

Proof: First we shall show that f is lower semicontinuous on A . For, let $\lambda \in \mathbf{R}$ and $B_\lambda = \{y \in A \mid f(y) \leq \lambda\}$. Let $\{y_\alpha\}_{\alpha \in \Gamma}$ be a net in B_λ and $y_0 \in A$ such that $y_\alpha \rightarrow y_0$ in $\delta\langle F, E \rangle$ topology. Therefore $f(y_\alpha) = \min_{x \in C} \operatorname{Re}\langle y_\alpha, x \rangle \leq \lambda$. Since C is a compact subset of E and $x \mapsto \operatorname{Re}\langle y_\alpha, x \rangle$ is continuous on C for each $\alpha \in \Gamma$, there exists $x_\alpha \in C$ such that $f(y_\alpha) = \operatorname{Re}\langle y_\alpha, x_\alpha \rangle \leq \lambda$. Therefore, $\{x_\alpha\}_{\alpha \in \Gamma}$ is a net in the compact set C . Hence there exist a subnet $\{x_{\alpha'}\}_{\alpha' \in \Gamma'}$ of $\{x_\alpha\}_{\alpha \in \Gamma}$ and $x_0 \in C$ (as C is closed in E) such that $x_{\alpha'} \rightarrow x_0$ in the relative vector topology on C . Thus $f(y_0) = \min_{x \in C} \operatorname{Re}\langle y_0, x \rangle \leq \operatorname{Re}\langle y_0, x_0 \rangle = \lim_{\alpha'} \operatorname{Re}\langle y_{\alpha'}, x_{\alpha'} \rangle = \lim_{\alpha'} f(y_{\alpha'}) \leq \lambda \Rightarrow y_0 \in B_\lambda \Rightarrow B_\lambda$ is closed. Hence f is lower semicontinuous on A .

Again, for each $x \in E$, if the map $g_x : A \rightarrow \mathbf{R}$ defined by $g_x(y) = \operatorname{Re}\langle y, x \rangle$ is continuous on A from the relative $\delta\langle F, E \rangle$ topology on A to \mathbf{R} , then $f(y) = \min_{x \in C} g_x(y) = \min_{x \in C} \operatorname{Re}\langle y, x \rangle$, for each $y \in A$, is upper semicontinuous on A from the relative $\delta\langle F, E \rangle$ topology on A to \mathbf{R} . Consequently f becomes continuous on A from the relative $\delta\langle F, E \rangle$ topology on A to \mathbf{R} . \square

When $F = E^*$ and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E , Lemma 3.3.2 reduces to the following result which is a modification of the Lemma 2.1.6 in [106]:

Corollary 3.3.3 *Let E be a Hausdorff topological vector space and E^* be the continuous dual of E equipped with the strong topology. Let A be a non-empty (strongly) bounded subset of E^* and C be a non-empty compact subset of E . Define $f : A \rightarrow \mathbf{R}$ by $f(y) = \min_{u \in C} \operatorname{Re}\langle y, u \rangle$ for all $y \in A$. Then f is strongly continuous on A .*

Definition 3.3.4 *Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty subset of F . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that for each fixed $p \in F$, the map $y \mapsto \operatorname{Re}\langle p, y \rangle$ is continuous on E . Equip F with the strong topology $\delta\langle F, E \rangle$ and let $T : X \rightarrow 2^E$ be a map. Then*

(a): *T is said to be weakly lower (respectively, upper) demi-continuous on X if and only if for each $p \in F$, the function $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ defined by*

$$f_p(z) = \sup_{u \in T(z)} \operatorname{Re}\langle p, u \rangle \quad \text{for each } z \in X,$$

is weakly lower (respectively, upper) semicontinuous on X when X is equipped with the relative $\sigma\langle F, E \rangle$ -topology on X (if and only if for each $p \in F$, the function $g_p : X \rightarrow \mathbf{R} \cup \{-\infty\}$ defined by

$$g_p(z) = \inf_{u \in T(z)} \operatorname{Re}\langle p, u \rangle \quad \text{for each } z \in X,$$

is weakly upper (respectively, lower) semicontinuous on X when X is equipped with the relative $\sigma\langle F, E \rangle$ -topology on X);

(b): T is said to be strongly lower (respectively, upper) demi-continuous on X if and only if for each $p \in F$, the function $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ defined by

$$f_p(z) = \sup_{u \in T(z)} \operatorname{Re}\langle p, u \rangle \quad \text{for each } z \in X,$$

is strongly lower (respectively, upper) semicontinuous on X when X is equipped with the relative $\delta\langle F, E \rangle$ -topology on X (if and only if for each $p \in F$, the function $g_p : X \rightarrow \mathbf{R} \cup \{-\infty\}$ defined by

$$g_p(z) = \inf_{u \in T(z)} \operatorname{Re}\langle p, u \rangle \quad \text{for each } z \in X,$$

is strongly upper (respectively, lower) semicontinuous on X when X is equipped with the relative $\delta\langle F, E \rangle$ -topology on X).

The definition of upper demi-continuous map is a generalization of the Definition 1 of upper hemi-continuous map in Section 4 of [4, p.59].

Note that if $M, T : X \rightarrow \mathfrak{P}^E$ are weakly lower (respectively, weakly upper) demi-continuous on X or strongly lower (respectively, strongly upper) demi-continuous on X and $\alpha \in \mathbf{R}$, then $M + T$ and αT are also weakly lower (respectively, weakly upper) demi-continuous on X or strongly lower (respectively, strongly upper) demi-continuous on X .

Proposition 3.3.5 *Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty subset of F . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that for each fixed $p \in F$, the map $y \mapsto \text{Re}\langle p, y \rangle$ is continuous on E . Equip F with the strong topology $\delta(F, E)$. Let $T : X \rightarrow 2^E$ be weakly (respectively, strongly) lower semicontinuous on X from relative weak (respectively, strong) topology on X to the weak topology $\sigma(E, E^*)$ on E . Then T is weakly (respectively, strongly) lower demi-continuous on X .*

Proof: For each $p \in F$, define $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$f_p(z) = \sup_{u \in T(z)} \text{Re}\langle p, u \rangle \quad \text{for each } z \in X.$$

Fix any $p \in F$. Let $\lambda \in \mathbf{R}$ be given and let $A = \{z \in X : f_p(z) > \lambda\}$. Take any $z_0 \in A$. Then $f_p(z_0) = \sup_{u \in T(z_0)} \text{Re}\langle p, u \rangle > \lambda$. Choose any $u_0 \in T(z_0)$ such that $\text{Re}\langle p, u_0 \rangle > \lambda$. Let $h : E \rightarrow \mathbf{R}$ be defined by $h(u) = \text{Re}\langle p, u \rangle$ for each $u \in E$. By hypothesis h is continuous on E .

Thus $h^{-1}(\lambda, +\infty) \cap T(z_0) \neq \emptyset$, where $h^{-1}(\lambda, +\infty)$ is an open set in E . Since T is weakly (respectively, strongly) lower semicontinuous at z_0 , there exists a $\sigma(F, E)$ -open (respectively, $\delta(F, E)$ -open) neighbourhood N_{z_0} of z_0 in X such that $T(z) \cap h^{-1}(\lambda, +\infty) \neq \emptyset$ for all $z \in N_{z_0}$. Hence $f_p(z) = \sup_{u \in T(z)} h(u) = \sup_{u \in T(z)} \text{Re}\langle p, u \rangle > \lambda$ for all $z \in N_{z_0}$. Thus $N_{z_0} \subset A$. Consequently, f_p is weakly (respectively, strongly) lower semicontinuous on X . Hence T is weakly (respectively, strongly) lower demi-continuous on X . \square

The converse of Proposition 3.3.5 is not true as can be seen in the following example which is similar to Example 3.2.7.

Example 3.3.6 *Let $E = \mathbf{R}$ and $F = E^*$. Since $E^* = \mathbf{R}$, we have $F = E^* = \mathbf{R}$. Let $X = [0, 1] \subset F$ and $T : X \rightarrow 2^E$ be defined by*

$$T(x) = \begin{cases} \{1, 3\}, & \text{if } x < 1, \\ \{1, 2, 3\}, & \text{if } x = 1. \end{cases}$$

The details of the rest of this example is similar to Example 3.2.7.

Proposition 3.3.7 *Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty subset of F . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that for each fixed $p \in F$, the map $y \mapsto \text{Re}\langle p, y \rangle$ is continuous on E . Equip F with the strong topology $\delta(F, E)$. Let $T : X \rightarrow 2^E$ be weakly (respectively, strongly) upper semicontinuous on X from relative weak (respectively, strong) topology on X to the weak topology $\sigma(E, E^*)$ on E . Then T is weakly (respectively, strongly) upper demi-continuous on X .*

Proof: For each $p \in F$, define $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$f_p(z) = \sup_{u \in T(z)} \text{Re}\langle p, u \rangle \text{ for each } z \in X.$$

Fix any $p \in F$. Let $\lambda \in \mathbf{R}$ be given and let $A = \{z \in X : f_p(z) < \lambda\}$. Take any $z_0 \in A$. Then $f_p(z_0) = \sup_{u \in T(z_0)} \text{Re}\langle p, u \rangle < \lambda$. Choose any $\epsilon > 0$ such that $f_p(z_0) < \lambda - \epsilon < \lambda$. Let $h : E \rightarrow \mathbf{R}$ be defined by $h(u) = \text{Re}\langle p, u \rangle$ for each $u \in E$. By hypothesis h is continuous on E . Since $h(u) < \lambda - \epsilon < \lambda$ for all $u \in T(z_0)$, $T(z_0) \subset h^{-1}(-\infty, \lambda - \epsilon)$ which is open in E . Since T is weakly (respectively, strongly) upper semicontinuous at z_0 , there exists a $\sigma(F, E)$ -open (respectively, $\delta(F, E)$ -open) neighbourhood N_{z_0} of z_0 in X such that $T(z) \subset h^{-1}(-\infty, \lambda - \epsilon)$ for all $z \in N_{z_0}$. Thus $h(u) < \lambda - \epsilon$ for all $u \in T(z)$ and for all $z \in N_{z_0}$. Hence $\sup_{u \in T(z)} h(u) \leq \lambda - \epsilon < \lambda$ for all $z \in N_{z_0}$; i.e., $f_p(z) = \sup_{u \in T(z)} \text{Re}\langle p, u \rangle \leq \lambda - \epsilon < \lambda$ for all $z \in N_{z_0}$. Therefore $N_{z_0} \subset A$ so that A is $\sigma(F, E)$ -open (respectively, $\delta(F, E)$ -open) in X . Consequently, f_p is weakly (respectively, strongly) upper semicontinuous on X . Hence T is weakly (respectively, strongly) upper demi-continuous on X . \square

The converse of Proposition 3.3.7 is not true as can be seen in the following example which is similar to Example 3.2.9.

Example 3.3.8 *Let $E = \mathbf{R}^2$ and $F = E^*$. Since $E^* = \mathbf{R}^2$, we have $F = E^* = \mathbf{R}^2$. Let $X = \{(x, y) \in F : x^2 + y^2 \leq 1 \text{ and } x, y > 0\}$. Define $f, g : X \rightarrow 2^E$ by*

$$f(r\cos\theta, r\sin\theta) = \{(t\cos\theta, t\sin\theta) : r \leq t \leq 2\} \text{ for all } r \in (0, 1), \theta \in (0, \frac{\pi}{2}),$$

and

$$g(x, y) = \{(z, 0) : z \geq x\} \text{ for all } (x, y) \in X.$$

The details of the rest of this example is similar to Example 3.2.9.

The following definition is a generalization of (2) of the Definition 3.2.10:

Definition 3.3.9 Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty subset of F . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional. Let $T : X \rightarrow 2^E$ be a map. Then T is said to be quasi-monotone if for each $x, y \in X$,

$$\inf_{w \in T(y)} \operatorname{Re}\langle y - x, w \rangle > 0$$

whenever

$$\sup_{u \in T(x)} \operatorname{Re}\langle y - x, u \rangle > 0.$$

In order to prove our main results of this section, we shall now establish the following result:

Lemma 3.3.10 Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty convex subset of F . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that for each fixed $y \in F$, the map $p \mapsto \operatorname{Re}\langle y, p \rangle$ is continuous on E . Equip F with the strong topology $\delta(F, E)$ and let $T : X \rightarrow 2^E$ be quasi-monotone (with respect to $\langle \cdot, \cdot \rangle$) and strongly lower demi-continuous along line segments in X . Let $\hat{y} \in X$. Then

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X \iff \sup_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X.$$

Proof: Suppose that $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$. Since T is quasi-monotone, we must have $\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$ for all $x \in X$.

Conversely, suppose $\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$ for all $x \in X$. Let $x \in X$ be arbitrarily fixed. Let $z_t = tx + (1 - t)\hat{y} = \hat{y} - t(\hat{y} - x)$ for all $t \in [0, 1]$; then $z_t \in X$ as X is convex. Let $L = \{z_t : t \in [0, 1]\}$. Thus $\sup_{u \in T(z_t)} \operatorname{Re}\langle \hat{y} - z_t, u \rangle \leq 0$ for all $t \in [0, 1]$. Therefore $\sup_{u \in T(z_t)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$ for all $t \in (0, 1]$.

Since T is strongly lower demi-continuous on L , the function $f_{\hat{y}-x} : L \rightarrow \mathbf{R} \cup \{+\infty\}$ defined by

$$f_{\hat{y}-x}(z_t) = \sup_{u \in T(z_t)} \operatorname{Re}\langle \hat{y} - x, u \rangle, \text{ for each } z_t \in L,$$

is strongly lower semicontinuous on L . Then the set $A = \{z_t \in L \mid f_{\hat{y}-x}(z_t) \leq 0\}$ is strongly closed in L . Now $z_t \rightarrow \hat{y}$ in L as $t \rightarrow 0^+$. Since $z_t \in A$ for all $t \in (0, 1]$ we have $\hat{y} \in A$. Hence $f_{\hat{y}-x}(\hat{y}) = \sup_{u \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$. Since $x \in X$ is arbitrary, we have $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$. \square

When $F = E^*$ and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E , we obtain the following result:

Corollary 3.3.11 *Let E be a topological vector space, E^* be the continuous dual of E equipped with the strong topology. Let X be a non-empty convex subset of E^* and $T : X \rightarrow 2^E$ be quasi-monotone and (strongly) lower demi-continuous along line segments in X . Let $\hat{y} \in X$. Then*

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X \iff \sup_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X.$$

Theorem 3.3.12 *Let E be a topological vector space over Φ , F be a vector space over Φ and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F . Equip F with the strong topology $\delta\langle F, E \rangle$ and let X be a non-empty $\sigma\langle F, E \rangle$ -compact convex subset of F . Suppose that for each fixed $p \in E$, the map $y \mapsto \operatorname{Re}\langle y, p \rangle$ is strongly continuous on X and for each fixed $y \in F$, the map $p \mapsto \operatorname{Re}\langle y, p \rangle$ is continuous on E . Let $T : X \rightarrow 2^E$ be quasi-monotone (with respect to $\langle \cdot, \cdot \rangle$) and (strongly) lower demi-continuous along line segments in X . Then there exists $\hat{y} \in X$ such that*

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X.$$

Proof: Let $\mathcal{F} = \{L : L \text{ is a finite dimensional subspace of } F \text{ such that } X \cap L \neq \emptyset\}$ and partially order \mathcal{F} by \subset . For each $L \in \mathcal{F}$, let $X_L = X \cap L$. Note that X_L is a compact convex subset of L . For each $x, y \in X_L$, define $\phi, \psi : X_L \times X_L \rightarrow \mathbf{R}$ by

$$\phi(x, y) = \sup_{u \in T(x)} \operatorname{Re}\langle y - x, u \rangle$$

and

$$\psi(x, y) = \inf_{w \in T(y)} \operatorname{Re}\langle y - x, w \rangle.$$

Then for each $x \in X_L$, $\psi(x, x) \leq 0$ and for each $x, y \in X_L$, since T is quasi-monotone $\phi(x, y) > 0$ implies $\psi(x, y) > 0$. For each fixed $x \in X_L$, $y \mapsto \phi(x, y)$ is (strongly) lower semicontinuous on X_L . For each fixed $y \in X_L$, the set $\{x \in X_L : \psi(x, y) > 0\}$ is convex. Hence by Theorem 3.3.1, there exists $\hat{y}_L \in X_L$ such that $\phi(x, \hat{y}_L) \leq 0$ for all $x \in X_L$; i.e.,

$$\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y}_L - x, u \rangle \leq 0 \text{ for all } x \in X_L.$$

Since $\{\hat{y}_L\}_{L \in \mathcal{F}}$ is a net in X which is $\sigma\langle F, E \rangle$ -compact, there is a subnet $\{\hat{y}_{L'}\}_{L' \in \mathcal{F}'}$ of $\{\hat{y}_L\}_{L \in \mathcal{F}}$ and $\hat{y} \in X$ such that $\hat{y}_{L'} \rightarrow \hat{y}$ in the relative $\sigma\langle F, E \rangle$ -topology on X .

Fix any $x \in X$. Choose $L_0 \in \mathcal{F}$ such that $x \in L_0$. Then for any $L \in \mathcal{F}$ with $L_0 \leq L$, we have

$$\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y}_L - x, u \rangle \leq 0,$$

so that there exists $L'_0 \in \mathcal{F}'$ such that

$$\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y}_{L'} - x, u \rangle \leq 0 \text{ for all } L' \in \mathcal{F}' \text{ with } L' \geq L'_0.$$

Since $\hat{y}_{L'} \rightarrow \hat{y}$ in the relative $\sigma\langle F, E \rangle$ -topology we have,

$$\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X. \quad (3.4)$$

Since T is (strongly) lower demi-continuous along line segments in X , by (3.4) and Lemma 3.3.10 we have,

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X. \square$$

When $F = E^*$ and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E , we obtain the following result:

Corollary 3.3.13 *Let E be a topological vector space, E^* be the continuous dual of E equipped with the strong topology, X be a non-empty weak* compact convex subset*

of E^* and $T : X \rightarrow 2^E$ be quasi-monotone and (strongly) lower demi-continuous along line segments in X . Then there exists $\hat{y} \in X$ such that

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \quad \text{for all } x \in X.$$

As an application of Theorem 3.3.12 we get the following result:

Theorem 3.3.14 *Let $(E, \|\cdot\|)$ be a non-reflexive Banach space and X be a non-empty unbounded, weak* closed and convex subset of E^* with $0 \in X$. Let $T : X \rightarrow 2^E$ be quasi-monotone and (strongly) lower demi-continuous along line segments in X . Suppose*

$$\liminf_{x \in X, \|x\| \rightarrow \infty} \sup_{u \in T(x)} \operatorname{Re}\langle x, u \rangle > 0.$$

Then there exists $\hat{y} \in X$ such that

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \quad \text{for all } x \in X.$$

Proof: Let $B(0, r)$ be the closed ball in E^* at center 0 with radius r . Let $B(0, r) \cap X = X_r$. Clearly $X_n \neq \emptyset$ as $0 \in X_n$ for every $n = 1, 2, \dots$. Fix an $n \in \mathbb{N}$. Then X_n is a weak* compact convex subset of E^* . Now $T|_{X_n} : X_n \rightarrow 2^E$ is quasi-monotone and (strongly) lower demi-continuous along line segments in X_n . Hence by Theorem 3.3.12, there exists $\hat{y}_n \in X_n$ such that

$$\sup_{w \in T(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n - x, w \rangle \leq 0 \quad \text{for all } x \in X_n. \quad (3.5)$$

Then by Lemma 3.3.10 we have

$$\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y}_n - x, u \rangle \leq 0 \quad \text{for all } x \in X_n. \quad (3.6)$$

Letting $x = 0$ in (3.5) we get $\sup_{w \in T(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n, w \rangle \leq 0$. Hence

$$\liminf_n \sup_{w \in T(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n, w \rangle \leq 0. \quad (3.7)$$

If $\{\hat{y}_n\}_{n=1}^\infty$ were unbounded, then $\|\hat{y}_n\| \rightarrow \infty$ as $n \rightarrow \infty$ for a subsequence $\{\hat{y}_{n_i}\}_{i=1}^\infty$ of $\{\hat{y}_n\}_{n=1}^\infty$ so that by hypothesis,

$$\liminf_{\|\hat{y}_n\| \rightarrow \infty} \sup_{w \in T(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n, w \rangle > 0,$$

which contradicts (3.7). Hence $\{\hat{y}_n\}_{n=1}^\infty$ is bounded, say $\|\hat{y}_n\| \leq M$ for all $n = 1, 2, \dots$. Without loss of generality we can assume that there exists a subnet $\{\hat{z}_\alpha\}_{\alpha \in \Gamma}$ of $\{\hat{y}_n\}_{n=1}^\infty$ and $\hat{y} \in E^*$ such that $\hat{z}_\alpha \rightarrow \hat{y}$ in weak*-topology. For each $\alpha \in \Gamma$, let $\hat{z}_\alpha = \hat{y}_{n_\alpha}$, where $n_\alpha \rightarrow \infty$. For each $x \in X$, there is $N \in \mathbb{N}$ with $\|x\| \leq N$ and $M \leq N$. Then for each $n_\alpha \geq N$, (3.6) shows that $\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y}_{n_\alpha} - x, u \rangle \leq 0$ so that $\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$. Hence

$$\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X.$$

Since T is (strongly) lower demi-continuous along line segments in X by Lemma 3.3.10 we have

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X. \square$$

The following is a result on surjectivity:

Theorem 3.3.15 *Let $(E, \|\cdot\|)$ be a non-reflexive Banach space and $T : E^* \rightarrow \mathcal{Z}^E$ be monotone and (strongly) lower demi-continuous along line segments in E^* . Suppose that*

$$\liminf_{\|x\| \rightarrow \infty} \sup_{w \in T(x)} \frac{\operatorname{Re}\langle x, w \rangle}{\|x\|} = \infty.$$

Then for each given $w_0 \in E$, there exists $y_0 \in E^$ such that $T(y_0) = \{w_0\}$.*

Proof: Let $w_0 \in E$ be given. By hypothesis

$$\begin{aligned} \liminf_{\|x\| \rightarrow \infty} \sup_{w \in T(x)} \frac{\operatorname{Re}\langle x, w - w_0 \rangle}{\|x\|} &= \liminf_{\|x\| \rightarrow \infty} \sup_{w \in T(x)} \left[\frac{\operatorname{Re}\langle x, w \rangle}{\|x\|} - \frac{\operatorname{Re}\langle x, w_0 \rangle}{\|x\|} \right] \\ &\geq \liminf_{\|x\| \rightarrow \infty} \sup_{w \in T(x)} \left[\frac{\operatorname{Re}\langle x, w \rangle}{\|x\|} - \|w_0\| \right] \\ &= \liminf_{\|x\| \rightarrow \infty} \sup_{w \in T(x)} \frac{\operatorname{Re}\langle x, w \rangle}{\|x\|} - \|w_0\| \\ &= \infty. \end{aligned}$$

Thus

$$\liminf_{\|x\| \rightarrow \infty} \sup_{w \in T(x)} \frac{\operatorname{Re}\langle x, w - w_0 \rangle}{\|x\|} = \infty. \quad (3.8)$$

We define $T_1 : E^* \rightarrow 2^E$ by $T_1(x) = T(x) - w_0$ for all $x \in E^*$. Then T_1 is monotone and hence quasi-monotone and (strongly) lower demi-continuous along line segments in E^* . Now by (3.8) we have

$$\liminf_{\|x\| \rightarrow \infty} \sup_{w' \in T_1(x)} \frac{\operatorname{Re}\langle x, w' \rangle}{\|x\|} = \infty.$$

Hence

$$\liminf_{\|x\| \rightarrow \infty} \sup_{w' \in T_1(x)} \operatorname{Re}\langle x, w' \rangle > 0.$$

Then by Theorem 3.3.14, there exists $y_0 \in E^*$ such that

$$\sup_{u \in T_1(y_0)} \operatorname{Re}\langle y_0 - x, u \rangle \leq 0 \text{ for all } x \in E^*.$$

Thus

$$\sup_{w \in T(y_0)} \operatorname{Re}\langle y_0 - x, w - w_0 \rangle \leq 0 \text{ for all } x \in E^*.$$

It follows that $w - w_0 = 0$ for all $w \in T(y_0)$ so that $T(y_0) = \{w_0\}$. □

In the first step of the above proof, we follow the argument of Shih and Tan in [98, pp.431-440].

3.4 Generalized Variational Inequalities for Quasi-Semi-Monotone and Upper Demi-Continuous Operators

In this section some results will be obtained in topological vector spaces for the existence of solutions for some generalized variational inequalities with quasi-semi-monotone and upper demi-continuous operators (see Definition 3.3.4). As applications, some results will be obtained in non-reflexive Banach spaces. A result on surjectivity will also be discussed.

The following definition is a generalization of (4) of the Definition 3.2.10:

Definition 3.4.1 *Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty subset of F . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional. Let $T : X \rightarrow 2^E$ be a map. Then T is said to be quasi-semi-monotone if for each $x, y \in X$,*

$$\inf_{w \in T(y)} \operatorname{Re}\langle y - x, w \rangle > 0$$

whenever

$$\inf_{u \in T(x)} \operatorname{Re}\langle y - x, u \rangle > 0.$$

We shall begin with the following:

Lemma 3.4.2 *Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty convex subset of F . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that for each fixed $p \in F$, the map $y \mapsto \operatorname{Re}\langle p, y \rangle$ is continuous on E . Equip F with the strong topology $\delta(F, E)$ and let $T : X \rightarrow 2^E$ be quasi-semi-monotone (with respect to $\langle \cdot, \cdot \rangle$) and strongly upper demi-continuous along line segments in X . Let $\hat{y} \in X$. Then*

$$\inf_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X \iff \inf_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X.$$

Proof: Suppose that $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$. Since T is quasi-semi-monotone, we must have $\inf_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$ for all $x \in X$.

Conversely suppose $\inf_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$ for all $x \in X$. Let $x \in X$ be arbitrarily fixed. Let $z_t = tx + (1-t)\hat{y} = \hat{y} - t(\hat{y} - x)$ for all $t \in [0, 1]$; then $z_t \in X$ as X is convex. Let $L = \{z_t : t \in [0, 1]\}$. Then

$$\inf_{u \in T(z_t)} \operatorname{Re}\langle \hat{y} - z_t, u \rangle \leq 0 \text{ for all } t \in [0, 1].$$

Therefore

$$\inf_{u \in T(z_t)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } t \in (0, 1].$$

Since T is strongly upper demi-continuous on L , the function $g_{\hat{y}-x} : L \rightarrow \mathbf{R} \cup \{+\infty\}$ defined by

$$g_{\hat{y}-x}(z_t) = \inf_{u \in T(z_t)} \operatorname{Re}\langle \hat{y} - x, u \rangle, \text{ for each } z_t \in L,$$

is strongly lower semicontinuous on L . Then the set $A = \{z_t \in L \mid g_{\hat{y}-x}(z_t) \leq 0\}$ is strongly closed in L . Now $z_t \rightarrow \hat{y}$ in L as $t \rightarrow 0^+$. Since $z_t \in A$ for all $t \in (0, 1]$ we have $\hat{y} \in A$. Hence $g_{\hat{y}-x}(\hat{y}) = \inf_{u \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$. Since $x \in X$ is arbitrary, we have $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$. \square

When $F = E^*$ and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E , we obtain the following result:

Corollary 3.4.3 *Let E be a topological vector space over Φ , E^* be the continuous dual of E equipped with the strong topology and X be a non-empty convex subset of E^* . Let $T : X \rightarrow 2^E$ be quasi-semi-monotone and (strongly) upper demi-continuous along line segments in X . Let $\hat{y} \in X$. Then*

$$\inf_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X \iff \inf_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X.$$

Theorem 3.4.4 *Let E be a Hausdorff topological vector space over Φ , F be a vector space over Φ and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F . Equip F with the strong topology $\delta(F, E)$ and let X be a non-empty $\sigma(F, E)$ -compact convex subset of F . Suppose that for each fixed $p \in F$, the map $y \mapsto \operatorname{Re}\langle p, y \rangle$ is continuous on E and $T : X \rightarrow 2^E$ is quasi-semi-monotone (with respect to $\langle \cdot, \cdot \rangle$) and (strongly) upper demi-continuous along line segments in X .*

Suppose further that each $T(x)$ is a compact subset of E . Then there exists $\hat{y} \in X$ such that

$$\min_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X.$$

If in addition, $T(\hat{y})$ is also convex, then there exists a point $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{y} - x, \hat{w} \rangle \leq 0 \text{ for all } x \in X.$$

Proof: Let $\mathcal{F} = \{L \mid L \text{ is a finite dimensional subspace of } F \text{ such that } X \cap L \neq \emptyset\}$ and partially order \mathcal{F} by \subset . For each $L \in \mathcal{F}$, let $X_L = X \cap L$. Define $\phi, \psi : X_L \times X_L \rightarrow \mathbf{R}$ by

$$\phi(x, y) = \inf_{u \in T(x)} \operatorname{Re}\langle y - x, u \rangle \text{ for all } x, y \in X_L$$

and

$$\psi(x, y) = \inf_{w \in T(y)} \operatorname{Re}\langle y - x, w \rangle \text{ for all } x, y \in X_L.$$

Note that X_L is a compact convex subset of L . For each $x \in X_L$, $\psi(x, x) \leq 0$ and for each $x, y \in X_L$, since T is quasi-semi-monotone, $\phi(x, y) > 0$ implies $\psi(x, y) > 0$. By Lemma 3.3.2, for each fixed $x \in X_L$, $y \mapsto \phi(x, y)$ is lower semicontinuous on X_L (from its relative $(\delta\langle F, E \rangle)$ -topology to the usual topology of \mathbf{R}). For each fixed $y \in X_L$, the set $\{x \in X_L : \psi(x, y) > 0\}$ is convex. Hence by Theorem 3.3.1, there exists $\hat{y}_L \in X_L$ such that $\phi(x, \hat{y}_L) \leq 0$ for all $x \in X_L$; i.e., $\inf_{u \in T(x)} \operatorname{Re}\langle \hat{y}_L - x, u \rangle \leq 0$ for all $x \in X_L$.

Since $\{\hat{y}_L\}_{L \in \mathcal{F}}$ is a net in X which is $\sigma\langle F, E \rangle$ -compact, there is a subnet $\{\hat{y}_{L'}\}_{L' \in \mathcal{F}'}$ of $\{\hat{y}_L\}_{L \in \mathcal{F}}$ and $\hat{y} \in X$ such that $\hat{y}_{L'} \rightarrow \hat{y}$ in the relative $\sigma\langle F, E \rangle$ topology on X .

Let $x \in X$ be arbitrarily fixed. Choose $L_0 \in \mathcal{F}$ such that $x \in L_0$. Then for any $L \in \mathcal{F}$ with $L_0 \leq L$ we have $\inf_{u \in T(x)} \operatorname{Re}\langle \hat{y}_L - x, u \rangle \leq 0$ so that there exists $L'_0 \in \mathcal{F}'$ such that $\inf_{u \in T(x)} \operatorname{Re}\langle \hat{y}_{L'} - x, u \rangle \leq 0$ for all $L' \in \mathcal{F}'$ with $L' \geq L'_0$. Since $\hat{y}_{L'} \rightarrow \hat{y}$ in the relative $\sigma\langle F, E \rangle$ topology we have

$$\inf_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X. \quad (3.9)$$

Since T is also (strongly) upper demi-continuous along line segments in X , by (3.9) and Lemma 3.4.2 we have $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$. But $T(\hat{y})$ is compact in E ; hence minimum is attained and therefore $\min_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$.

If $T(\hat{y})$ is also convex, we can apply Theorem 3.2.1 and find a $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{y} - x, \hat{w} \rangle \leq 0 \text{ for all } x \in X. \square$$

When $F = E^*$ and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E , we obtain the following result:

Corollary 3.4.5 *Let E be a Hausdorff topological vector space over Φ , E^* be the continuous dual of E equipped with the strong topology and X be a non-empty weak* compact convex subset of E^* . Let $T : X \rightarrow 2^E$ be quasi-semi-monotone and (strongly) upper demi-continuous along line segments in X . Suppose that each $T(x)$ is a compact subset of E . Then there exists $\hat{y} \in X$ such that $\min_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$.*

If in addition, $T(\hat{y})$ is also convex, then there exists a point $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{y} - x, \hat{w} \rangle \leq 0$ for all $x \in X$.

As an application of Theorem 3.4.4 we get the following result:

Theorem 3.4.6 *Let $(E, \|\cdot\|)$ be a non-reflexive Banach space and X be a non-empty unbounded, weak* closed and convex subset of E^* with $0 \in X$. Let $T : X \rightarrow 2^E$ be quasi-semi-monotone and (strongly) upper demi-continuous along line segments in X . Suppose that each $T(x)$ is a (norm) compact subset of E and*

$$\liminf_{x \in X, \|x\| \rightarrow \infty} \min_{u \in T(x)} \operatorname{Re}\langle x, u \rangle > 0.$$

Then there exists $\hat{y} \in X$ such that

$$\min_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X.$$

If in addition, $T(\hat{y})$ is also convex, then there exists $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{y} - x, \hat{w} \rangle \leq 0 \text{ for all } x \in X.$$

Proof: Let $B(0, r)$ be the closed ball in E^* at center 0 with radius r . Let $X_r = B(0, r) \cap X$. Clearly $X_n \neq \emptyset$ as $0 \in X_n$ for every $n = 1, 2, \dots$. Fix an $n \in \mathbb{N}$. Then X_n is a weak* compact convex subset of E^* .

Now, $T|_{X_n} : X_n \rightarrow 2^E$ is quasi-semi-monotone and (strongly) upper demi-continuous along line segments in X_n . Also for each $x \in X_n$, $T(x)$ is (norm) compact in E . Hence by Theorem 3.4.4 there exists $\hat{y}_n \in X_n$ such that

$$\min_{w \in T(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n - x, w \rangle \leq 0 \text{ for all } x \in X_n. \quad (3.10)$$

Then by Lemma 3.4.2 we have

$$\min_{u \in T(x)} \operatorname{Re}\langle \hat{y}_n - x, u \rangle \leq 0 \text{ for all } x \in X_n. \quad (3.11)$$

Letting $x = 0$ in (3.10) we get $\min_{w \in T(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n, w \rangle \leq 0$. Hence

$$\liminf_n \min_{w \in T(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n, w \rangle \leq 0. \quad (3.12)$$

If $\{\hat{y}_n\}_{n=1}^\infty$ were unbounded, then $\|\hat{y}_{n_i}\| \rightarrow \infty$ as $i \rightarrow \infty$ for a subsequence $\{\hat{y}_{n_i}\}_{i=1}^\infty$ of $\{\hat{y}_n\}_{n=1}^\infty$ so that by hypothesis,

$$\liminf_{\|\hat{y}_n\| \rightarrow \infty} \min_{w \in T(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n, w \rangle > 0,$$

which contradicts (3.12). Hence $\{\hat{y}_n\}_{n=1}^\infty$ is bounded, say $\|\hat{y}_n\| \leq M$ for all $n = 1, 2, \dots$. Without loss of generality we can assume that there exists a subnet $\{\hat{z}_\alpha\}_{\alpha \in \Gamma}$ of $\{\hat{y}_n\}_{n=1}^\infty$ and $\hat{y} \in E^*$ such that $\hat{z}_\alpha \rightarrow \hat{y}$ in weak*-topology. For each $\alpha \in \Gamma$, let $\hat{z}_\alpha = \hat{y}_{n_\alpha}$, where $n_\alpha \rightarrow \infty$. For each $x \in X$, there is $N \in \mathbb{N}$ with $\|x\| \leq N$ and $M \leq N$. Then for each $n_\alpha \geq N$, (3.11) shows that $\min_{u \in T(x)} \operatorname{Re}\langle \hat{y}_{n_\alpha} - x, u \rangle \leq 0$ so that $\min_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$. Hence

$$\min_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X.$$

Since T is (strongly) upper demi-continuous along line segments in X , by Lemma 3.4.2, we have $\min_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$.

If $T(\hat{y})$ is also convex, we can apply Theorem 3.2.1 and find a $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{y} - x, \hat{w} \rangle \leq 0$ for all $x \in X$. \square

The following is another result on surjectivity:

Theorem 3.4.7 Let $(E, \|\cdot\|)$ be a non-reflexive Banach space and $T : E^* \rightarrow 2^E$ be semi-monotone and (strongly) upper demi-continuous along line segments in E^* . Suppose that for each $x \in E^*$, $T(x)$ is a (norm) compact convex subset of E and

$$\liminf_{\|x\| \rightarrow \infty} \inf_{u \in T(x)} \frac{\operatorname{Re}\langle x, u \rangle}{\|x\|} = \infty.$$

Then for each given $w_0 \in E$, there exists $y_0 \in E^*$ such that $w_0 \in T(y_0)$.

Proof: Let $w_0 \in E$ be given. By hypothesis

$$\begin{aligned} \liminf_{\|x\| \rightarrow \infty} \min_{w \in T(x)} \frac{\operatorname{Re}\langle x, w - w_0 \rangle}{\|x\|} &= \liminf_{\|x\| \rightarrow \infty} \min_{w \in T(x)} \left[\frac{\operatorname{Re}\langle x, w \rangle}{\|x\|} - \frac{\operatorname{Re}\langle x, w_0 \rangle}{\|x\|} \right] \\ &\geq \liminf_{\|x\| \rightarrow \infty} \min_{w \in T(x)} \left[\frac{\operatorname{Re}\langle x, w \rangle}{\|x\|} - \|w_0\| \right] \\ &= \liminf_{\|x\| \rightarrow \infty} \min_{w \in T(x)} \frac{\operatorname{Re}\langle x, w \rangle}{\|x\|} - \|w_0\| \\ &= \infty. \end{aligned}$$

Thus

$$\liminf_{\|x\| \rightarrow \infty} \min_{w \in T(x)} \frac{\operatorname{Re}\langle x, w - w_0 \rangle}{\|x\|} = \infty. \quad (3.13)$$

We define $T_1 : E^* \rightarrow 2^E$ by $T_1(x) = T(x) - w_0$ for all $x \in E^*$. Then T_1 is semi-monotone and hence quasi-semi-monotone and (strongly) upper demi-continuous along line segments in E^* . Also, for each $x \in E^*$, $T_1(x) = T(x) - w_0$ is a (norm) compact convex subset of E . Now by (3.13) we have

$$\liminf_{\|x\| \rightarrow \infty} \min_{w' \in T_1(x)} \frac{\operatorname{Re}\langle x, w' \rangle}{\|x\|} = \infty.$$

Hence

$$\liminf_{\|x\| \rightarrow \infty} \min_{w' \in T_1(x)} \operatorname{Re}\langle x, w' \rangle > 0.$$

Then by Theorem 3.4.6, there exist $y_0 \in E^*$ and $\bar{w} \in T_1(y_0)$ such that $\operatorname{Re}\langle y_0 - x, \bar{w} \rangle \leq 0$ for all $x \in E^*$; but then there exists $\hat{w} \in T(y_0)$ such that $\bar{w} = \hat{w} - w_0$, so that $\operatorname{Re}\langle y_0 - x, \hat{w} - w_0 \rangle \leq 0$ for all $x \in E^*$. It follows that $w_0 = \hat{w} \in T(y_0)$. \square

In the first step of the above proof, we follow the argument of Shih and Tan in [98, pp.431-440].

3.5 Generalized Variational Inequalities in Non-Compact Settings using Escaping Sequences

In this section we obtain some generalized variational inequalities in non-compact settings using escaping sequences introduced by Border in [15].

Theorem 3.5.1 *Let E be a topological vector space over Φ , F be a vector space over Φ and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F . Equip F with the strong topology $\delta\langle F, E \rangle$ and let X be a non-empty (convex) subset of F such that $X = \bigcup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty $\sigma\langle F, E \rangle$ -compact convex subsets of X . Suppose that for each fixed $p \in E$, the map $y \mapsto \operatorname{Re}\langle y, p \rangle$ is strongly continuous on X and for each fixed $y \in F$, the map $p \mapsto \operatorname{Re}\langle y, p \rangle$ is continuous on E . Let $T : X \rightarrow 2^E$ be quasi-monotone (with respect to $\langle \cdot, \cdot \rangle$) and (strongly) lower demi-continuous along line segments in X such that*

(a) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n \in \mathbb{N}$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ there exist $n_0 \in \mathbb{N}$ and $x_0 \in C_{n_0}$ such that*

$$\sup_{w \in T(y_{n_0})} \operatorname{Re}\langle y_{n_0} - x_0, w \rangle > 0.$$

Then there exists $\hat{y} \in X$ such that

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0 \text{ for all } x \in X.$$

Proof: Fix an arbitrary $n \in \mathbb{N}$. Note that C_n is a non-empty $\sigma\langle F, E \rangle$ -compact convex subset of F . Define $T_n : C_n \rightarrow 2^E$ by $T_n(x) = T(x)$ for each $x \in C_n$; i.e., $T_n = T|_{C_n}$. Now clearly T_n is quasi-monotone and (strongly) lower demi-continuous along line segments in C_n . Then by Theorem 3.3.12, there exists $\hat{y}_n \in C_n$ such that

$$\sup_{w \in T_n(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n - x, w \rangle \leq 0 \text{ for all } x \in C_n. \quad (3.14)$$

Note that $\{\hat{y}_n\}_{n=1}^{\infty}$ is a sequence in $X = \bigcup_{n=1}^{\infty} C_n$ with $\hat{y}_n \in C_n$ for each $n \in \mathbb{N}$.

Case 1: $\{\hat{y}_n\}_{n=1}^{\infty}$ is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$.

Then by hypothesis (a), there exist $n_0 \in \mathbb{N}$ and $x_0 \in C_{n_0}$ such that

$$\sup_{w \in T_{n_0}(\hat{y}_{n_0})} \operatorname{Re}\langle \hat{y}_{n_0} - x_0, w \rangle > 0$$

which contradicts (3.14).

Case 2: $\{\hat{y}_n\}_{n=1}^\infty$ is not escaping from X relative to $\{C_n\}_{n=1}^\infty$.

Then there exist $n_1 \in \mathbb{N}$ and a subsequence $\{\hat{y}_{n_j}\}_{j=1}^\infty$ of $\{\hat{y}_n\}_{n=1}^\infty$ such that $\hat{y}_{n_j} \in C_{n_1}$ for all $j \in \mathbb{N}$. Since C_{n_1} is $\sigma(F, E)$ -compact in F , there exist a subnet $\{\hat{z}_\alpha\}_{\alpha \in \Gamma}$ of $\{\hat{y}_{n_j}\}_{j=1}^\infty$ and $\hat{y} \in C_{n_1} \subset X$ such that $\hat{z}_\alpha \rightarrow \hat{y}$ in the relative $\sigma(F, E)$ -topology.

For each $\alpha \in \Gamma$, let $\hat{z}_\alpha = \hat{y}_{n_\alpha}$, where $n_\alpha \rightarrow \infty$. Then according to the choice of \hat{y}_{n_α} in C_{n_α} we have

$$\sup_{w \in T_{n_\alpha}(\hat{y}_{n_\alpha})} \operatorname{Re}\langle \hat{y}_{n_\alpha} - x, w \rangle \leq 0 \text{ for all } x \in C_{n_\alpha}. \quad (3.15)$$

Now by Lemma 3.3.10 and (3.15) we have

$$\sup_{u \in T_{n_\alpha}(x)} \operatorname{Re}\langle \hat{y}_{n_\alpha} - x, u \rangle \leq 0 \text{ for all } x \in C_{n_\alpha}. \quad (3.16)$$

Let $x \in X$ be arbitrarily fixed. Let $n_{\alpha_0} \geq n_1$ be such that $x \in C_{n_{\alpha_0}}$. Thus $C_{n_{\alpha_0}} \subset C_{n_\alpha}$ for all $\alpha \geq \alpha_0$. Then by (3.16) we have

$$\sup_{u \in T_{n_\alpha}(x)} \operatorname{Re}\langle \hat{y}_{n_\alpha} - x, u \rangle \leq 0 \text{ for all } \alpha \geq \alpha_0. \quad (3.17)$$

Note that $\hat{y}_{n_\alpha} \in C_{n_1} \subset C_{n_{\alpha_0}}$ for all $n_\alpha \geq n_{\alpha_0}$ and $T_{n_{\alpha_0}}(x) = T(x) = T_{n_\alpha}(x)$ for each $\alpha \geq \alpha_0$. Now letting $n_\alpha \rightarrow \infty$ in (3.17) we have $\sup_{u \in T_{n_\alpha}(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$. Since $x \in X$ is arbitrary we have

$$\sup_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X. \quad (3.18)$$

Since T is (strongly) lower demi-continuous along line segments in X , by (3.18) and Lemma 3.3.10 we have $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$. \square

When $F = E^*$ and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E , we obtain the following result:

Corollary 3.5.2 *Let E be a topological vector space, E^* be the continuous dual of E equipped with the strong topology. Let X be a non-empty (convex) subset of E^* such that $X = \bigcup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty weak* compact convex subsets of X . Let $T : X \rightarrow \mathcal{Z}^E$ be quasi-monotone and (strongly) lower demi-continuous along line segments in X such that*

(a) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n \in \mathbf{N}$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ there exist $n_0 \in \mathbf{N}$ and $x_0 \in C_{n_0}$ such that*

$$\sup_{w \in T(y_{n_0})} \operatorname{Re}\langle y_{n_0} - x_0, w \rangle > 0.$$

Then there exists $\hat{y} \in X$ such that $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$.

Theorem 3.5.3 *Let E be a Hausdorff topological vector space over Φ , F be a vector space over Φ and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F . Equip F with the strong topology $\delta(F, E)$ and let X be a non-empty (convex) subset of F such that $X = \bigcup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty $\sigma(F, E)$ -compact convex subsets of X . Suppose that for each fixed $p \in F$, the map $y \mapsto \operatorname{Re}\langle p, y \rangle$ is continuous on E and $T : X \rightarrow \mathcal{Z}^E$ is quasi-semi-monotone (with respect to $\langle \cdot, \cdot \rangle$) and (strongly) upper demi-continuous along line segments in X such that each $T(x)$ is compact. Suppose further that the following condition is satisfied:*

(a') *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n \in \mathbf{N}$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ there exist $n_0 \in \mathbf{N}$ and $x_0 \in C_{n_0}$ such that*

$$\min_{w \in T(y_{n_0})} \operatorname{Re}\langle y_{n_0} - x_0, w \rangle > 0.$$

Then there exists $\hat{y} \in X$ such that $\min_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$.

If in addition, $T(\hat{y})$ is also convex then there exists a point $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{y} - x, \hat{w} \rangle \leq 0 \text{ for all } x \in X.$$

Proof: Fix an arbitrary $n \in \mathbf{N}$. Note that C_n is a non-empty $\sigma(F, E)$ -compact convex subset of F . Define $T_n : C_n \rightarrow \mathcal{Z}^E$ by $T_n(x) = T(x)$ for each $x \in C_n$; i.e., $T_n = T|_{C_n}$.

Now clearly, T_n is quasi-semi-monotone and (strongly) upper demi-continuous along line segments in C_n . Then by Theorem 3.4.4, there exists a $\hat{y}_n \in C_n$ such that

$$\min_{w \in T_n(\hat{y}_n)} \operatorname{Re}\langle \hat{y}_n - x, w \rangle \leq 0 \text{ for all } x \in C_n. \quad (3.19)$$

Note that $\{\hat{y}_n\}_{n=1}^\infty$ is a sequence in $X = \bigcup_{n=1}^\infty C_n$ with $\hat{y}_n \in C_n$ for each $n \in \mathbb{N}$.

Case 1: $\{\hat{y}_n\}_{n=1}^\infty$ is escaping from X relative to $\{C_n\}_{n=1}^\infty$.

Then by hypothesis (a'), there exist $n_0 \in \mathbb{N}$ and $x_0 \in C_{n_0}$ such that

$$\min_{w \in T_{n_0}(\hat{y}_{n_0})} \operatorname{Re}\langle \hat{y}_{n_0} - x_0, w \rangle > 0$$

which contradicts (3.19).

Case 2: $\{\hat{y}_n\}_{n=1}^\infty$ is not escaping from X relative to $\{C_n\}_{n=1}^\infty$.

Then there exist $n_1 \in \mathbb{N}$ and a subsequence $\{\hat{y}_{n_j}\}_{j=1}^\infty$ of $\{\hat{y}_n\}_{n=1}^\infty$ such that $\hat{y}_{n_j} \in C_{n_1}$ for all $j \in \mathbb{N}$. Since C_{n_1} is $\sigma(F, E)$ -compact in F there exist a subnet $\{\hat{z}_\alpha\}_{\alpha \in \Gamma}$ of $\{\hat{y}_{n_j}\}_{j=1}^\infty$ and $\hat{y} \in C_{n_1} \subset X$ such that $\hat{z}_\alpha \rightarrow \hat{y}$ in the relative $\sigma(F, E)$ -topology.

For each $\alpha \in \Gamma$ let $\hat{z}_\alpha = \hat{y}_{n_\alpha}$, where $n_\alpha \rightarrow \infty$. Then according to our choice of \hat{y}_{n_α} in C_{n_α} we have

$$\min_{w \in T_{n_\alpha}(\hat{y}_{n_\alpha})} \operatorname{Re}\langle \hat{y}_{n_\alpha} - x, w \rangle \leq 0 \text{ for all } x \in C_{n_\alpha}. \quad (3.20)$$

Now by Lemma 3.4.2 and (3.20) we have

$$\min_{u \in T_{n_\alpha}(x)} \operatorname{Re}\langle \hat{y}_{n_\alpha} - x, u \rangle \leq 0 \text{ for all } x \in C_{n_\alpha}. \quad (3.21)$$

Let $x \in X$ be arbitrarily fixed. Let $n_{\alpha_0} \geq n_1$ be such that $x \in C_{n_{\alpha_0}}$. Thus $C_{n_{\alpha_0}} \subset C_{n_\alpha}$ for all $\alpha \geq \alpha_0$. Then by (3.21) we have

$$\min_{u \in T_{n_\alpha}(x)} \operatorname{Re}\langle \hat{y}_{n_\alpha} - x, u \rangle \leq 0 \text{ for all } \alpha \geq \alpha_0. \quad (3.22)$$

Note that $\hat{y}_{n_\alpha} \in C_{n_1} \subset C_{n_{\alpha_0}}$ for all $n_\alpha \geq n_{\alpha_0}$ and $T_{n_{\alpha_0}}(x) = T(x) = T_{n_\alpha}(x)$ for each $\alpha \geq \alpha_0$. Now letting $n_\alpha \rightarrow \infty$ in (3.22) we have $\min_{u \in T_{n_\alpha}(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0$. Since $x \in X$ is arbitrary we have

$$\min_{u \in T(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0 \text{ for all } x \in X. \quad (3.23)$$

Since T is (strongly) upper demi-continuous along line segments in X , by (3.23) and Lemma 3.4.2 we have $\min_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$.

If $T(\hat{y})$ is also convex, we can apply Theorem 3.2.1 and find a $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{y} - x, \hat{w} \rangle \leq 0$ for all $x \in X$. \square

When $F = E^*$ and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E , we obtain the following result:

Corollary 3.5.4 *Let E be a Hausdorff topological vector space, E^* be the continuous dual of E equipped with the strong topology. Let X be a non-empty (convex) subset of E^* such that $X = \cup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty weak* compact convex subsets of X . Let $T : X \rightarrow \mathcal{P}E$ be quasi-semi-monotone and (strongly) upper demi-continuous along line segments in X such that each $T(x)$ is compact. Suppose that the following condition is satisfied:*

(a') *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n \in \mathbf{N}$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ there exist $n_0 \in \mathbf{N}$ and $x_0 \in C_{n_0}$ such that*

$$\min_{w \in T(y_{n_0})} \operatorname{Re}\langle y_{n_0} - x_0, w \rangle > 0.$$

Then there exists $\hat{y} \in X$ such that $\min_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - x, w \rangle \leq 0$ for all $x \in X$.

If in addition, $T(\hat{y})$ is also convex then there exists a point $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{y} - x, \hat{w} \rangle \leq 0$ for all $x \in X$.

3.6 Generalized Variational Inequalities for Pseudo-Monotone and Demi-Monotone Operators

In this section we shall obtain some existence theorems for generalized variational inequalities and generalized complementarity problems in topological vector spaces for pseudo-monotone and demi-monotone operators. Surjectivity of demi-monotone operators will also be discussed.

We shall begin with the following:

Definition 3.6.1 *Let E be a topological vector space, X be a non-empty subset of E and $T : X \rightarrow 2^{E^*}$. If $h : X \rightarrow \mathbf{R}$, then T is said to be an h -pseudo-monotone (respectively, h -demi-monotone) operator if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y (respectively, weakly to y) with*

$$\limsup_{\alpha} \left[\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right] \leq 0$$

we have

$$\limsup_{\alpha} \left[\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right] \geq \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)$$

for all $x \in X$.

T is said to be pseudo-monotone (respectively, demi-monotone) if T is h -pseudo-monotone (respectively, h -demi-monotone) with $h \equiv 0$.

Note that if T is single-valued and is pseudo-monotone in the sense of [16, p.297], then T is pseudo-monotone in the sense of Definition 3.6.1 above. Note also that every demi-monotone operator is a pseudo-monotone operator.

Proposition 3.6.2 *Let X be a non-empty subset of a topological vector space E . If $T : X \rightarrow E^*$ is monotone and continuous from the relative weak topology on X to the weak* topology on E^* , then T is both pseudo-monotone and demi-monotone.*

Proof: Suppose $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X and $y \in X$ with $y_\alpha \rightarrow y$ (respectively, $y_\alpha \rightarrow y$ weakly) (and $\limsup_\alpha \operatorname{Re}\langle Ty_\alpha, y_\alpha - y \rangle \leq 0$). Then for any $x \in X$ and $\epsilon > 0$, there are $\beta_1, \beta_2 \in \Gamma$ with $|\operatorname{Re}\langle Ty_\alpha, y_\alpha - y \rangle| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_1$ and $|\operatorname{Re}\langle Ty_\alpha - Ty, y - x \rangle| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_2$. Choose $\beta_0 \in \Gamma$ with $\beta_0 \geq \beta_1, \beta_2$. Thus

$$\begin{aligned} \operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle &= \operatorname{Re}\langle Ty_\alpha, y_\alpha - y \rangle + \operatorname{Re}\langle Ty_\alpha, y - x \rangle \\ &\geq \operatorname{Re}\langle Ty, y_\alpha - y \rangle + \operatorname{Re}\langle Ty_\alpha, y - x \rangle \\ &= \operatorname{Re}\langle Ty, y_\alpha - y \rangle + \operatorname{Re}\langle Ty_\alpha - Ty, y - x \rangle + \operatorname{Re}\langle Ty, y - x \rangle \\ &> -\frac{\epsilon}{2} - \frac{\epsilon}{2} + \operatorname{Re}\langle Ty, y - x \rangle \text{ for all } \alpha \geq \beta_0. \end{aligned}$$

Given $\gamma \in \Gamma$, choose any $\beta \in \Gamma$ such that $\beta \geq \gamma$ and $\beta \geq \beta_0$. Then for each $\alpha \geq \beta$, $\operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle > -\epsilon + \operatorname{Re}\langle Ty, y - x \rangle$ so that

$$\begin{aligned} \sup_{\alpha \geq \gamma} \operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle &\geq \sup_{\alpha \geq \beta} \operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle \\ &> -\epsilon + \operatorname{Re}\langle Ty, y - x \rangle. \end{aligned}$$

Therefore

$$\limsup_\alpha \operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle \geq -\epsilon + \operatorname{Re}\langle Ty, y - x \rangle.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_\alpha \operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle \geq \operatorname{Re}\langle Ty, y - x \rangle.$$

Hence T is pseudo-monotone (respectively, demi-monotone). \square

The converse is not true in general as can be seen in Example 3.2.11. In Example 3.2.11, we see that T is not monotone. But it is easy to show that T is both pseudo-monotone and demi-monotone.

We shall now prove the following lemma:

Lemma 3.6.3 *Let E be a Hausdorff topological vector space, $A \in \mathcal{F}(E)$, $X = \operatorname{co}(A)$ and $T : X \rightarrow 2^{E^*}$ be upper semicontinuous from X to the weak*-topology on E^* such that each $T(x)$ is weak*-compact. Let $f : X \times X \rightarrow \mathbf{R}$ be defined by $f(x, y) = \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle$ for all $x, y \in X$. Then for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semicontinuous on X .*

Proof: Let $\lambda \in \mathbf{R}$ be given and let $x \in X$ be arbitrarily fixed. Let $C_\lambda = \{y \in X : f(x, y) \leq \lambda\}$. Suppose $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in C_λ and $y_0 \in X$ such that $y_\alpha \rightarrow y_0$. Then for each $\alpha \in \Gamma$, $\lambda \geq f(x, y_\alpha) = \inf_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle$ so that by weak*-compactness of $T(y_\alpha)$, there exists $w_\alpha \in T(y_\alpha)$ such that $\lambda \geq \inf_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle = \operatorname{Re}\langle w_\alpha, y_\alpha - x \rangle$. Since T is upper semicontinuous from X to the weak*-topology on E^* , X is compact and each $T(z)$ is weak*-compact, $\cup_{z \in X} T(z)$ is also weak*-compact. Thus there is a subnet $\{w_{\alpha'}\}_{\alpha' \in \Gamma'}$ of $\{w_\alpha\}_{\alpha \in \Gamma}$ and $w_0 \in \cup_{z \in X} T(z) \subset E^*$ with $w_{\alpha'} \rightarrow w_0$ in the weak*-topology. Again, as T is upper semicontinuous with weak*-closed values, $w_0 \in T(y_0)$.

Suppose $A = \{a_1, \dots, a_n\}$ and let $t_1, \dots, t_n \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that $y_0 = \sum_{i=1}^n t_i a_i$. For each $\alpha' \in \Gamma'$, let $t_1^{\alpha'}, \dots, t_n^{\alpha'} \geq 0$ with $\sum_{i=1}^n t_i^{\alpha'} = 1$ such that $y_{\alpha'} = \sum_{i=1}^n t_i^{\alpha'} a_i$. Since E is Hausdorff and $y_{\alpha'} \rightarrow y_0$, we must have $t_i^{\alpha'} \rightarrow t_i$ for each $i = 1, \dots, n$. Thus $\lambda \geq \operatorname{Re}\langle w_{\alpha'}, y_{\alpha'} - x \rangle = \sum_{i=1}^n t_i^{\alpha'} \operatorname{Re}\langle w_{\alpha'}, a_i - x \rangle \rightarrow \sum_{i=1}^n t_i \operatorname{Re}\langle w_0, a_i - x \rangle = \operatorname{Re}\langle w_0, \sum_{i=1}^n t_i (a_i - x) \rangle = \operatorname{Re}\langle w_0, y_0 - x \rangle$ so that $\lambda \geq \inf_{w \in T(y_0)} \operatorname{Re}\langle w, y_0 - x \rangle$ and hence $y_0 \in C_\lambda$. Thus C_λ is closed in X for each $\lambda \in \mathbf{R}$. Therefore $y \mapsto f(x, y)$ is lower semicontinuous on X . \square

We remark here that in Lemma 3.6.3, T is only assumed to be upper semicontinuous from $X = \operatorname{co}(A)$ to the weak*-topology on E^* and T is weak*-compact valued. If X is any non-empty compact subset of E , the strong topology on E^* and a strongly-compact-valued mapping are generally required, see e.g. [100, Lemma 2].

We shall now establish the following result:

Theorem 3.6.4 *Let X be a non-empty convex subset of a Hausdorff topological vector space E and $h : E \rightarrow \mathbf{R}$ be convex. Let $T : X \rightarrow 2^{E^*}$ be h -pseudo-monotone (respectively, h -demi-monotone) and be upper semicontinuous from $\operatorname{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ and such that each $T(x)$ is weak*-compact convex. Suppose there exist a non-empty compact (respectively, weakly closed and weakly compact) subset K of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\min_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.*

Proof: We first note that for each $A \in \mathcal{F}(X)$, h is continuous on $\operatorname{co}(A)$ (see e.g. [87,

Corollary 10.1.1, p.83]). Define $\phi : X \times X \rightarrow \mathbf{R}$ by $\phi(x, y) = \min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle$, for each $x, y \in X$. Then we have the following.

(a) For each $A \in \mathcal{F}(X)$ and each fixed $x \in \operatorname{co}(A)$, since E is Hausdorff and $\operatorname{co}(A)$ is compact, and the relative weak topology on $\operatorname{co}(A)$ coincides with its relative topology; it follows that $y \mapsto \phi(x, y)$ is lower semicontinuous (respectively, weakly lower semicontinuous) on $\operatorname{co}(A)$, by Lemma 3.6.3.

(b) Clearly, for each $x \in X$, $\phi(x, x) = 0$ and for each fixed $y \in X$, $x \mapsto \phi(x, y)$ is quasi-concave. It follows that for each $A \in \mathcal{F}(X)$ and each $y \in \operatorname{co}(A)$, $\min_{x \in A} [\phi(x, y) + h(y) - h(x)] \leq 0$.

(c) Suppose $A \in \mathcal{F}(X)$, $x, y \in \operatorname{co}(A)$ and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X converging to y (respectively, weakly to y) with

$$\phi(tx + (1-t)y, y_\alpha) + h(y_\alpha) - h(tx + (1-t)y) \leq 0 \quad \text{for all } \alpha \in \Gamma \text{ and all } t \in [0, 1].$$

Then for $t = 0$ we have $\phi(y, y_\alpha) + h(y_\alpha) - h(y) \leq 0$ for all $\alpha \in \Gamma$, i.e., $\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y) \leq 0$ for all $\alpha \in \Gamma$. Hence

$$\limsup_\alpha \left[\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right] \leq 0.$$

Since T is h -pseudo-monotone (respectively, h -demi-monotone), we have

$$\begin{aligned} & \limsup_\alpha [\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x)] \\ & \geq \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x). \end{aligned} \quad (3.24)$$

For $t = 1$ we also have $\phi(x, y_\alpha) + h(y_\alpha) - h(x) \leq 0$ for all $\alpha \in \Gamma$, i.e., $\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \leq 0$ for all $\alpha \in \Gamma$. It follows that

$$\limsup_\alpha \left[\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right] \leq 0. \quad (3.25)$$

By (3.24) and (3.25), $\phi(x, y) + h(y) - h(x) \leq 0$.

(d) By assumption, K is a compact and therefore closed (respectively, weakly closed and weakly compact) subset of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\min_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$; it follows that for each $y \in X \setminus K$, $\phi(x_0, y) + h(y) - h(x_0) > 0$.

(If T is h -demi-monotone, we equip E with the weak topology.) Then ϕ satisfies all the hypotheses of Theorem 2.3.4. Hence by Theorem 2.3.4, there exists a point $\hat{y} \in K$ with $\phi(x, \hat{y}) \leq h(x) - h(\hat{y})$ for all $x \in X$; in other words, $\min_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$.

Define $f : X \times T(\hat{y}) \longrightarrow \mathbf{R}$ by

$$f(x, w) = \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \text{ for all } x \in X \text{ and for all } w \in T(\hat{y}).$$

Note that for each fixed $x \in X$, $w \longmapsto f(x, w)$ is weak* continuous and convex and for each fixed $w \in T(\hat{y})$, $x \longmapsto f(x, w)$ is concave. Thus by Theorem 3.2.1 we have

$$\min_{w \in T(\hat{y})} \sup_{x \in X} (\operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)) = \sup_{x \in X} \min_{w \in T(\hat{y})} (\operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)) \leq 0.$$

Hence there exists a point $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in X. \quad (3.26)$$

Since h is defined on all of E and is convex, by (3.26) and Lemma 3.2.21, we have $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$. \square

Remark 3.6.5 *If T is h -pseudo-monotone (respectively, h -demi-monotone), Theorem 3.6.4 generalizes (respectively, extends or improves) Application 3 in [16, p.297] in the following ways (1), (2) and (3) (respectively, following ways (1) and (2)):*

(1) *T is set-valued and upper semicontinuous from $\operatorname{co}(A)$ to the weak* topology on E^* for each $A \in \mathcal{F}(X)$ instead of single-valued and continuous on any finite dimensional subspace;*

(2) *h need not be lower semicontinuous on X ;*

(3) *As noted earlier, our definition of pseudo-monotonicity, even in the single-valued case, is more general.*

Theorem 3.6.6 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E and $h : E \rightarrow \mathbf{R}$ be convex. Let $T : X \rightarrow 2^{E^*}$ be h -demi-monotone and be upper semicontinuous from $\operatorname{co}(A)$ to the weak topology on E^* for each $A \in$*

$\mathcal{F}(X)$ such that each $T(x)$ is weakly compact convex. Suppose there is $x_0 \in X$ such that

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0. \quad (3.27)$$

Then there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.

Proof: Let $\alpha = \lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0)$. Then by (3.27), $\alpha > 0$. Let $M > 0$ be such that $\|x_0\| \leq M$ and $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > \frac{\alpha}{2}$ for all $y \in X$ with $\|y\| > M$. Let $K = \{x \in X : \|x\| \leq M\}$; then K is a non-empty weakly compact subset of X . Note that for any $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > \frac{\alpha}{2} > 0$. The conclusion now follows from Theorem 3.6.4. \square

By taking $h \equiv 0$ in Theorem 3.6.4 and applying Lemma 3.2.3 we have the following existence theorem of a generalized complementarity problem:

Theorem 3.6.7 *Let X be a cone in a Hausdorff topological vector space E . Let $T : X \rightarrow 2^{E^*}$ be pseudo-monotone (respectively, demi-monotone) and be upper semicontinuous from $\operatorname{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose there exist a non-empty compact (respectively, weakly closed and weakly compact) subset K of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\min_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle > 0$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} \rangle = 0$ and $\hat{w} \in \widehat{X}$.*

By taking $h \equiv 0$ in Theorem 3.6.6 and applying Lemma 3.2.3, we have the following existence theorem for a generalized complementarity problem:

Theorem 3.6.8 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a closed cone in E and $T : X \rightarrow 2^{E^*}$ be demi-monotone and be upper semicontinuous from $\operatorname{co}(A)$ to the weak topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weakly compact convex. Suppose there is $x_0 \in X$ such that*

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle > 0.$$

Then there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} \rangle = 0$ and $\hat{w} \in \widehat{X}$.

Finally in this section we shall prove a result on the surjectivity of demi-monotone operators.

Theorem 3.6.9 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E and $T : X \rightarrow 2^{E^*}$ be demi-monotone and be upper semicontinuous from $co(A)$ to the weak topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weakly compact convex. Suppose there is $x_0 \in X$ such that*

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T(y)} Re\langle w, y - x_0 \rangle / \|y\| = \infty.$$

Then for each given $w_0 \in E^*$, there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w} - w_0, \hat{y} - x \rangle \leq 0$ for all $x \in X$. In particular, if $X = E$, then T is surjective.

Proof: Let $w_0 \in E^*$ be given. Then

$$\begin{aligned} & \lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} (\inf_{w \in T(y)} Re\langle w - w_0, y - x_0 \rangle / \|y\|) \\ &= \lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} ((\inf_{w \in T(y)} Re\langle w, y - x_0 \rangle / \|y\|) - \|w_0\|) = \infty. \end{aligned}$$

Define $T^* : X \rightarrow 2^{E^*}$ by $T^*(x) = T(x) - w_0$ for all $x \in X$. Then T^* is upper semicontinuous from $co(A)$ to the weak topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T^*(x)$ is weakly compact convex and

$$\lim_{\substack{\|y\| \rightarrow \infty \\ y \in X}} \inf_{w \in T^*(y)} Re\langle w, y - x_0 \rangle / \|y\| = \infty.$$

Suppose $y \in X$ and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X converging weakly to y with

$$\limsup_{\alpha} [\inf_{u \in T^*(y_\alpha)} Re\langle u, y_\alpha - y \rangle] \leq 0.$$

It follows that

$$\begin{aligned} & \limsup_{\alpha} [\inf_{u \in T(y_\alpha)} Re\langle u, y_\alpha - y \rangle] \\ & \leq \limsup_{\alpha} [\inf_{u \in T(y_\alpha)} Re\langle u - w_0, y_\alpha - y \rangle] + \limsup_{\alpha} Re\langle w_0, y_\alpha - y \rangle \\ & = \limsup_{\alpha} [\inf_{u \in T^*(y_\alpha)} Re\langle u, y_\alpha - y \rangle] \leq 0. \end{aligned}$$

Since T is demi-monotone,

$$\limsup_{\alpha} [\inf_{u \in T(y_{\alpha})} \operatorname{Re}\langle u, y_{\alpha} - x \rangle] \geq \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle \quad \text{for all } x \in X.$$

Hence for each $x \in X$,

$$\begin{aligned} & \inf_{w \in T^*(y)} \operatorname{Re}\langle w, y - x \rangle \\ &= \inf_{w \in T(y)} \operatorname{Re}\langle w - w_0, y - x \rangle \\ &= \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle - \operatorname{Re}\langle w_0, y - x \rangle \\ &\leq \limsup_{\alpha} [\inf_{u \in T(y_{\alpha})} \operatorname{Re}\langle u, y_{\alpha} - x \rangle] - \operatorname{Re}\langle w_0, y - x \rangle \\ &\leq \limsup_{\alpha} [\inf_{u \in T(y_{\alpha})} \operatorname{Re}\langle u - w_0, y_{\alpha} - x \rangle + \operatorname{Re}\langle w_0, y_{\alpha} - x \rangle] - \operatorname{Re}\langle w_0, y - x \rangle \\ &\leq \limsup_{\alpha} [\inf_{u \in T^*(y_{\alpha})} \operatorname{Re}\langle u, y_{\alpha} - x \rangle] + \limsup_{\alpha} \operatorname{Re}\langle w_0, y_{\alpha} - x \rangle - \operatorname{Re}\langle w_0, y - x \rangle \\ &= \limsup_{\alpha} [\inf_{u \in T^*(y_{\alpha})} \operatorname{Re}\langle u, y_{\alpha} - x \rangle]. \end{aligned}$$

Therefore T^* is also demi-monotone.

Hence by Theorem 3.6.6 with $h \equiv 0$, there exist $\hat{y} \in X$ and $\bar{w} \in T^*(\hat{y})$ such that $\operatorname{Re}\langle \bar{w}, \hat{y} - x \rangle \leq 0$ for all $x \in X$. But then there exists $\hat{w} \in T(\hat{y})$ with $\bar{w} = \hat{w} - w_0$ so that $\operatorname{Re}\langle \hat{w} - w_0, \hat{y} - x \rangle \leq 0$ for all $x \in X$. Now if $X = E$, then $\hat{w} - w_0 = 0$ so that $w_0 = \hat{w} \in T(\hat{y})$. This shows that T is surjective. \square

Chapter 4

Applications

4.1 Introduction

In this chapter we shall give several applications of the generalized variational inequalities of Chapter 3 and the minimax inequalities of Chapter 2. We shall mainly apply the generalized variational inequalities of Chapter 3 in obtaining fixed point theorems in Hilbert spaces. Applying the minimax inequalities of Chapter 2, we shall obtain some existence theorems on generalized quasi-variational inequalities as well as some existence theorems on generalized bi-quasi-variational inequalities in locally convex Hausdorff topological vector spaces. The main applications in this chapter are listed as Theorems 4.2.3, 4.2.7, 4.2.11, 4.2.13, 4.2.20, Corollaries 4.2.5, 4.2.6, 4.2.9, 4.2.10, 4.2.17, 4.2.18 and Theorems 4.3.4, 4.3.10, 4.3.16, 4.3.20, 4.3.24, 4.4.6, 4.4.8, 4.4.11, 4.4.15, 4.4.19 and 4.4.26. We shall organize Chapter 4 as follows.

In Subsection 1 of Section 4.2, we shall investigate fixed point theorems for lower or upper hemi-continuous operators T such that $I - T$ are either quasi-monotone or quasi-semi-monotone operators in Hilbert spaces which extend or improve the corresponding fixed point theorems in the literature, e.g., see [6], [18], [27] and [91]. As special cases of these fixed point theorems, we shall also obtain fixed point theorems for quasi-nonexpansive or semi-nonexpansive operators.

In Subsection 2 of Section 4.2, we shall investigate some fixed point theorems in

Hilbert spaces H for set-valued operators T which are upper semicontinuous such that $I - T$ are pseudo-monotone or demi-monotone. These fixed point theorems will extend or improve the corresponding fixed point theorems in the literature, e.g., see [6], [27] and [91].

In Subsection 1 of Section 4.3, we shall obtain some results on existence theorems of generalized quasi-variational inequalities on paracompact sets for operators which are monotone and lower hemi-continuous along line segments.

In Subsection 2 of Section 4.3, we shall obtain some results on existence theorems of generalized quasi-variational inequalities on paracompact sets for operators which are semi-monotone and upper hemi-continuous along line segments.

In Subsection 3 of Section 4.3, we shall obtain some results on existence theorems of generalized quasi-variational inequalities on paracompact sets for upper semicontinuous operators. We shall obtain these results by applying a generalized version of Ky Fan's minimax inequality [48] due to Ding and Tan [39, Theorem 1]. Our results generalize the corresponding results in [70] and [92].

In Subsection 4 of Section 4.3, we shall obtain some results on existence theorems of generalized quasi-variational inequalities on paracompact sets for operators which are strong h -pseudo-monotone and which have some kind of upper semicontinuity. Our results extend the corresponding results in [70] and [92].

In Subsection 5 of Section 4.3, we shall obtain some results on existence theorems of generalized quasi-variational inequalities on paracompact sets for operators which are h -pseudo-monotone and which have some kind of upper semicontinuity. Our results extend or improve the corresponding results in [70] and [92].

In Section 4, we shall first obtain some results on existence theorems of generalized bi-quasi-variational inequalities in locally convex Hausdorff topological vector spaces on compact sets. Then as applications of these results and the results in [100], using the concept of escaping sequences introduced by Border in [15], we shall obtain some existence theorems on non-compact generalized bi-quasi-variational inequalities and generalized bi-complementarity problems for semi-monotone operators. Our results extend

the corresponding results in [100].

Finally, in Subsection 4 of this Section, we shall introduce the concept of bi-quasi-monotone operators. Then as applications of bi-quasi-monotone operators, we shall obtain some results on existence theorems for generalized bi-quasi-variational inequalities in locally convex Hausdorff topological vector spaces on compact sets.

4.2 Fixed Point Theorems in Hilbert spaces

Throughout Chapter 4, H denotes a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the corresponding induced norm $\| \cdot \|$ and $bc(H)$ denotes the family of all non-empty bounded closed subsets of H . If X is a non-empty subset of H , we shall denote by $\partial_H(X)$ the boundary of X in H . If $x \in H$ and $r > 0$, let $B_r(x) = \{y \in H : \|x - y\| < r\}$. If d is the metric on H induced by the norm $\| \cdot \|$, let D be the Hausdorff metric on $bc(H)$ induced by d . D is defined by

$$\begin{aligned} D(A_1, A_2) &= \inf\{r > 0 : A_1 \subset B_r(A_2) \text{ and } A_2 \subset B_r(A_1)\} \\ &= \max\{\sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1)\}, \end{aligned}$$

where $d(x, A) = \inf\{\|x - y\| : y \in A\}$ and $B_r(A) = \{x \in H : d(x, A) < r\}$ for any $A \in 2^H$ and $r > 0$. (If $A = \{y\}$, $B_r(A) = B_r(y)$.)

If X is a non-empty subset of H , a map $T : X \rightarrow 2^H$ is said to be *pseudo-contractive* [6] on X if for each $x, y \in X$, and for each $w \in T(y)$, there exists $u \in T(x)$ such that $\|x - y\| \leq \|(1 + r)(x - y) - r(u - w)\|$ for all $r > 0$.

A map $T : X \rightarrow bc(H)$ is said to be *nonexpansive* on X if for each $x, y \in X$, $D(T(x), T(y)) \leq \|x - y\|$.

We shall denote by I the identity operator on H ; i.e., $I(x) = x$ for all $x \in H$.

The following result is the Proposition in [6, pp.240-241].

Proposition 4.2.1 *Let X be a non-empty subset of H .*

(a) *If $T : X \rightarrow bc(H)$ is nonexpansive such that for each $x \in X, T(x)$ is either convex or compact, then T is pseudo-contractive on X .*

(b) *If $T : X \rightarrow 2^H$ is pseudo-contractive, then $I - T$ is semi-monotone on X .*

Let K be a non-empty closed convex subset of H . For each $x \in H$, there is a unique point $\pi_K(x)$ in K such that $\|x - \pi_K(x)\| = \inf_{z \in K} \|x - z\|$. $\pi_K(x)$ is called the projection of x on K .

The projection $\pi_K(x)$ of x on K is characterized as follows [71, Theorem 1.2.3, p.9]:

Proposition 4.2.2 *Let K be a non-empty closed convex subset of H . Then for each $x \in H$ and $y \in K$, $y = \pi_K(x)$ if and only if $\operatorname{Re}\langle x - y, z - y \rangle \leq 0$, for all $z \in K$.*

4.2.1 Some Fixed Point Theorems for Quasi-Monotone or Quasi-Semi-Monotone Operators

In this section, we shall apply the main results from Section 3.2, namely, Theorems 3.2.23, 3.2.28 and 3.2.33, to obtain fixed point theorems in Hilbert spaces H for set-valued operators T which are lower or upper hemi-continuous along line segments such that $I - T$ are either quasi-monotone, quasi-semi-monotone, quasi-nonexpansive or semi-nonexpansive.

As an application of Theorem 3.2.23, we have the following fixed point theorem:

Theorem 4.2.3 *Let X be a non-empty convex subset of H and $T : X \rightarrow 2^H$ be lower hemi-continuous along line segments in X such that each $T(x)$ is closed convex and $I - T$ is quasi-monotone. Suppose there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}\langle y - w, y - x_0 \rangle > 0$. Then there exists $\hat{y} \in K$ such that $\operatorname{Re}\langle \hat{y} - w, \hat{y} - x \rangle \leq 0$ for all $x \in \overline{I_X(\hat{y})}$ and for all $w \in T(\hat{y})$. Moreover, if either $\hat{y} \in \operatorname{int}_H(X)$ or $\pi_{T(\hat{y})}(\hat{y}) \in \overline{I_X(\hat{y})}$, then \hat{y} is a fixed point of T .*

Proof: Equip H with the weak topology. Since T is lower hemi-continuous along line segments in X , $I - T : X \rightarrow 2^H$ is also lower hemi-continuous along line segments in X and satisfies all the hypotheses of Theorem 3.2.23 with $h \equiv 0$. Hence by Theorem 3.2.23, there exists $\hat{y} \in K$ such that $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle \hat{y} - w, \hat{y} - x \rangle \leq 0$ for all $x \in I_X(\hat{y})$. By continuity, we have

$$\operatorname{Re}\langle \hat{y} - w, \hat{y} - x \rangle \leq 0 \quad \text{for all } x \in \overline{I_X(\hat{y})} \text{ and for all } w \in T(\hat{y}). \quad (4.1)$$

Case 1. Suppose $\hat{y} \in \operatorname{int}_H(X)$. Fix an arbitrary $\hat{w} \in T(\hat{y})$. Take any $r > 0$ such that $B_r(\hat{y}) \subset X$. Then for each $z \in H$ with $z \neq \hat{y}$, let $u = \hat{y} + \frac{r}{2} \cdot \frac{\hat{y} - z}{\|\hat{y} - z\|}$, then $u \in B_r(\hat{y}) \subset X \subset I_X(\hat{y})$. By (4.1), $\operatorname{Re}\langle \hat{y} - \hat{w}, \frac{r}{2} \cdot \frac{\hat{y} - z}{\|\hat{y} - z\|} \rangle \leq 0$ so that $\frac{r}{2\|\hat{y} - z\|} \operatorname{Re}\langle \hat{y} - \hat{w}, z - \hat{y} \rangle \leq 0$ and hence $\operatorname{Re}\langle \hat{y} - \hat{w}, z - \hat{y} \rangle \leq 0$ for all $z \in H$.

It follows that $\operatorname{Re}\langle \hat{y} - \hat{w}, z \rangle = 0$ for all $z \in H$ so that $\hat{y} = \hat{w} \in T(\hat{y})$. As $\hat{w} \in T(\hat{y})$ is arbitrary, we conclude that in fact $T(\hat{y}) = \{\hat{y}\}$.

Case 2. Suppose $p := \pi_{T(\hat{y})}(\hat{y}) \in \overline{I_X(\hat{y})}$. Fix an arbitrary $\hat{w} \in T(\hat{y})$. By Proposition 4.2.2, we have $p \in T(\hat{y})$ and $\operatorname{Re}\langle \hat{y} - p, \hat{w} - p \rangle \leq 0$. Thus

$$\begin{aligned} 0 &\leq \operatorname{Re}\langle p - \hat{y}, \hat{w} - p \rangle \\ &= \operatorname{Re}\langle p - \hat{y}, \hat{w} - \hat{y} + \hat{y} - p \rangle \\ &= \operatorname{Re}\langle p - \hat{y}, \hat{w} - \hat{y} \rangle - \|\hat{y} - p\|^2. \end{aligned}$$

Therefore $\|\hat{y} - p\|^2 \leq \operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - p \rangle \leq 0$ by (4.1) as $p \in \overline{I_X(\hat{y})}$. Thus $\hat{y} = p = \pi_{T(\hat{y})}(\hat{y}) \in T(\hat{y})$. \square

As seen in Example 3.2.11, if we define $T : \mathbf{R}^+ \rightarrow \operatorname{bc}(\mathbf{R})$ by $T(x) = [-x, 0]$ for all $x \in \mathbf{R}^+$, then $I - T$ is quasi-monotone but not monotone.

The following result follows from Theorem 4.2.3 and Proposition 3.2.16:

Corollary 4.2.4 *Let X be a non-empty convex subset of H and $T : X \rightarrow 2^H$ be quasi-nonexpansive and lower hemi-continuous along line segments in X such that each $T(x)$ is closed and convex. Suppose there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that (i) for each $y \in K \cap \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$ and (ii) for each $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}\langle y - w, y - x_0 \rangle > 0$. Then T has a fixed point in K .*

Corollary 4.2.5 *Let X be a non-empty bounded closed convex subset of H and $T : X \rightarrow 2^H$ be quasi-nonexpansive and lower hemi-continuous along line segments in X such that each $T(x)$ is closed and convex. If $\pi_{T(y)}(y) \in \overline{I_X(y)}$ for each $y \in \partial_H(X)$, then T has a fixed point in X .*

Corollary 4.2.6 *Let X be a non-empty bounded closed convex subset of H and $T : X \rightarrow 2^X$ be quasi-nonexpansive and lower hemi-continuous along line segments in X such that each $T(x)$ is closed and convex. Then T has a fixed point in X .*

As an application of Theorem 3.2.28, we have the following fixed point theorem:

Theorem 4.2.7 *Let X be a non-empty convex subset of H and $T : X \rightarrow 2^H$ be upper hemi-continuous along line segments in X such that each $T(x)$ is weakly compact*

convex and $I - T$ is quasi-monotone. Suppose there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}\langle y - w, y - x_0 \rangle > 0$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in \overline{I_X(\hat{y})}$. Moreover, if either $\hat{y} \in \operatorname{int}_H(X)$ or $\pi_{T(\hat{y})}(\hat{y}) \in \overline{I_X(\hat{y})}$, then \hat{y} is a fixed point of T , i.e., $\hat{y} \in T(\hat{y})$.

Proof: Equip H with the weak topology. Since T is upper hemi-continuous along line segments in X , $I - T : X \rightarrow 2^H$ is also upper hemi-continuous along line segments in X . Note that $I - T$ satisfies all the hypotheses of Theorem 3.2.28 with $h \equiv 0$. Thus by Theorem 3.2.28, there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in I_X(\hat{y})$. By continuity,

$$\operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0 \quad \text{for all } x \in \overline{I_X(\hat{y})}. \quad (4.2)$$

Case 1. Suppose $\hat{y} \in \operatorname{int}_H(X)$, then there exists $r > 0$ such that $B_r(\hat{y}) \subset X$. Then for each $z \in H$ with $z \neq \hat{y}$, let $u = \hat{y} + \frac{r}{2} \cdot \frac{\hat{y} - z}{\|\hat{y} - z\|}$, then $u \in B_r(\hat{y}) \subset X \subset I_X(\hat{y})$. Thus $\operatorname{Re}\langle \hat{y} - \hat{w}, \frac{r}{2} \cdot \frac{z - \hat{y}}{\|\hat{y} - z\|} \rangle \leq 0$ so that $\frac{r}{2\|\hat{y} - z\|} \operatorname{Re}\langle \hat{y} - \hat{w}, z - \hat{y} \rangle \leq 0$ and hence $\operatorname{Re}\langle \hat{y} - \hat{w}, z - \hat{y} \rangle \leq 0$ for all $z \in H$.

It follows that $\operatorname{Re}\langle \hat{y} - \hat{w}, z \rangle = 0$ for all $z \in H$ so that $\hat{y} = \hat{w} \in T(\hat{y})$.

Case 2. Suppose $p := \pi_{T(\hat{y})}(\hat{y}) \in \overline{I_X(\hat{y})}$. By Proposition 4.2.2, we have

$$p \in T(\hat{y}) \text{ and } \operatorname{Re}\langle \hat{y} - p, w - p \rangle \leq 0 \quad \text{for all } w \in T(\hat{y}). \quad (4.3)$$

Since $\hat{w} \in T(\hat{y})$, by (4.3) we have

$$\begin{aligned} 0 &\leq \operatorname{Re}\langle p - \hat{y}, \hat{w} - p \rangle \\ &= \operatorname{Re}\langle p - \hat{y}, \hat{w} - \hat{y} + \hat{y} - p \rangle \\ &= \operatorname{Re}\langle p - \hat{y}, \hat{w} - \hat{y} \rangle - \|\hat{y} - p\|^2. \end{aligned}$$

Therefore $\|\hat{y} - p\|^2 \leq \operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - p \rangle \leq 0$ by (4.2). Thus $\hat{y} = p = \pi_{T(\hat{y})}(\hat{y}) \in T(\hat{y})$. \square

The following result follows from Theorem 4.2.7 and Proposition 3.2.16:

Corollary 4.2.8 *Let X be a non-empty convex subset of H and $T : X \rightarrow 2^H$ be quasi-nonexpansive and upper hemi-continuous along line segments in X such that*

each $T(x)$ is weakly compact convex. Suppose there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that (i) for each $y \in K \cap \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$ and (ii) for each $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}\langle y - w, y - x_0 \rangle > 0$. Then T has a fixed point in K .

Corollary 4.2.9 *Let X be a non-empty bounded closed convex subset of H and $T : X \rightarrow 2^H$ be quasi-nonexpansive and upper hemi-continuous along line segments in X such that each $T(x)$ is weakly compact convex. If $\pi_{T(y)}(y) \in \overline{I_X(y)}$ for each $y \in \partial_H(X)$, then T has a fixed point in X .*

Corollary 4.2.10 *Let X be a non-empty bounded closed convex subset of H and $T : X \rightarrow 2^X$ be quasi-nonexpansive and upper hemi-continuous along line segments in X such that each $T(x)$ is weakly compact convex. Then T has a fixed point in X .*

As an application of Theorem 3.2.33 with $h \equiv 0$ we have the following fixed point theorem:

Theorem 4.2.11 *Let X be a non-empty convex subset of H , $T : X \rightarrow 2^H$ be upper hemi-continuous along line segments in X such that each $T(x)$ is compact convex and $I - T$ is quasi-semi-monotone. Suppose there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}\langle y - w, y - x_0 \rangle > 0$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in \overline{I_X(\hat{y})}$. Moreover, if either $\hat{y} \in \operatorname{int}_H(X)$ or $\pi_{T(\hat{y})}(\hat{y}) \in \overline{I_X(\hat{y})}$, then \hat{y} is a fixed point of T .*

Proof: Equip H with the weak topology. Since T is upper hemi-continuous along line segments in X , $I - T : X \rightarrow 2^H$ is also upper hemi-continuous along line segments in X and satisfies all the hypotheses of Theorem 3.2.33 with $h \equiv 0$. Hence by Theorem 3.2.33, there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in I_X(\hat{y})$. By continuity of \hat{w} , $\operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in \overline{I_X(\hat{y})}$. Now the rest of the proof is similar to that of Theorem 4.2.7 and the conclusion follows. \square

Remark 4.2.12 *Theorem 4.2.11 extends Theorem 6 of Bae-Kim-Tan in [6, pp.242-243] in the following ways:*

- (1) $I - T$ is quasi-semi-monotone instead of T is pseudo-contractive [6, p.240];
 (2) T is upper hemi-continuous along line segments instead of upper semicontinuous along line segments in X .

Note however that the coercive conditions in our Theorem 4.2.11 here and in Theorem 6 of [6] are not comparable.

The following result is an immediate consequence of Theorem 4.2.11:

Theorem 4.2.13 *Let X be a non-empty convex subset of H and $T : X \rightarrow 2^H$ be upper hemi-continuous along line segments in X such that each $T(x)$ is compact convex and $I - T$ is quasi-semi-monotone. Suppose there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that (i) for each $y \in K \cap \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$ and (ii) for each $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}\langle y - w, y - x_0 \rangle > 0$. Then T has a fixed point in K .*

As seen in Example 3.2.12, if we define $T : \mathbf{R}^+ \rightarrow bc(\mathbf{R})$ by

$$T(x) = \begin{cases} [\frac{x^2-1}{x}, 0], & \text{if } 0 < x < 1, \\ [0, \frac{x^2-1}{x}], & \text{if } x \geq 1, \end{cases}$$

then $I - T$ is quasi-semi-monotone but not semi-monotone.

Corollary 4.2.14 *Let X be a non-empty bounded closed convex subset of H and $T : X \rightarrow 2^H$ be semi-nonexpansive and upper hemi-continuous along line segments in X such that each $T(x)$ is compact convex. If $\pi_{T(y)}(y) \in \overline{I_X(y)}$ for each $y \in \partial_H(X)$, then T has a fixed point in X .*

Corollary 4.2.15 *Let X be a non-empty bounded closed convex subset of H and $T : X \rightarrow 2^X$ be semi-nonexpansive and upper hemi-continuous along line segments in X such that each $T(x)$ is compact convex. Then T has a fixed point in X .*

By Proposition 4.2.1 which is the Proposition in [6], if T is nonexpansive, then $I - T$ is semi-monotone; it follows from Proposition 3.2.18 that T is semi-nonexpansive. Also

observe that a set-valued nonexpansive operator is necessarily upper semicontinuous (and also lower semicontinuous) so that it is upper hemi-continuous by Proposition 3.2.8. Thus we have:

Corollary 4.2.16 *Let X be a non-empty convex subset of H and $T : X \rightarrow bc(H)$ be nonexpansive such that each $T(x)$ is compact convex. Suppose there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that (i) for each $y \in K \cap \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$ and (ii) for each $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}(y - w, y - x_0) > 0$. Then T has a fixed point in K .*

Corollary 4.2.17 *Let X be a non-empty bounded closed convex subset of H and $T : X \rightarrow 2^H$ be nonexpansive such that each $T(x)$ is compact convex. If $\pi_{T(y)}(y) \in \overline{I_X(y)}$ for each $y \in \partial_H(X)$, then T has a fixed point in X .*

Corollary 4.2.18 *Let X be a non-empty bounded closed convex subset of H and $T : X \rightarrow 2^X$ be nonexpansive such that each $T(x)$ is compact convex. Then T has a fixed point in X .*

Remark 4.2.19 *Theorem 1 of Browder [18] states that if X is a non-empty bounded closed convex subset of H and $f : X \rightarrow X$ is nonexpansive, then f has a fixed point in X . Thus Corollaries 4.2.6 and 4.2.10 (respectively, Corollary 4.2.18) generalize Browder's fixed point theorem [18, Theorem 1] to set-valued quasi-nonexpansive (respectively, nonexpansive) operators while Corollaries 4.2.5 and 4.2.9 (respectively, Corollary 4.2.17) generalize Browder's fixed point theorem [18, Theorem 1] to set-valued quasi-nonexpansive (respectively, nonexpansive) operators which need not be self-maps. Note that Corollary 4.2.6 generalizes Browder's fixed point theorem [18, Theorem 1] even for the single valued quasi-nonexpansive map T .*

Finally we observe that in all fixed point theorems stated above, (1) when T is lower hemi-continuous along line segments, T is required to have closed-convex values, (2) when T is upper hemi-continuous along line segments and quasi-monotone, T is required to have weakly-compact-convex values and (3) when T is upper hemi-continuous along line segments and quasi-semi-monotone, T is required to have compact-convex values.

4.2.2 Some Fixed Point Theorems for Pseudo-Monotone and Demi-Monotone Operators

In this section we shall obtain some fixed point theorems in Hilbert spaces H for set-valued operators T which have some kind of upper semicontinuity and such that $I - T$ is pseudo-monotone or demi-monotone.

As an application of Theorem 3.6.4, we have the following fixed point theorem:

Theorem 4.2.20 *Let X be a non-empty convex subset of H and $T : X \rightarrow 2^H$ be upper semicontinuous from $co(A)$ to the weak topology on H for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weakly compact convex and $I - T$ is pseudo-monotone (respectively, demi-monotone). Suppose there exist a non-empty compact (respectively, weakly compact) subset K of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\inf_{w \in T(y)} \operatorname{Re}\langle y - w, y - x_0 \rangle > 0$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that*

$$\operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0 \quad \text{for all } x \in \overline{I_X(\hat{y})}.$$

Moreover, if either \hat{y} is an interior point of X in H or $p(\hat{y}) \in \overline{I_X(\hat{y})}$, where $p(\hat{y})$ is the projection of \hat{y} on $T(\hat{y})$, then \hat{y} is a fixed point of T , i.e., $\hat{y} \in T(\hat{y})$.

Proof: (If $I - T$ is demi-monotone, we equip H with the weak topology.) Since T is upper semicontinuous from $co(A)$ to the weak topology on H for each $A \in \mathcal{F}(X)$, $I - T : X \rightarrow 2^H$ is also upper semicontinuous from $co(A)$ to the weak topology on H for each $A \in \mathcal{F}(X)$ and satisfies all the hypotheses of Theorem 3.6.4 with $h \equiv 0$. By Theorem 3.6.4, there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in I_X(\hat{y})$. By continuity,

$$\operatorname{Re}\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0 \quad \text{for all } x \in \overline{I_X(\hat{y})}. \quad (4.4)$$

Case 1. Suppose \hat{y} is an interior point of X in H , i.e., $\hat{y} \in \operatorname{int}_H X$, then there exists $r > 0$ such that $B_r(\hat{y}) \subset X$. Then for each $z \in H$ with $z \neq \hat{y}$, let $u = \hat{y} + \frac{r}{2} \cdot \frac{\hat{y} - z}{\|\hat{y} - z\|}$, then $u \in B_r(\hat{y}) \subset X \subset I_X(\hat{y})$. Thus $\operatorname{Re}\langle \hat{y} - \hat{w}, \frac{r}{2} \cdot \frac{z - \hat{y}}{\|\hat{y} - z\|} \rangle \leq 0$ so that $\frac{r}{2\|\hat{y} - z\|} \operatorname{Re}\langle \hat{y} - \hat{w}, z - \hat{y} \rangle \leq 0$ and hence $\operatorname{Re}\langle \hat{y} - \hat{w}, z - \hat{y} \rangle \leq 0$ for all $z \in H$.

It follows that $Re\langle \hat{y} - \hat{w}, z \rangle = 0$ for all $z \in H$ so that $\hat{y} = \hat{w} \in T(\hat{y})$.

Case 2. Suppose $p(\hat{y}) \in \overline{I_X(\hat{y})}$. By Proposition 4.2.2 of Section 4.2, the projection $p(\hat{y})$ of \hat{y} on $T(\hat{y})$ has the following property:

$$p(\hat{y}) \in T(\hat{y}) \text{ and } Re\langle \hat{y} - p(\hat{y}), w - p(\hat{y}) \rangle \leq 0 \text{ for all } w \in T(\hat{y}). \quad (4.5)$$

Since $\hat{w} \in T(\hat{y})$, by (4.5) we have

$$\begin{aligned} 0 &\leq Re\langle p(\hat{y}) - \hat{y}, \hat{w} - p(\hat{y}) \rangle \\ &= Re\langle p(\hat{y}) - \hat{y}, \hat{w} - \hat{y} + \hat{y} - p(\hat{y}) \rangle \\ &= Re\langle p(\hat{y}) - \hat{y}, \hat{w} - \hat{y} \rangle - \|\hat{y} - p(\hat{y})\|^2. \end{aligned}$$

Therefore

$$\|\hat{y} - p(\hat{y})\|^2 \leq Re\langle \hat{y} - \hat{w}, \hat{y} - p(\hat{y}) \rangle \leq 0 \text{ by (4.4).}$$

Thus $\hat{y} = p(\hat{y}) \in T(\hat{y})$. □

The following fixed point theorem is an immediate consequence of Theorem 4.2.20:

Theorem 4.2.21 *Let X be a non-empty convex subset of H and $T : X \rightarrow bc(H)$ be upper semicontinuous from $co(A)$ to the weak topology on H for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weakly compact convex and $I - T$ is pseudo-monotone (respectively, demi-monotone). Suppose there exist a non-empty compact (respectively, weakly compact) subset K of X and $x_0 \in K$ such that (i) for each $y \in K \cap \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$ and (ii) for each $y \in X \setminus K$, $\inf_{w \in T(y)} Re\langle y - w, y - x_0 \rangle > 0$. Then T has a fixed point in K .*

Corollary 4.2.22 *Let X be a non-empty compact (respectively, bounded closed) convex subset of H and $T : X \rightarrow bc(H)$ be upper semicontinuous from $co(A)$ to the weak topology on H for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weakly compact convex and $I - T$ is pseudo-monotone (respectively, demi-monotone). Suppose that for each $y \in \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$. Then T has a fixed point in X .*

Corollary 4.2.23 *Let X be a non-empty compact (respectively, bounded closed) convex subset of H and $T : X \rightarrow bc(X)$ be upper semicontinuous from $co(A)$ to the relative weak topology on X for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weakly compact convex and $I - T$ is pseudo-monotone (respectively, demi-monotone). Then T has a fixed point in X .*

4.3 Generalized Quasi-Variational Inequalities on Non-Compact Sets

Let X be a non-empty subset of a topological vector space E . Given the maps $S : X \rightarrow 2^X$ and $T : X \rightarrow E^*$, the quasi-variational inequality problem (QVI) is to find a point $\hat{y} \in S(\hat{y})$ such that $\text{Re}\langle T(\hat{y}), \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. The QVI was introduced by Bensoussan and Lions in 1973 (see, e.g., [13]) in connection with impulse control. Again, if we consider a set-valued map $T : X \rightarrow 2^{E^*}$, then the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $\text{Re}\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. The GQVI was introduced by Chan and Pang [23] in 1982 if $E = \mathbf{R}^n$ and by Shih and Tan [92] in 1985 if E is infinite dimensional.

If X is a topological space and $\{U_\alpha : \alpha \in \mathcal{A}\}$ is an open cover for X , then a partition of unity subordinated to the open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ is a family $\{\beta_\alpha : \alpha \in \mathcal{A}\}$ of continuous real-valued functions $\beta_\alpha : X \rightarrow [0, 1]$ such that (1) $\beta_\alpha(y) = 0$ for all $y \in X \setminus U_\alpha$, (2) $\{\text{support } \beta_\alpha : \alpha \in \mathcal{A}\}$ is locally finite and (3) $\sum_{\alpha \in \mathcal{A}} \beta_\alpha(y) = 1$ for each $y \in X$.

The following result is Lemma 1 of Shih and Tan in [92, pp.334-335]:

Lemma 4.3.1 *Let X be a non-empty subset of a Hausdorff topological vector space E and $S : X \rightarrow 2^E$ be an upper semicontinuous map such that $S(x)$ is a bounded subset of E for each $x \in X$. Then for each continuous linear functional p on E , the map $f_p : X \rightarrow \mathbf{R}$ defined by $f_p(y) = \sup_{x \in S(y)} \text{Re}\langle p, x \rangle$ is upper semicontinuous; i.e., for each $\lambda \in \mathbf{R}$, the set $\{y \in X : f_p(y) = \sup_{x \in S(y)} \text{Re}\langle p, x \rangle < \lambda\}$ is open in X .*

The following result is Lemma 3 of Takahashi in [103, p.177] (see also Lemma 3 in [100, pp.71-72]):

Lemma 4.3.2 *Let X and Y be topological spaces, $f : X \rightarrow \mathbf{R}$ be non-negative and continuous and $g : Y \rightarrow \mathbf{R}$ be lower semicontinuous. Then the map $F : X \times Y \rightarrow \mathbf{R}$, defined by $F(x, y) = f(x)g(y)$ for all $(x, y) \in X \times Y$, is lower semicontinuous.*

The following result is essentially Theorem 1 of Bae-Kim-Tan in [6, p.231]:

Theorem 4.3.3 *Let E be a topological vector space, X be a non-empty convex subset of E and $f, g : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ be such that*

(a) *$g(x, x) \leq 0$ for all $x \in X$ and $f(x, y) \leq g(x, y)$ for all $x, y \in X$:*

(b) *for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semicontinuous on non-empty compact subsets of X ;*

(c) *for each fixed $y \in X$, the set $\{x \in X : g(x, y) > 0\}$ is convex;*

(d) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \text{co}(X_0 \cup \{y\})$ with $f(x, y) > 0$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

4.3.1 Generalized Quasi-Variational Inequalities for Lower Hemi-Continuous Operators

In this section we shall obtain some existence theorems for generalized quasi-variational inequalities for monotone and lower hemi-continuous operators on paracompact sets.

We shall first establish the following result:

Theorem 4.3.4 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex subset of E . Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be monotone and lower hemi-continuous along line segments in X to the weak*-topology on E^* . Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} \sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0\}$$

is open in X . Suppose further that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in \operatorname{co}(X_0 \cup \{y\}) \cap S(y)$ with $\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Then there exists a point $\hat{y} \in K$ such that

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: We divide the proof into two steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\sup_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exist $x \in S(y)$ and $u \in T(x)$ such that $\operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $\operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0$. For each $y \in X$, set

$$\gamma(y) := \sup_{x \in S(y)} [\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)].$$

Let $V_0 := \{y \in X \mid \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$V_p := \{y \in X : \operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 4.3.1 and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X . Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$ (see, e.g., Theorem VIII.4.2 of Dugundji in [41]); that is for each $p \in E^*$, $\beta_p : X \rightarrow [0, 1]$ and $\beta_0 : X \rightarrow [0, 1]$ are continuous functions such that for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$ and $\beta_0(y) = 0$ for all $y \in X \setminus V_0$ and $\{\operatorname{support} \beta_0, \operatorname{support} \beta_p : p \in E^*\}$ is locally finite and $\beta_0(y) + \sum_{p \in E^*} \beta_p(y) = 1$ for each $y \in X$. Define $\phi, \psi : X \times X \rightarrow \mathbf{R}$ by

$$\phi(x, y) = \beta_0(y) \left[\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle,$$

and

$$\psi(x, y) = \beta_0(y) \left[\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle,$$

for each $x, y \in X$. Then we have the following.

(1) For each $x, y \in X$, since T is monotone, $\phi(x, y) \leq \psi(x, y)$ and $\psi(x, x) = 0$ for all $x \in X$.

(2) For each fixed $x \in X$ and each fixed $u \in T(x)$, the map

$$y \longmapsto \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)$$

is continuous on X and therefore the map

$$y \longmapsto \beta_0(y) \left[\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) \right]$$

is lower semicontinuous on X by Lemma 4.3.2. Also for each fixed $x \in X$,

$$y \longmapsto \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle$$

is continuous on X . Hence, for each fixed $x \in X$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on X .

(3) Clearly, for each $y \in X$, the set $\{x \in X : \psi(x, y) > 0\}$ is convex.

(4) By hypothesis, there exists a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in \text{co}(X_0 \cup \{y\}) \cap S(y)$ such that $\sup_{u \in T(x)} \text{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Thus $\beta_0(y)[\sup_{u \in T(x)} \text{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0$ whenever $\beta_0(y) > 0$. Also $\text{Re}\langle p, y - x \rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently, $\phi(x, y) = \beta_0(y)[\sup_{u \in T(x)} \text{Re}\langle u, y - x \rangle + h(y) - h(x)] + \sum_{p \in E^*} \beta_p(y) \text{Re}\langle p, y - x \rangle > 0$.

Then ϕ and ψ satisfy all the hypotheses of Theorem 4.3.3. Thus by Theorem 4.3.3, there exists $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$, i.e.,

$$\beta_0(\hat{y}) \left[\sup_{u \in T(x)} \text{Re}\langle u, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \text{Re}\langle p, \hat{y} - x \rangle \leq 0 \quad (4.6)$$

for all $x \in X$.

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0 = \Sigma$ so that $\gamma(\hat{y}) > 0$. Choose $\hat{x} \in S(\hat{y}) \subset X$ such that

$$\sup_{u \in T(\hat{x})} \text{Re}\langle u, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \geq \frac{\gamma(\hat{y})}{2} > 0;$$

it follows that

$$\beta_0(\hat{y}) \left[\sup_{u \in T(\hat{x})} \text{Re}\langle u, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] > 0.$$

If $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, then $\hat{y} \in V_p$ and hence

$$\text{Re}\langle p, \hat{y} \rangle > \sup_{x \in S(\hat{y})} \text{Re}\langle p, x \rangle \geq \text{Re}\langle p, \hat{x} \rangle$$

so that $\text{Re}\langle p, \hat{y} - \hat{x} \rangle > 0$. Then note that

$$\beta_p(\hat{y}) \text{Re}\langle p, \hat{y} - \hat{x} \rangle > 0 \quad \text{whenever} \quad \beta_p(\hat{y}) > 0 \quad \text{for} \quad p \in E^*.$$

Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, it follows that

$$\phi(\hat{x}, \hat{y}) = \beta_0(\hat{y}) \left[\sup_{u \in T(\hat{x})} \text{Re}\langle u, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \text{Re}\langle p, \hat{y} - \hat{x} \rangle > 0,$$

which contradicts (4.6). This contradiction proves Step 1.

Step 2.

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}).$$

Indeed, from Step 1, $\hat{y} \in S(\hat{y})$ which is a convex subset of X , and

$$\sup_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}).$$

Hence by Lemma 3.2.19, we have

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}). \square$$

If X is compact, Theorem 4.3.4 reduces to the following:

Theorem 4.3.5 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E . Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be monotone and lower hemi-continuous along line segments in X to the weak*-topology on E^* . Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} \sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0\}$$

is open in X . Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Remark 4.3.6 *Theorem 4.3.4 and Theorem 4.3.5 generalize Theorem 1 of Shih-Tan in [92, p.335].*

Note that if X is also bounded in Theorem 4.3.4 and the map $S : X \rightarrow 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0\}$, T is lower semicontinuous at some point x in $S(y)$ with $\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$, then the set Σ in Theorem 4.3.4 is always open in X as can be seen in the proof of the following:

Theorem 4.3.7 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex and bounded subset of E . Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be monotone and be lower hemi-continuous along line segments in X to the weak*-topology on E^* . Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0\}$, T is lower semicontinuous at some point x in $S(y)$ with $\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Suppose further that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in \operatorname{co}(X_0 \cup \{y\}) \cap S(y)$ with $\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Then there exists a point $\hat{y} \in K$ such that*

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: By virtue of Theorem 4.3.4, we need only show that the set

$$\Sigma := \{y \in X : \sup_{x \in S(y)} [\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Indeed, let $y_0 \in \Sigma$; then by hypothesis, T is lower semicontinuous at some point x_0 in $S(y_0)$ with $\sup_{u \in T(x_0)} \operatorname{Re}\langle u, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$. Hence there exists $u_0 \in T(x_0)$ such that $\operatorname{Re}\langle u_0, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$. Let

$$\alpha := \operatorname{Re}\langle u_0, y_0 - x_0 \rangle + h(y_0) - h(x_0).$$

Then $\alpha > 0$. Also let

$$U_1 := \{u \in E^* : \sup_{z_1, z_2 \in X} |\langle u - u_0, z_1 - z_2 \rangle| < \frac{\alpha}{6}\}.$$

Then U_1 is a strongly open neighborhood of u_0 in E^* . Since T is lower semicontinuous at x_0 and $U_1 \cap T(x_0) \neq \emptyset$, there exists an open neighborhood V_1 of x_0 in X such that $T(x) \cap U_1 \neq \emptyset$ for all $x \in V_1$.

As the map $x \mapsto \operatorname{Re}\langle u_0, x_0 - x \rangle + h(x_0) - h(x)$ is continuous at x_0 , there exists an open neighborhood V_2 of x_0 in X such that

$$|\operatorname{Re}\langle u_0, x_0 - x \rangle + h(x_0) - h(x)| < \frac{\alpha}{6} \quad \text{for all } x \in V_2.$$

Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X . Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semicontinuous at y_0 , there exists an open neighborhood N_1 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_1$.

Since the map $y \mapsto \operatorname{Re}\langle u_0, y - y_0 \rangle + h(y) - h(y_0)$ is continuous at y_0 , there exists an open neighborhood N_2 of y_0 in X such that

$$|\operatorname{Re}\langle u_0, y - y_0 \rangle + h(y) - h(y_0)| < \frac{\alpha}{6} \quad \text{for all } y \in N_2.$$

Let $N_0 := N_1 \cap N_2$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

- (i) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_1$; so we can choose any $x_1 \in S(y_1) \cap V_0$;
- (ii) $|\operatorname{Re}\langle u_0, y_1 - y_0 \rangle + h(y_1) - h(y_0)| < \frac{\alpha}{6}$ as $y_1 \in N_2$;
- (iii) $T(x_1) \cap U_1 \neq \emptyset$ as $x_1 \in V_1$; choose any $u_1 \in T(x_1) \cap U_1$ so that

$$\sup_{z_1, z_2 \in X} |\langle u_1 - u_0, z_1 - z_2 \rangle| < \frac{\alpha}{6};$$

- (iv) $|\operatorname{Re}\langle u_0, x_0 - x_1 \rangle + h(x_0) - h(x_1)| < \frac{\alpha}{6}$ as $x_1 \in V_2$.

It follows that

$$\begin{aligned} & \operatorname{Re}\langle u_1, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\ &= \operatorname{Re}\langle u_1 - u_0, y_1 - x_1 \rangle + \operatorname{Re}\langle u_0, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\ &\geq -\frac{\alpha}{6} + \operatorname{Re}\langle u_0, y_1 - y_0 \rangle + h(y_1) - h(y_0) \\ &\quad + \operatorname{Re}\langle u_0, y_0 - x_0 \rangle + h(y_0) - h(x_0) \\ &\quad + \operatorname{Re}\langle u_0, x_0 - x_1 \rangle + h(x_0) - h(x_1) \quad (\text{by (iii)}), \\ &\geq -\frac{\alpha}{6} - \frac{\alpha}{6} + \alpha - \frac{\alpha}{6} = \frac{\alpha}{2} > 0 \quad (\text{by (ii) and (iv)}); \end{aligned}$$

therefore

$$\sup_{x \in S(y_1)} \left[\sup_{u \in T(x)} \operatorname{Re}\langle u, y_1 - x \rangle + h(y_1) - h(x) \right] > 0$$

as $x_1 \in S(y_1)$ and $u_1 \in T(x_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$, so that Σ is open in X . This completes the proof. \square

If X is compact, Theorem 4.3.7 reduces to the following:

Theorem 4.3.8 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E . Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be monotone and be lower hemi-continuous along line segments in X to the weak*-topology on E^* . Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0\}$, T is lower semicontinuous at some point x in $S(y)$ with $\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Then there exists a point $\hat{y} \in X$ such that*

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Remark 4.3.9 *Theorem 4.3.7 and Theorem 4.3.8 generalize Theorem 2 of Shih-Tan in [92, p.338].*

4.3.2 Generalized Quasi-Variational Inequalities for Upper Hemi-Continuous Operators

In this section we shall obtain some existence theorems for generalized quasi-variational inequalities for semi-monotone and upper hemi-continuous operators on paracompact sets.

We shall now establish the following result:

Theorem 4.3.10 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex and bounded subset of E . Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be semi-monotone and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is strongly compact convex. Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Suppose further that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in \operatorname{co}(X_0 \cup \{y\}) \cap S(y)$ with $\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Then there exists a point $\hat{y} \in K$ such that

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: We divide the proof into three steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists

$p \in E^*$ such that $Re\langle p, y \rangle - \sup_{x \in S(y)} Re\langle p, x \rangle > 0$. For each $y \in X$, set

$$\gamma(y) := \sup_{x \in S(y)} \left[\inf_{u \in T(x)} Re\langle u, y - x \rangle + h(y) - h(x) \right].$$

Let $V_0 := \{y \in X \mid \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$V_p := \{y \in X : Re\langle p, y \rangle - \sup_{x \in S(y)} Re\langle p, x \rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 4.3.1 and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X . Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$. Define $\phi, \psi : X \times X \rightarrow \mathbf{R}$ by

$$\phi(x, y) = \beta_0(y) \left[\inf_{u \in T(x)} Re\langle u, y - x \rangle + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle,$$

and

$$\psi(x, y) = \beta_0(y) \left[\inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle$$

for each $x, y \in X$. Then we have the following.

(1) For each $x, y \in X$, since T is semi-monotone, $\phi(x, y) \leq \psi(x, y)$ and $\psi(x, x) = 0$ for all $x \in X$.

(2) For each fixed $x \in X$, the map

$$y \longmapsto \inf_{u \in T(x)} Re\langle u, y - x \rangle + h(y) - h(x)$$

is weakly lower semicontinuous (and therefore lower semicontinuous) on X by Lemma 3.2.4 and the fact that h is continuous; therefore the map

$$y \longmapsto \beta_0(y) \left[\inf_{u \in T(x)} Re\langle u, y - x \rangle + h(y) - h(x) \right]$$

is lower semicontinuous on X by Lemma 4.3.2. Also for each fixed $x \in X$,

$$y \mapsto \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle$$

is continuous on X . Hence, for each fixed $x \in X$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on X .

(3) Clearly, for each $y \in X$, the set $\{x \in X : \psi(x, y) > 0\}$ is convex.

(4) By hypothesis, there exists a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in \text{co}(X_0 \cup \{y\}) \cap S(y)$ such that $\inf_{u \in T(x)} \text{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Thus $\beta_0(y)[\inf_{u \in T(x)} \text{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0$ whenever $\beta_0(y) > 0$. Also $\text{Re}\langle p, y - x \rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently, $\phi(x, y) = \beta_0(y)[\inf_{u \in T(x)} \text{Re}\langle u, y - x \rangle + h(y) - h(x)] + \sum_{p \in E^*} \beta_p(y) \text{Re}\langle p, y - x \rangle > 0$.

Then ϕ and ψ satisfy all the hypotheses of Theorem 4.3.3. Thus by Theorem 4.3.3, there exists $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$, i.e.,

$$\beta_0(\hat{y}) \left[\inf_{u \in T(x)} \text{Re}\langle u, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \text{Re}\langle p, \hat{y} - x \rangle \leq 0 \quad (4.7)$$

for all $x \in X$.

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0 = \Sigma$ so that $\gamma(\hat{y}) > 0$. Choose $\hat{x} \in S(\hat{y}) \subset X$ such that

$$\inf_{u \in T(\hat{x})} \text{Re}\langle u, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \geq \frac{\gamma(\hat{y})}{2} > 0;$$

it follows that

$$\beta_0(\hat{y}) \left[\inf_{u \in T(\hat{x})} \text{Re}\langle u, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] > 0.$$

If $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, then $\hat{y} \in V_p$ and hence

$$\text{Re}\langle p, \hat{y} \rangle > \sup_{x \in S(\hat{y})} \text{Re}\langle p, x \rangle \geq \text{Re}\langle p, \hat{x} \rangle$$

so that $\text{Re}\langle p, \hat{y} - \hat{x} \rangle > 0$. Then note that

$$\beta_p(\hat{y}) \text{Re}\langle p, \hat{y} - \hat{x} \rangle > 0 \quad \text{whenever} \quad \beta_p(\hat{y}) > 0 \quad \text{for} \quad p \in E^*.$$

Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, it follows that

$$\phi(\hat{x}, \hat{y}) = \beta_0(\hat{y}) \left[\inf_{u \in T(\hat{x})} \text{Re}\langle u, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \text{Re}\langle p, \hat{y} - \hat{x} \rangle > 0,$$

which contradicts (4.7). This contradiction proves Step 1.

Step 2.

$$\inf_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all} \quad x \in S(\hat{y}).$$

Indeed, from Step 1, $\hat{y} \in S(\hat{y})$ which is a convex subset of X , and

$$\inf_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}).$$

Hence by Lemma 3.2.20, we have

$$\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}). \quad (4.8)$$

Step 3. There exist a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Indeed, from Step 2 we have

$$\sup_{x \in S(\hat{y})} \left[\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] \leq 0, \quad (4.9)$$

where $T(\hat{y})$ is a strongly compact convex subset of the Hausdorff topological vector space E^* and $S(\hat{y})$ is a convex subset of X .

Now, define $f : S(\hat{y}) \times T(\hat{y}) \rightarrow \mathbf{R}$ by $f(x, w) = \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ for each $x \in S(\hat{y})$ and each $w \in T(\hat{y})$. Note that for each fixed $x \in S(\hat{y})$, the map $w \mapsto f(x, w)$ is convex and continuous on $T(\hat{y})$ and for each fixed $w \in T(\hat{y})$, the map $x \mapsto f(x, w)$ is concave on $S(\hat{y})$. Thus by Theorem 3.2.1, we have

$$\min_{w \in T(\hat{y})} \sup_{x \in S(\hat{y})} [\operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] = \sup_{x \in S(\hat{y})} \min_{w \in T(\hat{y})} [\operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)].$$

Hence

$$\min_{w \in T(\hat{y})} \sup_{x \in S(\hat{y})} [\operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0, \quad \text{by (4.9).}$$

Since $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}). \square$$

If X is compact, Theorem 4.3.10 reduces to the following:

Theorem 4.3.11 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E . Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be semi-monotone and be upper*

hemi-continuous along line segments in X to the weak-topology on E^* such that each $T(x)$ is strongly compact convex. Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Then there exists a point $\hat{y} \in K$ such that

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Note that if the map $S : X \rightarrow 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, T is upper semicontinuous at some point x in $S(y)$ with $\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$, then the set Σ in Theorem 4.3.10 is always open in X as can be seen in the proof of the following:

Theorem 4.3.12 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex and bounded subset of E . Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^E$ be semi-monotone and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is strongly compact convex. Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at some point x in $S(y)$ with $\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Suppose further that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in \operatorname{co}(X_0 \cup \{y\}) \cap S(y)$ with $\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Then there exists $\hat{y} \in K$ such that*

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) there exist a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: By virtue of Theorem 4.3.10, it suffices to show that the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Indeed, let $y_0 \in \Sigma$; then by hypothesis, T is upper semicontinuous at some point x_0 in $S(y_0)$ with $\inf_{u \in T(x_0)} \operatorname{Re}\langle u, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$. Let

$$\alpha := \inf_{u \in T(x_0)} \operatorname{Re}\langle u, y_0 - x_0 \rangle + h(y_0) - h(x_0).$$

Then $\alpha > 0$. Also let

$$W := \{w \in E^* : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2 \rangle| < \alpha/6\}.$$

Then W is a strongly open neighborhood of 0 in E^* so that $U_1 := T(x_0) + W$ is an open neighborhood of $T(x_0)$ in E^* . Since T is upper semicontinuous at x_0 , there exists an open neighborhood V_1 of x_0 in X such that $T(x) \subset U_1$ for all $x \in V_1$.

As the map $x \mapsto \inf_{u \in T(x_0)} \operatorname{Re}\langle u, x_0 - x \rangle + h(x_0) - h(x)$ is continuous at x_0 , there exists an open neighborhood V_2 of x_0 in X such that

$$|\inf_{u \in T(x_0)} \operatorname{Re}\langle u, x_0 - x \rangle + h(x_0) - h(x)| < \alpha/6 \quad \text{for all } x \in V_2.$$

Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X . Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semicontinuous at y_0 , there exists an open neighborhood N_1 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_1$.

Since the map $y \mapsto \inf_{u \in T(x_0)} \operatorname{Re}\langle u, y - y_0 \rangle + h(y) - h(y_0)$ is continuous at y_0 , there exists an open neighborhood N_2 of y_0 in X such that

$$|\inf_{u \in T(x_0)} \operatorname{Re}\langle u, y - y_0 \rangle + h(y) - h(y_0)| < \alpha/6 \quad \text{for all } y \in N_2.$$

Let $N_0 := N_1 \cap N_2$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

- (i) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_1$; so we can choose any $x_1 \in S(y_1) \cap V_0$;
- (ii) $|\inf_{u \in T(x_0)} \operatorname{Re}\langle u, y_1 - y_0 \rangle + h(y_1) - h(y_0)| < \alpha/6$ as $y_1 \in N_2$;
- (iii) $T(x_1) \subset U_1 = T(x_0) + W$ as $x_1 \in V_1$;
- (iv) $|\inf_{u \in T(x_0)} \operatorname{Re}\langle u, x_0 - x_1 \rangle + h(x_0) - h(x_1)| < \alpha/6$ as $x_1 \in V_2$.

It follows that

$$\begin{aligned}
& \inf_{u \in T(x_1)} \operatorname{Re}\langle u, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\
& \geq \inf_{[u \in T(x_0) + W]} \operatorname{Re}\langle u, y_1 - x_1 \rangle + h(y_1) - h(x_1) \quad (\text{by (iii)}), \\
& \geq \inf_{u \in T(x_0)} \operatorname{Re}\langle u, y_1 - x_1 \rangle + h(y_1) - h(x_1) + \inf_{u \in W} \operatorname{Re}\langle u, y_1 - x_1 \rangle \\
& \geq \inf_{u \in T(x_0)} \operatorname{Re}\langle u, y_1 - y_0 \rangle + h(y_1) - h(y_0) \\
& \quad + \inf_{u \in T(x_0)} \operatorname{Re}\langle u, y_0 - x_0 \rangle + h(y_0) - h(x_0) \\
& \quad + \inf_{u \in T(x_0)} \operatorname{Re}\langle u, x_0 - x_1 \rangle + h(x_0) - h(x_1) + \inf_{u \in W} \operatorname{Re}\langle u, y_1 - x_1 \rangle \\
& \geq -\frac{\alpha}{6} + \alpha - \frac{\alpha}{6} - \frac{\alpha}{6} = \frac{\alpha}{2} > 0 \quad (\text{by (ii) and (iv)});
\end{aligned}$$

therefore

$$\sup_{x \in S(y_1)} \left[\inf_{u \in T(x)} \operatorname{Re}\langle u, y_1 - x \rangle + h(y_1) - h(x) \right] > 0$$

as $x_1 \in S(y_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$, so that Σ is open in X . This completes the proof. \square

If X is compact, Theorem 4.3.12 reduces to the following:

Theorem 4.3.13 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E . Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be semi-monotone and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is strongly compact convex. Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at some point x in $S(y)$ with $\inf_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle + h(y) - h(x) > 0$. Then there exists $\hat{y} \in X$ such that*

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

4.3.3 Generalized Quasi-Variational Inequalities for Upper Semi-Continuous Operators

In this section we shall obtain some existence theorems for generalized quasi-variational inequalities for upper semicontinuous operators on paracompact convex sets. In obtaining these results we shall mainly use the following generalized version of Ky Fan's minimax inequality [48] due to Ding and Tan [39, Theorem 1].

Theorem 4.3.14 *Let X be a non-empty convex subset of a topological vector space E and let $f : X \times X \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ be such that*

(a) *for each fixed $x \in X$, the map $y \mapsto f(x, y)$ is lower semicontinuous on each non-empty compact subset C of X ;*

(b) *for each $A \in \mathcal{F}(X)$ and for each $y \in \text{co}(A)$, $\min_{x \in A} f(x, y) \leq 0$;*

(c) *there exists a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists an $x \in \text{co}(X_0 \cup \{y\})$ with $f(x, y) > 0$.*

Then there exists a point $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

The following result is Lemma 2.2.7 of K.-K. Tan in [106] (see also the proof of Theorem 21 of Takahashi in [103]):

Lemma 4.3.15 *Let E be a topological vector space and E^* be the continuous dual of E equipped with the strong topology. Let X be a non-empty compact subset of E and $T : X \rightarrow 2^{E^*}$ be upper semicontinuous such that $T(x)$ is strongly compact for each $x \in X$. Define $f : X \times X \rightarrow \mathbf{R}$ by $f(x, y) = \inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle$, for each $x, y \in X$. Then f is lower semicontinuous on $X \times X$.*

By modifying the proof of Theorem 3 of Shih and Tan in [92], we have its generalization to a non-compact setting as follows:

Theorem 4.3.16 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex subset of E . Let $S : X \rightarrow 2^X$ be upper*

semicontinuous such that each $S(x)$ is a non-empty compact convex subset of X and $T : X \rightarrow 2^{E^*}$ be upper semicontinuous from the relative topology of X to the strong topology of E^* such that each $T(x)$ is a strongly compact convex subset of E^* . Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose that the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Suppose further that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in \operatorname{co}(X_0 \cup \{y\}) \cap S(y)$ with $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) > 0$. Then there exists $\hat{y} \in X$ such that

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) there exists $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: We divide the proof into two steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) > 0$; that is, $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $\operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0$. For each $y \in X$, set

$$\gamma(y) := \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)].$$

Let $V_0 := \{y \in X \mid \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$V_p := \{y \in X : \operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 4.3.1 and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X . Since X is paracompact,

there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$. Define $\phi : X \times X \rightarrow \mathbf{R}$ by

$$\phi(x, y) = \beta_0(y) \left[\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle,$$

for each $x, y \in X$. Then we have the following.

(1) For each fixed $x \in X$, the map

$$y \mapsto \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)$$

is lower semicontinuous on each non-empty compact subset of X by Lemma 4.3.15 and therefore the map

$$y \mapsto \beta_0(y) \left[\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) \right]$$

is lower semicontinuous on each non-empty compact subset of X by Lemma 4.3.2 of Section 4.3. Also for each fixed $x \in X$,

$$y \mapsto \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle$$

is continuous on X . Hence, for each fixed $x \in X$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on each non-empty compact subset of X .

(2) For each $A \in \mathcal{F}(X)$ and for each $y \in \operatorname{co}(A)$, $\min_{x \in A} \phi(x, y) \leq 0$. Indeed, if this were false, then for some $A = \{x_1, \dots, x_n\} \in \mathcal{F}(X)$ and some $y \in \operatorname{co}(A)$, say $y = \sum_{i=1}^n \lambda_i x_i$ where $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, such that $\min_{1 \leq i \leq n} \phi(x_i, y) > 0$. Then for each $i = 1, \dots, n$, $\beta_0(y) [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_i \rangle + h(y) - h(x_i)] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x_i \rangle > 0$ so that $0 = \phi(y, y) = \beta_0(y) [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - \sum_{i=1}^n \lambda_i x_i \rangle + h(y) - h(\sum_{i=1}^n \lambda_i x_i)] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - \sum_{i=1}^n \lambda_i x_i \rangle \geq \sum_{i=1}^n \lambda_i (\beta_0(y) [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_i \rangle + h(y) - h(x_i)] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x_i \rangle) > 0$, which is a contradiction.

(3) By hypothesis, there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in \operatorname{co}(X_0 \cup \{y\}) \cap S(y)$ such that $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) > 0$. Thus $\beta_0(y) [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0$ whenever $\beta_0(y) > 0$. Also $\operatorname{Re}\langle p, y - x \rangle > 0$

whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently, $\phi(x, y) = \beta_0(y)[\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle > 0$.

Then ϕ satisfies all hypotheses of Theorem 4.3.14. Hence by Theorem 4.3.14, there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$; i.e.,

$$\beta_0(\hat{y}) \left[\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \operatorname{Re}\langle p, \hat{y} - x \rangle \leq 0 \quad (4.10)$$

for all $x \in X$.

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0 = \Sigma$ so that $\gamma(\hat{y}) > 0$. Choose $\hat{x} \in S(\hat{y}) \subset X$ such that

$$\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \geq \frac{\gamma(\hat{y})}{2} > 0;$$

it follows that

$$\beta_0(\hat{y}) \left[\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] > 0.$$

If $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, then $\hat{y} \in V_p$ and hence

$$\operatorname{Re}\langle p, \hat{y} \rangle > \sup_{x \in S(\hat{y})} \operatorname{Re}\langle p, x \rangle \geq \operatorname{Re}\langle p, \hat{x} \rangle$$

so that $\operatorname{Re}\langle p, \hat{y} - \hat{x} \rangle > 0$. Then note that

$$\beta_p(\hat{y}) \operatorname{Re}\langle p, \hat{y} - \hat{x} \rangle > 0 \quad \text{whenever } \beta_p(\hat{y}) > 0 \text{ for } p \in E^*.$$

Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, it follows that

$$\phi(\hat{x}, \hat{y}) = \beta_0(\hat{y}) \left[\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \operatorname{Re}\langle p, \hat{y} - \hat{x} \rangle > 0,$$

which contradicts (4.10). This contradiction proves Step 1.

Step 2. There exists a point $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0 \text{ for all } x \in S(\hat{y}).$$

Note that for each fixed $x \in S(\hat{y})$, $w \mapsto \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ is convex and continuous on $T(\hat{y})$ and for each fixed $w \in T(\hat{y})$, $x \mapsto \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ is

concave on $S(\hat{y})$. Thus by Kneser's minimax theorem [73], i.e., by Theorem 3.2.1, we have

$$\min_{w \in T(\hat{y})} \max_{x \in S(\hat{y})} [Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] = \max_{x \in S(\hat{y})} \min_{w \in T(\hat{y})} [Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)].$$

Hence

$$\min_{w \in T(\hat{y})} \max_{x \in S(\hat{y})} [Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0 \quad \text{by Step 1.}$$

Since $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0 \quad \text{for all } x \in S(\hat{y}). \square$$

Note that if X is also bounded in Theorem 4.3.16 and the map $S : X \rightarrow 2^X$ is, in addition, lower semicontinuous, then the set Σ in Theorem 4.3.16 is always open in X as can be seen in the proof of the following:

Theorem 4.3.17 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty paracompact convex and bounded subset of E . Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is a non-empty compact convex subset of X and $T : X \rightarrow 2^{E^*}$ be upper semicontinuous from the relative topology of X to the strong topology of E^* such that each $T(x)$ is a strongly compact convex subset of E^* . Let $h : X \rightarrow \mathbf{R}$ be convex and continuous. Suppose further that there exists a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in co(X_0 \cup \{y\}) \cap S(y)$ with $\inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x) > 0$. Then there exists a point $\hat{y} \in X$ such that*

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: By virtue of Theorem 4.3.16, we need only show that the set

$$\Sigma := \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Indeed, let $y_0 \in \Sigma$; then there exists $x_0 \in S(y_0)$ such that $\alpha := \inf_{w \in T(y_0)} \operatorname{Re} \langle w, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$. Let

$$W := \left\{ w \in E^* : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2 \rangle| < \frac{\alpha}{6} \right\}.$$

Then W is a strongly open neighborhood of 0 in E^* so that $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in E^* . Since T is upper semicontinuous at y_0 , there exists an open neighborhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

As the map $x \mapsto \inf_{w \in T(y_0)} \operatorname{Re} \langle w, x_0 - x \rangle + h(x_0) - h(x)$ is continuous at x_0 , there exists an open neighborhood V_1 of x_0 in X such that

$$\left| \inf_{w \in T(y_0)} \operatorname{Re} \langle w, x_0 - x \rangle + h(x_0) - h(x) \right| < \frac{\alpha}{6} \text{ for all } x \in V_1.$$

Since $x_0 \in V_1 \cap S(y_0) \neq \emptyset$ and S is lower semicontinuous at y_0 , there exists an open neighborhood N_2 of y_0 in X such that $S(y) \cap V_1 \neq \emptyset$ for all $y \in N_2$.

Since the map $y \mapsto \inf_{w \in T(y_0)} \operatorname{Re} \langle w, y - y_0 \rangle + h(y) - h(y_0)$ is continuous at y_0 , there exists an open neighborhood N_3 of y_0 in X such that

$$\left| \inf_{w \in T(y_0)} \operatorname{Re} \langle w, y - y_0 \rangle + h(y) - h(y_0) \right| < \frac{\alpha}{6} \text{ for all } y \in N_3.$$

Let $N_0 := N_1 \cap N_2 \cap N_3$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

- (i) $T(y_1) \subset U_1 = T(y_0) + W$ as $y_1 \in N_1$;
- (ii) $S(y_1) \cap V_1 \neq \emptyset$ as $y_1 \in N_2$; so we can choose any $x_1 \in S(y_1) \cap V_1$;
- (iii) $\left| \inf_{w \in T(y_0)} \operatorname{Re} \langle w, y_1 - y_0 \rangle + h(y_1) - h(y_0) \right| < \frac{\alpha}{6}$ as $y_1 \in N_3$;
- (iv) $\left| \inf_{w \in T(y_0)} \operatorname{Re} \langle w, x_0 - x_1 \rangle + h(x_0) - h(x_1) \right| < \frac{\alpha}{6}$ as $x_1 \in V_1$.

It follows that

$$\begin{aligned}
& \inf_{w \in T(y_1)} \operatorname{Re}\langle w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\
& \geq \inf_{\{w \in T(y_0) + W\}} \operatorname{Re}\langle w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \quad (\text{by (i)}), \\
& \geq \inf_{w \in T(y_0)} \operatorname{Re}\langle w, y_1 - x_1 \rangle + h(y_1) - h(x_1) + \inf_{w \in W} \operatorname{Re}\langle w, y_1 - x_1 \rangle \\
& \geq \inf_{w \in T(y_0)} \operatorname{Re}\langle w, y_1 - y_0 \rangle + h(y_1) - h(y_0) \\
& \quad + \inf_{w \in T(y_0)} \operatorname{Re}\langle w, y_0 - x_0 \rangle + h(y_0) - h(x_0) \\
& \quad + \inf_{w \in T(y_0)} \operatorname{Re}\langle w, x_0 - x_1 \rangle + h(x_0) - h(x_1) \\
& \quad + \inf_{w \in W} \operatorname{Re}\langle w, y_1 - x_1 \rangle \\
& \geq -\frac{\alpha}{6} + \alpha - \frac{\alpha}{6} - \frac{\alpha}{6} = \frac{\alpha}{2} > 0 \quad (\text{by (iii) and (iv)});
\end{aligned}$$

therefore $\sup_{x \in S(y_1)} [\inf_{w \in T(y_1)} \operatorname{Re}\langle w, y_1 - x \rangle + h(y_1) - h(x)] > 0$ as $x_1 \in S(y_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$ so that Σ is open in X . This proves the theorem. \square

Theorem 4.3.17 generalizes a result of Shih-Tan-Kim ([92, Theorem 4] and [70, Theorem] which is a special case of Theorem 11 in [100]) to non-compact setting.

4.3.4 Generalized Quasi-Variational Inequalities for Strong Pseudo-Monotone operators

In this section we shall first introduce strong pseudo-monotone operators. As applications of strong pseudo-monotone operators, we shall obtain some general theorems on solutions of the GQVI in locally convex Hausdorff topological vector spaces. We shall obtain existence theorems for GQVI on paracompact sets X where the set-valued operators T are strong pseudo-monotone and are upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$.

We shall begin with the following:

Definition 4.3.18 *Let E be a topological vector space, X be a non-empty subset of E and $T : X \rightarrow 2^{E^*}$. If $h : X \rightarrow \mathbf{R}$, then T is said to be a strong h -pseudo-monotone operator if for each continuous function $\theta : X \rightarrow [0, 1]$, for each $y \in X$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y with*

$$\limsup_{\alpha} [\theta(y_\alpha) (\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - y \rangle + h(y_\alpha) - h(y))] \leq 0$$

we have

$$\begin{aligned} & \limsup_{\alpha} [\theta(y_\alpha) (\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - x \rangle + h(y_\alpha) - h(x))] \\ & \geq [\theta(y) (\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x))] \end{aligned}$$

for all $x \in X$.

T is said to be strong pseudo-monotone if T is strong h -pseudo-monotone with $h \equiv 0$.

Note that a strong h -pseudo-monotone operator is stronger than our Definition 3.6.1 of h -pseudo-monotone operator. Indeed, by choosing $\theta \equiv 1$ in the above definition, we see that a strong h -pseudo-monotone operator is also an h -pseudo-monotone operator. Thus every strong pseudo-monotone operator is also a pseudo-monotone operator.

Proposition 4.3.19 *Let X be a non-empty subset of a topological vector space E . If $T : X \rightarrow E^*$ is monotone and continuous from the relative weak topology on X to the weak* topology on E^* , then T is strong pseudo-monotone.*

Proof: Let us consider any arbitrary continuous function $\theta : X \rightarrow [0, 1]$. Suppose $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X and $y \in X$ with $y_\alpha \rightarrow y$ (and $\limsup_\alpha [\theta(y_\alpha)(\operatorname{Re}\langle Ty_\alpha, y_\alpha - y \rangle)] \leq 0$). Then for any $x \in X$ and $\epsilon > 0$, there are $\beta_1, \beta_2 \in \Gamma$ with $|\theta(y_\alpha)\operatorname{Re}\langle Ty, y_\alpha - y \rangle| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_1$ and $|\theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha - Ty, y - x \rangle| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_2$. Choose $\beta_0 \in \Gamma$ with $\beta_0 \geq \beta_1, \beta_2$. Thus for each $\alpha \geq \beta_0$,

$$\begin{aligned} \theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle &= \theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha, y_\alpha - y \rangle + \theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha, y - x \rangle \\ &\geq \theta(y_\alpha)\operatorname{Re}\langle Ty, y_\alpha - y \rangle + \theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha, y - x \rangle \\ &= \theta(y_\alpha)\operatorname{Re}\langle Ty, y_\alpha - y \rangle + \theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha - Ty, y - x \rangle \\ &\quad + \theta(y_\alpha)\operatorname{Re}\langle Ty, y - x \rangle \\ &> -\frac{\epsilon}{2} - \frac{\epsilon}{2} + \theta(y_\alpha)\operatorname{Re}\langle Ty, y - x \rangle \\ &= -\epsilon + \theta(y_\alpha)\operatorname{Re}\langle Ty, y - x \rangle. \end{aligned}$$

Given $\gamma \in \Gamma$, choose any $\beta \in \Gamma$ such that $\beta \geq \gamma$ and $\beta \geq \beta_0$. Then for each $\alpha \geq \beta$, $\theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle > -\epsilon + \theta(y_\alpha)\operatorname{Re}\langle Ty, y - x \rangle$ so that

$$\begin{aligned} \sup_{\alpha \geq \gamma} \theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle &\geq \sup_{\alpha \geq \beta} \theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle \\ &\geq -\epsilon + \sup_{\alpha \geq \beta} \theta(y_\alpha)\operatorname{Re}\langle Ty, y - x \rangle \\ &\geq -\epsilon + \limsup_\alpha \theta(y_\alpha)\operatorname{Re}\langle Ty, y - x \rangle \\ &= -\epsilon + \theta(y)\operatorname{Re}\langle Ty, y - x \rangle. \end{aligned}$$

Therefore $\inf_{\gamma \in \Gamma} \sup_{\alpha \geq \gamma} \theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle \geq -\epsilon + \theta(y)\operatorname{Re}\langle Ty, y - x \rangle$.

As $\epsilon > 0$ is arbitrary,

$$\limsup_\alpha \theta(y_\alpha)\operatorname{Re}\langle Ty_\alpha, y_\alpha - x \rangle \geq \theta(y)\operatorname{Re}\langle Ty, y - x \rangle.$$

Hence T is strong pseudo-monotone. \square

But the converse is not true in general as can be seen in Example 3.2.11. In Example 3.2.11, we see that T is not monotone. But it is easy to show that T is strong pseudo-monotone.

We shall now establish the following result:

Theorem 4.3.20 *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex subset of E and $h : X \rightarrow \mathbf{R}$ be convex. Let*

$S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be strong h -pseudo-monotone and be upper semicontinuous from $\text{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Suppose further that there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \text{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) there exists $\hat{w} \in T(\hat{y})$ with $\text{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: We divide the proof into two steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x) > 0$; that is, $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $\text{Re}\langle p, y \rangle - \sup_{x \in S(y)} \text{Re}\langle p, x \rangle > 0$. For each $y \in X$, set

$$\gamma(y) := \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)].$$

Let $V_0 := \{y \in X \mid \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$V_p := \{y \in X : \text{Re}\langle p, y \rangle - \sup_{x \in S(y)} \text{Re}\langle p, x \rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 4.3.1 of Section 4.3 and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X . Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated

to the open cover $\{V_0, V_p : p \in E^*\}$. Note that for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see e.g. [87, Corollary 10.1.1, p.83]). Define $\phi : X \times X \rightarrow \mathbf{R}$ by

$$\phi(x, y) = \beta_0(y) \left[\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle$$

for each $x, y \in X$. Then we have the following.

(1) Since E is Hausdorff, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map

$$y \mapsto \min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)$$

is lower semicontinuous on $co(A)$ by Lemma 3.6.3 and the fact that h is continuous on $co(A)$ and therefore the map

$$y \mapsto \beta_0(y) \left[\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x) \right]$$

is lower semicontinuous on $co(A)$ by Lemma 4.3.2 of Section 4.3. Also for each fixed $x \in X$,

$$y \mapsto \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle$$

is continuous on X . Hence, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on $co(A)$.

(2) For each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x \in A} \phi(x, y) \leq 0$. Indeed, if this were false, then for some $A = \{x_1, \dots, x_n\} \in \mathcal{F}(X)$ and some $y \in co(A)$, say $y = \sum_{i=1}^n \lambda_i x_i$ where $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, such that $\min_{1 \leq i \leq n} \phi(x_i, y) > 0$. Then for each $i = 1, \dots, n$, $\beta_0(y) [\min_{w \in T(y)} Re\langle w, y - x_i \rangle + h(y) - h(x_i)] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x_i \rangle > 0$ so that $0 = \phi(y, y) = \beta_0(y) [\min_{w \in T(y)} Re\langle w, y - \sum_{i=1}^n \lambda_i x_i \rangle + h(y) - h(\sum_{i=1}^n \lambda_i x_i)] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - \sum_{i=1}^n \lambda_i x_i \rangle \geq \sum_{i=1}^n \lambda_i (\beta_0(y) [\min_{w \in T(y)} Re\langle w, y - x_i \rangle + h(y) - h(x_i)] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x_i \rangle) > 0$, which is a contradiction.

(3) Suppose $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X converging to y with

$$\phi(tx + (1-t)y, y_\alpha) \leq 0 \quad \text{for all } \alpha \in \Gamma \text{ and all } t \in [0, 1].$$

Then for $t = 0$ we have $\phi(y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e., $\beta_0(y_\alpha) [\min_{w \in T(y_\alpha)} Re\langle w, y_\alpha -$

$y) + h(y_\alpha) - h(y)] + \sum_{p \in E} \beta_p(y_\alpha) \operatorname{Re}\langle p, y_\alpha - y \rangle \leq 0$ for all $\alpha \in \Gamma$. Hence

$$\begin{aligned} & \limsup_\alpha [\beta_0(y_\alpha)(\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y))] \\ & \quad + \liminf_\alpha [\sum_{p \in E} \beta_p(y_\alpha) \operatorname{Re}\langle p, y_\alpha - y \rangle] \\ & \leq \limsup_\alpha [\beta_0(y_\alpha)(\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y)) \\ & \quad + \sum_{p \in E} \beta_p(y_\alpha) \operatorname{Re}\langle p, y_\alpha - y \rangle] \leq 0 \end{aligned}$$

Therefore $\limsup_\alpha [\beta_0(y_\alpha)(\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y))] \leq 0$.

Since T is strong h -pseudo-monotone, we have

$$\begin{aligned} & \limsup_\alpha [\beta_0(y_\alpha)(\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x))] \\ & \geq \beta_0(y)(\min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)). \end{aligned} \quad (4.11)$$

Thus

$$\begin{aligned} & \limsup_\alpha [\beta_0(y_\alpha)(\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x))] \\ & \quad + \sum_{p \in E} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle \\ & \geq \beta_0(y)(\min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)) \\ & \quad + \sum_{p \in E} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle \end{aligned} \quad (4.12)$$

For $t = 1$ we have $\phi(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha) \left[\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right] + \sum_{p \in E} \beta_p(y_\alpha) \operatorname{Re}\langle p, y_\alpha - x \rangle \leq 0$$

for all $\alpha \in \Gamma$. Therefore

$$\begin{aligned} & \limsup_\alpha [\beta_0(y_\alpha)(\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x))] \\ & \quad + \liminf_\alpha [\sum_{p \in E} \beta_p(y_\alpha) \operatorname{Re}\langle p, y_\alpha - x \rangle] \\ & \leq \limsup_\alpha [\beta_0(y_\alpha)(\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x)) \\ & \quad + \sum_{p \in E} \beta_p(y_\alpha) \operatorname{Re}\langle p, y_\alpha - x \rangle] \\ & \leq 0. \end{aligned}$$

Thus

$$\begin{aligned} & \limsup_\alpha [\beta_0(y_\alpha)(\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x))] \\ & \quad + \sum_{p \in E} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle \\ & \leq 0. \end{aligned} \quad (4.13)$$

Hence by (4.12) and (4.13), we have $\phi(x, y) \leq 0$.

(4) By hypothesis, there exists a non-empty compact (and therefore also closed) subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for each $y \in X \setminus K$. Thus for each $y \in X \setminus K$, $\beta_0(y)[\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0)] > 0$ whenever $\beta_0(y) > 0$ and $\operatorname{Re}\langle p, y - x_0 \rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently, $\phi(x_0, y) = \beta_0(y)[\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0)] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x_0 \rangle > 0$ for all $y \in X \setminus K$.

Then ϕ satisfies all hypotheses of Theorem 2.3.5. Hence by Theorem 2.3.5, there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$; i.e.,

$$\beta_0(\hat{y}) \left[\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \operatorname{Re}\langle p, \hat{y} - x \rangle \leq 0 \quad (4.14)$$

for all $x \in X$.

Now, the rest of the proof of Step 1 is similar to the proof in Step 1 of Theorem 4.3.16. Thus Step 1 is proved.

Step 2. There exists a point $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0 \text{ for all } x \in S(\hat{y}).$$

Also the proof of Step 2 is similar to the proof of Step 2 of Theorem 4.3.16. Hence there exists $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$. \square

If X is compact, we obtain the following immediate consequence of Theorem 4.3.20:

Theorem 4.3.21 *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : X \rightarrow \mathbf{R}$ be convex. Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be strong h -pseudo-monotone and be upper semicontinuous from $\operatorname{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Then there exists $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Note that if X is also bounded in Theorem 4.3.20, the map $S : X \rightarrow 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, T is upper semicontinuous at y in X , then the set Σ in Theorem 4.3.20 is always open in X as can be seen in the proof of the following:

Theorem 4.3.22 *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : X \rightarrow \mathbf{R}$ be convex. Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be strong h -pseudo-monotone and be upper semicontinuous from $\text{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^* . Suppose further that there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \text{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that*

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists $\hat{w} \in T(\hat{y})$ with $\text{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: By virtue of Theorem 4.3.20, we need only show that the set

$$\Sigma := \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Indeed, let $y_0 \in \Sigma$; then there exists $x_0 \in S(y_0)$ such that $\alpha := \inf_{w \in T(y_0)} \text{Re}\langle w, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$.

Let $W := \{w \in E^* : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2 \rangle| < \frac{\alpha}{6}\}$. Then W is a strongly open neighborhood of 0 in E^* so that $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in E^* . Since T is upper semicontinuous at y_0 in X , there exists an open neighborhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

Now, the rest of the proof is similar to the proof of Theorem 4.3.17. Hence by the rest of the proof of Theorem 4.3.17, Σ is open in X . This proves the theorem. \square

If X is compact, we obtain the following immediate consequence of Theorem 4.3.22:

Theorem 4.3.23 *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : X \rightarrow \mathbf{R}$ be convex. Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be strong h -pseudo-monotone and be upper semicontinuous from $\text{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^* . Then there exists $\hat{y} \in X$ such that*

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists $\hat{w} \in T(\hat{y})$ with $\text{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

4.3.5 Generalized Quasi-Variational Inequalities for Pseudo-Monotone Operators

In this section we shall use Theorem 2.3.5 as a tool to obtain some general theorems on solutions of the GQVI in locally convex Hausdorff topological vector spaces. We shall obtain existence theorems for GQVI on paracompact sets X where the set-valued operators T are pseudo-monotone (see Definition 3.6.1) and are upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$.

We shall now establish the following result:

Theorem 4.3.24 *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : X \rightarrow \mathbf{R}$ be convex. Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be h -pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Suppose further that there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) there exists $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: We divide the proof into two steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) > 0$; that is, $y \notin S(y)$ or $y \in \Sigma$. If

$y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $Re\langle p, y \rangle - \sup_{x \in S(y)} Re\langle p, x \rangle > 0$. For each $y \in X$, set

$$\gamma(y) := \sup_{x \in S(y)} [\inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)].$$

Let $V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$V_p := \{y \in X : Re\langle p, y \rangle - \sup_{x \in S(y)} Re\langle p, x \rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 4.3.1 of Section 4.3 and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X . Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$. Note that for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see e.g. [87, Corollary 10.1.1, p.83]). Define $\phi : X \times X \rightarrow \mathbf{R}$ by

$$\phi(x, y) = \beta_0(y) [\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle$$

for each $x, y \in X$. Then we have the following.

(1) By following the same arguments as in (1) of the proof of Theorem 4.3.20, it follows that for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on $co(A)$.

(2) For each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x \in A} \phi(x, y) \leq 0$. The proof of this is similar to the proof of (2) of Theorem 4.3.20. Thus the conclusion follows.

(3) Suppose $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X converging to y with

$$\phi(tx + (1-t)y, y_\alpha) \leq 0 \quad \text{for all } \alpha \in \Gamma \quad \text{and all } t \in [0, 1].$$

Case 1: $\beta_0(y) = 0$.

Since β_0 is continuous and $y_\alpha \rightarrow y$, we have $\beta_0(y_\alpha) \rightarrow \beta_0(y) = 0$. Note that $\beta_0(y_\alpha) \geq 0$ for each $\alpha \in \Gamma$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a bounded net, it follows that

$$\limsup_{\alpha} [\beta_0(y_\alpha) (\min_{w \in T(y_\alpha)} Re\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x))] = 0. \quad (4.15)$$

Also

$$\beta_0(y) \left[\min_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x) \right] = 0.$$

Thus

$$\begin{aligned} & \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & + \sum_{p \in E} \beta_p(y) \operatorname{Re} \langle p, y - x \rangle \\ & = \sum_{p \in E} \beta_p(y) \operatorname{Re} \langle p, y - x \rangle \text{ by (4.15)} \\ & = \beta_0(y) [\min_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x)] \\ & + \sum_{p \in E} \beta_p(y) \operatorname{Re} \langle p, y - x \rangle. \end{aligned} \quad (4.16)$$

For $t = 1$ we have $\phi(x, y_{\alpha}) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_{\alpha}) \left[\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x) \right] + \sum_{p \in E} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - x \rangle \leq 0 \quad (4.17)$$

for all $\alpha \in \Gamma$. Therefore

$$\begin{aligned} & \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & + \liminf_{\alpha} [\sum_{p \in E} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - x \rangle] \\ & \leq \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & + \sum_{p \in E} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - x \rangle] \\ & \leq 0 \text{ by (4.17)}. \end{aligned}$$

Thus

$$\begin{aligned} & \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & + \sum_{p \in E} \beta_p(y) \operatorname{Re} \langle p, y - x \rangle \leq 0. \end{aligned} \quad (4.18)$$

Hence by (4.16) and (4.18), we have $\phi(x, y) \leq 0$.

Case 2: $\beta_0(y) > 0$.

Since β_0 is continuous, $\beta_0(y_{\alpha}) \rightarrow \beta_0(y)$. Again since $\beta_0(y) > 0$, there exists $\lambda \in \Gamma$ such that $\beta_0(y_{\alpha}) > 0$ for all $\alpha \geq \lambda$.

Then for $t = 0$ we have $\phi(y, y_{\alpha}) \leq 0$ for all $\alpha \in \Gamma$, i.e., $\beta_0(y_{\alpha}) [\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y)] + \sum_{p \in E} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - y \rangle \leq 0$ for all $\alpha \in \Gamma$. Thus

$$\begin{aligned} & \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y))] \\ & + \sum_{p \in E} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - y \rangle \leq 0. \end{aligned} \quad (4.19)$$

Hence

$$\begin{aligned} & \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y))] \\ & \quad + \liminf_{\alpha} [\sum_{p \in E} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - y \rangle] \\ & \leq \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y)) \\ & \quad + \sum_{p \in E} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - y \rangle] \leq 0 \text{ by (4.19).} \end{aligned}$$

Since $\liminf_{\alpha} [\sum_{p \in E} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - y \rangle] = 0$, we have

$$\begin{aligned} & \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y))] \\ & \leq 0. \end{aligned} \tag{4.20}$$

Since $\beta_0(y_{\alpha}) > 0$ for all $\alpha \geq \lambda$, it follows that

$$\begin{aligned} & \beta_0(y) \limsup_{\alpha} [\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y)] \\ & = \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y))]. \end{aligned} \tag{4.21}$$

Since $\beta_0(y) > 0$, by (4.20) and (4.21) we have

$$\limsup_{\alpha} [\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y)] \leq 0.$$

Since T is h -pseudo-monotone, we have

$$\begin{aligned} & \limsup_{\alpha} [\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)] \\ & \geq \min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x). \end{aligned} \tag{4.22}$$

Since $\beta_0(y) > 0$, we have

$$\begin{aligned} & \beta_0(y) [\limsup_{\alpha} (\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & \geq \beta_0(y) [\min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)]. \end{aligned} \tag{4.23}$$

Thus

$$\begin{aligned} & \beta_0(y) [\limsup_{\alpha} (\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & \quad + \sum_{p \in E} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle \\ & \geq \beta_0(y) [\min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] \\ & \quad + \sum_{p \in E} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle. \end{aligned} \tag{4.24}$$

For $t = 1$ we also have $\phi(x, y_{\alpha}) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_{\alpha}) [\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)] + \sum_{p \in E} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - x \rangle \leq 0$$

for all $\alpha \in \Gamma$. Therefore

$$\begin{aligned}
0 &\geq \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)) \\
&\quad + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - x \rangle] \\
&\geq \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\
&\quad + \liminf_{\alpha} [\sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - x \rangle] \\
&= \beta_0(y) [\limsup_{\alpha} (\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\
&\quad + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle.
\end{aligned} \tag{4.25}$$

Consequently, by (4.24) and (4.25), we have $\phi(x, y) \leq 0$.

Now, the rest of the proof of Step 1 is similar to the proofs in Step 1 of Theorems 4.3.20 and 4.3.16. Thus Step 1 is proved.

Step 2. There exists a point $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0 \text{ for all } x \in S(\hat{y}).$$

Also the proof of Step 2 is similar to the proof of Step 2 of Theorem 4.3.16. Hence there exists $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0 \text{ for all } x \in S(\hat{y}). \square$$

If X is compact, we obtain the following immediate consequence of Theorem 4.3.24:

Theorem 4.3.25 *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : X \rightarrow \mathbf{R}$ be convex. Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be h -pseudo-monotone and be upper semicontinuous from $\operatorname{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X .

Then there exists $\hat{y} \in X$ such that

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) there exists $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Note that if the map $S : X \rightarrow 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, T is upper semicontinuous at y in X , then the set Σ in Theorem 4.3.24 is always open in X as can be seen in the proof of the following:

Theorem 4.3.26 *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : X \rightarrow \mathbf{R}$ be convex. Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be h -pseudo-monotone and be upper semicontinuous from $\operatorname{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^* . Suppose further that there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that*

- (i) $\hat{y} \in S(\hat{y})$ and
- (ii) there exists $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof: By virtue of Theorem 4.3.24, we need only show that the set

$$\Sigma := \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X .

Now, following the same arguments as in the proofs of Theorems 4.3.22 and 4.3.17 we can similarly show that the set Σ is open in X . Hence by Theorem 4.3.24 the conclusion follows. \square

If X is compact, we obtain the following immediate consequence of Theorem 4.3.26:

Theorem 4.3.27 *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : X \rightarrow \mathbf{R}$ be convex. Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be h -pseudo-monotone and be upper semicontinuous from $\text{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^* .*

Then there exists $\hat{y} \in X$ such that

- (i) $\hat{y} \in S(\hat{y})$ and*
- (ii) there exists $\hat{w} \in T(\hat{y})$ with $\text{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

4.4 Generalized Bi-Quasi-Variational Inequalities

Let E and F be Hausdorff topological vector spaces over the field Φ , let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional, and let X be a non-empty subset of E . Given a set-valued map $S : X \rightarrow 2^X$ and two set-valued maps $M, T : X \rightarrow 2^F$, the *generalized bi-quasi-variational inequality* (GBQVI) problem is to find a point $\hat{y} \in X$ and a point $\hat{w} \in T(\hat{y})$ such that $\hat{y} \in S(\hat{y})$ and $\operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$ or to find a point $\hat{y} \in X$, a point $\hat{w} \in T(\hat{y})$ and a point $\hat{f} \in M(\hat{y})$ such that $\hat{y} \in S(\hat{y})$ and $\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. The generalized bi-quasi-variational inequality was introduced first by Shih and Tan [100] in 1989.

In this section we shall obtain some results on existence theorems for generalized bi-quasi-variational inequalities in locally convex topological vector spaces on compact sets.

Throughout Section 4.4, Φ denotes either the real field \mathbf{R} or the complex field \mathbf{C} . For other notations and preliminary concepts we shall refer to Section 3.3.

We now state the following result due to Yen in [112, pp.477-481] which is a generalization of Ky Fan's minimax inequality in [48, Theorem 1] and which can be easily derived from Theorem 4.3.3 (i.e., Theorem 1 of Bae-Kim-Tan in [6]):

Theorem 4.4.1 *Let E be a topological vector space, and X be a non-empty compact convex subset of E . Suppose that $f, g : X \times X \rightarrow \mathbf{R}$ are two mappings satisfying the following conditions:*

- (i) $f(x, y) \leq g(x, y)$ for all $x, y \in X$ and $g(x, x) \leq 0$ for all $x \in X$;
- (ii) for each fixed $x \in X$, the map $y \mapsto f(x, y)$ is lower semicontinuous on X ;
- (iii) for each fixed $y \in X$, the map $x \mapsto g(x, y)$ is quasi-concave on X ;

Then there exists a point $\hat{y} \in X$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

We remark here that in the original version of Theorem 4.4.1 as stated in [112], the topological vector space E is assumed to be Hausdorff.

The following definition generalizes Definition 3.2.5(b):

Definition 4.4.2 Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty subset of E . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional and $M : X \rightarrow 2^F$ be a map. Then M is said to be upper hemi-continuous on X if and only if for each $p \in E$, the function $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ defined by

$$f_p(z) = \sup_{u \in M(z)} \operatorname{Re}\langle u, p \rangle \quad \text{for each } z \in X,$$

is upper semicontinuous on X (if and only if for each $p \in E$, the function $g_p : X \rightarrow \mathbf{R} \cup \{-\infty\}$ defined by

$$g_p(z) = \inf_{u \in M(z)} \operatorname{Re}\langle u, p \rangle \quad \text{for each } z \in X,$$

is lower semicontinuous on X).

Note that the notion of upper hemi-continuity along line segments is independent of the topology τ on E as long as τ is Hausdorff and the vector space F over Φ remains unchanged. Note also that if $M, T : X \rightarrow 2^F$ are upper hemi-continuous on X and $\alpha \in \mathbf{R}$, then $M + T$ and αT are also upper hemi-continuous on X .

The following proposition generalizes Proposition 3.2.8:

Proposition 4.4.3 Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty subset of E . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that for each fixed $p \in E$, $u \mapsto \langle u, p \rangle$ is $\sigma(F, E)$ -continuous on F when F is equipped with the $\sigma(F, E)$ -topology. Let $M : X \rightarrow 2^F$ be upper semicontinuous from the relative topology on X to the weak topology $\sigma(F, E)$ on F . Then M is upper hemi-continuous on X .

Proof: For each $p \in E$, define $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$f_p(z) = \sup_{u \in M(z)} \operatorname{Re}\langle u, p \rangle \quad \text{for each } z \in X.$$

Fix any $p \in E$. Let $\lambda \in \mathbf{R}$ be given and let $A = \{z \in X : f_p(z) < \lambda\}$. Take any $z_0 \in A$. Then $f_p(z_0) = \sup_{u \in M(z_0)} \operatorname{Re}\langle u, p \rangle < \lambda$. Choose any $\epsilon > 0$ such that

$f_p(z_0) < \lambda - \epsilon < \lambda$. Let $h : F \rightarrow \mathbf{R}$ be defined by $h(u) = \operatorname{Re}\langle u, p \rangle$ for each $u \in F$. By hypothesis h is $\sigma(F, E)$ -continuous on F . Since $h(u) < \lambda - \epsilon < \lambda$ for all $u \in M(z_0)$, $M(z_0) \subset h^{-1}(-\infty, \lambda - \epsilon)$ which is $\sigma(F, E)$ -open in F .

Since M is upper semicontinuous at z_0 , there exists an open neighborhood N_{z_0} of z_0 in X such that $M(z) \subset h^{-1}(-\infty, \lambda - \epsilon)$ for all $z \in N_{z_0}$. Thus $h(u) < \lambda - \epsilon$ for all $u \in M(z)$ and for all $z \in N_{z_0}$. Hence $\sup_{u \in M(z)} h(u) \leq \lambda - \epsilon < \lambda$ for all $z \in N_{z_0}$; i.e., $f_p(z) = \sup_{u \in M(z)} \operatorname{Re}\langle u, p \rangle \leq \lambda - \epsilon < \lambda$ for all $z \in N_{z_0}$. Therefore $N_{z_0} \subset A$ so that A is open in X . Consequently, f_p is upper semicontinuous on X . Hence M is upper hemi-continuous on X . \square

Note that the converse of Proposition 4.4.3 is not true as can be seen in Example 3.2.9 which is Example 2.3 in [105, p.392]:

We shall now establish the following result:

Lemma 4.4.4 *Let E be a topological vector space over Φ , X be a non-empty compact subset of E and F be a Hausdorff topological vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional and $T : X \rightarrow 2^F$ be an upper semicontinuous map such that each $T(x)$ is compact. Let M be a non-empty compact subset of F , $x_0 \in X$ and $h : X \rightarrow \mathbf{R}$ be continuous. Define $g : X \rightarrow \mathbf{R}$ by $g(y) = [\inf_{f \in M} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x_0 \rangle] + h(y)$ for each $y \in X$. Suppose that $\langle \cdot, \cdot \rangle$ is continuous on the (compact) subset $[M - \bigcup_{y \in X} T(y)] \times X$ of $F \times E$. Then g is lower semicontinuous on X .*

Proof: Let $\lambda \in \mathbf{R}$ be given, and let $A_\lambda = \{y \in X : g(y) \leq \lambda\}$. Suppose that $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in A_λ such that $y_\alpha \rightarrow y_0 \in X$. Then $g(y_\alpha) = [\inf_{f \in M} \inf_{w \in T(y_\alpha)} \operatorname{Re}\langle f - w, y_\alpha - x_0 \rangle] + h(y_\alpha) \leq \lambda$ for each $\alpha \in \Gamma$. For each $\alpha \in \Gamma$, since the sets M and $T(y_\alpha)$ are compact, by continuity of $\langle \cdot, \cdot \rangle$, we can choose $f_\alpha \in M$ and $w_\alpha \in T(y_\alpha)$ such that $\operatorname{Re}\langle f_\alpha - w_\alpha, y_\alpha - x_0 \rangle + h(y_\alpha) = [\inf_{f \in M} \inf_{w \in T(y_\alpha)} \operatorname{Re}\langle f - w, y_\alpha - x_0 \rangle] + h(y_\alpha) = g(y_\alpha) \leq \lambda$. Since T is upper semicontinuous and each $T(x)$ is compact, $T(X) = \bigcup_{y \in X} T(y)$ is compact. Thus there are subnets $\{f_{\alpha'}\}_{\alpha' \in \Gamma'}$ of $\{f_\alpha\}_{\alpha \in \Gamma}$ and $\{w_{\alpha'}\}_{\alpha' \in \Gamma'}$ of $\{w_\alpha\}_{\alpha \in \Gamma}$ and $f_0 \in M$, $w_0 \in \bigcup_{y \in X} T(y)$ such that $f_{\alpha'} \rightarrow f_0$ and $w_{\alpha'} \rightarrow w_0$. As T is upper semicontinuous with closed values, T has a closed graph in $X \times F$ and hence $w_0 \in T(y_0)$.

Since $\langle \cdot, \cdot \rangle$ is continuous on the compact set $[M - \bigcup_{y \in X} T(y)] \times X$ and h is continuous on X , we have $g(y_0) = [\inf_{f \in M} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y_0 - x_0 \rangle] + h(y_0) \leq \operatorname{Re}\langle f_0 - w_0, y_0 - x_0 \rangle + h(y_0) = \lim_{\alpha'} [\operatorname{Re}\langle f_{\alpha'} - w_{\alpha'}, y_{\alpha'} - x_0 \rangle + h(y_{\alpha'})] = \lim_{\alpha'} g(y_{\alpha'}) \leq \lambda$. Thus $y_0 \in A_\lambda$ and hence A_λ is closed in the relative topology on X . Therefore g is lower semicontinuous on X . \square

When $h \equiv 0$ and $M = \{0\}$, replacing T by $-T$, Lemma 4.4.4 reduces to the Lemma 2 of Shih and Tan in [100, pp.70-71].

The following result generalizes Lemma 3.2.20:

Lemma 4.4.5 *Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty convex subset of E . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional. Equip F with the $\sigma(F, E)$ -topology. Let D be a non-empty $\sigma(F, E)$ -compact subset of F , $h : X \rightarrow \mathbf{R}$ be convex and $M : X \rightarrow 2^F$ be upper hemi-continuous along line segments in X . Suppose $\hat{y} \in X$ is such that $\inf_{f \in M(x)} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Then*

$$\inf_{f \in M(\hat{y})} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in X.$$

Proof: Suppose that $\inf_{f \in M(x)} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Fix an arbitrary $x \in X$. For each $t \in [0, 1]$, let $z_t = tx + (1 - t)\hat{y} = \hat{y} + t(x - \hat{y})$. Then $z_t \in X$ as X is convex. Thus for each $t \in (0, 1]$,

$$\begin{aligned} t \cdot \inf_{f \in M(z_t)} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle &= \inf_{f \in M(z_t)} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - z_t \rangle \\ &\leq h(z_t) - h(\hat{y}) \\ &\leq t(h(x) - h(\hat{y})), \end{aligned}$$

as h is convex so that $\inf_{f \in M(z_t)} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ and hence for all $t \in (0, 1]$, $\inf_{f \in M(z_t)} \operatorname{Re}\langle f, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) - \inf_{g \in D} \operatorname{Re}\langle g, x - \hat{y} \rangle$.

Let $L = \{z_t : t \in [0, 1]\}$ and $A = \{z \in L : \inf_{f \in M(z)} \operatorname{Re}\langle f, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) - \inf_{g \in D} \operatorname{Re}\langle g, x - \hat{y} \rangle\}$. Since M is upper hemi-continuous on L , $z_t \in A$ for all $t \in (0, 1]$ and $z_t \rightarrow \hat{y}$ as $t \rightarrow 0^+$, we have $\hat{y} \in A$ so that

$$\inf_{f \in M(\hat{y})} \operatorname{Re}\langle f, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) - \inf_{g \in D} \operatorname{Re}\langle g, x - \hat{y} \rangle.$$

It follows that $\inf_{f \in M(\hat{y})} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$. Since $x \in X$ is arbitrary, we have $\inf_{f \in M(\hat{y})} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. \square

4.4.1 Generalized Bi-Quasi-Variational Inequalities in Locally Convex Topological Vector Spaces

We shall now establish the following result:

Theorem 4.4.6 *Let E be a locally convex Hausdorff topological vector space over Φ , X be a non-empty compact convex subset of E and F be a Hausdorff topological vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that*

(a) $S : X \rightarrow 2^X$ is an upper semicontinuous map such that each $S(x)$ is closed and convex;

(b) $T : X \rightarrow 2^F$ is upper semicontinuous such that each $T(x)$ is compact convex;

(c) $h : X \rightarrow \mathbf{R}$ is convex and continuous;

(d) $M : X \rightarrow 2^F$ is upper hemi-continuous along line segments in X and semi-monotone (with respect to $\langle \cdot, \cdot \rangle$) such that each $M(x)$ is compact convex and

(e) the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} \inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re} \langle f - w, y - x \rangle + h(y) - h(x) > 0\} \text{ is open in } X.$$

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re} \langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex and if $T \equiv 0$, the continuity assumption on $\langle \cdot, \cdot \rangle$ can be weakened to the assumption that for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X .

Proof: We divide the proof into three steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} \left[\inf_{f \in M(x)} \inf_{w \in T(\hat{y})} \operatorname{Re} \langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that

$$\operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0.$$

For each $p \in E^*$, let

$$V(p) = \{y \in X : \operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0\}.$$

Then $V(p)$ is open by Lemma 4.3.1. Since $X = \Sigma \cup \bigcup_{p \in E^*} V(p)$, by compactness of X , there exist $p_1, p_2, \dots, p_n \in E^*$ such that $X = \Sigma \cup \bigcup_{i=1}^n V(p_i)$. For simplicity of notations, let $V_0 := \Sigma$ and $V_i = V(p_i)$ for $i = 1, 2, \dots, n$. Let $\{\beta_0, \beta_1, \dots, \beta_n\}$ be a continuous partition of unity on X subordinated to the covering $\{V_0, V_1, \dots, V_n\}$. Then $\beta_0, \beta_1, \dots, \beta_n$ are continuous non-negative real-valued functions on X such that β_i vanishes on $X \setminus V_i$, for each $i = 0, 1, \dots, n$ and $\sum_{i=0}^n \beta_i(x) = 1$ for all $x \in X$. Define $\phi, \psi : X \times X \rightarrow \mathbf{R}$ by

$$\phi(x, y) = \beta_0(y) \left[\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x) \right] + \sum_{i=1}^n \beta_i(y) \operatorname{Re}\langle p_i, y - x \rangle,$$

and

$$\psi(x, y) = \beta_0(y) \left[\inf_{g \in M(y)} \inf_{w \in T(y)} \operatorname{Re}\langle g - w, y - x \rangle + h(y) - h(x) \right] + \sum_{i=1}^n \beta_i(y) \operatorname{Re}\langle p_i, y - x \rangle,$$

for each $x, y \in X$. Then we have the following.

(1) $\psi(x, x) = 0$ for all $x \in X$.

(2) Since M is semi-monotone, for each $x, y \in X$ we have $\inf_{f \in M(x)} \operatorname{Re}\langle f, y - x \rangle \leq \inf_{g \in M(y)} \operatorname{Re}\langle g, y - x \rangle$. Then $\inf_{f \in M(x)} \operatorname{Re}\langle f - w, y - x \rangle \leq \inf_{g \in M(y)} \operatorname{Re}\langle g - w, y - x \rangle$ for all $w \in T(y)$. Hence $\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle \leq \inf_{g \in M(y)} \inf_{w \in T(y)} \operatorname{Re}\langle g - w, y - x \rangle$. Therefore $\phi(x, y) \leq \psi(x, y)$ for all $x, y \in X$.

(3) For each fixed $x \in X$, the map

$$y \longmapsto \inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x)$$

is lower semicontinuous on X by Lemma 4.4.4; therefore the map

$$y \longmapsto \beta_0(y) \left[\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x) \right]$$

is lower semicontinuous on X by Lemma 4.3.2. Hence for each fixed $x \in X$, the map $y \longmapsto \phi(x, y)$ is lower semicontinuous on X .

(4) Clearly, for each fixed $y \in X$, the map $x \mapsto \psi(x, y)$ is quasi-concave on X .

Then ϕ and ψ satisfy all the hypotheses of Theorem 4.4.1. Thus by Theorem 4.4.1, there exists $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$, i.e.,

$$\beta_0(\hat{y}) \left[\inf_{f \in M(x)} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] + \sum_{i=1}^n \beta_i(\hat{y}) \operatorname{Re}\langle p_i, \hat{y} - x \rangle \leq 0 \quad (4.26)$$

for all $x \in X$.

Choose $\hat{x} \in S(\hat{y})$ such that

$$\inf_{f \in M(\hat{x})} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) > 0 \text{ whenever } \beta_0(\hat{y}) > 0;$$

it follows that

$$\beta_0(\hat{y}) \left[\inf_{f \in M(\hat{x})} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] > 0 \text{ whenever } \beta_0(\hat{y}) > 0.$$

If $i \in \{1, \dots, n\}$ is such that $\beta_i(\hat{y}) > 0$, then $\hat{y} \in V_i = V(p_i)$ and hence

$$\operatorname{Re}\langle p_i, \hat{y} \rangle > \sup_{x \in S(\hat{y})} \operatorname{Re}\langle p_i, x \rangle \geq \operatorname{Re}\langle p_i, \hat{x} \rangle$$

so that $\operatorname{Re}\langle p_i, \hat{y} - \hat{x} \rangle > 0$. Then note that

$$\beta_i(\hat{y}) \operatorname{Re}\langle p_i, \hat{y} - \hat{x} \rangle > 0 \text{ whenever } \beta_i(\hat{y}) > 0 \text{ for } i = 1, \dots, n.$$

Since $\beta_i(\hat{y}) > 0$ for at least one $i \in \{0, 1, \dots, n\}$, it follows that

$$\beta_0(\hat{y}) \left[\inf_{f \in M(\hat{x})} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] + \sum_{i=1}^n \beta_i(\hat{y}) \operatorname{Re}\langle p_i, \hat{y} - \hat{x} \rangle > 0,$$

which contradicts (4.26). This contradiction proves Step 1.

Step 2.

$$\inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}).$$

Indeed, from Step 1, $\hat{y} \in S(\hat{y})$ which is a convex subset of X , and

$$\inf_{f \in M(x)} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}). \quad (4.27)$$

Hence by Lemma 4.4.5, we have

$$\inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}).$$

Step 3. There exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

From Step 2 we have

$$\sup_{x \in S(\hat{y})} \left[\inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] \leq 0;$$

i.e.,

$$\sup_{x \in S(\hat{y})} \inf_{(f, w) \in M(\hat{y}) \times T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0, \quad (4.28)$$

where $M(\hat{y}) \times T(\hat{y})$ is a compact convex subset of the Hausdorff topological vector space $F \times F$ and $S(\hat{y})$ is a convex subset of X .

Let $Q = M(\hat{y}) \times T(\hat{y})$ and the map $g : S(\hat{y}) \times Q \rightarrow \mathbf{R}$ be defined by $g(x, q) = g(x, (f, w)) = \operatorname{Re}\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ for each $x \in S(\hat{y})$ and each $q = (f, w) \in Q = M(\hat{y}) \times T(\hat{y})$. Note that for each fixed $x \in S(\hat{y})$, the map $(f, w) \mapsto g(x, (f, w))$ is lower semicontinuous from the relative product topology on Q to \mathbf{R} and also convex on Q . Clearly, for each fixed $q = (f, w) \in Q$, the map $x \mapsto g(x, q) = g(x, (f, w))$ is concave on $S(\hat{y})$. Then by Theorem 3.2.1 we have

$$\min_{(f, w) \in Q} \sup_{x \in S(\hat{y})} g(x, (f, w)) = \sup_{x \in S(\hat{y})} \min_{(f, w) \in Q} g(x, (f, w)).$$

Thus

$$\min_{(f, w) \in Q} \sup_{x \in S(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0, \quad \text{by (4.28).}$$

Since $Q = M(\hat{y}) \times T(\hat{y})$ is compact, there exists $(\hat{f}, \hat{w}) \in M(\hat{y}) \times T(\hat{y})$ such that

$$\sup_{x \in S(\hat{y})} \operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0.$$

Therefore

$$\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}).$$

In other words, there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with

$$\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}).$$

Next we note from the above proof that the requirement that E be locally convex is needed when and only when the separation theorem is applied to the case $y \notin S(y)$. Thus if $S : X \rightarrow 2^X$ is the constant map $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Finally, if $T \equiv 0$, in order to show that for each $x \in X$, $y \mapsto \phi(x, y)$ is lower semicontinuous, Lemma 4.4.4 is no longer needed and the weaker continuity assumption on $\langle \cdot, \cdot \rangle$ that for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X is sufficient. This completes the proof. \square

Theorem 4.4.7 *Let E be a locally convex Hausdorff topological vector space over Φ , X be a non-empty compact convex subset of E and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta(F, E)$. Suppose that*

(a) $S : X \rightarrow 2^X$ is a continuous map such that each $S(x)$ is closed and convex;

(b) $T : X \rightarrow 2^F$ is upper semicontinuous such that each $T(x)$ is strongly compact and convex;

(c) $h : X \rightarrow \mathbf{R}$ is convex and continuous;

(d) $M : X \rightarrow 2^F$ is upper hemi-continuous along line segments in X and semi-monotone (with respect to $\langle \cdot, \cdot \rangle$) such that each $M(x)$ is $\delta(F, E)$ -compact convex; also, for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x)] > 0\}$, M is upper semicontinuous at some point x in $S(y)$ with $\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x) > 0$.

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Proof: As $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ is a bilinear functional such that for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X and as F is equipped with the strong topology $\delta\langle F, E \rangle$, it is easy to see that $\langle \cdot, \cdot \rangle$ is continuous on compact subsets of $F \times X$. Thus by Theorem 4.4.6, it suffices to show that the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Indeed, let $y_0 \in \Sigma$; then by the last part of the hypothesis (d), M is upper semicontinuous at some point x_0 in $S(y_0)$ with $\inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$. Let

$$\alpha := \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y_0 - x_0 \rangle + h(y_0) - h(x_0).$$

Then $\alpha > 0$. Also let

$$W := \{w \in F : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2 \rangle| < \alpha/6\}.$$

Then W is an open neighborhood of 0 in F so that $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in F . Since T is upper semicontinuous at y_0 , there exists an open neighborhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

Let $U_2 := M(x_0) + W$, then U_2 is an open neighborhood of $M(x_0)$ in F . Since M is upper semicontinuous at x_0 , there exists an open neighborhood V_1 of x_0 in X such that $M(x) \subset U_2$ for all $x \in V_1$.

As the map $x \mapsto \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, x_0 - x \rangle + h(x_0) - h(x)$ is continuous at x_0 , there exists an open neighborhood V_2 of x_0 in X such that

$$|\inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, x_0 - x \rangle + h(x_0) - h(x)| < \alpha/6 \quad \text{for all } x \in V_2.$$

Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X . Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semicontinuous at y_0 , there exists an open neighborhood N_2 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_2$.

Since the map $y \mapsto \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y - y_0 \rangle + h(y) - h(y_0)$ is continuous at y_0 , there exists an open neighborhood N_3 of y_0 in X such that

$$\left| \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y - y_0 \rangle + h(y) - h(y_0) \right| < \alpha/6 \quad \text{for all } y \in N_3.$$

Let $N_0 := N_1 \cap N_2 \cap N_3$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

- (i) $T(y_1) \subset U_1 = T(y_0) + W$ as $y_1 \in N_1$;
- (ii) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_2$; so we can choose any $x_1 \in S(y_1) \cap V_0$;
- (iii) $\left| \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y_1 - y_0 \rangle + h(y_1) - h(y_0) \right| < \alpha/6$ as $y_1 \in N_3$;
- (iv) $M(x_1) \subset U_2 = M(x_0) + W$ as $x_1 \in V_1$;
- (v) $\left| \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, x_0 - x_1 \rangle + h(x_0) - h(x_1) \right| < \alpha/6$ as $x_1 \in V_2$.

It follows that

$$\begin{aligned} & \inf_{f \in M(x_1)} \inf_{w \in T(y_1)} \operatorname{Re}\langle f - w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\ & \geq \inf_{[f \in M(x_0) + W]} \inf_{[w \in T(y_0) + W]} \operatorname{Re}\langle f - w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \quad (\text{by (i) and (iv)}), \\ & \geq \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\ & \quad + \inf_{f \in W} \inf_{w \in W} \operatorname{Re}\langle f - w, y_1 - x_1 \rangle \\ & \geq \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y_1 - y_0 \rangle + h(y_1) - h(y_0) \\ & \quad + \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y_0 - x_0 \rangle + h(y_0) - h(x_0) \\ & \quad + \inf_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, x_0 - x_1 \rangle + h(x_0) - h(x_1) \\ & \quad + \inf_{f \in W} \operatorname{Re}\langle f, y_1 - x_1 \rangle + \inf_{w \in W} \operatorname{Re}\langle -w, y_1 - x_1 \rangle \\ & \geq -\alpha/6 + \alpha - \alpha/6 - \alpha/6 - \alpha/6 = \alpha/3 > 0 \quad (\text{by (iii) and (v)}); \end{aligned}$$

therefore

$$\sup_{x \in S(y_1)} \left[\inf_{f \in M(x)} \inf_{w \in T(y_1)} \operatorname{Re}\langle f - w, y_1 - x \rangle + h(y_1) - h(x) \right] > 0$$

as $x_1 \in S(y_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$, so that Σ is open in X . This proves the theorem. \square

4.4.2 Generalized Bi-Quasi-Variational Inequalities in Non-Compact Settings

In this section, we shall apply Theorem 4.4.7 together with the concept of escaping sequences to obtain an existence theorem on non-compact generalized bi-quasi-variational inequalities for semi-monotone operators.

Let E be a topological vector space over Φ , F be a vector space over Φ and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional.

If X is a cone in E and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ is a bilinear functional, then $\widehat{X} = \{w \in F : \text{Re}\langle w, x \rangle \geq 0 \text{ for all } x \in X\}$ is also a cone in F , called the dual cone of X (with respect to the bilinear functional $\langle \cdot, \cdot \rangle$).

We shall now establish the following result:

Theorem 4.4.8 *Let E be a locally convex Hausdorff topological vector space over Φ , X be a non-empty (convex) subset of E such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta(F, E)$. Suppose that*

- (1) $S : X \rightarrow 2^X$ is a continuous map such that
 - (a) for each $x \in X$, $S(x)$ is a closed convex subset of X and
 - (b) for each $n \in \mathbf{N}$, $S(x) \subset C_n$ for all $x \in C_n$;
- (2) $T : X \rightarrow 2^F$ is upper semicontinuous such that each $T(x)$ is $\delta(F, E)$ -compact convex;
- (3) $h : X \rightarrow \mathbf{R}$ is convex and continuous;
- (4) $M : X \rightarrow 2^F$ is upper hemi-continuous along line segments in X and semi-monotone (with respect to $\langle \cdot, \cdot \rangle$) such that each $M(x)$ is $\delta(F, E)$ -compact convex; also, for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{f \in M(x)} \inf_{w \in T(y)} \text{Re}\langle f - w, y - x \rangle + h(y) - h(x)] > 0\}$, M is upper semicontinuous at some point x in $S(y)$ with $\inf_{f \in M(x)} \inf_{w \in T(y)} \text{Re}\langle f - w, y - x \rangle + h(y) - h(x) > 0$ and M is upper semicontinuous on C_n for each $n \in \mathbf{N}$;

(5) for each sequence $\{y_n\}_{n=1}^{\infty}$ in X , with $y_n \in C_n$ for each $n \in \mathbf{N}$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, either there exists $n_0 \in \mathbf{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in S(y_{n_0})$ such that $\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \operatorname{Re}\langle f - w, y_{n_0} - x_{n_0} \rangle + h(y_{n_0}) - h(x_{n_0}) > 0$.

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with

$$\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}).$$

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Proof: Fix an arbitrary $n \in \mathbf{N}$. Note that C_n is a non-empty compact convex subset of E . Define $S_n : C_n \rightarrow 2^{C_n}$, $h_n : C_n \rightarrow \mathbf{R}$ and $M_n, T_n : C_n \rightarrow 2^F$ by $S_n(x) = S(x)$, $h_n(x) = h(x)$, $M_n(x) = M(x)$ and $T_n(x) = T(x)$ respectively for each $x \in C_n$; i.e., $S_n = S|_{C_n}$, $h_n = h|_{C_n}$, $M_n = M|_{C_n}$ and $T_n = T|_{C_n}$ respectively. By Theorem 4.4.7, there exists a point $\hat{y}_n \in C_n$ such that

(i)' $\hat{y}_n \in S_n(\hat{y}_n)$ and

(ii)' there exist a point $\hat{f}_n \in M(\hat{y}_n) = M_n(\hat{y}_n)$ and a point $\hat{w}_n \in T(\hat{y}_n) = T_n(\hat{y}_n)$ with $\operatorname{Re}\langle \hat{f}_n - \hat{w}_n, \hat{y}_n - x \rangle \leq h(x) - h(\hat{y}_n)$ for all $x \in S_n(\hat{y}_n)$.

Note that $\{\hat{y}_n\}_{n=1}^{\infty}$ is a sequence in $X = \bigcup_{n=1}^{\infty} C_n$ with $\hat{y}_n \in C_n$ for each $n \in \mathbf{N}$.

Case 1: $\{\hat{y}_n\}_{n=1}^{\infty}$ is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$.

Then by hypothesis (5), there exists $n_0 \in \mathbf{N}$ such that $\hat{y}_{n_0} \notin S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$, which contradicts (i)' or there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$ such that

$$\min_{f \in M(\hat{y}_{n_0})} \min_{w \in T(\hat{y}_{n_0})} \operatorname{Re}\langle f - w, \hat{y}_{n_0} - x_{n_0} \rangle + h(\hat{y}_{n_0}) - h(x_{n_0}) > 0,$$

which contradicts (ii)'.

Case 2: $\{\hat{y}_n\}_{n=1}^{\infty}$ is not escaping from X relative to $\{C_n\}_{n=1}^{\infty}$.

Then there exist $n_1 \in \mathbf{N}$ and a subsequence $\{\hat{y}_j\}_{j=1}^{\infty}$ of $\{\hat{y}_n\}_{n=1}^{\infty}$ such that $\hat{y}_j \in C_{n_1}$ for all $j = 1, 2, \dots$. Since C_{n_1} is compact, there exist a subnet $\{\hat{z}_\alpha\}_{\alpha \in \Gamma}$ of $\{\hat{y}_j\}_{j=1}^{\infty}$ and $\hat{y} \in C_{n_1} \subset X$ such that $\hat{z}_\alpha \rightarrow \hat{y}$.

For each $\alpha \in \Gamma$, let $\hat{z}_\alpha = \hat{y}_{n_\alpha}$, where $n_\alpha \rightarrow \infty$. Then according to our choice of \hat{y}_{n_α} in C_{n_α} , we have

(i)" $\hat{y}_{n_\alpha} \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$ and

(ii)" there exist a point $\hat{f}_{n_\alpha} \in M_{n_\alpha}(\hat{y}_{n_\alpha}) = M(\hat{y}_{n_\alpha})$ and a point $\hat{w}_{n_\alpha} \in T_{n_\alpha}(\hat{y}_{n_\alpha}) = T(\hat{y}_{n_\alpha})$ with $Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x \rangle + h(\hat{y}_{n_\alpha}) - h(x) \leq 0$ for all $x \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$. Since $n_\alpha \rightarrow \infty$, there exists $\alpha_0 \in \Gamma$ such that $n_\alpha \geq n_1$ for all $\alpha \geq \alpha_0$. Thus $C_{n_1} \subset C_{n_\alpha}$, for all $\alpha \geq \alpha_0$. From (i)" above we have $(\hat{y}_{n_\alpha}, \hat{y}_{n_\alpha}) \in G(S)$ for all $\alpha \in \Gamma$. Since S is upper semicontinuous with closed values, $G(S)$ is closed in $X \times X$; it follows that $\hat{y} \in S(\hat{y})$.

Moreover, since $\{\hat{f}_{n_\alpha}\}_{\alpha \geq \alpha_0}$ and $\{\hat{w}_{n_\alpha}\}_{\alpha \geq \alpha_0}$ are nets in the compact sets $\cup_{x \in C_{n_1}} M(x)$ and $\cup_{x \in C_{n_1}} T(x)$ respectively, without loss of generality, we may assume that the nets $\{\hat{f}_{n_\alpha}\}_{\alpha \in \Gamma}$ and $\{\hat{w}_{n_\alpha}\}_{\alpha \in \Gamma}$ converge to some $\hat{f} \in \cup_{x \in C_{n_1}} M(x)$ and some $\hat{w} \in \cup_{x \in C_{n_1}} T(x)$ respectively. Since M and T have closed graphs on C_{n_1} , $\hat{f} \in M(\hat{y})$ and $\hat{w} \in T(\hat{y})$.

Let $x \in S(\hat{y})$ be arbitrarily fixed. Let $n_2 \geq n_1$ be such that $x \in C_{n_2}$. Since S is lower semicontinuous at \hat{y} , without loss of generality we may assume that for each $\alpha \in \Gamma$, there is an $x_{n_\alpha} \in S(\hat{y}_{n_\alpha})$ such that $x_{n_\alpha} \rightarrow x$. By (ii)" we have, $Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle + h(\hat{y}_{n_\alpha}) - h(x_{n_\alpha}) \leq 0$ for all $\alpha \in \Gamma$. Note that $\hat{f}_{n_\alpha} - \hat{w}_{n_\alpha} \rightarrow \hat{f} - \hat{w}$ in $\delta\langle F, E \rangle$ and $\{\hat{y}_{n_\alpha} - x_{n_\alpha}\}_{\alpha \in \Gamma}$ is a net in the compact (and hence bounded) set $C_{n_2} - \cup_{y \in C_{n_2}} S(y)$. Thus, we have for each $\epsilon > 0$, there exists $\alpha_1 \geq \alpha_0$ such that $|Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha} - (\hat{f} - \hat{w}), \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle| < \epsilon/2$ for all $\alpha \geq \alpha_1$. Since $\langle \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle \rightarrow \langle \hat{f} - \hat{w}, \hat{y} - x \rangle$, there exists $\alpha_2 \geq \alpha_1$ such that $|Re\langle \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle - Re\langle \hat{f} - \hat{w}, \hat{y} - x \rangle| < \epsilon/2$ for all $\alpha \geq \alpha_2$. Thus for $\alpha \geq \alpha_2$,

$$\begin{aligned} & |Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle - Re\langle \hat{f} - \hat{w}, \hat{y} - x \rangle| \\ & \leq |Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha} - (\hat{f} - \hat{w}), \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle| + |Re\langle \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha} - (\hat{y} - x) \rangle| \\ & < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Thus

$$\lim_{\alpha} Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle = Re\langle \hat{f} - \hat{w}, \hat{y} - x \rangle.$$

By continuity of h , we have

$$\begin{aligned} & \operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \\ &= \lim_{\alpha} [\operatorname{Re}\langle \hat{f}_{n_{\alpha}} - \hat{w}_{n_{\alpha}}, \hat{y}_{n_{\alpha}} - x_{n_{\alpha}} \rangle + h(\hat{y}_{n_{\alpha}}) - h(x_{n_{\alpha}})] \\ &\leq 0. \quad \square \end{aligned}$$

Corollary 4.4.9 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta(F, E)$. Let $S : X \rightarrow 2^X$ be weakly continuous such that $S(x)$ is closed convex for each $x \in X$, $T : X \rightarrow 2^F$ be weakly upper semicontinuous such that each $T(x)$ is $\delta(F, E)$ -compact convex, $h : X \rightarrow \mathbf{R}$ be convex and (weakly) continuous and $M : X \rightarrow 2^F$ be (weakly) upper hemi-continuous along line segments in X and semi-monotone (with respect to $\langle \cdot, \cdot \rangle$) such that each $M(x)$ is $\delta(F, E)$ -compact convex. Also, for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x)] > 0\}$, M is weakly upper semicontinuous at some point x in $S(y)$ with $\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x) > 0$ and M is weakly upper semicontinuous on C_n for each $n \in \mathbf{N}$. Suppose that*

(1) *there exists an increasing sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers with $r_n \rightarrow \infty$ such that $S(x) \subset C_n$ for each $x \in C_n$ and each $n \in \mathbf{N}$ where $C_n = \{x \in X : \|x\| \leq r_n\}$:*

(2) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X , with $\|y_n\| \rightarrow \infty$, either there exists $n_0 \in \mathbf{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in S(y_{n_0})$ such that*

$$\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \operatorname{Re}\langle f - w, y_{n_0} - x_{n_0} \rangle + h(y_{n_0}) - h(x_{n_0}) > 0.$$

Then there exists $\hat{y} \in X$ such that

(a₁) $\hat{y} \in S(\hat{y})$ and

(b₁) there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with

$$\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}).$$

Proof: Equip E with the weak topology. Then C_n is weakly compact convex for each $n \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{\infty} C_n$. Now if $\{y_n\}_{n=1}^{\infty}$ is a sequence in X , with $y_n \in C_n$ for each $n = 1, 2, \dots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, then $\|y_n\| \rightarrow \infty$. By hypothesis (2), either there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in S(y_{n_0})$ such that $\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \operatorname{Re}\langle f - w, y_{n_0} - x_{n_0} \rangle + h(y_{n_0}) - h(x_{n_0}) > 0$. Thus all hypotheses of Theorem 4.4.8 are satisfied so that the conclusion follows. \square

We shall now obtain an existence theorem on non-compact generalized bi-complementarity problem for semi-monotone operators.

The proof of the result observed by S.C. Fang (e.g. see [23, p.213] and [94, p.59]) can show the following improvement of Lemma 3.2.3:

Lemma 4.4.10 *Let X be a cone in a Hausdorff topological vector space E over Φ and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional. Let $\hat{y} \in X$ and $\hat{g} \in F$. Then the following are equivalent:*

- (a) $\operatorname{Re}\langle \hat{g}, \hat{y} - x \rangle \leq 0$ for all $x \in X$.
- (b) $\operatorname{Re}\langle \hat{g}, \hat{y} \rangle = 0$ and $\hat{g} \in \widehat{X}$.

When X is a cone in E , by applying Lemma 4.4.10 and Theorem 4.4.8 with $h \equiv 0$ and $S(x) = X$ for all $x \in X$, we have immediately the following existence theorem of a generalized bi-complementarity problem:

Theorem 4.4.11 *Let E be a Hausdorff topological vector space over Φ , X be a cone in E such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$ the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta\langle F, E \rangle$. Suppose that*

- (1) $T : X \rightarrow 2^F$ is upper semicontinuous such that each $T(x)$ is $\delta\langle F, E \rangle$ -compact convex;

(2) $M : X \rightarrow 2^F$ is upper hemi-continuous along line segments in X and semi-monotone (with respect to $\langle \cdot, \cdot \rangle$) such that each $M(x)$ is $\delta\langle F, E \rangle$ -compact convex; also, for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle] > 0\}$, M is upper semicontinuous at some point x in $S(y)$ with $\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle > 0$ and M is upper semicontinuous on C_n for each $n \in \mathbb{N}$;

(3) for each sequence $\{y_n\}_{n=1}^\infty$ in X , with $y_n \in C_n$ for each $n \in \mathbb{N}$, which is escaping from X relative to $\{C_n\}_{n=1}^\infty$, there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in X$ such that

$$\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \operatorname{Re}\langle f - w, y_{n_0} - x_{n_0} \rangle > 0.$$

Then there exist a point $\hat{y} \in X$, a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} \rangle = 0$ and $\hat{f} - \hat{w} \in \widehat{X}$.

Corollary 4.4.12 Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a closed cone in E and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta\langle F, E \rangle$. Let $T : X \rightarrow 2^F$ be weakly upper semicontinuous such that each $T(x)$ is $\delta\langle F, E \rangle$ -compact convex and $M : X \rightarrow 2^F$ be (weakly) upper hemi-continuous along line segments in X and semi-monotone (with respect to $\langle \cdot, \cdot \rangle$) such that each $M(x)$ is $\delta\langle F, E \rangle$ -compact convex. Also, for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle] > 0\}$, M is weakly upper semicontinuous at some point x in $S(y)$ with $\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle > 0$ and M is weakly upper semicontinuous on C_n for each $n \in \mathbb{N}$. Let $\{r_n\}_{n=1}^\infty$ be an increasing sequence of positive numbers with $r_n \rightarrow \infty$ and $C_n = \{x \in X : \|x\| \leq r_n\}$ for each $n \in \mathbb{N}$. Suppose that for each sequence $\{y_n\}_{n=1}^\infty$ in X , with $\|y_n\| \rightarrow \infty$, there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in X$ such that $\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \operatorname{Re}\langle f - w, y_{n_0} - x_{n_0} \rangle > 0$. Then there exist $\hat{y} \in X$, $\hat{f} \in M(\hat{y})$ and $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} \rangle = 0 \text{ and } \hat{f} - \hat{w} \in \widehat{X}.$$

Proof: Equip E with the weak topology. Then C_n is weakly compact convex for each $n \in \mathbb{N}$ such that $X = \bigcup_{n=1}^\infty C_n$. Now if $\{y_n\}_{n=1}^\infty$ is a sequence in X , with $y_n \in C_n$

for each $n = 1, 2, \dots$, which is escaping from X relative to $\{C_n\}_{n=1}^\infty$, then $\|y_n\| \rightarrow \infty$. Hence by hypothesis, there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in X$ such that

$$\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \operatorname{Re}\langle f - w, y_{n_0} - x_{n_0} \rangle > 0.$$

Thus all hypotheses of Theorem 4.4.11 are satisfied so that the conclusion follows. \square

4.4.3 Further Results in Non-Compact Generalized Bi-Quasi-Variational Inequalities

In this section also we shall use the concept of escaping sequences introduced by Border to find an existence theorem on generalized bi-quasi-variational inequalities for monotone operators in non-compact settings. As an application, an existence theorem for generalized bi-complementarity problem will be given.

By modifying the proof of Theorem 2 of Shih and Tan in [100, pp.69-70] and by Theorem 3.2.1 which is Kneser's minimax theorem in [73, pp.2418-2420] (see also Aubin [2, pp.40-41]) we can similarly verify the following result whose proof is omitted:

Theorem 4.4.13 *Let E be a locally convex Hausdorff topological vector space over Φ , X be a non-empty compact convex subset of E and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta\langle F, E \rangle$. Suppose that*

(a) $S : X \rightarrow 2^X$ is a continuous map such that $S(x)$ is closed convex for each $x \in X$;

(b) $M : X \rightarrow 2^F$ is monotone (with respect to $\langle \cdot, \cdot \rangle$) and lower semicontinuous from the relative topology on X to the strong topology $\delta\langle F, E \rangle$ on F ;

(c) $T : X \rightarrow 2^F$ is an upper semicontinuous map from the relative topology on X to the strong topology $\delta\langle F, E \rangle$ on F such that $T(x)$ is strongly compact for each $x \in X$;

(d) $h : X \rightarrow \mathbf{R}$ is convex and continuous.

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

If $M(\hat{y})$ and $T(\hat{y})$ are also convex, then

(ii)' there exists $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

When $F = E^*$ and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E , by taking $M \equiv 0$ and replacing T by $-T$, Theorem 4.4.13 reduces to the following result:

Corollary 4.4.14 *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E . Let $S : X \rightarrow 2^X$ be continuous such that $S(x)$ is closed convex for each $x \in X$, $T : X \rightarrow 2^{E^*}$ be upper semicontinuous from the relative topology on X to the strong topology of E^* such that $T(x)$ is strongly compact for each $x \in X$ and $h : X \rightarrow \mathbf{R}$ be convex and continuous. Then there exists $\hat{y} \in X$ such that*

$$(1) \hat{y} \in S(\hat{y}) \text{ and}$$

$$(2) \inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}).$$

Moreover, if $T(\hat{y})$ is also convex, then

$$(2)' \text{ there exists } \hat{w} \in T(\hat{y}) \text{ such that } \operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}).$$

When $h \equiv 0$ and each $T(x)$ is also convex, the above result was observed by W.K. Kim in [70] which is an improvement of Theorem 4 of Shih and Tan in [92, pp.341-342].

We shall now apply Theorem 4.4.13 together with the concept of escaping sequences to obtain an existence theorem on generalized bi-quasi-variational inequalities in non-compact settings. As an application, we shall obtain an existence theorem on generalized bi-complementarity problem.

We shall now establish the following result:

Theorem 4.4.15 *Let E be a locally convex Hausdorff topological vector space over Φ , X be a non-empty (convex) subset of E such that $X = \cup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta(F, E)$. Suppose that*

$$(1) S : X \rightarrow 2^X \text{ is a continuous map such that}$$

$$(a) \text{ for each } x \in X, S(x) \text{ is a closed convex subset of } X \text{ and}$$

$$(b) \text{ for each } n \in \mathbf{N}, S(x) \subset C_n \text{ for all } x \in C_n;$$

(2) $M : X \rightarrow 2^F$ is monotone (with respect to $\langle \cdot, \cdot \rangle$) and lower semicontinuous from the relative topology on X to the strong topology $\delta\langle F, E \rangle$ on F such that each $M(x)$ is convex:

(3) $T : X \rightarrow 2^F$ is upper semicontinuous from the relative topology on X to the strong topology $\delta\langle F, E \rangle$ on F such that each $T(x)$ is a strongly compact convex subset of F ;

(4) $h : X \rightarrow \mathbf{R}$ is convex and continuous;

(5) for each sequence $\{y_n\}_{n=1}^\infty$ in X , with $y_n \in C_n$ for each $n \in \mathbf{N}$, which is escaping from X relative to $\{C_n\}_{n=1}^\infty$, either there exists $n_0 \in \mathbf{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbf{N}$, $x_{n_0} \in S(y_{n_0})$ and $f_{n_0} \in M(y_{n_0})$ such that

$$\min_{w \in T(y_{n_0})} \operatorname{Re}\langle f_{n_0} - w, y_{n_0} - x_{n_0} \rangle + h(y_{n_0}) - h(x_{n_0}) > 0.$$

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Proof: Fix an arbitrary $n \in \mathbf{N}$. Note that C_n is a non-empty compact convex subset of E . Define $S_n : C_n \rightarrow 2^{C_n}$, $h_n : C_n \rightarrow \mathbf{R}$ and $M_n, T_n : C_n \rightarrow 2^F$ by $S_n(x) = S(x)$, $h_n(x) = h(x)$, $M_n(x) = M(x)$ and $T_n(x) = T(x)$ for each $x \in C_n$; i.e., $S_n = S|_{C_n}$, $h_n = h|_{C_n}$, $M_n = M|_{C_n}$ and $T_n = T|_{C_n}$. By Theorem 4.4.13, there exists a point $\hat{y}_n \in C_n$ such that

(i)' $\hat{y}_n \in S_n(\hat{y}_n)$ and

(ii)' there exists a point $\hat{w}_n \in T(\hat{y}_n) = T_n(\hat{y}_n)$ with $\operatorname{Re}\langle f - \hat{w}_n, \hat{y}_n - x \rangle \leq h(x) - h(\hat{y}_n)$ for all $x \in S_n(\hat{y}_n)$ and for all $f \in M_n(\hat{y}_n)$.

Note that $\{\hat{y}_n\}_{n=1}^\infty$ is a sequence in $X = \bigcup_{n=1}^\infty C_n$ with $\hat{y}_n \in C_n$ for each $n \in \mathbf{N}$.

Case 1: $\{\hat{y}_n\}_{n=1}^\infty$ is escaping from X relative to $\{C_n\}_{n=1}^\infty$.

Then by (5), there exists $n_0 \in \mathbf{N}$ such that $\hat{y}_{n_0} \notin S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$, which contradicts (i)' or there exist $n_0 \in \mathbf{N}$, $x_{n_0} \in S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$ and $f_{n_0} \in M(\hat{y}_{n_0}) = M_{n_0}(\hat{y}_{n_0})$

such that $\min_{w \in T(\hat{y}_{n_0})} \operatorname{Re}\langle f_{n_0} - w, \hat{y}_{n_0} - x_{n_0} \rangle + h(\hat{y}_{n_0}) - h(x_{n_0}) > 0$, which contradicts (ii)'.

Case 2: $\{\hat{y}_n\}_{n=1}^\infty$ is not escaping from X relative to $\{C_n\}_{n=1}^\infty$.

Then there exist $n_1 \in \mathbb{N}$ and a subsequence $\{\hat{y}_{n_j}\}_{j=1}^\infty$ of $\{\hat{y}_n\}_{n=1}^\infty$ such that $\hat{y}_{n_j} \in C_{n_1}$, for all $j = 1, 2, \dots$. Since C_{n_1} is compact, there exist a subnet $\{\hat{z}_\alpha\}_{\alpha \in \Gamma}$ of $\{\hat{y}_{n_j}\}_{j=1}^\infty$ and $\hat{y} \in C_{n_1} \subset X$ such that $\hat{z}_\alpha \rightarrow \hat{y}$. For each $\alpha \in \Gamma$, let $\hat{z}_\alpha = \hat{y}_{n_\alpha}$, where $n_\alpha \rightarrow \infty$. Then according to our choice of \hat{y}_{n_α} in C_{n_α} , we have

(i)'' $\hat{y}_{n_\alpha} \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$ and

(ii)'' there exists a point $\hat{w}_{n_\alpha} \in T_{n_\alpha}(\hat{y}_{n_\alpha}) = T(\hat{y}_{n_\alpha})$ with $\operatorname{Re}\langle f - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x \rangle + h(\hat{y}_{n_\alpha}) - h(x) \leq 0$ for all $x \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$ and for all $f \in M_{n_\alpha}(\hat{y}_{n_\alpha}) = M(\hat{y}_{n_\alpha})$. Since $n_\alpha \rightarrow \infty$, there exists $\alpha_0 \in \Gamma$ such that $n_\alpha \geq n_1$ for all $\alpha \geq \alpha_0$. Thus $C_{n_1} \subset C_{n_\alpha}$, for all $\alpha \geq \alpha_0$. From (i)'' above we have $(\hat{y}_{n_\alpha}, \hat{y}_{n_\alpha}) \in G(S)$ for all $\alpha \in \Gamma$. Since S is upper semicontinuous with closed values, $G(S)$ is closed in $X \times X$; it follows that $\hat{y} \in S(\hat{y})$.

Also, since $\{\hat{w}_{n_\alpha}\}_{\alpha \geq \alpha_0}$ is a net in $\cup_{x \in C_{n_1}} T(x)$ which is compact, without loss of generality, we may assume that the net $\{\hat{w}_{n_\alpha}\}_{\alpha \in \Gamma}$ converges to some $\hat{w} \in \cup_{x \in C_{n_1}} T(x)$. Since T has a closed graph, $\hat{w} \in T(\hat{y})$.

Let $x \in S(\hat{y})$ and $f \in M(\hat{y})$ be arbitrarily fixed. Let $n_2 \geq n_1$ be such that $x \in C_{n_2}$. Since S and M are lower semicontinuous at \hat{y} , without loss of generality we may assume that for each $\alpha \in \Gamma$, there are $x_{n_\alpha} \in S(\hat{y}_{n_\alpha})$ and $f_{n_\alpha} \in M(\hat{y}_{n_\alpha})$ such that $x_{n_\alpha} \rightarrow x$ and $f_{n_\alpha} \rightarrow f$ respectively. By (ii)'' we have, $\operatorname{Re}\langle f_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle + h(\hat{y}_{n_\alpha}) - h(x_{n_\alpha}) \leq 0$ for all $\alpha \in \Gamma$. Note that $f_{n_\alpha} - \hat{w}_{n_\alpha} \rightarrow f - \hat{w}$ in $\delta(F, E)$ and $\{\hat{y}_{n_\alpha} - x_{n_\alpha}\}_{\alpha \in \Gamma}$ is a net in the compact (and hence bounded) set $C_{n_2} - \cup_{y \in C_{n_2}} S(y)$. Thus, we have for each $\epsilon > 0$, there exists $\alpha_1 \geq \alpha_0$ such that $|\operatorname{Re}\langle f_{n_\alpha} - \hat{w}_{n_\alpha} - (f - \hat{w}), \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle| < \epsilon/2$ for all $\alpha \geq \alpha_1$. Since $\langle f - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle \rightarrow \langle f - \hat{w}, \hat{y} - x \rangle$, there exists $\alpha_2 \geq \alpha_1$ such that $|\operatorname{Re}\langle f - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle - \operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle| < \epsilon/2$ for all $\alpha \geq \alpha_2$. Thus for $\alpha \geq \alpha_2$,

$$\begin{aligned} & |\operatorname{Re}\langle f_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle - \operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle| \\ & \leq |\operatorname{Re}\langle f_{n_\alpha} - \hat{w}_{n_\alpha} - (f - \hat{w}), \hat{y}_{n_\alpha} - x_{n_\alpha} \rangle| + |\operatorname{Re}\langle f - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha} - (\hat{y} - x) \rangle| \\ & < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Thus $\lim_{\alpha} \operatorname{Re}\langle f_{n_{\alpha}} - \hat{w}_{n_{\alpha}}, \hat{y}_{n_{\alpha}} - x_{n_{\alpha}} \rangle = \operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle$. By continuity of h , we have

$$\begin{aligned} & \operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \\ &= \lim_{\alpha} [\operatorname{Re}\langle f_{n_{\alpha}} - \hat{w}_{n_{\alpha}}, \hat{y}_{n_{\alpha}} - x_{n_{\alpha}} \rangle + h(\hat{y}_{n_{\alpha}}) - h(x_{n_{\alpha}})] \\ &\leq 0. \quad \square \end{aligned}$$

Corollary 4.4.16 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta\langle F, E \rangle$. Let $S : X \rightarrow 2^X$ be weakly continuous such that $S(x)$ is closed convex for each $x \in X$, $M : X \rightarrow 2^F$ be monotone (with respect to $\langle \cdot, \cdot \rangle$) and lower semicontinuous from the relative weak topology on X to the strong topology $\delta\langle F, E \rangle$ on F such that $M(x)$ is convex for each $x \in X$ and $T : X \rightarrow 2^F$ be upper semicontinuous from the relative weak topology on X to the strong topology $\delta\langle F, E \rangle$ on F such that $T(x)$ is a strongly compact convex subset of F for each $x \in X$. Suppose that $h : X \rightarrow \mathbf{R}$ is convex and weakly continuous. Suppose further that*

(1) *there exists an increasing sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers with $r_n \rightarrow \infty$ such that $S(x) \subset C_n$ for each $x \in C_n$ and each $n \in \mathbf{N}$ where $C_n = \{x \in X : \|x\| \leq r_n\}$;*

(2) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X , with $\|y_n\| \rightarrow \infty$, either there exists $n_0 \in \mathbf{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbf{N}$, $x_{n_0} \in S(y_{n_0})$ and $f_{n_0} \in M(y_{n_0})$ such that $\min_{w \in T(y_{n_0})} \operatorname{Re}\langle f_{n_0} - w, y_{n_0} - x_{n_0} \rangle + h(y_{n_0}) - h(x_{n_0}) > 0$.*

Then there exists $\hat{y} \in X$ such that

(a₁) $\hat{y} \in S(\hat{y})$ and

(b₁) *there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.*

Proof: Equip E with the weak topology. Then C_n is weakly compact convex for each $n \in \mathbf{N}$ such that $X = \bigcup_{n=1}^{\infty} C_n$. Now if $\{y_n\}_{n=1}^{\infty}$ is a sequence in X , with $y_n \in C_n$ for each $n = 1, 2, \dots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, then $\|y_n\| \rightarrow \infty$. By hypothesis (2), either there exists $n_0 \in \mathbf{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbf{N}$,

$x_{n_0} \in S(y_{n_0})$ and $f_{n_0} \in M(y_{n_0})$ such that $\min_{w \in T(y_{n_0})} \operatorname{Re}\langle f_{n_0} - w, y_{n_0} - x_{n_0} \rangle + h(y_{n_0}) - h(x_{n_0}) > 0$. Thus all hypotheses of Theorem 4.4.15 are satisfied so that the conclusion follows. \square

By taking $M \equiv 0$ and replacing T by $-T$ in Theorem 4.4.15, we obtain the following non-compact generalization of Corollary 4.4.14:

Corollary 4.4.17 *Let E be a locally convex Hausdorff topological vector space over Φ , X be a non-empty (convex) subset of E such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta(F, E)$. Suppose that*

(1) $S : X \rightarrow 2^X$ is a continuous map such that

(a) for each $x \in X$, $S(x)$ is a closed convex subset of X and

(b) for each $n \in \mathbf{N}$, $S(x) \subset C_n$ for all $x \in C_n$;

(2) $T : X \rightarrow 2^F$ is upper semicontinuous from the relative topology on X to the strong topology $\delta(F, E)$ on F such that $T(x)$ is a strongly compact convex subset of F for each $x \in X$;

(3) $h : X \rightarrow \mathbf{R}$ is convex and continuous;

(4) for each sequence $\{y_n\}_{n=1}^{\infty}$ in X , with $y_n \in C_n$ for each $n \in \mathbf{N}$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, either there exists $n_0 \in \mathbf{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in S(y_{n_0})$ such that $\min_{w \in T(y_{n_0})} \operatorname{Re}\langle w, y_{n_0} - x_{n_0} \rangle + h(y_{n_0}) - h(x_{n_0}) > 0$.

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

For other non-compact generalization of Corollary 4.4.14, we refer to Yuan [113].

The result observed by S.C. Fang (e.g. see [23, p.213] and [94, p.59]) can be modified below whose simple proof is omitted:

Lemma 4.4.18 *Let X be a cone in a Hausdorff topological vector space E over Φ and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional. Let $M, T : X \rightarrow 2^F$. Then the following are equivalent:*

(a) *There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that*

$$\operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle \leq 0 \text{ for all } x \in X \text{ and for all } f \in M(\hat{y}).$$

(b) *There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that*

$$\operatorname{Re}\langle f - \hat{w}, \hat{y} \rangle = 0 \text{ and } f - \hat{w} \in \widehat{X} \text{ for all } f \in M(\hat{y}).$$

When X is a cone in E , $h \equiv 0$ and $S(x) = X$ for all $x \in X$, by applying Lemma 4.4.18 and Theorem 4.4.15, we have immediately the following existence theorem of a generalized bi-complementarity problem:

Theorem 4.4.19 *Let E be a Hausdorff topological vector space over Φ , X be a cone in E such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta\langle F, E \rangle$. Suppose that*

(1) *$M : X \rightarrow 2^F$ is monotone (with respect to $\langle \cdot, \cdot \rangle$) and lower semicontinuous from the relative topology on X to the strong topology $\delta\langle F, E \rangle$ on F such that $M(x)$ is convex for each $x \in X$;*

(2) *$T : X \rightarrow 2^F$ is upper semicontinuous from the relative topology on X to the strong topology $\delta\langle F, E \rangle$ on F such that $T(x)$ is a strongly compact convex subset of F for each $x \in X$;*

(3) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X , with $y_n \in C_n$ for each $n \in \mathbf{N}$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$, $x_{n_0} \in X$ and $f_{n_0} \in M(y_{n_0})$ such that $\min_{w \in T(y_{n_0})} \operatorname{Re}\langle f_{n_0} - w, y_{n_0} - x_{n_0} \rangle > 0$.*

Then there exists a point $\hat{y} \in X$ and a point $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle f - \hat{w}, \hat{y} \rangle = 0 \text{ and } f - \hat{w} \in \widehat{X} \text{ for all } f \in M(\hat{y}).$$

Corollary 4.4.20 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a closed cone in E and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta\langle F, E \rangle$. Let $M : X \rightarrow 2^F$ be monotone (with respect to $\langle \cdot, \cdot \rangle$) and lower semicontinuous from the relative weak topology on X to the strong topology $\delta\langle F, E \rangle$ on F such that $M(x)$ is convex for each $x \in X$ and $T : X \rightarrow 2^F$ be upper semicontinuous from the relative weak topology on X to the strong topology $\delta\langle F, E \rangle$ on F such that $T(x)$ is a strongly compact convex subset of F for each $x \in X$. Let $\{r_n\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers with $r_n \rightarrow \infty$ and $C_n = \{x \in X : \|x\| \leq r_n\}$ for each $n \in \mathbf{N}$. Suppose that for each sequence $\{y_n\}_{n=1}^{\infty}$ in X , with $\|y_n\| \rightarrow \infty$, there exist $n_0 \in \mathbf{N}$, $x_{n_0} \in X$ and $f_{n_0} \in M(y_{n_0})$ such that $\min_{w \in T(y_{n_0})} \operatorname{Re}\langle f_{n_0} - w, y_{n_0} - x_{n_0} \rangle > 0$. Then there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle f - \hat{w}, \hat{y} \rangle = 0$ and $f - \hat{w} \in \widehat{X}$ for all $f \in M(\hat{y})$.*

Proof: Equip E with the weak topology. Then C_n is weakly compact convex for each $n \in \mathbf{N}$ such that $X = \bigcup_{n=1}^{\infty} C_n$. Now if $\{y_n\}_{n=1}^{\infty}$ is a sequence in X , with $y_n \in C_n$ for each $n = 1, 2, \dots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, then $\|y_n\| \rightarrow \infty$. Hence by hypothesis, there exist $n_0 \in \mathbf{N}$, $x_{n_0} \in X$ and $f_{n_0} \in M(y_{n_0})$ such that

$$\min_{w \in T(y_{n_0})} \operatorname{Re}\langle f_{n_0} - w, y_{n_0} - x_{n_0} \rangle > 0.$$

Thus all hypotheses of Theorem 4.4.19 are satisfied so that the conclusion follows. \square

4.4.4 Generalized Bi-Quasi-Variational Inequalities for Bi-Quasi-Monotone Operators in Compact Settings

In this section we shall first obtain some results on existence theorems of generalized bi-quasi-variational inequalities for bi-quasi-monotone operators in compact settings.

If we take $M = \{0\}$ and replace T by $-T$ in Lemma 4.4.4 of Section 4.4, we get the following result which slightly modifies Lemma 2 of Shih and Tan in [100, pp.70-71]:

Lemma 4.4.21 *Let E be a topological vector space over Φ , X be a non-empty compact subset of E and F be a Hausdorff topological vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional and $T : X \rightarrow 2^F$ be an upper semicontinuous map such that each $T(x)$ is compact. Let $x_0 \in X$ be arbitrarily fixed and $h : X \rightarrow \mathbf{R}$ be continuous. Define $g : X \rightarrow \mathbf{R}$ by $g(y) = [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle] + h(y)$ for each $y \in X$. Suppose that $\langle \cdot, \cdot \rangle$ is continuous on the (compact) subset $[\bigcup_{y \in X} T(y)] \times X$ of $F \times E$. Then g is lower semicontinuous on X .*

The following definition generalizes the Definition 3.2.5(a):

Definition 4.4.22 *Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty subset of E . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional and $M : X \rightarrow 2^F$ be a map. Then M is said to be lower hemi-continuous on X if and only if for each $p \in E$, the function $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ defined by*

$$f_p(z) = \sup_{u \in M(z)} \operatorname{Re}\langle u, p \rangle \quad \text{for each } z \in X,$$

is lower semicontinuous on X (if and only if for each $p \in E$, the function $g_p : X \rightarrow \mathbf{R} \cup \{-\infty\}$ defined by

$$g_p(z) = \inf_{u \in M(z)} \operatorname{Re}\langle u, p \rangle \quad \text{for each } z \in X,$$

is upper semicontinuous on X).

Note that the notions of lower hemi-continuity along line segments in X and upper hemi-continuity (see Definition 4.4.2) along line segments in X are independent of the topology τ on E as long as τ is Hausdorff and the vector space F over Φ remains unchanged. Note also that if $M, T : X \rightarrow 2^F$ are lower (respectively, upper) hemi-continuous on X and $\alpha \in \mathbf{R}$, then $M + T$ and αT are also lower (respectively, upper) hemi-continuous on X .

The following proposition generalizes Proposition 3.2.6:

Proposition 4.4.23 *Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty subset of E . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that for each $p \in E$, $u \mapsto \langle u, p \rangle$ is $\sigma\langle F, E \rangle$ -continuous on F when F is equipped with the $\sigma\langle F, E \rangle$ -topology. Let $M : X \rightarrow 2^F$ be lower semicontinuous from the relative topology on X to the weak topology $\sigma\langle F, E \rangle$ on F . Then M is lower hemi-continuous on X .*

Proof: For each $p \in E$, define $f_p : X \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$f_p(z) = \sup_{u \in M(z)} \operatorname{Re}\langle u, p \rangle \quad \text{for each } z \in X.$$

Fix any $p \in E$. Let $\lambda \in \mathbf{R}$ be given and let $A = \{z \in X : f_p(z) > \lambda\}$. Take any $z_0 \in A$. Then $f_p(z_0) = \sup_{u \in M(z_0)} \operatorname{Re}\langle u, p \rangle > \lambda$. Choose any $u_0 \in M(z_0)$ such that $\operatorname{Re}\langle u_0, p \rangle > \lambda$. Let $h : F \rightarrow \mathbf{R}$ be defined by $h(u) = \operatorname{Re}\langle u, p \rangle$ for each $u \in F$. By hypothesis h is $\sigma\langle F, E \rangle$ -continuous on F .

Thus $h^{-1}(\lambda, +\infty) \cap M(z_0) \neq \emptyset$, where $h^{-1}(\lambda, +\infty)$ is a $\sigma\langle F, E \rangle$ -open set in F . Since M is lower semicontinuous at z_0 , there exists an open neighborhood N_{z_0} of z_0 in X such that $M(z) \cap h^{-1}(\lambda, +\infty) \neq \emptyset$ for all $z \in N_{z_0}$. Hence $f_p(z) = \sup_{u \in M(z)} h(u) = \sup_{u \in M(z)} \operatorname{Re}\langle u, p \rangle > \lambda$ for all $z \in N_{z_0}$. Thus $N_{z_0} \subset A$. Consequently, f_p is lower semicontinuous on X . Hence M is lower hemi-continuous on X . \square

Note that the converse of Proposition 4.4.23 is not true as can be seen in Example 3.2.7.

Definition 4.4.24 Let E be a topological vector space, X be a non-empty subset of E . Let F be a vector space over Φ and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional. Let $M, T : X \rightarrow 2^F$ be two maps. Suppose $h : X \rightarrow \mathbf{R}$. Then M is said to be h - T -bi-quasi-monotone if for each $x, y \in X$, each finite set $\{\beta_j : j = 0, 1, \dots, n\}$ of non-negative real-valued functions and each finite set $\{p_k : k = 1, \dots, n\}$ of E^* ,

$$\beta_0(y) \left[\inf_{g \in M(y)} \inf_{w \in T(y)} \operatorname{Re} \langle g - w, y - x \rangle + h(y) - h(x) \right] + \sum_{k=1}^n \beta_k(y) \operatorname{Re} \langle p_k, y - x \rangle > 0$$

whenever

$$\beta_0(y) \left[\sup_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re} \langle f - w, y - x \rangle + h(y) - h(x) \right] + \sum_{k=1}^n \beta_k(y) \operatorname{Re} \langle p_k, y - x \rangle > 0.$$

M is said to be bi-quasi-monotone if M is h - T -bi-quasi-monotone with $h \equiv 0$ and $T \equiv 0$.

Clearly, a monotone operator is also an h - T -bi-quasi-monotone operator. But the converse is not true; because if $T \equiv 0$, $\beta_0 \equiv 1$ and each $p_k \equiv 0$, then an h - T -bi-quasi-monotone operator is an h -quasi-monotone operator which is not necessarily a monotone operator as shown in Example 3.2.11.

The following result generalizes Lemma 3.2.19:

Lemma 4.4.25 Let E be a topological vector space over Φ , F be a vector space over Φ and X be a non-empty convex subset of E . Let $\langle \cdot, \cdot \rangle : F \times E$ be a bilinear functional. Equip F with the $\sigma(F, E)$ -topology. Let D be a non-empty $\sigma(F, E)$ -compact subset of F , $h : X \rightarrow \mathbf{R}$ be convex and $M : X \rightarrow 2^F$ be lower hemi-continuous along line segments in X . Suppose $\hat{y} \in X$ is such that $\sup_{f \in M(x)} \inf_{g \in D} \operatorname{Re} \langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Then

$$\sup_{f \in M(\hat{y})} \inf_{g \in D} \operatorname{Re} \langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in X.$$

Proof: Suppose that $\sup_{f \in M(x)} \inf_{g \in D} \operatorname{Re} \langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Fix an arbitrary $x \in X$. For each $t \in [0, 1]$, let $z_t = tx + (1 - t)\hat{y} = \hat{y} + t(x - \hat{y})$. Then

$z_t \in X$ as X is convex. Thus for each $t \in (0, 1]$,

$$\begin{aligned} t \cdot \sup_{f \in M(z_t)} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle &= \sup_{f \in M(z_t)} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - z_t \rangle \\ &\leq h(z_t) - h(\hat{y}) \\ &\leq t(h(x) - h(\hat{y})), \end{aligned}$$

as h is convex so that $\sup_{f \in M(z_t)} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ and hence for all $t \in (0, 1]$, $\sup_{f \in M(z_t)} \operatorname{Re}\langle f, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) - \inf_{g \in D} \operatorname{Re}\langle g, x - \hat{y} \rangle$.

Let $L = \{z_t : t \in [0, 1]\}$ and $A = \{z \in L : \sup_{f \in M(z)} \operatorname{Re}\langle f, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) - \inf_{g \in D} \operatorname{Re}\langle g, x - \hat{y} \rangle\}$. Since M is lower hemi-continuous on L , $z_t \in A$ for all $t \in (0, 1]$ and $z_t \rightarrow \hat{y}$ as $t \rightarrow 0^+$, we have $\hat{y} \in A$ so that

$$\sup_{f \in M(\hat{y})} \operatorname{Re}\langle f, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) - \inf_{g \in D} \operatorname{Re}\langle g, x - \hat{y} \rangle.$$

It follows that $\sup_{f \in M(\hat{y})} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$. Since $x \in X$ is arbitrary, we have $\sup_{f \in M(\hat{y})} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. \square

We shall now establish the following result:

Theorem 4.4.26 *Let E be a locally convex Hausdorff topological vector space over Φ , X be a non-empty compact convex subset of E and F be a Hausdorff topological vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that*

(a) $S : X \rightarrow 2^X$ is an upper semicontinuous map such that each $S(x)$ is closed convex;

(b) $T : X \rightarrow 2^F$ is upper semicontinuous such that each $T(x)$ is compact convex;

(c) $h : X \rightarrow \mathbf{R}$ is convex and continuous;

(d) $M : X \rightarrow 2^F$ is lower hemi-continuous along line segments in X and h - T -bi-quasi-monotone (with respect to $\langle \cdot, \cdot \rangle$) such that each $M(x)$ is convex and

(e) the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\sup_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x)] > 0\} \text{ is open in } X.$$

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exist a point $\hat{w} \in T(\hat{y})$ with $Re\langle f - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex and if $T \equiv 0$, the continuity assumption on $\langle \cdot, \cdot \rangle$ can be weakened to the assumption that for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X .

Proof: We divide the proof into three steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} \left[\sup_{f \in M(x)} \inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exist $x \in S(y)$ and $f \in M(x)$ such that $\inf_{w \in T(y)} Re\langle f - w, y - x \rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that

$$Re\langle p, y \rangle - \sup_{x \in S(y)} Re\langle p, x \rangle > 0.$$

For each $p \in E^*$, let

$$V(p) = \{y \in X : Re\langle p, y \rangle - \sup_{x \in S(y)} Re\langle p, x \rangle > 0\}.$$

Then $V(p)$ is open by Lemma 4.3.1. Since $X = \Sigma \cup \bigcup_{p \in E^*} V(p)$, by compactness of X , there exist $p_1, p_2, \dots, p_n \in E^*$ such that $X = \Sigma \cup \bigcup_{i=1}^n V(p_i)$. For simplicity of notations, let $V_0 := \Sigma$ and $V_i = V(p_i)$ for $i = 1, 2, \dots, n$. Let $\{\beta_0, \beta_1, \dots, \beta_n\}$ be a continuous partition of unity on X subordinated to the covering $\{V_0, V_1, \dots, V_n\}$. Then $\beta_0, \beta_1, \dots, \beta_n$ are continuous non-negative real-valued functions on X such that β_i vanishes on $X \setminus V_i$, for each $i = 0, 1, \dots, n$ and $\sum_{i=0}^n \beta_i(x) = 1$ for all $x \in X$. Define $\phi, \psi : X \times X \rightarrow \mathbf{R}$ by

$$\phi(x, y) = \beta_0(y) \left[\sup_{f \in M(x)} \inf_{w \in T(y)} Re\langle f - w, y - x \rangle + h(y) - h(x) \right] + \sum_{i=1}^n \beta_i(y) Re\langle p_i, y - x \rangle,$$

and

$$\psi(x, y) = \beta_0(y) \left[\inf_{g \in M(y)} \inf_{w \in T(y)} \operatorname{Re} \langle g - w, y - x \rangle + h(y) - h(x) \right] + \sum_{i=1}^n \beta_i(y) \operatorname{Re} \langle p_i, y - x \rangle,$$

for each $x, y \in X$. Then we have the following.

(1) For each $x \in X$, $\psi(x, x) \leq 0$ and for each $x, y \in X$, since M is h - T -bi-quasi-monotone, $\phi(x, y) > 0$ implies $\psi(x, y) > 0$.

(2) For each fixed $x \in X$ and each fixed $f \in M(x)$, the map $T_f : X \rightarrow 2^F$ defined by

$$T_f(y) := f - T(y) \quad \text{for each } y \in X$$

is an upper semicontinuous map such that each $T_f(y)$ is a compact subset of F . Thus by Lemma 4.4.21, the map

$$y \longmapsto \left[\inf_{w \in T(y)} \operatorname{Re} \langle f - w, y - x \rangle \right] + h(y) - h(x) = \left[\inf_{w' \in T_f(y)} \operatorname{Re} \langle w', y - x \rangle \right] + h(y) - h(x)$$

is lower semicontinuous on X . By Lemma 4.3.2, the map

$$y \longmapsto \beta_0(y) \left[\sup_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re} \langle f - w, y - x \rangle + h(y) - h(x) \right]$$

is lower semicontinuous on X . Hence for each fixed $x \in X$, the map $y \longmapsto \phi(x, y)$ is lower semicontinuous on X .

(3) Clearly, for each fixed $y \in X$, the set $\{x \in X : \psi(x, y) > 0\}$ is convex.

Then ϕ and ψ satisfy all the hypotheses of Theorem 3.3.1. Thus by Theorem 3.3.1, there exists $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$, i.e.,

$$\beta_0(\hat{y}) \left[\sup_{f \in M(x)} \inf_{w \in T(\hat{y})} \operatorname{Re} \langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] + \sum_{i=1}^n \beta_i(\hat{y}) \operatorname{Re} \langle p_i, \hat{y} - x \rangle \leq 0 \quad (4.29)$$

for all $x \in X$.

Choose $\hat{x} \in S(\hat{y})$ such that

$$\sup_{f \in M(\hat{x})} \inf_{w \in T(\hat{y})} \operatorname{Re} \langle f - w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) > 0 \quad \text{whenever } \beta_0(\hat{y}) > 0;$$

it follows that

$$\beta_0(\hat{y}) \left[\sup_{f \in M(\hat{x})} \inf_{w \in T(\hat{y})} \operatorname{Re} \langle f - w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] > 0 \quad \text{whenever } \beta_0(\hat{y}) > 0.$$

If $i \in \{1, \dots, n\}$ is such that $\beta_i(\hat{y}) > 0$, then $\hat{y} \in V(p_i)$ and hence

$$Re\langle p_i, \hat{y} \rangle > \sup_{x \in S(\hat{y})} Re\langle p_i, x \rangle \geq Re\langle p_i, \hat{x} \rangle$$

so that $Re\langle p_i, \hat{y} - \hat{x} \rangle > 0$. Then note that

$$\beta_i(\hat{y})Re\langle p_i, \hat{y} - \hat{x} \rangle > 0 \quad \text{whenever} \quad \beta_i(\hat{y}) > 0 \quad \text{for} \quad i = 1, \dots, n.$$

Since $\beta_i(\hat{y}) > 0$ for at least one $i \in \{1, \dots, n\}$, it follows that

$$\beta_0(\hat{y}) \left[\sup_{f \in M(\hat{x})} \inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] + \sum_{i=1}^n \beta_i(\hat{y}) Re\langle p_i, \hat{y} - \hat{x} \rangle > 0,$$

which contradicts (4.29). This contradiction proves Step 1.

Step 2.

$$\sup_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all} \quad x \in S(\hat{y}).$$

Indeed, from Step 1, $\hat{y} \in S(\hat{y})$ which is a convex subset of X , and

$$\sup_{f \in M(x)} \inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all} \quad x \in S(\hat{y}). \quad (4.30)$$

Hence by Lemma 4.4.25, we have

$$\sup_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \text{for all} \quad x \in S(\hat{y}).$$

Step 3. There exist a point $\hat{w} \in T(\hat{y})$ with $Re\langle f - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

From Step 2 we have

$$\sup_{x \in S(\hat{y})} \left[\sup_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] \leq 0;$$

i.e.,

$$\sup_{(x, f) \in S(\hat{y}) \times M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0, \quad (4.31)$$

where $S(\hat{y}) \times M(\hat{y})$ is a convex subset of the Hausdorff topological vector space $E \times F$ and $T(\hat{y})$ is a compact convex subset of F .

Let $Q = S(\hat{y}) \times M(\hat{y})$ and the map $g : Q \times T(\hat{y}) \rightarrow \mathbf{R}$ be defined by $g(q, w) = g((x, f), w) = \operatorname{Re}\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ for each $q = (x, f) \in Q = S(\hat{y}) \times M(\hat{y})$ and each $w \in T(\hat{y})$. Note that for each fixed $q \in Q$, the map $w \mapsto g(q, w) = g((x, f), w)$ is lower semicontinuous from the relative topology on $T(\hat{y})$ to \mathbf{R} and also convex on $T(\hat{y})$. Clearly, for each fixed $w \in T(\hat{y})$, the map $(x, f) \mapsto g((x, f), w)$ is concave on Q . Then by Theorem 3.2.1 we have

$$\min_{w \in T(\hat{y})} \sup_{q \in Q} g(q, w) = \sup_{q \in Q} \min_{w \in T(\hat{y})} g(q, w).$$

Thus

$$\min_{w \in T(\hat{y})} \sup_{[(x, f) \in S(\hat{y}) \times M(\hat{y})]} [\operatorname{Re}\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0, \quad \text{by (4.31).}$$

Since $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that

$$\sup_{(x, f) \in S(\hat{y}) \times M(\hat{y})} [\operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0.$$

Therefore

$$\operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$$

for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

Next we note from the above proof that E is required to be locally convex when and only when the separation theorem is applied to the case $y \notin S(y)$. Thus if $S : X \rightarrow 2^X$ is the constant map $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Finally, if $T \equiv 0$, in order to show that for each $x \in X$, $y \mapsto \phi(x, y)$ is lower semicontinuous, Lemma 4.4.21 is no longer needed and the weaker continuity assumption on $\langle \cdot, \cdot \rangle$ that for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X is sufficient. This completes the proof. \square

Theorem 4.4.27 *Let E be a locally convex Hausdorff topological vector space over Φ , X be a non-empty compact convex subset of E and F be a vector space over Φ . Let $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in*

F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X . Equip F with the strong topology $\delta\langle F, E \rangle$. Suppose that

(a) $S : X \rightarrow 2^X$ is a continuous map such that each $S(x)$ is closed and convex;

(b) $T : X \rightarrow 2^F$ is upper semicontinuous such that each $T(x)$ is strongly compact and convex;

(c) $h : X \rightarrow \mathbf{R}$ is convex and continuous;

(d) $M : X \rightarrow 2^F$ is lower hemi-continuous along line segments in X and h - T -bi-quasi-monotone (with respect to $\langle \cdot, \cdot \rangle$) such that each $M(x)$ is convex; also, for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\sup_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x)] > 0\}$, M is lower semicontinuous at some point x in $S(y)$ with $\sup_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x) > 0$.

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle f - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Proof: As $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$ is a bilinear functional such that for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X and as F is equipped with the strong topology $\delta\langle F, E \rangle$, it is easy to see that $\langle \cdot, \cdot \rangle$ is continuous on compact subsets of $F \times X$. Thus by Theorem 4.4.26, it suffices to show that the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\sup_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Indeed, let $y_0 \in \Sigma$; then by the last part of the hypothesis (d), M is lower semicontinuous at some point x_0 in $S(y_0)$ with $\sup_{f \in M(x_0)} \inf_{w \in T(y_0)} \operatorname{Re}\langle f - w, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$. Hence there exists $f_0 \in M(x_0)$ such that $\inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$. Let

$$\alpha := \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y_0 - x_0 \rangle + h(y_0) - h(x_0).$$

Then $\alpha > 0$. Also let

$$W := \{w \in F : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2 \rangle| < \frac{\alpha}{6}\}.$$

Then W is an open neighborhood of 0 in F so that $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in F . Since T is upper semicontinuous at y_0 , there exists an open neighborhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

Let $U_2 := \{f \in F : \sup_{z_1, z_2 \in X} |\langle f - f_0, z_1 - z_2 \rangle| < \frac{\alpha}{6}\}$, then U_2 is an open neighborhood of f_0 in F . Since M is lower semicontinuous at x_0 and $U_2 \cap M(x_0) \neq \emptyset$, there exists an open neighborhood V_1 of x_0 in X such that $M(x) \cap U_2 \neq \emptyset$ for all $x \in V_1$.

As the map $x \mapsto \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, x_0 - x \rangle + h(x_0) - h(x)$ is continuous at x_0 , there exists an open neighborhood V_2 of x_0 in X such that

$$\left| \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, x_0 - x \rangle + h(x_0) - h(x) \right| < \frac{\alpha}{6} \quad \text{for all } x \in V_2.$$

Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X . Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semicontinuous at y_0 , there exists an open neighborhood N_2 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_2$.

Since the map $y \mapsto \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y - y_0 \rangle + h(y) - h(y_0)$ is continuous at y_0 , there exists an open neighborhood N_3 of y_0 in X such that

$$\left| \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y - y_0 \rangle + h(y) - h(y_0) \right| < \frac{\alpha}{6} \quad \text{for all } y \in N_3.$$

Let $N_0 := N_1 \cap N_2 \cap N_3$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

- (i) $T(y_1) \subset U_1 = T(y_0) + W$ as $y_1 \in N_1$;
- (ii) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_2$; so we can choose any $x_1 \in S(y_1) \cap V_0$;
- (iii) $\left| \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y_1 - y_0 \rangle + h(y_1) - h(y_0) \right| < \frac{\alpha}{6}$ as $y_1 \in N_3$;
- (iv) $M(x_1) \cap U_2 \neq \emptyset$ as $x_1 \in V_1$; choose any $f_1 \in M(x_1) \cap U_2$ so that

$$\sup_{z_1, z_2 \in X} |\langle f_1 - f_0, z_1 - z_2 \rangle| < \frac{\alpha}{6};$$

- (v) $\left| \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, x_0 - x_1 \rangle + h(x_0) - h(x_1) \right| < \frac{\alpha}{6}$ as $x_1 \in V_2$.

It follows that

$$\begin{aligned}
& \inf_{w \in T(y_1)} \operatorname{Re}\langle f_1 - w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\
& \geq \operatorname{Re}\langle f_1 - f_0, y_1 - x_1 \rangle + \inf_{w \in T(y_1)} \operatorname{Re}\langle f_0 - w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\
& \geq -\frac{\alpha}{6} + \inf_{[w \in T(y_0) + W]} \operatorname{Re}\langle f_0 - w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \quad (\text{by (i) and (iv)}), \\
& \geq -\frac{\alpha}{6} + \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\
& \quad + \inf_{w \in W} \operatorname{Re}\langle f_0 - w, y_1 - x_1 \rangle \\
& \geq -\frac{\alpha}{6} + \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y_1 - y_0 \rangle + h(y_1) - h(y_0) \\
& \quad + \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y_0 - x_0 \rangle + h(y_0) - h(x_0) \\
& \quad + \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, x_0 - x_1 \rangle + h(x_0) - h(x_1) \\
& \quad + \inf_{w \in W} \operatorname{Re}\langle -w, y_1 - x_1 \rangle \\
& \geq -\frac{\alpha}{6} - \frac{\alpha}{6} + \alpha - \frac{\alpha}{6} - \frac{\alpha}{6} = \frac{\alpha}{3} > 0 \quad (\text{by (iii) and (v)});
\end{aligned}$$

therefore

$$\sup_{x \in S(y_1)} \left[\sup_{f \in M(x)} \inf_{w \in T(y_1)} \operatorname{Re}\langle f - w, y_1 - x \rangle + h(y_1) - h(x) \right] > 0$$

as $x_1 \in S(y_1)$ and $f_1 \in M(x_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$, so that Σ is open in X . This proves the theorem. \square

Chapter 5

Concluding Remarks

In summary, in this thesis we have given a KKM type lemma, some generalizations of the Ky Fan's minimax inequality, several fixed point theorems in Hilbert spaces, and several existence theorems for non-compact generalized variational inequalities and non-compact generalized complementarity problems in topological vector spaces and non-compact generalized quasi-variational inequalities in locally convex Hausdorff topological vector spaces and several existence theorems for both compact and non-compact generalized bi-quasi-variational inequalities, and non-compact bi-complementarity problems in locally convex Hausdorff topological vector spaces.

Besides, we have introduced the concepts of h -quasi-monotone, quasi-monotone, bi-quasi-monotone, h -quasi-semi-monotone, quasi-semi-monotone, quasi-nonexpansive, semi-nonexpansive, lower hemi-continuous, upper hemi-continuous, weakly lower (respectively, upper) demi-continuous, strongly lower (respectively, upper) demi-continuous, strong h -pseudo-monotone, strong pseudo-monotone, h -pseudo-monotone, pseudo-monotone, h -demi-monotone, and demi-monotone operators.

Further, even though we have some results for demi-operators, generalized quasi-monotone, generalized quasi-semi-monotone, bi-quasi-semi-monotone, and hemi-continuous operators on generalized variational inequalities or generalized quasi-variational inequalities, they have not been included here. We have completed some work on these topics and wish to continue on these soon.

Note that, we have not covered the topics on generalized KKM (in short G-KKM) maps, minimax inequalities and existence theorems of equilibria for \mathcal{G}_C^L -majorized correspondences in generalized convex (or G-convex) spaces. But some work has been done by the author on these topics. In particular, the minimax inequalities of Chapter 2 have been generalized into G-convex spaces and as applications of some of these minimax inequalities, results on the existence theorems of equilibria have been obtained in G-convex spaces. For some detail on these topics, we refer to M. S. R. Chowdhury [26], M. S. R. Chowdhury and K.-K. Tan [28]-[29] and some references therein.

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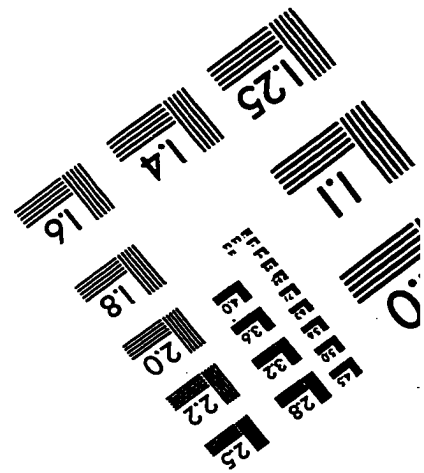
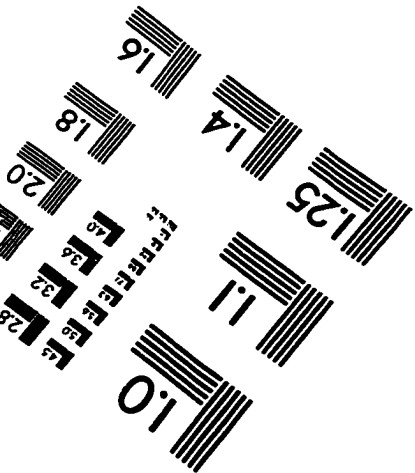
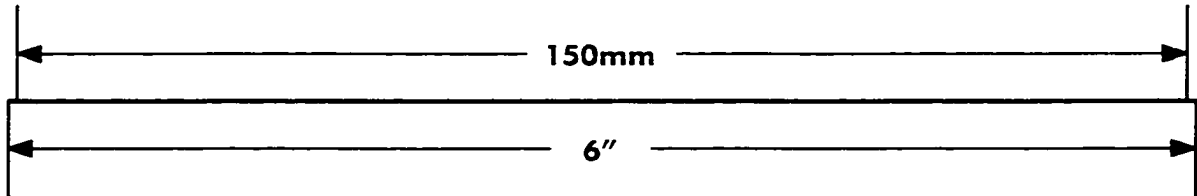
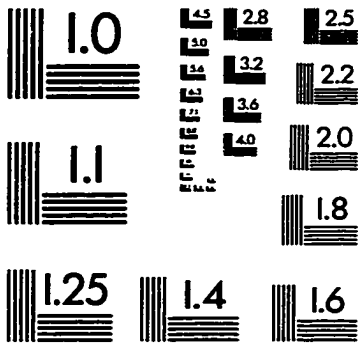
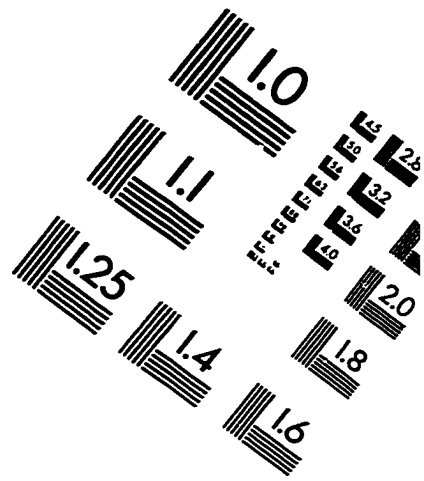
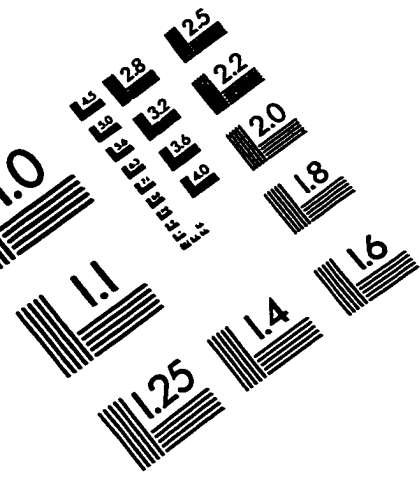
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IMAGE EVALUATION TEST TARGET (QA-3)



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