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ASPECTS OF DOMINATION

AND

DYNAMIC DOMINATION

By

Shannon L. Fitzpatrick

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<u>by</u>   <u>Shannon Fitzpatrick</u>

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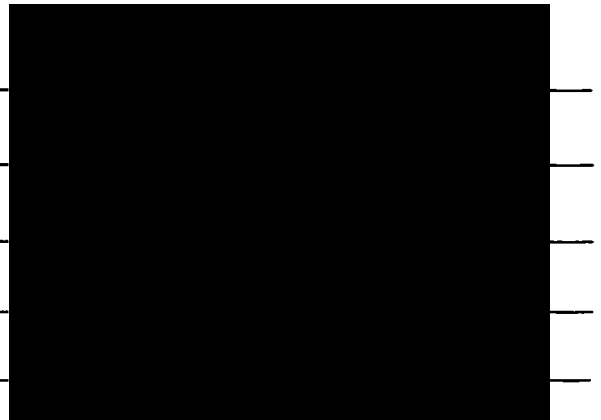
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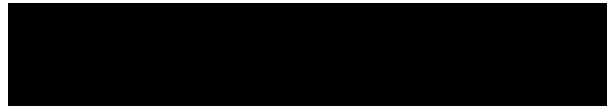
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Alice laughed: "There's no use in trying," she said; "one can't believe impossible things." "I daresay you haven't had much practice," said the Queen. "When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

Lewis Carroll

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Abstract

We continue the study of paired-domination, introduced by Haynes & Slater [27], and initiate the study of the related topic of paired-irredundance. In particular, we obtain results regarding paired-domination and paired-irredundance in products of graphs; characterize all well paired-dominated graphs of girth at least eight; and characterize all graphs of girth at least seven in which there is a minimum paired-dominating set which induces a maximal matching.

Our attention then turns toward dynamic domination. We study the game of Cops and Robber and introduce two variations of that game: the precinct game and the dragnet game. For both games we find upper bounds on the number of cops required to win the game, and for the precinct game, we find exactly the minimum number of cops required to win in such graphs as trees and grids. Finally, we examine isometric embeddings of graphs, and the relationship between the strong isometric dimension of a graph and the minimum number of cops required to win the game of cops and robber.

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I would like to thank Dr. Bert Hartnell who introduced me to the subject of graph theory while I was an undergraduate at St. Mary's University. The opportunities afforded me during that time are responsible for my study of the subject today. His more recent contributions to the development of my research career. including introducing me to conferences in warmer climates. is also greatly appreciated.

Finally. I extend all my love and thanks to my parents. Neil and Carol. and my brother Michael. They have encouraged all of my pursuits. especially academic. and taught me to be proud of my talents. None of this would have been possible without their love and support.

Introduction

Domination in graphs is a topic that has prompted much study. The original problem of finding a dominating set in a graph has inspired numerous variations of the problem. including the one presented in this thesis. In this thesis. the variation involves finding a dominating set in which the vertices in the set can be paired via a matching edge. Such a set is called a paired-dominating set.

A subject closely associated with the subject of domination is irredundance. It. therefore. seemed a natural progression to define a paired-irredundant set. While a paired-irredundant set is not necessarily an irredundant set. the relationship between paired-domination and paired-irredundance is analogous to that of domination and irredundance.

In Chapter 1. we give a brief survey of result in domination and irredundance and then present results for paired-domination and paired-irredundance. The problems examined include paired-domination and paired-irredundance in products of graphs and characterizing classes of graphs that contain a paired-dominating set which satisfies some additional criteria. The first criteria is that all minimal paired-dominating sets have the same cardinality. Graphs containing such a paired-dominating set are called well paired-dominated. We characterize all well paired-dominated graphs with girth at least eight. The second criteria is that there is a minimum paired-dominating set in the graph such that there is a maximal matching induced on the set. We show that every leafless graph of this type with girth at least seven belongs to an infinite family based on the 9-cycle.

We then move from these *static* dominating sets to consider *dynamic* domination.

Instead of finding a set of vertices which dominate the entire graph, we wish to find a series of vertex sets such that after some point in the series the vertex sets dominate some moving target on the graph. There are many possible interpretations of this problem of dynamic domination, subject to the conditions placed on the choice of sets and the movement of the target.

One dynamic domination problem we study takes the form of the game of Cops and Robber. This game has been studied extensively and we discuss its development in Chapter 2. We then propose variations of the game in which the cops are restricted to particular sets of vertices, or beats, in the graph. If each beat is both a retract and a copwin graph, then a single cop on each beat is eventually able to "protect" his beat. From that point forward, the robber will be immediately captured if he moves onto that beat.

In the precinct version of the game each cop can only move to vertices in his beat, and in the dragnet version each cop is restricted to his beat unless it is the final move of the game. By studying these games we gain information about the original game of cops and robber, but we also discover problems which are interesting outside of the context of the game. For example, the problem of finding the minimum number of cops required to win the precinct game is actually the minimum number of isometric paths required to cover the vertices of a graph. This is called the precinct number of the graph. We are able to find the exact precinct number for all $m \times n$ grids. We also find a lower bound on the precinct number for all d-dimensional grids.

This cops and robber theme continues in Chapter 3, where an attempt to find an upper bound on the number of cops required to win the game leads to a more algebraic study of the structure of a graph. The hope is that by isometrically embedding a graph into the strong product of paths we will be able to see possible "get-away" routes for the robber. This leads to the definition of the strong isometric dimension of a graph. It is the least number of paths such that the graph can be isometrically embedded in the strong product of those paths. For graphs with strong isometric dimension two, this proves to be a good technique for finding the minimum number of cops required.

Some progress has also been made for graphs with strong isometric dimension three. As for graphs with larger strong isometric dimension. while no good results are readily available. it is still thought to be a promising technique.

Determining the strong isometric dimension of a graph has also proved to be an interesting problem in its own right and is seen to have close ties to finding the injective hull of a graph. For some graphs. such as cycles and hypercubes. it is relatively easy to find the strong isometric dimension. For other graphs, such as trees, this problem is more difficult. We are. however, able to find upper and lower bounds on the strong isometric dimension of a tree that vary by a factor of two.

After studying cops and robber in a graph with strong isometric dimension two. we decided there was no reason to restrict ourselves to the strong products of paths. We show that if a graph can be isometrically embedded in the strong product of two trees. a cycle and a tree or two cycles. then good upper bounds can be found on the minimum number of cops necessary to win. These bounds are obtained by finding another means of dynamically dominating the graph. In this case the series of vertex sets required are cut sets.

Besides the relationship between the strong isometric dimension of a graph and its injective hull. there is little in Chapter 3 that can be related to any known results. Therefore. an account of the history behind the problems in the final chapter was too brief to warrant its own section.

Note that throughout this thesis. known results given for historical perspective only will be highlighted in quotation form. while those used in proving new results will be stated as Theorems. Lemmas. etc. and will credit the author(s) of the result. New results will also be displayed as Theorems, etc. but will not display a name nor a citation.

Definitions and Symbols

A **graph** $G = (V, E)$ is a set V of elements called **vertices**, together with a set E of two element subsets of V called **edges**. Two vertices, x and y, are said to be **adjacent** if they are joined by an edge. This is denoted by $x \sim y$. Otherwise they are **non-adjacent** which is denoted by $x \perp y$. A set of vertices is said to be **independent** if no two vertices in the set are adjacent. The set of all vertices adjacent to a vertex x is called the **neighbourhood** of x and is denoted $N(x)$. The cardinality of $N(x)$ is said to be the **degree** of x. The **closed neighbourhood** of x, $N[x]$, is the set $N(x) \cup \{x\}$. We say that the neighbourhood of a set S, $N(S)$, is the union of the neighbourhoods of all its elements. The closed neighbourhood of a set is defined similarly. An edge between vertices x and y is said to have x and y as **end vertices**. It is also said that the edge is **incident** with x (or with y). A set of edges is said to be independent if no two edges share a common end vertex.

A **walk** in G is a set of vertices $\{a_1, \ldots a_n\}$ such that $a_i a_{i+1} \in E$ for each $i = 1, \ldots n - 1$. This walk is called a **path** (or a path from a_1 to a_n) if all the vertices are distinct. An n-**path** is any path on n vertices. It is said to have **length** $n - 1$, denoted $l(P) = n - 1$. Two vertices are **connected** if there is a path between them. A graph G is connected if every pair of vertices in G are connected. The **distance** between connected vertices x and y in G is the minimum length of all paths from x to y in G. This is denoted $d(x, y)$ or $d_G(x, y)$ if we wish to emphasize that this occurs in the graph G. The **diameter** of a connected graph G, denoted $diam(G)$ is the maximum distance between two vertices in G. A set S in a connected graph G is called a **cut set** if for some pair of vertices x, y in G such that $x, y \notin S$ every path

from x to y contains a vertex of S.

A **cycle** is a set of vertices $\{a_1, a_2, \ldots, a_n\}$ such that $a_i a_{i+1} \in E$ for each $i = 1, 2, \ldots, n-1$ and $a_n \sim a_1$. A cycle on n vertices is also called an n-cycle. The **girth** of a graph is the minimum n such that the graph contains an n-cycle. A **tree** is a connected graph containing no cycles. A graph is **bipartite** if it contains no n-cycle for any odd n.

For other terms please see [10] and [11].

Chapter 1

Paired-Domination and Paired-Irredundance

1.1 Introduction

A **dominating set** in a graph. G. is a set of vertices. S. such that every vertex in G either belongs to S or is adjacent to a vertex in S. The minimum cardinality of a dominating set in a graph G is called the **domination number** of G and is denoted $\gamma(G)$ and the maximum cardinality of a minimal dominating set is denoted $\Gamma(G)$. If S is a dominating set. we say that every vertex of G is **dominated** by some vertex in S.

Figure 1.1 shows two minimal dominating sets in a graph G with minimum and maximum cardinality. respectively. (The solid vertices in each graph are those in the dominating set.) Hence. $\gamma(G) = 2$ and $\Gamma(G) = 3$.

Figure 1.1: A graph G in which $\gamma(G) = 2$ and $\Gamma(G) = 3$.

A set of vertices X is **irredundant** if for no vertex v in X is $N[v]$ a subset of $N[X \setminus \{v\}]$. If this is the case we say that each vertex of X has at least one **private neighbour**. That is. every vertex v in X has at least one neighbour. perhaps even v itself which is not adjacent or equal to any other vertex in X. Let $ir(G)$ and $IR(G)$ denote the size of the smallest and largest maximal irredundant sets. respectively. and call $ir(G)$ the **irredundance number** of G.

Figure 1.1 shows two maximal irredundant sets in G with minimum and maximum cardinality. respectively. Hence. $ir(G) = 2$ and $IR(G) = 3$.

Figure 1.2: A graph G in which $ir(G) = 2$ and $IR(G) = 3$.

A **matching** is defined to be a set of independent edges in a graph. We say that a vertex is **met by the matching** M if that vertex is an end vertex of some edge in the matching. and we let $V(M)$ denote the set of all vertices met by the matching M. Define a set of vertices S to be **saturated** if every vertex in S is met by the matching: that is. $S \subseteq V(M)$.

We will call a matching M a **dominating matching** or dom-matching for short. if the set $V(M)$ is a dominating set. Figure 1.3 shows two minimal dominating matchings in the path on the 4-path. P_4. The edges in the matching are represented by dashed lines.

Figure 1.3: The graph P_4 with dom-matchings containing one and two edges. respectively.

A set of vertices S is defined to be **paired-dominating set**. if there is a dom-matching M such that $S = V(M)$. A paired-dominating set S is minimal if there is

a minimal dom-matching M such that $V(M) = S$. For example. if we have the path $P_4 = \{a, b, c, d\}$, then both $S_1 = \{a, b, c, d\}$ and $S_2 = \{b, c\}$ are paired-dominating sets. Even though S_2 is strictly contained in S_1, they are both minimal paired-dominating sets. This is because the matchings $M_1 = \{ab, cd\}$ and $M_2 = \{bc\}$ are both minimal dom-matchings. Let $\gamma_p(G)$ and $\Gamma_p(G)$ denote the size of the smallest and largest minimal paired-dominating sets in G. Call any paired-dominating set with minimum cardinality a γ_p-set. So. $\gamma_p(P_4) = 2$ and $\Gamma_p(P_4) = 4$.

We will call a matching M an **irredundant matching** or ir-matching, if for no edge uv in M is $N[\{u, v\}]$ a subset of $N[V(M) \setminus \{u, v\}]$. Hence. for every pair in $V(M)$ at least one vertex in the pair has a private neighbour. We call the set $V(M)$ a **paired-irredundant set**. A set X is a paired-irredundant set if there is an ir-matching M such that $V(M) = X$. A paired-irredundant set X is maximal if there is a maximal ir-matching M such that $V(M) = X$. Let $ir_p(G)$ and $IR_p(G)$ denote the size of the smallest and largest maximal paired-irredundant sets. respectively, and call $ir_p(G)$ the **paired-irredundance number** of G. Figure 1.4 shows two maximal ir-matchings in a graph G. These are the smallest and largest maximal matchings. respectively, so $ir_p(G) = 2$ and $IR_p(G) = 4$.
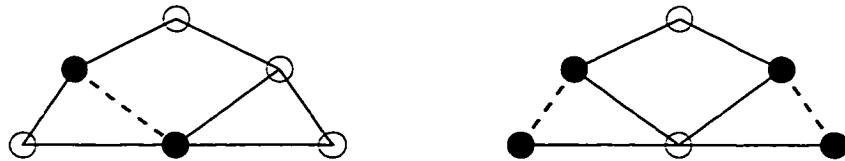
Figure 1.4: A graph G with $ir_p(G) = 2$ and $IR_p(G) = 4$.

In Section 1.2. we present known results in the theory of domination and irredundance and prove some analogous results for paired-domination and paired-irredundance. A well known result in the theory of domination and irredundance is the following string of inequalities:

$$ir \leq \gamma \leq \Gamma \leq IR.$$

We show a similar result for paired-domination and paired irredundance:

$$ir_p \leq \gamma_p \leq \Gamma_p \leq IR_p.$$

We define several graph products in Section 1.3 and investigate the parameters γ_p. Γ_p, ir_p and IR_p with respect to these products. We obtain various upper and lower bounds for these parameters.

In Section 1.4. we examine the class of graphs where every minimal paired-dominating set is a minimum paired-dominating set. We show that any such graph with girth at least eight is either a 9-cycle or its stems form an independent dominating set.

Finally. in Section 1.5, we characterize all leafless graphs of girth at least seven in which there exists a maximal matching. M, such that $V(M)$ is a γ_p-set. This family of graphs. denoted $\mathcal{G}$. turns out to be an infinite family based on the 9-cycle.

1.2 Historical Development and Preliminary Results

The theory of dominating sets was formally introduced by Ore [42] and Berge [8]. and has since received much attention. In [8]. Berge used dominating sets to model a set of radar stations where each station has itself and all adjacent vertices under surveillance. Another representation of dominating sets involves a communication network where the dominating set is a set of cities which. acting as transmitting stations. can transmit messages to every city in the network.

It was shown by Ore [42] that:

> If G is a graph with no isolated vertices and S is a minimal dominating set. then $V(G) \setminus S$ is also a dominating set.

This result gives the following corollary:

> If G is a connected graph with p vertices then $\gamma(G) + \Gamma(G) \leq p$.

To show this we let S be a minimal dominating set of size $\Gamma(G)$. Since $V(G) \setminus S$ is a dominating set, then $|V(G) \setminus S| \geq \gamma(G)$. The result follows because $|V(G) \setminus S| + |S| = p$.

A **total dominating set** is a set of vertices, S, such that every vertex in G is adjacent to a vertex in S. In other words, every vertex must be **externally dominated** by a vertex in S. Using the radar station scenario, this would represent a case where a station was incapable of monitoring itself, so it needs to monitored by some other station. Note that any graph with no isolated vertices can be totally dominated. The minimum cardinality of a total dominating set in a graph G containing no isolated vertices is denoted by $\gamma_t(G)$.

Total dominating sets were introduced by Cockayne, Dawes & Hedetniemi [13]. They showed, among other things, that:

If G is a connected graph with p vertices then $\gamma_t(G) \leq 2p/3$

and

If G has maximum degree $\Delta = \Delta(G)$ and p vertices, none of which are isolated, then $\gamma_t(G) \leq p - \Delta + 1$.

Now suppose we change our model slightly, such that instead of radar stations we have a set of guards where each guard protects all vertices adjacent to him. Hence, the guards must form a total dominating set in order to protect all the vertices. In addition, suppose each guard has a partner and they are assigned so each guard must protect his partner. Hence, we wish to find a dominating set where pairs of vertices in the set can be matched via edges in the graph. The set in question is the paired-dominating set which was introduced by Haynes & Slater [27]. They noted that every paired-dominating set is also a total dominating set and that a paired-dominating set would be at most twice the size of any dominating set. This resulted in the following bounds on γ_p:

If G is a graph with no isolated vertices then

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_p(G) \leq 2\gamma(G).$$

It was also shown in [27] that

$$\gamma_p(G) \leq 2\gamma_t(G) - 2.$$

This is due to the fact that at least two vertices in a total dominating set can be paired with each other. Hence, at most $\gamma_t(G) - 2$ vertices need to be added to a total dominating set to ensure a paired-dominating set is contained within it.

As previously noted. a paired-dominating set can always be found by taking the end vertices of the edges in a dom-matching. Every maximal matching is a dom-matching since every edge of G has at least one end vertex that is met by the matching. Let $\mu(G)$ denote the cardinality of the smallest maximal matching in G. Then, as shown in [27],

$$\gamma_p(G) \leq 2\mu(G).$$

Therefore. any graph with no isolated vertices contains a paired-dominating set.

This approach to the problem leads to a slightly different application. If we find a dominating matching. M. then the edges of M could be patrolled by $|M|$ guards. Although the set of guards do not form a dominating set. we are trading constant surveillance for a saving of half the number of guards. perhaps not an unrealistic approach.

Cockayne. Hedetniemi & Miller [15] were the first to introduce the concept of irredundance in graphs and provided the following preliminary result:

Theorem 1.2.0.1 ([15]) *A set of vertices is irredundant and dominating if and only if it is a minimal dominating set.*

This results in the following well known corollary:

Corollary 1.2.0.2 ([15]) *For any graph G. $ir(G) \leq \gamma(G) \leq \Gamma(G) \leq IR(G)$.*

Another inequality relating γ and ir was independently obtained by Allan & Laskar [2] and Bollobás & Cockayne [9]:

$$\gamma(G) \leq 2ir(G) - 1.$$

The next major contribution to the theory of irredundance came from Cockayne. Favaron. Payan & Thomason [14] who provided the first results obtained for the parameter $IR(G)$. They proved that:

> *If G has no isolated vertices and X is an irredundant set in G then $V(G) \setminus X$ is a dominating set.*

This provided the following inequalities as corollaries. where p is the number of vertices in G:

$$\gamma(G) + IR(G) \leq p$$

$$ir(G) + \Gamma(G) \leq p$$

and

$$ir(G) + IR(G) \leq p.$$

In [15]. Cockayne. Hedetniemi & Miller obtained a result which related $\gamma_t(G)$ and $ir(G)$ where G is a graph with no isolated vertices. It stated that

$$\gamma_t(G) \leq 2ir(G).$$

Using a technique similar to the one used in proving this inequality. we prove the following theorem:

Theorem 1.2.0.3 *For any graph G with no isolated vertices. $\gamma_p(G) \leq 2ir(G)$.*

Proof: Let $S = \{v_1. v_2. \ldots . v_m\}$ be a maximal irredundant set of size $ir(G)$ in G. Since S is irredundant, $N[v_i] \nsubseteq N[S \setminus \{v_i\}]$ for all $i = 1.2. \ldots . m$. For each i such that $N(v_i) \nsubseteq N[S \setminus \{v_i\}]$. let u_i be any vertex which is in $N(v_i) \setminus N[S \setminus \{v_i\}]$. Without loss of generality assume that this is the case for all $i = 1.2. \ldots . k$ for some $k \leq m$. Let $M' = \{u_i v_i : i = 1.2. \ldots . k\}$.

Now. for all $i = k + 1. k + 2. \ldots . m$ we have $v_i = N[v_i] \setminus N[S \setminus \{v_i\}]$. Let $H = G \setminus \{u_i. v_i : i = 1,2. \ldots . k\}$. Let M'' be the largest matching in H such that every edge in M'' has at least one end vertex in $T = \{v_{k+1}. v_{k+2}. \ldots . v_m\}$. Hence. every

vertex in T which is not met by M'' has all its neighbours met by either M' or M''. We claim that $M = M' \cup M''$ is a dom-matching in G.

If M is not a dom-matching then there is some vertex x which is not in $V(M)$ and x is not adjacent to any vertex in $V(M)$. Then $N(x) \cap V(M) = \emptyset$. Therefore, $N(x) \not\subseteq N[S]$, since any vertex in S not met by M is in T and has all its neighbours met by the matching.

Consider $S_1 = S \cup \{x\}$. Suppose $N[v_i] \subseteq N[S_1 \setminus \{v_i\}]$ for some $i = 1, 2, \ldots, k$, then $N[v_i] \setminus N[S \setminus \{v_i\}] \subseteq N[x]$ which implies that $u_i \in N[x]$. This contradicts the fact that x is not adjacent or equal to any vertex in $V(M)$. If $N[v_i] \setminus N[S \setminus \{v_i\}] \subseteq N[w]$ for some $i = k+1, k+2, \ldots, m$, then $v_i \in N[x]$. Since v_i is either met by the matching M or has all its neighbours met by M, this contradicts the fact x is not in $V(M)$ or adjacent to any vertex in $V(M)$.

Since $N[x] \not\subseteq N[S] = N[S_1 \setminus \{x\}]$ and $N[v_i] \not\subseteq N[S_1 \setminus \{v_i\}]$ for all $i = 1, 2, \ldots, m$, then S_1 is an irredundant set. This, however, contradicts the fact that S was a maximal irredundant set. Hence, it must be the case that $V(M)$ is a dominating set, and M is a dominating matching. Hence, $V(M)$ is a paired-dominating set such that $|V(M)| \leq 2|S|$. Since $|V(M)| \geq \gamma_p(G)$ and $|S| = ir(G)$, we have $\gamma_p(G) \leq 2ir(G)$. $\square$

We now turn our attention to the concept of paired-irredundance. Due to our definition of an ir-matching, the following result, similar to that of Theorem 1.2.0.1, is evident:

Theorem 1.2.0.4 *A matching is a dom-matching and an ir-matching if and only if it is a minimal dom-matching.*

This gives us a string of inequalities analogous to that in Corollary 1.2.0.2:

Corollary 1.2.0.5 *If G is a graph with no isolated vertices then*

$$ir_p(G) \leq \gamma_p(G) \leq \Gamma_p(G) \leq IR_p(G).$$

There are examples in which one or more of the inequalities are strict. The graph in Figure 1.5 has $ir_p = 4$ and $\gamma_p = 6$ while the graph in Figure 1.6 has $IR_p = 4$ and $\Gamma_p = 2$.
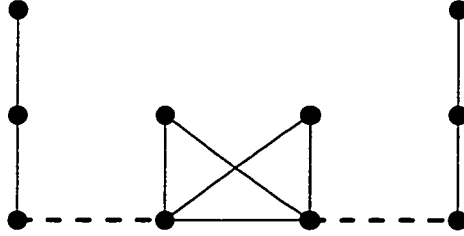
Figure 1.5: An maximal ir-matching of size 2 in a graph where $\gamma_p = 6$.
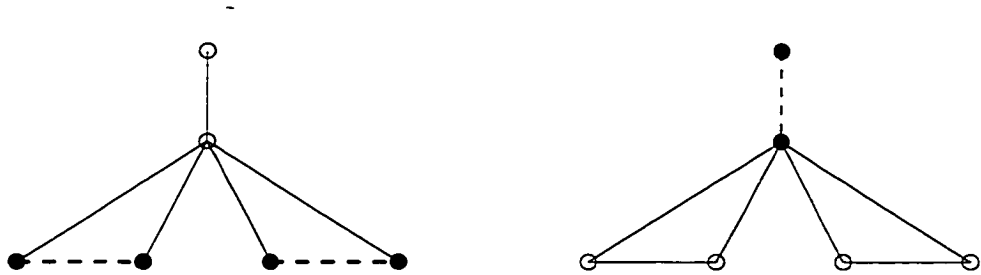
Figure 1.6: A maximal ir-matching of size 2 in a graph where $\Gamma_p = 2$.

Theorem 1.2.0.6 *If G is a graph with no isolated vertices then $IR_p(G) \le 2IR(G)$.*

Proof: Let M be an ir-matching in G such that $|V(M)| = IR_p(G)$. At least one end vertex of any edge, e, in M has a neighbour which is not a neighbour of any vertex met by $M \setminus \{e\}$. Therefore, we can construct an irredundant set by choosing the appropriate end vertex of each edge in M. If X is that set of vertices then $IR_p(G)/2 = |X| \le IR(G)$. Hence, $IR_p(G) \le 2IR(G)$. $\square$

1.3 Products

We use $\odot$ as the symbol for an arbitrary product where the product graph is defined by $V(G \odot H) = \{(x.y) : x \in V(G).y \in V(H)\}$ and whether two vertices in this

product are adjacent depends entirely on the adjacency relations in the factors. This can be represented by a 3×3 matrix. called the **edge matrix**. The rows (columns) are labeled by E which denotes adjacency of the vertices of the first (second) factor: N which denotes nonadjacency: and $\triangle$ which denotes the case where the vertices are the same. An E in the matrix indicates there is an edge between the vertices of the product: an N nonadjacency: and in the case where the relationship in both factors is $\triangle$ then the two vertices are the same and so the entry is $\triangle$.

$$
\begin{array}{c}
\begin{array}{ccc} E & \triangle & N \end{array} \\
\begin{array}{c} E \\ \triangle \\ N \end{array}
\left(
\begin{array}{ccc}
- & - & - \\
- & \triangle & - \\
- & - & -
\end{array}
\right)
\end{array}
$$

Since the rows and columns will always be labelled in this fashion we drop the labels from this point forward.

Since a graph can be defined in terms of non-edges. there is the notion of a complementary product. Specifically. if $\overline{G}$ is the complement of G the the **complementary product** $\odot^c$ to a product $\odot$ is given by $G \odot^c H = \overline{(\overline{G} \odot \overline{H})}$.

This "edge matrix" notation was introduced by Imrich & Izbicki [29]. They showed that out of 256 possible products there are 10 associative products which depend on the edge structure of both factors. Definitions of nine of these follow. and examples of the product of the 3-path. P_3. with itself are found in Figures 1.7 and 1.8. The tenth is the product whose edge matrix is the transpose of that of the lexicographic product. We do not consider this product. Note that of the nine products listed. all but the lexicographic product are commutative.

$$
\text{Categorical:} G \times H = \begin{pmatrix} E & N & N \\ N & \triangle & N \\ N & N & N \end{pmatrix}
\qquad
\text{Co-Categorical:} G \times^c H = \begin{pmatrix} E & E & E \\ E & \triangle & E \\ E & E & N \end{pmatrix}
$$

$$
\text{Cartesian:} G \square H = \begin{pmatrix} N & E & N \\ E & \triangle & N \\ N & N & N \end{pmatrix}
\qquad
\text{Co-Cartesian:} G \square^c H = \begin{pmatrix} E & E & E \\ E & \triangle & N \\ E & N & E \end{pmatrix}
$$

$$
\text{Strong:} G \boxtimes H = \begin{pmatrix} E & E & N \\ E & \triangle & N \\ N & N & N \end{pmatrix}
\qquad
\text{Disjunction:} G \boxtimes^c H = \begin{pmatrix} E & E & E \\ E & \triangle & N \\ E & N & N \end{pmatrix}
$$

$$
\text{Equivalence:} G \circledcirc H = \begin{pmatrix} E & E & N \\ E & \triangle & N \\ N & N & E \end{pmatrix}
$$

$$
\text{Symmetric Difference:} G \nabla H = \begin{pmatrix} N & E & E \\ E & \triangle & N \\ E & N & N \end{pmatrix}
$$

$$
\text{Lexicographic:} G \bullet H = \begin{pmatrix} E & E & E \\ E & \triangle & N \\ N & N & N \end{pmatrix}.
$$

Categorical   Cartesian

Strong   Lexicographic

Equivalence   Symmetric Difference
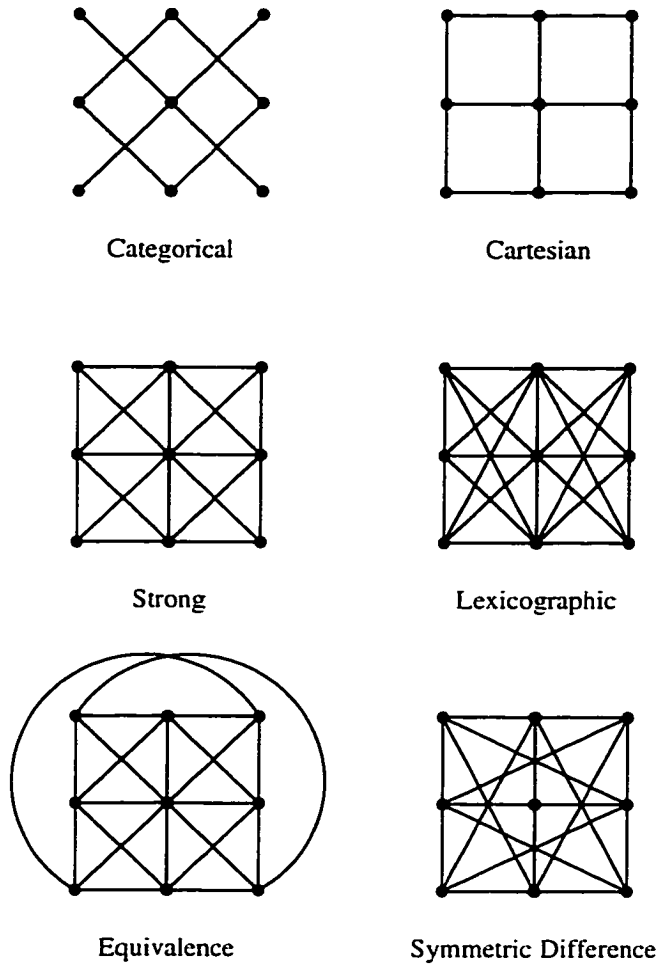
Figure 1.7: The edges of the product of P_3 with itself.

Disjunction   Co-Cartesian   Co-Categorical
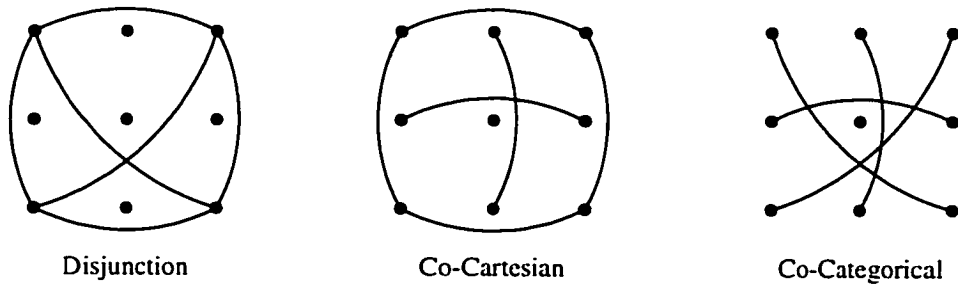
Figure 1.8: The nonedges of the product P_3 with itself.

These products can be ordered by "inclusion": that is. $\doteq \leq \supset$ if for each pair of graphs G and H. $E(G \doteq H) \subseteq E(G \odot H)$. For example, $\times \leq \boxtimes \leq \circledcirc \leq \square^c$. The suborder of all nine products of interest is shown in Figure 1.9.
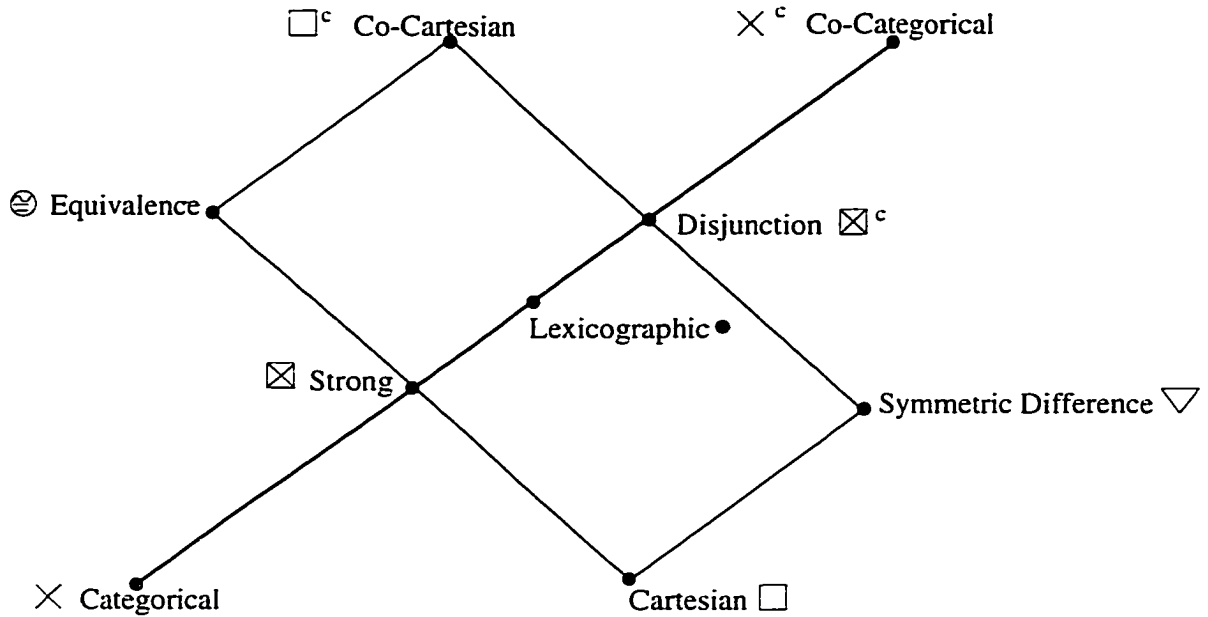
Figure 1.9: A partial ordering of products under inclusion.

1.3.1 Results

We call a graph product. $\supset$, **dominating (total-dominating, paired-dominating, irredundant, paired-irredundant) multiplicative** if for any two graphs G and H and any two dominating (total-dominating, paired-dominating, irredundant, paired-irredundant) sets $A \subset V(G)$ and $B \subset V(H)$, the set $A \times B$ is a dominating (total-dominating, paired-dominating, irredundant, paired-irredundant) set in $G \supset H$. This was investigated by Nowakowski & Rall [38] who gave the following result:

Lemma 1.3.1.1 (Nowakowski & Rall [38]) *Let* $\supset$ *be a graph product.*

(a) If $\supset \geq \boxtimes$ *then* $\supset$ *is dominating multiplicative.*

(b) If $\odot \geq \times$ then $\odot$ is total-dominating multiplicative.

(c) If $\odot \leq \boxtimes$ then $\odot$ is irredundant multiplicative.

We show a similar result for paired-dominating sets.

Lemma 1.3.1.2 *Let $\odot$ be a graph product. If $\odot \geq \times$ then $\odot$ is paired-dominating multiplicative.*

Proof: Suppose $\odot \geq \times$. Let $A = \{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$ and $B = \{b_1, b_2, \ldots, b_{2m-1}, b_{2m}\}$ be paired-dominating sets in graphs G and H, respectively, such that $M_A = \{a_{2i-1}a_{2i} : i = 1, 2, \ldots, n\}$ is a dom-matching in G and $M_B = \{b_{2j-1}b_{2j} : j = 1, 2, \ldots, m\}$ is a dom-matching in H.

Since A and B are paired-dominating sets, then they are also total-dominating sets. Hence, by Lemma 1.3.1.1, $A \times B$ is a total-dominating set in $G \odot H$. Furthermore, since $\times \leq \odot$, then $(a_{2i-1}, b_{2j-1}) \sim (a_{2i}, b_{2j})$ and $(a_{2i-1}, b_{2j}) \sim (a_{2i}, b_{2j-1})$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Therefore, $M_{A \times B} = \{(a_{2i-1}, b_{2j-1})(a_{2i}, b_{2j}), (a_{2i-1}, b_{2j})(a_{2i}, b_{2j-1}) : i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ is a matching in $G \odot H$. Since $V(M_{A \times B}) = A \times B$ then M is a dom-matching in $G \odot H$ and $A \times B$ is a paired-dominating set. Hence, $\odot$ is paired-dominating multiplicative. $\square$

Corollary 1.3.1.3 *Let G and H be connected graphs. If $\times \leq \odot$ then*

$$\gamma_p(G \odot H) \leq \gamma_p(G)\gamma_p(H).$$

Lemma 1.3.1.4 *Let $\odot$ be a graph product such that $\odot \geq \boxtimes$. If A is a paired-dominating set in G and B is a dominating set in H, then $A \times B$ is a paired-dominating set in $G \odot H$.*

Proof: Suppose $\odot \geq \boxtimes$. Let $A = \{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$ such that $M_A = \{a_{2i-1}a_{2i} : i = 1, 2, \ldots, n\}$ is a dom-matching in G, and let $B = \{b_1, b_2, \ldots, b_m\}$ be a dominating set in H. Since A and B are dominating sets in G and H, respectively, then by Lemma 1.3.1.1, $A \times B$ is a dominating set in $G \odot H$. Furthermore, $M = \{(a_{2i-1}, b_j)(a_{2i}b_j) :$

$i = 1, 2, \ldots, n$. $j = 1, 2, \ldots, m\}$ is a perfect matching in the subgraph induced on $A \times B$. Hence. $A \times B$ is a paired-dominating set in $G \odot H$. $\square$

Figure 1.10 shows a paired-dominating set in P_4. a dominating set in P_3 and the paired-dominating set in $P_4 \boxtimes P_3$ which results from the construction given in Lemma 1.3.1.4.
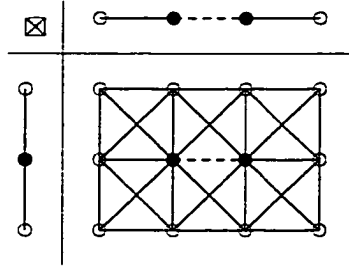
Figure 1.10: A paired-dominating set in $P_4 \boxtimes P_3$.

Lemma 1.3.1.5 *Suppose A is a paired-irredundant set in G and B is an irredundant set in H. Then $A \times B$ is a paired-irredundant set in $G \odot H$ if $\odot \in \{\square, \boxtimes\}$.*

Proof: Let $A = \{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$ such that $M_A = \{a_{2i-1}a_{2i} : i = 1, 2, \ldots, n\}$ is an ir-matching in G. and let $B = \{b_1, b_2, \ldots, b_m\}$ be an irredundant set in H. The matching $M = \{(a_{2i-1}, b_j)(a_{2i}b_j) : i = 1, 2, \ldots, n \ j = 1, 2, \ldots, m\}$ is a perfect matching in the subgraph induced on $A \times B$ in $G \odot H$. We claim that $V(M)$ is a paired-irredundant set under this matching. Since A and B are paired-irredundant and irredundant. respectively. there is a vertex. v. in $N[\{a_1, a_2\}]$ such that v is not in $N[A \setminus \{a_1, a_2\}]$. and there is a vertex. w. in $N[b_1]$ such that w is not in $N[B \setminus \{b_1\}]$

Case1: Let $\odot = \square$. Suppose $w \in N(b_1)$. Then the vertex (a_1, w) is not in the set $A \times B$. Suppose $(a_1, w) \sim (a, b)$ for some $(a, b) \in A \times B$. Since $w \neq b_j$ for any $j = 1, \ldots, m$ and $w \sim b_j$ only if $j = 1$. then $a = a_1$ and $b = b_1$. Hence. the only vertex in $A \times B$ adjacent to (a_1, w) is (a_1, b_1). Therefore. $(a_1, w) \in N[\{(a_1, b_1), (a_2, b_1)\}]$. but $(a_1, w) \notin N[A \times B \setminus \{(a_1, b_1), (a_2, b_1)\}]$.

Suppose the only vertex in $N[b_1]$ which is not in $N[B \setminus \{b_1\}]$ is b_1 itself. Suppose $(v.b_1) \simeq (a.b)$ for some $(a.b)$ in $A \times B$. Then $b = b_1$ and $v \simeq a$. Therefore. $a = a_1$ or $a = a_2$. So. $(v.b_1) \in N[\{(a_1.b_1).(a_2.b_1)\}]$, but $(v.b_1) \notin N[A \times B \setminus \{(a_1.b_1).(a_2.b_1)\}]$.

Case 2: Let $\odot = \boxtimes$. Suppose. $(v.w) \sim (a.b)$ for some $(a.b) \in A \times B$. Then either $v \sim a$ and $w = b$ or $v = a$ and $w \sim b$. Due to the choice of v and w. it must be the case that $b = b_1$ and $a = a_1$ or a_2. Therefore, $(v.w) \in N[(a_1.b_1).(a_2.b_1)]$. but $(v.w) \notin N[A \times B \setminus \{(a_1.b_1).(a_2.b_1)\}]$.

In either case, $N[(a_1.b_1).(a_2.b_1)] \nsubseteq N[A \times B \setminus \{(a_1.b_1).(a_2.b_1)\}]$. It can be similarly shown that $N[(a_{2i-1}.b_j).(a_{2i}.b_j)] \nsubseteq N[A \times B \setminus \{(a_{2i-1}.b_j).(a_{2i}.b_j)\}]$ for all $i = 1.2.....n$ and $j = 1.2.....m$. Hence. $A \times B$ is a paired-irredundant set in $G \odot H$ where $\odot \in \{\square. \boxtimes\}$. $\qquad\square$

Figure 1.11 shows a pair-irredundant set in P_4. an irredundant set in P_3 and the paired-irredundant sets in $P_4 \boxtimes P_3$ and $P_4 \square P_3$ which result from the construction given in Lemma 1.3.1.5.
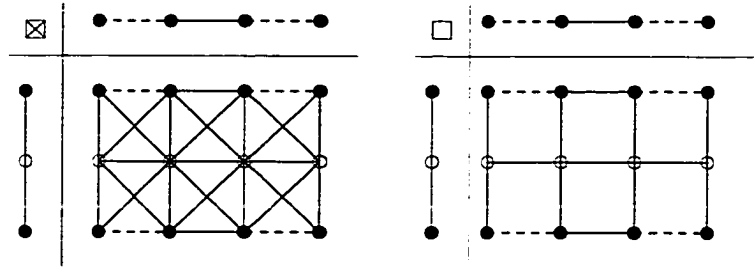
Figure 1.11: A paired-irredundant set in both $P_4 \boxtimes P_3$ and $P_4 \square P_3$.

Now. Lemma 1.3.1.4 and Lemma 1.3.1.5 together give the following corollary:

Corollary 1.3.1.6 *For any two connected graphs, G and H.*

(a) $IR_p(G \odot H) \geq IR_p(G)IR(H)$ if $\odot \in \{\square. \boxtimes\}$:

(b) $\gamma_p(G \boxtimes H) \leq \gamma_p(G)\gamma(H)$:

(c) $\Gamma_p(G \boxtimes H) \geq \Gamma_p(G)\Gamma(H)$.

Proof: (a) Suppose A is a maximum paired-irredundant set in G and B is maximum irredundant set in H. Then. by Lemma 1.3.1.5. $A \times B$ is a paired-irredundant set in $G \odot H$ if $\odot \in \{\Box, \boxtimes\}$. Therefore. $IR_p(G \odot H) \geq |A \times B| = IR_p(G)IR(H)$.

(b) (c) Suppose A is a minimal paired-dominating set in G with associated dom-matching M and B is a minimal dominating set in H. Since M is a minimal dom-matching then it is an ir-matching, by Theorem 1.2.0.4. and A is a paired-irredundant set. Furthermore, since every minimal dominating set is an irredundant set. by Theorem 1.2.0.1. then B is an irredundant set. So, $A \times B$ is a paired-irredundant set in $G \boxtimes H$, by Lemma 1.3.1.5. Furthermore. by Lemma 1.3.1.4. $A \times B$ is also a paired-dominating set in $G \boxtimes H$. Since $A \times B$ is both paired-dominating and paired-irredundant. then by Theorem 1.2.0.4 $A \times B$ is a minimal paired-dominating set. Hence. $\gamma_p(G \boxtimes H) \leq |A \times B| \leq \Gamma_p(G \boxtimes H)$. If we choose A and B such that $|A| = \gamma_p(G)$ and $B = \gamma(H)$ then $\gamma_p(G \boxtimes H) \leq \gamma_p(G)\gamma(H)$. If $|A| = \Gamma_p(G)$ and $|B| = \Gamma(H)$ then $\Gamma_p(G \boxtimes H) \geq \Gamma_p(G)\Gamma(H)$. $\quad\Box$

We now consider the lexicographic and disjunction products.

Lemma 1.3.1.7 *If* $\bullet \leq \odot$ *then* $\gamma_p(G \odot H) \leq \gamma_p(G)$. *If* $\boxtimes^c \leq \odot$ *then* $\gamma_p(G \odot H) \leq$ $\min\{\gamma_p(G), \gamma_p(H)\}$.

Proof: Suppose that $\bullet \leq \odot$. Let $M = \{a_{2i-1}a_{2i} : i = 1, 2, \ldots, n\}$ be a dom-matching in G and let v be any vertex in H. Then $\{(a_{2i-1}, v)(a_{2i}, v) : i = 1, 2, \ldots, n\}$ is a dom-matching in $G \odot H$. Therefore. $\gamma_p(G \odot H) \leq \gamma_p(G)$. The second part of the lemma follows since the lexicographic product and the product whose edge matrix is the transpose of the lexicographic matrix are both less than $\boxtimes^c$. $\quad\Box$

We now turn our attention to the Cartesian product. For two graphs G and H.

$$\gamma_p(G\Box H) \leq \gamma_p(G)|V(H)|.$$

This is because the subgraph induced on the vertices $\{(x, v)P : x \in V(G)\}$ for any $v \in V(H)$ is isomorphic to G. Therefore. $G\Box H$ can be partitioned into $|V(H)|$

subgraphs, each of which is isomorphic to G. By taking a paired-dominating set of size $\gamma_p(G)$ in each of these $|V(H)|$ subgraphs, we obtain a paired-dominating set in $G \square H$ of size $\gamma_p(G)|V(H)|$.

This can be improved, however, as shown in the next lemma.

Lemma 1.3.1.8 *Let M be a matching in G. Then*

$$\gamma_p(G \square H) \le 2|M|\gamma(H) + (|V(G)| - 2|M|)\gamma_p(H).$$

Proof: Let $M = \{a_{2i-1}a_{2i} : i = 1, 2, \ldots, m\}$ be a matching in G, and let $B = \{b_1, b_2, \ldots, b_k\}$ be a dominating set in H. The vertices met by the matching $M' = \{(a_{2i-1}, b_j)(a_{2i}, b_j) : i = 1, 2, \ldots, m \ \ j = 1, 2, \ldots, k\}$ in $G \square H$ dominate all the vertices in $V(M) \times H$. If S is the set of vertices not met by M, then the vertices in $S \times H$ can be dominated by the set $S \times C$ where C is a paired-dominating set in H. Furthermore, the vertices (s, c_1) and (s, c_2) in $S \times C$ are paired if c_1 and c_2 are paired in C. Hence, $(V(M') \times B) \cup (S \times C)$ is a paired-dominating set in $G \square H$. It follows that $\gamma_p(G \square H) \le 2|M|\gamma(H) + (|V(G)| - 2|M|)\gamma_p(H)$. $\square$

To demonstrate that this is an improvement, we consider the 5-path, P_5. The largest matching on P_5 consists of two edges, and therefore, meets four vertices. This matching is shown in Figure 1.12 along with a minimum dominating set of size two in P_5. The matching is also a minimum dom-matching in P_5. Therefore, we can find a paired-dominating set of size 12 in $P_5 \square P_5$ from the construction given in Lemma 1.3.1.8. This is also shown in Figure 1.12.
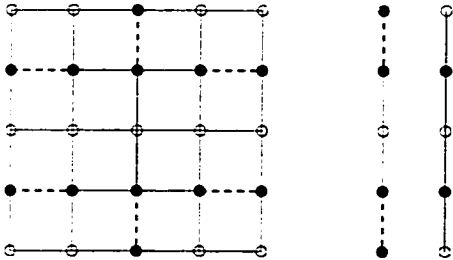
Figure 1.12: An paired-dominating set in $P_5 \square P_5$.

Since $\gamma_p(P_5)|V(P_5)| = 20$. the advantage of the construction given in Lemma 1.3.1.8 is clear. However. this does not give the minimum paired-dominating set for $P_5 \boxtimes P_5$ since there is a paired-dominating set of size ten, as shown in Figure 1.13. In fact. $\gamma_p(P_5 \square P_5) = 10$. This is because in any dom-matching of $P_5 \boxtimes P_5$. four distinct edges are required to dominate the corners. and none of these can also dominate the center vertex. Hence. at least five edges are required.
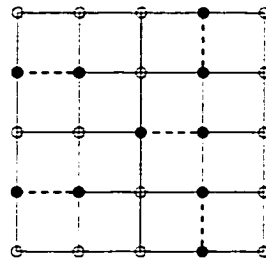
Figure 1.13: A minimum paired-dominating set in $P_5 \square P_5$.

There are also cases. however. in which Lemma 1.3.1.8 gives us a minimum paired dominating set. Consider $P_4 \square P_3$. A maximum matching in P_4 consists of two edges and $\gamma(P_3) = 1$. The resulting pair-dominating set in $P_4 \square P_3$ of size four is shown in Figure 1.14. In fact $\gamma_p(P_4 \square P_3) = 4$ since $P_4 \square P_3$ has no dom-matching of size one. Hence. Lemma 1.3.1.8 gives us $\gamma_p(P_4 \square P_3)$ exactly.
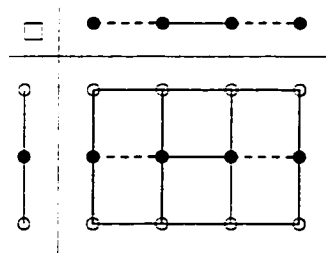
Figure 1.14: A minimum paired-dominating set in $P_4 \square P_3$.

1.3.2 Problems

We now pose some problems that, while not further addressed in this thesis, provide some potential topics for future investigations.

In Corollary 1.3.1.6 it was shown that for any two connected graphs G and H, $\gamma_p(G \boxtimes H) \leq \gamma_p(G)\gamma(H)$ and $\Gamma_p(G \boxtimes H) \geq \Gamma_p(G)\Gamma(H)$.

Problem 1.3.2.1 *Is there a pair of connected graphs G and H such that*

(a) $\gamma_p(G \boxtimes H) < \min\{\gamma_p(G)\gamma(H), \gamma_p(H)\gamma(G)\}$?

(b) $\Gamma_p(G \boxtimes H) > \max\{\Gamma_p(G)\Gamma(H), \Gamma_p(H)\Gamma(G)\}$?

We saw that for $P_5 \boxtimes P_5$, Theorem 1.3.1.8 did not give the actual paired-domination number.

Problem 1.3.2.2 *Can we find an improved upper bound for $\gamma_p(G \Box H)$ for any two connected graphs G and H?*

The graph $P_5 \Box P_5$ can not be dominated by four matching edges. Therefore, $\gamma_p(P_5 \Box P_5) > \gamma_p(P_5)\gamma(P_5)$.

Problem 1.3.2.3 *Is $\gamma_p(G \Box H) \geq \gamma_p(G)\gamma(H)$ for all connected graphs G and H?*

Note that this is similar to Vizing's Conjecture [49] which states that:

For all graphs G and H, $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.

1.4   Well Paired-Dominated

A graph is said to be **well dominated** if all of its minimal dominating sets are of the same cardinality: that is, every minimal dominating set is minimum. We will extend this terminology to paired-dominating sets and define a graph to be **well paired-dominated** if all its minimal paired dominating sets are γ_p-sets. For brevity, we will say that a well paired-dominated graph has **property W**. Since only graphs

containing no isolated vertices have a paired-dominating set. then no graph with isolated vertices has property W. Also note that for an unconnected graph. the graph has property W if and only if each of its components have property W.

We denote a **leaf** to be a vertex of degree one and a **stem** to be a vertex adjacent to any leaf. In the case of a single edge, we set one vertex to be the leaf and the other the stem. Hence. according to this terminology. a single edge does not have two adjacent stems.

In this section. we present a technique for reducing any graph with property W and girth at least six to obtain a smaller graph with property W. This gives a characterization of the graphs with property W and girth at least six that can be reduced to the empty graph via a particular reduction, as well as a complete characterization of graphs with property W and girth at least eight.

1.4.1 Preliminary Results

Lemma 1.4.1.1 *Suppose G is a graph with property W. If uv is an edge in G such that the graph $G' = G \setminus N[\{u, v\}]$ has no isolated vertices. then G' also has property W.*

Proof: Suppose G' does not have property W. Then G' has two minimal dom-matchings. say M_1 and M_2. such that $|M_1| < |M_2|$. Obviously. $M_1 \cup \{uv\}$ and $M_2 \cup \{uv\}$ are two dom-matchings in G of different sizes. Since M_i is also an ir-matching. by Theorem 1.2.0.4. and $N[u, v] \in N[x, y]$ for any edge $xy \in M_i$. then $M_i \cup \{uv\}$ is an ir-matching in G for $i = 1, 2$. Therefore. by Theorem 1.2.0.4 we have two minimal dom-matchings in G of different sizes. This contradicts the fact that G has property W. Hence. G' must also have property W. $\square$

Corollary 1.4.1.2 *Suppose G is a graph with girth at least six and no leaves. If G has property W. then for any edge uv in G. $G' = G \setminus N[\{u, v\}]$ has property W.*

Proof: Since G has no leaves and girth at least six. then every vertex has some

neighbour besides u or v. Hence. $G' = G \setminus N[\{u, v\}]$ has no isolated vertices and. by Lemma 1.4.1.1. also has property W. □

Call this process of removing an edge and its neighbours an **edge-reduction** on G.

Lemma 1.4.1.3 *Suppose G is a graph with property W. If u is a stem in G. then $G' = G \setminus N[u]$ has property W.*

Proof: Let uv be an edge in G such that u is a stem and v is a leaf. Since $N[u] = N[\{u, v\}]$ then $G' = G \setminus N[\{u, v\}]$. Hence, by Lemma 1.4.1.1. G' has property W if G' has no isolated vertices. Suppose this is not the case.

Let I be the set of all isolated vertices in G' and let H be the subgraph induced on the vertices $N[u] \cup I$. First. we take a minimal dom-matching in H which consists of the edge uv and edges from $N[u]$ to I. This matching must contain at least one edge. say ab. where $b \in I$ since no vertices in I are adjacent to either u or v. Call this matching M_1. Next. let M_2 be the matching of smallest size in $G' \setminus I$ such that $M_1 \cup M_2$ is a dom-matching. (We know that such an M_2 exists since any dom-matching in $G' \setminus I$ together with the edges in M_1 forms a dom-matching in G.) Let $M = M_1 \cup M_2$. This is a minimal dom-matching since no vertex in $I \cup \{v\}$ has a neighbour met by the matching M_2.

Now. let $M' = (M \setminus \{uv. ab\}) \cup \{ua\}$. Then $V(M') = V(M) \setminus \{v. b\}$. Since $V(M)$ is a dominating set then we need only verify that $V(M')$ dominates $N[v]$ and $N[b]$ to show that M' is a dom-matching. This is obviously the case since u is the only neighbour of v. all the vertices of $N(b)$ are in $N(u)$ and b is adjacent to a. However. $|M'| < |M|$. contradicting the assumption that G has property W. Hence. G' has no isolated vertices and. therefore. by Lemma 1.4.1.1. has property W. □

Corollary 1.4.1.4 *If G is a graph with property W. then G can not contain adjacent stems.*

Proof: If G contains two adjacent stems, say x and y, then the leaves adjacent to y would be isolated vertices in the graph $G \setminus N[x]$. Hence, by Lemma 1.4.1.3, if G has property W then G does not contain two adjacent stems. □

Call the procedure of removing a stem and its neighbours a **stem-reduction** on G. Let $G = G_0$. Suppose we perform a sequence of stem-reductions to obtain the graphs $\{G_0, G_1, \ldots, G_k\}$ where G_{i+1} is obtained by performing a stem-reduction on G_i and the graph G_k has no stems. Hence, G_k is either a graph with no vertices of degree one or the empty graph. If a series of stem-reductions on G always results in the empty graph, despite the choice of stems, then we will say that G is **stem-reducible**. Otherwise, say G is **stem-irreducible**.

1.4.2 Graphs with Property W and Girth at Least Eight

We now wish to characterize all graphs of girth at least eight which have property W. We do so by partitioning the graphs with property W into two sets: those which are stem-reducible and those which are stem-irreducible. The following theorem applies to all stem-reducible graphs of girth at least *six*:

Theorem 1.4.2.1 *Let G be a graph with girth at least six and let S be the set of stems in G. Then G is a stem-reducible graph with property W if and only if S is an independent dominating set in G.*

Proof: Let G be a stem-reducible graph with girth at least six and property W. Obviously, by Corollary 1.4.1.4, S must be an independent set. Suppose, however, that S is not a dominating set in G. Then the graph $G_0 = G \setminus N[S]$ is a non-empty, stem-reducible graph. Since G_0 is stem-reducible, then for some $k \geq 1$ there exists a collection of vertices $U = \{u_0, u_1, \ldots, u_{k-1}\}$ such that for each $i \in \{0, 1, \ldots, k-1\}$, u_i is a stem in G_i, $G_{i+1} = G_i \setminus N[u_i]$ and $G_k = \emptyset$. Let $S' = S \cup U$. Note that S' is a dominating set in G.

Let M_0 be a matching in G such that the set S is saturated by M_0 and every edge in M_0 is from a stem to a leaf. For $i = 0, \ldots, k-1$, let $M_{i+1} = M_i \cup \{u_i v_i\}$ where v_i

is some leaf adjacent to u_i in G_i. Let $M = M_k$. Then M is a dom-matching since $S' \subseteq V(M)$ and S' is a dominating set in G. The dom-matching M is also minimal. as every leaf in G is only externally dominated by its adjacent stem. and the vertex v_i is only externally dominated by u_i for all $i = 0, \ldots, k - 1$.

Since $G_k = \emptyset$ then every vertex in G_{k-1} is contained in $N[u_{k-1}]$ where u_{k-1} is a stem in G_{k-1}. Hence, $G_{k-1} = K_{1,n}$ for some $n \geq 1$.

Case 1: Suppose $n = 1$. Then G_{k-1} consists of the single edge $u_{k-1}v_{k-1}$. Since G_{k-1} contains none of the original leaves in G, then neither u_{k-1} nor v_{k-1} is a leaf in G. Hence. u_{k-1} and v_{k-1} have neighbours in G. say x and y, respectively. Due to girth restrictions we know that x and y are distinct. Since x and y were removed in a previous step. but all of their neighbours were not removed, then $x \in N(u)$ and $y \in N(u')$ for some $u. u' \in S'$. Let uv and $u'v'$ be the edges in M incident with u and u'. respectively. Note that the vertices u, u', v and v' are all distinct since the contrary would result in a cycle of length five or a pair of incident edges in the matching.

Now let $M' = M \setminus \{u_{k-1}v_{k-1}, uv, u'v'\} \cup \{xu. yu'\}$. Then $V(M') = V(M) \setminus \{u_{k-1}, v_{k-1}, v, v'\} \cup \{x, y\}$. Since $S' \setminus \{u_{k-1}\}$ dominates all vertices other than u_{k-1} and v_{k-1}. then $S' \cup \{x. y\}$ dominates all vertices in G. Therefore. M' is a dom-matching since $S' \cup \{x. y\} \subseteq V(M')$. Hence. G does not have property W since $|M'| < |M|$ and M is a minimal dom-matching.

Case 2: Suppose $G_{k-1} = K_{1,n}$ where $n \geq 2$. Let $\{w_1, w_2, \ldots, w_n\}$ be the leaves adjacent to the stem u_{k-1} where $w_1 = v_{k-1}$. Since $w_1, \ldots, w_n$ were not leaves in G. they have neighbours $x_1, \ldots, x_n$. respectively. in G. The vertices $\{x_1, \ldots, x_n\}$ must be distinct. due to girth restrictions.

Let $M' = M \setminus \{u_{k-1}w_1\} \cup \{w_1x_1, w_2x_2, \ldots, w_nx_n\}$. Then $V(M') = V(M) \setminus \{u_{k-1}\} \cup \{w_2, \ldots, w_n, x_1, \ldots, x_n\}$. Since $S' \setminus \{u_{k-1}, v_{k-1}\}$ dominates all vertices of G except $\{u_{k-1}, w_1, w_2, \ldots, w_n\}$. then $S' \setminus \{u_{k-1}, v_{k-1}\} \cup \{w_1, w_2, \ldots, w_n, x_1, \ldots, x_n\}$ is a dominating set in G. Hence. M' is a dom-matching. Since $|M'| > |M|$ and G has property W. then M' must not be minimal.

If M' is not minimal then for some edge $uv \in M'$ the matching $M' \setminus \{uv\}$ is also a dom-matching in G. If uv is an edge in $M \setminus \{u_{k-1}w_1\}$ then it either has one end vertex which is a leaf in the original graph or $uv = u_i v_i$ for some $i = 0, \ldots, k - 2$. Obviously, if the former case holds uv can not be removed. In the latter case, we know that v_i is a leaf in G_i which means it is not dominated be any vertex met by the matching $M \setminus \{u_i, v_i\}$. Hence, no edge in $M \setminus \{u_{k-1}w_1\}$ can be removed.

Since no edge in $M \setminus \{u_{k-1}w_1\}$ can be removed, then the edge in question must be $w_j x_j$ for some $j \in \{1, \ldots, n\}$. Consider the edge $w_1 x_1$. Since w_1 is not adjacent to any u_i or v_i for any $i = 1, \ldots, k - 2$, nor is it adjacent to w_j or x_j for any $j = 2, \ldots, n$. then its only neighbour in $V(M')$ is x_1. Hence, the edge $w_1 x_1$ can not be removed from the set M'. Similarly, no edge $w_j x_j$ for $j = 1, \ldots, n$ can be removed. Therefore, M' must be minimal and G does not have property W.

Hence, if G is stem-reducible and has property W then its stems must form an independent dominating set in G.

Conversely, suppose that S is an independent dominating set in G. Since the leaves of G can only be dominated by the stems, any dom-matching must saturate the set S. Since the stems are independent, no edge in M can have both end vertices in S. Hence, any dom-matching in G has at least $|S|$ edges. Since S is a dominating set, any edges in addition to those which saturate S are unnecessary in any dom-matching. Hence, every minimal dom-matching has exactly $|S|$ edges and such a dom-matching can be obtained by matching each stem with one of its leaves. Therefore, G has property W.

Since G has property W, then by Lemma 1.4.1.3, any stem reduction on G results in a graph with property W and, hence, no isolated vertices. Similarly, no stem-reduction in the series performed on G will result in an isolated vertex. Furthermore, a stem-reduction is possible as long as all the stems in S have not been selected. Since $G \setminus N[S]$ is the empty graph, we can conclude that G is stem-reducible. $\square$

Lemma 1.4.2.2 *Let P_n and C_n be the path and cycle on n vertices, respectively.*

(a) P_n has property W only if $n = 2, 3, 5$ or 6.

(b) C_n has property W only if $n = 3, 4, 5, 6, 7$ or 9.

Proof: (a) Suppose $n \not\equiv 1$ (mod 3) and $n \geq 5$. A stem reduction on P_n results in the path P_{n-3}. Hence, a series of stem reductions on P_n will eventually result in either P_2 or P_3, both of which are obviously stem-reducible. So, for all $n \not\equiv 1$ (mod 3) and $n \geq 2$, the path P_n is stem-reducible. Therefore, by Theorem 1.4.2.1, P_n has property W only if its stems are an independent dominating set in G. This only holds for $n = 2, 3, 5$ and 6.

Now suppose $n \equiv 1$ (mod 3). Obviously, P_1 does not have property W since it is an isolated vertex. Furthermore, for any P_n such that $n > 1$ and $n \equiv 1$ (mod 3), P_n can be reduced to P_1 by a series of stem-reductions. Hence, by Lemma 1.4.1.3, P_n does not have property W for any $n \equiv 1$ (mod 3).

(b) If $n = 3$ or 4, then we can see that one edge is always sufficient to dominate C_n. If $n = 5$ or 6 then two edges are always both necessary and sufficient. Hence, C_n has property W for $n = 3, 4, 5, 6$. If $n = 7$, then at least two edges are necessary and three edges will always result in a dom-matching that is not minimal. Hence, C_7 has property W. Similarly, for C_9 at least three edges are necessary and four edges results in a dom-matching that is not minimal. Hence, three edges are always sufficient and C_9 has property W. We can see in Figure 1.15 that both C_8 and C_{10} have two minimal matchings of different cardinalities. Hence, C_n does not have property W for $n = 8$ or 10. Finally C_n does not have property W for any $n \geq 11$, since an edge-reduction on C_n results in the path P_{n-4}, where $n - 4 \geq 7$. By Lemma 1.4.1.2, P_{n-4} does not have property W, and therefore, C_n does not have property W for any $n \geq 11$. $\square$

Suppose we have a graph G of girth at least six which is stem-irreducible. Then either G has no stems or a graph with no stems can be obtained via a series of stem-reductions on G. If the former case holds, let $G = H$. Otherwise let H be the stem-free graph obtained through stem-reductions. Hence, H consists of a set of components, each of which is either an isolated vertex or a graph with minimum degree two. Suppose H is not entirely composed of isolated vertices. Then we can perform an edge-reduction on H. Let $H' = H \setminus N[\{u, v\}]$ be the edge-reduction on H. Suppose
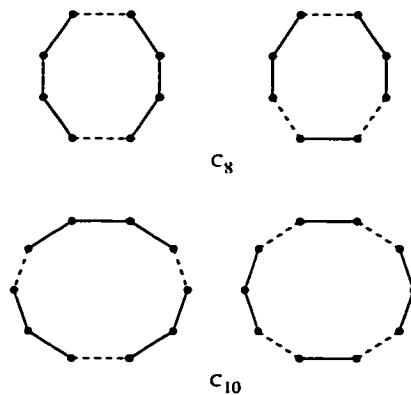
Figure 1.15: Two minimal dominating matchings in each of C_8 and C_{10}.

H has components $\{H_1, \ldots, H_k\}$ for some $k \geq 1$ and. without loss of generality. uv is an edge in H_1. Then $H' = \{H_1', H_2, \ldots, H_k\}$ where $H_1' = H_1 \setminus N[\{u, v\}]$. The graph H_1' is either a stem-reducible graph or contains a stem-irreducible component. If H_1' is stem-irreducible. then H' contains at least as many stem-irreducible components as H.

We wish to perform reductions on G to obtain a graph which is. in a sense. minimal with respect to being stem-irreducible. To obtain this graph we perform a series of stem- and edge-reductions where a stem-reduction is performed whenever possible. and an edge-reduction which does not decrease the number of stem-irreducible components is performed otherwise. This sequence of reductions on G will result in a graph $H = \{H_1, H_2, \ldots, H_k\}$ such that for any $i = 1, \ldots, k$. either H_i is an isolated vertex or $H_i \setminus N[\{u, v\}]$ is a stem-reducible graph for every edge uv in H_i.

Lemma 1.4.2.3 *Let $C = \{v_1, v_2, \ldots, v_9\}$ be a 9-cycle. For any pair of vertices in C there is a minimal dom-matching in C which meets both vertices.*

Proof: Without loss of generality. there are four pairs of vertices in C to consider. $(v_1, v_2.)$. (v_1, v_3). (v_1, v_4) and (v_1, v_5). The matching $M = \{v_1 v_2, v_3 v_4, v_7 v_8\}$ is a minimal dom-matching which meets v_1. v_2. v_3 and v_4. Hence. this is a matching which satisfies the lemma for the first three pairs. The minimal dom-matching $\{v_1 v_2, v_4 v_5, v_7 v_8\}$

meets both vertices in the final pair. Hence. for any pair of vertices in C there is a minimal dom-matching which meets both of them. □

Theorem 1.4.2.4 *Let G be a connected. stem-irreducible graph with girth at least eight. Then G has property W if and only if G is a 9-cycle.*

Proof: Suppose G is a stem-irreducible graph with property W and girth at least eight. Suppose we perform stem- and edge-reductions. as described above. to obtain the graph $H = \{H_1. H_2, \ldots, H_k\}$ which is "minimal" with respect to stem-irreducibility. Since G has property W and girth at least eight. then by Corollary 1.4.1.2 and Lemma 1.4.1.3, the graph H also has property W. Hence. no component of H is an isolated vertex. Furthermore. for any edge uv in H_1. for example. the graph $H_1 \setminus N[\{u. v\}]$ is stem-reducible.

The component H_1 has minimum degree two. Suppose there is a vertex. x. in H_1 which has degree at least three. Let y be a neighbour of x. Since H_1 is leafless. y has another neighbour u which in turn has a neighbour v. Due to girth restrictions. these are all distinct vertices and no vertex in $N[\{u. v\}]$ other than y is adjacent to x. Let $H_1' = H_1 \setminus N[\{u. v\}]$. Then x is in H_1' and has at least two neighbours in H_1'.

By Corollary 1.4.1.2. H_1' has property W since H_1 has property W. Since H' is stem-reducible. then by Theorem 1.4.2.1. the set of stems in H_1' must be an independent dominating set in H_1'. Therefore. x is either a stem in H_1' or is adjacent to a stem in H_1'. Hence. x is at most distance two from some vertex. say ℓ. which is a leaf in H_1'. The graph H_1 had no leaves. so the vertex ℓ is adjacent to a vertex in $N[\{u. v\}]$ in H_1. Hence. there is a path from x to u through ℓ in H_1 which has length at most five. This would result in a cycle of length at most seven in H_1 which is contrary to the girth restriction. Therefore. all vertices in H_1. and similarly all components of H. have degree two. Hence. every component is a cycle. Since the only cycle of girth at least eight with property W is the 9-cycle. then every component of H is a 9-cycle.

If G itself was not a 9-cycle. then there was some series of reductions to obtain H. Let K' be the first graph in the series to contain all of $\{H_1. H_2, \ldots, H_k\}$ as components. Since the number of stem-irreducible components can not decrease. then

K' contains no other stem-irreducible components. It may. however. contain stem-reducible components. say $R = \{R_1, \ldots, R_m\}$. Since $G \neq H$ and G was connected. then $G \neq K'$ and K' was obtained by some reduction on a graph K. Hence $K' = K \setminus \mathcal{N}[\{u, v\}]$ for some edge uv in K.

Since K does not have $\{H_1, H_2, \ldots, H_k\}$ as components. then there is some vertex $x \in (\mathcal{N}[u, v] \setminus \{u, v\})$ which is adjacent to a vertex in H_i for some $i = 1, 2, \ldots, k$. Without loss of generality suppose $x \in N(u) \setminus \{v\}$ and x is adjacent to a vertex in H_1. (See Figure 1.16.)
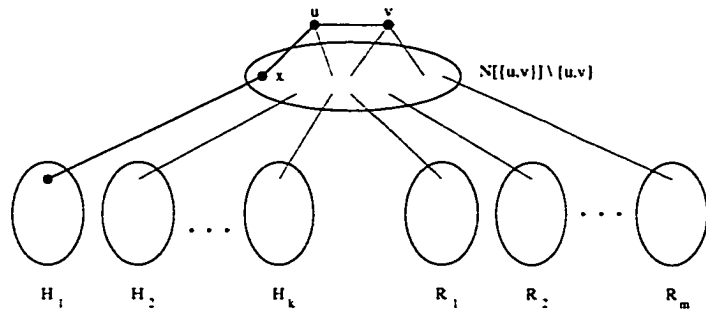
Figure 1.16: The graph K obtained by stem- and edge-reductions.

Case 1: Suppose K' was obtained from K by a stem-reduction. Then u is a stem and v is a leaf. Let $H_1 = \{v_1, v_2, \ldots, v_9\}$ and suppose that x is adjacent to v_1. Let M_i be any minimal dom-matching in H_i for $i = 1, 2, \ldots, k$ and let M_{k+1} be a minimal dom-matching in R. Hence $M = \{uv\} \cup M_1 \cup M_2 \cup \cdots M_{k+1}$ is a minimal dom-matching in K. Note that $|M_i| = 3$ for all $i = 1, 2, \ldots, k$ since H_i is a 9-cycle.

However. the matching $M' = \{ux\} \cup \{v_3v_4, v_7v_8\} \cup M_2 \cdots M_{k+1}$ is also a dom-matching in K. Since $|M'| = |M| - 1$. then M was not a minimum dom-matching and K does not have property W. Hence. by Lemmas 1.4.1.1 and 1.4.1.3. G does not have property W. Figure 1.17 shows the edges of M and M' contained in the subgraph induced on $H_1 \cup \mathcal{N}[u]$.

Case 2: Suppose K' was obtained by an edge-reduction. Then the graph K contains no stems. and therefore. no leaves. So. every vertex in $w \in \mathcal{N}(v) \setminus \{u\}$ has some neighbour besides v: that is $\mathcal{N}(w) \setminus \{v\}$ is nonempty. Due to the girth condition.
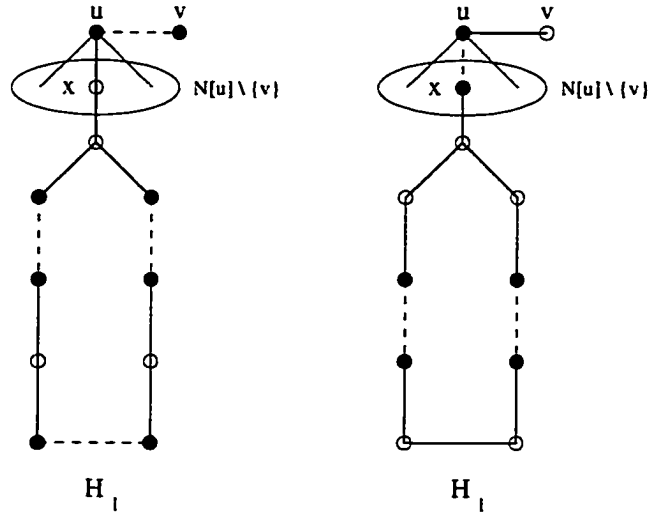
Figure 1.17: Two minimal dom-matchings in $H_1 \cup N[u]$.

$N(w) \setminus \{v\}$ must be in K'. We wish to find a minimal dom-matching in K' such that for every $w \in N(v) \setminus \{u\}$. all vertices in $N(w) \setminus \{v\}$ are met by the matching. Let $X = \bigcup_{w \in N(v) \setminus \{u\}} N(w) \setminus \{v\}$. Hence, every vertex in X must be met by the matching.

Consider $X \cap H_i$ for some $i = 1, 2, \ldots, k$. Suppose there are three vertices $a, b, c \in X \cap H_i$. Then some pair of $\{a, b, c\}$. say a and b are distance at most three apart on H_i. There is also a path from a to b through v which is length at most four. The result would be a cycle of length at most seven. which contradicts the girth restriction. Hence. $|X \cap H_i| \leq 2$ and. by Lemma 1.4.2.3. all the vertices in $X \cap H_i$ can be met by a minimal dom-matching M_i in H_i.

Now consider the set $X \cap R$. Suppose there is some vertex $a \in X \cap R_j$ which is not a leaf in R_j. Then a is distance at most two from a leaf ℓ in R_j, since the stems of R_j form a dominating set. The vertex ℓ was not a leaf in K. Therefore. ℓ is adjacent to some vertex in $N[\{u, v\}] \setminus \{u, v\}$ in K. Hence, there is a path from a to ℓ through v which has length at most five. This results in a cycle with length at most seven in K which contradicts the girth restriction. Hence. $X \cap R_j$ consists entirely of leaves in R_j. Then $X \cap R$ consists entirely of leaves in R.

Since each leave in R is adjacent to a vertex in $N[u.v]$ then no two leaves in R share a common stem. This would create a cycle of length at most seven. which contradicts the girth restriction. Hence, each stem in R is adjacent to exactly one leaf. Since the stems of R form an independent dominating set, then the set of edges which have a stem and its adjacent leaf as its end vertices form a minimal dom-matching in R. Call this matching M_{k+1}. Then all the vertices in $X \cap R$ are met by the matching M_{k+1}.

Hence. every vertex in X is met by the matching $M_1 \cup M_2 \cdots \cup M_k \cup M_{k+1}$ where each M_i. $i = 1.2.\ldots.k+1$ is as described above. Since each M_i is a minimal dom-matching in H_i for all $i = 1.2.\ldots.k$ and M_{k+1} is a minimal dom-matching in R. then $M = \{uv\} \cup M_1 \cup M_2 \cup \cdots \cup M_{k+1}$ is a minimal dom-matching in K and $|M| = 3k + |M_{k+1}| + 1$.

It was previously noted that there is a vertex x which is a neighbour of u in K such that x is adjacent to some vertex in H_1. Let $H_1 = \{v_1.\ldots.v_9\}$ where v_1 is adjacent to x. If there is a vertex $y \in N(v) \setminus \{u\}$ such that y is adjacent to H_1 then. without loss of generality. either $y \sim v_4$. $y \sim v_5$. $y \sim v_6$ or $y \sim v_7$. The cases $y \sim v_4$ and $y \sim v_7$ are symmetric. as are the cases $y \sim v_5$ and $y \sim v_6$. Therefore. we will only consider $y \sim v_4$ and $y \sim v_5$. (See Figure 1.18.) Note that there is no vertex in $N(v) \setminus \{u\}$ besides y is adjacent to H_1. due to girth restrictions.
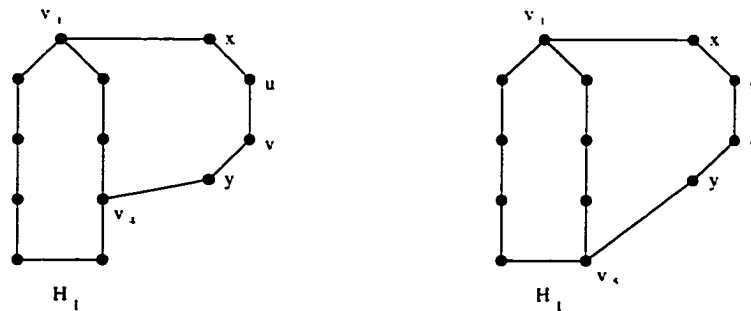
Figure 1.18: The two possible cases if a vertex in $N(v) \setminus \{u\}$ is adjacent to some vertex in H_1.

If $y \sim v_4$ or no such y exists then let $M' = \{ux. v_3v_4. v_7v_8\} \cup M_2 \cup \cdots \cup M_{k+1}$. Since v_4 is met by the matching. y is dominated. All the vertices in $N(u) \setminus \{v\}$

are dominated by u and all the vertices in $N(v) \setminus \{u, y\}$ are dominated by one of $M_2, M_3, \ldots, M_{k+1}$. Hence. M' is a dom-matching of size $3k + |M_{k+1}| < |M|$. This contradicts the fact that K has property W.

Suppose that $y \sim v_5$. If the vertex x is dominated by one of the matchings $M_2, M_3, \ldots, M_{k+1}$ then let $M' = \{vy, v_2v_3, v_7v_8\} \cup M_2 \cup \cdots \cup M_{k+1}$. Then M' is a dom-matching of size $3k + |M_{k+1}| < |M|$. This contradicts the assumption that K has property W. Hence. we may assume that x. and similarly y, are not adjacent to any vertex met by the matching $M_2 \cup \cdots \cup M_{k+1}$.

Let $M' = \{xv_1, vy, v_2v_3, v_5v_6, v_8v_9\} \cup M_2 \cup \cdots \cup M_{k+1}$. If K has property W then M' can not be a minimal dom-matching since it has cardinality $3k + |M_{k+1}| + 2$. Hence. some edge can be removed and the remaining matching is dominating. We see that. due to girth restrictions. the vertices v_3, v_6 and v_8 have no neighbours besides v_2, v_5 and v_9. respectively. which are met by the matching. Hence. none of v_2v_3, v_5v_6 and v_8v_9 can be removed from M'. Since u and v have no neighbours besides x and y. respectively. which are met by the matching. neither xv_1 nor vy can be removed from the matching. Suppose there is an edge ab in $M_2 \cup \cdots \cup M_k$ which can be removed. Without loss of generality. suppose ab is in M_2. If ab can be removed. then there is some vertex in $N(a) \setminus \{b\}$ which is met by the matching. Since a has no neighbours in H_i for any $i \neq 2$ nor is it adjacent to either x or y. then a must be dominated by another vertex met by the matching M_2. Similarly. b must be dominated by a vertex in H_2. other than a which is met by the matching M_2. Since $|M_2| = 3$ then M_2 consists of three consecutive edges in H_2. However. there is no such minimal dom-matching in H_2 since $H_2 = C_9$. Hence. no edge in $M_2 \cup \cdots \cup M_k$ can be removed.

Suppose an edge in M_{k+1} can be removed. All the edges in M_{k+1} are from a stem to a leaf in R. Let $s\ell$ be the edge to be removed where s is a stem and ℓ is a leaf in R. In order to remove $s\ell$. the vertex s must be adjacent to some vertex met by the matching. besides ℓ. This is impossible, however. since the stems in R are independent and s has no neighbour in $H \cup N[\{u, v\}]$. Therefore. no edge of M_{k+1} can be removed.

Hence. no edge in M' can be removed such that the result is a dom-matching. Therefore M' is a minimal dom-matching of size $3k + |M_{k+1}| + 2 > |M|$. This contradicts the fact that G has property W.

So, our assumption that H was obtained through reductions was incorrect and $G = H$. Since G was connected. then G consists of a single 9-cycle. Therefore. if G is stem-irreducible. has girth at least eight and has property W, then G is a 9-cycle. By Lemma 1.4.2.2. the 9-cycle has property W. The theorem. therefore, follows. $\square$

Hence. Theorem 1.4.2.1 and Theorem 1.4.2.4 together give us:

Corollary 1.4.2.5 *A connected graph G with girth at least eight has property W if and only if the stems of G form an independent dominating set in G or G is a 9-cycle.*

1.4.3 Problems

Now that the graphs of girth at least eight with property W have been characterized. the next step is to examine graphs of some girth less than eight. Since Corollary 1.4.1.2 and Theorem 1.4.2.1 apply to graphs with girth at least six. this seems the obvious choice. Since. by Theorem 1.4.2.1. those stem-reducible graphs with property W and girth at least six have been characterized, we need only consider stem-irreducible graphs.

Let G be a graph with property W and girth at least six. If G is stem-irreducible then we can reduce G to obtain H. Suppose there is a cut-edge uv in H_1. One minimal dom-matching in H_1 consists of uv and any minimal dom-matching in $H'_1 = H_1 \setminus N[\{u, v\}]$. Note that the stems of H'_1 must be independent and dominating by Theorem 1.4.2.1. Now consider the edges ux and vy for some $x \neq v$ and $y \neq u$. There is a minimal dom-matching in H'_1 which does not dominate either x or y. Therefore. this matching in H'_1 together with ux and vy is a minimal dom-matching in H_1 larger than that containing uv. This contradicts the fact that G has property W. Therefore. H does not contain a cut-edge.

Problem 1.4.3.1 *What do the components of H look like for a stem-irreducible graph G with property W and girth at least six?*

Problem 1.4.3.2 *Characterize those graphs with girth at least six and property W.*

1.5 Paired-Domination and Maximal Matchings

As previously discussed, γ_p is always bounded above by twice the size of any maximal matching (since the end vertices of a maximal matching form a paired-dominating set). In this section we shall focus on those graphs which have a maximal matching whose end vertices actually form a minimum paired-dominating set, or γ_p-set. That is, there is a maximal matching that is also a minimum dom-matching.

Let $\mathcal{G}$ denote the leafless graphs of this type which have girth at least seven. In this situation where there are no cycles of length six or smaller, we completely characterize these graphs showing they must belong to an infinite family based on the 9-cycle.

1.5.1 Preliminary Results

Let us consider a graph G in $\mathcal{G}$. That is, G has some maximal matching, say M, such that $V(M)$ forms a γ_p-set.

Lemma 1.5.1.1 *Let G be a leafless graph with girth six or more and let M be a maximal matching such that $V(M)$ is a γ_p-set in G. Then no end vertex of an edge in M can be adjacent to an end vertex of another edge in M.*

Proof: Suppose that G and M satisfy the hypothesis but assume there are two edges, pq and rs say, in M such that q is adjacent to r. Since G is leafless and has girth at least six, p must have a neighbour, say v, which is not on the 4-path $\{p, q, r, s\}$. Now v has a neighbour, say u, which is also not on the 4-path $\{p, q, r, s\}$ due to the girth restriction and the leafless situation. Since M is a maximal matching either u or v must be incident with some edge in M. Hence, any neighbour of p, other than q, is

adjacent to a vertex in $V(M)$ besides p. Similarly, any neighbour of s other than r is adjacent to a vertex in $V(M)$ besides s. Thus we could certainly interchange the two pairs represented by the edges pq and rs for the single pair q and r and have a paired-dominating set that is smaller than $2|M|$ which is a contradiction. $\qquad\square$

Observe that the girth restriction in Lemma 1.5.1.1 is sharp as illustrated by C_5 ($\gamma_p = 4$). The leafless property is also essential as shown by the graph in Figure 1.19 where $\{a, b, c, d\}$ is a γ_p-set and $\{ab, cd\}$ is the dom-matching.
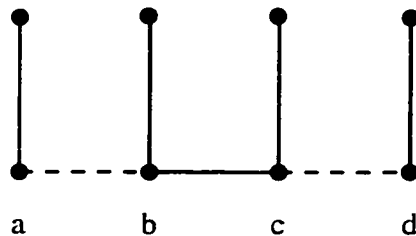
Figure 1.19: An example showing the necessity of the leafless property in Lemma 1.5.1.1.

We now proceed to show the importance of 9-cycles in these graphs.

Lemma 1.5.1.2 *Let G be a graph in $\mathcal{G}$, and let M be a maximal matching such that $V(M)$ is a γ_p-set. If there is an 8-path, say $\{a, b, c, d, e, f, g, h\}$, in which ab, de, and gh belong to M, then that 8-path must be part of a 9-cycle.*

Proof: Assume that G is a graph and M be a maximal matching satisfying the hypothesis of the lemma. Let $\{a, b, c, d, e, f, g, h\}$ be an 8-path where ab, de, and gh belong to M. Note that since M is maximal, every vertex not met by M has all its neighbours met by M.

Let $M' = (M \setminus \{ab, de, gh\}) \cup \{bc, fg\}$. Since $|M'| < |M|$ and M is a γ_p-set, then M' is not a dom-matching. Hence, there is a vertex v in G that is not dominated by $V(M')$. Note that v is neither in $V(M)$ nor on the 8-path, since all those vertices are dominated by $V(M')$. Hence, $N(v) \subseteq V(M)$ and $N[v] \cap V(M') = \emptyset$. So, $N(v) \subseteq$

$V(M) \setminus V(M') = \{a, d, e, h\}$. Since G has minimum degree two and girth at least seven then $N(v) = \{a, h\}$. Therefore, a and h share a neighbour that is not on the 8-path, implying that the 8-path is part of a 9-cycle. $\square$

Corollary 1.5.1.3 *If G belongs to $\mathcal{G}$, then the girth of G is at most nine.*

Proof: Suppose G is a graph in $\mathcal{G}$, and M is a maximal matching in G such that $V(M)$ is a γ_p-set. Let ab be any edge in M. Since G is leafless, every vertex has degree at least two. Then b has some neighbour, say c such that $a \neq c$. The vertex c can not be met by the matching M due to Lemma 1.5.1.1. Now, c has some neighbour d such that $d \neq b$. The vertex d is also distinct from a due to girth restrictions. The vertex d must be met by the matching since M is a maximal matching. Let de be the edge in M having d as an end vertex. Note that the vertices $\{a, b, c, d, e\}$ are all distinct. Now, e has a neighbour f which is not met by the matching, due to Lemma 1.5.1.1, and f has a neighbour g which is met by the matching since M is maximal. Due to girth restrictions, $\{a, b, c, d, e, f, g\}$ are all distinct. Hence, we have an 8-path such that ab, de and fg are all edges in M. By Lemma 1.5.1.2, this 8-path lies on a 9-cycle, and G has girth at most nine. $\square$

Since the edge ab was arbitrarily chosen from the edges of M, then every edge of M must lie on a 9-cycle. Since M is maximal, any edge not in M has one of its end vertices met by M. Hence, any edge not in M could be labelled bc where b is the end vertex met by the matching. Now, choose a to be the vertex such that ab is in M. Using these as the choices of $\{a, b, c\}$ in the above proof, we see that bc lies on a 9-cycle. Hence, *every* edge of a graph in $\mathcal{G}$ lies on a cycle of length nine.

Lemma 1.5.1.4 *If G belongs to $\mathcal{G}$ then G does not contain a 7-cycle.*

Proof: Assume that G satisfies the above hypothesis but does contain a 7-cycle, say $C = \{a, b, c, d, e, f, g\}$. Let M be a maximal matching of G such that $V(M)$ is a γ_p-set. Then, by Lemma 1.5.1.1, C contains at most two edges of M.

Suppose C contains two edges of M. We may assume, without loss of generality, that the edges ab and de are in M. Then no edge of M is incident with f or g, by Lemma 1.5.1.1. But then fg is neither in M nor incident with any edge in M. This contradicts the fact that M is maximal. Hence, C contains at most one edge of M.

Suppose C contains exactly one edge, say ab, of M. By Lemma 1.5.1.1, neither c nor g is incident with an edge in M. Hence, d and f must each be incident with an edge in M. Otherwise, cd and fg would neither be in M nor incident with an edge in M. Let xd and yf be the edges in M. We now have the path $\{y, f, g, a, b, c, d, x\}$ where yf, ab, and dx are in M. Therefore, by Lemma 1.5.1.2, the vertices x and y must have a common neighbour, say z. But this results in the 6-cycle $\{x, z, y, f, e, d\}$ which contradicts the girth restriction. Hence, C contains no edge of M. This too is impossible, however. Since M is maximal, at least one end vertex of each edge of C must be in $V(M)$. Because C is of odd length, this results in two edges of M having adjacent end vertices. We know this can not be the case due to Lemma 1.5.1.1. Hence the girth of G is at least eight. □

Therefore, by Lemmas 1.5.1.2 and 1.5.1.4, any graph in $\mathcal{G}$ must have either girth eight or girth nine.

Lemma 1.5.1.5 *Let G be in $\mathcal{G}$. If M is a maximal matching in G such that $V(M)$ is a γ_p-set, then no edge of M lies on an 8-cycle.*

Proof: Suppose G contains an 8-cycle, say $C = \{a, b, c, d, e, f, g, h\}$. Let M be a matching satisfying the hypothesis. By Lemma 1.5.1.1, C contains at most two edges in M.

Suppose C contains two edges of M. By Lemma 1.5.1.1, we may assume, without loss of generality, they are either the pair ab and de or the pair ab and ef. If the former case holds then g must be incident with an edge in M. Let xg be the edge in M. But then $\{x, g, h, a, b, c, d, e\}$ is an 8-path with xg, ab, and de in M. Therefore, by Lemma 1.5.1.2, x and e must have a common neighbour, say y. Now we have the 5-cycle $\{x, g, f, e, y\}$ which contradicts the girth restriction. Hence, the edges ab and

ef must be in M. However, this results in the edges cd and gh being neither in M nor incident with any edge in M, by Lemma 1.5.1.1. This contradicts the fact that M is maximal. Hence, C contains at most one edge of M.

Suppose C contains exactly one edge of M, say ab. Then both d and g must be incident with edges in M which are not in C. But then, by Lemma 1.5.1.1, the edge ef is neither in M nor incident with any edge in M. This is impossible since M is maximal. Hence, C contains no edge of M. □

Lemma 1.5.1.6 *Suppose G is in $\mathcal{G}$. If M is a maximal matching such that $V(M)$ is a γ_p-set, then any vertex of G that has degree at least three must be incident with an edge of M.*

Proof: Let G and M satisfy the hypothesis, and let v be a vertex having degree at least three. Assume that v is not incident with any edge of M.

Suppose v has neighbours a, b and c. They each must be incident with some edge of M, since M is maximal. Let ax, by and cz be the edges in M. Since G has no leaves and girth at least seven, x must have a neighbour, say u, which is adjacent to neither y nor z. The vertex u itself must have another neighbour, say p. The vertex p can not be adjacent to y nor z since this would create a 7-cycle in contradiction to Lemma 1.5.1.4. By Lemma 1.5.1.1, the vertex u is not incident with any edge in M. Hence, p must be incident with an edge in M, since M is maximal. Call this edge pq. By Lemma 1.5.1.1, q can not be adjacent to y nor z. However, $\{q, p, u, x, a, v, b, y\}$ is an 8-path with qp, xa and by in M. By Lemma 1.5.1.2, this path must be part of a 9-cycle. Hence, q and y share a common neighbour, say r. Similarly, q and z share a common neighbour, say s. The vertices r and s must be distinct due to girth restrictions. But now we have the 8-cycle $\{q, r, y, b, v, c, z, s\}$ containing two edges which are in M. This is impossible by Lemma 1.5.1.5. Hence, we conclude that v must be incident with an edge in M. □

1.5.2 The Characterization of All Graphs in $\mathcal{G}$

We now wish to characterize all the graphs contained in the set $\mathcal{G}$.

Define the infinite family of graphs. $\mathcal{F}$. to be the set of those graphs H which can be obtained from three nonempty sets of parallel edges. $\{u_r v_r : r = 1 \ldots \ldots k\}$. $\{w_s x_s : s = 1 \ldots \ldots l\}$ and $\{y_t z_t : t = 1 \ldots \ldots m\}$. by connecting each of the pairs of vertices $(v_r. w_s)$. $(x_s. y_t)$ and $(z_t. u_r)$ with a path of length two. Hence. for each such pair of vertices a new vertex is introduced which is a common neighbour of these vertices. We will call the original set of $k + l + m$ parallel edges the **associated matching** of H. Figure 1.20 shows the graph in $\mathcal{F}$ where $k = l = m = 2$. The dashed edges indicate the associated matching.
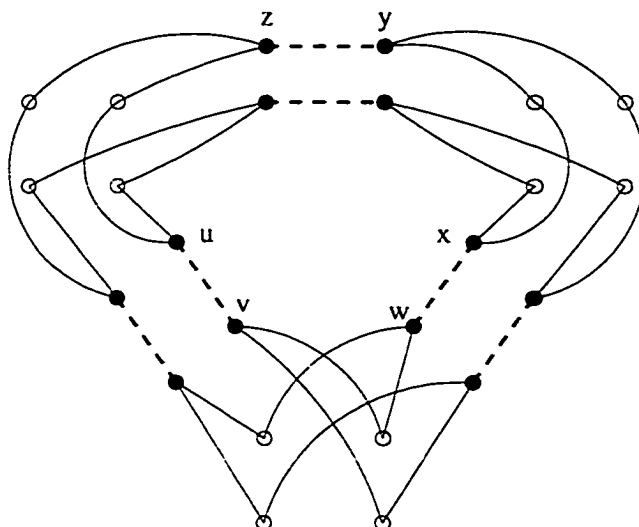
Figure 1.20: The graph in $\mathcal{F}$ in which $k = l = m = 2$.

The graphs in $\mathcal{F}$ are obviously leafless and have girth at least eight. Furthermore. the associated matching of any graph in $\mathcal{F}$ is a maximal matching. Note that if $k = l = m = 1$. then H is the 9-cycle and the end vertices of associated matching form a γ_p-set. Hence. the 9-cycle is in $\mathcal{G}$.

We are prepared to show:

Theorem 1.5.2.1 *A graph G is in $\mathcal{F}$ if and only if G is also in $\mathcal{G}$.*

Proof: First, we show that every graph in $\mathcal{F}$ is also in $\mathcal{G}$. Let G be any graph in $\mathcal{F}$. Partition the edges in the associated matching into the sets $UV = \{u_r v_r : r = 1,\ldots,k\}$, $WX = \{w_s x_s : s = 1,2,\ldots,l\}$, and $YZ = \{y_t z_t : t = 1,2,\ldots,m\}$, as previously described. We wish to show that $\gamma_p \geq 2(k + l + m)$. That is, any matching (not necessarily maximal), say M, such that $V(M)$ is a paired-dominating set contains at least $k + l + m$ edges.

Let M be a matching such that $V(M)$ is a paired-dominating set. Since every u_r and z_t share a common neighbour of degree two, then either all the vertices $\{u_r : r = 1,2,\ldots,k\}$ or all the vertices $\{z_t : t = 1,2,\ldots,m\}$ must be incident with an edge in M. Similarly, either all of $\{v_r : r = 1,2,\ldots,k\}$ or all of $\{w_s : s = 1,2,\ldots,l\}$ must be incident with an edge in M, and either all of $\{x_s : s = 1,2,\ldots,l\}$ or all of $\{y_t : t = 1,2,\ldots,m\}$ must be incident with an edge in M. Without loss of generality assume that all the vertices $\{u_r : r = 1,2,\ldots,k\}$ are met by the matching.

Case 1: Suppose that all the edges of UV are in M. The end vertices of these edges are not adjacent to any of the vertices $\{w_s : s = 1,2,\ldots,l\}$ nor the vertices $\{z_t : t = 1,2,\ldots,m\}$. Since no pair of vertices in $\{w_s, z_t : s = 1,\ldots,l, t = 1,2,\ldots,m\}$ are adjacent or have a common neighbour, each vertex in this set requires a unique vertex to dominate it. Furthermore, no such set of dominating vertices contains an adjacent pair. Hence, another $l + m$ edges are required and $|M| \geq l + k + m$.

This also includes the case where $WX \subseteq M$ or $YZ \subseteq M$.

Case 2: Suppose that $UV \not\subseteq M$, $WX \not\subseteq M$ and $YZ \not\subseteq M$. By Lemma 1.5.1.1, no two edges of M can have adjacent end vertices. Therefore, both u_r and v_r are met by M only if $u_r v_r$ is in M. Since all of the vertices $\{u_r : r = 1,2,\ldots,k\}$ are met by M and $UV \not\subseteq M$, then some v_r is not met by the matching. Therefore, all of $\{w_s : s = 1,2,\ldots,l\}$ must be met by the matching. Since $WX \not\subseteq M$, then by Lemma 1.5.1.1 there is some x_s not met by the matching. Therefore, all of $\{y_t : t = 1,2,\ldots,m\}$ must be met by the matching. This gives us a total of $k + l + m$ vertices all of which must be met by the matching, but none of which are adjacent. Therefore, at least $k + l + m$ edges are required and $|M| \geq l + k + m$.

So. at least $(l + k + m)$ edges are required in any matching. M. where $V(M)$ is a paired-dominating set. Therefore. $\gamma_p \geq 2(k + l + m)$. But we know the edge set $\{u_r v_r. w_s x_s. y_t z_t : r = 1, \ldots, k, s = 1, 2, \ldots, l, t = 1, \ldots, m\}$ is a maximal matching of size $k + l + m$. Hence. $\gamma_p = 2(k + l + m)$. and the vertex set of this maximal matching is a γ_p-set. Therefore. any graph in $\mathcal{F}$ must also be in $\mathcal{G}$. $\qquad \square$

It has been shown that for any graph in $\mathcal{F}$. the vertex set of the associated matching is a γ_p-set. In fact. for every graph in $\mathcal{F}$, other than the 9-cycle. the associated matching is the only maximal matching with this property. This can be easily verified using Lemma 1.5.1.1 and Lemma 1.5.1.5.

We will now show that every graph in $\mathcal{G}$ is also in $\mathcal{F}$. We proceed by induction on γ_p. Suppose G is a graph in $\mathcal{G}$ and $\gamma_p = 6$. Then G has a maximal matching. say M. such that $|M| = 3$ and $V(M)$ is a paired-dominating set. By Lemma 1.5.1.2. we know that G contains a 9-cycle. say $C = \{a. b. c. d. e. f. g. h. i\}$. If $G = C$ then we are done since C is obviously in $\mathcal{F}$. Suppose $G \neq C$. Since M is maximal and C has odd length. according to Lemma 1.5.1.1 there must be at least one edge of the matching which lies on C. Without loss of generality. let ab be that edge. Due to Lemma 1.5.1.1. the vertex c is not incident with any edge in M. Hence. d must be incident with an edge in M. since M is maximal. Similarly. h must be incident with an edge in M. If de is not in M. then the vertex f must be incident with an edge in M. This. however. is impossible since $|M| = 3$. Hence. de and. similarly. gh are in M. Hence $V(M) = \{a. b. d. e. g. h\}$ is a paired-dominating set in G. Since $G \neq C$ there is some vertex. say v. which is not on C. This vertex must be adjacent to at least one of $\{a. b. d. e. g. h\}$. Without loss of generality. we can assume that v is adjacent to a. Since G is leafless. v has another neighbour. say w. which must also be adjacent to one of $\{a. b. d. e. g. h\}$. This. however. results in a cycle of length at most seven. This is a contradiction due to the girth restriction together with Lemma 1.5.1.5. Hence. the only graph in $\mathcal{G}$ with $\gamma_p = 6$ is the 9-cycle.

Let M be a maximal matching in G such that $V(M)$ is a γ_p-set and $|V(M)| = 2n > 6$. Choose any edge. say uv. in M. Let $N(u) = \{v. u_1. u_2. \ldots. u_k\}$ and let

$N(v) = \{u, v_1, v_2, \ldots, v_l\}$. We know from Lemma 1.5.1.1 that no u_i, $i = 1, \ldots, k$, or v_j, $j = 1, \ldots, l$, is incident with an edge in M. Hence, by Lemma 1.5.1.6, each u_i and v_j must have degree two. Let w_i be adjacent to u_i for each $i = 1, 2, \ldots, k$. Similarly, let x_j be adjacent to v_j for each $j = 1, 2, \ldots, l$. Note that each w_i and x_j must be incident with an edge in M, since M is maximal. Let $\{w_i y_i : i = 1, \ldots, k\}$, and $\{x_j z_j : j = 1, 2, \ldots, l\}$, be the edges in M. Now, by Lemma 1.5.1.2, it must be the case that y_i and z_j have a common neighbour, say q_{ij}, for all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$. Let $S = \{u, v, u_i, w_i, y_i, v_j, x_j, z_j, q_{ij} : i = 1, 2, \ldots, k, j = 1, 2, \ldots, l\}$. Note that due to girth restrictions and Lemma 1.5.1.5, all the vertices in S must be distinct.

Case 1: Suppose that at least one of the w_i's or x_j's has a neighbour not in S. Without loss of generality assume that w_1 has a neighbour, say a, not in S. By Lemma 1.5.1.1 and Lemma 1.5.1.6, a has degree two and is not incident with an edge in M. In addition, a is not adjacent to any vertex in S due to girth restrictions. Lemma 1.5.1.4 and Lemma 1.5.1.5. Hence, a has another neighbour, say b, where b is incident with an edge in M. Let bc be that edge.

For all $j = 1, 2, \ldots, l$, we have the path $\{x_j, z_j, q_{1j}, y_1, w_1, a, b, c\}$ where $x_j z_j$, $y_1 w_1$ and bc are in M. Hence, by Lemma 1.5.1.2, x_j and c have a common neighbour, say r_j, for all $i = 1, 2, \ldots, l$. Furthermore, for all $i = 1, 2, \ldots, k$, we have the path $\{w_i, y_i, q_{i1}, z_1, x_1, r_1, c, b\}$. Hence, by Lemma 1.5.1.2, w_i and b have a common neighbour, say s_i, for all $i = 1, 2, \ldots, k$. Note that each r_j and s_i has degree two, due to Lemma 1.5.1.1 and Lemma 1.5.1.6, and that $deg(u) \leq deg(b)$ and $deg(v) \leq deg(c)$. Now we wish to show that equality holds for both of these inequalities.

Suppose this is not the case, and that c has another neighbour, say d. Since M is maximal, d has a neighbour, say e, which is incident with an edge in M. Note that neither d nor e is in $S \cup \{a, b, c\} \cup \{s_i, r_j : i = 1, \ldots, k, j = 1, 2, \ldots, l\}$ due to girth restrictions. Now, let ef be the edge in M. By Lemma 1.5.1.2, the path $\{f, e, d, c, b, a, w_1, y_1\}$ implies that f and y_1 must have a common neighbour, say g. This gives us the path $\{e, f, g, y_1, w_1, u_1, u, v\}$ which, by Lemma 1.5.1.2, implies that

e and v have a common neighbour. Hence. e is adjacent to v_j for some $j = 1.2.....l$. Without loss of generality, assume that e is adjacent to v_1. However. we now have the 6-cycle $\{e.d.c.r_1.x_1.v_1\}$, which is impossible due to the girth restriction. Hence $deg(v) = deg(c)$. It can be similarly shown that $deg(u) = deg(b)$.

Let us now consider the graph. $H = G \setminus (N(u) \cup N(v))$. This graph is connected. leafless. and has girth at least seven. Furthermore, the matching $M' = M \setminus \{uv\}$ is a maximal matching in H such that $V(M')$ is a γ_p-set in H. Hence. H is in $\mathcal{G}$ where $\gamma_p(H) = 2n - 2$. Then. by the induction hypothesis. H must be in $\mathcal{F}$ and M' is its associated matching. The edges of the associated matching of H can be partitioned into three sets of parallel edges. since H is in $\mathcal{F}$. The graph G is obtained from H by adding one more edge. uv. to the set of parallel edges containing bc. Therefore. G is also be in $\mathcal{F}$.

Case 2: Suppose that all of the neighbours of the w_i's and x_j's lie in S. Hence. the only vertices which may be adjacent to some vertex not in S are the y_i's and z_j's. Suppose y_1 has a neighbour which is not in S. Let a be that neighbour. Since G is leafless a has another neighbour, say b. Note that b is neither in nor adjacent to any vertex in S due to Lemma 1.5.1.4. Lemma 1.5.1.5 and the girth restriction. Since M is maximal. b must be incident with some edge in M. Let bc be the edge in M. Now we have the path $\{v.u.u_1.w_1.y_1.a.b.c\}$ where vu. w_1y_1 and bc are in M. Hence. by Lemma 1.5.1.2. the vertices v and c have a common neighbour. Since $N(v) = \{u.v_1.v_2.....v_l\}$. we can assume. without loss of generality. that v_1 is adjacent to both v and c. This is impossible. however. since the only neighbours of v_1 are v and x_1 which are both in S. Hence. y_1 has no neighbours other than those in S. Similarly. no y_i or z_j has a neighbour outside of S. Hence. $V(G) = S$.

Obviously. if both u and v have degree 2 then $G = C_9$ and $\gamma_p = 6$. Therefore. at least one of u and v must have degree at least three. Without loss of generality. assume that u has degree at least three. Let $H = G \setminus \{u_1.w_1.y_1.q_{1j} : j = 1.....l\}$. Then H is connected. leafless. and in $\mathcal{G}$. Furthermore. the edge w_1y_1 is parallel to w_2y_2. which is contained in the associated matching of H. By the induction hypothesis. H is in

$\mathcal{F}$. and. therefore. the graph G is also in $\mathcal{F}$.

Therefore. the sets $\mathcal{G}$ and $\mathcal{F}$ are equal. □

Corollary 1.5.2.2 *If G belongs to $\mathcal{G}$ and contains an 8-cycle, then G contains the graph F. shown in Figure 1.21. as an induced subgraph.*
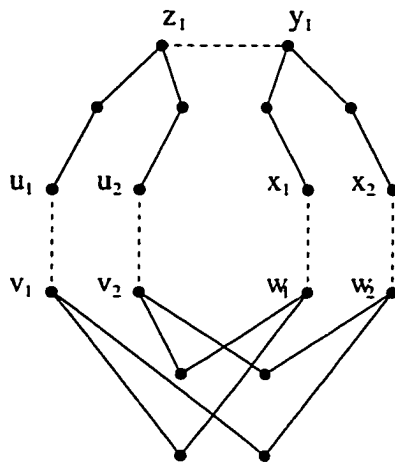
Figure 1.21: The graph $F \in \mathcal{F}$ in which $k = l = 2$ and $m = 1$.

Proof: Suppose $G \in \mathcal{G}$ and F is not a subgraph of G. Then at least two of k. l and m are are equal to one. Such graphs clearly do not contain any 8-cycles. □

1.5.3 Problems

Suppose we define an infinite family $\mathcal{E}$ to be the set of those graphs that can be obtained from two nonempty set of parallel edges $UV = \{u_r v_r : r = 1.2.....k\}$ and $WX = \{w_s x_s : s = 1.2.....l\}$ by connecting each of the pairs $(v_r. w_s)$ and $(x_r. u_s)$ with a path of length two. If $k = l = 1$. then the graph is the 6-cycle. Hence. $\mathcal{E}$ is an infinite family based on the 6-cycle. The set $UV \cup WX$ is a maximal matching which is also a minimum dom-matching. The proof of this is similar to part one of the proof of Theorem 1.5.2.1 $(\mathcal{F} \subseteq \mathcal{G})$.

Problem 1.5.3.1 *Let $\mathcal{H}$ be the set of all leafless graphs of girth at least six which have a maximal matching whose end vertices form a γ_p-set. Does $\mathcal{H} = \mathcal{E} \cup \mathcal{F}$?*

To this point we have only investigated leafless graphs. since Lemma 1.5.1.1 only applies to leafless graphs of girth at least six.

Problem 1.5.3.2 *What complications arise when leaves are added?*

Chapter 2

Cops and Robbers

2.1 Introduction

The game of Cops and Robber is a pursuit game played on a graph. The game was introduced by Nowakowski & Winkler [37] and independently by Quilliot [45]. The cop side consists of some set of k cops and the robber side consists of a single robber. The rules of the game are as follows: first each cop chooses a vertex and then the robber chooses a vertex then they move alternately. The cops' move consists of some (possibly empty) subset of the cops moving to an adjacent vertex and the robber's move is to move to an adjacent vertex or stay at his current position. The game is played with perfect information: that is, the cops and robber are aware of the other's position at all times. The cops win if one or more cops manage to occupy the same vertex as the robber. The robber wins if he manages to avoid this situation forever.

Since k cops can always win if k is large enough (for example $k = \gamma(G)$), then the obvious question is how few cops are sufficient to ensure the apprehension of the robber. The minimum number required to catch a robber on a graph G is called the **copnumber** of G and is denoted $c(G)$. Graphs in which one cop is sufficient are called **copwin graphs**. These graphs were characterized by Nowakowski & Winkler [37].

In Section 2.2, we examine known results, including the characterization of copwin

graphs. upper and lower bounds on cop numbers and the cop numbers of products of graphs. Using the traditional game as inspiration, we then propose new variations of the game in which the cops' movements are restricted. In Section 2.3, we introduce the **precinct** version of the game. The vertices of the graph are partitioned into subsets, or **beats**, then one or more cops is assigned to each beat and no cop is ever permitted to move off of his beat. The second variation, introduced in Section 2.4 is the **dragnet** version of the game. This is similar to the precinct game in that each cop is assigned a beat, but in this case the cop is allowed to move to a vertex outside of his beat if it is the final move of the game. As it was for the game of cops and robber, the problem is to determine the minimum number of cops required to win. For both games, we establish upper and lower bounds on this number for various types of graphs, as well as compare these with the actual cop numbers of the graphs.

2.2   Historical Development and Known Results

Regard all graphs as reflexive. An induced subgraph H of G is a **retract** of G if there is an edge preserving map f from G to H such that $f|_H$ is the identity map on H. It was shown by Nowakowski and Winkler [37] that:

> *The strong product of a finite collection of copwin graphs is also a copwin graph*

and

> *Any retract of a copwin graph is also a copwin graph.*

First suppose we have the strong product, H, of a finite collection $\{G_1, \ldots, G_n\}$ of copwin graphs. There is a projection of H onto each G_i which is edge preserving. Hence, the projection of the cop and robber on each G_i is a game in itself. The cop takes his winning strategy on each G_i and plays the composition of those n moves on H. Once he captures a robber on one projection he stays with the robber in that

projection until he catches the robber in the other projections. Eventually, the robber will be caught on all G_i and, hence, on H.

Now, let f be a retraction map of a copwin graph G onto a graph H. The cop has a winning strategy on G no matter how the robber moves. When playing on the retract H the cop considers his winning strategy on the larger graph G and then plays the image of that winning strategy under the retraction map, f. Hence, the cop can catch the image of the robber and then stay with the image. Since the robber is restricted to the subgraph H and f is the identity on H then the cop catches the robber.

If a graph G is copwin then there is a final move where the cop moves onto the robber. Consider the robber's position prior to his last move. Call this vertex p. The robber obviously had no move which would put him on a vertex nonadjacent to the cop's position. Therefore, the vertex, say d, occupied by the cop was adjacent to every vertex in the closed neighbourhood of p: $N[p] \subseteq N[d]$. We will call the vertices p and d a **pitfall** and its **dominating vertex**, respectively. Obviously, any graph with no pitfalls is not a copwin graph. However, if a pitfall is present we can remove it and not change the status of the graph in terms of being copwin, as shown in:

Lemma 2.2.0.3 (Nowakowski & Winkler [37]) *Let p be a pitfall in a graph G and let $G' = G \setminus \{p\}$. Then G is copwin if and only if G' is copwin.*

Let p be a pitfall, and d be its dominating vertex in G. The graph $G' = G \setminus \{p\}$ is a retract of G where $f(p) = d$ and f is the identity on G'. The cop can, therefore, take his winning strategy on G and use it in G', except that whenever p is to be occupied in G, d is occupied in G'. This is a winning strategy in G'.

Now suppose G' is copwin. If f is the map of G onto G' then the cop can catch the image of the robber on G'. Suppose the image is caught on a vertex x. If $x \neq d$ then the robber is on his image and the robber has been caught. If $x = d$ then the robber is on p or d. If he is on d then he has been caught, and if he is on p then every possible move will leave him adjacent to the cop who is on d. Hence, the robber will be caught on the next move.

Using Lemma 2.2.0.3. Nowakowski and Winkler [37] showed that:

A graph is copwin if and only if there is an ordering of the vertices of G. say $\{v_1, v_2, \ldots, v_n\}$. such that v_i is a pitfall in $G \setminus \{v_1, v_2, \ldots, v_{i-1}\}$ for all $i = 1, 2, \ldots, n$.

Call this a **copwin ordering** of the vertices of G.

Aigner and Fromme [1] noted that if a copwin ordering exists. it can be obtained by choosing v_i to be any pitfall in $G \setminus \{v_1, v_2, \ldots, v_{i-1}\}$. Hence, the characterization of copwin graphs could be restated as:

A graph is copwin if and only if the successive removal of pitfalls results in a single vertex.

A graph G is **bridged** if for every cycle C of length greater than three in G there are a pair of vertices. say x and y on C such that $d_C(x, y) > d_G(x, y)$. This means that on each cycle of length greater than three there is a "shortcut" between two vertices on the cycle. A **chordal** graph is one in which there are no induced cycles of length greater than three. Hence. every chordal graph is also a bridged graph. It was shown by Anstee & Farber [5] that every nontrivial bridged graph contains a pitfall and that the removal of that pitfall results in another bridged graph. Hence. they concluded that:

Every bridged graph is a copwin graph.

Furthermore. it was shown by Chepoi [12] that the pitfalls in a bridged graph can be found by performing a breadth first search on the graph. If we let v_n be the first vertex labelled by the search. v_{n-1} be the second. v_{n-2} the third and so on. then the ordering $\{v_1, v_2, \ldots, v_n\}$ is a copwin ordering.

With copwin graphs completely characterized. Aigner and Fromme [1]. turned their investigation toward establishing upper and lower bounds on the cop number of certain classes of graphs. One such result:

For any graph G with minimum degree $\delta(G)$ and girth at least five.
$c(G) \geq \delta(G)$

showed that there are graphs which require an arbitrary number of cops.

Noting that every planar graph with minimum degree at least four contains a 3-cycle or a 4-cycle. they proved that:

For any planar graph G. $c(G) \leq 3$.

They also conjectured that for every increase of one in the genus at most two additional cops need to be added. This extension of the planar result was proved by Quilliot [46] who showed that:

If G is a graph with a given genus k. then $c(G) \leq 2k + 3$.

Aigner and Fromme's work also inspired Frankl [21] who investigated the cop numbers of graphs with large girth. He showed that:

If a graph has girth $g \geq 8t - 3$ then $c(G) > (\delta(G) - 1)^t$. where $\delta(G)$ is the minimum degree of the vertices of G.

For $t = 1$ this result reduces to the lower bound given by Aigner and Fromme [1] for graphs of girth at least five.

Seeing that $c(G)$ could be bounded below by $\delta(G)$ if G's girth was sufficiently large. Neufeld [35] posed the question *what is the minimum girth that ensures $c(G) > \delta(G)$?* He proved that:

If a graph G has girth $g \geq 9$ then $c(G) > \delta(G)$.

Another line of investigation into the cop numbers of graphs has used the properties of retracts. It has been shown that the retract of any copwin graph is copwin. Using a similar argument. Berarducci and Intrigila [7] showed that:

If H is a retract of a graph G. then $c(H) \leq c(G)$.

Suppose f is a retraction map of G onto H. Then $c(G)$ cops can win on H by taking their winning strategy in the graph G and then playing the image of each move under the retraction map f. Furthermore, if $c(H)$ cops move on the subgraph H in G, then after a finite number of moves the robber will be caught if he ever moves onto a vertex of H. This is due to the fact that every vertex on H is its own image. Therefore, it can be shown, as in [7], that:

Lemma 2.2.0.4 *If H is a retract of G then $c(H)$ cops moving on H can, after a finite number of moves, guarantee the robber's capture if he moves onto H.*

The $c(H)$ cops can catch the image of the robber under the retraction map f on H. Once this is done, one cop continues to move on the image, thus preventing the robber from moving onto H.

Berarducci and Intrigila [7] noted that once some cop was on the robber's image in H under the retraction map then $c(G \setminus H)$ cops could catch the robber on the subgraph $G \setminus H$ to which the robber is now restricted. They gave the following corollary:

If H is a retract of a graph G then $c(G) \leq \max\{c(H), c(G \setminus H) + 1)\}$.

If H is a retract of G then it is necessarily an **isometric subgraph** of G. An isometric subgraph of G is defined to be a subgraph H of G such that for all $x, y \in V(H)$, $d_H(x, y) = d_G(x, y)$. While it is the case that every retract is an isometric subgraph, not every isometric subgraph is a retract. There are, however, certain classes of graph which are retracts whenever they are isometric subgraphs. These graphs are called **absolute retracts**. Paths are contained in the set of absolute retracts, as are complete graphs. It has also been shown by Nowakowski and Rival [39] that trees and certain cycles are absolute retracts.

Lemma 2.2.0.5 (Nowakowski & Rival [39]) *Let G be a graph.*

a) Every cycle of minimum order in G is a retract.

b) If T is a connected subgraph of G without cycles then T is a retract of G if an only if T is isometric in G.

In Figure 2.1. G is an induced subgraph of both H and I (indicated by the larger circles) but G is only an isometric subgraph of H, whereas in Figure 2.2. G is an isometric subgraph of both H and I but is only a retract of I.
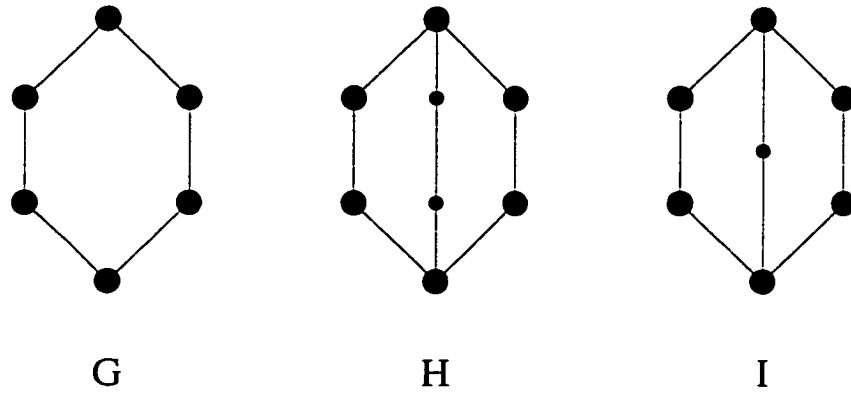
Figure 2.1: Graph G is an Isometric Subgraph of H but not of I.
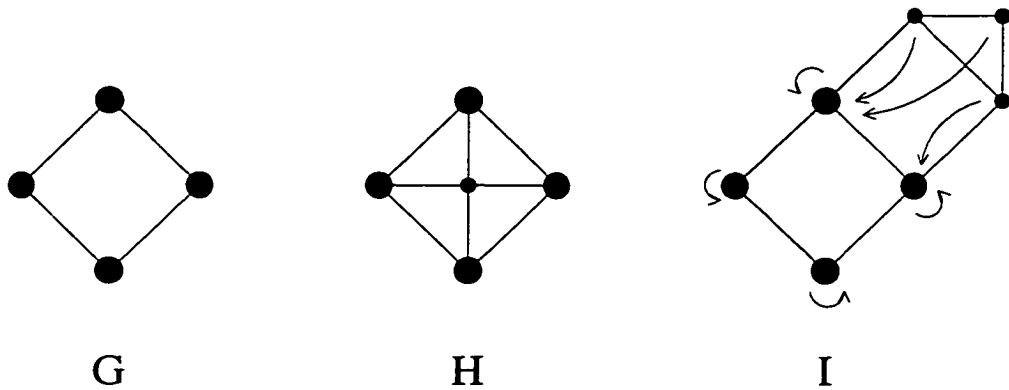
Figure 2.2: G is an Isometric Subgraph of H and a Retract of I.

The use of absolute retracts played a central role in Aigner and Fromme's proof that all planar graphs have cop number at most three. The retraction used was a map of G onto an isometric path in G. Knowing that paths are copwin graphs. the following result becomes evident.

Lemma 2.2.0.6 (Aigner & Fromme [1]) *Let P be an isometric path in a graph G. A single cop moving on P can guarantee that after a finite number of moves the robber will be immediately caught if he moves onto P.*

In this case. the retraction map $f : G \to P$. where $P = \{a_0. a_1 \ldots\ldots a_n\}$ is an isometric path in G. was defined by

$$f(v) = \begin{cases} a_k & \text{if } k = d(a_0, v) \text{ and } k \leq n \\ a_n & \text{otherwise} \end{cases}$$

For future reference. we will call this particular retraction map onto an isometric path the **canonical retraction**, and once the cop catches and stays with the robber's image under this mapping. he will be said to be **shadowing** the robber.

Another area of investigation has been into the cop numbers of products of graphs. The game of cops and robber on Cartesian products of graphs was first explored by Tošić [48] and Maamoun & Meyniel [34]. Tošić [48] showed that for two graphs G and H.

$$c(G \Box H) \leq c(G) + c(H).$$

This can be extended to the more general result for any finite collection of graphs $\{G_1. G_2 \ldots\ldots G_k\}$.

$$c(\Box_{i=1}^k G_i) \leq \sum_{i=1}^k c(G_i).$$

Maamoun and Meyniel [34] found the copnumber of the Cartesian product of trees:

If $\{T_1. T_2 \ldots\ldots T_k\}$ is a collection of trees then $c(\Box_{i=1}^k T_i) = \lceil (k+1)/2 \rceil$.

The examination of Cartesian products continued with Neufeld [35] who looked at products of cycles and trees. He showed that:

If $\{C_1. C_2 \ldots\ldots C_k\}$ is a collection of cycles. each with length of at least four. then $c(\Box_{i=1}^k C_i) = k + 1$.

He also proved that:

If $G = \Box_{i=1}^k C_i$ and $H = \Box_{i=1}^j T_i$ where $\{C_1. C_2, \ldots. C_k\}$ is a collection of cycles each with length of at least four and $\{T_1, T_2, \ldots. T_j\}$ is a collection of trees. then $c(G \Box H) = c(G) + c(H) - 1 = k + \lceil (j+1)/2 \rceil$.

Another product which has received attention has been the strong product. We saw that the strong product of two copwin graphs is also a copwin graph. Neufeld & Nowakowski [36] found the following generalized result for the strong product of graphs with arbitrary cop numbers:

For any graphs G and H, $c(G \boxtimes H) \leq c(G) + c(H) - 1$.

Suppose we take the projection map of $G \boxtimes H$ onto $G \boxtimes w$ for some vertex w in H. It takes $c(G)$ cops to catch the image of the robber under this projection. Once the robber is caught one cop is left on his image. Add one more cop to the remaining $c(G) - 1$ cops and catch the image of the robber again. Now two cops can be left to move on the image of the robber. Another cop is added and the image of the robber is caught again. This is repeated until there are at least $c(H)$ cops on the image of the robber under this projection. Hence, at most $c(G) + c(H) - 1$ cops have been used. Now, those $c(H)$ can win by playing a composition of moves so that they stay on the robber in the projection onto G and play their winning strategy in the projection onto H.

2.3 Precincts

The Aigner & Fromme result in Lemma 2.2.0.6 prompted the idea of playing a game of cops and robber where each cop is assigned a "beat" and his movements are restricted to that beat. If each beat is an isometric path, then the minimum number of cops required to apprehend the robber is exactly the minimum number of isometric paths required to cover all the vertices of the graph. Call this number the **precinct number** of G and denote it $pn(G)$. Say a set of isometric paths **covers** G if every vertex of G lies in at least one of the paths in the set.

We first give a lower bound on the precinct number that is dependent on diameter and show some classes of graphs in which this lower bound is met. We then show that for any tree T, $pn(T) = \lceil \ell/2 \rceil$ where ℓ is the number of leaves in the tree. In Sections 2.3.3 and 2.3.4, we examine grid graphs. We show the precinct number for

an $m \times n$ grid is exactly $\left\lceil 2/3(m + n - \sqrt{m^2 + n^2 - mn}) \right\rceil$. For higher dimensional grids we establish a lower bound on precinct number by examine subgraphs of the grid. We also show a scheme for coving larger grids by "blowing up" a covering of a smaller grid. We can, therefore, determine an upper bound on the precinct numbers of some grids.

We then consider the problem of covering a graph with types of beats other than isometric paths, such as trees and complete graphs. The motivation for doing so is, in part, to establish an upper bound on the cop number of certain graphs. Obviously, if a set of cops can win the precinct version of the game, they can win the regular game. Unfortunately, this can be very wasteful in terms of manpower, as we see when the beats are isometric paths. However, we can often reduce the number of cops needed if we change the beats, and come closer to the actual cop number of the graph. Finally, we pose some problems.

2.3.1 Preliminary Results

The **diameter** of a graph G, denoted $diam(G)$, is defined to be the length of the longest isometric path in G. Therefore, a single isometric path in G contains at most $diam(G) + 1$ vertices. This gives us the following lower bound on the precinct number of G:

Theorem 2.3.1.1 *Let G be any connected graph with vertex set V. Then*

$$pn(G) \geq \left\lceil \frac{|V|}{diam(G) + 1} \right\rceil.$$

For some common graph families, Theorem 2.3.1.1 is exact.

Theorem 2.3.1.2 *Let P_n C_n be defined as previously, and let K_n and E_n be the complete graph and edgeless graph, respectively, on n vertices. Then*

(a) for all $n \geq 1$, $pn(P_n) = 1$, $pn(K_n) = \lceil n/2 \rceil$, and $pn(E_n) = n$:

(b) for all $n \geq 3$, $pn(C_n) = 2$:

(c) for any $k \geq 1$ and $1 \leq n_1 \leq n_2 \cdots \leq n_k$, $pn(\boxtimes_{i=1}^{k} P_{n_i}) = \Pi_{i=1}^{k-1} n_i$.

Proof: The proofs of (a) and (b) are straight forward, so we leave them out.

For (c), let $G = \boxtimes_{i=1}^{k} P_{n_i}$ for some $k \geq 1$ and $1 \leq n_1 \leq n_2 \cdots \leq n_k$. Then $diam(G) = \max\{n_i : i = 1,2,\ldots,k\} - 1 = n_k - 1$. Since $|V| = \Pi_{i=1}^{k} n_i$ then by Theorem 2.3.1.1. $pn(G) \geq \Pi_{i=1}^{k-1} n_i$.

Now, let $H = \boxtimes_{i=1}^{k-1} P_{n_i}$ and $P_{n_k} = \{x_1, x_2, \ldots, x_{n_k}\}$. Then $V = \{(v, x_i) : v \in V(H)$ and $i = 1,2,\ldots,n_k\}$. For every $v \in V(H)$, let $P_v = \{(v, x_1), (v, x_2), \ldots (v, x_{n_k})\}$. This is an isometric path in G. Hence, the set $\mathcal{P} = \{P_v : v \in V(H)\}$ is a set of $|V(H)| = \Pi_{i=1}^{k-1} n_i$ isometric paths which cover G. Therefore, $pn(G) = \Pi_{i=1}^{k-1} n_i$. □

There is also a relationship between the precinct number of a graph and the precinct numbers of its isometric subgraphs.

Theorem 2.3.1.3 *Let G be any graph and H be any isometric subgraph of G. Then $pn(H) \leq pn(G)$.*

Proof: Suppose $pn(G) = n$ and G can be covered by isometric paths $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$. Let H be an isometric subgraph in G and suppose $H = G \setminus S$, for some set of vertices S. For any path P in $\mathcal{P}$, the set of vertices in $P \cap H$, if nonempty, constitute a set of paths in H. Hence, $P \cap H = Q^1 \cup Q^2 \cup \cdots \cup Q^k$ for some $k \geq 1$ where each Q^i is a path in H for $i = 1,2,\ldots,k$. Since H is an isometric subgraph of G, there is a path in H from the last end vertex in Q^i to the first end vertex in Q^{i+1} that is isometric in both H and G. This path contains no vertices of $P \cap H$ other than its end vertices, due to the isometry of P. Hence, by adding the isometric paths joining Q^i to Q^{i+1} for each $i = 1,2,\ldots,k-1$ to the set $\{Q^1, Q^2, \ldots, Q^k\}$ we obtain a path, Q. Since P was an isometric path in G, then Q is an isometric path in H, and Q contains all the vertices in $P \setminus S$. Therefore, we can find a set of isometric paths $\{Q_1, Q_2, \ldots, Q_n\}$, some of which may be empty, which cover all the vertices of $G \setminus S = H$. Hence, $pn(H) \leq pn(G)$. □

Such a relationship does not exist between the precinct number of a graph and its induced subgraphs. Consider the graphs Figure 2.3. The graph G can be covered

with two isometric paths. while the graph H, which is an induced subgraph of G. requires three.
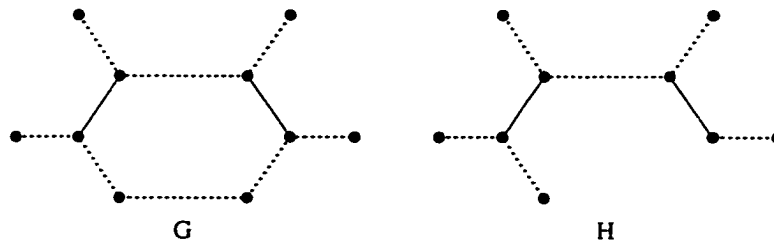
Figure 2.3: H is an induced subgraph of G such that $pn(G) < pn(H)$.

The removal of an edge can either increase or decrease the precinct number, as well. For example. the precinct number of a cycle is two. If we remove any edge of a cycle we obtain a path which has precinct number one. However. if we take a path of length at least two and remove an edge the result is a disconnected graph consisting of two paths. Such a graph has precinct number two.

2.3.2 Trees

Since every path is a tree is necessarily an isometric path. then the problem is to find a set of paths that covers all the vertices in the tree.

Theorem 2.3.2.1 *If T is any tree then $pn(T) = \lceil \ell(T)/2 \rceil$ where $\ell(T)$ is the number of leaves in T.*

Proof: Any path in a tree. T. may contain at most two leaves. Therefore. if T has $\ell(T)$ leaves. then $pn(T) \geq \lceil \ell(T)/2 \rceil$. Hence. it suffices to show that there exists a set of $\lceil \ell(T)/2 \rceil$ isometric paths which cover all the vertices of T. We will show that the edges. and hence the vertices of T can be covered with $\lceil \ell(T)/2 \rceil$ isometric paths. While the vertices of a tree can be covered without covering all the edges. a covering of the edges is required for the induction which follows.

We now proceed by induction on the number of vertices in T. If T is a tree with two vertices then $T = K_2$ and one isometric path covers all edges of T. Inductively

assume that for some $n > 2$ and all k such that $2 \leq k < n$. the edges of a tree. T. with k vertices can be covered with $\lceil \ell(T)/2 \rceil$ isometric paths.

Now. suppose that T is a tree with n vertices. Since T has more than two vertices. it must have a diameter of at least two. If the diameter of T is exactly two then $T = K_{1,n-1}$ and $\ell(T) = n - 1$. We will now find a set of isometric paths which cover the edges of T. If T has an even number of leaves then pair each leaf with another leaf. If T has an odd number of leaves then pair $n - 2$ of the leaves with each other and pair the remaining leaf with the vertex of degree $n - 1$. Now. take the path between each pair. This gives us a set of $\lceil \ell(T)/2 \rceil$ isometric paths which cover all the edges of T.

Suppose T has a vertex. v. of degree two. and suppose u_1 and u_2 are distinct neighbours of v. Let $T' = T \circ u_1 v$ be the tree obtained by contracting the edge $u_1 v$ and let u_1' be the vertex created by associating u_1 and v. Then T' is a tree with $n - 1$ vertices and $\ell(T) = \ell(T')$. By induction. the edges of T' can be covered with a set of $\lceil \ell(T')/2 \rceil = \lceil \ell(T)/2 \rceil$ isometric paths. Let $\{P_1', P_2', \ldots, P_{\lceil \ell(T)/2 \rceil}'\}$ be those paths. For $k = 1, 2, \ldots, \lceil \ell(T)/2 \rceil$. let

$$P_k = \begin{cases} P_k'. & \text{if } u_1' u_2 \notin P_k' \\ P_k' + u_1 v + u_2 v - u_1' u_2. & \text{if } u_1' u_2 \in P_k'. \end{cases}$$

Then $\{P_1, P_2, \ldots, P_{\lceil \ell(T)/2 \rceil}\}$ is a set of isometric paths in T which cover all the edges of T.

This leaves one final case where T has diameter at least three and no vertices of degree two. Then T must contain two leaves. say v_1 and v_2. with distinct neighbors u_1 and u_2. respectively. Then $T' = T \setminus \{v_1, v_2\}$ is a tree with $n - 2$ vertices and $\ell(T') = \ell(T) - 2$. (Since T has no vertices of degree two then u_1 and u_2 have degree at least two in T'.) Thus. the edges of T' can be covered with $\lceil \ell(T')/2 \rceil = \lceil (\ell(T) - 2)/2 \rceil = \lceil \ell(T)/2 \rceil - 1$ isometric paths. These paths along with a path from v_1 to v_2 cover the edges of T. Therefore. the edges of T can be covered with a set of $\lceil \ell(T)/2 \rceil$ isometric paths. Hence. for any tree. T. with $\ell(T)$ leaves $pn(T) = \lceil \ell(T)/2 \rceil$. $\square$

2.3.3 Grids

We now determine the precinct number of a grid. An $m \times n$ **grid** is the Cartesian product of an m-path and an n-path. We denote it by $G_{m,n}$ and label the vertices of $G_{m,n}$ as coordinates on the grid. Hence. $V(G_{m,n}) = \{(i,j)|i = 1,2,\ldots,m, j = 1,\ldots,n\}$ and the distance between any pair of vertices (a,b) and (c,d) in $G_{m,n}$ is given by $d((a,b),(c,d)) = |c-a| + |d-b|$. We also wish to assign a direction to every isometric path. P. in $G_{m,n}$. The **first end point** of P will be that endpoint of P with the minimum first coordinate. or if the first coordinates are the same then with minimum second coordinate. Hence. $P = \{(a_0,b_0),(a_1,b_1),\ldots,(a_n,b_n)\}$ implies that either $a_0 \leq a_n$ or $a_0 = a_n$ and $b_0 \leq b_n$. Let $P[(a_i,b_i),(a_j,b_j)]$ denote the subpath of P from (a_i,b_i) to (a_j,b_j) for any $i \leq j$.

Lemma 2.3.3.1 *Suppose* $P = \{(a_0,b_0),\ldots,(a_k,b_k)\}$ *is an isometric path in* $G_{m,n}$. *then*

(a) $a_i \leq a_{i+1}$ *for all* $i = 0,1,\ldots,k-1$;

(b) if $b_0 \leq b_k$ *then* $b_i \leq b_{i+1}$ *for all* $i = 0,1,\ldots,k-1$;

(c) if $b_0 > b_k$ *then* $b_i \geq b_{i+1}$ *for all* $i = 0,1,\ldots,k-1$.

Proof: (a) Let $P = \{(a_0,b_0),(a_1,b_1),\ldots,(a_k,b_k)\}$ be a path in $G_{m,n}$. Now suppose that for some i. $a_i > a_{i+1}$. Hence. $a_{i+1} = a_i - 1$ and $b_{i+1} = b_i$. Since P is an isometric path. each subpath of P is also isometric. Hence.

$$
\begin{aligned}
d((a_0,b_0),(a_k,b_k)) &= d((a_0,b_0),(a_i,b_i)) + d((a_i,b_i),(a_{i+1},b_{i+1})) + \\
&\quad\ d((a_{i+1},b_{i+1}),(a_k,b_k)) \\
&= |a_i - a_0| + |b_i - b_0| + 1 + |a_k - a_{i+1}| + |b_k - b_{i+1}| \\
&\geq |a_i - a_0 + a_k - a_i + 1| + |b_i - b_0 + b_k - b_i| + 1 \\
&= d((a_0,b_0),(a_k,b_k)) + 2.
\end{aligned}
$$

This contradiction implies that $a_i \leq a_{i+1}$ for all $i = 0,1,\ldots,k$.

Parts (b) and (c) can be shown similarly. □

For any path $P = \{(a_0, b_0), \ldots, (a_k, b_k)\}$ if (a). (b) and (c) hold then P is an isometric path. Therefore. we may categorize each isometric path in $G_{m,n}$ as one of two types. Call the path **rising** if $b_0 \leq b_k$ and **sinking** if $b_0 > b_k$. Note that rising paths include those in which $b_i = b_{i+1}$ for all $i = 0, \ldots, k - 1$.

Suppose $P = \{(a_0, b_0), \ldots, (a_k, b_k)\}$ is a path with end points $(1, 1)$ and (m, n). Then P cuts the grid into two components. one which lies "above" the path and one which lies "below". Therefore, if Q is a path with vertices in both components of the grid, as determined by P. then Q must intersect P at some vertex.

Theorem 2.3.3.2 (Fisher & Fitzpatrick [20]) *Let $G_{m,n}$ be an $m \times n$ grid. Then*

$$pn(G_{m,n}) \geq \left\lceil 2/3 \left(m + n - \sqrt{m^2 + n^2 - mn} \right) \right\rceil.$$

Proof: Suppose $p = pn(G_{m,n})$ where $1 \leq m \leq n$. Obviously. m isometric paths are sufficient to cover all the vertices of $G_{m,n}$. Hence. $p \leq m$. Let $\mathcal{P}$ be a set of p isometric paths which cover all the vertices of $G_{m,n}$. The set $\mathcal{P}$ can be partitioned into a set of rising paths and a set of sinking paths. denoted $\mathcal{R}$ and $\mathcal{S}$. respectively. Suppose $p = r + s$ where $r = |\mathcal{R}|$ and $s = |\mathcal{S}|$.

We first consider the paths in $\mathcal{R}$. Without loss of generality. we can assume that each path in $\mathcal{R}$ has end points $(1, 1)$ and (m, n). (If this is not the case. the path can be extended to contain these vertices by adding a rising path from $(1, 1)$ to the first end point of the path and a rising path from the last end point to (m, n)). We wish to show that the maximum number of vertices covered by the paths in $\mathcal{R}$ is $(m + n)r - r^2$. We proceed by induction on m.

Suppose $G = G_{1,n}$ for some $n \geq 1$. Then G is a path. and obviously one rising path covers at most n vertices in G. If $G = G_{2,n}$ then one rising path on G covers at most $n + 1$ vertices and two rising paths cover at most $2n$ vertices. Hence. the result holds for $m = 1$ and $m = 2$.

Assume that for all k such that $1 \leq k < m$. the maximum number of vertices covered by a set of r rising paths in $G = G_{k,n}$. where $r < k$. is $(k + n)r - r^2$. Also, assume that for any set of r rising paths in G there is another set of rising paths $\{P_1', \ldots, P_r'\}$. which covers the same vertices as the original set. but has the additional

property that each P'_i has first end point $(i, 1)$ and last end point $(k, n - i + 1)$ for all $i = 1, \ldots, r$.

Let $G = G_{m,n}$ and let $\mathcal{R}$ be a set of r rising paths in G. Let $(1, \overline{b})$ be the vertex covered by $\mathcal{R}$ such that no vertex $(1, j)$ is covered by $\mathcal{R}$ for any $j > \overline{b}$. Similarly, let $(\underline{a}, n)$ be the vertex covered by $\mathcal{R}$ such that no vertex (i, n) is covered by $\mathcal{R}$ for any $i < \underline{a}$.

Suppose no single path in $\mathcal{R}$ contains both $(1, \overline{b})$ and $(\underline{a}, n)$. Then $\underline{a} > 1$ and $n > \overline{b}$. Let P and Q be distinct paths in $\mathcal{R}$ containing $(1, \overline{b})$ and $(\underline{a}, n)$, respectively. The paths P and Q must intersect on the subpaths $P[(1, \overline{b}), (m, n)]$ and $Q[(1, 1), (\underline{a}, n)]$.

Let the vertex (x, y) be on both $P[(1, \overline{b}), (m, n)]$ and $Q[(1, 1), (\underline{a}, n)]$. We wish to consider new paths $P^* = P[(1, 1), (x, y)] \cup Q[(x, y), (m, n)]$ and $Q^* = Q[(1, 1), (x, y)] \cup P[(x, y), (m, n)]$. These paths are both rising and, hence, isometric. Replace P and Q with P^* and Q^* in $\mathcal{R}$. The new set, $\mathcal{R}^*$, of rising paths is the same size and covers the same vertices as the set $\mathcal{R}$.

If a single path, say P, contains both $(1, \overline{b})$ and $(\underline{a}, n)$, then $P^* = P$ and $\mathcal{R}^* = \mathcal{R}$.

In either case, $\mathcal{R}^*$ is a set of r paths which cover the same vertices as the paths in $\mathcal{R}$. Furthermore, the path P^* in $\mathcal{R}^*$ contains all the vertices covered by $\mathcal{R}$ which have either 1 as a first coordinate or n as a second coordinate. Therefore, the vertices covered by $\mathcal{R}^* \setminus P^*$ but not by P^* lie in the set $\{(i, j) | 1 < i \le m, 1 \le j < n\}$. The subgraph induced by these vertices is the graph $G_{m-1, n-1}$.

Therefore, by the induction hypothesis, the maximum number of vertices of the subgraph covered by the $r - 1$ paths in $\mathcal{R}^* \setminus P^*$ is $(m - 1 + n - 1)(r - 1) - (r - 1)^2 = (m + n)(r - 1) - r^2 + 1$. Since P^* covers $m + n - 1$ vertices, the total number of vertices covered by the paths in $\mathcal{R}$ is at most $(m + n - 1) + (m + n)(r - 1) - r^2 + 1 = (m + n)r - r^2$.

Also by induction, the vertices covered by any set of $r - 1$ in $G_{m-1, n-1}$ can also be covered by a set of $r - 1$ paths which have $\{(i, 1) : i = 1, 2, \ldots, r - 1\}$ as the set of all first end points of the paths and $\{(m - 1, n - i) : i = 1, 2, \ldots, r - 1\}$ as the set of all final endpoints. This was, however, assuming that the grid was labelled so that the first coordinates ranged from 1 to $m - 1$ and the second coordinates ranged

from 1 to $n - 1$. Since the $(m - 1) \times (n - 1)$ subgrid of $G_{m,n}$ has coordinates from 2 to m and 1 to $n - 1$, it follows that there is a set of $r - 1$ paths, say $\{P_2', P_3', \ldots, P_r'\}$ which cover the same vertices as $\mathcal{R}^* \setminus P^*$ such that P_i' has end coordinates $(i, 1)$ and $(m, n - i + 1)$ for $i = 2, 3, \ldots, r$. Since P^* has end points $(1, 1)$ and (n, n), if we let $P_1' = P^*$, then $\{P_1', P_2', \ldots, P_r'\}$ a set of r paths which have the end points required for the induction.

Hence, for any $1 \leq m \leq n$, the maximum number of vertices covered by a set, $\mathcal{R}$, of r rising paths in $G_{m,n}$ is $(m + n)r - r^2$. Furthermore, by induction, there is a set of r rising paths which cover the same vertices as $\mathcal{R}$ and can be ordered $\{P_1, P_2, \ldots, P_r\}$ such that P_i has end vertices $(i, 1)$ and $(m, n - i + 1)$ for all $i = 1, \ldots, r$. Call this set $\mathcal{R}'$.

Similarly, we can find a set of sinking paths, $\mathcal{S}' = \{Q_1, \ldots, Q_s\}$, such that each Q_j has end vertices (j, n) and (m, j) for all $j = 1, \ldots, s$, and the set $\mathcal{S}'$ covers the same vertices as the original set, $\mathcal{S}$. Hence, the number of vertices covered by $\mathcal{S}'$ alone is at most $(m + n)s - s^2$. However, each path in $\mathcal{S}'$ must intersect every path in $\mathcal{R}'$. To demonstrate, consider a rising path, P_r with end vertices $(i, 1)$ and $(m, n - i + 1)$ and a sinking path, P_s with end vertices (j, n) and (m, j), where $i \leq r$ and $j \leq s$. If P_r contains either (j, n) or (m, j), then P_r and P_s obviously intersect. Otherwise, P_r cuts the grid into two components, one containing (j, n) and the other containing (m, j). Hence, P_s must intersect P_r at least once.

Therefore, for each $j = 1, \ldots, k$, there are r vertices in Q_j which have already been counted in the set of vertices covered by the rising paths. Therefore, for $j = 1, \ldots, s$, the path Q_j contributes at most $|Q_j| - r$ new vertices to our total. Hence, the number of vertices covered by $\mathcal{S}'$ but not by $\mathcal{R}'$ is at most $(m + n)s - s^2 - rs$.

Therefore, the total number of vertices covered by $\mathcal{R}' \cup \mathcal{S}'$, and thus by $\mathcal{P} = \mathcal{R} \cup \mathcal{S}$, is at most $((m + n)r - r^2) + ((m + n)s - s^2) - rs = (m + n)p - p^2 + rs$, where $s + r = p$ and $rs = s(p - s) = sp - s^2$. We maximize rs at $s = p/2$ and, hence, at most $(m + n)p - 3p^2/4$ vertices in $G_{m,n}$ are covered by $\mathcal{P}$. If p paths are sufficient to cover

all the vertices of $G_{m,n}$, then $mn \leq (m+n)p - 3p^2/4$. Hence.

$$3p^2/4 - (m+n)p + mn \leq 0$$

$$p \geq 2/3 \left(m + n - \sqrt{m^2 + n^2 - mn} \right)$$

Hence. $pn(G_{m,n}) \geq \left\lceil 2/3 \left(m + n - \sqrt{m^2 + n^2 - mn} \right) \right\rceil$. $\square$

Now. suppose we have a set of r rising path and s sinking paths which cover the vertices of $G_{m,n}$. Define this cover to be **normal** if the rising paths cover the vertices

$$\{(r,1), (r-1,2), \ldots, (1,r), (m-r+1,n), (m-r+2,n-1), \ldots, (m,n-r+1)\}$$

and the sinking paths cover the vertices

$$\{(m-s+1,1), (m-s+2,2), \ldots, (m,s), (1,n-s+1), (2,n-s+2), \ldots, (s,n)\}.$$
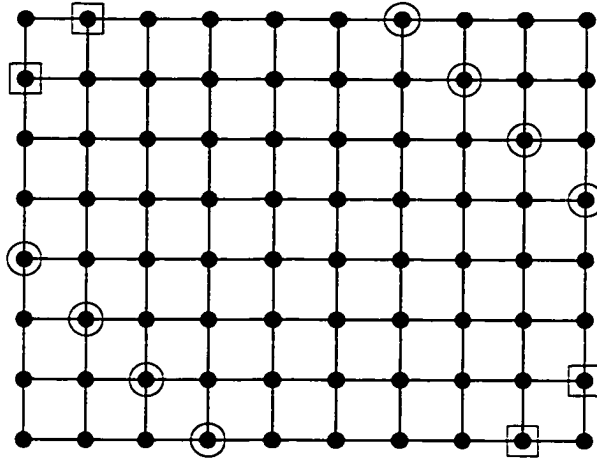
Figure 2.4: A cover of $G_{8,10}$ with four rising paths and two sinking paths is normal if the rising paths cover the circled vertices and the sinking paths cover the boxed vertices.

Theorem 2.3.3.3 (Fisher & Fitzpatrick [20]) *Let r, s, m, n be nonnegative integers. If*

$$(m - r - s)(n - r - s) \leq rs$$

then r rising paths and s sinking paths are sufficient to cover all the vertices of $G_{m,n}$.

Proof: Suppose $r + s < \min(m, n)$. Without loss of generality, assume $r \leq s$ and $m \geq n$ (if $r > s$, flip the graph to interchange the bottom and top; if $m < n$, flip along the $i = j$ diagonal). Also assume that $r \leq n$, since there is an obvious cover of $G_{m,n}$ otherwise.

Then

$$(m - r - s)(n - r - s) \leq rs$$

$$(n - r - s)^2 \leq s^2$$

$$n \leq r + 2s.$$

Note that if $r + s < \min(m, n)$ then the inequality $(m - r - s)(n - r - s) \leq rs$ is not satisfied for $r = 0$. Hence, we may assume $1 \leq r \leq s$.

Suppose $r + s \geq \min(m, n)$. Then for $1 \leq k \leq r$, let the kth rising path on $G_{m,n}$ go from $(1, k)$ to $(m - k + 1, k)$ to $(m - k + 1, n)$. And for $1 \leq \ell \leq s$, let the ℓth sinking path go from $(1, n - \ell + 1)$ to $(m - \ell + 1, n - \ell + 1)$ to $(m' - \ell + 1, 1)$. Hence, in this case there is a normal cover of $G_{m,n}$ with r rising and s sinking paths. (See Figure 2.5 for an example).
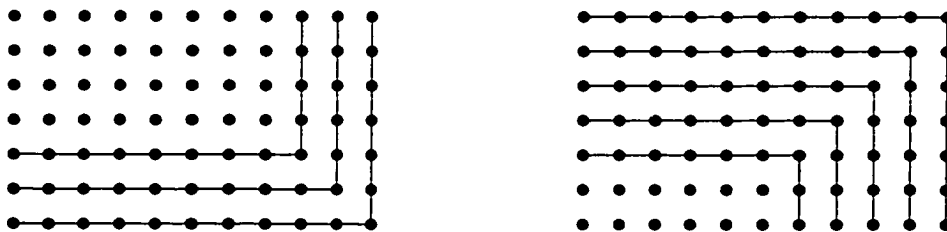
Figure 2.5: A Normal Cover of $G_{m,n}$ when $r + s \geq n$. This shows a normal cover of $G_{11,7}$ with 3 rising paths (on left) and 5 sinking paths (on right).

Suppose $r + s < \min(m, n)$. We proceed by induction on n. Let $G = G_{m,3}$. Then $r = s = 1$ are the only possible values that need to be tested. Hence, $(m - r - s)(n - r - s) = (m - 2)(3 - 2) = m - 2$. Hence, $(m - r - s)(n - r - s) \leq rs$ only if $m \leq 3$. Since $m \geq n$, then $m = 3$. Therefore, $m = n = 3$ and $r = s = 1$ are the only values

which satisfy the inequality, and in fact there is a normal cover of $G_{3,3}$ with one rising and one sinking path (see Figure 2.6).
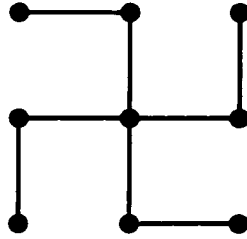
Figure 2.6: A normal cover of $G_{3,3}$ with one rising path and one sinking path.

For induction purposes. assume there is a normal cover of $G_{m',n'}$ with r' rising and s' sinking paths if $s' \leq (m' - r' - s')(n' - r' - s') \leq r's'$. $r' + s' < \min(m'. n')$. and $3 \leq n' < n$. We now cover part of $G_{m,n}$ as follows:

- For $1 \leq k \leq r$. the kth rising path goes from vertex $(k. 1)$ to $(k. n - s - k + 1)$ to $(n - s + k. n - s - k + 1)$.

- For $1 \leq j \leq r + 2s - n$. the jth sinking path goes from vertex $(1. n - j + 1)$ to $(r + s - j + 1. n - j + 1)$. For $r + 2s - n < j \leq s$. the jth sinking path goes from vertex $(1. n - j + 1)$ to $(r + s - j + 1. n - j + 1)$. then to $(r + s - j + 1. s - j + 1)$ and finally to $(m. s - j + 1)$.

This set of paths covers all the vertices in the first $n - s$ columns and the first $n - r - s$ rows of $G_{m,n}$. This leaves an $(m - (n - s)) \times (n - (n - r - s))$ grid to cover. However. the r rising and the first $r + 2s - n$ sinking paths described above enter this "subgrid" in such a way that a normal cover of $G_{m,n}$ can be completed if there is a normal cover of the $(m - n + s) \times (r + s)$ grid with r rising paths and $r + 2s - n$ sinking paths.

Let $m' = m - n + s$. $n' = r + s$. $r' = r$ and $s' = r + 2s - n$. Suppose first that $r' + s' < \min(m. n)$. Then. by induction. there is a normal cover of $G_{m',n'}$ if $(m' - r' - s')(n - r' - s') - r's' \leq 0$. This is true because

$$(m' - r' - s')(n - r' - s') - r's' = (m - 2r - s)(n - r - s) - r(r + 2s - n)$$
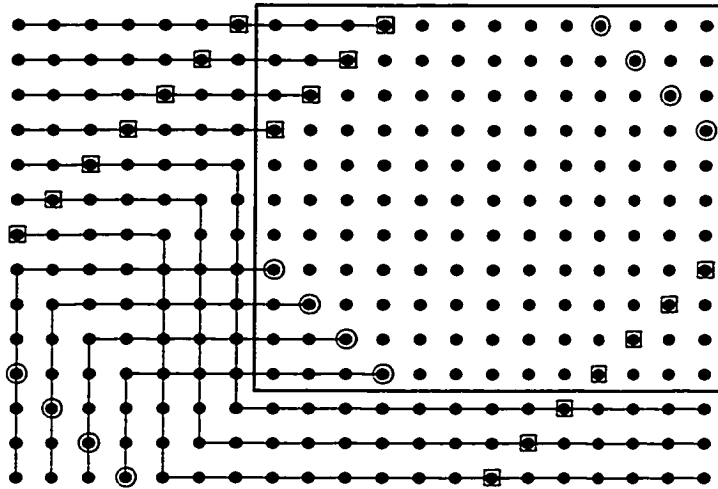
Figure 2.7: The problem of covering $G_{20,14}$ with 4 rising paths and 7 sinking paths is reduced to the problem to finding a normal cover for $G_{13,11}$ with 4 rising paths and 4 sinking paths.

$$= (m - r - s)(n - r - s) - rs$$

$$\leq 0.$$

Suppose $r' + s' \geq \min(m'.n')$. Note that $\max(r'.s') \leq s \leq \min(m'.n')$. Hence. as previously shown. there is a normal cover of $G_{m',n'}$ with r' rising and s' sinking paths.

Therefore. there is a normal cover of $G_{m,n}$ with r rising and s sinking paths where $(m - r - s)(n - r - s) \leq rs$. □

Figure 2.8 illustrates the normal cover of $G_{20,14}$ with four rising and seven sinking paths given by the construction in the proof of Theorem 2.3.3.3. This requires a normal cover of $G_{13,11}$ with four rising and four sinking paths which in turn requires a normal cover of $G_{6,8}$ with four rising paths and one sinking path which in turn requires a normal cover of $G_{5,6}$ with three rising paths and one sinking path which in turn requires a normal cover of $G_{4,4}$ with two rising path and one sinking paths which in turn requires a normal cover of $G_{3,2}$ with one rising and one sinking path. This last cover uses the construction in Figure 2.5 resulting in an edge on the right

side that is in both a rising and a sinking path.
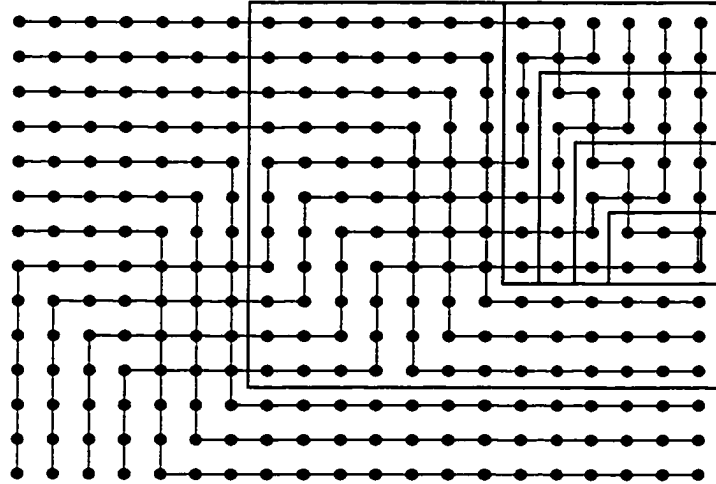
Figure 2.8: A normal cover of $G_{20,14}$ with four rising paths and seven sinking paths.

Theorem 2.3.3.4 (Fisher & Fitzpatrick [20]) *For all $m, n \geq 1$, we have*

$$pn(G_{m,n}) \leq \left\lceil \frac{2}{3} \left(m + n - \sqrt{m^2 + n^2 - mn} \right) \right\rceil.$$

Proof: Suppose $p = \left\lceil \frac{2}{3} \left(m + n - \sqrt{m^2 - mn + n^2} \right) \right\rceil$. Then

$$\frac{2}{3} \left(m + n - \sqrt{m^2 + n^2 - mn} \right) \leq p < 1 + \frac{2}{3} \left(m + n - \sqrt{m^2 + n^2 - mn} \right).$$

Since $m, n \geq 1$, then $\sqrt{m^2 + n^2 - mn} \geq 1$. Therefore,

$$\frac{2}{3} \left(m + n - \sqrt{m^2 + n^2 - mn} \right) \leq p < \frac{2}{3} \left(m + n + \sqrt{m^2 + n^2 - mn} \right).$$

This means that p lies between the roots of the quadratic equation $\frac{3}{4}x^2 - (m+n)x + mn$. Therefore,

$$\frac{3}{4}p^2 - (m+n)p + mn \leq 0$$

$$p^2 - (m+n)p + mn \leq \frac{p^2}{4}$$

$$(m-p)(n-p) \leq \frac{p^2}{4}$$

If $m \geq n$, we have

$$
\begin{aligned}
p \quad &< \quad 1 + \frac{2}{3}\left(m + n - \sqrt{m^2 - mn + n^2}\right) \\
&< \quad 1 + \frac{2}{3}\left(m + n - \sqrt{m^2 - mn + n^2/4}\right) \\
&= \quad 1 + \frac{2}{3}(m + n - (m - n/2)) \\
&= \quad 1 + n.
\end{aligned}
$$

Since p is integer, $p \leq n$ if $n \leq m$. Similarly, $p \leq m$ if $m \leq n$. Therefore, $p \leq \min(m, n)$.

Suppose p is even. If we let $r = \frac{p}{2}$ and $s = \frac{p}{2}$ then

$$
\begin{aligned}
(m - r - s)(n - r - s) \quad &= \quad (m - p)(n - p) \\
&\leq \quad \frac{p^2}{4} \\
&= \quad rs
\end{aligned}
$$

and

$$
r + s = p \leq \min(m, n).
$$

Hence, by Theorem 2.3.3.3, $G_{m,n}$ can be covered with $p/2$ rising paths and $p/2$ sinking paths.

Suppose p is odd. Note that the left side of $(m - p)(n - p) \leq p^2/4$ is integer, while the right side is an integer plus a quarter. Therefore, $(m - p)(n - p) \leq (p^2 - 1)/4$. So, if we let $r = (p - 1)/2$ and $s = (p + 1)/2$ then

$$
\begin{aligned}
(m - r - s)(n - r - s) \quad &= \quad (m - p)(n - p) \\
&\leq \quad \frac{p^2 - 1}{4} \\
&= \quad rs
\end{aligned}
$$

and

$$
r + s = p \leq \min(m, n).
$$

Hence. by Theorem 2.3.3.3. $G_{m,n}$ can be covered with $(p-1)/2$ rising paths and $(p+1)/2$ sinking paths. Therefore. $G_{m,n}$ can always be covered with

$$r + s = p = \left\lceil \frac{2}{3} \left(m + n - \sqrt{m^2 - mn + n^2} \right) \right\rceil$$

isometric paths. and

$$pn(G_{m,n}) \leq \left\lceil \frac{2}{3} \left(m + n - \sqrt{m^2 - mn + n^2} \right) \right\rceil .$$

$\square$

Hence. Theorem 2.3.3.2 and Theorem 2.3.3.4 together give us:

Corollary 2.3.3.5 *If $G_{m,n}$ is an $m \times n$ grid for some integers $m.n \geq 2$ then*

$$pn(G_{m,n}) = \left\lceil \frac{2}{3} \left(m + n - \sqrt{m^2 - mn + n^2} \right) \right\rceil .$$

2.3.4 Higher Dimensional Grids

The **d-dimensional** $n \times n \times \cdots \times n$ **grid** is the Cartesian product of d n-paths and is denoted G_n^d. Let the vertex set be $\{(a_1.a_2. \ldots .a_d) : 1 \leq a_i \leq n.i = 1.2. \ldots .d\}$. We will call any vertex $(a_1.a_2. \ldots .a_d)$ in G_n^d a **corner** if $a_i = 1$ or n for all $i = 1. \ldots .d$. and specifically. we will call the vertex $(1.1. \ldots .1)$ the 1-**corner** of G_n^d.

Lemma 2.3.4.1 *For all integers $d.n$ and k such that $d.n \geq 2$ and $k \leq n - 1$. the number of vertices distance k from a particular corner of the d-dimensional grid G_n^d is*

$$\left(\begin{array}{c} k + d - 1 \\ d - 1 \end{array} \right) .$$

Proof: Since the grid G_n^d is symmetric. the number of vertices at distance k from each corner is the same. Therefore. we will determine the number of vertices at distance k from the 1-corner of G_n^d.

Let $(a_1.a_2. \ldots .a_d)$ be a vertex at distance k from the 1-corner. Then $\sum_{i=1}^{m} |a_i - 1| = k$ and $\sum_{i=1}^{m} a_i = k + d$. So. there are as many vertices at distance k from the 1-corner as

there are positive integer solutions of $\sum_{i=1}^{m} x_i = k+d$. Hence. there are $\begin{pmatrix} k+d-1 \\ d-1 \end{pmatrix}$

vertices at distance k from the 1-corner of G_n^d for $d \geq 2$. $\qquad\square$

Lemma 2.3.4.2 *Let G_n^d be the d-dimensional grid for some integers $d \geq 2$ and $n \geq 3$. If k is an integer such that $k \leq n/2 - 1$ then no vertex in G_n^d is distance at most k from more than one corner.*

Proof: Without loss of generality, suppose $b = (b_1, b_2, \ldots, b_d)$ is a vertex at distance k from the 1-corner of G_n^d for some $k \leq n/2 - 1$. Note that $1 \leq b_i \leq n$ for all $i = 1.2.\ldots.d$. Since $k = \sum_{i=1}^{d}(b_i - 1)$, then it must be the case that $b_i \leq k+1 \leq n/2$ for all $i = 1.2.\ldots,d$.

Now suppose the same vertex b is distance at most k from another corner of G_n^d. Let $c = (c_1.c_2.\ldots.c_n)$ be that corner. Since. $d(b.c) = \sum_{i=1}^{d} |c_i - b_i| \leq n/2 - 1$ and $1 \leq b_i \leq n$. then $b_i > n/2 + 1$ for all i such that $c_i = n$. We see that this can never be the case. and b is distance at most k from only one corner. $\qquad\square$

Let H_t be the subgraph obtained from G_n^d by removing all the vertices in G_n^d which are distance less than t from some corner. For example. H_1 is the subgraph obtained by removing all the corners from G_n^d. and H_1 contains exactly $(n^d - 2^d)$ vertices.

The number of vertices at distance less than t from one corner for $t \leq n$ is

$$\sum_{k=0}^{t-1} \begin{pmatrix} k+d-1 \\ d-1 \end{pmatrix} = \begin{pmatrix} t+d-1 \\ d \end{pmatrix}.$$

By Lemma 2.3.4.2. for all $t \leq n/2$ no vertex is distance less than t from more than one corner. Then H_t has exactly $n^d - 2^d \begin{pmatrix} t+d-1 \\ d \end{pmatrix}$ vertices. Note that for any grid G_n^d. where $n = 2$. all the vertices are corners and H_1 is empty.

Lemma 2.3.4.3 *For any integers $t.n$ and d such that $1 \leq t \leq n/2. n \geq 3$ and $d \geq 2$. let a and b be two vertices in the subgraph H_t of G_n^d. Then the distance from a to b in G_n^d is at most $dn - d - 2t$.*

Proof: Suppose a and b are two vertices in H_t where $a = (a_1, a_2, \ldots, a_d)$ and $b = (b_1, b_2, \ldots, b_d)$. We will assume that $b_i \geq a_i$ for all $i = 1, 2, \ldots, d$. (If this is not the case we could simply relabel the grid.) Then $d(a, b) = \sum_{i=1}^{d} |b_i - a_i| = \sum_{i=1}^{d}(b_i - a_i)$. Since a is in H_t, then a is distance at least t from the 1-corner. Hence, $\sum_{i=1}^{d}(a_i - 1) \geq t$. Similarly, b is distance at least t from the corner $(n, n, \ldots, n)$. So, $\sum_{i=1}^{d}(n - b_i) \geq t$. Therefore,

$$\sum_{i=1}^{d}(b_i - a_i) = -\left(\sum_{i=1}^{d}(a_i - 1)\right) - d - \left(\sum_{i=1}^{d}(n - b_i)\right) + dn \leq -2t - d + dn.$$

Hence, $d(a, b) \leq dn - d - 2t$. $\qquad\qquad\square$

In fact, this upper bound is actually attained. Suppose $1 \leq t \leq n/2$ and $n \geq 3$. Let $a = (t, 2, 1, 1, \ldots, 1)$ and let $c = (c_1, c_2, \ldots, c_d)$ be any corner in G_n^d. Since $c_2 = 1$ or n,

$$d(a, c) = \sum_{i=1}^{d} |c_i - a_i| \geq |c_1 - t| + |c_2 - 2| \geq |c_1 - t| + 1.$$

If $c_1 = 1$ then $|c_1 - t| + 1 = t$ and $d(a, c) \geq t$. If $c_1 = n$ then $|c_1 - t| + 1 = |n - t| + 1 \geq t + 1$ since $n \geq 2t$. Therefore, a is in H_t. Let $b = (n - t + 1, n - 1, n, n, \ldots, n)$. Then, similarly, $d(b, c) \geq |c_1 - n + t - 1| + |c_2 - n + 1| \geq |c_1 - n + t - 1| + 1$. If $c_1 = n$ then $|c_1 - n + t - 1| + 1 = t$. If $c_1 = 1$ then $|c_1 - n + t - 1| + 1 = |n - t| + 1 \geq t + 1$. Hence, b is also in H_t. Finally,

$$\begin{aligned}
d(a, b) &= (n - t + 1 - t) + (n - 1 - 2) + (d - 2)(n - 1) \\
&= 2n - 2t - 2 + dn - 2n - d + 2 \\
&= dn - d - 2t.
\end{aligned}$$

So, there are two vertices in H_t which are distance $dn - d - 2t$ apart.

Theorem 2.3.4.4 *If n, d and t are integers such that $n, d \geq 2$ and $0 \leq t \leq n/2$, then*

$$pn(G_n^d) \geq \frac{n^d - 2^d \dbinom{t + d - 1}{d}}{dn - d - 2t + 1}.$$

Proof: It has been shown that for any $1 \leq t \leq n/2$ and $n \geq 3$. the subgraph H_t of G_n^d has exactly $n^d - 2^d \begin{pmatrix} t+d-1 \\ d \end{pmatrix}$ vertices. In Lemma 2.3.4.3. it was shown that the distance between any two vertices in H_t is at most $dn - d - 2t$. Therefore. no isometric path in G_n^d can contain more than $dn - d - 2t + 1$ of the vertices contained in the subgraph H_t. Therefore.

$$pn(G_n^d) \geq \frac{n^d - 2^d \begin{pmatrix} t+d-1 \\ d \end{pmatrix}}{dn - d - 2t + 1}.$$

If $t = 0$ and $n \geq 2$. then the result is exactly the result of Theorem 2.3.1.1. $\square$

If $d = 3$ then $pn(G_n^3) \geq \frac{n^3 - 4/3(t^3 + 3t^2 + 2t)}{3n - 2t - 2}$. By substituting different values of t into this inequality. we obtain lower bounds for some small $n \times n \times n$ grids shown in Table 2.1.

n	2	3	4	5	6	7	8	9	10	11	12	13
$pn(G_n^3) \geq$	2	4	7	11	16	21	27	35	42	51	61	71

n	14	15	16	17	18	19	20	21	22	23	24	25
$pn(G_n^3) \geq$	82	95	107	122	136	152	167	185	202	222	241	262

Table 2.1: Lower Bounds on $pn(G_n^3)$ for small values of n.

If we let $n = 2^k$ and $t = 2^{k-1} - 1$ then the lower bound of $pn(G_n^3)$ is

$$\frac{5(4^{k-1}) + 1}{3}.$$

This always gives an integer since $5(4^{k-1}) + 1 \equiv 0 \pmod 3$. Therefore. to achieve this lower bound every path in G_n^3 must use $2n - 2t - 1$ vertices of H_t and there can be no intersection of paths in H_t. The lower bounds for $pn(G_n^3)$ where $n = 2^k$ are given in Table 2.2.

n	2	4	8	16	32	64	128	256	512
$pn(G_n^3) \geq$	2	7	27	107	427	1707	6827	27307	109227

Table 2.2: Lower Bounds on G_n^3 for $n = 2^k$.

For the small values of n we have considered. $\frac{n^3-4/3(t^3+3t^2+2t)}{3n-2t-2}$ has been maximized at either $t = \lceil \frac{n}{2} \rceil - 1$ or $t = \lfloor \frac{n}{2} \rfloor - 1$. However, for larger values of n these values of t do not give the bests lower bound for precinct number. We can see this by maximizing the function $f(t) = \frac{n^3-4/3(t^3+3t^2+2t)}{3n-2t-2}$ over all values of t where $0 \leq t \leq t/2$. First, we find the derivative of $f(t)$:

$$f'(t) = \frac{(-4t^2 - 8t - \frac{8}{3})(3n - 2t - 2) + 2(n^3 - \frac{4}{3}t^3 - 4t^2 - \frac{8}{3})}{(3n - 2t - 2)^2}.$$

Note that

$$f'(0) = \frac{2n^3 - 8n + \frac{16}{3}}{(3n - 2)^2}$$

and

$$f'\left(\frac{n}{2}\right) = \frac{-\frac{1}{3}n^3 - 8n^2 + \frac{16}{3}}{(2n - 2)^2}.$$

Therefore. $f'(0) > 0$ and $f'(\frac{n}{2}) < 0$ for all $n \geq 3$. So. there is exactly one root of $f'(t)$ which lies between 0 and $\frac{n}{2}$. In fact. $f'(t) = 0$ when $t \approx .4574n$. Hence. if $t = .4572n$ is an integer. this will give the best lower bound on precinct number.

We now turn our attention toward establishing an upper bound on $pn(G_n^3)$. For $n = 2$ and $n = 4$. the $n \times n \times n$ grid can be covered with 2 and 7 isometric paths. respectively (see Figure 2.9). Therefore, $pn(G_2^3) = 2$ and $pn(G_4^3) = 7$. We can improve the lower bound for $n = 3$ by simply taking into account that all grids are bipartite graphs. If we partition the vertices of G_3^3 into a set of 14 "black" vertices and a set of 13 "white" vertices. then any path in G_3^3 must alternate between black and white vertices. If a path contains four white vertices then it must contain at least three black vertices as well. However. every isometric path with seven vertices begins and

ends at a corner. all of which are black. Hence. no isometric path contains more than three white vertices. Therefore. $pn(G_3^3) \geq \lceil 13/3 \rceil = 5$. In fact. $pn(G_3^3) = 5$ since G_3^3 can be covered with exactly five isometric paths. as in Figure 2.9.
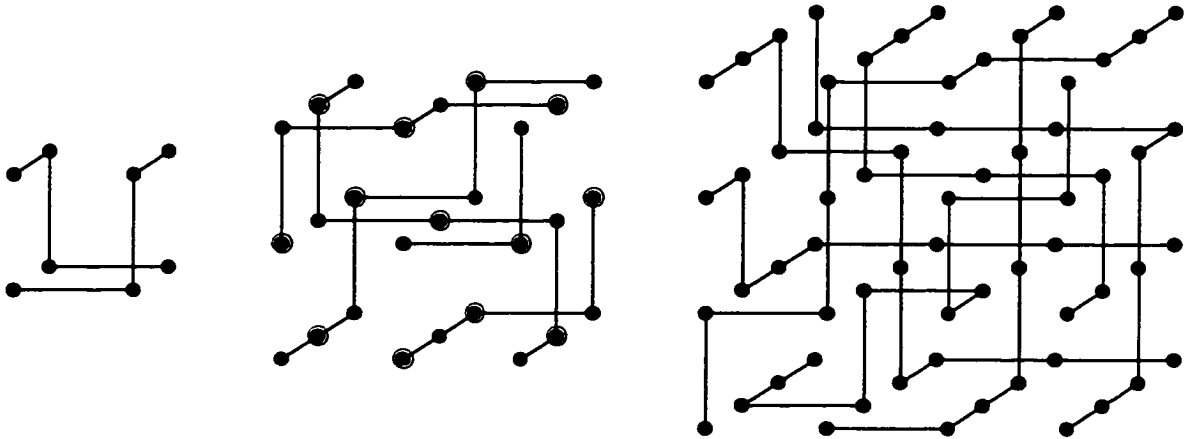
Figure 2.9: Covers of G_2^3. G_3^3, and G_4^3 with 2, 5, and 7 isometric paths. respectively.
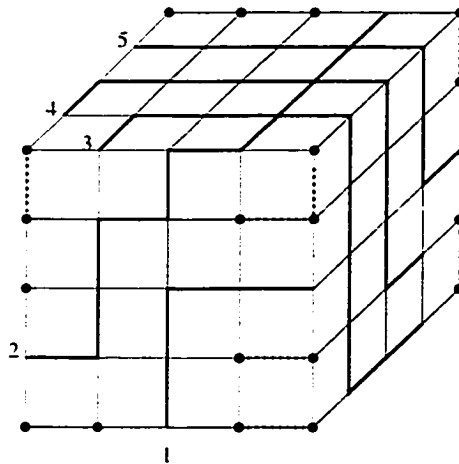
Figure 2.10: A covering of the outer "walls" of G_5^3

The grid G_5^3 can be covered with 12 isometric paths. Consider Figure 2.9. This shows three of the outer "walls" of the $5 \times 5 \times 5$ cube. If we sit the $4 \times 4 \times 4$ cube inside of these walls and connect the appropriate vertices we obtain the entire cube G_5^3. The

paths which cover the grid G_4^3 in Figure 2.9 can be extended to include the vertices of G_5^3 which are shown in Figure 2.10. Some extensions are to single vertices. others are to paths. indicated by dotted lines. The remaining vertices can be covered with 5 isometric paths. giving a total of 12 isometric paths. Hence. $11 \leq pn(G_5^3) \leq 12$.

The next theorem shows that if we have a way of coving the cube G_n^3 then we can use this as a guide to find a cover for G_{mn}^3 for any $m \geq 1$. While this may not be the optimal cover. it does provide an upper bound for $pn(G_{mn}^3)$.

Theorem 2.3.4.5 *If G_n^3 can be covered by p isometric paths then G_{mn}^3 can be covered by pm^2 isometric paths.*

Proof: Consider the cube G_n^3 and suppose it can be covered with p isometric paths. A vertex $x = (x_i, x_j, x_k)$ in G_n^3 corresponds to an $m \times m \times m$ subcube of G_{mn}^3 by $(x_i, x_j, x_k) \rightarrow \{(mx_i + \alpha, mx_j + \beta, mx_3 + \delta) : 0 \leq \alpha, \beta, \delta \leq m - 1\}$.

Let $P = \{\overline{x^0}, \overline{x^1}, \ldots, \overline{x^n}\}$ be one of the p isometric paths used to cover G_n^3. and assign a direction to the path from $\overline{x_0}$ to $\overline{x_n}$. We can assume that for any pair of vertices x^s and x^t on P such that $s \leq t$. $x_i^s \leq x_i^t$, $x_j^s \leq x_j^t$ and $x_k^s \leq x_k^t$. If this is not the case. then we can simply change the orientation of the coordinates of G_n^3. Therefore. for each $0 \leq s \leq n - 1$. x^{s+1} is one of the following:

1. $(x_i^s + 1, x_j^s, x_k^s)$

2. $(x_i^s, x_j^s + 1, x_k^s)$

3. $(x_i^s, x_j^s, x_k^s + 1)$

Now we will construct a set of m^2 paths in G_{mn}^3 associated with P which cover all the subcubes associated with the vertices $\{\overline{x^0}, \overline{x^1}, \ldots, \overline{x^n}\}$. First. suppose that the vertex $\overline{x^1}$ is of the form $(x_i^1, x_j^1, x_k^1) = (x_i^0 + 1, x_j^0, x_k^0)$. Then for each $0 \leq \beta, \delta \leq m - 1$. the vertices $\{(mx_i^0 + \ell, mx_j^0 + \beta, mx_k^0 + \delta) : 0 \leq \ell \leq m - 1\}$ form an isometric path in G_{mn}^3. Therefore. there is a set of m^2 isometric paths in G_{mn}^3 which cover all the vertices of the subcube associated with $\overline{x^0}$ and have the set $\{(x_i^0 + m - 1, x_j^0, x_k^0\}$ as end vertices.

There are similar sets of paths in G_{mn} which cover the same vertices and have either $\{(mx_i^0 + \beta.mx_j^0 + m - 1.mx_k^0 + \delta) : 0 \leq \beta, \delta \leq m-1\}$ or $\{(mx_i^0 + \beta, mx_j^0 + \delta.mx_k^0 + m - 1) : 0 \leq \beta, \delta \leq m - 1\}$ as end vertices.

Without loss of generality, assume that $x_i^{r+1} = x_i^r + 1$ for some $0 \leq r \leq n - 1$. Inductively, assume that there is a set of m^2 isometric paths covering all the subcubes associated with $\{\overline{x^0}, \overline{x^1}, \ldots, \overline{x^r}\}$ such that $\{(mx_i^r + m - 1.mx_j^r + \beta.mx_k^r + \delta) : 0 \leq \beta, \delta \leq m - 1\}$ is the set of end vertices of the paths. Also assume that for each path, the coordinates of the vertices are increasing along the path.

Let $P_{\beta\delta}$ be the path ending at $(mx_i^r + m - 1.mx_j^r + \beta.mx_k^r + \delta)$ for each $0 \leq \beta, \delta \leq m - 1$. Let the next vertex on $P_{\beta\delta}$ be $(mx_i^r + m, mx_j^r + \beta.mx_k^r + \delta) = (mx_i^{r+1}, mx_j^{r+1}, mx_k^{r+1})$.

If $x_i^{r+2} = x_i^{r+1} + 1$ or $\overline{x^{r+2}} = \overline{x^n}$ then let the next section of the path $P_{\beta\delta}$ include all the vertices $\{(mx_i^{r+1} + \ell.mx_j^{r+1} + \beta.mx_k^{r+1} + \delta) : 0 \leq \ell \leq m - 1\}$. The vertices of the subcube corresponding to $\overline{x^{r+1}}$ are now covered and the end vertices of the paths are $\{(mx_i^{r+1} + m - 1.mx_j^{r+1} + \beta.mx_k^{r+1} + \delta) : 0 \leq \beta, \delta \leq m - 1\}\}$.

If $x_j^{r+2} = x_j^{r+1} + 1$ then let $P_{\beta\delta}$ include the vertices $\{(mx_i^{r+1} + \ell.mx_j^{r+1} + \beta.ms_k^{r+1} + \delta) : 0 \leq \ell \leq m - \beta - 1\}$, and then continue on to include the vertices $\{mx_i^{r+1} + m - \beta - 1.mx_j^r + \ell.ms_k^r + \delta : 0 \leq \ell \leq m - 1\}$. The vertices of the subcube have been covered and the ends of the paths are $\{(x_i^{r+1} + m - \beta - 1.mx_j^r + m - 1.mx_k^r + \delta : 0 \leq \ell \leq m - 1) : 0 \leq \beta, \delta \leq m - 1\} = \{(mx_i^{r+1} + \beta.mx_j^{r+1} + m - 1.mx_k^{r+1} + \delta\} : 0 \leq \beta, \delta \leq m - 1\}$.

Similarly, if $x_k^{r+2} = x_k^{r+1} + 1$ then the paths can be extended to cover the subcube associated with $\overline{x^{r+1}}$ so that the the end vertices of the paths are $\{(mx_i^{r+1} + \beta.mx_j^{r+1} + \delta.mx_k^{r+1} + m - 1)\} : 0 \leq \beta, \delta \leq m - 1\}$.

In all cases, the paths have been extended in such a way that no coordinate decreases. Therefore, each path $P_{\beta\delta}$ is isometric. Hence, by induction, we have a set of m^2 isometric paths which cover all the subcubes associated with P. Therefore, if a set of p paths cover the vertices of G_n^3, then a set of pm^2 paths cover all the vertices of G_{mn}^3. $\square$

This result can be extended to d-dimensional graphs for all $d \geq 3$.

Corollary 2.3.4.6 *If G_n^d can be covered by p isometric paths then G_{mn}^d can be covered with pm^{d-1} isometric paths.*

If we let a vertex $x = (x_1, x_2, \ldots, x_d)$ in G_n^d correspond to a subgrid of G_{mn}^d by $(x_1, x_2, \ldots, x_d) \rightarrow \{(mx_1 + \alpha_1, mx_2 + \alpha_2, \ldots, mx_d + \alpha_d) : 0 \leq \alpha_i \leq m - 1, i = 1, 2, \ldots, d\}$, then the remainder of the proof is similar to that of Theorem 2.3.4.5.

Since we have obtained upper bounds on $pn(G_n^3)$ for $n = 2, 3, 4, 5$, we can use Theorem 2.3.4.5 to obtain upper bounds for $n = 2m, 3m, 4m$ or $5m$ for any $m \geq 1$. We can obtain upper bounds on other values of n by using the following lemma:

Lemma 2.3.4.7 *For any integer $n \geq 2$, $pn(G_{n-1}^3) \leq pn(G_n^3)$*

Proof: Obviously, G_{n-1}^3 is an isometric subgraph of G_n^3. Therefore, by Theorem 2.3.1.3, $pn(G_{n-1}^3) \leq pn(G_n^3)$. □

Since we have an upper bound on $pn(G_n^3)$ for each of $n = 2, 3, 4, 5$, we can use Theorem 2.3.4.5 and Lemma 2.3.4.7 to obtain an upper bound on $pn(G_n^3)$ for all $n \geq 2$. This is done in Table 2.3 for small values of n.

n	2	3	4	5	6	7	8	9	10	11	12	13
$pn(G_n^3) \leq$	2	5	7	12	18	28	28	45	48	63	63	98

n	14	15	16	17	18	19	20	21	22	23	24	25
$pn(G_n^3) \leq$	98	108	112	162	162	175	175	245	252	252	252	300

Table 2.3: Upper Bounds on $pn(G_n^3)$ for small values of n.

2.3.5 Other Beats

We have only investigated one possibility for assigning beats: the one in which beats are isometric paths. We see that in many cases the precinct number obtained from

using this beat is very different from the cop number of a graph. For example. we can make the precinct number of a tree arbitrarily large by adding leaves. while the cop number of a tree is always one. If we wish to establish reasonable upper bounds on the cop numbers of particular graphs using this technique. we need to investigate the use of other types of beats. Some obvious choices are those graphs which are absolute retracts. Let $\mathcal{F}$ represent a particular family in the set of absolute retracts. such as trees or complete graphs. Let the $\mathcal{F}$-**precinct number** of G be the minimum number of cops required to capture the robber if each cop is restricted to an isometric subgraph of G which is also in $\mathcal{F}$. Denote the $\mathcal{F}$-precinct number of G. $\mathcal{F}pn(G)$.

Theorem 2.3.5.1 *Let G be any graph and let $\mathcal{T}$ and $\mathcal{K}$ be the sets of all trees and complete graphs. respectively.*

1) If m is the minimum number of isometric trees required to cover the vertices of G. then $\mathcal{T}pn(G) = m$.

2) If m is the minimum number of complete graphs required to cover the vertices of G. then $\mathcal{K}pn(G) = m$.

Proof: Since a minimum of one cop is required on each beat. then $m \leq \mathcal{F}pn(G)$ for $\mathcal{F} = \mathcal{T}$ and $\mathcal{F} = \mathcal{K}$. If a graph H is a retract of G then $c(H)$ cops moving on H can. after a finite number of moves. ensure the robber's capture if he ever moves onto H. Since $c(T) = 1$ for any tree. T and $c(K_n) = 1$ for any complete graph K_n. then $\mathcal{T}pn(G) \leq m$ and $\mathcal{K}pn(G) \leq m$. □

By chosing copwin graphs such as paths. trees. and complete graphs which are isometric subgraphs of G. we have guarenteed a retraction map of G onto each of the subgraphs. and therefore. a strategy for keeping the robber off of that subgraph. Depending on the graph G. there may be other copwin graphs which are retracts in G which are not absolute retracts.

Suppose H_1 is a retract of G such that H_1 is a copwin graph. Once a cop catches the image of the robber under the retraction of G onto H_1, the robber is restricted to moving in $G \setminus H_1$ if he is to avoid capture. Obviously. if we find another copwin

graph, H_2, that is a retract of G, one cop can, after a finite number of moves. prevent the robber from moving onto H_2. However, if H_2 is a retract of $G \setminus H_1$, then one cop can catch the image of the robber under the retraction map of $G \setminus H_1$ onto H_2. This is due to the fact that the robber can not move out of $G \setminus H_1$ without being captured by the cop in H_1. In fact, if H_2 is a retract of any induced subgraph of G containing $G \setminus H_1$, a single cop can, after a finite number of moves, capture the robber if he moves onto H_2. Therefore, we have the following result:

Lemma 2.3.5.2 *Suppose* $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$ *is a set of subgraphs of a graph* G *such that* H_1 *is a retract of* G *and* H_{i+1} *is a retract of an induced subgraph of* G *containing* $G \setminus \{H_1, H_2, \dots, H_i\}$ *for all* $i = 1, 2, \dots, n - 1$. *If the subgraphs in* $\mathcal{H}$ *together cover the vertices of* G *and* H_i *a is copwin for all* $i = 1, 2, \dots, n$. *then* $c(G) \leq n$.

So, the problem is to cover a graph with copwin subgraphs such that a single cop on each subgraph can prevent the robber from moving onto his subgraph.

2.3.6   Problems

We were able to establish lower bounds for the precinct numbers of d-dimensional grids and a means of finding upper bounds if we have small examples to begin with. While finding the precinct number of G_n^d for all values of n seems to be a difficult problem. the asymptotic problem may be more reasonable. We saw that for $d = 3$ the lower bound on $pn(G_n^3)$ was best when $t \approx .4574n$. If we let n approach infinity we have $pn(G_n^3) \geq .4183n^2$.

Problem 2.3.6.1 *As n approaches infinity. does the precinct number of G_n^d for $d \geq 3$ approach the maximum of the lower bound given in Theorem 2.3.4.4 over all values of t?*

2.4 Dragnet

We wish to extend the idea of each cop being assigned to a particular beat. but perhaps give them some freedom to move off of their beat. In this section a cop will be allowed to move off his assigned beat if he can immediately apprehend the robber. Hence. the cop's move from his beat will be the final move of the game.

Again we will assume that each beat is an isometric path and that more than one cop can walk part or all of the same beat. Call the minimum number of cops required to apprehend the robber in this version of the game the **dragnet** number of the graph and denote this by $dn(G)$. Hence. the dragnet number of G is at least the cop number of G.

We will first consider the set of all graphs that contain an isometric path whose vertices form a dominating set. We show that the dragnet number of such graphs is at most four and determine the upper bound on the dragnet number of various classes of graphs in this set. We also characterize all those graphs with a dragnet number of one. We then look at graphs in which the vertices of no less than d isometric paths dominate the graph. We show that at most $5d - 1$ cops are required to win. and we examine conditions that will allow us to decrease the number of cops to $4d$ and d. respectively. Finally. we pose some problems.

2.4.1 One Isometric Path Dominates

Suppose we are given a graph G and an isometric path $P = \{x_0. x_1. .... x_n\}$ such that P is a dominating set. Call P an **isometric dominating** path. The path P will serve as the beat for the cops. We wish to partition the vertices off the path according to their neighbours on the path. Vertices off the path will be placed in the same set if their "first" neighbour on the path is the same. Specifically. for all $i = 0. 1. .... n$ let

$$G_i = \{v : i = \min_{j=0.1....n} (v \sim x_j). v \notin P\}.$$
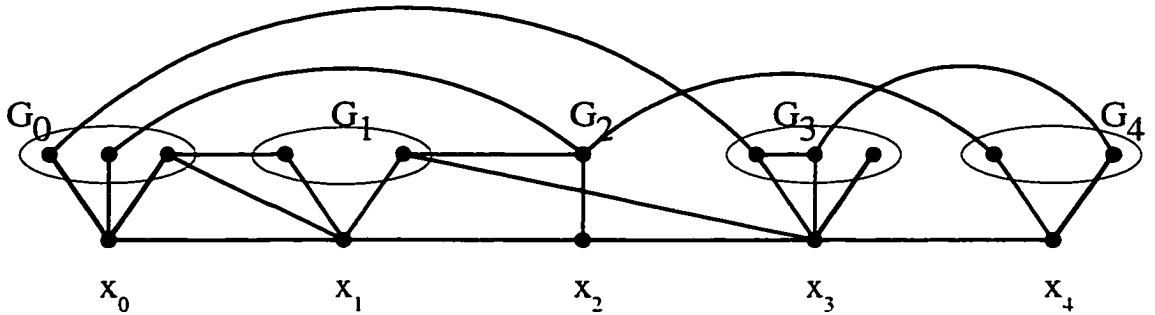
Figure 2.11: An example of a graph with an isometric dominating path and the resulting partition of the vertices off the path.

Now, define a function, r, which maps the set V onto the set $\{0, 1, \ldots, n+1\}$ as follows:

$$r(v) = \begin{cases} i, & \text{if } v \in G_i \\ i+1, & \text{if } v = x_i \end{cases}$$

For each edge xy such that x is not in P we define the **stretch** of xy, denoted $st(xy)$ as follows:

$$st(xy) = r(y) - r(x).$$

For example, in Figure 2.12 $st(ab) = r(b) - r(a) = 2 - 0 = 2$ and $st(ax_2) = r(x_2) - r(a) = 3 - 0 = 3$.
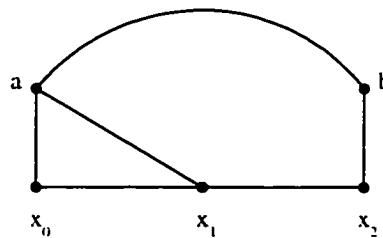
Figure 2.12: An example of a graph in which $st(ab) = 2$ and $st(ax_2) = 3$.

Suppose we have a path $Q = \{v_0, v_1, \ldots, v_m\}$ for some $m \geq 1$ such that Q does not intersect P except possibly at the final vertex, v_m. We define the stretch of the path Q to be

$$st(Q) = \sum_{i=1}^{m} st(v_{i-1}, v_i) = r(v_m) - r(v_0).$$

Because P is isometric, the stretch of the path is bounded above. as seen in the following lemma:

Lemma 2.4.1.1 *If G contains a path $Q = \{v_0, v_1, \ldots, v_m\}$ for some $m \geq 1$ such that $\{v_0, v_1, \ldots, v_{m-1}\} \subseteq V \setminus P$ then $|st(Q)| \leq m + 2$.*

Proof: Let Q be a path as described above.

Case 1: Suppose v_m is not on the path P. Then $v_0 \in G_i$ and $v_m \in G_j$ for some $i, j \in \{0, 1, \ldots, n\}$. Hence, $|st(Q)| = |r(v_m) - r(v_0)| = |j - i|$. Now consider the path (x_i, Q, x_j). This path has length $m + 2$. Therefore, $d(x_i, x_j) = |j - i| \leq m + 2$ and $|st(Q)| \leq m + 2$.

Case 2: If v_m is in P then $v_m = x_k$ for some $k \in \{0, 1, \ldots, n\}$. Hence, $r(v_m) = k + 1$. Furthermore, v_0 is in G_i for some i and $r(v_0) = i$. Hence, $|st(Q)| = |k + 1 - i| \leq |k - i| + 1$. The path (x_i, Q) has length $m + 1$. Hence, $d(x_i, x_k) = |k - i| \leq m + 1$ and $|st(Q)| \leq m + 2$. $\square$

Corollary 2.4.1.2 *Let G be a graph with dominating path $P = \{x_0, x_1, \ldots, x_n\}$. If $u \in G_i$ for some $i = 1, \ldots, n$ then*

a) if $u \sim v$ for some $v \notin P$ then there is some j such that $v \in G_j$ and $|j - i| \leq 3$.

b) if $u \sim x_j$ then $i \leq j \leq i + 2$.

These are merely specific cases of Lemma 2.4.1.1 where $m = 1$.

As the game is played, each cop moves along the path $P = \{x_0, x_1, \ldots, x_n\}$. We will say that a cop moves **up the path** if he moves from some vertex x_i to vertex x_{i+1}. Similarly, he moves **down the path** if he moves from x_i to x_{i-1}. We also wish to define $A_i = \bigcup_{j=i+1}^{n}(G_j \cup x_j)$ to be the set of all vertices **ahead** of x_i and $B_i = \bigcup_{j=0}^{i-1}(G_j \cup x_j)$ to be the set of vertices **behind** x_i. Hence, if a cop occupies vertex x_i and the robber occupies some vertex in A_i (B_i) then the robber is ahead of (behind) the cop. Note that there are no vertices ahead of x_n or behind x_0.

We now wish to consider what happens if we have c cops which initially occupy the first c vertices of the isometric path and then either move up the path on their

turn or move to apprehend the robber whenever possible. Obviously. the robber will initially occupy some vertex ahead of the cops. However. he cannot stay ahead of the cops forever because he will simply run out of vertices. Hence. his only hope of avoiding capture is to move behind the cops at some point. If the cops wish to win using this simple strategy. then it must be the case that the robber's move behind them is to a vertex adjacent to at least one of them. This is proved in the following lemma:

Lemma 2.4.1.3 *Let G be a graph with isometric dominating path $P = \{x_0. x_1. \ldots . x_n\}$. Suppose that for some $c \geq 1$ and all $k = 0, \ldots . n - c$. any vertex $w \in B_{k+1}$ with a neighbour in A_{k+c-1} also has a neighbour in $\{x_{k+1}. x_{k+2}. \ldots . x_{k+c}\}$. Then $dn(G) \leq c$.*

Proof: Let $P = \{x_0. x_1, \ldots, x_n\}$ be an isometric dominating path in G and assume that G satisfies the hypothesis of the lemma. Place the cops on vertices $\{x_0. x_1. \ldots . x_{c-1}\}$ of the path. To avoid being caught. the robber must choose a vertex which is not adjacent to any of the cops' positions. Hence. the robber will choose a vertex which is ahead of all the cops. He. therefore. chooses a vertex in A_{c-1}.

Now suppose the cops' strategy is for each cop to move up the path on their turn unless the robber is adjacent to the current position of one of them. In that case. the cop in question will move onto the robber. Due to this strategy. the robber cannot always move onto a vertex ahead of the cops. We know this because there are no vertices ahead of x_n. Suppose that for some $k \geq 0$. the robbers' first k moves are ahead of the cops. After their k^{th} moves the cops occupy the vertices $\{x_k. \ldots . x_{k+c-1}\}$ and the robber occupies some vertex in A_{k+c-1}. The cops now move up the path to $\{x_{k+1}. \ldots . x_{k+c}\}$. Suppose the robber now moves behind the cops. then he moves to some vertex $w \in B_{k+1}$. However. w must also be adjacent to one of $\{x_{k+1}. x_{k+2}. \ldots . x_{k+c}\}$. Hence. any move behind the cops will result in the robber's immediate capture. Therefore. the cops' strategy will ensure the apprehension of the robber and $dn(G) \leq c$.   □

Theorem 2.4.1.4 *If G is a graph containing an isometric dominating path then $dn(G) \leq 4$.*

Proof: Consider any vertex $w \in B_{k+1}$ for some $0 \leq k \leq n - 4$. By Corollary 2.4.1.2. w has no neighbours in A_{k+3}. Hence. by Lemma 2.4.1.3. $dn(G) \leq 4$. □
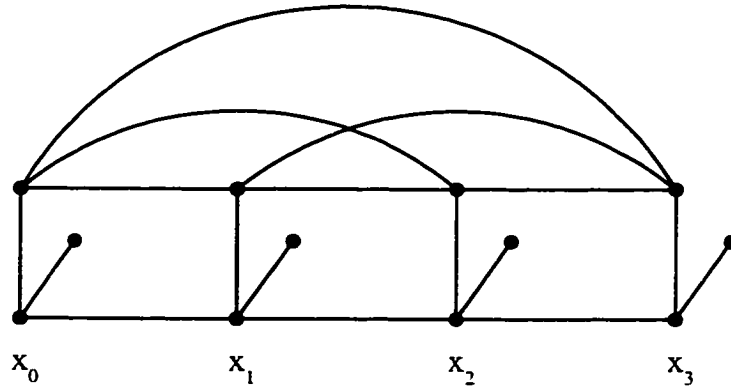
Figure 2.13: An example of a graph in which $dn(G) = 4$.

Note that the number of cops required is sharp. Consider the graph in Figure 2.13. There are four vertices. one in each G_i for $i = 0.1.2.3$. such that each has exactly one neighbour on the path $P = \{x_0, x_1. x_2. x_3\}$. and together they form an induced K_4. If there are at most three cops on P then the robber can always avoid capture by moving on the above mentioned K_4. Furthermore. P is the only isometric dominating path in the graph. since each leaf must be dominated.

In this example there was only one choice of an isometric dominating path in the graph. However. there are graphs which contain more than one isometric dominating path. and the number of cops required on a single beat may vary depending on the choice of beat. For example. in Figure 2.14 we have a graph with isometric dominating path $\{x_0. x_1. x_2. x_3\}$. If the cops move on this beat then four cops are required due to the K_4 formed by the vertices $a. b. c$ and d. But. if we take the same graph and use $\{x_0. a. d. x_3\}$ as our beat. only three cops are required. (Place one cop one each of $x_0. a$ and x_3.)

A **comparability graph** is one in which the edges can be assigned a transitive orientation. It was shown in [23] that a graph is a comparability graph if and only if each odd cycle has at least one triangulating chord. This gives the following result:
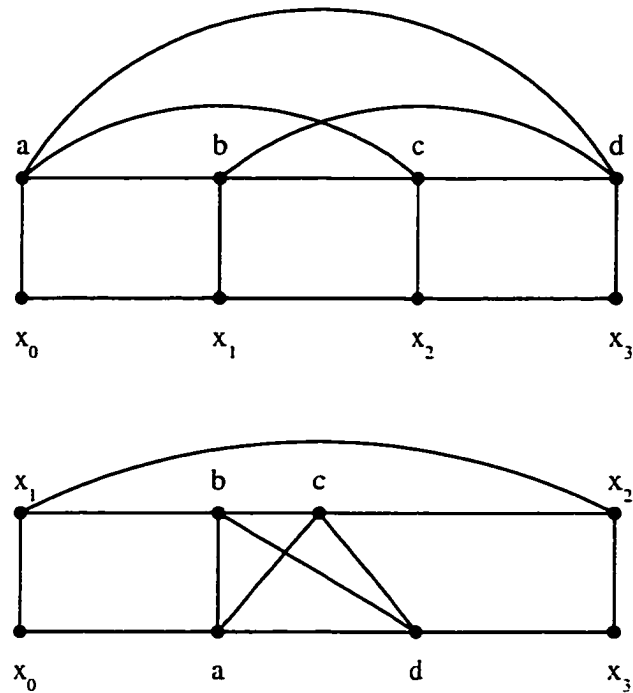
Figure 2.14: An example of a graph with two different isometric dominating paths.

Lemma 2.4.1.5 *Let G be a comparability graph with an isometric dominating path $P = \{x_0, x_1, \ldots, x_n\}$. If there is an edge uv such that $u \in G_i$ and $v \in G_{i+2}$ for some $i = 0, 1, \ldots, n-2$, then u is adjacent to either x_{i+1} or x_{i+2}.*

Proof: Let G be a comparability graph with a dominating path $P = \{x_0, x_1, \ldots, x_n\}$, and suppose there some edge uv where $u \in G_i$ and $v \in G_{i+2}$. Then $\{i, x, y, i+2, i+1\}$ forms a cycle of length five. Since G is a comparability graph then this cycle must have a triangulating chord. Therefore, either $u \sim x_{i+1}$ or $u \sim x_{i+2}$. $\square$

We can now show the following:

Theorem 2.4.1.6 *If G is a comparability graph with an isometric dominating path then $dn(G) \leq 3$.*

Proof: Let $P = \{x_0, x_1, \ldots, x_n\}$ be an isometric dominating path in G. Place the cops on $\{x_0, x_1, x_2\}$. To avoid capture the robber must choose a vertex in A_2. Suppose

the cops' strategy is for each of them to move up the path if the robber is ahead of them and to move onto the robber if he is ever adjacent to one of them. Obviously. the robber cannot always stay ahead of the cops because he will simply run out of vertices. At some point he will be forced onto a vertex adjacent to one of the cops or he will have to move behind all of them. We will consider what happens the first time the robber moves to a vertex which is not ahead of the cops.

Suppose that for some $k \geq 1$, the robber is ahead of the cops for his first $k - 1$ moves, but does not stay ahead of them on his k^{th} move. Hence. after their $k - 1^{st}$ move the cops occupy the vertices $\{x_{k-1}, x_k, x_{k+1}\}$ and the robber occupies a vertex in A_{k+1}. On their. k^{th} move the cops move to $\{x_k, x_{k+1}, x_{k+2}\}$. Now assume that the robber can avoid moving onto a vertex adjacent to one of the cops. Then the robber must move to a vertex $w \in B_k \setminus \{x_{k-1}\}$.

Since $v \in A_{k+1}$ and $w \in B_k \setminus \{x_{k-1}\}$ then by the definition. $r(v) \geq k + 2$ and $r(w) \leq k - 1$. By Corollary 2.4.1.2. $|st(vw)| = |r(v) - r(w)| \leq 3$. Hence. $r(v) = k + 2$ and $r(w) = k - 1$. This implies $v \in G_{k+2}$ and $w \in G_{k-1}$. We may assume that $w \perp x_k$ and $w \perp x_{k+1}$. since the robber could be immediately captured otherwise.

Now move the cops to the vertices $\{x_{k-1}, x_k, x_{k+2}\}$ and suppose the robber moves to some vertex x. By Lemma 2.4.1.5. x is not in G_{k+1} since a 5-cycle with no triangulating chord would otherwise result. Also assume that x is not adjacent to any vertex occupied by one of the cops. Hence. x is either in A_{k+2} or $B_{k-1} \setminus \{x_{k-2}\}$. If $x \in A_{k+2}$ then $r(x) \geq k + 3$. Since $r(w) = k - 1$ this implies $|st(wx)| \geq 4$ which contradicts Corollary 2.4.1.2. So. it must be the case that $x \in B_{k-1} \setminus \{x_{k-2}\}$ and $r(x) \leq k - 2$. Now. let $Q = \{x, w, v\}$. By Lemma 2.4.1.1. $|st(Q)| = r(v) - r(x) = k + 2 - r(x) \leq 4$. which gives $r(x) \geq k - 2$. Hence. $r(x) = k - 2$. Since x is adjacent to the vertex w in G_{k-1}. then by the definition of each G_i. x can not be on the path P. Hence. by the definition of $r(x)$, $x \in G_{k-2}$.

The cops now move to $\{x_{k-2}, x_{k-1}, x_{k+1}\}$. Since $x \perp x_{k-1}, x_k$. then by Lemma 2.4.1.5. x has no neighbour in G_k. The robber is in a similar situation in that the only vertices he can safely move to are those in B_{k-2}. Furthermore if he moves to

a vertex $y \in B_{k-2}$ it must be the case that $y \in G_{k-3}$ since $|st(Q')| \leq 5$ where $Q' = \{y, x, w, v\}$. This continues, by induction until the robber occupies a vertex in G_0. The cops then move to $\{x_0, x_1, x_3\}$. The robber is then forced to move to a vertex which is adjacent to at least one of the cops and he is immediately apprehended. $\square$

Corollary 2.4.1.7 *If G is a bipartite graph with an isometric dominating path then $dn(G) \leq 3$.*

Proof: A graph is a comparability graph if every odd cycle has a triangulating chord. Since any bipartite graph, G, has no odd cycles, then it is also a comparability graph. Hence, by Theorem 2.4.1.6, $dn(G) \leq 3$. $\square$

In Figure 2.15, we see that the upper bound given in Theorem 2.4.1.6 is strict. The graph in this figure is a bipartite, and therefore, comparability graph. There are two choices for an isometric dominating path, each of which must include x_0 and x_3, and in either case three cops are necessary.
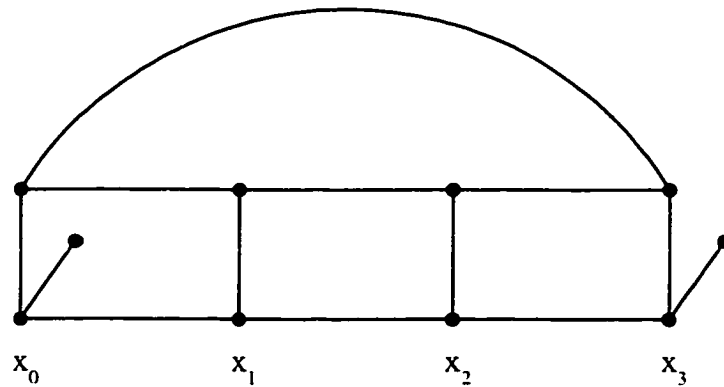
Figure 2.15: An example of a bipartite graph in which $dn(G) = 3$.

2.4.2 Onebeat Graphs

We can characterize those graphs G for which $dn(G) = 1$. Call these graphs **onebeat** graphs. Obviously, onebeat graphs contain an isometric dominating path. Let $P = \{x_0, x_1, \ldots, x_n\}$ be an isometric dominating path in G. We define G_i for all $i = 0, \ldots, n$

as previously. Now for each i we wish to consider those vertices in G_i which have a "forward" edge in this layout. That is, any vertex in G_i which has a neighbour ahead of x_i. Let $F_i = \{x \in G_i : N(x) \cap A_i \neq \emptyset\}$ be that set of vertices in G_i.

Theorem 2.4.2.1 *Let G be a finite connected graph. Then G is a onebeat graph if and only if G contains an isometric dominating path $P = \{x_0, x_1, \ldots, x_n\}$ and for all $i = 0, 1, \ldots, n-1$ each $u \in F_i$ satisfies the following:*

(a) $u \sim x_{i+1}$

(b) *if $u \sim v$ such that $v \in G_{i+2}$ then $u \sim x_{i+2}$*

(c) $u \perp v$ *for all $v \in G_{i+3}$*

Proof: Suppose that for some $k \geq 0$ there is a vertex $w \in B_{k+1}$ such that w has a neighbour $v \in A_k$. We wish to show that $w \sim x_{k+1}$ Suppose v is on the path. P. Then $v = x_j$ for some $j \geq k+1$. If $j = k+1$, then w is obviously adjacent to x_{k+1}. Suppose $j \geq k+2$. Then, by Corollary 2.4.1.2, it must be the case that $v = x_{k+2}$ and $w \in G_k$. Since $w \in F_k$. condition (a) gives us $w \sim x_{k+1}$.

Suppose v is not on the path. Then $v \in G_i$ for some $i \geq k+1$. By the definition of G_i it must be the case that w is not on the path either. Hence. $w \in G_j$ for some $j \leq k$. By Corollary 2.4.1.2. $j \geq k-2$ and by condition (c) of the theorem. $j \neq k-2$. Hence. $j = k-1$ or $j = k$. By applying conditions (a) and (b). we conclude that $w \sim x_{k+1}$.

Hence. for all $k = 0, \ldots, n-1$. any vertex $w \in B_{k+1}$ with a neighbour in A_k is also adjacent to x_{k+1}. Therefore. by Lemma 2.4.1.3. G is a onebeat graph.

Now suppose G has an isometric path $P = \{x_0, x_1, \ldots, x_n\}$. If condition (b) or (c) is not satisfied. then there is a vertex $x \in F_i$ such that x has a neighbour y. and moreover, x and y have no common neighbour in P. Therefore. no matter where the cop is on P the robber can safely occupy either x or y. since the cop can never be adjacent to both vertices at once.

Suppose condition (a) is not satisfied. but conditions (b) and (c) are satisfied. Then there are two cases left to consider : (1) $u \sim v$ where $u \in F_i$ and $v \in G_{i+1}$ and (2) $u \sim x_{i+2}$ where $u \in F_i$. Suppose we have case (1). Then the robber can safely occupy either u or v no matter where the cop is.

If case (2) applies. then let the robber occupy vertex u until the cop moves to a vertex adjacent to u. This only occurs when the cop moves onto either x_i or x_{i+2}. If the cop moves onto x_i then the robber can safely move from u onto x_{i+2}. If the cop moves to x_{i+2} then the robber can move from u to x_i. In either case. this is a safe move for the robber. The robber now waits until the cop moves to an adjacent vertex. This will only occur when the cop moves onto vertex x_{i+1}. The robber then moves back to vertex u. This is a safe move since $u \perp x_{i+1}$. The entire process then repeats itself. Hence. the robber will never be caught. $\square$

Recall that a graph is bridged if it contains no isometric cycles of length greater than three. This means that each cycle of length greater than three has a shortcut. Hence. every chordal graph is also a bridged graph. It was shown by Anstee & Farber [5] that a bridged graph is copwin.

Theorem 2.4.2.2 *If G is a bridged graph with an isometric dominating path then G is a onebeat graph.*

Proof: Suppose G has a dominating path $P = \{x_0, x_1, \ldots, x_n\}$ and we partition the vertices not on the path into G_i's. We wish to show that G satisfies the conditions of Theorem 2.4.2.1.

Case 1: Suppose we have a vertex $u \in G_i$ and a vertex $v \in G_{i+1} \cup \{x_{i+2}\}$ for some $i \in \{0, 1, \ldots, n-1\}$ such that $u \sim v$. Hence. we have the 4-cycle $\{x_i, x_{i+1}, v, u\}$. Since G is bridged G can not contain an induced cycle of length four. Therefore. there must be a chord and the only possibility is the edge from u to x_{i+1}. Hence $u \sim x_{i+1}$ in G.

Case 2: Suppose we have a vertex $u \in G_i$ and a vertex $v \in G_{i+2}$ for some $i \in \{0, 1, \ldots, n-2\}$ such that $u \sim v$. Hence. we have the cycle $\{x_i, x_{i+1}, x_{i+2}, v, u\}$ of length five. Since G is bridged G can not contain an induced cycle of length five.

Therefore. there must be a chord and the only possibilities are the edge from u to x_{i+1} or the edge from u to x_{i+2}. In fact. both edges are required since the absence of either creates an induced cycle of length four. Hence $u \sim x_{i+1}$ and $u \sim x_{i+2}$ in G.

Case 3: Suppose we have a vertex $x \in G_i$ and a vertex $v \in G_{i+3}$ for some $i \in \{0. 1, \ldots . n-3\}$ such that $u \sim v$. Hence. we have the 6-cycle $\{x_i, x_{i+1}. x_{i+2}. x_{i+3}. v. u\}$. Since G is bridged there must be a "shortcut" between two of the vertices of the cycle. This takes the form of a chord between two vertices on the cycle or a path of length two between two vertices which are distance three on the cycle.

Suppose there is a chord on the cycle. The only possibilities are an edge from u to x_{i+1} or an edge from u to x_{i+2}. If $u \sim x_{i+1}$ then there is a 5-cycle $\{u. x_{i+1}. x_{i+2}. x_{i+3}. v\}$. Hence. this cycle must contain a chord. Since. $u \perp i+3$ the edge from u to x_{i+2} must be in G. Hence. $u \sim x_{i+2}$ in G. This creates the 4-cycle $\{u. x_{i+2}. x_{i+3}. v\}$. Since G is bridged. this cycle must have a chord. However, there is no candidate for the chord since $u \perp x_{i+3}$. Due to this contradiction we can conclude that there can not be a chord on this 6-cycle. Therefore. if G is bridged there must be a path of length two between two vertices which are distance three on the cycle.

The only possibilities are a path of length two from u to x_{i+2} or from x_{i+1} to v. Suppose there is a path $\{u. w. x_{i+2}\}$. Then we have the 5-cycle $\{x_i. x_{i+1}. x_{i+2}. w. u\}$. As seen previously. $u \perp x_{i+1}$ and $u \perp x_{i+2}$ Therefore. the chords on the five cycle must be the edges $w x_i$ and $w. x_{i+1}$. Hence $w \in G_i$. We also have the 5-cycle $\{x_{i+2}. x_{i+3}. v. u. w\}$. This cycle must have two chords. Since $u \perp x_{i+3}. w \perp x_{i+3}$ by Corollary 2.4.1.2. the only possible chord is from w to v. However. this creates a 4-cycle $\{x_{i+2}. x_{i+3}. v. w\}$ which has no chord. By this contradiction we conclude there is no path of length two from u to x_{i+2}.

Hence. there is a path $\{x_{i+1}. w. v\}$. Then we have the 5-cycle $\{x_{i+1}. x_{i+2}. x_{i+3}. v. w\}$. Since G is bridged this cycle must have two chords. The only possibilities are the edges $w x_{i+2}$ and $w x_{i+3}$. We also have the 5-cycle $\{x_i. x_{i+1}. w. v. u\}$. This cycle must also have two chords. Since $w \sim x_{i+3}$ it must be the case that $w \perp x_i$. by Corollary 2.4.1.2. Since $u \perp x_{i+1}$ there is only possible chord on this cycle is from u to z. Hence.

there can not be two chords on this 5-cycle which contradicts the assumption that G is bridged. Hence. if G is bridged then there is no edge from a vertex in G_i to a vertex in G_{i+3}.

Since a bridged graph. G. with an isometric dominating path satisfies the conditions of Theorem 2.4.2.1. then G is onebeat. $\square$

Corollary 2.4.2.3 *If G is a chordal graph with an isometric dominating path then G is onebeat.*

This is an obvious consequence of Theorem 2.4.2.2, since all chordal graphs are bridged graphs.

2.4.3   Asteroidal Triple - Free

A class of graphs which seems a good choice to use in the study of this dragnet game are the **asteroidal triple-free** graphs. This is due to the fact that these graphs always have an isometric dominating path.

An asteroidal triple is an independent set of vertices x, y, z such that for each pair of vertices there exists a path joining them that avoids the neighbourhood of the third. This concept was first introduced by Lekkerkerker & Boland [33]. It was later shown by Corneil. Olariu & Stewart [16] that every asteroidal triple-free graph has a **dominating pair**. That is. a pair of vertices such that every path between them is a dominating path. Hence. any shortest path between such a pair will serve as an isometric dominating path. In [17] a linear time algorithm was given for finding all dominating pairs in a connected asteroidal triple-free graph.

Theorem 2.4.3.1 *If G is an asteroidal triple-free graph then $dn(G) \leq 3$.*

Proof: Suppose that G is an asteroidal triple-free graph such that there is an edge vw such that $v \in A_{k+2}$. $w \in B_{k+1}$. and w is not adjacent to any of x_{k+1}, x_{k+2} or x_{k+3} for some $0 \leq k \leq n - 3$. By Corollary 2.4.1.2 it must be the case that $w \in G_k$ and $v \in G_{k+3}$. Hence. we have the 6-cycle $\{w, x_k, x_{k+1}, x_{k+2}, x_{k+3}, v\}$. This cycle has no

chords due to the isometry of P. the definitions of each G_i and the assumption that w is not adjacent to $x_{k+1}.x_{k+2}$ or x_{k+3}. This, however results in the asteroidal triple $\{w, x_{k+1}, x_{k+3}\}$. which contradicts the fact that G is asteroidal triple-free. Hence, it must be the case that w is adjacent to one of $x_{k+1}, x_{k+2}, x_{k+3}$. Hence, by Lemma 2.4.1.3. $dn(G) \leq 3$. $\square$

This bound is sharp. Consider the graph in Figure 2.16. This graph is asteroidal triple-free and $P = \{x_0, x_1, x_2\}$ is the only isometric dominating path in the graph. If there are at most two cops on P then the robber can always avoid capture by moving on the induced K_3 that lies off the path.

$$X_0 \qquad X_1 \qquad X_2$$
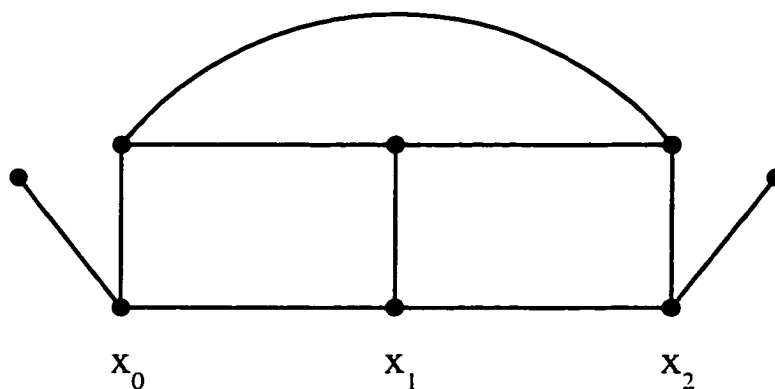
Figure 2.16: An example of an asteroidal triple-free graph in which $dn(G) = 3$.

Hence, if G is an asteroidal triple-free graph then $c(G) \leq 3$.

A **co-comparability** graph is a graph whose complement is a comparability graph. Golumbic, Monma & Trotter [24] showed that the set of co-comparability graphs is strictly contained in the set of asteroidal triple-free graphs.

An **interval** graph is the intersection graph of intervals on a line. Recall that a **chordal** graph is one that contains no induced cycle of length greater than three: i.e. every cycle has a triangulating chord. It was shown in [33] that a graph G is an interval graph if and only if it is both chordal and asteroidal triple-free. Interval graphs are also precisely those which lie in the intersection of co-comparability graphs and chordal graphs.

The class of co-comparability graphs also includes **permutation graphs** which includes all graphs which are both comparability and co-comparability graphs. See Figure 2.17 for a partial ordering of these classes of graphs under inclusion. Figure 2.18 gives an example of a graph in each of these classes.
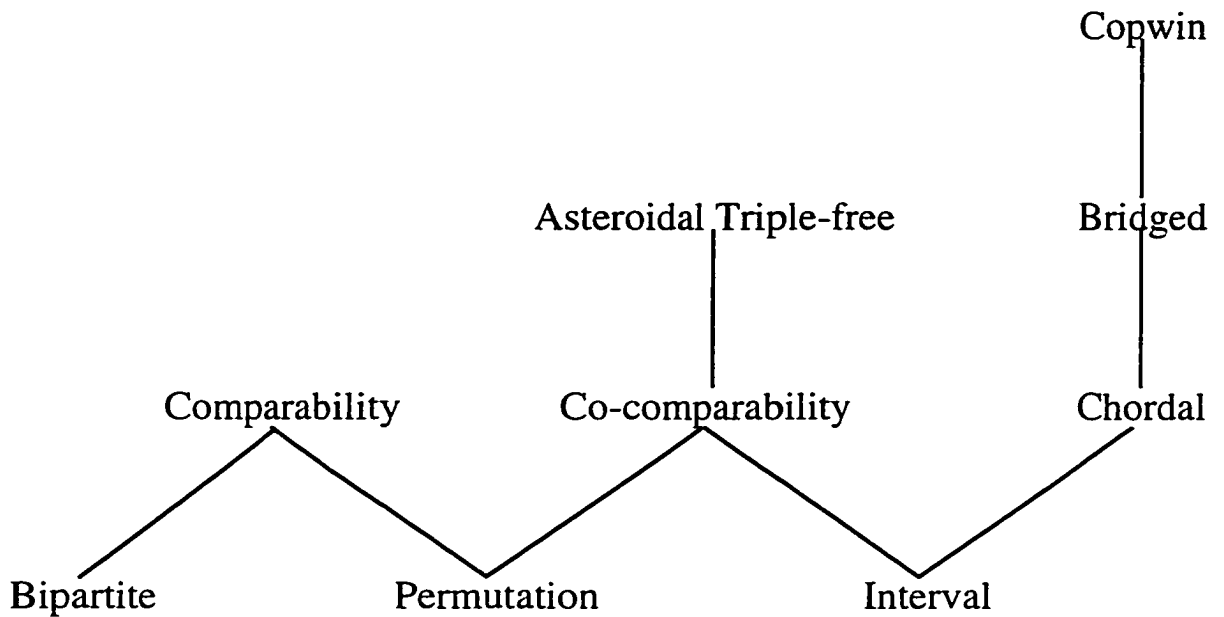
Figure 2.17: A partial ordering of classes of graphs under inclusion.
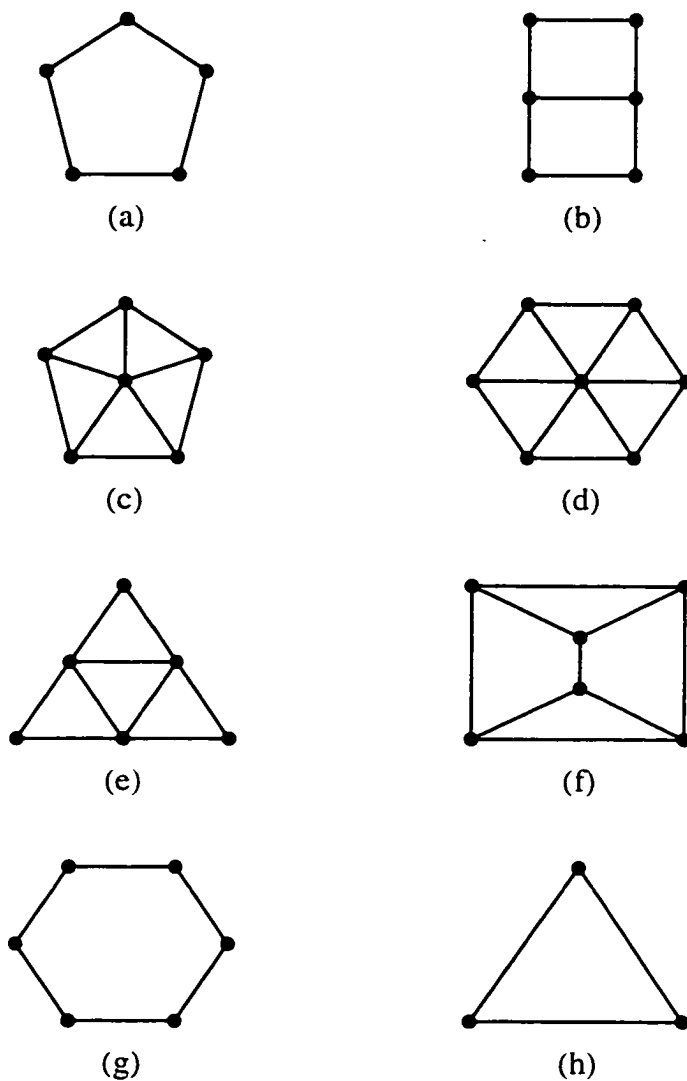
Figure 2.18: (a) Asteroidal Triple-free, but not Co-Comparability. (b) Co-Comparability. but not Interval. (c) Copwin. but not Bridged. (d) Bridged. but not Chordal. (e) Chordal. but not Interval. (f) Co-comparability. but not Permutation. (g) Comparability. but not Permutation. (h) Comparability. but not Bipartite.

Lemma 2.4.3.2 *Suppose G is a co-comparability graph. Then G does not contain an induced cycle of length five.*

Proof: Suppose G has an induced cycle of length 5. Then the complement of G. denoted G^c. contains an induced cycle of length 5. Hence. G^c has an odd cycle without a triangulating chord and is not a comparability graph. Therefore. G is not a co-comparability graph and no co-comparability graph contains an induced cycle of length five. □

Theorem 2.4.3.3 *If G is a co-comparability graph then $dn(G) \leq 2$.*

Proof: Suppose G is a co-comparability graph. Since G is asteroidal triple-free. it has an isometric dominating path. say $P = \{x_0. x_1, ... x_n\}$. By Lemma 2.4.1.3. if every vertex in B_{k+1} with a neighbour in A_{k+1} is also adjacent to one of x_{k+1} and x_{k+2}. for $k = 0. 1. .... n - 2$. then $dn(G) \leq 2$.

Suppose that for some $0 \leq k \leq n - 2$ there is an edge vw such that $v \in A_{k+1}$. $w \in B_{k+1}$. By the definition of G_i and the isometry of P. w is not on P. Hence. $w \in G_i$ for some $i \leq k$.

Suppose v is on the path P. Then by Corollary 2.4.1.2. it must be the case that $v = x_{k+2}$ and $w \sim x_{k+2}$. If v is not on P then it is in G_j for some $j \geq k + 2$. By Corollary 2.4.1.2. either $i = k - 1$ and $j = k + 2$: $i = k$ and $j = k + 3$: or $i = k$ and $j = k + 2$.

Case 1: Suppose $v \in G_{k+2}$ and $w \in G_{k-1}$. Then we have the 6-cycle $\{w. x_{k-1}. x_k. x_{k+1}. x_{k+2}. v\}$. Since G is asteroidal triple-free it contains no induced 6-cycles. Therefore. this cycle contains a chord, and the only candidate is an edge from w to one of x_k. x_{k+1}. or x_{k+2}. However. by Corollary 2.4.1.2. $w \perp x_{k+2}$. Hence. w must be adjacent to x_{k+1} or x_{k+2}.

Case 2: Suppose $v \in G_{k+3}$ and $w \in G_k$. Then we have the 6-cycle $\{w. x_k. x_{k+1}. x_{k+2}. x_{k+3}. v\}$. and w must be adjacent to one of $x_{k+1}. x_{k+2}. x_{k+3}$. However. $w \perp x_{k+3}$ due to Corollary 2.4.1.2. Hence. w is adjacent to x_{k+1} or x_{k+2}.

Case 3: If $v \in G_{k+2}$ and $w \in G_k$ then we have the 5-cycle $\{w, x_k, x_{k+1}, x_{k+2}, v\}$. By Lemma 2.4.3.2, this cycle is not induced. Hence, there must be chord, and the only possibilities are an edge from w to either x_{k+1} or x_{k+2}.

Therefore, if there is an edge vw with $v \in A_{k+1}$, $w \in B_{k+1}$, then w must be adjacent to at least one of the vertices x_{k+1} and x_{k+2}. Hence, by Lemma 2.4.1.3, $dn(G) \leq 2$. $\qquad \square$

Corollary 2.4.3.4 *If G is a permutation graph then $dn(G) \leq 2$.*

This is an obvious consequence of Theorem 2.4.3.3, since all permutation graphs are co-comparability graphs.

The upper bound for the dragnet number of co-comparability and permutation graphs is sharp in both instances. Consider the graph G and its complement, G^c, in Figure 2.19. The edges of G and G^c have been assigned a transitive orientation. Therefore, both G and G^c are comparability graphs, and therefore, both permutation graphs. The graph G has no pitfalls, so it is not copwin. The graph G^c is not copwin since the removal of the pitfalls b and e results in a 4-cycle which is not copwin. If a graph is not copwin, then it is obviously not onebeat. Therefore, we have two examples of permutation graphs, and thus co-comparability graphs, with dragnet number two.
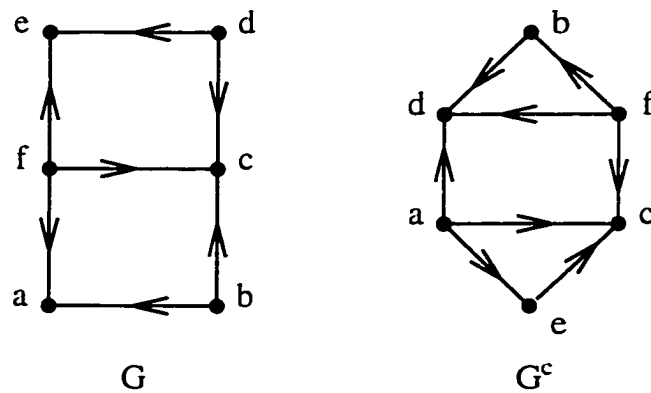
Figure 2.19: A transitive orientation assigned to the edges of a graph and its complement.

The following is a corollary of Theorem 2.4.2.2.

Corollary 2.4.3.5 *If G is an interval graph then G is onebeat.*

Proof: It has been previously stated that interval graphs are both chordal and asteroidal triple-free. Hence, if G is an interval graph then it is a bridged graph which contains an isometric dominating path. Hence, by Theorem 2.4.2.2, $dn(G) = 1$. □

2.4.4 More than One Path Required to Dominate

There are obviously graphs in which a single isometric path can not dominate all the vertices. We will now examine graphs in which some set of isometric paths are required to dominate. That is, there is some set $\mathcal{P} = \{P_1, P_2, \ldots, P_d\}$ of isometric paths in G such that every vertex of G is either in P_i or adjacent to P_i for some $i = 1, \ldots, d$. Since we must have at least one cop on each path in a dominating set, we have a natural lower bound on the dragnet number of a graph:

Theorem 2.4.4.1 *If d is the minimum number of isometric paths required to dominate G then $dn(G) \geq d$.*

We will now examine various upper bounds on the dragnet number.

Lemma 2.4.4.2 *Suppose $P = \{x_0, x_1, \ldots, x_n\}$ is an isometric path in G and $f : G \to P$ is the canonical retraction map of G onto P. If there is an edge uv such that $f(u) = x_i$ and $v \sim x_j$ for some $i, j = 1, \ldots, n$, then $i - 2 \leq j \leq i + 2$.*

Proof: Since $f(x_j) = x_j$, $x_j \sim v$ and f is edge-preserving, then $f(v) \in \{x_{j-1}, x_j, x_{j+1}\}$. Similarly, since $u \sim v$ then $f(v) \in \{x_{i-1}, x_i, x_{i+1}\}$. Therefore, $i - 2 \leq j \leq i + 2$. □

Theorem 2.4.4.3 *If G is a graph such that a set of d isometric paths, say $\{P_1, P_2, \ldots, P_d\}$, dominates G then $dn(G) \leq 5d - 1$.*

Proof: Place five cops on each of the paths $\{P_1, P_2, \ldots, P_{d-1}\}$ and place four cops on P_d. After a finite number of moves the cops on the first $d-1$ paths can prevent the robber from moving onto any vertex on the paths or adjacent to the paths.

To demonstrate. let $P = \{x_0, x_1, \ldots, x_n\}$ be an isometric path in G and let f be the canonical retraction map of G onto P. Suppose that we have five cops and that P has at least six vertices. (If P has any less then the result is obvious). Place the cops on five consecutive vertices in P. Let C be the third. or middle. cop. Let C catch the shadow of the robber. The other cops will move such that the vertices one and two moves up the path from C. as well as the vertices one and two moves down the path from C are all occupied. Once C is shadowing the robber. the robber will be caught if he ever moves onto the path. Suppose the robber moves from a vertex u to a vertex v once he is shadowed. Then the cop is on $f(u) = x_i$ for some $i = 1, \ldots, n$. If $v \sim x_j$ for some $x_j \in P$ then. by Lemma 2.4.4.2. $i - 2 \leq j \leq i + 2$. Due to the cops' strategy. we know that there must be a cop on the vertex x_j. and therefore. the robber can be captured on the cops' next move. So. if the robber is to avoid capture he must avoid moving onto any vertex in the closed neighbourhood of P.

Therefore. after a finite number of moves. the robber can be prevented from moving onto a vertex in or adjacent to any of the paths $\{P_1, P_2, \ldots, P_{d-1}\}$. Hence. the robber is restricted to moving in the subgraph induced on $N[P_d]$. By Lemma 2.4.1.4. only four cops are required to capture the robber in this subgraph. Therefore. $dn(G) \leq 5d - 1$. $\square$

We can show that this bound is obtained if we are required to use a particular set of isometric paths. Consider the graph. G. in Figure 2.20 with isometric paths $P_1 = \{x_0, x_1, x_2, x_3, x_4\}$ and $P_2 = \{y_0, y_1, y_2, y_3, y_4\}$. The vertices not on either of these paths form the graph $K_{5,5}$. If we have four cops on each of the paths. the robber will obviously be able to avoid capture by moving on the subgraph $K_{5,5}$. Hence. if we use these two beats. exactly $5(2) - 1 = 9$ cops are required. We can similarly construct graphs with d isometric paths which require $5d - 1$ cops. Instead of two paths and the complete bipartite graph $K_{5,5}$. we use d paths and a complete d-partite graph

with five vertices in each partition where each partition is dominated by a different isometric path.
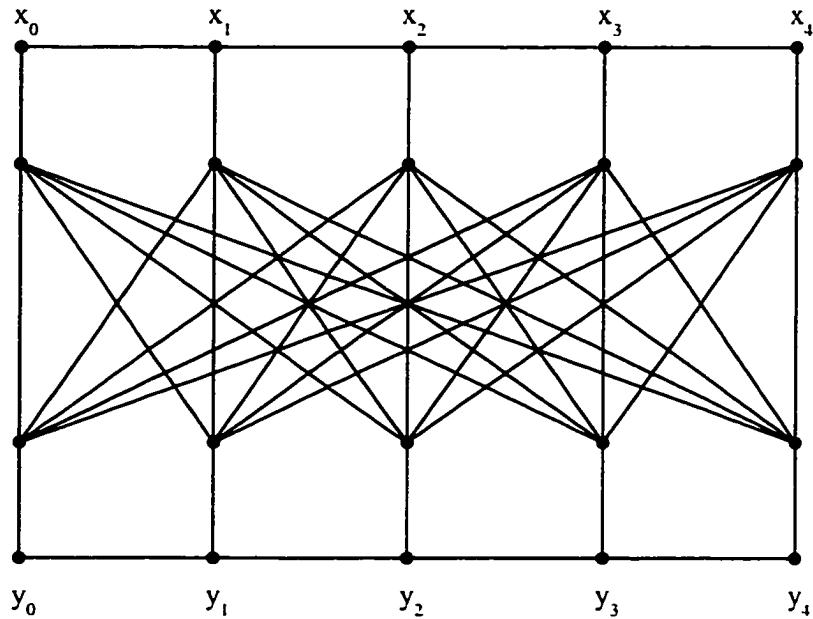
Figure 2.20: An example of a graph in which nine cops moving on given beats are necessary to capture a robber.

The remaining results in this section show situations in which the upper bound on the dragnet number can be decreased from $5d - 1$. In most instances this will depend on which paths intersect the neighbourhood of a particular vertex. So. for each vertex x in $G \setminus P$. we denote $DP(x)$ to be the set of paths in P which dominate x. That is. the set of paths such that x is adjacent to at least one vertex on each of the paths.

Theorem 2.4.4.4 *Given a graph G and a set of d isometric dominating paths P. suppose for every edge xy such that neither vertex is on a path in P. $|DP(x)| + |DP(y)| > |P|$. Then $dn(G) \leq 4d$.*

Proof: Let $P = \{P_1. P_2. \ldots, P_d\}$ be the set of isometric dominating paths. For every path $P \in P$ place four cops. say $(C_1. C_2. C_3. C_4)$. on P. Let f be the canonical retraction map of G onto P and let C_3 shadow the robber on P. The cops C_1 and C_2

occupy the vertices one and two moves down the path from C_3 and let C_4 occupy the vertex one move up the path from C_3. If, on some move, no such vertices exist then the cop in question can simply share a vertex with another cop. With this strategy, after a finite number of moves by the cops, the robber cannot move onto any of the paths.

Suppose once he is shadowed on each of the paths, the robber moves from a vertex u onto a vertex v, which is adjacent to some path $P \in \mathcal{P}$ where $P = \{x_0, x_1, \ldots, x_n\}$. If $f(u) = x_i$ then, by Lemma 2.4.4.2, $v \sim x_j$ where $i - 2 \leq j \leq i + 2$. So, for the robber to avoid capture it must be the case that $i + 2 \leq n$ and $j = i + 2$.

Since P is isometric and $v \sim x_{i+2}$, then $d(x_0, v) \geq i + 1$. In fact, $f(v) = x_{i+1}$ since f is edge-preserving and $u \sim v$. So C_3 moves to x_{i+1} and the other cops move accordingly. Hence, whenever the robber moves onto a vertex adjacent to a path P, C_3 moves up the path and the other cops move accordingly. Note that if C_3 is occupying the vertex x_{n-1} in P then the robber has no safe move adjacent to P. Therefore, if this situation occurs for all the paths in $\mathcal{P}$ then the robber has no safe move and will be immediately apprehended.

Let C_3^k be the third cop on P_k for $k = 1, 2, \ldots, d$. Suppose on the ith move after C_3^k has started to shadow the robber on P_k, the cop C_3^k occupies vertex x_{c_k}. Let $S(i) = \sum_{k=1}^{d} c_k$. If the robber now moves to vertex x then $DP(x)$ is the set of paths which dominate x. Assuming the robber can not be caught on this move then, as previously shown, the third cops' positions on each path in $DP(x)$ will increase by one. On paths which are not in $DP(x)$, the third cop on that path will, at worst, move down that path. Therefore,

$$S(i + 1) \geq S(i) + |DP(x)| - (|\mathcal{P}| - |DP(x)|)$$

Suppose the robber now moves on to a vertex y. Then

$$S(i + 2) \geq S(i + 1) + |DP(y)| - (|\mathcal{P}| - |DP(y)|)$$

Since $|DP(x)| + |DP(y)| > |\mathcal{P}|$ then

$$S(i + 2) \geq S(i) + 2|DP(x)| + 2|DP(y)| - 2|\mathcal{P}| > S(i)$$

Therefore. after every two moves, more of the C_3^k cops have moved up the path than down the path. If the robber remains uncaptured. then for $k = 1, \ldots, d$. each C_3^k will move to occupy either the last or second to last vertex of P_k. Once this is done. there are no vertices that the robber can safely occupy. Therefore. the robber will be forced to move to a vertex dominated by one of the cops and he will be captured. Hence. $dn(G) \leq 4d$. □

Lemma 2.4.4.5 *Let* $P = \{x_0, x_1, \ldots, x_n\}$ *be an isometric path in* G *and let* $f : G \to P$ *be the canonical retraction map of* G *onto* P. *If* $f(u) = x_i$ *and* $u \sim x_j$ *for some vertex* u *and some* $i, j = 1, \ldots, n$. *then* $i - 1 \leq j \leq i + 1$.

Proof: Suppose $f(u) = x_i$ and $u \sim x_j$ for some $i, j = 1, \ldots, n$. Then

$$j = d(x_0, x_j) \leq d(x_0, u) + d(u, x_j) = i + 1$$

and

$$i = d(x_0, u) \leq d(x_0, x_j) + d(x_j, u) = j + 1.$$

Therefore. $i - 1 \leq j \leq i + 1$. □

Theorem 2.4.4.6 *If there are d isometric paths* $\mathcal{P} = \{P_1, P_2, \ldots, P_d\}$ *which dominate a graph* G *such that for every cycle* C *in* $G \setminus \mathcal{P}$. $\bigcap_{x \in C} DP[x] \neq \emptyset$. *then* $dn(G) \leq 4d$.

Proof: For each $P \in \mathcal{P}$ place four cops. say C_1, C_2, C_3, C_4 on P. Let f be the canonical retraction map of G onto P. and let cop C_2 shadow the robber on P. The cops C_1. C_3 and C_4 occupy the vertices one move down the path. one move up the path. and two moves up the path from C_2, respectively.

Suppose that the robber is being shadowed by C_2 on each path $P \in \mathcal{P}$. We now change the strategy. If the robber's image moves up the path then let C_3 shadow the robber. If the image moves down the path let C_2 shadow the robber. And if the image does not move then the cops do not move either. Hence, every time the robber changes direction. we change the cop who shadows him.

By using this strategy we accomplish two things. First. we prevent the robber from moving onto any vertices on the paths. Second. we prevent the robber from using the same edge twice in a row without being caught. This follows because if u is a vertex in G such that u is adjacent to a path $P \in \mathcal{P}$ and $f(u) = x_i$ then. by Lemma 2.4.4.5. $u \sim x_j$ for some $j = i - 1, i, i + 1$. Now suppose the robber moves to a vertex v. Then $f(v) = x_{i-1}, x_i$ or x_{i+1}. If $f(v) = x_{i-1}$ then C_2 moves to the vertex x_{i-1}. Therefore. C_3 and C_4 occupy x_i and x_{i+1}, respectively. Since u is adjacent to at least one of these. then the robber can not safely move back to u. Similarly. if $f(v) = x_i$ or x_{i+1}. the vertices x_{i-1}, x_i and x_{i+1} will be occupied by the cops. Therefore, the robber can not safely move back to u.

Now. in order for the robber to win he must be able to move onto some vertex more than once. (Otherwise. he simply runs out of vertices.) Since he can not use the same edge twice in a row. this can only be accomplished by moving around some cycle in the graph. By assumption. every cycle in the graph is dominated by a common path. Suppose the robber moves around a cycle, $R = \{u_0, \ldots, u_r\}$. which is dominated by a path P. Without loss of generality. suppose $f(u_0) = x_i$ for some $i = 1, \ldots, n$. and the cop C_3 moved onto x_i. If the robber's move from u_0 to u_1 was a safe move. then $u_1 \sim x_{i+2}$ and $u_1 \perp x_j$ for all $j < i + 2$. Therefore. $f(u_1) = x_{i+1}$ and C_3 moves up the path onto x_{i+1}. Similarly. if the move from u_1 to u_2 was a safe move then $u_2 \sim x_{i+3}$. $u_2 \perp x_j$ for $j < i + 3$. Therefore. $f(u_2) = x_{i+2}$ and C_3 moves to x_{i+2}. By induction. we see that in order to make safe moves on vertices adjacent to P the robber must move to a distinct vertex on each move. If he moves around the cycle R he must move onto some vertex which is adjacent to one of the cops. Therefore. the robber can not avoid capture and $dn(G) \leq 4d$. $\square$

Lemma 2.4.4.7 *Suppose G can be dominated by a set of d isometric paths. $\mathcal{P} = \{P_1, P_2, \ldots, P_d\}$. and G has girth greater than $2(d + 2)$. Let x and y be two distinct vertices which are not on any path in $\mathcal{P}$ such that $DP(x) \cap DP(y) \neq \emptyset$. Then there is no path from x to y in $G \setminus \mathcal{P}$.*

Proof: Let $\mathcal{P} = \{P_1, P_2, \ldots, P_d\}$ be a set of isometric paths which dominate G. Suppose G has girth greater than $2(d + 2)$. Suppose there exists at least one pair of distinct vertices which are dominated by a common path and are in the same component of $G \setminus \mathcal{P}$. Let x and y be such a pair whose distance in $G \setminus \mathcal{P}$ is minimum. Let $Q = (x = v_1, v_2, \ldots, v_n = y)$ be the path of minimum length between x and y in $G \setminus \mathcal{P}$.

Without loss of generality, suppose $P = \{x_0, x_1, \ldots, x_n\}$ is a path in $\mathcal{P}$ that dominates both x and y. It must be the case that no vertex $\{v_2, v_3, \ldots, v_n\}$ is dominated by P nor does any pair in the set share a common dominating path. Otherwise, the choice of x and y as the required pair with minimal distance would be contradicted. Hence, there must be $n - 1$ distinct paths in $\mathcal{P}$, i.e. $d \geq n - 1$.

Since x and y are adjacent to P, then $x \sim x_i$ and $y \sim x_j$. Since $\{x_i, Q, x_j\}$ is a path of length $n + 1$ from x_i to x_j and P is isometric, then $|j - i| \leq n + 1 \leq d + 2$. This means there is a cycle of length at most $2(d + 2)$ in G. This contradicts the fact that the girth of G is greater than $2(d + 2)$.

Hence, no two distinct vertices dominated by a common path in $\mathcal{P}$ are in the same component of $G \setminus \mathcal{P}$. □

This means that there are at most d vertices in any component of $G \setminus \mathcal{P}$ and, due to the girth restriction, all the components are trees.

Theorem 2.4.4.8 *Suppose d isometric paths are required to dominate a graph G and G has girth greater than $2(d + 2)$. Then $dn(G) = d$.*

Proof: Let $\mathcal{P} = \{P_1, P_2, \ldots, P_d\}$ be a set of isometric paths which dominate G. Suppose G has girth greater than $2(d + 2)$. By assumption, no fewer than d isometric paths will dominate G. Therefore, $dn(G) \geq d$. Place one cop on each path in $\mathcal{P}$. Let each cop shadow the robber on his respective path. Once this is done the robber is forced to move onto a vertex which is not on any of the paths.

Suppose the robber occupies a vertex, x, which is not on any path in $\mathcal{P}$, and the cops shadow the robber on each of their respective paths. The robber now moves to

a vertex. y. If $P \in DP(x)$ then move the cop on P to the vertex adjacent to x on P. There is only one such vertex due to girth restrictions. and this move is available to the cop by Lemma 2.4.4.5. For every path not in $DP(x)$, let the cop on that path continue to shadow the robber. The cops on $DP(x)$ will now remain stationary unless they can immediately capture the robber. Repeat this strategy with each following move.

The vertex x was contained in some component H in $G \setminus \mathcal{P}$. We know by the previous lemma that H is a tree with at most d vertices. The cops' strategy ensures that two moves after the robber initially occupies a vertex in H, he can never occupy that vertex again. Hence. the robber must leave the component H in order to avoid capture. To do so he must move onto a path in $\mathcal{P}$. Let v be the vertex occupied by the robber just prior to his first move onto a path. P, in $\mathcal{P}$. Therefore. $P \in DP(v)$. The robber can not move onto any path where the cop is on his image. He must. therefore. move onto a path in $DP(w)$ where w is some vertex in $G \setminus \mathcal{P}$ which he has previously occupied. Hence. $P \in DP(w) \cap DP(v)$. If v and w are distinct vertices then. by Lemma 2.4.4.7. there is no path from v to w in $G \setminus \mathcal{P}$. Therefore. $v = w$ and the robber's move was actually a pass. Following the robber's pass. the cop on P moved onto the only vertex in P adjacent to v. Hence. if the robber moves from v onto the path. he will be moving onto the cop. So. the robber has no escape and $dn(G) = d$. $\qquad \square$

Theorem 2.4.4.9 *Let G be a chordal graph. Suppose d isometric paths are required to dominate G. Then $dn(G) = d$.*

Proof: Let $\mathcal{P} = (P_1. P_2. \ldots . P_d)$ be the d paths which dominate G. and suppose we have d cops. $(C^1. C^2. \ldots , C^d)$. For $i = 1. \ldots . d$. place the cop C^i on the first vertex of the path P_i. From this point on if the robber moves to a vertex which is adjacent to P_i then C^i moves up the path. otherwise C^i stays where it is.

Let H_i be the subgraph induced on $N[P_i]$. Hence. $P_i = \{x_1. \ldots . x_n\}$ is an isometric dominating path in H_i. Suppose through some series of moves the robber occupies

distinct vertices u and v in H_i. Let $\{u = u_0. u_1, \ldots u_k = v\}$ be the shortest path from u to v among those vertices which the robber has used, and assume that u_0 and u_n are the only vertices on this path which lie in H_i. Suppose that u is ahead of C^i in H_i and v is behind C^i. Let $P = \{x_0, x_1, \ldots, x_n\}$. Suppose after the robber moved onto u the cop moved from x_{k-1} to x_k. This is assuming that $u \perp x_{k-1}$. since the robber could be apprehended otherwise. Since G is chordal and no vertices of $\{u_1, \ldots, u_{n-1}\}$ are adjacent to P then it must be the case that $u = v$. Therefore, as in the case when one path dominated a chordal graph. $v \sim x_k$. So. there is in fact no safe move from a vertex ahead of C^i to a vertex behind C^i for all $i = 1, \ldots, d$.

Note that for each $i = 1, \ldots, d$, the robber's first move into H_i is to a vertex ahead of C^i. Since the robber can never safely move to a vertex behind the cop C^i in H_i. then if he is adjacent to P_i he must be ahead of C^i. This means that he will eventually run out of safe vertices and be captured. Therefore. $dn(G) \leq d$. and since no fewer than d isometric paths will dominate G. $dn(G) = d$. $\quad\square$

2.4.5 Problems

If a graph is bridged and has an isometric dominating path then it has dragnet number one.

Problem 2.4.5.1 *What is the dragnet number of a copwin graph containing an isometric dominating path?*

We saw that in Figure 2.20 that nine cops were needed if the beats were $P_1 = \{x_0. x_1. x_2. x_3. x_4\}$ and $P_2 = \{y_0. y_1. y_2. y_3. y_4\}$. The number of cops could be reduced to five. for example. by taking isometric paths from x_i to y_i for $i = 0. 1, \ldots, 4$ that together cover all the vertices of the graph. Since only one cop per path is necessary. then $dn \leq 5$.

Problem 2.4.5.2 *Is there an example of a graph G in which d paths are required to dominate and $dn(G) = 5d - 1$?*

For asteroidal triple-free graphs, there is a linear time algorithm for finding an isometric dominating path.

Problem 2.4.5.3 *What is the complexity for determining whether d isometric paths are sufficient to dominate a graph G?*

When investigating precincts, we noted that in some cases there were distinct advantages to changing the beats from isometric paths to other subgraphs, such as isometric trees and complete graphs. While the fact that there is a retraction map onto an isometric path was not always put to use in the dragnet game, for certain graphs it was important to the strategy.

Problem 2.4.5.4 *Are there graphs in which a good strategy for playing the dragnet game can be found if every beat is*
- *(a) an isometric tree?*
- *(b) a complete graph?*
- *(c) any retract that is also a copwin graph?*

Chapter 3

Isometric Embeddings

3.1 Introduction

The **isometric dimension** of a graph G has been defined as the least number of paths needed so as to be able to isometrically embed G in the Cartesian product of the paths (see [50]). This is not always possible unless there is a relaxation of the isometry condition. (For example C_3 can not be isometrically embedded in any Cartesian product of paths.) We are interested in finding when a given graph is an isometric subgraph of the *strong* product of paths. The **strong isometric dimension** of a graph G is the least number k such that there is a set of k paths $\{P_1, P_2, \ldots, P_k\}$ with G an isometric subgraph of $\boxtimes_{i=1}^k P_i$. We denote this by $idim(G) = k$.

Figure 3.1 shows that $idim(C_4) \leq 2$ and since C_4 is not an induced subgraph of any path then $idim(C_4) = 2$. Cycles require a lot of space. indeed. in Lemma 3.2.2.10 we show that $idim(C_n) = \lceil n/2 \rceil$. In contrast. the strong product of n edges is the complete graph K_{2^n}. thus $idim(K_m) = \lceil \log_2 m \rceil$.

In Section 3.2 we also investigate the relationship between strong isometric dimension and the categorical notion of the injective hull. We then show that $idim(G)$ exists for any finite. reflexive graph G and establish upper and lower bounds for $idim(G)$. These bounds are used to find the strong isometric dimension of particular graphs such as cycles and hypercubes.
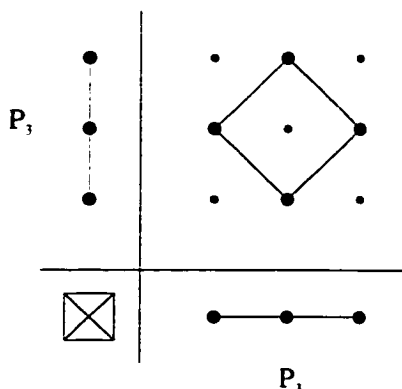
Figure 3.1: An Isometric Embedding of C_4 in $P_3 \boxtimes P_3$.

The motivation for examining the strong isometric dimension of a graph comes in part from the game of cops and robber. The representation of G in the strong product of paths allows the 'holes' of G to appear. These are structures which the robber can use to evade the cops. Generally speaking. the greater the complexity of the 'hole' the more options the robber has and therefore the more cops are needed to capture the robber. However. the degrees of the vertices in a product of k paths are bounded by 3^k and so the size of the strong isometric dimension bounds the possible complexity of the 'holes'. This question is addressed is Section 3.3 where we determine the cop numbers of graphs with strong isometric dimension two.

In Sections 3.4 and 3.5. we extend this idea of embedding graphs in the strong product of paths to the strong products of other graphs. In the former section we look at the case when one of the graphs in the product is a tree and in the latter we consider the case when one of the graphs is a cycle. We see that by embedding a graph in such products. we are able to cover the vertices of the graph with subgraphs where each subgraph is a retract.

We have seen throughout this thesis that retracts have proved useful in devising winning strategies for the game of cops and robber. In Section 3.6. we find that this continues to be the case. If we are given a graph G which is an isometric subgraph of $H \boxtimes I$ where I is either a tree or a cycle. we can find a winning strategy in G by playing a new "roadblock" game on H. This gives us upper bounds on the cop

numbers of isometric subgraphs of $T \boxtimes T$. $T \boxtimes C$ and $C \boxtimes C$. where T is any tree and C is any cycle.

3.2 The Strong Isometric Dimension of a Graph

As previously noted. one of the motivations for examining the strong isometric dimension of a graph is to find upper bounds on the cop number of a graph. The other is its relationship to the concept of the injective hull of a graph [30. 31. 40. 44]. We investigate this notion and give two examples of the construction of the injective hull of a graph.

We then proceed to establish upper and lower bounds on $idim(G)$. One such bound shows that if we have k vertices such that the distance between each pair is $diam(G)$ then $idim(G) \geq k$. This allows us to determine the strong isometric dimension of cycles and hypercubes.

We also show that the embedding of a graph in the strong product of paths can be represented by a set of orientations on some subset of edges of the graph. This technique proves useful in finding the strong isometric dimension of a tree to within a factor of two. Finally. we pose some problems.

3.2.1 Injective Hull

Every graph can be isometrically embedded in an absolute retract. In fact. for any graph G all minimal absolute retracts into which G can be isometrically embedded are isomorphic. as shown in [44]. This graph is called the **injective hull** of G. One way of finding the injective hull for a graph G is to embed G in a strong product of paths then take the smallest retract of the product that contains that image of G. This is due to the following result by Pesch [44]:

A graph G is an absolute retract of reflexive graphs if and only if it is a retract of some finite strong product of reflexive graphs.

So, if we isometrically embed a graph G in the strong product of the fewest paths. $F : G \rightarrow \boxtimes_{i=1}^{k} P_i$. and retract what we can while leaving $F(G)$ fixed. then we obtain the smallest absolute retract containing G. Hence, we have obtained the injective hull of G.

In [30]. Isbel gives a means of constructing the injective hull of a graph by regarding a graph G as a metric space X. this construction gives the injective hull of a graph. We now reproduce this method of construction for the injective hull.

First. define an **extremal** function $f : X \rightarrow Z$ as a integer-valued function which is pointwise minimal subject to

$$f(x) + f(y) \geq d(x, y)$$

for all x and y in X. (Pointwise minimal means that for every $x \in X$ there is some $y \in X$ such that $x \neq y$ and $f(x) + f(y) = d(x, y)$.) The difference between any two extremal functions is bounded. thus the set ϵX of all extremal functions is a metric space with the distance between any two extremal functions f and g defined by $d(f, g) = \sup |f(x) - g(x)|$. If we define the mapping $\epsilon : X \rightarrow \epsilon X$ by $(\epsilon(x))(y) = d(x, y)$ then ϵ is an isometric embedding.

For example. if we use as our set X the vertices of the 4-cycle. $C = \{a, b, c, d\}$. There are exactly five point-wise minimal functions. f. which satisfy $f(x) + f(y) \geq d(x, y)$. They are listed in Table 3.1.

f	$f(a)$	$f(b)$	$f(c)$	$f(d)$
A	1	1	1	1
B_a	0	1	2	1
B_b	1	0	1	2
B_c	2	1	0	1
B_d	1	2	1	0

Table 3.1: Extremal functions on $C = \{a, b, c, d\}$.

Now we can calculate the distance between each pair of extremal functions. For example. $d(A. B_a) = \sup\{1 - 0.1 - 1.2 - 1.1 - 1\} = 1$. We find the other distances similarly and obtain the distance matrix presented in Table 3.2.

d	A	B_a	B_b	B_c	B_d
A	0	1	1	1	1
B_a		0	1	2	1
B_b			0	1	2
B_c				0	1
B_d					0

Table 3.2: Distances between pairs of extremal functions.
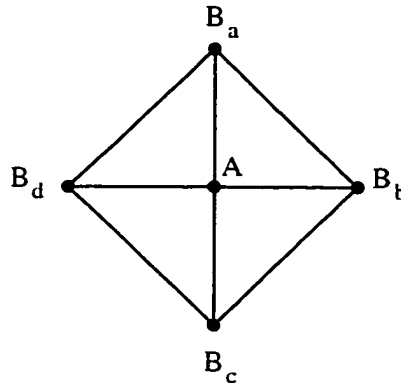
Figure 3.2: The injective hull of the 4-cycle.

We can represent the metric space ϵX by the graph in Figure 3.2. where the vertex set is the set of extremal functions and there is an edge between any two vertices which are at distance one. as given in Table 3.2.

Finally. we can map $\{a. b. c. d\}$ to $\{A. B_a, B_b. B_c. B_d\}$. For example. $(\epsilon(a))(x) = d(a.x)$. Since $d(a.x) = B_a(c)$ for all $x \in \{a. b. c. d\}$. then $\epsilon(a) = B_a$. Similarly. $\epsilon(b) = B_b$. $\epsilon(c) = B_c$ and $\epsilon(d) = B_d$. This gives an isometric embedding of the 4-cycle into the graph in Figure 3.2.

Now suppose X is the set of vertices of the 6-cycle $C = \{a.b.c.d.e.f\}$. There are exactly fourteen point-wise minimal functions. g, which satisfy $g(x) + g(y) \geq d(x.y)$. They are listed in Table 3.3.

g	$g(a)$	$g(b)$	$g(c)$	$g(d)$	$g(e)$	$g(f)$
A_{abc}	1	1	1	2	2	2
A_{abe}	1	1	2	2	1	2
A_{abf}	1	1	2	2	2	1
A_{ace}	1	2	1	2	1	2
A_{bcd}	2	1	1	1	2	2
A_{bdf}	2	1	2	1	2	1
A_{cde}	2	2	1	1	1	2
A_{def}	2	2	2	1	1	1
B_a	0	1	2	3	2	1
B_b	1	0	1	2	3	2
B_c	2	1	0	1	2	3
B_d	3	2	1	0	1	2
B_e	2	3	2	1	0	1
B_f	1	2	3	2	1	0

Table 3.3: Extremal functions on $C = \{a.b.c.d.e.f\}$.

We see that any pair of distinct "A" functions are distance one apart. The distances between each of the "B" functions are given in Table 3.4. The distances between the "A"s and "B"s are presented in Table 3.5.

d	B_a	B_b	B_c	B_d	B_e	B_f
B_a	0	1	2	3	2	1
B_b		0	1	2	3	2
B_c			0	1	2	3
B_d				0	1	2
B_e					0	1
B_f						0

Table 3.4: Distances between pairs of extremal "B" functions.

d	B_1	B_b	B_c	B_d	B_e	B_f
A_{abc}	1	1	1	2	2	2
A_{abe}	1	1	2	2	1	2
A_{abf}	1	1	2	2	2	1
A_{ace}	1	2	1	2	1	2
A_{bcd}	2	1	1	1	2	2
A_{bdf}	2	1	2	1	2	1
A_{cde}	2	2	1	1	1	2
A_{def}	2	2	2	1	1	1

Table 3.5: Distances between extremal "A" functions and extremal "B" functions

Finally. the isometric embedding is given by $e(x) = B_x$ for all $x \in \{a.b.c.d.e.f\}$. This is an isometric embedding of C into the graph shown in Figure 3.3. Note that the "A" functions form the complete graph on eight vertices which is drawn as a cube in the figure. and the "B" functions are situated around the cube.

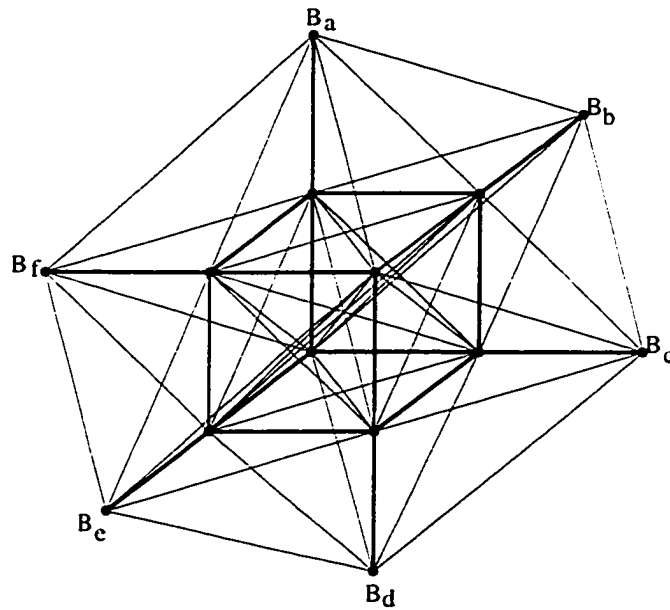

Figure 3.3: The injective hull of the 6-cycle.

While finding the injective hull of a graph does not provide its strong isometric dimension. a graph G can only be embedded in the strong product of k paths if its

injective hull can be embedded. Hence, the lower bounds on $idim(G)$ to be presented can also be applied to the injective hull of G, which may prove to be more useful.

3.2.2 Bounds on the Strong Isometric Dimension of a Graph

A **projection** of $H \subseteq \boxtimes_{i=1}^{k} G_i$ onto G_i is a map $\pi_i : H \rightarrow G_i$ defined as $\pi_i(a_1, a_2, \dots a_k) = a_i$. A **realizer** of G is a set of paths $\{P_i : i = 1, \dots, k\}$ with $k = idim(G)$ and an edge-preserving map $F : G \rightarrow \boxtimes_{i=1}^{k} P_i$ such that $F(G)$ is an isometric subgraph of $\boxtimes_{i=1}^{k} P_i$. We will put $\bar{a} = F(a)$. The vertices of a path in the realizer will be a range of consecutive integers. This will allow us to refer to the next and previous vertex along a path as $\pi_i(\bar{a}) + 1$ and $\pi_i(\bar{a}) - 1$.

We say that vertices $a, b \in V(G)$ are **separated** by (H, f) if H is a graph and $f : G \rightarrow H$ is an edge-preserving map where $d(a, b) = d(f(a), f(b))$. Often the separating graph H will be a path P_i from a realizer and the projection onto P_i will be the corresponding map. In this case, we have $d(a, b) = d(\pi_i(\bar{a}), \pi_i(\bar{b}))$ and we say that a, b ($\bar{a}, \bar{b}$) are separated in the ith coordinate.

Let G be a graph and let $P_v = \{v = v_0, v_1, \dots, v_k\}$ be an isometric path of G. Let $P_v^* = \{0, 1, 2, \dots, k\}$ be a path disjoint from G. The **distance retraction** map $f_v^* : G \rightarrow P_v^*$ is defined by $f_v^*(x) = d(v, x)$ if $d(v, x) \leq k$ else $f_v^*(x) = k$. Note the sense of direction with these maps.

Lemma 3.2.2.1 *Let $P_v = \{v = v_0, v_1, \dots, v_k\}$ be an isometric path of G. Then P_v is a retract of G, moreover v is separated by (P_v^*, f_v^*) from $x \in V(G)$ if $d(v, x) \leq k$.*

Proof: Define $g : P_v^* \rightarrow P_v$ by $g(i) = v_i$ and it is easy to verify that g is edge-preserving. Also, if $x \sim y$ in G then $|d(v, x) - d(v, y)| \leq 1$ and thus $f_v^*(x) \simeq f_v^*(y)$ so f_v^* is edge-preserving. Now, $g \circ f_v^*$ maps G onto P_v and $g \circ f_v^*$ is the identity map on P_v. Therefore P_v is a retract.

If $d(v, x) = j \leq k$ then $d(f_v^*(v), f_v^*(x)) = j$ and thus v and x are separated by (P_v^*, f_v^*). □

It is necessary to separate all pairs of vertices to find the strong isometric dimension.

Lemma 3.2.2.2 *If every pair of vertices in G is separated by at least one of (P_1, f_1), $(P_2, f_2), \ldots, (P_k, f_k)$, then $idim(G) \le k$.*

Proof: Let $H = \boxtimes_{i=1}^k P_i$ and define the map $F : V(G) \to V(H)$ by $F(x) = (f_i(x))_{i=1}^k$.

We claim that $F(G)$ is an isometric subgraph of H. Consider vertices v and w in $V(G)$ with $d_G(v, w) = d$. Since each f_i is edge preserving, $d_{P_i}(f_i(v), f_i(w)) \le d$ for all $i = 1, \ldots, k$. Furthermore, since v and w are separated by at least one path, $d_{P_i}(f_i(v), f_i(w)) = d$ for some i. Therefore $d_H(F(v), F(w)) = d$ and $F(G)$ is an isometric subgraph of H. $\square$

The next result not only shows that the strong isometric dimension exists for every finite, connected, reflexive graph but also gives the first upper bound.

Theorem 3.2.2.3 *Let G be a finite, connected, reflexive graph, then $idim(G) \le |V(G)|$.*

Proof: For each $v \in V(G)$ let v' be a vertex such that $d(v, v')$ is maximum. Let P_v be a shortest path in G from v to v'. Clearly, P_v is isometric. Now consider a pair of vertices v and w in $V(G)$ with $d_G(v, w) = d$. Since P_v is a longest isometric path starting at v, $l(P_v) \ge d_G(v, w)$. Hence, with the distance-retraction map $f_v^* : G \to P_v^*$, we have $d_G(v, w) = d_{P_v^*}(f_v^*(v), f_v^*(w))$, and v and w are separated by (P_v^*, f_v^*). Hence, $\{(P_v^*, f_v^*) : v \in V(G)\}$ separate every pair of vertices in $V(G)$ and, by Lemma 3.2.2.2, $idim(G) \le |V(G)|$. $\square$

The construction in the previous result is inefficient. A slightly better result is:

Corollary 3.2.2.4 *Let G be a finite, connected, reflexive graph, then*

$$idim(G) \le |V(G)| - diam(G).$$

Proof: Find $\{P_v^- : v \in V(G)\}$ as in the previous theorem. Now choose a vertex x such that $l(P_x^-) = diam(G)$. If $v \neq x$ and $v \in V(P_x)$ then eliminate P_v^- from the collection of paths. This new collection of paths also separates every pair of vertices in G. This follows since for any v, P_v^- separates v from $V(G) \setminus \{v\}$ and also separates a and b for all $a, b \in P_v$. Thus P_x^- separates all pairs of vertices on P_x: also if $y \in P_x$ and $z \notin P_x$ then y and z are separated on P_z^-. Thus the paths P_y^-, $y \in P_x \setminus \{x\}$ are unnecessary. Hence, by Lemma 3.2.2.2, $idim(G) \leq |V(G)| - diam(G)$. $\square$

As expected. there is a relationship between the strong isometric dimension of a graph and its isometric subgraphs. as well as a pair of graphs and their strong product.

Theorem 3.2.2.5 *If H is an isometric subgraph of G then $idim(H) \leq idim(G)$.*

Proof: If G is a graph such that $idim(G) = k$ then G is an isometric subgraph of $\boxtimes_{i=1}^{k} P_i$ for some collection of paths $\{P_1, P_2, \ldots, P_k\}$. If H is an isometric subgraph of G then it is also an isometric subgraph of $\boxtimes_{i=1}^{k} P_i$. Hence. $idim(H) \leq k$. $\square$

This is not necessarily true if H is only an induced subgraph of G. For example. C_6 is an induced subgraph of $P_4 \boxtimes P_3$. as seen in Figure 3.4. It is later shown in Lemma 3.2.2.10 that $idim(C_6) = 3$. Hence. $idim(C_6) > idim(P_3 \boxtimes P_4) = 2$.
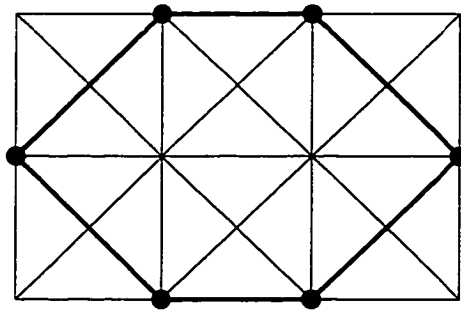

Figure 3.4: C_6 is an induced subgraph of $P_4 \boxtimes P_3$.

Theorem 3.2.2.6 *If G and H are any two graphs then*

$$idim(G \boxtimes H) \leq idim(G) + idim(H).$$

Proof: Suppose G and H are two graphs such that $idim(G) = k$ and $idim(H) = l$. Then there are distance-preserving maps F_1 and F_2 such that $F_1 : G \to \boxtimes_{i=1}^{k} P_i$ and $F_2 : H \to \boxtimes_{i=k+1}^{k+l} P_i$. for some collection of paths $\{P_1, P_2, \ldots, P_{k+l}\}$. The graph $G \boxtimes H$ has vertex set $\{(u, v) : u \in V(G), v \in V(H)\}$. Let $F : G \boxtimes H \to \boxtimes_{i=1}^{k+l} P_i$ be defined by $F(u, v) = (F_1(u), F_2(v))$ for all $(u, v) \in V(G \boxtimes H)$. F is distance-preserving since

$$
\begin{aligned}
& d(F(u, v), F(u', v')) \\
={} & d((F_1(u), F_2(v)), (F_1(u'), F_2(v'))) \\
={} & \max(d(F_1(u), F_1(u')), d(F_2(v), F_2(v'))) \\
={} & \max(d(u, u'), d(v, v')) \\
={} & d((u, v), (u', v')).
\end{aligned}
$$

Therefore. $G \boxtimes H$ is an isometric subgraph of $\boxtimes_{i=k+1}^{k+l} P_i$. and $idim(G \boxtimes H) \leq k + l = idim(G) + idim(H)$. □

Equality need not hold. For example, $idim(K_3 \boxtimes K_5) = idim(K_{15}) = 4$ but $idim(K_3) + idim(K_5) = 2 + 3$.

The idea of direction in the distance-retraction maps is generalized in the next result which will be used later in evaluating the strong isometric dimension of hypercubes and trees.

From any graph G we can obtain a directed graph by specifying a direction on each edge of $E(G)$. Such a directed graph is called an **orientation** of G. If an orientation is placed on a subset of the edges of $E(G)$ this called a **sub-orientation** of G.

Suppose we have a walk $W = \{v_0, v_1, \ldots, v_n\}$. We say an edge $(v_{i-1} v_i)$ is **forward directed** on W if $v_{i-1} \to v_i$, **backward directed** on W if $v_{i-1} \leftarrow v_i$. and **undirected** otherwise. Forward and backward directed edges on the closed walk $X = \{v_0, v_1, \ldots, v_n = v_0\}$ are defined similarly. Define the **edge-sum** of a (closed) walk to be the number of forward edges minus the number of backward edges on that (closed) walk.

Lemma 3.2.2.7 *Suppose G is a finite connected graph. Then $idim(G) \leq k$ if and only if there is a set of k sub-orientations of G. $\{G_1, G_2, \ldots, G_k\}$. such that for every pair of vertices in $V(G)$ there is a directed isometric path between them in at least one of the k sub-orientations. and for each $i \in \{1, \ldots, k\}$ the edge-sum of every cycle in G_i is zero.*

Proof: Let $idim(G) = k$ and let $\{P_1, \ldots, P_k\}$ be a realizer for G. For $1 \leq i \leq k$. construct G_i as follows: for each edge $ab \in E(G)$ let $a \to b$ if $\pi_i(\bar{b}) - \pi_i(\bar{a}) = 1$. let $b \to a$ if $\pi_i(\bar{b}) - \pi_i(\bar{a}) = -1$. and leave ab undirected otherwise.

Now consider a pair of vertices $\{x, y\}$ in $V(G)$. Let $P(x, y) = \{x = x_0, x_1, \ldots, x_d = y\}$ be an isometric path from x to y in G. Suppose that x and y are separated in the ith coordinate and that $\pi_i(\bar{y}) > \pi_i(\bar{x})$. Then. for all $j = 1, \ldots, d$. $\pi_i(\bar{x}_j) - \pi_i(\bar{x}_{j-1}) = 1$ and so $x_{j-1} \to x_j$. Hence. $P(x, y)$ is a directed isometric path from x to y in G_i.

Let $C = \{v_0, v_1, \ldots, v_n = v_0\}$ be a cycle in G_i for some $i = 1, \ldots, k$. Since $v_o = v_n$. we have $\pi_i(v_0) = pi_i(v_n) = 0$. Hence.

$$\pi_i(\bar{v}_n) - \pi_i(\bar{v}_0)$$

$$= (\pi_i(\bar{v}_n) - \pi_i(\bar{v}_{n-1})) + (\pi_i(\bar{v}_{n-1}) - \pi_i(\bar{v}_{n-2})) + \cdots + (\pi_i(\bar{v}_1) - \pi_i(\bar{v}_0))$$

$$= 0$$

and the edge-sum of C is zero.

To prove the converse. suppose that $\{G_1, G_2, \ldots, G_k\}$ is a set of sub-orientations of G such that for every pair of vertices in G there is a directed path between them in at least one of the k sub-orientations and for each $i = 1, \ldots, k$ every cycle in G_i has edge-sum zero. This latter fact implies that every closed walk in G_i has an edge-sum of zero. Therefore. for any pair of vertices $x, y \in V(G_i)$ the edge-sums of every walk from x to y are equal.

Let $d = diam(G)$ and let $\{P_1, P_2, \ldots, P_k\}$ be a set of disjoint paths where for every $i = 1, 2, \ldots, k$. $P_i = \{-d, -d+1, \ldots, d\}$. We now define a set of maps $\{f_1, f_2, \ldots, f_k\}$ where $f_i : G \to P_i$. Choose a vertex $v \in V(G)$ and set $f_i(v) = 0$ for $1 \leq i \leq k$. Now for each vertex $x \in V(G)$ and $1 \leq i \leq k$. let $f_i(\bar{x})$ equal the edge sum of any path

from v to x in G_i. Choose any pair of vertices $x, y \in V(G_i)$. Then $f_i(y) - f_i(x)$ is the edge sum of any path from x to y. Since there is a path of length $d(x, y)$ we have $|f_i(\bar{y}) - f_i(\bar{x})| \leq d(x, y)$ for all $i = 1, \ldots, k$. Since there is a directed path between x and y in G_i for some i then $|f_i(\bar{y}) - f_i(\bar{x})| = d(x, y)$ for at least one value of i.

Finally, let $F(x) = (f_i(x))_{i=1}^{k}$. Then $F(x)$ maps $V(G)$ into $\boxtimes_{i=1}^{k} P_i$. Note that F is edge preserving and any pair of vertices x, y is separated on P_i for those i in which there is a directed path between them in G_i. Hence, by Lemma 3.2.2.2, $idim(G) \leq k$. $\square$

In the case that G is a tree, the cycle condition is not required.

Corollary 3.2.2.8 *Suppose T is a tree. Then $idim(T) \leq k$ if and only if there is a set of k orientations of T, $\{T_1, T_2, \ldots, T_k\}$, such that for every pair of vertices in $V(T)$ there is a directed path between them in at least one of the k orientations of T.*

Proof: This follows from the proof above except that when defining each T_i let $a \rightarrow b$ if $\pi_i(\bar{b}) - \pi_i(\bar{a}) \geq 0$ and let $a \leftarrow b$ otherwise. Since every edge of T is given a direction, we have an orientation of T rather than a sub-orientation $\square$

We say that G has a **diameter n-tuple** if there exists n distinct points $\{a_1, a_2, \ldots, a_n\}$ where $d(a_i, a_j) = diam(G)$ whenever $i \neq j$. For convenience, a 2-tuple will be called a pair and a 3-tuple a triple. For example, the 4-cycle $C = \{a, b, c, d\}$ has two diameter pairs, (a, c) and (b, d). The leaves of the graph $K_{1,n}$ form a diameter n-tuple.

The next result gives a good lower bound in many cases and allows us to find the strong isometric dimension of cycles and hypercubes exactly. These are given in the subsequent two lemmas.

Theorem 3.2.2.9 *Let G be a graph which contains no diameter 4-tuples or triples. If there are p distinct diameter pairs then $idim(G) \geq p$.*

Proof: Let $\{P_i : i = 1, \ldots, k\}$ be a realizer for G. Let the diameter pairs be (a_i, b_i) for $1 \leq i \leq k$. Let $diam(G) = d$. Thus, $l(P_i) \leq d$ for all $i = 1, \ldots, k$. Recall that the

distance between two vertices in the product is the maximum distance in a coordinate (i.e. the projection onto a path) over all coordinates. If two diameter pairs $(a_i. b_i)$ and (a_j, b_j) are separated in the same coordinate then they produce a diameter triple or 4-tuple. Thus. each pair must be separated in a distinct coordinate and so at least p paths are required and $idim(G) = k \geq p$. $\square$

Lemma 3.2.2.10 *For $n \geq 3$. $idim(C_n) = \lceil n/2 \rceil$.*

Proof: A cycle has no diameter triples or 4-tuples. Let $C = \{c_0, \ldots, c_{n-1}\}$. There are $\lfloor n/2 \rfloor$ distinct diameter pairs, specifically $c_i, c_{i+\lfloor n/2 \rfloor}$, $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$. Hence. by the previous lemma $idim(C_n) \geq \lfloor n/2 \rfloor$ for all $n \geq 3$. So for n even we have $idim(C_n) \geq \lceil n/2 \rceil$

In the case of odd cycles we can improve the lower bound by one. We label the vertices as $\{c_0, \ldots, c_{2m}\}$. We may assume that c_0 and c_m are separated by the first coordinate with c_0 being mapped to 0 and c_m to m. Now c_i. $0 \leq i \leq m$. is mapped to i. Consider c_1 and c_{m+1}. They can not be separated in the first coordinate and thus require a second coordinate. Again. in the second coordinate the vertex c_i. $1 \leq i \leq m + 1$ is mapped to $i - 1$. Inductively. consider c_j and c_{m+j}. These vertices can not be separated in the first $j - 1$ coordinates. since c_j is not mapped to 0 or m in any of these coordinates. Thus they must be separated in say the jth coordinate and c_{j+i} is mapped to i for $0 \leq i \leq m$.

For $0 \leq j \leq m - 1$. this is just making specific the proof of the preceding lemma. We now continue. Consider c_m and c_{2m}. The m coordinates of c_m are completely specified and are $m. m - 1, \ldots. 1$. If only m coordinates were to be used then this pair must be separated in the first coordinate and the first coordinate of c_{m+i} is $m - i$. Inductively again. consider c_{m+j} and c_{j-1}. $j < m$. For c_{m+j} only the $j + 1^{st}$ coordinate is m and none is 0. Thus. this pair must be separated in the $j + 1^{st}$ coordinate. Therefore. the $j + 1^{st}$ coordinate of c_{m+j+i} is $m - i$. Finally. consider c_{2m}. Now all its coordinates are specified and are $0. 1, \ldots, m - 1$. and so are the coordinates of c_{m-1} specifically $m - 1. m - 2, \ldots. 0$. But then the distance of the image of c_{2m} from

c_{m-1} is less than $m-1$ which is impossible. Hence. another coordinate is required to separate this pair and $idim(C_n) \geq \lceil n/2 \rceil$ when n is odd.

To show that $idim(C_n) \leq \lceil n/2 \rceil$, let $C = \{c_1, \ldots, c_n\}$ and also let $\{P_1, P_2, \ldots, P_{\lceil n/2 \rceil}\}$ be the set of paths of C such that $P_i = \{c_i, c_{i+1}, \ldots, c_{i+\lfloor n/2 \rfloor}\}$ for $i = 1, \ldots, \lceil \frac{n}{2} \rceil$. Also let f_i^* be the distance retraction map of G onto P_i^*. We will now show that every pair of vertices is separated on at least one of these paths.

Choose any two vertices c_a and c_b on C where $1 \leq a < b \leq n$. If $a \leq \lceil n/2 \rceil$ then c_a and c_b are separated on the path P_a^*. If $\lceil n/2 \rceil < a < b$ then both c_a and c_b lie on the path $P_{\lceil n/2 \rceil}$ and are therefore separated on $P_{\lceil n/2 \rceil}^*$. Hence. by Lemma 3.2.2.2. $idim(C_n) \leq \lceil n/2 \rceil$ and we have $idim(C_n) = \lceil n/2 \rceil$. $\qquad\square$

Note that $idim(C_n) = \lceil n/2 \rceil = |V(C_n)| - diam(C_n)$. This satisfies the upper bound on strong isometric dimension given in Corollary 3.2.2.4.

Let **upper girth** of a graph G, denoted by $ug(G)$. be the cardinality of the longest isometric cycle in G. Since the strong isometric dimension of a graph is at least as big as the strong isometric dimension of any isometric subgraph the preceding result can be used to show:

Corollary 3.2.2.11 *Let G be a finite. connected. reflexive graph. then*

$$idim(G) \geq \lceil ug(G)/2 \rceil.$$

Lemma 3.2.2.12 *Let Q_k be the hypercube with 2^k vertices. Then $idim(Q_k) = 2^{k-1}$.*

Proof: Since $Q_2 = C_4$ by Lemma 3.2.2.10 we have $idim(Q_2) = 2$. Furthermore. Q_2 has two diameter pairs.

Now. inductively. assume that Q_{k-1} has 2^{k-2} diameter pairs and that $idim(Q_{k-1}) = 2^{k-2}$. Let $Q_k = Q_{k-1} \square P_2$ where $P_2 = \{a, b\}$. Then $V(Q_k) = A \cup B$ where $A = \{(v, a) = v_a : v \in V(Q_{k-1})\}$ and $B = \{(v, b) = v_b : v \in V(Q_{k-1})\}$. Note that $d_{Q_k}(x_a, y_b) = d_{Q_{k-1}}(x, y) + 1$ and $d_{Q_k}(x_a, y_a) = d_{Q_k}(x_b, y_b) = d_{Q_{k-1}}(x, y)$.

If x and y are diameter pairs in Q_{k-1} then x_a and y_b are diameter pairs in Q_k. as are x_b and y_a. Hence, Q_k has 2^{k-1} distinct diameter pairs. Since Q_k has no diameter 4-tuples or triples then by Lemma 3.2.2.9. $idim(Q_k) \geq 2^{k-1}$.

For each $v_a \in A$ let P_{v_a} be the longest isometric path starting at v_a and let $f_{v_a}^* : Q_k \to P_{v_a}^*$ be the distance retraction map. Then each vertex in A is separated from all vertices in $A \cup B$ by at least one of $(P_{v_a}^*, f_{v_a}^*)$. Furthermore, for any $x_b, y_b \in B$ we have $f_{x_a}(y_b) - f_{x_a}(x_b) = d(x_a, y_b) - d(x_a, x_b) = d(x, y) + 1 - 1 = d(x_b, y_b)$. Hence, every pair of vertices in B are separated on at least one path. Since every pair of vertices in $V(Q_k)$ are separated on at least one P_{v_a}. by Lemma 3.2.2.2 we have $idim(Q_k) \leq 2^{k-1}$ and thus $idim(Q_k) = 2^{k-1}$. $\qquad\square$

The last few lower bounds are based mainly on neighbourhood considerations. They are in terms of the maximum degree $\Delta(G)$. the chromatic number $\chi(G)$ and the independence number of the neighbourhood of a vertex $\beta(N(v))$. For our purposes we define $\beta^N(G) = \max\{\beta(N(v)) | v \in V(G)\}$.

Theorem 3.2.2.13 *Let G be a finite, connected, reflexive graph. Then*

(a) $idim(G) \geq \lceil \log_3(\Delta(G) + 1) \rceil$ *;*

(b) $idim(G) \geq \lceil \log_2 \beta^N(G) \rceil$ *;*

(c) $idim(G) \leq \lceil \log_2(\chi(G)) \rceil$ *.*

Proof: Throughout this proof let $\{P_1, P_2, \ldots, P_k\}$ be a realizer for G. Let $F : G \to \boxtimes_{i=1}^{k} P_i$ be an isometric embedding of G in $\boxtimes_{i=1}^{k} P_i$. Recall that we denote by $\bar{v} = (v_1, v_2, \ldots, v_k)$ the vertices of $\boxtimes_{i=1}^{k} P_i$.

(a) Consider a vertex $a \in F(G)$ such that $\bar{a} = (a_1, a_2, \ldots, a_k)$. For any $\bar{x} \in N(\bar{a})$ we have $|\pi_i(\bar{x}) - \pi_i(\bar{a})| \leq 1$ for every $i = 1, \ldots, k$. Hence, $\pi_i(\bar{x}) \in \{a_i - 1, a_i, a_i + 1\}$. Since $\bar{x} \neq \bar{a}$. there are $3^k - 1$ possible vertices onto which x may be mapped. Since each vertex in $N(a)$ must be mapped to a unique vertex in $\boxtimes_{i=1}^{k} P_i$ then $|N(a)| \leq 3^k - 1$. Therefore, $\Delta(G) \leq 3^k - 1$ and thus $k \geq \log_3(\Delta(G) + 1)$.

(b)If $idim(G) = 2$ then it is easy to see that $\beta^N(G) \leq 4$.

Choose any $\bar{a} \in F(G)$ and where $\bar{a} = (a_1.a_2.\ldots a_k)$. Then for each $\bar{x} \in N(\bar{a})$. $d_{P_i}(\pi_i(\bar{x}).\pi_i(\bar{a})) \leq 1$ for each $i = 1.\ldots.k$. If $I \subseteq N(\bar{a})$ is an independent set then each pair of vertices in I must be separated by at least two on some P_i.

We proceed by induction on k assuming that in a product of $k-1$ paths $\beta^N(G) \leq 2^{k-1}$.

Let I be an independent set in $N(\bar{a})$ with $I = A \cup B \cup C$ where $A = \{\bar{x} : \pi_1(\bar{x}) = a_1 - 1\}$. $B = \{\bar{x} : \pi_1(\bar{x}) = a_1\}$. and $C = \{\bar{x} : \pi_1(\bar{x}) = a_1 + 1\}$. Since any two vertices of $A \cup B$ are not separated on P_1 they must be separated on at least one of the other $k - 1$ paths. Let R be the set of vertices in $\boxtimes_{i=2}^k P_i$ obtained from $A \cup B$ by dropping the first coordinate. Thus R is an independent set and $|R| = |A \cup B|$. By induction we have $|A \cup B| = |R| \leq 2^{k-1}$. Similarly. we have $|B \cup C| \leq 2^{k-1}$. Thus. $|A \cup B \cup C| = |I| \leq 2^k$.

(c) For each path P_i. $i = 1.\ldots.k$.in the realizer. let v_i be an end vertex of P_i. Consider vertices $x.y \in \boxtimes_{i=1}^k P_i$. We wish to place x and y in the same colour class if $d_{P_i}(v_i.\pi_i(x)) \equiv d_{P_i}(v_i.\pi_i(y))$ (mod 2) for every $i = 1.\ldots.k$. Hence. there are at most 2^k colour classes. We must now show that no two adjacent vertices have been placed in the same colour class.

Suppose x and y are two adjacent vertices in $\boxtimes_{i=1}^k P_i$. Then $\pi_i(x)$ and $\pi_i(y)$ are adjacent in P_i for at least one i. Therefore. $d_{P_i}(v_i.\pi_i(x)) - d_{P_i}(v_i.\pi_i(y)) \equiv 1$ (mod 2) for some i. Hence. x and y are in different colour classes and $\chi(\boxtimes_{i=1}^k P_i) \leq 2^k$. In fact. since $\boxtimes_{i=1}^k P_i$ contains the complete graph K_{2^k} as a subgraph. $\chi(\boxtimes_{i=1}^k P_i) = 2^k$.

Since G is a subgraph of $\boxtimes_{i=1}^k P_i$ then $\chi(G) \leq \chi(\boxtimes_{i=1}^k P_i)$ and $\log_2(\chi(G)) \leq \log_2(\boxtimes_{i=1}^k P_i) = k = idim(G)$. $\square$

Note that in the proof of part b). we have that $|A| + |B| \leq 2^{k-1}$ and. similarly. $|B| + |C| \leq 2^{k-1}$. Thus $|A| + 2|B| + |C| \leq 2^k$. Thus. for $|I| = |A| + |B| + |C| = 2^k$ we must have $|B| = 0$. Therefore. an independent set of this size is unique and is $\{(a_i + \epsilon_i)_{i=1}^k : \epsilon_i = 1 \text{ or } -1\}$.

3.2.3 The Strong Isometric Dimension of Trees

We know show that the strong isometric dimension of trees is bounded in terms of the number of leaves. The main result gives the bounds and the proof follows by showing that for every tree T there are two associated trees T_1 and T_2, obtained from T by contraction or subdivision of edges such that $idim(T_1) \leq idim(T) \leq idim(T_2)$.

Theorem 3.2.3.1 *Let T be a tree with k leaves. Then*

$$\lceil \log_2 k \rceil \leq idim(T) \leq 2 \lceil \log_2 k \rceil.$$

The theorem is proved by a series of lemmas. The first result is the basic manipulation technique giving us a means of associating with a tree two other trees whose strong isometric dimension is easier to calculate.

If ab is an edge of G then $G \circ ab$ denotes the graph after the edge has been contracted.

Lemma 3.2.3.2 *Let G be a graph and $ab \in E(G)$ be a cut edge of G. Then*

(a) $idim(G \circ ab) \leq idim(G)$; and

(b) $idim(H) = idim(G)$ where H is the graph obtained by subdividing the edge ab.

Proof: There are some general results needed in proving both parts of the lemma. Let $\{P_i : 1 \leq i \leq k\}$ be a realizer for G. Let $\bar{a} = (a_1, a_2, \ldots, a_k)$ and $\bar{b} = (b_1, b_2, \ldots, b_k)$ and put $\Delta(ab) = \bar{b} - \bar{a} = (b_1 - a_1, \ldots, b_k - a_k)$. Relabelling P_i if necessary allows to assume that $b_i \geq a_i$ for all i. If $P_i = \{p_1, p_2, \ldots, p_m\}$ then let $Q_i = \{p_1, p_2, \ldots, p_m, 1 + p_m\}$

Let G_a, G_b be the connected components containing a and b, respectively, when the edge ab is deleted. Suppose that $x \in G_a$ and $y \in G_b$. Let $P(x,y) = \{x = x_0, x_1, \ldots, x_d = y\}$ be an isometric path between x and y in G. Then we have $\pi_i(\bar{y}) - \pi_i(\bar{x}) = (\pi_i(\bar{x}_d) - \pi_i(\bar{x}_{d-1})) + (\pi_i(\bar{x}_{d-1}) - \pi_i(\bar{x}_{d-2})) + \cdots + (\pi_i(\bar{x}_1) - \pi_i(\bar{x}_0))$. Since every path from x to y passes through the edge ab then $\Delta(ab)_i = \pi_i(\bar{x}_j) - \pi_i(\bar{x}_{j-1})$ for some $0 \leq j \leq d$. Hence.

$$-(d-1) + \Delta(ab)_i \leq \pi_i(\bar{y}) - \pi_i(\bar{x}) \leq d - 1 + \Delta(ab)_i. \tag{1}$$

If $\bar{x}$ and $\bar{y}$ are separated by the ith coordinate then $|\pi_i(\bar{y}) - \pi_i(\bar{x})| = d$. From the inequality above, we see that this is true only if $\pi_i(\bar{y}) > \pi_i(\bar{x})$ and $\Delta(ab)_i = 1$.

(a) Note that the distance between $x \in G_a$ and $y \in G_b$ is reduced by one after the edge is contracted. Let $g : G \circ ab \rightarrow \boxtimes_{i=1}^{k} P_i$ be defined by $g(v) = \bar{v}$ if $v \in G_a$ otherwise $g(v) = \bar{v} - \Delta(ab)$. Note that $g(ab)$ is well defined: $g(ab) = \bar{a}$ by the first part of the definition of g and equals $\bar{b} - \Delta(ab)$ by the second part, but this latter is equal to $\bar{a}$. We claim that g is an isometric embedding of $G \circ ab$.

If $x, y \in G_a \setminus \{a, b\}$ or $x, y \in G_b \setminus \{a, b\}$ then $d(x, y) = d(g(x), g(y))$ since $\pi_i(g(x)) - \pi_i(g(y)) = \pi_i(\bar{x}) - \Delta(ab)_i - \pi_i(\bar{y}) + \Delta(ab)_i = \pi_i(\bar{x}) - \pi_i(\bar{y})$.

If $x \in G_a$ and $y \in G_b$ then by equation (1), $|\pi_i(g(y)) - \pi_i(g(x))| = |\pi_i(\bar{y}) - \Delta(ab)_i - \pi_i(\bar{x})| \leq d - 1$ for all $i = 1, \ldots, k$. Therefore, if i is a separating coordinate of $\bar{x}$ and $\bar{y}$ we have $\pi_i(g(y)) - \pi_i(g(x)) = d - 1$. Hence, i is also a separating coordinate of $g(x)$ and $g(y)$.

Since g is an isometric embedding and thus an edge-preserving map and since every pair of vertices are separated in at least one coordinate then by Lemma 3.2.2.2, $idim(G \circ ab) \leq k = idim(G)$.

(b) H is obtained by removing the edge ab in G and then adding a vertex, c, and the edges ac and bc. Let G_a and G_b be the connected components containing a and b, respectively, when the edge ab is deleted in G. Note that the distance between $x \in G_a$ and $y \in G_b$ is increased by one once the edge is subdivided.

Let $g : H \rightarrow \boxtimes_{i=1}^{k} Q_i$ be defined by $g(v) = \bar{v}$ if $v \in G_a$, $g(v) = \bar{v} + \Delta(ab)$ if $v \in G_b$, and $g(c) = \bar{b}$. We claim that g is an isometric embedding of H.

If $x, y \in G_a$ or $x, y \in G_b$ then $d(x, y) = d(g(x), g(y))$ since $\pi_i(g(x)) - \pi_i(g(y)) = \pi_i(\bar{x}) + \Delta(ab)_i - \pi_i(\bar{y}) - \Delta(ab)_i = \pi_i(\bar{x}) - \pi_i(\bar{y})$.

Now suppose that $x \in G_a$ and $y \in G_b$. Then from (1) we have $|\pi_i(g(y)) - \pi_i(g(x))| = |\pi_i(\bar{y}) + \Delta(ab)_i - \pi_i(\bar{x})| \leq d(x, y) + 1$. If i is a separating coordinate of $\bar{x}$ and $\bar{y}$ then $\pi_i(\bar{y}) > \pi_i(\bar{x})$ and $\Delta(ab)_i = 1$. Therefore $\pi_i(g(y)) - \pi_i(g(x)) = \pi_i(\bar{y}) + \Delta(ab)_i - \pi_i(\bar{x}) = d + 1$ and so i is also a separating coordinate of $g(x)$ and $g(y)$.

For any $x \in G_a$. $d(g(c).g(x)) = d(\bar{b}.\bar{x}) = d(b.x)$. For any $y \in G_b$. $d(g(c).g(y)) = d(\bar{b}.\bar{y} + \Delta(ab)) = d(\bar{a} + \Delta(ab).\bar{y} + \Delta(ab)) = d(\bar{a}.\bar{y}) = d(a.y)$.

Since g is edge-preserving and every pair of vertices are separated in at least one coordinate $idim(H) \leq k = idim(G)$. Furthermore. $G = H \circ ac$. Hence. by part (a) of the lemma $idim(H) = idim(G)$. $\square$

Proof of first inequality of Theorem 3.2.3.1. Let $T = T_0$ be a tree with k leaves. We apply Lemma 3.2.3.2 a). to an interior edge of T and call the result T_1. We continue this and produce a sequence of trees ending with T_j. a star with the same number of leaves as T_0. From Theorem 3.2.2.13 b), we know that $idim(T_j) = \lceil \log_2 k \rceil$. Since T_j has the same number of leaves as T, we therefore have what we require. i.e.

$$idim(T) \geq idim(T_1) \geq \cdots \geq idim(T_k) = \lceil \log_2 k \rceil$$

Proof of second inequality of Theorem 3.2.3.1. We first construct from T the required associated tree which has maximum degree three and no vertices of degree two and whose isometric dimension is at least that of T.

Suppose there is a vertex v in T such that $N(v) = \{v_1.v_2.\ldots.v_n\}$ and $n \geq 4$. Let T_1 be the graph obtained by removing the vertex v and adding the vertices $x.y$ and the edges $\{xv_1.xv_2.xy.yv_3.\ldots.yv_n\}$. Note that $deg(x) = 3$ and $deg(y) = n - 1$. Since $T = T_1 \circ xy$ then from Lemma 3.2.3.2 a). we have $idim(T) \leq idim(T_1)$. We continue this producing a sequence of trees ending with T_j. a tree with maximum degree three. If T has no vertices of degree greater than three. let $T_j = T$. In either case.

$$idim(T) \leq idim(T_1) \leq \cdots \leq idim(T_j).$$

Now suppose that T_j has a degree two vertex. v. Let $T_{j+1} = T_j \circ (vw)$ for some $vw \in E(T)$. Note that by Lemma 3.2.3.2 b). we have that $idim(T_{j+1}) = idim(T_j)$. We can continue this thereby obtaining a sequence of trees ending with T_{j+r} such that

$$idim(T_j) = idim(T_{j+1}) = \cdots = idim(T_{j+r}).$$

Hence. there is a tree $T' = T_{j+r}$ with the same number of leaves as T such that all vertices of T' are degree one or three and $idim(T') \geq idim(T)$.

Suppose S is a tree such that all vertices have degree one or three. Obviously. if S has only two leaves it is an edge and $idim(S) = 1$. Suppose that S has four leaves. There is only one case to consider. We can find two orientations of S such that there is a directed path between every pair of vertices. Hence. $idim(S) \leq 2$. (In fact, $idim(S) = 2$.)
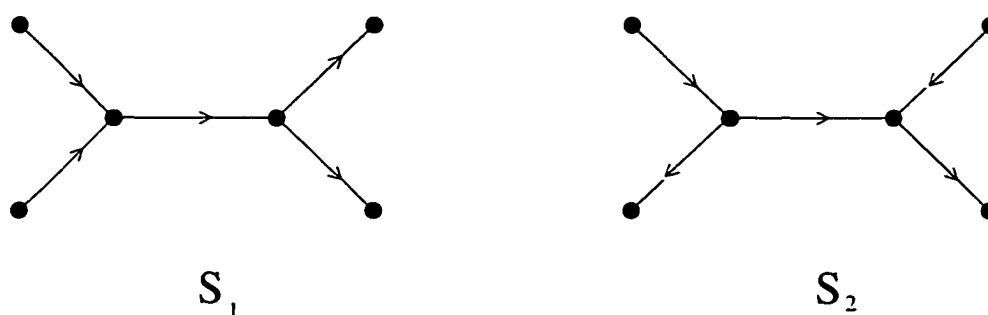

$$S_1 \qquad\qquad S_2$$

Figure 3.5: The Orientations of the Associated Four-leaved Tree.

Lemma 3.2.3.3 *Let S be a tree with 2^n leaves. $n \geq 1$ and all vertices of degree one or three. Then there exists a degree three vertex. v. such that each connected component of $S \setminus \{v\}$ has at most 2^{n-1} leaves.*

Proof: Suppose for every vertex. $v \in V(S)$ there is one component in $S \setminus \{v\}$ with more than 2^{n-1} leaves. Let x be a vertex such that the number of leaves in that component is minimized. Let A. B. and C be the three components of $S \setminus \{x\}$ where A has more than 2^{n-1} leaves. Let y be the vertex in A which is adjacent to x. Obviously. $deg(y) = 3$ since A contains more than one vertex. Consider the components of $S \setminus \{y\}$. The component containing x is simply $B \cup C \cup \{x\}$. Since A has greater than 2^{n-1} leaves. $B \cup C \cup \{x\}$ has less than 2^{n-1} leaves. Furthermore. the other two components of $S \setminus \{y\}$. together with y form A. Hence. each of these two components has fewer leaves than A. This contradicts the choice of x. Hence.

there exists a vertex v such that the components of $S \setminus \{v\}$ each contain at most 2^{n-1} leaves. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

We now continue the proof and assume that if S is a tree with 2^m leaves where $m \leq n - 1$ and all vertices of degree one or three, then $idim(S) \leq 2m$.

Now consider a tree S with 2^n leaves and all vertices of degree one or three. Let v be a vertex such that all components of $S \setminus \{v\}$ have at most 2^{n-1} leaves. Each component has an associated tree with all vertices of degree one or three. By induction, each of these associated trees has strong isometric dimension at most $2n - 2$. Hence, by construction of the associated tree, each component also has strong isometric dimension at most $2n - 2$. Then, by Corollary 3.2.2.8, each component has a set of $2n - 2$ orientations such that each pair of vertices in that component has a directed path between them in at least one of the orientations. Let A, B, and C be the three components. The component A has orientations $\{A_1, A_2, \ldots, A_{2n-2}\}$. Let the orientations of B and C be denoted similarly.

For $i = 1, \ldots, 2n - 2$ we can define the orientation S_i as follows: if ϵ is an edge in A (respectively, B, C) assign ϵ the same direction in S_i as it has in A_i (B_i, C_i). Direct the three edges incident with v arbitrarily.

For S_{2n-1} we wish to have directed paths from all the vertices in A to v and from v to all the vertices in $B \cup C$. This can be accomplished by directing the edge connecting v with A toward v, and then for each edge xy in $E(A)$ direct $x \to y$ if $d(v, x) > d(v, y)$ and $y \to x$ otherwise. Then direct the other two edges incident with v away from v and for each edge (xy) in $B \cup C$ direct $x \to y$.

For S_{2n} we wish to have directed paths from all the vertices in $A \cup B$ to v and from v to all the vertices in C. This orientation is achieved in a manner similar to the construction of S_{2n-1}.

We now verify that there is a directed path between every pair of vertices in at least one of these $2n$ orientations of S. Suppose we have two vertices x, y such that both are in A (respectively, B, C). Obviously there is a path between the two in one of the first $2n - 2$ orientations of S. Suppose that $x \in A$ and $y \in B \cup C$. Then there

is a path from x to y in S_{2n-1}. Suppose $x \in B$ and $y \in C$. Then there is a path from x to y in S_{2n}. Finally, there is a directed path between $x \in A \cup B \cup C$ and v in both S_{2n-1} and S_{2n}. Hence, there is a directed path between every pair of vertices. Therefore, by Corollary 3.2.2.8 $idim(S) \leq 2n$.

To complete the proof, note that the given tree T with k leaves, where $2^{m-1} < k \leq 2^m$, can be isometrically embedded in a tree S with 2^m leaves by adding the extra leaves at any interior vertex. There is also a tree S' associated with S which only has vertices of degree one and degree three. We know that $idim(T) \leq idim(S)$. Lemma 3.2.3.2 part b, gives that $idim(S) \leq idim(S')$ and the preceding argument shows that $idim(S') \leq 2m$. Putting this together we obtain the desired result:

$$idim(T) \leq 2\lceil \log_2 k \rceil.$$

$\square$

3.2.4 Problems.

Both cycles and hypercubes have an strong isometric dimension of $\lceil |V(G)|/2 \rceil$.

Problem 3.2.4.1 *Is there a graph G such that $idim(G) > \lceil |V(G)|/2 \rceil$?*

Is the upper bound for trees given in Theorem 3.2.3.1 the correct one? Both binary trees and caterpillars have a strong isometric dimension of $\lceil \log_2 t \rceil$ where t is the number of leaves.

Problem 3.2.4.2 *Is there a tree T such that $idim(T) > \lceil \log_2 t \rceil$?*

3.3 Cop Number of Graphs with Strong Isometric Dimension Two

In this section we show that any graph with strong isometric dimension two has cop number at most two. The strong product of two paths is known to have cop

number one while the 4-cycle has cop number two. Hence. this bound is sharp. In proving this bound we use the canonical retraction map. We use this result to obtain an upper bound of $diam(G) + 3$ on the cop number of any graph G with strong isometric dimension three. We then show that there are graphs with arbitrarily high genus which have strong isometric dimension two. Therefore. high genus does not necessarily translate to a high cop number. Finally. we pose a problem concerning the possibility of extending the result for strong isometric dimension two graphs. to graphs with higher strong isometric dimension.

3.3.1 Results

Theorem 3.3.1.1 *Let G be a finite connected graph. If $idim(G) = 2$ then $c(G) \leq 2$.*

Proof: Suppose $idim(G) = 2$. then G is an isometric subgraph of $P_n \boxtimes P_m$ for some $n. m \geq 1$. We may assume that $n + m$ is minimum. Let $V(P_n) = \{1.....n\}$ and let $f : V(G) \rightarrow \{(i.j) : 1 \leq i \leq n. 1 \leq j \leq m\}$ be an edge preserving map such that $f(G)$ is an isometric subgraph of $P_n \boxtimes P_m$. We identify G with this subgraph.

First. we partition the vertices of G according to their first coordinate. Let $A_i = \{v \in V(G) : \pi_1(v) = i\}$ for all $i = 1.2.....n$. We claim that for each $i = 1.2.....n$ there exists an isometric path of G which contains all the vertices of A_i. To see this. let $A = A_i$ for some $1 \leq i \leq n$. Order the vertices of A by their second coordinate. i.e. $A = \{v_1. v_2.....v_k\}$ where $\pi_2(v_j) < \pi_2(v_{j+1})$ for all $j = 1.2....k - 1$. For each $j = 1.2.....k - 1$. let Q_j be an isometric path in G from v_j to v_{j+1}. Now let $R = Q_1 \cup \cdots \cup Q_{k-1}$. Note that for any pair of vertices u and v in $A. d(u. v) = |\pi_2(u) - \pi_2(v)|$. Therefore. $\sum_{j=1}^{k-1} d(v_j. v_{j+1}) = \sum_{j=1}^{k-1}(\pi_2(v_{j+1}) - \pi_2(v_j)) = \pi_2(v_k) - \pi_2(v_1) = d(v_1. v_k)$. and the path R is isometric.

Hence. there exists a set of isometric paths $R_1.....R_n$ such that for each $i = 1.....n$ all the vertices of A_i are on the path R_i. By Lemma 2.2.0.6. we know that one cop moving on R_i can. after a finite number of moves. prevent the robber from moving onto R_i. Therefore. one cop can "protect" the vertices in A_i by moving on the larger set R_i. Now suppose that we have two cops. Place one cop on a vertex in

A_1 and the second on a vertex in A_2. After a finite number of moves the cops can prevent the robber from moving onto A_1 and A_2. Therefore, once the cops have A_1 and A_2 protected, the robber, if uncaptured, must occupy a vertex in A_i for some $i > 2$.

By induction, assume that for some $k \leq n - 1$ the cops have the sets A_{k-1} and A_k protected and the robber occupies a vertex in A_i for some $i > k$. The cop protecting A_{k-1} now leaves R_{k-1} and moves to protect the set A_{k+1}. Meanwhile, the other cop continues to protect the vertices of A_k. The robber can obviously not move from a vertex in A_i for some $i > k$ to a vertex in A_j for $j < k$ without moving through a vertex in A_k. Since a move onto A_k would result in his immediate capture, we can assume that the robber restricts his movement to the vertices in $A_{k+1} \cup A_{k+2} \cup \cdots \cup A_n$. Once the set A_{k+1} is protected, the robber's movement is further restricted to $A_{k+2} \cup A_{k+3} \cup \cdots \cup A_n$.

If we continue in this manner, the cops will eventually protect the sets A_{n-1} and A_n and the robber will have no vertex which he can safely occupy. Therefore, two cops are sufficient to apprehend the robber. □

We can extend this result to obtain an upper bound for the cop number of a graph of strong isometric dimension three.

Theorem 3.3.1.2 *Let G be a finite connected graph. If $idim(G) = 3$ and G is an isometric subgraph of $P_n \boxtimes P_m \boxtimes P_l$ where $n \leq m \leq l$ then $c(G) \leq n + 2$.*

Proof: Let $f : V(G) \rightarrow \{(i,j,k) : 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq l\}$ be an edge preserving map such that $f(G)$ is an isometric subgraph of $P_n \boxtimes P_m \boxtimes P_l$. Identify G with this subgraph. We wish to partition the vertices of G according to their first and second coordinates. First, let $B_j = \{v \in V(G) : \pi_2(v) = j\}$ and then let $A_{i,j} = \{v \in B_j : \pi_1(v) = i\}$. Hence, $A_{i,j}$ is the set of all vertices in $V(G)$ with i as their first coordinate and j as their second coordinate. As in Theorem 3.3.1.1, there is an isometric path of G, $R_{i,j}$, that contains all the vertices of $A_{i,j}$. If $A_{i,j}$ is empty for some i, j then let $R_{i,j}$ be a path with zero vertices. Any cop assigned to an empty

set may occupy any vertex in G and still be considered to be protecting that set. Hence. the set of paths $S_j = \{R_{1,j}, R_{2,j}, \ldots, R_{n,j}\}$ covers all the vertices of B_j. and a set of n cops can prevent a robber from moving onto B_j.

Now suppose we have a set of $n+2$ cops. $\{c_1, \ldots, c_n, c_{n+1}, c_{n+2}\}$. Let the cop c_i protect the set $A_{i,1}$ for $i = 1, \ldots, n$. Let c_{n+1} protect $A_{1,2}$ and c_{n+2} protect $A_{2,2}$. The cops now force the robber to occupy some vertex in $B_2 \cup B_3 \cdots \cup B_m$. Suppose. by induction. that for some j, $n+1$ cops protect the sets in $A_p = \{A_{1,j+1}, A_{2,j+1}, \ldots, A_{p,j+1}\} \cup \{A_{p,j}, A_{p+1,j}, \ldots, A_{n,j}\}$. The $(n+2)^{nd}$ cop may be protecting a set or may be in transition. If the robber is on some vertex. x, in B_{j+1} then he must be in $A_{i,j+1}$ for some $i \geq p+1$. Hence. any vertex in B_j adjacent to x is protected by one of the cops. Therefore. if the robber occupies a vertex in the set $B_{j+1} \cup \cdots \cup B_m$. then it is impossible for him move out of that set. The $(n+2)^{nd}$ cop can now move to protect the set $A_{p+1,j+1}$ and thus. A_p can be replaced with A_{p+1}. Eventually all of B_{j+1} is protected by cops in B_{j+1} and the robber will be confined to vertices in $B_{j+2} \cup \cdots \cup B_m$. Note this requires only n cops. Move the other two cops to protect $A_{1,j+2}$ and $A_{2,j+2}$ and repeat this procedure.

By induction. the cops will eventually protect B_m and the robber will have no vertex which he can safely occupy. Hence. the robber will be apprehended by one of the cops and $c(G) \leq n+2$. $\qquad\square$

Corollary 3.3.1.3 *If* $idim(G) = 3$ *then* $c(G) \leq diam(G) + 3$.

Proof: Suppose G is an isometric subgraph of $P_n \boxtimes P_m \boxtimes P_l$ where $n \leq m \leq l$ and $n + m + l$ is minimized. Then $l - 1 = diam(G)$ and. by Theorem 3.3.1.2. $c(G) \leq n+2 \leq l+2 = diam(G) + 3$. $\qquad\square$

We believe that the results in Theorem 3.3.1.2 and Corollary 3.3.1.3 are not the best possible.

3.3.2 Genus

We saw in the previous section that if $idim(G) = 2$ then $c(G) \leq 2$. We can contrast this result with the genus result of Quilliot [46], $c(G) \leq 2g + 3$. It is the case that for arbitrarily high genus. g. there is a graph of that genus with strong isometric dimension two. and. thus. restricted clique size. To demonstrate this we use the following theorem. where $g(G)$ denotes the genus of a graph G.

Theorem 3.3.2.1 (Battle, Harary, Kodama & Youngs [6]) *If two graphs G and H are joined at a vertex v (called $G *_v H$) then $\rho(G *_v H) = \rho(G) + \rho(H)$.*
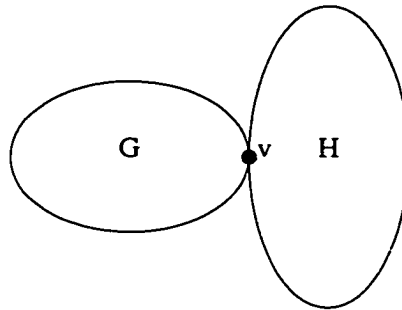

Figure 3.6: $G *_v H$

Now consider the graph $P_5 \boxtimes P_4$ in Figure 3.7. It has a vertex set of size 20 and an edge set of size 55. For any planar graph. G. $|E(G)| \leq 3|V(G)| - 6$. Hence. this graph has genus at least 1. If we take n copies of this graph and associate vertices as in Figure 3.8. then. by Theorem 3.3.2.1. the resulting graph has genus at least n. By adding vertices and edges to this graph we can obtain an isometric subgraph of $P_{5n} \boxtimes P_{4n}$ that has genus at least n and cop number at most 2.
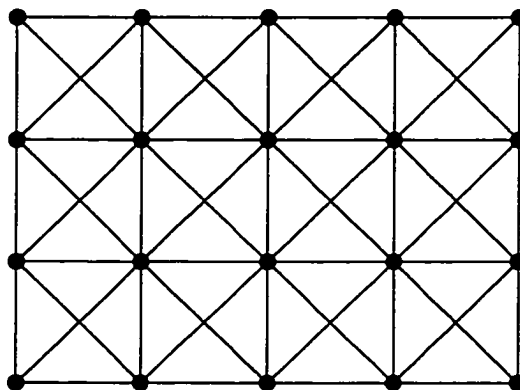
Figure 3.7: $P_5 \boxtimes P_4$
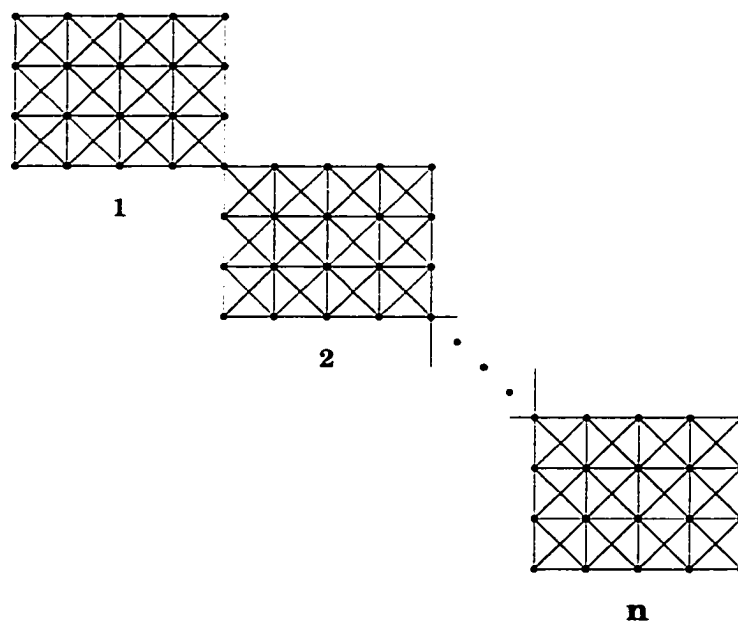
Figure 3.8: n copies of $P_5 \boxtimes P_4$ with associated vertices.

3.3.3 Problems

We saw that for any graph with strong isometric dimension two. we could partition the vertices into sets $A_1, A_2, \ldots$ and then find an isometric path containing each A_i. Hence. the graph could be covered with subgraphs of strong isometric dimension one where each subgraph "cut" the graph into two parts. This resulted in a winning strategy for two cops in the graph. If this technique is to be extended to graphs of higher strong isometric dimension we must first answer the following questions:

Problem 3.3.3.1 *Suppose G is an isometric subgraph of $P_n \boxtimes P_m \boxtimes P_l$. For each $i = 1, 2, \ldots, n$. does there exist an isometric subgraph H_i in G such that $idim(H_i) \le 2$ and H_i contains all the vertices of $A_i = \{v \in V(G) : \pi_1(v) = i\}$?*

Problem 3.3.3.2 *If the answer to Problem 3.3.3.1 is yes for graphs with strong isometric dimension three. then can we cover a graph G where $dim(G) = d$ with similar isometric subgraphs $\{H_1, H_2, \ldots\}$ such that $idim(H_i) \le d - 1$?*

3.4 Isometric Subgraphs of the Strong Product of a Graph and a Tree

We now show that if G is an isometric subgraph of $H \boxtimes T$ for any graph H and tree T. then we can cover the graph G with isometric trees. In Section 2.3 we defined the minimum number of isometric trees required to cover a graph to be the tree-precinct number of a graph. We determine that $|V(H)|$ is an upper bound for $Tpn(G)$ for any isometric subgraph G of $H \boxtimes T$. More important. however. is how we can use these retracts to establish an upper bound on the cop number of G. We know that $c(G) \le Tpn(G)$. but we will show an improved upper bound in Section .

Suppose H is a graph such that $V(H) = \{v_1, \ldots, v_n\}$ and T is a tree. Suppose there is an edge-preserving map $f : V(G) \to V(H \boxtimes T) = \{(v_i, w) | w \in V(T). i = 1, \ldots, n\}$ such that $f(G)$ is an isometric subgraph of $H \boxtimes T$. We will associate G with the subgraph $f(G)$.

Lemma 3.4.0.3 *Suppose G is an isometric subgraph of $H \boxtimes T$ for some graph H and tree T. Let $A_i = \{v \in V(G) | \pi_1(v) = v_i\}$ for all $1 \le i \le n$. Then for each $i = 1, 2, \ldots, n$ there exists a tree T_i containing all the vertices in A_i that is an isometric subgraph of G. Moreover, $\pi_2(T_i)$ is an isometric embedding of T_i in T.*

Proof: Let $A = A_i$ for any $i = 1, \ldots, n$ and let $\{B_1, B_2, \ldots, B_m\}$ be a set of subgraphs of G defined as follows:

1. $B_1 = \{x\}$ for some $x \in A$.

2. $A \subseteq B_m$, but $A \not\subseteq B_k$ for any $k < m$.

3. $B_{k+1} = B_k \cup P(u, v)$ where $P(u, v)$ is an isometric path from u to v in G such that $u \in B_k$, $v \in A \setminus B_k$ and the distance $d_T(\pi_2(u), \pi_2(v))$ is the minimum among all such pairs of vertices.

We claim that the graph, B_m, is an isometric subgraph of G and $\pi_2(B_m)$ is an isometric embedding of B_m in T.

We need to show that for every pair of vertices, x and y, in $V(B_m)$ at least one isometric path from x to y in G is also in B_m and $d_T(\pi_2(x), \pi_2(y)) = d(x, y)$. The subgraph B_1 is obviously an isometric subgraph of G and $\pi_2(B_1)$ is an isometric embedding of B_1 in T. If $|A| \ge 2$ then B_2 is an isometric path in G. The end vertices of the path B_2 are both in A, and therefore, separated on T. Since B_2 is an isometric path, then every pair of vertices in B_2 must be separated on T.

By induction, assume that for some $k \ge 1$ and all $1 \le j \le k$, B_j is an isometric subgraph of G and every pair of vertices in B_j is separated on T. For any pair of vertices x, y in G we will let $d(x, y)$ denote their distance in G, $d_i(x, y)$ denote their distance in B_i for all $i = 1, 2, \ldots, m$ and let $d(\pi_2(x), \pi_2(y))$ denote the distance between $\pi_2(x)$ and $\pi_2(y)$ in T.

Case 1: Let x and y be a pair of vertices such that $x, y \in P(u, v)$. Obviously, $d_{k+1}(x, y) = d(x, y)$ since $P(u, v)$ contains an isometric path from x to y in G. If the vertices u and v are separated on T then x and y will also be separated on T. Suppose

u and v are not separated on T. Then $d(u,v) = d(\pi_1(u),\pi_1(v)) > d(\pi_2(u),\pi_2(v))$. Since every pair of vertices in A are separated on T. then it must be the case that u is not in A. Therefore, for some $j = 1,2,\ldots,k-1$. $u \notin B_j$ and $u \in B_{j+1}$. Suppose $B_{j+1} = B_j \cup P(r,s)$ where $r \in B_j$ and $s \in A$. Then u lies on the path $P(r,s)$. but u does not equal either r or s.

Since every pair of vertices in B_j is separated on T. then $d(u,s) = d(\pi_2(u),\pi_2(s)) \geq d(\pi_1(u),\pi_1(s))$. Since $s,v \in A$ we have $\pi_1(s) = \pi_1(v)$. Hence. $d(\pi_1(u),\pi_1(s)) = d(\pi_1(u),\pi_1(v)) > d(\pi_2(u),\pi_2(v))$. Therefore, $d(\pi_2(u),\pi_2(s)) > d(\pi_2(u),\pi_2(v))$. and

$$
\begin{aligned}
d(\pi_2(r),\pi_2(s)) &= d(\pi_2(r),\pi_2(u)) + d(\pi_2(u),\pi_2(s)) \\
&> d(\pi_2(r),\pi_2(u)) + d(\pi_2(u),\pi_2(v)) \\
&\geq d(\pi_2(r),\pi_2(v)).
\end{aligned}
$$

But $d(\pi_2(r),\pi_2(s)) > d(\pi_2(r),\pi_2(v))$ contradicts the choice of s and r as the pair with minimum separation on T. Hence. all vertices on $P(u,v)$ are separated on T. and $d_{k+1}(x,y) = d(\pi_2(x),\pi_2(y))$.

Case 2: Now consider a pair x,y such that $x \in B_k$ and $y \in P(u,v)$. Let P_1 be an isometric path from x to u in B_k and let P_2 be the isometric path from u to y on $P(u,v)$. We know the pairs (x,u) and (u,y) are separated on the tree by induction and Case 1. respectively. Note that $P_1 \cup P_2$ is a path in B_{k+1} since no P_1 is contained in B_k and the path P_2 only has the vertex u in B_k.

Now. suppose that the path $P = P_1 \cup P_2$ is not separated on the tree. Since π_2 is edge-preserving. it will map the path P onto a path in T. Since P is not separated on T. this means that P is mapped to a shorter path in T. Hence. for some pair of vertices (a,b) in P. $\pi_2(a) = \pi_2(b)$. Since P_1 and P_2 are separated on T it must be the case that where $a \in P_1$ and $b \in P_2$ and neither a nor b is equal to u. Hence. $d(\pi_2(a),\pi_2(v)) = d(\pi_2(b),\pi_2(v)) < d(\pi_2(u),\pi_2(v))$. Since $a \neq u$ and $a \in B_k$. this contradicts the choice of u and v as the pair with minimum separation on T. Hence. P is separated on T which means that $d(\pi_2(x),\pi_2(v)) = l(P) \geq d(x,y)$. Since $d(x,y) \geq d(\pi_2(x),\pi_2(y))$ then $l(P) = d(x,y) = d(\pi_2(x),\pi_2(v))$. Hence. P is an

isometric path and x and y are separated on T.

Hence. for every pair of vertices x and y in B_{k+1} at least one isometric path from x to y in G also lies in B_{k+1}. Therefore. B_{k+1} is an isometric subgraph of G. Furthermore. every pair of vertices in B_{k+1} is separated on T. This means that $d(x,y) = d(\pi_2(x),\pi_2(y))$. Therefore. $\pi_2(B_{k+1})$ is an isometric embedding of B_{k+1} in T.

By induction. we conclude that $T_i = B_m$ is an isometric subgraph of G containing all the vertices in $A = A_i$ for some $i = 1,2,\ldots.n$. It is also the case that $\pi_2(T_i)$ is an isometric embedding of T_i in T. $\qquad\square$

Obviously. each T_i is a tree since there is an isometric embedding of T_i in T.

Lemma 3.4.0.4 *If G is an isometric subgraph of $H \boxtimes T$ and $\{T_1.T_2.\ldots.T_n\}$ are the subgraphs of G described above. then T_i is a retract of G for all $i = 1.2.\ldots.n$.*

Proof: The tree T_i is an isometric subgraph of G for all $i = 1.2.\ldots.n$. and by Lemma 2.2.0.5. whenever a tree is an isometric subgraph of G it is a retract of T. Therefore. T_i is a retract of G for all $i = 1.2.\ldots.n.$. $\qquad\square$

In order to find a retraction mapping of G onto T_i we first consider a retraction mapping on T onto $\pi_2(T_i)$. One possible retraction map. $g_i : T \to \pi_2(T_i)$ maps each vertex in T to the nearest vertex in $\pi_2(T_i)$. Hence. we put $g_i(x) = y$ whenever

$$d(x,y) = \min_{u \in \pi_2(T_i)}(d(x,u)).$$

Then g_i is the identity on $\pi_2(T_i)$ and maps every vertex in $T \setminus \pi_2(T_i)$ to the nearest leaf in $\pi_2(T_i)$.

Let $T' = \pi_2(T_i)$ and $g = g_i$ for some $i = 1.2.\ldots.n$. The map g is well defined since $d(x,y) = d(x,z) = \min_{u \in T'}(d(x,u))$ for some x in T, and some pair y and z in T' implies that $y = z$. To show this. we let $P(x,y)$ and $P(x,z)$ be the paths from x to y and from x to z in T. respectively. Since y and z are at minimum distance from x. then $P(x,y) \cap T' = \{y\}$ and $P(x,z) \cap T' = \{z\}$. If x and y are distinct vertices

in T' then there is a path $P(y,z)$ from y to z which lies entirely in T'. This implies that $P(x,y) \cup P(x,z) \cup P(y,z)$ contains a cycle. This is impossible since T is a tree. Hence, $y = z$.

Now, we verify that g_i is a retraction. Let $T' = \pi_2(T_i)$ and $g = g_i$ for some $i = 1,2,\dots,n$. Since g is the identity on T', it suffices to show that g is edge-preserving. Consider two vertices x and y in T such that $x \sim y$. If x and y are both in T' then $g(x) \sim g(y)$ since g is the identity on T'. Suppose x is in T', but y is not. Then x is the closest vertex to y in T'. Since the map is well defined then $g(y) = x$. Suppose that neither x nor y is in T' and $g(x) \neq g(y)$. Then there is a path $P(g(x),g(y))$ from $g(x)$ to $g(y)$ which lies entirely in T'. If x and y are both in T' then $g(x) \sim g(y)$ since g is the identity on T'. Suppose x is in T', but y is not. Then x is the closest vertex to y in T'. Since the map is well defined then $g(y) = x$. Suppose that neither x nor y is in T' and $g(x) \neq g(y)$. Then there is a path $P(g(x),g(y))$ from $g(x)$ to which lies entirely in T'. The paths $P(x,g(x))$ and $P(y,g(y))$ intersect T' at $g(x)$ and $g(y)$, respectively. This implies that there is a cycle contained in $P(x,g(x)) \cup \{xy\} \cup P(y,g(y)) \cup P(g(x),g(y))$. This contradicts the fact that T is a tree. Hence, $g(x) = g(y)$ and g is edge-preserving.

Since g is edge-preserving and the identity on T' it is, therefore, a retraction. Hence, $\pi_2(T_i)$ is a retract of T for all $i = 1,2,\dots,n$.

Then, similarly, a retraction map, $f_i : G \to T_i$, for all $i = 1,2,\dots,n$ can be defined as follows: for $v \in V(G)$ put $f(v) = w$ where

$$d(\pi_2(v),\pi_2(w)) = \min_{u \in B}(d(\pi_2(v),\pi_2(u))).$$

Now, the map f_i can be written as the composition of maps $f_i(G) = h_i(g_i(\pi_2(G)))$ where $h_i : \pi_2(T_i) \to T_i$ is the inverse of π_2 restricted to T_i ($h_i(v) = w$ where $\pi_2(w) = v$). The map h_i is edge-preserving since T_i is separated on T. Since π_2, g_i and h_i are all edge-preserving, then f_i is edge-preserving. Furthermore, $f_i(T_i) = T_i$ since g_i is the identity on $\pi_2(T_i)$ and h_i is the inverse of π_2 restricted to T_i. Therefore, f_i is a retraction map of G onto T_i.

Since the vertices of an isometric subgraph, G of $H \boxtimes T$ can be covered with

$|V(H)|$ isometric trees. we have an upper bound for the tree-precinct number of G.

Lemma 3.4.0.5 *If G is an isometric subgraph of $H \boxtimes T$ for any graph H and tree T. then $Tpn(G) \leq |V(H)|$.*

This follows since the tree-precinct number of G is. by Lemma 2.3.5.1. the minimum number of isometric trees required to cover G.

3.5 Isometric Subgraphs of the Strong Product of a Graph and a Cycle

As in Section 3.4. we show that any isometric subgraph G of $H \boxtimes C$. where H is any graph and C is any cycle. can be covered with a set of subgraphs of G which are also retracts. In this case each subgraph is either an isometric path or an isometric cycle. Our motivation is to use these retracts to establish an upper bound on the cop number of G. This is addressed in Section 3.6.

Suppose G is an isometric subgraph of $H \boxtimes C$ for some graph H and cycle C. Let $A_i = \{v \in V(G)|\pi_1(v) = v_i\}$ for all $1 \leq i \leq n$. As for trees. for each $i = 1.2.\ldots.n$. we can construct an isometric subgraph of G which contains all the vertices of A_i.

Lemma 3.5.0.6 *Suppose G is an isometric subgraph of $H \boxtimes C$ for some graph H and cycle C. Then for each $i = 1.2.\ldots.n$ there exists a graph W containing all the vertices in A_i that is an isometric subgraph of G. Moreover. $\pi_2(W)$ is either an isometric path in G or $\pi_2(W) = C$.*

Proof: Let C be a cycle on m vertices and let $A = A_i$ for some $i = 1.2.\ldots.n$. If A consists of a single vertex. then A is obviously an isometric path in G. If $|A| = 2$ then let P be an isometric path joining the two vertices of A. Hence. P is an isometric path in G containing all the vertices of A.

Suppose $|A| \geq 3$ and let x and y be two vertices in A such that $d(x.y) = \max_{u.v \in A}(d(u.v))$. Then $d(x.y) \leq m/2$. Since A contains at least three vertices

then we are able to choose a vertex $z \in A$ such that $d(x,z) < m/2$. Let $k = d(x,z) = d(\pi_2(x),\pi_2(z))$. Since x and z are separated on C, we can label the vertices of C with integers $\{0,1,\ldots,m-1\}$, such that $\pi_2(x) = 0$ and $\pi_2(z) = k$ and $0 \sim 1 \sim \cdots \sim m - 1 \sim 0$. Now, let $A = \{x = w_0,w_1,\ldots,w_t\}$ such that $0 = \pi_2(w_0) < \pi_2(w_1) < \cdots < \pi_2(w_t)$. Let $a_i = \pi_2(w_i)$. Note that $a_0 = 0$.

Let P_i be an isometric path in G from w_{i-1} to w_i for each $i = 1,2,\ldots,t$. For every pair of vertices w_{i-1} and w_i, $d(w_{i-1},w_i) = d(a_{i-1},a_i) < m/2$. This is true for w_0 and w_1 due to the labelling of the vertices of C. It is true for all other w_{i-1} and w_i, since $d(a_{l-1},a_l) = m/2$ for some $l > 1$ implies there is no vertex $w \in A$ such that $\pi_2(w) = a_j$ for any a_j such that $a_{l-1} < a_j < a_l$. Hence, $\max_{i=1}^{t}(d(0,a_i)) = \max(d(0,a_{l-1}),d(0,a_l)) < m/2$ since $d(0,a_{l-1}) + d(0,a_l) = m/2$. This contradicts the choice of labelling of x,y which were chosen to be a pair at maximum distance.

There is only one isometric path on C between two vertices at distance less than $m/2$ apart. Since the projection π_2 of a path between two vertices separated on C is a one-one mapping, then $\pi_2(P_i)$ is an isometric path from a_{i-1} to a_i on C. Due to the labelling of C, the isometric path $\pi_2(P_i)$ on C must be $\pi_2(P_i) = \{a_{i-1},a_{i-1}+1,a_{i-1}+2,\ldots,a_i\}$.

Let $P = \cup_{i=1}^{t} P_i$. Since the projection of two paths P_i and P_j on C intersect only at one end vertex, if at all, then the paths themselves are either disjoint or intersect only at a common end vertex. Hence P is a path and $l(P) = a_t$.

If $l(P) = d(w_0,w_t)$ then P is an isometric path in G containing all the vertices in A. Hence, $W = P$ is the desired isometric subgraph. If P is not isometric then let P_{t+1} be an isometric path in G from w_t to w_0. We claim that $W = P \cup P_{t+1}$ is a cycle on m vertices. Since $d(w_t,w_0) = d(a_t,0) < a_t$ then $d(w_t,w_0) = m - a_t$. Hence, the isometric path P_{t+1} must have the projection $\pi_2(P_{t+1}) = \{a_t,a_t+1,\ldots,m-1,0\}$. Hence, P_{t+1} is disjoint from all vertices in P except w_0 and w_t. Hence, W is a cycle on m vertices. Furthermore, $\pi_2(W) = C$. □

Let W_i be an isometric subgraph of G which contains all the vertices in A_i such that W_i is either a path or a cycle on m vertices and $\pi_2(W_i)$ is an isometric embedding

of W in C.

Lemma 3.5.0.7 *If G is an isometric subgraph of $H \boxtimes C$ for some graph H and cycle C and $\{W_1, W_2, \ldots, W_n\}$ are the subgraphs of G described above. then W_i is a retract of G for all $i = 1, 2, \ldots, n$.*

Proof: It was shown in Lemma 3.5.0.6 that W_i for each $i = 1, 2, \ldots, n$ is either an isometric path or a cycle. If W_i is an isometric path, then it is a retract and the canonical retraction maps G onto W_i. If W_i is a cycle then $\pi_2(W_i) = C$. as shown in Lemma 3.5.0.6. Let $f : G \to W_i$ where $f(u) = w$ whenever $\pi_2(u) = \pi_2(w)$. We can write $f(G) = g(\pi_2(G))$ where g is the inverse of π_2 restricted to W_i. Since π_2 and g are both edge preserving, and f is obviously the identity on W_i. then f is a retraction of G onto W_i. Hence. W_i is always a retract of G. $\square$.

3.6 Road Blocks and Cop Number

If we are playing a game of cops and robber on the graph G which is an isometric subgraph of $H \boxtimes T$ then. by Lemma 2.2.0.4. after a finite number of moves. a single cop moving on T_i can guarantee the robber's capture if he ever moves onto a vertex in T_i. and hence. A_i. This follows since T_i is a retract of G and $c(T_i) = 1$ for all $i = 1, 2, \ldots, n$. We will say that A_i is protected when the cop on T_i has captured the image of the robber in T_i.

Note that the strategy for one cop to win a game of cop and robber on a tree T is for him to move toward the robber on the path joining their two current positions. Then the cop makes at most $diam(T)$ moves in capturing the robber. Since each T_i can be isometrically embedded in T. it takes at most $diam(T)$ moves for the cop to catch the image of the robber on each T_i.

We will find a winning strategy for a set of cops in G by considering a strategy in H. We know that if the robber is moving in G then $c(H)$ cops can catch the image of the robber projected onto H. Suppose we attempt to translate the winning strategy from H onto G by letting a cop in G protect A_i whenever a cop in H moves onto v_i.

If A_i could be protected after a single move. then $c(H)$ cops have a winning strategy in G. Unfortunately. we can not alway protect A_i in one move. since it may take a series of moves to reach a vertex in T_i and up to $diam(T)$ moves to catch the image once we are there. Meanwhile. the robber is moving as usual. During the time the cop on T_i was moving to protect A_i, any moves made by the associated cop on H have not been translated onto G. Since the cop on G will "fall behind" the cop on H in such instances. it is unlikely that this strategy on G will be successful.

However, if we can find a winning strategy for a set of cops on H where the "pace" of the game is irrelevant to their strategy. then by translating that strategy to the graph G in the manner previously described, we also have a winning strategy in G.

Suppose the cops occupy a set of vertices S. Let the **robber space** be the component of $H \setminus S$ containing the vertex currently occupied by the robber. Since we are looking for a strategy that does not depend on the pace of the game. we will assume that the robber can reach any vertex in the robber space on his turn. We. therefore. need to reduce the robber space to a single vertex and then capture the robber.

We now introduce a process which begins with a cut-set S_1 and gives a series of cut-sets which will reduce the robber space.

The Process:

0)Let $i = 1$ and let R_1 be any component of $H \setminus S_1$.

1)If R_i is not a single vertex then find a vertex $y \in R_i$ and a vertex $x \in S_i$ such that no vertex in $R_i \setminus \{y\}$ is adjacent to x. If such a pair exist then let $S_{i+1} = (S_i \setminus \{x\}) \cup \{y\}$. and let R_{i+1} be any component in $H \setminus S_{i+1}$ such that $R_{i+1} \cap R_i \neq \emptyset$. Replace i with $i + 1$ and repeat (1).

The process terminates when R_i is a single vertex or S_{i+1} can not be formed. Let $S_1. S_2. \ldots. S_n$ be all the cut-sets formed in the process.

We claim that $R_n \subset R_{n-1} \subset R_{n-2} \subset \cdots \subset R_1$. Suppose this is not the case. Then there a vertex u in R_i which is not in R_{i-1}. Hence. $u \in H \setminus R_{i-1}$. Let v be a vertex in $R_i \cap R_{i-1}$ such that the path from u to v in R_i has minimum length. Then the

path from u to v goes through S_1 but not S_2. Hence, the only path in R_i from u to v includes x. Since v was at minimum distance from u, then $v \sim x$. Hence, $v = y$. However, $y \notin R_i$ since $y \in S_i$. Therefore, the claim is correct.

If a cut-set S_1 will always generate a set $S_2, S_3, \ldots, S_n$ as described above such that R_n is a single vertex then call S_1 an **initial roadblock** and call $\{S_1, S_2, \ldots, S_n\}$ **roadblocks**. Let the minimum $|S|$ for all initial roadblocks, S, be the **roadblock number** of H, denoted $rb(H)$. We chose this terminology because each roadblock prevents the robber from moving out of his current robber space, just as an actual road block would prevent a robber from leaving town.

Consider the graph H in Figure 3.9. Let $S_1 = \{a, b\}$. The graph $H \setminus S_1$ consists of two single vertices and a 3-path. According to the process above, if we chose R_1 to be a single vertex we are done. Suppose R_1 is the 3-path $\{f, c, g\}$. Then we can let $S_2 = \{b, c\}$ since no vertex in $R_1 \setminus \{c\} = \{f, g\}$ is adjacent to a. The components of $H \setminus S_2$ are the single vertices f and g and the 3-path $\{d, a, e\}$. Hence, R_2 must be either f or g according to the process. Hence, the process terminates with R_2 being a single vertex. Since the choice of $S_1 = \{a, b\}$ as our initial cut-set will always result in R_n as a single vertex, S_1 is an initial roadblock for this graph.
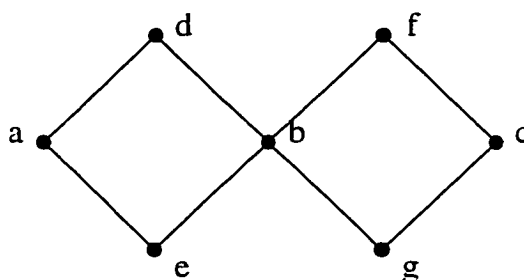

Figure 3.9: A graph H where $S_1 = \{a, b\}$ is an initial roadblock.

The following lemma is proved by finding appropriate initial roadblocks:

Lemma 3.6.0.8 *If T is any tree and C is any cycle, then*

(a) $rb(T) = 1$;

(b) $rb(C) = 2$.

Proof: (a) Choose a vertex which is not a leaf in T and call that vertex v_1. Then $S_1 = \{v_1\}$ is a cut-set. Let R_1 be any component of $T \setminus \{v_1\}$. Then there is a vertex v_2 adjacent to v_1 such that v_2 is in R_1. If $R_2 \neq v_2$ then let $S_2 = \{v_2\}$ it is obviously a cut set and v_2 is the only vertex in R_1 adjacent to v_1. Then R_2 is any component of $T \setminus S_2$ contained in R_1. Since starting with S_2 as our initial roadblock is similar to starting with S_1, we can find S_3 similarly. Since all S_i can be found in this manner and $R_i \subset R_{i-1}$ the process will continue until we obtain R_n which is a leaf.

(b) Let $C = \{v_1, v_2, \ldots, v_n\}$ be a cycle on n vertices. Let $S_1 = \{v_1, v_3\}$. If $R_1 = v_2$ then we are done. Otherwise. $R_1 = \{v_4, v_5, \ldots, v_n\}$. The only vertex adjacent to v_3 in R_1 is v_4, so we let $S_2 = \{v_1, v_4\}$. Then $R_2 = \{v_5, v_6, \ldots, v_n\}$. Then we have the roadblocks $\{S_1, S_2, \ldots S_{n-3}\}$ where $S_i = \{v_1, v_{i+2}\}$ and $R_i = \{v_{i+3}, v_{i+4}, \ldots, v_n\}$ for all $i = 1, 2, \ldots, n-3$. Therefore. R_{n-3} is an single vertex and $rb(C) \leq 2$. Any vertex v_i in C is adjacent to two vertices in $C \setminus \{v_i\}$. Therefore. if S_1 is a single vertex there is no choice for S_2. Hence. $rb(C) \neq 1$ and $rb(C) = 2$. $\square$

These roadblocks can now be used to give a winning strategy for a set of $rb(H) + 1$ cops to win on H. Furthermore. this strategy can be translated to a winning strategy in G. as previously described.

Theorem 3.6.0.9 *If G is an isometric subgraph of $H \boxtimes T$ where H is any graph and T is any tree. then $c(G) \leq rb(H) + 1$.*

Proof: The strategy for the $rb(H) + 1$ cops to win on H is as follows: First. they occupy an initial roadblock S_1 where $|S_1| = rb(H)$. Two cops will occupy a single vertex. Deem one of these cops the "spare". Then R_1 is chosen to be the component of $H \setminus S_1$ containing the robber. The second roadblock S_2 is determined accordingly. The spare cop then moves to the vertex $y_1 = S_2 \setminus S_1$. Once he occupies y_1. the cop on $x_1 = S_1 \setminus S_2$ is the spare. Since S_2 is protected. the robber is now confined to some component R_2 is $H \setminus S_2$ where $R_2 \subset R_1$. This process repeats until the robber is confined to a single vertex and then the spare cop moves on him.

Now. the winning strategy on G is to take the winning strategy for the game played on the projection of G onto H and whenever a cop would move to a vertex v

in H. we will move a cop to $(v \boxtimes T) \cap G$ and the vertex v in H is considered to be occupied once $(v \boxtimes T) \cap G$ is protected.

Therefore. the robber will be confined to the set $(R_1 \boxtimes T) \cap G$. then $(R_2 \boxtimes T) \cap G$. and so on. until he is confined to $(R_n \boxtimes T) \cap G$. where R_n is a single vertex. Then the spare cop moves to capture the robber in $(R_n \boxtimes T) \cap G$. □

The following is a consequence of Theorem 3.6.0.9 and Lemma 3.6.0.8:

Corollary 3.6.0.10 *Let T_1 and T_2 be any trees and let C be any cycle.*

(a)If G is an isometric subgraph of $T_1 \boxtimes T_2$ for trees T_1 and T_2 then $c(G) \leq 2$.

(b) If G is an isometric subgraph of $C \boxtimes T$ where C is a cycle and T is a tree then $c(G) \leq 3$.

While our discussion has centered around G as an isometric subgraph of $H \boxtimes T$ where T is a tree. the same arguments can be used to give an upper bound for the cop number of an isometric subgraph $H \boxtimes C$ where C is any cycle.

Theorem 3.6.0.11 *If G is an isometric subgraph of $H \boxtimes C$ where H is a graph and C is any cycle then $c(G) \leq rb(H) + 2$.*

Proof: The winning strategy on G is the same as in Theorem 3.6.0.9 except that we require $rb(H)+2$ cops to protect $(S_1 \boxtimes C) \cap G$. Let $S_1 = \{v_1. v_2. \ldots. v_{rb(H)}\}$. Without loss of generality suppose $\{W_1. W_2. \ldots. W_l\}$ are cycles and $\{W_l. W_{l+1}. \ldots. W_{rb(H)}\}$ are paths. Place one cop on each W_i for $i = 1. 2. \ldots. rb(H)$. Add a second cop to W_1 to capture the image of the robber on W_1. Once this is done. one cop stays on the image and the second cop moves to W_2 and the two cops catch the image of the robber on W_2. This continues until the image of the robber is caught on W_i for all $i = 1. 2. \ldots. l$. For $l + 1 \leq i \leq rb(H)$. only a single cop is needed to catch the image of the robber. Once each of the W_i's is protected. a total of $rb(H)$ cops are protecting $(S_1 \boxtimes C) \cap G$ and we have two spare cops. The cops continue to move as in Theorem 3.6.0.9 except in this case we always have two spare cops to catch the robber on the next retract. W. □

This gives us the following corollary which is evident since. by Lemma 3.6.0.8. $rb(C) = 2$.

Corollary 3.6.0.12 *If G is an isometric subgraph of $C_1 \boxtimes C_2$ for any cycles C_1 and C_2 then $c(G) \leq 4$.*

3.6.1 Problems

For trees and cycles the roadblock number is small and the initial roadblocks are easy to find.

Problem 3.6.1.1 *Are there other classes of graphs in which small initial roadblocks can be found?*

Finding a roadblock in graph H was an effective way of finding a strategy for G where G was an isometric subgraph of either $H \boxtimes T$ or $H \boxtimes C$ because we could cover the graph G with retracts.

Problem 3.6.1.2 *Suppose G is an isometric subgraph of $I \boxtimes H$ for some graphs I and H. Is there family of graphs besides trees and cycles such that if I is a member of that family then we can find a set of retracts which cover all the vertices of G?*

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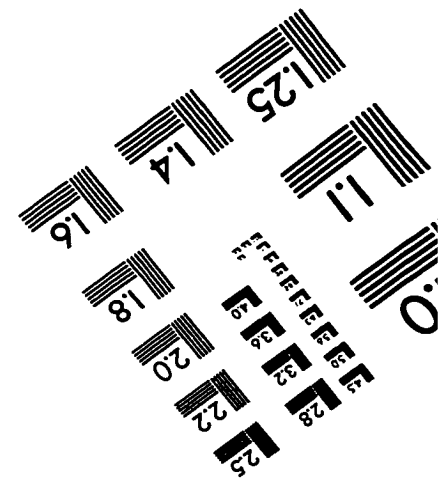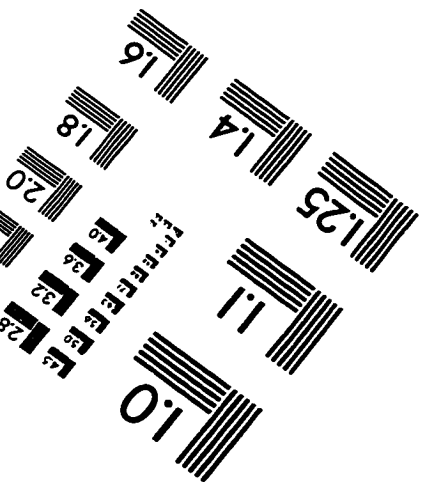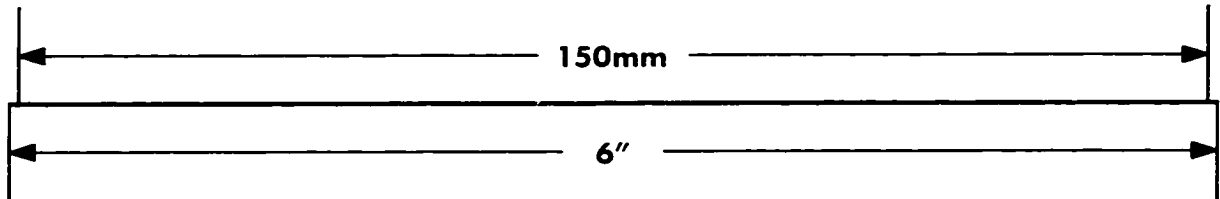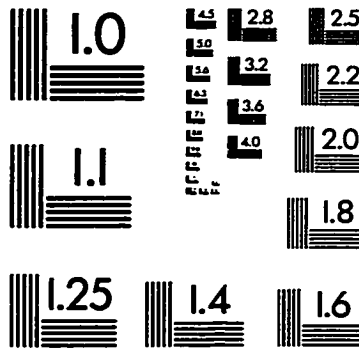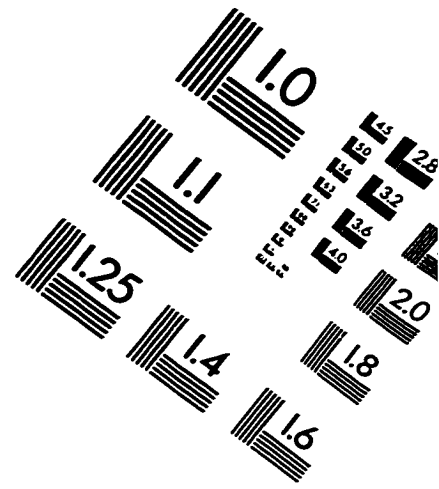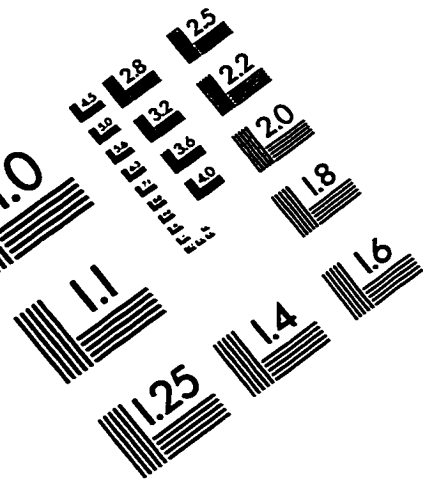
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IMAGE EVALUATION
TEST TARGET (QA-3)

1.0 1.1 1.25 1.4 1.6 1.8 2.0 2.2 2.5 2.8 3.2 3.6 4.0 4.5 5.0 5.6

150mm

6"