KINKS
IN GENERAL RELATIVITY

by

Tina A. Harriott

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

at

Dalhousie University
Halifax, Nova Scotia, July, 1991

DALHOUSIE UNIVERSITY

DEPARTMENT OF MATHEMATICS

ISBN 0-315-71532-4

TABLE OF CONTENTS

Chapter 7: The Schwarzschild Kink Solution

Chapter 8: Kink Solutions in 1+1 dimensions

Chapter 9: The Friedmann-LeMaitre-Robertson-Walker Kink and Minkowski Solutions

Chapter 10: Other Kink Solutions

Chapter 11: Conclusions

LIST OF ILLUSTRATIONS

ABSTRACT

This thesis is a study of kinks in general relativity. The kink spacetimes are topologically non-trivial and possess other interesting features such as tumbling light cones and a non-zero conserved quantity, now called the kink number.

Skyrme first noted the existence of kinks in certain non-linear scalar field theories. Finkelstein and Misner were the first to recognize the existence of similar structures in general relativity. This thesis begins with a review of past work on kinks.

The general form of a kink metric is discussed and a formula to calculate the kink number of any metric is derived.

Several exact kink solutions of the Einstein field equations are found. The relationship of these solutions to well known (zero kink) metrics, such as the de Sitter and Friedmann-LeMaitre-Robertson-Walker metrics is discussed. Possible interpretations of the kink solutions are suggested. Analogous solutions in a (1+1)-dimensional theory of gravity are also presented. Finally, work in progress and areas for future work are mentioned.

SYMBOL TABLE

In what follows the metric is chosen to have signature

$$(- + + +)$$

and

Greek letters	$\alpha, \beta, \tau \ldots$ run from 0 to 3 with summation over repeated indices unless otherwise specified.
Roman letters	a,b,c .... run from 1 to 3 with summation over repeated indices unless otherwise specified.
$T_{\alpha\beta,\tau}$	refers to the ordinary partial derivative of any tensor $T_{\alpha\beta}$.
$T_{\alpha\beta;\tau}$	refers to the 4-covariant derivative of any tensor $T_{\alpha\beta}$ with respect to the metric tensor $g_{\alpha\beta}$.

ACKNOWLEDGEMENTS

I would like to thank Dr. Ken Dunn and Dr. Jeff Williams for all their help and encouragement.

CHAPTER ONE

THE SKYRMIONIC KINKS

Introduction

This chapter introduces the notion of kinks in nonlinear scalar field theories and in theories such as general relativity (1.1). These structures, in the nonlinear scalar field theories, are now called skyrmions in recognition of the work of Skyrme. The term "kink" is now for the most part reserved for similar structures that occur in theories such as general relativity. In honour of the pioneering work of Skyrme a review of his work is given (1.2).

(1.1) Introduction.

Skyrme was the first person to construct a non-linear scalar field theory that possessed certain conserved quantities which today may be called kinks. Kinks can be considered as twists in the field variables. Skyrme referred to the kinks as "singularities" in the field variables, by this he meant they had a twist, not an infinity. The term "kink" was first introduced by Finkelstein (Finkelstein, 1966) and is now for the most part reserved for the similar structures that arise in theories such as general relativity that were first considered by Finkelstein and others several decades ago (Finkelstein and Misner, 1959; Finkelstein, 1966).

In recognition of Skyrme's work, the kinks arising in certain non-linear scalar field theories are now called "skyrmions". There are many similiarities in the mathematics of skyrmions and kinks but also some important differences. A common aim of both types of field theories in which kinks arose was to find a unified description for fermions and bosons. The particles that make up ordinary matter, characterized by their fractional spin quantum number m_m = 1/2, 3/2,.... are called fermions. The fermions interact through the exchange of bosons which are characterized by their integer spin quantum number

$m_m = 0, 1, \ldots$ Skyrme achieved his goal of unification when he found a common description for fermions and bosons, but the kinks of general relativity were found not to describe fermions and other physical interpretations have been suggested (Finkelstein and McCollum, 1975; Harriott and Williams, 1988). The skyrmions do possess half odd intrinsic angular momentum (spin) as is required for the description of a fermion (Skyrme 1962) but the metrical kinks do not (Williams and Finkelstein, 1984). The skyrmionic kinks are scalar quantities. The metrical kinks of general relativity are described by a quantity that is neither a scalar nor a tensorial quantity.

The focus of this thesis is the metrical kinks of general relativity but as a historical introduction to the development of kink theory the rest of this chapter will briefly review the work of Skyrme.

(1.2) Skyrmions

On November 17th-18th, 1984 a workshop was held at Cosener's House, Abingdon, United Kingdom on the topic of skyrmions. Skyrme made some historical remarks at the beginning of this workshop and was due to repeat this talk at the beginning of another workshop on skyrmions in 1987 but sadly he died a week before this latter symposium. His earlier talk was reconstructed by Ian Aitcheson and published in 1988 (Skyrme, 1988; Dalitz and Stinchcombe, Editors, 1988).

In this talk Skyrme explains the three main ideas that motivated his study of the nonlinear scalar field theories that possessed skyrmions. His first goal was to construct a self interacting boson field theory. That is, he wished to construct a unified field theory that eliminated the need for two separate fields to describe bosons and fermions.

Skyrme's second goal was to address the renormalisation problem in quantum field theory. This is a problem with infinities of physical variables, such as density, that arise in quantum field theories when particles are described as point-like objects. Skyrme's approach was to describe particles as extended objects, thus avoiding the infinities. This differed from the usual approach which was to attempt to mathematically remove the infinities, by renormalization, from any theory that contained point particles. Skyrme was certainly not alone in his unease with the notion of elementary particles as point-like objects - Kelvin (1824-1907) did not like the idea of infinitely rigid point-like atoms (Thompson, 1910). Skyrme discusses in his 1984 talk, the "smoke ring" model studied extensively by Kelvin and Tait but finally abandoned in favour of Maxwell's Theory (Thompson, 1910).

Skyrme's third aim was to find a theory that naturally reproduced the behaviour of fermions rather than imposing the necessary mathematics into the theory to reproduce the required properties. The fermions arise in quantised theories but have no obvious classical analogue. This led Skyrme to believe that the fermion should not be regarded as a fundamental particle. He was attracted to nonlinear theories such as general relativity in which the "sources" (particles possessing mass) of gravitation might themselves be produced by the field equations - arising as some kind of **singularity*** in the fields - instead of having to add them to the theory.

* (It is important to note here that Skyrme often referred to a kink as a **singularity** in the field variables. By this, he meant that the field variables had a **twist** and

not that they had an infinity.)

Particles can be classified on the basis of whether or not they are affected by the strong nuclear force, which is the force that binds atomic nuclei together. Particles that are affected by the strong nuclear force are known as hadrons and those that are not affected are called leptons. The hadrons are then subclassified as to whether or not they are fermions or bosons. Baryons are hadrons that are also fermions. Mesons are hadrons that are also bosons. Skyrme hoped that the fermionic sources would occur as **singularities*** in some nonlinear classical meson field theory arising as a conserved quantity that might be identified with the baryon number. In this way the fermions might naturally arise from the meson fields instead of having to be imposed on the theory.

Skyrme began his work by first turning his attention to nuclear physics. Nuclear matter is very homogeneous and Skyrme found the description of the nuclear matter as a fluid very attractive. He suggested that the individual nucleons could be described as local twists in this fluid that described the nucleus. (Skyrme, 1958, 1959, 1961, 1962). This idea is identical to the current concept of a preferred direction in a theory with a spontaneously broken internal symmetry.

To quantify these ideas of describing a nucleon as a twist in a fluid Skyrme began by looking at the analogous problem in (1 + 1)-dimensional spacetime which is now called the sine-Gordon equation:

$$a_{,tt} - a_{,xx} = -m^2\sin(a) \tag{1.1}$$

where a(x,t) is the (single) field variable. The well known time-independent (one-kink) solution of this equation is (Skyrme, 1958; Rogers and Shadwick, 1982)

$$a = 4\tan^{-1}[\exp(x)]$$

A vacuum state, obtained when the right hand side of equation (1.1) is zero, is clearly

$$\cos(a) = 1$$

However, a is only determined up to a factor of 2π, and so it is possible to have a time independent situation in which a(x) changes from one vacuum state at $x = -\infty$, say $a(-\infty) = 0$, to another at $x = \infty$, say $a(\infty) = 2\pi$. This situation describes a (sine-Gordon) one-kink solution and may be illustrated as shown below in Fig. (1.1).

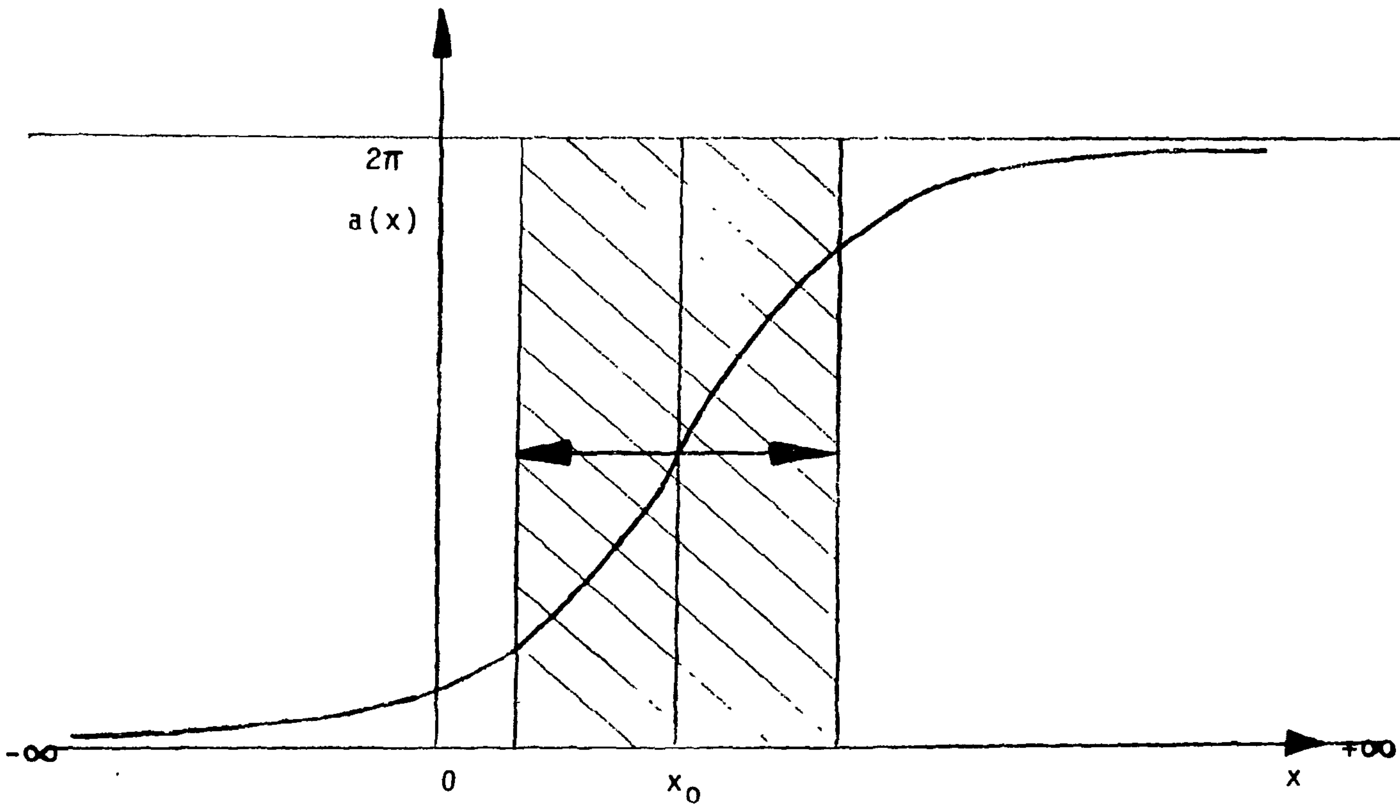

Fig. (1.1) A skyrmion kink.

A simple kink centred at $x = x_o$. The arrows show it's width, the x range over which most of the variation of a(x) occurs.

In general, choosing boundary conditions such as $a(-\infty) = 0$ and $a(\infty) = 2N\pi$ describes an N-kink solution.

If all physical variables depend only on a mod(2π) and if under the boundary conditions a tends to a multiple of 2π at $x = \pm\infty$, then R^1 is effectively compactified to S^1 and a(x) defines a mapping from the physical space S^1 to the field space S^1. The number of times the circle is covered is called the winding number of the map. This winding number is equal to 1 for the boundary conditions $a(-\infty) = 0$ and $a(\infty) = 2\pi$. For the the boundary conditions $a(-\infty) = 0$ and $a(\infty) = 2N\pi$ the winding number is N. The situation is illustrated below in Fig. (1.2).

The winding number is a topologically conserved quantity and so may be identified with a physical conserved quantity such as the baryon number. This model generalizes to 3 + 1 spacetime, mapping S^3 to S^3 with a conserved winding number (and, in fact, to n + 1 dimensions with a mapping from S^n to S^n). The specific equations needed to calculate this conserved number, now known as the (skyrmionic) kink number, were first published by Skyrme (Skyrme 1961) and have since been extended (Dunn, Harriott and Williams, 1990) to include general relativity.

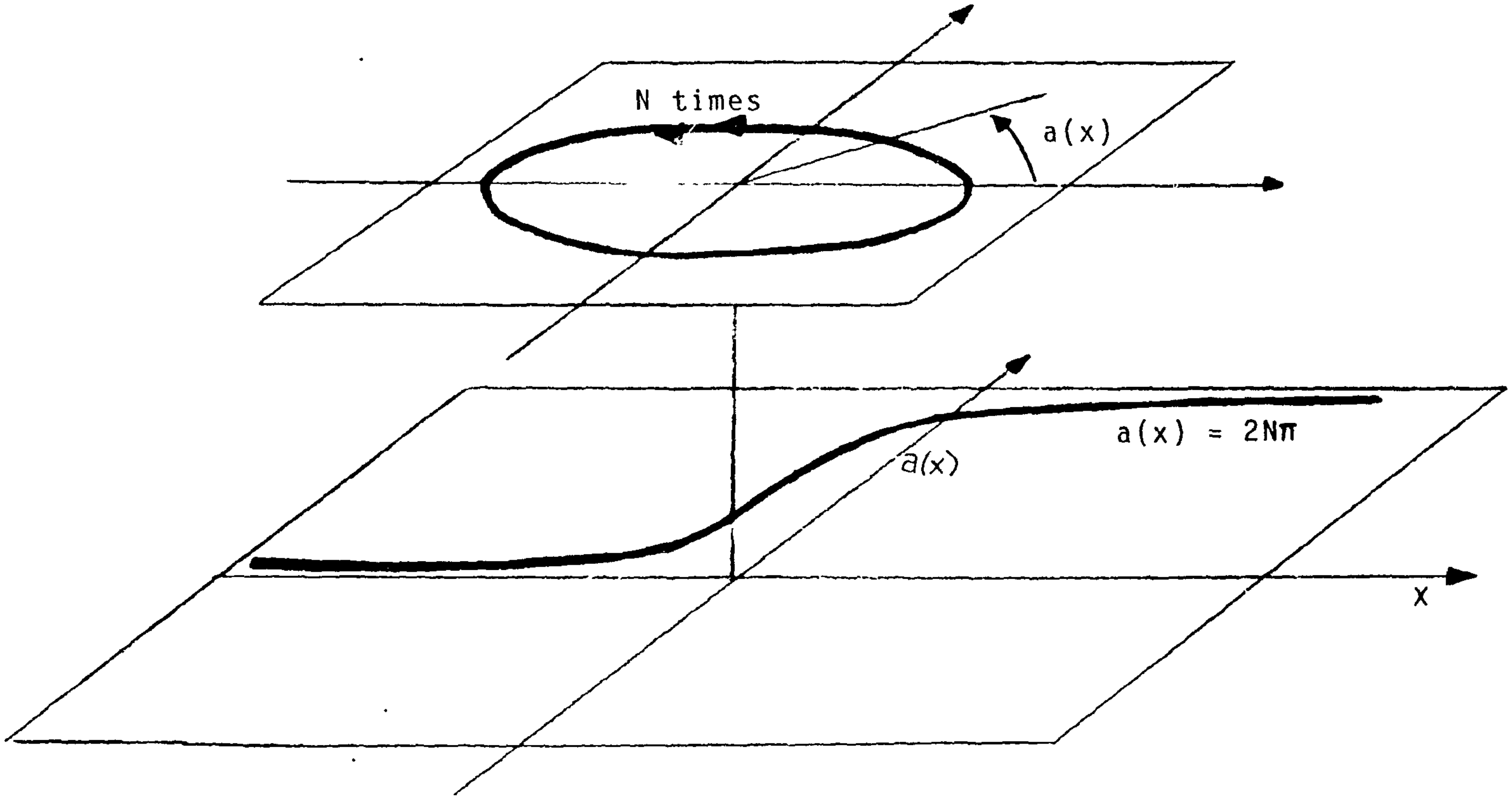

Fig. (1.2). The mapping a(x).

The mapping from real space to field space provided by a(x). On the lower plane, the function a(x) is plotted vs. x, for the case $a(+\infty) - a(-\infty) = 2N\pi$ On the upper plane a(x) is plotted as the angle of a point on a unit circle, which winds around N times as x runs from $-\infty$ to $+\infty$. Such an a(x) has a winding number of N.

At the time of his death, Skyrme still hoped that it might be possible to construct a nonlinear theory that allows a semi-classical visualization of elementary particles. All hadrons are now believed to have a more elemental substructure composed of two or three fundamental constituents called quarks. His goal was that the quarks and leptons, usually introduced as sources in these theories, be regarded as helpful ways of describing the situation rather than as fundamental particles.

Skyrme was also investigating the possibility of topologically interesting field configurations in gauge field theories as opposed to his original work which was with nonlinear scalar field theories. If such configurations existed, he questioned what the affect of the constraints of the gauge invariance might be. In particular, he was considering the possibility that a certain choice of gauge that admits stable topological structures may require the addition of source particles to make it consistent and that these source particles may correspond to physical states. A similar situation is known to arise in the work of Faddeev and Popov (Taylor, 1976). Unfortunately Skyrme had not found any interesting specific examples of such a gauge theory by the time he died in 1987.

CHAPTER TWO

THE GENERAL RELATIVISTIC KINKS

Introduction

The homotopic classification of the metrics of general relativity is discussed to introduce the notion of a kink metric (2.1). A detailed discussion of previous work concerning kinks in general relativity, in particular the work of Finkelstein, is presented (2.2). The general conditions that lead to a kink metric are derived (2.3). A formula to find the kink number of any metric is discussed (2.4). This extends earlier work of Skyrme. The exact form of the general spherically symmetric metric is derived (2.5). Possible interpretations for kink metrics are discussed (2.6).

(2.1) Homotopic Classification of Metrics.

Field theories that admit configurations that are topologically distinct have applications in several areas of physics. A field configuration represents a mapping Φ from the domain X of field variables into the range Y which is assumed to be a connected manifold

$$\Phi : X \rightarrow Y \,.$$

It is usual to assume that only maps, Φ, that map fixed base point(s) $x_o \in X$ into a fixed base point $y_o \in Y$ are considered. The set of all topologically distinct classes of base-point preserving maps are called homotopy classes and are denoted by [X,Y]. In many field theories $X = R^3$, and the infinite boundary of R^3 is mapped into some fixed point y_o, so that R^3 may be compactified to S^3. In such theories there is interest in calculating $[S^3,Y]$ for different Y.

Consider such a situation, when $X = R^3$, with boundary conditions $\Phi(\mathbf{x}) \rightarrow y_o$ (fixed) as $|\mathbf{x}| \rightarrow \infty$, so that R^3 may be replaced by S^3. Such a field theory can therefore be described in terms of mappings

$$\Phi : S^3 \rightarrow Y.$$

These maps can now be classified by computing the homotopy class $[S^3, Y] = \pi_3(Y)$, which is the third homotopy group. The homotopy classes can be labelled by Q_n where Q_o is the group identity containing the constant map, Φ_o, which maps the whole of R^3 into the fixed point y_o. If Q_o is not the only element of $\pi_3(Y)$ then the field theory is said to admit kinks (Finkelstein, 1966). For example:

(i) $Y = V$, any vector space, then

$$\pi_3(V) = Q_o$$

and there are no kinks.

(ii) $Y = S^3$, as in the case of Skyrme's theory of strong interactions (Williams, 1970; Skyrme, 1971), then

$$\pi_3(Y) = Z.$$

The homotopy classes, $\ldots Q_{-2}, Q_{-1}, Q_o, Q_1, Q_2 \ldots$ can be labelled by a single integer $n \in Z$. The classes are generated by Q_1 and Q_{-1}. When $n > 0$ mappings belonging to Q_n are called n-kink maps and when $n < 0$ they are called n-anti-kink maps. A specific example of such a 1-kink map is

$$\Phi: R^3 \rightarrow S^3$$

such that

$$\Phi(\mathbf{x}) = (\phi_1, \phi_2, \phi_3, \phi_4)$$

and

$$\phi_i = 2ax_i(r^2 + a^2)^{-1}$$

$$\phi_4 = (r^2 - a^2)(r^2 + a^2)^{-1}$$

where $r = |\mathbf{x}|$ and a is a non-zero constant. This is the usual stereographic projection and the homotopy class of Φ_1 is Q_1.

These 1-kink mappings of Skyrme's theory are degree-1 maps. These degree-1 kinks are now called skyrmions. In the 1-dimensional case they may be pictured as 2π twists in an infinitely long strip (Finkelstein and Misner, 1959) as illustrated in Fig. (2.1).

(i)

(ii)

(iii)

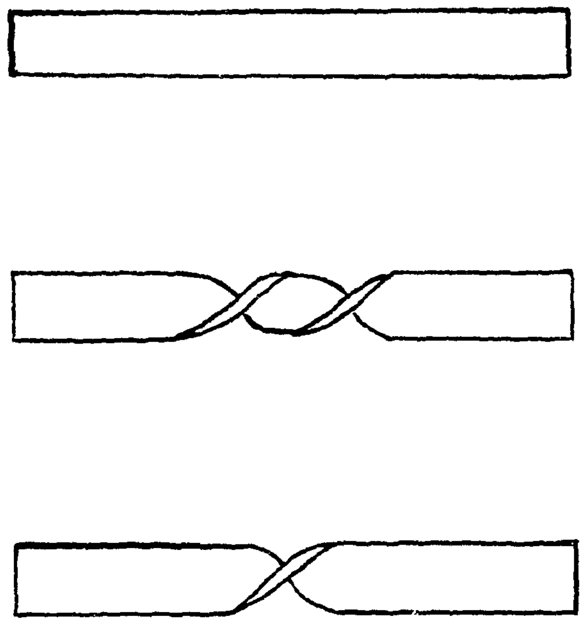

Fig. (2.1). (1+1)-dimensional Kinks viewed as twists

(i) A zero kink.

(ii) A skyrmion kink.

(iii) A gravitational kink.

(iii) $Y = P^3$, where P^3 is real projective 3-space (S^3 with the antipodal points identified). Then

$$\pi_3(P^3) = Z,$$

and so the theory does admit kinks. If

$$K: S^3 \to P^3$$

is the usual double cover map which identifies the antipodal points of S^3 then the homotopy class of K generates $\pi_3(P^3)$ and so K is a 1-kink map but deg(K) = 2.

Such kinks correspond to a half twist (through an angle π) and not a full twist because the mapping characterizing the homotopy behaviour from M^n into P^n is not the double cover map. For example in the 1+1 case it is the map that is obtained by mapping S^1 into "half of S^1" and then identifying the antipodal points to obtain P^1. The 1+1 case can therefore be pictured as Möbius strip type of twist and is illustrated in Fig. (2.1).

(iv) $Y = SO(4) = S^3 \times P^3$ then

$$\pi_3(SO(4)) = Z \oplus Z$$

and so such a theory admits two types of kinks.

General relativity was one of the first field theories to be studied from the point of view of homotopy theory. It can be shown (Shastri, Wiliams and Zvengrowski, 1980) that for general relativity in 3+1 dimensions $Y = P^3$ and it is of interest to find $[X,Y]$ for different spacetime manifolds, X. Finkelstein and Misner (1959) showed that when $X = R^1 \times R^3$ or $X = R^1 \times S^3$ then the homotopy classes can be specified by a single integer, which has now become known as the kink number of the metric. The kink number of the metric can therefore not be altered by any coordinate transformation that is non-singular or does not involve a change in the global hypersurface foliation of the manifold, as this would involve a change in the homotopy class. Shastri, Williams and Zvengrowksi (1980) also discuss the classification problem for more general parallelizable spacetime manifolds.

More generally, gravitational kinks can be shown to occur in (n+1)-dimensional spacetimes for any $n > 0$. In particular if $M^{n+1} = R^1 \times M^n$ and the hypersurface M^n is assumed connected, orientable and compactifiable (in the sense that the metric is required to take the same value everywhere on the boundary of M^n so that the boundary could be identified to a point) then the relevant homotopy classes are $[M^n, P^n]$, where P^n is real projective n-space. This result will be shown below for the usual case of M^4. Manifolds are classified as to whether there exist mappings

$$\Phi: M^n \to P^n$$

which have degree 1 (called Type 1) or whether no such mappings are possible (called Type 2). Generally the degree of a map can be defined between any two orientable manifolds of the **same** dimension. If

$$\Phi: X^n \to Y^n$$

then $\deg(\Phi)$ is the number of times that Φ wraps X^n around Y^n. A simple example was illustrated in Fig. (1.2).

In 1+1 dimensions, Type 1 is the only possibility, and $M^1 = S^1$. The homotopy classes are $[S^1, P^1] = \pi_1(P^1) = Z$, where $\pi_1(P^1)$ is the first homotopy group. A 1-kink metric is associated with a degree 1 mapping

$$\Phi: S^1 \to P^1 \ .$$

Some examples of such metrics will be given later. In 2+1 dimensions Williams and Zvengrowski (1991) show that Type 2 is the only possibility. In 3+1 dimensions both Type 1 and Type 2 are possible (Shastri, Williams and Zvengrowski, 1980). An example of a manifold that admits Type 1 is $M^3 = P^3$, and an example of a manifold which admits Type 2 is $M^3 = S^3$. The latter is the main focus of this thesis.

The 4-dimensional spacetime manifold $X = M^4$ of general relativity and the problem of classifying the Lorentz metrics is now considered in more detail. The manifold is assumed to be parallelizable. Examples are $M^4 = R^1 \times M^3$, where M^3 is connected and orientable. If asymptotic flatness is also assumed then M^3 can be compactified to form a closed, connected, orientable three-manifold. A Lorentz metric is a cross section, Σ, of the Lorentz metric tensor bundle $T_{0,2}(M^4)$. Shastri, Williams and Zvengrowski show that the parallelizability of M^4 means that classifying cross sections is equivalent to classifying the maps, Φ, from M^4 to the fiber of $T_{0,2}(M^4)$. This fiber is called $S_{4,1}$ by Steenrod (1951) and is the set of all 4 x 4 real symmetric matrices of signature (- + + +)

$$\Phi: M^4 \to S_{4,1} .$$

This thesis assumes that $M^4 = R^1 \times M^3$ and in particular that $M^3 = R^3$ or S^3. For these choices for M^3 the homotopy classes will be shown to be labelled by a single integer, now called the kink number. Kinks for manifolds with more complicated topologies have been studied by Shastri, Williams and Zvengrowski (1980), who consider M^3 compact, closed, connected and orientable. Their work has been generalized somewhat by Whiston (1981) and Bugajska (1989). The corresponding problem in 2+1 dimensions has been analyzed by Williams and Zvengrowski (1991) and the problems involved in adding kinks for spacetime manifolds that are not simply connected have recently been addressed by Shastri and Zvengrowski (1991) for the (3+1)-dimensional case.

Since R^1 is topologically trivial, the assumption that $M^4 = R^1 \times M^3$ and in particular that $M^3 = R^3$ or S^3, means that the homotopy classes to be found are $[S^3, S_{4,1}]$. Since the set of all Lorentz metrics has a dimension greater than 3 it is not immediately clear how the classification can be achieved using the concept of a degree of mapping, since this is not necessarily defined between spaces of different dimension. The classification can be achieved however, as outlined below.

To find the homotopy classes $[S^3, S_{4,1}]$ let $G \in S_{4,1}$, then it will be shown later that G can be written uniquely as the product of two matrices S and Q. That is,

$$G = SQ = QS$$

where S is positive definite and Q is symmetric and orthogonal. The matrix S is homotopically trivial since it can be deformed into the identity matrix and so the classification depends only on Q. Steenrod (1951) shows that the set of all Q is homeomorphic to the Grassman manifold $M_{4,1}$ which in turn is homeomorphic to real projective 3-space, P^3. The homotopy classes which classify the metrics of general relativity if M^4 is assumed to be $R^1 \times R^3$ (or $R^1 \times S^3$) are therefore $[S^3, P^3]$, which as described in the example (iii) may be labelled by a single integer, the kink number.

It will also be shown later that Q may be written as the following product

$$Q = P \operatorname{diag}(-1, 1, 1, 1) P^T ,$$

where P is orthogonal. Since P is orthogonal, each of its rows or columns represents maps Φ: $R^3 \to S^3$. The kink number is the degree of this map.

(2.2) Historical Introduction to Kinks in General Relativity.

Finkelstein and Misner (1959) first drew attention to a class of nonlinear field theories that they called "intrinsic". If, as in the section (2.1), Φ is a mapping from a domain X into a range Y, the term "intrinsic" was used to emphasise the fact that Y was not homeomorphic to a vector space as was the case with most field theories in the literature at that time. Their interest in this class of field theories was due to the fact that they possessed a non-trivial conservation law in which the conserved quantity assumed only a set of discrete values, even in unquantized theories, as described in the above section. Finkelstein and Misner noted that general relativity was such a theory and that in general relativity this conserved quantity could assume only the values 0, ± 1, ± 2, ± 3, ... This quantity therefore possessed many of the properties of a classical particle number even though it arose from the continuity of the basic fields. The authors suggested that in the case of general relativity this quantity might be interpreted as a particle number. They named the new particle associated with this conserved quantity the M-geon. The term "geon" was coined by Wheeler as a corruption of "geometrical entity" and seemed appropriate since the quantity arose purely from the metric.

The authors also showed in this 1959 paper that the set possible metrics on a manifold may be divided into homotopy classes. Two metrics are said to belong to the same homotopy class if and only if they can be continuously deformed into each other. They noted that this set of homotopy classes is in one-one correspondence with the group of integers. Any metric can therefore be associated with an integer, which cannot be changed by any non-singular coordinate transformation, as this would involve a change of homotopy class. This integer has since become known as the kink number, N, of the metric and is a conserved quantity associated with any metric. Finkelstein and Misner were also the first to recognize the phenomenon of tumbling light cones which is associated with these kink metrics. This may be illustrated as shown below in Fig. (2.2).

This initial work was continued by Finkelstein (1966) and it was Finkelstein who first introduced the term "kink" (replacing the term M-geon of the earlier work) for the conserved objects arising in these nonlinear field

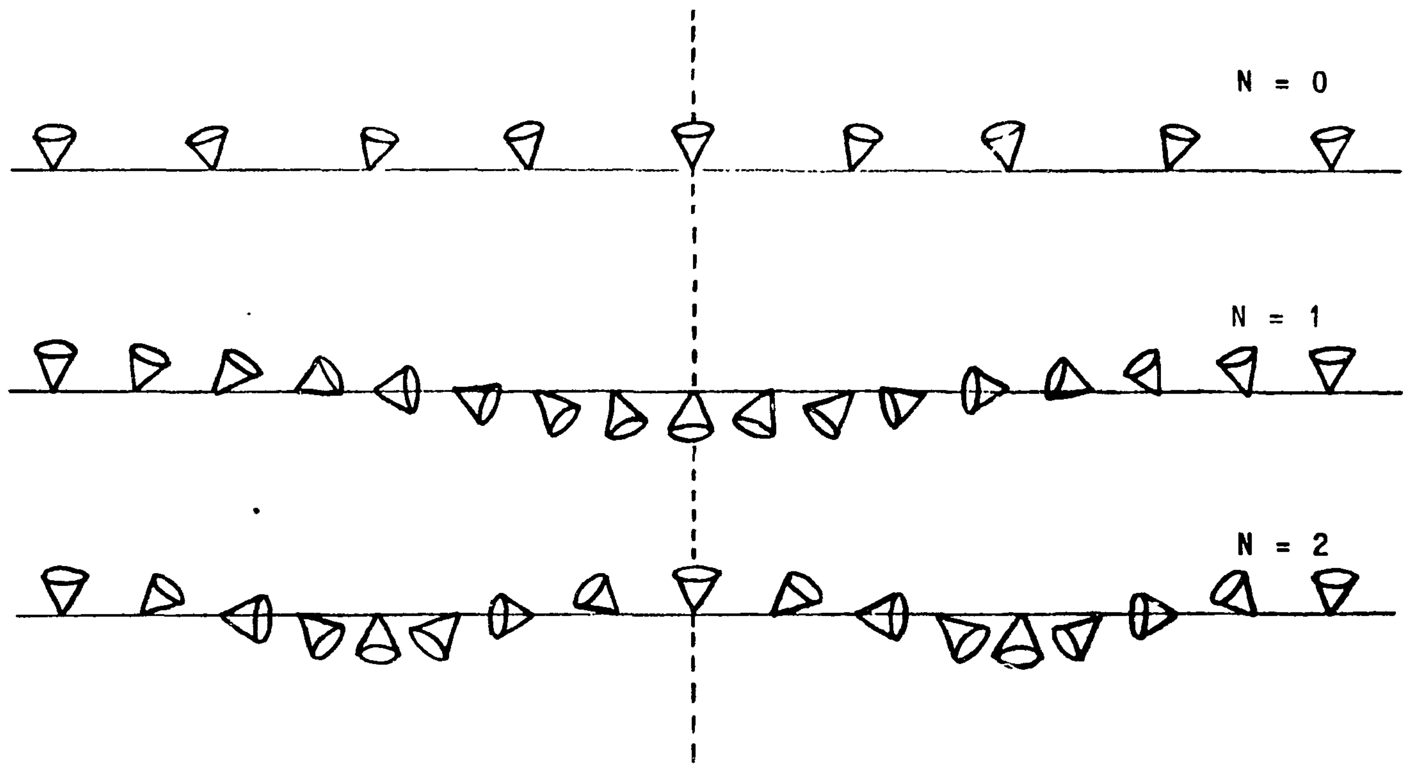

Fig. (2.2). Light cone behaviour for metrics belonging to the homotopy classes labelled by N = 0, N = 1, N = 2.

theories. The purpose of this 1966 paper was to deduce the properties that must be possessed by the underlying field for kinks to exist and to possess half-integer spin and Fermi-Dirac statistics. That is, Finkelstein hoped the kinks would provide a description of fermions (particles with intrinsic angular momentum (spin) 1/2, such as electrons). He noted the connection between the kinks of general relativity and those investigated by Skyrme (1959). Finkelstein also discussed the fact that kinks do not describe point particles but rather particles that possess internal structure, distributed over a finite volume.

Some confusion arose in the literature concerning the angular momentum properties of the kink spacetimes. There are two types of angular momentum: intrinsic and extrinsic. Intrinsic angular momentum is usually called spin, it refers to internal variables and distinguishes between the fermions and bosons. For example the gravitational field (graviton) has spin 2; the Dirac field has spin 1/2. Extrinsic angular momentum is usually called orbital angular momentum.

It was hoped that the spin properties of the kinks would allow them to be identified as fermions. However, Williams and Finkelstein (1984) showed that the usual kinks of general relativity cannot have orbital angular momentum of 1/2. They considered group fields which are field theories for which the field or mapping Φ maps R^3 into a Lie group G, with Φ mapping the infinite boundary of R^3 into the group identity. Orbital angular momentum of 1/2 can arise if and only if the 2π rotation loops in the field space are not deformable to a single point, that is if they are nontrivial. To determine the existence of half-integer orbital angular momentum, the transformation properties of the fields under rotation must be considered. Using the notation of Williams and Finkelstein (1984), if the range of the mapping Φ is chosen to be a 3-sphere S^3, then S^3 may be parametrized by four variables $(\phi_1,\phi_2,\phi_3,\phi_4)$ subject to $\Sigma\phi_\alpha\phi_\alpha = 1$. They use as an example of a 1-kink mapping:

$$\Phi(\mathbf{x}) = (\phi_1,\phi_2,\phi_3,\phi_4)$$

with $\phi_\alpha = f_\alpha(\mathbf{x})$ where $f_\alpha(\mathbf{x})$ are some functions of $\mathbf{x}$ that satsify $\Sigma\phi_\alpha\phi_\alpha = 1$. The value of the orbital angular momentum now depends on the **transformation** properties of the f_α. In particular if the rotation loop is trivial then orbital angular momentum of 1/2 is not possible.

Williams and Finkelstein (1984) showed that for skyrmions, where the f_α transform as scalars, this loop is not a single point so the kinks of Skyrme's theory do have orbital angular momentum of 1/2. The skyrmion fields considered by Williams (1970) also possess such nontrivial loops.

For the gravitational field, which can be considered as a group field, the $g_{\alpha\beta}$ are second rank tensors and the authors showed that the 2π rotation loop for a 1-kink metric was a single point and hence orbital angular momentum of 1/2 was not possible. This contradicted the earlier work of Williams (1971) and that of Shastri, Williams and Zvengrowski (1980) where the ϕ_α were treated as scalars. Their analysis would only be valid in a theory such as that of Skyrme's, where the ϕ_α refer to internal variables that transform as scalars.

Williams (1985) showed that if the dimension of the usual spacetime manifold is increased to allow inner degrees of freedom then it is possible to have half-integral orbital angular momentum. The metric studied is of the form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{ij}dx^i dx^j$$

where

$$g_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_\alpha\phi_\beta.$$

The indicies α and β run over time and the inner dimensions. The indicies i and j are the usual spatial indices. The indices of the $g_{\alpha\beta}$ thus refer to internal variables and so transform as scalars under spatial rotation. They therefore have non-trivial rotation loops and allow half-integer orbital angular momentum.

The simple one-kink metric, in the usual 3+1 (time and three spatial) dimensions

$$g_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_\alpha\phi_\beta$$

where the ϕ_α may be chosen to be

$$\phi_i = x^i \sin\alpha$$
$$\phi_o = \cos\alpha$$

was first introduced by Williams and Zia (1973). This metric will be discussed in detail in future chapters.

Williams and Zia (1973) also were the first to discuss what type of mass distribution might give rise to such a spacetime. Their discussions showed that it is possible to recover Newton's Inverse Square (force) Law by considering the asymptotic behaviour of the metric. This part of the paper will also be discussed in more detail in future chapters of this thesis.

Williams (1974) introduced the following generalization of the simple one-kink metric

$$ds^2 = (e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha)dt^2 - 2(e^{\sigma} + e^{\Omega})\sin\alpha\cos\alpha\, dr\, dt$$
$$+ [e^{\sigma} - (e^{\sigma} + e^{\Omega})\sin^2\alpha]dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\Phi^2$$

where all functions α, σ, and Ω are functions of r alone. This metric and some further generalizations will be discussed in detail in further chapters.

Williams also notes that because the g_{tt} term will be zero at least once if there is a kink present, the usual transformation to remove the drdt term will be singular and therefore invalid. A singularity even at a single point will render the transformation inadmissable. This point has been noted in a different context by Rosen (1983, 1985). Such transformations will be examined in detail, with specific examples, in future chapters.

The interpretation of kink structures in spacetimes was extended in the work of Finkelstein and McCollum (1975) to include the internal structure of black holes that have no curvature singularities and obey the weak energy condition. This work also demonstrated, in more detail than Finkelstein and Misner (1959), the feature common to all kink metrics: that of tumbling light cones. This feature is illustrated below in Fig. (2.3).

Spacetime is assumed Minkowski at infinity. Gradually, the lightcones tumble towards the centre so they have future toward -t at r = 0. They then turn back up symmetrically on the other side. Such a spacetime has a background topology of R^4 and no curvature singularites. Following a procedure similar to Williams and Zia (1973), which will be discussed in more detail in later a chapter, Finkelstein and McCollum showed that one example of a physical source that may lead to such a light cone configuration is one with an energy density, μ, given by

$$\mu = 9(2 + r^{-1})^{-3}.$$

However, no exact metric that satisfies the Einstein field equations with such a form of μ used in the stress-energy tensor was presented by the authors.

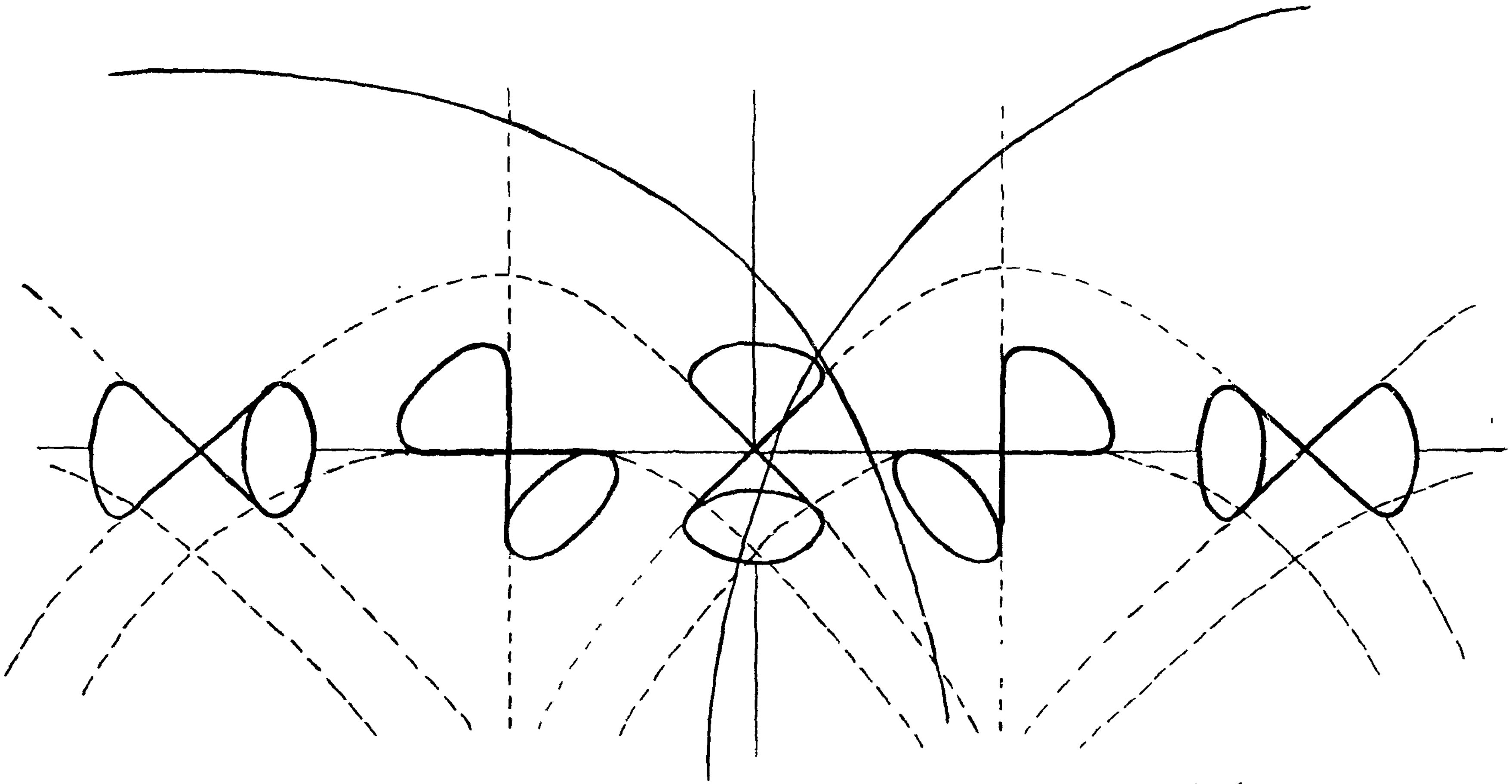

Fig. (2.3). Light cones for a one-kink spacetime. -------- = null geodesic.
———— = timelike geodesic.

A feature common to many spacetimes is that of incomplete geodesics. The main result presented by Finkelstein and McCollum is that kink metrics and all other spherically symmetric stationary spacetimes have incomplete geodesics approaching every root of g_{tt} in the metric. In general relativity the geodesics represent paths of test particles and so incompleteness is a problem because it means that test particles reach the edge of the manifold in finite proper time. Such a manifold can therefore not be the complete physical manifold even though this is not evident from the field equations themselves. The authors discussed how to extend the geodesics by various methods depending on the nature of the roots of g_{tt}. For simple roots, they used the Kruskal extension method (Kruskal, 1960) and showed that the topology around each root is that of the Kruskal manifold. For multiple roots they illustrated two methods one based on symmetry and another based on topology.

In particular, for the kink shown in Fig. (2.3) there is at least one set of incomplete geodesics. At the radii r_1 and r_2, where the light cones are turned so that one branch is parallel to the t axis, there are null geodesics parallel to this t axis. These are therefore one-way surfaces. For example, nothing can get out of the region where $|r| < |r_1|$. Near either of these one-way surfaces there are three sets of null geodesics: those that cross the surface, those that stay in the surface and those that approach but do not cross the surface. The authors showed that those geodesics that approach but do not cross the surface are all incomplete; the other two sets may or may not be. The Kruskal extension method was used to extend this kink manifold. This method assumes that the full null manifold may be made up of two copies of the given part joined along a surface in a smooth manner. This extended manifold may be illustrated as shown below in Fig. (2.4).

The top two diagrams are of the same patch shown in different coordinates. Each patch in the top two diagrams is half of the patch in the lower diagram. The U and V axes are at the radius of the one-way surface, r_1 or r_2. The three sets of geodesics transform as follows: Null geodesics parallel to the $\tilde{t}$ axis transform to the V axis. Null geodesics crossing the one-way surface transform to the lines crossing the V axis, parallel to the U axis. Incomplete null geodesics approaching the one-way surface but not crossing it transform into lines crossing the U axis into the new region.

These results are obtained by Finkelstein and McCollum as a special case of the Kruskal method described for all spherically symmetric stationary spacetimes. To construct

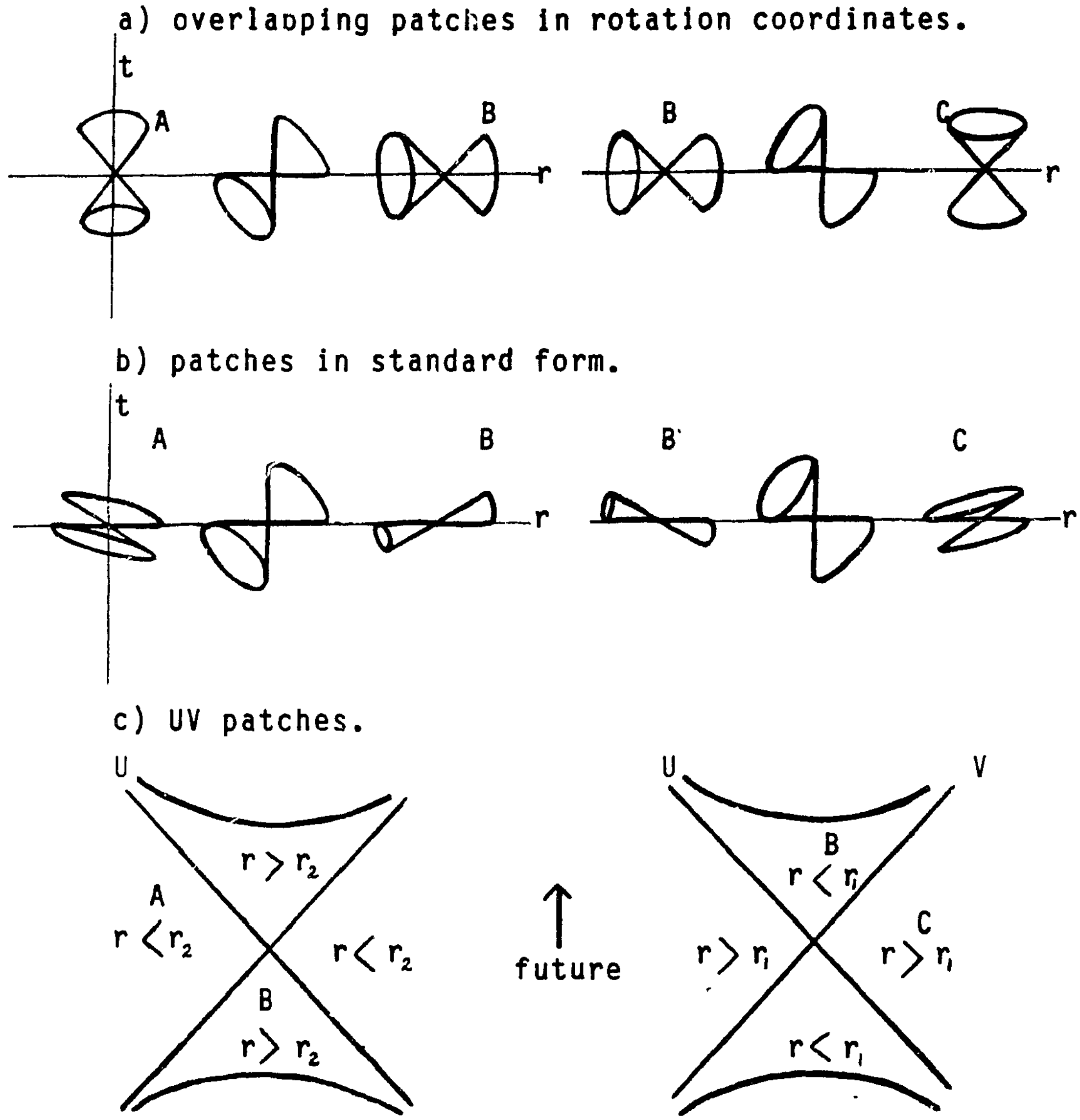

Fig. (2.4). Kink extended.

the extensions, they considered the most general spherically symmetric metric, which, following their notation is

$$ds^2 = g_{oo}(dx^o)^2 + 2g_{01}dx^0dx^1 + g_{11}(dx^1)^2 + g_{22}dw^2$$

where

$$dw^2 = d\theta^2 + \sin^2\theta d\Phi^2.$$

This metric is transformed to what the authors call standard form

$$ds^2 = e^{2\beta}(\Gamma dt^2 + 2K_i dtdr) + g_{22}dw^2$$

where β and Γ are functions of r, $K_i = \pm 1$ and $g_{22} = -r^2$ except in regions of extremal r.

To achieve this standard form, a transformation is first made to the form

$$ds^2 = e^{2\beta}(\Gamma(dx^0)^2 + 2Kdx^0dr + Ldr^2) + g_{22}dw^2$$

where $\Gamma L - K^2 = -1$ and all functions are functions of r only. The standard form is then obtained by a transformation

$$x^0 = t - \int_t^r \frac{K - K_i}{\Gamma} dr$$

where K_i is the sign of K at a root r_i of Γ, where the transformation becomes singular.

One coordinate patch surrounds each root of Γ. Zero kink number metrics may or may not be covered by one patch. Metrics of a non-zero kink number will require more than one patch to cover them. Each root of Γ defines a surface called a root surface. The standard form draws attention to the root surfaces, which are trapped surfaces, and the incomplete geodesics approach root surfaces.

The authors showed that three quantities are needed to count the kink number of the metric. These are, K_i which is the sign of g_{11} at a root of g_{00}; f_i, the sign of x^0 in the future direction; and $\Delta\Gamma_i$ which is the change in Γ at r_i. To qualitatively construct the extension near a root it is sufficient to know K_i, f_i, $\Delta\Gamma_i$ and the order of the root. These four quantities are invariants and so, therefore, is the topology of the extended patch which depends only on the order of the root.

These four quantities give further information about the spacetime. In particular, Finkelstein and McCollum showed that they indicate whether the spacetime describes a black or a white hole. A trapped surface may be defined by considering a shell of light emitted normal to a spherical surface. The behaviour of this shell is determined by the behaviour of $\sqrt{-g_{22}}$ in the direction in which the light is emitted. For example, if $\sqrt{-g_{22}}$ has a maximum, the shell shrinks in surface area in whatever direction it is emitted If $\sqrt{-g_{22}}$ has a minimum, then the shell can only grow in surface area. A trapped surface is one from which such a surface area of a shell of light can either never grow or can never shrink. The surface of extremal $\sqrt{-g_{22}}$ is always a trapped surface. Trapped surfaces also occur where the future part of the light cone points only in one direction in r. Matter can only flow in one direction there, thus determining a black or white hole situation. The direction in r of the positive t half of the lightcone at a root of Γ is given by K_i. If f_i is the sign of the chosen future and $\sqrt{-g_{22}}$ is not extremal at a root of Γ then $fK_i = 1$ indicates a white hole and $fK_i = -1$ indicates a black hole.

Finkelstein and McCollum illustrate a manifold extension for the kink metric in what they call rotation coordinates

$$ds^2 = \cos 2\alpha(dt^2 - dr^2) \pm 2\sin 2\alpha dtdr - r^2dw^2.$$

The authors did not demonstrate a form for $\cos 2\alpha$ that satisfied the Einstein field equations. They showed however that this metric can be transcribed into their standard form

$$ds^2 = \cos 2\alpha\ dt^2 \pm dtdr - r^2dw^2$$

where $\Gamma = \cos 2\alpha$, $g_{22} = -r^2$. There are two roots, at r_1 and r_2. Both root surfaces are black (matter trapped inside). In patch 1, f = 1 and $K_1 = -1$. In patch 2, -t is the future direction at r_2 and so f = -1, $K_2 = 1$. Incomplete geodesics approach each root. The geodesics can be completed by transforming the metric to standard form and then to the form

$$ds^2 = e^{2\beta}(2f^2dUdV) + g_{22}dw^2$$

where f and r are functions of U and V. (Kruskal, 1960). The authors observed that the original tr plane covers only half the UV plane. The tr plane is now extended by letting it cover the other half of the UV plane by reversing t but keeping the r the same. The geodesics crossing the U axis are now continuous. In this way, two new sheets are attached. The light cone behaviour for this metric in rotation coordinates, standard form and UV form is illustrated in Fig. (2.4).

Finkelstein and McCollum suggest that n-kink metrics, where $n > 1$, may need more extensions. Finkelstein and McCollum also noted that it is possible to construct a black kink (matter trapped inside) and a white kink (matter can only flow outwards) from an originally kinkless spacetime by a continuous process. They called this structure an "onion" as the formation of such a spacetime arises from "a pulling apart or as a nesting of spheres within spheres like an onion" as illustrated below in Fig. (2.5). In later chapters of this thesis the concept of an "onion" will be further explored.

The interest in kink spacetimes initially arose from the possibility that they might contain a conserved quantity that could be interpreted as an elementary particle number. When this was found not to be the case, because of their incorrect spin properties, interest remained because the spacetimes were found to be topologically non-trivial and to possess other interesting features such as tumbling light cones. As discussed previously, Williams and Zia (1973), Williams (1974) and Finkelstein and McCollum (1975) introduced various kink

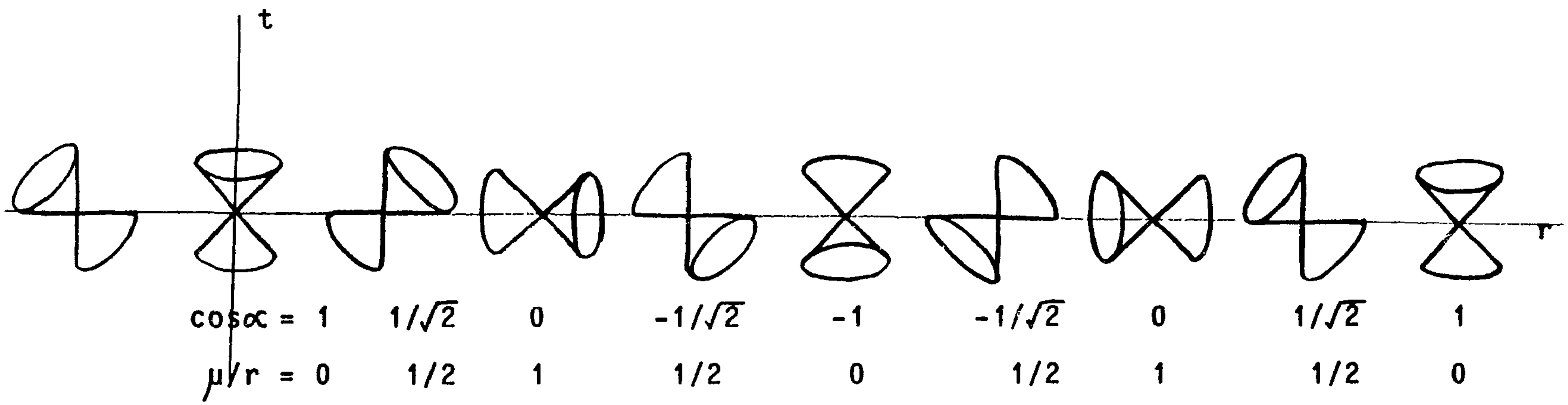

Fig. (2.5). Extension to an n-kink manifold: an "onion".

metrics and suggested possible mass distributions that might give rise to such spacetimes. No exact kink solutions of the Einstein field equations were demonstrated by any authors however. Also, the exact conditions that lead to a kink metric were unclear at this time. The integral counting number, also called the kink number, introduced by Finkelstein and Misner (1959) was not expressed in a covariant form. For this, and other reasons, it could not be used in general relativity to unambiguously define a kink metric as a metric with a non-zero integral counting number.

This thesis first extends the work of Williams and Zia (1973) and Williams (1974), who suggested the general conditions that lead to kink metrics. Several exact kink solutions of the Einstein field equations are then presented. The tumbling light cone behaviour of all kink metrics, noted previously by Finkelstein and Misner (1959) and Finkelstein and McCollum (1975), is illustrated for these solutions. The relationship of these exact kink solutions to well known solutions, such as the de Sitter and Friedmann-LeMaitre-Robertson-Walker solutions, is also investigated. Finkelstein and McCollum (1975) suggested that the kink metrics might serve as suitable metrics to describe the interiors of black holes. This idea, and other possible interpretations for kink metrics, are discussed for the various solutions presented. Modifications to the integral counting number formula, introduced by Finkelstein and Misner (1959), are suggested. These modifications allow the kink number to be interpreted, in a well defined way, for any spacetime whose manifold is $R^1 \times S^n$.

(2.3) Particular Form of the Kink Metrics

To derive the general conditions that lead to a kink metric consider, the set of all possible metric tensors of general relativity that make up the set of 4x4 real symmetric matrices of signature (-+++). This set is called $S_{4,1}$ by Steenrod (1951). Any 4x4 real symmetric matrix G of signature (-+++) can be written as a product of a positive definite symmetric matrix S, which is a member of $S_{4,0}$ and a 4x4 symmetric, orthogonal matix Q of signature (-+++) which is a member of $O_{4,1}$, that commute with each other (Birkhoff and MacLane, 1965)

$$G = SQ = QS \tag{2.1}$$

This decomposition of the matrix G is unique (Chevalley, 1946) and is proved as follows.

The matrix G is real and symmetric and can therefore be diagonalized by an orthogonal matrix R,

$$R^{-1}GR = \mathrm{diag}(\Omega_0,\Omega_1,\Omega_2,\Omega_3).$$

R is the matrix of eigenvectors of G and Ω_α, $\alpha = 0,1,2,3$ are the eigenvalues of G.

Consider the matrix G^2 which is symmetric and positive definite. Then R also diagonalizes G^2

$$\begin{aligned} R^{-1}GRR^{-1}GR &= R^{-1}G^2R \\ &= \mathrm{diag}(\Omega_0{}^2,\Omega_1{}^2,\Omega_2{}^2,\Omega_3{}^2). \end{aligned}$$

Define

$$T^2 = \mathrm{diag}(\Omega_0{}^2,\Omega_1{}^2,\Omega_2{}^2,\Omega_3{}^2)$$

so that

$$G^2 = RT^2R^{-1}.$$

The matrix T^2 is positive definite, since it is the matrix of the squares of the eigenvalues of G. A matrix T can therefore be extracted easily by taking the positive square roots of all the diagonal entries of T^2. Define

$$S = RTR^{-1} \tag{2.2}$$

and

$$Q = S^{-1}G$$

so that

$$SQ = G$$

as required.

It remains to show that S is symmetric, Q is orthogonal, that the decomposition is unique and that S and Q commute.

$$S^T = (RTR^{-1})^T$$
$$= (R^{-1})^T T^T R^T$$
$$= RTR^{-1}$$

since R is orthogonal

$$= S ,$$

hence S is symmetric.

The definition of S given by (2.2) implies that

$$S^2 = RTR^{-1}RTR^{-1}$$
$$= RT^2R^{-1}$$

and therefore that

$$S^2 = G^2.$$

Using this last result

$$QQ^T = (S^{-1}G)(S^{-1}G)^T$$
$$= S^{-1}GG^T(S^{-1})^T$$

$$= S^{-1}S^2(S^{-1})^T$$
$$= S(S^{-1})^T$$
$$= I.$$

This shows that Q is orthogonal. The uniqueness of the decomposition is shown as follows:

Let

$$G = SQ = S'Q'$$

where S and S' are symmetric positive definite matrices and Q and Q' are orthogonal. Define

$$Q'' = Q(Q')^T$$

then Q" is also orthogonal since

$$Q''(Q'')^T = Q(Q')^T[Q(Q')^T]^T$$
$$= Q(Q')^TQ'Q^T$$
$$= I.$$

Also

$$S' = SQ''$$

since

$$SQ'' = SQ(Q')^T$$
$$= S'Q'(Q')^T$$
$$= S'.$$

The matrix S' is symmetric, therefore it is also true that

$$S' = (S')^T$$
$$= (SQ'')^T$$
$$= (Q'')^T S^T$$
$$= (Q'')^T S.$$

These last two results show that

$$(S')^2 = SQ''(Q'')^T S$$
$$= S^2.$$

Any positive definte matrix can be written uniquely (Chevalley, 1946) as the exponent of some matrix. Define

$$S = \exp A$$
$$S' = \exp A'$$

then $S^2 = (S')^2$ implies that

$$\exp 2A = \exp 2A'$$

and hence that

$$A = A'.$$

Clearly now

$$S = S'$$

and

$$Q = Q'.$$

The decomposition is therefore unique as required. The commutativity of S and Q is shown as follows:

The matrix G = SQ is symmetric and therefore

$$SQ = (SQ)^T = Q^T S.$$

Hence, using this last result and the fact that Q is orthogonal,

$$S = QQ^T S = QSQ = QSQ^T Q^2.$$

The matrices Q and Q^T have the same eigenvalues and so QSQ^T is clearly positive definite. Also

$$(QSQ^T)^T = QS^TQ^T = QSQ^T$$

and so the product QSQ^T is symmetric and positive definite.

$$\begin{aligned} Q^2(Q^2)^T &= QQQ^TQ^T \\ &= QQ^T \\ &= I \end{aligned}$$

therefore Q^2 is orthogonal. The decomposition (2.1) for any matrix is unique. Therefore since the decomposition for any symmetric positive definite matrix S is S = SI it must be true that

$$Q^2 = I$$

and

$$QSQ^T = S$$

or equivalently

$$SQ = SQ.$$

The matrices S and Q therefore commute as required and Q is symmetric as well as orthogonal.

Many of the known metrics of general relativity have a trivial Q matrix and a non-trivial matrix S. For example, for the Schwarzschild metric

$$ds^2 = -(1 - 2mr^{-1})dt^2 + (1 - 2mr^{-1})^{-1}dr^2$$
$$+ r^2d\theta^2 + r^2\sin^2\theta d\Phi^2$$

$$Q = \mathrm{diag}(-1,1,1,1)$$

$$S = \begin{pmatrix} 1-2mr^{-1} & 0 & 0 & 0 \\ 0 & (1-2mr^{-1})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} .$$

However, the Gödel metric

$$ds^2 = -dt^2 - 2\exp(\sqrt{2}wx)dydt + dx^2 - 2^{-1}\exp(2\sqrt{2}wx)dy^2 + dz^2$$

has both S and Q non-trivial. This may be shown as follows: Define

$$B = -2\exp(\sqrt{2}wx)$$
$$C = 8^{-1}B^2$$

so that the matrix G representing the metric can be written as

$$G = \begin{pmatrix} -1 & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ B & 0 & C & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The S matrix is

$$S = \begin{pmatrix} W & 0 & X & 0 \\ 0 & 1 & 0 & 0 \\ X & 0 & Z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$X = \pm B(1 - C)\{[(1 + C)^2 + 4B^2]^{1/2}\}^{-1}$$
$$W = B^2(C - 1) - X^2(C + 1)(2BX)^{-1}$$
$$Z = B^2(C - 1) + X^2(C + 1)(2BX)^{-1}.$$

The matrix Q is

$$Q = \begin{pmatrix} M & 0 & N & 0 \\ 0 & 1 & 0 & 0 \\ N & 0 & -M & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $M = [8^{-1}B^2 - 1]\{[(1 - 8^{-1}B^2)^2 + 4B^2]^{1/2}\}^{-1}$

$N = 2B\{[(1 - 8^{-1}B^2)^2 + 4B^2]^{1/2}\}^{-1}.$

Any positive definite matrix can be continuously deformed into the unit matrix. The matrix S, which is symmetric and positive definite can therefore be deformed into the unit matrix and is said to be homotopically trivial. The matrices that reduce S to the unit matrix will be constructed in section (2.4). The kink nature of the metric must therefore be found in the matrix Q. The matrix Q can also be written as (Roman, 1961)

$$Q = P \operatorname{diag}(-1,1,1,1)\ P^T \tag{2.3}$$

where P is an orthogonal matrix, given by

$$P = \begin{pmatrix} \phi_0 & -\phi_1 & -\phi_2 & -\phi_3 \\ \phi_1 & \phi_0 & \phi_3 & -\phi_2 \\ \phi_2 & -\phi_3 & \phi_0 & \phi_1 \\ \phi_3 & \phi_2 & -\phi_1 & \phi_0 \end{pmatrix} \tag{2.4}$$

and where

$$\Sigma\phi_\alpha\phi_\alpha = 1.$$

Using equation (2.4) with equation (2.3) shows that Q may also be written as

$$Q_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_\alpha\phi_\beta.$$

The matrix Q can be reduced to the matrix diag(-1, 1, 1, 1) by its matrix of eigenvectors, because Q is orthogonal and therefore has eigenvalues ± 1 (Horn and Johnson, 1985). It should be noted however that P is not the matrix of eigenvectors of Q. The required form of P can easily be found if the metric is spherically symmetric as will be demonstrated in section (2.5). For **any** symmetric, orthogonal Q the result is more difficult and lengthy to establish. A proof can be found however in Roman (1961). The kink metrics considered in this thesis are spherically symmetric metrics and therefore the proof for more general matrices Q is omitted.

Williams (1971) showed that the ϕ_α define a mapping from R^3 into S^3. This mapping is said to be non-trivial if it does not belong to the homotopy class that contains the group identity, that is, the class that contains the constant map, that maps R^3 into a single point. If this mapping is non-trivial, then the metric represented by the matrix G will be a non-trivial kink metric and the degree of the map will be equal to the kink number. For example, for the Schwarzschild metric, the ϕ_α are

$$\phi_o = 1$$

$$\phi_i = 0$$

which is clearly the constant map. This mapping is therefore trivial and confirms that the Schwarzschild metric is not a kink metric.

The factorization of any matrix G, representing a metric, into the matrices S and Q is unique in a given coordinate system but the factorization is not covariant. That is, if the metric is transformed to a different coordinate system, then it will be represented by a different matrix G' that has a factorization S' and Q', and there is no simple relationship between S' and S or between Q' and Q. This is because the $\{\phi_\alpha\}$ are not the components of a vector or a tensor.

(2.4) Derivation of the Kink Counting Number.

Finkelstein and Misner (1959) were the first to demonstrate the existence of an integral counting number N that could be used to classify the metrics of general relativity up to a homotopy. The number N is now also called the kink number of the metric. They also demonstrated that metrics whose kink numbers differ cannot be continuously deformed into one another. The kinks of general relativity and skyrmion kinks are both characterized by mappings from a three-dimensional space into a three-sphere. This kink number N, corresponding to the homotopy class that contains the metric $g_{\alpha\beta}$, is related to the degree of this mapping, as is the counting number of skyrmion kinks. The skyrmion kink counting number is obtained when the N^0 component of the skyrmionic current N^α of strong interaction theory (Skyrme, 1961) is integrated over three-space.

$$N^\alpha = (12\pi^2)^{-1}\epsilon^{\alpha\cap\Omega\Gamma}\epsilon_{\sigma\beta\tau\delta}\phi^\sigma\phi^\beta{}_{,\cap}\phi^\tau{}_{,\Omega}\phi^\delta{}_{,\Gamma} \qquad (2.5)$$

It is important to note that the ϕ_α in the above equation due to Skyrme are **scalar** fields such that

$$\Sigma\phi_\alpha\phi_\alpha = 1$$

The set $\{\phi_\alpha\}$ therefore defines a map from three-space into the three-sphere

$$\Phi : R^3 \rightarrow S^3$$

such that

$$\Phi(\mathbf{x}) = (\phi_0, \phi_1, \phi_2, \phi_3)$$

The skyrmionic current is therefore proportional to the Jacobian of Φ, and the skyrmion integral counting number N_S is equal to the degree of the map, $\deg(\Phi)$.

$$N_S = \deg(\Phi) = \int N^0 d^3x \qquad (2.6)$$

As noted by Felsager (1981), this winding number, $\deg(\Phi)$, is only well-defined for smooth maps between compact manifolds and even if it is defined for non-smooth, non-compact manifolds it need not be an integer or even constant. It is therefore necessary to be able to compactify R^3 to form a three-sphere. This is possible when Φ maps the infinite boundary of R^3 into a single fixed point in S^3, say (1,0,0,0). This means the set of maps Φ is restricted to satisfy

$$(\phi_0, \phi_1, \phi_2, \phi_3) \rightarrow (1, 0, 0, 0)$$

at the boundary at spatial infinity.

Such a restriction leads to asymptotic flatness

$$\lim_{|x|\to\infty} g_{\alpha\beta} = \eta_{\alpha\beta}$$

and may be interpreted as "preventing" the kink from escaping at infinity. That is, if there is no matter at infinity the kink cannot be "pushed out" to infinity or "flattened out" under some coordinate transformation.

Several examples of general relativistic kink metrics have been discussed (Finkelstein and McCollum, 1975; Harriott and Williams, 1988; Dunn and Williams 1989). All of these examples arise from metrics of the form

$$g_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_\alpha\phi_\beta \qquad (2.7)$$

where the ϕ_α are functions satisfying $\Sigma\phi_\alpha\phi_\alpha = 1$. This metric given by equation (2.7), in the notation of section (2.2), corresponds to the choice S = I and G = Q. The kink number of such metrics is again related to the degree of the mapping defined by the $\{\phi_\alpha\}$ and it appears that it could be calculated by using the formulae given in equations (2.5) and (2.6). However, there are several problems with actually using these formulae to do such a calculation. These are:

(i) The ϕ_α of Skyrme's theory are scalars, whereas those arising in the general relativistic metrics are not scalars, vectors or tensors of any kind. Also, the relation they obey, namely $\Sigma\phi_\alpha\phi_\alpha = 1$, is not covariant. It is therefore not clear how these quantities transform under a change of coordinates.

(ii) To be useful, it is desirable that the kink number could be calculated for **any** metric of general relativity. In general, it is not possible to transform a metric into a form such as (2.7) in order to extract the ϕ_α to calculate the kink number. That is, in general it is not true that $S = I$.

(iii) The concept of degree is only defined for mappings between spaces of the same dimension. The set of all Lorentz metrics, $S_{4,1}$, has a dimension greater than 3. The mapping used to define the kink number must therefore be shown to be between spaces of the same dimension.

To overcome these problems it is therefore necessary to generalise the process used above to calculate a kink number so that it will be well defined for all metrics. This is achieved by specifying how to define the ϕ_α for any metric even if it cannot be expressed in the form of equation (2.7) and then showing how to define a vector quantity from the ϕ_α that will allows a covariant form of the kink number to be defined.

The ϕ_α for any general relativistic metric can always be found as follows: It was demonstrated in section (2.3) that any metric can be represented by the product of two matrices S and Q where S is positive definite and symmetric and Q is orthogonal and symmetric. It was also demonstrated that the matrix Q can always be written in the form

$$P\ \mathrm{diag}(-1,1,1,1)\ P^T,$$

where P is also orthogonal, so that each of its columns or rows represents a mapping from R^3 into S^3. In particular, it was shown that the elements of Q can always be expressed in the form

$$q_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_\alpha\phi_\beta,$$

where the ϕ_α are the elements of P, and

$$P = \begin{pmatrix} \phi_0 & -\phi_1 & -\phi_2 & -\phi_3 \\ \phi_1 & \phi_0 & \phi_3 & -\phi_2 \\ \phi_2 & -\phi_3 & \phi_0 & \phi_1 \\ \phi_3 & \phi_2 & -\phi_1 & \phi_0 \end{pmatrix}. \qquad (2.8)$$

The ϕ_α satisfy

$$\Sigma\phi_\alpha\phi_\alpha = 1.$$

Thus it is demonstrated that such ϕ_α can always be found for any metric in a given coordinate system.

It is a well known result (Geroch and Horowitz, 1979) that any Lorentz metric $g_{\alpha\beta}$ in a particular coordinate system can be written in terms of an arbitrary positive definite metric $h_{\alpha\beta}$ and a vector field X^α that is unique up to a sign

$$g_{\alpha\beta} = h_{\alpha\beta} - 2h_{\alpha\tau}h_{\beta\Theta}X^\tau X^\Theta(h_{\sigma\mu}X^\sigma X^\mu)^{-1}.$$

It is this vector field X^α that will be used to replace the ϕ_α in the formula for the kink number.

Let the matrix G represent the metric $g_{\alpha\beta}$ and let H represent the metric $h_{\alpha\beta}$ in some coordinate system. According to Perlis (1964), it is always possible to find an invertible matrix C that simultaneously diagonalizes M and K where M and K are any symmetric matrices and K is positive definite. G and H are two such matrices and it will be shown that, following Perlis, there exists C such that

$$C^T G C = \mathrm{diag}(\eta_o, \eta_1, \eta_2, \eta_3)$$

$$C^T H C = I$$

and

$$G c^\alpha = \eta_\alpha H c^\alpha$$

where c^α are the columns of C. These results are proved as follows:

H is a real, symmetric matrix and therefore there exists a real, orthogonal matrix that diagonalizes it. Let this matrix be U, and let the eigenvalues be μ_α, $\alpha = 0,1,2,3$ so that

$$U^T H U = \mathrm{diag}(\mu_0, \mu_1, \mu_2, \mu_3).$$

The eigenvalues μ_α of H are all positive because H is positive definite and therefore it is possible to construct a symmetric matrix, D, from the inverse square roots of the eigenvalues. That is

$$D = \mathrm{diag}(\mu_0^{-1/2},\ \mu_1^{-1/2},\ \mu_2^{-1/2},\ \mu_3^{-1/2}).$$

Clearly

$$\begin{aligned} D^T U^T H U D &= \mathrm{diag}(\mu_\alpha^{-1/2})\mathrm{diag}(\mu_\alpha)\mathrm{diag}(\mu_\alpha^{-1/2}) \\ &= I. \end{aligned}$$

A symmetric matrix, F, can now be constructed from D, G and U:

$$F = D^T U^T G U D.$$

The matrix F is easily seen to be symmetric because

$$\begin{aligned} F^T &= (D^T U^T G U D)^T \\ &= D^T U^T G^T (U^T)^T (D^T)^T \\ &= D^T U^T G U D \\ &= F. \end{aligned}$$

The symmetric nature of the matrix F means that it can be diagonalized by a real orthogonal matrix. Let this matrix be V and let the eigenvalues be η_α, $\alpha = 0,1,2,3$ so that

$$V^T F V = \mathrm{diag}(\eta_0,\ \eta_1,\ \eta_2,\ \eta_3).$$

Since V is orthogonal it is clear that

$$\begin{aligned} V^T D^T U^T H U D V &= V^T I V \\ &= V^T V \\ &= I \end{aligned}$$

This result shows that a matrix C = UDV will diagonalize H, in particular it will reduce H to the unit matrix. Such a matrix C can be shown to also diagonalize G as follows:

$$\begin{aligned} C^T G C &= (UDV)^T G (UDV) \\ &= V^T (D^T U^T G U D) V \\ &= V^T R V \\ &= \mathrm{diag}(\eta_0,\ \eta_1,\ \eta_2,\ \eta_3). \end{aligned}$$

The matrix C = UDV is therefore the required matrix that simultaneously diagonalizes G and H. These last two results show that

$$GC = (C^T)^{-1} \mathrm{diag}(\eta_0, \eta_1, \eta_2, \eta_3)$$

and

$$HC = (C^T)^{-1} I$$

so that

$$GC = HC\mathrm{diag}(\cap_0, \cap_1, \cap_2, \cap_3)$$
$$= \mathrm{diag}(\cap_0, \cap_1, \cap_2, \cap_3)HC.$$

This last result shows that for any choice of C the $\cap_\alpha$ will satisfy

$$\det(G - \cap_\alpha H) = 0.$$

There will therefore be non-trivial solutions to the eigenvalue equation

$$g_{\alpha\beta}X^\beta = \cap h_{\alpha\beta}X^\beta. \qquad (2.9)$$

Recalling that $G = SQ$, it is therefore possible to make a special choice for $h_{\alpha\beta}$, namely let

$$H = S.$$

This choice means that

$$GX = SQX = \cap SX.$$

It therefore follows that

$$q_{\alpha\beta}X^{\beta} = \cap X^{\beta}$$

or equivalently,

$$QX = \cap X.$$

The signature of Q means that one of the eigenvalues $\cap_{\alpha}$, say $\cap_{o}$, will be negative. The eigenvalues of Q must be ± 1, because Q is orthogonal, so that $\cap_{O} = -1$ and $\cap_{i} = +1$. If $\cap = \cap_{O}$, equation (2.9) can now be solved for X^{β}, which determines a vector field $\underline{\mathbf{X}}$, that is timelike and unique up to a sign and normalisation. The normalisation may be chosen so that $\Sigma X^{\alpha}X^{\alpha} = 1$. However, with these choices, the X^{α} are equal to the ϕ_{α} in this coordinate system. This can be shown as follows:

Let

$$\mathrm{col}(\phi) = \text{zeroth column of } P$$

$$= \begin{pmatrix} \phi_o \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

and

$$\begin{aligned} \text{col}(n) &= \text{zeroth column of } \eta \\ &= \text{diag}(-1,1,1,1) \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

It is now easy to show that

$$\begin{aligned} G\ \text{col}(\phi) &= SQ\ \text{col}(\phi) \\ &= SP\ \text{diag}(-1,1,1,1)P^T\ \text{col}(\phi) \\ &= SP\ \text{col}(n) \\ &= -S\ \text{col}(\phi) \end{aligned}$$

Comparing this result to (2.9) shows that X^α can be identified with the ϕ_α.

This vector field X^α is now used in the equation (2.5) replacing the scalar ϕ^α of Skyrme's formula. The ordinary derivatives are replaced by the covariant derivatives with respect to the tensor $s_{\alpha\beta}$, leading to the following covariant form of the kink number formula

$$N = (12\pi^2)^{-1} \int \epsilon^{oijk}\ \epsilon_{\alpha\beta\tau\delta}\ X^\alpha X^\beta{}_{|i} X^\tau{}_{|j} X^\delta{}_{|k}\ d^3x.$$

The | denotes differentiation with respect to the tensor $s_{\alpha\beta}$. Whenever the matrix S is the unit matrix, the covariant derivatives reduce to ordinary partial derivatives and the formula returns to the original form proposed by Skyrme.

This form of the kink number will be used later to demonstrate the that some metric solutions of the Einstein field equations are kink metrics. The agreement between this formula and Skyrme's original formula for these solutions will be demonstrated. This is achieved by calculating the kink number in the chosen coordinate system of the metric and then transforming the metric into coordinates where the S matrix is the unit matrix I and showing that the kink number remains unchanged. The kink number will remain unchanged in **any** coordinate system chosen provided the transformation is one allowed by homotopy theory. That is, provided the transformation is not singular at any point and provided that there is no change in the foliation of the manifold into hypersurfaces as a result of the transformation.

(2.5) The General Spherically Symmetric Kink Metric

Consider the decomposition of the matrix G, representing the most general spherically symmetric metric, into its S and Q matrices. That is, as shown in section (2.3)

$$G = SQ$$

where S is symmetric and positive definite and Q is orthogonal. The most general spherically symmetric metric may be written in the coordinates $\{x^\alpha\}$ as (Bergmann 1942)

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta \\ &= Adt^2 + 2B(x^i r^{-1})dtdx^i + [C\delta_{ij} + D(x^i x^j r^{-2})]dx^i dx^j \end{aligned}$$

where $A = A(r,t)$, $B=B(r,t)$, $C=C(r,t)$, $D=D(r,t)$,
$t = x^0$, $r = (x^i x^i)^{1/2}$.

The matrix representing this metric is

$$G = \begin{pmatrix} A & Bx^1r^{-1} & Bx^2r^{-1} & Bx^3r^{-1} \\ Bx^1r^{-1} & C + Dx^1x^1r^{-2} & Dx^1x^2r^{-2} & Dx1x^3r^{-2} \\ Bx^2r^{-1} & Dx^2x^1r^{-2} & C + Dx^2x^2r^{-2} & Dx^2x^3r^{-2} \\ Bx^3r^{-2} & Dx^3x^1r^{-2} & Dx^3x^2r^{-2} & C + Dx^3x^3r^{-2} \end{pmatrix} \tag{2.10}$$

Let R be the orthogonal matrix that diagonalizes G and let the eigenvalues of G be Ω_0, Ω_1, Ω_2, Ω_3, so that

$$R^T G R = \mathrm{diag}(\Omega_0, \Omega_1, \Omega_2, \Omega_3)$$

where $\Omega_0 < 0$ and $\Omega_1, \Omega_2, \Omega_3 > 0$ to satisfy the signature condition. Then R simultaneously diagonalizes S and Q. This result is shown as follows:

S is defined from equation (2.2) by

$$S = RTR^{-1}$$

where T is the diagonal matrix whose elements are obtained by taking the positive square roots of the squares of the eigenvalues of G. To satisfy the signature conditions of G, exactly one of it's eigenvalues will be negative. This negative eigenvalue can be chosen to be Ω_0 without loss of generality. Clearly

$$S = R\ \mathrm{diag}(|\Omega_0|, \Omega_1, \Omega_2, \Omega_3)\ R^T$$

and

$$R^T S R = \mathrm{diag}(|\Omega_0|, \Omega_1, \Omega_2, \Omega_3)$$

$$R^T G R = \mathrm{diag}(\Omega_0,\ \Omega_1,\ \Omega_2,\ \Omega_3).$$

The matrix Q can be diagonalized as follows.

$$\begin{aligned}
Q &= S^{-1}G \\
&= \{R\ \mathrm{diag}(|\Omega_0|,\Omega_1,\Omega_2,\Omega_3)\ R^T\}^{-1}\{R\ \mathrm{diag}(\Omega_0,\Omega_1,\Omega_2,\Omega_3)\ R^T\} \\
&= (R^T)^{-1}\ \mathrm{diag}(|\Omega_0|^{-1},(\Omega_1)^{-1},(\Omega_2)^{-1},(\Omega_3)^{-1})R^{-1}R \\
&\qquad\qquad \mathrm{diag}(\Omega_0,\Omega_1,\Omega_2,\Omega_3)R^T \\
&= R\ \mathrm{diag}(|\Omega_0|^{-1},(\Omega_1)^{-1},(\Omega_2)^{-1},(\Omega_3)^{-1}) \\
&\qquad\qquad \mathrm{diag}(\Omega_0,\Omega_1,\Omega_2,\Omega_3)R^T \\
&= R\ \mathrm{diag}(-1,\ 1,\ 1,\ 1)\ R^T.
\end{aligned}$$

Therefore R simultaneously diagonalizes S and Q as required.

It is important to note that R is the matrix of eigenvectors of G, and P is the specific matrix defined in equation (2.4). Both of these matrices reduce Q to the matrix diag(-1,1,1,1) but they are clearly distinct matrices, except in the special case when G = Q and G is diagonal. In the case where S = I and so G = Q, then R is clearly the matrix of eigenvectors for Q as well as G. For the spherically symmetric case, the form of these two matrices, R and P, will be found below.

The eigenvalues of G are

$$\Omega_0 = \{A + C + D - [(A - C - D)^2 + 4B^2]^{1/2}\}/2$$
$$\Omega_1 = \Omega_2 = C$$
$$\Omega_3 = \{A + C + D + [(A - C - D)^2 + 4B^2]^{1/2}\}/2.$$

To ensure that $\Omega_0 < 0$ and $\Omega_1, \Omega_2, \Omega_3 > 0$, the following inequalities must hold

$$C > 0 \; ; \quad B^2 > A(C + D)$$

The matrix of eigenvectors of G, R, is

$$R = \begin{pmatrix} -(C+D-\Omega_0)R_o^{-1} & 0 & 0 & -(C+D-\Omega_3)R_3^{-1} \\ Bx^1(rR_o)^{-1} & 0 & -S_{23}r^{-1} & Bx^1(rR_3)^{-1} \\ Bx^2(rR_o)^{-1} & x^3S_{23}^{-1} & x^1x^2(rS_{23})^{-1} & Bx^2(rR_3)^{-1} \\ Bx^3(rR_o)^{-1} & -x^2S_{23}^{-1} & x^1x^3(rS_{23})^{-1} & Bx^3(rR_3)^{-1} \end{pmatrix}$$

where

$$R_o = [B^2 + (C + D - \Omega_0)^2]^{1/2}$$
$$R_3 = [B^2 + (C + D - \Omega_3)^2]^{1/2}$$
$$S_{23} = [(x^2)^2 + (x^3)^2]^{1/2}.$$

To simplify the form of the matrices S and Q, consider the following relabelling

$$\Omega_0 = -e^{\Omega}$$
$$\Omega_1 = \Omega_2 = e^{\tau}$$
$$\Omega_3 = e^{\sigma}$$
$$\sin 2\alpha = -2B[(A - C - D)^2 + 4B^2]^{-1/2} .$$

The matrix S is now

$$\begin{pmatrix} e^{\Omega}c^2\alpha+e^{\sigma}s^2\alpha & e^{-}s2\alpha x^1(2r)^{-1} & e^{-}s2\alpha x^2(2r)^{-1} & e^{-}s2\alpha x^3(2r)^{-1} \\ e^{-}s2\alpha x^1(2r)^{-1} & e^{\tau}+F(x^1r^{-1})^2 & Fx^1x^2r^{-2} & Fx^1x^3r^{-2} \\ e^{-}s2\alpha x^2(2r)^{-1} & Fx^1x^2r^{-2} & e^{\tau}+F(x^2r^{-1})^2 & Fx^2x^3r^{-2} \\ e^{-}s2\alpha x^3(2r)^{-1} & Fx^1x^3r^{-2} & Fx^2x^3r^{-2} & e^{\tau}+F(x^3r^{-1})^2 \end{pmatrix} \tag{2.11}$$

where

$$c\alpha = \cos\alpha, \; s\alpha = \sin\alpha$$
$$e^{-} = e^{\Omega} - e^{\sigma}$$
$$F = e^{\Omega}\sin^2\alpha + e^{\sigma}\cos^2\alpha - e^{\tau}$$

and the matrix Q is

$$\begin{pmatrix} -c2\alpha & -x^1r^{-1}s2\alpha & -x^2r^{-1}s2\alpha & -x^3r^{-1}s2\alpha \\ -x^1r^{-1}s2\alpha & 1-2(x^1r^{-1})^2s^2\alpha & -2x^1x^2r^{-2}s^2\alpha & -2x^1x^3r^{-2}s^2\alpha \\ -x^2r^{-1}s2\alpha & -2x^1x^2r^{-2}s^2\alpha & 1-2(x^2r^{-1})^2s^2\alpha & -2x^2x^3r^{-2}s^2\alpha \\ -x^3r^{-1}s2\alpha & -2x^1x^3r^{-2}s^2\alpha & -2x^2x^3r^{-2}s^2\alpha & 1-2(x^3r^{-1})^2s^2\alpha \end{pmatrix} \tag{2.12}$$

where $c2\alpha = \cos 2\alpha$ and $s2\alpha = \sin 2\alpha$. The matrix that represents the metric $g_{\alpha\beta}$ can now be expressed as:

$$G = R\mathrm{diag}(-e^{\Omega},\ e^{\tau},\ e^{\tau},\ e^{\sigma})R^T.$$

Under this relabelling, the matrix R is

$$R = \begin{pmatrix} -\cos\alpha & 0 & 0 & -\sin\alpha \\ -x^1r^{-1}\sin\alpha & 0 & -S_{23}r^{-1} & x^1r^{-1}\cos\alpha \\ -x^2r^{-1}\sin\alpha & x^3S_{23}{}^{-1} & x^1x^2(rS_{23})^{-1} & x^2r^{-1}\cos\alpha \\ -x^3r^{-1}\sin\alpha & -x^2S_{23}{}^{-1} & x^1x^3/(rS_{23}) & x^3r^{-1}\cos\alpha \end{pmatrix} \tag{2.13}$$

and the matrix G, given previously by equation (2.10), is

$$\begin{pmatrix} e^{\sigma}s^2\alpha-e^{\Omega}c^2\alpha & -2^{-1}x^1r^{-1}s2\alpha e^+ & -2^{-1}x^2r^{-1}s2\alpha e^+ & -2^{-1}x^3r^{-1}s2\alpha e^+ \\ -2^{-1}x^1r^{-1}s2\alpha e^+ & -H(x^1r^{-1})^2+e^{\tau} & -Hx^1x^2r^{-2} & -Hx^1x^3r^{-2} \\ -2^{-1}x^2r^{-1}s2\alpha e^+ & -Hx^1x^2r^{-2} & -H(x^2r^{-1})^2+e^{\tau} & -Hx^2x^3r^{-2} \\ -2^{-1}x^3r^{-1}s2\alpha e^+ & -Hx^1x^3r^{-2} & -Hx^2x^3r^{-2} & -H(x^3r^{-1})^2+e^{\tau} \end{pmatrix}$$

where

$$c\alpha = \cos\alpha, \; s\alpha = \sin\alpha$$
$$H = e^{\Omega}\sin^2\alpha - e^{\sigma}\cos^2\alpha + e^{\tau}$$
$$e^{+} = e^{\Omega} + e^{\sigma}.$$

It is clear that the matrix Q depends on α but not on Ω, τ or σ. It can be written in a more compact form by introducing the functions $\{\phi_\alpha\}$:

$$\begin{aligned} \phi_o &= \cos\alpha \\ \phi_i &= x^i r^{-1}\sin\alpha. \end{aligned} \tag{2.14}$$

These are the functions that were first introduced by Skyrme (1961) defining the hedgehog of skyrmionic gauge theory. They clearly obey

$$\Sigma\phi_\alpha\phi_\alpha = 1$$

and so these $\{\phi_\alpha\}$ represent a mapping from R^3 into S^3.

These ϕ_α as defined above are the ϕ_α defined in Section (2.3) as components of the matrix P where

$$Q = P\text{diag}(-1,1,1,1)P^T.$$

Here the form of P is

$$\begin{pmatrix} \cos\alpha & -x^1r^{-1}\sin\alpha & -x^2r^{-1}\sin\alpha & -x^3r^{-1}\sin\alpha \\ x^1r^{-1}\sin\alpha & -\cos\alpha & x^3r^{-1}\sin\alpha & -x^2r^{-1}\sin\alpha \\ x^2r^{-1}\sin\alpha & -x^3r^{-1}\sin\alpha & -\cos\alpha & -x^1r^{-1}\sin\alpha \\ x^3r^{-1}\sin\alpha & -x^2r^{-1}\sin\alpha & -x^1r^{-1}\sin\alpha & -\cos\alpha \end{pmatrix} \tag{2.15}$$

Equivalently, these are the ϕ_α that lead to the following form for the tensor represented by the matrix Q

$$q_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_\alpha\phi_\beta.$$

It is now easy to see that when S = I, so that G = Q, the matrix R given by equation (2.13) is the matrix of eigenvectors of Q which is clearly distinct from the matrix P given by equation (2.15).

One way to find kink metrics is therefore to find a suitable form for the angle α (in the above metric) that satisfies the Einstein field equations. To be of physical interest, this form should lead to a physically acceptable equation of state in the stress-energy tensor. The question of how to choose a stress-energy tensor will be addressed in the next chapter.

The various choices that can be made for S and Q are:

(i) If $\alpha = 0$ then

$$\phi_o = 1$$
$$\phi_i = 0$$

$$Q = \mathrm{diag}(-1, 1, 1, 1)$$

$$S =$$

$$\begin{pmatrix} e^{\Omega} & 0 & 0 & 0 \\ 0 & e^{\tau}+(e^{\sigma}-e^{\tau})(x^1r^{-1})^2 & (e^{\sigma}-e^{\tau})x^1x^2r^{-2} & (e^{\sigma}-e^{\tau})x^1x^3r^{-2} \\ 0 & (e^{\sigma}-e^{\tau})x^2x^1r^{-2} & e^{\tau}+(e^{\sigma}-e^{\tau})(x^2r^{-1})^2 & (e^{\sigma}-e^{\tau})x^2x^3r^{-2} \\ 0 & (e^{\sigma}-e^{\tau})x^3x^1r^{-2} & (e^{\sigma}-e^{\tau})x^3x^2r^{-2} & e^{\tau}+(e^{\sigma}-e^{\tau})(x^3r^{-1})^2 \end{pmatrix}$$

This choice can be shown to lead to one of the well known spherically symmetric solutions, such as the Schwarzchild solution or the de Sitter solution, which are not kink metrics.

(ii) $\Omega = \tau = \sigma = 0$ and $\alpha \neq 0$, which imply that $e^{-} = e^{\Omega} - e^{\sigma} = 0$, $F = 0$, $e^{+} = e^{\Omega} + e^{\sigma} = 2$, $H = 2\sin^2\alpha$ so that

$$S = \mathrm{diag}(1, 1, 1, 1)$$

and

$$Q = G$$

is non trivial. The elements of Q or G are given by the matrix given in equation (2.12), equivalently,

$$q_{\alpha\beta} = g_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_{\alpha}\phi_{\beta}.$$

Under the coordinate transformation

$$\begin{aligned} x^0 &= t \\ x^1 &= r\cos\Phi\cos\theta \\ x^2 &= r\sin\Phi\cos\theta \\ x^3 &= r\sin\theta \end{aligned} \tag{2.16}$$

the metric becomes

$$\begin{aligned} ds^2 = &-\cos 2\alpha dt^2 - 2\sin 2\alpha dtdr + \cos 2\alpha dr^2 \\ &+ r^2 d\theta^2 + r^2\sin^2\theta d\Phi^2. \end{aligned} \tag{2.17}$$

It is easy to show that in these new coordinates the S and Q matrices are

$$S = \mathrm{diag}(1,\ 1,\ r^2,\ r^2\sin^2\theta)$$

$$Q = \begin{pmatrix} -\cos 2\alpha & -\sin 2\alpha & 0 & 0 \\ -\sin 2\alpha & \cos 2\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix P, where $Q = P\mathrm{diag}(-1, 1, 1, 1)P^T$, is

$$P = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

The form of P shows that

$$\phi_t = \cos\alpha$$
$$\phi_r = \sin\alpha$$
$$\phi_\theta = \phi_\Phi = 0.$$

These matrices do not arise from the transformations of equations (2.11) (2.12) (2.13) because the factorization of G into the matrices S and Q is not covariant. They are found from the decomposition process described in section (2.3).

For particular values of α, this is a simple one-kink metric which has been studied by several authors (Clément, 1984 a,b,c, 1986; Finkelstein and McCollum, 1975; Finkelstein and Williams, 1984; Harriott and Williams, 1986, 1988a; Williams, 1985 and Williams and Zia, 1973) and will be further discussed in Chapter 4.

(iii) If S and Q keep their most general form, then the metric tensor has components:

$$
\begin{aligned}
g_{oo} &= e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\
g_{oi} &= -2^{-1}x^i r^{-1}\sin 2\alpha(e^{\Omega} + e^{\sigma}) \\
g_{ij} &= e^{\tau}\delta ij - x^i x^j r^{-2}(e^{\tau} + e^{\Omega}\sin^2\alpha - e^{\sigma}\cos^2\alpha)
\end{aligned}
$$

Under the coordinate transformation (2.16) the metric tensor components become

$$
\begin{aligned}
g_{tt} &= e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\
g_{tr} &= -2^{-1}(e^{\sigma} + e^{\Omega})\sin 2\alpha \\
g_{rr} &= e^{\sigma}\cos^2\alpha - e^{\Omega}\sin^2\alpha \\
g_{\theta\theta} &= e^{\tau}r^2 \\
g_{\Phi\Phi} &= e^{\tau}r^2\sin^2\theta
\end{aligned}
\tag{2.18}
$$

and the S and Q matrices are now

$$S = \begin{pmatrix} s_{tt} & s_{tr} & 0 & 0 \\ s_{rt} & s_{rr} & 0 & 0 \\ 0 & 0 & e^{\tau}r^2 & 0 \\ 0 & 0 & 0 & e^{\tau}r^2\sin^2\theta \end{pmatrix}$$

where

$$s_{tt} = e^{\sigma}\sin^2\alpha + e^{\Omega}\cos^2\alpha$$
$$s_{rt} = \sin\alpha\cos\alpha(e^{\Omega} - e^{\sigma})$$
$$s_{rr} = e^{\sigma}\sin^2\alpha + e^{\Omega}\cos^2\alpha$$

and

$$Q = \begin{pmatrix} -\cos 2\alpha & -\sin 2\alpha & 0 & 0 \\ -\sin 2\alpha & \cos 2\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Therefore the ϕ_α of the P matrix, where

$$Q = P\mathrm{diag}(-1, 1, 1, 1)P^T,$$

are

$$\phi_t = \cos\alpha$$
$$\phi_r = \sin\alpha \qquad (2.19)$$
$$\phi_\theta = \phi_\Phi = 0.$$

For certain choices of α, this metric can be shown to have one kink present (Harriott and Williams, 1986b) This metric will be further discussed in Chapters 3 and 5.

It should be noted that n-kink metrics can easily be generated from any known one kink metric by construction of a new metric tensor represented by the matrix G_n (Williams and Zia, 1973)

$$G_n = SQ_n$$

where G = SQ is the one kink metric

$$Q = P\ \mathrm{diag}(-1,\ 1,\ 1,\ 1)\ P^T$$

and

$$Q_n = (P)^n\ \mathrm{diag}(-1,\ 1,\ 1,\ 1)(P^T)^n.$$

Such metrics may not, however, satisfy the Einstein field equations. For the metric given by equation (2.16), the matrix P is

$$P = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}.$$

It is easy to show that

$$P_n = (P)^n = \begin{pmatrix} \cos(n\alpha) & -\sin(n\alpha) & 0 & 0 \\ \sin(n\alpha) & \cos(n\alpha) & 0 & 0 \\ 0 & 0 & \cos(n\alpha) & \sin(n\alpha) \\ 0 & 0 & -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix}$$

and that the matrix Q_n is

$$Q_n = \begin{pmatrix} -\cos(2n\alpha) & -\sin(2n\alpha) & 0 & 0 \\ -\sin(2n\alpha) & \cos(2n\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

The new values of the ϕ_α are

$$\phi_t = \cos(n\alpha)$$
$$\phi_r = \sin(n\alpha).$$

A possible example of an n-kink metric is therefore

$$ds^2 = -\cos(2n\alpha)dt^2 - 2\sin(2n\alpha)drdt + \cos(2n\alpha)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\Phi^2 ,$$

but such an n-kink metric that satisfies the Einstein field equations has not yet been found. Extension of the spacetime manifold to construct n-kink metrics from known one kink metrics, that do satisfy the field equations, will be discussed in future chapters.

(2.6) Interpretations of Kink Solutions.

The kinks, now called skyrmions, arising in the nonlinear scalar field theories were developed by Skyrme (Skyrme, 1962) with the hope that they would provide a unified description of fermions and bosons. The similar structures arising in general relativity were first hoped to have a similar interpretation. However, the latter cannot usually describe half-odd intrinsic spin particles (Finkelstein and Williams, 1984). If the number of dimensions is extended to include inner degrees of freedom then Williams (1985) showed that half-integral spin is possible. Such solutions may have a particle interpretation but the usual metrics of general relativity will not.

Features of a kink spacetime include tumbling light cones. They have no global timelike Killing vector, no global timelike coordinate. There are therefore no spacelike (Cauchy) hypersurfaces and so the Singularity Theorems (Hawking and Ellis, 1973) do not apply to these spacetimes. A future direction is well defined at each point of the spacetime manifold by the light cones and so the manifold is in general time orientable. (This may not be the case if the infinite boundary of a single kink metric, where the light cones tumble through an angle π, is identified).

Finkelstein and McCollum (1975) stated that the one kink metrics of general relativity (with spherical symmetry) that they considered do not have closed timelike curves and so causality will not be violated. Their arguement will extend almost unchanged to n-kink spacetimes and is as follows: Consider a particle fired from a large value of r, within its light cone. Without loss of generality assume the particle moves to the left (to smaller r) as illustrated in Fig. (2.3). The particle moves to different slices labelled by t = k where k is constant. For greater values of k as r decreases, the light cones are tipping more and more and so the particle, confined within its light cone at each point is constrained to move along the timelike geodesic sketched in Fig. (2.3), which is parabolic in shape. The particle cannot return to the large r values and so there are no closed timelike curves.

Finkelstein and McCollum (1975) suggested that kinks may provide a description of the internal structure of black holes that possess no curvature singularities. These kink spacetimes have trapped surfaces: at certain radii the lightcones are turned so that one branch is parallel to the t axis at those radii. In one direction nothing can cross these radii. Kink metrics are global in nature and therefore any structures they describe must also be global.

The global nature of the kink solutions found becomes evident from the fact that locally they will be shown to be transformable to non-kink metrics such as the de Sitter metric. The equation of state of the de Sitter kink metric will be shown to be

$$p = - \mu.$$

This equation of state has been found to be associated with conditions in the early universe and, as shown in a later section, the associated scalar expansion is exponential. It is possible therefore that kink metrics may also have some relevance in the early universe.

It is possible that there may be other interpretations for kink metrics. Williams (1974) briefly discussed using the stress energy tensor of the electromagnetic field in the field equations, however no solutions were found. Another approach may be to consider vacuum expectation energy solutions but no work has yet been done in this area.

CHAPTER THREE

THE GENERAL SPHERICALLY SYMMETRIC KINK METRIC

Introduction

This chapter derives the Christoffel symbols, Ricci tensor components, scalar curvature, Einstein tensor components (3.1) and Killings equations (3.2) for the most general form of the spherically symmetric metric discussed in Chapter 2. The form of the stress-energy tensor is presented (3.3). The choice of an appropriate velocity vector for this metric is discussed (3.4). The form of various hydrodynamical quantities appearing in the stress-energy tensor are also found (3.5) - (3.7).

(3.1) The Christoffel Symbols, Ricci Tensor, Scalar Curvature and the Einstein Tensor.

The metric to be discussed is the most general form of the spherically symmetric metric that may admit kinks. It was shown in Chapter 2 that this metric could be written as

$$
\begin{aligned}
g_{tt} &= e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\
g_{tr} &= -(e^{\Omega} + e^{\sigma})\sin\alpha\cos\alpha \\
g_{rr} &= e^{\sigma}\cos^2\alpha - e^{\Omega}\sin^2\alpha \\
g_{\theta\theta} &= e^{\tau}r^2 \\
g_{\Phi\Phi} &= e^{\tau}r^2\sin^2\theta.
\end{aligned}
\tag{3.1}
$$

The contravariant components of this metric are

$$
\begin{aligned}
g^{tt} &= e^{-\sigma}\sin^2\alpha - e^{-\Omega}\cos^2\alpha \\
g^{tr} &= -\,(e^{-\sigma} + e^{-\Omega})\sin\alpha\cos\alpha \\
g^{rr} &= e^{-\sigma}\cos^2\alpha - e^{-\Omega}\sin^2\alpha \\
g^{\theta\theta} &= (e^{\tau}r^2)^{-1} \\
g^{\Phi\Phi} &= (e^{\tau}r^2\sin^2\theta)^{-1}
\end{aligned}
$$

and the determinant of the matrix $||g_{\alpha\beta}||$ is

$$\det(g_{\alpha\beta}) = -r^4\sin^2\theta\ \exp(\Omega + 2\tau + \sigma).$$

Writing

$$E = \exp(\Omega + \sigma),$$

it is clear that

$$g^{tt} = -E^{-1}g_{rr}$$
$$g^{rr} = -E^{-1}g_{tt}$$
$$g^{tr} = E^{-1}g_{tr}$$

and

$$(g_{rt})^2 - g_{tt}g_{rr} = E.$$

The Christoffel symbols are calculated from the equation (Misner, Thorne and Wheeler, 1973)

$$\Gamma^{\alpha}{}_{\beta\tau} = 2^{-1}g^{\alpha\mu}(g_{\beta\mu,\tau} + g_{\tau\mu,\beta} - g_{\beta\tau,\mu}).$$

For this general metric, the equations for the non-zero Christoffel symbols are rather lengthy if written explicity in terms of the metric functions, α, σ, Ω, τ. Therefore, where convenient, they are stated here in terms of the metric components g_{tt}, g_{tr} and g_{rr} and their derivatives.

$$\Gamma^{t}{}_{tt} = -2^{-1}E^{-1}(g_{rr}g_{tt,t} - 2g_{tr}g_{tr,t} + g_{tr}g_{tt,r})$$

$$\Gamma^{t}{}_{tr} = -2^{-1}E^{-1}(g_{rr}g_{tt,r} - g_{tr}g_{rr,t})$$

$$\Gamma^{t}{}_{rr} = 2^{-1}E^{-1}(g_{tr}g_{rr,r} - 2g_{rr}g_{tr,r} + g_{rr}g_{rr,t})$$

$$\Gamma^{t}{}_{\theta\theta} = 2^{-1}E^{-1}e^{\tau}r^{2}[g_{rr}\tau_{,t} - g_{tr}(\tau_{,r} + 2r^{-1})]$$

$$\Gamma^{t}{}_{\Phi\Phi} = \sin^{2}\theta\Gamma_{t\theta\theta}$$

$$\Gamma^{r}{}_{tt} = 2^{-1}E^{-1}(g_{tr}g_{tt,t} - 2g_{tt}g_{tr,t} + g_{tt}g_{tt,r})$$

$$\Gamma^{r}{}_{tr} = 2^{-1}E^{-1}(g_{tr}g_{tt,r} - g_{tt}g_{rr,t})$$

$$\Gamma^{r}{}_{rr} = -2^{-1}E^{-1}(g_{tt}g_{rr,r} - 2g_{tr}g_{tr,r} + g_{tr}g_{rr,t})$$

$$\Gamma^{r}{}_{\theta\theta} = 2^{-1}E^{-1}e^{\tau}r^{2}[g_{tt}(\tau_{,r} + 2r^{-1}) - g_{tr}\tau_{,t}]$$

$$\Gamma^{r}{}_{\Phi\Phi} = \sin^{2}\theta\Gamma^{r}{}_{\theta\theta}$$

$$\Gamma^{\theta}{}_{t\theta} = \Gamma^{\Phi}{}_{t\Phi} = 2^{-1}\tau_{,t}$$

$$\Gamma^{\theta}{}_{\Phi\Phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\Phi}{}_{r\Phi} = 2^{-1}\tau_{,r} + r^{-1}$$

$$\Gamma^{\Phi}{}_{\theta\Phi} = \cot\theta$$

The above results imply that

$$\Gamma^t{}_{tt} + \Gamma^r{}_{tr} = -2^{-1}E^{-1}(g_{rr}g_{tt,t} + g_{tt}g_{rr,t} - 2g_{tr}g_{tr,t})$$
$$= 2^{-1}E^{-1}(g_{tr}{}^2 - g_{tt}g_{rr})_{,t}$$
$$= 2^{-1}(\Omega_{,t} + \sigma_{,t})$$

and

$$\Gamma^t{}_{rt} + \Gamma^r{}_{rr} = -2^{-1}E^{-1}(g_{rr}g_{tt,r} + g_{tt}g_{rr,r} - 2g_{tr}g_{tr,r})$$
$$= 2^{-1}E^{-1}(g_{tr}{}^2 - g_{tt}g_{rr})_{,r}$$
$$= 2^{-1}(\Omega_{,r} + \sigma_{,r}).$$

The Ricci tensor is the contraction of the Riemann curvature tensor and is defined by (Misner, Thorne and Wheeler, 1973)

$$R_{\alpha\beta} = g^{\tau\delta}R_{\tau\alpha\beta\delta} = \Gamma^{\tau}{}_{\alpha\beta,\tau} - \Gamma^{\tau}{}_{\alpha\tau,\beta} + \Gamma^{\tau}{}_{\alpha\beta}\Gamma^{\delta}{}_{\tau\delta} - \Gamma^{\tau}{}_{\alpha\delta}\Gamma^{\delta}{}_{\tau\beta}.$$

The non-zero Ricci components for this metric are stated below. For this general metric they are expressed in terms of the non-zero Christoffel symbols and their derivatives.

$$R_{tt} = \Gamma^r{}_{tt,r} - \Gamma^r{}_{tr,t} - 2\Gamma^\theta{}_{t\theta,t} + \Gamma^t{}_{tt}\Gamma^\alpha{}_{t\alpha} + \Gamma^r{}_{tt}\Gamma^\alpha{}_{r\alpha}$$
$$- (\Gamma^t{}_{tt})^2 - (\Gamma^r{}_{tr})^2 - 2(\Gamma^\theta{}_{t\theta})^2 - 2\Gamma^r{}_{tt}\Gamma^t{}_{tr}$$

$$= \Gamma^r{}_{tt,r} - \Gamma^r{}_{tr,t} - \tau_{,tt}$$
$$+ 2^{-1}\Gamma^t{}_{tt}(\Omega_{,t} + \sigma_{,t} + 2\tau_{,t} + 2r^{-1})$$
$$+ 2^{-1}\Gamma^r{}_{tt}[\Omega_{,r} + \sigma_{,r} + 2\tau_{,r}] - (\Gamma^t{}_{tt})^2 - (\Gamma^r{}_{tr})^2 -$$
$$2^{-1}(\tau_{,t})^2 - 2\Gamma^r{}_{tt}\Gamma^t{}_{tr}$$

$$R_{tr} = \Gamma^t{}_{tr,t} + \Gamma^r{}_{tr,r} - (\Gamma^\alpha{}_{t\alpha})_{,r} + 2\Gamma^t{}_{tr}\Gamma^\theta{}_{t\theta} + \Gamma^r{}_{tr}\Gamma^t{}_{tr}$$
$$+ 2\Gamma^r{}_{tr}\Gamma^\theta{}_{r\theta} - \Gamma^r{}_{tt}\Gamma^t{}_{rr} - 2\Gamma^\theta{}_{t\theta}\Gamma^\theta{}_{r\theta}$$

$$= \Gamma^t{}_{tr,t} + \Gamma^r{}_{tr,r} - 2^{-1}(\Omega_{,tr} + \sigma_{,tr} + 2\tau_{,tr})$$
$$+ \tau_{,t}\Gamma^t{}_{tr} + \Gamma^r{}_{tr}\Gamma^t{}_{tr} - \Gamma^r{}_{tt}\Gamma^t{}_{rr}$$
$$- \tau_{,t}(2^{-1}\tau_{,r} + r^{-1}) + \Gamma^r{}_{tr}(\tau_{,r} + 2r^{-1})$$

$$R_{rr} = \Gamma^t{}_{rr,t} - \Gamma^t{}_{tr,r} - 2\Gamma^\theta{}_{r\theta,r} + \Gamma^t{}_{rr}\Gamma^t{}_{tt}$$
$$+ 2\Gamma^t{}_{rr}\Gamma^\theta{}_{t\theta} + \Gamma^r{}_{rr}\Gamma^t{}_{tr} + 2\Gamma^r{}_{rr}\Gamma^\theta{}_{r\theta} - (\Gamma^t{}_{tr})^2$$
$$- 2(\Gamma^\theta{}_{r\theta})^2 - \Gamma^r{}_{tr}\Gamma^t{}_{rr}$$

$$= \Gamma^t{}_{rr,t} - \Gamma^t{}_{tr,t} - 2(2^{-1}\tau_{,rr} - r^{-2}) + \tau_{,t}\Gamma^t{}_{rr}$$
$$+ \Gamma^t{}_{rr}(\Gamma^t{}_{tt} - \Gamma^r{}_{tr}) - \Gamma^t{}_{tr}(\Gamma^t{}_{tr} - \Gamma^r{}_{rr})$$
$$+ 2\Gamma^r{}_{rr}(2^{-1}\tau_{,r} + r^{-1}) - 2(2^{-1}\tau_{,r} + r^{-1})^2$$

$$R_{\theta\theta} = \Gamma^{t}{}_{\theta\theta,t} + \Gamma^{r}{}_{\theta\theta,r} - \Gamma^{\Phi}{}_{\theta\Phi,\theta} - (\Gamma^{\Phi}{}_{\theta\Phi})^2 + \Gamma^{t}{}_{\theta\theta}(\Gamma^{t}{}_{tt} + \Gamma^{r}{}_{tr}) + \Gamma^{r}{}_{\theta\theta}(\Gamma^{t}{}_{tr} + \Gamma^{r}{}_{rr})$$

$$= 1 + \Gamma^{t}{}_{\theta\theta,t} + \Gamma^{r}{}_{\theta\theta,r} + 2^{-1}\Gamma^{t}{}_{\theta\theta}(\Omega_{,t} + \sigma_{,t}) + 2^{-1}\Gamma^{r}{}_{\theta\theta}(\Omega_{,r} + \sigma_{,r})$$

$$R_{\Phi\Phi} = \sin^2\theta\, R_{\theta\theta}.$$

The curvature scalar is defined by

$$R = g^{\alpha\beta}R_{\alpha\beta} ,$$

and for this metric it is

$$R = g^{tt}R_{tt} + 2g^{tr}R_{tr} + g^{rr}R_{rr} + 2g^{\theta\theta}R_{\theta\theta}$$

$$= e^{-\sigma}\{\cos^2\alpha R_{rr} - 2\sin\alpha\cos\alpha R_{tr} + \sin^2\alpha R_{tt}\} - e^{-\Omega}\{\sin^2\alpha R_{rr} + 2\sin\alpha\cos\alpha R_{tr} + \cos^2\alpha R_{tt}\} + 2e^{-\Omega}r^{-2}R_{\theta\theta}.$$

The Einstein tensor is defined by

$$G_{\alpha\beta} = R_{\alpha\beta} - 2^{-1}g_{\alpha\beta}R$$

or

$$G^{\alpha}{}_{\beta} = g^{\alpha\tau}G_{\tau\beta} = R^{\alpha}{}_{\beta} - 2^{-1}R\delta^{\alpha}{}_{\beta}.$$

For this metric

$$G^t{}_t = g^{tt}R_{tt} + g^{tr}R_{tr} - 2^{-1}R$$

$$G^r{}_r = g^{rr}R_{rr} + g^{tr}R_{tr} - 2^{-1}R$$

$$G^r{}_t = g^{rr}R_{tr} + g^{tr}R_{tt}$$

$$G^t{}_r = g^{tr}R_{rr} + g^{tt}R_{tr}$$

$$G^\theta{}_\theta = G^\Phi{}_\Phi = g^{\theta\theta}R_{\theta\theta} - 2^{-1}R$$

$$G^t{}_\theta = G^t{}_\Phi = G^\theta{}_t = G^\Phi{}_t = G^r{}_\theta = 0$$

$$G^r{}_\Phi = G^\theta{}_r = G^\Phi{}_r = G^\theta{}_\Phi = G^\Phi{}_\theta = 0.$$

The components of the Ricci tensor and the Einstein tensor will not be further simplified until certain restrictions are placed on the functions σ, Ω, τ, α . These restrictions will simplify the equations so that solutions to the field equations can be found.

(3.2) Killing's Equations

The symmetries of a metric space are described by the number of Killing vectors it possesses (Schutz, 1980). The Killing vectors are vectors satisfying the equations

$$\mathcal{L}_{\mu} g = \mu_{\alpha,\beta} + \mu_{\beta,\alpha} = 0$$

For this metric, the Killing equations are

$$\mu_{t,t} - \Gamma^{t}{}_{tt}\mu_{t} - \Gamma^{r}{}_{tt}\mu_{r} = 0$$

$$\mu_{t,r} + \mu_{r,t} - 2\Gamma^{t}{}_{tr}\mu_{t} = 0$$

$$\mu_{t,\theta} + \mu_{\theta,t} - 2\Gamma^{\theta}{}_{t\theta}\mu_{\theta} = 0$$

$$\mu_{t,\Phi} + \mu_{\Phi,t} - 2\Gamma^{\Phi}{}_{t\Phi}\mu_{\Phi} = 0$$

$$\mu_{r,r} - \Gamma^{t}{}_{rr}\mu_{t} - \Gamma^{r}{}_{rr}\mu_{r} = 0$$

$$\mu_{r,\theta} + \mu_{\theta,r} - 2\Gamma^{\theta}{}_{r\theta}\mu_{\theta} = 0$$

$$\mu_{r,\Phi} + \mu_{\Phi,r} - 2\Gamma^{\Phi}{}_{r\Phi}\mu_{\Phi} = 0$$

$$\mu_{\theta,\theta} - \Gamma^{t}{}_{\theta\theta}\mu_{\theta} - \Gamma^{r}{}_{\theta\theta}\mu_{r} = 0$$

$$\mu_{\theta,\phi} + \mu_{\phi,\theta} - 2\Gamma^{\phi}{}_{\theta\phi}\mu_{\phi} = 0$$

$$\mu_{\phi,\phi} - \Gamma^{t}{}_{\phi\phi}\mu_{t} - \Gamma^{r}{}_{\phi\phi}\mu_{r} - \Gamma^{\theta}{}_{\phi\phi}\mu_{\theta} = 0.$$

These equations will be solved later for the specific solutions of the field equations that are found for this metric. However these general equations will clearly admit the three Killing vectors that are the generators of the rotation group SO(3), since the spacetime is spherically symmetric.

(3.3) The Stress-Energy Tensor

The stress-energy tensor for an imperfect fluid can be written (Ellis, 1971)

$$T^{\alpha}{}_{\beta} = (\mu + p)u^{\alpha}u_{\beta} + (q^{\alpha}u_{\beta} + q_{\beta}u^{\alpha}) + p\delta^{\alpha}{}_{\beta} + \pi^{\alpha}{}_{\beta} \tag{3.2}$$

where μ is the total energy density measured by an observer moving with 4-velocity u^{α} and q^{α} is the energy flux relative to u^{α} and represents physical processes such as diffusion and heat conduction. The vector q^{α} obeys

$$q^{\alpha}u_{\alpha} = 0.$$

The isotropic pressure is given by p and the trace-free anisotropic matter pressure is $\pi_{\alpha\beta}$ and represents processes such as viscosity. It satisfies

$$\pi_{\alpha\beta}u^{\beta} = 0.$$

It is usual to assume that certain phenomenological equations of state hold (Ellis, 1971). These arise from comparison with Newtonian theory and the condition that entropy must never be negative. They are

$$\pi^{\alpha}{}_{\beta} = -\eta\sigma^{\alpha}{}_{\beta}$$

$$q_{\alpha} = -\chi h^{\alpha}{}_{\beta}(T_{,\beta} + u_{\beta}T)$$

$$p - p_T = -\Sigma\theta$$

where T is the temperature, p_T is the thermodynamic pressure, $\eta(p_T,v)$ is the coefficient of viscosity, $\chi(p_T,v)$ is the heat conduction coefficient, and $\Sigma(p_T,v)$ is the bulk viscosity coefficient. These last three coefficients are functions of the thermodynamic pressure and the specific volume v. The specific volume is defined by

$$v = \Gamma^{-1}$$

where Γ is the rest mass density measured by an observer moving with 4-velocity u^{α}. The rest mass is related to the total energy density μ by

$$\mu = \Gamma(1 + \epsilon)$$

where ϵ is the specific internal energy. Therefore these coefficients may be regarded as functions of p_T and μ. The coefficients η, χ, and Σ obey the restrictions

$$\cap \geqslant 0$$

$$X \geqslant 0$$

$$\Sigma \geqslant 0.$$

The stress-energy tensor can therefore be written

$$T^{\alpha}{}_{\beta} = (\mu + p_T)u^{\alpha}u_{\beta} + p_T\delta^{\alpha}{}_{\beta} - \Sigma\Theta h^{\alpha}{}_{\beta} - 2\cap\sigma^{\alpha}{}_{\beta} + q^{\alpha}u_{\beta} + q_{\beta}u^{\alpha}. \tag{3.3}$$

The perfect fluid approximation is obtained from equation (3.2) when

$$q_{\alpha} = \pi_{\alpha\beta} = 0,$$

and then the stress-energy tensor reduces to

$$T^{\alpha}{}_{\beta} = (\mu + p)u^{\alpha}u_{\beta} + p\delta^{\alpha}{}_{\beta}. \tag{3.4}$$

(3.4) Choice of a Velocity Vector

Finkelstein and McCollum (1975) and Williams and Zia, 1973) discussed possible mass distributions that might lead to kink solutions, but did not consider the exact form of the stress-energy tensor, the velocity vector components or what the equation of state might be. Therefore, these earlier suggestions for possible mass distributions did not necessarily satisfy the Einstein field equations. Exact solutions of the Einstein field equations identified as kink solutions were found by Harriott and Williams (1988a). For the most generally spherically symmetric metric of section (2.4), given by equation (3.1), which is

$$
\begin{aligned}
g_{tt} &= e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\
g_{tr} &= -2^{-1}(e^{\Omega} + e^{\sigma})\sin 2\alpha \\
g_{rr} &= e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\
g_{\theta\theta} &= e^{\tau}r^2 \\
g_{\Phi\Phi} &= e^{\tau}r^2\sin^2\theta
\end{aligned}
$$

the ϕ_α defined by equation (2.17) are

$$
\begin{aligned}
\phi_t &= \cos\alpha \\
\phi_r &= \sin\alpha \\
\phi_\theta &= \phi_\Phi = 0
\end{aligned}
$$

Solutions of the Einstein field equations are sought for which the components of the velocity vector are generalizations of the ϕ_α, in that (Harriott and Williams, 1986)

$$
\begin{aligned}
u^t &= e^{-\Omega/2}\cos\alpha \\
u^r &= e^{-\Omega/2}\sin\alpha \\
u^\Theta &= u^\Phi = 0.
\end{aligned}
\tag{3.5}
$$

For the simple kink metric,

$$g_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_\alpha\phi_\beta$$

for which $\Omega = \sigma = \tau = 0$, it will be shown later that the **only** acceptable choice for the velocity vector is that given by equation (3.5). MacCallum (1973) who develops general relativity in tetrad formalism notes that the timelike tetrad may be identified with the velocity vector. For the simple kink metric given above, the four tetrads can be shown to be identical to the four columns of the matrix P, given by equation (2.8). The first column of the matrix P is

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

and the ϕ_α obey

$$\phi^\alpha \phi_\alpha = -1.$$

The covariant components of the velocity vector are

$$\begin{aligned} u_t &= g_{t\alpha}u^\alpha \\ &= g_{tt}u^t + g_{tr}u^r \\ &= (e^\sigma \sin^2\alpha - e^\Omega \cos^2\alpha)e^{-\Omega/2}\cos\alpha \\ &\quad +[-(e^\sigma + e^\Omega)\sin\alpha\cos\alpha]e^{-\Omega/2}\sin\alpha \\ &= -e^{\Omega/2}\cos\alpha \end{aligned}$$

$$\begin{aligned} u_r &= g_{rt}u^t + g_{rr}u^r \\ &= -(e^\sigma + e^\Omega)\sin\alpha\cos\alpha e^{-\Omega/2}\cos\alpha \\ &\quad +(e^\sigma \cos^2\alpha - e^\Omega \sin^2\alpha)e^{-\Omega/2}\sin\alpha \\ &= -e^{\Omega/2}\sin\alpha \end{aligned}$$

$$u_\theta = u_\Phi = 0.$$

This choice for the velocity components guarantees that

$$u^{\alpha}u_{\alpha} = -1$$

as required. It also leads to various kink solutions of the field equations which have a physically acceptable stress-energy tensor. These solutions will be found and discussed in later chapters of this thesis.

(3.5) The Acceleration Vector.

Before stating the acceleration vector components, the following results should be noted:

$$g_{rr}\sin\alpha + g_{tr}\cos\alpha = -e^{\Omega}\sin\alpha$$
$$g_{tt}\cos\alpha + g_{tr}\sin\alpha = -e^{\Omega}\cos\alpha$$
$$g_{rr}\cos\alpha - g_{tr}\sin\alpha = e^{\sigma}\cos\alpha$$
$$g_{tr}\cos\alpha - g_{tt}\sin\alpha = -e^{\sigma}\sin\alpha$$

These results may be used to deduce the following expressions for the non-zero components of the covariant derivatives of the velocity vector, which are

$$u^{t}{}_{;t} = u^{t}{}_{,t} + \Gamma^{t}{}_{tt}u^{t} + \Gamma^{t}{}_{tr}u^{r}$$

$$\begin{aligned} &= (e^{-\Omega/2}\cos\alpha)_{,t} + 2^{-1}e^{-\sigma-\Omega/2}\sin\alpha g_{tt,r} + \\ &\quad 2^{-1}E^{-1}e^{-\Omega/2}g_{tr}[2g_{tr,t}\cos\alpha + g_{rr,t}\sin\alpha] - \\ &\quad 2^{-1}E^{-1}e^{-\Omega/2}g_{rr}g_{tt,t}\cos\alpha \end{aligned}$$

$$u^{r}{}_{;r} = u^{r}{}_{,r} + \Gamma^{r}{}_{tr}u^{t} + \Gamma^{r}{}_{rr}u^{r}$$

$$\begin{aligned} &= (e^{-\Omega/2}\sin\alpha)_{,r} + 2^{-1}e^{-\sigma-\Omega/2}\sin\alpha g_{rr,t} + \\ &\quad 2^{-1}E^{-1}e^{-\Omega/2}g_{tr}[g_{tt,r}\cos\alpha + 2g_{tr,r}\sin\alpha] - \\ &\quad 2^{-1}E^{-1}e^{-\Omega/2}g_{tt}g_{rr,r}\sin\alpha \end{aligned}$$

$$u^t{}_{;r} = u^t{}_{,r} + \Gamma^t{}_{tr}u^t + \Gamma^t{}_{rr}u^r$$

$$= (e^{-\Omega/2}\cos\alpha)_{,r} + 2^{-1}E^{-1}e^{\Omega/2}g_{tr}g_{rr,t}\cos\alpha$$
$$+ 2^{-1}E^{-1}e^{-\Omega/2}[g_{tr}g_{rr,r}\sin\alpha$$
$$- 2g_{rr}g_{tr,r}\sin\alpha - g_{rr}g_{tt,r}\cos\alpha]$$

$$u^r{}_{;t} = u^r{}_{,t} + \Gamma^r{}_{tt}u^t + \Gamma^r{}_{tr}u^r$$

$$= (e^{-\Omega/2}\sin\alpha)_{,t} - 2^{-1}e^{-\sigma-\Omega/2}\cos\alpha g_{tt,r}$$
$$+ 2^{-1}E^{-1}e^{-\Omega/2}[g_{tr}g_{tt,t}\cos\alpha$$
$$- 2g_{tt}g_{tr,t}\cos\alpha - g_{tt}g_{rr,t}\sin\alpha]$$

$$u^\theta{}_{;\theta} = u^\Phi{}_{;\Phi} = \Gamma^\theta{}_{t\theta}u^t + \Gamma^\theta{}_{r\theta}u^r$$

$$= 2^{-1}e^{-\Omega/2}[\tau_{,t}\cos\alpha + (\tau_{,r} + 2r^{-1})\sin\alpha].$$

and

$$u_{t;t} = u_{t,t} - \Gamma^t{}_{tt}u_t - \Gamma^r{}_{tt}u_r$$

$$= (-e^{\Omega/2}\cos\alpha)_{,t} + 2^{-1}e^{-\Omega/2}g_{tt,r}\sin\alpha$$
$$- 2^{-1}e^{-\Omega/2}g_{tt,t}\cos\alpha - e^{-\Omega/2}g_{tr,t}\sin\alpha$$

$$u_{t;r} = u_{t,r} - \Gamma^{t}{}_{tr}u_t - \Gamma^{r}{}_{tr}u_r$$

$$= (-e^{\Omega/2}\cos\alpha)_{,r} - 2^{-1}e^{-\Omega/2}g_{tt,r}\cos\alpha$$
$$- 2^{-1}e^{-\Omega/2}g_{rr,t}\sin\alpha$$

$$u_{r;t} = u_{r,t} - \Gamma^{t}{}_{tr}u_t - \Gamma^{r}{}_{tr}u_r$$

$$= (-e^{\Omega/2}\sin\alpha)_{,t} - 2^{-1}e^{-\Omega/2}g_{tt,r}\cos\alpha$$
$$- 2^{-1}e^{-\Omega/2}g_{rr,t}\sin\alpha$$

$$u_{r;r} = u_{r,r} - \Gamma^{t}{}_{rr}u_t - \Gamma^{r}{}_{rr}u_r$$

$$= (-e^{\Omega/2}\sin\alpha)_{,r} - 2^{-1}e^{-\Omega/2}g_{rr,r}\sin\alpha$$
$$- e^{-\Omega/2}g_{tr,r}\cos\alpha + 2^{-1}e^{-\Omega/2}g_{rr,t}\cos\alpha$$

$$u_{\theta;\theta} = -\Gamma^{t}{}_{\theta\theta}u_t - \Gamma^{r}{}_{\theta\theta}u_r$$

$$= 2^{-1}e^{-\Omega/2+\tau}r^2\tau_{,t}\cos\alpha$$
$$+ 2^{-1}e^{-\Omega/2+\tau}r^2[\tau_{,r} + 2r^{-1}]\sin\alpha$$

$$u_{\Phi;\Phi} = -\Gamma^t{}_{\Phi\Phi}u_t - \Gamma^r{}_{\Phi\Phi}u_r = \sin^2\theta u_{\theta;\theta}.$$

The acceleration vector is defined as the covariant derivative of **u** along the particle world lines and is given by

$$\dot{u}^\alpha = u^\alpha{}_{;\beta}u^\beta.$$

Long calculations show that the components of this vector are

$$\dot{u}^t = -E^{-1}e^{\Omega/2}\sin\alpha(e^{\Omega/2}\cos\alpha)_{,r} + E^{-1}e^{\Omega/2}(e^{\Omega/2}\sin\alpha)_{,t}$$

$$\dot{u}^r = E^{-1}e^{\Omega/2}\cos\alpha(e^{\Omega/2}\cos\alpha)_{,r} - E^{-1}e^{\Omega/2}(e^{\Omega/2}\sin\alpha)_{,t}$$

$$\dot{u}^\theta = \dot{u}^\Phi = 0.$$

It can also be shown that

$$\dot{u}_t = -e^{-\Omega/2}\sin\alpha(e^{\Omega/2}\cos\alpha)_{,r} - e^{-\Omega/2}\cos\alpha(e^{\Omega/2}\cos\alpha)_{,t} + 2^{-1}\Omega_{,t}$$

$$\dot{u}_r = -e^{-\Omega/2}\cos\alpha(e^{\Omega/2}\sin\alpha)_{,t} - e^{-\Omega/2}\sin\alpha(e^{\Omega/2}\sin\alpha)_{,r} + 2^{-1}\Omega_{,r}$$

$$\dot{u}_\theta = \dot{u}_\Phi = 0.$$

(3.6) The Projection Tensor.

The projection tensor is defined by

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta .$$

It has the following non-zero covariant and contravariant components for this metric.

$$\begin{aligned}
h^{tt} &= e^{-\sigma}\sin^2\alpha = e^{-2\sigma}h_{tt} \\
h^{rr} &= e^{-\sigma}\cos^2\alpha = e^{-2\sigma}h_{rr} \\
h^{tr} &= -e^{-\sigma}\sin\alpha\cos\alpha = e^{-2\sigma}h_{tr} \\
h^{\theta\theta} &= e^{-\tau}r^{-2} \;\; = (h_{\theta\theta})^{-1} \\
h^{\Phi\Phi} &= e^{-\tau}r^{-2}\sin^2\alpha = (h_{\Phi\Phi})^{-1}.
\end{aligned}$$

The mixed components are

$$\begin{aligned}
h^t{}_t &= \sin^2\alpha \\
h^r{}_r &= \cos^2\alpha \\
h^r{}_t = h^t{}_r &= -\sin\alpha\cos\alpha \\
h^\theta{}_\theta = h^\Phi{}_\Phi &= 1.
\end{aligned}$$

These definitions clearly show that

$$h^\alpha{}_\beta u^\beta = 0.$$

(3.7) **Other Hydrodynamic Quantities.**

The covariant derivative of the velocity vector can be completely determined in terms of various dynamical quantities known, respectively, as the vorticity tensor $w_{\alpha\beta}$, the shear tensor $\sigma_{\alpha\beta}$, the velocity vector u^α, the acceleration vector $\dot{u}^\alpha$ and the scalar expansion θ (Ellis, 1971):

$$u_{\alpha;\beta} = w_{\alpha\beta} + \sigma_{\alpha\beta} + 3^{-1}h_{\alpha\beta}\theta - \dot{u}_\alpha u_\beta$$

The shear tensor $\sigma_{\alpha\beta}$ and the term involving the expansion θ are the trace and trace free parts of the expansion tensor $\theta_{\alpha\beta}$.

$$\theta_{\alpha\beta} = \sigma_{\alpha\beta} + 3^{-1}h_{\alpha\beta}\theta.$$

The expansion tensor and the vorticity tensor are the symmetric and antisymmetric parts of a tensor $v_{\alpha\beta}$, that is the spatial gradient of the velocity vector defined by

$$v_{\alpha\beta} = h^\Gamma{}_\alpha h^\mu{}_\beta u_{\Gamma;\mu}.$$

The definition of $v_{\alpha\beta}$ and the relation $h^\alpha{}_\beta u^\beta = 0$ clearly show that

$$\theta_{\alpha\beta}u^{\beta} = w_{\alpha\beta}u^{\beta} = \sigma_{\alpha\beta}u^{\beta} = 0$$

$$\theta = u^{\alpha}{}_{;\alpha}$$

$$\sigma^{\alpha}{}_{\beta} = 2^{-1}(u^{\alpha}{}_{;\tau}h^{\tau}{}_{\beta} + u_{\beta;\tau}h^{\alpha\tau}) - 3^{-1}h^{\alpha}{}_{\beta}\theta. \qquad (3.6)$$

From the shear tensor, a shear scalar σ^2, may be defined:

$$\sigma^2 = 2^{-1}\sigma^{\alpha}{}_{\beta}\sigma^{\beta}{}_{\alpha}. \qquad (3.7)$$

Justification for the names "expansion", "shear" and "vorticity" tensors for the tensors, $\theta_{\alpha\beta}$, $\sigma_{\alpha\beta}$, and $w_{\alpha\beta}$ respectively and for the expansion scalar, θ, can be found as follows.

It can be shown (Ellis, 1971) that the expansion tensor determines an expression for the rate of change of relative distance of neighbouring fluid particles

$$\begin{aligned}(\delta s)^{\cdot}/\delta s &= \sigma_{\alpha\beta}n^{\alpha}n^{\beta} + 3^{-1}\theta \\ &= \theta_{\alpha\beta}n^{\alpha}n^{\beta}\end{aligned}$$

where δs is a relative distance in direction n^{α}. This clearly justifies the na of expansion tensor for $\theta_{\alpha\beta}$. The isotropic part is completely determined by θ, the scalar volume expansion. The shear tensor measures the distortion of the volume element in all directions except those of the principal axes of the shear as determined by the eigenvectors of $\sigma_{\alpha\beta}$, while leaving the volume unchanged.

Ellis (1971) also shows that

$$h^{\alpha}{}_{\beta}(n^{\beta})^{\cdot} = (w^{\alpha}{}_{\beta} + \sigma^{\alpha}{}_{\beta} - \sigma_{\Gamma\pi}n^{\Gamma}n^{\pi}h^{\alpha}{}_{\beta})n^{\beta}.$$

This last equation shows that the action of the vorticity tensor alone is that of a rigid rotation of the fluid particles with respect to a local inertial rest frame.

With the above definitions, for this metric (3.1), the expansion scalar $\theta = u^{\alpha}{}_{;\alpha}$ is given by

$$\begin{aligned}
\theta &= u^{t}{}_{,t} + u^{r}{}_{,r} + u^{t}\Gamma^{\alpha}{}_{t\alpha} + u^{r}\Gamma^{\alpha}{}_{r\alpha} \\
&= (e^{-\Omega/2}\cos\alpha)_{,t} + (e^{-\Omega/2}\sin\alpha)_{,r} \\
&\quad + 2^{-1}e^{-\sigma/2}\cos\alpha(\Omega,t + \sigma_{,t} + 2\tau_{,t}) \\
&\quad + 2^{-1}e^{-\Omega/2}\sin\alpha(\Omega_{,r} + \sigma_{,r} + 2\tau_{,r} + 4r^{-1}) \\
&= E^{-1/2}\{ (e^{\sigma/2}\cos\alpha)_{,t} + (e^{\sigma/2}\sin\alpha)_{,r} \} \\
&\quad + \tau_{,t}\cos\alpha e^{-\Omega/2} + (\tau_{,r} + 2r^{-1})\sin\alpha e^{-\Omega/2}.
\end{aligned} \tag{3.8}$$

The shear tensor components and the shear scalar will be calculated from equations (3.6) and (3.7) when specific solutions have been found.

The spherical symmetry of this metric clearly implies that the vorticity tensor

$$w_{\alpha\beta} = 0.$$

CHAPTER FOUR

DERIVATION OF KINK SOLUTIONS I

Introduction

The metric of Chapter 3 is now simplified by setting $\Omega = \sigma = \tau = 0$. Using the notation of Section (2.2), in which the matrix G, representing the metric tensor, is decomposed into a symmetric positive definite matrix S and an orthogonal matrix Q, this simplification is equivalent to the choice, $S = I$, so that $G = Q$. The various curvature quantities arising from this simplified form of the metric are presented (4.1). The form of the Einstein field equations for this metric are stated (4.2) and several perfect fluid, one-kink solutions to the field equations are found (4.3). Imperfect fluid versions of these solutions are discussed (4.4).

(4.1) The Christoffel Symbols, Ricci Tensor and Curvature Scalar.

The field equations of Chapter 3 can be considerably simplified by setting

$$\Omega = \sigma = \tau = 0.$$

With these simplifications the metric becomes

$$g_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_\alpha\phi_\beta.$$

It can be seen, using the notation of section (2.3), that this is equivalent to choosing S = I so that G = Q. This simplification allows several solutions to be found. As defined by equation (2.17), the ϕ_α are

$$\phi_t = \cos\alpha$$
$$\phi_r = \sin\alpha$$
$$\phi_\theta = \phi_\Phi = 0.$$

The metric components, frcm equation (3.1), now reduce to

$$g_{tt} = -\cos 2\alpha = g^{tt}$$
$$g_{tr} = -\sin 2\alpha = g^{tr}$$
$$g_{rr} = \cos 2\alpha = g^{rr}$$

$$g_{\theta\theta} = r^2 = (g^{\theta\theta})^{-1}$$
$$g_{\Phi\Phi} = r^2\sin^2\theta = (g^{\Phi\Phi})^{-1}.$$

The Christoffel symbols for this metric are

$$\Gamma^t{}_{tt} = -\Gamma^r{}_{tr} = \sin^2 2\alpha\ \alpha_{,r} + 2^{-1}\sin 4\alpha\ \alpha_{,t}$$

$$\Gamma^t{}_{tr} = -\Gamma^r{}_{rr} = -2^{-1}\sin 4\alpha\ \alpha_{,r} + \sin^2 2\alpha\ \alpha_{,t}$$

$$\Gamma^r{}_{tt} = -2^{-1}\sin 4\alpha\ \alpha_{,r} - (1 + \cos^2 2\alpha\ )\alpha_{,t}$$

$$\Gamma^t{}_{rr} = (1 + \cos^2 2\alpha)\alpha_{,r} - 2^{-1}\sin 4\alpha\ \alpha_{,t}$$

$$\Gamma^t{}_{\theta\theta} = r\sin 2\alpha$$

$$\Gamma^t{}_{\Phi\Phi} = \sin^2\theta\Gamma^t{}_{\theta\theta}$$

$$\Gamma^r{}_{\theta\theta} = -r\cos 2\alpha$$

$$\Gamma^r{}_{\Phi\Phi} = \sin^2\theta\Gamma^r{}_{\theta\theta}$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\Phi{}_{r\Phi} = r^{-1}$$

$$\Gamma^\theta{}_{\Phi\Phi} = -\sin\theta\cos\theta$$

$$\Gamma^\Phi{}_{\theta\Phi} = \cot\theta.$$

From the above expressions it is clear that

$$\Gamma^{\alpha}{}_{t\alpha} = \Gamma^{\alpha}{}_{\Phi\alpha} = 0$$

$$\Gamma^{\alpha}{}_{r\alpha} = r^{-1}$$

$$\Gamma^{\alpha}{}_{\theta\alpha} = \cot\theta .$$

The Ricci tensor components are

$$\begin{aligned} R_{tt} = {} & 2^{-1}\sin 4\alpha\ \alpha_{,tt} - 2^{-1}\sin 4\alpha\ \alpha_{,rr} + 2\cos^2 2\alpha\ (\alpha_{,t})^2 \\ & - 2\cos^2 2\alpha\ (\alpha_{,r})^2 + 2\sin 4\alpha\ \alpha_{,r}\ \alpha_{,t} - 2\cos^2 2\alpha\ \alpha_{,tr} \\ & - 2r^{-1}(1 + \cos^2 2\alpha)\alpha_{,t} - r^{-1}\sin 4\alpha\ \alpha_{,r} \end{aligned}$$

$$\begin{aligned} R_{tr} = {} & \sin^2 2\alpha\ \alpha_{,tt} - \sin^2 2\alpha\ \alpha_{,rr} + \sin 4\alpha\ (\alpha_{,t})^2 \\ & - \sin 4\alpha\ (\alpha_{,r})^2 - \sin 4\alpha\ \alpha_{,tr} + 4\sin^2 2\alpha\ \alpha_{,r}\ \alpha_{,t} \\ & - 2r^{-1}\sin^2 2\alpha\ \alpha_{,r} - r^{-1}\sin 4\alpha\ \alpha_{,t} \end{aligned}$$

$$\begin{aligned} R_{rr} = {} & -2^{-1}\sin 4\alpha\ \alpha_{,tt} + 2^{-1}\sin 4\alpha\ \alpha_{,rr} - 2\cos^2 2\alpha\ (\alpha_{,t})^2 \\ & + 2\cos^2 2\alpha\ (\alpha_{,r})^2 + 2\cos^2 2\alpha\ \alpha_{,tr} - 2\sin 4\alpha\ \alpha_{,r}\ \alpha_{,t} \\ & - 2r^{-1}\sin^2 2\alpha\ \alpha_{,t} + r^{-1}\sin 4\alpha\ \alpha_{,t} \end{aligned}$$

$$R_{\theta\theta} = 2r\cos 2\alpha\ \alpha_{,t} + 2r\sin 2\alpha\ \alpha_{,r} + 2\sin^2\alpha$$

$$R_{\Phi\Phi} = \sin^2\theta\ R_{\theta\theta}$$

and the scalar curvature is

$$\begin{aligned} R = {} & 4\cos 2\alpha\,(\alpha_{,r})^2 - 4\cos 2\alpha\,(\alpha_{,t})^2 + 4\cos 2\alpha\,\alpha_{,rt} \\ & + 2\sin 2\alpha\,\alpha_{,rr} - 2\sin 2\alpha\,\alpha_{,tt} - 8\sin 2\alpha\,\alpha_{,r}\,\alpha_{,t} \\ & + 8r^{-1}\sin 2\alpha\,\alpha_{,r} + 8r^{-1}\cos 2\alpha\,\alpha_{,t} + 4r^{-2}\sin^2\alpha. \end{aligned} \tag{4.1}$$

(4.2) The Perfect Fluid Field Equations

The components of the velocity vector can be found from equation (3.5). With this simplified form of the metric they are

$$u^t = \cos\alpha = -u_t$$
$$u^r = \sin\alpha = -u_r$$
$$u^\theta = u^\Phi = 0$$

This choice was made following standard practice (MacCallum, 1973). For this simple kink metric, assuming the perfect fluid approximation it can be shown to be the **only** acceptable choice for the velocity vector, unless the equation of state is $p + \mu = 0$. This result is shown as follows. Let

$$u^t = f_1 \; ; \; u^r = f_2 \; ; \; u^\theta = f_3 \; ; \; u^\Phi = f_4$$

where $f_a = f(t,r,\theta,\Phi)$, $a = 1,2,3,4$. Then

$$u_t = -f_1\cos 2\alpha - f_2\sin 2\alpha$$
$$u_r = -f_1\sin 2\alpha + f_2\cos 2\alpha$$
$$u_\theta = r^2 f_3$$
$$u_\Phi = r^2\sin^2\theta f_4.$$

The Einstein tensor components $G^t{}_r$ and $G^r{}_t$ are not identically zero therefore f_1 and f_2 must be non-zero. The components $G^t{}_\theta$ and $G^t{}_\phi$ are identically zero. Hence,

$$f_3 = f_4 = 0.$$

The non zero velocity components must obey $u^\alpha u_\alpha = -1$, which may be expressed as

$$(f_2{}^2 - f_1{}^2)\cos 2\alpha - 2f_1f_2\sin 2\alpha = -1.$$

The Einstein tensor component, $G^t{}_r = G^r{}_t$ and therefore the stress energy components $T^t{}_r$ and $T^r{}_t$ must also be equal. This implies that

$$2f_1f_2\cos 2\alpha + (f_2{}^2 - f_1{}^2)\sin 2\alpha = 0.$$

These last two equations may be solved to give

$$\cos 2\alpha = (f_1{}^2 - f_2{}^2)(f_1{}^2 + f_2{}^2)^{-2}$$

$$\sin 2\alpha = 2f_1f_2(f_1{}^2 + f_2{}^2)^{-2} .$$

The relation $\sin^2 2\alpha + \cos^2 2\alpha = 1$ now shows that

$$f_1{}^2 + f_2{}^2 = 1.$$

These functions may now be written as

$$f_1 = \cos\theta$$
$$f_2 = \sin\theta,$$

but substitution back into the above equations shows that $\theta = \alpha$. The velocity components are therefore identified with the ϕ_α.

Using these velocity components in the stress-energy tensor, the perfect fluid field equations are

$$G^t{}_t = -2r^{-2}(r\sin^2\alpha)_{,r} = -(\mu + p)\cos^2\alpha + p \tag{4.2}$$

$$\begin{aligned} G^t{}_r = G^r{}_t &= 2r^{-1}\sin 2\alpha\ \alpha_{,t} \\ &= -(\mu + p)\sin\alpha\cos\alpha \end{aligned} \tag{4.3}$$

$$\begin{aligned} G^r{}_r &= -2r^{-2}(r\sin^2\alpha)_{,r} - 4r^{-1}\cos 2\alpha\ \alpha_{,t} \\ &= -(\mu + p)\sin^2\alpha + p \end{aligned} \tag{4.4}$$

$$\begin{aligned} G^\theta{}_\theta = G^\Phi{}_\Phi &= -r^{-1}(r\sin^2\alpha)_{,rr} - 2r^{-1}(r\cos 2\alpha\ \alpha_{,t})_{,r} \\ &\quad + (\sin^2\alpha)_{,tt} \\ &= p\ . \end{aligned} \tag{4.5}$$

(4.3) Three Perfect Fluid Solutions.

To solve these equations, first note that substituting equations (4.2) and (4.3) into the left hand side of equation (4.4) gives

$$-(\mu + p)\cos^2\alpha + p + \cos 2\alpha(\mu + p) = -(\mu + p)\sin^2\alpha + p.$$

Equation (4.4) is therefore consistent with equations (4.2) and (4.3). There are now three equations to satisfy, namely equations (4.2), (4.3) and (4.5).

Equation (4.3) clearly reduces to

$$4r^{-1}\,\alpha_{,t} = -(\mu + p) \qquad (4.6)$$

and equation (4.2) can be written as

$$-4r^{-1}\sin\alpha\cos\alpha\,\alpha_{,r} - 2r^{-2}\sin^2\alpha = -(\mu + p)\cos^2\alpha + p\,. \qquad (4.7)$$

Substituting equation (4.6) into equation (4.7), the following expression for p is obtained

$$p = -4r^{-1}\sin\alpha\cos\alpha\,\alpha_{,r} - 2r^{-2}\sin^2\alpha - 4r^{-1}\cos^2\alpha\,\alpha_{,t} \qquad (4.8)$$

Combining equations (4.8) with (4.5) and rearranging, a differential equation for α is obtained. **All** perfect fluid solutions of this metric must satisfy this equation.

$$\sin 2\alpha\ \alpha_{,tt} - \sin 2\alpha\ \alpha_{,rr} + 2\cos 2\alpha\ (\alpha_{,t})^2 - 2\cos 2\alpha\ (\alpha_{,r})^2 + 4\sin 2\alpha\ \alpha_{,t}\ \alpha_{,r} - 2\cos 2\alpha\ \alpha_{,tr} + 2r^{-2}\sin^2\alpha + 2r^{-1}\ \alpha_{,t} = 0 \tag{4.9}$$

Three solutions of this equation can be readily obtained. If $\alpha_{,t} = 0$ then equation (4.9) becomes

$$-\sin 2\alpha\ \alpha_{,rr} - 2\cos 2\alpha\ (\alpha_{,r})^2 + 2r^{-2}\sin^2\alpha = 0 \tag{4.10}$$

Note that for this metric, when $\alpha_{,t} = 0$ is chosen, the equation of state of **any** solution must be

$$p = -\mu$$

because the left hand side of equation (4.6) is identically zero. Solutions of the equation (4.10) are found by noting that if

$$\sin^2\alpha = f(r) \tag{4.11}$$

where f(r) is some function of r, then differentiating with respect to r gives

$$\sin 2\alpha(\alpha_{,r}) = f(r)_{,r}.$$

Differentiating again shows

$$2\cos 2\alpha(\alpha_{,r})^2 + \sin 2\alpha(\alpha_{,rr}) = f(r)_{,rr}.$$

These results mean that equation (4.10) can be written as

$$2r^{-2}f(r) - f(r)_{,rr} = 0 \; . \qquad (4.12)$$

This last equation is a second order, linear, homogeneous equation in normal form and clearly has particular solutions of the form

$$f(r) = Ar^n \qquad (4.13)$$

where A and n are constants. Substitution of this equation (4.13) into equation (4.12) shows that solutions are obtained when

$$n = 2$$

or when

$$n = -1.$$

One solution, when $n = 2$, of equation (4.10) and therefore of the field equations is

$$\sin\alpha = Kr \tag{4.14}$$

where $K = A^{1/2}$ is a constant. This solution is the de Sitter kink, which will be discussed further in Chapter 6.

The other solution of equation (4.10), when $n = -1$, is

$$\sin^2\alpha = Ar^{-1} \tag{4.15}$$

where A is constant. This is the Schwarzchild kink, which will be discussed in Chapter 7.

A third solution of equation (4.9), where α is a function of r and t, is

$$\tan\alpha = rt^{-1}. \tag{4.16}$$

To prove this result, note that $\tan\alpha = rt^{-1}$ implies that

$$\sin 2\alpha = 2rt(r^2 + t^2)^{-1}$$
$$\cos 2\alpha = (t^2 - r^2)(t^2 + r^2)^{-1}$$

$$\sin^2\alpha = r^2(t^2 + r^2)^{-1}$$

$$\alpha_{,r} = t(t^2 + r^2)^{-1}$$

$$\alpha_{,t} = -r(t^2 + r^2)^{-1}$$

$$\alpha_{,rr} = -2tr(r^2 + t^2)^{-2}$$

$$\alpha_{,tt} = 2tr(r^2 + t^2)^{-2}$$

$$\alpha_{,rt} = (r^2 - t^2)(r^2 + t^2)^{-2} \ .$$

These results substituted into the left hand side of (4.9) give

$$\begin{aligned} &4r^2t^2(r^2 + t^2)^{-3} - (-4r^2t^2)(r^2 + t^2)^{-3} \\ &+ 2r^2(t^2 - r^2)(r^2 + t^2)^{-3} - 2t^2(t^2 - r^2)(r^2 + t^2)^{-3} \\ &+ 4(-2r^2t^2)(r^2 + t^2)^{-3} - 2(t^2 - r^2)(r^2 - t^2)(r^2 + t^2)^{-3} \\ &- 2(r^2 + t^2)^{-1} + 2(t^2 + r^2)^{-1} \\ &= \ 0, \end{aligned}$$

as required.

This solution is called the Friedmann-LeMaitre-Robertson -Walker (FLRW) kink and will be discussed further in Chapter 9.

Equation (4.9) becomes an ordinary differential equation when the substitution $\alpha = \alpha(rt^{-1}) = \alpha(u)$ is made. Solutions of equation (4.9) of this form are being sought.

(4.4) Imperfect fluid solutions

The stress-energy tensor discussed previously in section (3.3) can be expressed as given by equation (3.3)

$$T^{\alpha}{}_{\beta} = (\mu + p_T)u^{\alpha}u_{\beta} + p_T\delta^{\alpha}{}_{\beta} - \Sigma\theta h^{\alpha}{}_{\beta} - 2\cap\sigma^{\alpha}{}_{\beta} + q^{\alpha}u_{\beta} + q_{\beta}u^{\alpha}$$

where μ is the total energy density measured by an observer moving with 4-velocity u^{α} and q^{α} is the energy flux relative to u^{α}. The thermodynamic pressure is given by p_T, Σ is the bulk viscosity coefficient, $\cap$ is the coefficient of viscosity. The shear tensor is $\sigma^{\alpha}{}_{\beta}$ and the expansion scalar is θ.

In all the previous solutions, the coefficient of bulk viscosity Σ, the coefficient of dynamic viscosity $\cap$, and the heat conduction q_{α}, that appear in the stress energy tensor have been assumed to be zero. It is possible that more solutions may be found by now allowing these quantities to be non zero.

For both the de Sitter and FLRW kink solutions, using equation (3.6), it can be shown that the shear tensor, as expected, satisfies

$$\sigma_{\alpha\beta} = 0.$$

It can also be shown that for the simple kink metric

$$g_{\alpha\beta} = \delta_{\alpha\beta} - 2\phi_\alpha\phi_\beta$$

assuming the form of the velocity vector, $u^\alpha = \phi_\alpha$, that the heat conduction vector q^α must be zero. This result is shown as follows. The Einstein tensor components $G^t{}_\theta$ and $G^t{}_\Phi$ are zero. Therefore

$$q_\theta = q_\Phi = q^\theta = q^\Phi = 0 \ .$$

The heat conduction vector obeys

$$q^\alpha u_\alpha = 0.$$

This result reduces to

$$q^t u_t + q^r u_r = 0,$$

and substituting for the velocity vector components this can be expressed as

$$q^t = - q^r \tan\alpha.$$

It is easily shown that

$$q_t = q^t$$

and

$$q_r = q^r.$$

The Einstein tensor components $G^t{}_r$ and $G^r{}_t$ are equal. Therefore equating the stress-energy components $T^t{}_r$ and $T^r{}_t$ it follows that

$$q^t u_r + q_r u^t = q^r u_t + q_t u^r \ .$$

Substituting for the velocity components in terms of α and for q^t in terms of q^r and α, this last expression reduces to

$$2q^r[\tan\alpha\sin\alpha + \cos\alpha] = 0.$$

Hence

$$q^t = q^r = 0.$$

Any solution of the field equations for this simple kink metric which has a vanishing shear tensor must also have a zero heat conduction vector.

For the de Sitter kink solution, using equation (3.8), it can be shown that the expansion scalar Θ is given by

$$\Theta = 3K.$$

Using this result, the equation of state for the de Sitter solution may be expresesed as

$$p = -\mu = -6K^2 = p_T - 3K\Sigma.$$

For the FLRW kink, the expansion scalar Θ is given by

$$\Theta = 3(r^2+t^2)^{-1/2}.$$

The equation of state for the FLRW kink may be written as

$$p = p_T - 3\Sigma(r^2 + t^2)^{-1/2} = -3^{-1}\mu .$$

CHAPTER FIVE

DERIVATION OF KINK SOLUTIONS II

Introduction

The metric of Chapter 3 is simplified by setting $\alpha_{,t} = 0$. The various curvature quantities arising from this simplified form of the metric are presented (5.1). The Einstein field equations are calculated (5.2). Several perfect fluid (5.3) and imperfect fluid (5.4) and (5.5), one-kink solutions to the field equations are found.

(5.1) The Christoffel Symbols, Ricci Tensor and Curvature Scalar.

(5.2) The Einstein Tensor.

(5.3) Perfect Fluid Solutions.

(5.4) Imperfect Fluid Solutions I.

(5.5) Imperfect Fluid Solutions II.

(5.1) The Christoffel Symbols, Ricci Tensor and Scalar Curvature.

The metric discussed in Chapter 3, namely

$$
\begin{aligned}
g_{tt} &= e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\
g_{tr} &= -(e^{\sigma}+e^{\Omega})\sin\alpha\cos\alpha \\
g_{rr} &= e^{\sigma}\cos^2\alpha - e^{\Omega}\sin^2\alpha \\
g_{\theta\theta} &= e^{\tau}r^2 \\
g_{\phi\phi} &= e^{\tau}r^2\sin^2\theta,
\end{aligned}
$$

can also be simplified assuming that the functions α, σ, Ω, and τ are functions of r only. With this simplification the Christoffel symbols and Ricci tensor components can be written compactly in terms of the metric tensor and its derivatives. Substitution for the metric tensor components is made only after the field equations have been constructed and further simplifications have been made. The Christoffel symbols are

$$\Gamma^{t}{}_{tt} = -\Gamma^{r}{}_{tr} = -2^{-1}g^{tr}g_{tt,r}$$

$$\Gamma^{t}{}_{tr} = 2^{-1}g^{tt}g_{tt,r}$$

$$\Gamma^{t}{}_{rr} = g^{tt}g_{tr,r} + 2^{-1}g^{tr}g_{rr,r}$$

$$\Gamma^t{}_{\theta\theta} = -2^{-1}g^{tr}g_{\theta\theta,r}$$
$$= -2^{-1}e^{\tau}r^2g^{tr}[2r^{-1} + \tau_{,r}]$$
$$= -e^{\tau}r^2g_{tr}F$$

$$\Gamma^t{}_{\Phi\Phi} = \sin^2\theta\ \Gamma^t{}_{\theta\theta}$$

$$\Gamma^r{}_{tt} = -2^{-1}g^{rr}g_{tt,r}$$

$$\Gamma^r{}_{rr} = 2^{-1}g^{rr}g_{rr,r} + g^{tr}g_{tr,r}$$

$$\Gamma^r{}_{\theta\theta} = -2^{-1}g^{rr}g_{\theta\theta,r}$$
$$= -2^{-1}e^{\tau}r^2g^{rr}[2r^{-1} + \tau_{,r}]$$
$$= -e^{\tau}r^2g^{rr}F$$

$$\Gamma^r{}_{\Phi\Phi} = \sin^2\theta\Gamma^r{}_{\theta\theta}$$

$$\Gamma^\theta{}_{\Phi\Phi} = -\sin\theta\cos\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\Phi{}_{r\Phi} = 2^{-1}\tau_{,r} + r^{-1}$$
$$= F$$

$$\Gamma^\Phi{}_{\theta\Phi} = \cot\theta$$

where

$$F = 2^{-1}\tau_{,r} + r^{-1}$$

and, as before,

$$E = e^{\Omega+\sigma}.$$

The non-zero Ricci components are now

$$R_{tt} = (4E)^{-1} g_{tt} g_{tt,r}\{4F + 2g_{tt,rr}(g_{tt,r})^{-1} - E_{,r}{}^{E}-1\}$$

$$R_{tr} = (4E)^{-1} g_{tr} g_{tt,r}\{4F + 2g_{tt,rr}(g_{tt,r})^{-1} - E_{,r}E^{-1}\}$$

$$R_{rr} = (4E)^{-1} g_{rr} g_{tt,r}\{4F + 2g_{tt,rr}(g_{tt,r})^{-1} - E_{,r}E^{-1}\} \\ - 2F_{,r} - FE_{,r}E^{-1} - 2F^2$$

$$R_{\theta\theta} = 1 + e^{\tau} r^2 (2E)^{-1} g_{tt} F\{4F + 2g_{tt,r}(g_{tt})^{-1} \\ + 2F_{,r}F^{-1} - E^{-1}E_{,r}\}$$

$$R_{\Phi\Phi} = \sin^2\theta R_{\theta\theta}$$

and the scalar curvature is

$$R = E^{-1}g_{tt,rr} - 2^{-1}E^{-2}g_{tt,r}E_{,r} + 2e^{-\tau}r^2 + 6E^{-1}g_{tt}F^2 \\ + 4E^{-1}g_{tt}F_{,r} + 4E^{-1}Fg_{tt,r} - 2E^{-2}Fg_{tt}E_{,r}.$$

(5.2) The Einstein tensor components.

The Einstein tensor components are now

$$G^t{}_t = -e^{-\tau}r^{-2} - 3g_{tt}F^2E^{-1} - 2g_{tt}F_{,r}E^{-1}$$
$$- Fg_{tt,r}E^{-1} + Fg_{tt}E_{,r}E^{-2}$$

$$= -e^{-\tau}r^{-2} - g_{tt}F^2E^{-1} - Fg_{tt,r}E^{-1}$$
$$- 2g_{tt}E^{-1}\{F_{,r} - (2E)^{-1}FE_{,r} + F^2\}$$

$$G^r{}_t = 0$$

$$G^t{}_r = -2g_{tr}E^{-1}\{F_{,r} - (2E)^{-1}FE_{,r} + F^2\}$$

$$G^r{}_r = -e^{-\tau}r^{-2} - g_{tt}F^2E^{-1} - Fg_{tt,r}E^{-1}$$

$$G^\theta{}_\theta = G^\Phi{}_\Phi$$
$$= -(2E)^{-1}g_{tt,rr} + 4E^{-2}g_{tt,r}E_{,r} - E^{-1}Fg_{tt,r}$$
$$- g_{tt}E^{-1}\{F_{,r} - (2E)^{-1}FE_{,r} + F^2\}$$

$$G^t{}_\theta = G^\theta{}_t = G^t{}_\Phi = G^\Phi{}_t = G^r{}_\theta = 0$$

$$G^\theta{}_r = G^r{}_\Phi = G^\Phi{}_r = G_{\theta\Phi} = G_{\Phi\theta} = 0$$

(5.3) Perfect Fluid Solutions.

The stress-energy tensor for a perfect fluid is given in equation (3.4) by

$$T^{\alpha}{}_{\beta} = (\mu + p)u^{\alpha}u_{\beta} + p\delta^{\alpha}{}_{\beta}.$$

By comparison with the velocity components for the simple kink metric (for which $\Omega = \sigma = \tau = 0$) solutions of the field equations are sought for which

$$\begin{aligned} u^{t} &= e^{-\Omega/2}\cos\alpha = -u_{t} \\ u^{r} &= e^{-\Omega/2}\sin\alpha = -u_{r} \\ u^{\theta} &= u^{\Phi} = u_{\theta} = u_{\Phi} = 0. \end{aligned}$$

It is therefore clear that

$$T^{r}{}_{t} = T^{t}{}_{r} = -(\mu + p)\sin\alpha\cos\alpha.$$

Recalling that $G^{r}{}_{t} = 0$, this result implies that the equation of state for **any** perfect fluid solution of the field equations resulting from this form of the metric must be

$$p = -\mu.$$

Fluids satisfying this equation of state arise in inflationary cosmological models (Guth, 1981; Guth 1983; Guth, 1984) and in certain particle models (Rosen,1983).

The stress energy tensor therefore must have the form

$$T^{\alpha}{}_{\beta} = p\delta^{\alpha}{}_{\beta} \ .$$

and the Bianchi identities show that the pressure and density must be constant. The form of the stress energy tensor implies that

$$G^{t}{}_{r} = -g_{tr}E^{-1}\{F_{,r} - (2E)^{-1}FE_{,r} + F^{2}\} = 0 \qquad (5.1)$$

It is necessary for (sperically symmetric) kink solutions of this metric that

$$g_{tr} \neq 0,$$

because $g_{rt} = 0$ would imply that $\alpha = 0$. Clearly, it is also true that

$$E = e^{\sigma+\Omega} \neq 0.$$

This field equation (5.1) can therefore be satisfied if either

$$F = 2^{-1}\tau_{,r} + r^{-1} = 0 \tag{5.2}$$

or if

$$F \neq 0 \quad \text{but} \quad F^{-1}F_{,r} - (2E)^{-1}E_{,r} + F = 0 \tag{5.3}$$

Both of these possibilities lead to solutions of the field equations.

Consider first the case given by equation (5.2) where

$$F = 2^{-1}\tau_{,r} + r^{-1} = 0 \ .$$

This equation (5.2) implies that

$$e^{\tau} = cr^{-2} \tag{5.4}$$

where c is a positive constant.

When F = 0 the remaining field equations reduce to

$$\begin{aligned} G^t{}_t &= G^r{}_r \\ &= -e^{-\tau}r^{-2} \\ &= p \end{aligned}$$

$$
\begin{aligned}
G^{\theta}{}_{\theta} &= G^{\Phi}{}_{\Phi} \\
&= -2^{-1}E^{-1}g_{tt,rr} + 4^{-1}E^{-2}g_{tt,r}E_{,r} \\
&= p
\end{aligned}
$$

These last three equations imply that any solution must have

$$p = -c^{-1}$$

and satisfy

$$2^{-1}E^{-1}g_{tt,rr} - 4^{-1}E^{-2}g_{tt,r}E_{,r} = c^{-1} = C \tag{5.5}$$

where C is also a positive constant.

A general solution of this equation (5.5) is quite difficult to find since

$$
\begin{aligned}
g_{tt} &= e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\
E &= e^{\sigma+\Omega}
\end{aligned}
\tag{5.6}
$$

and σ, Ω, and α all depend on r.

However, if $\sigma = \Omega = 0$ or $\sigma = -\Omega$ is selected, so that

$$E = 1,$$

then solutions are easily found.

With either of these choices for σ and Ω, equation (5.5) reduces to

$$g_{tt,rr} = 2C$$

and this last equation clearly has a solution

$$g_{tt} = Cr^2 + Dr + H \tag{5.7}$$

where D and H are arbitrary constants.

First consider the case where $\sigma = \Omega = 0$. Using equation (5.6), it is clear that this choice of σ and Ω implies that

$$g_{tt} = -\cos 2\alpha.$$

Equivalently this last result combined with equation (5.7) shows that

$$\sin\alpha = 2^{-1/2}\{1 + Cr^2 + Dr + H\}^{1/2}.$$

There is considerable freedom in choosing the values of these constants, while ensuring that $|\sin\alpha| \leqslant 1$. The resulting solution will be discussed further in section (10.1).

To summarize, this solution of the field equations for the metric given in equation (3.1) has

$$\sigma = \Omega = 0$$
$$e^{\tau} = C^{-1}r^2$$

where C is a positive constant and

$$\sin\alpha = 2^{-1/2}(1 + Cr^2 + Dr + H)^{-1/2} \tag{5.8}$$

where D is any constant. The equation of state is $p = -\mu$. It will be shown in section (10.1) that the metric can be written as

$$ds^2 = -(1 - r - r^2)dt^2 - 2(2r + r^2 - 2r^3 - r^4)^{1/2}\, dtdr$$
$$+ (1 - r - r^2)^{dr}{}_2 + d\theta^2 + \sin^2\theta d\Phi^2.$$

valid for $0 \leqslant r \leqslant 1$.

The case where $\sigma = -\Omega$ is now considered in seeking another solution of equation (5.5) and hence of the field equations. This choice for σ and Ω, using equation (5.6), means that

$$E = 1$$

and

$$g_{tt} = -e^{\Omega} + (e^{\Omega} + e^{-\Omega})\sin^2\alpha.$$

Equivalently, using equation (5.7) to substitute for g_{tt}, this choice for Ω requires

$$\sin^2\alpha = (Cr^2 + Dr + H + e^{\Omega})(e^{\Omega} + e^{-\Omega})^{-1}.$$

This last result can be written

$$\sin^2\alpha = \tanh\Omega,$$

if Ω is chosen such that

$$-e^{-\Omega} = Cr^2 + Dr + H.$$

The constants C, D, H must be restricted to ensure that

$$|\sin\alpha| \leqslant 1.$$

but there is considerable freedom in doing this. To summarize, this solution of the field equations for the metric given in equation (3.1) has

$$\sigma = -\Omega\, e^{\tau} = C^{-1}r^{2}$$

where C is a positive constant and

$$\sin^{2}\alpha = (Cr^{2} + Dr + H + e^{\Omega})(e^{\Omega} + e^{-\Omega})^{-1}. \qquad (5.9)$$

where D and H are positive constants. The resulting solution will be discussed further in section (10.2). It will be shown that the following metrics are solutions of the field equations

$$\begin{aligned} ds^{2} = {} & (r^{2} - 2r + 1)dt^{2} + 2(8r^{3} + 8r - 12r^{2} - 2r^{4})^{1/2}\, dtdr \\ & - [1 - 2(r^{2} - 2r -+ 1)^{2}][r^{2} - 2r + 1]^{-1}\, dr^{2} \\ & + d\theta^{2} + \sin^{2}\theta\, d\Phi^{2} \end{aligned}$$

and

$$\begin{aligned} ds^{2} = {} & -(1 - r^{2})\, dt^{2} + 2r(4 - 2r^{2})^{1/2}\, dtdr \\ & + [\, 1 - 2(r^{2} - 1)^{2}][r^{2} - 1]^{-1}\, dr^{2} \\ & + r^{2}\, d\theta^{2} + \sin^{2}\theta\, d\Phi^{2} \end{aligned}$$

both valid for $0 \leqslant r \leqslant 1$.

The second possible solution of the $G^t{}_r$ field equation (5.1) was shown in equation (5.3) to occur when

$$F \neq 0 \quad \text{but} \quad F^{-1}F_{,r} - (2E)^{-1}E_{,r} + F = 0 \ .$$

This last equation can be integrated to obtain

$$\ln F - 2^{-1}\ln E + 2^{-1}\tau + \ln r = k \tag{5.10}$$

for some constant k. Rearranging, equation (5.10) can be written as

$$rFE^{-1/2} = Ke^{-1/2\tau} \tag{5.11}$$

where $K = e^k > 0$. Equivalently, substituting for $F = 2^{-1}\tau_{,r} + r^{-1}$, equation (5.11) can also be written as

$$2^{-1}r\tau_{,r}e^{1/2\tau} + e^{1/2\tau} = KE^{1/2}$$

or, by substituting for $E = e^{\sigma+\Omega}$, it can be written in terms of the metric functions τ, σ, Ω as

$$[re^{1/2\tau}]_{,r} = Ke^{(\sigma+\Omega)/2}.$$

There are four remaining field equations to satisfy. These are

$$G^t{}_t = T^t{}_t$$
$$G^r{}_r = T^r{}_r$$
$$G^\theta{}_\theta = T^\theta{}_\theta$$
$$G^\Phi{}_\Phi = T^\Phi{}_\Phi .$$

The general forms of the field equations are listed in section (5.2) and these equations now simplify to

$$\begin{aligned} G^t{}_t &= G^r{}_r \\ &= -e^{-\tau} r^{-2} - g_{tt} F^2 E^{-1} - F g_{tt,r} E^{-1} \\ &= p \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} G^\theta{}_\theta &= G^\Phi{}_\Phi \\ &= -(2E)^{-1} g_{tt,rr} + 4E^{-2} g_{tt,r} E_{,r} \\ &\quad - F g_{tt,r} E^{-1} \\ &= p \,. \end{aligned} \tag{5.13}$$

Comparing these two field equations it is clear that for a consistent solution the following equation must hold

$$-e^{-\tau} r^2 - F^2 E^{-1} g_{tt} - 4E^{-2} E_{,r} g_{tt,r} + (2E)^{-1} g_{tt,rr} = 0 \,.$$

(5.14)

General solutions of this last equation (5.14) will be hard to find since

$$g_{tt} = e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha$$
$$E = e^{\sigma+\Omega}$$
$$F = 2^{-1}\tau_{,r} + r^{-1}$$

and σ, τ, Ω and α are undetermined functions of r. However, solutions can be found if it is assumed that

$$\tau = 0.$$

The fact that $\tau = 0$ implies that

$$F = r^{-1} \tag{5.15}$$

Using equation (5.15) with equation (5.3) shows that the simplification $\tau = 0$ implies

$$E = e^{\sigma+\Omega} = K^{-2}.$$

Assuming, also for simplicity, that $K = 1$, it is clear that

$$\sigma = -\Omega.$$

The metric component g_{tt} is now

$$g_{tt} = e^{-\Omega}\sin^2\alpha - e^{\Omega}\cos^2\alpha.$$

These simplifications, $E = 1$, $F = r^{-1}$, and $\tau = 0$, reduce the remaining field equations to

$$\begin{aligned} G^t{}_t = G^r{}_r &= -r^{-2} - r^{-2}g_{tt} - r^{-1}g_{tt,r} \\ &= -r^{-2}[1 + (rg_{tt})_{,r}] \\ &= p \ , \end{aligned} \tag{5.16}$$

$$\begin{aligned} G^\theta{}_\theta = G^\Phi{}_\Phi &= -2^{-1}g_{tt,rr} - r^{-1}g_{tt,r} \\ &= -(2r)^{-1}[rg_{tt,rr} + 2g_{tt,r}] \\ &= -(2r)^{-1}[1 + (rg_{tt})_{,r}]_{,r} \\ &= (2r)^{-1}[r^2G^t{}_t]_{,r} \\ &= p \ . \end{aligned} \tag{5.17}$$

These equations (5.16) and (5.17) now show that

$$(2r)^{-1}[r^2G^t{}_t]_{,r} = G^t{}_t$$

or equivalently

$$(r^2G^t{}_t)^{-1}[r^2G^t{}_t]_{,r} = 2r^{-1}.$$

This last equation can be integrated to give

$$G^t{}_t = -L$$

where L is a positive constant. Recalling that

$$G^t{}_t = p = -\mu,$$

it is clear that the constant L must be chosen to be positive to ensure that the energy density is positive. Substituting $G_{tt} = -L$ back into the $G^t{}_t$ field equation (5.16) also shows that

$$\begin{aligned} g_{tt} &= e^{-\Omega}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\ &= \sin^2\alpha(e^{\Omega} + e^{-\Omega}) - e^{\Omega} \\ &= 3^{-1}Lr^2 - 1 + Mr^{-1} \end{aligned}$$

where M is an arbitrary constant. Equivalently, rearranging the above equation, a consistent solution requires that

$$\sin^2\alpha = [3^{-1}Lr^2 - 1 + Mr^{-1} + e^{\Omega}][e^{\Omega} + e^{-\Omega}]^{-1} .$$

There is considerable freedom in choosing Ω, L and M while still ensuring the required behaviour of the function $\sin\alpha$.

To summarize, this solution of the field equations for the metric given by equation (3.1) requires

$$\tau = 0 \quad \sigma = -\Omega$$

and

$$\sin^2\alpha = [3^{-1}Lr^2 - 1 + Mr^{-1} + e^{\Omega}][e^{\Omega} + e^{-\Omega}]^{-1} \ . \tag{5.18}$$

This solution will be discussed in section (10.3) It will be shown that the metric for this solution is

$$\begin{aligned} ds^2 = {} & (r^2 - 1)\ dt^2 + 2[2(1 - (r^2 - 1)^2]^{1/2}\ drdt \\ & + [1 - 2(r^2 - 1)^2][r^2 - 1]^{-1}\ dr^2 + r^2\ d\theta^2 \\ & + r^2\sin^2\theta\ d\Phi^2 \end{aligned}$$

valid for $0 \leqslant r \leqslant 1$.

(5.4) Imperfect Fluid Solutions I.

For the perfect fluid solutions previously found in this chapter the shear scalar can be shown to be non-zero. If the heat conduction vector components q^{α} are still assumed zero, the stress-energy tensor, from equation (3.3), is

$$T^{\alpha}{}_{\beta} = (\mu + p_T)u^{\alpha}u_{\beta} + p_T\delta^{\alpha}{}_{\beta} - \Sigma\Theta h^{\alpha}{}_{\beta} - 2\Omega\sigma^{\alpha}{}_{\beta} \qquad (5.19)$$

where μ is the total energy density, p_T is the thermodynamic pressure Σ is the coefficient of bulk viscosity, and Ω is the coefficient of dynamic viscosity. As shown in section (5.2), for all the solutions in which $\alpha_{,t} = 0$, the field equation $G^{r}{}_{t}$ is identically zero. It can also be shown from equation (3.6), that when $\alpha_{,t} = 0$, the $\sigma^{r}{}_{t}$ component of the shear tensor is

$$\sigma^{r}{}_{t} = \sigma^{t}{}_{r} = -2\sigma\ 3^{-1/2}\sin\alpha\cos\alpha$$

where, using equation (3.7), the shear scalar σ is

$$\sigma = 3^{-1/2}r\exp\{2^{-1}(-\Omega+\tau-\sigma)\}[r^{-1}\exp\{2^{-1}(-\tau+\sigma)\}\sin\alpha]_{,r}$$

The $T^{r}{}_{t}$ component of equation (5.19) therefore indicates that **any** solution of the field equations with $\alpha_{,t} = 0$ and $q^{\alpha} = 0$ must have an equation of state given by

$$\mu + p_T - \Sigma\theta - 4\Omega\sigma\, 3^{-1/2} = 0. \qquad (5.20)$$

The fact that $T^r{}_t = T^t{}_r$ and $G^r{}_t = 0$, means that any solution for which $\alpha_{,t} = 0$ and $q_\alpha = 0$ must have

$$G^t{}_r = -\, 2g_{tr}E^{-1}\{F_{,r} - (2E)^{-1}FE_{,r} + F^2\} = 0$$

Therefore, as in the perfect fluid case, either

$$F = 2^{-1}\tau_{,r} + r^{-1} = 0$$

or

$$F \neq 0, \quad \text{but} \qquad F^{-1}F_{,r} - (2E)^{-1}E_{,r} + F = 0$$

In either case, using the previously listed forms of the Einstein tensor components in section (5.2), it is clear that

$$G^t{}_t = G^r{}_r$$

and

$$G^\theta{}_\theta = G^\phi{}_\phi.$$

The shear tensor components $\sigma^{t}{}_{t}$, $\sigma^{r}{}_{r}$, $\sigma^{\theta}{}_{\theta}$, and $\sigma^{\Phi}{}_{\Phi}$ are calculated from equations (3.6) and (3.7), they are

$$\sigma^{t}{}_{t} = 2\sigma\ 3^{-1/2}\sin^{2}\alpha$$
$$\sigma^{r}{}_{r} = 2\sigma\ 3^{-1/2}\cos^{2}\alpha$$
$$\sigma^{\theta}{}_{\theta} = \sigma^{\Phi}{}_{\Phi} = -3^{1/2}\sigma.$$

Substituting for these shear tensor components from the above equations and for the energy density, μ, from the equation of state (5.20), gives the following expressions for the stress energy tensor components

$$\begin{aligned} T^{t}{}_{t} &= -\mu\cos^{2}\alpha + (p_{T} - \Sigma\theta)\sin^{2}\alpha - 4\eta 3^{-1/2}\sigma\sin^{2}\alpha \\ &= p_{T} - \Sigma\theta - 4\eta\ 3^{-1/2}\sigma \end{aligned}$$

$$\begin{aligned} T^{r}{}_{r} &= -\mu\sin^{2}\alpha + (p_{T} - \Sigma\theta)\cos^{2}\alpha - 4\eta 3^{-1/2}\sigma\cos^{2}\alpha \\ &= p_{T} - \Sigma\theta - 4\eta\ 3^{-1/2}\sigma \end{aligned}$$

$$T^{\theta}{}_{\theta} = T^{\Phi}{}_{\Phi} = p_{T} - \Sigma\theta + 2\eta\ 3^{-1/2}\sigma\ .$$

There are therefore two independent field equations. These are

$$G^{t}{}_{t} = G^{r}{}_{r} = p_{T} - \Sigma\theta - 4\eta\ 3^{-1/2}\sigma$$

$$G^{\theta}{}_{\theta} = G^{\Phi}{}_{\Phi} = p_{T} - \Sigma\theta + 2\eta\ 3^{-1/2}\sigma\ .$$

If F = 0, these field equations reduce to

$$G^t{}_t = G^r{}_r = -e^{-\tau} r^{-2}$$
$$= p_T - \Sigma\theta - 4\Omega\, 3^{-1/2}\sigma$$

$$G^\theta{}_\theta = G^\Phi{}_\Phi = -2^{-1}E^{-1}g_{tt,rr} + 4^{-1}E^{-2}g_{tt,r}E_{,r}$$
$$= p_T - \Sigma\theta + 2\Omega\, 3^{-1/2}\sigma.$$

If $F \neq 0$ but $F^{-1}F_{,r} - (2E)^{-1}E_{,r} + F = 0$, the field equations become

$$G^t{}_t = G^r{}_r$$
$$= -e^{-\tau}r^{-2} - g_{tt}F^2E^{-1} - Fg_{tt,r}E^{-1}$$
$$= p_T - \Sigma\theta - 4\Omega\, 3^{-1/2}\sigma$$

$$G^\theta{}_\theta = G_{\Phi\Phi}$$
$$= -(2E)^{-1}g_{tt,rr} + 4E^{-2}g_{tt,r}E_{,r} - Fg_{tt,r}E^{-1}$$
$$= p_T - \Sigma\theta + 2\Omega\, 3^{-1/2}\sigma.$$

General solutions of these field equations are clearly quite difficult to find because

$$g_{tt} = e^\sigma \sin^2\alpha - e^\Omega \cos^2\alpha,$$
$$E = e^{\sigma+\Omega}$$

$$\theta = e^{-(\sigma+\Omega)/2}(e^{\sigma/2})_{,r} \qquad (5.21)$$
$$+ 2(2^{-1}\tau_{,r} + r^{-1})(e^{-\Omega/2}\sin\alpha)$$
$$\sigma = 3^{-1/2}re^{(-\Omega+\tau-\sigma)/2}[r^{-1}e^{(-\tau+\sigma)/2}\sin\alpha]_{,r}$$

and α, σ, Ω, τ are all unknown functions of r. In the special case $\tau = \sigma = \Omega = 0$, so that $F = r^{-1}$, and from equations (5.21), $E = 1$ and $g_{tt} = -\cos 2\alpha$, it can be shown that a solution of the field equations is

$$\sin\alpha = -2\Omega\, 3^{-1}r + K^{1/2}(1 + K)r^{-1/2}$$

where K is a constant (Harriott and Williams, 1988a). However, this solution has negative energy density for certain values of r and will not be discussed further.

(5.5) Imperfect Fluid solutions II

To find a further solution, the heat conduction vector q^{α} is chosen to be non-zero. The stress-energy tensor is now given by equation (3.3) which is

$$T^{\alpha}{}_{\beta} = (\mu + p_T)u^{\alpha}u_{\beta} + p_T\delta^{\alpha}{}_{\beta} - \Sigma\Theta h^{\alpha}{}_{\beta} - 2\eta\sigma^{\alpha}{}_{\beta} + q^{\alpha}u_{\beta} + u^{\alpha}q_{\beta} \tag{5.22}$$

The components of q_{α} must satisfy (Ellis, 1971)

$$q_{\alpha}u^{\alpha} = 0.$$

The velocity components, as discussed previously in section (3.4), are chosen to be

$$\begin{aligned} u_t &= -e^{\Omega/2}\cos\alpha & \qquad u^t &= e^{-\Omega/2}\cos\alpha \\ u_r &= -e^{\Omega/2}\sin\alpha & u^r &= e^{-\Omega/2}\sin\alpha \\ u_{\theta} &= u_{\phi} = 0 & u^{\theta} &= u^{\phi} = 0. \end{aligned}$$

As shown in section (5.2), some of the Einstein tensor components are identically zero. These are

$$G^{r}{}_{\theta} = G^{\theta}{}_{r} = G^{\phi}{}_{r} = G^{r}{}_{\phi} = 0$$

These results together with the velocity components and equation (5.22) for the stress energy tensor, show that

$$q^{\theta} = q^{\Phi} = 0$$

$$q_{\theta} = q_{\Phi} = 0$$

and suggest the following form for the remaining components of the heat conduction vector

$$q_t = Qe^{\sigma/2}\sin\alpha$$

$$q_r = -Qe^{\sigma/2}\cos\alpha$$

where Q is an unknown function of r. The contravariant components are now

$$\begin{aligned} q^t &= g^{tt}q_t + g^{tr}q_r \\ &= Qe^{-\sigma/2}\sin\alpha \end{aligned}$$

$$\begin{aligned} q^r &= g^{rt}q_t + g^{rr}q_r \\ &= -Qe^{-\sigma/2}\cos\alpha. \end{aligned}$$

These results show that the square of the length of the heat conduction vector is given by

$$q^{\alpha}q_{\alpha} = Q^2 \ .$$

The term $[q_{\alpha}u^{\beta} + u_{\alpha}q^{\beta}]$ appears in the following four non-zero terms in the stress energy tensor components

$$q_t u^t + u_t q^t = Qe^{(\sigma-\Omega)/2}\sin\alpha\cos\alpha - Qe^{(\Omega-\sigma)/2}\cos\alpha\sin\alpha$$

$$\begin{aligned} q_r u^r + u_r q^r &= -Qe^{(\sigma-\Omega)/2}\sin\alpha\cos\alpha + Qe^{(\Omega-\sigma)/2}\cos\alpha\sin\alpha \\ &= -(q_t u^t + u_t q^t) \end{aligned}$$

$$q_t u^r + u_t q^r = Qe^{(\sigma-\Omega)/2}\sin^2\alpha + Qe^{(\Omega-\sigma)/2}\cos^2\alpha$$

$$\begin{aligned} q_r u^t + u_r q^t &= -Qe^{(\sigma-\Omega)/2}\cos^2\alpha - Qe^{(\Omega-\sigma)/2}\sin^2\alpha \\ &= -(q_t u^r + u_t q^r). \end{aligned}$$

Some restrictions are now made on the metric. These restrictions are to assist in finding a solution, because the general forms for the field equations are difficult to solve. It is assumed that

$$\sigma = \Omega.$$

This assumption means that

$$q_t u^r + u_t q^r = Q$$

$$q_t u^t + u_t q^t = q_r u^r + u_r q^r = 0.$$

The Einstein tensor component $G^r{}_t$ is identically zero and so the stress energy tensor component $T^r{}_t$ now gives the following expression for Q.

$$Q = [\mu + p_T - \Sigma\theta - 4\Omega\ 3^{-1/2}\sigma]\sin\alpha\cos\alpha. \qquad (5.23)$$

Using this result for Q, the field equation

$$\begin{aligned} G^t{}_r &= -2g_{tr}E^{-1}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\} \\ &= -[\mu + p_T - \Sigma\theta - 4\Omega\ 3^{-1/2}]\sin\alpha\cos\alpha - Q \end{aligned}$$

where $F = 2^{-1}\tau_{,r} + r^{-1}$ and $E = e^{\sigma+\Omega} = e^{2\Omega}$, reduces to

$$G^t{}_r = -2Q.$$

This last result gives another expression for Q, which is

$$\begin{aligned} Q &= g_{tr}E^{-1}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\} \\ &= -\sin 2\alpha\ e^{-\Omega}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\} \qquad (5.24) \end{aligned}$$

Therefore, for the heat conduction vector to be non-zero, it must hold that

$$\alpha \neq 0$$

and

$$F_{,r} - F(2E)^{-1}E_{,r} + F^2 \neq 0.$$

The first of these equations, namely $\alpha \neq 0$ is also a requirement for a kink solution. Comparing the expressions for Q, given in equations (5.23) and (5.24), it is clear that the non-zero hydrodynamic quantities must obey the following equation

$$\mu + p_T - \Sigma\theta - 4\eta\, 3^{-1/2}\sigma = -2e^{-\Omega}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\}. \tag{5.25}$$

The remaining field equations to be satisfied are

$$G^t{}_t = T^t{}_t$$

$$G^r{}_r = T^r{}_r$$

$$G^\theta{}_\theta = G^\phi{}_\phi = T^\phi{}_\phi = T^\theta{}_\theta.$$

From equation (5.22),

$$\begin{aligned} T^t{}_t &= -\mu\cos^2\alpha + (p_T - \Sigma\theta)\sin^2\alpha - 4\eta\, 3^{-1/2}\sigma\sin^2\alpha \\ &= -\mu + [\mu + p_T - \Sigma\theta - 4\eta\, 3^{-1/2}\sigma]\sin^2\alpha. \end{aligned}$$

Using equation (5.25), the above expression for $T^t{}_t$ can be written

$$T^t{}_t = -\mu - 2e^{-\Omega}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\}\sin^2\alpha.$$

Similarly,

$$\begin{aligned} T^r{}_r &= -\mu\sin^2\alpha + (p_T - \Sigma\theta)\cos^2\alpha - 4\cap\, 3^{-1/2}\sigma\cos^2\alpha \\ &= -\mu + [\mu + p_T - \Sigma\theta - 4\cap\, 3^{-1/2}\sigma]\cos^2\alpha \\ &= -\mu - 2e^{-\Omega}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\}\cos^2\alpha. \end{aligned}$$

The Einstein tensor components $G^t{}_t$ and $G^r{}_r$ are

$$\begin{aligned} G^t{}_t = &-e^{-\tau}r^{-2} - g_{tt}F^2E^{-1} - Fg_{tt,r}E^{-1} \\ &- 2g_{tt}E^{-1}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\} \end{aligned}$$

$$G^r{}_r = -e^{-\tau}r^{-2} - g_{tt}F^2E^{-1} - Fg_{tt,r}E^{-1}.$$

These last four equations can now be used to show that although the field equations $G^t{}_t = T^t{}_t$ and $G^r{}_r = T^r{}_r$ are no longer equal, they are dependent and it is hence convenient to consider

$$\begin{aligned} G^r{}_r - G^t{}_t &= 2g_{tt}E^{-1}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\} \\ &= e^{-\Omega}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\}\cos 2\alpha \\ &= 2g_{tt}\{F_{,r} - F(2E)^{-1}E_{,r} + F^2\} \\ &= T^r{}_r - T^t{}_t \end{aligned}$$

The remaining independent field equation is $G^\theta{}_\theta = T^\theta{}_\theta$ which is

$$G^\theta{}_\theta = p_T - \Sigma\theta + 2\cap\, 3^{-1/2}\sigma \ .$$

So far, the only simplifications made to the metric to assist in finding a solution are to assume that $\alpha_{,t} = 0$ and $\Omega = \sigma$. To demonstrate an exact solution, several further simplifications are made. These simplifications are to choose two of the exponential functions, σ and Ω to be zero; setting the coefficient of viscosity, $\cap$, to zero and selecting a specific form for the remaining nonzero exponential function, τ. That is

$$\Omega = 0,$$

$$\cap = 0,$$

and

$$F = 2^{-1}\tau_{,r} + r^{-1} = -N \ ,$$

where N is constant and the negative sign is included for convenience. Choosing $\Omega = 0$ leads to

$$E = e^{\sigma+\Omega} = 1$$

and

$$g_{tt} = -\cos 2\alpha.$$

It is clear that choosing $F = -N$ is equivalent to the choice of

$$\tau = -2Nr - 2\ln r$$

or, equivalently,

$$e^{\tau} r^{2} = e^{-2Nr}$$

where the constant of integration has been set to zero for convenience. It should also be noted that choosing $F = -N$ ensures that

$$F_{,r} - F(2E)^{-1}E_{,r} + F^{2} \neq 0$$

as is required if the heat conduction is to be nonzero. In fact, this choice of F means that

$$F_{,r} - F(2E)^{-1}E_{,r} + F^2 = N^2. \qquad (5.26)$$

These choices now allow the field equations to be simplified to

$$\begin{aligned} G^t{}_t &= -N(\cos 2\alpha)_{,r} + 3N^2\cos 2\alpha - e^{2Nr} \\ &= -\mu + [\mu + p_T - \Sigma\theta]\sin^2\alpha \end{aligned}$$

$$\begin{aligned} G^r{}_r &= -N(\cos 2\alpha)_{,r} + N^2\cos 2\alpha - e^{2Nr} \\ &= -\mu + [\mu + p_T - \Sigma\theta]\cos^2\alpha \end{aligned}$$

$$\begin{aligned} G^\theta{}_\theta &= 2^{-1}(\cos 2\alpha)_{,rr} - N(\cos 2\alpha)_{,r} + N^2\cos 2\alpha \\ &= p_T - \Sigma\theta. \end{aligned}$$

The $G^t{}_t$ and $G^r{}_r$ equations can be combined to give

$$-2N(\cos 2\alpha)_{,r} + 4N^2\cos 2\alpha - 2e^{2Nr} = -\mu + p_T - \Sigma\theta.$$

Equivalently, substituting for μ from equation (5.25) and using equation (5.26) shows that

$$-N(\cos 2\alpha)_{,r} + 2N^2\cos 2\alpha - e^{2Nr} = p_T - \Sigma\theta + N^2 .$$

This last equation and the $G^\theta{}_\theta$ field equation are the remaining independent equations to solve. By inspection, a solution to these equations exists of the form

$$\cos 2\alpha = P\, e^{2Nr} + T + f(r)$$

where P and T are constants and f(r) is an unknown function of r which satisfies

$$N^2 P\, e^{2Nr} + 2^{-1} f(r)_{,rr} - N^2 T - N^2 f(r) + e^{2Nr} + N^2 = 0.$$

This last equation is valid if

$$N^2 P + 1 = 0$$

$$2^{-1} f(r)_{,rr} - N^2 f(r) - N^2 T + N^2 = 0 .$$

It is possible to choose

$$P = -N^{-2}$$

$$T = 1$$

so that

$$f(r) = U\exp(\pm 2^{1/2} Nr)$$

where U is an arbitrary constant. These choices show that

$$\cos 2\alpha = -N^{-2}e^{2Nr} + 1 + U\exp(\pm 2^{1/2}Nr)$$

or equivalently that

$$\sin^2\alpha = (2N^2)^{-1}e^{-2Nr} - 2^{-1}U\exp(\pm 2^{1/2}Nr).$$

Some restrictions must be made on these unknown constants to ensure the correct behaviour of $\sin\alpha$. To summarize, for this solution of the metric given by equation (3.1),

$$\Omega = \sigma = 0$$

$$\cap = 0$$

$$\tau = -2Nr - 2\ln r$$

or, equivalently,

$$e^{\tau}r^2 = e^{-2Nr}$$

where N is a positive constant and

$$\sin^2\alpha = (2N^2)^{-1}e^{-2Nr} - 2^{-1}U\exp(\pm 2^{1/2}Nr) \tag{5.27}$$

where U is a constant. This solution will be discussed in section (10.4). It will be shown that the equation of state is

$$\mu + p_T - \Sigma\theta + 2N^2 = 0$$

where

$$\mu = \exp(2Nr) - (2 - 2^{1/2})\exp(\pm 2^{1/2}Nr) - 2N^2$$

and N may assume any value in the range

$$N^2 < (2^{1/2} - 1)3^{-1}.$$

The metric will be shown to be

$$\begin{aligned} ds^2 = &-[1 - N^{-2}\exp(2Nr) + N^{-2}\exp(\pm 2^{1/2}Nr)]\ dt^2 \\ &- 2[-N^{-4}\exp(4Nr) - N^{-4}\exp(2Nr2^{1/2}) + 2N^{-2}\exp(2Nr) \\ &- 2\exp(\pm 2^{1/2}Nr) + 2N^{-4}\exp(2Nr \pm 2^{1/2}Nr)]\ dtdr \\ &+ [1 - N^{-2}\exp(2Nr) + N^{-2}\exp(\pm 2^{1/2}Nr)]\ dr^2 \\ &+ \exp(-2Nr)d\theta^2 + \exp(-2Nr)\sin^2\theta\ d\Phi^2 \end{aligned}$$

valid for $0 \leqslant r \leqslant R_1$ where R_1 is defined from

$$\exp(2NR_1) - \exp(\pm 2^{1/2}NR_1) = 2N^2 .$$

CHAPTER SIX

THE DE SITTER KINK SOLUTION

Introduction

The form of metric and various hydrodynamical quantities for this kink solution are discussed (6.1) and the transformation to the familiar de Sitter form is demonstrated via a singular transformation (6.2). The Killing vectors for this spacetime are found (6.3). The kink number is calculated for this metric and shown to be equal to one (6.4), and the feature common to all kink metrics, the tipping light cone behaviour, is illustrated for this solution (6.5). Extension of the manifold to produce an n-kink solution is demonstrated (6.6).

(6.1) The Form of the Metric and Solution Properties.

The solution obtained in Chapter 4,

$$ds^2 = -\cos 2\alpha dt^2 - 2\sin 2\alpha dt dr + \cos 2\alpha dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta d\Phi^2 \tag{6.1}$$

which may be called the de Sitter kink metric was given by equation (4.14) which shows that

$$\sin\alpha = Kr \tag{6.2}$$

for constant K. If the constant K is chosen to be positive, then this solution is valid for

$$0 \leqslant r \leqslant K^{-1}$$
$$0 \leqslant t < \infty.$$

Justification for naming this solution the de Sitter kink metric and proof that it is a one-kink metric will be given in future sections.

Analogous de Sitter kink solutions exist in both 1+1 and 2+1 dimensions (Dunn, Harriott and Williams, 1991a; Williams and Zvengrowski, 1991). The 1+1 case will be discussed further in chapter 8.

To allow the presence of one complete kink, the variable r can be allowed to be negative, so that the range of r is

$$-K^{-1} \leqslant r \leqslant K^{-1}.$$

The metric can now be written as

$$ds^2 = -(1 - 2K^2r^2)dt^2 - 4Kr(1 - K^2r^2)^{1/2}dtdr + (1 - 2K^2r^2)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\Phi^2. \tag{6.3}$$

Note that for $-(2K^2)^{-1/2} \leqslant r \leqslant (2K^2)^{-1/2}$, t is the timelike coordinate, but for $(2K^2)^{-1/2} < r < K^{-1}$, and $-K^{-1} < r \leqslant -(2K^2)^{-1/2}$, r is the timelike coordinate.

The equation of state for this solution was shown in section (4.3) to be

$$p = -\mu.$$

Also, from equation (4.8) the isotropic pressure can be found to be constant, as expected, and in particular

$$p = - 6K^2.$$

It is also easy to see that this solution may be interpreted in a different way: as an empty space solution with a non-zero cosmological constant. If the choice $p = \mu = 0$ is made, then a cosmological constant

$$\Omega = 6K^2$$

must be introduced. The usual de Sitter universe is usually regarded as an empty space solution with a cosmological constant.

The scalar expansion, θ, can be found from equation (3.8). For this solution it is

$$\theta = 3K.$$

The expansion factor for this solution is therefore exponential. This may be shown as follows. From the definition of the scalar expansion

$$(\delta s)^{\cdot}/\delta s = \theta = 3K \ .$$

This equation may be integrated to obtain

$$\delta s = A\exp(3Kt)$$

for constant A.

The shear scalar and shear tensor can be shown, using equations (3.6) and (3.7), to be zero for this solution.

The scalar curvature, R, can be calculated from equation (4.1). For this solution it is found to be

$$R = 24K^2.$$

The scalar curvature is therefore a positive constant for this solution and the spacetime manifold is well behaved at all points.

The velocity vector components are found from equation (3.5). They are

$$\begin{aligned} u^t &= (1 - K^2r^2)^{1/2} = -u_t \\ u^r &= Kr = -u_r \\ u^\theta &= u_\theta = 0 \\ u^\Phi &= u_\Phi = 0 \end{aligned}$$

The 4-velocity vector is (1, 0, 0, 0) at $r = 0$ and continuously turns to become (0, 1, 0, 0) at $r = \pm K^{-1}$.

The acceleration vector has components which are found from the equations listed in section (3.5). For this solution,

$$\dot{u}^t = -K^2 r$$

$$\dot{u}^r = K(1 - K^2 r^2)^{1/2}$$

$$\dot{u}^\theta = \dot{u}^\Phi = 0.$$

At $r = 0$ the acceleration vector is $(0, K, 0, 0)$. This vector turns continuously, reaching $(\pm K, 0, 0, 0)$ at $r = \pm K^{-1}$.

(6.2) Transformation to Usual de Sitter Form.

This solution can be transformed to the de Sitter solution by means of a **singular** transformation. This was first demonstrated by Dunn and Williams (1989). The cross term dtdr can be removed by introducing a new coordinate

$$t' = t + f(r).$$

Under such a transformation

$$g_{t'r} = g_{tr} - g_{tt}df/dr$$

where f(r) must be chosen so that

$$df/dr = \tan 2\alpha = 2Kr(1 - K^2r^2)^{1/2}(1 - 2K^2r^2)^{-1}$$

Note that this transformation is singular at $r = \pm(2K^2)^{-1/2}$, which are points within the range of allowed values for r.

The metric is now

$$ds^2 = -(1 - 2K^2r^2)dt'^2 + (1 - 2K^2r^2)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\Phi^2$$

which is a form of the de Sitter metric. The usual form of the de Sitter metric is

$$ds^2 = -dT^2 + a^2\cosh^2(a^{-1}T)[dX^2 + \sin^2X(d\theta^2 + \sin^2\theta d\Phi^2)] \tag{6.4}$$

and can be obtained by the transformation

$$r = a\ \sin X\ \cosh(a^{-1}T)$$

$$\sinh(a^{-1}t') = \pm\sinh(a^{-1}T)[1 - \sin^2X\ \cosh^2(a^{-1}T)]^{1/2}$$

where $a = (2K^2)^{-1/2}$, and is valid for

$$0 < X < \pi$$
$$-\infty < T < \omega.$$

The existence of this transformation justifies the name "de Sitter kink" for the solution given in equation (6.3). It also shows that the kinks are a global feature because locally the metric given in equation (6.3) is identical to the familiar de Sitter metric.

(6.3) The Killing Vectors.

The Killing vectors are used to find the symmetries of the spacetime. For this metric, the Killing equations stated in section (3.2) reduce to

$$\mu_{t,t} - 4K^3r^2(1 - K^2r^2)^{1/2}\mu_t + 2K^2r(1 - 2K^2r^2)\mu_t = 0$$

$$\mu_{t,r} + \mu_{r,t} + 4K^2r(1 - 2K^2r^2)\mu_t + 8K^3r^2(1 - K^2r^2)\mu_r = 0$$

$$\mu_{t,\theta} + \mu_{\theta,t} = 0$$

$$\mu_{t,\Phi} + \mu_{\Phi,t} = 0$$

$$\mu_{r,r} - 2K(1 - K^2r^2)^{-1/2}(1 - 2K^2r^2 + 2K^2r^4)\mu_t - 2K^2r(1 - 2K^2r^2)\mu_r = 0$$

$$\mu_{r,\theta} + \mu_{\theta,r} - 2r^{-1}\mu_\theta = 0$$

$$\mu_{r,\Phi} + \mu_{\Phi,r} - 2r^{-1}\mu_\Phi = 0 \quad .$$

$$\mu_{\theta,\theta} - 2Kr^2(1 - K^2r^2)^{1/2}\mu_t + r(1 - 2K^2r^2)\mu_r = 0$$

$$\mu_{\theta,\Phi} + \mu_{\Phi,\theta} - 2\cot\theta\ \mu_\Phi = 0$$

$$\mu_{\Phi,\Phi} - 2Kr^2\sin^2\theta(1 - K^2r^2)^{1/2}\mu_t + r^2\sin^2\theta(1 - 2K^2r^2)\mu_r + \sin\theta\cos\theta\ \mu_\theta = 0\ .$$

The usual de Sitter spacetime admits a full set of ten Killing vectors, these are the timelike vector $\delta/\delta T$, three vectors which are the (spacelike) generators of the rotation group SO(3), and the six (spacelike)vectors which are the translations. The above equations clearly admit the vector

$$\mu_1 = \delta/\delta t.$$

The length of this vector is

$$||\mu_1|| = 2K^2r^2 - 1\ .$$

This Killing vector μ_1 is timelike for

$$-2^{-1/2}K^{-1} < r < 2^{-1/2}K^{-1}.$$

It becomes null at $r = \pm 2^{-1/2}K^{-1}$, and is spacelike for

$$-K^{-1} < r < -2^{-1/2}K^{-1} \quad \text{and} \quad 2^{-1/2}K^{-1} < r < K^{-1}.$$

Therefore there is no global timelike Killing vector and so the spacetime is not globally stationary.

(6.4) Kink Number Calculation.

The kink number N for this solution can be calculated using the formula stated previously in section (2.3), namely

$$N = (12\pi^2)^{-1}\int N^0 d^3x \qquad (6.5)$$

where

$$N^0 = \epsilon^{0ijk}\epsilon_{\alpha\beta\tau\delta}\, X^\alpha X^\beta{}_{|i} X^\tau{}_{|j} X^\delta{}_{|k}. \qquad (6.6)$$

and $|$ denotes differentiation with respect to the S matrix in the G = SQ decomposition for any matrix G representing a metric tensor. Any metric given by equation (6.1), can be represented by the matrix

$$G = \begin{pmatrix} -\cos 2\alpha & -\sin 2\alpha & 0 & 0 \\ -\sin 2\alpha & \cos 2\alpha & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

where $\alpha = \alpha(r)$.

Following the procedure outlined in section (2.3) to find the kink number, the ϕ_α are first found for this metric. As stated in section (2.4), any metric given by equation (6.1) is a metric for which the matrices S and Q in the decomposition of the matrix representing the metric are

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and

$$Q = \begin{pmatrix} -\cos 2\alpha & -\sin 2\alpha & 0 & 0 \\ -\sin 2\alpha & \cos 2\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Writing

$$Q = P\ \mathrm{diag}(-1,\ 1,\ 1,\ 1)\ P^T$$

where P is the matrix defined in equation (2.4), then

$$P = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

so that the required ϕ_α are

$$\phi_t = \cos\alpha$$

$$\phi_r = \sin\alpha$$

$$\phi_\theta = \phi_\Phi = 0.$$

These ϕ_α are the X^α used in the formula given in equation (6.6). The covariant derivatives of the X^α, with respect to the tensor $s_{\alpha\beta}$ represented by the S matrix, are also needed to calculate the kink number from the formula given by equations (6.5) and (6.6). The nonzero Christoffel symbols of the tensor $s_{\alpha\beta}$ are

$$\Gamma^r{}_{\theta\theta} = -r$$

$$\Gamma^r{}_{\Phi\Phi} = -r\sin^2\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\Phi{}_{r\Phi} = r^{-1}$$

$$\Gamma^\theta{}_{\Phi\Phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\Phi}{}_{\theta\Phi} = \cot\theta \ .$$

The non-zero covariant derivatives of the $X^{\alpha} = \phi_{\alpha}$ are

$$\begin{aligned} X^{r}{}_{|r} &= X^{r}{}_{,r} \\ X^{r}{}_{|t} &= X^{r}{}_{,t} \\ X^{t}{}_{|r} &= X^{t}{}_{,r} \\ X^{\theta}{}_{|\theta} &= \Gamma^{\theta}{}_{\theta r}X^{r} \\ X^{\Phi}{}_{|\Phi} &= \Gamma^{\Phi}{}_{\Phi r}X^{r} \ . \end{aligned}$$

Therefore the non-zero terms in the equation (6.6) will be those of the kind

$$X^{t}X^{r}{}_{|r}X^{\theta}{}_{|\theta}X^{\Phi}{}_{|\Phi} = \cos\alpha r^{-2}\sin^{2}\alpha(\sin\alpha)_{,r}$$

and

$$X^{r}X^{t}{}_{|r}X^{\theta}{}_{|\theta}X^{\Phi}{}_{|\Phi} = \sin\alpha r^{-2}\sin^{2}\alpha(\cos\alpha)_{,r}$$

These groups of terms, actually six terms of each kind, will combine to give a non-zero answer because of the antisymmetrization. In particular, using the above results in equation (6.5) shows that

$$N = (12\pi^{2})^{-1}\int 6[r^{-2}\cos^{2}\alpha\sin^{2}\alpha\alpha_{,r} + r^{-2}\sin^{4}\alpha\alpha_{,r}]\ d^{3}x$$

$$= (2\pi^2)^{-1} \int_{\alpha_2}^{\alpha_1} \int_0^{2\pi} \int_0^{\pi} \sin^2\alpha \; \alpha_{,r} \; \sin\Phi dr d\Phi d\theta \qquad (6.7)$$

where α_1 and α_2 are the values of α at the limiting values of r. For the de Sitter kink,

$$\alpha(r = -K^{-1}) = -\pi/2 \quad \alpha(r = K^{-1}) = \pi/2 \ ,$$

so that the integral given in equation (6.7) reduces to

$$N = (2\pi^2)^{-1} \left(\int_{-\pi/2}^{\pi/2} \sin^2\alpha d\alpha \right) \left(\int_0^{\pi} \sin\Phi d\Phi \right) \left(\int_0^{2\pi} d\theta \right)$$

$$= (2\pi^2)^{-1} \left[\int_{-\pi/2}^{\pi/2} 2^{-1}(1 - \cos 2\alpha) d\alpha \right] (2) (2\pi)$$

$$= (2\pi^2)^{-1}(2^{-1}\pi)(2)(2\pi)$$

$$= 1.$$

In x^α coordinates, the S matrix for this solution is the identity matrix and so the covariant derivatives with respect to the tensor $s_{\alpha\beta}$ reduce to ordinary derivatives and the equation (6.6) reduces to Skryme's original formula given earlier as equation (2.5). It can be shown (Skyrme, 1961) that in this case

$$N = (\pi)^{-1} \int_{\alpha_2}^{\alpha_1} (1 - \cos 2\alpha)\, d\alpha$$

$$= (\pi)^{-1}[\alpha(0) - \alpha(\infty)],$$

provided that $\sin 2\alpha$ is zero at the origin and at infinity.

For the de Sitter kink, $-K^{-1} \leq r \leq K^{-1}$, $\sin\alpha = Kr$, $\alpha(0) = 0$, $\alpha(K) = \pi/2$. If r is allowed to be negative, so the complete kink is present, then $\alpha(-K) = -\pi/2$, and this integral becomes

$$= \pi^{-1} \int_{-\pi/2}^{\pi/2} (1 - \cos 2\alpha)\, d\alpha$$

$$= 1.$$

Therefore the above solution is a one kink solution.

If, however, the usual form, given by equation (6.4), of the de Sitter solution is considered, then the kink number is found to be 0. This is shown as follows.

For the usual form of the de Sitter metric

$$G = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2\cosh^2(a^{-1}T) & 0 & 0 \\ 0 & 0 & a^2\cosh^2(a^{-1}T)\sin^2 X & 0 \\ 0 & 0 & 0 & a^2\cosh^2(a^{-1}T)\sin^2 X\sin^2\theta \end{pmatrix} \tag{6.8}$$

It is easy to show that

$$Q = \mathrm{diag}(-1, 1, 1, 1)$$

and hence that

$$\phi_T = 1$$
$$\phi_i = 0 \quad \text{for } i = 1, 2, 3.$$

The vector field X^α is therefore (1, 0, 0, 0) and so all the terms $X^\alpha{}_{,\beta}$ will be zero. The matrix S in the usual G = SQ decomposition of the matrix G given in equation (6.8) is

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2\cosh^2(a^{-1}T) & 0 & 0 \\ 0 & 0 & a^2\cosh^2(a^{-1}T)\sin^2 X & 0 \\ 0 & 0 & 0 & a^2\cosh^2(a^{-1}T)\sin^2 X\sin^2\theta \end{pmatrix}$$

The non-zero Christoffel symbols (with respect to the tensor $s_{\alpha\beta}$) are

$$\begin{aligned}
\Gamma^{T}{}_{\theta\theta} &= -a\sin^2 X\sinh(2Ta^{-1}) \\
\Gamma^{t}{}_{\Phi\Phi} &= \sin^2\theta\,\Gamma^{T}{}_{\theta\theta} \\
\Gamma^{X}{}_{XT} &= a^{-1}\tanh(Ta^{-1}) \\
\Gamma^{X}{}_{\theta\theta} &= -2^{-1}\sin 2X \\
\Gamma^{X}{}_{\Phi\Phi} &= \sin^2\theta\,\Gamma^{X}{}_{\theta\theta} \\
\Gamma^{\theta}{}_{\Phi\Phi} &= -2^{-1}\sin 2\theta .
\end{aligned}$$

Recalling that $X^T = 1$, $X^X = X^\theta = X^\Phi = 0$, it is hence clear that the only non-zero covariant derivative

$$X^{\alpha}{}_{|\beta} = X^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\beta\tau}X^{\tau}$$

is

$$X^{X}{}_{|X} = \Gamma^{X}{}_{XT} = a^{-1}\tanh(Ta^{-1}).$$

Therefore, there are no non-zero terms in equation (6.6) and the kink number for the de Sitter metric is zero.

(6.5) Light Cone Behaviour.

The kink properties of this solution can be more readily seen by looking at the light cone behaviour of the metric. The light cone behaviour is demonstrated by considering the equation

$$g_{\mu\pi}(dx^{\mu}/ds)(dx^{\pi}/ds) = 0 \qquad (6.9)$$

This equation for the de Sitter kink of equation (6.2) is

$$-(1- 2K^2r^2)(dt/ds)^2 - 4Kr(1 - K^2r^2)^{1/2}dt/dsdr/ds$$
$$+ (1 - 2K^2r^2)(dr/ds)^2 + r^2(d\theta/ds)^2 + r^2\sin^2\theta(d\Phi/ds)^2 = 0$$

For convenience, the plane $\theta = \Phi =$ constant is considered. Along the line,

$$r = 0, \alpha = \pi,$$

this equation reduces to

$$-(t - t_0)^2 + (r - r_o)^2 = 0$$

or, equivalently,

$$r - r_0 = \underline{+}(t - t_0).$$

The light cones are therefore bounded by lines of slope ± 1 so that their axes point up the t axis. Along the line

$$r = (2K^2)^{-1/2} , \ \alpha = 3\pi/4$$

$$(r - r_0)(t - t_0) = 0$$

or, equivalently,

$$r = r_0 \ , \ t = t_0$$

The light cones are therefore bounded by lines parallel to the r and t axes and the axes of the cones are at an angle $\pi/4$ to either coordinate axis. Along the lines

$$r = K^{-1}, \ \alpha = \pi/2$$

$$(t - t_0)^2 - (r - r_0)^2 = 0.$$

The cones are therefore again bounded by lines of slope ± 1 but now the axis is parallel to the r axis.

The complete light cone behaviour in the planes $\theta = 0$ and $\Phi = 0$ is shown below in Fig. (6.1).

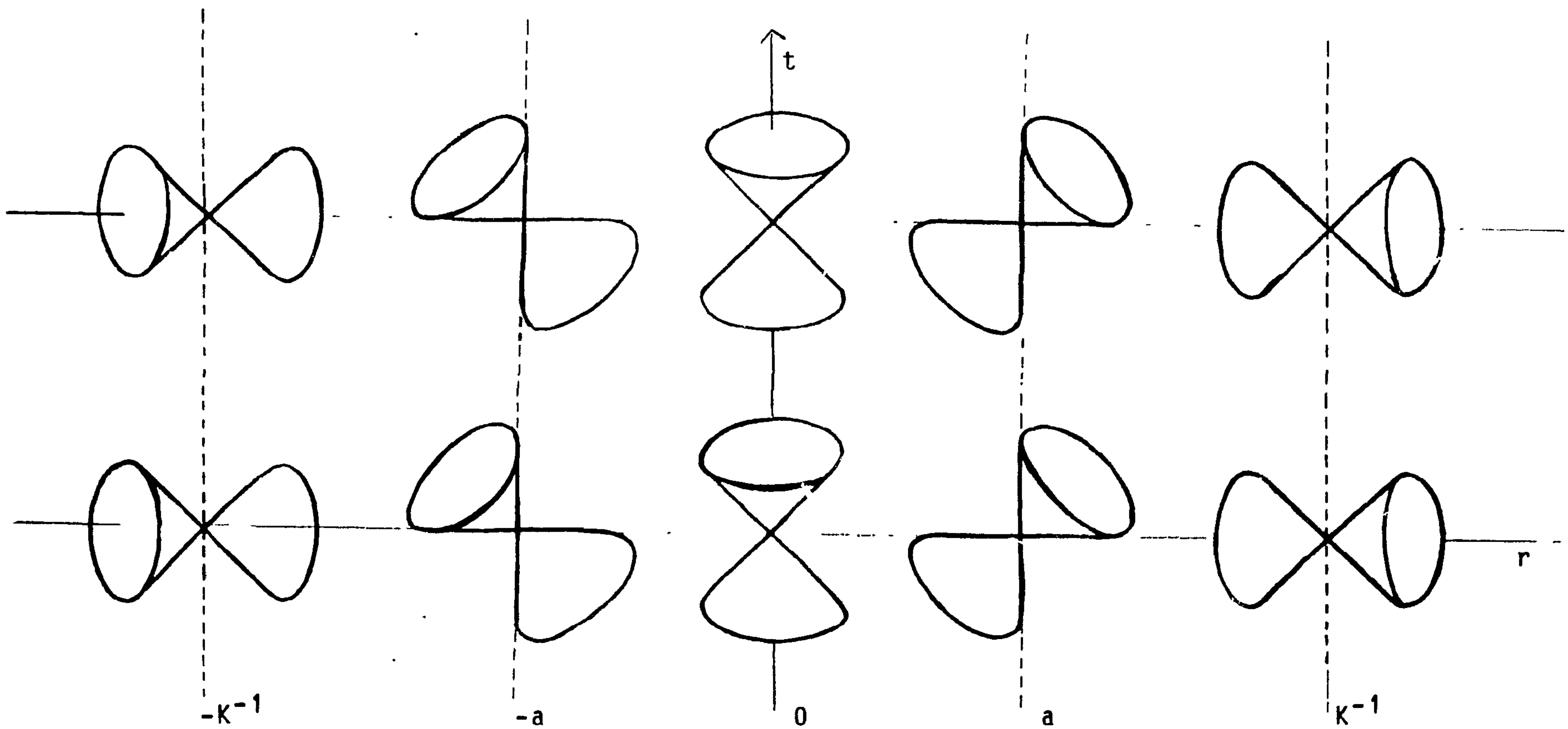

Fig. (6.1). Light cone picture for the de Sitter kink spacetime.

For the usual form of the de Sitter solution, as given in equation (6.4), the equation (6.9) in the $\Theta = \Phi = 0$ plane reduces to

$$-(T - T_0)^2 + a^2\cosh^2(a^{-1}T)(X - X_0)^2 = 0.$$

Hence everywhere in this plane, the cones are bounded by lines whose slopes are

$$\pm a\cosh^2(a^{-1}T).$$

These cones therefore have a changing vertex angle as T varies but do not alter their orientation with respect to the coordinate axes for different coordinate values in the T,X plane. The light cone picture for the usual de Sitter solution is therefore as shown below in Fig. (6.2).

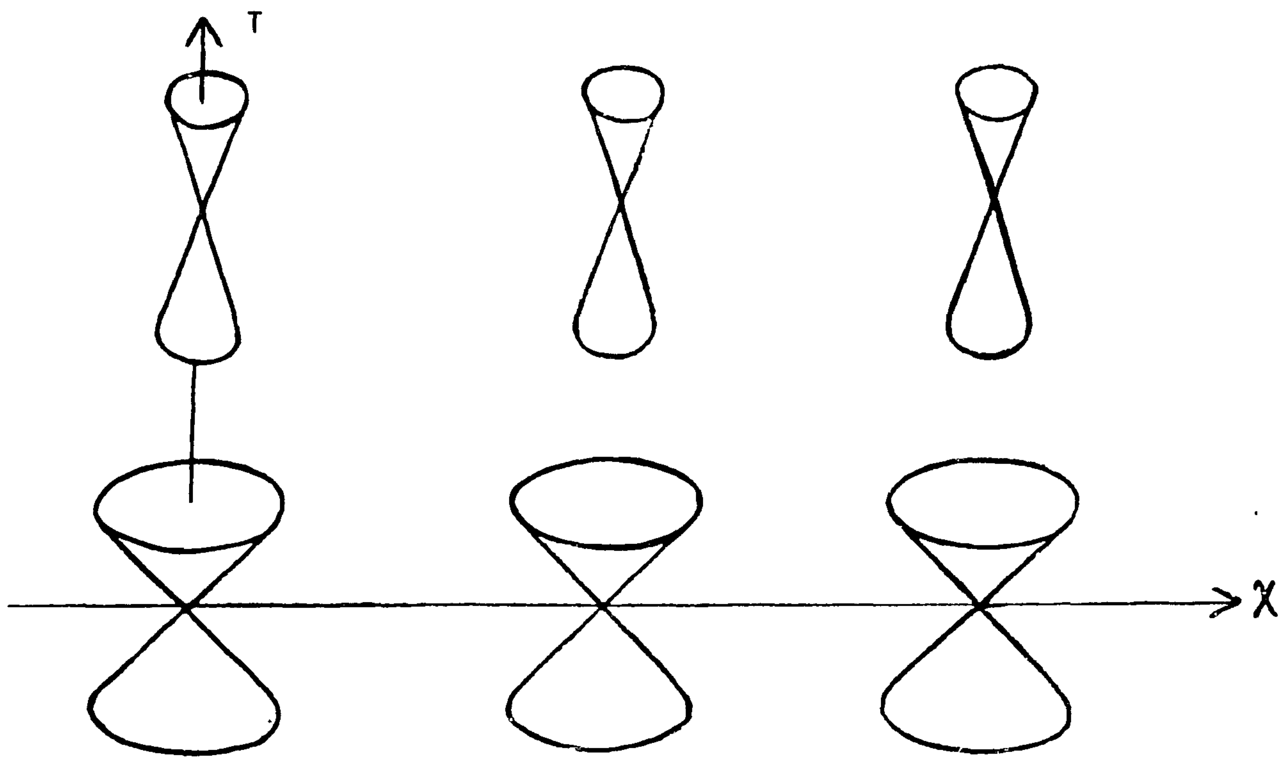

Fig. (6.2). Light cone picture for the usual de Sitter spacetime.

(6.6) Extension of the Manifold

The kink form of the de Sitter metric given in equation (6.3) by

$$ds^2 = -(1 - 2K^2r^2)dt^2 - 4Kr(1 - K^2r^2)^{1/2}dtdr$$
$$+ (1 - 2K^2r^2)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\Phi^2$$

assumes the same value at $r = \pm K^{-1}$, therefore these points may be identified. Under a change of coordinates

$$r = K^{-1}\sin(2^{-1}\beta),$$

valid for $-K^{-1} < r < K^{-1}$ and $-\pi < \beta < \pi$, the metric becomes

$$ds^2 = -\cos\beta dt^2 - K^{-1}\sin\beta\cos(2^{-1}\beta)dtd\beta$$
$$+ 4^{-1}\cos\beta\cos^2(2^{-1}\beta)d\beta^2$$
$$+ K^{-2}\sin^2(2^{-1}\beta)d\theta^2 + K^{-2}\sin^2(2^{-1}\beta)\sin^2\theta d\Phi^2.$$

In these new coordinates it is still not possible to globally introduce a new time variable t' that will remove the $g_{t\beta}$ term because the transformation

$$t' = t + f(\beta)$$

leads to

$$g_{t'\beta} = g_{t\beta} - g_{tt}\, f(\beta)_{,\beta}$$

so that

$$f(\beta)_{,\beta} = (2K)^{-1}\tan\beta\,\cos(2^{-1}\beta)$$

which is undefined at $\beta = \pi/2$.

It is possible to assume that the manifold, over which the de Sitter kink given above is defined, is not the whole of spacetime. Equivalently, the coordinate system can be regarded as unnecessarily restricting the solution because the coordinate patch ends at $r = \pm K$, $\beta = \pm\pi$. The larger manifold covering the whole of spacetime can be constructed by attaching other coordinate patches. This procedure follows that suggested by Finkelstein and McCollum (1975) and describes what they call an "onion". Construction of such an "onion" can be achieved here by a change of coordinates

$$r = K^{-1}\sin(2^{-1}n\beta)$$

valid for

$$-K^{-1} \leqslant r \leqslant K^{-1} \text{ and } -\pi \leqslant \beta \leqslant \pi$$

Under this coordinate change the metric becomes

$$
\begin{aligned}
ds^2 = &-\cos(n\beta)\ dt^2 - nK^{-1}\sin(n\beta)\cos(2^{-1}n\beta)\ dt\ d\beta \\
&+ n^2(2K)^{-2}\cos(n\beta)\cos^2(2^{-1}n\beta)\ d\beta^2 + \\
&K^{-2}\sin^2(2^{-1}n\beta)\ d\theta^2 + K^{-2}\sin^2(2^{-1}n\beta)\sin^2\theta\ d\Phi^2.
\end{aligned}
$$

The metric is still defined on the same manifold because the points $\beta = \pm\pi$ where the metric takes the same value can still be identified. The above metric now represents an n-kink metric rather than a one-kink metric. This can be shown by calculating the kink number. This is achieved by considering again equation (6.7), which is

$$
N = (2\pi^2)^{-1} \int_{\alpha_2}^{\alpha_1} \int_0^{\pi} \int_0^{2\pi} \sin^2\alpha\ \alpha_{,r}\ \sin\Phi\ dr\ d\Phi\ d\theta.
$$

Under the coordinate transformation

$$
r = K^{-1}\sin(2^{-1}n\beta)
$$

the integral becomes

$$N = (2\pi^2)^{-1}[2^{-1}n \int_{-\pi}^{\pi} \sin^2(2^{-1}n\beta)\ d\beta] \int_0^{\pi} \sin\Phi\ d\Phi \int_0^{2\pi} d\theta$$

$$= 2^{-1}n\ (2\pi^2)^{-1}\{2^{-1} \int_{-\pi}^{\pi} [1 - \cos(n\beta)]\ d\beta\}(2\pi)(2)$$

$$= 2^{-1}n\ (2\pi^2)^{-1}\ 2^{-1}(2\pi)(2\pi)(2)$$

$$= n.$$

However, the transformation that achieved this change is not one-to-one thoughout the range $-K^{-1} \leqslant r \leqslant K^{-1}$. The transformation extends the manifold from the one original coordinate patch. Within each patch, the metric is locally tranformable to the usual de Sitter metric. This transformation is clearly not globally possible.

CHAPTER SEVEN

THE SCHWARZSCHILD KINK SOLUTION

Introduction

The form of metric and other properties of this kink solution are discussed (7.1) and the transformation to the familiar Schwarschild form is demonstrated via a singular transformation (7.2). The Killing vectors for this spacetime are found (7.3). The kink number is calculated for this metric and shown to be equal to one (7.4), and the feature common to all kink metrics, the tipping light cone behaviour, is illustrated for this solution (7.5).

(7.1) The Form of the Metric and Solution Properties.

A solution of the Einstein field equations for the metric

$$ds^2 = -\cos 2\alpha dt^2 - 2\sin 2\alpha dtdr + \cos 2\alpha dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\Phi^2 \qquad (7.1)$$

was shown in Chapter 4 to be

$$\sin^2\alpha = Ar^{-1}$$

where A is a constant. If A is chosen to be positive then the solution is valid for $A \leqslant r < \infty$, and

$$\sin\alpha = \pm(Ar^{-1})^{1/2}.$$

The metric can therefore be written as

$$ds^2 = -(1 - 2Ar^{-1})dt^2 - 4A^{1/2}r^{-1/2}(1 - Ar^{-1})^{1/2}\, dtdr + (1 - 2Ar^{-1})dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\Phi^2 \; . \qquad (7.2)$$

It is clear from equation (7.2) that, for $A \leqslant r < 2A$, r is the timelike coordinate and, for $r > 2A$, t is the timelike coordinate. It can also be seen that, for $A \leqslant r < \infty$, the metric is non-singular everywhere.

The coordinate r may be allowed to be negative provided that A is now also chosen to be negative. That is, a solution is

$$\sin^2\alpha = ar^{-1}$$

where $a < 0$. It is valid for $-\infty < r \leqslant a$ and

$$\sin\alpha = \pm(ar^{-1})^{1/2}.$$

The equation of state for this solution is

$$p = -\mu$$

but equation (4.8) shows that

$$p = 0.$$

Clearly for this particlular solution the choice of a velocity vector has been made redundant.

Substituting for $\sin\alpha$ into equation (4.1) shows that the scalar curvature for this solution is zero. This solution is therefore a vacuum solution which may be called the Schwarzschild kink because it can be transformed into the usual Schwarzschild solution via a **singular** transformation. This will be demonstrated in the next section. It will also be shown in a future section that the kink number is non-zero.

(7.2) Transformation to the Schwarzschild Metric.

The spherical symmetry of this vacuum solution means that according to Birkhoff's Theorem (Misner, Thorne and Wheeler, 1973) it must locally be transformable to the Schwarzschild metric, which may be written as

$$ds^2 = -(1 - 2Ar^{-1})\, dT^2 + (1 - 2Ar^{-1})^{-1}\, dr^2$$
$$+ r^2\, d\theta^2 + r^2 \sin^2\theta\, d\Phi^2. \qquad (7.3)$$

The transformation that changes the kink metric given by equation (7.2) into the metric given above in equation (7.3) is

$$T = t + f(r)$$

where

$$\begin{aligned} f'(r) &= g_{tr}(g_{tt})^{-1} \\ &= \tan 2\alpha \\ &= 4A^{1/2} r^{-1/2} (1 - Ar^{-1})^{1/2} (1 - 2Ar^{-1})^{-1} \end{aligned}$$

so that the cross term drdt is removed.

The required form of f(r) can be shown to be given by

$$f(r) = 4A^{1/2}(r - A)^{1/2} - 2A\ln\{[(A^{-1}r - 1)^{1/2} + 1][(A^{-1}r - 1)^{1/2} - 1]^{-1}\} .$$

The existence of this transformation justifies the metric of equation (7.2) being called the Schwarzschild kink metric. The transformation is not valid globally since it is singular at $r = 2A$. Rosen (1985) also found a number of spherically symmetric vacuum solutions that could not be transformed to the Schwarzschild solution via nonsingular transformations. The solutions found by Rosen are not kink solutions.

The above solution was first found by Harriott and Williams (1988). This solution and transformation is similar to that noted by Dunn (1990) for the metric

$$ds^2 = \cos 2\alpha dt^2 - 2\sin 2\alpha dtdr - \cos 2\alpha dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\Phi^2 \tag{7.4}$$

where

$$\cos^2\alpha = Ar^{-1}$$

The difference between this metric and the metric given in equation (7.1) is due to a different definition of the ϕ_α. The metric of equation (7.1) arises from defining

$$\phi_t = \cos\alpha$$
$$\phi_r = \sin\alpha$$
$$\phi_\theta = \phi_\Phi = 0$$

where the ϕ_α are the terms in the orthogonal matrix P that diagonalizes the matrix Q in the G = SQ decomposition of the metric as discussed previously. The metric given in equation (7.4) arises from defining

$$\phi_t = \sin\alpha$$
$$\phi_r = \cos\alpha$$
$$\phi_\theta = \phi_\Phi = 0.$$

This difference in the definitions of the ϕ_α means that the angles defined as α in the two metrics given in equations (7.1) and (7.4) differ by $\pi/2$. The angle α relates to the tipping of the light cones. However, both choices for the ϕ_α lead to the solution given in equation (7.2).

(7.3) The Killing Vectors.

Killing's Equations for this metric are

$$\mu_{t,t} + 2A^{3/2}r^{-5/2}(1 - Ar^{-1})\mu_t - Ar^{-2}(1 - 2Ar^{-1})\mu_r = 0$$

$$\mu_{t,r} + \mu_{r,t} - 2Ar^{-2}(1 - 2Ar^{-1})\mu_t$$
$$- 4A^{3/2}r^{-5/2}(1 - Ar^{-1})\mu_r = 0$$

$$\mu_{t,\theta} + \mu_{\theta,t} = 0$$

$$\mu_{t,\Phi} + \mu_{\Phi,t} = 0$$

$$\mu_{r,r} + A^{1/2}r^{-3/2}(1 - 2Ar^{-2} + 2A^2r^{-2})(1 - Ar^{-1})^{-1/2}\mu_t$$
$$+ Ar^{-2}(1 - 2Ar^{-1})\mu_r = 0$$

$$\mu_{r,\theta} + \mu_{\theta,r} - 2r^{-1}\mu_\theta = 0$$

$$\mu_{r,\Phi} + \mu_{\Phi,r} - 2r^{-1}\mu_\Phi = 0$$

$$\mu_{\theta,\theta} - 2A^{1/2}r^{1/2}(1 - Ar^{-1})^{1/2}\mu_t$$
$$+ r(1 - 2Ar^{-1})\sin^2\theta\ \mu_r = 0$$

$$\mu_{\theta,\Phi} + \mu_{\Phi,\theta} - 2\cot\theta\ \mu_\Phi = 0$$

$$\mu_{\Phi,\Phi} - 2r(1 - Ar^{-1})^{1/2}\sin^2\theta\ \mu_t + r(1 - 2Ar^{-1})\sin^2\theta\ \mu_r + \sin\theta\cos\theta\ \mu_\theta = 0.$$

These equations have solutions

$$\underline{\mu}_1 = \delta/\delta t$$

$$\underline{\mu}_2 = \delta/\delta\Phi$$

$$\underline{\mu}_3 = \sin\Phi\delta/\delta\theta + \cot\theta\cos\Phi\delta/\delta\Phi$$

$$\underline{\mu}_4 = \cos\Phi\delta/\delta\theta - \cot\theta\sin\Phi\delta/\delta\Phi.$$

As stated previously $\underline{\mu}_2$, $\underline{\mu}_3$, and $\underline{\mu}_4$ are the generators of the rotation group SO(3).

The length of the Killing vector $\underline{\mu}_1$ is

$$||\underline{\mu}_1|| = -(1 - Ar^{-1})$$

which is spacelike for $A < r < 2A$. It becomes null at $r = 2A$ and is timelike for $r > 2A$. There is therefore no global timelike Killing vector.

(7.4) Kink Number Calculation

The kink number formula as derived in section (2.3) is

$$N = (12\pi^2)^{-1} \int N^0 d^3x \qquad (7.5)$$

where

$$N^0 = \epsilon^{0ijk} \epsilon_{\alpha\beta\tau\delta} X^\alpha X^\beta{}_{|i} X^\tau{}_{|j} X^\delta{}_{|k} \ , \qquad (7.6)$$

and the | denotes differentiation with respect to the S matrix in the G = SQ decomposition of the matrix representing the metric. It was shown in section (6.4) that for any metric of the form given by equation (7.1) this kink number formula reduces to the following integral, given in equation (6.7)

$$N = (2\pi^2)^{-1} \int_{\alpha_2}^{\alpha_1} \int_0^{2\pi} \int_0^{\pi} \sin^2\alpha \ \alpha_{,r} \ \sin\Phi \ dr d\theta d\Phi \qquad (7.7)$$

where α_1 and α_2 are the limiting values of α for the solution. For the Schwarschild kink,

$$\alpha_1(r = A) = \pi/2 \ \alpha_2(r = \infty) = 0$$

The integral given in equation (7.7) therefore reduces to

$$N = (2\pi^2)^{-1} \int_0^{\pi/2} [\ 2^{-1}(1 - \cos 2\alpha)d\alpha](2)(2\pi)$$

$$= 2^{-1}.$$

This result is not unexpected as the angle only turns through $\pi/2$ radians for this range of r.

A complete kink can be constructed by attaching the de Sitter kink solution discussed in chapter 6 for the interior range of r

$$0 \leqslant r \leqslant K^{-1} = A.$$

The complete range of r is now

$$0 \leqslant r < \infty$$

and

$$\sin\alpha = Kr \qquad \text{valid for } 0 \leqslant r \leqslant K^{-1}$$
$$\sin\alpha = +(Ar^{-1})^{1/2} \quad \text{valid for } A = K^{-1} \leqslant r < \infty .$$

The solution is continuous at $r = K^{-1}$ but not differentiable.

It is also possible to allow r to be negative and to construct an extension of the solution as follows

$$\sin\alpha = Kr \qquad \text{valid for} \quad -K^{-1} \leqslant r \leqslant 0$$
$$\sin\alpha = -(ar^{-1})^{1/2} \quad \text{valid for} \quad -\infty < r \leqslant -K^{-1} = a\ .$$

The complete solution, valid for $-\infty < r < \infty$, has two kinks, and is illustrated below in Fig. (7.1).

After transformation to the Schwarzschild metric, whose standard form is

$$ds^2 = -(1 - 2Ar^{-1})dT^2 + (1 - 2Ar^{-1})^{-1}dr^2$$
$$+ r^2d\theta^2 + r^2\sin^2\theta d\Phi^2$$

via the singular transformation described in section (7.1), the kink number is zero. This can be shown as follows. For the standard form of the Schwarzschild metric the S and Q matrices in the decomposition of the matrix representing the metric are easily found to be

$$Q = \mathrm{diag}(-1,\ 1,\ 1,\ 1)$$

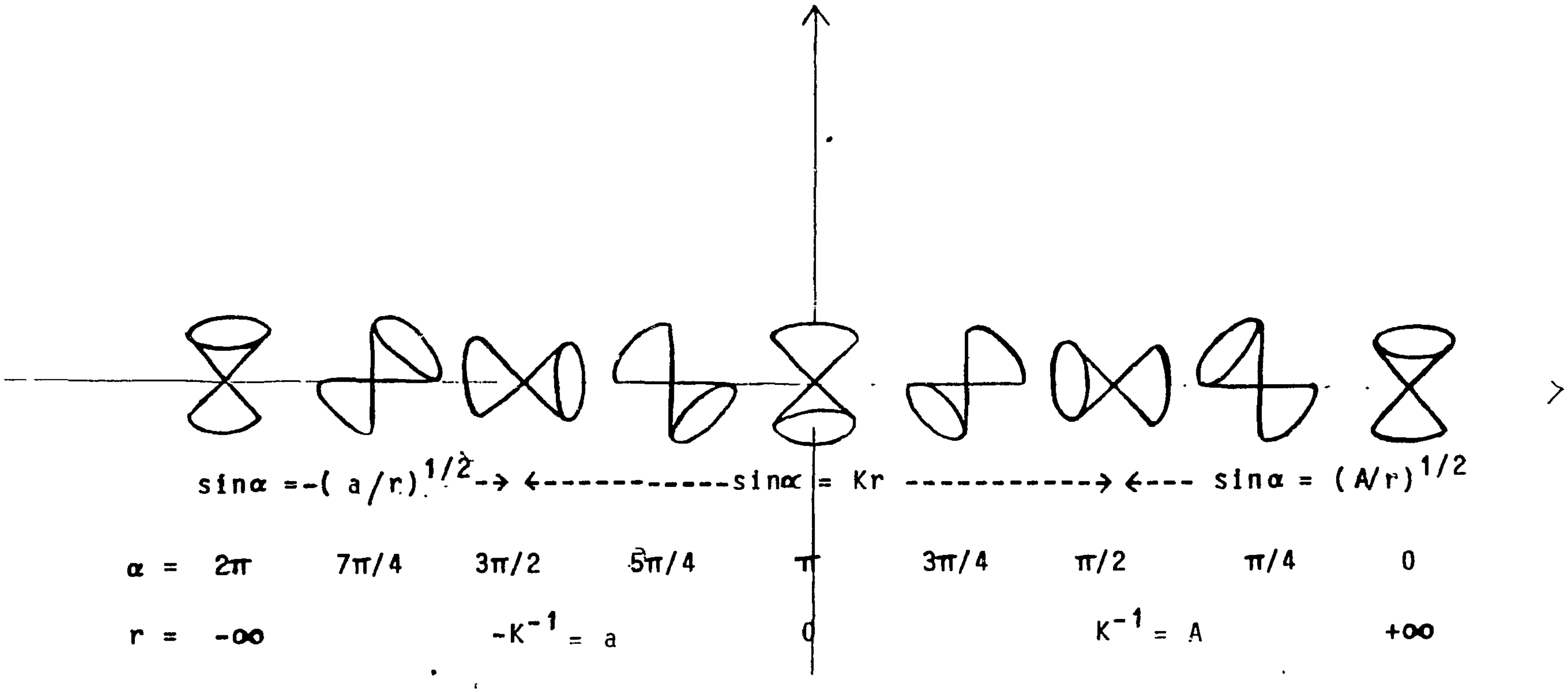

Fig. (7.1). A two-kink solution, with an interior de Sitter type kink and an exterior Schwarzschild type kink.

and

$$S = \begin{pmatrix} (1 - 2Ar^{-1}) & 0 & 0 & 0 \\ 0 & (1 - 2Ar^{-1})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The $X^\alpha = \phi_\alpha$ are therefore clearly constant and so all terms $X^\alpha{}_{,\beta}$ will be zero. The only nonzero Christoffel symbols for the tensor represented by the matrix S are

$$\begin{aligned}
\Gamma^T{}_{Tr} &= Ar^{-2}(1 - 2Ar^{-1})^{-1} \\
\Gamma^r{}_{rr} &= -Ar^{-2}(1 - 2Ar^{-1})^{-1} \\
\Gamma^r{}_{\theta\theta} &= -r(1 - 2Ar^{-1}) \\
\Gamma^r{}_{\Phi\Phi} &= -\sin^2\theta r(1 - 2Ar^{-1}) \\
\Gamma^\theta{}_{\theta r} &= r^{-1} \\
\Gamma^\theta{}_{\Phi\Phi} &= -\sin\theta\cos\theta \\
\Gamma^\Phi{}_{\Phi r} &= r^{-1} \\
\Gamma^\Phi{}_{\Phi\theta} &= \cot\theta
\end{aligned}$$

The covariant derivatives of X^α, which are given by

$$X^\alpha{}_{|\beta} = X^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\tau}X^\tau ,$$

are therefore all zero except for

$$X^T{}_{|r} = \Gamma^T{}_{Tr}X^T = Ar^{-2}(1 - 2Ar^{-1})^{-1}$$

There is thus no non-zero contribution to equation (7.6). The kink number is clearly zero.

(7.5) Light Cone Behaviour

The light cone behaviour is determined by solving the equation.

$$g_{\alpha\beta}(dx^{\alpha}/ds)(dx^{\beta}/ds) = 0 \tag{7.8}$$

For the solution given by equation (7.2), the above equation reduces to

$$-(1 - 2Ar^{-1})(dt/dr)^2 - 4A^{1/2}r^{-1/2}(1 - Ar^{-1})^{1/2}\, dt/dsdr/ds + (1 - 2Ar^{-1})(dr/ds)^2 + r^2(d\theta/ds)^2 + r^2\sin^2\theta(d\Phi/ds)^2 = 0 \,. \tag{7.9}$$

Consider the plane $\theta = \Phi =$ constant. Along the line $r = A$, the above equation (7.9) becomes

$$(t - t_o)^2 = (r - r_o)^2.$$

The light cones have their axes parallel to the $\underline{r}$ coordinate axis.

Along the line $r = 2A$, equation (7.9) becomes

$$(t - t_o)(r - r_o) = 0$$

and the light cones are tipped so that the bounding curves lie parallel to the coordinate axes.

As $r \to \infty$, the light cones continue to tip until their axes become parallel to the t axis at infinity. This behaviour was first noted by Finkelstein and McCollum (1975) and is illustrated below in Fig. (7.2).

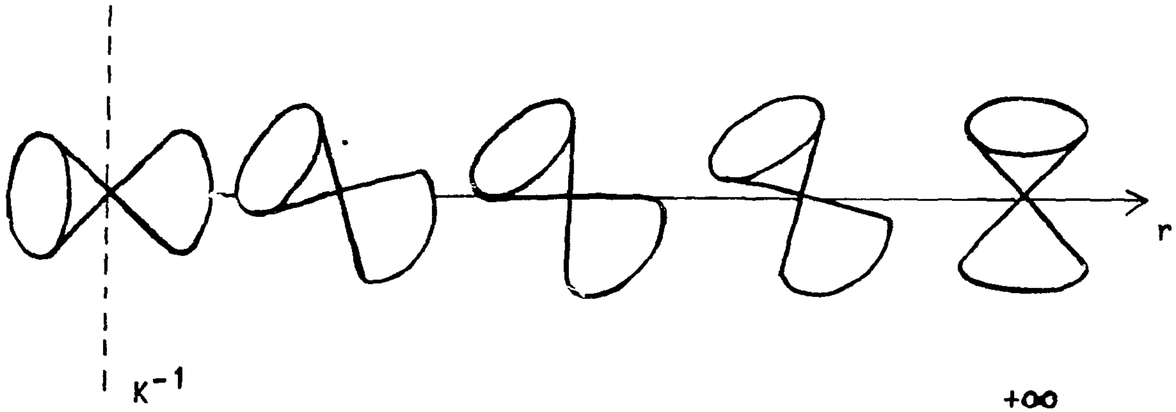

Fig. (7.2). Light cone picture for the Schwarzschild kink solution.

CHAPTER EIGHT

KINK SOLUTIONS IN 1+1 DIMENSIONS

Introduction

A (1+1)-dimensional theory of gravity is discussed (8.1). A kink metric in 1+1 dimensions is constructed and its curvature quantitites calculated (8.2). The energy conservation conditions are also constructed (8.2). De Sitter kink solutions are found, analagous to those previously discussed in chapter 6, (10.3).

(8.1) A (1+1)-Dimensional Theory of Gravity.

Recently there has been interest in 1+1 theories of gravity, primarily for pedagogical reasons. However, interest in these theories also exists because it is usually easier to work in two dimensions rather than four, so if a theory can be shown to duplicate the features of general relativity, at least qualitatively, it will provide a useful investigative tool for testing new ideas. It is in this capacity that such a theory is included here: to study kink solutions in two dimensions.

The Einstein tensor is identically zero in 1+1 dimensions and so to construct a (1+1)-dimensional theory of gravity, a theory to replace general relativity must be proposed. Several such theories have been suggested (Teitelboim, 1983 and 1984; Banks and Susskind, 1984; Jackiw 1984 and 1985; Brown, Henneaux and Teitelboim, 1986; Sanchez, 1986 and 1987; Brown, 1988; Gegenberg, Kelly, Mann and Vincent, 1988) and most recently Mann et al (Mann, 1989; Mann, Shiekh and Tarasov, 1990; Mann and Steele, 1990; Mann, Morsink, Sikkema and Steele, 1990; Kelly and Mann, 1991; Sikkema and Mann, 1991a and b; Mann, 1992) have proposed a (1+1)-dimensional theory of gravity with the following field equation

$$R - \Omega = 8\pi GT \tag{8.1}$$

where Ω is the usual (Einstein) cosmological constant, G is the universal (Newton) constant of gravitation and T is the trace of the stress-energy tensor $T^{\alpha\beta}$,

$$T = g_{\alpha\beta}T^{\alpha\beta}.$$

An additional postulate of the theory is that

$$T^{\alpha\beta}{}_{;\beta} = 0. \tag{8.2}$$

This is to ensure energy conservation. In general relativity the latter follows directly from the field equations, but in (1+1)-dimensional theory it is an added requirement. This particular theory has field equations that are simple to solve but rich in structure. Einstein's principal notion of equating curvature to matter is retained in this theory whose field equations are formally similar to the Einstein equations. It has been shown (Sikkema and Mann, 1991b) that this theory qualitatively reproduces many of the features of general relativity and also reduces to Newtonian gravity in the weak field and low velocity limit.

The stress-energy tensor for a perfect fluid is

$$T^{\alpha}{}_{\beta} = (\mu + p)u^{\alpha}u_{\beta} + p\delta^{\alpha}{}_{\beta}$$

where, as usual, μ is the total energy density and p is the isotropic pressure. The trace of $T_{\alpha\beta}$ is therefore

$$T = p - \mu \tag{8.3}$$

Given any metric, a relationship between μ and p will therefore be determined by equation (8.1), and actual solutions of this 1+1 theory will be those metrics that, in addition, allow p and μ to satisfy the energy conservation equation (8.2).

It is easy to show, following the general scheme of Shastri, Williams and Zvengrowski (1980), that these 1+1 dimensional space-times do admit kink-like structures. By analogy with higher dimensions, the matrix G, representing the metric, can be written as the product G = SQ = QS. The matrix S is symmetric and positiye definite and Q is symmetric and orthogonal. It must therefore be possible to write

$$G = SQ$$

$$= \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} -\cos\beta & -\sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$$

for some functions A, B, C and angle β. By analogy with higher dimensions, the matrix Q can also be expressed as

$$P \operatorname{diag}(-1, 1) P^T$$

where P is the matrix

$$\begin{pmatrix} \phi_t & -\phi_x \\ \phi_x & \phi_t \end{pmatrix}$$

and

$$\phi_t = \cos\alpha$$
$$\phi_x = \sin\alpha.$$

Comparison of the two expressions for Q shows that

$$\beta = 2\alpha.$$

The commutivity of S and Q implies

$$-B \cos\beta - C \sin\beta = -A \sin\beta + B \cos\beta,$$

or equivalently

$$B = 2^{-1}(A - C)\tan\beta.$$

The matrix S is homotopically similar to the identity matrix and so a transformation exists that transforms S to a diagonal matrix whose non-zero (positive) elements may be written as e^{τ}. Equivalently, because S is positive definite, it must have two real positive eigenvalues. It can therefore be diagonalized by its matrix of eigenvectors. The matrix representing G can therefore be written as

$$e^{\tau} \begin{pmatrix} -\cos\beta & -\sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$$

and the metric is

$$ds^2 = -e^{-\tau}\cos\beta\, dt^2 - 2e^{-\tau}\sin\beta\, dtdx + e^{-\tau}\cos\beta\, dx^2 . \tag{8.4}$$

If $\beta = 0$, this metric is conformally flat and reduces to the case studied by Jackiw (1985). When $\tau = 0$, the metric reduces to the 1+1 version of the simple kink metric discussed previously in Chapters 4, 6, and 7.

The kink number formula discussed previously in section (2.4) for metrics in 3+1 dimensions is

$$N = (12\pi^2)^{-1} \int \epsilon^{0ijk} \epsilon_{\alpha\beta\tau\delta} \, X^\alpha X^\beta{}_{|i} X^\tau{}_{|j} X^\delta{}_{|k} \, d^3x \, .$$

In 1+1 dimensions it will reduce to

$$N = (2\pi)^{-1} \int \delta^{0i} \delta_{\alpha\beta} \, X^\alpha \, X^\beta{}_{|i} \, dx \, .$$

The | denotes covariant differentiation with respect to the positive definite metric represented by the S matrix. In the (1+1)-dimensional case, when the S matrix is diag(1,1) and the X^α are chosen equal to the ϕ^α, the formula further simplifies to

$$N = \pi^{-1} \int_{-\infty}^{\infty} (\phi_t \, \phi_{x,x} - \phi_x \phi_{t,x}) \, dx \, . \qquad (8.5)$$

(8.2) Curvature and Hydrodynamic Quantities.

If τ is assumed to be zero in the metric given by equation (8.4), then the metric becomes

$$ds^2 = -\cos\beta\, dt^2 - 2\sin\beta\, dtdx + \cos\beta\, dx^2. \qquad (8.6)$$

The non-zero Christoffel symbols are

$$\Gamma^t{}_{tt} = 2^{-1}\{\sin\beta\cos\beta\ \beta_{,t} + \sin^2\beta\ \beta_{,x}\} = -\Gamma^x{}_{tx}$$

$$\Gamma^t{}_{tx} = 2^{-1}\{\sin^2\beta\ \beta_{,t} - \sin\beta\cos\beta\ \beta_{,x}\} = -\Gamma^x{}_{xx}$$

$$\Gamma^x{}_{tt} = 2^{-1}\{-(1 + \cos^2\beta)\ \beta_{,t} - \sin\beta\cos\beta\ \beta_{,x}\}$$

$$\Gamma^t{}_{xx} = 2^{-1}\{\sin\beta\cos\beta\ \beta_{,t} + (1 + \cos^2\beta)\ \beta_{,x}\}.$$

There is only one independent Ricci tensor component (Weinberg, 1972),

$$R_{tt} = 2^{-1}\cos\beta\{(\beta_{,x})^2 - (\beta_{,t})^2 + \sin\beta\ \beta_{,tt}$$
$$- \sin\beta\ \beta_{,xx} + 2\sin\beta\ \beta_{,t}\beta_{,x} - 2\cos\beta\ \beta_{,tx}\},$$

and

$$R_{xx} = -\ R_{tt}$$

$$R_{tx} = \tan\beta \, R_{tt} \, .$$

The scalar curvature is

$$R = \sin\beta[\beta_{,xx} - \beta_{,tt}] + 2\cos\beta \, \beta_{,tx} + \cos\beta[(\beta_{,x})^2 - (\beta_{,t})^2] - 2\sin\beta \, \beta_{,t} \, \beta_{,x}$$

which can be written more compactly as

$$R = (\cos\beta)_{,tt} - (\cos\beta)_{,xx} + 2(\sin\beta)_{,tx} \, . \qquad (8.7)$$

As in higher dimensions, the velocity vector is chosen to be

$$u^t = \cos\alpha$$
$$u^x = \sin\alpha.$$

It can now be easily shown that, for the metric given by equation (10.6), the energy conservation equations, $T^{\alpha\beta}{}_{;\beta} = 0$, reduce to

$$T^{\alpha}{}_{\beta,\beta} = 0 \qquad (8.8)$$

Substituting for the velocity components, these conservation conditions given by (8.8) can be expressed as

$$\alpha_{,t}(\mu + p) + p_{,x} + \sin\alpha \cos\alpha\ (\mu - p)_{,t} + \sin^2\alpha\ (\mu - p)_{,x} = 0 \tag{8.9}$$

$$\alpha_{,x}(\mu + p) + p_{,t} + \cos^2\alpha\ (\mu - p)_{,t} + \sin\alpha \cos\alpha\ (\mu - p)_{,x} = 0. \tag{8.10}$$

(8.3) The De Sitter Kink Solution.

Using equation (8.7) to substitute for the scalar curvature, the field equation (8.1) can be written as

$$(\cos\beta)_{,tt} - (\cos\beta)_{,xx} + 2(\sin\beta)_{,tx} - \Omega = 8\pi G(\mu - p) \ .$$

If $p = -\mu$ is chosen as the equation of state, the energy conservation equations (8.9) and (8.10) reduce to

$$\mu_{,t} = \mu_{,x} = 0.$$

Therefore any solution with an equation of state $p = -\mu$ must have p and μ constant. The field equation is now

$$(\cos\beta)_{,tt} - (\cos\beta)_{,xx} + 2(\sin\beta)_{,tx} - \Omega = -16\pi G\mu < 0. \tag{8.11}$$

The scalar curvature must therefore be constant. That is,

$$R = (\cos\beta)_{,tt} - (\cos\beta)_{,xx} + 2(\sin\beta)_{,tx} = \text{constant}.$$

This last equation is satisfied by any of the following

$$\sin\alpha = Kx \qquad \text{valid for } -K^{-1} \leqslant x < K^{-1}$$
$$\sin\alpha = Kt \qquad \text{valid for } -K^{-1} \leqslant t \leqslant K^{-1}$$

$$\cos\alpha = Kx \qquad \text{valid for } -K^{-1} \leq x \leq K^{-1}$$

$$\cos\alpha = Kt \qquad \text{valid for } -K^{-1} \leq t \leq K^{-1},$$

where K is a constant.

If $\sin\alpha = Kx$ or $\cos\alpha = Kt$, then $R = 4K^2$, and the field equation (8.11) becomes

$$4K^2 - \Omega = -16\pi G\mu.$$

This solution therefore requires a positive cosmological constant

$$\Omega = 4K^2 + 16\pi G\mu.$$

If $\cos\alpha = Kx$ or $\sin\alpha = Kt$, then $R = -4K^2$, and the field equation (8.11) reduces to

$$-4K^2 - \Omega = - 16\pi G\mu$$

The cosmological constant may be set to zero if the density is chosen to be

$$\mu = K^2(4\pi G)^{-1}.$$

Alternatively, if the solution is regarded as an empty space ($\mu = 0$) solution, the cosmological constant must be negative

$$\Omega = -\ 4K^2.$$

Comparing these solutions to the de Sitter kink solution in higher dimensions discussed in Chapter 6, it is clear the first is the (1+1)-dimensional analogue of the de Sitter kink and the second can be considered as the (1+1)-dimensional analogue of the negative constant curvature space-time, the anti-de Sitter kink, discussed previously by Williams and Zvengrowski (1990).

For the de Sitter kink solution, $\sin\alpha = Kx$, the metric is

$$ds^2 = -(1 - 2K^2x^2)dt^2 - 4Kx(1 - K^2x^2)^{1/2}dtdx + (1 - 2K^2x^2)dx^2 \ , \qquad (8.12)$$

valid for $-K^{-1} \leqslant x \leqslant K^{-1}$ and $0 \leqslant t < \infty$. For this de Sitter kink metric the kink number, given by equation (8.5), can be shown to be equal to one.

$$\phi_t = \cos\alpha$$
$$\phi_x = \sin\alpha$$

and so

$$\phi_t \phi_{x,x} - \phi_x \phi_{t,x} = (\cos^2\alpha + \sin^2\alpha)\, \alpha_{,x}$$
$$= \alpha_{,x}.$$

Substituting this last result into the kink number formula (8.5) gives

$$N = \pi^{-1} \int_{\alpha(-\infty)}^{\alpha(\infty)} d\alpha$$

$$= \pi^{-1} \{ \pi \}$$

$$= 1.$$

The kink metric given by equation (8.12) can be transformed to the usual form of the de Sitter metric

$$ds^2 = -(1 - x^2 a^{-2}) dT^2 + (1 - x^2 a^{-2})^{-1} dx^2 ,$$

where a is a constant, via a **singular** transformation, as was the case in higher dimensions. The manifold can be extended, in an analogous manner to the procedure described for 3+1 dimensions, to form an n-kink metric by writing

$$x = K^{-1}\sin(n\theta/2)$$

where $-\pi \leq \theta \leq \pi$. The metric now becomes

$$ds^2 = -\cos(n\theta)\ dt^2 - nK^{-1}\sin(n\theta)\cos(n\theta/2)\ dtd\theta + 4^{-1}n^2K^{-2}\cos(n\theta)\cos^2(n\theta/2)\ d\theta^2.$$

This is analogous to the "onion" extension suggested by Finkelstein and McCollum (1975) in 3+1 dimensions.

CHAPTER NINE

FRIEDMANN-LEMAITRE-ROBERTSON-WALKER KINK SOLUTIONS

Introduction

A solution in 1+1 dimensions that is locally transformable to one of the (1+1)-dimensional analogues of the Freidmann-LeMaitre-Robertson-Walker (FLRW) solutions is demonstrated (9.1). The actual transformation of this 1+1 solution to one of the Friedmann-LeMaitre-Robertson-Walker metrics is presented and the properties of this solution are discussed (9.2). Analagous solutions in higher dimensional situations are found (9.3) and the transformation to FLRW form for the 3+1 case is discussed (9.4). The kink number for the 3+1 kink metric and the corresponding FLRW metric are found (9.5). The Riemann tensor components (9.6) and the Killing Vectors (9.7) are also found for this 3+1 case.

(9.1) A Friedmann-LeMaitre-Robertson-Walker and Minkowski Kink Solution in 1+1 Dimensions.

A solution found previously in 3+1 dimensions was given by equation (4.16). The solution is

$$\tan\alpha = rt^{-1}.$$

This (3+1)-dimensional solution will be discussed in section (9.3). By analogy with this previously found solution, a possible solution of the 1+1 theory of gravity discussed in chapter 8 is

$$\tan\alpha = xt^{-1}. \tag{9.1}$$

The angle α is only well defined for all x when t = 0. The spacetime manifold is therefore assumed to be the upper half plane, t > 0. The scalar curvature can be found from equation (8.7)

$$R = (\cos\beta)_{,tt} - (\cos\beta)_{,xx} + 2(\sin\beta)_{,tx} .$$

When $\tan\alpha = xt^{-1}$, the scalar curvature can easily be shown to be zero. The field equation, using equations (8.1) and (8.3) can be written as

$$R - \Omega = 8\pi G(p - \mu). \tag{9.2}$$

Equation (9.2) shows that this possible solution must have

$$p = \mu$$

if Ω is chosen to be zero.

The energy conservation conditions for a perfect fluid are given by (8.9) and (8.10). When $p = \mu$ they reduce to

$$p_{,x} - 2p\ \alpha_{,t} = 0$$
$$p_{,t} + 2p\ \alpha_{,x} = 0.$$

If $\tan\alpha = xt^{-1}$, it is easy to show that

$$\alpha_{,t} = -r(t^2 + r^2)^{-1}$$
$$\alpha_{,x} = \ t(t^2 + x^2)^{-1}$$

and so the energy conservation equations can be written as

$$p^{-1}\ p_{,t} = -t(t^2 + x^2)^{-1}$$
$$p^{-1}\ p_{,x} = -x(t^2 + x^2)^{-1}\ .$$

These last two equations clearly have a solution

$$p = (t^2 + x^2)^{-1} .$$

It is also clear that $p = \mu = 0$ will also satisfy both the field equation and the energy conservation equations. The solution $\tan\alpha = xt^{-1}$ may therefore be regarded in two ways: as a flat empty space ($p = \mu = 0$) solution or as a flat stiff matter ($p = \mu = (t^2 + x^2)^{-1}$) solution.

The metric for this solution is

$$ds^2 = -(t^2 - x^2)(t^2 + x^2)^{-1}\, dt^2 - 4tx(t^2 + x^2)^{-1}\, dt\, dx + (t^2 - x^2)(t^2 + x^2)^{-1}\, dx^2, \tag{9.3}$$

valid for $-\infty < x < \infty$ and $0 < t < \infty$. This metric describes four distinct regions as shown below in Fig. (9.1). These regions are bounded by the lines $x = \pm t$ along which $\delta/\delta t$ is a null vector. In the regions labelled II and IV, where

$$|t| > |x|,$$

t is the timelike coordinate. In regions I and III where

$$|t| < |x|,$$

x is the timelike coordinate.

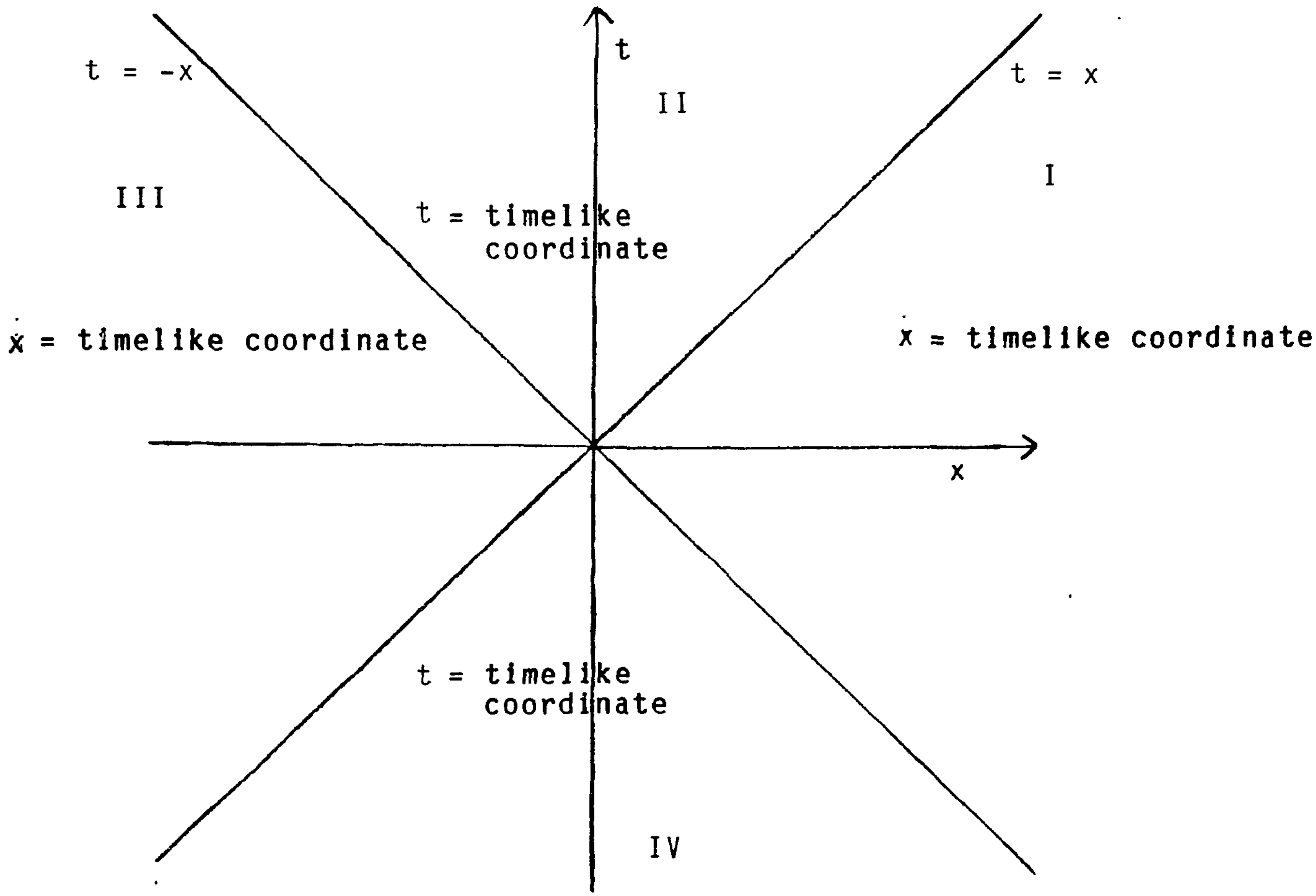

Fig. (9.1). The Friedmann-LeMaitre-Robertson-Walker kink spacetime.

The angle α determines the orientation of the light cones. As x varies from $-\infty$ to $+\infty$, for any fixed value of t, the angle α of the light cones changes by π, and so there is a complete kink present. This can be confirmed by calculation of the kink number. Integrating along any hypersurface t = constant, the kink number formula given by equation (8.5) shows N = 1 as expected. This is proved as follows: For this solution

$$\begin{aligned}\phi_t &= \cos\alpha \\ &= t(t^2 + x^2)^{-1/2}\end{aligned}$$

$$\begin{aligned}\phi_x &= \sin\alpha \\ &= x(t^2 + x^2)^{-1/2}\end{aligned}$$

and hence

$$\phi_t\phi_{x,x} - \phi_x\phi_{t,x} = t(t^2 + x^2)^{-1} .$$

The kink number formula is

$$N = \pi^{-1} \int_{-\infty}^{\infty} (\phi_t\phi_{x,x} - \phi_x\phi_{t,x})\, dx.$$

For this solution it can therefore be written as

$$N = \pi^{-1} \int_{-\infty}^{\infty} t(t^2 + x^2)^{-1} \, dx$$

$$= \pi^{-1} \tan^{-1} (xt^{-1}) \Big|_{-\infty}^{\infty}$$

$$= 1.$$

The light cone behaviour of this solution is illustrated in Fig. (9.2). It is interesting to note that, unlike previous solutions, the light cone behaviour changes with the value of t. The range of x needed to tip through any particular angle increases as t increases. Along the lines $t = ax$, for any constant a, the light cones do not alter their orientation with respect to the axes. However, the orientation changes as the constant a changes. This can be seen by substituting $t = ax$ into equation (9.3), which is

$$ds^2 = -(t^2 - x^2)(x^2 + t^2)^{-1}dt^2 - 4xt(x^2 + t^2)^{-1}dxdt + (t^2 - x^2)(x^2 + t^2)^{-1}dx^2 ,$$

so that the metric becomes

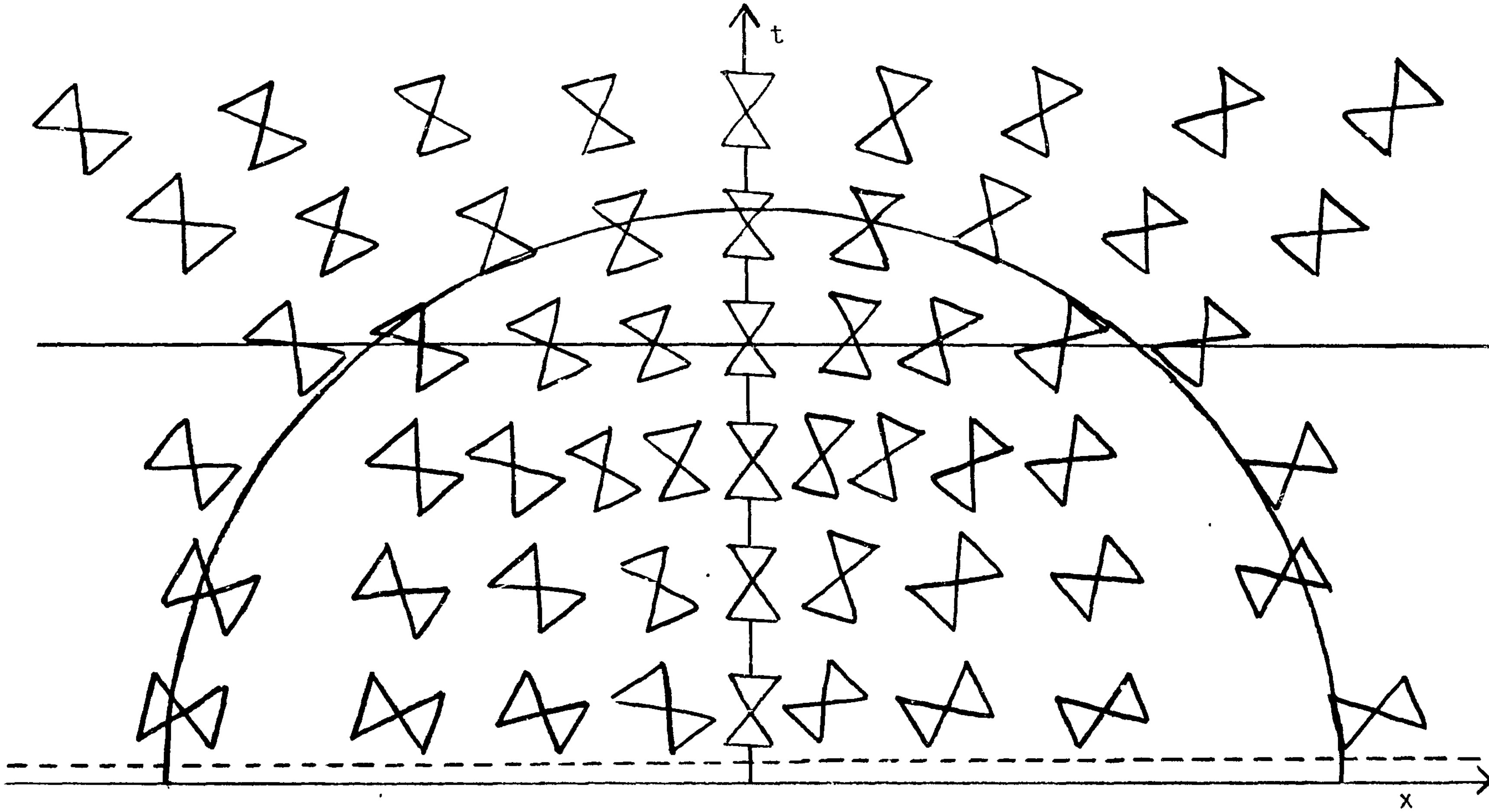

Fig. (9.2) The light cone picture for the Friedman-LeMaitre-Robertson-Walker 1+1 kink solution.

$$ds^2 = -(a^2 - 1)(a^2 + 1)^{-1}dt^2 - 4a(a^2 + 1)^{-1}dxdt$$
$$+ (a^2 - 1)(a^2 + 1)^{-1}dx^2.$$

For $a = 0$ the metric reduces to

$$ds^2 = dt^2 - dx^2.$$

For $a = \pm 1$ it reduces to

$$ds^2 = \pm 2dxdt,$$

and as $a \to \pm\infty$ it reduces to

$$ds^2 = -dt^2 + dx^2.$$

It will be shown in the next section that the stiff matter solution is locally transformable to a (1+1)-dimensional analogue of one of the Friedmann-LeMaitre-Robertson-Walker (FLRW) solutions. These 1+1 FLRW solutions have been discussed by Mann and Sikkema (1991a). The empty flat space solution must be locally transformable to the Minkowski metric. Globally, however, this will not be possible.

(9.2) Transformation of the 1+1 Solution to the FLRW and Minkowski Forms.

Consider first the stiff matter solution. Under the coordinate transformation

$$T^2 = t^2 + x^2$$
$$X = \tan^{-1}(xt^{-1})$$

the kink metric given by equation (9.3), defined on the upper half plane, $t > 0$, becomes

$$ds^2 = -dT^2 + T^2 dX^2, \qquad (9.4)$$

valid for $0 < T < \infty$, $-\pi/2 \leq X \leq \pi/2$. If the boundary points $X = \pm\pi/2$ are identified, the spacetime manifold of this metric given by equation (9.4) is the whole plane except for the origin. This metric is easily shown to have no kink, when integration is performed along the hypersurfaces T = constant: By inspection it is clear that for this metric (9.4) the matrices S and Q, in the decomposition of the matrix G representing the metric, are

$$S = \begin{pmatrix} 1 & 0 \\ 0 & T^2 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly therefore,

$$\phi_T = 1$$
$$\phi_X = 0$$

and the kink number formula, equation (8.5), shows the kink number is zero. This discrepancy in the kink number is explained by the fact that the manifold over which this last metric is defined has been obtained from the original kink manifold by some "cutting and pasting" which homotopy theory does not allow.

The original kink metric, given by equation (9.3) is defined on the upper half plane $R^1 \times R^1$, or, if the boundary points $x = \pm\,\infty$, where the metric takes the same value, are identified, on the cylinder $R^1 \times S^1$. The manifold is foliated by the hypersurfaces $t = k$, for any positive constant k, each with the topology of R^1, or of S^1 if the boundary points are identified. Along any such slice it is clear from Fig. (9.2) that the angle α of the light cones turns through π radians and, integrating with respect to x, the kink number N = 1. The S^1 hypersurfaces are **not everywhere spacelike.**

The metric given by equation (9.4) is defined on a **different** $R^1 \times S^1$ manifold, obtained by identifying $X = \pm\pi/2$ for each T, and is foliated by the hypersufaces T = k, for any constant k. Integrating now with respect to X gives a kink number N = 0. Again, inspection of Fig. (9.2) shows that around the circles

$$T^2 = t^2 + x^2 = \text{constant},$$

the orientation of the light cones does not alter. The S^1 hypersurfaces are **everywhere spacelike** and such spacetimes do not have kinks (Finkelstein, 1978).

It is important to note that changing the family of S^1 hypersurfaces will only change the kink number **if some cutting** of the original manifold occurs. A change in the hypersurface decomposition without a change in the manifold will not affect the kink number.

It is also important that the hypersurfaces over which the integral is taken to find the kink number are those labelled by t = constant. This is because the manifold was specifically chosen with $t \in R^1$ and $x \in S^1$ so that the integration is over a hypersurface that is clearly compactifiable. This will guarantee that the degree of mapping is well defined. It is not clear that the hypersurfaces labelled by $T = (t^2 + x^2)^{1/2}$ = constant will be compactifiable.

To summarize, the original kink manifold was chosen to be $R^1 \times S^1$ with $t \in R^1$ and $x \in S^1$, in particular the upper half plane, $t > 0$. It is foliated by the family of hypersurfaces $t = k$ which are not everywhere spacelike, and has a kink number of one. The FLRW manifold is $R^1 \times S^1$ with $T \in R^1$ and $X \in S^1$, in particular the whole of the TX plane except the origin. It is foliated by the hypersurfaces $T = k$ which are everywhere spacelike, and has a kink number of zero.

The transformation

$$T' = T \cosh X$$
$$X' = T \sinh X$$

will transform the FLRW metric

$$ds^2 = -dT^2 + T^2 dX^2$$

to the 1+1 Minkowski metric

$$ds^2 = -dT'^2 + dX'^2.$$

The above transformation was first noted by Bondi (1965). The empty space flat kink solution given by equation (9.3) may therefore be transformed locally to the Minkowski (zero kink) metric via the two transformations

$$T = (t^2 + x^2)^{1/2}$$
$$X = \tan^{-1}(xt^{-1})$$

followed by

$$T' = T \cosh X$$
$$X' = T \sinh X.$$

It is also important to note that this solution differs from the previous solutions discussed in that the change of kink number under a certain coordinate transformation does not arise from the transformation becoming singular (infinite) at some finite point within the coordinate patch. Instead, the change in the kink number is because of the change in hypersurface foliation of the manifold resulting from the transformation. A knowledge of both the metric and

the manifold in terms of its globally defined hypersurface decomposition is therefore needed to determine the kink number of a given spacetime. This has been noted previously by Unruh (1971).

(9.3) A Two-Kink Minkowski Solution in 1+1 Dimensions.

The Minkowski metric expressed in its usual coordinates is

$$ds^2 = - dt^2 + dx^2.$$

The above metric is defined on the whole xt plane and clearly has a decomposition into matrices $S = \mathrm{diag}(1,1)$ and $Q = \mathrm{diag}(-1,1)$. The kink number is zero when integration is performed over hypersurfaces labelled by t = constant.

Consider now the change of variables

$$t = T\cos X$$
$$x = T\sin X .$$

Under this transformation the metric becomes

$$ds^2 = -\cos 2X\, dT^2 + 2T\sin 2X\, dT dX + T^2 \cos 2X\, dX^2 \;.$$

It is defined on the whole plane except the origin. Integration now along hypersurfaces labelled by T = constant will give a kink number of 2 and the light cones will rotate through 2π. This result is seen most easily by considering the decomposition of the matrix G, representing the metric, as follows

$$G = \begin{pmatrix} -\cos 2X & T\sin 2X \\ T\sin 2X & T^2\cos 2X \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} -\cos 2X & \sin 2X \\ \sin 2X & \cos 2X \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}.$$

The matrices diag(1,T) are homotopically trivial as they can be continuously deformed to the identity matrix. The "kink nature" of the metric therefore resides in the other matrix. As the angle X varies from $-\pi$ to π the light cones will tip continuously through 2π.

This solution again shows the importance of the hypersurface decomposition of the manifold when determining the kink number for a given metric.

(9.4) FLRW Kink Solutions in Higher Dimensions.

In 3+1 dimensions a solution of the metric

$$ds^2 = -\cos 2\alpha dt^2 - 2\sin 2\alpha dtdr + \cos 2\alpha dt^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\Phi^2 \qquad (9.5)$$

was given by equation (4.16), namely

$$\tan\alpha = rt^{-1}. \qquad (9.6)$$

whence

$$ds^2 = -(t^2 - r^2)(t^2 + r^2)^{-1} dt^2 - 4rt(t^2 + r^2)^{-1} drdt + (t^2 - r^2)(t^2 + r^2)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\Phi^2. \qquad (9.7)$$

Equation (9.6) shows that the angle α is well defined for all values of r when $t \neq 0$. By analogy with the (1+1)-dimensional solution it is clear that this solution may be transformed to one of the usual Friedmann-LeMaitre-Robertson-Walker solutions, justifying the name for this kink solution. Equation (9.6) also shows that the angle α of the light cones changes by π as r varies from $-\infty$ to ∞ (for any value of t) as illustrated in Fig. (9.3). Therefore it is expected that the solution will have a kink

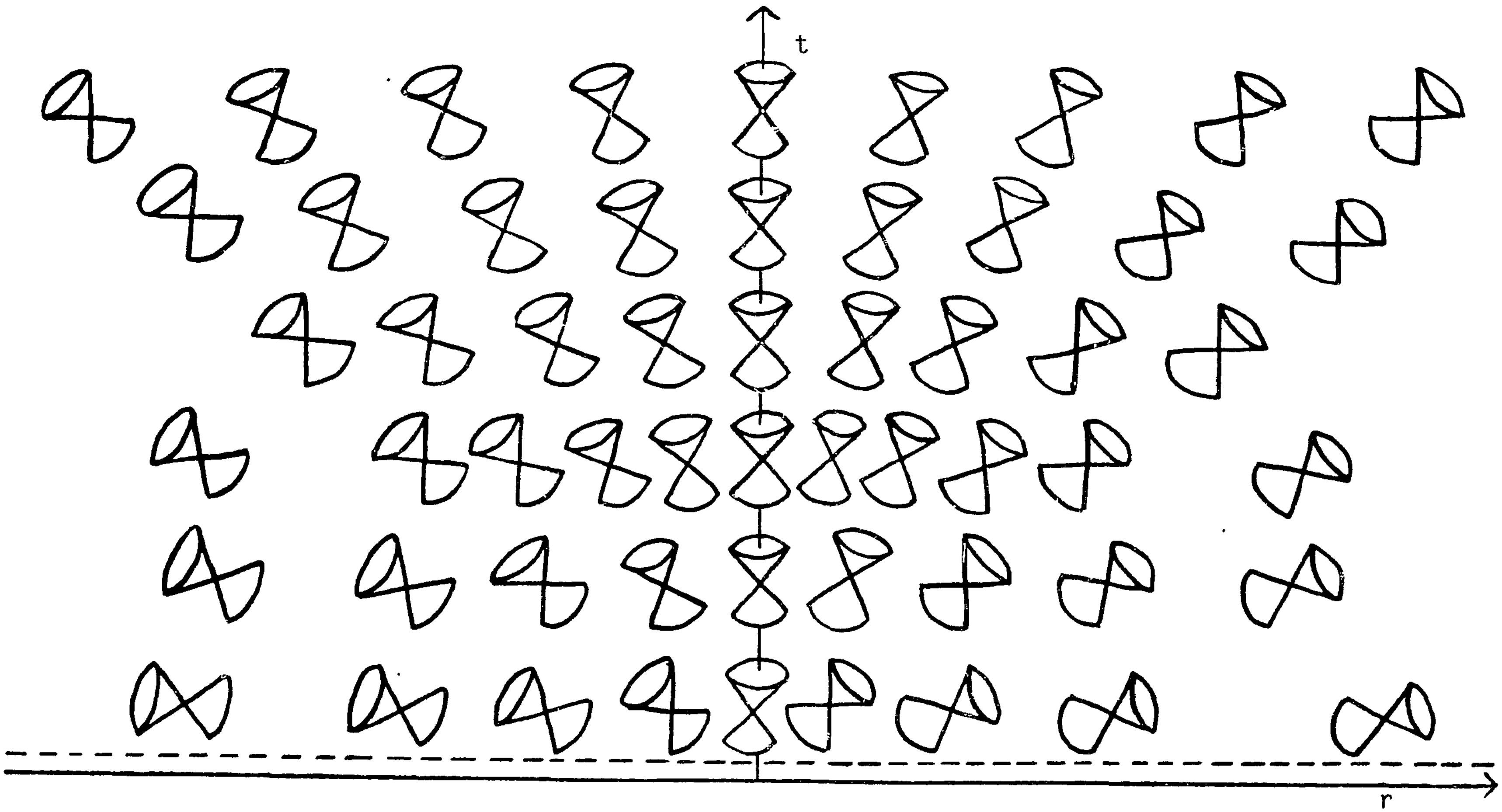

Fig. (9.3). The light cone picture for the Friedmann-LeMaitre-Robertson-Walker 3 + 1 kink solution.

number of one. This will be confirmed in the section (9.5). The manifold can be regarded as $R^1 \times R^3$, or if the boundary points $r = \pm\infty$ are identified, as $R^1 \times S^3$, where $t \in R^1$. The S^3 hypersurfaces are not everywhere spacelike.

The fluid pressure can be found from equation (4.8) and for this solution it is

$$p = -2(r^2 + t^2)^{-1}.$$

Equation (4.6) now shows that the equation of state is

$$p = -3^{-1}\mu.$$

It is easy to show that the eigenvalues of the matrix representing the tensor $T^{\alpha\beta}$ are μ, p, p, and p. This solution therefore obeys the weak energy condition, which may be stated as (Hawking and Ellis, 1973)

$$\mu + 3p \geq 0.$$

Unlike the previously discussed (3+1)-dimensional solutions, this solution has non-constant energy density and pressure. However, the pressure is still negative, as was the case for the previously discussed 3+1 solutions.

The scalar curvature is found from equation (4.1):

$$R = 12(r^2 + t^2)^{-1},$$

and the scalar expansion, given by equation (3.8), becomes

$$\theta = 3(r^2 + t^2)^{-1/2}.$$

Again, this solution differs from those previously discussed in 3+1 dimensions because the curvature and expansion scalars are non-constant.

The velocity components, given by equation (3.5), are

$$u^t = t(r^2 + t^2)^{-1/2}$$
$$u^r = r(r^2 + t^2)^{-1/2}$$
$$u^\theta = u^\Phi = 0$$

The components u^t and u^r have no limit as $(r,t) \to (0,0)$. Also it can be seen that as $r \to \infty$ for any fixed value of t that

$$u_t \to 0 \quad u_r \to 1$$

The acceleration vector components listed in section (3.5) can be shown to be zero for this solution.

A similar solution also exists in 2+1 dimensions (Dunn, Harriott and Williams, 1991b). In 2+1 dimensions the metric is

$$ds^2 = -(t^2 - r^2)(t^2 + r^2)^{-1}\,dt^2 - 4rt(t^2 + r^2)^{-1}dtdr + (t^2 - r^2)(t^2 + r^2)^{-1}dr^2 + r^2 d\theta^2 \ ,$$

where $r = (x^2 + y^2)^{1/2}$. This solution is a dust solution, that is, the equation of state is

$$p = 0 \ ,$$

and the energy density is

$$\mu = 2(t^2 + r^2)^{-1}.$$

(9.5) Transformation of the 3+1 Solution to FLRW Form.

Under the transformation

$$r = T \sin X$$
$$t = T \cos X,$$

the kink metric given by equation (9.7) becomes

$$ds^2 = -dT^2 + T^2 dX^2 + T^2 \sin^2 X \, d\theta^2 + T^2 \sin^2 X \sin^2\theta \, d\Phi^2 \tag{9.8}$$

which is the usual form of the closed FLRW metrics with expansion factor $L(T) = T$ and curvature constant $K = +1$. It will be shown later that this form of the metric, given by equation (9.8), has a kink number of zero. The spacetime manifold of this FLRW solution is $R^1 \times S^3$. It has compact spacelike hypersurfaces given by

$$T = \text{constant}.$$

This solution is a rather unusual form of the closed FLRW models however because unlike most closed models the universe it describes does not expand to a certain radius and then collapse back on itself. This solution expands forever but the expansion tends to zero as $T \to \infty$. This can be easily seen by considering

$$\dot{L}/L = T^{-1}.$$

(9.6) Kink Number Calculation for the 3+1 Solution.

As shown previously, the kink number formula for the metric given in equation (9.5) reduces to the form given in equation (6.7), namely

$$N = (2\pi^2)^{-1} \int_{\alpha_2}^{\alpha_1} \int_0^{2\pi} \int_0^{\pi} \sin^2\alpha \ \alpha_{,r} \ \sin\Phi dr d\Phi d\Theta$$

where α_1 and α_2 are the limiting values of α. For the solution given by equation (9.7),

$$\alpha_1 = \pi/2, \alpha_2 = -\pi/2 \ .$$

The kink number therefore equals one.

If the metric is transformed to the usual FLRW coordinates, the ϕ_α are seen to be

$$\phi_T = 1$$
$$\phi_i = 0$$

because the matrix Q is diag(-1, 1, 1, 1). The S matrix is

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & T^2 & 0 & 0 \\ 0 & 0 & T^2\sin^2 X & 0 \\ 0 & 0 & 0 & T^2\sin^2 X\sin^2\theta \end{pmatrix}$$

and so the Christoffel symbols with respect to the tensor represented by the matrix S are

$$\begin{aligned}
&\Gamma^X{}_{TX} = \Gamma^\theta{}_{T\theta} = \Gamma^\Phi{}_{T\Phi} = T^{-1} \\
&\Gamma^T{}_{XX} = T \\
&\Gamma^\theta{}_{X\theta} = \Gamma^\Phi{}_{X\Phi} = \cot X \\
&\Gamma^X{}_{\theta\theta} = -\sin X\cos X \\
&\Gamma^\Phi{}_{\theta\Phi} = \cot\theta \\
&\Gamma^T{}_{\Phi\Phi} = T\sin^2\theta\sin^2 X \\
&\Gamma^T{}_{\theta\theta} = T\sin^2 X \\
&\Gamma^X{}_{\Phi\Phi} = -\sin X\cos X\sin^2\theta \\
&\Gamma^\theta{}_{\Phi\Phi} = -\sin\theta\cos\theta .
\end{aligned}$$

All the terms $X^\alpha{}_{,\beta}$ will be zero. However

$$X^\alpha{}_{|\beta} = X^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\tau}X^\tau$$

will have the following non-zero contributions

$$X^X{}_{|X} = X^\theta{}_{|\theta} = X^\Phi{}_{|\Phi} = T^{-1}.$$

Non-zero contributions to the kink number formula come from terms of the form

$$\epsilon^{0ijk}\epsilon_{\alpha\beta\tau\delta}X^{\alpha}X^{\beta}{}_{|i}X^{\tau}{}_{|j}X^{\delta}{}_{|k}.$$

However, due to the antisymmetrization these terms sum to zero and so the kink number is zero.

This discrepancy in the kink number is explained in an analagous way to the 1+1 case as discussed in section (9.2).

(9.7) The Riemann Tensor for the 3+1 Solution.

The components of the Riemann tensor can be calculated from the equation (Misner, Thorne and Wheeler, 1973)

$$R^{\alpha}{}_{\beta\tau\delta} = \Gamma^{\alpha}{}_{\beta\delta,\tau} - \Gamma^{\alpha}{}_{\beta\tau,\delta} + \Gamma^{\sigma}{}_{\beta\delta}\Gamma^{\alpha}{}_{\sigma\tau} - \Gamma^{\sigma}{}_{\beta\tau}\Gamma^{\alpha}{}_{\sigma\delta}.$$

This tensor has the following symmetries

$$R^{\alpha}{}_{\beta\tau\delta} = -R^{\alpha}{}_{\beta\delta\tau}$$

$$R^{\alpha}{}_{[\beta\tau\delta]} = R^{\alpha}{}_{\beta\tau\delta} + R^{\alpha}{}_{\delta\beta\tau} + R^{\alpha}{}_{\tau\delta\beta} = 0,$$

and so the number of independant components of the Riemann tensor is 80.

The kink form of the 3+1 FLRW metric is given by equation (9.6) and is

$$\begin{aligned} ds^2 = &-(t^2 - r^2)(t^2 + r^2)^{-1}dt^2 - 4rt(t^2 + r^2)^{-1}drdt \\ &+ (t^2 - r^2)(t^2 + r^2)^{-1}\,dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\Phi^2. \end{aligned}$$

For this metric, the Christoffel symbols are

$$\Gamma^{t}{}_{tt} = -\Gamma^{t}{}_{tr} = 2tr^2(t^2 + r^2)^{-2}$$
$$\Gamma^{t}{}_{tr} = -\,\Gamma^{r}{}_{rr} = -2rt^2(t^2 + r^2)^{-2}$$

$$\Gamma^{r}{}_{tt} = 2r^3(t^2 + r^2)^{-2}$$
$$\Gamma^{t}{}_{rr} = 2t^3(t^2 + r^2)^{-2}$$
$$\Gamma^{t}{}_{\theta\theta} = 2r^2t(t^2 + r^2)^{-1}$$
$$\Gamma^{t}{}_{\Phi\Phi} = \sin^2\theta\Gamma^{t}{}_{\theta\theta}$$
$$\Gamma^{r}{}_{\theta\theta} = -r(t^2 - r^2)(t^2 + r^2)^{-1}$$
$$\Gamma^{r}{}_{\Phi\Phi} = \sin^2\theta\Gamma^{r}{}_{\theta\theta}$$
$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\Phi}{}_{r\Phi} = r^{-1}$$
$$\Gamma^{\theta}{}_{\Phi\Phi} = -\sin\theta\cos\theta$$
$$\Gamma^{\Phi}{}_{\theta\Phi} = \cot\theta.$$

Using these Christoffel symbols, the non-zero Riemann components are

$$R^{\Phi}{}_{tt\Phi} = -2r^2(r^2 + t^2)^{-2}$$
$$R^{\Phi}{}_{rt\Phi} = 2tr(r^2 + t^2)^{-2}$$
$$R^{\Phi}{}_{rr\Phi} = -2t^2(r^2 + t^2)^{-2}$$
$$R^{\Phi}{}_{\theta\theta\Phi} = -2r^2(r^2 + t^2)^{-1}$$
$$R^{\theta}{}_{rr\theta} = -2t^2(r^2 + t^2)^{-2}$$
$$R^{\theta}{}_{tt\theta} = -2r^2(r^2 + t^2)^{-2}$$
$$R^{\theta}{}_{rt\theta} = 2tr(r^2 + t^2)^{-2}$$
$$R^{\theta}{}_{\Phi\theta\Phi} = 2r^2\sin^2\theta(r^2 + t^2)^{-1}$$
$$R^{r}{}_{\theta r\theta} = 2r^2t^2(r^2 + t^2)^{-2}$$
$$R^{r}{}_{\theta t\theta} = -2r^3t(r^2 + t^2)^{-2}$$
$$R^{r}{}_{\Phi r\Phi} = \sin^2\theta R^{r}{}_{\theta r\theta}$$
$$R^{r}{}_{\Phi t\Phi} = \sin^2\theta R^{r}{}_{\theta t\theta}$$
$$R^{t}{}_{\theta r\theta} = -2r^3t(r^2 + t^2)^{-2}$$

$$R^{t}{}_{\theta t\theta} = 2r^4(r^2 + t^2)^{-2}$$
$$R^{t}{}_{\Phi r\Phi} = \sin^2\theta R^{t}{}_{\theta r\theta}$$
$$R^{t}{}_{\Phi t\Phi} = \sin^2\theta R^{t}{}_{\theta t\theta}.$$

If the kink form of the metric, given by equation (9.7), is transformed to the usual FLRW coordinates it becomes

$$ds^2 = -dT^2 + T^2dX^2 + T^2\sin^2Xd\theta^2 + T^2\sin^2X\sin^2\theta d\Phi^2.$$

In these coordinates, the nonzero Christoffel symbols are

$$\Gamma^{X}{}_{TX} = \Gamma^{\theta}{}_{T\theta} = \Gamma^{\Phi}{}_{T\Phi} = T^{-1}$$
$$\Gamma^{T}{}_{XX} = T$$
$$\Gamma^{\theta}{}_{X\theta} = \Gamma^{\Phi}{}_{X\Phi} = \cot X$$
$$\Gamma^{X}{}_{\theta\theta} = -\sin X\cos X$$
$$\Gamma^{\Phi}{}_{\theta\Phi} = \cot\theta$$
$$\Gamma^{T}{}_{\Phi\Phi} = T\sin^2\theta\sin^2X$$
$$\Gamma^{T}{}_{\theta\theta} = T\sin^2X$$
$$\Gamma^{X}{}_{\Phi\Phi} = -\sin X\cos X\sin^2\theta$$
$$\Gamma^{\theta}{}_{\Phi\Phi} = -\sin\theta\cos\theta$$

and the non-zero Riemann components are

$$R^{\Phi}{}_{\theta\theta\Phi} = -2\sin^2X$$
$$R^{\Phi}{}_{XX\Phi} = -2$$

$$R^{\theta}{}_{\chi\chi\theta} = -2$$
$$R^{\theta}{}_{\phi\theta\phi} = 2\sin^2\theta\sin^2\chi$$
$$R^{\chi}{}_{\phi\chi\phi} = 2\sin^2\chi\sin^2\theta$$
$$R^{\chi}{}_{\theta\chi\theta} = 2\sin^2\chi.$$

All the nonzero Riemann tensor components are well behaved for all points of the manifold.

(9.8) The Killing Vectors for the 3+1 Solution.

Killing's Equations for the 3+1 FLRW kink metric of equation (9.6) are

$$\mu_{t,t} - 2tr^2(r^2 + t^2)^{-2}\,\mu_t - 2r^3(r^2 + t^2)^{-2}\,\mu_r = 0$$

$$\mu_{t,r} + \mu_{r,t} + 4rt^2(r^2 + t^2)^{-2}\,\mu_t + 4tr^2(r^2 + t^2)^{-2}\,\mu_r = 0$$

$$\mu_{t,\theta} + \mu_{\theta,t} = 0$$

$$\mu_{t,\Phi} + \mu_{\Phi,t} = 0$$

$$\mu_{r,r} - 2t^3(r^2 + t^2)^{-2}\,\mu_t - 2rt^2(r^2 + t^2)^{-2}\,\mu_r = 0$$

$$\mu_{r,\theta} + \mu_{\theta,r} - 2r^{-1}\,\mu_\theta = 0$$

$$\mu_{r,\Phi} + \mu_{\Phi,r} - 2r^{-1}\,\mu_\Phi = 0$$

$$\mu_{\theta,\theta} - 2r^2t(r^2 + t^2)^{-1}\,\mu_t + r(t^2 - r^2)(r^2 + t^2)^{-1}\,\mu_r = 0$$

$$\mu_{\theta,\Phi} + \mu_{\Phi,\theta} - 2\cot\theta\,\mu_\Phi = 0$$

$$\mu_{\Phi,\Phi} + \sin\theta\cos\theta\,\mu_\theta - 2\sin^2\theta\, r^2t(r^2 + t^2)^{-1}\,\mu_t + \sin^2\theta\, r(t^2 - r^2)(r^2 + t^2)^{-1}\,\mu_r = 0.$$

These equations admit six Killing vectors

$$\mu_1 = \delta/\delta\Phi$$

$$\mu_2 = \sin\Phi\ \delta/\delta\Theta + \cot\Theta\cos\Phi\ \delta/\delta\Phi$$

$$\mu_3 = \cos\Phi\ \delta/\delta\Theta - \cot\Theta\sin\Phi\ \delta/\delta\Phi$$

$$\mu_4 = -r\cos\Theta\ \delta/\delta t + t\cos\Theta\ \delta/\delta r - tr^{-1}\sin\Theta\ \delta/\delta\Theta$$

$$\mu_5 = -r\sin\Theta\sin\Phi\ \delta/\delta t + t\sin\Theta\sin\Phi\ \delta/\delta r + tr^{-1}\cos\Theta\sin\Phi\ \delta/\delta\Theta + tr^{-1}\mathrm{cosec}\Theta\cos\Phi\ \delta/\delta\Phi$$

$$\mu_6 = -r\sin\Theta\cos\Phi\ \delta/\delta t + t\sin\Theta\cos\Phi\ \delta/\delta r + tr^{-1}\cos\Theta\cos\Phi\ \delta/\delta\Theta - tr^{-1}\mathrm{cosec}\Theta\sin\Phi\ \delta/\delta\Phi.$$

The generators of the rotation group SO(3) are μ_1, μ_2, μ_3, and μ_4, μ_5, μ_6 are translations.

The lengths of these Killing vectors are

$$||\mu_1|| = r^2\sin^2\Theta$$

$$||\mu_2|| = r^2\sin^2\Phi + r^2\cos^2\theta\cos^2\Phi$$

$$||\mu_3|| = r^2\cos^2\Phi + r^2\cos^2\theta\sin^2\Phi$$

$$||\mu_4|| = t^2 + r^2\cos^2\theta$$

$$||\mu_5|| = t^2 + r^2\sin^2\theta\sin^2\Phi$$

$$||\mu_6|| = t^2 + r^2\sin^2\theta\cos^2\Phi.$$

These vectors are therefore clearly spacelike everywhere.

Under the coordinate transformation

$$r = T\sin X$$
$$t = T\cos X,$$

which transforms the kink metric of equation (9.6) to the usual FLRW form which is

$$ds^2 = -dT^2 + T^2dX^2 + T^2\sin^2X\,(d\theta^2 + \sin^2\theta\, d\Phi^2),$$

the Killing Equations are

$$\mu_{T,T} = 0$$

$$\mu_{T,X} + \mu_{X,T} - 2T^{-1}\mu_X = 0$$

$$\mu_{T,\theta} + \mu_{\theta,T} - 2T^{-1}\mu_\theta = 0$$

$$\mu_{T,\Phi} + \mu_{\Phi,T} - 2T^{-1}\mu_\Phi = 0$$

$$\mu_{X,X} - T\mu_T = 0$$

$$\mu_{X,\theta} + \mu_{\theta,X} - 2\cot X\,\mu_\theta = 0$$

$$\mu_{X,\Phi} + \mu_{\Phi,X} - 2\cot X\,\mu_\Phi = 0$$

$$\mu_{\theta,\theta} - T\sin^2 X\,\mu_T + \sin X\cos X\,\mu_X = 0$$

$$\mu_{\theta,\Phi} + \mu_{\Phi,\theta} - 2\cot\theta\,\mu_\Phi = 0$$

$$\mu_{\Phi,\Phi} - T\sin^2 X\sin^2\theta\,\mu_T + \sin X\cos X\sin^2\theta\,\mu_X + \sin\theta\cos\theta\,\mu_\theta = 0$$

The six Killing vectors are

$$\underline{\mu}_1 = \delta/\delta\Phi$$

$$\underline{\mu}_2 = \cos\Phi\,\delta/\delta\theta - \cot\theta\sin\Phi\,\delta/\delta\Phi$$

$$\underline{\mu}_3 = \sin\Phi\ \delta/\delta\Theta + \cot\Theta\cos\Phi\ \delta/\delta\Phi$$

$$\underline{\mu}_4 = \cos\Theta\ \delta/\delta X - \cot X\sin\Theta\ \delta/\delta\Theta$$

$$\underline{\mu}_5 = \sin\Theta\sin\Phi\ \delta/\delta X + \cot X\cos\Theta\sin\Phi\ \delta/\delta\Theta + \cot X\operatorname{cosec}\Theta\cos\Phi\ \delta/\delta\Phi$$

$$\underline{\mu}_6 = \sin\Theta\cos\Phi\ \delta/\delta X + \cot X\cos\Theta\cos\Phi\ \delta/\delta\Theta - \cot X\operatorname{cosec}\Theta\sin\Phi\ \delta/\delta\Phi$$

and their lengths are

$$||\underline{\mu}_1|| = T^2\sin^2 X\sin^2\Theta$$

$$||\underline{\mu}_2|| = T^2\sin^2 X\cos^2\Phi + T^2\sin^2 X\cos^2\Theta\sin^2\Phi$$

$$||\underline{\mu}_3|| = T^2\sin^2 X\sin^2\Phi + T^2\sin^2 X\cos^2\Theta\cos^2\Phi$$

$$||\underline{\mu}_4|| = T^2\cos^2\Theta + T^2\cos^2 X\sin^2\Theta$$

$$||\underline{\mu}_5|| = T^2\sin^2\Theta\sin^2\Phi + T^2\cos^2 X\cos^2\Theta\sin^2\Phi + T^2\cos^2 X\cos^2\Phi$$

$$||\underline{\mu}_6|| = T^2\sin^2\Theta\cos^2\Phi + T^2\cos X\cos^2\Theta\cos^2\Phi + T^2\cos^2 X\sin^2\Phi.$$

As required, these lengths agree with those found in the (t, r, θ, Φ) coordinate system. For example,

$$
\begin{aligned}
||\mu 6|| &= t^2 + r^2\sin^2\Phi \\
&= T^2\cos^2 X + T^2\sin^2 X\sin^2\theta\sin^2\Phi \\
&= T^2\cos^2 X + T^2(1 - \cos^2 X)\sin^2\theta\sin^2\Phi \\
&= T^2\sin^2\theta\sin^2\Phi + \\
&\quad T^2\cos^2 X(1 - [1 - \cos^2\theta][1 - \cos^2\Phi]) \\
&= T^2\sin^2\theta\sin^2\Phi + T^2\cos^2 X\cos^2\Phi + \\
&\quad T^2\cos^2 X\cos^2\theta\sin^2\Phi.
\end{aligned}
$$

The linear independence of these Killing vectors is easily established.

(9.9) A Minkowski Two-Kink Solution in Higher Dimensions

The two-kink Minkowski solution discussed previously clearly extends to the (2+1) and (3+1)-dimensional situations. The usual 3+1 Minkowski metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ .$$

Under the coordinate transformation

$$\begin{aligned} t &= t' \\ x &= r\cos\Phi\cos\theta \\ y &= r\sin\Phi\cos\theta \\ z &= r\sin\theta \end{aligned}$$

it becomes

$$ds^2 = -dt'^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\Phi \cdot$$

Under the transformation

$$\begin{aligned} t' &= T\cos X \\ r &= T\sin X \end{aligned}$$

it becomes

$$ds^2 = -\cos 2X\, dT^2 + 2T \sin 2X\, dTdX + T^2 \cos 2X\, dX^2$$
$$+ T^2 \sin^2 X\, d\theta^2 + T^2 \sin^2 X \sin^2 \theta\, d\Phi^2.$$

As discussed previously for the (1+1)-dimensional case, this metric has two kinks.

CHAPTER TEN

OTHER KINK SOLUTIONS

Introduction

Kink solutions found in Chapter 5 are briefly discussed. The first of these is another form of the de Sitter kink (10.1). Two further perfect fluid solutions (10.2) and (10.3) and an imperfect fluid solution (10.4) are also illustrated.

(10.1) Solution I

Solutions of the field equations for the metric given by

$$\begin{aligned} g_{tt} &= e^{\sigma}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\ g_{tr} &= -(e^{\sigma} + e^{\Omega})\sin\alpha\cos\alpha \\ g_{rr} &= e^{\sigma}\cos^2\alpha - e^{\Omega}\sin^2\alpha \\ g_{\theta\theta} &= e^{\tau}r^2 \\ g_{\Phi\Phi} &= e^{\tau}r^2\sin^2\theta \end{aligned} \tag{10.1}$$

were found in chapter five. Such a solution was given by equation (5.8) for which

$$\sigma = \Omega = 0 \quad e^{\tau} = C^{-1}r^{-2}$$

and

$$\sin\alpha = 2^{-1/2}\{1 + Cr^2 + Dr + H\}^{1/2} \tag{10.2}$$

where C, D, and H are constants, with C > 0. There is considerable freedom in choosing the values of these constants. For example, requiring that

$$\sin\alpha = 0 \text{ at } r = 0$$

means that

$$H = -1$$

and requiring that

$$\sin\alpha = 1 \text{ at } r = R_o > 0$$

forces

$$CR_o^{\,2} + DR_o - 2 = 0.$$

This last quadratic has solutions

$$R_o = (2C)^{-1}[-D \pm \{D^2 + 8C\}^{1/2}].$$

The constant $C > 0$ and so a positive and real R_o will be obtained if the positive square root is chosen.

For example, selecting

$$D = 1 \text{ and } C = 1$$

and substituting into equation (10.2) gives

$$\sin\alpha = (2)^{-1/2}[r + r^2]^{1/2}.$$

To ensure that $|\sin\alpha| \leqslant 1$, this last result leads to the restriction

$$0 \leqslant r \leqslant 1,$$

where

$$\sin\alpha = 0 \qquad \text{at} \qquad r = 0$$

and

$$\sin\alpha = 1 \qquad \text{at} \qquad r = R_o = 1.$$

With these choices of D = 1 and C = 1, the metric is now

$$\begin{aligned} ds^2 = &-(1 - r - r^2)dt^2 - 2(2r + r^2 - 2r^3 - r^4)^{1/2}dtdr \\ &+ (1 - r - r^2)dr^2 + d\theta^2 + \sin^2\theta d\Phi^2. \end{aligned}$$

For this solution .

$$\begin{aligned} g^{tt} &= g_{tt} = r^2 + r - 1 \\ g^{tr} &= g_{tr} = -(2r + r^2 - 2r^3 - r^4)^{1/2} \\ g^{rr} &= g_{rr} = 1 - r - r^2 = -g_{tt} \\ g_{\theta\theta} &= g^{\theta\theta} = 1 \\ g_{\Phi\Phi} &= (g^{\Phi\Phi})^{-1} = \sin^2\theta . \end{aligned}$$

The non-zero Christoffel symbols are

$$\Gamma^{t}{}_{tt} = \Gamma^{r}{}_{tr} = 2^{-1}(1 + 2r)(2r + r^2 - 2r^3 - r^4)^{1/2}$$
$$\Gamma^{t}{}_{tr} = -\Gamma^{r}{}_{rr} = \Gamma^{r}{}_{tt} = 2^{-1}(2r^3 + 3r^2 - r - 1)$$
$$\Gamma^{t}{}_{rr} = 2^{-1}(2 + 2r - 5r^2 + 5r^4 + 2r^5)(2r + r^2 - 2r^3 - r^4)^{-1/2}$$

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$
$$\Gamma^{\phi}{}_{\theta\phi} = \cot\theta.$$

It follows that

$$R_{tt} = r^2 + r - 1 = g_{tt} = -g_{rr}$$

$$R_{tr} = -(2r + r^2 - 2r^3 - r^4)^{1/2} = g_{tr}$$

$$R_{rr} = 1 - r - r^2 = g_{rr}$$

$$R_{\theta\theta} = 1 = g_{\theta\theta}$$

$$R_{\phi\phi} = \sin^2\theta = g_{\phi\phi} ,$$

so that the solution is an Einstein space.

The Ricci scalar, calculated from the equation listed in section (5.1), is

$$R = (g_{tt})^2 + 2(g_{tr})^2 + (g_{rr})^2 + (g_{\theta\theta})^2 + g^{\Phi\Phi}g_{\Phi\Phi}$$
$$= 4$$

The equation of state for this metric is

$$p = - \mu = -C .$$

With the above choice of C = 1, this equation of state is

$$p = -\mu = -1$$

The fact that this solution is an Einstein space and has a constant scalar curvature and constant energy density suggest that it may be a form of the de Sitter kink. The usual transformation to remove the cross term drdt is obtained by finding a function f(r) such that

$$t' = t + f(r)$$

$$f'(r) = g_{tr}(g_{tt})^{-1}.$$

For this solution,

$$f'(r) = (2r + r^2 - 2r^3 - r^4)^{1/2}(r^2 + r - 1)^{-1} \tag{10.3}$$

where $0 \leq r \leq 1$.

This transformation will be singular because

$$r^2 + r - 1 = 0$$

at

$$r = (2)^{1/2}[-1 + (5)^{1/2}] = 0.7$$

which is within the allowed range of r.

(10.2) Solution II

Another solution of the field equations was given by equation (5.9) for the metric of equation (10.1). This solution is

$$\sigma = -\Omega$$
$$e^{\tau} = C^{-1}r^{-2}$$

and

$$\sin^2\alpha = (Cr^2 + Dr + H + e^{\Omega})(e^{\Omega} + e^{-\Omega})^{-1}. \tag{10.4}$$

This last result can be written

$$\sin^2\alpha = \tanh\Omega,$$

with Ω is chosen such that

$$-e^{-\Omega} = Cr^2 + Dr + H.$$

The constants C, D, H must be restricted to ensure that

$$|\sin\alpha| \leq 1,$$

but there is considerable freedom in doing this. For example, if

$$\sin\alpha = 0 \text{ at } r = 0,$$

then the constant H must be chosen so that

$$H = 1 \text{ or } H = -1.$$

If

$$\sin\alpha \to 1 \text{ as } r \to 1$$

then

$$C + D = -1 \quad \text{if} \quad H = 1$$

and

$$C + D = 1 \quad \text{if} \quad H = -1.$$

The constant C can assume any positive value. Therefore, without loss of generality, C = 1 can be selected. Substituting into equation (10.4), C = 1, H = 1 and D = -2 gives

$$\sin^2\alpha = [1 - (r^2 - 2r + 1)^2][1 + (r^2 - 2r + 1)^2]^{-1}$$

valid for

$$0 \leq r \leq 1.$$

With these choices for $\sin\alpha$, H, D, σ and τ, the metric given by equation (10.1) becomes

$$\begin{aligned} ds^2 = {} & (r^2 - 2r + 1)dt^2 + 2(8r^3 + 8r - 12r^2 - 2r^4)^{1/2}dtdr \\ & - [1 - 2(r^2 - 2r + 1)^2][r^2 - 2r + 1]^{-1}dr^2 \\ & + d\theta^2 + \sin^2\theta d\Phi^2. \end{aligned}$$

which is valid for $0 \leq r \leq 1$.

The other possibility, when C = 1, H = -1 and D = 0 are selected, gives

$$\sin^2\alpha = [1 - (r^2 - 1)^2][1 + (r^2 - 1)^2]^{-1}$$

valid for

$$0 \leq r \leq 1.$$

For this last form of $\sin\alpha$ the metric is

$$ds^2 = -(1 - r^2)dt^2 + 2r(4 - 2r^2)^{1/2}drdt$$
$$+ [1 - 2(r^2 - 1)^2][r^2 - 1]^{-1}dr^2 + r^2d\theta^2$$
$$+ r^2\sin^2\theta d\Phi^2.$$

The equation of state for both of the above metrics is

$$p = -\mu = -1$$

(10.3) Solution III

Another solution of the field equations for the metric given in equation (10.1) was found in chapter five. As stated in equation (5.18) this solution is

$$
\begin{aligned}
\tau &= 0 \\
\sigma &= -\Omega \qquad\qquad (10.5) \\
g_{tt} &= e^{-\Omega}\sin^2\alpha - e^{\Omega}\cos^2\alpha \\
&= \sin^2\alpha(e^{\Omega} + e^{-\Omega}) - e^{\Omega} \\
&= 3^{-1}Lr^2 - 1 + Mr^{-1}
\end{aligned}
$$

where L is an arbitrary constant. Equivalently, rearranging the above equation, it can be seen that a consistent solution requires

$$\sin^2\alpha = [3^{-1}Lr^2 - 1 + Mr^{-1} + e^{\Omega}][e^{\Omega} + e^{-\Omega}]^{-1} .$$

There is considerable freedom in choosing Ω, L and M while still ensuring the required behaviour of the function $\sin\alpha$. For example, choosing $\Omega = 0$ leads to the de Sitter kink (if $M = 0$) and Schwarzschild kink (if $L = 0$) both of which were discussed previously. However if

$$-e^{-\Omega} = Lr^2 - 1 + Mr^{-1}$$

is chosen then

$$\sin^2\alpha = \tanh\Omega.$$

With this choice, the constants must now be restricted to ensure that $|\sin\alpha| \leq 1$. Selecting

$$\sin\alpha = 0 \text{ at } r = 0$$

requires

$$M = 0 ,$$

and choosing

$$\sin\alpha \to 1 \text{ as } r \to 1$$

requires

$$L = 3.$$

Therefore, with these choices,

$$\sin^2\alpha = \tanh\Omega = [1 - (1 - r^2)^2][1 + (1 - r^2)^2]^{-1}$$

valid for

$$0 \leq r \leq 1$$

and the metric is

$$ds^2 = (r^2 - 1)dt^2 + 2[2(1 - (r^2 - 1)^2]^{1/2}drdt + [1 - 2(r^2 - 1)^2][r^2 - 1]^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\Phi^2.$$

(10.4) Imperfect Fluid solution I

An imperfect fluid solution of the metric given by equation (10.1) was found in chapter five and given by equation (5.27). This solution is

$$e^{\tau}r^2 = e^{-2Nr}$$

$$\sigma = \Omega = 0$$

where N is a positive constant and

$$\sin^2\alpha = (2N^2)^{-1}e^{-2Nr} - 2^{-1}U\exp(\pm 2^{1/2}Nr) \ ,$$

where U is any constant.

To ensure $\sin\alpha = 0$ at $r = 0$

$$U = N^{-2}$$

is required, and now

$$\sin^2\alpha = (2N^2)^{-1}[\exp(2Nr) - \exp(\pm 2^{1/2}Nr)]$$

$$\cos 2\alpha = 1 + N^{-2}\exp(\pm 2^{1/2}Nr) - N^{-2}\exp(2Nr).$$

The expression for $\sin\alpha$ is an increasing function of r and reaches 1, at some value of $r = R_1$, when

$$\exp(2NR_1) - \exp(\pm 2^{1/2}NR_1) = 2N^2$$

The equation of state with $n = \Omega = 0$ is now

$$\mu + p_T - \Sigma\Theta + 2N^2 = 0 \ .$$

Substituting for $p_T - \Sigma\Theta$ from the $G^{\theta}{}_{\theta}$ field equation produces an expression for μ

$$\begin{aligned}\mu &= -2N^2 - 2^{-1}(\cos 2\alpha)_{,rr} + N(\cos 2\alpha)_{,r} - N^2\cos 2\alpha \\ &= \exp(2Nr) - (2 - 2^{1/2})\exp(\pm 2^{1/2}Nr) - 3N^2\end{aligned}$$

This is an increasing function of r and so will be positive for all r if it is positive at $r = 0$. This function is positive at $r = 0$ if

$$-1 + 2^{1/2} - 3N^2 > 0.$$

That is, if,

$$N^2 < (2^{1/2} - 1)3^{-1}.$$

The metric is now

$$
\begin{aligned}
ds^2 = &-[1 - N^{-2}\exp(2Nr) + N^{-2}\exp(\pm 2^{1/2}Nr)]dt^2 \\
&- 2\{-N^{-4}\exp(4Nr) - N^{-4}\exp(2Nr2^{1/2}) + 2N^{-2}\exp(2Nr) \\
&-2\exp(\pm 2^{1/2}Nr) + 2N^{-4}\exp(2Nr \pm 2^{1/2}Nr)\}dtdr \\
&+ [1 - N^{-2}\exp(2Nr) + N^{-2}\exp(\pm 2^{1/2}Nr)]dr^2 \\
&+ \exp(-2Nr)d\theta^2 + \exp(-2Nr)\sin^2\theta d\Phi^2
\end{aligned}
$$

which is valid for $0 < r < R_1$ with R_1 is defined by

$$\exp(2NR_1) - \exp(\pm 2^{1/2}NR_1) = 2N^2.$$

The constant N may assume any value in the range

$$N^2 < (2^{1/2} - 1)3^{-1}.$$

CHAPTER ELEVEN

CONCLUSIONS

Introduction

The work of this thesis is summarized (11.1) and a discussion of future work in the area is presented (11.2).

(11.1) Summary.

This thesis reviewed the work of Skyrme, who first noted the kink-like structures in certain non-linear scalar field theories. Previous work on kinks in general relativity was also reviewed.

The general form of a kink metric was derived and various kink solutions were found using this form. These are the first exact kink solutions of the Einstein field equations published. Several of these solutions were discussed in detail, and their relationship to familiar solutions of the field equations was demonstrated.

The formula to calculate the kink number for any metric, introduced by Skryme was elaborated. The formula was stated in a covariant form and the kink number of various solutions were found. The kink number of these solutions was demonstrated to be the same in several coordinate systems.

Analagous kink solutions to those found in the usual 3+1 dimensions were shown to exist in a (1+1)-dimensional theory of gravity. The lower dimensional theory was introduced so that in this simpler situation the kink properties might be more easily illustrated.

(11.2) Future Work.

At least four areas for future study are easily identified. These are:

(i) There is a need to find more exact solutions of the Einstein field equations which are kink solutions. In particular solutions which have more than one kink and solutions that have equations of state with positive pressure in 3+1 dimensions. Other solutions related to familiar zero kink solutions via singular coordinate transformations or surgery on the manifold should also be sought.

Solutions of the general kink metric for which $\alpha = \alpha(r,t)$ are being investigated. If $\alpha = \alpha(r)$ only, it was shown that if a perfect fluid form is assumed then the equation of state must be $p = -\mu$. To find perfect fluid, positive pressure solutions with this form of the metric, $\alpha = \alpha(r,t)$ is required.

The solutions listed in chapter 10 are being further investigated to see if they are related to well known solutions.

(ii) This thesis concentrated on kinks in general relativity on manifolds that are assumed to be $R^1 \times R^3$ or on $R^1 \times S^3$ if the boundary points are identified. Kinks are known to exist on more complicated manifolds. The kinks studied in this thesis are known as kinks of type 2. Kinks of type 1 on more complicated manifolds also exist and have not been studied in detail.

(iii) More work may need to be done on the kink number formula. It is not clear that in its current form it will be sufficient to calculate the kink number of kinks defined on more complicated manifolds than those studied here.

(iv) Solutions of the Einstein field equations using the stress energy tensor of the electromagnetic field might be investigated. Also vacuum expectation energy solutions might be of interest.

BIBLIOGRAPHY

A.P. Balachandran, "The Skyrme soliton - A review", in Solitons in Nuclear and Elementary Particle Physcic, Proceedings of the Lewes Workshop, 2-16 June 1984, edited by A.Chodos, E. Hadjimichael and C. Tze (World Scientific, Singapore, 1984).

T. Banks and L. Susskind, International Journal of Theoretical Physics, **23**, 475 (1984).

P.G. Bergmann, Introduction to the Theory of Relativity, (New Jersey, Prentice Hall, 1959), chapter 13.

P.G. Bergmann, Physical Review, **107**, 2, (1957).

P.G. Bergmann, M. Cahen and A.B. Komar, Journal of Mathematical Physics, **6**, 1, (1965).

Birkhoff and MacLane, A Survey of Modern Algebra (MacMillan, New York, 1965).

S.K. Blau and A.H. Guth, "Inflationary Cosmology", in Three Hundred Years of Gravitation, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987).

H. Bondi, "Some Special Solutions of the Einstein Equations", in Lectures on General Relativity - 1964 Brandeis Summer Institute in Theoretical Physics, Volume 1, edited by A. Trautman, F.A.E. Pirani and H. Bondi (Prentice-Hall, Englewood Cliffs, NJ, 1965).

J.D. Brown, M. Henneaux and C. Teitelboim, Physics Review D, **33**, 319 (1986).

J.D. Brown, Lower Dimensional Gravity (World Scientific, Singapore, 1988)

K. Bugajska, International Journal of Theoretical Physics, **26**, 623, (1987).

K. Bugajska, Journal of Mathematical Physics, **30**, 619 (1989).

Chevalley, Theory of Lie Groups, (Princeton University Press, 1946).

G. Clément, General Relativity and Gravitation, **16**, 131, (1984a).

G. Clément, General Relativity and Gravitation, **16**, 477, (1984b).

G. Clément, General Relativity and Gravitation, **16**, 491, (1984c).

G. Clément, International Journal of Theoretical Physics, **24**, 3, (1985).

G. Clément, General Relativity and Gravitation, **18**, 137, (1986).

K.A. Dunn, General Relativity and Gravitation, **23**, 507, (1990).

K.A. Dunn, T.A. Harriott, and J.G. Williams, Journal of Mathematical Physics, **32**, 476, (1991).

K.A. Dunn, T.A. Harriott and J.G. Williams, "Toy model for gravitational kinks", Preprint 1991a.

K.A. Dunn, T.A. Harriott and J.G. Williams, "An FLRW kink metric in 2+1 and 3+1 dimensions", Preprint 1991b.

K.A. Dunn, T.A. Harriott and J.G. Williams, "Kinks in 1+1, 2+1 and 3+1 Dimensions", to appear in proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, edited by G. Kunstatter et al. (World Scientific, Singapore, 1992)

K.A. Dunn and J.G. Williams, Journal of Mathematical Physics, **30**, 87, (1989).

G.F.R. Ellis, "Relativistic Cosmology", in Proceedings of the International School of Physics "Enrico Fermi", Course XLVII - General Relativity and Cosmology, held at Varenno, Lake Como, 30 June - 12 July 1969, edited by B.K. Sachs (Academic Press, New York, 1971).

B. Felsager, Geometry, Particles and Fields (Odense University Press, Odense, Denmark, 1981), pages 568-569 and 324.

D. Finkelstein, Physical Review, **110**, 965, (1958).

D. Finkelstein, Journal of Mathematical Physics, **7**, 1218, (1966).

D. Finkelstein, "The Delinearisation of Physics", in the Proceedings of the Symposium on the Foundations of Modern Physics, edited by V. Karimaki (University of Joensuu, Joensuu, Finland, 1978).

D. Finkelstein and C.W. Misner, Annals of Physics, **6**, 230, (1959).

D. Finkelstein and C.W. Misner, "Further Results in Topological Relativity", in les Théories Relativistes de la Gravitation: Royaumont, June 21 - 27 1959, edited by A. Lichnerowicz and M.A. Tonnelat (Centre National de la Recherche Scientifique, Paris, 1962).

D. Finkelstein and G. McCollum, Journal of Mathematical Physics, **16**, 2250, (1975).

D. Finkelstein and J. Rubinstein, Journal of Mathematical Physics, **9**, 1762, (1968).

J.L. Freidman and R.D. Sorkin, Physics Review Letters, **44**, 1100 (1980).

J. Gegenberg, P.F. Kelly, R.B. Mann and D. Vincent, Physics Review D, **37**, 3463 (1988).

R. Geroch and G.T. Horowitz, "Global Structures of Spacetimes", in General Relativity - An Einstein Centenary, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).

A. Guth, Physical Review D, **23**, 347, (1981).

A. Guth, "Phase Transitions in the Very Early Universe", in The Very Early Universe, Proceedings of the Nuffield Workshop, held at Cambridge, 21 June - 9 July, 1982, edited by G.W. Gibbons, S.W. Hawking and S.T.C. Siklos (Cambridge University Press, 1983).

A. Guth, "The New Inflationary Universe", in Inner Space/Outer Space, held at Fermilab, Chicago, 1982, edited by E.W. Kolb, M.S. Turner, D. Lindley, K. Olive and P. Seckel (University of Chicago Press, Chicago, 1986).

T.A. Harriott and J.G. Williams, Journal of Mathematical Physics, **27**, 2706, (1986).

T.A. Harriott and J.G. Williams, Journal of Mathematical Physics, **29**, 179, (1988a).

T.A. Harriott and J.G. Williams, "Imperfect Fluid Solution for a Kinked Metric", in proceedings of the fifth Marcel Grossman Conference, held at the University of Western Australia, 8 - 13 August, 1988, edited by D.G. Blair and M.J. Buckingham (World Scientific, Singapore, 1989).

T.A. Harriott and J.G. Williams, "Homotopically Nontrivial Metric for a Perfect Fluid", in Proceedings of the 2nd Canadian Conference on General Relativity and Relativistic Astrophysics. edited by A. Coley, C. Dyer, and B.Tupper, (World Scientific, Singapore, 1988b).

T.A. Harriott and J.G. Williams, International Journal of Physics, **28**, 511, (1989).

S.W. Hawking and G.R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press, Cambridge, 1973).

R.A. Horn and C.A. Johnson, Matrix Analysis (Cambridge University Press, Cambridge, 1985)

C.J. Isham, "Quantum Gravity 2 - An Overview", in Quantum Gavity 2 - A Second Oxford Symposium, edited by C.J. Isham, R. Pensrose and D.W. Sciama (Oxford University Press, Oxford, 1981), pages. 28-29.

R. Jackiw, "Louiville Field Theory: A Two-Dimensional Model for Gravity", in Quantum Theory of Gravity - Essays in honor of the 60th birthday of Bryce S. De Witt, edited by S.M. Christensen (Adam Hilger, Bristol, UK, 1984)

R. Jackiw, Nuclear Physics B, **252**, 343, (1985).

P.F. Kelly and R.B. Mann, "Exact Solutions of a Dynamical Theory of Gravity in 1+1 Dimensions", Toronto-Waterloo preprint, August 1990, to appear in Physical Review D.

M.D. Kruskal, Physical Review, **119**, 1743, (1960).

M.A.H. MacCallum, "Cosmological Models from a Geometric Point of View", in Cargèse Lectures in Physics, Volume 6, edited by E. Schatzman (Gordon and Breach, N.Y, 1973).

R.B. Mann, "The Simplest Black Holes", University of Waterloo preprint, October 1989, submitted to American Journal of Physics.

R.B. Mann, "Lowest Dimensional Gravity", in Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, University of Winnipeg, 16-18 May 1991, edited by G. Kunstatter et. al. (World Scientific, Singapore) to appear 1992.

R.B. Mann, A. Shiekh and L. Tarasov, Nuclear Physics B, **341**, 134 (1990).

R.B. Mann and T.G. Steele, "Thermodynamics and Quantum Aspects of Black Holes in 1+1 Dimensions", University of Waterloo preprint, June 1990, submitted to Nuclear Physics B.

R.B. Mann, S.M. Morsink, A.E. Sikkhema and T.G. Steele, Physics Review D, **43**, 3948 (1991).

C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation (Freeman, San Francisco, 1973).

M. Nakahara, Geometry, Topology and Physics (Adam Hilger, New York, 1990)

S. Perlis, Theory of Matrices (Addison-Wesley, Reading, MA, 1952), page 187.

D.C. Ravenal and A. Zee, Communications in Mathematical Physics, **98**, 239 (1985).

C. Rogers and W. F. Shadwick, Backlund Transformations and their Applications, Mathematics in Science and Engineering, Volume 161 (Academic Press, 1982).

P.Roman, Theory of Elementary Particles (North-Holland, Amsterdam, 1961).

N. Rosen, "The Field of a Particle in General Relativity Theory", in Unified Field Theories of more than 4 Dimensions including exact solutions, Eighth course of the International School of Cosmology and Gravitation of the Ettore Majorana International Centre for Scientific Culture, held in Erice, 1982, edited by V. de Sabbata and E. Schmutzer (World Scientific, Singapore, 1983).

N. Rosen, Foundations of Physics, **15**, 517, (1985).

N. Sanchez, "Semiclassical Quantum Gravity in Two and Four Dimensions", in Gravitation in Astrophysics - Proceedings of the 1986 Cargèse Summer Institute, Cargèse, France, 15 - 31 July 1986, edited by B. Carter and J.B. Hartle (Plenum Press, NY, 1987)

N. Sanchez, Nuclear Physics B, **266**, 487 (1986).

E. Schrodinger, Expanding Universes (Cambridge University Press, London, 1957), chapter 1.

B.F. Schutz, Geometrical Methods of Mathematical Physics (Cambridge University Press, Cambridge, 1980).

A.R. Shastri, J.G. Williams and P.Zvengrowski, International Journal of Theoretical Physics, **19**, 1, (1980).

A.R. Shastri and P. Zvengrowski, "Types of 3-Manifolds and Addition of Relativistic Kinks", (1991) to appear in Reviews of Mathematical Physics.

A.E. Sikkema and R.B. Mann, Classical and Quantum Gravity, **8**, 219 (1991a).

A.E. Sikkema and R.B. Mann, Physics in Canada bf 47, 78 (1991b).

T.H.R. Skryme, Proceedings of the Royal Society of London, Series A, **247**,260, (1958).

T.H.R. Skyrme, Proceedings of Royal Society of London, Series A, **260**, 127, (1959).

T.H.R. Skyrme, Proceedings of the Royal Society, Series A, **260**, 127, (1961).

T.H.R. Skyrme, Nuclear Physics **31**, 556, (1962).

T.H.R. Skyrme, Journal of Mathematical Physics, **12**, 1735, (1971).

T.H.R. Skyrme, International Journal of Modern Physics A, Vol. 3, No. 12, 2745-51, 1988 and "The origins of Skrymions", in the Proceedings of the Peierls 80th Birthday Symposium, A Breadth of Physics, held at Oxford University, 27 June 1987, edited by R.H. Dalitz, R.B. Stinchombe (World Scientific, Singapore, 1988).

N. Steenrod, The Topology of Fibre Bundles (Princeton University Press, Princeton, New Jersey, 1951), chapter 40.

J.C. Taylor, Gauge Theories of Weak Interactions (Cambridge University Press, Cambridge, 1976)

C. Teitelboim, "The Hamiltonian Structure of Two-Dimensional Spacetime and its Relation with the Conformal Anomaly", in Quantum Theory of Gravity - Essays in honor of the 60th birthday of Bryce S. DeWitt, edited by S.M. Christensen (Adam Hilger, Bristol, UK, 1984).

C. Teitelboim, Physics Letters B, **126**, 41 (1983).

S.P. Thompson, Life of Lord Kelvin (MacMillan, London, 1910)

W.G. Unruh, "Dirac Particles and Geometrodynamical Charge in Curved Geometries", PhD thesis, Princeton University, 1971, Dissertation No. 72-2755, University Microfilms International, Ann Arbor, Michigan.

R.M. Wald, General Relativity (University of Chicago Press, 1984), chapter 6.

S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (Wiley, New York, 1972), page 143.

G.S. Whiston, Journal of Physics A, **14**, 2861, 1981.

J.G. Williams, Journal of Mathematical Physics, **11**, 2611, (1970).

J.G. Williams, Journal of Mathematical Physics, **12**, 308, (1971).

J.G. Williams, Journal of Physics A, **7**, 1871, (1974).

J.G. Williams, Lettere al Nuovo Cimento, **43**, 282, (1985).

J.G. Williams and D. Finkelstein, International Journal of Theoretical Physics, **23**, 61, (1984).

J.G. Williams and R.K.P. Zia, Journal of Physics A, **6**, 1, (1973).

J.G. Williams and P. Zvengrowski, International Journal of Theoretical Physics, **16**, 755 (1977).

J.G. Williams and P. Zvengrowski, "Homotopy and Lorentz Metrics in 2+1 dimensions", in Proceedings of the 3rd Canadian Conference on General Relativity and Relativistic Astrophysics, University of Victoria 4-6 May 1989, edited by A.A. Coley, F.I. Cooperstock and B.O.J. Tupper (World Scientific, Singapore, 1990)

J.G. Williams and P. Zvengrowski, "Kink metrics in (2+1)-dimensional spacetime", Preprint 1991.

J.G. Williams and P. Zvengrowski, "2+1 Gravity Kinks for Multiply Connected Spacetime Manifolds", to appear in the Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, edited by G. Kunstatter et al. (World Scientific, Singapore, 1992)