DEVELOPMENT OF THE INVARIANT THEORY OF KILLING TENSORS DEFINED ON PSEUDO-RIEMANNIAN SPACES OF CONSTANT CURVATURE

by

Jin Yue

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Abstract

The thesis is devoted to the development of certain aspects of the invariant theory of Killing tensors (ITKT) defined on pseudo-Riemannian spaces of constant curvature. A systematic study of ITKT began in 2001 by incorporating the underlying ideas of the classical invariant theory (CIT) of homogeneous polynomials into the geometric study of Killing tensors. One of the main problems in this study is the development of effective algorithms that can be used to determine the invariants, covariants and joint invariants. The methods of infinitesimal generators (which can be traced back to A. Cayley) and moving frames (as recently reformulated by P. Olver and others) have proven to be most effective in tackling this task.

We begin by presenting comprehensive reviews of pseudo-Riemannian geometry and CIT that are blended together in ITKT. We review the notion of an isometry group invariant of Killing tensors and then, in complete analogy with the corresponding notions in CIT, introduce the new concepts of a covariant and a joint invariant of Killing tensors. We use the method of moving frames, in particular the inductive technique introduced by I. Kogan and used in the study of differential invariants, to compute fundamental invariants and covariants of certain vector spaces of Killing tensors.

Our next goal is to formulate and prove an analogue of the result known in CIT as the 1856 lemma of Cayley. More specifically, we establish a Lie algebra representation of the isometry group on the vector space of Killing tensors of arbitrary valence defined on the Minkowski plane. The result is extended by solving the corresponding problem of the determination of a set of fundamental invariants.

We apply these results to solve the problems of group invariant classification of the orthogonal coordinate webs generated by Killing two-tensors defined on the Euclidean and Minkowski planes. Our results compare well with the solutions obtained previously by other methods. In addition, we study the Drach potentials and show that the ten Killing tensors of valence three that define the corresponding first integrals cubic in the momenta are isometrically distinct.

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Chapter 1

Introduction

Classical invariant theory was conceived in the first half of the nineteenth century and was flourishing as one of the most active areas of mathematics by the end of the century. Many eminent mathematicians of that time contributed to its success, including Sylvester, Cayley, Hermite, Clebsch, Gordan, Hilbert, as well as many others. The theory has been resurrected to its present glory in the 20th century through the works of Weyl, Mumford, Rota, Howe, Popov, Vinberg, Olver and many others (see the references in the monograph by Olver [67]). The underlying ideas of the classical invariant theory of homogeneous polynomials have inspired scientists working in other areas of mathematics and physics to look for applications where the theory can form a theoretical foundation leading to fruitful research.

Note that in the 1965 paper [87] Winternitz and Friš studied second order differential operators that commute with the Schrödinger operator and derived two isometry group invariants of Killing two tensors on the Euclidean plane. They were then used to classify orthogonal coordinate webs that afford separation of variables in the two-dimensional non-relativistic equation. A systematic approach to the group invariant study of Killing tensors was initiated independently in 2001 by McLenaghan, Smirnov and The [52] (see also [53]) by planting the underlying ideas of classical invariant theory into the field of geometric study of Killing tensors defined on pesudo-Riemannian spaces of constant curvature. This idea has proven its worth in various applications to the study of Hamiltonian systems, notably the Hamilton-Jacobi theory of orthogonal separation of variables and superintegrability (see Benenti (2004) [4], Kalnins (1975, 1986) [37, 38], Kalnins and Miller (1981, 1983) [44, 45, 46], Kalnins, Kress and Miller (2005) [39, 40, 41], Kalnins, Kress, Pogosyan and Miller (2001) [43], Miller (1977) [58], as well as the relevant references therein).

Before we proceed with a discussion of the basic details of the invariant theory of Killing tensors (to be given in Chapter 4), we first review in Chapter 2 pseudo-Riemannian geometry and establish the requisite language to be used later on. In particular, following the definition and basic properties of pseudo-Riemannian manifolds of constant curvature, we define the Schouten bracket as a natural generalization of the Lie bracket of two vector fields. In turn, the Schouten bracket is employed to define the concept of (generalized) contravariant Killing tensors. We choose to work with contravariant Killing tensors in view of the fact that they appear naturally in Hamiltonian mechanics in this form. The chapter is concluded with a discussion on Killing vectors as generators of the Lie algebra of the isometry Lie group acting on a given pseudo-Riemannian manifold of constant curvature.

Chapter 3 is devoted to a review of the underlying notions, ideas and results of the classical invariant theory of homogeneous polynomials which inspire the next chapter. Thus, we consider the concepts of an invariant and a covariant of binary forms, and a basic technique based on infinitesimal group action that can be used to determine the fundamental sets of invariants and covariants. These ideas come together in the result known as the 1856 lemma of Cayley. This result will inspire us in Chapter 5 to formulate and prove its analogue in the invariant study of Killing tensors defined on two-dimensional pseudo-Riemannian manifolds. Next, we describe the basic ideas of E. Cartan's method of moving frames as was generalized recently by M. Fels and P. Olver, and then further developed by I. Kogan and others (see, [67, 22, 23, 50, 13, 79, 80, 81]).

Recall that the concept of an isometry group invariant of Killing tensors, introduced in [52], can be employed to solve the problems of equivalence and finding canonical forms. Systematically, it can be introduced in complete analogy with the corresponding notion in classical invariant theory discussed in Chapter 3. However, unfortunately, it is not always the case. Difficulties arising from the properties of the group action may preclude one from solving the above problems by the invariants alone. In complete analogy with the corresponding concept of classical invariant theory we introduce the concept of a covariant. This is accomplished by considering the isometry group action on the product of two spaces, namely the vector spaces of Killing tensors in question and the underlying pseudo-Riemannian manifold. It is demonstrated that the invariants and covariants together form a powerful tool to solve the equivalence type problems in the invariant theory of Killing tensors. By considering the action of the isometry group on products of vector spaces of Killing tensors, we proceed to introduce the concept of a joint invariant. We shall not dwell on this concept in the thesis but there are promising indications that the joint invariants can be employed in the study of superintegrable Hamiltonian systems whose integrability is assured by the existence of first integrals which are polynomials in the momenta. Finally we show how the moving frames method can be naturally incorporated into the group invariant study of Killing tensors.

In Chapter 5 we apply the results and concepts obtained and introduced in Chapter 4 to formulate and prove an analogue of the 1856 lemma of Cayley. The result is obtained by means of a representation of the infinitesimal action of the isometry group of the Minkowski plane in the vector space of Killing tensors of arbitrary valence n. Furthermore, for such vector spaces we solve the problem of the determination of the fundamental invariants by making use of the method of moving frames.

Chapter 6 is devoted to applications. We employ the invariants and covariants of Killing two-tensors defined on the Euclidean and Minkowski planes to solve the problem of classification of the orthogonal coordinate webs generated by nontrivial Killing two-tensors. The results compare well with the results obtained previously by Winternitz and Friš [87], Kalnins [37], Rastelli [71], McLenaghan *et al* [52, 54] to solve these problems. Using the results describing invariants of Killing tensors of valence greater than two we study the so-called Drach potentials [17]. Furthermore, we use invariant theory to complete an invariant classification of the ten valence three Killing tensors that define the leading terms of the first integrals isolated by Drach in 1935.

Some of the results presented in this thesis have been published in [79, 80, 89].

Chapter 2

A Review of Pseudo-Riemannian Geometry

The main goal of this chapter is to review the underlying ideas and notions of pseudo-Riemannian geometry that are used in the following chapters. We restrict our attention to the pseudo-Riemannian manifolds of constant curvature which provide the geometric framework for our theory. They are described in Section 2.1. Section 2.2 investigates the isometry groups of pseudo-Riemannian spaces of constant curvature. Then we discuss the intimately related concepts of the Lie derivative and the Schouten bracket. The remaining two sections deal with generalized Killing vectors fields and Killing tensors defined on pseudo-Riemannian spaces of constant curvature, which form the main objects of the study in this thesis.

2.1 Pseudo-Riemannian manifolds of constant curvature

The first fundamental object that we are going to discuss is that of a differentiable manifold. Roughly speaking, a *differentiable manifold* is a Hausdorff topological space which *locally* looks like the Euclidean m-space $\mathbb{E}^m$, although it may be different from $\mathbb{E}^m$ *globally*. A curve and a surface are considered *locally* homeomorphic respectively to $\mathbb{E}$ and $\mathbb{E}^2$ respectively, where $\mathbb{E}$ and $\mathbb{E}^2$ denote Euclidean one and two-space. Calculus on a manifold is assured by the existence of smooth coordinate systems. A manifold may carry other structures that determine its geometry. For example, it can be furnished with a metric tensor, which is a natural generalization of the usual inner product of two vectors on the Euclidean space.

In view of the above, pseudo-Riemannian geometry is the study of a differentiable manifold equipped with a metric tensor of arbitrary signature. The principal special case is Riemannian geometry, in which the metric is positive-definite. Another special case is Lorentzian geometry, which is characterized by an indefinite metric.

In what follows we elaborate on these notions.

Definition 2.1 Let M be an m-dimensional smooth manifold. A *metric* g is a symmetric nondegenerate $(0,2)$ tensor field of constant signature (see Definition 2.3 below).

In other words, g assigns smoothly to each $\mathrm{x} \in M$ a scalar product g_{x} on the tangent space $T_{\mathrm{x}}M$ and the signature of g_{x} is the same for all $\mathrm{x} \in M$. Let $(x^1, \ldots, x^m)$ be a local coordinate system, then the components of the metric tensor g area given by $g_{ij} = g_{\mathrm{x}}(\partial_i, \partial_j)$, where $\partial_i = \frac{\partial}{\partial x^i}, i = 1, \ldots, m$. Thus, in terms of local coordinates, $g = g_{ij}dx^i dx^j$. It follows that for any pair of vectors $\mathrm{U} = U^i\partial_i, \mathrm{V} = V^j\partial_j$

$$g(\mathrm{U}, \mathrm{V}) = g_{ij}U^iV^j. \tag{2.1}$$

The symmetry of the metric means that the matrix g_{ij} is symmetric, while the nondegeneracy condition means that the matrix g_{ij} is nonsingular. We denote its inverse by g^{ij}.

Definition 2.2 A *pseudo-Riemannian manifold* is a manifold M provided with a metric g. Thus, it is an ordered pair (M, g).

Remark 2.1 For simplicity we often denote a pseudo-Riemannian manifold by M rather than (M, g), meaning that the metric under consideration is known and there is no danger of confusion.

Since the matrix g_{ij} is symmetric and real, it can be diagonalized. We define the notion of *signature* of a pseudo-Riemannian manifold as follows,

Definition 2.3 The *signature* of a manifold M is the common value ν, which is the number of negative eigenvalues of the diagonalized metric g_{ij} for every point $\mathrm{x} \in M$.

Example 2.1 The simplest example is the Euclidean m-space $\mathbb{E}^m$, where the metric, with respect to the Cartesian coordinates, is given by

$$g_{ij} = \delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j. \end{cases} \tag{2.2}$$

The space $\mathbb{E}^m = (\mathbb{R}^m, \delta)$ is called the Euclidean m-space and the Riemannian geometry of this space is the metric Euclidean geometry.

Example 2.2 Consider any two tangent vectors $\mathrm{X} = X^i\partial_i, \mathrm{Y} = Y^j\partial_j$, defined on (M, g). If the inner product is defined by

$$g(\mathrm{X_x}, \mathrm{Y_x}) = -\sum_{i=1}^{\nu} X^iY^i\big|_{\mathrm{x}} + \sum_{i=\nu+1}^{m} X^iY^i\big|_{\mathrm{x}}, \tag{2.3}$$

then the metric g, denoted in this case by η_ν is called a Minkowski metric and we call $(\mathbb{R}^m, \eta_\nu)$ a *pseudo-Euclidean space*, denoted by $\mathbb{E}^m_\nu$. If $m \geq 2$, $\mathbb{E}^m_1$ is called *the Minkowski m-space*; if $m = 2$, $\mathbb{E}^2_1$ is called *the Minkowski plane.* A considerable part of this thesis is devoted to the study of Killing tensors defined on the Minkowski plane.

Definition 2.4 A pseudo-Riemannian manifold M is said to be of *constant curvature* if its sectional curvature is constant.

A pseudo-Riemannian manifold for which the curvature tensor vanishes is called *flat.* Simple examples are the Euclidean plane $\mathbb{E}^2$ and the Minkowski plane $\mathbb{E}^2_1$.

Example 2.3 [68] For $m \geq 2, 0 \leq \nu \leq m$, the pseudo-sphere of radius $r > 0$ in the pseudo-Euclidean space $\mathbb{E}^{m+1}_\nu$ is the hyperquadric

$$\mathbb{S}^m_\nu(r) = \{\mathrm{x} \in \mathbb{E}^{m+1}_\nu | g(\mathrm{x}, \mathrm{x}) = r^2\}. \tag{2.4}$$

For $\nu = 0$, $\mathbb{S}^m_0(r)$ is the standard sphere $\mathbb{S}^m(r)$ in $\mathbb{E}^m$. Note the pseudo-sphere $\mathbb{S}^m_\nu(r)$ is of positive constant curvature $K = \frac{1}{r^2}$.

Example 2.4 [68] For $m \geq 2$, $0 \leq \nu \leq m$, the pseudo-hyperbolic space of radius $r > 0$ in pseudo-Euclidean space $\mathbb{E}^{m+1}_{\nu+1}$ is the hyperquadric

$$\mathbb{H}^m_\nu(r) = \{\mathrm{x} \in \mathbb{E}^{m+1}_{\nu+1} | g(\mathrm{x}, \mathrm{x}) = -r^2\}. \tag{2.5}$$

Also, the pseudo-hyperbolic space $\mathbb{H}^m_\nu(r)$ is of negative constant curvature $K = -\frac{1}{r^2}$.

2.2 The Lie derivative and the Schouten bracket

Recall, the set of all isometries of a pseudo-Riemannian manifold M forms a subgroup of the group of all diffeomorphisms of M and carries the natural open topology. Furthermore, it may be shown that this group has the structure of a Lie group (A brief description of Lie group theory will be given in Chapter 2) with respect to the above topology and acts as a Lie transformation group on M.

Definition 2.5 Let (M, g) be a pseudo-Riemannian manifold, a diffeomorphism $\phi\colon M \to M$ is an *isometry* if it preserves the metric

$$\phi^\star g_{\phi(\mathrm{x})} = g_{\mathrm{x}}, \tag{2.6}$$

where $\phi^\star$ is the pull-back map induced by ϕ. That is, $g_{\phi(\mathrm{x})}(\phi_\star u, \phi_\star v) = g_{\mathrm{x}}(u, v)$ for any $u, v \in T_{\mathrm{x}}M$.

In components this can be given by the following formula

$$\frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j} g_{rs}(\phi(\mathrm{x})) = g_{ij}(\mathrm{x}), \tag{2.7}$$

where $x^1, \ldots, x^m$ and $y^1, \ldots, y^m$ are the coordinates of x and $\phi(\mathrm{x})$ respectively. We say that a mapping $\phi\colon M \to M$ is a *local isometry* at a point $\mathrm{x} \in M$ if there is a neighborhood $U \subset M$ of x such that $\phi\colon U \to \phi(U)$ is a diffeomorphism that satisfies (2.7).

It is easy to see that the identity map, the compositions of isometries and the inverse of an isometry are also isometries. Therefore the isometries form a group, called the *isometry group* of the manifold, denoted throughout this thesis by $I(M)$. Since an isometry preserves the length of a vector, and in particular that of an infinitesimal displacement vector, it can be regarded as a *rigid motion*. For instance, the Euclidean group $I(\mathbb{E}^m)$, which is the set of the mappings of the form

$$\phi\colon \mathrm{x} \mapsto A\mathrm{x} + \mathrm{t}, \quad A \in \mathrm{SO}(n), \quad \mathrm{t} \in \mathbb{E}^m,$$

is the isometry group of $\mathbb{E}^m$.

Let X be a vector field on M, X^i the components of the vector field with respect to a local coordinate system $x^1, \ldots, x^m$. The flow generated by X is denoted by $\sigma_t(\mathrm{x})$, which is a map $\sigma\colon \mathbb{R} \times M \to M$, satisfying the following three axioms.

- $\sigma_0(\mathrm{x}) = \mathrm{x}$,
- the map $t \mapsto \sigma_t(\mathrm{x})$ is a solution to differential equations determined by the vector field,
- $\sigma_{t_2}(\sigma_{t_1}(\mathrm{x})) = \sigma_{t_1+t_2}(\mathrm{x})$.

Example 2.5 Let $\mathrm{X} = -y\partial_x + x\partial_y$. Then the flow generated by X is given by

$$\sigma_t(\mathrm{x}) = (x\cos t - y\sin t, x\sin t + y\cos t), \quad \mathrm{x} = (x, y). \tag{2.8}$$

Example 2.6 Let $\mathrm{X} = y\partial_x + x\partial_y$. In this case the corresponding flow is

$$\sigma_t(\mathrm{x}) = (x\cosh t + y\sinh t, x\sinh t + y\cosh t), \quad \mathrm{x} = (x, y). \tag{2.9}$$

With the notion of the flow of vector fields, one can define the concept of the *Lie derivative*, which evaluates the change of a vector field with respect to another vector field.

Let $\mathrm{X}, \mathrm{Y} \in \mathcal{X}(M)$. Let $\mathcal{X}(M)$ denote the set of vector fields on M and $\mathcal{F}(M)$ the set of smooth functions defined on M. Denote by $\sigma_t(\mathrm{x})$ the flow corresponding to X, $\mathrm{y} = \sigma_t(\mathrm{x})$. We will compare the change of Y from x to y. Note that we can not simply take the difference of these two vectors, because they belong to different tangent spaces. We need to employ the push-forward map $(\sigma_{-t})_*$ to map $Y|_{\mathrm{y}} \in T_{\mathrm{y}}M$ back to $(\sigma_{-t})_*(Y|_{\mathrm{y}}) \in T_{\mathrm{x}}M$, now the latter belongs to the same tangent space as $Y|_{\mathrm{x}}$, we can make the difference of these two and then take the limit of quotient of this quantity with t.

Definition 2.6 The *Lie derivative* of a vector field Y with respect to a vector X is defined by

$$\mathcal{L}_{\mathrm{X}}\mathrm{Y}|_{\mathrm{x}} = \lim_{t\to 0}\frac{1}{t}((\sigma_{-t})_*(Y|_{\mathrm{y}}) - Y|_{\mathrm{x}}), \tag{2.10}$$

where $\mathrm{y} = \sigma_t(\mathrm{x})$.

Remark 2.2 One can immediately verify that the following formulas hold true

$$\begin{aligned}\mathcal{L}_{\mathrm{X}}\mathrm{Y}|_{\mathrm{x}} &= \lim_{t\to 0}\frac{1}{t}(Y|_{\mathrm{x}} - (\sigma_t)_*(Y|_{\mathrm{z}})) \\ &= \lim_{t\to 0}\frac{1}{t}(Y|_{\mathrm{y}} - (\sigma_t)_*(Y|_{\mathrm{x}})),\end{aligned}$$

where $\mathrm{y} = \sigma_t(\mathrm{x})$ and $\mathrm{z} = \sigma_{-t}(\mathrm{x})$.

Definition 2.7 Let $\mathrm{X}, \mathrm{Y} \in \mathcal{X}(M)$ and $f \in \mathcal{F}(M)$, we define *Lie bracket* as follows.

$$[\mathrm{X}, \mathrm{Y}](f) = \mathrm{X}(\mathrm{Y}(f)) - \mathrm{Y}(\mathrm{X}(f)). \tag{2.11}$$

It is straightforward to show that

$$\mathcal{L}_X Y = [X, Y]. \tag{2.12}$$

The Lie bracket or Lie derivative operator has the following important properties, which are widely used in differential geometry and mathematical physics.

Proposition 2.1 *Suppose* $X, Y, Z, X_1, X_2, Y_1, Y_2 \in \mathcal{X}(M)$, $k_1, k_2 \in \mathbb{R}$, *and* $\phi : M \to N$ *is a diffeomorphism from* M *to a manifold* N. *Then the following properties hold true.*

- *bilinearity*

$$\begin{aligned} [X, k_1 Y_1 + k_2 Y_2] &= k_1 [X, Y_1] + k_2 [X, Y_2], \\ [k_1 X_1 + k_2 X_2, Y] &= k_1 [X_1, Y] + k_2 [X_2, Y], \end{aligned}$$

- *skew-symmetry*

$$[X, Y] = -[Y, X],$$

- *Jacobi identity*

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0.$$

Also one can prove directly the following useful formulas:

$$\begin{aligned} \mathcal{L}_{fX} Y &= f[X, Y] - Y(f)X, \\ \mathcal{L}_X (fY) &= f[X, Y] + X(f)Y, \\ \phi_*[X, Y] &= [\phi_* X, \phi_* Y]. \end{aligned} \tag{2.13}$$

Now one can extend the definition of the Lie derivative to general tensor fields. This operator satisfy the following properties.

Proposition 2.2

$$\begin{aligned} \mathcal{L}_X (t_1 + t_2) &= \mathcal{L}_X t_1 + \mathcal{L}_X t_2, \\ \mathcal{L}_X (t_1 \otimes t_2) &= (\mathcal{L}_X t_1) \otimes t_2 + t_1 \otimes (\mathcal{L}_X t_2), \\ \mathcal{L}_{[X,Y]} t &= \mathcal{L}_X \mathcal{L}_Y t - \mathcal{L}_Y \mathcal{L}_X t. \end{aligned}$$

In components one can write

$$\mathcal{L}_{\mathbf{X}} f = X^k f_{,k}, \quad f \in \mathcal{F}(M), \tag{2.14}$$

$$[\mathbf{X}, \mathbf{Y}]^k = X^j Y^k_{,j} - Y^j X^k_{,j}. \tag{2.15}$$

To derive a formula for $\mathcal{L}_{\mathbf{X}}\mathbf{t}$ one needs to use the Leibniz rule and the following fundamental formulas,

$$\mathcal{L}_{\partial_i}(\partial_j) = 0, \quad \mathcal{L}_{\partial_i}(\mathrm{d}x^j) = 0, \quad \mathcal{L}_{\mathbf{X}}(\mathrm{d}x^j) = X^j_{,k}\mathrm{d}x^k. \tag{2.16}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \quad f_{,i} = \frac{\partial f}{\partial x^i}, \quad i = 1, \dots m.$$

Therefore

$$\begin{aligned} (\mathcal{L}_{\mathbf{X}}\mathbf{t})^{i_1 \dots i_r}_{j_1 \dots j_s} &= t^{i_1 \dots i_r}_{j_1 \dots j_s, k} X^k \\ &\quad - t^{k i_2 \dots i_r}_{j_1 \dots j_s} X^{i_1}_{,k} - \dots - t^{i_1 \dots i_{r-1} k}_{j_1 \dots j_s} X^{i_r}_{,k} \\ &\quad + t^{i_1 \dots i_r}_{k j_2 \dots j_s} X^k_{,j_1} + \dots + t^{i_1 \dots i_r}_{j_1 \dots j_{s-1} k} X^k_{,j_s}. \end{aligned} \tag{2.17}$$

Definition 2.8 Let $\mathbf{X} \in \mathcal{X}(M)$. If an infinitesimal displacement given by $\epsilon\mathbf{X}$, (ϵ being infinitesimal) generates an isometry, the vector field is called a *Killing vector field*. Thus, $\mathbf{X}$ is a Killing vector field if and only if

$$\mathcal{L}_{\mathbf{X}} g = 0. \tag{2.18}$$

Now using Lie derivative formulas

$$\begin{aligned} \mathcal{L}_{\mathbf{X}} g &= (\mathcal{L}_{\mathbf{X}} g_{ij}) dx^i \otimes dx^j + g_{ij}(\mathcal{L}_{\mathbf{X}} dx^i) \otimes dx^j + g_{ij} dx^i \otimes (\mathcal{L}_{\mathbf{X}} dx^j) \\ &= X^k g_{ij,k} dx^i \otimes dx^j + g_{ij} X^i_{,k} dx^k \otimes dx^j + g_{ij} dx^i \otimes X^j_{,k} \otimes dx^k \\ &= \left(X^k g_{ij,k} + g_{kj} X^k_{,i} + g_{ik} X^k_{,j}\right) dx^i \otimes dx^j. \end{aligned} \tag{2.19}$$

Since $\mathcal{L}_{\mathbf{X}} g$ is a symmetric tensor field (see (2.19)) of type $(0, 2)$, $\mathbf{X}$ is a Killing vector field if and only if the following system of PDEs hold.

$$X^k g_{ij,k} + g_{kj} X^k_{,i} + g_{ik} X^k_{,j} = 0, \quad i, j, k = 1, \dots, m. \tag{2.20}$$

A natural generalization of Killing vector fields is the notion of Killing tensor fields, which turn out to be very important in problems arising in mathematical physics, including the problem of integrable and superintegrable Hamiltonian systems and the problem of orthogonal separability in the Hamilton-Jacobi theory of separation of variables (see for example [62, 4, 14, 18, 19, 44, 46, 34, 46, 39, 40, 41, 43, 52, 53, 54, 55, 56, 59, 79, 80] as well as the relevant references therein). Killing tensor fields can be defined via the *Schouten bracket* [76]), which, we describe as follows.

Definition 2.9 [76] Let **P** and **Q** be two contravariant tensors of valence $p+1$ and $q+1$, then the Schouten bracket $[\mathbf{P}, \mathbf{Q}]$ is a contravariant $a+b-1$-tensor whose components are specified in terms of local coordinates as follows.

$$\begin{aligned}
[\mathbf{P}, \mathbf{Q}]^{i_1 \cdots i_{p+q-1}} = & \left(\sum_{k=1}^{p} P^{(i_1 \cdots i_{k-1}|\mu|i_k \cdots i_{(p-1)}}\right) \partial_\mu Q^{i_p \cdots i_{(p+q-1)})} + \\
& \left(\sum_{k=1}^{p} (-1)^k P^{[i_1 \cdots i_{k-1}|\mu|i_k \cdots i_{(p-1)}}\right) \partial_\mu Q^{i_p \cdots i_{(p+q-1)}]} - \\
& \left(\sum_{j=1}^{q} Q^{(i_1 \cdots i_{j-1}|\mu|i_j \cdots i_{(q-1)}}\right) \partial_\mu P^{i_q \cdots i_{(p+q-1)})} - \\
& \left(\sum_{j=1}^{p} (-1)^{pq+p+q+j} Q^{[i_1 \cdots i_{j-1}|\mu|i_j \cdots i_{(q-1)}}\right) \partial_\mu P^{i_q \cdots i_{(p+q-1)}]}, \quad (2.21)
\end{aligned}$$

where (,) and [,] are the symmetrizer and skew symmetrizer respectively, and the notation $|\mu|$ means that the signature μ is excluded from the symmetrization or skew symmetrization.

Thus, if **P**, **Q** are symmetric, the second and last terms of (2.21) vanish, while if **P**, **Q** are skew symmetric, then the first and third terms of (2.21) vanish.

Example 2.7 The Schouten bracket of two symmetric contravariant tensors **P**, **Q** of valence two is given by (assume $m = 2, x^1 = x, x^2 = y$.)

$$\begin{aligned}
[\mathbf{P}, \mathbf{Q}]^{abc} &= P^{(ab}_{,d} Q^{c)d} - Q^{(ab}_{,d} P^{c)d} \\
&= P^{(ab}_{,x} Q^{c)1} + P^{(ab}_{,y} Q^{c)2} - Q^{(ab}_{,x} P^{c)1} - Q^{(ab}_{,y} P^{c)2}.
\end{aligned}$$

Thus, for example

$$
\begin{aligned}
[\mathrm{P},\mathrm{Q}]^{112} &= \frac{1}{3}(P^{11}_{,d}Q^{d2} + P^{12}_{,d}Q^{d1} + P^{21}_{,d}Q^{d1} - Q^{11}_{,d}P^{d2} - Q^{12}_{,d}P^{d1} - Q^{21}_{,d}P^{d1}) \\
&= \frac{1}{3}(P^{11}_{,x}Q^{12} + P^{11}_{,y}Q^{22} + P^{12}_{,x}Q^{11} + P^{12}_{,y}Q^{21} + P^{21}_{,x}Q^{11} + P^{21}_{,y}Q^{21} \\
&\quad -Q^{11}_{,x}P^{12} - Q^{11}_{,y}P^{22} - Q^{12}_{,x}P^{11} - Q^{12}_{,y}P^{21} - Q^{21}_{,x}P^{11} - Q^{21}_{,y}P^{21}) \\
&= \frac{1}{3}(P^{11}_{,x}Q^{12} + P^{11}_{,y}Q^{22} + 2P^{12}_{,x}Q^{11} + 2P^{12}_{,y}Q^{12} \\
&\quad -Q^{11}_{,x}P^{12} - Q^{11}_{,y}P^{22} - 2Q^{12}_{,x}P^{11} - 2Q^{12}_{,y}P^{12}).
\end{aligned}
$$

We remark that (2.21) for the case $p = q = 0$ is equivalent to (2.15).

2.3 Generalized Killing tensor fields

Let (M, g) be an m-dimensional pseudo-Riemannian manifold of constant curvature.

Definition 2.10 A symmetric contravariant tensor K of valence n defined in (M, g) is said to be a *generalized Killing tensor* (GKT) of order p if and only if

$$[[\dots[K, g], g], \dots, g] = 0 \quad (p+1 \text{ brackets}), \tag{2.22}$$

where [,] denotes the Schouten bracket (2.21).

It follows immediately from the $\mathbb{R}$-bilinear properties of Schouten bracket that GKTs of the same valence and order constitute a *vector space*. Note that GKTs of order zero are the standard Killing tensors defined by the system of over-determined PDEs

$$[\mathrm{K}, g] = 0. \tag{2.23}$$

The concept of a generalized Killing tensor defined on the Minkowski space $\mathbb{E}^m_1$ was introduced by Nikitin and Prilipko [62] as a generalized symmetry of Klein-Gordon-Fock equation. The authors also derived the formulas for the dimensions of the vector spaces that GKTs constitute. More recently, these differential geometric objects were independently re-introduced in a more general setting by Eastwood [18, 19] within the framework of the study of over-determined systems of PDEs by the methods of representation theory.

More specifically, it has been shown that the vector space of the solutions of the over-determined system of PDEs (2.22) is preserved by the action induced by $\mathrm{SL}(m+1,\mathbb{R})$. From this perspective the author defines such a vector space as an irreducible representation of $sl(m+1,\mathbb{R})$ (see also [51]), and then derives the formula for the dimension of the vector space $\mathcal{K}_p^n(M)$ of the generalized Killing tensors of valence n and order p defined on (M,g). The dimension d of the vector space $\mathcal{K}_p^n(M), n \geq 0, p \geq 1$ is determined by *Nikitin-Prilipko-Eastwood (NPE) formula*:

$$d = \dim \mathcal{K}_p^n(M) = \frac{p+1}{m}\binom{n+m-1}{m-1}\binom{p+n+m}{m-1}, \tag{2.24}$$

where $m = \dim M$. We immediately recognize that for $p = 0$ the formula (2.24) reduces to the *Delong-Takeuchi-Thompson (DTT) formula* (see [51] and [54] as well as the relevant references therein for more details). Moreover the elements of $\mathcal{K}_p^n(M)$ are specified by d arbitrary parameters $a_1, \ldots, a_d$, where d is given by (2.24), with respect to a given basis.

Example 2.8 Consider the contravariant Killing tensors of valence two on the Euclidean plane $\mathbb{E}^2$, the equation (2.23) with respect to the Cartesian coordinates (x,y) amounts to

$$K^{ij}_{,k} + K^{jk}_{,i} + K^{ki}_{,j} = 0, \quad x^1 = x, x^2 = y, \tag{2.25}$$

which is a system of over-determined PDEs

$$\begin{aligned} &\partial_y K^{22} = 0, \quad \partial_x K^{22} + 2\partial_y K^{12} = 0, \\ &\partial_x K^{11} = 0, \quad \partial_y K^{11} + 2\partial_x K^{11} = 0. \end{aligned} \tag{2.26}$$

Solving (2.26), we obtain the following general formula

$$\mathrm{K} = \begin{pmatrix} a_1 + 2a_4 y + a_6 y^2 & a_3 - a_4 x - a_5 y - a_6 xy \\ a_3 - a_4 x - a_5 y - a_6 xy & a_2 + 2a_5 x + a_6 x^2 \end{pmatrix}. \tag{2.27}$$

Note that the parameters a_i, $i = 1, \ldots, 6$, appear as constants of integration. The number of these constants represents the dimension of the vector space of Killing tensors of valence two defined on the Euclidean plane.

If $n = 1, p = 0$, the equation (2.22) reduces to $\mathcal{L}_{\mathrm{X}} g = 0$, i.e. (2.18). The solutions are Killing vector fields, which will be the subjects of the following section.

2.4 Killing vector fields

Recall Killing vector fields (see Example 2.8) are the vector fields whose infinitesimal displacements generate isometries, that is, the metric remains constant with respect the flow generated by the vector field. Thus, the local geometry does not change as one moves along the flow generated by a Killing vector field. In this sense, the Killing vector fields represent the direction of the symmetry of a manifold.

There may be more independent Killing vector fields than the dimension of the manifold. Note that the number of independent symmetries has no direct connection with the dimension of M, the maximum number, however, does. Indeed, when M is of constant curvature, we know that this number is $\frac{1}{2}m(m+1)$, where $m = \mathrm{dim}M$.

Example 2.9 Consider Killing vector fields on the Euclidean plane $\mathbb{E}^2$. The Killing equation (2.20) reads, with respect to the Cartesian coordinates (x, y)

$$X^1_{,1} = 0, \quad X^2_{,2} = 0, \quad X^2_{,1} + X^1_{,2} = 0, \quad x^1 = x, x^2 = y. \tag{2.28}$$

Solving (2.28), one arrives at the general solution

$$\mathrm{X} = (a_1 + a_3 y)\partial_x + (a_2 - a_3 x)\partial_y. \tag{2.29}$$

We note that there are three independent Killing vectors

$$\mathrm{X} = \partial_x, \quad \mathrm{Y} = \partial_y, \quad \mathrm{R} = -y\partial_x + x\partial_y, \tag{2.30}$$

corresponding to the translations and the rotations respectively. Their commutator relations are given by

$$[\mathrm{X}, \mathrm{Y}] = 0, \quad [\mathrm{X}, \mathrm{R}] = \mathrm{Y}, \quad [\mathrm{Y}, \mathrm{R}] = -\mathrm{X}. \tag{2.31}$$

Note also that these basic Killing vector fields generate the Lie algebra $i(\mathbb{E}^2)$ of the isometry group $I(\mathbb{E}^2)$.

Proposition 2.3 [69] *A vector field* X *defined on a manifold* M *is a Killing vector field if and only if the mapping* $\mathrm{Y} \mapsto \nabla_{\mathrm{Y}}\mathrm{X}, (\mathrm{Y} \in \mathcal{X}(M))$ *is a skew-symmetric tensor of type* $(1, 1)$.

Proof Using the properties of Lie derivative (see Proposition 2.1) and covariant differentiation, one can write

$$\begin{aligned}(\mathcal{L}_{\mathrm{X}}g)(\mathrm{Y},\mathrm{Z}) &= (\nabla_{\mathrm{X}}g)(\mathrm{Y},\mathrm{Z}) + (\nabla_{\mathrm{Y}}\mathrm{X},\mathrm{Z}) + (\mathrm{Y},\nabla_{\mathrm{Z}}\mathrm{X}) \\ &= (\nabla_{\mathrm{Y}}\mathrm{X},\mathrm{Z}) + (\mathrm{Y},\nabla_{\mathrm{Z}}\mathrm{X}). \end{aligned} \tag{2.32}$$

Therefore $\mathcal{L}_{\mathrm{X}}g = 0$ if and only if $(\nabla_{\mathrm{Y}}\mathrm{X},\mathrm{Z}) = -(\mathrm{Y},\nabla_{\mathrm{Z}}\mathrm{X})$ for all Y,Z. This means that the mapping $\mathrm{Y} \mapsto \nabla_{\mathrm{Y}}\mathrm{X}$, $(\mathrm{Y} \in \mathcal{X}(M))$ is a skew-symmetric tensor of type $(1,1)$. ∎

Example 2.10 [60] Consider the Killing vectors defined on the Minkowski space-time $\mathbb{E}^4_1$, the Killing equation reads (with respect to the pseudo-Cartesian coordinates)

$$X^j_{,i} + X^i_{,j} = 0, \tag{2.33}$$

solving which one sees that X^i is at most the first order in x, with the constant solutions

$$X^j_{(i)} = \delta^j_i, \qquad 0 \le i \le 3, \tag{2.34}$$

which correspond to space-time translations. Now let $X_i = a_{ij}x^j$, where a_{ij} is constant. It follows from (2.33) that a_{ij} is skew-symmetric with respect to i,j. Since $\binom{4}{2} = 6$, there are 6 independent solutions of this form, three of which are given by

$$X_{(i)0} = 0, \; X_{(i)m} = \epsilon_{imn}x^n, \qquad 1 \le i,m,n \le 3 \tag{2.35}$$

corresponding to spatial rotations about the x^i-axis, while the others are

$$X_{(j)0} = x^j, \; X_{(j)k} = -\delta_{jk}x^0, \qquad 1 \le j,k \le 3. \tag{2.36}$$

which correspond to Lorentz boosts along the x^j-axis.

On the m-dimensional ($m \ge 2$) Minkowski space-time, there are $\frac{m(m+1)}{2}$ independent Killing vector fields, m of which correspond to translations, $m-1$ to boosts and $\frac{(m-1)(m-2)}{2}$ generate space rotations. This is an example of a maximally symmetric space in the sense that an m-dimensional space admits at most $\frac{m(m+1)}{2}$ independent Killing vectors.

Proposition 2.4 [69] *The set of Killing vector fields forms a Lie algebra $i(M)$ of dimension $\le \frac{1}{2}m(m+1)$.*

Let us look at another example.

Example 2.11 [60] Let $\mathbb{S}^2$ be the unit sphere in the Euclidean 3-space with the metric given by $g = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \, \mathrm{d}\phi \otimes \mathrm{d}\theta$. Then the Killing equations are

$$\begin{aligned} \partial_\theta X_\theta &= 0, \\ 2\partial_\phi X_\phi + 2\sin\theta \, \cos\theta X_\theta &= 0, \\ \partial_\theta X_\phi + \partial_\phi X_\theta - 2\cot\theta X_\phi &= 0. \end{aligned} \tag{2.37}$$

Solving the system of PDEs (2.37), one arrives at the following general formula for a Killing vector field

$$\begin{aligned} \mathrm{X} &= X^\theta \, \partial_\theta + X^\phi \, \partial_\phi \\ &= a_1(\sin\phi \, \partial_\theta + \cos\phi \, \cot\theta \, \partial_\phi) + a_2\partial_\phi \\ &\quad a_3(\cos\phi \, \partial_\theta - \sin\phi \, \cot\theta \, \partial_\phi). \end{aligned} \tag{2.38}$$

The three basic Killing vectors,

$$\begin{aligned} \mathrm{K}_x &= -\cos\phi \, \partial_\theta + \cot\theta \, \sin\phi \, \partial_\phi, \\ \mathrm{K}_y &= \sin\phi \, \partial_\theta + \cot\theta \, \cos\phi \, \partial_\phi, \\ \mathrm{K}_z &= \partial_\phi \end{aligned} \tag{2.39}$$

generate rotations round the x, y and z axes, respectively. Also notice that these Killing vectors generate the Lie algebra $so(3)$, which reflects the fact that $\mathbb{S}^2$ is the homogeneous space SO(3)/SO(2) and the metric on $\mathbb{S}^2$ retains this SO(3) symmetry. In general we have $\mathbb{S}^m = \mathrm{SO}(m+1)/\mathrm{SO}(m)$, which, with respect to the usual metric, has $\frac{1}{2}m(m+1)$ independent Killing vectors. They, in turn, generate the Lie algebra $so(m+1)$.

Chapter 3

Classical Invariant Theory

In this chapter we briefly review the basic ideas of the *classical invariant theory* (CIT) of homogeneous polynomials. Ultimately, this will establish a prerequisite language to be used later in the *invariant theory of Killing tensors* (ITKT) defined on pseudo-Riemannian spaces of constant curvature. In particular, in Section 3.1 we describe the notion of a vector space of binary forms, as well as the corresponding notions of an invariant and covariant of binary forms that form the back bone of CIT. The concept of an invariant is also extended to general Lie group actions. In the next section we proceed to introduce *the method of infinitesimal generators* and *the method of moving frames* in its modern formulation and show how they can be effectively employed to compute invariants, covariants and joint invariants. As an illustration of the method of infinitesimal generators, we review the classical result known as the 1856 lemma of Cayley. It concerns the problem of the determination of the infinitesimal action of the special linear group $\mathrm{SL}(2, \mathbb{R})$ on the vector space of binary forms of arbitrary degree. An analogue of this classical result in ITKT has been obtained in [89] and will be presented in Chapter 5.

As is well-known, CIT studies intrinsic properties of polynomials, that is, those properties that remain fixed with respect to changes of coordinates. A number of eminent mathematicians, including Gauss, Hilbert, Cayley, Gordan, Sylvester and others worked in this area and made significant contributions that shaped CIT. In recent years CIT has reinvented itself once again through new aspects of Lie group theory (notably, the generalizations of the moving frames method due to Fels and Olver [22, 23] and Kogan [50], see also the relevant references therein), the rise of modern computer algebra and new applications in other areas of mathematics (see Hilbert [31] and Olver [67] for a complete review and related references).

3.1 Binary forms, invariants and covariants

One of the main goals of CIT is to solve the problem of the determination of complete sets of the *invariants* and *covariants* of the vector spaces of homogeneous polynomials. These sets can be used to solve the intimately related problems of equivalences and canonical forms. They, in turn, can in principle completely characterize the underlying intrinsic properties of a given polynomial. Of particular importance are the vector spaces of homogeneous polynomials of arbitrary degree in two variables, called *binary forms* (referred to by Cayley [10] as *quantics*).

Definition 3.1 *A binary form of degree* n defined over reals is given by

$$P(x,y)=\sum_{i=0}^{n}\binom{n}{i}a_i x^i y^{n-i},\quad x,y\in\mathbb{R}. \tag{3.1}$$

The set of all binary forms of degree n constitutes a vector space, denoted by $\mathcal{P}^n(\mathbb{R}^2)$. To consider the intrinsic properties of forms, one introduces a change of variables given by a (real) general linear transformation:

$$\bar{x}=\alpha x+\beta y,\quad \bar{y}=\gamma x+\delta y,\qquad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{GL}(2,\mathbb{R}). \tag{3.2}$$

Define

$$\bar{P}(\bar{x},\bar{y})=\bar{P}(\alpha x+\beta y,\gamma x+\delta y)=P(x,y). \tag{3.3}$$

This means that $P(x,y)$ is mapped into $\bar{P}(\bar{x},\bar{y})$ under the linear transformation (3.2). In this case the transformation (3.2) induces the corresponding change for the coefficients of the binary forms. The general formula is presented in [67] as follows.

$$a_i=\sum_{k=0}^{n}\tilde{a}_k\sum_{j=\max\{0,i+k-n\}}^{\min\{i,k\}}\binom{i}{j}\binom{n-i}{k-j}\alpha^j\beta^{k-j}\gamma^{i-j}\delta^{n+j-i-k},\quad i=0,\dots,n. \tag{3.4}$$

Denote by Σ the parameter space of the vector space $\mathcal{P}^n(\mathbb{R}^2)$ spanned by $a_0,\dots,a_n$, then $\dim\mathcal{P}^n(\mathbb{R}^2)=n+1$. We define now the concepts of an invariant and a covariant of binary forms.

Definition 3.2 [67] A $\mathrm{GL}(2,\mathbb{R})$-*invariant* I of weight k of $\mathcal{P}^n(\mathbb{R}^2)$ is a function $\Sigma \to \mathbb{R}$ that satisfies

$$I(a_0,\ldots,a_n) = (\alpha\delta - \beta\gamma)^k I(\tilde{a}_0,\ldots,\tilde{a}_n) \tag{3.5}$$

under the transformation (3.4).

More generally, we can consider the action of a Lie group G on an m-dimensional manifold M. We first briefly review the fundamental notations regarding Lie group, Lie transformation groups and Lie group actions. More details can be found in the Monograph by Olver [67].

Definition 3.3 An analytic Lie group G is a group which carries the structure of an analytic manifold in such a way that the group multiplication $(a,b) \mapsto a \cdot b,\ a,b \in G$ and inversion $a \mapsto a^{-1}, a \in G$ define analytic maps.

Definition 3.4 A *Lie transformation group* is a Lie group G which acts on an m-dimensional vector space X analytically. This means that the group action $\omega(g,\mathrm{x}) = g \cdot \mathrm{x}$ defines an analytic map $G \times X \to X$.

A Lie group of dimension m is often referred to as an m-parameter group. The parameters here are the local coordinates that the group elements depend on.

Example 3.1 The simplest example of an m-parameter Lie group is the Euclidean m-space $\mathbb{E}^m$, one system of coordinates is the usual Cartesian coordinates $(x^1,\ldots,x^m) \in \mathbb{E}^m$. It is trivial to see that the group operations, namely vector addition $\mathrm{x}+\mathrm{y}$ and inversion $-\mathrm{x}$, depend analytically on the coordinates.

Example 3.2 [67] The general linear group $\mathrm{GL}(2,\mathbb{R})$ that consists of all 2×2 matrices of nonzero determinant forms an analytic 4-parameter Lie group. The (global) parameters are the entries $\alpha,\beta,\gamma,\delta$ of the general nonsingular matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We can identify $\mathrm{GL}(2,\mathbb{R}) = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},\ \alpha\delta - \beta\gamma \neq 0 \right\} \subset \mathbb{R}^4$ as an open subset of a four-dimensional manifold. Clearly matrix multiplication and inversion are analytic functions in the entries.

We remark that $GL(2,\mathbb{R})$ consists of two connected open subsets of $\mathbb{R}^4$, distinguished by the sign of the determinant. The subset $GL(2,\mathbb{R})^+ = \{A \in GL(2,\mathbb{R}),\ \det A > 0\}$ forms a 4-parameter subgroup: the group of orientation-preserving planar linear maps.

Example 3.3 [67] The special linear group $SL(2,\mathbb{R})$ forms a three-dimensional submanifold of $\mathbb{R}^4$, defined by the condition $\alpha\delta - \beta\gamma = 1$, which is a regular level set of the determinant function. Since the group operations of matrix multiplication and inversion are still analytic when restricted to $SL(2,\mathbb{R})$, it is a three-parameter analytic Lie group. Note there is no convenient global parametrization, one can introduce local parameters on specific open subsets. For instance, on the subset where $\alpha \neq 0$, the matrix can take the following form $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha^{-1}(1+\beta\gamma) \end{pmatrix}$. Thus we use α, β and γ to parametrize the group. Similarly, if $\gamma \neq 0$, one can use the same condition to solve for $\beta = \gamma^{-1}(\alpha\delta - 1)$ and therefore use α, γ, δ as local coordinates.

We now define the concept of an invariant of a Lie group action.

Definition 3.5 Let G be a Lie group acting on a vector space X. A real-valued function $I : X \to \mathbb{R}$ is a *G-invariant* if and only if $I(g \cdot \mathrm{x}) = I(\mathrm{x})$ for all $\mathrm{x} \in X$ and all $g \in G$. If the above condition holds only for all $g \in H$, where $H \subset G$ is some neighborhood near the identity I of G, then the function I is called a *local invariant*.

Example 3.4 Let X be the Euclidean plane $\mathbb{E}^2$ and G the rotation group $SO(2,\mathbb{R})$.

$$\bar{x} = x\cos t - y\sin t, \quad \bar{y} = x\sin t + y\cos t, \qquad t \in \mathbb{R}. \tag{3.6}$$

Then the distance function r from the origin to the point (x, y) is an invariant (so is any function $F(r)$ thereof), since

$$(x\cos t - y\sin t)^2 + (x\sin t + y\cos t)^2 = x^2 + y^2.$$

Thus a circle centered at the origin is a rotationally invariant subset of the Euclidean plane. Since the circle contains no other rotationally invariant subsets, it is an orbit of $SO(2,\mathbb{R})$. Any other invariant subset of $SO(2,\mathbb{R})$ must be the union of circles that are centered at the origin. Note that the only fixed point of the rotation is the origin. Two points that lie in the same circle (centered at the origin) are equivalent, and the origin itself is an equivalence class. For each equivalence class, we can, for example, choose $(r, 0)$ $(r \geq 0)$ as its canonical form. We conclude that, by using the above invariant, the problem of equivalences and canonical forms can be completely solved.

In CIT, sometimes it is not sufficient to solve the equivalence problem in question by just knowing invariants. In such a case one needs to consider those functions depending on both the parameters and the coordinate variables that remain unchanged. This leads to the definition of a covariant of binary forms.

Definition 3.6 [67] A $\mathrm{GL}(2,\mathbb{R})$-*covariant* C of weight k of $\mathcal{P}^n(\mathbb{R}^2)$ is a function $\Sigma \times \mathbb{R}^2 \to \mathbb{R}$ that satisfies

$$C(a_0, \ldots, a_n, x, y) = (\alpha\delta - \beta\gamma)^k C(\tilde{a}_0, \ldots, \tilde{a}_n, \tilde{x}, \tilde{y}) \tag{3.7}$$

under the transformation formulas (3.3) and (3.4).

Remark 3.1 It is easy to see that, an invariant is necessarily a covariant - it is just a covariant which is independent of the coordinates. The simplest example of a covariant is the binary form itself.

With the above notions in mind, we now recall a well-known example [67], which serves to demonstrate the main features of CIT.

Example 3.5 The general quadratic binary form is of the form

$$Q(x, y) = a_2 x^2 + 2a_1 xy + a_0 y^2, \quad x, y \in \mathbb{R}. \tag{3.8}$$

The action of $\mathrm{GL}(2,\mathbb{R})$ on the parameter space Σ spanned by a_0, a_1 and a_2 is given by

$$\begin{aligned} a_2 &= \alpha^2\tilde{a}_2 + 2\alpha\gamma\tilde{a}_1 + \gamma^2\tilde{a}_0, \\ a_1 &= \alpha\beta\tilde{a}_2 + (\alpha\gamma + \beta\gamma)\tilde{a}_1 + \gamma\delta\tilde{a}_0, \\ a_0 &= \beta^2\tilde{a}_2 + 2\beta\delta\tilde{a}_1 + \delta^2\tilde{a}_0. \end{aligned} \tag{3.9}$$

One claims that the quantity $\Delta = a_0 a_2 - a_1^2$ is an $\mathrm{GL}(2,\mathbb{R})$-invariant according to Definition 3.2. Indeed, we have the following formula

$$\Delta = (\alpha\gamma - \beta\delta)^2\tilde{\Delta}. \tag{3.10}$$

This can be verified as follows.

Note

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.11}$$

and the quadratic forms can be represented as matrix products

$$\begin{aligned} Q(x,y) &= (x,y) \begin{pmatrix} a_2 & a_1 \\ a_1 & a_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \\ \tilde{Q}(\tilde{x},\tilde{y}) &= (\tilde{x},\tilde{y}) \begin{pmatrix} \tilde{a}_2 & \tilde{a}_1 \\ \tilde{a}_1 & \tilde{a}_0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \\ Q(x,y) &= \tilde{Q}(\tilde{x},\tilde{y}). \end{aligned} \tag{3.12}$$

In view of (3.11) and (3.12) one can write

$$\begin{pmatrix} a_2 & a_1 \\ a_1 & a_0 \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \tilde{a}_2 & \tilde{a}_1 \\ \tilde{a}_1 & \tilde{a}_0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{3.13}$$

and the formula (3.10) follows easily from (3.13).

This is not surprising, since $\Delta = a_2 a_0 - a_1^2$ is the discriminant of a quadratic form. The equivalence classes and canonical forms of homogeneous quadratic binary forms are solved in the following Table 3.1(see [67]).

Equivalence class	Canonical form	Invariant
I	$x^2 + y^2$	$\Delta > 0, Q(x,y) \geq 0$
II	$-x^2 - y^2$	$\Delta > 0, Q(x,y) \leq 0$
III	xy, or $x^2 - y^2$	$\Delta < 0$
IV	x^2	$\Delta = 0, Q(x,y) \geq 0$
V	$-x^2$	$\Delta = 0, Q(x,y) \leq 0$
VI	0	$Q(x,y) \geq 0$

Table 3.1: Canonical forms for real quadratic forms

One can use the fundamental invariant Δ to describe the underlying geometry which is defined by (3.8). Note that all centrally symmetric conic sections in $\mathbb{E}^2$ are given by

$$Q(x,y) = a_2 x^2 + 2a_1 xy + a_0 y^2 = \text{constant}, \tag{3.14}$$

they can be classified in the following.

- if Δ, a_0 and $Q(x, y)$ are of the same sign, then (3.14) defines an ellipse,
- if $\Delta \cdot a_2 < 0$, then (3.14) defines a hyperbola,
- if $\Delta = 0$ and $a_2 \neq 0$, then (3.14) defines a pair of straight lines.

Remark 3.2 If we consider the action of the special linear group $\mathrm{SL}(2, \mathbb{R})$ on the vector space $\mathcal{P}^n(\mathbb{R}^2)$, then the invariants appear to be of weight zero due to the fact $\alpha\delta - \beta\gamma = 1$. This is the case we will consider later in this chapter.

We observe that the above example shows that group invariants are very effective in classification-type problems. This will be further illustrated with more advanced examples in the study of Killing tensors.

The main problem of CIT can be summarized as follows [54].

Problem 3.1 *Determine the linear action of a group G in a K-vector space V. Then in the ring of polynomial functions $K[V]$ describe the subring $K[V]^G$ of all functions in V that remain unchanged under the action of the group G.*

To solve a more general problem of determining the space of invariants, one needs to find a set of *fundamental invariants* in the sense that any other invariant can be expressed as a function of the fundamental invariants. Thus, the fundamental invariants are necessarily functionally independent. For our purposes, we concentrate on the determination of invariants, covariants and joint invariants of Lie group actions on vector spaces. To find a complete set of fundamental invariants, one first determines the number of fundamental invariants, which, in the case of a regular Lie group action, can be specified by the fundamental theorem on invariants (Theorem 3.1 below).

Theorem 3.1 [67] *Let G be a Lie group acting regularly on an m-dimensional manifold M with s-dimensional orbits (see Definition 3.8 for semi regular and regular Lie group actions). Then, in a neighborhood U of each point $\mathrm{x}_0 \in M$, there exist $m - s$ functionally independent G-invariants $\Delta_1, \ldots, \Delta_{m-s}$. Any other G-invariant $\mathcal{I}$ defined near x_0 can be locally uniquely expressed as an analytic function of the fundamental invariants through*

$$\mathcal{I} = F(\Delta_1, \ldots, \Delta_{m-s}). \tag{3.15}$$

The fundamental invariants can be employed to distinguish between the orbits near a point in the sense that, two points $\mathrm{x}, \mathrm{x}_0 \in U$ are equivalent (meaning that they belong to the same orbit) if and only if all the fundamental invariants agree:

$$\Delta_i(\mathrm{x}) = \Delta_i(\mathrm{x}_0), \quad i = 1, \ldots, m - s.$$

To determine s and the subspaces of Σ where the group acts with orbits of the same dimension, one employs the following proposition [67].

Proposition 3.1 *Let G be a Lie group acting on an m-dimensional manifold M, $\mathfrak{g}$ is the corresponding Lie algebra and let $\mathrm{x} \in M$. The vector space $S|_{\mathrm{x}} = \mathrm{Span}\{\mathrm{V}_i(\mathrm{x}) \,|\, \mathrm{V}_i \in \mathfrak{g}\}$ spanned by all vector fields determined by the infinitesimal generators at x coincides with the tangent space to the orbit $\mathcal{O}_{\mathrm{x}}$ of G that passes through x, so $S|_{\mathrm{x}} = T\mathcal{O}_{\mathrm{x}}|_{\mathrm{x}}$. In particular, the dimension of $\mathcal{O}_{\mathrm{x}}$ equals the dimension of $S|_{\mathrm{x}}$. Moreover, the isotropy subgroup $G_{\mathrm{x}} \subset G$ has dimension $\dim G - \dim \mathcal{O}_{\mathrm{x}} = r - s$.*

There are many methods available for computing invariants. We are mainly interested in *the method of infinitesimal generators* and *the method of moving frames*, both of which prove potent in the invariant theory of Killing tensors.

3.2 The method of infinitesimal generators

The method of infinitesimal generators is a powerful method that can be used to compute group invariants. In this section we briefly describe this method as applied to the 1856 lemma of Cayley in CIT.

3.2.1 One-parameter subgroups of transformations

As has been found by Sophus Lie, the most important group actions that appear in geometry and invariant theory are those acting analytically on a manifold.

Example 3.6 The action of $\mathrm{GL}(m, \mathbb{R})$ on $\mathbb{E}^m$ is given by the usual matrix multiplication

$$\omega(A, \mathrm{x}) = A \cdot \mathrm{x}, \quad A \in \mathrm{GL}(m, \mathbb{R}), \quad \mathrm{x} \in \mathbb{E}^m. \tag{3.16}$$

Note the action of a subgroup of $\mathrm{GL}(m, \mathbb{R})$ can be considered similarly, which may act on a smaller space. For instance, the orthogonal group $\mathrm{O}(m)$ acts in $S^{m-1}(r)$, a sphere of radius r.

Definition 3.7 [67] An *orbit* of a transformation group G is a minimal nonempty G-invariant subset. In particular, a *fixed point* is a G-invariant point.

In general orbits can have differing dimensions, including fixed points, which have dimension 0.

To have a better picture of a transformation group, one needs to deal with semi-regular and regular Lie group actions.

Definition 3.8 [67] Let G be an r-parameter Lie group acting on a manifold M. The group action is said to *semi-regular* if all its orbits have the same dimension. A semi-regular group action is *regular* if, in addition, each point $p \in M$ admits a system of arbitrarily small neighborhoods N_α whose intersections with each orbit $\mathcal{O}$ are (pathwise) connected subsets $N_\alpha \cap \mathcal{O}$ of the orbit.

A Lie group $G \subset \mathrm{GL}(m, \mathbb{R})$ that forms an analytical submanifold of the general linear group $\mathrm{GL}(m, \mathbb{R})$ is called a matrix Lie group. Each Lie group contains different kinds of subgroups, among which are the *one-parameter subgroups*.

Definition 3.9 Let G be a Lie group. A *one-parameter subgroup* of G is a group homomorphism $\sigma : \mathbb{R} \to G$ that satisfies

$$\sigma(t)(\sigma(s)) = \sigma(t+s). \tag{3.17}$$

Example 3.7 [67] Consider the one-parameter subgroups of $\mathrm{GL}(m, \mathbb{R})$. Any nonzero matrix $A \in \mathrm{GL}(m, \mathbb{R})$ defines a one-parameter subgroup as follows:

$$H_A = \{e^{tA}, t \in \mathbb{R}\}. \tag{3.18}$$

The matrix A is called an *infinitesimal generator* of the one-parameter subgroup H_A. Indeed, since $\left.\frac{\mathrm{d}e^{tA}}{\mathrm{d}t}\right|_{t=0} = A$, if we look at the one-parameter subgroup H_A as a matrix-valued curve, say $\sigma(t)$, then A can be identified as the tangent to the curve at the identity $I = \sigma(0)$. Thus, H_A is the unique solution to the initial value problem

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = A\sigma, \quad \sigma(0) = \mathrm{I}. \tag{3.19}$$

Notation: Throughout the thesis, I stands for the identity element of the underlying transformation group G.

Suppose $G \subset \mathrm{GL}(m, \mathbb{R})$ is a matrix Lie group, then the set

$$g = \{A |\, e^{tA} \in G\} \tag{3.20}$$

of all matrices such that their one-parameter subgroups are contained in G is a subspace of $gl(m, \mathbb{R})$. This matrix Lie algebra is called the space of infinitesimal generators of G and $g \simeq TG|_I$. Taking into account the commutator relations of the infinitesimal generators, one sees that not every subspace of the matrix Lie algebra $gl(m, \mathbb{R})$ is a matrix Lie algebra. It turns out that a subspace of $gl(m, \mathbb{R})$ which is closed under the matrix commutator

$$[A, B] = AB - BA$$

is a Lie subalgebra.

Example 3.8 [67] The Lie algebra $gl(2, \mathbb{R})$ of the Lie group $\mathrm{GL}(2, \mathbb{R})$ is generated by

$$A^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.21}$$

The first three, namely A^-, A^0 and A^+ span the Lie algebra $sl(2, \mathbb{R})$ of the special linear group $\mathrm{SL}(2, \mathbb{R})$, with the commutator relations:

$$[A^-, A^0] = -2A^- \quad [A^-, A^+] = A^0, \quad [A^+, A^0] = 2A^+. \tag{3.22}$$

The corresponding one-parameter subgroups are

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \tag{3.23}$$

3.2.2 Infinitesimal invariance criteria

If we identify the infinitesimal generators of a transformation groups with differential operators, we can then employ analytical methods to compute the group invariants. Let $G \subset \mathrm{GL}(m, \mathbb{R})$ be a matrix Lie group that acts on an m-dimensional manifold M, $g \subset gl(m, \mathbb{R})$ is the corresponding Lie algebra. Then for each infinitesimal generator A of a one-parameter subgroup $\{e^{At}\}$, the associated infinitesimal generator V_A of the action G in M can be realized as a first order differential operator such that

$$\mathrm{V}_A(F(\mathrm{x})) = \frac{\mathrm{d}}{\mathrm{d}t} F(e^{-At} \cdot \mathrm{x})\big|_{t=0} \tag{3.24}$$

where $F \in \mathcal{F}(M)$. Thus, if F is an invariant under the one-parameter group action defined by A, it is necessarily annihilated by the vector field $\mathbf{V}_A$. In fact, F is G-invariant if and only if [67]

$$\mathbf{V}_{A_i}(F) = 0, \quad i = 1, \dots, r, \tag{3.25}$$

where $\mathbf{V}_{A_i}, i = 1, \dots, r$ are defined by (3.24) and $A_i, i = 1, \dots r$ are generators of those one parameter subgroups of G that span g.

Thus the problem of the determination of G-invariants is now reduced to solving a system of first order linear homogeneous PDEs (3.25). For specific group action, one can find the infinitesimal generators by making use of formula (3.24).

This idea was first introduced and employed by Cayley [10] to compute the invariants of binary forms. Now let us consider a simple but elucidating example.

Example 3.9 [67] Consider the action of the special linear group $\mathrm{SL}(2, \mathbb{R})$ on the vector space $\mathcal{Q}^2(\mathbb{R}^2)$. The three dimensional Lie group $\mathrm{SL}(2, \mathbb{R})$ acting on $\mathbb{R}^2$ induces the corresponding action on the parameters (3.9).

The three infinitesimal generators of $sl(2, \mathbb{R})$ and their corresponding one-parameter subgroups are given in Example 3.8 . Employing (3.24), one obtains the three (3.24) infinitesimal generators of the group action on the parameter space.

$$\begin{aligned} \mathbf{V}^- &= 2a_1\partial_{a_0} + a_2\partial_{a_1}, \\ \mathbf{V}^0 &= -2a_0\partial_{a_0} + 2a_2\partial_{a_2}, \\ \mathbf{V}^+ &= a_0\partial_{a_1} + 2a_1\partial_{a_2}. \end{aligned} \tag{3.26}$$

It is easy to very that their commutator relations are

$$[\mathbf{V}^-, \mathbf{V}^0] = -2\mathbf{V}^-, \quad [\mathbf{V}^-, \mathbf{V}^+] = \mathbf{V}^0, \quad [\mathbf{V}^+, \mathbf{V}^0] = 2\mathbf{V}^+, \tag{3.27}$$

which are the same as (3.22). Thus an $\mathrm{SL}(2, \mathbb{R})$-invariant I of $\mathcal{Q}^2(\mathbb{R}^2)$ must satisfy simultaneously

$$\mathbf{V}^-(I) = 0, \quad \mathbf{V}^0(I) = 0, \quad \mathbf{V}^+(I) = 0. \tag{3.28}$$

Solving the system of PDEs (3.28) by the method of characteristics, one recovers the fundamental invariant $\Delta = a_0a_2 - a_1^2$.

Remark 3.3 Looking at the three vector fields (3.26), we see that the group acts with two-dimensional orbits, namely, the space spanned by (3.26) is two-dimensional. Thus, according to the fundamental theorem on invariants (Theorem 3.1) the expected number of fundamental invariants is $3-2=1$.

3.2.3 The 1856 lemma of Cayley

Arthur Cayley (1821-1895), an eminent English mathematician of the 19th century, made many significant contributions to a number of areas of mathematics, including algebra, invariant theory, projective geometry and group theory. Note that Lie's infinitesimal method allows one to replace rather complicated nonlinear group transformations by simpler infinitesimal counterparts, leading to a system of linear first order homogeneous PDEs. The key step is to identify the infinitesimal generators of the Lie group action as the form of differential operators. A. Cayley was the first to observe the importance of these operators in the context of binary forms, which was many years before S. Lie arrived at the same scene. In spite of the fact that Cayley thought of this as of something pertinent only to the general linear group and its subgroups, his results in this area may be considered as a precursor to Sophus Lie's theory of abstract Lie groups that was developed later in the 19th century. In his "second memoir on quantics" [10] Cayley considers (in modern mathematical language) the problem of the determination of the action of the Lie group $\mathrm{SL}(2,\mathbb{R})$ on the vector space $\mathcal{P}^n(\mathbb{R}^2)$ in conjunction with the problem of computing the invariants. The main result is the subject of the following lemma (Cayley [10], see also Olver [67]).

Lemma 3.1 (Cayley, 1856) *The action of* $\mathrm{SL}(2,\mathbb{R})$ *on the vector space* $\mathcal{P}^n(\mathbb{R}^2)$ *defined by (3.1) has the following infinitesimal generators on the parameter space* Σ*:*

$$\begin{aligned}
\mathbf{V}^- &= na_1\partial_{a_0} + (n-1)a_2\partial_{a_1} + \cdots + 2a_{n-1}\partial_{a_{n-2}} + a_n\partial_{a_{n-1}},\\
\mathbf{V}^0 &= -na_0\partial_{a_0} + (2-n)a_1\partial_{a_1} + \cdots + (n-2)a_{n-1}\partial_{a_{n-1}} + na_n\partial_{a_n},\\
\mathbf{V}^+ &= a_0\partial_{a_1} + 2a_1\partial_{a_2} + \cdots + (n-1)a_{n-2}\partial_{a_{n-1}} + na_{n-1}\partial_{a_n},
\end{aligned} \tag{3.29}$$

where

$$\partial_{a_i} = \frac{\partial}{\partial a_i}, \quad i = 0, \ldots, n.$$

Observe that the vector fields (3.29) enjoy the following commutator relations

$$[\mathbf{V}^-, \mathbf{V}^0] = -2\mathbf{V}^-, \quad [\mathbf{V}^-, \mathbf{V}^+] = \mathbf{V}^0, \quad [\mathbf{V}^+, \mathbf{V}^0] = 2\mathbf{V}^+, \tag{3.30}$$

which (comparing with (3.22)) confirm that the generators (3.29) represent the action of $\mathrm{SL}(2,\mathbb{R})$ on the parameter space Σ. In view of the above, solving the problem of the determination of the $\mathrm{SL}(2,\mathbb{R})$-invariants of $\mathcal{P}^n(\mathbb{R}^2)$ now amounts to solving the corresponding system of PDEs determined by the vector fields (3.29):

$$\mathbf{V}^-(F) = 0, \quad \mathbf{V}^0(F) = 0, \quad \mathbf{V}^+(F) = 0 \tag{3.31}$$

for an analytic function F defined on Σ.

We immediately notice that the result of Example 3.9 agrees with Cayley's lemma, that is, the three infinitesimal generators can also be derived by directly using Cayley's lemma. Furthermore, one can employ Cayley's lemma to compute the covariants of binary forms of degree n.

Corollary 3.1 [67] *A Function C is an $\mathrm{SL}(2,\mathbb{R})$-covariant of $\mathcal{P}^n(\mathbb{R}^2)$ if and only if*

$$\mathbf{U}^-(C) = 0, \quad \mathbf{U}^0(C) = 0, \quad \mathbf{U}^+(C) = 0. \tag{3.32}$$

Where

$$\mathbf{U}^- = \mathbf{V}^- - y\,\partial_x, \; q\mathbf{U}^+ = \mathbf{V}^+ - x\,\partial_y, \quad \mathbf{U}^0 = \mathbf{V}^0 - x\,\partial_x + y\,\partial_y. \tag{3.33}$$

To solve the systems (3.31) or (3.32), one normally employs the method of characteristics. It works in many cases, but not always, due to computational difficulties. More specifically, for a one-parameter group action, we only need to solve a first order, linear PDE,

$$V^1 \frac{\partial I}{\partial x^1} + \cdots V^m \frac{\partial I}{\partial x^m} = 0, \tag{3.34}$$

where $\mathbf{V} = (V^1, \ldots, V^m)$ is the infinitesimal generator. PDE (3.34) can be solved by integrating the corresponding characteristic system [65]

$$\frac{\mathrm{d}x^1}{V^1} = \cdots = \frac{\mathrm{d}x^m}{V^m}. \tag{3.35}$$

We thus obtain $m-1$ functionally independent solutions

$$\begin{aligned} I_1(x^1,\ldots,x^m) &= c_1, \\ &\vdots \\ I_{m-1}(x^1,\ldots,x^m) &= c_{m-1}, \end{aligned} \tag{3.36}$$

where $c_1,\ldots,c_{m-1}$ are arbitrary constants. The functions $I_1,\ldots,I_{m-1}$ on the left hand side of (3.36) form a complete set of the *fundamental G-invariants*. For a multi-parameter group action, one has to deal with a *system* of first order, homogeneous PDEs resulting from those infinitesimal generators. That is, one will have to look for those functions that are annihilated by all the infinitesimal generators simultaneously. This can be computationally very challenging. One way to proceed is to solve for the invariants of one generator (a vector field). Using these invariants as new coordinates one rewrites the remaining generators. We continue this process until the required number of fundamental invariants are found. An interesting and elucidating example in ITKI is Example 4.1 in Chapter 4. For more details and other examples, see, for example, Olver [65].

Generally, computer algebra may be used to alleviate the difficulty. Indeed, in the invariant theory of Killing tensors the combination of a computer algebra package (for example MAPLE) with the *method of undetermined coefficients* has been very successful in solving the problem of the determination of fundamental invariants (see, for example, Horwood *et al* [34]).

3.3 The method of moving frames

The method of moving frames, introduced originally by Cartan [9], is a powerful technique that can be employed to solve a wide range of classification-type problems. This method was recently reformulated by Fels and Olver [22, 23, 67, 66] and has seen many applications in various areas of mathematics and mathematical physics. See also Kogan [50], Bouten [7]. We very briefly review the basic definitions and results of the moving frames theory in its modern formulation (see Olver [67] for a complete review).

The simplest example of a moving frame is the Frenet frame $\{\mathbf{t},\mathbf{n}\}$ of a regular curve $\gamma\in\mathbb{E}^2$ parametrized by its arc length. In this case the equivariant map assigns to each point on the curve $\gamma(s)$ the corresponding frame $\{\mathbf{t}(s),\mathbf{n}(s)\}$. Clearly, the moving frame along γ

can be obtained from a fixed frame via a combination of rotations and/or translations. This puts in evidence that there is a natural isomorphism between the moving frame and the orientation-preserving isometry group (the Euclidean group) $I(\mathbb{E}^2)$. This is the essence of the later generalizations of the moving frame method [28, 29, 30], where the moving frame was viewed as *an equivariant map* from the space of submanifolds to the group itself. In recent works [22, 23, 50] the classical moving frame method was further generalized to completely general transformation groups, including infinite-dimensional Lie pseudo-groups . Ultimately, the authors have succeeded in bringing the theory up to the level where the bundle of frames is no longer needed.

Definition 3.10 Let G be a Lie group acting on a vector space X. A map $\rho : M \to G$ is said *equivariant* if $\rho(g \cdot \mathrm{x}) = \rho(\mathrm{x}) \cdot g^{-1}$.

Definition 3.11 [67] A *moving frame* is a smooth, *G-equivariant* map $\rho : M \to G$, where G is an r-dimensional group acting smoothly on an m-dimensional underlying manifold M.

Theorem 3.2 *A moving frame exists in a neighborhood of a point* $\mathrm{x} \in M$ *if and only if G acts freely and regularly near* x.

To construct a moving frame, one employs Cartan's *normalization method* [9].

Theorem 3.3 [67] *Let G act freely and regularly in an m-dimensional manifold M and $K \subset M$ be a (local) cross-section to the group orbits. Given $\mathrm{x} \in M$, let $\mathrm{g} = \rho(\mathrm{x})$ be the unique group element that maps* x *to the cross-section:*

$$\mathrm{g} \cdot \mathrm{x} = \rho(\mathrm{x}) \cdot \mathrm{x} \in K.$$

Then

$$\rho : M \to G$$

is a right moving frame.

More specifically, let $\mathrm{x} = (x^1, \ldots, x^m) \in M$ be local coordinates. Consider the explicit formulas for the coordinate transformations induced by the action of G: $\omega(\mathrm{g}, \mathrm{x}) = \mathrm{g} \cdot \mathrm{x}$.

$$\begin{aligned} \bar{x}^1 &= \omega_1(x^1, \ldots, x^m; g_1, \ldots, g_r), \\ &\vdots \\ \bar{x}^m &= \omega_m(x^1, \ldots, x^m; g_1, \ldots, g_r). \end{aligned} \tag{3.37}$$

The right moving frame $\mathrm{g} = \rho(\mathrm{x})$ can be constructed by making use of a *coordinate cross-section*

$$K = \{x^1 = c_1, \ldots, x^r = c_r\},$$

where $c_i, i = 1, \ldots, r$ are some constants and solving the corresponding *normalization equations*

$$\begin{aligned} \omega_1(\mathrm{g}, \mathrm{x}) &= c_1, \\ &\vdots \\ \omega_r(\mathrm{g}, \mathrm{x}) &= c_r \end{aligned} \tag{3.38}$$

for the group G locally parametrized by $\mathrm{g} = (g_1, \ldots, g_r)$ in terms of the local coordinates $(x^1, \ldots, x^m)$. Substituting the resulting expressions for $g_1, \ldots, g_r$ in terms of the local coordinates $(x^1, \ldots x^m)$ into the left hand side of the remaining $m - r$ formulas of (3.37) yields a complete set of fundamental G-invariants.

Theorem 3.4 [67] *If* $\mathrm{g} = \rho(\mathrm{x})$ *is the moving frame solution to the normalization equations (3.38), then the functions*

$$\begin{aligned} \Delta_1 &= \omega_{r+1}(\rho(\mathrm{x}), \mathrm{x}), \\ &\vdots \\ \Delta_{m-r} &= \omega_m(\rho(\mathrm{x}), \mathrm{x}) \end{aligned} \tag{3.39}$$

form a complete system of fundamental G-invariants.

Remark 3.4 There may be more than one way of choosing a cross-section.

Definition 3.12 Let G be a group acting on a space X. The action is called *transitive* if for any two points $\mathrm{x}, \mathrm{y} \in X$ there is an element $g \in G$ such that $g(\mathrm{x}) = \mathrm{y}$.

Obviously, if the group action is transitive, then there is only one orbit.

Definition 3.13 Let G act on X. For a given point $\mathrm{x} \in X$, the symmetry group $G_{\mathrm{x}} = \{g \in G, g(\mathrm{x}) = \mathrm{x}\}$ is called its local isotropy subgroup. The action is called *free* if all of the local isotropy subgroups are trivial.

Example 3.10 [67] Consider the planar Euclidean group $\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{E}^2$ acting on the Euclidean plane $\mathbb{E}^2$. The group action is transitive and there is only one orbit. It follows that there is no invariant. We can, however, extend the action to find differential invariants. Let Γ be the curve denoting the graph of a function $y = f(x)$ and let $\bar{\mathrm{x}} = R\mathrm{x}+\mathrm{a}$ ($\mathrm{x} = (x, y), \mathrm{a} = (a, b) \in \mathbb{E}^2, R \in \mathrm{SO}(2)$) be an Euclidean transformation, which is given by

$$\begin{aligned} \bar{x} &= x\cos t - f(x)\sin t + a, \\ \bar{y} &= x\sin t + f(x)\cos t + b. \end{aligned} \tag{3.40}$$

Denote the transformed curve of Γ by $\bar{\Gamma} = R\Gamma + \mathrm{a}$, then $\bar{\Gamma}$ will be the graph of $\bar{y} = \bar{f}(\bar{x})$.

The transformed function $\bar{f}$ can be derived by eliminating x from (3.40). The first prolongation of this group action maps the tangent line to curve Γ at point x to the tangent line to $\bar{\Gamma}$ at the corresponding point $\bar{\mathrm{x}}$, (that is consider the action on the space $(x, f(x), f'(x))$) and the second prolongation will be defined to map the osculating circle at x to the osculating circle at $\bar{\mathrm{x}}$ (that is consider the action on the space $(x, f(x), f'(x), f''(x))$) and so on. The explicit formulas for the transformations can be obtained by using the implicit differentiation of (3.40),

$$\begin{aligned} \bar{f}'(\bar{x}) &= \frac{\sin t + f'(x)\cos t}{\cos t - f'(x)\sin t}, \\ \bar{f}''(\bar{x}) &= \frac{f''(x)}{(\cos t - f'(x)\sin t)^3}. \end{aligned} \tag{3.41}$$

Now we can introduce new variables $v = f'(x)$ and $w = f''(x)$. In view of (3.40) and (3.41) we arrive at the prolonged action of SE(2) on $\mathbb{E}^4$

$$\bar{x} = x\cos t - y\sin t + a, \tag{3.42}$$

$$\bar{y} = x\sin t + y\cos t + b, \tag{3.43}$$

$$\bar{v} = \frac{\sin t + v\cos t}{\cos t - v\sin t}, \tag{3.44}$$

$$\bar{w} = \frac{w}{(\cos t - v\sin t)^3}, \tag{3.45}$$

where t, a, b are the coordinates that parametrize the group SE(2). The dimension of the orbits of this group action ≤ 3 and the action is not *transitive*. According to the fundamental theorem on invariants of regular Lie group action (Theorem 3.1) one can expect $4 - 3 = 1$ fundamental invariant. We can employ the method of moving frames.

Consider the cross-section $K = \{x = y = v = 0\}$, which defines the corresponding normalization equation

$$\begin{aligned} x\cos t - y\sin t + a &= 0, \\ x\sin t + y\cos t + b &= 0, \\ \frac{\sin t + v\cos t}{\cos t - v\sin t} &= 0. \end{aligned} \tag{3.46}$$

Solving (3.46) leads to the following moving frames map

$$\begin{aligned} t &= -\tan^{-1} v, \\ a &= -\frac{x + yv}{\sqrt{1+v^2}}, \\ b &= \frac{xv - y}{\sqrt{1+v^2}}. \end{aligned} \tag{3.47}$$

Substituting the moving frame map (3.47) into the right hand side of (3.45), we obtain the fundamental invariant

$$\Delta = \frac{w}{(1+v^2)^{3/2}}. \tag{3.48}$$

We see that the above invariant is indeed the curvature function of the curve. This shows that the group action preserves the curvature, which agrees with the results in the classical differential geometry of curves and surfaces.

This method will be employed again in the computation of isometry group invariants of Killing tensors in the following chapters.

Chapter 4

Invariant Theory of Killing Tensors

4.1 Introduction

In this chapter, we describe in detail the *invariant theory of Killing tensors* (ITKT) defined on pseudo-Riemannian spaces of constant curvature, including the introduction of the theory, the concept of an invariant, the new concepts of a covariant and a joint invariant of Killing tensors. We will also give the main problems and main methods for computing invariants.

In Section 4.2 we incorporate the basic ideas of ITKT from CIT, and pose the main problems of ITKT. To be more specific, we deal with invariants, covariants and joint invariants. As a new development of ITKT, we introduce the concepts of a covariant and a joint invariant of Killing tensors and determine the complete sets of invariants and joint invariants for several vector spaces of Killing tensors. The method of infinitesimal generators, which has been described in the previous chapter, will be used again here, and it works well for vector spaces of Killing tensors of small valences and defined on constant curvature spaces of low dimension.

Section 4.3 will be devoted to the application of the method of moving frames described in Chapter 3 to the invariant theory of Killing tensors. It can be effectively employed to compute the fundamental sets of invariants, covariants and joint invariants. We determine the complete sets of covariants for two vector spaces of Killing tensors. Some of the results presented here appeared in a paper that has been published [79]. We should mention that a new technique of the method of moving frames developed by Kogan [50] has been incorporated in ITKT, several important cases have been worked out with this new technique. One advantage of this method is, among others, that one does not need any computer algebra to complete the computation. Thus, we solve the problem of the determination of the complete sets of fundamental invariants of Killing tensors of valence three defined on the Minkowski plane and the Euclidean plane, respectively.

4.2 Invariants, covariants and joint invariants of Killing tensors

The second half of the 19th century saw the development of the post-"Theorema Egregium of Gauss" differential geometry heading in two major directions. Bernhard Riemann [72] generalized Gauss' geometry of surfaces in the Euclidean 3-space by introducing the concept of a differentiable manifold of arbitrary dimension and defining the metric tensor on tangent spaces. This remarkable work has evolved into what is today's (Riemannian) differential geometry.

The other direction originated in the celebrated "Erlangen Program" of Felix Klein [47, 48], the main idea of which is that any geometry can be interpreted as an invariant theory with respect to a specific transformation group, and the main goal of any geometry is the determination of those properties of geometrical figures that remain unchanged under the action of a transformation group. One of the main contributions of Élie Cartan to differential geometry, in particular with his moving frames method [9], is the blending of these two directions into a single theory. An excellent exposition of this fact can be found in Sharpe [75] (see also, for example, Arvanitoyeorgos [2]). The following diagram presented in [75] elucidates the relations among these approaches to geometry described above.

$$
\begin{array}{ccc}
\text{Euclidean Geometry} & \xrightarrow{\text{generalization}} & \text{Klein Geometries} \\
\downarrow \text{generalization} & & \text{generalization} \downarrow \\
\text{Riemannian Geometry} & \xrightarrow{\text{generalization}} & \text{Cartan Geometries}
\end{array}
\tag{4.1}
$$

As an analogue of the classical invariant theory of homogeneous polynomials, the invariant theory of Killing tensors defined on pseudo-Riemannian spaces of constant curvature (ITKT) formed recently a new area of research [88, 34, 51, 89, 90, 79, 53, 80, 13, 54, 55, 56, 52, 57], which, in view of the above, can be placed in the theory by Cartan. This is especially evident in the study of vector spaces of Killing tensors of valence two. Indeed, a number of vector spaces of Killing tensors have been investigated from this viewpoint by means of determining the corresponding sets of fundamental *invariants* and, much like in CIT, using them to solve the problem of equivalences and canonical forms in each case.

These results have been employed in applications arising in the *theory of orthogonally separable coordinate webs* [3, 4, 5, 12, 20, 21, 64, 4, 37, 38, 58, 52, 54, 79, 34], where Killing tensors of valence two play a pivotal role. An orthogonal separable coordinate web which gives rise to an orthogonally separable coordinate system is an integral part of the geometry of the underlying pseudo-Riemannian manifold. Therefore the problem of group invariant classification of these webs in a specific pseudo-Riemannian space of constant curvature is a problem of F. Klein's approach to geometry, as well as that of Riemann, both leading to the theory due to Cartan (see the diagram (4.1)).

Let (M, g) be an m-dimensional pseudo-Riemannian manifold of constant curvature. Recall that a standard Killing tensor K of valence n defined on M is a symmetric $(n, 0)$ tensor satisfying the Killing tensor equation (2.23).

When $n = 1$, K is said to be a *Killing vector (infinitesimal isometry)* and the equation (2.23) reads $\mathcal{L}_{\mathrm{K}} g = 0$, that is (2.18) in Chapter 1, where $\mathcal{L}$ denotes the Lie derivative operator.

The $\mathbb{R}$-bilinear properties of the Schouten bracket immediately indicate that the Killing tensors of the same valence n defined on M constitute a vector space, denoted by $\mathcal{K}^n(M)$ (Note here and below we only deal with standard Killing tensors, that is Killing tensors of order 0, see Definition 2.22 in Chapter 1). The dimension d of the vector space $\mathcal{K}^n(M)$ is determined by the *Delong-Takeuchi-Thompson (DTT) formula* [14, 82, 84]

$$d = \dim \mathcal{K}^n(M) = \frac{1}{n}\binom{m+n}{n+1}\binom{m+n-1}{n}, \quad n \geq 1, \tag{4.2}$$

which is a special case of the NPE-formula (2.24) given in Chapter 1. Thus, a Killing tensor of valence $n \geq 1$ defined on (M, g) can be viewed as an algebraic object. Indeed, each Killing tensor of valence n defined on M is determined by its d arbitrary parameters $(\alpha_1, \ldots, \alpha_d)$ with respect to a given coordinate system. This approach to the study of Killing tensors introduced in [52] differs significantly from the conventional approach based on the property that Killing tensors defined in pseudo-Riemannian spaces of constant curvature are sums of symmetrized tensor products of Killing vectors (see, for example, [84]). Moreover, the idea leads to a natural link between the study of Killing tensors and the classical invariant theory of homogeneous polynomials, which in the last decade has become an active area of research once again (see Olver [67] and the references therein). Thus, it has been shown in a series of recent papers [13, 52, 53, 55, 56, 57, 88, 79, 80, 89, 90] that one can utilize the basic ideas of classical invariant theory in the study of Killing tensors defined in pseudo-Riemannian spaces of constant curvature. The concept of an isometry group *invariant* of Killing tensors was introduced in [52] in the study of non-trivial Killing tensors of valence two that generate orthogonal coordinate webs on the Euclidean plane.

4.2.1 Invariants

It was observed [52] that the isometry group $I(M)$ acting on M preserves the Killing tensors defined on M. Thus the isometry group also induces an action on the d-dimensional parameter space $\Sigma \simeq \mathbb{R}^d$ defined by the parameters $\alpha_1, \ldots, \alpha_d$. The corresponding transformations for the parameters $(\alpha_1, \ldots, \alpha_d)$ can be obtained in each case by performing the standard tensor transformation laws:

$$\begin{aligned} \tilde{\alpha}_1 &= \tilde{\alpha}_1(\alpha_1, \ldots, \alpha_d, g_1, \ldots, g_r), \\ &\vdots \\ \tilde{\alpha}_d &= \tilde{\alpha}_d(\alpha_1, \ldots, \alpha_d, g_1, \ldots, g_r), \end{aligned} \tag{4.3}$$

where $g_1, \ldots, g_r$ are local coordinates on $I(M)$.

The notion of *isometry group invariant of Killing tensors* was proposed in McLenaghan *et al* [52]. In essence, invariants of Killing tensors are functions of the parameters that remain unchanged under the induced action 4.3 induced by the isometry group action.

An observation is now in need. The induced action of $I(M)$ on the vector space $\mathcal{K}^n(M)$ defined by the push-forward map g_* (where $g \in I(M)$) is linear invertible transformation, furthermore, $(gh)_* = g_* h_*$. Thus, the correspondence between the elements of $I(M)$ and their induced action on $\mathcal{K}^n(M)$ is a homomorphism and therefore defines a representation of $I(M)$ on $\mathcal{K}^n(M)$ and isometry group invariants of Killing tensors can be considered as fixed points of the induced representation of $I(M)$ on the function space $\mathcal{F}(\mathcal{K}^n(M))$. See McLenaghan, Milson and Smirnov [51] and Horwood, McLenaghan, and Smirnov [34] for more details.

We note that the action of $I(M)$ can be considered on the spaces M and Σ concurrently.

One of the main problems of invariant theory is to describe the space of invariants (covariants, joint invariants) for a given vector space under the action of a transformation group. To solve this problem one has to find a set of *fundamental invariants* (*covariants, joint invariants*) with the property that any other invariant (covariant, joint invariant) is a (analytic) function of the fundamental invariants (covariants, joint invariants). The fundamental theorem on invariants of a regular Lie group action (Theorem 3.1) determines the number of fundamental invariants that are required to describe the space of $I(M)$-invariants.

We assume that $I(M)$ acts on a subspace Σ_r of Σ regularly with r-dimensional orbits, then, according to Theorem 3.1, the number of fundamental invariants required to describe the whole space of $I(M)$-invariants of $\mathcal{K}^n(M)$ is $d - r$, where d is given by (4.2) (note $d \geq r$). This has been shown to be the case for the vector spaces $\mathcal{K}^2(\mathbb{E}^2)$ [52], $\mathcal{K}^2(\mathbb{E}^2_1)$ [53], $\mathcal{K}^3(\mathbb{E}^2)$ [13] and $\mathcal{K}^2(\mathbb{E}^3)$ [34], where $\mathbb{E}^2$, $\mathbb{E}^2_1$ and $\mathbb{E}^3$ denote the Euclidean plane, the Minkowski plane and the Euclidean 3-space, respectively.

Remark 4.1 The dimension of the orbits of the isometry group $I(M)$ acting on Σ is not always the same as the dimension of the group, and the dimension of orbits can vary. For example, this is the case for the vector space $\mathcal{K}^1(\mathbb{E}^3)$ [13, 88].

To determine a complete set of fundamental invariants one can use the method of infinitesimal generators.

In the following we describe a procedure that was designed to determine the infinitesimal generators of the isometry group $I(M)$ on Σ. The procedure was first introduced in McLenaghan, Smirnov and The [52] and so we call it the MST-procedure.

Let $\mathrm{X}_1, \ldots, \mathrm{X}_r \in \mathcal{X}(M)$ be the infinitesimal generators (Killing vector fields) of the Lie algebra of the Lie group $I(M)$ acting on M. Note Span$\{\mathrm{X}_1, \ldots, \mathrm{X}_r\} = \mathcal{K}^1(M) = i(M)$, where $i(M)$ is the Lie algebra of the Lie group $I(M)$. For a fixed $n \geq 1$, consider the corresponding vector space $\mathcal{K}^n(M)$. The aim is to find the infinitesimal generators of $i(M)$ on Σ. Consider Diff Σ, it defines the corresponding space Diff $\mathcal{K}^n$(M), whose elements are determined by the elements of Diff Σ in an obvious way. Let $\mathrm{K}^0 \in$ Diff $\mathcal{K}^n$(M). Note K^0 is determined by d parameters $\alpha_i^0(\alpha_1, \ldots, \alpha_d)$, $i = 1, \ldots, d$, which are functions of $\alpha_1, \ldots, \alpha_d$ - the parameters of Σ. Define now a map π : Diff $\mathcal{K}^p$(M) $\to \mathcal{X}(\Sigma)$, the set of smooth vector fields defined on Σ, given by

$$\mathrm{K}^0 \to \sum_{i=1}^{d} \alpha_i^0(\alpha_1, \ldots, \alpha_d)\frac{\partial}{\partial \alpha_i}. \tag{4.4}$$

To specify the action of $I(M)$ on Σ, we have to find the counterparts of the generators $\mathrm{X}_1, \ldots, \mathrm{X}_r$ on $\mathcal{X}(\Sigma)$.

Consider the composition $\pi \circ \mathcal{L}$, where π is defined by (4.4) and $\mathcal{L}$ is the Lie derivative operator. Let K be the general element of $\mathcal{K}^n(M)$, in other words K is the general solution to the Killing tensor equation (2.23). Next, define

$$\mathrm{V}_i = \pi \circ \mathcal{L}_{\mathrm{X}_i} \mathrm{K}, \quad i = 1, \ldots r. \tag{4.5}$$

The composition $\pi \circ \mathcal{L} : i(M) \to \mathcal{X}(\Sigma)$ maps the generators $\mathrm{X}_1, \ldots, \mathrm{X}_r$ to $\mathcal{X}(\Sigma)$.

Theorem 4.1 *Suppose the generators* $\mathrm{X}_1, \ldots, \mathrm{X}_r$ *of* $i(M)$ *satisfy the following commutator relations:*

$$[\mathrm{X}_i, \mathrm{X}_j] = c_{ij}^k \mathrm{X}_k, \quad i, j, k = 1, \ldots, r, \tag{4.6}$$

where c_{ij}^k, $i, j, k = 1, \ldots, r$ *are the structural constants. Then the corresponding vector fields* $\mathrm{V}_i \in \mathcal{X}(\Sigma)$, *defined by (4.5) satisfy the same commutator relations:*

$$[\mathrm{V}_i, \mathrm{V}_j] = c_{ij}^k \mathrm{V}_k, \quad i, j, k = 1 \ldots, r. \tag{4.7}$$

Therefore the map $F_* := \pi \circ \mathcal{L} : i(M) \to i_\Sigma(M)$ *is a Lie algebra isomorphism, where* $i_\Sigma(M)$ *is the Lie algebra generated by* $\mathrm{V}_1, \ldots, \mathrm{V}_r$.

Remark 4.2 The commutator relations (4.7) can be confirmed directly on a case by case basis, provided that the general form of a Killing tensor $\mathrm{K} \in \mathcal{K}^n(M)$ is available. For a proof of Theorem 4.1 see McLenaghan *et al* [51]. Also See Horwood *et al* [34] for another approach for determining the infinitesimal generators.

We remark that the technique of the Lie derivative deformations used here is a very powerful tool. It was used before, for example, in [78] to generate compatible Poisson bi-vectors in the theory of bi-Hamiltonian systems.

Remark 4.3 Alternatively, the generators (4.5) can be obtained from the formulas for the action of the group (4.3) in the usual way taking into account that a Lie algebra is the tangent space at the identity of the corresponding Lie group. We note, however, that in this way the formulas (4.3) are not easy to derive in general [89, 81].

In view of the isomorphism exhibited in Theorem 4.1 and the fact that invariance of a function under an entire Lie group is equivalent to the infinitesimal invariance under the infinitesimal generators of the corresponding Lie algebra one can determine a set of fundamental invariants by solving the system of PDEs

$$\mathrm{V}_i(F) = 0, \quad i = 1, \ldots, r \tag{4.8}$$

for an analytic function $F : \Sigma \to \mathbb{R}$, where the vector fields V_i, $i = 1, \ldots, r$ are the generators defined by (4.5). As is specified by Theorem 3.1, the general solution to the system (4.8) is an analytic function F of the fundamental invariants. The number of fundamental invariants is $d - s$, where d is specified by the DTT-formula (4.2) and s the dimension of the orbits of $I(M)$ that acts regularly on the parameter space Σ.

The system of PDEs (4.8) can be solved by the method of characteristics in many cases, but not always. The determination of fundamental invariants by solving (4.8) adapting the method of characteristics is the key idea used in [52].

Example 4.1 Consider the action of the isometry group $I(\mathbb{E}^3)$ on the vector space $\mathcal{K}^1(\mathbb{E}^3)$ of Killing vectors defined on the Euclidean 3-space.

Solving the Killing vector equation (2.18) with respect to the Cartesian coordinates (x, y, z) leads to the following general representation for Killing vectors defined on the Euclidean 3-space.

$$\mathrm{K} = (a_1 + a_5 z - a_6 y)\, \partial_x + (a_2 + a_6 x - a_4 z)\, \partial_y + (a_3 + a_4 y - a_5 x)\, \partial_z. \tag{4.9}$$

The six generators of the Lie algebra of the Lie group $I(\mathbb{E}^3)$ are

$$\begin{aligned} \mathrm{X}_1 &= \partial_x, & \mathrm{X}_4 &= y\,\partial_z - z\,\partial_y, \\ \mathrm{X}_2 &= \partial_y, & \mathrm{X}_5 &= z\,\partial_x - x\,\partial_z, \\ \mathrm{X}_3 &= \partial_z, & \mathrm{X}_6 &= x\,\partial_y - y\,\partial_x. \end{aligned} \tag{4.10}$$

Now we employ the MST-procedure to obtain the six infinitesimal generators that can be used to compute the isometry group invariants.

$$
\begin{aligned}
\mathbf{V}_1 &= a_6\,\partial_{a_2} - a_5\,\partial_{a_3}, \\
\mathbf{V}_2 &= -a_6\,\partial_{a_1} + a_4\,\partial_{a_3}, \\
\mathbf{V}_3 &= a_5\,\partial_{a_4} - a_4\,\partial_{a_2}, \\
\mathbf{V}_4 &= a_3\,\partial_{a_2} - a_2\,\partial_{a_3} + a_6\,\partial_{a_5} - a_5\,\partial_{a_6}, \\
\mathbf{V}_5 &= -a_3\,\partial_{a_1} + a_1\,\partial_{a_3} - a_6\,\partial_{a_4} + a_4\,\partial_{a_6}, \\
\mathbf{V}_6 &= a_2\,\partial_{a_1} - a_1\,\partial_{a_2} + a_5\,\partial_{a_4} - a_4\,\partial_{a_5}.
\end{aligned}
\tag{4.11}
$$

According to the method of infinitesimal generators, any invariant I must satisfy simultaneously the equations

$$
\mathbf{V}_i(I) = 0, \quad i = 1, \ldots 6. \tag{4.12}
$$

To find the fundamental invariants, one has to solve the over determined system of PDEs (4.12). We employ the method of characteristics as follows. First find the invariants corresponding to vector field $\mathbf{V}_1$. Integrating the characteristic system leads to the 5 fundamental solutions:

$$
a_1, \quad a_4, \quad a_5, \quad a_6, \quad r_1 = a_2 a_5 + a_3 a_6. \tag{4.13}
$$

Thus any invariant is of the form

$$
F = F(a_1, a_4, a_5, a_6, r_1). \tag{4.14}
$$

Using the chain rule when differentiating we rewrite the following five vectors:

$$
\begin{aligned}
\mathbf{V}_2 &= -a_6\,\partial_{a_1} + a_4 a_6\,\partial_{r_1}, \\
\mathbf{V}_3 &= a_5\,\partial_{a_1} - a_4 a_5\,\partial_{r_1}, \\
\mathbf{V}_4 &= a_6\,\partial_{a_5} - a_5\,\partial_{a_6}, \\
\mathbf{V}_5 &= -a_3\,\partial_{a_1} - a_6\,\partial_{a_4} + a_4\,\partial_{a_6} + (a_1 a_6 + a_3 a_4)\,\partial_{r_1}, \\
\mathbf{V}_6 &= a_2\,\partial_{a_1} + a_5\,\partial_{a_4} - a_4\,\partial_{a_5} - (a_1 a_5 + a_2 a_4)\,\partial_{r_1}.
\end{aligned}
\tag{4.15}
$$

We proceed to find the fundamental solutions corresponding to the new $\mathbf{V}_2$: Again integrating the characteristic system, we arrive at the following 4 functionally independent invariants:

$$
a_4, \quad a_5, \quad a_6, \quad r_2 = a_1 a_4 + r_1. \tag{4.16}
$$

Now use (4.16) to rewrite the remaining 4 vector fields in (4.15):

$$\begin{aligned} \mathbf{V}_3 &= 0, \\ \mathbf{V}_4 &= a_6\,\partial_{a_5} - a_5\,\partial_{a_6}, \\ \mathbf{V}_5 &= -a_6\,\partial_{a_4} + a_4\,\partial_{a_6}, \\ \mathbf{V}_6 &= a_5\,\partial_{a_4} - a_4\,\partial_{a_5}. \end{aligned} \tag{4.17}$$

In view of (4.17), $a_4, a_5, a_6, r_2 (= a_1a_4 + r_1)$ are also functionally independent invariants of the new $\mathbf{V}_3$.

The three functionally independent invariants corresponding to the new $\mathbf{V}_4$ (see (4.17)) are found to be

$$a_4, \quad r_2, \quad r_3 = a_5^2 + a_6^2. \tag{4.18}$$

Based on (4.18), one rewrites the remaining two vectors in (4.17):

$$\begin{aligned} \mathbf{V}_5 &= -a_6\,\partial_{a_4} + 2a_4a_6\,\partial_{r_3}, \\ \mathbf{V}_6 &= a_5\,\partial_{a_4} - 2a_4a_5\,\partial_{r_3}. \end{aligned} \tag{4.19}$$

In view of (4.19), we expect $3 - 1 = 2$ functionally independent invariants corresponding to the new field $\mathbf{V}_5$:

$$r_2, \quad r_4 = r_3 + a_4^2. \tag{4.20}$$

It remains now to deal with the last field given in (4.19). Rewriting it using (4.20) we see

$$\mathbf{V}_6 = 0, \tag{4.21}$$

which means the functions $r_2, r_4 = r_3 + a_4^2$ are also killed by the new $\mathbf{V}_6$, and we finally obtain two fundamental isometry group invariants of Killing vectors defined on the Euclidean 3-space:

$$\begin{aligned} \Delta_1 &= a_1a_4 + a_2a_5 + a_3a_6, \\ \Delta_2 &= a_4^2 + a_5^2 + a_6^2. \end{aligned} \tag{4.22}$$

The above example shows how the method of characteristics can be employed to compute isometry group invariants of Killing tensors. Note also that two vectors vanish in the process, which means that the dimension of the orbits of the group action is 4 rather than 6. Indeed, the coefficient matrix of the generators (4.11)

$$\begin{pmatrix} 0 & a_6 & -a_5 & 0 & 0 & 0 \\ -a_6 & 0 & a_4 & 0 & 0 & 0 \\ a_5 & -a_4 & 0 & 0 & 0 & 0 \\ 0 & a_3 & -a_2 & 0 & a_6 & -a_5 \\ -a_3 & 0 & a_1 & -a_6 & 0 & a_4 \\ a_2 & -a_1 & 0 & a_5 & -a_4 & 0 \end{pmatrix} \tag{4.23}$$

is of rank 4 almost everywhere, thus according to the fundamental theorem on invariants (Theorem 3.1) one expects exactly two fundamental invariants, which forms the following theorem.

Theorem 4.2 *Any $I(\mathbb{E}^3)$-invariant of Killing vectors defined on the Euclidean 3-space is locally uniquely represented by*

$$\mathcal{I} = F(\Delta_1, \Delta_2), \tag{4.24}$$

where Δ_1, and Δ_2 are given in (4.22).

We remark that in [13, 34] the *method of undetermined coefficients* was developed to solve the over determined system of PDEs with the aid of a computer algebra package such as MAPLE. We briefly describe the idea of the method [13, 34]. To find the solution to the system of PDEs (4.12), one constructs a trial function as the solution to the PDEs, which is a polynomial in the parameters $a_1, \ldots, a_d$ of some fixed order. Substituting the trial function into the PDEs leads to a system of linear equations in the coefficients of the polynomial and any nontrivial solution to this system leads to a solution to the system of PDEs, and hence an invariant. We repeat this process with higher degree polynomial trial functions until the required number of invariants are found.

Example 4.2 [79] Consider the action of the isometry group $I(\mathbb{E}^2_1)$ on the vector space $\mathcal{K}^2(\mathbb{E}^2_1)$ of Killing tensors of valence two defined on the Minkowski plane $\mathbb{E}^2_1$. The general

Killing tensor in terms of the standard pseudo-Cartesian coordinates (t, x) is given by

$$\begin{aligned} \mathrm{K} \;=\; & (\alpha_1 + 2\alpha_4 x + \alpha_6 x^2)\,\partial_t \odot \partial_t \\ & +(\alpha_3 + \alpha_4 t + \alpha_5 x + \alpha_6 tx)\,\partial_t \odot \partial_x \\ & +(\alpha_2 + 2\alpha_5 t + \alpha_6 t^2)\,\partial_x \odot \partial_x, \end{aligned} \tag{4.25}$$

or in matrix form

$$\mathrm{K} = \begin{pmatrix} \alpha_1 + 2\alpha_4 x + \alpha_6 x^2 & \alpha_3 + \alpha_4 t + \alpha_5 x + \alpha_6 tx \\ \alpha_3 + \alpha_4 t + \alpha_5 x + \alpha_6 tx & \alpha_2 + 2\alpha_5 t + \alpha_6 t^2 \end{pmatrix}. \tag{4.26}$$

The isometry group $I(\mathbb{E}^2_1)$ acts on the Minkowski plane $\mathbb{E}^2_1$ with respect to the same coordinate system as follows

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \quad \phi, a, b \in \mathbb{R}. \tag{4.27}$$

The Lie algebra $i(\mathbb{E}^2_1)$ of the isometry group is generated by

$$\mathrm{T} = \partial_t, \quad \mathrm{X} = \partial_x, \quad \mathrm{H} = x\,\partial_t + t\,\partial_x \tag{4.28}$$

corresponding to the t- and x-translations and the hyperbolic rotations. Note the generators (4.28) of the Lie algebra $i(\mathbb{E}^2_1)$ enjoy the following commutator relations:

$$[\mathrm{T}, \mathrm{X}] = 0, \quad [\mathrm{T}, \mathrm{H}] = \mathrm{X}, \quad [\mathrm{X}, \mathrm{H}] = \mathrm{T}. \tag{4.29}$$

In view of (4.25), (4.27) the transformations (4.3) for the parameters α_i, $i = 1, \ldots, 6$ take the following form (see also [44, 56]).

$$
\begin{aligned}
\tilde{\alpha}_1 &= \alpha_1 \cosh^2\phi + 2\alpha_3 \cosh\phi \sinh\phi + \alpha_2 \sinh^2\phi + \alpha_6\, b^2 \\
&\quad -2(\alpha_4 \cosh\phi + \alpha_5 \sinh\phi)\, b, \\
\tilde{\alpha}_2 &= \alpha_1 \sinh^2\phi + 2\alpha_3 \cosh\phi \sinh\phi + \alpha_2 \cosh^2\phi + \alpha_6\, a^2 \\
&\quad -2(\alpha_5 \cosh\phi + \alpha_4 \sinh\phi)\, a, \\
\tilde{\alpha}_3 &= \alpha_3(\cosh^2\phi + \sinh^2\phi) + (\alpha_1 + \alpha_2)\cosh\phi \sinh\phi \\
&\quad -(a\,\alpha_4 + b\,\alpha_5)\cosh\phi - (a\,\alpha_5 + b\,\alpha_4)\sinh\phi + \alpha_6\, ab, \\
\tilde{\alpha}_4 &= \alpha_4 \cosh\phi + \alpha_5 \sinh\phi - \alpha_6\, b, \\
\tilde{\alpha}_5 &= \alpha_4 \sinh\phi + \alpha_5 \cosh\phi - \alpha_6\, a, \\
\tilde{\alpha}_6 &= \alpha_6.
\end{aligned} \tag{4.30}
$$

We note that the corresponding transformations for the parameters obtained in [53] were derived for *covariant* Killing tensors. Accordingly, they differ somewhat from (4.30) presented above (compare with (7.6) in [53]).

To determine the fundamental $I(\mathbb{E}_1^2)$-invariants, we first employ the MST-procedure to derive the infinitesimal generators.

$$
\begin{aligned}
\mathbf{V}_1 &= \alpha_4\, \partial_{\alpha_3} + 2\alpha_5\, \partial_{\alpha_2} + \alpha_6\, \partial_{\alpha_5}, \\
\mathbf{V}_2 &= \alpha_5\, \partial_{\alpha_3} + 2\alpha_4\, \partial_{\alpha_1} + \alpha_6\, \partial_{\alpha_4}, \\
\mathbf{V}_3 &= -2\alpha_3\, \partial_{\alpha_1} - \alpha_5\, \partial_{\alpha_4} - (\alpha_1 + \alpha_2)\, \partial_{\alpha_3} - 2\alpha_3\, \partial_{\alpha_2} - \alpha_4\, \partial_{\alpha_5}.
\end{aligned} \tag{4.31}
$$

Remark 4.4 The infinitesimal generators 4.31 can be obtained directly from the transformation formulas 4.30 by differentiating with respect to the group parameters a, b, ϕ evaluated at $a = 0, b = 0, \phi = 0$.

Note the vector fields $-\mathbf{V}_i$, $i = 1, 2, 3$ satisfy the same commutator relations as (4.28) (see (4.29)), which confirms Theorem 4.1. Solving the system of PDEs determined by the vector fields (4.31), we arrive at the following theorem.

Theorem 4.3 *An $I(\mathbb{E}_1^2)$-invariant $\mathcal{I}$ of the subspace of the parameter space Σ where the orbits of the group action are 3-dimensional can be (locally) uniquely expressed as an analytic function*

$$\mathcal{I} = F(\Delta_1, \Delta_2, \Delta_3), \tag{4.32}$$

where the fundamental invariants $\mathcal{I}_i$, $i = 1, 2, 3$ *are given by*

$$\begin{aligned} \Delta_1 &= (\alpha_4^2 + \alpha_5^2 - \alpha_6(\alpha_1 + \alpha_2))^2 - 4(\alpha_3\alpha_6 - \alpha_4\alpha_5)^2, \\ \Delta_2 &= \alpha_6(\alpha_1 - \alpha_2) - \alpha_4^2 + \alpha_5^2, \\ \Delta_3 &= \alpha_6. \end{aligned} \tag{4.33}$$

The fact that $\mathcal{I}_3 = \alpha_6$ is a fundamental $I(\mathbb{E}_1^2)$ -invariant of the vector space $\mathcal{K}^2(\mathbb{E}_1^2)$ trivially follows from the transformation (4.30). The fundamental $I(\mathbb{E}_1^2)$-invariant $\mathcal{I}_1$ was derived in [53, 57] in the study of the five-dimensional subspace of non-trivial Killing tensors defined on the Minkowski plane.

4.2.2 Covariants

Just like the classical invariant theory, sometimes it is not enough to just know the information about the isometry invariants of Killing tensors. We can introduce the concept of isometry group covariants of Killing tensors.

Consider the action of the isometry group $I(M)$ on the product space $\mathcal{K}^n(M) \times M$. The corresponding action on the extended parameter space $\Sigma \times M$ can be found explicitly, where Σ is the parameter space of $\mathcal{K}^n(M)$.

$$\begin{aligned} \tilde{\alpha}_1 &= \tilde{\alpha}_1(\alpha_1, \ldots, \alpha_d, g_1, \ldots, g_r), \\ &\vdots \\ \tilde{\alpha}_d &= \tilde{\alpha}_d(\alpha_1, \ldots, \alpha_d, g_1, \ldots, g_r), \\ \tilde{x}^1 &= \tilde{x}^1(x^1, \ldots, x^m, g_1, \ldots, g_r), \\ &\vdots \\ \tilde{x}^m &= \tilde{x}^m(x^1, \ldots, x^m, g_1, \ldots, g_r). \end{aligned} \tag{4.34}$$

Observe that the transformation formulas for the Killing tensor parameters are linear in $\alpha_1, \ldots, \alpha_d$.

We now introduce the concept of an isometry group *covariant* of Killing tensors.

Definition 4.1 An $I(M)$-*covariant* of the vector space $\mathcal{K}^n(M)$, $n \geq 1$ is a function C : $\Sigma \times M \to \mathbb{R}$ satisfying the condition

$$C = F(\alpha_1, \ldots, \alpha_d, x^1, \ldots, x^m) = F(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_d, \tilde{x}^1 \ldots, \tilde{x}^m) \tag{4.35}$$

under the transformation laws (4.34).

Theorem 4.1 entails the following corollary.

Corollary 4.1 *Consider the product vector space $\mathcal{K}^n(M) \times M$, $n \geq 1$. Define the vector fields*

$$\mathrm{V}'_i := \mathrm{V}_i + \mathrm{X}_i \quad i = 1, \dots, r, \tag{4.36}$$

where V_i, $i = 1, \dots, r$ are described in (4.5) and X_i, $i = 1, \dots, r$ are the generators of $i(M)$. Then the vector fields $\mathrm{V}'_1, \dots, \mathrm{V}'_r$ enjoy the same commutator relations as the generators $\mathrm{X}_1, \dots, \mathrm{X}_r$:

$$[\mathrm{V}'_i, \mathrm{V}'_j] = c^k_{ij}\mathrm{V}'_k, \quad i, j, k = 1, \dots, r, \tag{4.37}$$

where the structural constants c^k_{ij} are given in (4.6).

Therefore, an $I(M)$-covariants C of $\mathcal{K}^n(M)$ can be obtained by solving the system of PDEs determined by the vector fields (4.36):

$$\mathrm{V}'_i(C) = 0, \quad i = 1, \dots, r. \tag{4.38}$$

Remark 4.5 Alternatively, one can employ the method of moving frames, we will be using this method to compute the isometry group-covariants of Killing tensors of valence two defined on the Euclidean plane and the Minkowski plane, respectively. See Example 4.4 and Example 4.5 later in this Chapter.

4.2.3 Joint invariants

Consider the action of the isometry group $I(M)$ on the product space

$$\mathcal{K}^\ell(M) \times \mathcal{K}^n(M) \times \ldots \times \mathcal{K}^q(M), \quad \ell, n, \dots, q \geq 1.$$

Let $\alpha_1, \dots, \alpha_d$, $\beta_1, \dots, \beta_e$, ..., $\gamma_1, \dots, \gamma_f$ be the parameters of the vector spaces $\mathcal{K}^\ell(M)$, $\mathcal{K}^n(M)$, ..., $\mathcal{K}^q(M)$ respectively, where $d, e, \dots, f$ are the corresponding dimensions determined by (4.2). Then the action of the isometry group $I(M)$ induces the corresponding transformation laws for the parameters $\alpha_1, \dots, \alpha_d$, $\beta_1, \dots, \beta_e$, ..., $\gamma_1, \dots, \gamma_f$.

$$
\begin{aligned}
\tilde{\alpha}_1 &= \tilde{\alpha}_1(\alpha_1, \ldots, \alpha_d, g_1, \ldots, g_r), \\
&\vdots \\
\tilde{\alpha}_d &= \tilde{\alpha}_d(\alpha_1, \ldots, \alpha_d, g_1, \ldots, g_r), \\
\tilde{\beta}_1 &= \tilde{\beta}_1(\beta_1, \ldots, \beta_e, g_1, \ldots, g_r), \\
&\vdots \\
\tilde{\beta}_e &= \tilde{\beta}_e(\beta_1, \ldots, \beta_e, g_1, \ldots, g_r), \\
&\vdots \\
\tilde{\gamma}_1 &= \tilde{\gamma}_1(\gamma_1, \ldots, \gamma_f, g_1, \ldots, g_r), \\
&\vdots \\
\tilde{\gamma}_f &= \tilde{\gamma}_f(\gamma_1, \ldots, \gamma_f, g_1, \ldots, g_r).
\end{aligned}
\tag{4.39}
$$

Note again that the transformations are linear in the Killing tensor parameters.

We now introduce the concept of a *joint* $I(M)$*-invariant.*

Definition 4.2 A *joint* $I(M)$*-invariant* J of the product space

$$\mathcal{K}^{\ell}(M) \times \mathcal{K}^{m}(M) \times \ldots \times \mathcal{K}^{q}(M)$$

is a function $J : \Sigma^{\ell} \times \Sigma^{m} \times \ldots \times \Sigma^{q} \to \mathbb{R}$ satisfying the condition

$$
\begin{aligned}
J &= F(\alpha_1, \ldots, \alpha_d, \beta_1 \ldots, \beta_e, \ldots, \gamma_1 \ldots, \gamma_f) \\
&= F(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_d, \tilde{\beta}_1 \ldots, \tilde{\beta}_e, \ldots, \tilde{\gamma}_1 \ldots, \tilde{\gamma}_f)
\end{aligned}
\tag{4.40}
$$

under the transformation laws (4.39) induced by the isometry group $I(M)$.

In this case again Theorem 4.1 entails the following corollary.

Corollary 4.2 *Define the vector fields*

$$\mathbf{V}_i := \mathbf{V}_i^{\ell} + \mathbf{V}_i^{n} + \ldots + \mathbf{V}_i^{q}, \quad i = 1, \ldots, r, \tag{4.41}$$

where $\{\mathbf{V}_i^{\ell}\}, \{\mathbf{V}_i^{n}\}, \ldots, \{\mathbf{V}_i^{q}\}$*,* $i = 1, \ldots, r$ *are the sets of infinitesimal generators of the Lie algebra* $i(M)$ *on the parameter spaces* Σ^{ℓ}*,* $\Sigma^{m}, \ldots, \Sigma^{q}$ *of the vector spaces* $\mathcal{K}^{\ell}(M)$*,* $\mathcal{K}^{n}(M)$*, …,* $\mathcal{K}^{q}(M)$*, respectively, obtained by the MST-procedure. Then the vector fields* $\mathbf{V}_1, \ldots, \mathbf{V}_r$ *enjoy the same commutator relations as the generators* $\mathbf{X}_1, \ldots, \mathbf{X}_r$*:*

$$[\mathbf{V}_i, \mathbf{V}_j] = c_{ij}^{k} \mathbf{V}_k, \quad i, j, k = 1, \ldots, r, \tag{4.42}$$

where the structural constants c_{ij}^{k} *are as in (4.6).*

Example 4.3 [79] Consider the product vector space $\mathcal{K}^1(\mathbb{E}^2) \times \mathcal{K}^2(\mathbb{E}^2)$. A general Killing vector with respect to the Cartesian coordinates takes the following form.

$$\mathbf{K}^1 = (\alpha_1 + \alpha_3 y)\,\partial_x + (\alpha_2 - \alpha_3 x)\,\partial_y, \tag{4.43}$$

while with respect to the same coordinate system the elements of $\mathcal{K}^2(\mathbb{E}^2)$ assume

$$\begin{aligned} \mathbf{K}^2 &= (\beta_1 + 2\beta_4 y + \beta_6 y^2)\,\partial_x \odot \partial_x \\ &\quad + (\beta_3 - \beta_4 x - \beta_5 y - \beta_6 xy)\,\partial_x \odot \partial_y \\ &\quad + (\beta_2 + 2\beta_5 x + \beta_6 x^2)\,\partial_y \odot \partial_y. \end{aligned} \tag{4.44}$$

Here $\odot$ denotes the symmetric tensor product. The formulas (4.43) and (4.44) put in evidence that the corresponding parameter spaces Σ^1 and Σ^2 are determined by the three parameters α_i, $i = 1, \dots, 3$ and the six parameters β_j, $j = 1, \dots, 6$ respectively.

The isometry group acts on $\mathbb{E}^2$ as follows.

$$\begin{aligned} \tilde{x} &= x\cos\theta - y\sin\theta + a, \\ \tilde{y} &= x\sin\theta + y\cos\theta + b, \end{aligned} \qquad \theta, a, b \in \mathbb{R}. \tag{4.45}$$

Note, the generators of $i(\mathbb{E}^2) = \mathcal{K}^1(\mathbb{E}^2)$, which is the Lie algebra of the Lie group $I(\mathbb{E}^2)$, are given with respect to the Cartesian coordinates by

$$\mathbf{X} = \partial_x, \quad \mathbf{Y} = \partial_y, \quad \mathbf{R} = x\,\partial_y - y\,\partial_x. \tag{4.46}$$

The corresponding flows are translations and rotations respectively. Employing the construction (4.5), we derive two triples of the vector fields representing the generators (4.46) on Σ^1

$$\begin{aligned} \mathbf{V}_1^1 &= -\alpha_3\,\partial_{\alpha_2}, \\ \mathbf{V}_2^1 &= \alpha_3\,\partial_{\alpha_1}, \\ \mathbf{V}_3^1 &= \alpha_1\,\partial_{\alpha_2} - \alpha_2\,\partial_{\alpha_1}, \end{aligned} \tag{4.47}$$

and (4.46) on Σ^2

$$\begin{aligned} \mathbf{V}_1^2 &= -2\beta_5\,\partial_{\beta_2} - \beta_4\,\partial_{\beta_3} + \beta_6\,\partial_{\beta_5}, \\ \mathbf{V}_2^2 &= 2\beta_4\,\partial_{\beta_1} - \beta_5\,\partial_{\beta_3} + \beta_6\,\partial_{\beta_6}, \\ \mathbf{V}_3^2 &= -2\beta_3\left(\partial_{\beta_1} - \partial_{\beta_2}\right) + (\beta_1 - \beta_2)\,\partial_{\beta_3} + \beta_5\,\partial_{\beta_4} - \beta_4\,\partial_{\beta_5}, \end{aligned} \tag{4.48}$$

respectively. We note that in view of Conjecture 4.1 both the vector fields (4.47) and the vector fields (4.48) satisfy the same commutator relations as the generators of $i(\mathbb{E}^2)$ (4.46). By Corollary 4.2 this fact entails immediately that the vector fields $\mathbf{V}_i$, $i = 1, 2, 3$ defined by

$$\mathbf{V}_i := \mathbf{V}_i^1 + \mathbf{V}_i^2, \quad i = 1, 2, 3 \tag{4.49}$$

also enjoy the same commutator relations. Therefore we have determined the infinitesimal action of $I(\mathbb{E}^2)$ on the product space $\Sigma^1 \times \Sigma^2$. The orbits of the isometry group $I(\mathbb{E}^2)$ acting in $\Sigma^1 \times \Sigma^2$ are three-dimensional in the subspace $\mathcal{S}_3 \subset \Sigma^1 \times \Sigma^2$, where the generators (4.49) are linearly independent. According to Theorem 3.1, the number of fundamental invariants in $\mathcal{S}_3$ is 9 (dimension of $\Sigma^1 \times \Sigma^2$) - 3 (dimension of the orbits in $\mathcal{S}_3$) = 6. Some of these fundamental invariants may be the fundamental invariants of the group action on the vector spaces $\mathcal{K}^1(\mathbb{E}^2)$ and $\mathcal{K}^2(\mathbb{E}^2)$. Indeed, it is instructive to look at the induced transformations of the 9 parameters $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$.

$$\begin{aligned}
\tilde{\alpha}_1 &= \alpha_1 \cos\theta - \alpha_2 \sin\theta - b\,\alpha_3, \\
\tilde{\alpha}_2 &= \alpha_1 \sin\theta + \alpha_2 \cos\theta + a\,\alpha_3, \\
\tilde{\alpha}_3 &= \alpha_3, \\
\tilde{\beta}_1 &= \beta_1 \cos^2\theta - 2\beta_3 \cos\theta \sin\theta + \beta_2 \sin^2\theta - 2b\,\beta_4 \cos\theta - 2b\,\beta_5 \sin\theta \\
&\quad + \beta_6\, b^2, \\
\tilde{\beta}_2 &= \beta_1 \sin^2\theta - 2\beta_3 \cos\theta \sin\theta + \beta_2 \cos^2\theta - 2a\,\beta_5 \cos\theta + 2a\,\beta_4 \sin\theta \\
&\quad + \beta_6\, a^2, \\
\tilde{\beta}_3 &= (\beta_1 - \beta_2) \sin\theta \cos\theta + \beta_3(\cos^2\theta - \sin^2\theta) + (a\,\beta_4 + b\,\beta_5) \cos\theta \\
&\quad + (a\,\beta_5 - b\,\beta_4) \sin\theta - \beta_6\, ab, \\
\tilde{\beta}_4 &= \beta_4 \cos\theta + \beta_5 \sin\theta - \beta_6\, b, \\
\tilde{\beta}_5 &= \beta_5 \cos\theta - \beta_4 \sin\theta - \beta_6\, a, \\
\tilde{\beta}_6 &= \beta_6.
\end{aligned} \tag{4.50}$$

Hence, the dimension of the orbits in this subspace coincides with the dimension of the group. We also observe that α_3 and β_6 are fundamental joint $I(\mathbb{E}^2)$-invariants of Killing tensors.

To determine the remaining four fundamental invariants we use the method of characteristics to solve the system of linear PDEs

$$\mathbf{V}_i(J) = 0, \quad i = 1, 2, 3, \tag{4.51}$$

where $J : \Sigma^1 \times \Sigma^2 \to \mathbb{R}$ and the vector fields $\mathbf{V}_i$, $i = 1, 2, 3$ are given by (4.49). Solving the system of PDEs (4.51), we arrive at the following result.

Theorem 4.4 *Any joint $I(\mathbb{E}^2)$-invariant J of $\mathcal{K}^1(\mathbb{E}^2) \times \mathcal{K}^2(\mathbb{E}^2)$ where the vector fields (4.49) are linearly independent can be locally uniquely expressed as an analytic function*

$$J = F(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{J}_1, \mathcal{J}_2), \tag{4.52}$$

where the fundamental joint $I(\mathbb{E}^2)$-invariants $\mathcal{I}_i$, $\mathcal{J}_j$, $i = 1, \ldots, 4$, $j = 1, 2$ are given by

$$\begin{aligned}
\mathcal{I}_1 &= [\beta_6(\beta_1 - \beta_2) + \beta_5^2 - \beta_4^2]^2 + 4(\beta_3\beta_6 + \beta_4\beta_5)^2, \\
\mathcal{I}_2 &= \beta_6(\beta_1 + \beta_2) - \beta_4^2 - \beta_5^2, \\
\mathcal{I}_3 &= \beta_6, \\
\mathcal{I}_4 &= \alpha_3, \\
\mathcal{J}_1 &= (\beta_6\alpha_2 + \beta_5\alpha_3)^2 + (\beta_6\alpha_1 - \beta_4\alpha_3)^2, \\
\mathcal{J}_2 &= (\beta_6\alpha_2 + \alpha_3\alpha_5)(\beta_6\beta_2 - \beta_5^2) + 2(\beta_3\beta_6 + \beta_4\beta_5)(\beta_6\alpha_1 - \beta_4\alpha_3).
\end{aligned} \tag{4.53}$$

The fundamental joint $I(\mathbb{E}^2)$-invariants $\mathcal{I}_i$, $i = 1, 2, 3$ are the fundamental $I(\mathbb{E}^2)$-invariants of the vector space $\mathcal{K}^2(\mathbb{E}^2)$ ($\mathcal{I}_1$ was derived in [52]), while $\mathcal{I}_4$ is the fundamental $I(\mathbb{E}^2)$-invariant of the vector space $\mathcal{K}^1(\mathbb{E}^2)$. Note the fundamental $I(\mathbb{E}^2)$-invariants $\mathcal{J}_1$ and $\mathcal{J}_2$ are "truly" joint $I(\mathbb{E}^2)$-invariants of $\mathcal{K}^1(\mathbb{E}^2) \times \mathcal{K}^2(\mathbb{E}^2)$.

Remark 4.6 It will be interesting to investigate the geometric interpretation of the joint invariants of Killing tensors.

4.3 The method of moving frames

It was first observed in [13] that the method of moving frames could be used to solve the problem of the determination of fundamental invariants of vector spaces of Killing tensors under the action of the isometry group. In some cases this method seems much more efficient than the method of infinitesimal generators. Note also that recently a new technique of the method of moving frames due to Kogan [50] has been added to the literature of ITKT [80, 81, 90].

4.3.1 Covariants of Killing tensors of valence two

Example 4.4 [79] Consider first the extended vector space $\mathcal{K}^2(\mathbb{E}^2) \times \mathbb{E}^2$. The corresponding extended parameter space $\Sigma \times \mathbb{E}^2$ is determined by the parameters $\beta_1, \ldots, \beta_6, x, y$, where β_i, $i = 1, \ldots, 6$ are as in (4.44) and x, y are Cartesian coordinates. The isometry group $I(\mathbb{E}^2)$ induces the corresponding action (4.34) on the parameter space $\Sigma \times \mathbb{E}^2$.

$$
\begin{aligned}
\tilde{\beta}_1 &= \beta_1 \cos^2\theta - 2\beta_3 \cos\theta \sin\theta + \beta_2 \sin^2\theta - 2b\,\beta_4 \cos\theta - 2b\,\beta_5 \sin\theta \\
&\quad + \beta_6\, b^2, \\
\tilde{\beta}_2 &= \beta_1 \sin^2\theta - 2\beta_3 \cos\theta \sin\theta + \beta_2 \cos^2\theta - 2a\,\beta_5 \cos\theta + 2a\,\beta_4 \sin\theta \\
&\quad + \beta_6\, a^2, \\
\tilde{\beta}_3 &= (\beta_1 - \beta_2)\sin\theta\cos\theta + \beta_3(\cos^2\theta - \sin^2\theta) + (a\,\beta_4 + b\,\beta_5)\cos\theta \\
&\quad + (a\,\beta_5 - b\,\beta_4)\sin\theta - \beta_6\, ab, \\
\tilde{\beta}_4 &= \beta_4 \cos\theta + \beta_5 \sin\theta - \beta_6\, b, \\
\tilde{\beta}_5 &= \beta_5 \cos\theta - \beta_4 \sin\theta - \beta_6\, a, \\
\tilde{\beta}_6 &= \beta_6, \\
\tilde{x} &= x\cos\theta - y\cos\theta + a, \\
\tilde{y} &= x\sin\theta + y\cos\theta + b.
\end{aligned}
\tag{4.54}
$$

Next, we construct a moving frame by using the cross-section (for example)

$$
K = \{\beta_3 = \beta_4 = \beta_5 = 0\}, \tag{4.55}
$$

which yields the normalization equations

$$\begin{aligned} 0 &= (\beta_1-\beta_2)\sin\theta\cos\theta+\beta_3(\cos^2\theta-\sin^2\theta) \\ &\quad +(a\,\beta_4+b\,\beta_5)\cos\theta+(a\,\beta_5-b\,\beta_4)\sin\theta-\beta_6\,ab, \\ 0 &= \beta_4\cos\theta+\beta_5\sin\theta-\beta_6\,b, \\ 0 &= \beta_5\cos\theta-\beta_4\sin\theta-\beta_6\,a. \end{aligned} \tag{4.56}$$

Solving (4.56) for the parameters a, b and θ, we obtain the moving frame map provided that $\beta_6 \neq 0$.

$$\begin{aligned} a &= \frac{\beta_5\cos\theta-\beta_4\sin\theta}{\beta_6}, \\ b &= \frac{\beta_4\cos\theta+\beta_5\sin\theta}{\beta_6}, \\ \theta &= \frac{1}{2}\arctan\frac{2(\beta_3\beta_6+\beta_4\beta_5)}{\beta_6(\beta_1-\beta_2)-\beta_4^2+\beta_5^2}. \end{aligned} \tag{4.57}$$

Having derived the moving frame map (4.57) and the transformation laws (4.54), we can now make use of the result of Theorem 3.1 and determine a complete set of fundamental $I(\mathbb{E}^2)$-covariants of $\mathcal{K}^2(\mathbb{E}^2)$. Substituting (4.57) into (4.54), by Theorem 3.4 we arrive at the following result.

Theorem 4.5 *Consider the vector space $\mathcal{K}^2(\mathbb{E}^2)$. Any algebraic $I(\mathbb{E}^2)$-covariant C defined over the subspace of $\Sigma\times\mathbb{E}^2$ where the isometry group $I(\mathbb{E}^2)$ acts freely and regularly with three-dimensional orbits (provided that $\beta_6\neq 0$) can be locally uniquely expressed as an analytic function*

$$C = F(\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3,\mathcal{C}_1,\mathcal{C}_2), \tag{4.58}$$

where the fundamental $I(\mathbb{E}^2)$-covariants $\mathcal{I}_i$, $\mathcal{C}_j$, $i=1,2,3$, $j=1,2$ are given by

$$\begin{aligned} \mathcal{I}_1 &= (\beta_6(\beta_1-\beta_2)+\beta_5^2-\beta_4^2)^2+4(\beta_3\beta_6+\beta_4\beta_5)^2, \\ \mathcal{I}_2 &= \beta_6(\beta_1+\beta_2)-\beta_4^2-\beta_5^2, \\ \mathcal{I}_3 &= \beta_6, \\ \mathcal{C}_1 &= (\beta_6x+\beta_5)^2+(\beta_6y+\beta_4)^2, \\ \mathcal{C}_2 &= ((\beta_6x+\beta_5)^2-(\beta_6y+\beta_4)^2)(\beta_5^2-\beta_4^2+\beta_6(\beta_1-\beta_2)) \\ &\quad +4(\beta_6x+\beta_5)(\beta_6y+\beta_4)(\beta_6\beta_3+\beta_4\beta_5), \end{aligned} \tag{4.59}$$

where Σ is the parameter space of $\mathcal{K}^2(\mathbb{E}^2)$.

We immediately observe that the functions $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ constitute in fact a set of fundamental $I(\mathbb{E}^2)$-invariants of the vector space $\mathcal{K}^2(\mathbb{E}^2)$, while the functions $\mathcal{C}_1$ and $\mathcal{C}_2$ are "truly" fundamental $I(\mathbb{E}^2)$-covariants of the vector space $\mathcal{K}^2(\mathbb{E}^2)$. We also observe that the fundamental covariant $\mathcal{C}_1$ can be expressed as

$$\mathcal{C}_1 = \mathcal{I}_3 \mathrm{tr}\hat{\mathbf{K}} - \mathcal{I}_2,$$

where the $(1,1)$-tensor $\hat{\mathbf{K}}$ is given by $\hat{\mathbf{K}} = \mathbf{K}\mathbf{g}^{-1}$. This observation immediately suggests that $\mathrm{tr}\hat{\mathbf{K}}$ is a fundamental $I(\mathbb{E}^2)$-covariant of $\mathcal{K}^2(\mathbb{E}^2)$. We note, however, that the function $\det\hat{\mathbf{K}}$ is not a fundamental $I(\mathbb{E}^2)$-covariant of $\mathcal{K}^2(\mathbb{E}^2)$.

Example 4.5 [79] Let $\mathcal{K}^2(\mathbb{E}^2_1) \times \mathbb{E}^2_1$ be the extended vector space of $\mathcal{K}^2(\mathbb{E}^2_1)$. The action of the isometry group $I(\mathbb{E}^2_1)$ on the Minkowski plane $\mathbb{E}^2_1$ is given by (4.27), while the corresponding action on the parameter space Σ of $\mathcal{K}^2(\mathbb{E}^2_1)$ is given by (4.30). The transformation laws (4.30) combined with the transformations (4.27) yield an analogue of (4.54). Next, we proceed as in Example 4.4. The resulting moving frame map $\rho : \Sigma \times \mathbb{E}^2_1 \to I(\mathbb{E}^2_1)$ (provided that $\alpha_6 \neq 0$) is given by

$$\begin{aligned} a &= \frac{\alpha_4 \sinh\phi + \alpha_5 \cosh\phi}{\alpha_6}, \\ b &= \frac{\alpha_4 \cosh\phi + \alpha_5 \sinh\phi}{\alpha_6}, \\ \phi &= \frac{1}{2}\mathrm{arctanh}\,\frac{2(\alpha_3\alpha_6 - \alpha_4\alpha_5)}{\alpha_4^2 + \alpha_5^2 - \alpha_6(\alpha_1 + \alpha_2)}. \end{aligned} \tag{4.60}$$

Now we can continue as in the previous example to determine a set of fundamental $I(\mathbb{E}^2_1)$-covariants of $\mathcal{K}^2(\mathbb{E}^2_1)$.

Theorem 4.6 *Any algebraic $I(\mathbb{E}^2_1)$-covariant of $\mathcal{K}^2(\mathbb{E}^2_1)$ C defined over the subspace of $\Sigma \times \mathbb{E}^2_1$ where the isometry group $I(\mathbb{E}^2_1)$ acts freely and regularly with three-dimensional orbits (provided that $\alpha_6 \neq 0$) can be locally uniquely expressed as an analytic function*

$$C = F(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{C}_1, \mathcal{C}_2),$$

where the fundamental covariants $\mathcal{I}_i$, $\mathcal{C}_j$, $i = 1, 2, 3$, $j = 1, 2$ are given by

$$\begin{aligned}
\mathcal{I}_1 &= (\alpha_4^2 + \alpha_5^2 - \alpha_6(\alpha_1 + \alpha_2))^2 - 4(\alpha_3\alpha_6 - \alpha_4\alpha_5)^2, \\
\mathcal{I}_2 &= (\alpha_1 - \alpha_2)\alpha_6 - \alpha_4^2 + \alpha_5^2, \\
\mathcal{I}_3 &= \alpha_6, \\
\mathcal{C}_1 &= (\alpha_6 t + \alpha_5)^2 - (\alpha_6 x + \alpha_4)^2, \\
\mathcal{C}_2 &= ((\alpha_6 t + \alpha_5)^2 + (\alpha_6 x + \alpha_4)^2)(\alpha_4^2 + \alpha_5^2 - \alpha_6(\alpha_1 + \alpha_2)) \\
&\quad +4(\alpha_6 t + \alpha_5)(\alpha_6 x + \alpha_4)(\alpha_3\alpha_6 - \alpha_4\alpha_5).
\end{aligned} \tag{4.61}$$

The conclusion here is similar to that following Theorem 4.5. Thus, we observe again that the functions $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ constitute in fact a set of fundamental $I(\mathbb{E}_1^2)$-invariants of the vector space $\mathcal{K}^2(\mathbb{E}_1^2)$, while the functions $\mathcal{C}_1$ and $\mathcal{C}_2$ are "truly" fundamental $I(\mathbb{E}_1^2)$-covariants of the vector space $\mathcal{K}^2(\mathbb{E}_1^2)$.

4.3.2 Killing tensors of valence three on the Minkowski plane

As has been demonstrated above, the method of moving frames is effective in computing isometry group of invariants. As one increases the valence of the Killing tensor under investigation, however, one will face the difficulty of finding the solutions to the normalization equations and thus it is very difficult to proceed to determine the fundamental invariants, see for example [83]. In [50] an inductive version of the method of moving frames is developed which can be effectively used in ITKT. To be more specific, if the action of the group can be factorized as two subgroups and the intersection of the two is discrete, then one can employ the moving frame method to find the fundamental invariants of the first group action. Then, using these fundamental invariants as new coordinates, consider the action of the second group, again employ the moving frame method to find the fundamental invariants, which will be the fundamental invariants of the total group action. We shall make use of this technique to solve the problem of the determination of a complete set of fundamental isometry group invariants of Killing tensors of valence three defined On the Minkowski plane and the Euclidean plane, respectively. These will take up the following two subsections.

Consider Killing tensors of valence three defined on the Minkowski plane. Using the null coordinates (u, v) with respect to which the metric is of the form $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note here the relation between the null coordinates (u, v) and the pseudo-Cartesian coordinates (t, x) is given by

$$u = \frac{1}{\sqrt{2}}(t + x), \quad v = \frac{1}{\sqrt{2}}(t - x). \tag{4.62}$$

Solving the corresponding Killing tensor equations, we arrive at a representation for the Killing tensors

$$\begin{aligned} K^{111} &= a_1 - 3a_5 u - 3a_7 u^2 - a_{10} u^3, \\ K^{112} &= (a_3 + a_9 u + a_8 u^2) + v(a_5 + 2a_7 u + a_{10} u^2), \\ K^{122} &= (a_4 - a_9 v - a_7 v^2) + u(a_6 - 2a_8 v - a_{10} v^2), \\ K^{222} &= a_2 - 3a_6 v + 3a_8 v^2 + a_{10} v^3, \end{aligned} \tag{4.63}$$

Alternatively the Killing tensor can be represented as sum of symmetrized product of Killing vectors, see Horwood *et al*[34] and Horwood *et al*[36] for more details. The infinitesimal generators determined via the MST-procedure are

$$\begin{aligned} \mathbf{U}_1 &= -3a_5\,\partial_{a_1} + a_9\,\partial_{a_3} + a_6\,\partial_{a_4} + 2a_7\,\partial_{a_5} + 2a_8\,\partial_{a_9} + a_{10}\,\partial_{a_7}, \\ \mathbf{U}_2 &= -3a_6\,\partial_{a_2} - a_9\,\partial_{a_4} + a_5\,\partial_{a_3} - 2a_8\,\partial_{a_6} + 2a_7\,\partial_{a_9} + a_{10}\,\partial_{a_8}, \\ \mathbf{V} &= -3a_1\,\partial_{a_1} + 3a_2\,\partial_{a_2} - a_3\,\partial_{a_3} + a_4\,\partial_{a_4} - 2a_5\,\partial_{a_5} + 2a_6\,\partial_{a_6} \\ &\quad -a_7\,\partial_{a_7} + a_8\,\partial_{a_8}, \end{aligned} \tag{4.64}$$

where $\partial_{a_i} = \frac{\partial}{\partial a_i}, \quad i = 1, \ldots, 10.$
To determine a complete set of fundamental $I(\mathbb{E}_1^2)$-invariants (or covariants) of $\mathcal{K}^3(\mathbb{E}_1^2)$ one can attempt to solve the system of PDEs determined by the vector fields (4.64) for invariants and covariants. Unfortunately, the method of characteristics fails in this case. Employing the method of moving frames for the *whole group* $I(\mathbb{E}_1^2)$ leads to equally frustrating results (see [83] for more details) due to insurmountable computational difficulties.

We make use of the construction of moving frames introduced recently by Kogan [50] to the theory of moving frames [22, 22] (see also the references therein), which has been successfully applied to the theory of invariant differential forms on jet bundles. The application of the algorithm due to Kogan for our purposes in ITKT is especially simple, effective and greatly simplifies the procedure.

We first observe that the rank of the system of linear PDEs (4.64) is three almost everywhere. Therefore there is an open subspace of Σ^3 where the group acts with 3-dimensional orbits. The latter means that in view of Theorem 3.1 we have to derive 10 (dimension of Σ^3) -3 (dimension of the orbits) $= 7$ fundamental invariants in the subspace of Σ^3 where the group acts freely and regularly with 3-dimensional orbits. Observe next that the isometry group $I(\mathbb{E}_1^2)$ is a product $I(\mathbb{E}_1^2) = \mathrm{TH}$ of the subgroup T of translations and the subgroup H of hyperbolic rotations. Moreover, $\mathrm{T} \cap \mathrm{H}$ is discrete. The two-parameter subgroup T acts on $\mathbb{E}_1^2$ as follows:

$$\tilde{u} = u + b, \quad \tilde{v} = v + c, \qquad b, c \in \mathbb{R}. \tag{4.65}$$

The action induces the action on the Killing tensors.

$$\begin{aligned}
\tilde{a}_1 &= a_1 + 3b\,a_5 - 3b^2 a_7 + b^3 a_{10}, \\
\tilde{a}_2 &= a_2 + 3c\,a_6 + 3c^2 a_8 - c^3 a_{10}, \\
\tilde{a}_3 &= a_3 - b\,a_9 - c\,a_5 + b^2 a_8 + 2bc\,a_7 - b^2 c\,a_{10}, \\
\tilde{a}_4 &= a_4 + c\,a_9 - b\,a_6 - c^2 a_7 - 2bc\,a_8 + bc^2 a_{10}, \\
\tilde{a}_5 &= a_5 - 2b\,a_7 + b^2 a_{10}, \\
\tilde{a}_6 &= a_6 + 2c\,a_8 - c^2 a_{10}, \\
\tilde{a}_7 &= a_7 - b\,a_{10}, \\
\tilde{a}_8 &= a_8 - c\,a_{10}, \\
\tilde{a}_9 &= a_9 - 2b\,a_8 - 2c\,a_7 + 2bc\,a_{10}, \\
\tilde{a}_{10} &= a_{10}.
\end{aligned} \tag{4.66}$$

We are now using (4.66) together with the method of moving frames to find a set of fundamental T-invariants.

Indeed, choosing the cross-section $K = \{a_7 = 0, a_8 = 0\}$ leads to the moving frame solution to the normalization equations provided that $a_{10} \neq 0$.

$$b = \frac{a_7}{a_{10}}, \quad c = \frac{a_8}{a_{10}}. \tag{4.67}$$

Substituting these expressions into the remaining eight equations of (4.66), we obtain the following fundamental T-invariants of $\mathcal{K}^3(\mathbb{E}_1^2)$ (provided that $a_{10} \neq 0$):

$$\begin{aligned}
I_1 &= a_1a_{10}^2 + 3a_5a_7a_{10} - 2a_7^3, \\
I_2 &= a_2a_{10}^2 + 3a_6a_8a_{10} + 2a_8^3, \\
I_3 &= a_3a_{10}^2 - a_7a_9a_{10} - a_5a_8a_{10} + 2a_7^2a_8, \\
I_4 &= a_4a_{10}^2 + a_8a_9a_{10} - a_6a_7a_{10} - 2a_7a_8^2, \\
I_5 &= a_5a_{10} - a_7^2, \\
I_6 &= a_6a_{10} + a_8^2, \\
I_7 &= a_9a_{10} - 2a_7a_8, \\
I_8 &= a_{10}.
\end{aligned} \tag{4.68}$$

Next, we determine the action of the subgroup H on the space spanned by the eight T-invariants (4.68).

The subgroup H acts on $\mathbb{E}_1^2$ by

$$\tilde{u} = ue^t, \quad \tilde{v} = ve^{-t}, \qquad t \in \mathbb{R}, \tag{4.69}$$

where t is the parameter of H. We derive the action of H on Σ^3

$$\begin{aligned}
\tilde{a}_1 &= a_1e^{3t}, & \tilde{a}_2 &= a_2e^{-3t}, \\
\tilde{a}_3 &= a_3e^{t}, & \tilde{a}_4 &= a_4e^{-t}, \\
\tilde{a}_5 &= a_5e^{2t}, & \tilde{a}_6 &= a_6e^{-2t}, \\
\tilde{a}_7 &= a_7e^{t}, & \tilde{a}_8 &= a_8e^{-t}, \\
\tilde{a}_9 &= a_9, & \tilde{a}_{10} &= a_{10},
\end{aligned} \tag{4.70}$$

and hence the action on the space spanned by the T-invariants (4.68).

$$\begin{aligned}
\tilde{I}_1 &= e^{3t}I_1, & \tilde{I}_2 &= e^{-3t}I_2, \\
\tilde{I}_3 &= e^{2t}I_3, & \tilde{I}_4 &= e^{-2t}I_4, \\
\tilde{I}_5 &= e^{t}I_5, & \tilde{I}_6 &= e^{-t}I_6, \\
\tilde{I}_7 &= I_7, & \tilde{I}_8 &= I_8.
\end{aligned} \tag{4.71}$$

We immediately observe that I_7 and I_8 are fundamental $I(\mathbb{E}_1^2)$-invariants of $\mathcal{K}^3(\mathbb{E}_1^2)$, while $I_1, \ldots, I_6$ are *conformal* $I(\mathbb{E}_1^2)$-invariants of $\mathcal{K}^3(\mathbb{E}_1^2)$ (see [54] for more details). Moreover, it easily follows that

Proposition 4.1 *The functions*

$$Z_i := \operatorname{sign} I_i, \quad i = 1, \ldots, 6$$

are discrete *$I(\mathbb{E}_1^2)$-invariants of $\mathcal{K}^3(\mathbb{E}_1^2)$, where $I_1, \ldots, I_6$ are given by (4.68).*

To determine the remaining five continuous invariants, we employ the method of moving frames again, taking into account the action (4.71) of the subgroup H on $I_1, \ldots, I_8$. Indeed, we note that no cross-section of the form $K = \{I_i = 0\}, i = 1, \ldots, 6$ intersects the orbits transversally. Hence, we choose the cross-section $K = \{I_5 = c\}$, $c = \text{const}$, $c \neq 0$, leading to the moving frames map $e^t = c/I_5$. Substituting the latter expression into the remaining five formulas (4.71), after some algebra we arrive at the following result.

Theorem 4.7 *Any $I(\mathbb{E}_1^2)$-invariant $\mathcal{I}$ of $\mathcal{K}^3(\mathbb{E}_1^2)$ defined over the open submanifold of Σ^3 where the isometry group $I(\mathbb{E}_1^2)$ acts freely and regularly with three-dimensional orbits can be locally uniquely expressed as an analytic function provided that $a_{10} \neq 0$.*

$$\mathcal{I} = F(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7),$$

where the fundamental $I(\mathbb{E}_1^2)$-invariants Δ_j, $j = 1, \ldots, 7$ are given by

$$\begin{aligned} \Delta_1 &= I_8, & \Delta_2 &= I_7, & \Delta_3 &= I_5 I_6, \\ \Delta_4 &= I_3 I_4, & \Delta_5 &= I_1 I_2, & \Delta_6 &= I_1 I_6^3, \\ \Delta_7 &= I_3 I_6^2, \end{aligned} \tag{4.72}$$

where $I_1, \ldots, I_8$ are given by (4.68).

One can apply the same technique to determine the fundamental $I(\mathbb{E}_1^2)$-covariants of $\mathcal{K}^3(\mathbb{E}_1^2)$, the results are presented in the following theorem.

Theorem 4.8 *Any $I(\mathbb{E}_1^2)$-covariant $\mathcal{C}$ defined over the open submanifold of $\Sigma^3 \times \mathbb{E}_1^2$ where the isometry group $I(\mathbb{E}_1^2)$ acts freely and regularly with three-dimensional orbits can be locally uniquely expressed as an analytic function provided that $a_{10} \neq 0$.*

$$\mathcal{C} = F(\Delta_1^c, \Delta_2^c, \Delta_3^c, \Delta_4^c, \Delta_5^c, \Delta_6^c, \Delta_7^c, \Delta_8^c, \Delta_9^c),$$

where the fundamental $I(\mathbb{E}_1^2)$-covariants Δ_i^c, $i = 1, \dots, 9$ are given by

$$
\begin{aligned}
\Delta_1^c &= a_9 + 2(a_8 u + a_7 v + a_{10} uv), \\
\Delta_2^c &= (a_1 - 3a_5 u - 3a_7 u^2 - a_{10} u^3)(a_2 - 3a_6 v + 3a_8 v^2 + a_{10} v^3), \\
\Delta_3^c &= (a_3 + a_9 u + a_5 v + a_8 u^2 + 2a_7 uv + a_{10} u^2 v) \\
&\quad \times (a_4 - a_9 v + a_6 u - a_7 v^2 - 2a_8 uv - a_{10} uv^2), \\
\Delta_4^c &= (a_5 + 2a_7 u + a_{10} u^2)(a_6 - 2a_8 v - a_{10} v^2), \\
\Delta_5^c &= (a_7 + a_{10} u)(a_8 + a_{10} v), \\
\Delta_6^c &= (a_1 - 3a_5 u - 3a_7 u^2 - a_{10} u^3) \\
&\quad \times (-a_4 + a_9 v - a_6 u + a_7 v^2 + 2a_8 uv + a_{10} uv^2)^3, \\
\Delta_7^c &= (a_2 - 3a_6 v + 3a_8 v^2 + a_{10} v^3) \\
&\quad \times (-a_3 - a_9 u - a_5 v - a_8 u^2 - 2a_7 uv - a_{10} u^2 v)^3, \\
\Delta_8^c &= (a_5 + 2a_7 u + a_{10} u^2)(a_8 + a_{10} v)^2, \\
\Delta_9^c &= (a_6 - 2a_8 v - a_{10} v^2)(a_7 + a_{10} u)^2.
\end{aligned}
\tag{4.73}
$$

We remark that the fundamental $I(\mathbb{E}_1^2)$-invariants of $\mathcal{K}^3(\mathbb{E}_1^2)$ are *syzygies* for the fundamental $I(\mathbb{E}_1^2)$-covariants (4.73). Indeed, it is easy to check that, for example,

$$\Delta_2 = \Delta_1 \Delta_1^c - 2\Delta_5^c.$$

The results of Theorems 6.1 and 6.2 constitute an analogue of the corresponding result presented in [36] for the vector space $\mathcal{K}^3(\mathbb{E}^2)$. We note, however, that the latter result was obtained by solving a system of PDEs analogous to (4.64) by the method of undetermined coefficients with the aid of MAPLE. In this work, based on an inductive version of the moving frames technique we have solved a very similar problem in what appears to be a much more efficient way, in particular, without the aid of a computer algebra system.

Following the idea exhibited in [36], we observe that the elements of $\mathcal{K}^3(\mathbb{E}_1^2)$ can be expressed by the general formula

$$\begin{aligned}
\mathrm{K}^3 &= a^{ijk}\mathrm{X}_i \odot \mathrm{X}_j \odot \mathrm{X}_k + 3b^{ij}\mathrm{X}_i \odot \mathrm{X}_j \odot \mathrm{H} + 3c^i\mathrm{X}_i \odot \mathrm{H} \odot \mathrm{H} \\
&\quad + e\mathrm{H} \odot \mathrm{H} \odot \mathrm{H} \\
&= a^{ijk}\mathrm{X}_i \odot \mathrm{X}_j \odot \mathrm{X}_k + (3b^{ij}\mathrm{X}_i \odot \mathrm{X}_j + \mathrm{K}^1_0 \odot \mathrm{H}) \odot \mathrm{H} \\
&= a^{ijk}\mathrm{X}_i \odot \mathrm{X}_j \odot \mathrm{X}_k + \mathrm{K}^2_0 \odot \mathrm{H}, \quad i,j,k = 1,2,
\end{aligned} \tag{4.74}$$

where $\mathrm{X}_1 = \partial_u, \mathrm{X}_2 = \partial_v$, and $\mathrm{H} = u\partial_u + v\partial_v$, while K^1 and K^2 represent the elements of the vector spaces $\mathcal{K}^1(\mathbb{E}^2_1)$ and $\mathcal{K}^2(\mathbb{E}^2_1)$ respectively. Furthermore, the components of the tensors a^{ijk}, b^{ij}, c^i and e are given by $a^{111} = a_1$, $a^{112} = a_3$, $a^{221} = a_4$, $a^{222} = a_2$, $b^{11} = -a_5$, $b^{12} = b^{21} = a_9$, $b^{22} = -a_6$, $c^1 = -a_7$, $c^2 = a_8$ and $e = -a_{10}$ in terms of the parameters $a_i, i = 1, \ldots, 10$ that appear in (4.63). Taking into account the formulas (4.72), (4.73) and (4.74), we arrive at the following two results.

Corollary 4.3 *Any algebraic $I(\mathbb{E}^2_1)$-invariant $\mathcal{I}$ of the vector space $\mathcal{K}^2(\mathbb{E}^2_1)$ defined over the open submanifold of the corresponding parameter space Σ^2 where the isometry group $I(\mathbb{E}^2_1)$ acts freely and regularly with three-dimensional orbits can be locally uniquely expressed as an analytic function $\mathcal{I} = F(\Delta_1, \Delta_2, \Delta_3)$, where the fundamental $I(\mathbb{E}^2_1)$-invariants Δ_i, $i = 1, \ldots, 3$ are given by the formulas (4.72).*

Corollary 4.4 *Any algebraic $I(\mathbb{E}^2_1)$-covariant C of $\mathcal{K}^2(\mathbb{E}^2_1)$ defined over the open submanifold of $\Sigma^2 \times \mathbb{E}^2_1$ where the isometry group $I(\mathbb{E}^2_1)$ acts freely and regularly with three-dimensional orbits can be locally uniquely expressed as an analytic function*

$$C = F(\Delta^c_1, \Delta^c_4, \Delta^c_5, \Delta^c_8, \Delta^c_9),$$

where the fundamental $I(\mathbb{E}^2_1)$-covariants $\Delta^c_1, \Delta^c_4, \Delta^c_5, \Delta^c_8$ and Δ^c_9 are given by the formulas (4.73).

Remark 4.7 We note finally that the action of the isometry group $I(\mathbb{E}^2_1)$ on $\mathcal{K}^3(\mathbb{E}^2_1)$ is not free globally due to the existence of isotropy subgroups (see (4.74)). This is why it is very difficult to employ only these fundamental invariants to classify the ten cases of Drach potentials. More delicate rank analysis and group theory will be needed. This problem is completely solved and will be presented in Section 6.4.

4.3.3 Killing tensors of valence three on the Euclidean plane

Consider now the contravariant Killing tensors of valence three defined on the Euclidean plane. Solving the Killing tensor equations (2.23) with respect to the Cartesian coordinates (x, y) yields the following general formulas for the components of the Killing tensors.

$$\begin{aligned} K^{111} &= a_1 + 3a_5y + 3a_8y^2 + a_{10}y^3, \\ K^{112} &= a_2 + a_6y - a_5x - 2a_8xy + a_9y^2 - a_{10}xy^2, \\ K^{122} &= a_3 - a_6y - a_7y - 2a_9xy + a_8x^2 + a_{10}yx^2, \\ K^{222} &= a_4 + 3a_7x + 3a_9x^2 - a_{10}x^3. \end{aligned} \tag{4.75}$$

Let $I(\mathbb{E}^2)$ denote the Lie group of isometries of $\mathbb{E}^2$ which consists of translations and rotations. Its corresponding Lie algebra is generated by the following three Killing vector fields

$$\mathbf{X} = \partial_x, \quad \mathbf{Y} = \partial_y, \quad \mathbf{R} = y\,\partial_x - x\,\partial_y, \tag{4.76}$$

which satisfy the commutator relations

$$[\mathbf{X}, \mathbf{Y}] = 0, \quad [\mathbf{X}, \mathbf{R}] = -\mathbf{Y}, \quad [\mathbf{Y}, \mathbf{R}] = \mathbf{X}. \tag{4.77}$$

Our goal is to determine a complete set of fundamental $I(\mathbb{E}^2)$-invariants of $\mathcal{K}^3(\mathbb{E}^2)$.

One method available is that of infinitesimal generators. We first find a representation of the infinitesimal action of $I(\mathbb{E}^2)$ on the parameter space Σ. Using the MST-procedure one arrives at

$$\begin{aligned} \mathbf{V}_1 &= -a_5\,\partial_{a_2} - a_6\,\partial_{a_3} + 3a_7\,\partial_{a_4} - 2a_8\,\partial_{a_6} + 2a_9\,\partial_{a_7} - a_{10}\,\partial_{a_9}, \\ \mathbf{V}_2 &= 3a_5\,\partial_{a_1} + a_6\,\partial_{a_2} - a_7\,\partial_{a_3} + 2a_8\,\partial_{a_5} + 2a_9\,\partial_{a_6} + a_{10}\,\partial_{a_8}, \\ \mathbf{V}_3 &= 3a_2\,\partial_{a_1} + (2a_3 - a_1)\,\partial_{a_2} + (a_4 - 2a_2)\,\partial_{a_3} - 3a_3\,\partial_{a_4} \\ &\quad + a_6\,\partial_{a_5} - 2(a_5 + a_7)\,\partial_{a_6} + a_6\,\partial_{a_7} + a_9\,\partial_{a_8} - a_8\,\partial_{a_9}. \end{aligned} \tag{4.78}$$

Note that they satisfy the commutation relations

$$[\mathbf{V}_1, \mathbf{V}_2] = 0, \quad [\mathbf{V}_1, \mathbf{V}_3] = -\mathbf{V}_2, \quad [\mathbf{V}_2, \mathbf{V}_3] = \mathbf{V}_1, \tag{4.79}$$

which verify the isomorphism between the Lie algebra generated by $\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3$ and the one by X,Y,R. Then any $I(\mathbb{E}^2)$-invariants $\mathcal{I}$ of $\mathcal{K}^3(\mathbb{E}^2)$ must satisfy the following infinitesimal invariance criteria

$$\mathrm{V}_1(\mathcal{I}) = 0, \quad \mathrm{V}_2(\mathcal{I}) = 0, \quad \mathrm{V}_3(\mathcal{I}) = 0. \tag{4.80}$$

In principle one can find all fundamental invariants by solving the system PDEs (4.80) via the method of characteristics. We note, however, in this case again the method of characteristics fails.

The other method one can use to achieve the goal is the *method of moving frames*. But it does not work if we use the moving frame method for the whole isometry group action [83]. Thus one can try the inductive version of the moving frame method [50], which we mentioned earlier and applied with success in the previous section.

First the isometry group $I(\mathbb{E}^2)$ factors as a discrete product of subgroup T of *translations* and R of *rotations*. We will proceed by first determining the T-invariants and then using them as new coordinates, performing the tensor transformation laws for these new coordinates under the action of the subgroup R. Thus the invariants of the latter group action will be the invariants for the whole group action [50].

Consider first the translation-invariants (i.e. T-invariants) of $\mathcal{K}^3(\mathbb{E}^2)$:

Subgroup T acts on $\mathbb{E}^2$ as follows.

$$\tilde{x} = x + a, \quad \tilde{y} = y + b, \qquad a, b \in \mathbb{R}, \tag{4.81}$$

which induces the action on the Killing tensors.

$$
\begin{aligned}
\tilde{a}_1 &= a_1 - 3b\, a_5 + 3b^2 a_8 - b^3 a_{10}, \\
\tilde{a}_2 &= a_2 + a\, a_5 - b a_6 - 2ab\, a_8 + b^2 a_9 + ab^2 a_{10}, \\
\tilde{a}_3 &= a_3 + a\, a_6 + b\, a_7 - 2ab\, a_9 + a^2 a_8 - b\, a^2 a_{10}, \\
\tilde{a}_4 &= a_4 - 3a\, a_7 + 3a^2 a_9 + a^3 a_{10}, \\
\tilde{a}_5 &= a_5 - 2b\, a_8 + b^2 a_{10}, \\
\tilde{a}_6 &= a_6 + 2a\, a_8 - 2b\, a_9 - 2ab\, a_{10}, \\
\tilde{a}_7 &= a_7 - 2a\, a_9 - a^2 a_{10}, \\
\tilde{a}_8 &= a_8 - b\, a_{10}, \\
\tilde{a}_9 &= a_9 + a\, a_{10}, \\
\tilde{a}_{10} &= a_{10}.
\end{aligned}
\tag{4.82}
$$

We now employ the method of moving frames to find the group invariants. Solving the following normalization equations

$$a_8 - b\, a_{10} = 0, \quad a_9 + a\, a_{10} = 0 \tag{4.83}$$

leads to the moving frame map

$$a = -\frac{a_9}{a_{10}}, \quad b = \frac{a_8}{a_{10}}. \tag{4.84}$$

Substituting the moving frame map into the remaining transformations, one obtains the following fundamental T-invariants:

$$
\begin{aligned}
I_1 &= a_1 a_{10}^2 - 3a_5 a_8 a_{10} + 2a_8^3, \\
I_2 &= a_2 a_{10}^2 - a_5 a_9 a_{10} - a_6 a_8 a_{10} + 2a_8^2 a_9, \\
I_3 &= a_3 a_{10}^2 - a_6 a_9 a_{10} + a_7 a_8 a_{10} + 2a_8 a_9^2, \\
I_4 &= a_4 a_{10}^2 + 3a_7 a_9 a_{10} + 2a_9^3, \\
I_5 &= a_5 a_{10} - a_8^2, \\
I_6 &= a_6 a_{10} + a_9^2, \\
I_7 &= a_6 a_{10} - 2a_8 a_9, \\
I_8 &= a_{10}.
\end{aligned}
\tag{4.85}
$$

Now consider the action of the subgroup R.

$$
\tilde{x} = x\cos(t) - y\sin(t), \quad \tilde{y} = x\sin(t) + y\cos(t), \qquad t \in \mathbb{R}. \tag{4.86}
$$

This leads to the following parameter transformations.

$$
\begin{aligned}
\tilde{a}_1 &= a_1 \cos^3(t) - 3a_2 \sin(t)\cos^2(t) + 3a_3 \cos(t)\sin^2(t) - a_4 \sin^3(t), \\
\tilde{a}_2 &= a_1 \cos^2(t)\sin(t) + a_2 \cos(t)(\cos^2(t) - 2\sin^2(t)) \\
&\quad + a_3 \sin(t)(\sin^2(t) - 2\cos^2(t)) + a_4 \cos(t)\sin^2(t), \\
\tilde{a}_3 &= a_1 \sin^2(t)\cos(t) + a_2 \sin(t)(2\cos^2(t) - \sin^2(t)) \\
&\quad + a_3 \cos(t)(\cos^2(t) - 2\sin^2(t)) - a_4 \sin(t)\cos^2(t), \\
\tilde{a}_4 &= a_1 \sin^3(t) + 3a_2 \cos(t)\sin^2(t) + 3a_3 \sin(t)\cos^2(t) + a_4 \cos^3(t), \\
\tilde{a}_5 &= a_5 \cos^2(t) - a_6 \cos(t)\sin(t) - a_7 \sin^2(t), \\
\tilde{a}_6 &= 2(a_5 + a_7)\cos(t)\sin(t) + a_6(\cos^2(t) - \sin^2(t)), \\
\tilde{a}_7 &= -a_5 \sin^2(t) - a_6 \cos(t)\sin(t) + a_7 \cos^2(t), \\
\tilde{a}_8 &= a_8 \cos(t) - a_9 \sin(t), \\
\tilde{a}_9 &= a_8 \sin(t) + a_9 \cos(t), \\
\tilde{a}_{10} &= a_{10}.
\end{aligned}
\tag{4.87}
$$

Now we consider the actions on the fundamental T-invariants of the subgroup R.

$$
\begin{aligned}
\tilde{I}_1 &= I_1 \cos^3(t) - I_4 \sin^3(t) - 3I_2 \cos^2(t)\sin(t) + 3I_3 \sin^2(t)\cos(t), \\
\tilde{I}_2 &= I_2 \cos^3(t) + I_3 \sin^3(t) + (I_1 - 2I_3)\cos^2(t)\sin(t) + (I_4 - 2I_2)\sin^2(t)\cos(t), \\
\tilde{I}_3 &= I_3 \cos^3(t) - I_2 \sin^3(t) + (2I_2 - I_4)\cos^2(t)\sin(t) + (I_1 - 2I_3)\sin^2(t)\cos(t), \\
\tilde{I}_4 &= I_4 \cos^3(t) + I_1 \sin^3(t) + 3I_3 \cos^2(t)\sin(t) + 3I_2 \sin^2(t)\cos(t), \\
\tilde{I}_5 &= I_5 \cos^2(t) - I_6 \sin^2(t) - I_7 \cos(t)\sin(t), \\
\tilde{I}_6 &= -I_5 \sin^2(t) + I_6 \cos^2(t) - I_7 \cos(t)\sin(t), \\
\tilde{I}_7 &= (I_5 + I_6)\sin(2t) + I_7 \cos(2t), \\
\tilde{I}_8 &= I_8.
\end{aligned}
\tag{4.88}
$$

Again we employ the method of moving frames. Choosing the cross-section

$$K = \{I_7 = 0\}$$

leads to the normalization equation

$$(I_5 + I_6)\sin(2t) + I_7\cos(2t) = 0. \tag{4.89}$$

Solving (4.89) we arrive at the moving frame map provided that $I_5 + I_6 \neq 0$.

$$t = \arctan\frac{-I_7}{I_5 + I_6}. \tag{4.90}$$

Now substituting the moving frame map into the remaining transformations for the T-invariants (4.88), after some algebras we arrive at a complete set of fundamental $I(\mathbb{E}^2)$-invariants.

$$\begin{aligned}
\Delta_1 &= I_8, \\
\Delta_2 &= I_5 - I_6, \\
\Delta_3 &= I_7^2 + (I_5 + I_6)^2, \\
\Delta_4 &= (2\,\Delta_3 - 4I_7^2)(I_1 I_2 - I_3 I_4) \\
&\quad - I_7(I_5 + I_6)\,\{I_1^2 - 3I_2^2 - 3I_3^2 + I_4^2 - 2I_1 I_3 - 2I_2 I_4\}, \\
\Delta_5 &= -2I_7\,\Delta_3\,\{I_1^2 - I_4^2 - 3(I_2^2 - I_3^2) - 2(I_1 I_3 - I_2 I_4)\} \\
&\quad + I_7^3\,\{(I_1 - 3I_3)^2 - (I_4 - 3I_2)^2\} \\
&\quad + 2(I_5 + I_6)\,\{2(I_1 I_2 + I_3 I_4)\,\Delta_3 + I_7^2(I_1 - 3I_3)((I_4 - 3I_2)\}, \\
\Delta_6 &= (I_5 + I_6)\,\{4\,\Delta_3\,(I_1^2 - I_4^2) - I_7^2((I_1 - 3I_3)^2 - (I_4 - 3I_2)^2)\} \\
&\quad + 12I_7\,\Delta_3\,(I_1 I_2 + I_3 I_4) + 2I_7^3(I_1 - 3I_3)(I_4 - 3I_2), \\
\Delta_7 &= (I_5 + I_6)\,\{4\,\Delta_3\,(I_2^2 - I_3^2) + I_7^2((I_1 - 3I_3)^2 - (I_4 - 3I_2)^2)\} \\
&\quad - 4I_7\,\Delta_3\,(I_1 I_2 + I_3 I_4 - 4I_2 I_3) - 2I_7^3(I_1 - 3I_3)(I_4 - 3I_2),
\end{aligned}$$

where $I_1, \ldots, I_8$ are given by (4.85).

Remark 4.8 All of the seven fundamental invariants are homogeneous polynomials in the ten parameters $a_i, \quad i = 1, \ldots, 10$, with Δ_1 of order 1, Δ_2 of order 2, Δ_3 of order 4, Δ_4 of order 10, while $\Delta_5, \Delta_6, \Delta_7$ are of order 12.

Remark 4.9 Similarly, one can employ this technique to determine a complete set fundamental $I(\mathbb{E}^2)$-covariants of $\mathcal{K}^3(\mathbb{E}^2)$.

Note that recently Horwood *et al* [36] employed the method of undetermined coefficients and the method of infinitesimal generators to determine a complete set of fundamental isometry group invariants of Killing three tensors defined on the Euclidean plane.

The inductive version of the method of moving frames will be applied again in Chapter 5 to the problem of the determination of a complete set of fundamental isometry group-invariants of Killing tensors of arbitrary valence defined on the Minkowski plane [81].

Chapter 5

An Analogue of the 1856 Lemma of Cayley

5.1 Introduction

In light of the fact that "Mathematics is the study of analogies between analogies" [73], we wish to continue developing ITKT by establishing more analogies with CIT. In this Chapter, we formulate and prove an analogue of the 1856 lemma of Cayley in CIT described in Chapter 3. To be more specific, we completely determine the infinitesimal generators of the action of the isometry group on the Killing tensors of arbitrary valence defined on the Minkowski plane. Theoretically, this result allows one to determine the isometry group invariants and covariants of Killing tensors of high valences on the Minkowski plane, except that one may need computer algebra together with *the method of undetermined coefficients*. This is an original contribution to the invariant theory of Killing tensors and has potential applications in Lie group theory and mathematical physics. A paper [89] based on this result has been published.

As a second part of the Chapter, by making use of a new technique of the method of moving frames [50], we solve the problem of the determination of a complete set of the fundamental invariants of Killing tensors of arbitrary valence defined on the Minkowski plane [81].

5.2 The formulation of the problem

Since Cayley's problem concerns binary forms of arbitrary degree, it will be natural to investigate the Killing tensors of arbitrary valence defined on a pseudo-Riemannian manifold of dimension two, for example, the Minkowski plane $\mathbb{E}_1^2$. That is, the object of study here is the vector space $\mathcal{K}^n(\mathbb{E}_1^2)$ of Killing tensors of valence n (where n is arbitrary) defined on the Minkowski plane, while as before the group that acts on it is the isometry group of the underlying manifold. A comparison of the two sister problems is given in Table 5.1.

Theory	Vector Space	Group	Dimension of the Space	Dimension of the orbits
CIT	$\mathcal{P}^n(\mathbb{R}^2)$	$SL(2,\mathbb{R})$	$n+1$	≤ 3
ITKT	$\mathcal{K}^n(\mathbb{R}^2_1)$	$I(\mathbb{E}^2_1)$	$(n+1)(n+2)/2$	≤ 3

Table 5.1: The settings for the corresponding problems in CIT and ITKT

Having made these observations, we are now in the position to formulate the ITKT version of the problem considered by Cayley in 1856 [10].

Problem 5.1 *Consider the action of $I(\mathbb{E}^2_1)$ on $\mathcal{K}^n(\mathbb{E}^2_1)$. Determine a representation of the corresponding Lie algebra $i(\mathbb{E}^2_1)$ on the parameter space Σ of $\mathcal{K}^n(\mathbb{E}^2_1)$.*

5.3 Infinitesimal generators

In this section we solve Problem 5.1 presented above. As a first step, we need to derive a general representation for an element of the vector space $\mathcal{K}^n(\mathbb{E}^2_1)$, (i.e., an analogue of (3.1) of CIT). As is well known, each Killing tensor of valence n on a 2-dimensional manifold is determined by $(n+1)(n+2)/2$ parameters that appear in the $n+1$ components, which take the following form.

$$K^{i_1\ldots i_p j_1 \ldots j_{n-p}}, \tag{5.1}$$

where

$$i_1 = \ldots = i_p = 1, \quad j_1 = \ldots = j_{n-p} = 2, \quad p = 0, 1, \ldots, n.$$

The Killing tensor equation (2.23) with respect to the pseudo-Cartesian coordinates (t, x) reduce to a system of PDEs:

$$\begin{gathered} \partial_t K^{i_1\ldots i_n} = 0, \qquad \partial_x K^{j_1 \ldots j_n} = 0, \\ (n-p+1)\partial_x K^{i_1\ldots i_p j_1\ldots j_{n-p}} = p\partial_t K^{i_1 \ldots i_{p-1} j_1 \ldots j_{n-p+1}}, \end{gathered} \tag{5.2}$$

where

$$p = 0, 1, \ldots, n, \quad \partial_t = \frac{\partial}{\partial t}, \quad \partial_x = \frac{\partial}{\partial x}.$$

As a consequence of (5.2), we readily obtain the necessary differential conditions:

$$\begin{aligned} &(\partial_x)^{p+1} K^{i_1\ldots i_p j_1 \ldots j_{n-p}} = 0, \\ &(\partial_t)^{n-p+1} K^{i_1\ldots i_p j_1 \ldots j_{n-p}} = 0. \end{aligned} \tag{5.3}$$

Solving the above PDEs one arrives at the following general formulas (Note again here the parameters in the formulas are constants of integration when solving over determined system of PDEs)

$$K^{i_1\cdots i_p j_1\cdots j_{n-p}} = \begin{cases} \displaystyle\sum_{i=0}^{n-p}\left[\binom{n-p}{i} t^i \sum_{j=0}^{p}\binom{p}{j} a_{pij} x^j\right], & \text{if} \quad n \geq p \geq \left[\frac{n+1}{2}\right], \\ \displaystyle\sum_{i=0}^{p}\left[\binom{p}{i} x^i \sum_{j=0}^{n-p}\binom{n-p}{j} b_{pij} t^j\right], & \text{if} \quad 0 \leq p \leq \left[\frac{n+1}{2}\right], \end{cases} \tag{5.4}$$

where the parameters a_{pij}, b_{pij} are to be determined (or arranged) a little later (at this stage they are merely inserted for our convenience). We immediately recognize that the formula (5.4) is an ITKT analogue of the general formula (3.1) that was considered by Cayley in CIT. The parameters a_{pij}, b_{pij} can be determined by following the general procedure of solving the system of PDEs (5.2). We consider separately two cases: n is even and n is odd. The parameters of each of the $n+1$ components can be organized into groups in such a way that the parameters of one group are completely determined by the parameters of the other (see the illustrative examples below). After relabelling the parameters, we obtain the following two schemes which will specify the arrangements of the parameters of the first groups of the components. Once they are specified, the parameters of the other groups and hence the formula for all of the components can be determined completely.

1. n is even.

$$\begin{array}{llllllll}
\text{Step } 1: & a_0^1 & a_1^1 & \cdots & a_{n-2}^1 & a_{n-1}^1 & a_n^1, \\
 & b_0^1 & b_1^1 & \cdots & b_{n-2}^1 & b_{n-1}^1 & a_n^1 \\
\text{Step } 2: & a_0^2 & a_1^2 & \cdots & a_{n-3}^2 & a_{n-2}^2 & b_{n-1}^1 \\
 & b_0^2 & b_1^2 & \cdots & b_{n-3}^2 & a_{n-2}^2 & a_{n-1}^1 \\
 & & & \vdots & & & \\
\text{Step } \frac{n}{2}: & a_0^{\frac{n}{2}} & a_1^{\frac{n}{2}} & a_2^{\frac{n}{2}} & b_1^{\frac{n-2}{2}} & \cdots & b_{\frac{n+2}{2}}^1 \\
 & b_0^{\frac{n}{2}} & b_1^{\frac{n}{2}} & a_2^{\frac{n}{2}} & a_1^{\frac{n-2}{2}} & \cdots & a_{\frac{n+2}{2}}^1 \\
\text{Step}\frac{n+2}{2}: & a_0^{\frac{n+2}{2}} & b_1^{\frac{n}{2}} & b_2^{\frac{n-2}{2}} & b_3^{\frac{n-4}{2}} & \cdots & b_{\frac{n}{2}}^1
\end{array} \tag{5.5}$$

2. n is odd.

$$\begin{array}{lcccccc}
\text{Step 1}: & a_0^1 & a_1^1 & \dots & a_{n-2}^1 & a_{n-1}^1 & a_n^1, \\
 & b_0^1 & b_1^1 & \dots & b_{n-2}^1 & b_{n-1}^1 & a_n^1 \\
\text{Step 2}: & a_0^2 & a_1^2 & \dots & a_{n-3}^2 & a_{n-2}^2 & b_{n-1}^1 \\
 & b_0^2 & b_1^2 & \dots & b_{n-3}^2 & a_{n-2}^2 & a_{n-1}^1 \\
 & & & \vdots & & & \\
\text{Step}\dfrac{n-1}{2}: & a_0^{\frac{n-1}{2}} & a_1^{\frac{n-1}{2}} & a_2^{\frac{n-1}{2}} & a_3^{\frac{n-1}{2}} & \dots & b_{\frac{n+3}{2}}^1 \\
 & b_0^{\frac{n-1}{2}} & b_1^{\frac{n-1}{2}} & b_2^{\frac{n-1}{2}} & a_3^{\frac{n-1}{2}} & \dots & a_{\frac{n+3}{2}}^1 \\
\text{Step}\dfrac{n+1}{2}: & a_0^{\frac{n+1}{2}} & a_1^{\frac{n+1}{2}} & b_2^{\frac{n-1}{2}} & b_3^{\frac{n-3}{2}} & \dots & b_{\frac{n+1}{2}}^1 \\
 & b_0^{\frac{n+1}{2}} & a_1^{\frac{n+1}{2}} & a_2^{\frac{n-1}{2}} & a_3^{\frac{n-3}{2}} & \dots & a_{\frac{n+1}{2}}^1
\end{array} \tag{5.6}$$

We now show by examples how these schemes work.

- we first give $2(n+1)-1$ parameters

$$a_0^1, \dots, a_{n-1}^1, a_n^1, \qquad b_0^1, \dots, b_{n-1}^1, a_n^1,$$

and write down the first and the last components of $\mathrm{K} \in \mathcal{K}^n(\mathbb{E}_1^2)$:

$$\begin{aligned}
K^{11\ldots 11} &= \left[a_0^1 + \binom{n}{1} a_1^1 x + \binom{n}{2} a_2^1 x^2 + \ldots + \binom{n}{n-1} a_{n-1}^1 x^{n-1} + a_n^1 x^n\right], \\
K^{22\ldots 22} &= \left[b_0^1 + \binom{n}{1} b_1^1 t + \binom{n}{2} b_2^1 t^2 + \ldots + \binom{n}{n-1} b_{n-1}^1 t^{n-1} + a_n^1 t^n\right].
\end{aligned}$$

- Next, for $2(n-1)-1$ new parameters

$$a_0^2, \dots, a_{n-3}^2, a_{n-2}^2, \qquad b_0^2, \dots, b_{n-3}^2, a_{n-2}^2,$$

we present second and penultimate components of K (see (5.4)), each of which is the sum of two polynomials, the first having been determined by the newly specified parameters and the other - by the parameters determined previously.

$$\begin{aligned} K^{11\ldots 12} &= \left[a_0^2 + \binom{n-1}{1} a_1^2 x + \ldots + \binom{n-1}{n-2} a_{n-2}^2 x^{n-2} + b_{n-1}^1 x^{n-1}\right] \\ &\quad + t\left[a_1^1 + \binom{n-1}{1} a_2^1 x + \ldots + \binom{n-1}{n-2} a_{n-1}^1 x^{n-2} + a_n^1 x^{n-1}\right], \\ K^{22\ldots 21} &= \left[b_0^2 + \binom{n-1}{1} b_1^2 t + \ldots + \binom{n-1}{n-2} a_{n-2}^2 t^{n-2} + a_{n-1}^1 t^{n-1}\right] \\ &\quad + x\left[b_1^1 + \binom{n-1}{1} b_2^1 t + \ldots + \binom{n-1}{n-2} b_{n-1}^1 t^{n-2} + a_n^1 t^{n-1}\right]. \end{aligned}$$

- To clarify the process more, let us consider the next step (if any): given $2(n-3)-1$ new parameters

$$a_0^3, \ldots, a_{n-5}^3, a_{n-4}^3, \qquad b_0^3, \ldots, b_{n-5}^3, a_{n-4}^3,$$

'one can specify the next two components:

$$\begin{aligned} &K^{11\ldots 122} = \\ &\left[a_0^3 + \binom{n-2}{1} a_1^3 x + \ldots + \binom{n-2}{n-4} a_{n-4}^3 x^{n-4} + \binom{n-2}{n-3} b_{n-3}^2 x^{n-3} + b_{n-2}^1 x^{n-2}\right] \\ &+2t\left[a_1^2 + \binom{n-2}{1} a_2^2 x + \ldots + \binom{n-2}{n-4} a_{n-3}^2 x^{n-4} + \binom{n-2}{n-3} a_{n-2}^2 x^{n-3} + b_{n-1}^1 x^{n-2}\right] \\ &+t^2\left[a_2^1 + \binom{n-2}{1} a_3^1 x + \ldots + \binom{n-2}{n-4} a_{n-2}^1 x^{n-4} + \binom{n-2}{n-3} a_{n-1}^1 x^{n-3} + a_n^1 x^{n-2}\right], \end{aligned}$$

$$\begin{aligned} &K^{22\ldots 211} = \\ &\left[b_0^3 + \binom{n-2}{1} b_1^3 t + \ldots + \binom{n-2}{n-4} a_{n-4}^3 t^{n-4} + \binom{n-2}{n-3} a_{n-3}^2 t^{n-3} + a_{n-2}^1 t^{n-2}\right] \\ &+2x\left[b_1^2 + \binom{n-2}{1} b_2^2 t + \ldots + \binom{n-2}{n-4} b_{n-3}^2 t^{n-4} + \binom{n-2}{n-3} a_{n-2}^2 t^{n-3} + a_{n-1}^1 t^{n-2}\right] \\ &+x^2\left[b_2^1 + \binom{n-2}{1} b_3^1 t + \ldots + \binom{n-2}{n-4} b_{n-2}^1 t^{n-4} + \binom{n-2}{n-3} b_{n-1}^1 t^{n-3} + a_n^1 t^{n-2}\right]. \end{aligned}$$

- We continue this process in both directions (i.e., going "downwards" and "upwards") until it is terminated in the middle of (5.4).

Remark 5.1 In this view, counting the steps in both cases (n is even and n is odd), it is easy to see that the dimension of the space $d = \dim \mathcal{K}^n(\mathbb{E}_1^2) = \frac{1}{2}(n+1)(n+2), \quad n \geq 1$ gets decomposed as follows.

$$d = \begin{cases} [2(n+1)-1] + [2(n-1)-1] + \ldots + [2 \times 1 - 1] & \text{if } n \text{ is even,} \\ [2(n+1)-1] + [2(n-1)-1] + \ldots + [2 \times 2 - 1] & \text{if } n \text{ is odd.} \end{cases} \tag{5.7}$$

The auxiliary problem of finding the general form for the elements $\mathbf{K} \in \mathcal{K}^n(\mathbb{R}_1^2)$ is therefore completely solved. We immediately notice that the general solution (4.26) can be relabeled following the scheme (5.5) as follows:

$$\mathbf{K} = \begin{pmatrix} a_0^1 + 2a_1^1 x + a_2^1 x^2 & (a_0^2 + b_1^1 x) + t(a_1^1 + a_2^1 x) \\ (a_0^2 + b_1^1 x) + t(a_1^1 + a_2^1 x) & b_0^1 + 2b_1^1 t + a_2^1 t^2 \end{pmatrix}. \tag{5.8}$$

To further illustrate our results we consider the following two examples.

Example 5.1 $\mathcal{K}^4(\mathbb{R}_1^2)$. Note $d = \dim \mathcal{K}^4(\mathbb{R}_1^2) = 15$. Following the formula (5.4) and the coefficient scheme (5.5).

$$\begin{aligned} K^{1111} &= a_0^1 + 4a_1^1 x + 6a_2^1 x^2 + 4a_3^1 x^3 + a_4^1 x^4, \\ K^{1112} &= (a_0^2 + 3a_1^2 x + 3a_2^2 x^2 + b_3^1 x^3) + t(a_1^1 + 3a_2^1 x + 3a_3^1 x^2 + a_4^1 x^3), \\ K^{1122} &= (a_0^3 + 2b_1^2 x + b_2^1 x^2) + 2t(a_1^2 + 2a_2^2 x + b_3^1 x^2) + t^2(a_2^1 + 2a_3^1 x + a_4^1 x^2), \\ K^{1222} &= (b_0^2 + 3b_1^2 t + 3a_2^2 t^2 + a_3^1 t^3) + x(b_1^1 + 3b_2^1 t + 3b_3^1 t^2 + a_4^1 t^3), \\ K^{2222} &= b_0^1 + 4b_1^1 t + 6b_2^1 t^2 + 4b_3^1 t^3 + a_4^1 t^4. \end{aligned} \tag{5.9}$$

Example 5.2 $\mathcal{K}^5(\mathbb{R}^2_1)$. $d = \dim\, \mathcal{K}^5(\mathbb{R}^2_1) = 21$. We use the scheme (5.6).

$$
\begin{aligned}
K^{11111} &= a^1_0 + 5a^1_1 x + 10a^1_2 x^2 + 10a^1_3 x^3 + 5a^1_4 x^4 + a^1_5 x^5, \\
K^{11112} &= (a^2_0 + 4a^2_1 x + 6a^2_2 x^2 + 4a^2_3 x^3 + b^1_4 x^4) + t(a^1_1 + 4a^1_2 x + 6a^1_3 x^2 + 4a^1_4 x^3 + a^1_5 x^4), \\
K^{11122} &= (a^3_0 + 3a^3_1 x + 3b^2_2 x^2 + b^1_3 x^3) + 2t(a^2_1 + 3a^2_2 x + 3a^2_3 x^2 + b^1_4 x^3) \\
&\quad + t^2(a^1_2 + 3a^1_3 x + 3a^1_4 x^2 + a^1_5 x^3), \\
K^{11222} &= (b^3_0 + 3a^3_1 t + 3a^2_2 t^2 + a^1_3 t^3) + 2x(b^2_1 + 3b^2_2 t + 3a^2_3 t^2 + a^1_4 x^3) \\
&\quad + x^2(b^1_2 + 3b^1_3 t + 3b^1_4 t^2 + a^1_5 t^3), \\
K^{12222} &= (b^2_0 + 4b^2_1 t + 6b^2_2 t^2 + 4a^2_3 t^3 + a^1_4 t^4) + x(b^1_1 + 4b^1_2 t + 6b^1_3 t^2 + 4b^1_4 t^3 + a^1_5 t^4), \\
K^{22222} &= b^1_0 + 5b^1_1 t + 10b^1_2 t^2 + 10b^1_3 t^3 + 5b^1_4 t^4 + a^1_5 t^5.
\end{aligned}
\tag{5.10}
$$

We see that, following the parameter schemes given above, i.e. using the formulas (5.4), (5.5) and (5.6), one can now write down explicitly the formula for a Killing tensor of valence n on the Minkowski plane with respect to the pseudo-Cartesian coordinates (t, x), without any difficulty.

To solve Problem 5.1, we employ the MST-procedure [52] outlined and used in Chapter 3. Using the formulas (4.5), (5.4), (5.5) and (5.6), we obtain two triples of the vector fields that represent the infinitesimal action of the isometry group $I(\mathbb{E}^2_1)$ on the parameter space Σ of the vector space Killing tensors of valence n on the Minkowski plane.

We distinguish two cases.

1. n is even.

$$
\begin{aligned}
\mathbf{V}_1 \quad = \quad & a_1^1\partial_{a_0^2} + a_2^1\partial_{a_1^2} + \ldots + a_{n-1}^1\partial_{a_{n-2}^2} \\
& + 2a_1^2\partial_{a_0^3} + 2a_2^2\partial_{a_1^3} + \ldots + 2a_{n-3}^2\partial_{a_{n-4}^3} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \frac{n}{2}a_1^{\frac{n}{2}}\partial_{a_0^{\frac{n+2}{2}}} \\
& + \frac{n+2}{2}b_1^{\frac{n}{2}}\partial_{b_0^{\frac{n}{2}}} + \frac{n}{2}a_2^{\frac{n}{2}}\partial_{b_1^{\frac{n}{2}}} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + (n-1)b_1^2\partial_{b_0^2} + (n-2)b_2^2\partial_{b_1^2} + \ldots 2b_{n-2}^2\partial_{b_{n-3}^2} \\
& + nb_1^1\partial_{b_0^1} + (n-1)b_2^1\partial_{b_1^1} + \ldots + a_n^1\partial_{b_{n-1}^1},
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
\mathbf{V}_2 \quad = \quad & b_1^1\partial_{b_0^2} + b_2^1\partial_{b_1^2} + \ldots + b_{n-1}^1\partial_{a_{n-2}^2} \\
& + 2b_1^2\partial_{b_0^3} + 2b_2^2\partial_{b_1^3} + \ldots + 2b_{n-3}^2\partial_{a_{n-4}^3} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \frac{n}{2}b_1^{\frac{n}{2}}\partial_{a_0^{\frac{n+2}{2}}} \\
& + \frac{n+2}{2}a_1^{\frac{n}{2}}\partial_{a_0^{\frac{n}{2}}} + \frac{n}{2}a_2^{\frac{n}{2}}\partial_{a_1^{\frac{n}{2}}} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + (n-1)a_1^2\partial_{a_0^2} + (n-2)a_2^2\partial_{a_1^2} + \ldots + 2a_{n-2}^2\partial_{a_{n-3}^2} \\
& + na_1^1\partial_{a_0^1} + (n-1)a_2^1\partial_{a_1^1} + \ldots + a_n^1\partial_{a_{n-1}^1},
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
\mathbf{V}_3 \quad = \quad & -na_0^2\partial_{a_0^1} - (n-1)a_1^2\partial_{a_1^1} - \ldots - 2a_{n-2}^2\partial_{a_{n-2}^1} - b_{n-1}^1\partial_{a_{n-1}^1} \\
& - [(n-1)a_0^3 + a_0^1]\partial_{a_0^2} - \ldots - [2b_{n-3}^2 + a_{n-3}^1]\partial_{a_{n-3}^2} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& - \frac{n}{2}[a_0^{\frac{n}{2}} + b_0^{\frac{n}{2}}]\partial_{a_0^{\frac{n+2}{2}}} - \ldots - [a_{n-2}^1 + b_{n-2}^1]\partial_{a_{n-2}^2} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& - [(n-1)b_0^3 + b_0^1]\partial_{b_0^2} - \ldots - [2a_{n-3}^2 + b_{n-3}^1]\partial_{b_{n-3}^2} \\
& - nb_0^2\partial_{b_0^1} - (n-1)b_1^2\partial_{b_1^1} - \ldots - 2a_{n-2}^2\partial_{b_{n-2}^1} - a_{n-1}^1\partial_{b_{n-1}^1}.
\end{aligned}
\tag{5.13}
$$

2. n is odd.

$$
\begin{aligned}
\mathbf{V}_1 \;=\;& a_1^1\partial_{a_0^2} + a_2^1\partial_{a_1^2} + \dots + a_{n-1}^1\partial_{a_{n-2}^2} \\
& +2a_1^2\partial_{a_0^3} + 2a_2^2\partial_{a_1^3} + \dots + 2a_{n-3}^2\partial_{a_{n-4}^3} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& +\frac{n+1}{2}a_1^{\frac{n+1}{2}}\partial_{b_0^{\frac{n+1}{2}}} \\
& +\frac{n+3}{2}b_1^{\frac{n-1}{2}}\partial_{b_0^{\frac{n-1}{2}}} + \frac{n+1}{2}b_1^{\frac{n-1}{2}}\partial_{b_1^{\frac{n-1}{2}}} + \frac{n-1}{2}a_3^{\frac{n-1}{2}}\partial_{b_2^{\frac{n-1}{2}}} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& +(n-1)b_1^2\partial_{b_0^2} + (n-2)b_2^2\partial_{b_1^2} + \dots 2a_{n-2}^2\partial_{b_{n-3}^2} \\
& +nb_1^1\partial_{b_0^1} + (n-1)b_2^1\partial_{b_1^1} + \dots + 2b_{n-1}^1\partial_{b_{n-2}^1} + a_n^1\partial_{b_{n-1}^1},
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
\mathbf{V}_2 \;=\;& b_1^1\partial_{b_0^2} + b_2^1\partial_{b_1^2} + \dots + b_{n-1}^1\partial_{a_{n-2}^2} \\
& +2b_1^2\partial_{b_0^3} + 2b^2\partial_{b_1^3} + \dots + 2b_{n-3}^2\partial_{a_{n-4}^3} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& +\frac{n+1}{2}a_1^{\frac{n+1}{2}}\partial_{a_0^{\frac{n+1}{2}}} \\
& +\frac{n+3}{2}a_1^{\frac{n-1}{2}}\partial_{a_0^{\frac{n-1}{2}}} + \frac{n+1}{2}a_2^{\frac{n+1}{2}}\partial_{b_1^{\frac{n-1}{2}}} + \frac{n-1}{2}a_3^{\frac{n-1}{2}}\partial_{a_2^{\frac{n-1}{2}}} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& +(n-1)a_1^2\partial_{a_0^2} + (n-2)a_2^2\partial_{a_1^2} + \dots + 2a_{n-2}^2\partial_{a_{n-3}^2} \\
& +na_1^1\partial_{a_0^1} + (n-1)a_2^1\partial_{a_1^1} + \dots + 2a_{n-1}^1\partial_{a_{n-2}^1} + a_n^1\partial_{a_{n-1}^1},
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
\mathbf{V}_3 \;=\;& -na_0^2\partial_{a_0^1} - (n-1)a_1^2\partial_{a_1^1} - \dots 2a_{n-2}^2\partial_{a_{n-2}^1} - b_{n-1}^1\partial_{a_{n-1}^1} \\
& -\left[(n-1)a_0^3 + a_0^1\right]\partial_{a_0^2} - \dots - \left[2b_{n-3}^2 + a_{n-3}^1\right]\partial_{a_{n-3}^2} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& -\left[\frac{n+1}{2}b_0^{\frac{n+1}{2}} + \frac{n-1}{2}a_0^{\frac{n-1}{2}}\right]\partial_{a_0^{\frac{n+1}{2}}} \\
& -\frac{n-1}{2}[a_1^{\frac{n-1}{2}} + b_1^{\frac{n-1}{2}}]\partial_{a_1^{\frac{n+1}{2}}} - \dots - [a_{n-2}^1 + b_{n-2}^1]\partial_{a_{n-2}^2} \\
& -\left[\frac{n+1}{2}a_0^{\frac{n+1}{2}} + \frac{n-1}{2}b_0^{\frac{n-1}{2}}\right]\partial_{b_0^{\frac{n+1}{2}}} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& -[(n-1)b_0^3 + b_0^1]\partial_{b_0^2} - \dots - [2a_{n-3}^2 + b_{n-3}^1]\partial_{b_{n-3}^2} \\
& -nb_0^2\partial_{b_0^1} - (n-1)b_1^2\partial_{b_1^1} - \dots - 2a_{n-2}^2\partial_{b_{n-2}^1} - a_{n-1}^1\partial_{b_{n-1}^1}.
\end{aligned}
\tag{5.16}
$$

We remark that the vector fields $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ correspond to the generators $\mathbf{T}$, $\mathbf{X}$ and $\mathbf{H}$ given by (4.28) respectively. Moreover, it is easy to verify directly that the vector fields $-\mathbf{V}_1$, $-\mathbf{V}_2$ and $-\mathbf{V}_3$ satisfy the same commutator relations as $\mathbf{T}$, $\mathbf{X}$ and $\mathbf{H}$ (see (4.29)).

We thus conclude that the vector fields $\mathbf{V}_i$, $i = 1, 2, 3$ represent the infinitesimal action of the isometry group $I(\mathbb{E}_1^2)$ on the parameter space Σ defined by $\mathcal{K}^n(\mathbb{E}_1^2)$ and we obtain an ITKT analogue of 1856 lemma of Cayley (Lemma 3.1) [10].

Theorem 5.1 *The action of the isometry group $I(\mathbb{E}_1^2)$ on the vector space $\mathcal{K}^n(\mathbb{E}_1^2)$ has the infinitesimal generators (5.11),(5.12) and (5.13) when n is even and (5.14),(5.15) and (5.16) when n is odd.*

To illustrate our results, consider the following examples.

Example 5.3 $\mathcal{K}^4(\mathbb{E}_1^2)$. Using Theorem 5.1, the infinisimal generators are found to be

$$\begin{aligned}
\mathbf{V}_1 = \; & a_1^1\partial_{a_0^2} + a_2^1\partial_{a_1^2} + a_3^1\partial_{a_2^2} \\
& +2a_1^2\partial_{a_0^3} \\
& +3b_1^2\partial_{b_0^2} + 2a_2^2\partial_{b_1^2} \\
& +4b_1^1\partial_{b_0^1} + 3b_2^1\partial_{b_1^1} + 2b_3^1\partial_{b_2^1} + a_4^1\partial_{b_3^1},
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\mathbf{V}_2 = \; & b_1^1\partial_{b_0^2} + b_2^1\partial_{b_1^2} + b_3^1\partial_{a_2^2} \\
& +2b_1^2\partial_{a_0^3} \\
& +3a_1^2\partial_{a_0^2} + 2a_2^2\partial_{a_1^2} \\
& +4a_1^1\partial_{b_0^1} + 3a_2^1\partial_{b_1^1} + 2a_3^1\partial_{b_2^1} + a_4^1\partial_{a_3^1},
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\mathbf{V}_3 = \; & -4a_0^2\partial_{a_0^1} - 3a_1^2\partial_{a_1^1} - 2a_2^2\partial_{a_2^1} - b_3^1\partial_{a_3^1} \\
& -(3a_0^3 + a_0^1)\partial_{a_0^2} - (2b_1^2 + a_1^1)\partial_{a_1^2} \\
& -2(a_0^2 + b_0^2)\partial_{a_0^3} - (a_2^1 + b_2^1)\partial_{a_2^2} \\
& -(3a_0^3 + b_0^1)\partial_{b_0^2} - (2a_1^2 + b_1^1)\partial_{b_1^2} \\
& -4b_0^2\partial_{b_0^1} - 3b_1^2\partial_{b_1^1} - 2a_2^2\partial_{b_2^1} - a_3^1\partial_{b_3^1}.
\end{aligned} \tag{5.19}$$

Example 5.4 $\mathcal{K}^5(\mathbb{E}_1^2)$. Using again Theorem 5.1, one arrives at the following infinitesimal generators

$$\begin{aligned}
\mathbf{V}_1 \;=\;& a_1^1\partial_{a_0^2} + a_2^1\partial_{a_1^2} + a_3^1\partial_{a_2^2} + a_4^1\partial_{a_3^2}\\
&+2a_1^2\partial_{a_0^3} + 2a_2^2\partial_{a_1^3}\\
&+3a_1^3\partial_{b_0^3} \qquad\qquad (5.20)\\
&+4b_1^2\partial_{b_0^2} + 3b_2^2\partial_{b_1^2} + 2a_3^2\partial_{b_2^2}\\
&+5b_1^1\partial_{b_0^1} + 4b_2^1\partial_{b_1^1} + 3b_3^1\partial_{b_2^1} + 2b_4^1\partial_{b_3^1} + a_5^1\partial_{b_4^1},
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_2 \;=\;& b_1^1\partial_{b_0^2} + b_2^1\partial_{b_1^2} + b_3^1\partial_{b_2^2} + b_4^1\partial_{a_3^2}\\
&+2b_1^2\partial_{b_0^3} + 2a_2^2\partial_{a_1^3}\\
&+3a_1^3\partial_{a_0^3} \qquad\qquad (5.21)\\
&+4a_1^2\partial_{a_0^2} + 3a_2^2\partial_{a_1^2} + 2a_3^2\partial_{a_2^2}\\
&+5a_1^1\partial_{a_0^1} + 4a_2^1\partial_{a_1^1} + 3a_3^1\partial_{a_2^1} + 2a_4^1\partial_{a_3^1} + a_5^1\partial_{a_4^1},
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_3 \;=\;& -5a_0^2\partial_{a_0^1} - 4a_1^2\partial_{a_1^1} - 3a_2^2\partial_{a_2^1} - 2a_3^2\partial_{a_3^1} - b_4^1\partial_{a_4^1}\\
&-(4a_0^3 + a_0^1)\partial_{a_0^2} - (3a_1^3 + a_1^1)\partial_{a_1^2} - (2b_2^2 + a_2^1)\partial_{a_2^2}\\
&-(3b_0^3 + 2a_0^2)\partial_{a_0^3}\\
&-2(b_1^2 + a_1^2)\partial_{a_1^3} - (a_3^1 + b_3^1)\partial_{a_3^2} \qquad\qquad (5.22)\\
&-(3a_0^3 + 2b_0^2)\partial_{b_0^3}\\
&-(4b_0^3 + b_0^1)\partial_{b_0^2} - (3a_1^3 + b_1^1)\partial_{b_1^2} - (2a_2^2 + b_2^1)\partial_{b_2^2}\\
&-5b_0^2\partial_{b_0^1} - 4b_1^2\partial_{b_1^1} - 3b_2^2\partial_{b_2^1} - 2a_3^2\partial_{b_3^1} - a_4^1\partial_{b_4^1}.
\end{aligned}$$

One of the main consequences of Theorem 5.1 is that one can employ the infinitesimal invariance criteria to determine the $I(\mathbb{E}_1^2)$-invariants of Killing tensors.

Theorem 5.2 *A function $I\,:\,\Sigma \to \mathbb{R}$ is an $I(\mathbb{E}_1^2)$-invariant of $\mathcal{K}^n(\mathbb{E}_1^2)$ if and only if it satisfies the infinitesimal invariance criteria*

$$\mathbf{V}_i(I) = 0, \qquad i = 1, 2, 3 \tag{5.23}$$

where Σ is the parameter space and $\mathbf{V}_i$, $i = 1, 2, 3$ are given in Theorem 5.1.

Corollary 5.1 *The parameter a_n^1 is a fundamental $I(\mathbb{E}_1^2)$-invariant of $\mathcal{K}^n(\mathbb{E}_1^2)$.*

In view of Theorem 5.2 the problem of the determination of the space of $I(\mathbb{E}_1^2)$-invariants now amounts to solving a system of linear homogeneous, first order PDEs resulting from the generators in Theorem 5.1. For large values of n the problem becomes very challenging computationally. One may employ the method of undetermined coefficients in conjuncture with the result of the fundamental theorem on invariants (see Theorem 3.1) of a regular Lie group action, as well as computer algebra.

The concept of a isometry group covariant of Killing tensors has been introduced in Definition 4.1 (see also [79, 80]). Theorem 5.1 also entails the corresponding criteria for $I(\mathbb{E}_1^2)$-covariants (see [79] for more details) of $\mathcal{K}^n(\mathbb{E}_1^2)$.

Theorem 5.3 *A function $C : \Sigma \times \mathbb{E}_1^2 \to \mathbb{R}$ is an $I(\mathbb{E}_1^2)$-covariant of $\mathcal{K}^n(\mathbb{E}_1^2)$ if and only if it satisfies the infinitesimal invariance conditions*

$$\begin{gathered} \mathrm{U}_i(C) = 0, \qquad i = 1, 2, 3, \\ \mathrm{U}_1 = \mathrm{V}_1 + \mathrm{T}, \quad \mathrm{U}_2 = \mathrm{V}_2 + \mathrm{X}, \quad \mathrm{U}_3 = \mathrm{V}_3 + \mathrm{H}, \end{gathered} \tag{5.24}$$

where Σ is the parameter space, V_i, $i = 1, 2, 3$ are specified in Theorem 5.1, T,X *and* H *are given in (4.28).*

5.4 Fundamental invariants

Following the ideas presented in the previous section (see also Yue [89]) we now describe the space of isometry group invariants of Killing tensors of valence n on the Minkowski plane. This will take a few steps.

5.4.1 General Killing tensors

We first work out a general Killing tensor of arbitrary valence n defined on the Minkowski plane using again the null coordinates (u, v). Compared with the case of the pseudo-Cartesian coordinates, the Killing tensors will have the same structure as shown in (5.4)(see also [89]), except that

1. one has to switch t to v and x to u in (5.4);

2. the sign of the coefficients a_{pij}, b_{qij} in the first groups (and hence the following groups) of the components will change in a way depending on whether the valence n is even or odd.

We specify the pattern in the following.

- n is even.

$$
\begin{array}{llllll}
\text{step1} & a_0^1 & a_1^1 & \cdots & a_{n-1}^1 & a_n^1 \\
 & b_0^1 & b_1^1 & \cdots & b_{n-1}^1 & a_n^1 \\
\text{step2} & a_0^2 & \cdots & a_{n-3}^2 & a_{n-2}^2 & -b_{n-1}^1 \\
 & b_0^2 & \cdots & b_{n-3}^2 & a_{n-2}^2 & -a_{n-1}^1 \\
\text{step3} & a_0^3 & \cdots & a_{n-4}^3 & -b_{n-3}^2 & b_{n-2}^1 \\
 & b_0^3 & \cdots & a_{n-4}^3 & -a_{n-3}^2 & a_{n-2}^1 \\
\vdots & & & & & \\
\text{step}\frac{n}{2} & a_0^{\frac{n}{2}} & a_1^{\frac{n}{2}} & a_2^{\frac{n}{2}} & \cdots & (-1)^{\frac{n-2}{2}} b^1_{\frac{n+2}{2}} \\
 & b_0^{\frac{n}{2}} & b_1^{\frac{n}{2}} & b_2^{\frac{n}{2}} & \cdots & (-1)^{\frac{n-2}{2}} a^1_{\frac{n+2}{2}} \\
\text{step}\frac{n+2}{2} & a_0^{\frac{n+2}{2}} & -b_1^{\frac{n}{2}} & b_2^{\frac{n-2}{2}} & \cdots & (-1)^{\frac{n}{2}} b^1_{\frac{n}{2}}
\end{array}
\tag{5.25}
$$

- n is odd.

$$
\begin{array}{lllllll}
\text{step1} & a_0^1 & a_1^1 & \cdots & a_{n-2}^1 & a_{n-1}^1 & a_n^1 \\
 & b_0^1 & b_1^1 & \cdots & b_{n-2}^1 & b_{n-1}^1 & -a_n^1 \\
\text{step2} & a_0^2 & a_1^2 & \cdots & a_{n-3}^2 & a_{n-2}^2 & b_{n-1}^1 \\
 & b_0^2 & b_1^2 & \cdots & b_{n-3}^2 & -a_{n-2}^2 & a_{n-1}^1 \\
\text{step3} & a_0^3 & \cdots & a_{n-5}^3 & a_{n-4}^3 & b_{n-3}^2 & -b_{n-2}^1 \\
 & b_0^3 & \cdots & b_{n-5}^3 & -a_{n-4}^3 & a_{n-3}^2 & -a_{n-2}^1 \\
\vdots & & & & & & \\
\text{step}\frac{n+1}{2} & a_0^{\frac{n+1}{2}} & a_1^{\frac{n+1}{2}} & b_2^{\frac{n-1}{2}} & -b_3^{\frac{n-3}{2}} & \cdots & (-1)^{\frac{n+1}{2}} b^1_{\frac{n+1}{2}} \\
 & b_0^{\frac{n+1}{2}} & -a_1^{\frac{n+1}{2}} & a_2^{\frac{n-1}{2}} & -a_3^{\frac{n-3}{2}} & \cdots & (-1)^{\frac{n+1}{2}} a^1_{\frac{n+1}{2}}
\end{array}
\tag{5.26}
$$

These schemes are now illustrated by several examples.

Example 5.5 Consider $\mathcal{K}^4(\mathbb{E}_1^2)$, the five components are given by

$$
\begin{aligned}
K^{1111} &= a_0^1 + 4a_1^1 u + 6a_2^1 u^2 + 4a_3^1 u^3 + a_4^1 u^4,\\
K^{1112} &= (a_0^2 + 3a_1^2 u + 3a_2^2 u^2 - b_3^1 u^3) + v(-a_1^1 - 3a_2^1 u - 3a_3^1 u^2 - a_4^1 u^3),\\
K^{1122} &= (a_0^3 - 2b_1^2 u + b_2^1 u^2) + 2v(-a_1^2 - 2a_2^2 u + b_3^1 u^2)\\
&\quad +v^2(a_2^1 + 2a_3^1 u + a_4^1 u^2),\\
K^{1222} &= (b_0^2 + 3b_1^2 v + 3a_2^2 v^2 - a_3^1 v^3) + u(-b_1^1 - 3b_2^1 v - 3b_3^1 v^2 - a_4^1 v^3),\\
K^{2222} &= b_0^1 + 4b_1^1 v + 6b_2^1 v^2 + 4b_3^1 v^3 + a_4^1 v^4.
\end{aligned}
\tag{5.27}
$$

Example 5.6 Consider $\mathcal{K}^5(\mathbb{E}_1^2)$, the six components are given by

$$
\begin{aligned}
K^{11111} &= a_0^1 + 5a_1^1 u + 10a_2^1 u^2 + 10a_3^1 u^3 + 5a_4^1 u^4 + a_5^1 u^5,\\
K^{11112} &= (a_0^2 + 4a_1^2 u + 6a_2^2 u^2 + 4a_3^2 u^3 - b_4^1 u^4) + v(-a_1^1 - 4a_2^1 u - 6a_3^1 u^2\\
&\quad -4a_4^1 u^3 - a_5^1 u^4),\\
K^{11122} &= (a_0^3 + 3a_1^3 u + 3b_2^2 u^2 - b_3^1 u^3) + 2v(-a_1^2 - 3a_2^2 u - 3a_3^2 u^2 + b_4^1 u^3)\\
&\quad +v^2(a_2^1 + 3a_3^1 u + 3a_4^1 u^2 + a_5^1 u^3),\\
K^{11222} &= (b_0^3 - 3a_1^3 v + 3a_2^2 v^2 - a_3^1 v^3) + 2u(-b_1^2 - 3b_2^2 v + 3a_3^2 v^2 - a_4^1 v^3)\\
&\quad +u^2(b_2^1 + 3b_3^1 v + 3b_4^1 v^2 - a_5^1 v^3),\\
K^{12222} &= (b_0^2 + 4b_1^2 v + 6b_2^2 v^2 - 4a_3^2 v^3 + a_4^1 v^4) + u(-b_1^1 - 4b_2^1 v - 6b_3^1 v^2\\
&\quad -4b_4^1 v^3 + a_5^1 v^4),\\
K^{22222} &= b_0^1 + 5b_1^1 v + 10b_2^1 v^2 + 10b_3^1 v^3 + 5b_4^1 v^4 - a_5^1 v^5.
\end{aligned}
\tag{5.28}
$$

5.4.2 Fundamental invariants

With the general formulae of Killing tensors available, we now proceed to employ the inductive version of the method of moving frames to determine a complete set of fundamental isometry group invariants (see [50, 80, 81]).

Step 1. Translation invariants

Assume the valence be even. We first consider a change of coordinates corresponding to translations

$$\bar{u} = u + b, \quad \bar{v} = v + c, \qquad b, c \in \mathbb{R}. \tag{5.29}$$

This will induce an action on the Killing tensors.

$$\begin{aligned}
\bar{a}_0^1 &= \sum_{k=0}^{n} a_k^1 \binom{n}{k} (-b)^k, \\
\bar{b}_0^1 &= \sum_{k=0}^{n} b_k^1 \binom{n}{k} (-c)^k, \\
\bar{a}_1^1 &= \sum_{k=0}^{n} a_{k+1}^1 \binom{n-1}{k} (-b)^k, \\
\bar{b}_1^1 &= \sum_{k=0}^{n} b_{k+1}^1 \binom{n-1}{k} (-c)^k, \\
&\vdots \\
\bar{a}_{n-1}^1 &= a_{n-1}^1 - b a_n^1, \\
\bar{b}_{n-1}^1 &= b_{n-1}^1 - c a_n^1, \\
\bar{a}_n^1 &= a_n^1, \\
&\vdots \\
\bar{a}_0^{\frac{n+2}{2}} &= \left[a_0^{\frac{n+2}{2}} + \sum_{k=1}^{\frac{n}{2}} b_k^{\frac{n-2k+2}{2}} \binom{\frac{n}{2}}{k} b^k \right] + \cdots + c^{\frac{n}{2}} \sum_{k=0}^{\frac{n}{2}} a_{\frac{n+2k}{2}}^1 \binom{\frac{n}{2}}{k} (-b)^k.
\end{aligned} \tag{5.30}$$

Choosing the cross-section $K = \{a_{n-1}^1 = b_{n-1}^1 = 0\}$ leads to the moving frame map (provided that $a_n^1 \neq 0$)

$$b = \frac{a_{n-1}^1}{a_n^1}, \qquad c = \frac{b_{n-1}^1}{a_n^1}. \tag{5.31}$$

Substituting (5.31) into the remaining equations in (5.30), we obtain the fundamental translation invariants (provided that $a_n^1 \neq 0$), denoted by

$$\begin{array}{llllll} A_0^1 & A_1^1 & \dots & A_{n-3}^1 & A_{n-2}^1 & A_n^1 \\ B_0^1 & B_1^1 & \dots & B_{n-3}^1 & B_{n-2}^1 & \\ A_0^2 & A_1^2 & \dots & A_{n-3}^2 & A_{n-2}^2 & \\ B_0^1 & B_1^1 & \dots & B_{n-3}^2 & & \\ & & \vdots & & & \\ A_0^{\frac{n}{2}} & A_1^{\frac{n}{2}} & A_2^{\frac{n}{2}} & & & \\ B_0^{\frac{n}{2}} & B_1^{\frac{n}{2}} & & & & \\ A_0^{\frac{n+2}{2}} & & & & & \end{array} \tag{5.32}$$

Note that the number of functionally independent translation invariants is $\frac{(n+1)(n+2)}{2} - 2$.

Step 2. Rotational invariants - fundamental invariants

Now consider the action of the subgroup of rotations, the change of coordinates is

$$\bar{u} = ue^{\lambda}, \quad \bar{v} = ve^{-\lambda}, \qquad \lambda \in \mathbb{R}. \tag{5.33}$$

where λ is the coordinate that parametrizes the group.

This change of coordinates induces the corresponding coefficient transformations as follows.

$$\begin{array}{lllll}
\bar{a}_0^1 = a_0^1 e^{n\lambda} & \bar{a}_1^1 = a_1^1 e^{(n-1)\lambda} & \cdots & \bar{a}_{n-1}^1 = a_{n-1}^1 e^{\lambda} & \\
\bar{b}_0^1 = b_0^1 e^{-n\lambda} & \bar{b}_1^1 = b_1^1 e^{-(n-1)\lambda} & \cdots & \bar{b}_{n-1}^1 = b_{n-1}^1 e^{-\lambda} & \bar{a}_n^1 = a_n^1 \\
\bar{a}_0^2 = a_0^2 e^{(n-2)\lambda} & \bar{a}_1^2 = a_1^2 e^{(n-3)\lambda} & \cdots & \bar{a}_{n-3}^2 = a_{n-3}^2 e^{\lambda} & \\
\bar{b}_0^2 = b_0^2 e^{-(n-2)\lambda} & \bar{b}_1^2 = b_1^2 e^{-(n-3)\lambda} & \cdots & \bar{b}_{n-3}^2 = b_{n-3}^2 e^{-\lambda} & \bar{a}_{n-2}^2 = a_{n-2}^2 \\
 & & \vdots & &
\end{array}$$

$$\begin{array}{lll}
\bar{a}_0^{\frac{n}{2}} = a_0^{\frac{n}{2}} e^{2\lambda} & \bar{a}_1^{\frac{n}{2}} = a_1^{\frac{n}{2}} e^{\lambda} & \\
\bar{b}_0^{\frac{n}{2}} = b_0^{\frac{n}{2}} e^{-2\lambda} & \bar{b}_1^{\frac{n}{2}} = b_1^{\frac{n}{2}} e^{-\lambda} & \bar{a}_2^{\frac{n}{2}} = a_2^{\frac{n}{2}} \\
\bar{a}_0^{\frac{n+2}{2}} = a_0^{\frac{n+2}{2}} & &
\end{array}$$

In view of (5.34), we determine the action of the subgroup H on the vector space spanned by the fundamental translation invariants 5.32.

$$\begin{array}{lllll}
\bar{A}_0^1 = e^{n\lambda} A_0^1 & \bar{A}_1^1 = e^{(n-1)\lambda} A_1^1 & \cdots & \bar{A}_{n-2}^1 = e^{2\lambda} A_{n-2}^1 & \\
\bar{B}_0^1 = e^{-n\lambda} B_0^1 & \bar{B}_1^1 = e^{-(n-1)\lambda} B_1^1 & \cdots & \bar{B}_{n-2}^1 = e^{-2\lambda} B_{n-2}^1 & \bar{A}_n^1 = A_n^1 \\
\bar{A}_0^2 = e^{(n-2)\lambda} A_0^2 & \bar{A}_1^2 = e^{(n-3)\lambda} A_1^2 & \cdots & \bar{A}_{n-3}^2 = e^{\lambda} A_{n-3}^2 & \\
\bar{B}_0^2 = e^{-(n-2)\lambda} B_0^2 & \bar{B}_1^2 = e^{-(n-3)\lambda} B_1^2 & \cdots & \bar{B}_{n-3}^2 = e^{-\lambda} B_{n-3}^2 & \bar{A}_{n-2}^1 = A_{n-2}^1 \\
 & & \vdots & &
\end{array}$$

$$\begin{array}{lll}
\bar{A}_0^{\frac{n}{2}} = e^{2\lambda} A_0^{\frac{n}{2}} & \bar{A}_1^{\frac{n}{2}} = e^{\lambda} A_1^{\frac{n}{2}} & \\
\bar{B}_0^{\frac{n}{2}} = e^{-2\lambda} B_0^{\frac{n}{2}} & \bar{B}_1^{\frac{n}{2}} = e^{-\lambda} B_1^{\frac{n}{2}} & \bar{A}_2^{\frac{n}{2}} = A_2^{\frac{n}{2}} \\
\bar{A}_0^{\frac{n+2}{2}} = A_0^{\frac{n+2}{2}} & &
\end{array}$$

Now we again employ the method of moving frames.

Choosing the cross section

$$K = \{A_1^{\frac{n}{2}} = c, \quad \text{where } c \neq 0 \text{ is a constant}\}$$

leads to the moving frame map

$$\lambda = \ln \frac{c}{A_1^{\frac{n}{2}}}. \tag{5.36}$$

Combining (5.36) and (5.35), after some algebra one arrives at the following fundamental invariants (provided that $a_n^1 \neq 0$): Invariants of type I

$$\mathcal{I}_0^1 = A_0^1 B_0^1, \quad \mathcal{I}_1^1 = A_0^1 B_0^1, \quad \cdots, \quad \mathcal{I}_{n-2}^1 = A_{n-2}^1 B_{n-2}^1, \quad \mathcal{I}_n^1 = A_n^1,$$

$$\mathcal{I}_0^2 = A_0^2 B_0^2, \quad \mathcal{I}_1^2 = A_1^2 B_1^2, \quad \cdots, \quad \mathcal{I}_{n-3}^2 = A_{n-3}^2 B_{n-3}^2, \quad \mathcal{I}_{n-2}^2 = A_{n-2}^2,$$

$$\vdots$$

$$\mathcal{I}_0^{\frac{n}{2}} = A_0^{\frac{n}{2}} B_0^{\frac{n}{2}}, \quad \mathcal{I}_1^{\frac{n}{2}} = A_1^{\frac{n}{2}} B_1^{\frac{n}{2}}, \quad \mathcal{I}_2^{\frac{n}{2}} = A_2^{\frac{n}{2}},$$

$$\mathcal{I}_0^{\frac{n+2}{2}} = A_0^{\frac{n+2}{2}},$$

Invariants of type J

$$\mathcal{J}_0^1 = A_0^1 (B_1^{\frac{n}{2}})^n, \quad \mathcal{J}_1^1 = A_1^1 (B_1^{\frac{n}{2}})^{n-1}, \quad \cdots, \quad \mathcal{J}_{n-2}^1 = A_{n-2}^1 (B_1^{\frac{n}{2}})^2,$$

$$\mathcal{J}_0^2 = A_0^2 (B_1^{\frac{n}{2}})^{n-2}, \quad \mathcal{J}_1^2 = A_1^2 (B_1^{\frac{n}{2}})^{n-3}, \quad \cdots, \quad \mathcal{J}_{n-3}^2 = A_{n-3}^2 (B_1^{\frac{n}{2}}),$$

$$\vdots$$

$$\mathcal{J}_0^{\frac{n}{2}} = A_0^{\frac{n}{2}} (B_1^{\frac{n}{2}})^2.$$

Remark 5.2 We remark that the number of invariants found above is exactly

$$\frac{(n+1)(n+2)}{2} - 3.$$

We thus obtain Theorem 5.4 which follows from Theorem 3.4.

Theorem 5.4 *Assuming that $a_n^1 \neq 0$, the type I invariants and type J invariants listed above together form a complete set of fundamental isometry group-invariants of Killing tensors of valence n defined on the Minkowski plane.*

Remark 5.3 The case where the valence is odd is similar and we omit the process.

Chapter 6

Applications

6.1 Introduction

One of the main features of any invariant theory is that the invariants can be employed to solve the classification problem. It is well known that the elements of the vector subspaces of non-trivial Killing tensors of valence two generate *orthogonal coordinate webs* on the Euclidean plane $\mathbb{E}^2$ and the Minkowski plane $\mathbb{E}^2_1$ respectively, provided the Killing tensors in question have distinct (and real) eigenvalues. From the invariant theory point of view the problem of classification of orthogonal coordinate webs (equivalence problem) and the related canonical form problem are intimately related to the problem of the determination of fundamental isometry group invariants (covariants, joint invariants) of Killing tensors. In the following two sections, we present an invariant classification of orthogonal coordinate webs on the Euclidean plane and on the Minkowski plane that are generated by non-trivial Killing tensors of valence two. For the classification on the Euclidean plane, the problem was frist solved by Winternitz and Friš in 1965 [87] and then by McLenaghan, Smirnov and The independently in 2002 [52]. Here we resolve the problem by employing the fundamental covariants. For the corresponding problem on the Minkowski plane, we use the fundamental covariants of valence two Killing tensors. We note that the complexity of the solution is significantly reduced compared to [54].

The second part of the chapter deals with integrable and super-integrable Hamiltonian systems, using the invariant theory of Killing tensors. We are particularly interested in the potentials of Drach.

6.2 Orthogonal coordinate webs on the Euclidean plane

Let $\mathcal{K}^2_{nt}(\mathbb{E}^2)$ be the vector space of non-trivial Killing two tensors of valence two defined on the Euclidean plane $\mathbb{E}^2$. By "non-trivial" we mean that the Killing tensor in question is not a multiple of the metric. Thus such a Killing tensor will be determined by 5 independent parameters, that is $\dim \mathcal{K}^2_{nt}(\mathbb{E}^2) = 5$. It is established in [55, 56, 52, 54] that the fundamental invariants $\mathcal{I}_1$ and $\mathcal{I}_3$ given by (4.73) are the fundamental $I(\mathbb{E}^2)$-invariants of $\mathcal{K}^2_{nt}(\mathbb{E}^2)$. The two fundamental invariants divide $\mathcal{K}^2_{nt}(\mathbb{E}^2)$ into four equivalence classes and they can be used to completely solve the problem of the classification of orthogonal coordinate webs on the Euclidean plane. The results are summarized in Table 6.1.

We note that the same classification can be done by means of the fundamental $I(\mathbb{E}^2)$-covariants $\mathcal{C}_1$ and $\mathcal{C}_2$ (see (4.59)). The details are given in Table 6.2. The four orthogonal coordinate webs and given in Figure 6.1-6.4.

Without loss of generality we can assume that the general non-trivial Killing tensor of valence two defined on the Euclidean plane with respect to the Cartesian coordinates (x, y) takes the following form.

$$\mathbf{K}^2_{nt} = \begin{pmatrix} \beta_1' + 2\beta_4 y + \beta_6 y^2 & \beta_3 - \beta_4 x - \beta_5 y - \beta_6 xy \\ \beta_3 - \beta_4 x - \beta_5 y - \beta_6 xy & 2\beta_5 x + \beta_6 x^2 \end{pmatrix}. \tag{6.1}$$

where the parameters β_i, $i = 1, \ldots, 6$ are given in (4.44) and $\beta_1' = \beta_1 - \beta_2$.

The Killing tensors within each of the four equivalence class generate the same orthogonal coordinate web (up to the action of the isometry group $I(\mathbb{E}^2)$).

One can use this fact to choose appropriate canonical forms for each of the four equivalence classes. We consider the Killing tensors with respect to the orthogonal coordinates (u, v) (see [52]) and then use the standard coordinate transformations to the Cartesian coordinates (x, y) in order to determine the corresponding canonical forms for EC1-4. Alternatively, one can proceed by using the coordinate cross-sections, which, we demonstrate in the following.

EC1. The parameter space Σ' defined by the five parameters of (6.1) can be intersected by the coordinate cross-section

$$K_1 = \{\beta_3 = \beta_4 = \beta_5 = 0\}. \tag{6.2}$$

In view of (6.1) and the formulas for $\mathcal{I}_1$ and $\mathcal{I}_3$ given by (4.59), we see that all but one (β_1') parameters vanish in this case. The parameter β_1' remains arbitrary. Without loss of generality we can set $\beta_1' = 1$, leading to the canonical form

$$\mathbf{K}_I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.3}$$

Alternatively, we could have used the coordinate cross-section

$$K_1' = \{\beta_1' = \beta_4 = \beta_5 = 0\}, \tag{6.4}$$

which would have led to the canonical form

$$\mathbf{K}_I' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.5}$$

Note the canonical forms (6.3) and (6.5) are equivalent up to a rotation.

EC2. Same argument as in EC1. Either of the coordinate cross-sections (6.2) and (6.4) leads to the canonical form

$$\mathbf{K}_{II} = \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}. \tag{6.6}$$

EC3. First, note that the condition $\mathcal{I}_1 \neq 0, \mathcal{I}_3 = 0$ (see Table 6.1) prompts $\beta_4^2 + \beta_5^2 \neq 0$. Therefore the coordinate cross-sections that can be used here are:

$$K_3 = \{\beta_1' = \beta_3 = \beta_4 = 0\} \tag{6.7}$$

and

$$K_4 = \{\beta_1' = \beta_3 = \beta_5 = 0\}, \tag{6.8}$$

which lead to the canonical forms

$$\mathbf{K}_{III} = \begin{pmatrix} 0 & -y \\ -y & 2x \end{pmatrix}, \tag{6.9}$$

and

$$\mathbf{K}_{III}' = \begin{pmatrix} 2y & -x \\ -x & 0 \end{pmatrix}, \tag{6.10}$$

respectively.

Note the canonical forms (6.9) and (6.10) are equivalent up to a rotation.

EC4. In this case we can use either of the coordinate cross-sections (6.2) and (6.4). Intersecting the common level set defined by $\mathcal{I}_1 \neq 0$, $\mathcal{I}_3 \neq 0$ (see Table 6.1) with (6.2) yields the canonical form

$$\mathrm{K}_{IV} = \begin{pmatrix} \beta_1' + y^2 & -xy \\ -xy & x^2 \end{pmatrix}, \tag{6.11}$$

while with (6.4) we arrive at the canonical form

$$\mathrm{K}_{IV}' = \begin{pmatrix} y^2 & \beta_3 - xy \\ \beta_3 - xy & x^2 \end{pmatrix}. \tag{6.12}$$

Note the canonical forms (6.11) and (6.12) are equivalent up to a rotation and rescaling.

6.3 Orthogonal coordinate webs on the Minkowski plane

The problem of classification of the ten orthogonal coordinate webs on the Minkowski plane was initially solved by Kalnins (1975) [37]. The approach in [37] is based on the property that the Killing tensors defined on pseudo-Riemannian spaces of constant curvature are the sums of symmetrized tensor products of Killing vectors, and different combinations (as symmetric tensor products) of the basic Killing vectors (4.28) were analyzed modulo the action of the eight-dimensional discrete group $\mathcal{R}$ of permutations of coordinates and reflections of the signature of the Minkowski metric $g = \mathrm{diag}(1, -1)$ with respect to the pseudo-Cartesian coordinates (t, x).

A different approach was used in Rastelli [71], where the ten orthogonal webs were classified based on the algebraic properties of the non-trivial Killing tensors of valence two defined on the Minkowki plane. More specifically, the author made use of the points where the eigenvalues of such Killing tensors coincide (singular points). Recently, McLenaghan *et al* [54] employed a set of the fundamental isometry group-invariants of Killing tensors to classify the ten orthogonal coordinate webs defined on the Minkowski plane. The problem appears to be much more challenging than the corresponding problem of classification on $\mathbb{E}^2$ [55, 52]. The reason is simple: In both cases one has two fundamental invariants available, while the number of orthogonal coordinate webs is four on the Euclidean plane and ten on the Minkowski plane. In the latter case the problem was solved in [54] by introducing the concept of a *conformal* $I(\mathbb{E}_1^2)$*-invariant*, which was used to generate additional *discrete* $I(\mathbb{E}_1^2)$-invariants. The authors also investigate the effect of the eight dimensional discrete group $\mathcal{R}$ on the discrete $I(\mathbb{E}_1^2)$-invariants. Unordered pairs (as the objects preserved by the discrete group) of discrete invariants along with one of the fundamental invariants were used to solve the problem. In the following we present a simpler solution based on the fundamental $I(\mathbb{E}_1^2)$-covariants that are obtained in Chapter 3. The ten orthogonal coordinate webs are given in Figure 5.5-5.14.

Let $\mathcal{K}_{nt}^2(\mathbb{E}_1^2)$ be the vector space of non-trivial Killing tensors of valence two defined on the Minkowski plane. Again dim $\mathcal{K}_{nt}^2(\mathbb{E}_1^2) = 5$ and we assume without loss of generality that in terms of the pseudo-Cartesian coordinates (t, x) the non-trivial Killing tensor of valence two is of the form

$$\mathrm{K} = \begin{pmatrix} \alpha_1' + 2\alpha_4 x + \alpha_6 x^2 & \alpha_3 + \alpha_4 t + \alpha_5 x + \alpha_6 tx \\ \alpha_3 + \alpha_4 t + \alpha_5 x + \alpha_6 tx & 2\alpha_5 t + \alpha_6 t^2 \end{pmatrix}, \tag{6.13}$$

where the parameters α_i, $i = 1, \ldots, 6$ are as in (4.25) and $\alpha_1' = \alpha_1 + \alpha_2$.

Note that by Theorem 4.6 any $I(\mathbb{E}_1^2)$-covariant of $\mathcal{K}_{nt}^2(\mathbb{E}_1^2)$ enjoys the form

$$C = F(\mathcal{I}_1, \mathcal{I}_3, \mathcal{C}_1, \mathcal{C}_2),$$

where $\mathcal{I}_1$, $\mathcal{I}_3$, $\mathcal{C}_1$ and $\mathcal{C}_2$ are given by (4.61). As in the case of $\mathcal{K}_{nt}^2(\mathbb{E}^2)$ we can use $\mathcal{I}_1$, $\mathcal{I}_3$, $\mathcal{C}_1$ and $\mathcal{C}_2$ to classify the ten orthogonal webs. However, in view of the number of webs we have to use these functions concurrently. Before doing so, we check the effect of $\mathcal{R}$ on $\mathcal{I}_1$, $\mathcal{I}_3$, $\mathcal{C}_1$ and $\mathcal{C}_2$. Recall [37, 54] that the group (under composition) $\mathcal{R} =< R_1, R_2 >$ consists of eight discrete transformations generated by

$$\begin{aligned} R_1 :\ & \tilde{t} = t, \quad \tilde{x} = -x \quad \text{(spatial reflections)}, \\ R_2 :\ & \tilde{t} = x, \quad \tilde{x} = t \qquad \text{(permutation)}. \end{aligned} \tag{6.14}$$

The group $\mathcal{R}$ (along with the isometry group $I(\mathbb{E}_1^2)$) preserves the geometry of the orthogonal webs defined on the Minkowski plane. Next, it is observed in [54] that R_1 and R_2 induce the following transformations on the parameters α_i, $i = 1, \ldots, 6$ of $\mathcal{K}^2(\mathbb{E}_1^2)$ (see

(4.44)):

$$\begin{aligned} R_1: &\quad \tilde{\alpha}_1 = \alpha_1, \quad \tilde{\alpha}_2 = \alpha_2, \quad \tilde{\alpha}_3 = -\alpha_3, \quad \tilde{\alpha}_4 = -\alpha_4, \quad \tilde{\alpha}_5 = \alpha_5, \quad \tilde{\alpha}_6 = \alpha_6, \\ R_2: &\quad \tilde{\alpha}_1 = \alpha_2, \quad \tilde{\alpha}_2 = \alpha_1, \quad \tilde{\alpha}_3 = \alpha_3, \quad \tilde{\alpha}_4 = \alpha_5, \quad \tilde{\alpha}_5 = \alpha_4, \quad \tilde{\alpha}_6 = \alpha_6. \end{aligned} \tag{6.15}$$

It follows immediately that the fundamental covariants $\mathcal{I}_1, \mathcal{I}_3, \mathcal{C}_1$ and $\mathcal{C}_2$ remain unchanged under the transformations (6.15). We conclude that we can use them in the classification of the ten orthogonal webs.

Recall that the space $\mathcal{K}^2_{nt}(\mathbb{E}^2_1)$ can be divided into ten equivalence classes EC1-10 within each of which the corresponding elements generate the *same orthogonal coordinate web*.

We consider next the ten *canonical elements* determined in [54], which represents each class of EC1-10 by transforming them to contravariant form and making them compatible with the general form (6.13) by adding multiples of the metric if necessary. The latter operation does not affect the geometry of the coordinate webs generated by the canonical elements. We arrive at the following list.

$$\text{EC1} \quad \mathrm{K}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.16}$$

$$\text{EC2} \quad \mathrm{K}_2 = \begin{pmatrix} x^2 & tx \\ tx & t^2 \end{pmatrix}, \tag{6.17}$$

$$\text{EC3} \quad \mathrm{K}_3 = \begin{pmatrix} \frac{1}{2} - x & \frac{1}{4} - \frac{1}{2}t + \frac{1}{2}x \\ \frac{1}{4} - \frac{1}{2}t + \frac{1}{2}x & t \end{pmatrix}, \tag{6.18}$$

$$\text{EC4} \quad \mathrm{K}_4 = \begin{pmatrix} 0 & x \\ x & 2t \end{pmatrix}, \tag{6.19}$$

$$\text{EC5} \quad \mathrm{K}_5 = \begin{pmatrix} 2k^2 - \frac{1}{4}x^2 & -\frac{1}{4}tx \\ -\frac{1}{4}tx & -\frac{1}{4}t^2 \end{pmatrix}, \tag{6.20}$$

$$\text{EC6} \quad \mathrm{K}_6 = \begin{pmatrix} \frac{1}{4} + \frac{1}{4}x^2 & \frac{1}{4} + \frac{1}{4}tx \\ \frac{1}{4} + \frac{1}{4}tx & \frac{1}{4}t^2 \end{pmatrix}, \tag{6.21}$$

$$\text{EC7} \quad \mathrm{K}_7 = \begin{pmatrix} -\frac{1}{2}+\frac{1}{4}x^2 & -\frac{1}{4}+\frac{1}{4}tx \\ -\frac{1}{4}+\frac{1}{4}tx & \frac{1}{4}t^2 \end{pmatrix}, \tag{6.22}$$

$$\text{EC8} \quad \mathrm{K}_8 = \begin{pmatrix} \frac{1}{4}x^2 & -k^2+\frac{1}{4}tx \\ -k^2+\frac{1}{4}tx & \frac{1}{4}t^2 \end{pmatrix}, \tag{6.23}$$

$$\text{EC9} \quad \mathrm{K}_9 = \begin{pmatrix} 2k^2+\frac{1}{4}x^2 & \frac{1}{4}tx \\ \frac{1}{4}tx & \frac{1}{4}t^2 \end{pmatrix}, \tag{6.24}$$

$$\text{EC10} \quad \mathrm{K}_{10} = \begin{pmatrix} -2k^2+\frac{1}{4}x^2 & \frac{1}{4}tx \\ \frac{1}{4}tx & \frac{1}{4}t^2 \end{pmatrix}, \tag{6.25}$$

where the parameter k is a $I(\mathbb{E}_1^2)$-invariant of $\mathcal{K}_{nt}^2(\mathbb{E}_1^2)$. In view of Theorem 4.3 (see also Theorem 4.6), it can be represented via the fundamental $I(\mathbb{E}_1^2)$-invariants. Indeed, the corresponding formulas were found in [54]:

$$\begin{aligned} \text{EC5, EC9, EC10}: \ k^2 &= \frac{\sqrt{\mathcal{I}_1}}{\mathcal{I}_3}, \quad (\mathcal{I}_1 > 0), \\ \text{EC8}: \ k^2 &= \frac{\sqrt{-\mathcal{I}_1}}{\mathcal{I}_3}, \quad (\mathcal{I}_1 < 0). \end{aligned} \tag{6.26}$$

Note the canonical forms (6.16)-(6.25) are compatible with the general form given by (6.13).

Following the procedure devised in [54], we use the canonical forms (6.16-6.25) to evaluate the corresponding values of the fundamental $I(\mathbb{E}_1^2)$-covariants $\mathcal{I}_1$, $\mathcal{I}_3$, $\mathcal{C}_1$, $\mathcal{C}_2$ (see formulae (4.61) in Chapter 3) and employ the results to distinguish the elements belonging to different equivalence classes EC1-10. The elements of $\mathcal{K}_{nt}^2(\mathbb{E}_1^2)$ must have the same values of $\mathcal{I}_1$, $\mathcal{I}_3$, $\mathcal{C}_1$ and $\mathcal{C}_2$. We note however that these functions do not distinguish EC1 from EC3 and EC6 from EC8. Therefore we have to derive some auxiliary $I(\mathbb{E}_1^2)$-invariants to complete the classification scheme. Indeed, consider the vector space $\mathcal{K}^2(\mathbb{E}_1^2)$ under the action of the isometry group $I(\mathbb{E}_1^2)$. Since $\mathcal{I}_3$ is a fundamental $I(\mathbb{E}_1^2)$-invariant, we can consider the level set

$$\mathcal{S}_{\mathcal{I}_3} = \{(\alpha_1, \ldots, \alpha_5) \in \Sigma \mid \ \mathcal{I}_3 = 0\}. \tag{6.27}$$

Note $\mathcal{S}_{\mathcal{I}_3}$ is an $I(\mathbb{E}_1^2)$-invariant submanifold in Σ defined by the parameters α_i, $i = 1, \ldots, 5$. Next we prove the following result by using the techniques exhibited in Section 2.

Lemma 6.1 *Any algebraic $I(\mathbb{E}_1^2)$-invariant I of the $I(\mathbb{E}_1^2)$-invariant submanifold $\mathcal{S}_{\mathcal{I}_3}$ defined by (6.27) can be (locally) uniquely expressed as an analytic function $I = F(\mathcal{I}_1', \mathcal{I}_2')$, where the fundamental invariants $\mathcal{I}_i'$, $i = 1, 2$ are given by*

$$\mathcal{I}_1' = \alpha_4^2 - \alpha_5^2, \quad \mathcal{I}_2' = 2\alpha_3\alpha_4\alpha_5 - \alpha_2\alpha_4^2 - \alpha_1\alpha_5^2, \tag{6.28}$$

provided the group acts in $\mathcal{S}_{\mathcal{I}_3}$ with three-dimensional orbits.

We note that the fundamental $I(\mathbb{E}_1^2)$-invariants $\mathcal{I}_1'$ and $\mathcal{I}_2'$ still cannot be used in the problem of classification of the elements of $\mathcal{K}_{nt}^2(\mathbb{E}_1^2)$. In particular, $\mathcal{I}_2'$ appears to be a function of α_1, α_2, α_3, α_4 and α_5 (not α_1', α_3, α_4, α_5). However, under the additional *invariant* condition

$$\mathcal{I}_1' = \alpha_4^2 - \alpha_5^2 = 0, \tag{6.29}$$

it assumes the following form:

$$\mathcal{I}_2' = 2\alpha_3\alpha_4\alpha_5 - \alpha_1'\alpha_4^2, \tag{6.30}$$

where $\alpha_1' = \alpha_1 + \alpha_2$. We immediately recognize the $I(\mathbb{E}_1^2)$-invariant (6.30) to be an $I(\mathbb{E}_1^2)$-invariant of the submanifold in $\mathcal{S}_{\mathcal{I}_3}$ determined by the condition (6.29).

Thus, $\mathcal{I}_2'$ given by (6.30) can be used to distinguish between EC1 and EC3.

Next, in order to distinguish between the elements of EC6 and EC8, introduce the following auxiliary $I(\mathbb{E}_1^2)$-invariant. $\mathcal{I}^* := k^4\mathcal{I}_3 + \mathcal{I}_1$, where k is given by (6.26) (the formula for EC8). We note that $\mathcal{I}^*$ is an $I(\mathbb{E}_1^2)$-invariant. The values of $\mathcal{I}_1$ and $\mathcal{I}_3$ evaluated with respect to the parameters of the canonical form EC8 given by (6.23) are $\mathcal{I}_1 = -\dfrac{k^4}{4}$, $\mathcal{I}_3 = \dfrac{1}{4}$. Therefore the $I(\mathbb{E}_1^2)$-invariant $\mathcal{I}^* = 0$, whenever the Killing tensor in question belongs to EC8. The classification scheme is now complete. We summarize the results in Table 6.3.

Using the results obtained we can devise a general algorithm of classification the elements of the vector spaces $\mathcal{K}^2(\mathbb{E}^2)$ and $\mathcal{K}^2(\mathbb{E}_1^2)$. It consists of the following two steps. Let $\mathbf{K} \in \mathcal{K}^2(\mathbb{E}^2)$ ($\mathcal{K}^2(\mathbb{E}_1^2)$).

(i) If $\mathbf{K}$ has arbitrary constants, decompose $\mathbf{K}$ as follows:

$$\mathbf{K} = \ell_0 g + \sum_{i=1}^{5} \ell_i \mathbf{K}_i, \tag{6.31}$$

where ℓ_i $i = 1, \ldots, 5$ are the arbitrary constants. Note $\sum_{i=1}^{5} \ell_i \mathbf{K}_i \in \mathcal{K}_{nt}^2(\mathbb{E}^2)$ ($\mathcal{K}_{nt}^2(\mathbb{E}_1^2)$). Clearly, $\mathbf{K} \in \mathcal{K}_{nt}^2(\mathbb{E}^2)$ ($\mathcal{K}_{nt}^2(\mathbb{E}_1^2)$) iff $\ell_0 = 0$.

(ii) Each Killing tensor in the representation (6.31) represents one of the equivalence classes (and thus, - an orthogonal coordinate web), provided it has real eigenvalues in the case of the vector space being $\mathcal{K}^2(\mathbb{E}^2_1)$. We can determine which one by evaluating the corresponding $I(\mathbb{E}^2)$ and $I(\mathbb{E}^2_1)$-invariants and covariants and then using the information provided in Table 6.1 or Table 6.2 for the Killing tensors defined in the Euclidean plane and Table 6.3 - the Minkowski plane.

The problem of classification is therefore solved.

Remark 6.1 We remark that EC5 and EC10 are characterized by the same values of the fundamental $I(\mathbb{E}^2_1)$-covariants. It agrees with the geometry of the corresponding orthogonal coordinate webs, namely they determine two distinct orthogonal coordinate systems that cover two disjoint areas of the same space [58].

The ten webs are given in Figure 6.5-6.14.

Recently Horwood *et al* considered the problem of the invariant classification of the orthogonal coordinate webs on the Euclidean 3-space $\mathbb{E}^3$. The problem seems more complicated than the corresponding problems on the Euclidean plane or Minkowski plane. The eleven orthogonal coordinate webs on $\mathbb{E}^3$ are generated by Killing 2-tensors with normal eigenvectors, while not every Killing 2-tensors on $\mathbb{E}^3$ with two distinct eigenvalues has normal eigenvetors (which is the case on $\mathbb{E}^2$ or $\mathbb{E}^2_1$). One has to verify the normality condition, which is essentially a system of nonlinear PDEs. For more details see [34].

6.4 Drach potentials

As is well-known, Killing tensors naturally appear in the study of Hamiltonian systems. It is established [3] that the non-orthogonal separation of variables in the Hamilton-Jacobi equation defined by a natural Hamiltonian (see (6.32)) can be intrinsically characterized by Killing tensors of valence two. In this section, we consider the Hamiltonian systems with two degrees of freedom whose complete integrability is afforded by first integrals cubic in the momenta and thus determined by Killing tensors of valence three. More specifically, we examine from this viewpoint the potentials of Drach [17].

Let (M, g) be an m-dimensional pseudo-Riemannian manifold of constant curvature. Consider a Hamiltonian system defined by a natural Hamiltonian of the form (we adopt the Einstein summation convention)

$$H = \frac{1}{2}g^{ij}(\mathbf{q})p_i p_j + V(\mathbf{q}), \quad i, j = 1, \ldots, m, \tag{6.32}$$

via the canonical Poisson bi-vector $\mathbf{P}_0 = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$, where g^{ij} are the components of the metric tensor g and $(\mathbf{q}, \mathbf{p}) \in T^*M$ (cotangent bundle) are the standard position-momenta coordinates.

The Hamiltonian system with m degrees of freedom is a system of autonomous PDEs

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \qquad i = 1, \ldots, m. \tag{6.33}$$

Definition 6.1 A constant of motion of the system (6.33) is a function F in the position-momenta coordinates $(\mathbf{q}, \mathbf{p})$ that remains constant along any trajectory on the phase space.

It follows immediately from Definition 6.1 and the formula (6.33) that

Theorem 6.1 *A function F is a constant of motion of the Hamiltonian system (6.33) iff the Poisson bracket of F and H vanishes, that is* $\{F, H\} = \sum_{i=1}^{m} \left(\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} \right) = 0.$

In general the Hamiltonian system with m-degrees of freedom defined by (6.32) may admit constants of motion F which are polynomials in the momenta.

$$F = K_r^{i_1 i_2 \ldots i_r} p_{i_1} p_{i_2} \ldots p_{i_r} + \cdots + K_1^{i_1} p_{i_1} + U, \quad i_1, \ldots, i_r = 1, \ldots, m. \tag{6.34}$$

Definition 6.2 [43] A Hamiltonian system with m degrees of freedom defined by (6.32) is said to be *completely integrable* if there are m *functionally independent* constants of motion which are mutually in involution; The system is called *superintegrable* if there exist more than m functionally independent constants of motion, but not necessarily in involution. If there are exactly $2m - 1$ such constants of motion, the system is called *maximally superintegrable.*

Suppose a Hamiltonian system defined by a natural Hamiltonian (6.32) admits a constant of motion that is cubic in the momenta $F = L^{ijk}p_i p_j p_k + K^{ij}p_i p_j + B^i p_i + U$, then the vanishing of the Poisson bracket $\{H, F\} = 0$ takes the following equations expressed in both component and coordinate-free forms respectively [53].

$$L^{(ijk}_{,f} g^{\ell)f} - \frac{3}{2} L^{f(ij} g^{k\ell)}_{,f} = 0 \;\Leftrightarrow\; [\mathbf{L}, g] = 0, \tag{6.35}$$

$$K^{(ij}{}_{,f} g^{k)f} - K^{f(i} g^{jk)}_{,f} = 0 \;\Leftrightarrow\; [\mathbf{K}, g] = 0, \tag{6.36}$$

$$B^{(i}_{,f} g^{j)f} - \frac{1}{2} B^f g^{ij}_{,f} - 3L^{fij} V_{,f} = 0 \;\Leftrightarrow\; [\mathbf{B}, g] = 3\mathbf{L}dV, \tag{6.37}$$

$$U_{,f} g^{fi} - 2K^{fi} V_{,f} = 0 \; (U_{,i} = 2K^j{}_i V_{,j}) \;\Leftrightarrow\; dUg = 2\mathbf{K}dV, \tag{6.38}$$

$$B^f V_{,f} = 0 \;\Leftrightarrow\; \mathbf{B}(V) = 0, \tag{6.39}$$

where $_{,f}$, $(\ ,\ )$ and $[\ ,\ ]$ denote partial differentiation, symmetrization and the Schouten bracket, respectively. Note $[\mathbf{B}, g] = -\mathcal{L}_{\mathbf{B}} \mathrm{g}$, where $\mathcal{L}$ denotes the Lie derivative operator. It follows immediately from (6.35) and (6.36) that $\mathbf{L}$ and $\mathbf{K}$ are Killing tensors of valence three and two respectively, while $\mathbf{B}$ - in general - is not a Killing vector field.

Equation (6.39) reveals that the potential function V is preserved by the vector field $\mathbf{B}$. It is observed [53] next those Equations (6.35)-(6.39) separate into two groups, namely (6.35), (6.37), (6.39) and (6.36), (6.38), involving only components of F which are polynomials of odd order and even order in the momenta respectively. That is, constant of motion can be written as

$$F = F_{odd} + F_{even}, \tag{6.40}$$

where

$$\begin{aligned} F_{odd} &= L^{ijk}(\mathrm{q}) p_i p_j p_k + B^k(\mathrm{q}) p_k, \\ F_{even} &= K^{ij}(\mathrm{q}) p_i p_j + U(\mathrm{q}), \\ \{H, F_{odd}\} &= \{H, F_{even}\} = 0. \end{aligned} \tag{6.41}$$

One immediately arrives at the following result.

Theorem 6.2 *A Hamiltonian system with two degrees of freedom determined by a natural Hamiltonian (6.32) which admits a constant of motion (6.34) of order $r \geq 3$ having both even and odd terms in the momenta is necessarily superintegrable, provided that the odd order term and the even order term are functionally independent.*

The above observation can be extended [53] to the case in which the constant of motion is of the order $r > 3$ in the momenta.

$$
\begin{aligned}
[\mathrm{K}_r, g] &= 0, \\
[\mathrm{K}_{r-1}, g] &= 0, \\
[\mathrm{K}_{r-2}, g] &= r\mathrm{K}_r dV, \\
&\vdots \\
[\mathrm{K}_i, g] &= (i+2)\mathrm{K}_{i+2} dV, \\
&\vdots \\
[\mathrm{K}_1, g] &= 3\mathrm{K}_3 dV, \\
dUg = [U, g] &= 2\mathrm{K}_2 dV, \\
\mathrm{K}_1(V) &= 0,
\end{aligned}
\tag{6.42}
$$

where the tensorial quantities K_r, $r \geq 1$ are determined by the corresponding components of (6.34). That is, the constant of motion F can also be written as $F_r = F_{r(odd)} + F_{r(even)}$, implying that $\{H, F_{r(odd)}\} = \{H, F_{r(even)}\} = 0$.

Remark 6.2 *Given a constant of motion of order $r \geq 3$ in the momenta of the type 6.34), the only the tensorial quantities K_r and K_{r-1} that define the first two terms of (6.34) are Killing tensors: $\mathrm{K}_r \in \mathcal{K}^r(M), \mathrm{K}_{r-1} \in \mathcal{K}^{r-1}(M)$.*

In the following we restrict our attention to a natural Hamiltonian defined on the Minkowski plane $\mathbb{E}_1^2$.

Recall that in 1935 Drach [17] listed ten completely integrable Hamiltonian systems defined on the Minkowski plane $\mathbb{E}_1^2$. Their complete integrability is afforded by the existence of an additional constant of motion which are cubic in the momenta. In recent years the ten integrable systems of Drach have received much attention from the viewpoint of the theory of superintegrable systems [25, 43, 42, 39, 40, 41] (see also the references therein). Thus, it has been shown [70, 85] that seven of the ten Drach potentials are, in fact, superintegrable, admitting in addition a constant of motion that is quadratic in the momenta.

Drach [17] considered the above problem and posed an ansatz in which the Hamiltonian and an additional constant of motion assume the following forms.

$$H = p_1 p_2 + U, \quad F = -6w\frac{\partial H}{\partial v}p_1 + 6w\frac{\partial H}{\partial u}p_2 - K^{ijk}p_i p_j p_k, \quad q^1 = u, q^2 = v. \tag{6.43}$$

The vanishing of the Poisson bracket indicates that

$$[\mathbf{L}, g] = 0, \tag{6.44}$$

$$[\mathbf{B}, g] = 6\mathbf{L}dU, \tag{6.45}$$

$$\mathbf{B}(U) = 0. \tag{6.46}$$

The solution to (6.44) is given in section 3 of Chapter 4, that is (4.63)

With the above ansatz, in 1935 Drach [17] derived ten potentials [85].

- Case 1

$$\begin{aligned} U &= \frac{\alpha}{uv} + \beta u^{r_1}v^{r_2} + \gamma u^{r_2}v^{r_1}, \quad \text{where} \quad r_j^2 + 3r_j + 3 = 0, \\ P &= (up_1 - vp_2)^3, \quad w = u^2v^2/2 \end{aligned} \tag{6.47}$$

- Case 2

$$\begin{aligned} U &= \frac{\alpha}{\sqrt{uv}} + \frac{\beta}{(v-\mu v)^2} + \frac{\gamma(v+\mu u)}{\sqrt{uv}(v-\mu u)^2}, \\ P &= 3(up_1 - vp_2)^2(p_1 + \mu p_2), \quad w = uv(v - \mu u), \end{aligned} \tag{6.48}$$

- Case 3

$$\begin{aligned} U &= \alpha uv + \frac{\beta}{(v-au)^2} + \frac{\gamma}{(v+au)^2}, \\ P &= 3(up_1 - vp_2)^2(p_1^2 - a^2p_2^2), \quad w = (v^2 - a^2u^2)/2, \end{aligned} \tag{6.49}$$

- Case 4

$$\begin{aligned} U &= \frac{\alpha}{\sqrt{v(u-a)}} + \frac{\beta}{\sqrt{v(u+a)}} + \frac{\gamma u}{\sqrt{u^2 - v^2}}, \\ P &= 3p_2[(up_1 - vp_2)^2 - a^2p_1^2], \quad w = -v(u^2 - a^2), \end{aligned} \tag{6.50}$$

- Case 5

$$\begin{aligned} U &= \frac{\alpha}{\sqrt{uv}} + \frac{\beta}{\sqrt{u}} + \frac{\gamma}{\sqrt{v}}, \\ P &= 3p_1p_2(up_1 - vp_2), \quad w = -2uv, \end{aligned} \tag{6.51}$$

- Case 6

$$U = \alpha uv + \beta v\frac{2u^2+c}{\sqrt{u^2+c}} + \frac{\gamma u}{\sqrt{u^2+c}},$$
$$P = 3p_2^2(up_1 - vp_2), \quad w = (u^2 - a^2)/2, \tag{6.52}$$

- Case 7

$$U = \frac{\alpha}{(v+3mu)^2} + \beta(v-3mu) + \gamma(v-mu)(v-9mu),$$
$$P = (up_1 + 3mvp_2)^2(p_1 - 3mp_2), \quad w = -m(v+3mu), \tag{6.53}$$

- Case 8

$$U = (v+mu/3)^{-2/3}[\alpha + \beta(v-mu/3) + \gamma(v^2 - 14muv/3 + m^2u^2/9)],$$
$$P = (p_1 - mp_2/3)(p_1^2 + 10mp_1p_2/3 + m^2p_2^2/9), \quad w = -m(v+mu/3), \tag{6.54}$$

- Case 9

$$U = \alpha v^{-1/2} + \beta uv^{-1/2} + \gamma u,$$
$$P = 3p_1^2p_2, \quad w = -v, \tag{6.55}$$

- Case 10

$$U = \alpha(v - \rho u/3) + \beta u^{-1/2} + \gamma u^{-1/2}(v - \rho u),$$
$$P = 3p_1^2p_2 + \rho p_2^3, \quad w = u. \tag{6.56}$$

In the following we show that the Killing tensors that define the leading terms of the first integrals above are biometrically different. For the corresponding coefficients of those Killing tensors, see Table 6.4, we plan to employ the invariant theory of Killing tensors of valence three defined on the Minkowski plane, including the techniques based on invariants and covariants as well as the analysis based on the dimensions of the corresponding orbits.

We first note that in view of the ten Killing tensors (refer to (??)) corresponding to the ten cases listed by Drach, the action of the isometry group $I(\mathbb{E}_1^2)$ on the vector space $\mathcal{K}_0^3(\mathbb{E}_1^2)$ is not semi-regular everywhere. Indeed, using the result of Proposition 3.1, it is easy to check the rank of the following matrix (with respect to the basis $\partial_{a_i}, i = 1, \ldots, 10$) resulting from the three infinitesimal generators (4.64) at each point,

$$\begin{pmatrix} -3a_5 & 0 & a_9 & a_6 & 2a_7 & 0 & a_{10} & 0 & 2a_8 & 0 \\ 0 & -3a_6 & a_5 & -a_9 & 0 & -2a_8 & 0 & a_{10} & 2a_7 & 0 \\ -3a_1 & 3a_2 & -a_3 & a_4 & -2a_5 & 2a_6 & -a_7 & a_8 & 0 & 0 \end{pmatrix}. \tag{6.57}$$

As a result, we obtain the rank of each orbits containing the corresponding Killing tensors above respectively. See Table 6.5 for more details. Thus, for example, one cannot simply use the fundamental invariants (4.72) to solve the classification problem. A more elaborate scheme is required.

We begin by considering Case 1 (refer to Table 6.5). Since a_{10} is an invariant of the full isometry group $I(\mathbb{E}_1^2)$, it can be used to distinguish Case 1 from the rest. Observe next that Case 1 and Case 5 are the only cases where the corresponding orbits of group action are two-dimensional, this immediately distinguishes Case 5 from the remaining eight cases. Indeed, the invariant a_{10}) distinguishes the Killing tensor of Case 5 from that of Case 1 and the fact that its orbit is two-dimensional shows that it is isometrically distinct from the rest. Since Cases 2,3,4 and 6 belong to three-dimensional orbits, while Case 7-10 have one-dimensional orbits, these two groups are immediately distinguished. It remains to distinguish the four cases in the first group and the four cases in the second group.

For Cases 2,3,4 and 6, one can use invariant $\Delta_2 = a_9a_{10} - 2a_7a_8$ to distinguish Case 2 from the other three, since it does not vanish for Case 2 while it does for Case 3, 4 and 6. Now consider the invariant submanifold defined by

$$S = \{a_7 = a_8 = a_{10} = 0\}.$$

One employs the method of infinitesimal generators to find one of the reduced invariants is

$$\Delta_8 = a_5a_6, \tag{6.58}$$

which can be used to distinguish Case 3 from Case 4 and 6. To distinguish Case 4 from Case 6, one computes the fundamental covariants (see (4.73)).

$$\text{Case 4.}\quad \Delta_3^C = 2uv(a^2 - u^2),\ \Delta_3^C = 3v^2(1 + u^2),\ \Delta_i^C = 0, i \neq 3, 7,$$

$$\text{Case 6.}\quad \Delta_i^C = 0, i = 1, \dots, 9.$$

This distinguishes Case 4 from Case 6.

Finally, for the last 4 cases, note the only non-vanishing generator is $\mathbf{V}$, which means that in the subspace where these four Killing tensors are located (i.e., the subspace characterized by the condition $a_i = 0$, $i \neq 1, 2, 3, 4$) the subgroup of translation is in fact an isotropy subgroup (see (4.66) in Chapter 4 for the transformation laws) at each point. The

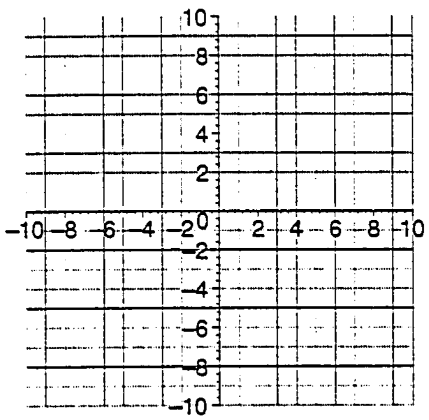

Figure 6.1: Euclidean plane - Cartesian coordinate web

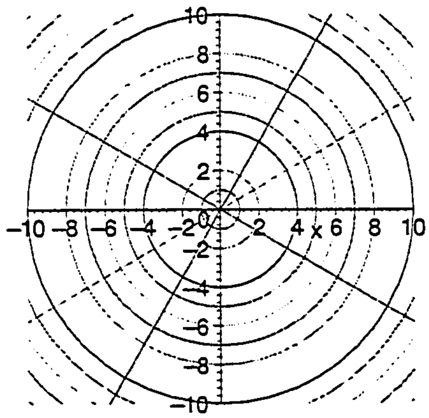

Figure 6.2: Euclidean plane - Polar coordinate web

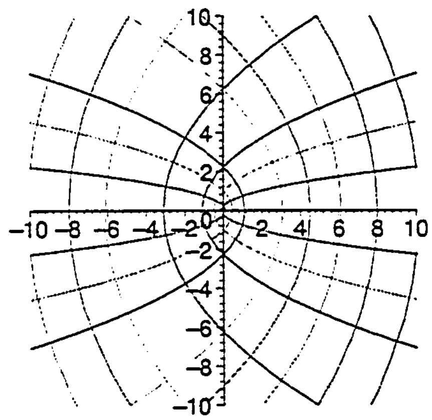

Figure 6.3: Euclidean plane - Parabolic coordinate web

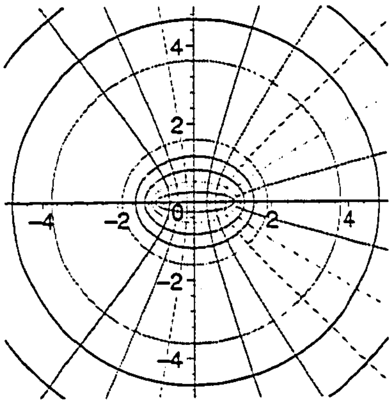

Figure 6.4: Euclidean plane - Elliptic-hyperbolic coordinate web

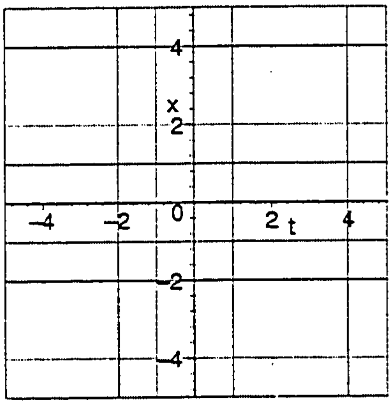

Figure 6.5: Minkowski plane - Orthogonal coordinate web 1

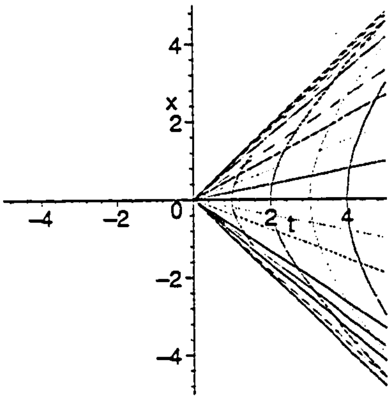

Figure 6.6: Minkowski plane - Orthogonal coordinate web 2

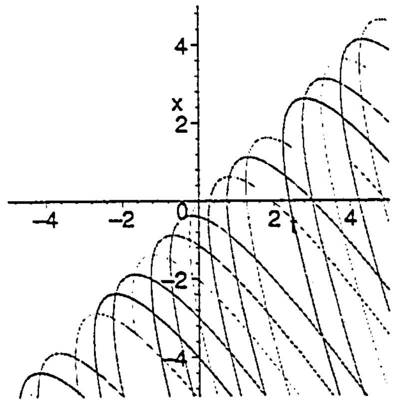

Figure 6.7: Minkowski plane - Orthogonal coordinate web 3

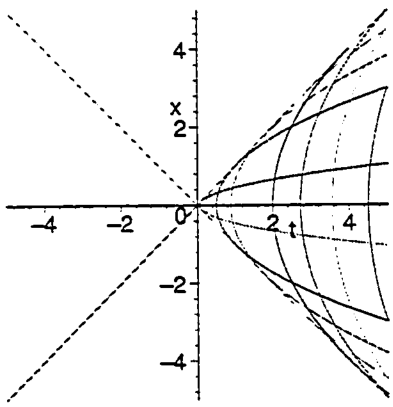

Figure 6.8: Minkowski plane - Orthogonal coordinate web 4

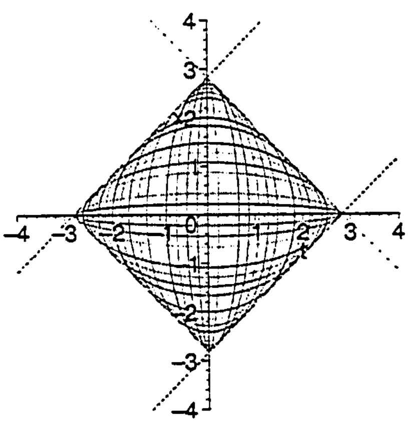

Figure 6.9: Minkowski plane - Orthogonal coordinate web 5

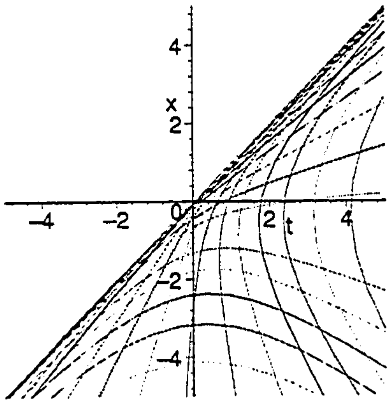

Figure 6.10: Minkowski plane - Orthogonal coordinate web 6

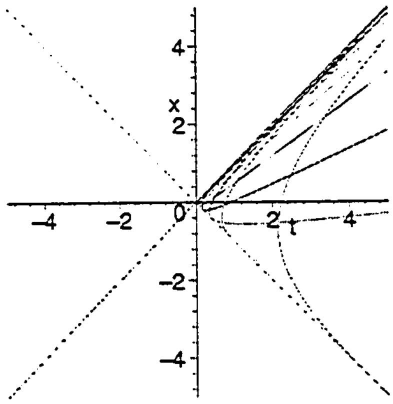

Figure 6.11: Minkowski plane - Orthogonal coordinate web 7

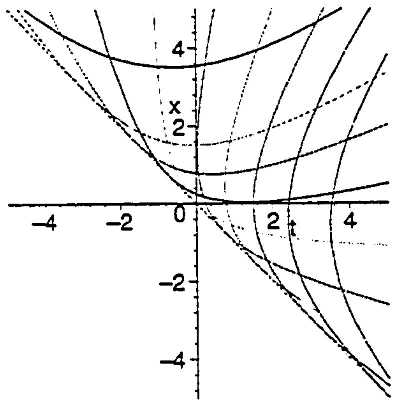

Figure 6.12: Minkowski plane - Orthogonal coordinate web 8

Equivalence class	$\mathcal{I}_1$	$\mathcal{I}_3$	Orthogonal web
EC1	0	0	Cartesian
EC2	0	$\neq 0$	Polar
EC3	$\neq 0$	0	Parabolic
EC4	$\neq 0$	$\neq 0$	Elliptic-hyperbolic

Table 6.1: Orthogonal coordinate webs on $\mathbb{E}^2$ by invariants

Equivalence class	$\mathcal{C}_1$	$\mathcal{C}_2$	Orthogonal web
EC1	0	0	Cartesian
EC2	positive-definite	0	Polar
EC3	1	1	Parabolic
EC4	positive-definite	indefinite	Elliptic-hyperbolic

Table 6.2: Orthogonal coordinate webs on $\mathbb{E}^2$ by covariants

only transformations that matter are the rotations (4.70). This can easily distinguish Case 7-10 from each other. The classification is now complete.

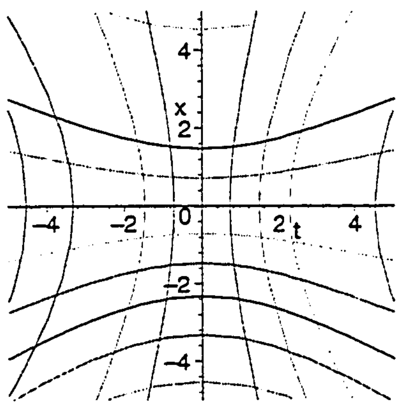

Figure 6.13: Minkowski plane - Orthogonal coordinate web 9

Equivalence class	$\mathcal{I}_1$	$\mathcal{I}_3$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{I}'_1$	$\mathcal{I}'_2$	$\mathcal{I}^*$
EC1	0	0	0	0	0	0	
EC2	0	$\neq 0$	indefinite	0			
EC3	0	0	0	0	0	$\neq 0$	
EC4	$\neq 0$	0	1	1			
EC5	$\neq 0$	$\neq 0$	indefinite	positive-definite			
EC6	$\neq 0$	$\neq 0$	indefinite	indefinite			$\neq 0$
EC7	0	$\neq 0$	indefinite	positive-definite			
EC8	$\neq 0$	$\neq 0$	indefinite	indefinite			0
EC9	$\neq 0$	$\neq 0$	indefinite	negative-definite			
EC10	$\neq 0$	$\neq 0$	indefinite	positive-definite			

Table 6.3: Orthogonal coordinate webs on $\mathbb{E}_1^2$ by invariants and covariants

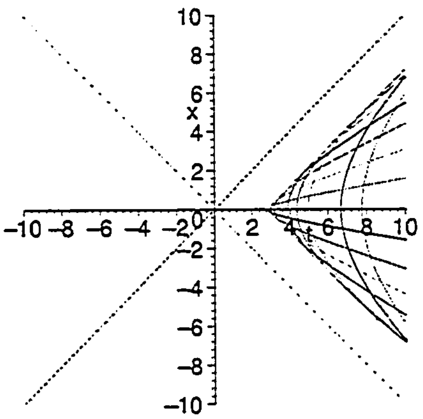

Figure 6.14: Minkowski plane - Orthogonal coordinate web 10

Case #	Killing tensor	Non-vanishing coefficients
Case 1	$(up_1 - vp_2)^3$	$a_{10} = 1$
Case 2	$3(up_1 - vp_2)^2(p_1 + \mu p_2)$	$a_7 = 1, a_8 = -\mu$
Case 3	$3(up_1 - vp_2)^2(p_1^2 - a^2p_2^2)$	$a_5 = 1, a_6 = a^2$
Case 4	$3p_2[(up_1 - vp_2)^2 - a^2p_1^2]$	$a_3 = a^2, a_8 = -1$
Case 5	$3p_1p_2(up_1 - vp_2)$	$a_9 = -1$
Case 6	$3p_2^2(up_1 - vp_2)$	$a_6 = -1$
Case 7	$(up_1 + 3mvp_2)^2(p_1 - 3mp_2)$	$a_1 = -1, \quad a_2 = 27m^3,$ $a_3 = -3m, \quad a_4 = 9m^2$
Case 8	$(p_1 - \frac{m}{3}p_2)(p_1^2 + \frac{10}{3}mp_1p_2 + \frac{m^2}{9}p_2^2)$	$a_1 = -1, \quad a_2 = \frac{m^3}{27},$ $a_3 = -m, \quad a_4 = m^2$
Case 9	$3p_1^2p_2$	$a_3 = -1$
Case 10	$3p_1^2p_2 + \rho p_2^3$	$a_2 = -\rho, a_4 = -1$

Table 6.4: Drach's Killing tensors cubic in the momenta

Case #	Rank of generators	Non-vanishing generators
Case 1	2	$\mathrm{U}_1, \mathrm{U}_2$
Case 2	3	$\mathrm{U}_1, \mathrm{U}_2, \mathrm{V}$
Case 3	3	$\mathrm{U}_1, \mathrm{U}_2, \mathrm{V}$
Case 4	3	$\mathrm{U}_1, \mathrm{U}_2, \mathrm{V}$
Case 5	2	$\mathrm{U}_1, \mathrm{U}_2$
Case 6	3	$\mathrm{U}_1, \mathrm{U}_2, \mathrm{V}$
Case 7	1	V
Case 8	1	V
Case 9	1	V
Case 10	1	V

Table 6.5: Rank analysis of the infinitesimal generators

Chapter 7

Conclusions

In this thesis we have furthered the development of the invariant theory of Killing tensors (ITKT) defined on pseudo-Riemannian spaces of constant curvature which originated in Winternitz and Friš [87] and then systematically developed by McLenaghan *et al* [13, 34, 36, 51, 52, 53, 54, 55, 56, 57, 88, 79, 80, 81, 89, 90].

The results presented in the thesis have already been used by other researchers working in the area. Thus, the new concepts of a covariant of Killing tensors have been employed in Horwood *et al* [35] to classify orthogonal coordinate webs in the Minkowski space (see also [36]). In addition, the concept of a joint invariant has been used by Adlam [1] in the study of superintegrable Hamiltonian systems. Another significant and novel contribution to the development of ITKT is the implementation of the inductive method of moving frames developed by Kogan [50] in the theory of differential invariants. Using this version of the moving frames method we have managed to solve a number of problems that would have been difficult to solve with the aid of other methods.

We have also have found new solutions to the problems of group invariant classifications of the orthogonal coordinate webs defined on the Euclidean and Minkowski planes by making use of covariants.

An analogue of the 1856 Lemma of Cayley is formulated and solved here by making use of the method based on the Lie derivative deformations of Killing tensors introduced in [52]. This result makes it possible to determine the complete sets of fundamental invariants and covariants of Killing tensors defined on the Minkowski plane via the method of infinitesimal generators. The inductive method of moving frames has then been used to solve the problem of finding sets of fundamental invariants and covariants for vector spaces of Killing tensors of arbitrary valences defined on the Minkowski plane.

These results have been employed to show that the ten integrable cases of Drach [17] are in fact distinct.

Our next goal is to determine whether or not the Drach list is exhaustive. If not, that is there are other integrable systems of the Drach type, we wish to determine whether or not they are superintegrable.

We notice that recently much work has been done to develop further the theory of superintegrable Hamiltonian systems. See, for example, [39, 40, 41]. These papers lay the ground work for a structure and classification theory of 2D or 3D second-order superintegrable systems, both classical and quantum defined on conformally flat spaces. In the latest paper [41] it is proven that there exists a standard structure for the above systems, based on the algebra of 3×3 symmetric matrices, and that the quadratic algebra always closes at order six. These ideas and others might work together with the invariant theory of Killing tensors to answer the question of whether or not the Drach list is exhaustive.

Another important and natural development of our results presented in the thesis will be an extension to other pseudo-Riemannian spaces of constant (non-zero) curvature. The work in this direction is underway.

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