Scale disparity and spectral transfer in anisotropic numerical turbulence

Ye Zhou
Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, Virginia 23681

P. K. Yeung
School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332

James G. Brasseur
Department of Mechanical Engineering, The Pennsylvania State University, University Park, Pennsylvania 16802
(Received 17 January 1995; revised manuscript received 19 May 1995)

To study the effects of cancellations within long-range interactions on isotropy at small scales, we calculate explicitly the degree of cancellation in long-range, or "distant," triadic interactions in the simulations of Yeung and Brasseur [Phys. Fluids A **3**, 884 (1991)] and Yeung, Brasseur, and Wang [J. Fluid Mech. **283**, 43 (1995)] using the single scale disparity parameter s developed by Zhou [Phys. Fluids A **5**, 1092 (1993); **5**, 2511 (1993)]. In the simulations initially isotropic turbulence was subjected to coherent anisotropic forcing at the large scales and the smallest scales were found to become anisotropic as a consequence of direct large-small scale couplings and then to return towards isotropy. We verify here that the most nonlocal interactions do not cancel out under summation, that the observed small-scale anisotropy is indeed a direct result of the distant triadic group, and that the reduction of anisotropy at later times follows from the influences towards isotropy of more local energy-cascading triadic interactions. We find that as the scale separation s increases beyond about 10, the net energy transfer to or from high-wave-number shells within the distant triadic group goes asymptotically to zero, while the long-range anisotropic influences *increase* monotonically, indicating that long-range dynamics persists to larger scale separations and hence higher Reynolds numbers.

PACS number(s): 47.10.+g, 47.27.Gs

PRELIMINARIES

Underlying the high Reynolds number Kolmogorov similarity theory [1] of 1941 is the implication that interactions among motions at different length scales are statistically dominated by interactions that scale on a single local length scale r, so long as r is much smaller than the integral scale l. If this is the case, then (a) the net energy transfer from motions surrounding scale $r \ll l$ would not be directly influenced by integral-scale motions and (b) the local structure of the motions at scales $r \ll l$ would be isotropic. Correspondingly, in Fourier space the net *energy* transfer to or from a spectral shell surrounding the inverse scale $k \sim 1/r$ would not be directly influenced by large-scale motions, and when $k \gg 1/l$, the *distributions* of the energy and phase of the Fourier modes in shell k would be independent of the large-scale structure.

Full knowledge of the velocity field in direct numerical simulations of homogeneous turbulence allows the calculation of contributions to the total energy transfer between different scales from predefined classes of the nonlinear triadic interactions in Fourier space [2–6]. In the inertial range, Zhou [5,6] found that for stationary isotropic turbulence, consistent with the classical Kolmogorov hypotheses, the *net* energy flux is local (occurring between similar scale sizes), but is dominated by "local-to-nonlocal" interactions among scales separated by a decade or less (with the strongest contribution from interactions separated in scale by approximately 1.8–5). Consistent with Zhou [5,6], Brasseur and Wei [7] also found that the net energy transfer within isolated "chains" of triads are dominated by the local-to-nonlocal triadic group. Moreover, it was found that the distant triadic group tends to redistribute energy among Fourier modes *within* spectral shells in a manner directly related to the structure of the more energetic large-scale motions. Brasseur and Wei [7] argued, therefore, that, *in principle*, direct long-range inertial couplings can lead to departures from local isotropy at the smallest dynamical scales.

To study the direct influence of the large scales on the distributions of energy and phase within motions surrounding inverse scale k far removed from $1/l$, Yeung and Brasseur [3] (YB) and Yeung, Brasseur, and Wang [4] (YBW) performed numerical experiments in which fully developed isotropic turbulence was subjected to sustained anisotropic forcing in the energy-containing wave-number range. In Fourier space nearly all the forcing energy is added to two pairs of Fourier modes with wave vectors $\mathbf{k_F}=(\pm 2, \pm 2, 0)$ with magnitude $k_F = 2\sqrt{2}$ and the Fourier velocity coefficients $\hat{\mathbf{u}}(\mathbf{k_F})$ are in the k_x-k_y plane. After forcing for about one initial eddy-turnover time (T_E), the smallest scales rapidly become anisotropic, followed by the next smallest scales and so on. However, the structure of this small-scale anisotropy is such that at the high wave numbers it is u_3 that has the most energy, in contrast to u_1 and u_2 in the forced low wave-number modes. The highest wave-number shell attains maximum anisotropy in the component spectra at $1.915T_E$ after initiating forcing. The component anisot-

1063-651X/96/53(1)/1261(4)/$06.00

ropy subsequently decreases and appears to disappear when $t^* = t/T_E \gtrsim 3.5$, even though forcing continues and the large scales are still significantly anisotropic [8].

The triadic energy transfer is defined [2] by

$$T(k|p,q) = \sum_{\mathbf{k}'\in \text{shell } k} \sum_{\substack{\mathbf{p}'\in \text{shell } p \\ \mathbf{q}'=\mathbf{k}'-\mathbf{p}'\in \text{shell } q}} T(\mathbf{k}'|\mathbf{p}',\mathbf{q}') , \quad (1)$$

where

$$T(\mathbf{k}'|\mathbf{p}',\mathbf{q}') = \tfrac{1}{2}\mathrm{Im}[\hat{u}_i(-\mathbf{k}')P_{ilm}(\mathbf{k}')\hat{u}_l(\mathbf{p}')\hat{u}_m(\mathbf{q}')] \quad (2)$$

is the energy transfer into or out of mode $\mathbf{k}'$ due to a single triadic interaction with modes $\mathbf{p}',\mathbf{q}'$. The component energy transfers $T_{\alpha\alpha}(k|p,q)$ (no sum on α) are obtained from Eq. (1) by replacing i with α in (2). $T(k,p,q)$ and $T_{\alpha\alpha}(k|p,q)$ are the net energy transfers to or from all Fourier modes in a wave-number shell centered on radius k due to all interactions within triads $\mathbf{k}'=\mathbf{p}'+\mathbf{q}'$ with one leg in a shell centered on radius p and the other leg in a shell of radius q, where the shells are of specified thickness. Here $\hat{u}_i(\mathbf{k}')$ is the Fourier velocity coefficient for the wave vector $\mathbf{k}'$ in shell k and the projection tensor P_{ilm} is given by $P_{ilm}(\mathbf{k}) = k_m(\delta_{il} - k_i k_l/k^2) + k_l(\delta_{im} - k_i k_m/k^2)$. The summation in (2) is over repeated Roman subscripts and in (1) over all triads $\mathbf{k}'=\mathbf{p}'+\mathbf{q}'$, where $\mathbf{k}'$, $\mathbf{p}'$, and $\mathbf{q}'$ lie in shells k, p, and q, respectively.

The distant triadic group is given in the limit $k/p \sim k/q \to \infty$. Analyzing $T(k|p,q)$ and $T_{\alpha\alpha}(k|p,q)$, YB [3] and YBW [4] found that the marginally distant triadic interactions coupling the most disparate scales in the simulation produced, over time, anisotropic redistributions of the energy and phase in high wave-number spectral shells and consequently an anisotropic small-scale structure.

Although $T(k|p,q)$ and $T_{\alpha\alpha}(k|p,q)$ are fundamental building blocks in the energy transfer process, Waleffe [9] and Zhou [5,6] have pointed out that identifying the effects of the local, nonlocal, or distant triadic groups through $T(k|p,q)$ and $T_{\alpha\alpha}(k|p,q)$ does not provide the net contributions from these triadic groups. In particular, they observed that the net contributions of *all* highly nonlocal or distant triadic groups to the net energy transfer $T(k)$ to or from high-wave-number shells k nearly cancel in isotropic turbulence. However, the extent to which distant triadic interactions cancel out was not *directly* analyzed by YB [3] and YBW [4].

To explicitly take into account cancellations in net energy transfer to or from shell k resulting from all triadic interactions within a given triadic group based on scale disparity alone, Zhou [5,6] applied the scale disparity parameter

$$s \equiv \frac{\max(k',p',q')}{\min(k',p',q')} , \quad (3)$$

which is the ratio of the longest to the shortest leg in a triad, giving a direct measure of the disparity of the interacting scales. We introduce in this paper the net *component energy transfer*

$$T_{\alpha\alpha}(k,s) = \sum_{\mathbf{k}'\in \text{shell } k} \sum_{\substack{\mathbf{p}',\mathbf{q}'|s \\ \mathbf{q}'=\mathbf{k}'-\mathbf{p}'}} T_{\alpha\alpha}(\mathbf{k}'|\mathbf{p}',\mathbf{q}') , \quad (4)$$

which is the partial sum of all triadic interactions involving wave vectors $\mathbf{k}'$ in shell k and all other wave vectors $\mathbf{p}'$ and $\mathbf{q}'=\mathbf{k}'-\mathbf{p}'$, where the scale disparity s falls into a prescribed range. Contributions of triadic interactions at a given scale disparity parameter s to the net energy transfer at a scale k are given by $T(k,s) \equiv \frac{1}{2}\sum_{\alpha=1}^{3} T_{\alpha\alpha}(k,s)$. Because $T_{\alpha\alpha}(k,s)$ and $T(k,s)$ are partial sums over all triadic interactions with given scale disparity s, these variables provide a more direct measure of scale disparity in the net component energy transfer within triadic interactions [6,7]. Note that $T(k,s)$ is *half* the trace of $T_{\alpha\beta}(k,s)$, and $T_{\alpha\alpha}(k) \equiv \sum_{p,q} T_{\alpha\alpha}(k|p,q) \equiv \sum_s T_{\alpha\alpha}(k,s)$.

RESULTS

We focus on the development of small-scale anisotropy from large-scale forcing by reanalyzing the numerical database of YBW [4], but using the quantities $T_{\alpha\alpha}(k,s)$ and $T(k,s)$ at the *later* time $t^*=3.83$ when the energy spectra and transfer *appear* to have returned to isotropy [4]. (Results for $t^*=1.915$ are given in Ref. [10].) We emphasize the distinction between net *energy transfer* to or from a shell k within the distant triadic group and the *redistribution of energy* within the shell k due to distant interactions, which can occur in the absence of net energy transfer into or out of the shell [7].

Consider the high-wave-number shell $k=50$, separated in scale from the forced modes by a factor of about 18 and influenced by highly nonlocal (marginally distant [7]) triads. In Fig. 1 we plot the net energy transfer $T(k,s)$ for this shell as a function only of scale separation s at the

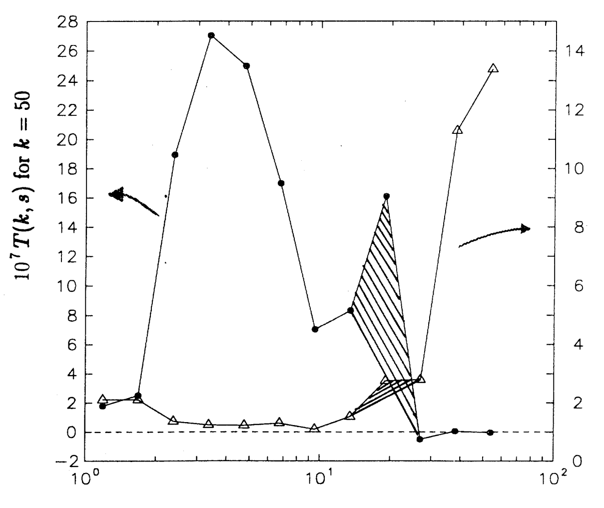

FIG. 1. Net energy transfer $T(k,s)$ (●) and anisotropy in component energy transfer $|T_{\alpha\alpha}|_{\max}/|T_{\alpha\alpha}|_{\min}$ (△), for the shell $k=50$ within different triadic groups with scale disparity s in half octaves, at the later time $t^*=3.83$. The only s band that contains the forced triads is shown in the cross-hatched areas for both quantities plotted.

time of reduced anisotropy ($t^*=3.83$). To improve resolution, s is divided into half octave bands ($2^{N/2}$, $N=0,12$). Except for the only s band, which contains the forced triads (shown in the cross-hatched areas), the net energy transfer is into shell k and is dominated by local-to-nonlocal interactions up to scale disparity roughly 10–15 (consistent with Brasseur and Wei [7]) and with the strongest contributions from the range $s\sim 2-8$ (consistent with Zhou [5,6]). The modest height of the spike in Fig. 1 indicates that the forced triadic group contributes only a relatively small proportion of the net energy flux into the high-wave-number shell.

We now introduce the anisotropy in $T_{\alpha\alpha}(k,s)$ as a *marker* of anisotropy development in the high-wave-number shell $k=50$, which results from the process of both energy and phase redistribution in the shell by long-range interactions (in contrast to classical cascade-type arguments). For any s band we quantify anisotropy in component energy transfer by the ratio of maximum to minimum energy transfers among the components, in absolute value (unity if the transfer is isotropic). In contrast to energy transfer, Fig. 1 also shows the anisotropy in component energy transfer as a function of scale separation s for shell $k=50$ at the same time as $T(k,S)$. Because of the forcing, the s band containing the forced triads (cross-hatched areas) stands out as a spike in the anisotropy of component energy transfer in shell k. Around this spike, anisotropy begins to increase at scale separation $s\approx 10$ and then rapidly increases at higher s. It is particularly interesting that the increase in anisotropy begins at roughly the same scale separation where the dominance of cascading local-to-nonlocal triads in $T(k,s)$ ends. We conclude that the influences towards anisotropy of the large scales on smaller scales are most strongly felt within the distant triadic group and that the more distant the triadic group, the stronger is their influence towards anisotropy. We further conclude that, with cancellation among triads taken into account, the strongest long-range influences towards anisotropy (redistribution of energy and phase in high-k shells) occur in the absence of direct energy transfer.

As might be expected, forcing enhances the influence towards anisotropy of the highly nonlocal distant group within the forced triads. For this long-range effect to persist to higher Reynolds numbers, net *energy transfer* within the forced triads should *decrease* with scale separation s while the *anisotropy* in component energy transfer should *increase* with s. This is seen to be the case in Figs. 2 and 3 below. The range of s for those triadic interactions with the forced shell are given by s_F, determined from

$$\frac{(k-\frac{1}{2})}{k_F}\le s_F\le\frac{(k+\frac{1}{2})+k_F}{k_F}\ . \tag{5}$$

Note that the maximum value of s_F occurs when the wave vector $\mathbf{k}$ in shell k (of unit thickness $\Delta k=1$) is the intermediate mode in a triadic interaction, whereas the minimum s_F occurs when $\mathbf{k}$ is the highest-wave-number mode in a triad.

In Fig. 2 we plot $T(k,s)$ against s only for the s_F bands

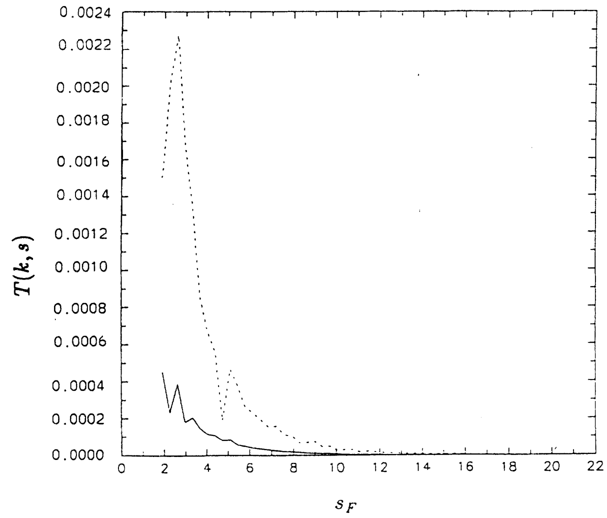

FIG. 2. Net energy transfer $T(k,s)$ within the *forced* triadic groups as a function of scale disparity s_F, at the time of peak small scale anisotropy $t^*=1.915$ (solid line) and at the later time $t^*=3.83$ (dashed line).

that contain forced triads, using Eq. (5) for each k shell. Consequently, each s band plotted corresponds to a different k shell, and because the k shells are narrow ($\Delta k=1$), so are the s_F bands shown in the figure. Note that, whereas the magnitude of $T(k,s)$ is higher at the later time due to energy input from forcing, at both $t^*=1.915$ and 3.83 the net energy transfer within the forced triadic groups decreases rapidly with increasing scale separation, approaching zero asymptotically at large s.

By contrast, Fig. 3 shows that the anisotropy within the forced triadic group increases within scale separation. As in Fig. 2, we plot the anisotropy measure of Fig. 1 only for the forced s bands. Because the s_F bands are narrow, the variation of this ratio with s_F is very noisy,

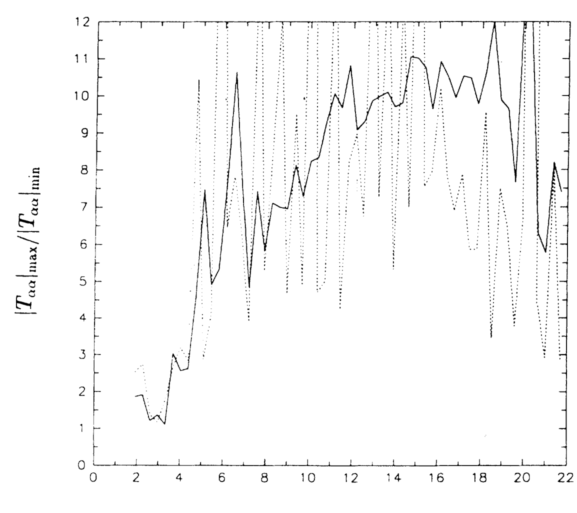

FIG. 3. Same as Fig. 2, but for anisotropy in component energy transfer $|T_{\alpha\alpha}|_{\max}/|T_{\alpha\alpha}|_{\min}$.

particularly at $t^* = 3.83$ when the influence of the distant triadic group was weakened substantially. Nevertheless, it is apparent that the anisotropy in the component energy transfer increases overall with s_F even as the net energy transfer $T(k,s)$ decreases. This result indicates that the influence towards anisotropy of the forced triads on the small scales remains strong within the distant triadic group as the net energy transfer decreases asymptotically to zero with increasing scale separation s. Furthermore, although the net anisotropy in the component energy transfer decreases with time [4,10] the influence towards anisotropy of the distant group remains strong at later times. These results suggest that the influences towards anisotropy of the distant triadic group do not diminish at higher scale separations (suggesting higher Reynolds numbers), even as the scale separation becomes sufficiently high to block direct energy transfer between the forced large scales and the smallest scales.

To show the net effects of cancellation more completely, the cumulative contributions of the dominant nonlocal classes of $T(k,s)$ are shown for all three velocity components in Fig. 4 at time $t^* = 3.83$, for the range $40 \leq k < 60$. Specifically, the *complete* component energy transfers $T_{\alpha\alpha}(k)$ are compared with net contributions from triads with scale separation $s \sim 8-32$. Note that whereas the complete energy transfer spectrum strongly favors the z component at $t^* = 1.915$ [10], at the later time $t^* = 3.83$ the complete energy transfer is isotropically distributed within components. The more nonlocal triadic interactions, on the other hand, redistribute energy anisotropically at both times, providing u_3 with more energy than u_1 and u_2 at the small scales. A reduction of anisotropy at $t^* > 1.915$ is a consequence of the influences towards anisotropy of more local energy-cascading triadic interactions [4].

In summary, the development of small-scale anisotropy in response to large-scale forcing previously reported by YB [3] and YBW [4] has been studied by an analysis of triadic energy transfer using the $T(k,s)$ formalism introduced by Zhou [5,6]. This analysis shows explicitly that, whereas highly nonlocal and distant triadic interactions do not contribute to the net energy transfer and the forward cascade (as argued by Zhou [5,6] and Brasseur and Wei [7]), their *dynamical* effects on the small scales do not cancel out completely. These interactions, in fact, lead directly to the anisotropic redistribution of the small-scale energy and phase in response to large-scale anisotropic forcing. Furthermore, we found that the influence towards anisotropy of the distant triadic group (under forcing) increases with scale disparity s, suggesting that the dynamical influences of long-range interactions would persist at higher Reynolds numbers.

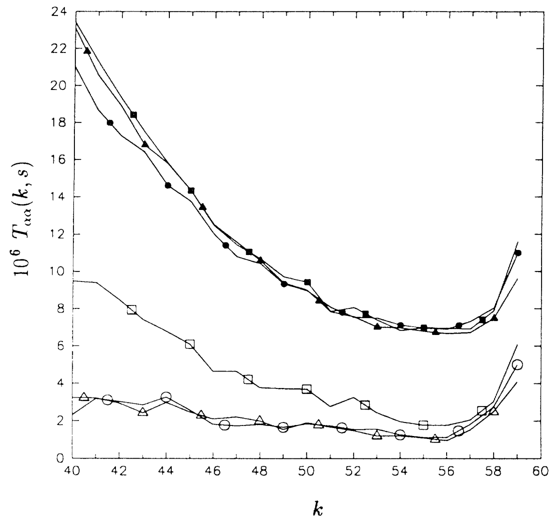

FIG. 4. Component transfers $T_{11}(k,s)$ (△), $T_{22}(k,s)$ (○), $T_{33}(k,s)$ (□), summed $8 \leq s < 32$ (open symbols) in the range $40 \leq k < 60$ and compared with the total transfers $[T_{11}(k), T_{22}(k), T_{33}(k)]$ (closed symbols) at $t^* = 3.83$.

ACKNOWLEDGMENTS

This research was partly supported by NASA under Contract No. NAS1-19480 while Y.Z. was at ICASE. P.K.Y. is supported by NSF Grant No. CTS 93-07973. J.G.B. was supported by AFOSR/URI Grant No. AFOSR-90-0113, ONR Grant No. N00014-92-J-1417, and currently by ARO/URI Grant No. DAAL03-92-G-0117.

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 301 (1941).
[2] J. A. Domaradzki and R. S. Rogallo, Phys. Fluids A **2**, 413 (1990).
[3] P. K. Yeung and J. G. Brasseur, Phys. Fluids A **3**, 884 (1991).
[4] P. K. Yeung, J. G. Brasseur, and Q. Wang, J. Fluid Mech. **283**, 43 (1995).
[5] Y. Zhou, Phys. Fluids A **5**, 1092 (1993).
[6] Y. Zhou, Phys. Fluids A **5**, 2511 (1993).
[7] J. G. Brasseur and C.-H. Wei, Phys. Fluids **6**, 842 (1994).
[8] Reference [4] showed that, although the level of anisotropy decreases along with the relative energy level of the forced modes, at $t^* = 3.83$ third-order moments and three-dimensional visualizations reveal that the small scales are still anisotropic.
[9] F. Waleffe, Phys. Fluids A **4**, 350 (1992).
[10] Y. Zhou, P. K. Yeung, and J. G. Brasseur, NASA-ICASE Report No. 94-64, 1994 (unpublished).