ON HÖLDER CONTINUITY OF WEAK SOLUTIONS TO DEGENERATE LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

by

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Submitted in partial fulfillment of the requirements
for the degree of Master of Science

at

Dalhousie University
Halifax, Nova Scotia
August 2013

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Abstract

For degenerate elliptic partial differential equations, it is often desirable to show that a weak solution is smooth. The first and most difficult step in this process is establishing local Hölder continuity. Sufficient conditions for establishing continuity have already been documented in [FP], [SW1], and [MRW], and their necessity in [R]. However, the complexity of the equations discussed in those works makes it difficult to understand the core structure of the arguments employed. Here, we present a harmonic-analytic method for establishing Hölder continuity of weak solutions in context of a simple linear equation

$$\operatorname{div}(Q\nabla u) = f$$

in a homogeneous space structure in order to showcase the form of the argument. Additionally, we correct an oversight in the adaptation of the John-Nirenberg inequality presented in [SW1], restricting it to a much smaller class of balls.

Acknowledgements

I would like to sincerely thank everyone who helped to make this thesis a reality. This extends to my partner Emily, and to my family and friends whose support I simply could not live without. I also feel I owe a great debt to the students and faculty at Dalhousie University and at Cape Breton University. In particular, I want to thank my supervisor Dr. Scott Rodney for the countless hours of lecture and discussion he has provided. Without his undying support and boundless enthusiasm, this would not have been possible.

Chapter 1

Introduction

1.1 Notation

In this thesis we attempt to follow the notational conventions of the modern literature. As an aid to the reader, we provide here a list of symbols which will not be defined in the body of the paper.

- $\mathbb{N}$: the set of natural numbers.
- $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$
- $\mathbb{Z}$: the ring of integers.
- $\mathbb{R}$: the field of real numbers.
- $\mathbb{R}^n := \prod_{i=1}^n \mathbb{R}$: Euclidean n-space.

For the remainder of the section, let $\Omega, E \subset \mathbb{R}^n$, let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, let μ be a measure on $\mathbb{R}^n$, let $f : \Omega \to \mathbb{R}$ be a function and let $\mathbf{f} = (f_1, \ldots, f_n)$ be an $\mathbb{R}^n$-valued vector field.

- log: the natural logarithm.
- $|x - y| := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ denotes the Euclidean distance between x and y in $\mathbb{R}^n$.
- $D(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$, the Euclidean ball centred at x of radius r in $\mathbb{R}^n$. Note that when working on subsets of $\mathbb{R}^n$, the definition is adapted accordingly.
- $\operatorname{dist}(x, E) := \inf_{y \in E} |x - y|$ is the Euclidean distance from x to the set E.
- $\partial\Omega$: the Euclidean boundary of Ω.

- $\overline{\Omega}$: the Euclidean closure of Ω.

- $E \Subset \Omega$ means $\overline{E} \subset \Omega$

- Given $i \in \{1, \ldots, n\}$, $D_i f := \dfrac{\partial f}{\partial x_i}$. Note that D_i may be taken in the sense of distributions where appropriate.

- Given $i \in \{1, \ldots, n\}, k \in \mathbb{N}$, $D_i^k f := \dfrac{\partial^k f}{\partial x_i^k}$.

- $\nabla f := (D_1 f, \ldots, D_n f)$ is the gradient of the function f. Note that ∇ may be taken in the sense of distributions where appropriate. We will most often refer to ∇f as the *derivative* of f.

- Given $k \in \mathbb{N}$, $\nabla^k(f) := (D_1^k f, \ldots, D_n^k f)$. We will most often refer to $\nabla^k f$ as the k^{th} derivative of f.

- $\mathrm{div}(\mathbf{f}) := \nabla \cdot \mathbf{f} = \sum_{i=1}^n D_i f_i$ is the divergence of the vector field $\mathbf{f}$.

- $\mathrm{supp} f := \{x \in \Omega : f \neq 0\}$ is the support of f.

- $C(\Omega)$: the set of all continuous functions on Ω.

- Given $k \in \mathbb{N}$, $C^k(\Omega)$ is the set of all functions on Ω with continuous k^{th} derivatives.

- $C^\infty(\Omega)$: the set of all functions on Ω with continuous k^{th} derivatives for every $k \in \mathbb{N}$.

- $C_0(\Omega)$: the set of all continuous functions on Ω with compact support.

- $Lip(\Omega)$: the set of all functions on Ω that are Lipschitz with respect to the Euclidean metric.

- $Lip_0(\Omega)$: the set of all Lipschitz functions with compact support in Ω.

- $Lip_{loc}(\Omega)$: the set of all functions on Ω that are Lipschitz when restricted to any compact subset of Ω.

- Given $p \in (0, \infty)$, $L^p(\Omega)$ is the space of functions $g : \Omega \to \mathbb{R}$ on Ω such that the norm $||g||_{L^p(\Omega),d\mu} = \left(\int_\Omega |g|^p d\mu\right)^{\frac{1}{p}}$ is finite, modulo the equivalence relation $g \sim h$ if and only if $||g - h||_{L^p(\Omega)} = 0$. Note that when μ is Lebesgue measure, we often omit the $d\mu$ on the norm and simply write $||g||_{L^p(\Omega)}$.

- $L^\infty(\Omega)$: the space of essentially bounded functions on Ω with respect to μ, with norm $||f||_{L^\infty(\Omega),d\mu} = \operatorname{ess\,sup}_{x\in\Omega} |f(x)|$.

- Given $p \in (0, \infty]$, $L^p_{loc}(\Omega)$ is the space of all functions $g : \Omega \to \mathbb{R}$ such that $||g||_{L^p(\Theta)} < \infty$ for all compact sets $\Theta \subset \Omega$.

- Given $p \in (0, \infty]$, $(L^p(\Omega))^n := \prod_{i=1}^n L^p(\Omega)$ is the n-dimensional L^p space with norm $||\mathbf{f}||_{(L^p(\Omega))^n,d\mu} = \left(\sum_{i=1}^n \int_\Omega |f_i|^p d\mu\right)^{\frac{1}{p}}$ if $p < \infty$ and $||\mathbf{f}||_{(L^\infty(\Omega))^n,d\mu} = \max ||f_i||_{L^\infty(\Omega),d\mu}$ if $p = \infty$.

- $|\Omega|$: the Lebesgue measure of the set Ω.

- $|\Omega|_\mu$: the μ-measure of the set Ω.

- Depending on the situation, we may write the average value of a function over Ω, given by $\frac{1}{|\Omega|_\mu} \int_\Omega f d\mu$ in one of two ways: f_Ω or $⨍_\Omega f d\mu$.

- $||f||_{L^p(\Omega),\overline{d\mu}} := \left(\frac{1}{|\Omega|_\mu} \int_\Omega |f|^p\right)^{\frac{1}{p}}$ is the normalized L^p norm of f.

Remark 1.1.1. *The bulk of this paper consists of proofs of complicated estimates, where the constants are changing frequently and are only important insofar as their dependencies on other quantities. We denote the dependance of a constant C on a quantity α by writing $C(\alpha)$. Moreover, the sheer number of such constants can be quite cumbersome if one attempts to keep track of all of them. As such, the constant C will be abused and allowed to change from line to line, and we will only keep track of particular constants if they must be used later. If the dependencies of C change from one line to the next, we will write that dependency explicitly.*

1.2 Motivation and an Historical Overview

The theorey of linear elliptic second order partial differential equations began with the study of the classical equation of Poisson:

$$\Delta u = \sum_{i=1}^{n} D_i^2 u = f \text{ in } \Omega \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. It is well known that if f is α-Hölder continuous for some $\alpha \in (0,1]$, then (1.1) has a unique solution in Ω

$$u(x) = \int_{\Omega} \Gamma(x-y) f(y) dy \tag{1.2}$$

where Γ is the fundamental solution of Laplace's equation

$$\Delta u = 0 \text{ in } \Omega \tag{1.3}$$

given by

$$\Gamma(x) = \begin{cases} \dfrac{1}{n(2-n)w_n}|x|^{2-n} & \text{if } n > 2 \\ \dfrac{1}{2\pi}\log|x| & \text{if } n = 2 \end{cases} \tag{1.4}$$

and where w_n is the volume of the unit ball in $\mathbb{R}^n$. To summarize, we have the following:

Theorem 1.2.1. *Let f be a bounded and locally α-Hölder continuous function on Ω for some $\alpha \in (0,1]$. Then if $u(x)$ is as in* (1.2)*, we have that $u \in C^2(\Omega)$, u is α-Hölder continuous on Ω and u is the unique solution to* (1.1) *in Ω.*

For a proof and more details, see chapter 4 of [GT].

Physical applications led to the study of generalized versions of Poisson's equation, which we now say are elliptic equations. Employing Einstein notation, these are equations of the form

$$Lu = D_j(a^{ij}(x)D_i u) + b_i(x)D_i u + c(x)u = f \text{ in } \Omega \tag{1.5}$$

where

- a^{ij}, b_i and c are α-Hölder continuous on $\overline{\Omega}$ for some $\alpha \in (0,1]$
- $a^{ij} = a^{ji}$

- there exist $\lambda > 0$ such that for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$.

Note that (1.1) is indeed a special case of (1.5) with $b_i = c = 0$ and $a^{ij} = \delta^{ij}$.

The generalization of Theorem 1.2.1 to the broad class of equations represented by (1.5) was published in 1937 by Leray and Schauder in their famous paper [LS]. We state the main result here without proof.

Theorem 1.2.2. *Let L be elliptic on Ω and let $c(x) \leq 0$. Let f and the coefficients of L be α-Hölder continuous on $\overline{\Omega}$ for some $\alpha \in (0,1]$. Then if $\varphi \in C^2(\overline{\Omega})$ and $\nabla^2\varphi$ is α-Hölder continuous on Ω, then the Dirichlet problem*

$$Lu = f \text{ in } \Omega$$

$$u = \varphi \text{ on } \partial\Omega$$

has a unique solution $u \in C^2(\overline{\Omega})$ and $\nabla^2 u$ is α-Hölder continuous on $\overline{\Omega}$.

With this result, the study of linear elliptic equations was thought to have been concluded. However, through the study of non-linear problems it became apparent that this was not the case. As it turns out, it is often necessary to understand linear equations whose coefficients are only assumed to be measurable functions in Ω. This is due to methods used to solve quasi-linear equations where we are required to solve linear elliptic equations whose coefficients depend on the weak solution and hence cannot be assumed to be smooth. Since the proofs of Theorems 1.2.1 and 1.2.2 rely heavily on the Hölder continuity of the coefficients, new methods were needed.

In 1957, De Giorgi published [D], where he proved a Harnack inequality and a Hölder estimate for weak solutions to homogeneous linear elliptic equations with rough coefficients in divergence form

$$\text{div}(A\nabla u) = 0 \text{ in } \Omega$$

where $A = A(x) = [a^{ij}(x)]_{1\leq i,j\leq n}$ is a positive-definite bounded symmetric matrix. This result was reproved by Moser in 1960 in [M]. The new method employed in [M], which has since been coined Moser iteration, lent itself to generalization more easily than did the original method used by De Giorgi.

In this thesis, we present the Moser method, adapted for establishing Hölder continuity of weak solutions to *degenerate* linear elliptic equations with rough coefficients in divergence form

$$\operatorname{div}(A\nabla u) = f$$

in the context of a homogeneous space. Such equations have been studied at length in [FP], [CW], [MRW], [R], [R2], [SW1], [SW2] and [SW3]. The degeneracy of such equations is represented by the weakening of the restriction on the matrix A from positive-definite to non-negative definite, allowing that the quadratic form $\mathcal{A}(x,\xi) = \xi^\top A\xi$ may vanish for non-zero ξ. As we will see, this gives rise to some interesting challenges when defining weak solutions to such equations.

Our work here is expository. We present special cases of arguments from [SW1], [SW2] and [MRW], where the Moser method has been adapted to progressively larger classes of equations. The goal is to present the form of the argument in a readable, easy-to-follow manner, so that the reader may apply his or her understanding to more general equations currently being studied by, for example, Monticelli, Rodney and Wheeden. This is applicable, since the arguments used for more general equations still have the same essential structure. Additionally, in Theorem 4.1.3 we greatly expand Sawyer and Wheeden's proof of the John-Nirenberg inequality for homogeneous spaces from [SW1] and remedy an oversight that caused the theorem to be stated incorrectly in that work. This is important, as proving more general versions of the John-Nirenberg inequality (particularly versions where the homogeneous space structure has a weakened doubling condition) has been a significant obstacle in current work. We hope that access to a less condensed version of the proof will make it easier to adapt the argument to classes of equations currently being studied.

Chapter 2

Preliminaries

The goal of this chapter is twofold. First, we define rigorously the homogeneous space structure which we will impose on a set Ω in which we will be working and present the necessary spacial assumptions as outlined in [MRW]. Second, we give a detailed account of the background necessary to understand the argument proving Hölder continuity of weak solutions to degenerate linear elliptic partial differential equations given later in the paper. While we intend that this work be as self-contained as possible, we still assume a basic understanding of analysis, topology and PDE on the part of the reader.

2.1 Quasi-metric Spaces

We begin by imposing a weakened notion of distance on our space.

Definition 2.1.1. Given a set X, a quasi-metric is a function $d : X \times X \to \mathbb{R}$ with a constant $\kappa \geq 1$ such that

$$d(x, y) \geq 0 \tag{2.1}$$

$$d(x, y) = 0 \iff x = y \tag{2.2}$$

$$d(x, z) \leq \kappa(d(x, z) + d(z, y)) \tag{2.3}$$

for all $x, y, z \in X$. The ordered pair (X, d) is called a quasi-metric space.

As with metric spaces, we define the family of quasi-metric balls $\mathcal{B} = \{B(x, r)\}_{\substack{x \in X \\ r>0}}$ where

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Example 2.1.2. The function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_2 - x_1| + |y_2 - y_1| & \text{if } y_2 \geq y_1 \\ \frac{1}{3}(|x_1| + |y_2 - y_1| + |x_2|) & \text{if } y_2 < y_1 \end{cases}$$

is a quasi-metric on $\mathbb{R}^2$ with constant $\kappa = 3$. We may think of this metric as being similar to the Manhattan (taxi-cab) metric, with the added conditions that one may only travel south when on the y-axis, and that the speed-limit of the entire trip is tripled if one is forced to move south at all. The non-symmetry is clear from the definition, and the axioms of a quasi-metric are easily seen to be satisfied. We also include several illustrations of balls to various centres and radii to showcase the non-symmetry and the necessary constant on the triangle inequality.

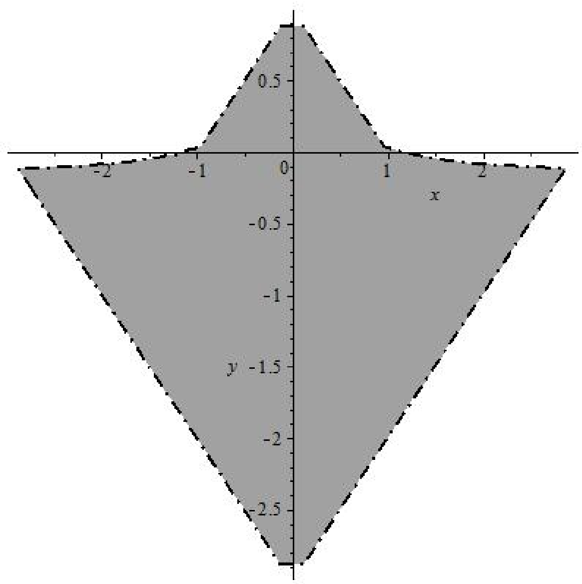

Figure 2.1: $B((0,0),1)$ relative to d

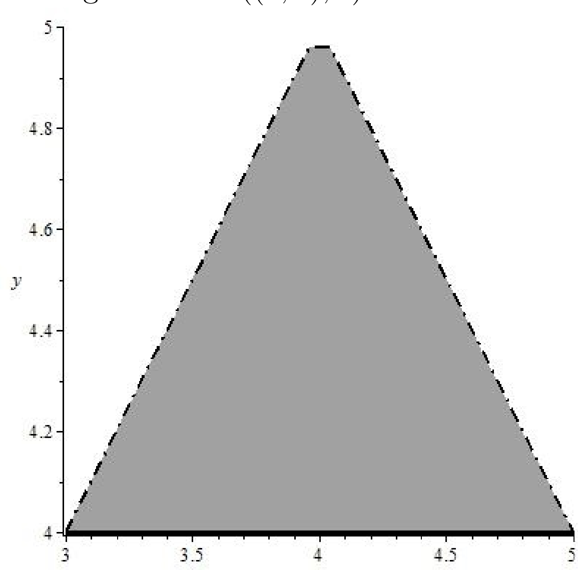

Figure 2.2: $B((4,4),1)$ relative to d

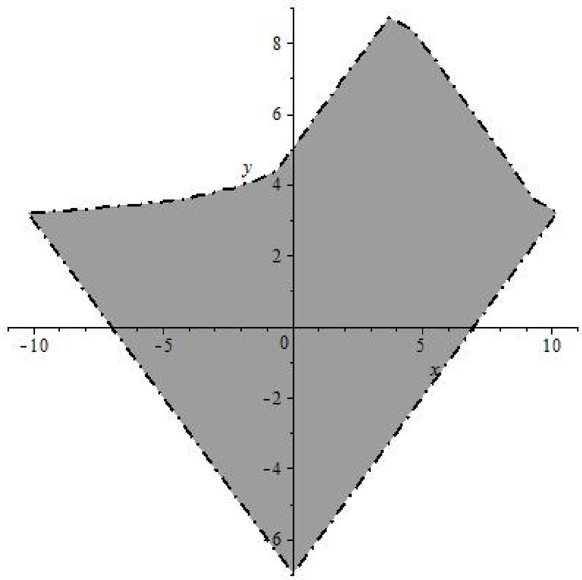

Figure 2.3: $B((4,4),5)$ relative to d

The previous example clearly shows that not all quasi-metrics are symmetric. However, we can always symmetrize a quasi-metric as follows:

Proposition 2.1.3. *Given a quasi-metric space (X, d), the function*

$$d_{sym}(x,y) = \frac{1}{2}\left(d(x,y) + d(y,x)\right)$$

is again a quasi-metric, and is also symmetric. That is, $d_{sym}(x,y) = d_{sym}(y,x)$.

Proof. Clearly d_{sym} satisfies the first two axioms in the above definition. It is also clearly symmetric, so only the triangle inequality remains. Given $x, y, z \in X$

$$\begin{aligned} d_{\mathrm{sym}}(x,y) &= \frac{1}{2}\left(d(x,y) + d(y,x)\right) \\ &\leq \frac{1}{2}\left(\kappa(d(x,z) + d(z,y)) + \kappa(d(y,z) + d(z,x))\right) \\ &= \kappa\left(\frac{1}{2}\left(d(x,z) + d(z,x)\right) + \frac{1}{2}\left(d(z,y) + d(y,z)\right)\right) \\ &= \kappa(d_{\mathrm{sym}}(x,z) + d_{\mathrm{sym}}(z,y)) \end{aligned}$$

which completes the proof. □

The weakened axioms of a quasi-metric still entail the swallowing property of balls familiar to us:

Proposition 2.1.4. *Let (X, d) be a quasi-metric space. Then there exists a constant $\gamma = \gamma(\kappa) > 1$ such that for all $x, y \in X$ and $s \geq r > 0$, if $B(x, r) \cap B(y, s) \neq \emptyset$ then*

$$B(x, r) \subset B(y, \gamma s). \tag{2.4}$$

Proof. Let $x, y \in X$ and $s \geq r > 0$ such that $B(x, r) \cap B(y, s) \neq \emptyset$. Let $\gamma = \kappa(2\kappa + 1)$ where κ is as in (2.3). Let $w \in B(x, r)$ and let $z \in B(x, r) \cap B(y, s)$. Then

$$\begin{aligned} d(y, w) &\leq \kappa(d(y, z) + d(z, w)) \\ &\leq \kappa(s + \kappa(d(z, x) + d(x, w))) \\ &\leq \kappa(s + 2\kappa r) \\ &\leq \kappa(s + 2\kappa s) \\ &= \kappa(2\kappa + 1)s \\ &= \gamma s, \end{aligned}$$

which shows that $w \in B(y, \gamma s)$, and thus that $B(x, r) \subset B(y, \gamma s)$. □

Because of the weakening of the conditions of a quasi-metric space as opposed to a metric space, quasi-metric balls are not quite as well behaved as we might like. In particular, they are not necessarily open in X. In figure 2.3, for example, we see that the lower edge of the ball (the edge parallel to the x-axis) is included in the ball, while the upper sides are not, making the ball neither open nor closed in the Euclidean sense. This can cause problems, since we expect our spaces to be domains. An added assumption can remedy this, however.

Definition 2.1.5. Let X be a quasi-metric space endowed with some topology $\mathcal{T}$. A function $f : X \to \mathbb{R}$ is called upper semicontinuous with respect to $\mathcal{T}$ if the set $\{x \in X : f(x) < r\}$ is open with respect to $\mathcal{T}$ for every $r \in \mathbb{R}$.

Lemma 2.1.6. *Let (X, d) be a quasi-metric space endowed with a topology $\mathcal{T}$. If d is upper semicontinuous in the second variable (that is, if for each $x \in X$ the function $d_x : X \to \mathbb{R}$ defined by $d_x(y) = d(x, y)$ is upper semicontinuous with respect to $\mathcal{T}$), then $B(x, r)$ is open for all $x \in X$ and $r > 0$.*

Remark 2.1.7. *The only issue with assuming upper semicontinuity in the second variable is that it is not preserved when constructing a symmetric quasi-metric as in*

Proposition 2.1.3. For this reason, we shall simply assume that our quasi-metrics are both upper semicontinuous and symmetric for most of the important results. Note that the assumption that the quasi-metric is upper semicontinuous in both variables is also sufficient, since this would allow d_{sym} to preserve the upper semicontinuity in the second variable, but this point is a small one.

2.2 Generalized Dyadic Cubes

We are now ready to construct a generalized grid of dyadic cubes in the context of a quasi-metric space, an extremely useful partitioning which will later be key to proving the John-Nirenberg inequality. There are currently two ways to go about this. We present the construction from [SW2], which defines the cubes based on a pre-chosen smallest level m. Each level of cubes created in this case partitions the space entirely. The drawback is that when decreasing the minimum level, we must construct an entirely new collection of cubes unrelated to those generated for higher minimum levels. In [C], M. Christ presents an alternative where there is a single grid constructed for all levels, but he allows that a set of measure 0 not be included in any cubes.

In order to construct the grid of dyadic cubes, we require a further topological property of our space:

Definition 2.2.1. A quasi-metric space (X, d) endowed with a topology $\mathcal{T}$ is called separable if it has a countable dense subset. That is, if there exists a set $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that for every set $U \in \mathcal{T}$

$$U \cap \{x_n\}_{n\in\mathbb{N}} \neq \emptyset.$$

Lemma 2.2.2. *Let (X, d) be a separable quasi-metric space with some topology $\mathcal{T}$. Suppose d is upper semicontinuous in the second variable with respect to $\mathcal{T}$. Then there exists $\lambda = \lambda(\kappa) > 0$ such that for all $m \in \mathbb{Z}$, there are points $x_{j;k} \in X$ and Borel sets (relative to the topology $\mathcal{T}$) $Q_{j;k}$ for $1 \leq j < n_k \in \overline{N}$ with $k \geq m$ such that*

(i) $B(x_{j;k}, \lambda^k) \subset Q_{j;k} \subset B(x_{j;k}, \lambda^{k+1})$ *for all* $1 \leq j < n_k$ *with* $k \geq m$

(ii) $Q_{i;k} \cap Q_{j;k} = \emptyset$ *for* $k \geq m$ *and* $i \neq j$

(iii) $X = \cup_j Q_{j;k}$ *for* $k \geq m$

(iv) Either $Q_{j;k} \subset Q_{i;\ell}$ *or* $Q_{j;k} \cap Q_{i,\ell} = \emptyset$ *for* $1 \leq j < n_k$, $1 \leq i < n_\ell$, $m \leq k < \ell$

Given $m \in \mathbb{Z}$ *the collection of all sets* $Q_{j;k}$ *with* $k \geq m$ *is called a grid of dyadic cubes for the space* (X, d)*, and is denoted* $\mathcal{D}_m$*. A set* $Q_{j;k} \in \mathcal{D}_m$ *is the* j^{th} *dyadic cube of order k.*

Proof. Set $\lambda = 8\kappa^5$ where κ is as in (2.3). For any $k \in \mathbb{Z}$ we may define a maximal collection of balls $\{B(x_{j;k}, 3\kappa^2\lambda^k)\}_{j\in I}$ such that the balls are pairwise disjoint. The separability of the space implies that either $I \cong \mathbb{N}$ or that $I \cong \{1, \ldots, p\}$ for some $p \in \mathbb{N}$, since each ball must contain at least one element of the countable dense subset. Moreover,

$$\bigcup_{j\in I} B(x_{j;k}, 6\kappa^3\lambda^k) = X. \tag{2.5}$$

since for any $x \in X$, $B(x, 3\kappa^2\lambda^k)$ must intersect one of the balls $B(x_{j;k}, 3\kappa^2\lambda^k)$ in the maximal collection (or be part of the collection itself). If y is a common point, then

$$d(x, x_{j;k}) \leq \kappa(d(x, y) + d(y, x_{j;k})) \leq 6\kappa^3\lambda^k.$$

Fix $m \in \mathbb{Z}$. We construct the cubes of order m as follows:

$$\begin{aligned}
Q_{1;m} &= B(x_{1;m}, 6\kappa^3\lambda^m) \setminus \bigcup_{i\neq 1} B(x_{i;m}, \lambda^m) \\
Q_{2;m} &= B(x_{2;m}, 6\kappa^3\lambda^m) \setminus \bigcup_{i\neq 2} B(x_{i;m}, \lambda^m) \setminus Q_{1;m} \\
&\vdots \\
Q_{j;m} &= B(x_{j;m}, 6\kappa^3\lambda^m) \setminus \bigcup_{i\neq j} B(x_{i;m}, \lambda^m) \setminus \bigcup_{i<j} Q_{i;m} \\
&\vdots
\end{aligned}$$

Let us show that the first three properties hold for $k = m$. To show (i), for the upper inclusion we have that

$$Q_{j;m} \subset B(x_{j;m}, 6\kappa^3\lambda^m) \subset B(x_{j;m}, \lambda^{m+1})$$

since $\lambda > 6\kappa^3$. For the lower inclusion, note that since the $\{B(x_{j;m}, 3\kappa^3\lambda^m)\}_{j\in I}$ are pairwise disjoint, so must be the balls $\{B(x_{j;m}, \lambda^m\}$, (being subsets of their larger counterparts). This allows us to write

$$B(x_{j;m},\lambda^m) \subset B(x_{j;m},6\kappa^3\lambda^m) \setminus \bigcup_{i\neq j} B(x_{i;m},\lambda^m).$$

Moreover, $B(x_{j;m},\lambda^m) \cap Q_{i;m} = \emptyset$ for $i \neq j$ by definition of $Q_{i;m}$ (since $B(x_{j;m},\lambda^m)$ is one of the balls subtracted in the definition of $Q_{i;m}$ for $i \neq j$). Hence

$$B(x_{j;m},\lambda^m) \subset B(x_{j;m},6\kappa^3\lambda^m) \setminus \bigcup_{i\neq j} B(x_{i;m},\lambda^m) \setminus \bigcup_{i<j} Q_{i;m} = Q_{j;m} \tag{2.6}$$

as desired.

That property (ii) holds is clear from the definition of the cubes (the subtractions force non-intersection), so we now turn to property (iii). Let $x \in X$. If $x \in B(x_{j;m},\lambda^m)$ for some j, then (iii) holds by (2.6). If $x \notin \cup_{j\in I} B(x_{j;m},\lambda^m)$, then by (2.5) there exists j such that $x \in B(x_{j;m},6\kappa^3\lambda^m)$. Let j_0 be the smallest such j. Then by definition, $x \in Q_{j_0,m}$. Hence (iii) holds, and since (iv) is vacuous with only one order of cubes to consider, we are done.

We proceed by induction on k. Let $\ell > m$ and suppose that the sets $Q_{j;k}$ have been defined for all $1 \leq j < n_k$, $m \leq k < \ell$ and that these sets satisfy all four properties. Define

$$B^*(x_{j;\ell},r) = \cup_i \{Q_{i;\ell-1} : Q_{i;\ell-1} \cap B(x_{j;\ell},r) \neq \emptyset\}. \tag{2.7}$$

We construct the $Q_{j;\ell}$ similarly to the $Q_{j;m}$:

$$\begin{aligned}
Q_{1;\ell} &= B^*(x_{1;\ell},6\kappa^3\lambda^\ell) \setminus \bigcup_{i\neq 1} B^*(x_{i;\ell},\lambda^\ell)\\
Q_{2;\ell} &= B^*(x_{2;\ell},6\kappa^3\lambda^\ell) \setminus \bigcup_{i\neq 2} B^*(x_{i;\ell},\lambda^\ell) \setminus Q_{1;\ell}\\
&\vdots\\
Q_{j;\ell} &= B^*(x_{j;\ell},6\kappa^3\lambda^\ell) \setminus \bigcup_{i\neq j} B^*(x_{i;\ell},\lambda^\ell) \setminus \bigcup_{i<j} Q_{i;\ell}\\
&\vdots
\end{aligned}$$

Before we continue, let us first note that

$$B(x_{j;\ell},r) \subset B^*(x_{j;\ell},r) \subset B(x_{j;\ell},\kappa^2 r + (\kappa^2+\kappa)\lambda^\ell) \tag{2.8}$$

for $r > 0$. The first inclusion follows from (2.7) and property (iii), which is satisfied by the $Q_{j;\ell-1}$ by the induction hypothesis. For the second inclusion, if $x \in B^*(x_{j;\ell}, r)$ then $x \in Q_{i;\ell-1}$ intersecting $B(x_{j;\ell}, r)$ for some i. Let $y \in Q_{i;\ell-1} \cap B(x_{j;\ell}, r)$. Then since $Q_{i;\ell-1}$ satisfies (i), we have that

$$\begin{aligned} d(x, x_{j;\ell}) &\leq \kappa(d(x, x_{i;\ell-1}) + d(x_{i;\ell-1}, x_{j;k})) \\ &\leq \kappa(\lambda^\ell + \kappa(d(x_{i;\ell-1}, y) + d(y, x_{j;k}))) \\ &= \kappa(\lambda^\ell + \kappa(\lambda^\ell + r)) \\ &= \kappa^2 r + (\kappa^2 + \kappa)\lambda^\ell. \end{aligned}$$

Now we show that property (i) holds for $k = \ell$. For the upper inclusion we show that

$$Q_{j;\ell} \subset B^*(x_{j;\ell}, 6\kappa^3\lambda^\ell) \subset B(x_{j;\ell}, (6\kappa^5 + \kappa^2 + \kappa)\lambda^\ell) \subset B(x_{j;\ell}, \lambda^{\ell+1}). \tag{2.9}$$

The first inclusion is clear from the definition of $Q_{j;\ell}$, the second follows from (2.8) with $r = 6\kappa^3\lambda^\ell$, and the third comes since $6\kappa^5 + \kappa^2 + \kappa \leq 8\kappa^5 = \lambda$ (note that $\kappa \geq 1$ by (2.3)). For the lower inclusion of (i), we have that

$$B^*(x_{i;\ell}, \lambda^\ell) \subset B(x_{i;\ell}, (2\kappa^2 + \kappa)\lambda^\ell) \subset B(x_{i;\ell}, 3\kappa^2\lambda^\ell) \tag{2.10}$$

for any i by (2.8) with $r = \lambda^\ell$. By construction, $B(x_{i;\ell}, 3\kappa^2\lambda^\ell) \cap B(x_{j;\ell}, 3\kappa^2\lambda^\ell) = \emptyset$ if $i \neq j$, so by (2.10) $B^*(x_{i;\ell}, \lambda^\ell) \cap B^*(x_{j;\ell}, \lambda^\ell) = \emptyset$ for $i \neq j$. Hence by (2.7) and by definition of $Q_{j;\ell}$

$$B(x_{j;\ell}, \lambda^\ell) \subset Q_{j;\ell}. \tag{2.11}$$

As before, (ii) is clear by construction, so we now turn to (iii).

$$X = \bigcup_i B^*(x_{j;\ell}, 6\kappa^3\lambda^\ell) \tag{2.12}$$

by (2.5) and the first inclusion in (2.8) with $r = 6\kappa^3\lambda^\ell$. Moreover,

$$B^*(x_{j;\ell}, \lambda^\ell) \subset B^*(x_{j;\ell}, 6\kappa^3\lambda^\ell) \setminus \bigcup_{i \neq j} B^*(x_{i;\ell}, \lambda^\ell) \tag{2.13}$$

by (2.10) and since the $B(x_{j;\ell}, \lambda^\ell)$ are disjoint. Hence

$$\bigcup_j B^*(x_{j;\ell}, \lambda^\ell) \subset \bigcup_j Q_{j;\ell} \tag{2.14}$$

by (2.13) and definition of the $Q_{j;k}$. Now, to show (iii), let $x \in X$. We may assume by (2.12) that $x \in B^*(x_{j_0,\ell}, 6\kappa^3\lambda^\ell)$ for some smallest j_0. If $x \in \cup_j B^*(x_{j;\ell}, \lambda^\ell)$, then we are done by (2.14) and if not, then $x \in Q_{j_0;\ell}$ by definition. Thus (iii) holds for $k = \ell$.

Finally, for property (iv), it is sufficient by the induction hypothesis to show that if $Q_{i_0;k} \cap Q_{j_0;\ell} \neq \emptyset$ for $m \leq k < \ell$, then $Q_{i_0;k} \subset Q_{j_0;\ell}$. By (2.7) and the induction hypothesis, we know that each $B^*(x_{j;\ell}, 6\kappa^3\lambda^\ell)$ is a disjoint union of $Q_{j;\ell-1}$'s. Hence by definition, each cube $Q_{j;\ell}$ is also a disjoint union of $Q_{j;\ell-1}$'s, and thus that any $Q_{j;\ell-1}$ intersecting $Q_{j;k}$ must be contained therein. So if $Q_{i_0;k} \cap Q_{j_0;\ell} \neq \emptyset$, it follows that $Q_{i_0;k} \cap Q_{i;\ell-1} \neq \emptyset$ for some i with $Q_{i;\ell-1} \subset E_{j_0;\ell}$. But then $Q_{i_0;k} \subset Q_{i;\ell-1}$ by hypothesis, and thus $Q_{i_0;k} \subset Q_{j_0;\ell}$ as desired, proving (iv) for $k = \ell$. Gathering results, we have that (i), (ii), (iii) and (iv) hold for $k = \ell$, which concludes the proof by induction. □

Remark 2.2.3. *While the lack of a true metric prevents us from thinking of these objects as cubes in a classical geometric sense, they still have many of the desired properties, hence the name. For each order k, we may think of those cubes as lining up side-by-side (though not necessarily in order) to partition X. They are all disjoint from one another (a property that was not present in [SM], where the cubes were allowed to intersect on the boundary) but fill the space completely. Also, no cube of a lower order may sit on the boundary of a cube of higher order. They must fall completely inside, or completely outside of the higher-order cube.*

The theory we have developed in this section is very powerful, even in the amount of generality in which it has been presented. Often however, we want to be slightly more particular about which balls and cubes we consider, since we will be working in the subspace topology on $\Omega \subset \mathbb{R}^n$.

Definition 2.2.4. Given a quasi-metric space (Ω, d) where $\Omega \subset \mathbb{R}^n$ and $\delta > 0$, a ball $B = B(x, r)$ is called δ-local if $0 < r < \delta \text{dist}(x, \partial\Omega)$ where dist represents Euclidean distance. Similarly, a dyadic cube $Q_{j;k}$ is called δ-local if the ball $B(x_{j;k}, \lambda^{k+1})$ containing it is δ-local.

The idea is that it is very often useful to be able to ensure that our balls do not intersect the boundary of Ω and become deformed. For sufficiently small δ, this condition of δ-localness not only ensures this, but gives us a lower bound on the (Euclidean) distance from our balls to the boundary of Ω.

2.3 Homogeneous Spaces

Having developed an appropriate notion of distance for our spaces, we now further endow them with a special type of measure:

Definition 2.3.1. Let (X, d) be a quasi-metric space. Let μ be a measure on X such that every ball $B \in \mathcal{B}$ is μ-measurable. Then μ is called a doubling measure if there exists $C > 0$ such that for any ball $B(x, r) \in \mathcal{B}$

$$|B(x, 2r)|_\mu \leq C\, |B(x, r)|_\mu \tag{2.15}$$

The ordered triple (X, d, μ) is called a quasi-metric doubling measure space.

It is important to realize that there is nothing special about 2 in (2.15), though that particular presentation is the one for which the property was named. The idea of a doubling condition is simply that any two quasi-metric balls which are located near enough to one another within the space will have comparable volumes. This is formalized in the following way:

Proposition 2.3.2. *Let (X, d, μ) be a quasi-metric measure space. The measure μ is a doubling measure if and only if there exist $D, E > 0$ such that for all $x, y \in X$ and $s \geq r > 0$ if $B(x, r) \cap B(y, s) \neq \emptyset$, then*

$$|B(y, s)|_\mu \leq D \left(\frac{s}{r}\right)^E |B(x, r)|_\mu.$$

Proof. For the forward direction, let C and γ be as in (2.15) and (2.4), respectively. Let $x, y \in X$ and $s \geq r > 0$ such that $B(x, r) \cap B(y, s) \neq \emptyset$. There exists $k \in \mathbb{Z}$ such that $2^k \leq \frac{\gamma s}{r} \leq 2^{k+1}$, which implies that

1. $\dfrac{\gamma s}{2^{k+1}} \leq r$

2. $k \leq \log_2 \left(\dfrac{\gamma s}{r}\right).$

These taken together with (2.15) and (2.4) yield

$$\begin{aligned}
|B(y,s)|_\mu &\le |B(x,\gamma s)|_\mu \\
&\le C^{k+1}\left|B\left(x,\frac{\gamma s}{2^{k+1}}\right)\right|_\mu \\
&\le C^{k+1}|B(x,r)|_\mu \\
&\le C^{\log_2 \frac{\gamma s}{r}+1}|B(x,r)|_\mu \\
&= C^{\log_2 C \log_C \frac{\gamma s}{r}+1}|B(x,r)|_\mu \\
&= C^{\left(\log_C \frac{\gamma s}{r}\right)^{\log_2 C}+1}|B(x,r)|_\mu \\
&= C\left(\frac{\gamma s}{r}\right)^{\log_2 C}|B(x,r)|_\mu \\
&= \left(C\gamma^{\log_2 C}\right)\left(\frac{s}{r}\right)^{\log_2 C}|B(x,r)|_\mu
\end{aligned}$$

which completes the proof, since the converse is trivial. □

Amalgamating all of the structure developed in the last two sections, we now define the type of space in which we will be working:

Definition 2.3.3. A quasi-metric measure space (Ω, d, μ) where $\Omega \subset \mathbb{R}^n$ is open and μ is a Borel measure (with respect to the standard topology on $\mathbb{R}^n$) is called a homogeneous space if d is upper semicontinuous in the second variable, μ satisfies a doubling condition as in (2.15) and there exists a constant $C_{euc} \geq 1$ such that for all $x \in \Omega$ and $r > 0$,

$$B(x,r) \subset D(x, C_{euc}r) \tag{2.16}$$

where D is a Euclidean ball. If d is also symmetric the space is called a symmetric homogeneous space.

The Euclidean containment condition (2.16) is added in order to ensure that the quasi-metric balls do not stretch too much. It is necessary in order to prove the John-Nirenberg inequality, which requires that subcubes of δ-local dyadic cubes are again δ-local. For an example showcasing the utility of the properties of such spaces, see Section 3.4.

Chapter 3

Degenerate Linear Elliptic Partial Differential Equations

3.1 Degenerate Linear Elliptic PDEs, Degenerate Sobolev Spaces and Weak Solutions

Fix a symmetric homogeneous space (Ω, d, μ) where μ is Lebesgue measure. Given a measurable matrix $Q(x) = [q_{i,j}(x)]_{1\leq i,j\leq n}$, we denote by $\mathcal{Q}(x,\xi)$ the quadratic form related to Q given by

$$\mathcal{Q}(x,\xi) = \xi^\top Q(x)\xi \tag{3.1}$$

for every $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega$. We say that Q is bounded if the norm

$$||Q||_{L^\infty(\Omega)} = \sup_{x\in\Omega} \max_{1\leq i,j\leq n} |q_{i,j}(x)|$$

is finite, or equivalently if there exists $C > 0$ such that

$$\mathcal{Q}(x,\xi) \leq C|\xi|^2 \tag{3.2}$$

for every $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega$.

Definition 3.1.1. Given a matrix $Q = Q(x)$ and a measurable function f, a second order linear partial differential equation of the form

$$\operatorname{div}(Q\nabla u) = f \text{ almost everywhere in } \Omega \tag{3.3}$$

is said to be *degenerate* elliptic if

$$\begin{cases} Q(x) \text{ is symmetric for almost every } x \in \Omega, \\ Q(x) \text{ is non-negative definite for almost every } x \in \Omega, \\ Q \text{ is bounded .} \end{cases} \tag{3.4}$$

Equation (3.3) is called *elliptic* if there also exists $c > 0$ such that

$$c|\xi|^2 \leq \mathcal{Q}(x,\xi) \leq C|\xi|^2. \tag{3.5}$$

We intend to study the class of degenerate elliptic equations using techniques inspired by the elliptic case as in [GT].

Fix a matrix Q satisfying (3.4) and a measurable function f. This defines a degenerate elliptic equation as in (3.3). If $Q \in C^1(\Omega)$ (that is, if $q_{i,j} \in C^1(\Omega)$ for each $i, j = 1, \dots, n$), a classical solution to this equation is simply a twice-differentiable function u that satisfies (3.3) when substituted therein. Our focus will instead be on continuity properties of weak solutions. Before we give the precise definition of a weak solution to (3.3) we define the spaces to which weak solutions, if they exist, belong.

Definition 3.1.2. For $w \in Lip_{loc}(\Omega)$, define the (possibly infinite) norm

$$||w||_{H^1(\Omega)} = \left(||w||^2_{L^2(\Omega)} + ||\nabla w||^2_{(L^2(\Omega))^n}\right)^{\frac{1}{2}}.$$

Note that ∇w exists almost everywhere by the Rademacher-Stepanov theorem (see §9.1 of [BL]). The *classical* Sobolev space $H^1(\Omega)$ is defined as the completion of the space

$$\{w \in Lip_{loc}(\Omega) : ||w||_{H^1(\Omega)} < \infty\}$$

with respect to $|| \cdot ||_{H^1(\Omega)}$.

It is a famous result of Meyers and Serrin (see [MS]) that $H^1(\Omega)$ is isomorphic to the space

$$W^{1,2}(\Omega) = \{w \in L^2(\Omega) : \nabla w \in (L^2(\Omega))^n\}$$

where ∇w is taken in the weak sense or in the sense of distributions. For a more thorough account of this space, see the appendix.

If our equation is *elliptic* (i.e. if Q also satisfies (3.5)) in Ω, then given sufficient smoothness conditions on Q and f (see [GT]) we could find objects $u \in W^{1,2}(\Omega)$ that would satisfy the integral equation

$$\int_\Omega (\nabla u)^\top Q \nabla \varphi d\mu = -\int_\Omega f\varphi$$

for all $\varphi \in C_c^\infty$. Such objects are called weak solutions to (3.3). In degenerate spaces however, this is not necessarily the case (see [R], [MRW], [SW3]). The solutions instead reside in the larger *degenerate* Sobolev space, which we will now aim to define. First, we require a weighted vector-valued L^2 space comparable with the quadratic form (3.1).

Definition 3.1.3. Given $\mathbf{f} : \Omega \to \mathbb{R}^n$, define

$$||\mathbf{f}||_{\mathcal{L}^2(\Omega,\mathcal{Q})} = \left[\int_\Omega \mathcal{Q}(x,\mathbf{f}(x))d\mu\right]^{\frac{1}{2}}. \tag{3.6}$$

Define also an equivalence relation R on the set $\{\mathbf{f} : \Omega \to \mathbb{R}^n : ||\mathbf{f}||_{\mathcal{L}^2(\Omega,\mathcal{Q})} < \infty\}$ by ${}_{\mathbf{f}}R_{\mathbf{g}}$ if and only if $||\mathbf{f}-\mathbf{g}||_{\mathcal{L}^2(\Omega,\mathcal{Q})} = 0$. Then we define the form-weighted vector-valued L^2 space $\mathcal{L}^2(\Omega,\mathcal{Q})$ by

$$\mathcal{L}^2(\Omega,\mathcal{Q}) = \{\mathbf{g} : \Omega \to \mathbb{R}^n : ||\mathbf{g}||_{\mathcal{L}^2(\Omega,\mathcal{Q})} < \infty\}/R \tag{3.7}$$

whose elements are equivalence classes of $\mathbb{R}^n$-valued vector fields modulo R. Note that $||\cdot||_{\mathcal{L}^2(\Omega,\mathcal{Q})}$ defines a norm on $\mathcal{L}^2(\Omega,\mathcal{Q})$.

Definition 3.1.4. For $w \in Lip_{loc}(\Omega)$, define

$$||w||_{QH^1(\Omega)} = \left(||w||^2_{L^2(\Omega)} + ||\nabla w||^2_{\mathcal{L}^2(\Omega,\mathcal{Q})}\right)^{\frac{1}{2}}. \tag{3.8}$$

This defines a norm on $Lip_{loc}(\Omega)$. We define the degenerate Sobolev space $QH^1(\Omega)$ as the completion of the linear space

$$Lip_Q(\Omega) = \{w \in Lip_{loc}(\Omega) : ||w||_{QH^1(\Omega)} < \infty\} \tag{3.9}$$

in the metric induced by the norm. Similarly, we define $QH^1_0(\Omega)$ as the completion of the space

$$\{w \in Lip_0(\Omega) : ||w||_{QH^1(\Omega)} < \infty\}.$$

Remark 3.1.5.

1. *If* (3.3) *is elliptic (that is, if Q satisfies* (3.5)*), then we would obtain from* (3.5) *that $\mathcal{Q}(x,\mathbf{f}) \le C|\mathbf{f}|^2$ and that $|\mathbf{f}|^2 \le c^{-1}\mathcal{Q}(x,\mathbf{f})$. Hence $\mathcal{L}^2(\Omega,\mathcal{Q}) = (L^2(\Omega))^n$ and $QH^1(\Omega) = H^1(\Omega)$. So in the elliptic case, the notions of degenerate and classical Sobolev spaces are equivalent.*

2. *Analogously to $H^1(\Omega)$, $QH^1(\Omega)$ is a Banach space (see [SW3]) consisting of equivalence classes of Cauchy sequences of functions in $Lip_Q(\Omega)$. We will often abuse notation and simply write $\{w_i\} \in QH^1(\Omega)$, with the understanding that $\{w_i\}$ is a representative of the equivalence class in $QH^1(\Omega)$ containing $\{w_i\}$.*

Given a particular Cauchy sequence $\{w_i\} \in QH^1(\Omega)$, there exist unique functions $w \in L^2(\Omega)$ and $\mathbf{v} \in \mathcal{L}^2(\Omega, \mathcal{Q})$ such that

$$w_i \to w \text{ in } L^2 \qquad \text{and} \qquad \nabla w_i \to \mathbf{v} \text{ in } \mathcal{L}^2(\Omega, \mathcal{Q}).$$

That is, for each element $\{w_i\} \in QH^1(\Omega)$ there exists a unique pair $(w, \mathbf{v}) \in L^2(\Omega) \times \mathcal{L}^2(\Omega, \mathcal{Q})$, so that $QH^1(\Omega)$ is isomorphic to a closed subspace $\mathcal{W}_Q^{1,2}(\Omega)$ of $L^2(\Omega) \times \mathcal{L}^2(\Omega, \mathcal{Q})$. Similarly, $QH_0^1(\Omega)$ is isomorphic to a closed subspace $(\mathcal{W}_Q^{1,2})_0(\Omega)$.

It is the pair $(u, \mathbf{v}) \in \mathcal{W}_Q^{1,2}(\Omega)$ that we will most often refer to explicitly in our work, thinking of our weak solutions as elements of the appropriate Banach space obtained by isomorphism. One issue does arise, however. Given an element $(w, \mathbf{v}) \in \mathcal{W}_Q^{1,2}(\Omega)$, the vector-valued function $\mathbf{v}$ need *not* depend on the function w. Rather, it depends on the equivalence class of sequences that defines the pair. This phenomenon arises due to the degeneracy of the quadratic form (3.1). Since it is degenerate there may be sequences $\{\varphi_i\}$, $\{\xi_i\}$ in $\mathcal{W}_Q^{1,2}(\Omega)$ such that

1. $\varphi_i \to w$ and $\xi_i \to w$ in $L^2(\Omega)$
2. $\nabla \varphi_i \to \mathbf{v}$ in $\mathcal{L}^2(\Omega, \mathcal{Q})$
3. $\nabla \xi_i \to \mathbf{v'}$ in $\mathcal{L}^2(\Omega, \mathcal{Q})$
4. $\mathbf{v}$ and $\mathbf{v'}$ are not in the same $\mathcal{L}^2(\Omega, \mathcal{Q})$ equivalence class

so that $(w, \mathbf{v}), (w, \mathbf{v'}) \in \mathcal{W}_Q^{1,2}$. In other words, the projection of $\mathcal{W}_Q^{1,2}(\Omega)$ onto $L^2(\Omega)$ is not, in general, injective. A truly spectacular example of this can be found on page 92 of [FKS]. There, the authors define a 1×1 matrix with quadratic form q on $[0,1] \times \mathbb{R}$ such that $(0,1) \in \mathcal{W}_q^{1,2}([0,1])$. That is, in their weighted space a possible "derivative" of the the constant function 0 is the constant function 1. Moreover, by Lemma 3.2.1, this implies that $(0, \varphi) \in \mathcal{W}_q^{1,2}([0,1])$ *for every* $\varphi \in Lip_0([0,1])$.

Remark 3.1.6. *In our work we will only ever be dealing with one particular weak solution at a time. As such, it is permissible to follow the conventions of [MRW], [R] and [SW3] and refer to a weak solution as* $(u, \nabla u)$*, with the understanding that* ∇u *does not in general depend on* u*, but is simply one of the vector-valued functions in* $\mathcal{L}^2(\Omega, \mathcal{Q})$ *for which* $(u, \nabla u) \in \mathcal{W}_Q^{1,2}(\Omega)$.

With the conclusion of this discussion, we are now ready to define weak solutions:

Definition 3.1.7. A weak solution to the degenerate linear elliptic partial differential equation (3.3) is a pair $(u, \nabla u) \in \mathcal{W}_Q^{1,2}(\Omega)$ satisfying the equation

$$\int_\Omega (\nabla u)^\top Q \nabla \varphi d\mu = -\int_\Omega f \varphi d\mu \tag{3.10}$$

for all $\varphi \in Lip_0(\Omega)$.

3.2 Some Calculus in Degenerate Sobolev Spaces

In this section, we prove that under certain conditions, the product and chain rules still hold in degenerate Sobolev spaces. In addition to their own instrumental value, the proofs of these results will also allow us to examine the underlying sequential structure of these new spaces much more closely.

Lemma 3.2.1 (The Product Rule)**.** *Suppose Ω is bounded and let $(u, \nabla u) \in \mathcal{W}_Q^{1,2}(\Omega)$. If $\varphi \in Lip_0(\Omega)$ then $(\varphi u, \varphi \nabla u + u \nabla \varphi) \in \mathcal{W}_Q^{1,2}(\Omega)$.*

Proof. Since $(u, \nabla u) \in \mathcal{W}_Q^{1,2}(\Omega)$, there exists a sequence $\{u_i\} \subset Lip_Q(\Omega)$ such that $u_i \to u$ in $L^2(\Omega)$ and $\nabla u_i \to \nabla u$ in $\mathcal{L}^2(\Omega, \mathcal{Q})$ (recall that ∇u need not be uniquely determined by u, but rather by the sequence $\{u_i\}$).

We first prove that for all $i \in \mathbb{N}$, $\varphi u_i \in Lip_Q(\Omega)$. First, since $\varphi, u_i \in Lip_{loc}(\Omega)$ and products of locally Lipschitz functions are again locally Lipschitz we have that $\varphi u_i \in Lip_{loc}(\Omega)$. Second, note that $\varphi \in L^\infty(\Omega)$ since $\varphi \in Lip_0(\Omega)$, so by Hölder's inequality $\varphi u_i \in L^2(\Omega)$ since $u_i \in L^2(\Omega)$. We also need to show that $\nabla(\varphi u_i) \in \mathcal{L}^2(\Omega, \mathcal{Q})$. Note first that since $\varphi u_i \in Lip_{loc}(\Omega)$, it is almost everywhere differentiable by the Rademacher-Stepanov theorem. Thus the (classical) product rule yields $\nabla \varphi u_i = \varphi \nabla u_i + u_i \nabla \varphi$. So we obtain

$$\begin{aligned} ||\nabla \varphi u_i||_{\mathcal{L}^2(\Omega,\mathcal{Q})} &= ||\varphi \nabla u_i + u_i \nabla \varphi||_{\mathcal{L}^2(\Omega,\mathcal{Q})} \\ &\leq ||\varphi \nabla u_i||_{\mathcal{L}^2(\Omega,\mathcal{Q})} + ||u_i \nabla \varphi||_{\mathcal{L}^2(\Omega,\mathcal{Q})}. \end{aligned} \tag{3.11}$$

Using the fact that φ is bounded on Ω, we have

$$\begin{aligned}||\varphi\nabla u_i||_{\mathcal{L}^2(\Omega,\mathcal{Q})} &= \left(\int_\Omega (\varphi\nabla u_i)^\top Q(\varphi\nabla u_i)d\mu\right)^{\frac{1}{2}} \\ &= \left(\int_\Omega \varphi^2\mathcal{Q}(x,\nabla u_i)d\mu\right)^{\frac{1}{2}} \\ &\le ||\varphi||_{L^\infty(\Omega)}||\nabla u_i||_{\mathcal{L}^2(\Omega,\mathcal{Q})}.\end{aligned}$$

For the second term, since Q is bounded and φ has bounded derivative, we have that $(\nabla\varphi)^\top Q\nabla\varphi \le C$ for some constant $C > 0$. Hence

$$||u_i\nabla\varphi||_{\mathcal{L}^2(\Omega,\mathcal{Q})} \le C||u_i||_{L^2(\Omega)} < \infty$$

since the sequence (or a tail thereof if necessary) $\{u_i\}$ is bounded in $L^2(\Omega)$ as a convergent sequence in $L^2(\Omega)$. Combining the last two estimates, we see that (3.11) is finite, which shows that $\varphi u_i \in Lip_Q(\Omega)$.

Thus $\{\varphi u_i\} \subset Lip_Q(\Omega)$. We now show that this sequence converges to $(\varphi u, \varphi\nabla u + u\nabla\varphi)$ in the norm $||\cdot||_{QH^1(\Omega)}$. This is equivalent to showing that $\varphi u_i \to \varphi u$ in $L^2(\Omega)$ and $\nabla(\varphi u_i) \to \varphi\nabla u + u\nabla\varphi$ in $\mathcal{L}^2(\Omega,\mathcal{Q})$. For the first part, we have that

$$||\varphi u_i - \varphi u||_{L^2(\Omega)} = ||\varphi(u_i - u)||_{L^2(\Omega)} \le ||\varphi||_{L^\infty(\Omega)}||u_i - u||_{L^2(\Omega)} \to 0$$

since $u_i \to u$ in $L^2(\Omega)$. For the second part,

$$\begin{aligned}||\nabla(\varphi u_i) - \varphi\nabla u + u\nabla\varphi||_{\mathcal{L}^2(\Omega,\mathcal{Q})} &= ||\varphi(\nabla u_i - \nabla u) + (u - u_i)\nabla\varphi||_{\mathcal{L}^2(\Omega,\mathcal{Q})} \\ &\le ||\varphi(\nabla u_i - \nabla u)||_{\mathcal{L}^2(\Omega,\mathcal{Q})} + ||\nabla\varphi(u_i - u)||_{\mathcal{L}^2(\Omega,\mathcal{Q})}.\end{aligned} \tag{3.12}$$

We estimate the two norms separately. For the first, analogously to the above we have that

$$||\varphi(\nabla u_i - \nabla u)||_{\mathcal{L}^2(\Omega,\mathcal{Q})} \le ||\varphi||_{L^\infty(\Omega)}||\nabla u_i - \nabla u||_{\mathcal{L}^2(\Omega,\mathcal{Q})} \to 0 \tag{3.13}$$

since $\nabla u_i \to \nabla u$ in $\mathcal{L}^2(\Omega,\mathcal{Q})$. For the second, since $\varphi \in Lip_0(\Omega)$ we have that $|\nabla\varphi| \in L^\infty(\Omega)$. Then since Q is bounded, we obtain by (3.2) that

$$\mathcal{Q}(x,\nabla\varphi) \le C||\nabla\varphi||^2_{L^\infty(\Omega)}.$$

This allows us to write

$$||\nabla\varphi(u_i-u)||_{\mathcal{L}^2(\Omega,\mathcal{Q})} = \left(\int_\Omega (u_i-u)^2\mathcal{Q}(x,\nabla\varphi)d\mu\right)^{\frac{1}{2}}$$
$$\leq \left(C||\nabla\varphi||^2_{L^\infty(\Omega)}\int_\Omega (u_i-u)^2 d\mu\right)^{\frac{1}{2}} \tag{3.14}$$
$$= C^{\frac{1}{2}}||\nabla\varphi||_{L^\infty(\Omega)}||u_i-u||_{L^2(\Omega)} \to 0$$

since $u_i \to u$ in $L^2(\Omega)$. Combining (3.12), (3.13) and (3.14), we conclude that $||\nabla(\varphi u_i) - \varphi\nabla u + u\nabla\varphi||_{\mathcal{L}^2(\Omega,\mathcal{Q})} \to 0$, completing the proof. □

Lemma 3.2.2 (The Chain Rule)**.** *Suppose Ω is bounded. If $(u,\nabla u) \in W^{1,2}_Q(\Omega)$ and $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$, then $(f\circ u, (f'\circ u)\nabla u) \in \mathcal{W}^{1,2}_Q(\Omega)$.*

Proof. Again, since $(u,\nabla u) \in \mathcal{W}^{1,2}_Q(\Omega)$, there exists a sequence $\{u_i\} \subset Lip_Q(\Omega)$ such that $u_i \to u$ in $L^2(\Omega)$ and $\nabla u_i \to \nabla u$ in $\mathcal{L}^2(\Omega,\mathcal{Q})$.

We note first that $w_i = f\circ u_i \in Lip_Q(\Omega)$ for all $i \in \mathbb{N}$. That w_i is Lipschitz at all follows since f is Lipschitz (being continuous with bounded derivative), and compositions of Lipschitz maps are again Lipschitz. Moreover, the mean value theorem yields the pointwise inequality $|(f(u_i(x)) - f(u(x))| \leq ||f'||_{L^\infty(\mathbb{R})}|u_i(x)-u(x)|$ for almost every $x \in \Omega$, whence

$$||w_i - w||_{L^2(\Omega)} \leq ||f'||_{L^\infty(\mathbb{R})}||u_i-u||_{L^2(\Omega)} \to 0 \tag{3.15}$$

since $u_i \to u$ in $L^2(\Omega)$. Thus w_i converges in $L^2(\Omega)$ and must therefore be bounded (up to a subsequence) in $L^2(\Omega)$. Lastly, we have that $\nabla w_i = f'(u_i)\nabla u_i$. Thus we compute

$$||\nabla w_i||_{\mathcal{L}^2(\Omega,\mathcal{Q})} = ||f'(u_i)\nabla u_i||_{\mathcal{L}^2(\Omega,\mathcal{Q})} \leq ||f'||_{L^\infty(\mathbb{R})}||\nabla u_i||_{\mathcal{L}^2(\Omega,\mathcal{Q})} < \infty$$

since $u_i \in Lip_Q(\Omega)$.

Hence $\{w_i\} \subset Lip_Q(\Omega)$, and we have already shown that $w_i \to w$ in $L^2(\Omega)$. Thus we need only show that $\nabla w_i \to (f'\circ u)\nabla u$ in $\mathcal{L}^2(\Omega,\mathcal{Q})$. To that end, we first compute

$$||f'(u_i)\nabla u_i - f'(u)\nabla u||_{\mathcal{L}^2(\Omega,\mathcal{Q})} = ||f'(u_i)(\nabla u_i - \nabla u) - \nabla u(f'(u_i)-f'(u))||_{\mathcal{L}^2(\Omega,\mathcal{Q})}$$
$$\leq ||f'(u_i)(\nabla u_i-\nabla u)||_{\mathcal{L}^2(\Omega,\mathcal{Q})} + ||\nabla u(f'(u_i)-f'(u)||_{\mathcal{L}^2(\Omega,\mathcal{Q})}. \tag{3.16}$$

For the first term of (3.16), we have that

$$||f'(u_i)(\nabla u_i - \nabla u)||_{\mathcal{L}^2(\Omega,\mathcal{Q})} \to 0 \tag{3.17}$$

analogously to (3.13). For the second, we first write

$$\begin{aligned} ||\nabla u(f'(u_i) - f'(u))||_{\mathcal{L}^2(\Omega,\mathcal{Q})} &= \left(\int_\Omega (f'(u_i) - f'(u))^2 \mathcal{Q}(x,\nabla u) d\mu\right)^{\frac{1}{2}} \\ &= \left(\int_\Omega (f'(u_i)\sqrt{\mathcal{Q}(x,\nabla u)} - f'(u)\sqrt{\mathcal{Q}(x,\nabla u)})^2 d\mu\right)^{\frac{1}{2}}. \end{aligned}$$

For the sequence $\{f'(u_i)\sqrt{\mathcal{Q}(x,\nabla u)}\}$, since $u_i \to u$ in $L^2(\Omega)$, we have that $u_i \to u$ pointwise almost everywhere. Moreover, since f' is continuous and pointwise convergence is preserved under composition with a continuous function, we have that $f'(u_i)\sqrt{\mathcal{Q}(x,\nabla u)} \to f'(u)\sqrt{\mathcal{Q}(x,\nabla u)}$ pointwise almost everywhere. Additionally, for each $i \in \mathbb{N}$,

$$|f'(u_i)\sqrt{\mathcal{Q}(x,\nabla u)}| \leq ||f'||_{L^\infty(\mathbb{R})}\sqrt{\mathcal{Q}(x,\nabla u)}$$

and

$$||f'||_{L^\infty(\mathbb{R})}||\sqrt{\mathcal{Q}(x,\nabla u)}||_{L^2(\Omega)} = ||f'||_{L^\infty(\mathbb{R})}||\nabla u||_{\mathcal{L}^2(\Omega,\mathcal{Q})} < \infty.$$

Thus by Lebesgue's dominated convergence theorem, we get that

$$||\nabla u(f'(u_i) - f'(u))||_{\mathcal{L}^2(\Omega,\mathcal{Q})} \to 0. \tag{3.18}$$

Combining (3.16), (3.17) and (3.18), we conclude that $\nabla w_i \to (f' \circ u)\nabla u$ in $\mathcal{L}^2(\Omega,\mathcal{Q})$, which completes the proof. □

3.3 Assumptions

Our main goal is to prove Hölder continuity of weak solutions to (3.3) in as much generality as possible by establishing a strong Harnack inequality via the inequality of John and Nirenberg. The sufficiency of the structural requirements of this method are well documented in [FP], [SW1] and [MRW], and the necessity of the first two is dealt with in [R]. Thus we make the following assumptions.

Assumption 3.3.1. Suppose that a Sobolev inequality holds on Ω. That is, suppose there exist constants $\sigma > 1$ and $C_1 > 0$ and $\delta > 0$ such that for every δ-local ball $B(x,r)$ (i.e, every ball $B(x,r)$ with $0 < r < \delta \text{dist}(x, \partial\Omega)$), the inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |w|^{2\sigma} d\mu\right)^{\frac{1}{2\sigma}} \leq C_1 \left[r\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} \mathcal{Q}(x,\nabla w) d\mu\right)^{\frac{1}{2}} + \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |w|^2 d\mu\right)^{\frac{1}{2}}\right] \quad (3.19)$$

holds for all $(w, \nabla w) \in \left(\mathcal{W}_Q^{1,2}\right)_0(\Omega)$.

This is a local estimate which tells us that the first component of a weak solution $(w, \nabla w)$ satisfies a stronger integrability condition than membership in $L^2(\Omega)$. The constant σ is referred to as the Sobolev gain factor, and determines precisely how much the integrability increases.

Assumption 3.3.2. Suppose that a Poincaré inequality holds on Ω. That is, there are constants $C_2 > 0, b \geq 1$ and $\delta > 0$ so that for every δ-local ball $B(x,r)$, the inequality

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |w - w_{B(x,r)}| d\mu \leq C_2 r \left(\frac{1}{|B(x,br)|}\int_{B(x,br)} \mathcal{Q}(x,\nabla w) d\mu\right)^{\frac{1}{2}} \quad (3.20)$$

holds for all $(w, \nabla w) \in \mathcal{W}_Q^{1,2}(\Omega)$, where

$$w_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} w d\mu$$

is the average value of w on $B(x,r)$.

As another local estimate, the Poincaré inequality ensures that a weak solution w does not deviate too much from its mean value on a ball. Note that the right-hand side may be rewritten as

$$C_2 r \frac{1}{|B(x,br)|^{\frac{1}{2}}} ||\nabla w||_{\mathcal{L}^2(\Omega,\mathcal{Q})},$$

so that the Poincaré estimate may be seen as saying that the mean oscillation of a weak solution depends on the $\mathcal{L}^2(\Omega, \mathcal{Q})$ norm of its gradient. It is presented as in (3.20) since that form will be most useful in the coming sections.

Assumption 3.3.3. Suppose there exist positive constants C_3, τ, δ and N such that for each δ-local ball $B(x,r)$, there is an accumulating sequence $\{\eta_j\}_{j\in\mathbb{N}}$ of Lipschitz cutoff functions (ASLCOF) on $B(x,r)$ satisfying the following properties:

$$\begin{cases} \text{supp}(\eta_1) \subset B(x,r), \\ 0 \leq \eta_j \leq 1 \quad \text{for all } j, \\ \text{supp}(\eta_{j+1}) \subset \{y \in B(x,r) : \eta_j(x) = 1\} \ \text{ for all } j, \\ B(x,\tau r) \subset \{y \in B(x,r) : \eta_j(x) = 1\} \quad \text{for all } j, \\ ||\sqrt{Q}\nabla\eta_j||_{(L^\infty(B(x,r)))^n} \leq C_3\frac{N^j}{r} \ \text{ for all } j. \end{cases} \tag{3.21}$$

The first four conditions in (3.21) are what give the sequence that name. The idea, as will be illustrated in an example to follow, is that as n increases, the functions bunch up on the inner ball $B(x,\tau r)$, with their supports approaching precisely the boundary of that ball. The fifth condition is one we add so that the quadratic forms with respect to the ASLCOF functions (which are almost everywhere differentiable by the Rademacher-Stepanov theorem) are bounded in a very specific way.

Example 3.3.4. Define the sequenece $\{a_n\} \subset \mathbb{R}$ by

$$a_n = \frac{1+2^{n-1}}{2^{n-1}}.$$

For each $n \in \mathbb{N}$, define also $f_n, g_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = 2^{2n+1}(x+a_n)^2 \qquad \text{and} \qquad g_n(x) = -2^{2n+1}(x+a_{n+1})^2 + 1.$$

Then the sequence of functions

$$h_n(x) = \begin{cases} 0 & \text{if } |x| \geq a_n \\ f_n(x) & \text{if } -a_n \leq x \leq -\frac{a_n+a_{n+1}}{2} \\ g_n(x) & \text{if } -\frac{a_n+a_{n+1}}{2} \leq x \leq -a_{n+1} \\ 1 & \text{if } |x| \leq a_{n+1} \\ f_n(-x) & \text{if } a_{n+1} \leq x \leq \frac{a_n+a_{n+1}}{2} \\ g_n(-x) & \text{if } \frac{a_n+a_{n+1}}{2} \leq x \leq a_n \end{cases}$$

is an accumulating sequence of Lipschitz cutoff functions on the ball $B(0,2) \subset \mathbb{R}$ with $\tau = \frac{1}{2}$. It is easy to check that for each n, h_n is continuous and differentiable.

Moreover, the derivatives are clearly bounded, making h_n Lipschitz for all $n \in \mathbb{N}$ and the conditions on the supports of the functions are easily seen to be satisfied from the definition. For the sake of clarity, we also present below an illustration of $h_1, \ldots, h_5$.

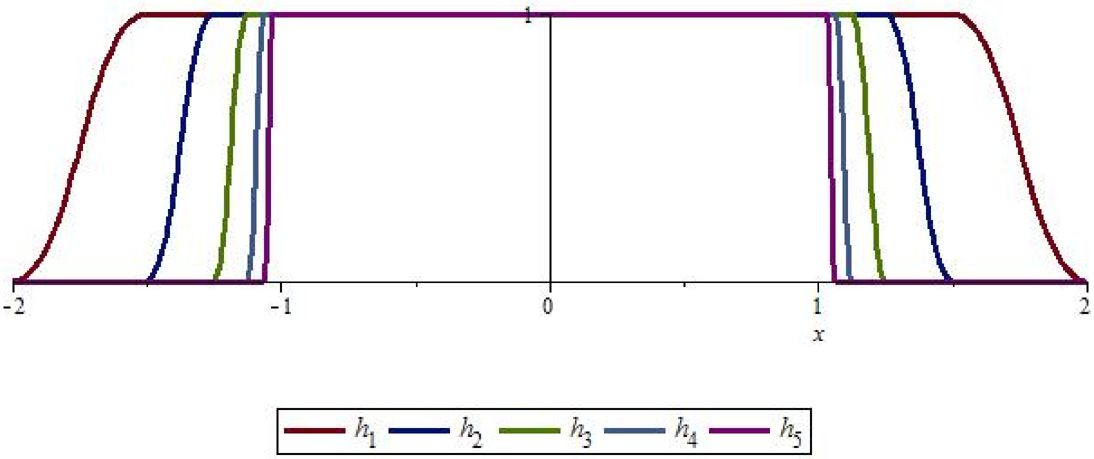

Figure 3.1: A graph of $h_1, \ldots, h_5$

3.4 An Example

To illustrate the connection between the matrix $Q(x)$ and the quasimetric d we give an example from [HK]. Let Ω_0 be a bounded domain in $\mathbb{R}^2$, fix a domain Ω with $\Omega \Subset \Omega_0$, and define

$$Q(x) = \begin{pmatrix} 1 & 0 \\ 0 & x^4 \end{pmatrix}.$$

Notice that Q degenerates on the y-axis. Consider the related degenerate elliptic partial differential equation

$$\mathrm{Div}(Q\nabla u) = f \text{ in } \Omega \tag{3.22}$$

where it is useful to note that $\mathrm{div}(Q\nabla u) = \frac{\partial^2 u}{\partial x^2} + x^4 \frac{\partial^2 u}{\partial y^2}$. Then, this equation satisfies our hypotheses as outlined. There is a large body of general results in [HK] connected to the verification of this and we will only give an overview of them in this specific case and omit the proofs. The interested reader is strongly encouraged to read chapters 11 and 13 of [HK] for a thorough treatment.

Connected to the matrix $Q(x)$ and the partial differential equation (3.22) is a collection of $C^\infty(\Omega)$ vector fields $\mathcal{X} = \{X, Y\}$ given by the square root of the columns of $Q(x)$. That is,

$$\begin{aligned} X = X(x,y) &= (1,0)\cdot\nabla, \text{ and} \\ Y = Y(x,y) &= (0,x^2)\cdot\nabla. \end{aligned}$$

$\mathcal{X}$ acts on any differentiable function f via the identity $\mathcal{X}f = (Xf, Yf) = (\frac{\partial f}{\partial x}, x^2\frac{\partial f}{\partial y})$ producing an $\mathcal{X}$-gradient adapted to Q. We say that $\mathcal{X}f$ is adapted to Q since a simple calculation shows that $|\mathcal{X}f|^2 = \mathcal{Q}(x,\nabla f) = (\nabla f)^T Q(x)\nabla f$.

The important item for this collection $\mathcal{X}$ of vector fields is that it satisfies what is known as the Hörmander condition. That is, the collection of commutators of X, Y of length $\ell \le 3$ span $\mathbb{R}^2$ at every point $x \in \mathbb{R}^2$ and, in particular, for all $x \in \Omega_0$. See [HK, Ch. 11.4] for more on this condition. This property is the key to verifying the hypotheses of the thesis where our quasi metric is to be chosen as the famous Carnot-Carathéodory control metric ρ. The Carnot-Carathéodory control metric $\rho(x,y)$ is defined as

$$\begin{aligned} \rho(x,y) &= \inf\{T > 0 \ : \ \exists \text{ an admissible curve } \gamma \text{ so that} \\ &\qquad \gamma(0) = x \text{ and } \gamma(T) = y\}. \end{aligned} \tag{3.23}$$

A curve γ is called admissible with respect to $\mathcal{X}$ if both of the following hold.

- $\gamma = (\gamma_1(t), \gamma_2(t)) : [a,b] \to \mathbb{R}^2$ is absolutely continuous.
- There are measurable functions $c_j(t),\ a \le t \le b$ satisfying $c_1^2(t) + c_2^2(t) \le 1$ and $\frac{d\gamma}{dt} = c_1(t)X(\gamma(t)) + c_2(t)Y(\gamma(t)) = (c_1(t), c_2(t)\gamma_2^2(t))$.

That such curves exist in this context is left to the reader; see [HK]. Theorems 11.19, 11.20, and 13.1 of [HK] give that the collection of metric balls $\mathcal{B} = \{B(x,r)\}_{x\in\Omega, r>0}$ where

$$B(x,r) = \{y \in \Omega \ : \ \rho(x,y) < r\} \tag{3.24}$$

has the following properties.

1. ρ is equivalent to the Euclidean metric. More precisely, there are positive constants C_1, C_2 so that

$$C_1|x-y| \leq \rho(x,y) \leq C_2|x-y|^{1/3}$$

for all $x, y \in \Omega$.

2. Lebesgue measure is doubling on $\mathcal{B}$. That is, there are positive constants C, r_0 so that $|B(x,2r)| \leq C|B(x,r)|$ for all $x \in \Omega$ and $0 < r \leq r_0$.

3. A Poincaré inequality holds for Lipschitz functions defined on a fixed metric ball. There is a positive constant C_3 so that

$$\begin{aligned} \fint_B |f - f_B| dx &\leq C_3 r\Big(\fint_{2B} |\mathcal{X}f|^2 dx\Big)^{1/2} \\ &= C_3 r\Big(\fint_{2B} |(\frac{\partial f}{\partial x}, x^2\frac{\partial f}{\partial y})|^2 dx\Big)^{1/2} \\ &= C_3 r\Big(\fint_{2B} \mathcal{Q}(x, \nabla f) dx\Big)^{1/2} \end{aligned}$$

for all $f \in Lip(2B)$ where $B = B(x,r)$ with $r \leq r_0$ and $2B = B(x,2r)$.

4. A Sobolev inequality holds. There is a $C_4 > 0$ and a $\sigma > 1$ so that

$$\begin{aligned} \Big(\fint_B |g|^{2\sigma} dx\Big)^{1/2\sigma} &\leq C_4 r\Big(\fint_B |\mathcal{X}g|^2 dx\Big)^{1/2} \\ &= C_4 r\Big(\fint_B \mathcal{Q}(x, \nabla g) dx\Big)^{1/2} \end{aligned}$$

for all $g \in Lip_0(B)$ where $B = B(x,r)$ with $r \leq r_0$. Note that $\sigma > 1$ is related to the doubling constant C in item (2).

For the reader's convenience we mention where each result may be found in [HK]. Items (1) and (2) are the content of theorem 11.19. Item (3) is gleaned from theorem 11.20. Item (4) is due to theorem 13.1 but it is important to put this in context with the definition of 2-admissible weights on p. 79 where the weight $w(x)$ is chosen to be 1 so that the resulting measure μ (as in [HK] theorem 13.1) is Lebesgue measure. Note that theorem 13.1 requires both items (2) and (3).

Lastly, we mention that the existence of the accumulating sequence of Lipschitz cut-off functions related to a control ball $B(x,r)$ with $r > 0$ sufficiently small is also assured. We will not state this explicitly but mention that this result is found in [SW1, Proposition 51] as the quadratic form $\mathcal{Q}(x)$ is continuous.

Chapter 4

Harnack's Inequality and Hölder Continuity of Weak Solutions

Suppose $(u, \nabla u) \in \mathcal{W}_Q^{1,2}(\Omega)$ is a weak solution to (3.3) with $u \geq 0$. In this chapter, we will prove that under the appropriate conditions, u is locally Hölder continuous on Ω. We begin by establishing a generalization of the classical inequality of John and Nirenberg in a homogeneous space setting.

4.1 The John-Nirenberg Inequality and an A_2 Weight Corollary

In the preceeding chapters, we placed many restrictions on the space Ω on which our weak solution $(u, \nabla u)$ is defined. We now begin the process of restricting the behavior of the weak solution itself, as well as the data function f. To that end, we give several new definitions. Please note that for this section only, the measure μ is considered only to be some suitable doubling measure, of which Lebesgue measure is a special case.

Definition 4.1.1. Given $g \in L^1_{\text{loc}}(\Omega)$, we say that $g \in \text{BMO}(\Omega)$, or that g is of bounded mean oscillation in Ω, if there exists $C > 0$ such that

$$\sup_{B \in \mathcal{B}} \frac{1}{|B|_\mu} \int_B |g - g_B| \, d\mu \leq C \tag{4.1}$$

where

$$g_B = \frac{1}{|B|_\mu} \int g d\mu \tag{4.2}$$

is the average value of g on the ball B. If $g \in \text{BMO}(\Omega)$, we write

$$\|g\|_{\text{BMO}(\Omega)} = \inf \left\{ C > 0 : \sup_{B \in \mathcal{B}} \frac{1}{|B|_\mu} \int_B |g - g_B| \, d\mu \leq C \right\}.$$

We also say that $g \in \delta\text{-BMO}(\Omega)$ if g is as above, but while only considering δ-local balls B (see Definition 2.2.4).

Definition 4.1.2. Let $g \in L^1_{loc}(\Omega)$ be a non-negative measurable function. We say that $g \in A_2(\Omega)$, or that g is an A_2 weight on Ω if for some $C > 0$

$$\sup_{B \in \mathcal{B}} \left(\frac{1}{|B|_\mu} \int_B g d\mu \right) \left(\frac{1}{|B|_\mu} \int_B g^{-1} d\mu \right) \leq C. \tag{4.3}$$

If $g \in A_2(\Omega)$, we write

$$\|g\|_{A_2(\Omega)} = \inf \left\{ C > 0 : \sup_{B \in \mathcal{B}} \left(\frac{1}{|B|_\mu} \int_B g d\mu \right) \left(\frac{1}{|B|_\mu} \int_B g^{-1} d\mu \right) \leq C \right\}.$$

δ-$A_2(\Omega)$ is defined analogously to δ-BMO(Ω).

Ultimately, it is this A_2 condition that, when combined with Lemmas 4.2.1 and 4.2.2 will allow us to establish a Harnack inequality and so prove Hölder continuity. Showing directly that $(u, \nabla u)$ is an A_2 weight is very difficult, however. The following results allow us to circumvent this issue.

Lemma 4.1.3 (The John-Nirenberg Inequality)**.** *Given*

$$0 < \delta \leq \frac{8\kappa^5 - 1}{8\kappa^5 C_{euc}},$$

there exist constants $\delta_0, C_1, c_2 > 0$ *such that*

$$|\{x \in B_0 : |g(x) - g_B| > \alpha\}|_\mu \leq C_1 e^{-\frac{c_2 \alpha}{\|g\|_{\delta\text{-}BMO}}} |B_0|_\mu \tag{4.4}$$

for all $\alpha > 0$*,* $g \in \delta$*-BMO*(Ω) *and balls* $B_0 = B_0(x, r) \subset \Omega$ *with* $0 < r < \delta_0 \operatorname{dist}(x, \partial\Omega)$.

Remark 4.1.4. *While Lemma 4.1.3 and its proof are largely taken from [SW1] (though the proof has been expanded significantly), one small but important correction has been made. In the original publication, the theorem was stated to hold for all* $\delta > 0$*, but the argument fails without the above restriction on* δ*. The problem is that for* $\delta > \frac{8\kappa^5 - 1}{8\kappa^5 C_{euc}}$*, subcubes of* δ*-local cubes are not necessarily again* δ*-local, and the proof depends heavily on this assumption.*

Corollary 4.1.5. *Given* $0 < \delta \leq \frac{8\kappa^5 - 1}{8\kappa^5}$*, there are positive constants* δ_0*,* C_1 *and* c_2 *such that* $e^g \in \delta_0$*-*$A_2(\Omega)$ *with* $\|e^g\|_{\delta_0\text{-}A_2(\Omega)} \leq \left(1 + \frac{C_1 \|g\|_{\delta\text{-}BMO(\Omega)}}{c_2 - \|g\|_{\delta\text{-}BMO(\Omega)}}\right)^2$ *whenever* $\|g\|_{\delta\text{-}BMO(\Omega)} < c_2$.

Proof. Let $0 < \delta < \frac{8\kappa^5 - 1}{C_{euc} 8\kappa^5}$ and let δ_0, C_1 and c_2 be as in Lemma 4.1.3. Fix a δ_0-local ball B_0. Then

$$\begin{aligned}\frac{1}{|B_0|_\mu}\int_{B_0} e^{|g-g_{B_0}|}d\mu &= \frac{1}{|B_0|_\mu}\int_{-\infty}^{\infty} e^{\alpha}|\{x\in B_0 : |g-g_{B_0}|>\alpha\}|_\mu d\alpha\\ &\le 1 + C_1\int_0^\infty e^{\alpha\left(1-\frac{c_2}{\|g\|_{\delta\text{-BMO}(\Omega)}}\right)}d\alpha\\ &= 1 + C_1\frac{\|g\|_{\delta\text{-BMO}(\Omega)}}{c_2-\|g\|_{\delta\text{-BMO}(\Omega)}},\end{aligned}$$

assuming that $\|g\|_{\delta\text{-BMO}(\Omega)} < c_2$. The first equality is obtained from Theorem 8.16 of [R3]; the inequality uses (4.4) and the fact that

$$\frac{1}{|B_0|_\mu}\int_{-\infty}^{0} e^{\alpha}|\{x\in B_0 : |g-g_{B_0}|>\alpha\}|_\mu d\alpha = \int_{-\infty}^{0} e^{\alpha}d\alpha = 1.$$

Hence

$$\begin{aligned}&\left(\frac{1}{|B_0|_\mu}\int_{B_0} e^{g}d\mu\right)\left(\frac{1}{|B_0|_\mu}\int_{B_0} e^{-g}d\mu\right)\\ &= \left(\frac{1}{|B_0|_\mu}\int_{B_0} e^{(g-g_{B_0})}d\mu\right)\left(\frac{1}{|B_0|_\mu}\int_{B_0} e^{-(g-g_{B_0})}d\mu\right)\\ &\le \left(\frac{1}{|B_0|_\mu}\int_{B_0} e^{|g-g_{B_0}|}d\mu\right)\left(\frac{1}{|B_0|_\mu}\int_{B_0} e^{|g-g_{B_0}|}d\mu\right)\\ &\le \left(1 + C_1\frac{\|g\|_{\delta\text{-BMO}(\Omega)}}{c_2-\|g\|_{\delta\text{-BMO}(\Omega)}}\right)^2\end{aligned}$$

which completes the proof. □

To summarize, we may show that a function $g \in \delta_0\text{-}A_2(\Omega)$ by showing that $\log g \in \delta\text{-BMO}(\Omega)$. For the rest of the paper, our angle of attack will be to apply this result to modified versions of our weak solution $(u, \nabla u)$. Before we continue, however, we provide a fully detailed proof of Lemma 4.1.3.

Proof of Lemma 4.1.3. Let $0 < \delta \le \frac{8\kappa^5 - 1}{8\kappa^5 C_{euc}}$. Fix $m \in \mathbb{Z}$ and denote by $\mathcal{D}_m$ the collection of dyadic cubes as in Lemma 2.2.2 with $k \ge m$, and by $\delta\text{-}\mathcal{D}_m$ the collection of δ-local dyadic cubes with $k \ge m$. Note that if $Q_{j;k} \in \delta\text{-}\mathcal{D}_m$, then all subcubes $Q_{i,\ell}$ of $Q_{j;k}$ are as well. This is clear if $\ell = k$ and if not, we use the fact that $Q_{j;k} \in \delta\text{-}\mathcal{D}_m$

to obtain

$$\begin{aligned}
\lambda^{\ell+1} &= \frac{\lambda^{k+1}}{\lambda^{k-\ell}} \\
&< \frac{\delta \mathrm{dist}(x_{j;k}\partial\Omega)}{\lambda^{k-\ell}} \\
&\leq \frac{\delta}{\lambda^{k-\ell}} \left(\mathrm{dist}(x_{j;k}, x_{i,\ell}) + \mathrm{dist}(x_{i,\ell}, \partial\Omega)\right) \\
&\leq \frac{\delta C_{euc}\lambda^{k+1}}{\lambda^{k-\ell}} + \frac{\delta \mathrm{dist}(x_{i;m}, \partial\Omega)}{\lambda^{k-\ell}} \\
&\leq \frac{8\kappa^5 - 1}{8\kappa^5}\lambda^{\ell+1} + \frac{\delta \mathrm{dist}(x_{i;m}, \partial\Omega)}{\lambda^{k-\ell}},
\end{aligned}$$

whence

$$\frac{1}{8\kappa^5}\lambda^{\ell+1} < \frac{\delta \mathrm{dist}(x_{i,\ell}, \partial\Omega)}{\lambda^{k-\ell}}.$$

Thus

$$\lambda^{\ell+1} \leq \delta \mathrm{dist}(x_{i,\ell}, \partial\Omega)$$

since $\lambda = 8\kappa^5$.

Part 1: Fix $g \in \delta$-BMO(Ω) and suppose $||g||_{\delta\text{-BMO}(\Omega)} = 1$. We begin by establishing the dyadic distribution inequality: there exist $C, c_2 > 0$ (both independent of m and δ) such that

$$\left|\{x \in Q_0 : |g^m(x) - g_{Q_0}| > \alpha\}\right|_\mu < Ce^{-\frac{c_2\alpha}{||g||_{\delta\text{-BMO}}}} |Q_0|_\mu \tag{4.5}$$

for all $\alpha > 0$ and cubes $Q_0 \in \delta$-$\mathcal{D}_m$. Here

$$g^m(x) = \sum_j g_{Q_{j;m}}\chi_{Q_{j;m}}(x) = \sum_j \left(\frac{1}{|Q_{j;m}|_\mu}\int_{Q_{j;m}} g d\mu\right)\chi_{Q_{j;m}}(x) \tag{4.6}$$

is the expectation of g on the dyadic decomposition of Ω. Define M_μ^Δ, the dyadic maximal operator by

$$M_\mu^\Delta h(x) = \sup_{Q \ni x} \frac{1}{|Q|_\mu}\int_Q |h| d\mu \tag{4.7}$$

where we consider only cubes $Q \in \delta$-$\mathcal{D}_m$. This new maximal operator is weak type (1,1) with constant 1. That is, for $h \in L^1$

$$\left|\{x : M_\mu^\Delta h(x) > \alpha\}\right|_\mu \leq \frac{1}{\alpha}\int_\Omega |h| d\mu. \tag{4.8}$$

The proof of this is analogous to that of part (a) of Lemma B.0.3 in the appendix.

Before we can get to the main section of the proof, we establish some basic inequalities. For any $Q_{j;k} \in \delta\text{-}\mathcal{D}_m$ we have by Lemma 2.2.2 that

$$\begin{aligned}\left|g_{B(x_{j;k},\lambda^{k+1})} - g_{Q_{j;k}}\right| &= \left|\frac{1}{|Q_{j;k}|_\mu}\int_{Q_{j;k}} g d\mu - g_{B(x_{j;k},\lambda^{k+1})}\right| \\ &\leq \frac{1}{|Q_{j;k}|_\mu}\int_{Q_{j;k}} \left|g - g_{B(x_{j;k},\lambda^{k+1})}\right| d\mu \\ &\leq \frac{1}{|B(x_{j;k},\lambda^k)|_\mu}\int_{B(x_{j;k},\lambda^{k+1})} \left|g - g_{B(x_{j;k},\lambda^{k+1})}\right| d\mu\end{aligned} \tag{4.9}$$

Then since $\|g\|_{\delta-\mathrm{BMO}} = 1$, and by (4.9) and the doubling condition we obtain,

$$\begin{aligned}&\frac{1}{|Q_{j;k}|_\mu}\int_{Q_{j;k}} \left|g - g_{Q_{j;k}}\right| d\mu \\ &\leq \frac{1}{|Q_{j;k}|_\mu}\int_{Q_{j;k}} \left|g - g_{B(x_{j;k},\lambda^{k+1})}\right| d\mu + \frac{1}{|Q_{j;k}|_\mu}\int_{Q_{j;k}} \left|g_{B(x_{j;k},\lambda^{k+1})} - g_{Q_{j;k}}\right| d\mu \\ &\leq \frac{1}{|B(x_{j;k},\lambda^k)|_\mu}\int_{B(x_{j;k},\lambda^{k+1})} \left|g - g_{B(x_{j;k},\lambda^{k+1})}\right| d\mu + \left|g_{B(x_{j;k},\lambda^{k+1})} - g_{Q_{j;k}}\right| \\ &\leq \frac{2}{|B(x_{j;k},\lambda^k)|_\mu}\int_{B(x_{j;k},\lambda^{k+1})} \left|g - g_{B(x_{j;k},\lambda^{k+1})}\right| d\mu \\ &\leq \frac{2C_0}{|B(x_{j;k},\lambda^{k+1})|_\mu}\int_{B(x_{j;k},\lambda^{k+1})} \left|g - g_{B(x_{j;k},\lambda^{k+1})}\right| d\mu \\ &\leq 2C_0 \|g\|_{\delta\text{-BMO}(\Omega)} = 2C_0\end{aligned} \tag{4.10}$$

where $C_0 = D\lambda^E$ is a doubling constant as in Proposition 2.3.2.

Fix $Q_0 \in \delta\text{-}\mathcal{D}_m$ and let $h = (g - g_{Q_0})\chi_{Q_0}$. Then for all cubes $Q \supset Q_0$,

$$\frac{1}{|Q|_\mu}\int_Q |h| d\mu \leq \frac{1}{|Q_0|_\mu}\int_{Q_0} |g - g_{Q_0}| d\mu \leq 2C_0 \tag{4.11}$$

by (4.10). For each $\alpha > 0$, define

$$\Omega_\alpha = \{x \in \Omega : M_\mu^\Delta h(x) > \alpha\}. \tag{4.12}$$

If $\alpha \geq 2C_0$, then $\Omega_\alpha \subset Q_0$. To see this, note that for $x \in \Omega_\alpha$

$$M_\mu^\Delta h(x) = \sup_{Q \ni x}\left\{\frac{1}{|Q|_\mu}\int_Q |h| d\mu\right\} > \alpha \geq 2C_0 > 0.$$

So if $x \notin Q_0$, then for any $Q \ni x$, either $Q \supset Q_0$ or $Q \cap Q_0 = \emptyset$ by Lemma 2.2.2. In the first case, this would imply by (4.11) that

$$\frac{1}{|Q|_\mu}\int_Q |h| d\mu \leq 2C_0 \leq \alpha,$$

and in the second case we would simply have that

$$\frac{1}{|Q|_\mu}\int_Q |h|d\mu = \frac{1}{|Q|}\int_Q |f - f_{Q_0}|\chi_{Q_0}d\mu = 0 < \alpha, \tag{4.13}$$

which together would give that $M_\mu^\Delta h(x) \leq \alpha$, a contradiction.

Let

$$\mathcal{C}_\alpha = \left\{ Q \in \delta\text{-}\mathcal{D}_m : \frac{1}{|Q|_\mu}\int_Q |h|d\mu > \alpha \right\}. \tag{4.14}$$

By Lemma 2.2.2 and equations (4.11) and (4.13), for $\alpha > 2C_0$ the cubes in $\mathcal{C}_\alpha$ are all proper subcubes of Q_0. Now, let $\{Q_{\alpha,j}\}_j$ be the maximal collection of cubes in $\mathcal{C}_\alpha$. That is, let $Q_{\alpha,n}$ be a cube of largest measure in $\mathcal{C}_\alpha$ not intersecting any $Q_{\alpha,m}$ for $1 \leq m < n$. Then the collection $\{Q_{\alpha,j}\}_j$ satisfies the following properties:

(a) for all $\alpha \geq 2C_0$, the $Q_{\alpha,j}$ are pairwise disjoint,

(b) for all $\alpha > 2C_0$, $\cup_j Q_{\alpha,j} = \Omega_\alpha \subset Q_0$,

(c) if $2C_0 \leq \alpha < \beta$ and i, j are fixed, then either $Q_{\beta,j} \subset Q_{\alpha,i}$ or $Q_{\alpha,j} \cap Q_{\alpha,i} = \emptyset$.

The first property holds by construction. For property (b), we already have that $\cup_j Q_{\alpha,j} \subset \Omega_\alpha$ since each $Q \in \{Q_{\alpha,\ell}\}_\ell$ satisfies (4.13), so that each $x \in Q$ is as in (4.12). Conversely, for $x \in \Omega_\alpha$, we have that $M_\mu^\Delta(x) > \alpha$. Hence there exists $Q \in \mathcal{C}_\alpha$ such that $x \in Q$ and $\frac{1}{|Q|_\mu}\int_Q |h|d\mu > \alpha$. Now, clearly either $Q \in \{Q_{\alpha,\ell}\}_\ell$ (in which case there is nothing to show) or $Q \cap \cup_\ell Q_{\alpha,\ell} \neq \emptyset$. In this case, let Q' be the largest cube in $\{Q_{\alpha,\ell}\}_\ell$ intersecting Q (that is, the cube with the smallest ℓ). If $x \in Q'$ then we are done, but if $x \notin Q'$ then $Q \supset Q'$ by dyadic structure, contradicting the maximality of $\{Q_{\alpha,\ell}\}_\ell$ since Q should have been chosen instead of Q'. Hence $x \in \cup_\ell Q_{\alpha,\ell}$ in all cases.

For property (c), suppose $Q_{\beta,j} \cap Q_{\alpha,i} \neq \emptyset$. Then by dyadic structure, either $Q_{\beta,j} \subset Q_{\alpha,i}$ or $Q_{\alpha,i} \subset Q_{\beta,j}$. Suppose the latter for the sake of contradiction. Then since $\beta > \alpha$, we have that $Q_{\beta,j} \in \mathcal{C}_\alpha$. So by maximality of $\{Q_{\alpha,\ell}\}_\ell$, it must be that $Q_{\beta,j} \cap Q_{\alpha,i'} \neq \emptyset$ for some $i' < i$ (else $Q_{\beta,j}$ would have been chosen instead of $Q_{\alpha,i}$). Thus by dyadic structure, either $Q_{\beta,j} \subset Q_{\alpha,i'}$ or $Q_{\alpha,i'} \subset Q_{\beta,j}$. The former case would imply that $Q_{\alpha,i} \subset Q_{\alpha,i'}$, which is impossible, so $Q_{\beta,j} \supset Q_{\alpha,i'}$. Finitely many iterations of this process gives that $Q_{\beta,j} \supset Q_{\alpha,1}$ whence $Q_{\alpha,j} = Q_{\alpha,1}$. Thus $Q_{\alpha,i} \subset Q_{\alpha,1}$, a contradiction. Thus $Q_{\beta,j} \subset Q_{\alpha,i}$.

Now, let $\Lambda : \mathbb{R} \to \mathbb{R}$ be the distribution function of $M_\mu^\Delta h(x)$. That is,

$$\Lambda(\alpha) = |\Omega_\alpha|_\mu = |\{x \in \Omega : M_\mu^\Delta h(x) > \alpha\}|_\mu. \tag{4.15}$$

For all $Q = Q_{j;k} \in \delta\text{-}\mathcal{D}_m$, let $\widetilde{Q}$ denote the dyadic predecessor of Q. That is, $\widetilde{Q} = Q_{i;k+1}$ is the unique cube of order $k+1$ containing Q. Then for all cubes Q,

$$|\widetilde{Q}|_\mu \leq \left|B\left(x_{i;k+2}, \lambda^{k+2}\right)\right|_\mu \leq C_0^2 \left|B\left(x_{j;k}, \lambda^k\right)\right|_\mu \leq C_0^2 |Q|_\mu \tag{4.16}$$

by doubling.

Next, we show that

$$|\Omega_{\zeta\alpha} \cap Q_{\alpha,j}|_\mu \leq \frac{1}{2}|Q_{\alpha,j}|_\mu \tag{4.17}$$

if $\zeta = 1 + \frac{4C_0^3}{\alpha}$ and $\alpha \geq 2C_0$. Note first that $\widetilde{Q_{\alpha,j}} \subset Q_0$ since every cube in $\mathcal{C}_\alpha$ is properly contained in Q_0, and therefore is of order at least one less than that of Q_0. The dyadic predecessor of a cube must be of order one higher than the cube itself, forcing it by dyadic structure to be either contained in Q_0, or to be Q_0 itself. Note also that $\left|h_{\widetilde{Q_{\alpha_j}}}\right| \leq |h|_{\widetilde{Q_{\alpha,j}}} \leq \alpha$, since if $\widetilde{Q_{\alpha,j}} \in \mathcal{C}_\alpha$, a similar argument to that used to prove property (c) above would imply that $Q_{\alpha,1} = \widetilde{Q_{\alpha,j}} \supset Q_{\alpha,j}$.

Let $\varphi = \chi_{\widetilde{Q_{\alpha,j}}}\left(h - h_{\widetilde{Q_{\alpha,j}}}\right)$, $\zeta \geq 1$ and $x \in \Omega_{\zeta\alpha} \cap Q_{\alpha_j}$. By properties (b) and (c), there exists a cube $Q_{\zeta\alpha,i} \ni x$ such that $Q_{\zeta\alpha,i} \subset Q_{\alpha,j}$. So

$$\begin{aligned}
\zeta\alpha &< \frac{1}{|Q_{\zeta\alpha,i}|_\mu} \int_{Q_{\zeta\alpha,i}} |h| d\mu \\
&= \frac{1}{|Q_{\zeta\alpha,i}|_\mu} \int_{Q_{\zeta\alpha,i}} \chi_{\widetilde{Q_{\alpha,j}}} |h| d\mu \\
&\leq \frac{1}{|Q_{\zeta\alpha,i}|_\mu} \int_{Q_{\zeta\alpha,i}} |k| d\mu + \left|h_{\widetilde{Q_{\alpha,j}}}\right| \\
&\leq \frac{1}{|Q_{\zeta\alpha,i}|_\mu} \int_{Q_{\zeta\alpha,i}} |k| d\mu + \alpha
\end{aligned}$$

which implies that

$$\frac{1}{|Q_{\zeta\alpha,i}|_\mu} \int_{Q_{\zeta\alpha,i}} |k| d\mu > \zeta\alpha - \alpha = (\zeta - 1)\alpha$$

and thus that for any $x \in \Omega_{\zeta\alpha} \cap Q_{\alpha,j}$,

$$M_\mu^\Delta \varphi(x) \geq (\zeta - 1)\alpha. \tag{4.18}$$

Now, since $\widetilde{Q_{\alpha,j}} \subset Q_0$, we get that

$$\varphi(x) = h(x) - h_{\widetilde{Q_{\alpha,j}}} = (g(x) - g_{Q_0}) - (g - g_{Q_0})_{\widetilde{Q_{\alpha,j}}} = g(x) - g_{\widetilde{Q_{\alpha,j}}}.$$

Hence

$$\begin{aligned}
|\Omega_{\zeta\alpha} \cap Q_{\alpha,j}|_\mu &\leq \left|\left\{M_\mu^\Delta k > (\zeta-1)\alpha\right\}\right|_\mu \\
&\leq \frac{1}{(\zeta-1)\alpha}\int_\Omega |k| d\mu \\
&= \frac{1}{(\zeta-1)\alpha}\int_{\widetilde{Q_{\alpha,j}}} \left|g - g_{\widetilde{Q_{\alpha,j}}}\right| d\mu \\
&\leq \frac{1}{(\zeta-1)\alpha} 2C_0 \left|\widetilde{Q_{\alpha,j}}\right|_\mu \\
&\leq \frac{1}{(\zeta-1)\alpha} 2C_0^3 |Q_{\alpha,j}|_\mu
\end{aligned}$$

by (4.10), (4.16) (4.18) and (4.8), which proves (4.17) with $\zeta = 1 + \frac{4C_0^3}{\alpha}$ and $\alpha > 2C_0$.

We now obtain from (4.17) and properties (a) and (b) of the $\{Q_{\alpha,j}\}_j$ that

$$\Lambda(\alpha + 4C_0^3) = \Lambda(\zeta\alpha) = |\Omega_{\zeta\alpha}|_\mu = \sum_j |\Omega_{\zeta\alpha} \cap Q_{\alpha,j}|_\mu \leq \sum_j \frac{1}{2}|Q_{\alpha,j}|_\mu \leq \frac{1}{2}|\Omega_\alpha|_\mu = \frac{1}{2}\Lambda(\alpha).$$

Moreover, for $\alpha \geq 4C_0^3 + 2C_0$, the above inequality yields

$$\Lambda(\alpha) = \Lambda((\alpha - 4C_0^3) + 4C_0^3) \leq \frac{1}{2}\Lambda(\alpha - 4C_0^3). \tag{4.19}$$

Thus for $\alpha \geq 8C_0^3$, we get by iterating (4.19) $\lfloor \frac{\alpha}{4C_0^3} \rfloor - 1$ times that

$$\begin{aligned}
|\Omega_\alpha|_\mu &= \Lambda(\alpha) \\
&\leq \left(\frac{1}{2}\right)^{\left\lfloor \frac{\alpha}{4C_0^3} \right\rfloor - 1} \Lambda\left(\alpha - \left(\left\lfloor \frac{\alpha}{4C_0^3} \right\rfloor - 1\right) 4C_0^3\right) \\
&\leq \left(\frac{1}{2}\right)^{\frac{\alpha}{4C_0^3} - 2} \Lambda\left(\alpha - \left(\frac{\alpha}{4C_0^3} - 1\right) 4C_0^3\right) \\
&\leq 4\left(\frac{1}{2}\right)^{\frac{\alpha}{4C_0^3}} \Lambda(2C_0) \\
&\leq 4\left(\frac{1}{2}\right)^{\frac{\alpha}{4C_0^3}} |Q_0|_\mu
\end{aligned}$$

since $|\Omega_{2C_0}|_\mu \leq |Q_0|_\mu$. This holds also for $2C_0 \leq \alpha \leq 8C_0^3$ since

$$|\Omega_\alpha| \leq |Q_0|_\mu = 4\left(\frac{1}{2}\right)^{\frac{8C_0^3}{4C_0^3}} \leq 4\left(\frac{1}{2}\right)^{\frac{\alpha}{4C_0^3}} |Q_0|_\mu.$$

Hence we conclude that $|\Omega_\alpha| \leq 4\left(\frac{1}{2}\right)^{\frac{\alpha}{4C_0^3}} |Q_0|_\mu$ for all $\alpha \geq 2C_0$, or equivalently that

$$|\{x \in Q_0 : M_\mu^\Delta h(x) > \alpha\}|_\mu \leq 4e^{-c_2\alpha}|Q_0|_\mu$$

where $c_2 = \frac{\ln 2}{4C_0^3}$. A similar calculation to the above shows that if $\alpha < 2C_0$, then

$$\left|\left\{x \in Q_0 : M_\mu^\Delta h(x) > \alpha\right\}\right|_\mu \leq \left(e^{2C_0c_2}\right)\left(e^{-c_2\alpha}\right)|Q_0|_\mu$$

since $\left\{x \in Q_0 : M_\mu^\Delta h(x) > \alpha\right\} \subset Q_0$. So with $C = \max\left\{e^{2C_0c_2}, 4\right\}$, we obtain

$$\left|\left\{x \in Q_0 : M_\mu^\Delta\left[\left(g - g_{Q_0}\right)\chi_{Q_0}\right](x) > \alpha\right\}\right|_\mu \leq Ce^{-c_2\alpha}|Q_0|_\mu \tag{4.20}$$

for all $\alpha > 0$.

We now wish to show that $M_\mu^\Delta[(g - g_{Q_0})\chi_{Q_0}] \geq |g^m - g_{Q_0}|\chi_{Q_0}$, which when taken together with (4.20) will at last prove (4.5). But for every $Q_{i,m} \subset Q_0$, we have that $Q_{i,m} \in \delta\text{-}\mathcal{D}_m$. Thus given a cube $Q_{i;m} \subset Q_0$,

$$\begin{aligned} M_\mu^\Delta[(g - g_{Q_0})\chi_{Q_0}] &\geq \left|\frac{1}{|Q_{i,m}|_\mu}\int_{Q_{i;m}}(g - g_{Q_0})\chi_{Q_0}d\mu\right| \\ &= \left|\frac{1}{|Q_{i,m}|_\mu}\int_{Q_{i;m}} g\chi_{Q_0}d\mu - \frac{1}{|Q_{i,m}|_\mu}\int_{Q_{i;m}} g_{Q_0}\chi_{Q_0}d\mu\right| \\ &= \left|g^m - g_{Q_0}\right|\chi_{Q_0}. \end{aligned}$$

Hence

$$|\{x \in Q_0 : |g^m(x) - g_{Q_0}|\chi_{Q_0} > \alpha\}|_\mu < \left|\left\{x \in Q_0 : M_\mu^\Delta\left[\left(g - g_{Q_0}\right)\chi_{Q_0}\right](x) > \alpha\right\}\right|_\mu,$$

and (4.5) is proved.

Part 2: We now wish to extend our result from dyadic cubes to quasi-metric balls. That is, we wish to establish

$$|\{x \in B_0 : |g^m(x) - g_{B_0}| > \alpha\}|_\mu < Ce^{\frac{-c_2}{2}\alpha}|B_0|_\mu \tag{4.21}$$

for all δ_0-local balls B_0. Fix $B_0 = B(x, r)$ and $m \in \mathbb{Z}$ with $\lambda^{m+1} < r$. Then there exists $k > m$ such that $\lambda^k < r \leq \lambda^{k+1}$. This gives rise to a collection of dyadic cubes $\{Q_{j;k}\}_{j \in F} \subset \mathcal{D}_m$ where F is an index set and $Q_{j,k} \cap B_0 \neq \emptyset$ for all $j \in F$ such that

$$B_0 \subset \cup_{j \in F} Q_{j;k} \subset B(x_0, (\lambda\gamma)r) = B_0^*,$$

where γ is as in Proposition 2.1.4. We want to show that the cardinality of F, denoted by $\#F$, is bounded above with no dependence on k. But

$$|B_0^*|_\mu \le |B(x_{j;k}, (\lambda\gamma)^2\lambda^k)|_\mu \le D\left(\frac{(\lambda\gamma)^2\lambda^k}{\lambda^k}\right)^E |B(x_{j;k}, \lambda^k)|_\mu \le D(\lambda\gamma)^{2E}|Q_{j;k}|_\mu$$

for each $j \in F$, by doubling and engulfing where D and E are as in Proposition 2.3.2. We will denote $D(\lambda\gamma)^{2E}$ by C'. Summing these inequalities yields

$$(\#F)|B_0^*|_\mu \le C' \sum_{j\in F} |Q_{j;k}|_\mu \le C'|B_0^*|_\mu,$$

so that $\#F \le C'$, as desired..

Let $\delta_0 = \frac{\delta}{C'}$. Then B_0^* is δ-local whenever B_0 is δ_0-local. Then if $B_0 = B(x_0, r)$ is δ_0-local and E is any subset of B_0^* with $|E|_\mu \ge \frac{|B_0^*|}{C'}$, we have

$$|g_{B_0^*} - g_E| = \left|\frac{1}{|E|_\mu}\int_E (g - g_{B_0^*})d\mu\right| \le \frac{C'}{|B_0^*|_\mu}\int_{B_0^*} |g - g_{B_0^*}|d\mu \le C'$$

since B_0^* is δ-local and $\|g\|_{\delta\text{-BMO}(\Omega)} = 1$. In particular, we obtain by doubling that B_0 and $Q_{j;k}$ are sufficiently large for all j, so

$$|g_{B_0} - g_{Q_{j;k}}| \le |g_{B_0^*} - g_{Q_{j;k}}| + |g_{B_0^*} - g_{B_0}| \le 2C' \tag{4.22}$$

for all $j \in F$. Thus for $\alpha > 4C'$, if $|g^m(x) - g_{B_0}| > \alpha$ then

$$\alpha < |g^m(x) - g_{B_0}| \le |g^m(x) - g_{Q_{j;k}}| + 2C' < |g^m(x) - g_{Q_{j;k}}| + \frac{\alpha}{2},$$

by (4.22) implying that $|g^m(x) - g_{Q_{j;k}}| > \frac{\alpha}{2}$. Hence

$$\begin{aligned}
|\{x \in B_0 : |g^m(x) - g_{B_0}| > \alpha\}|_\mu &\le \sum_{j\in F} \left|\left\{x \in Q_{j;k} : \left|g^m(x) - g_{Q_{j;k}}\right| > \frac{\alpha}{2}\right\}\right|_\mu \\
&\le \sum_{j\in F} Ce^{-c_2\frac{\alpha}{2}}|Q_{j;k}|_\mu \\
&\le Ce^{-\frac{c_2}{2}\alpha}|B_0^*|_\mu \\
&\le C'Ce^{-\frac{c_2}{2}\alpha}|B_0|_\mu
\end{aligned}$$

by (4.5) and doubling, where C is as in (4.5). As in part 1, we may extend this result to all $\alpha > 0$. In particular, for $\alpha < 4C'$ we obtain that

$$|\{x \in B_0 : |g^m(x) - g_{B_0}| > \alpha\}|_\mu \le (e^{2c_2C'})e^{-\frac{c_2}{2}\alpha}|B_0|_\mu,$$

so (4.21) is proved with $C_1 = \max\{C'C, e^{2c_2C'}\}$.

Part 3: For the final portion of the argument, we let $m \to -\infty$ and remove our dependence of g^m to obtain (4.4). Note now that given $x \in \Omega$, we can construct a sequence $\{Q_{i_m,m}\}_{m=1}^{-\infty}$ of the unique dyadic cubes of order m containing x. By definition of a homogeneous space these cubes are of bounded eccentricity and their Euclidean diameters tend to 0 as $m \to -\infty$. Thus by Theorem B.0.5, we have that

$$\lim_{m\to-\infty} g^m(x) = \lim_{m\to-\infty} \frac{1}{|Q_{i_m;m}|_\mu} \int_{Q_{i_m,m}} g d\mu = g(x)$$

for almost every $x \in B_0$, as desired.

Finally, we employ Fatou's Lemma (Lemma 1.28 from [R3]) to obtain that

$$\begin{aligned} |\{x \in B_0 : |g(x) - g_{B_0}| > \alpha\}|_\mu &= \int \chi_{\{x\in B_0:|g(x)-g_{B_0}|>\alpha\}} d\mu \\ &\leq \int \liminf_{m\to-\infty} \chi_{\{x\in B_0:|g^m(x)-g_{B_0}|>\alpha\}} d\mu \\ &\leq \liminf_{m\to-\infty} \int \chi_{\{x\in B_0:|g^m(x)-g_{B_0}|>\alpha\}} d\mu \\ &= \liminf_{m\to-\infty} \{x \in B_0 : |g^m(x) - g_{B_0}| > \alpha\} \\ &\leq C_1 e^{-\frac{c_2}{2}\alpha} |B_0|_\mu, \end{aligned}$$

for all $\alpha > 0$. This proves (4.4) for g with $||g||_{\delta\text{-BMO}(\Omega)} = 1$. The general case follows upon replacing g with $\frac{g}{||g||_{\delta\text{-BMO}(\Omega)}}$ and α with $\frac{\alpha}{||g||_{\delta\text{-BMO}(\Omega)}}$. □

4.2 The Mean-value Estimates and Local Boundedness of Weak Solutions

In this section, we briefly present special cases of three powerful results (including the main result in the case of Theorem 4.2.4) from [MRW]. We omit the proofs of these results as they are beyond the scope of this thesis, but they are accessible in [MRW] as indicated.

We begin with the mean-value estimates.

Lemma 4.2.1. *Let $\delta > 0$ be such that* (3.19) *holds for some $\sigma > 0$ and* (3.21) *holds for some $\tau > 0$. Fix a ball $B(x, r)$ with $0 < r < \tau^2\delta\text{dist}(x, \partial\Omega)$. Let $k, \alpha_1 > 0$ be* *given and set* $\bar{u} = u + k$*. Then there exists* $C = C(\alpha_1, \sigma, k, f)$ *such that*

$$\operatorname{ess}\sup_{B(x,\tau r)} \bar{u} \le C \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \bar{u}^{\alpha_1} d\mu \right)^{\frac{1}{\alpha_1}}. \tag{4.23}$$

Lemma 4.2.2. *Let* $\delta > 0$ *be such that* (3.19) *holds for some* $\sigma > 0$ *and* (3.21) *holds for some* $\tau > 0$*. Fix a ball* $B(x,r)$ *with* $0 < r < \tau^2 \delta \operatorname{dist}(x, \partial\Omega)$*. Let* $k > 0$ *and* $\alpha_2 < 0$ *be given and set* $\bar{u} = u + k$*. Then there exists* $C = C(\alpha_2, \sigma, k, f)$ *such that*

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \bar{u}^{\alpha_2} d\mu \right)^{\frac{1}{\alpha_2}} \le C \operatorname{ess}\inf_{B(x,\tau r)} \bar{u}. \tag{4.24}$$

Remark 4.2.3. *(1) Given* $k, \alpha_1 > 0$*, we may choose* $\alpha_2 = -\alpha_1$*.*

(2) If the α_2 *is chosen this way, then the constant* C *appearing in* (4.24) *is identical to that appearing in* (4.23)*.*

Note that as of yet, the integrals on the right-hand side of (4.23) and the left-hand side of (4.24) may not be finite. This third result guarantees that they are:

Theorem 4.2.4. *Let* $\delta > 0$ *be such that* (3.19) *holds for some* $\sigma > 0$ *and* (3.21) *holds for some* $\tau > 0$*. Fix a ball* $B(x,r)$ *with* $0 < r < \tau \delta \operatorname{dist}(x, \partial\Omega)$*. Let* $k > 0$ *and* $\bar{u} = u + k$*. Then* $\bar{u} \in L^\infty(B(x, \tau r))$*.*

The local boundedness of weak solutions given in the previous theorem is an extremely powerful and versatile property which we will exploit later in the paper.

4.3 The Log Estimate and Harnack's Inequality

In Section 4.1, we proved that, given sufficiently small $\delta > 0$ if $\log g \in \delta$-BMO(Ω) and has sufficiently small BMO norm, then $g \in \delta_0$-$A_2(\Omega)$ where δ_0 is as in Lemma 4.1.3. Here, we provide conditions under which a modified version of $(u, \nabla u) \in \delta$-BMO(Ω) in a local sense. Indeed, for the remainder of the paper our arguments will be local in nature, as we will require stricter conditions in terms of boundedness and structure of our sets.

Lemma 4.3.1. *Suppose* Ω *is bounded. Let*

$$0 < \delta < \frac{8\kappa^5 - 1}{C_{euc} 8\kappa^5}$$

be sufficiently small so that (3.20) *holds for some* $b > 0$ *and* (3.21) *holds for some* $\tau > 0$, *and suppose that* $f \in L^\infty(\Omega)$. *Let* $k > 0$ *and define* $\bar{u} = u + k$ *and* $w = \log \bar{u}$. *Fix a ball* $B(x, r)$ *with* $0 < r < \frac{\tau}{b}\delta\text{dist}(x, \partial\Omega)$. *Then there exists a positive constant* $C = C(\tau, N, f, \Omega, \delta)$ *such that*

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |w - w_{B(x,r)}| d\mu \leq C \tag{4.25}$$

where N *is as in* (3.21). *That is,* $w \in \frac{\tau}{b}\delta$-BMO$(\Omega)$.

Proof. Let $\eta \in Lip_0(\Omega)$ with $0 \leq \eta \leq 1$ and let $v = \eta^2\bar{u}^{-1}$. Then by Lemmas 3.2.1 and 3.2.2, we have that $(v, \nabla v) \in \mathcal{W}^{1,2}_Q(\Omega)$ where

$$\nabla v = 2\eta\bar{u}^{-1}\nabla v - \eta^2\bar{u}^{-2}\nabla\bar{u}. \tag{4.26}$$

Moreover,

$$\begin{aligned} \nabla v \cdot Q\nabla\bar{u} + vf(x) &= (2\eta\bar{u}^{-1}\nabla\eta - \eta^2\bar{u}^{-2}\nabla\bar{u}) \cdot Q\nabla\bar{u} + \eta^2\bar{u}^{-1}f(x) \\ &= 2\eta\bar{u}^{-1}\sqrt{Q}\nabla\eta \cdot \sqrt{Q}\nabla\bar{u} - \eta^2\bar{u}^{-2}\nabla u \cdot Q\nabla\bar{u} + \eta^2\bar{u}^{-1}f(x) \\ &\leq 2\eta\bar{u}^{-1}|\sqrt{Q}\nabla\eta||\sqrt{Q}\nabla\bar{u}| - \eta^2\bar{u}^{-2}|\sqrt{Q}\nabla\bar{u}|^2 + \eta^2\frac{|f|}{k}. \end{aligned} \tag{4.27}$$

The last step in (4.27) comes from the Cauchy-Schwarz inequality and from

$$\eta^2\bar{u}^{-1}f(x) \leq \eta^2(u+k)^{-1}\frac{u+k}{k}|f| = \eta^2\frac{|f|}{k}.$$

We apply Young's inequality with $\theta = 2$ to the first term on the right-hand side of (4.27) to obtain

$$2\eta\bar{u}^{-1}|\sqrt{Q}\nabla\eta||\sqrt{Q}\nabla\bar{u}| \leq 8|\sqrt{Q}\nabla\eta|^2 + \frac{|\eta\bar{u}^{-1}\sqrt{Q}\nabla\bar{u}|^2}{2}. \tag{4.28}$$

Combining (4.27) and (4.28), we get that

$$\nabla v \cdot Q\nabla\bar{u} + vf(x) \leq 8|\sqrt{Q}\nabla\eta|^2 - \frac{|\eta\bar{u}^{-1}\sqrt{Q}\nabla\bar{u}|^2}{2} + \eta^2\frac{|f|}{k} \tag{4.29}$$

and using the fact that $(u, \nabla u)$ is a weak solution to (3.3) in Ω and that $Lip_0(\Omega)$ is dense in $Lip_Q(\Omega)$, we integrate (4.29) over $B = B(x, \frac{b}{\tau}r)$ (noting that $B \Subset \Omega$ by construction) and move the second term to the left side, which gives

$$\int_B |\eta\bar{u}^{-1}\nabla\bar{u}|^2 d\mu \leq C\left(\int_B |\sqrt{Q}\nabla\eta|^2 d\mu + \int_B \eta^2\frac{|f|}{k}d\mu\right). \tag{4.30}$$

Applying Hölder's inequality to the second term, we get

$$\int_B \eta^2 \frac{|f|}{k} d\mu \leq \frac{1}{k} ||\eta^2||_{L^1(B)} ||f||_{L^\infty(B)} \leq |B| \frac{||f||_{L^\infty(\Omega)}}{k}$$

since $0 \leq \eta \leq 1$. Setting $D = r^2 \frac{||f||_{L^\infty(\Omega)}}{k}$, we normalize (4.30) to obtain

$$⨍_B |\eta \bar{u}^{-1} \sqrt{Q} \nabla \bar{u}|^2 d\mu \leq C \left(⨍_B |\sqrt{Q} \nabla \eta|^2 d\mu + \frac{D}{r^2} \right). \tag{4.31}$$

Choose $\eta = \eta_1$ from (3.21) relative to the ball $B(x, \frac{b}{\tau} r)$ (which is indeed still δ-local by construction) so that $\eta \equiv 1$ on $B(x, br)$. Then applying the Poincaré inequality (3.20) and Lemma 3.2.2 to $w = \log \bar{u}$, we obtain

$$\begin{aligned}
\frac{1}{|B(x,r)|_\mu} \int_{B(x,r)} |w - w_{B(x,r)}| d\mu &\leq C_2 r \left(\frac{1}{|B(x,br)|_\mu} \int_{B(x,br)} |\sqrt{Q} \nabla w|^2 d\mu \right)^{\frac{1}{2}} \\
&\leq C_2 r \left(\frac{1}{|B(x,br)|_\mu} \int_{B(x,\frac{b}{\tau} r)} \eta^2 |\sqrt{Q} \nabla w|^2 d\mu \right)^{\frac{1}{2}} \\
&= C_2 r \left(\frac{|B(x,\frac{b}{\tau} r)|_\mu}{|B(x,br)|_\mu} \right)^{\frac{1}{2}} \left(⨍_{B(x,\frac{b}{\tau} r)} |\eta \bar{u}^{-1} \sqrt{Q} \nabla u|^2 d\mu \right)^{\frac{1}{2}} \\
&\leq C(\tau) r \left(⨍_B |\sqrt{Q} \nabla \eta|^2 d\mu + \frac{D}{r^2} \right)^{\frac{1}{2}} \\
&\leq C(\tau) r \left(||\sqrt{Q} \nabla \eta||_{L^2(B), \overline{d\mu}} + \frac{D^{\frac{1}{2}}}{r} \right) \\
&\leq C(\tau) r \left(||\sqrt{Q} \nabla \eta||_{(L^\infty(B))^n} + \frac{D^{\frac{1}{2}}}{r} \right) \\
&\leq C(\tau) r \left(\frac{N}{r} + \frac{D^{\frac{1}{2}}}{r} \right) \\
&\leq C(\tau, N)(1 + D^{\frac{1}{2}})
\end{aligned}$$

by doubling and the ASLCOF condition, where N is as in (3.21). Finally, recall that $0 < r < \frac{\tau}{b} \delta \mathrm{dist}(x, \partial\Omega) < \frac{\tau}{b} \delta \mathrm{diam}(\Omega)$ where $\mathrm{diam}(\Omega) = \sup_{y,y' \in \Omega} |y - y'| < \infty$ since Ω is bounded. Hence

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |w - w_{B(x,r)}| d\mu \leq C,$$

where C is independent of x, r. Since $B(x,r)$ was an arbitrary $\frac{\tau}{b}\delta$-local ball, we conclude that $w \in \frac{\tau}{b}\delta\text{-BMO}(\Omega)$. □

The following corollary combines this with some of our other results.

Corollary 4.3.2. *Suppose Ω is bounded. Let $0 < \delta' < \frac{8\kappa^5-1}{8\kappa^5 C_{euc}}$ be sufficiently small that (3.20) holds for some $b > 0$ and (3.21) holds for some $\tau > 0$ and suppose that $f \in L^\infty(\Omega)$. Let $\delta = \frac{\tau}{b}\delta'$ and $k > 0$. Define $\bar{u} = u + k$ and $w = \log \bar{u}$. Then*

(a) For all $\alpha_1 > 0$, $\log \bar{u}^{\alpha_1} \in \delta\text{-BMO}(\Omega)$.

(b) If $\alpha_1 || \log \bar{u}||_{\delta\text{-BMO}(\Omega)} < c_2$, then $\log \bar{u}^{\alpha_1} \in \delta_0\text{-}A_2(\Omega)$, where δ_0, c_2 are as in Lemma 4.1.3.

Proof. Let $\alpha_1 > 0$. For part (a), we have by the above and Lemma 4.3.1 that $\log \bar{u} \in \delta\text{-BMO}(\Omega)$. Moreover, $\log \bar{u}^{\alpha_1} = \alpha_1 \log \bar{u}$ so that $|| \log \bar{u}^{\alpha_1}||_{\delta\text{-BMO}(\Omega)} \leq \alpha_1 || \log \bar{u}||_{\delta\text{-BMO}(\Omega)} < \infty$. Hence $\log u^{\alpha_1} \in \delta\text{-BMO}(\Omega)$.

For part (b), suppose α_1 is sufficiently small such that $\alpha_1 || \log \bar{u}||_{\delta\text{-BMO}(\Omega)} < c_2$. Then $|| \log \bar{u}^{\alpha_1}||_{\delta\text{-BMO}(\Omega)} < c_2$. Hence by Corollary 4.1.5, $e^{\log \bar{u}^{\alpha_1}} = \bar{u}^{\alpha_1} \in \delta_0\text{-}A_2(\Omega)$ as desired. $\square$

With this corollary, we are at last ready to combine all of our results to establish a strong Harnack inequality:

Theorem 4.3.3 (Harnack's Inequality)**.** *Suppose Ω is bounded. Let*

$$0 < \delta' \leq \frac{8\kappa^5 - 1}{8\kappa^5 C_{euc}}$$

be sufficiently small so that (3.19) holds for some $\sigma > 1$, (3.20) holds for some $b > 0$ and (3.21) holds for some $\tau > 0$, and suppose that $f \in L^\infty(\Omega)$. Let $\delta = \frac{\tau^2}{b}\delta'$ and fix a ball $B(x, r)$ with $0 < r < \delta \text{dist}(x, \partial\Omega)$. Let $m(r) = r^2 ||f||_{L^\infty(\Omega)}$ and let $\bar{u} = u + m(r)$. Then there exists $C_{har} > 0$ such that $\bar{u}$ satisfies the strong Harnack inequality

$$\text{ess} \sup_{B(x,\tau r)} \bar{u} \leq C_{har} \text{ess} \inf_{B(x,\tau r)} \bar{u} \tag{4.32}$$

Proof. By Corollary 4.3.2, we may choose a sufficiently small $\alpha_1 > 0$ such that $\bar{u} \in \delta_0 - A_2(\Omega)$ where δ_0 is as in Lemma 4.1.3. Hence

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \bar{u}^{\alpha_1} d\mu \right)^{\frac{1}{\alpha_1}} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \bar{u}^{-\alpha_1} d\mu \right)^{\frac{1}{\alpha_1}} \leq C_{har}$$

where $C_{har} = ||\bar{u}||_{A_2(\Omega)}^{\frac{1}{\alpha_1}}$ is independent of $B(x,r)$. Equivalently,

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} \bar{u}^{\alpha_1} d\mu\right)^{\frac{1}{\alpha_1}} \leq C_{har}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} \bar{u}^{-\alpha_1} d\mu\right)^{\frac{1}{-\alpha_1}}. \tag{4.33}$$

The hypotheses of Lemmas 4.2.1 and 4.2.2 are satisfied, so combining (4.33) with (4.23) and (4.24) and taking into account Remark 4.2.3, we obtain

$$C\text{ess}\sup_{B(x\tau r)} \bar{u} \leq C_{har} C\text{ess}\inf_{B(x\tau r)} \bar{u}$$

whence

$$\text{ess}\sup_{B(x\tau r)} \bar{u} \leq C_{har}\text{ess}\inf_{B(x\tau r)} \bar{u},$$

proving (4.32). □

4.4 Hölder Continuity of Weak Solutions

Having established a Harnack inequality in Theorem 4.3.3, we now use this result to establish local Hölder continuity of our weak solutions using the methods outlined in [D], and later in [M] and [SW1].

Definition 4.4.1. Given $\alpha > 0$, we say that a function $g : \Omega \to \mathbb{R}$ is α-Hölder continuous on Ω with respect to the quasi-metric d if there exists $C > 0$ such that for every $x, x' \in \Omega$,

$$|g(x) - g(x')| \leq Cd(x, x')^{\alpha}. \tag{4.34}$$

It is important to notice that if $\alpha = 1$, then g is Lipschitz with respect to the quasi-metric d.

As we will see in a moment, the machinery from the previous sections is sufficient to establish local Hölder continuity with respect to the quasi-metric d. We often wish, however, to establish Hölder continuity in the classical Euclidean sense in order to test for differentiability of our weak solution. To that end, we shall follow [FP] and impose one final structural assumption on our space:

Definition 4.4.2. Ω is said to satisfy a Fefferman-Phong containment condition if there exist positive constants C, ε and δ such that for every $x \in \Omega$ and $0 < r < \delta\text{dist}(x, \partial\Omega)$

$$D(x,r) \subset B(x, Cr^{\varepsilon}) \tag{4.35}$$

where D is a Euclidean ball.

Lemma 4.4.3. *Let*

$$0 < \delta < \frac{8\kappa^5 - 1}{C_{euc} 8\kappa^5}$$

. *Fix* $y \in \Omega$ *and choose* $0 < \rho < \frac{1}{\gamma}\delta \mathrm{dist}(y, \partial\Omega)$ *where* γ *is as in* (2.1.4). *Then*

1. *For every* $x \in B(y, \rho)$, $B(x, 2\rho)$ *is also* δ*-local.*

2. *If* (Ω, d, μ) *satisfies the containment condition* (4.35), *then for all* $x, x' \in B(y, \rho)$, $d(x, x') \leq C|x - y|^{\varepsilon}$.

Proof. For part 1, given $x \in B(y, \rho)$ we have (since $\gamma \geq 3$) that

$$\begin{aligned} 3\rho &\leq \gamma\rho \\ &< \delta \mathrm{dist}(y, \partial\Omega) \\ &\leq \delta(|y - x| + \mathrm{dist}(x, \partial\Omega)) \\ &\leq \delta(C_{euc}\rho + \mathrm{dist}(x, \partial\Omega)) \\ &\leq \rho + \delta \mathrm{dist}(x, \partial\Omega) \end{aligned}$$

whence

$$2\rho < \delta \mathrm{dist}(x, \Omega).$$

For part 2, suppose (Ω, d, μ) satisfies the containment condition (4.35). Let $x, x' \in B(y, \rho)$. By part 1, the ball $B(x, 2\rho)$ is δ-local, so (4.35) holds for all $B(x, r) \subset B(x, 2\rho)$. Suppose for the sake of contradiction that $d(x, x') > C|x - x'|^{\varepsilon}$. Then $\left(\frac{d(x,x')}{C}\right)^{\frac{1}{\varepsilon}} > |x - x'|$. Set $r = \frac{1}{2}\left(\left(\frac{d(x,x')}{C}\right)^{\frac{1}{\varepsilon}} + |x - x'|\right)$. Then $r > |x - x'|$, so $x' \in D(x, r)$. But $x' \notin B(x, Cr^{\varepsilon}) = B(x, d(x, x'))$, a contradiction. □

Theorem 4.4.4. *Suppose* Ω *is bounded. Let*

$$0 < \delta' \leq \frac{8\kappa^5 - 1}{8\kappa^5 C_{euc}}$$

be sufficiently small that (3.19) *holds for some* $\sigma > 0$, (3.20) *holds for some* $b > 0$ *and* (3.21) *holds for some* $\tau > 0$ *and suppose that* $f \in L^{\infty}(\Omega)$. *Let* $\delta = \frac{\tau^2}{b}\delta'$ *and fix* $y \in \Omega$ *and choose* $0 < \rho < \frac{1}{\gamma}\delta \mathrm{dist}(y, \partial\Omega)$ *where* γ *is as in* (2.1.4). *Then there exists* $\alpha > 0$ *such that* $(u, \nabla u)$ *is* α*-Hölder continuous on* $B(y, \rho)$ *with respect to the quasi-metric* d. *Moreover, if the containment condition* (4.35) *holds, then* $(u, \nabla u)$ *is* $\alpha\varepsilon$*-Hölder continuous in the Euclidean metric.*

Proof. Let $0 < r \le \rho$ and define

$$\omega_y(r) = \operatorname{ess}\sup_{x\in B(y,r)} u(x) - \operatorname{ess}\inf_{x\in B(y,r)} u(x)$$

If $\omega_y(r) = 0$, then u is constant on $B(y,r)$, so there is nothing to show. So assume that $\omega_y(r) > 0$. Moreover, by Theorem 4.2.4, $u \in L^\infty(B(y,r))$, so $\omega_y(r) < \infty$. Now, define

$$M = -\frac{1}{2}\left(\operatorname{ess}\sup_{x\in B(y,r)} u(x) + \operatorname{ess}\inf_{x\in B(y,r)} u(x)\right).$$

Then we have that

$$\begin{aligned}
&\operatorname{ess}\sup_{x\in B(y,r)} (u(x)+M)\\
&= \operatorname{ess}\sup_{x\in B(y,r)} u(x) - \frac{1}{2}\operatorname{ess}\sup_{x\in B(y,r)} u(x) - \frac{1}{2}\operatorname{ess}\inf_{x\in B(y,r)} u(x)\\
&= \frac{1}{2}\omega_y(r)
\end{aligned}$$

and similarly that

$$-\operatorname{ess}\inf_{x\in B(y,r)}(u(x)+M) = \frac{1}{2}\omega_y(r).$$

Let $u_+ = 1 + \frac{u+M}{\frac{1}{2}\omega_y(r)}$ and $u_- = 1 - \frac{u+M}{\frac{1}{2}\omega_y(r)}$. Then since $(u,\nabla u)$ is a weak solution to (3.3) in Ω, we obtain by Lemma 3.2.2 with $f(x) = 1 + \frac{x+M}{\frac{1}{2}\omega_y(r)}$ that $(u_+,\nabla u_+)$ is a weak solution to

$$\operatorname{div}(Q\nabla u_+) = \frac{f}{\frac{1}{2}\omega_y(r)}$$

in Ω, and similarly $(u_-,\nabla u_-)$ is a weak solution to

$$\operatorname{div}(Q\nabla u_-) = -\frac{f}{\frac{1}{2}\omega_y(r)}$$

in Ω. Note that all results that held for (3.3) will also hold for these equations as well upon replacing f with $\frac{f}{\frac{1}{2}\omega_y(r)}$. (Note that in the case of the log estimate, Lemma 4.3.1, the quantity D will not depend on $\omega_y(r)$ when $k = r^2\frac{||f||_{L^\infty(\Omega)}}{\omega_y(r)}$). In particular, the Harnack inequality holds for these equations. Both u_+ and u_- are nonnegative on $B(y,r) \supset B(y,\tau r)$ (where τ is as in (4.32)) by construction. Moreover, since $u_+ + u_- = 2$, we have that either $\operatorname{ess}\sup_{x\in B(y,\tau r)} u_+(x) \ge 1$ or $\operatorname{ess}\sup_{x\in B(y,\tau r)} u_-(x) \ge 1$. Suppose the former holds. Then

$$\operatorname{ess}\sup_{x\in B(y,\tau r)} (u_+(x) + 2\omega_y(r)^{-1}m(r)) \ge \operatorname{ess}\sup_{x\in B(y,\tau r)} u_+(x) \ge 1$$

where $m(r)$ is as in Theorem 4.3.3. Applying an adapted version of the Harnack inequality (4.32), we see that

$$1 \leq C_{har} \text{ess} \inf_{x \in B(y,\tau r)} (u_+(x) + 2\omega_y(r)^{-1} m(r))$$

or equivalently that

$$C' \leq \text{ess} \inf_{x \in B(y,\tau r)} (u_+(x) + 2\omega_y(r)^{-1} m(r))$$

where $C' = C_{har}^{-1}$. Hence

$$0 < C' \leq u_+(x) + 2\omega_y(r)^{-1} m(r)$$

for almost every $x \in B(y, \tau r)$. Expanding this, we have

$$C' \leq 1 + \frac{u(x) + M}{\frac{1}{2}\omega_y(r)} + 2\omega_y(r)^{-1} m(r)$$

or equivalently

$$-\frac{1}{2}\omega_y(r)(1 - C') - m(r) \leq u(x) + M \leq \frac{1}{2}\omega_y(r)$$

for almost every $x \in B(x, \tau r)$. This means that

$$\text{ess} \inf_{x \in B(y,\tau r)} u(x) \geq -\frac{1}{2}\omega_y(r)(1 - C') - m(r)$$

and

$$\text{ess} \sup_{x \in B(y,\tau r)} (u(x) + M) \leq \frac{1}{2}\omega_y(r).$$

Thus we obtain at last

$$\begin{aligned} \omega_y(\tau r) &= \text{ess} \sup_{x \in B(y,\tau r)} u(x) - \text{ess} \inf_{x \in B(y,\tau r)} u(x) \\ &= \text{ess} \sup_{x \in B(y,\tau r)} (u(x) + M) - \text{ess} \inf_{x \in B(y,\tau r)} (u(x) + M) \\ &\leq \frac{1}{2}\omega_y(r) + \frac{1}{2}\omega_y(r)(1 - C') + m(r) \\ &= \left(1 - \frac{1}{2}C'\right) \omega_y(r) + m(r) \end{aligned} \tag{4.36}$$

for $0 < r \leq \rho$. A similar calculation shows that the estimate (4.36) also holds if $u_- \geq 1$. We now apply Lemma 8.23 from [GT], which we present here:

Lemma 4.4.5. *Let ω_y be a nondecreasing function on an interval $(0, R_0]$ satisfying, for all $R \leq R_0$, the inequality*

$$\omega_y(\tau R) \leq \gamma \omega_y(R) + \sigma(R)$$

where σ is also nondecreasing and $0 < \gamma, \tau < 1$. Then for any $\theta \in (0,1]$ and $R \leq R_0$, we have

$$\omega_y(R) \leq C\left(\left(\frac{R}{R_0}\right)^{\alpha} \omega_y(R_0) + \sigma\left(R^{\theta} R_0^{1-\theta}\right)\right)$$

where $C = C(\gamma)$ and $\alpha = \alpha(\gamma, \tau, \theta)$ are positive constants.

The application is obvious with $\sigma = m$ and $\gamma = (1 - \frac{1}{2}C')$ (by conincidence, the τ and ω_y in the lemma and the rest of the proof coincide). Hence we obtain

$$\omega_y(r) \leq C\left(\left(\frac{r}{\rho}\right)^{\alpha} \omega_y(\rho) + m(r^{\theta}\rho^{1-\theta})\right). \tag{4.37}$$

Examining the proof of Lemma 8.23 of [GT], we find the precise definition of α:

$$\begin{aligned}\alpha &= (1-\theta)\frac{\log \gamma}{\log \tau} \\ &= (\theta - 1)\log_{\frac{1}{\tau}}\left(1 - \frac{1}{2}C'\right) > 0.\end{aligned}$$

Now, note that

$$m(r^{\theta}\rho^{1-\theta}) = (r^{\theta}\rho^{1-\theta})||f||_{\frac{q}{2}} = \left(\frac{r}{\rho}\right)^{\theta} \rho ||f||_{\frac{q}{2}} = \left(\frac{r}{\rho}\right)^{\theta} m(\rho). \tag{4.38}$$

Substituting (4.38) into (4.37), we obtain

$$\omega_y(r) \leq C\left(\left(\frac{r}{\rho}\right)^{\alpha} \omega_y(\rho) + \left(\frac{r}{\rho}\right)^{\theta} m(\rho)\right) \tag{4.39}$$

for all $0 < r \leq \rho < 1$ and $\theta \in (0,1)$, where C depends only on C_{har}. Choosing θ such that

$$\alpha = (\theta - 1)\log_{\frac{1}{\tau}}\left(1 - \frac{1}{2}C'\right) < \theta,$$

(4.39) becomes

$$\omega_y(r) \leq C(\omega_y(\rho) + m(\rho))\left(\frac{r}{\rho}\right)^{\alpha} \tag{4.40}$$

for $0 < r \leq \rho < 1$, where C depends only on C_{har}.

Fix $x, x' \in B(y,\rho)$. We wish to use (4.40) to deduce Hölder continuity of u on $B(y,\rho)$. Note that the estimate (4.40) holds for any δ-local ball $B(y', r)$ with $r < \rho$, so by Lemma 4.4.3, it holds for $B(y',\rho)$ for any $y' \in B(y,\rho)$. If $d(x,x') \geq \rho$, then

$$\begin{aligned} |u(x) - u(x')| &\leq \omega_y(\rho) \\ &\leq C(\omega_y(\rho) + m(\rho)) \left(\frac{\rho}{\rho}\right)^{\alpha} \\ &\leq C(\omega_y(\gamma\rho) + m(\rho)) \left(\frac{d(x,x')}{\rho}\right)^{\alpha}. \end{aligned}$$

So suppose $d(x,x') < \rho$. Let $\beta_0 = \frac{\rho}{d(x,x')} > 1$. Then for every $1 < \beta \leq \beta_0$ we have both that $x' \in B(x, \beta d(x,x'))$ and, since $\beta d(x,x') \leq \rho$, that $B(x, \beta d(x,x'))$ is δ-local by Lemma 4.4.3. Hence

$$\begin{aligned} |u(x) - u(x')| &\leq \omega_x(\beta d(x,x')) \\ &\leq C(\omega_x(\rho) + m(\rho)) \left(\frac{\beta d(x,x')}{\rho}\right)^{\alpha} \\ &\leq C(\omega_y(\gamma\rho) + m(\rho)) \left(\frac{\beta d(x,x')}{\rho}\right)^{\alpha} \end{aligned}$$

for every $1 < \beta \leq \beta_0$. Taking the limit as $\beta \to 1^+$, we obtain once again that

$$|u(x) - u(x')| \leq C(\omega_y(\gamma\rho) + m(\rho)) \left(\frac{d(x,x')}{\rho}\right)^{\alpha}. \tag{4.41}$$

Thus u is α-Hölder continuous with respect the the d on $B(y,\rho)$. Moreover, if (Ω, d, θ) satisfies the containment condition (4.35), then by Lemma 4.4.3, we may replace $d(x,x')$ by $C|x - x'|^{\varepsilon}$, so that u is $\alpha\varepsilon$-Hölder continuous in the Euclidean sense on $B(y,\rho)$.

□

Chapter 5

Discussion

In this thesis we have been following primarily the work of Sawyer and Wheeden in [SW1], which in turn follows the work of DeGiorgi and Moser in [D] and [M]. Although we have dealt soley with the simple equation (3.3), the methods used here generalize to more complicated equations. In particular, they apply to general second order degenerate linear elliptic equations of the form

$$\operatorname{div}(Q\nabla u) + \mathbf{HR}u + \mathbf{S'G}u + Fu = f \text{ in } \Omega \tag{5.1}$$

where $\mathbf{R} = (R_1, \ldots, R_n)$ and $\mathbf{S} = (S_1, \ldots, S_n)$ are n-tuples of first order vector fields ($\mathbf{S'}$ denotes the formal adjoint of $\mathbf{S}$) subunit with respect to $\mathcal{Q}(x, \xi)$ in Ω, $\mathbf{H}$ and $\mathbf{G}$ are measurable $\mathbb{R}^n$-valued functions on Ω and F, f are measurable real-valued functions on Ω. A first-order vector field $V(x) = \sum_{i=1}^{n} v_j(x) D_j$ identified with the vector $\mathbf{v}(x) = (v_1(x), \ldots, v_n(x))$ is called subunit with respect to $\mathcal{Q}(x, \xi)$ in Ω if

$$(\mathbf{v}(x)^\top \cdot \xi)^2 \leq \mathcal{Q}(\mathbf{v}(x), \xi) \text{ for almost every } x \in \Omega, \xi \in \mathbb{R}^n.$$

Regularity and existence of solutions to equations of this type have been treated in [SW1] and [R]. There, the core of the arguments are identical to those found above. The lower-order terms $\mathbf{H}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{G}$, and F are estimated separately in all of the important inequalities.

More recently, Montecelli, Rodney and Wheeden have treated a still-more general class of second order degenerate quasi-linear elliptic equations in [MRW] and a sequel currently in pre-publication. Given $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, these are equations of the form

$$\operatorname{div}(A(x, u(x), \nabla u(x))) = B(x, u(x), \nabla u(x)) \text{ for } x \in \Omega. \tag{5.2}$$

In (5.2), the functions A, B are assumed to satisfy the following: there exists a vector

$\tilde{A}(x,z,\xi)$ such that for almost every $x \in \Omega$ and every $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$, the conditions

$$\begin{cases} A(x,z,\xi) = \sqrt{Q(x)}\tilde{A}(x,z,\xi) \\ \xi \cdot A(x,z,\xi) \geq a^{-1}|\sqrt{Q(x)}\xi|^p - h(x)|z|^\gamma - g(x) \\ |\tilde{A}(x,z,\xi)| \leq a|\sqrt{Q(x)}\xi|^{p-1} + b(x)|z|^{\gamma-1} + e(x) \\ |B(x,z,\xi)| \leq c(x)|\sqrt{Q(x)}\xi|^{\psi-1} + d(x)|z|^{\delta-1} + f(x) \end{cases}$$

hold relative to some particular nonnegative definite symmetric matrix $Q(x)$ with $|Q(x)| \in L^1_{loc}(\Omega)$, where $a, \gamma, \psi, \delta > 1$ are constants and b, c, d, e, f, g, h are nonnegative functions of x in Ω. There, the same basic structure of argument is used: showing that a weak solution is an A_2 weight using a (now greatly modified) version of the John-Nirenberg inequality and combining that with mean-value estimates to establish a local Harnack inequality. Given all this, it appears that a large nail has been put in the coffin, so to speak, for the regularity of degenerate (quasi)-linear elliptic equations. The reader is strongly encouraged to investigate this further. The modifications required to adapt the theory presented here to the more general cases just described are non-trivial and very interesting.

Appendix A

The Sobolev Space $W^{1,2}(\Omega)$

Let (Ω, d, μ) be a homogeneous space. In order to more rigorously define $W^{1,2}(\Omega)$, we must first weaken our notion of the derivative of a function $u : \Omega \to \mathbb{R}$. Recall that if u is differentiable and $\varphi \in C_c^\infty(\Omega)$, then the integration by parts formula yields

$$\int_\Omega u D_i \varphi d\mu = \int_{\partial\Omega} u\varphi ds - \int_\Omega D_i u \varphi d\mu = -\int_\Omega D_i u \varphi d\mu. \tag{A.1}$$

This equivalence motivates the following definitions:

Definition A.0.1. Let $u \in L^1_{loc}(\Omega)$. Then for $1 \le i \le n$ we say that v is the weak partial derivative of u in the direction x_i, (or the partial derivative of u in the sense of distributions in the direction x_i) if

$$\int_\Omega u D_i \varphi d\mu = -\int_\Omega v \varphi d\mu$$

for every $\varphi \in C_c^\infty(\Omega)$. If for each $i = 1, \ldots, n$, u has a weak derivative in direction x_i, we simply call u weakly differentiable on Ω.

Definition A.0.2. Let $u \in L^1_{loc}(\Omega)$ be weakly differentiable. Then we define the weak gradient of u by

$$\nabla u = (v_1, \ldots, v_n) \tag{A.2}$$

where v_i is the weak partial derivative of u in the direction x_i.

Remark A.0.3. *Note that derivatives defined in this way are unique if they exist (see Chapter 1 of [AF]). Hence if u is differentiable, its weak derivatives will coincide with its partial derivatives by* (A.1). *That is, weak-derivatives are generalizations of derivatives. Similarly, the weak gradient ∇ is a generalization of the classical gradient, since for a differentiable function, the two are identical.*

Definition A.0.4. Let $\Omega \subset \mathbb{R}^n$ be open. We define the Sobolev space $W^{1,2}(\Omega)$ by

$$W^{1,2}(\Omega) = \{u \in L^2(\Omega) : \nabla u \in (L^2(\Omega))^n\}$$

where ∇u is as in (A.2).

Appendix B

Lebesgue's Differentiation Theorem for Homogeneous Spaces

Here, we take a moment to generalize Lebesgue's differentiation theorem to a homogeneous space (Ω, d, μ). The following covering lemma is standard, and is available on page 843 of [SW2], so we present it without proof. The proofs of the other results are adapted from arguments used in [F] and [SM].

Lemma B.0.1 (Vitali)**.** *Suppose* $\{B_\alpha\}_{\alpha\in A}$ *is a family of balls contained in some fixed ball* $B \subset \Omega$*. Then there is a countable subcollection* $\{B_i\}_{i\in I}$ *of these balls such that*

(i) $B_i \cap B_j = \emptyset \ \text{ if } i \neq j$,

(ii) Every B_α *is contained in some* B_i^**, where* B_i^* *is the ball concentric with* B_i *and radius* $\kappa + 4\kappa^2$ *times that of* B_i*,*

(iii) $|\cup_{\alpha\in A} B_\alpha|_\mu \leq C' \sum_{i\in I} |B_i|_\mu$,

and where $C' = C'(\kappa, C)$*,* κ *is as in* (2.3) *and* C *is as in* (2.15)*.*

Definition B.0.2. Let $f \in L^1_{loc}(\Omega)$. Define the d-Hardy-Littlewood maximal function $M^d(f)$ by

$$M^d(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|_\mu} \int_{B(x,r)} |f| d\mu. \tag{B.1}$$

Define also the *uncentred* d-Hardy-Littlewood maximal function by

$$U^d(f)(x) = \sup_{B\ni x} \frac{1}{|B|_\mu} \int_B |f| d\mu. \tag{B.2}$$

Lemma B.0.3 (The Maximal Theorem)**.** *Let* $f : \Omega \to \mathbb{R}$*. Then*

(a) If $f \in L^1(\Omega)$*, then for every* $\alpha > 0$*,*

$$|\{x \in \Omega : (M^d(f)(x) > \alpha\}|_\mu \leq \frac{C'}{\alpha} \int_\Omega |f| d\mu.$$

(b) If $f \in L^p(\Omega)$, $1 < p \leq \infty$, then $M^d(f) \in L^p(\Omega)$ and

$$||M^d(f)||_p \leq C''||f||_p.$$

(c) If $f \in L^p$, $1 \leq p \leq \infty$, then $M^d(f)$ is finite almost everywhere.

The constant C is as in (2.15), *$C' = C'(C, \kappa)$ and $C'' = C''(C, p)$.*

Proof. We first remark that $M^d < U^d$, since U^d simply considers more balls, so we will prove the theorem for U^d, which will imply that it holds for M^d as well. To prove (a), let $\alpha > 0$, let $E_\alpha = \{x \in \Omega : U^d(f)(x) > \alpha\}$ and let $E \subset E_\alpha$ be compact. By definition of U^d, for each $x \in E$ there exists a ball B_x so that $x \in B_x$ and

$$|B_x|_\mu \leq \frac{1}{\alpha} \int_{B_x} |f| d\mu. \tag{B.3}$$

The collection $\{B_x\}_{x \in E}$ forms an open (since Ω was a homogeneous space) cover of E, so there exists a finite subcollection of $\{B_x\}_{x \in E}$ which also covers E. By Lemma B.0.1, we may select a disjoint subcollection of the finite subcollection $B_1, \ldots, B_n$ such that

$$|E|_\mu \leq C' \sum_{i=1}^{n} |B_i|_\mu. \tag{B.4}$$

Combining (B.3) and (B.4), we obtain

$$|E|_\mu \leq C' \sum_{i=1}^{n} \frac{1}{\alpha} \int_{B_x} |f| d\mu \leq \frac{C'}{\alpha} \int_\Omega |f| d\mu$$

since the balls are disjoint. Taking the supremum over all compact subsets E proves (a).

Moving on to part (b), define $f_1(x) = f(x)$ if $|f(x)| > \frac{\alpha}{2}$ and 0 otherwise. Then we have that $U^d(f) \leq U^d(f_1) + \frac{\alpha}{2}$. This yields that $\{x \in \Omega : U^d(f) > \alpha\} \subset \{x \in \Omega : U^d(f_1) > \frac{\alpha}{2}\}$. Applying part (a), we obtain

$$\begin{aligned} |\{x : U^d(f)(x) > \alpha\}|_\mu &\leq \left| \left\{ x : U^d(f_1)(x) > \frac{\alpha}{2} \right\} \right|_\mu \\ &\leq \frac{C}{\alpha} \int_\Omega |f_1| d\mu \\ &= \frac{C}{\alpha} \int_{\{x : |f| > \frac{\alpha}{2}\}} |f| d\mu. \end{aligned} \tag{B.5}$$

By Theorem 8.16 of [R3], we have that

$$\int_{\Omega} (U^d(f))^p d\mu = p \int_0^{\infty} |\{U^d(f) > \alpha\}|_{\mu} \alpha^{p-1} d\alpha, \tag{B.6}$$

and combining (B.5) and (B.6), we obtain

$$\begin{aligned} ||U^d(f)||_p^p &\leq Cp \int_0^{\infty} \left[\int_{\{|f| > \frac{\alpha}{2}\}} |f| d\mu \right] \alpha^{p-2} d\alpha \\ &= Cp \int_{\Omega} \int_0^{2|f|} |f| \alpha^{p-2} d\alpha d\mu \\ &= \frac{Cp}{p-1} 2^{p-1} ||f||_p^p \end{aligned}$$

which proves part (b). Part (c) is immediate from the preceding parts, so we are done. □

Lemma B.0.4. *If $f \in L^1_{loc}(\Omega)$, then*

$$\lim_{r \to 0} \frac{1}{|B(x,r)|_{\mu}} \int_{B(x,r)} f d\mu = f(x)$$

for almost every $x \in \Omega$.

Proof. Let $f \in L^1_{loc}(\Omega)$. Since the estimate we are trying to prove is local in nature (that is, since we will be integrating over smaller and smaller balls) we may assume that $f \equiv 0$ outside of some large ball, so that $f \in L^1(\Omega)$. We proceed by proving that the set

$$A_{\alpha} = \left\{ x \in \Omega : \lim_{r \to 0} \frac{1}{|B(x,r)|_{\mu}} \int_{B(x,r)} |f - f(x)| d\mu > \alpha \right\}$$

is of measure 0 for all $\alpha > 0$.

Fix $\alpha > 0$, $\varepsilon > 0$. By the density of $C(\Omega)$ in $L^1(\Omega)$, there exists $g \in C(\Omega)$ such that $\int_{\Omega} |f - g| d\mu < \varepsilon$. Now, we obtain

$$\begin{aligned} &\frac{1}{|B(x,r)|_{\mu}} \int_{B(x,r)} |f - f(x)| d\mu \\ &\leq \frac{1}{|B(x,r)|_{\mu}} \int_{B(x,r)} |f - g| d\mu + \frac{1}{|B(x,r)|_{\mu}} \int_{B(x,r)} |g - g(x)| d\mu + |f(x) - g(x)| \\ &\leq M^d(f - g)(x) + \delta(r) + |f(x) - g(x)| \end{aligned}$$

for every $x \in \Omega$, where $\delta(r)$ is some function such that $|g(y) - g(x)| < \delta(r)$ whenever $|y - x| < r$ (note that $\delta(r)$ is well-defined since g is continuous). Note also that $\delta(r) \to 0$ as $r \to 0$.

By the above, if $x \in A_\varepsilon$, then either $M^d(f-g)(x) > \frac{\varepsilon}{2}$ or $|f(x)-g(x)| > \frac{\alpha}{2}$. But by Lemma B.0.3, we have that

$$\left|\left\{x \in \Omega : M^d(f-g)(x) > \frac{\alpha}{2}\right\}\right|_\mu \leq \frac{C}{\alpha}||f-g||_1 < \frac{C}{\alpha}\varepsilon$$

and by Chebychev's inequality (Theorem 6.17 of [F])

$$\left|\left\{x \in \Omega : |f(x)-g(x)| > \frac{\alpha}{2}\right\}\right|_\mu \leq \frac{||f-g||_1}{\alpha} < \frac{\varepsilon}{\alpha}.$$

Thus both of these sets are of measure 0 and we are done. □

Theorem B.0.5 (Lebesgue's Differentiation Theorem for Homogeneous Spaces)**.** *Let (Ω, d, μ) be a homogeneous space and let $f \in L^1_{loc}(\Omega)$. For every $x \in \Omega$ such that*

$$\lim_{r\to 0} \frac{1}{|B(x,r)|_\mu} \int_{B(x,r)} f d\mu = f(x) \tag{B.7}$$

holds (in particular, for almost every $x \in \Omega$), given a family $\{E_r\}_{r>0}$ such that

- *$E_r \subset B(x,r)$ for every $r > 0$*
- *there exists $C > 0$ independent of r such that $C|E_r|_\mu > |B(x,r)|_\mu$ for every $r > 0$*

we have that

$$\lim_{r\to 0} \frac{1}{|E_r|_\mu} \int_{E_r} f d\mu = f(x).$$

Proof. Choose $x \in \Omega$ such that (B.7) holds and let $\{E_r\}_{r>0}$ satisfy the above hypotheses. Then

$$\begin{aligned} \left|\frac{1}{|E_r|_\mu} \int_{E_r} f d\mu - f(x)\right| &\leq \frac{1}{|E_r|_\mu} \int_{E_r} |f - f(x)| d\mu \\ &\leq \frac{C}{|B(x,r)|_\mu} \int_{B(x,r)} |f - f(x)| d\mu \end{aligned}$$

which tends to 0 as $r \to 0$ by Lemma B.0.4. □

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