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VSI_i space–times and the ϵ-property

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We investigate Lorentzian space–times where all zeroth and first order curvature invariants vanish and discuss how this class differs from the one where all curvature invariants vanish (VSI). We show that for VSI space–times all components of the Riemann tensor and its derivatives up to some fixed order can be made arbitrarily small. We discuss this in more detail by way of examples. *© 2005 American Institute of Physics.* [DOI: 10.1063/1.1904707]

I. INTRODUCTION

Recently it was proven that in four-dimensional Lorentzian space–times all of the scalar invariants constructed from the Riemann tensor and its covariant derivatives are zero if and only if the space–time is of Petrov (P)-type III, N or O, all eigenvalues of the Ricci tensor are zero and hence of Plebański–Petrov (PP)-type N or O (Ref. 1) and the common multiple null eigenvector of the Weyl and Ricci tensors is geodesic, shear-free, expansion-free, and twist-free; let us refer to these space–times as vanishing scalar invariant (VSI) space–times. VSI space–times include the well-known *pp*-wave space–times.[2]

Since all of the scalar curvature invariants vanish, all VSI space–times are exact solutions of higher-order Lagrangian based theories (in which the action is given by higher order scalar corrections to the usual general relativistic action based on the Ricci scalar). It has subsequently been argued that, as in the case of *pp*-waves, VSI space–times are exact solutions in string theory,[3–5] when supported by appropriate bosonic massless fields of the string (such as, for example, a dilaton and an antisymmetric massless field). Solutions of classical field equations for which the counter terms required to regularize quantum fluctuations vanish (i.e., they suffer no quantum corrections to all loop orders) are also of importance because they offer insights into the behavior of the full quantum theory.[6]

In particular fundamental field theories only certain specific types of higher order corrections occur (cf. Refs. 7–9), and so for a space–time to be a solution of a particular field theory to all orders, with a specific effective action containing only certain higher order correction terms, it may not be necessary for *all* curvature invariants to vanish. Consequently it is also of interest to determine the set of spacetimes for which (only) the zeroth order curvature invariants vanish (i.e., algebraic scalar invariants constructed from the Riemann tensor), denoted VSI_0, those space–times for which (only) the zeroth *and* first order curvature invariants vanish (i.e., scalar invariants constructed from the Riemann tensor and its first covariant derivative), denoted VSI_1, and so on. In fact, it was proven in Ref. 1 that if all of the zeroth, first, and second order curvature invariants vanish, then necessarily all scalar curvature invariants vanish; so that VSI_2 is equivalent to the set of VSI space–times.

Let us first recall some properties of VSI space–times. Utilizing a complex null tetrad in the Newman–Penrose (NP) formalism it was shown that for P-types III and N the repeated null vector of the Weyl tensor ℓ^α is geodesic, shear-free, expansion-free, and twist-free (and the NP coefficients κ,σ, and ρ are consequently zero), and the Ricci tensor has the form

$$R_{\alpha\beta} = -2\Phi_{22}\ell_\alpha\ell_\beta + 4\Phi_{21}\ell_{(\alpha}m_{\beta)} + 4\Phi_{12}\ell_{(\alpha}\bar{m}_{\beta)}, \tag{1}$$

in terms of the nonzero Ricci components Φ_{ij}. For P-type O, the Weyl tensor vanishes and so it suffices that the Ricci tensor has the form (1). All of these space–times belong to Kundt's class,[10] and the metrics for all VSI space–times are displayed in Ref. 1. The generalized *pp*-wave solutions are of P-type N, PP-type O (so that the Ricci tensor has the form of null radiation) with $\tau=0$, and admit a covariantly constant null vector field.[2] The Ricci tensor (1) has four vanishing eigenvalues, and the PP-type is N for $\Phi_{12}\neq 0$ or O for $\Phi_{12}=0$. It is known that the energy conditions are violated in the PP-type N models[11] and hence attention is usually concentrated on the more physically interesting PP-type O case, which in the nonvacuum case corresponds to pure radiation.

It is well known that the necessary and sufficient conditions for space–times for which the zeroth order algebraic scalar curvature invariants vanish (VSI_0) are of P-type III, N or O and PP-type N or O. Moreover, the repeated principal null direction of Weyl must be aligned with an eigenvector of the Ricci tensor. The last condition follows from the vanishing of the mixed invariants (see Sec. 3.1 of Ref. 1). Next we determine the VSI_1 space–times.

II. VSI_1

We begin by assuming VSI_0 and determine the conditions which imply VSI_1. From the Bianchi identities it follows for VSI_0 that $\kappa=0$. The invariants used here are all constructed from spinors that are symmetrized before and after contractions. Since contractions are always performed with symmetrized spinors we need only give the number of indices contracted between any two spinors. In particular, we shall make use of the following invariant, $I_1 \equiv (\nabla\Psi)^2(\nabla\bar{\Psi})^2$. Here $(\nabla\Psi)^2$ is used to indicate the contraction over four indices of two copies of $\nabla_{(A\dot{A}}\Psi_{BCDE)}$. The result is then symmetrized and contracted with its conjugate to give I_1.

A. Petrov-type III

Using $\Psi_3\neq 0$ with PP-type N or O, we have from the Bianchi identities that $\sigma\Psi_3=\rho\Phi_{12}$ and $\kappa=0$. Applying $\kappa=0$ throughout, we find that two of the Bianchi identities yield the following relation:

$$D\Psi_3 = \rho\Phi_{21} + \bar{\sigma}\Phi_{12} + 2(\rho-\varepsilon)\Psi_3. \tag{2}$$

Computing I_1 and using (2), we obtain

$$I_1 = \tfrac{576}{625}[81(\sigma\bar{\sigma}\Psi_3\bar{\Psi}_3)^2 + \sigma\bar{\sigma}\Psi_3\bar{\Psi}_3X\bar{X} + (X\bar{X})^2], \tag{3}$$

where $X=\rho\Phi_{21}+\bar{\sigma}\Phi_{12}+5\rho\Psi_3$.

The vanishing of I_1 necessarily implies that $\sigma=0$, thus from the Bianchi identities $\rho\Phi_{12}=0$. If $\rho=0$ we get VSI. If $\Phi_{12}=0$ then (3) becomes $I_1=576(\rho\bar{\rho}\Psi_3\bar{\Psi}_3)^2$ which vanishes when $\rho=0$, giving VSI with PP-type O (null radiation).

B. Petrov-type N

Using $\kappa=0$ in the Bianchi identities we find that $\rho\Phi_{21}=-\bar{\sigma}\Phi_{12}$ and $\rho\Phi_{12}=0$. Therefore if $\Phi_{12}\neq 0$ then $\rho=0$ implies that $\sigma=0$, hence we recover VSI. If $\Phi_{12}=0$ then two of the Bianchi identities combine to yield $\sigma\Psi_4=\rho\Phi_{22}$. The conditions $\kappa=\Phi_{12}=0$ and $\sigma\Psi_4=\rho\Phi_{22}$ are necessary to characterize the VSI_1 PP-O null radiation models. Suppose that $\sigma=0$; then either $\rho=0$ and we have VSI, or $\rho\neq 0$ and $\Phi_{22}=0$ which necessarily characterizes the vacuum VSI_1 models. (See Ref. 12.)

To show sufficiency, we assume $\kappa=\Phi_{12}=0$ and then note that the remaining curvature components, Ψ_4 and Φ_{22}, both have boost weight -2. In the compacted (GHP) formalism[13] the relevant operators have boost weight 0 or 1 and the only spin coefficients with positive boost weight are σ and ρ with weights 1; it follows that the covariant derivative of either Ψ_{ABCD} or $\Phi_{AB\dot{A}\dot{B}}$ will have components with only negative boost weight. Therefore, all zeroth and first order curvature invariants vanish, implying VSI_1.

C. Petrov-type O

The freedom in the frame can be used here to consider PP-type N and PP-type O null radiation separately, and it follows trivially from the Bianchi identities that $\kappa=\sigma=\rho=0$, so that we obtain VSI. Therefore all Petrov type O VSI_0 are VSI from the Bianchi identities.

In summary, the only space–times in the class VSI_1 that are not VSI are of P-type N and all have $\kappa=\Phi_{12}=0$. The first of these VSI_1 models have $\sigma\Psi_4=\rho\Phi_{22}$; exact solutions were found by Plebański.[11] The second of the VSI_1 models have $\sigma=\Phi_{22}=0$, and these are the vacuum Petrov-type N solutions with $\rho=\Theta+i\omega\neq 0$. If $\omega=0$ these solutions belong to the Robinson–Trautman class and all are known.[11] If $\omega\neq 0$ then the only twisting, vacuum, P-type N solution known is that of Hauser.[11]

There are other cases that may also be of interest. Notice the example in Ref. 14 in which there are scalar curvature invariants that are nonzero (constant, depending on a cosmological constant) while all higher order scalar curvature invariants are zero.

III. ε-PROPERTY

A scalar invariant for a matrix is a polynomial of the matrix entries that is invariant with respect to all changes of basis. It is easy to characterize all such invariants. Let M be an $n\times n$ matrix. The characteristic polynomial of M is given by

$$p_M(x) = \det(xI - M) = x^n + \sum_{j=1}^{n} (-1)^j \sigma_j(M) x^{n-j}.$$

The expressions $\sigma_j(M)$ are called the elementary symmetric polynomials of M and are the scalar invariants of M [$\sigma_1(M)$ is just the trace of M and $\sigma_n(M)$ is the determinant]. All other scalar invariants can be given as polynomials of $\sigma_1(M),\sigma_2(M),\ldots,\sigma_n(M)$. A matrix M for which the characteristic polynomial is just x^n is nilpotent. Now a matrix with the ε-property, i.e., the property that all entries can be made smaller than every given ε by a change of basis, must be nilpotent.[15] The converse is also true, that is, every nilpotent matrix necessarily possesses the ε-property. Therefore, a matrix is VSI_0 if and only if it is nilpotent. Hence we anticipate that VSI space–times will have the ε-property, and this is what we prove next.

Theorem: For and only for VSI space–times (in arbitrary dimension D and C^∞ metric) one can find, for arbitrarily large N and for arbitrarily small ε, a tetrad in which all components of the Riemann tensor and its derivatives up to order N are smaller than ε.

Proof: For non-VSI space–times there always exist a nonvanishing curvature invariant. Its value of course does not depend on the choice of the tetrad and thus there does not exist a tetrad with the desired property. It was proven in Ref. 1 that in four-dimensional VSI space–times the boost weight of all components of the Riemann tensor and its derivatives is negative. Thus with an appropriate boost we can make all components of the Riemann tensor and its derivatives up to a desired order N arbitrarily small.[16,17] □

It was pointed out by Penrose in Ref. 18 that P-types III and N have "the property that gravitational density can be made as small as we please by a suitable choice of time axis (following the wave)." It turns out that for VSI_0 space–times, not only the gravity density but the energy-momentum tensor can be made arbitrarily small by an appropriate boosting of the frame. In the case of VSI_1 space–times we can also make the first derivatives of the Riemann tensor essentially undetectable, and for VSI space–times it is possible to do this for arbitrarily large derivatives as well. Since experiments measure tetrad components of the Riemann tensor and as every experiment has some sensitivity limit, we can effectively, by an appropriate boost, "locally transform away" the Riemann tensor and its derivatives.

It is of interest to consider if any of the VSI space–times satisfy the following stronger ε-property. We shall say that the Riemann tensor has the uniform ε-property if, given an arbitrarily small ε, there exists a tetrad in which the components of the Riemann tensor and all of its derivatives are smaller than ε. Not all VSI space–times satisfy the uniform ε-property; this is shown by considering P-type N vacuum VSI space–times with $\tau \neq 0$. Let us denote

$$X_k = C_{abcd;e_1\cdots e_k} n^a \overline{m}^b n^c \overline{m}^d m^{e_1}\cdots m^{e_k}, \quad Y_k = C_{abcd;e_1\cdots e_k}\delta(n^a \overline{m}^b n^c \overline{m}^d m^{e_1}\cdots m^{e_k}). \tag{4}$$

By induction on k we shall show that the component $C_{2424;3\cdots 3}=X_k=-k!\,\tau^k\Psi_4$ for all orders k. From Ref. 1 we have the following relations:

$$\kappa = \sigma = \rho = \epsilon = 0, \quad \tau = \pi = 2\beta = 2\alpha, \quad \lambda = \mu = (2/3)\gamma, \tag{5}$$

where all of these spin coefficients are real, and ν is nonzero and complex as well. The Bianchi identities and NP equations then give

$$\delta\Psi_4 = -\tau\Psi_4, \quad D\Psi_4 = 0, \quad \delta\tau = \tau^2, \quad D\tau = 0. \tag{6}$$

It can be shown directly that $X_1 = C_{2424;3} = -\tau\Psi_4$, and using strong induction we assume that X_k has the required form. In general, the following recursive relation holds $X_k = \delta X_{k-1} - Y_{k-1}$, consequently this implies that $Y_{k-1} = 2(k-1)!\,\tau^k\Psi_4$. Similarly, $X_{k+1} = \delta X_k - Y_k$, and on expanding Y_k we observe that it is composed of terms with boost weight -2 and -1, but the boost weight -1 terms vanish as a result of a similar proof found in Ref. 1. To show this we note that in this case we have

$$\text{þ}\Psi_4 = 0, \quad \text{þ}\tau = 0, \quad \text{þ}\rho' = -2\tau^2 = \text{þ}\sigma', \quad \text{þ}\kappa' = 6\tau\rho' \tag{7}$$

with commutators [19]

$$\text{þ}\check{\eth} - \check{\eth}\text{þ} = \tau\text{þ}, \quad \text{þ}\text{þ}' - \text{þ}'\text{þ} = 2\tau(\check{\eth} + \check{\eth}') - (p+q)\tau^2.$$

Assuming that η is a tetrad component of the Weyl tensor of arbitrary order k with boost weight -2 such that $\text{þ}\eta = 0$, it is straightforward to show that the following boost weight -1 scalars,

$$\text{þ}^3(\kappa'\eta), \quad \text{þ}^2(\sigma'\eta), \quad \text{þ}^2(\rho'\eta), \quad \text{þ}(\tau\eta), \quad \text{þ}(\tau'\eta), \quad \text{þ}\check{\eth}\eta, \quad \text{þ}\check{\eth}'\eta, \quad \text{þ}^2\text{þ}'\eta$$

all vanish. Therefore Y_k consists of only the boost weight -2 term, hence we have that $Y_k = -2\tau X_k$ and thus $X_{k+1} = -(k+1)!\,\tau^{k+1}\Psi_4$. Since the component $C_{2424;3\cdots 3}$ can be made arbitrarily large by increasing the order, in this case the Riemann tensor cannot therefore satisfy the uniform ε-property.

A subclass of the VSI space–times for which the uniform ε-property is satisfied are those in which $\nabla^{(N)} R_{abcd} = 0$, where (N) denotes N covariant derivatives. Since only a finite set of components of the Riemann tensor and its derivatives are nonzero, then by an appropriate boost all components of the Riemann tensor and its derivatives can be made smaller than ε. In the case of $N=1$ we have the VSI symmetric spaces in which $\nabla_e R_{abcd} = 0$ (cases in which $N > 1$ will be referred to as higher order symmetric spaces); we shall show that this class is nonempty. We consider the following line-element:

$$\mathrm{d}s^2 = 2h\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 \tag{8}$$

and solve $\nabla_e R_{abcd} = 0$, assuming that $h = h(u,x,y)$. After an appropriate coordinate transformation, which preserves the form of the metric, we find that $h = k(x^2+y^2) + c^2(x^2-y^2)$ where k and c are arbitrary constants. Using the NP tetrad $\ell^a = \delta^a_v, n^a = \delta^a_u - h\delta^a_v$ and $m^a = (i\delta^a_x - \delta^a_y)/\sqrt{2}$ it follows that the only nonvanishing spin coefficient is ν with Φ_{22} and Ψ_4 being constants. If $k=0$ and $c \neq 0$ we recover the P-type N vacuum symmetric space,[11] if $k \neq 0$ and $c=0$ we obtain the P-type O, PP-type O null radiation symmetric space.[11] These VSI symmetric spaces clearly satisfy the uniform ε-property. In P-type III it is known that no symmetric spaces exist;[11] however, the possibility remains that P-type III VSI space–times satisfying the uniform ε-property may exist (for example, if $\nabla^{(N)}R_{abcd}=0$ for $N>1$).

To illustrate a higher order symmetric space, consider (8) with $h=g(u)(x^2-y^2)$, a subclass of the P-type N vacuum VSI space–times with $\tau=0$. Next, apply a boost so that $l'=Al$ and $n'=A^{-1}n$ where the boost parameter $A=Cg'(u)$ with C constant. Dropping the primes and working in the boosted frame we have the following nonvanishing scalars, $\nu=-\sqrt{2}g(y+ix)/A^2$, $\gamma=A'/(2A^2)$, and $\Psi_4=-2g/A^2$. It follows that the Weyl tensor has the form [20]

$$C_{abcd} = \tfrac{1}{2}C_{2i2j}\{\ell_a m_b^{(i)}\ell_c m_d^{(j)}\}, \tag{9}$$

where $i,j=3,4, m^{(3)}=\overline{m}$, and $m^{(4)}=m$, the only nonvanishing Weyl tetrad components are $C_{2i2i}=2g/A^2$. Let $X_0=C_{2i2i}$, then (9) is $C_{abcd}=\frac{1}{2}X_0\{\ell_a m_b^{(i)}\ell_c m_d^{(i)}\}$ and

$$\nabla_e C_{abcd} = \tfrac{1}{2}X_1\ell_e\{\ell_a m_b^{(i)}\ell_c m_d^{(i)}\}, \tag{10}$$

where $X_1=\Delta X_0+4\gamma X_0$. It can be shown that the nth order covariant derivative of the Weyl tensor has the following simple form:

$$\nabla_{e_n}\cdots\nabla_{e_1}C_{abcd} = \tfrac{1}{2}X_n\ell_{e_n}\cdots\ell_{e_1}\{\ell_a m_b^{(i)}\ell_c m_d^{(i)}\}. \tag{11}$$

Proceeding inductively, we obtain the following recurrence relation $X_n=\Delta X_{n-1}+2(n+1)\gamma X_{n-1}$. From (11) we have that the only nonvanishing nth order tetrad components of the Weyl tensor will be $C_{2i2i;2\cdots 2}$. Again, by induction, one can show that $X_n=2A^{(n-1)}/(CA^{n+2})$ for all $n\geqslant 1$ (denoting the $n-1$ derivative of A as $A^{(n-1)}$ and $A^{(0)}=A$).

We now have an expression for the nth order derivatives of the tetrad components of the Weyl tensor

$$C_{2i2i;2\cdots 2} = \frac{2g^{(n)}}{(Cg')^{n+2}}, \tag{12}$$

where it is assumed that $g'\neq 0$, otherwise the boost is degenerate. Therefore, for any $n\geqslant 2$ we can obtain an nth order symmetric space simply by setting $g(u)$ to be any polynomial in u of degree $n-1$. All of these VSI space–times will satisfy the uniform ε-property; more generally this is also satisfied if there exists a constant M such that $|g^{(n)}|\leqslant M$ for all n and $g'\neq 0$. On the other hand, we can use (12) to find examples of VSI space–times that do not satisfy the uniform ε-property. It is known [21] that every geodesic of (8) is either of type 1 or type 2, where type 1 refers to geodesics in the 2-surface u and v constant and type 2 refers to geodesics in the 2-surface x and y constant. Let us consider type 2 geodesics, and set $x=x_0$, $y=y_0$. We find that the tangent vectors are given by $w^a=(a,b/(2a)-ag(u)(x_0^2-y_0^2),0,0)$ and parametrized by u. Here, $\dot{u}=a$ is a constant and $b=1$ or 0 for timelike or null geodesics, respectively. The NP tetrad defined above is parallel propagated along such geodesics, hence from (12) if the uniform ε-property is not satisfied at some order k then we obtain a parallel propagated curvature singularity of order k. That is, the curvature components of order k in a parallel propagated frame become unbounded along the geodesic; when $k=0$ we recover the definition[22] of a parallel propagated curvature singularity. In Ref. 23, geodesic motion in vacuum Kundt-type N solutions with $\tau\neq 0$ have revealed the existence of parallel propagated curvature singularities of order 0.

IV. CONCLUSION

We have determined the necessary and sufficient conditions that characterize VSI$_1$ space–times. Assuming VSI$_0$, we have shown that in P-type III, VSI$_1$ implies VSI and in P-type O, VSI$_0$ implies VSI. The only proper VSI$_1$ space–times occur in P-type N and PP-type O with $\kappa=\Phi_{12}=0$. In addition, the nonvacuum VSI$_1$ space–times are further characterized by $\sigma\Psi_4=\rho\Phi_{22}$, and the vacuum space–times have $\sigma=\Phi_{22}=0$. It has been shown that the ε-property offers an alternative characterization of the VSI space–times, in the sense that only for VSI space–times can a tetrad be found in which the Riemann tensor and its derivatives up to any fixed order can be made arbitrarily small. A strengthening of the ε-property leads us to define the uniform ε-property; this condition determines a subclass of the VSI space–times where there exists a tetrad in which the components of the Riemann tensor and all of its derivatives can be made arbitrarily small. Some examples of VSI space–times satisfying the uniform ε-property have been presented.

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[24] GRTensorII is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World Wide Web from the address http://grtensor.org