Generalizations of *pp*-wave spacetimes in higher dimensions

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We investigate D-dimensional Lorentzian spacetimes in which all of the scalar invariants constructed from the Riemann tensor and its covariant derivatives are zero. These spacetimes are higher-dimensional generalizations of D-dimensional pp-wave spacetimes, which have been of interest recently in the context of string theory in curved backgrounds in higher dimensions.

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I. INTRODUCTION

Higher-dimensional pp-wave spacetimes are of current interest in string theory in curved backgrounds, particularly since these Lorentzian spacetimes are exact solutions in string theory and their spectrum can therefore be explicitly determined. In this paper we shall discuss D-dimensional Lorentzian spacetimes in which all of the scalar invariants constructed from the Riemann tensor and its covariant derivatives are zero. These spacetimes can be regarded as higher-dimensional generalizations of D-dimensional pp-wave spacetimes.

This research follows on from the recent work of [1] in four dimensions, in which it was proven that in Lorentzian spacetimes all of the scalar invariants constructed from the Riemann tensor and its covariant derivatives are zero if and only if the spacetime is of Petrov type III, N or O, all eigenvalues of the Ricci tensor are zero and the common multiple null eigenvector l^a of the Weyl and Ricci tensors is geodesic, shearfree, nonexpanding, and nontwisting [1] [i.e., the Newman-Penrose (NP) coefficients κ, σ, and ρ are zero]; we shall refer to these spacetimes as vanishing scalar invariant (VSI) spacetimes. The Ricci tensor has the form

$$R_{ab}=-2Al_al_b+4A_il_{(a}m^i_{b)} \tag{1}$$

$(i=1,2)$. The Plebański-Petrov (PP) type is N for $A_i\neq 0$ or O for $A_i=0$. We note that for PP-type N, using a null rotation, a boost and a spatial rotation we can transform away the Ricci component A and set $A_i=1$. For PP-type O it is possible to set $A=1$ by performing a boost.

It is known that the energy-momentum tensor for a spacetime corresponding to PP-type N cannot satisfy the weak energy conditions [2], and hence such spacetimes are not regarded as physical in classical general relativity (however, see [3]). Therefore, attention is usually restricted to PP-type O models, in which the energy-momentum tensor corresponds to a pure null radiation field [2]. All of these spacetimes belong to Kundt's class, and hence the metric of these spacetimes can be expressed in an appropriate form in adapted coordinates [2,4]. The metrics for all VSI spacetimes are displayed in [1]. The generalized pp-wave solutions are of Petrov-type N, PP-type O (so that the Ricci tensor has the form of null radiation) with $\tau=0$, and admit a covariantly constant null vector field [5]. The vacuum spacetimes, which are obtained by setting $A=0$, are the well-known pp-wave spacetimes (or plane-fronted gravitational waves with parallel rays).

II. HIGHER ORDER THEOREM

This theorem can be readily generalized to higher dimensions. We shall study Lorentzian VSI spacetimes in arbitrary D-dimensions (not necessary even, but $D=10$ is of particular importance from string theory) with signature $D-2$. In principle we could study other signatures; for example, manifolds with signature $D-4$ with $D\geqslant 5$ may also be of physical interest [6].

Let the tetrad be $l,n,m^1,m^2,\ldots,m^i$ (l,n null with $l^al_a=n^an_a=0$, $l^an_a=1$, m^i real and spacelike), so that

$$g_{ab}=2l_{(a}n_{b)}+\delta_{jk}m^j_am^k_b. \tag{2}$$

Using the notation

$$\begin{aligned}\{[w_px_q][y_rz_s]\}\equiv{}& w_px_qy_rz_s-w_px_qz_ry_s-x_pw_qy_rz_s\\&+x_pw_qz_ry_s+y_pz_qw_rx_s-y_pz_qx_rw_s\\&-z_py_qw_rx_s+z_py_qx_rw_s,\end{aligned} \tag{3}$$

if all zeroth order invariants vanish then there exists a tetrad (2) l, n, m^i $(i=1,\ldots,N=D-2)$ such that [7]

$$\begin{aligned}R_{abcd}={}&A_i\{[l_an_b][l_cm^i_d]\}+B_{[ij]k}\{[m^i_am^j_b][l_cm^k_d]\}\\&+C_{(ij)}\{[l_am^i_b][l_cm^j_d]\}.\end{aligned} \tag{4}$$

We still have the freedom to "choose the frame" and simplify further, using boosts, spins and null rotations, depending on the algebraic structure of the Ricci and Weyl tensors (a generalization of Petrov and Petrov-Plebanski classifications).

From Eq. (4) we obtain the Ricci tensor:

$$R_{bd}=2[-A_i+2B_{[ij]k}\delta^{jk}]l_{(b}m^i_{d)}+Al_bl_d, \tag{5}$$

where $A\equiv 2C_{jk}\delta^{jk}$. We can further simplify R_{bd} depending on its algebraic type. If the energy conditions are satisfied,

$$A_i-2B_{[ij]k}\delta^{jk}=0, \tag{6}$$

we shall refer to this as type $PP\bar{O}$. In this case we have that

$$R_{bd}=Al_b l_d. \tag{7}$$

If this condition is not satisfied, we can use boosts, spins and null rotations to set $A=0$, which we shall refer to as type $PP\bar{N}$.

From Eq. (4) we obtain the Weyl tensor:

$$C_{abcd}=\Psi_i\{[l_a n_b][l_c m^i_d]\}+\Psi_{\{ijk\}}\{[m^i_a m^j_b][l_c m^k_d]\} + \Psi_{\langle ij\rangle}\{[l_a m^i_b][l_c m^j_d]\}, \tag{8}$$

where

$$\Psi_i=2\Psi_{\{ijk\}}\delta^{jk}\equiv C_{abcd}n^a l^b n^c m^d_i = \frac{1}{D-2}[(D-3)A_i+2B_{[ij]k}\delta^{jk}], \tag{9}$$

$$\Psi_{\{ijk\}}\equiv\frac{1}{2}C_{abcd}m^a_i m^b_j n^c m^d_k = B_{[ij]k}+\frac{1}{D-2}(A_{[i}\delta_{j]k}-2B_{[im|n|}\delta^{mn}\delta_{j]k}) \tag{10}$$

and

$$\Psi_{\langle ij\rangle}\equiv\frac{1}{2}C_{abcd}n^a m^b_i n^c m^d_j = C_{(ij)}-\frac{1}{2(D-2)}A\delta_{ij}. \tag{11}$$

In analogy with the Petrov classification, we shall say that spacetimes with $\Psi_{\{ijk\}}\neq 0$ are of type $P\overline{III}$ (in some instances we can use the remaining tetrad freedom in this case to set $\Psi_{\langle ij\rangle}=0$). Spacetimes with $\Psi_{\{ijk\}}=0$ will be referred to as of type $P\bar{N}$. Conformally flat spacetimes with $\Psi_{\{ijk\}}=0$ and $\Psi_{\langle ij\rangle}=0$ will be referred to as type $P\bar{O}$. [$\Psi_{\{ijk\}}$ and $\Psi_{\langle ij\rangle}$ are higher-dimensional analogues of the complex NP coefficents $\bar{\Psi}_3$ and $\bar{\Psi}_4$ in 4 dimensions. A comprehensive higher-dimensional Petrov classification, which is not necessary here, will be discussed elsewhere.]

For spacetimes of type $PP\bar{O}$ and type $P\bar{N}$, the Ricci tensor is given by Eq. (7) and the Weyl tensor is given by

$$C_{abcd}=\left[C_{(ij)}-\frac{1}{2(D-2)}A\delta_{ij}\right]\{[l_a m^i_b][l_c m^j_d]\}. \tag{12}$$

III. GENERALIZED KUNDT SPACETIMES

Using the Bianchi and Ricci identities, it is possible to prove [7] that all curvature invariants of all orders vanish for spacetimes with Riemann tensor of the form of Eq. (4) that satisfy the following conditions on the covariant derivative of the uniquely defined null vector $l_{a;b}$ namely

$$l^a l_a=0,\quad l^a{}_{;b}l^b=0,\quad l^a{}_{;a}=0,\quad l_{(a;b)}l^{a;b}=0,$$

$$l_{[a;b]}l^{a;b}=0.$$

In general, the covariant derivative then has the form

$$l_{a;b}=L_{11}l_a l_b+L_{1i}l_a m^i_b+L_{i1}m^i_a l_b.$$

We are consequently led to study spacetimes which admit a geodesic, shear-free, divergence-free, irrotational null congruence $l=\partial_v$, and hence belong to the "generalized Kundt" class in which the metric can be written as

$$ds^2=-2du[Hdu+dv+W_i dx^i]+g_{ij}dx^i dx^j \tag{13}$$

where $i,j,k=1,\ldots,N$ and the metric functions

$$H=H(u,v,x^i),\quad W_i=W_i(u,v,x^i),\quad g_{ij}=g_{ij}(u,x^i)$$

satisfy the remaining vanishing invariant conditions and the Einstein field equations (see [2,4]). We may, without loss of generality, use the remaining coordinate freedom [e.g., transformations of the form $x^{i'}=x^{i'}(u,x^j)$] to simplify g_{ij}. For the spacetimes considered here we shall diagonalize g_{ij}, and in the particular examples below we shall take $g_{ij}=\delta_{ij}$. The null tetrad is then

$$l=-\partial_v,\quad n=\partial_u-\left(H+\frac{1}{2}W^2\right)\partial_v+W_i\partial_{x^i},\quad m^i=\partial_{x^i}, \tag{14}$$

where $W^2\equiv W_i W_j\delta^{ij}$. (Note that in 4D the uniquely defined null vector given by $l=\partial_v$ is the repeated Weyl eigenvector.)

All of the exact higher-dimensional solutions will be discussed in detail in [7]. Let us present a subclass of type $PP\bar{O}$ and type $P\bar{N}$ exact solutions explicitly here, in which the Ricci tensor is given by Eq. (7) and the Weyl tensor is given by Eq. (12) in the local coordinates above, and in which

$$g_{ij}=\delta_{ij},\quad W_1=-\epsilon\frac{v}{x_1},\quad W_i=0\quad (i\neq 1),$$

$$H=-\epsilon\frac{v^2}{4x_1^2}+H_0(u,x^k), \tag{15}$$

where $\epsilon=1$ corresponds to the case "$\tau\neq 0$" (see [1]); higher-dimensional pp-wave spacetimes have $\epsilon=0$ ("$\tau=0$"). In these spacetimes all of the scalar invariants constructed from the Riemann tensor and its covariant derivatives are zero. In the case of vacuum the function $H_0(u,x^k)$ satisfies a differential equation.

A second example of a higher-dimensional VSI spacetime is given by type $P\overline{III}$ ("$\tau=0$") solutions

$$g_{ij}=\delta_{ij},\quad W_i=\epsilon W_i(u,x^k),\quad H=\epsilon v H_1(u,x^k)+H_0(u,x^k). \tag{16}$$

In general these spacetimes are of type $PP\bar{N}$ (and the remaining tetrad freedom can be employed to simplify the metric further). In the case of type $PP\bar{O}$ (null radiation) the functions $W_i(u,x^k)$ and $H_1(u,x^k)$ satisfy additional differential equations. The higher-dimensional type $P\bar{N}$ pp-wave spacetimes again occur as a subcase with $\epsilon=0$ [$W_i(u,x^k)=0$, $H_1(u,x^k)=0$].

IV. DISCUSSION

The VSI spacetimes have a number of important physical applications. In particular, in four dimensions a wide range of VSI spacetimes (in addition to the pp-wave spacetimes) are exact solutions in string theory to all perturbative orders in the string tension (even in the presence of the RR five-form field strength) [8] (cf. [9]). As a result, these models are expected to provide some hints for the study of superstrings on more general backgrounds [10]. String theory in pp-wave backgrounds has been studied by many authors [11], partly in a search for a connection between quantum gravity and gauge theory dynamics. Solutions of classical field equations for which the counter terms required to regularize quantum fluctuations vanish are also of importance because they offer insights into the behavior of the full quantum theory. A subclass of Ricci flat VSI 4-metrics, which includes the pp-wave spacetimes and some special Petrov type III or N spacetimes, have vanishing counter terms up to and including two loops and thus VSI suffer no quantum corrections to all loop orders [12].

Finding new string models with Lorentzian signature which are exact in α' and whose spectrum can be explicitly determined is of great interest in the context of string theory in curved backgrounds in higher dimensions and, indeed, higher dimensional generalizations of pp-wave backgrounds have been considered by a number of authors [11]. In particular, it was recently realized [13–15] this solvability property applies to string models corresponding not only to the Neveu-Schwarz–Neveu-Schwarz (NS-NS) but also to certain Ramond-Ramond (RR) plane-wave backgrounds. (See also [16], and a general discussion of pp waves in $D=10$ supergravity appeared in [17].)

There is also an interesting connection between pp-wave backgrounds and gauge field theories. It is known that any solution of Einstein gravity admits plane-wave backgrounds in the Penrose limit [18]. This was extended to solutions of supergravities in [19]. It was shown that the super-pp-wave background can be derived by the Penrose limit from the $\mathrm{AdS}_p\times \mathrm{S}^q$ backgrounds in [15]. The Penrose limit was recognized to be important in an exploration of the AdS conformal field theory (CFT) correspondence beyond massless string modes in [20,13]. Maximally supersymmetric pp-wave backgrounds of supergravity theories in eleven- and ten-dimensions have also attracted interest [21].

Recently the idea that our universe is embedded in a higher-dimensional world has received a great deal of renewed attention [22]. Because of the importance of branes in understanding the nonperturbative dynamics of string theories, a number of classical solutions of branes in the background of a pp-wave have been studied; in particular a new brane-world model has been introduced in which the bulk solution consists of outgoing plane waves (only) [23].

For example, a class of pp-wave string solutions with nonconstant NS-NS or RR field strengths, which are exact type II superstring solutions to all orders in α' since all corrections to the leading-order field equations naturally vanish, were discussed recently [14] (see also [24]). The metric ansatz and NS-NS 2-form potential in 10-dimensional superstring theory is given by

$$\begin{aligned} ds^2 &= -du dv - K(x^k)du^2 + dx_i^2 + dy_m^2 \\ B_2 &= b_m(x^k)du\wedge dy_m \\ H_3 &= \partial_i b_m(x^k)dx_i\wedge du\wedge dy_m\,, \end{aligned} \tag{17}$$

where $i=1,\ldots,d$ and $m=d+1,\ldots,8$ [and a dilaton of the form $\phi=\rho_i x_i+\tilde{\phi}(u)$ can be included]. In particular, it was found [14] that the only nonzero component of the generalized curvature is

$$R_{uiuj} = -\frac{1}{2}\partial_i\partial_j K - \frac{1}{2}\partial_i b_m \partial_j b_m\,. \tag{18}$$

These solutions are consequently of type $PP\overline{O}$ and type $P\overline{N}$ [see Eqs. (7) and (12)]. There are several special cases. For $b_m=0$ the standard higher-dimensional generalized pp-wave solution is recovered with $K=K_0(x)$ being a harmonic function. Wess-Zumino-Witten (WZW) models [25] result when the b_m are linear, corresponding to homogeneous plane-wave backgrounds with constant H_3 field. The Laplace equation for b_m can also be solved by choosing b_m to be the real part of complex holomorphic functions. The RR counterparts of these string models are direct analogs of the pp-wave solution [24] supported by a non-constant 5-form background. Note that lifts of the above solutions to 11 dimensions belong to a class of $D=11$ pp-wave backgrounds first considered in [26].

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