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LOWER BOUNDS FROM TILE COVERS FOR THE CHANNEL ASSIGNMENT PROBLEM*

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Abstract. A method to generate lower bounds for the channel assignment problem is given. The method is based on the reduction of the channel assignment problem to a problem of covering the demand in a cellular network by preassigned blocks of cells called tiles. This tile cover approach is applied to networks with a cosite constraint and two different constraints between cells. A complete family of lower bounds is obtained, which include a number of new bounds that improve or include almost all known clique bounds. When applied to an example from the literature, the new bounds give better results.

Key words. channel assignment, graph labeling, polyhedral lower bounds

AMS subject classifications. 05C78, 05C90, 90C57

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1. Introduction.

Finding an optimal assignment of communication channels in a cellular network is a difficult combinatorial optimization problem which has received considerable attention over the last decade. This is due to the explosive growth of wireless communications and the scarcity of the radio spectrum. The channel assignment problem (CAP) is NP-complete even in a drastically simplified form, and, consequently, most efforts have gone toward the development of good heuristics. (Recently, integer programming techniques which can lead to exact solutions have been used. See, for example, [12].) Lower bounds play an important role in the evaluation of any heuristic or approximation algorithm. Moreover, lower bounds can help to identify the structures that form the bottleneck for a particular instance, and this information can, in turn, be used to find better assignments.

A basic model for a cellular network describes it in terms of the *demand* for channels in each cell and a set of *reuse constraints* which prescribe minimal separations that must exist between channels assigned to certain cells in order to avoid interference. The goal of the CAP is to assign channels (represented by integers) to the cells such that each cell receives as many channels as its demand requires while respecting the reuse constraints. Here, the objective is to minimize the *span* of the assignment, which is the difference between the highest and the lowest channel assigned. (An alternative objective, when a limited span is given, can be to minimize the number of violated interference constraints.)

Cellular networks can be modeled as graphs where the nodes of the graph represent the cells, and two nodes are adjacent precisely when there exists a (nonzero) reuse constraint between them. The demands are given by a *weight vector* indexed by the nodes, and the reuse constraints are given by a vector indexed by the nodes and

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edges. When all reuse constraints are 1, the CAP reduces to the problem of finding a coloring of a weighted graph.

The minimal span needed for any assignment will generally be determined by the cells with highest demand. It is reasonable to assume that these cells will often be geographically close, corresponding, for example, to a business district or a city center. Since interference also tends to be highest between cells that are close, these cells will often form a clique in the underlying graph.

Most lower bounds for the CAP are therefore based on cliques. The simplest clique bound, mentioned in [6] but generally considered folklore, is found by assuming that all edge constraints and cosite constraints are equal to the lowest constraint in the clique. A first refinement was obtained in [6] by considering two different constraints. A second refinement, similar to the situation studied here, was considered in [18]. In all of these cases, bounds were obtained using ad hoc methods.

In this paper, we study networks where the reuse constraint between different cells can take only three values, one of which is reserved for the *cosite constraint*. The cosite constraint is the reuse constraint between channels assigned to the same cell, or node. Naturally, any bounds obtained from this approach can also be used in networks with more general constraints by reducing the constraints in any particular set of edges to the lowest constraint in that set.

We describe how lower bounds can be generated from an approach based on reducing the CAP to a covering problem. The crucial step is to show that any channel assignment can be broken down into small blocks called *tiles*. A *tile cover* is a collection of tiles such that the number of tiles covering a node equals the number of channels assigned to that node. The conversion of the CAP to a tile cover problem brings the advantage that tile covers can be easily analyzed using linear programming (LP) duality and polyhedral methods. A similar tile cover method, applied to the simpler case of cliques with one cosite constraint and one edge constraint, can be found in [10]. This particular result is used in our paper as the base case for the induction which forms the proof of our main theorem. In [13], heuristic channel assignment methods using preassigned "tiles" of assigned channels are applied successfully to a number of CAP instances.

We apply the tile cover approach to configurations which we call *nested cliques*. These are cliques consisting of an *inner* clique and an *outer* clique where all edge constraints involving an inner clique node take the larger constraint value, while all edge constraints containing only nodes from the outer clique take the smaller value (see section 2 for a more precise definition). Nested cliques arise naturally from the geographical layout of cellular networks and from the fact that interference levels are generally lower between transmitters that are at greater distance from each other. Hence, it will be common to find a cluster of cells with high interference constraints between them surrounded by an outer shell of cells at greater distance and thus with weaker interference constraints. Such a situation will form a nested clique in the interference graph.

Using the tile cover approach on nested cliques, we derive a comprehensive family of general "second generation" clique bounds. This family includes all bounds from [6] and improves the bound obtained in [18]. We also show, using an example, how the approach can be used directly to obtain specific lower bounds for any specific set of parameters.

There are two types of clique bounds that cannot be derived directly from our approach. In [15], [8], and [16], it was shown how the traveling salesman problem

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and its linear program relaxation can be used to derive lower bounds for cliques. This approach is most effective when the cosite constraint is relatively low. In [2], an integer programming approach for obtaining upper and lower bounds is given, which is based on *d-walks*, i.e., walks that cover each node more than once. This method is somewhat related to the tile cover method, since paths between successive visits of a node in the walk can be seen as tiles.

In [19] a lower bounding method is described which is based on network flows. We will show that our tile cover bounds give an improvement of 13% when applied to the example given in this paper.

Since it is NP-hard to find a maximum weight clique in a graph, it will also be hard to find the nested clique that gives the best bound. However, clique enumeration procedures such as the *Carraghan–Pardalos* algorithm (see [5]) give good performance in practice. The reduction of the CAP to a tile cover problem leads to an easy way of computing the lower bound for any particular clique by way of a linear program. Alternatively, any particular network can be analyzed in advance using our method, and a complete family of easily computable lower bounds can be obtained. Therefore, we expect the computation of the best tile cover clique bound to be feasible and realistic.

The layout of the paper is at follows. After introducing some formal definitions related to channel assignment in section 2, we introduce and define the concepts involved in the tile cover method in section 3. At the end of this section we also state our main result, namely, that each channel assignment can be reduced to a tile cover, such that the cost of the cover is no larger than the span of the assignment. In section 4, we develop lower bounds for tile covers using an LP formulation and we show how they translate into bounds for the CAP. In section 5, the proof of the main theorem is given.

2. Preliminaries. For the basic definitions of graph theory we refer to [4]. A (simple) graph G is a pair (V, E) of a node set V and an edge set E, where each edge $e \in E$ is an unordered pair of nodes. A *clique* in a graph is a set of nodes of which every pair is adjacent.

In this paper, we will use the following notation for integer vectors: if $y \in \mathbb{Z}^V$ for some set V, then $y(v)$ is the coordinate of y indexed by v. Sets will often be represented by their characteristic vectors. Given a set V and $A \subseteq V$, the characteristic vector $\chi^A \in \mathbb{Z}_+^V$ is defined as follows:

$$\chi^A(v) = \begin{cases} 1 & \text{if } v \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, given a vector $y \in \mathbb{Z}_+^V$, the *support* of y, denoted by $V(y)$, is the set of all nodes in V indexing nonzero coordinates of y, so

$$V(y) = \{v \in V : y(v) > 0\}.$$

A *constrained graph* $G = (V, E, s, e)$ is a graph with node set V, edge set E, and positive integer constraint vectors $s \in \mathbb{Z}_+^V$, $e \in \mathbb{Z}_+^E$. Vectors s and e represent the channel reuse constraints: vector s represents the *cosite constraints*, the required separation between channels assigned to the same node, and e represents the *edge constraints*, the required separation between channels assigned to the two endpoints of an edge.

A *constrained, weighted graph* is a pair (G, w) where G is a constrained graph and w is a positive integral weight vector indexed by the nodes of G. The coordinate of

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w corresponding to node u is denoted by $w(u)$ and called the *weight* of node u. The weight of node u represents the number of channels needed at node u.

A *channel assignment* for a constrained, weighted graph (G, w) where $G = (V, E, s, e)$ is an assignment f of sets of nonnegative integers (which will represent the channels) to the nodes of G that satisfies the conditions

$$\begin{array}{ll} |f(u)| = w(u) & (u \in V), \\ i \in f(u) \text{ and } j \in f(v) \Rightarrow |i - j| \geq e(uv) & (uv \in E,\, u \neq v), \\ i, j \in f(u) \text{ and } i \neq j \Rightarrow |i - j| \geq s(u) & (u \in V). \end{array}$$

For reasons of brevity, throughout this paper we will use the notation $f(V)$ to denote $f(V) = \bigcup_{u \in V} f(u)$, in deviation from the standard definition of $f(V) = \{f(u) \,|\, u \in V\}$.

The *span* $S(f)$ of a channel assignment f of a constrained weighted graph is the difference between the lowest and the highest channel assigned by f, in other words, $S(f) = \max_{v \in V} f(v) - \min_{v \in V} f(v)$. The span $S(G, w)$ of a constrained, weighted graph G and a positive integer vector w indexed by the nodes of G is the minimum span of any channel assignment for (G, w).

We will consider complete graphs with constraints that have a special, nested structure. A constrained graph $G = (V, E, s, e)$ is a *nested clique* with parameters (k, u, a), where $k \geq u > a$ if $s(v) \geq k$ for all $v \in V$, and V can be partitioned into two sets Q and R such that $e(vw) \geq a$ if $v, w \in R$, and $e(vw) \geq u$ otherwise. The parameters k, u, and a are always assumed to be positive integers.

3. Tile covers. In this paper, we reduce the channel assignment problem for nested cliques to a tile covering problem. The tiles that may be used for a tile cover are defined in this section. We can think of these tiles as partial assignments, or "building blocks," from which any possible assignment can be constructed.

We assume that a particular nested clique G with node partition (Q, R) and parameters (k, u, a) is given. We define the set $\mathcal{T}$ of all possible tiles that may be used in a tile cover of G. All tiles are defined as vectors indexed by the nodes of G. For reasons of brevity we will sometimes identify a tile with its support and thus think of tiles as node sets. It is this representation that allows mention of "the nodes in tile t."

In order to facilitate the definition and the proof of Theorem 5.1, we distinguish various categories of tiles. So

$$\mathcal{T} = \mathcal{T}_Q \cup \mathcal{T}_R \cup \mathcal{T}_{QR} \cup \mathcal{T}_{QR}^{big}.$$

The tiles in each category are defined as

$$\begin{array}{lll} \mathcal{T}_Q & = & \{\chi^A : A \subseteq Q\}, \\ \mathcal{T}_R & = & \{\chi^B : B \subseteq R\}, \\ \mathcal{T}_{QR} & = & \{\chi^A + \chi^B : A \subseteq Q,\, B \subseteq R, \text{ where } A \neq \emptyset,\, B \neq \emptyset\}, \\ \mathcal{T}_{QR}^{big} & = & \{\chi^{A \cup B} + \chi^{A_2 \cup B_2} : A_2 \subseteq A \subseteq Q,\, B_2 \subseteq B \subseteq R,\, A_2 \neq \emptyset,\, B_2 \neq \emptyset\}. \end{array}$$

The tiles in $\mathcal{T}_{QR}^{big}$ will be called *big* tiles. Note that all coefficients of tiles in $\mathcal{T}_Q$, $\mathcal{T}_R$, and $\mathcal{T}_{QR}$ have value either zero or 1, while for tiles in $\mathcal{T}_{QR}^{big}$, the coefficients indexed by nodes in A_2 and B_2 have value 2.

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A *tiling* is a collection of tiles from $\mathcal{T}$ (multiplicities are allowed). We represent a tiling by a nonnegative integer vector $y \in \mathbb{Z}_+^{\mathcal{T}}$, where $y(t)$ represents the number of copies of tile t present in the tiling. A *tile cover* of a weighted nested clique (G, w) is a tiling y such that $\sum_{t\in\mathcal{T}} y(t)t(v) \geq w(v)$ for each node v of G.

With each tile $t \in \mathcal{T}$ we associate a cost $c(t)$. The costs of the tiles in each category are given in Table 1. The cost of each tile t is derived from the span of a channel assignment for (G, t) plus a "link-up" cost of connecting the assignment to a following tile. This link-up cost is calculated using the assumption that the same assignment will be repeated. For example, $t = \chi^A$, where $A = \{v_0, \dots, v_{j-1}, v_j\}$, is a tile of $j + 1$ distinct vertices in Q. Then the minimum span of (G, t) is u, and an assignment of minimum span would be $f(v_i) = iu$ for all i. However, if this assignment is repeated, the next channel that can be assigned will be $(j + 1)u$, which is u more than the highest channel in the assignment. Hence the link-up cost of this assignment equals u.

It will follow from Theorem 5.1 that our choice of the costs is justified.

TABLE 1
Costs of tiles.

Category	Number of nodes in Q	Number of nodes in R	Cost
$\mathcal{T}_Q$	n	0	$\max\{k, nu\}$
$\mathcal{T}_R$	0	m	$\max\{k, ma\}$
$\mathcal{T}_{QR}$	n	m	$\max\{k, nu + ma + u - a\}$
$\mathcal{T}_{QR}^{big}$	n, of which n_2 have value 2	m, of which m_2 have value 2	$\max\{k, nu\} + \max\{k, ma\} + n_2u + m_2a + u - a$

Formally, the cost of a tile t is such that for any constant α the minimum span of $(G, \alpha t)$ equals $\alpha c(t)$ minus a small constant, or

$$\frac{S(G, \alpha t)}{\alpha} \to c(t) \text{ as } \alpha \to \infty.$$

The *cost* of a tiling y, denoted by $c(y)$, is the sum of the cost of the tiles in the tiling. So $c(y) = \sum_{t\in\mathcal{T}} y(t)c(t)$. The minimum cost of a tile cover of a weighted nested clique (G, w) will be denoted by $\tau(G, w)$.

4. Polyhedral bounds from tile covers. In section 5 we will prove the following theorem.

THEOREM 5.1. *Let G be a nested clique with node partition (Q, R) and parameters (k, u, a). Then for any weight vector w for G,*

$$S(G, w) \geq \tau(G, w) - k.$$

In this section, we will demonstrate how this theorem, combined with polyhedral methods, leads to new lower bounds for $S(G, w)$.

The problem of finding a minimum cost tile cover of (G, w) can be formulated as an integer program (IP):

$$\begin{aligned} &\text{Minimize} \quad && \sum_{t\in\mathcal{T}} c(t)y(t) && \\ &\text{subject to:} \quad && \sum_{t\in\mathcal{T}} t(v)y(t) \geq w(v) \quad && (v \in V), \\ & && y(t) \geq 0 && (t \in \mathcal{T}), \\ & && y \text{ integer}. && \end{aligned}$$

We obtain the LP relaxation of this IP by removing the requirement that y must be integral. Any feasible solution to the resulting linear program is called a *fractional* tile cover. The minimum cost of a fractional tile cover gives a lower bound on the minimum cost of a tile cover. The dual of this LP is formulated as follows:

$$\begin{array}{lll} \text{Maximize} & \sum_{v\in V} w(v)x(v) & \\ \text{subject to:} & \sum_{v\in V} t(v)x(v) \le c(t) & (t \in \mathcal{T}), \\ & x(v) \ge 0 & (v \in V). \end{array}$$

By LP duality, the maximum of the dual is equal to the minimum cost of a fractional tile cover. Thus, any vector that satisfies the inequalities of the dual program gives a lower bound on the cost of a minimum fractional tile cover, and therefore also on the span of the corresponding complete constrained, weighted graph. The maximum is achieved by one of the vertices of the polytope $TC(G)$ defined as follows:

$$TC(G) = \left\{ x \in \mathbb{Q}_+^V : \sum_{v\in V} t(v)x(v) \le c(t) \text{ for all } t \in \mathcal{T} \right\}.$$

A classification of the vertices of this polytope will therefore lead to a comprehensive set of lower bounds that can be obtained from fractional tile covers. The next theorem demonstrates the strength of the tile cover approach, by giving a family of bounds for nested cliques with parameters $(k, u, 1)$.

THEOREM 4.1. *Let G be a nested clique with node partition (Q, R) and parameters $(k, u, 1)$. Let $w \in \mathbb{Z}_+^V$ be a weight vector for G, and let w_{Qmax} be the maximum weight of any node in Q, and w_{Rmax} the maximum weight of any node in R. Then*

$$\tau(G, w) \ge (\lambda_1 - \lambda_2)w_{Qmax} + \lambda_2 \sum_{v\in Q} w(v) + (\lambda_3 - \lambda_4)w_{Rmax} + \lambda_4 \sum_{v\in R} w(v)$$

for each 4-tuple $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_1, λ_2, λ_3, and λ_4 can take the following values:

λ_1	λ_2	λ_3	λ_4	Case
k	0	0	0	(1)
0	0	k	0	(2)
$k-(\mu-1)\delta$	δ	δ	0	(3)
δ	δ	$k-(\mu-1)\delta$	0	(4)
$k-(\mu-1)\delta$	δ	ϵ	ϵ	(5)
u	u	1	1	(6)
u	u	u	$\frac{k-u}{k-1}$	(7)
$2u-1$	ν	1	1	(8)

where

$$\mu = \lfloor \tfrac{k}{u} \rfloor,$$
$$\delta = (\mu+1)u - k,$$
$$\epsilon = \begin{cases} 1 & \text{if } \mu = 1, \\ \min\left\{ \frac{\delta}{k-2u+1}, \frac{2u+\mu\delta-\delta}{k+1}, 1 \right\} & \text{otherwise,} \end{cases}$$
$$\nu = \begin{cases} 1 & \text{if } \mu = 1, \\ u - \max\left\{ \frac{u-1}{\mu}, \frac{\delta-1}{\mu-1} \right\} & \text{otherwise.} \end{cases}$$

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Proof. For the proof we consider feasible points in $TC(G)$ that are of the form

$$\lambda_1\chi^{\{q\}} + \lambda_2\chi^{Q-\{q\}} + \lambda_3\chi^{\{r\}} + \lambda_4\chi^{R-\{r\}}, \quad \text{where } q \in Q,\ r \in R,\ \lambda_1 \geq \lambda_2,\ \lambda_3 \geq \lambda_4.$$

For such points, the inequality system that defines $TC(G)$ reduces to the following form:

$$\lambda_1 + (\mu-1)\lambda_2 \leq k, \tag{1}$$
$$\lambda_1 + \mu\lambda_2 \leq (\mu+1)u, \tag{2}$$
$$\lambda_3 + (k-1)\lambda_4 \leq k, \tag{3}$$
$$\lambda_1 + (\mu-2)\lambda_2 + \lambda_3 + (k-\mu u)\lambda_4 \leq k, \tag{4}$$
$$\lambda_1 + (\mu-1)\lambda_2 + \lambda_3 \leq (\mu+1)u, \tag{5}$$
$$2\lambda_1 + (\mu-1)\lambda_2 + 2\lambda_3 + (k-1)\lambda_4 \leq 2k+2u, \tag{6}$$
$$2\lambda_1 + \mu\lambda_2 + 2\lambda_3 + (k-1)\lambda_4 \leq k + (\mu+3)u, \tag{7}$$
$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0. \tag{8}$$

Inequalities (1) and (2) are obtained by choosing tiles of size μ and $\mu+1$, respectively, from $\mathcal{T}_Q$. Inequality (3) is derived from a tile of size k from $\mathcal{T}_R$.

Inequalities (4) and (5) are derived from tiles in $\mathcal{T}_{QR}$. Inequality (4) is derived from a tile with $\mu-1$ nodes in Q and $k-\mu u-u+1$ nodes in R, and inequality (5) from a tile with μ nodes in Q and one node in R.

Inequalities (6) and (7) are obtained by choosing tiles from $\mathcal{T}_{QR}^{big}$, where nodes q and r have weight 2, all other nodes have weight 1, $m=k$, and $n=\mu$ or $n=\mu+1$, respectively.

Note that inequalities (2) and (3) imply that $\lambda_2 \leq u$ and $\lambda_4 \leq 1$. Using this fact, it is easy to see that all inequalities that correspond to tiles other than those mentioned are implied by inequalities (1)–(7).

It can be verified that each of the points provided in the statement of the theorem provides a feasible solution to the system. Note that each of the feasible solutions satisfies at least one inequality with equality. So, for each vector $x = \lambda_1\chi^{\{q\}} + \lambda_2\chi^{Q-\{q\}} + \lambda_3\chi^{\{r\}} + \lambda_4\chi^{R-\{r\}}$ with $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ as given, and q and r any nodes in Q and R, respectively, it holds that $x \in TC(G)$. Therefore, $\tau(G,w) \geq \sum_{v\in V} w(v)x(v)$. Since $\lambda_1 \geq \lambda_2$ and $\lambda_3 \geq \lambda_4$, $\sum_{v\in V} w(v)x(v)$ is maximized when we choose q and r to be the nodes of maximum weight in Q and R, respectively. With this choice of q and r, $\sum_{v\in V} w(v)x(v) = (\lambda_1-\lambda_2)w_{Qmax} + \lambda_2\sum_{v\in Q} w(v) + (\lambda_3-\lambda_4)w_{Rmax} + \lambda_4\sum_{v\in R} w(v)$, and the result follows. ∎

Theorem 4.1 leads to a family of bounds, since each case of values for the parameters $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ as given in the table leads to a different bound. Some of these bounds are new, while others have been obtained before by conventional methods.

The bounds derived from cases (5), (7), and (8) are new. From case (7), where $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (u,u,u,\frac{k-u}{k-1})$, we obtain the bound

$$S(G,w) \geq u\left(\sum_{v\in Q} w(v) + w_{R\max}\right) + \frac{k-u}{k-1}\sum_{v\in R, v\neq v_{Rmax}} w(v) - k.$$

This bound strengthens the bound $S(G,w) \geq u\sum_{v\in C} w(v) - u$ (first mentioned in [6]), which holds for any clique C, where all edge constraints have value at least u.

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From case (8), which uses the point $(2u-1, \nu, 1, 1)$, we obtain the new bound

$$S(G,w) \geq (2u-1)w_{Qmax} + \nu \sum_{v \in Q, v \neq v_{Qmax}} w(v) + \sum_{v \in R} w(v) - k.$$

In [17] a bound of $(2u-1)w_{Qmax} + \sum_{v \in R} w(v) - \kappa$ (where κ is a small constant) is given for nested cliques with the special property that $|Q| = 1$. The bound resulting from case (8) can be seen as a generalization of this bound for nested cliques where Q contains more than one node.

Case (5) uses the point $(k-(\mu-1)\delta, \delta, \epsilon, \epsilon)$ and leads to the bound

$$S(G,w) \geq (k-\mu\delta)w_{Qmax} + \delta \sum_{v \in Q} w(v) + \epsilon \sum_{v \in R} w(v) - k.$$

The new bound from case (5) can be seen as an extension of the bound $S(G,w) \geq (k-\mu\delta)w_{max} + \delta \sum_{v \in C} w(v) - \kappa$ (κ is a small constant) that was given for cliques with cosite constraint k and uniform edge constraint u in [6].

Using the clique $Q \cup \{v_{Rmax}\}$ (with edge constraint at least u), our method also gives the bound

$$S(G,w) \geq (k-\mu\delta)w_{max} + \delta \left(\sum_{v \in Q} w(v) + w_{Rmax} \right) - k.$$

We simply use case (3) or (4), depending on whether $w_{max} = w_{Qmax}$ or $w_{max} = w_{Rmax}$, respectively.

The bound from case (6), namely,

$$S(G,w) \geq u \sum_{v \in Q} w(v) + \sum_{v \in R} w(v) - k,$$

was the first bound treating nested cliques specifically. It was derived in [6] using ad hoc methods.

The bound derived from cases (1) and (2) is the well-known bound

$$S(G,w) \geq kw_{max} - k.$$

In all these results, we have used the general rule, stated in Theorem 5.1, that $S(G,w) \geq \tau(G,w) - k$. A careful reading of the proof of Theorem 5.1 will show that in most cases the extra term k is too pessimistic. In principle, it is possible to find a more precise additive term by a more precise, and hence more complicated, analysis. Since our main interest here lies in showing a *method* by which lower bounds can be derived rather than in finding the best possible lower bounds, we content ourselves with the additive factor of k. However, this may cause our bounds to differ slightly from the older bounds.

The preceding theorems show how new lower bounds can be generated for any particular choice of parameters. In practice, it will often be useful to apply the tile cover method directly to the exact parameters of the particular network. For any specific nested clique, a classification of all extreme points of $TC(G)$ can be obtained by using vertex enumeration software, for example, the package `lrs`, developed by Avis [3]. In general, we can use the dual program to obtain families of vertices, and hence bounds, for certain choices of parameters.

This approach is demonstrated in the following example. The example is taken from [19], where it was used to demonstrate a lower bound derived from network flows. We will see that our tile cover approach gives a significant improvement.

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Example 4.1. Consider the cellular network layout as shown Figure 1. The circled number in each cell represents the label of the cell; the node associated with the cell with label i is called v_i. The larger number in each cell gives the demand in the cell, i.e., the weight of the associated node. The particular hexagonal cell layout of this example is that of the "Philadelphia problem" [1], which has frequently been used as a benchmark for algorithms and lower bounds for the channel assignment problem (see, for example, [6, 7, 9, 13, 14, 20, 2]).

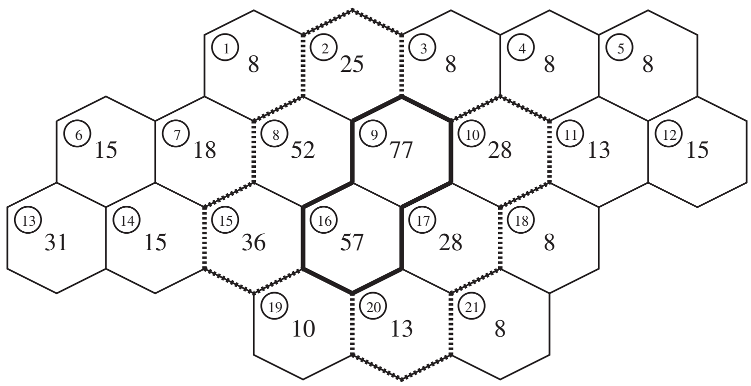

FIG. 1. *The layout of the example.*

The cosite constraint $s(v_i) = 5$ for each node v_i. The edge constraints are described in terms of the distance d_{ij} between the centers of cells v_i and v_j, where the unit is the distance between the centers of adjacent cells:

$$e(v_i, v_j) = \begin{cases} 0 & \text{if } d_{ij} > 3, \\ 1 & \text{if } \sqrt{3} < d_{ij} \leq 3, \\ 2 & \text{if } 0 < d_{ij} \leq \sqrt{3}. \end{cases}$$

This layout contains nested cliques of size 8, with 2 nodes in Q and 6 nodes in R, and nested cliques of size 7, with 1 node in Q and 6 nodes in R. The nested cliques have parameters $(5, 2, 1)$.

For a nested clique with bipartition (Q, R), where $|Q| = 2$ and $|R| = 6$, we derived a set of lower bounds using the software `lrs`. We looked for points of the form $(x_1, x_2, y_1, y_2, y_3, y_4, y_5, y_6)$, where x_1 and x_2 correspond to nodes of Q and $x_1 \geq x_2$, and $y_1, \ldots, y_6$ correspond to the nodes of R and $y_1 \geq y_2 \geq \cdots \geq y_6$. The inequality system that defines $TC(G)$ reduces to the following:

$$\begin{aligned}
&x_1 + x_2 \leq 5, \\
&y_1 + y_2 + y_3 + y_4 + y_5 \leq 5, \\
&x_1 + y_1 + y_2 \leq 5, \\
&x_1 + y_1 + y_2 + y_3 \leq 6, \\
&x_1 + y_1 + y_2 + y_3 + y_4 \leq 7, \\
&x_1 + x_2 + y_1 \leq 6, \\
&x_1 + x_2 + y_1 + y_2 \leq 7, \\
&x_1 + x_2 + y_1 + y_2 + y_3 \leq 8, \\
&x_1 + x_2 + y_1 + y_2 + y_3 + y_4 \leq 9, \\
&x_1 \geq x_2 \geq 0,\ y_1 \geq y_2 \geq \cdots \geq y_6 \geq 0.
\end{aligned}$$

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Given this system, `lrs` returned a set of vertices, 14 of which could be used to generate lower bounds (the other vertices could be obtained from those 14 by dropping some coordinates to zero).

We applied these bounds to the nested clique formed by the cells as indicated in Figure 1. Here $Q = \{v_9, v_{16}\}$, and $R = \{v_2, v_8, v_{10}, v_{15}, v_{17}, v_{20}\}$. To obtain the best possible results, the nodes of larger weight in Q and R were matched with larger coordinates x_i or y_i, respectively. The best result was obtained by the point $(3, 2, 1, 1, 1, 1, 1, 1)$. The corresponding lower bound is

$$\begin{aligned} S(G, w) &\geq 3w(v_9) + 2w(v_{16}) + \sum_{v \in R} w(v) - 5 \\ &= 3 \cdot 77 + 2 \cdot 57 + (52 + 36 + 28 + 28 + 25 + 13) - 5 \\ &= 522. \end{aligned}$$

This improves by 13% the lower bound of 460 obtained in [19].

Example 4.2. Our second example also involves a variation of the Philadelphia problem. It should be noted that this example incorporates many properties of real-life problems: a regular planar layout of the base stations, derived from the ideal packing formed by the hexagonal grid, as well as edge constraints that diminish as the distance between base stations increases. This example again uses the layout of Figure 1. The cosite constraint for this example is 7, while the edge constraints are as follows:

$$e(v_i, v_j) = \begin{cases} 0 & \text{if } d_{ij} > 3, \\ 1 & \text{if } \sqrt{7} d_{ij} \leq 3, \\ 2 & \text{if } \sqrt{3} < d_{ij} \leq 2, \\ 3 & \text{if } 0 < d_{ij} \leq \sqrt{3}. \end{cases}$$

In this case, the network contains a nested clique (Q, R), where $Q = \{v_8.v_9, v_{16}\}$ and $R = \{v_2, v_{15}, v_{17}\}$, with parameters $(7, 3, 2)$ (other nested cliques exist in similar configurations). Assume that the demand in this nested clique is as follows:

Node	v_2	v_8	v_9	v_{15}	v_{16}	v_{17}
Demand	10	15	30	30	15	10

Consider the following dual solution to the tile cover problem: $x = (2, 2, 4, 3, 2, 3)$ (the ordering of the components in the vector refers to the order of the nodes as given in the table above), and a tile cover consisting of 10 copies of the tile $(1, 1, 2, 2, 1, 1) \in \mathcal{T}_{QR}^{big}$, and 5 copies each of tiles $(0, 1, 1, 1, 0, 0)$ and $(0, 0, 1, 1, 1, 0)$, both in $\mathcal{T}_Q$. It can be easily checked that these primal/dual solutions have the same value, namely, 310. This leads to a lower bound for the span of 303. This lower bound can be refined to 307 if the tile cover method is extended to include patches, as explained later in this paper. Moreover, the optimal tile cover can be converted into a matching channel assignment (in the last line, i takes values from 0 to 9):

v_2	v_8	v_9	v_{15}	v_{16}	v_{17}
	6,15,...,42	0,9,18,...,81	3,12,21,...,84	51,60,...,87	
$104 + 22i$	$93 + 22i$	$90, 99 + 22i$	$102, 109 + 22i$	$96 + 22i$	$106 + 22i$

Example 4.3. In [2], a small assignment problem of only 7 nodes is presented (instances M1 and M2). The problem was formed to test the limits of the method

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proposed in the paper. Indeed, for this instance there is a gap of 3 between upper and lower bounds found by the method of Avenali, Mannino, and Sassano [2]. The cosite constraint is 5 for each node, while the edge constraints are as given in the following table.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	5	2	3	4	1	0	0
v_2		5	1	4	1	4	2
v_3			5	1	1	2	0
v_4				5	0	1	0
v_5					5	1	1
v_6						5	2
v_7							5

This example is highly irregular, but it does contain some small nested cliques. For example, there is a nested clique (Q, R) with parameters $(5, 4, 2)$, where $Q = \{v_6\}$ and $R = \{v_1, v_2\}$. One of the tiles for this nested clique is the tile from $\mathcal{T}_{QR}$ consisting of all vertices of (Q, R), with cost $2 \cdot 4 + 2 = 10$. In the examples of [2], the demand of all nodes is equal. Combining 10 such tiles gives a tile cover of cost 100, which leads to a lower bound of 95 for the case where the demand on all nodes equals 10. Following the more precise method outlined later in this paper, we can replace one of the tiles by a patch with cost 6, which gives a lower bound of 96 when the demand on all nodes equals 10, and 106 when the demand on all nodes equals 11. This reduces the gap between upper and lower bounds to 1. (Note that this particular case, where $|Q| = 1$, can also be solved with the bound from [17].)

5. From channel assignments to tile covers. In this section we give the proof of the following theorem.

THEOREM 5.1. *Let G be a nested clique with node partition (Q, R) and parameters (k, u, a). Then for any weight vector w for G,*

$$S(G, w) \geq \tau(G, w) - k.$$

This theorem will follow as a corollary from a more technical lemma. The lemma reduces any channel assignment to a tiling that uses only tiles from $\mathcal{T}$, except for one extra tile called a *patch*. (In subsequent proofs, we will specify a specific tile to act as the patch of any given tiling.) A patch is added to take care of the highest channels assigned, for which there is no link-up cost. Patches are defined as follows.

Given a nested clique G with node bipartition (Q, R) and constraints (k, u, a), the patch set $\mathcal{P}$ is defined as

$$\mathcal{P} = \mathcal{P}_Q \cup \mathcal{P}_R \cup \mathcal{P}_{QR} \cup \mathcal{P}_{QR}^{big}.$$

The patches in each category are defined below:

$$\begin{aligned}
\mathcal{P}_Q &= \{\chi^A : A \subseteq Q\},\\
\mathcal{P}_R &= \{\chi^B : B \subseteq R\},\\
\mathcal{P}_{QR} &= \{\chi^A + \chi^B : A \subseteq Q,\, B \subseteq R,\, A \neq \emptyset,\ B \neq \emptyset\},\\
\mathcal{P}_{QR}^{big} &= \{\chi^{A \cup B} + \chi^{A_2 \cup B_2} : A_2 \subseteq A \subseteq Q,\, B_2 \subseteq B \subseteq R,\, A_2 \neq \emptyset,\, B_2 \neq \emptyset\}.
\end{aligned}$$

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The cost of a patch p is denoted by $c'(p)$. The definition of the cost of a tile cover $y \in \mathbb{Z}^{\mathcal{T}\cup\mathcal{P}}$ is adjusted to account for patch cost:

$$c(y) = \sum_{t\in\mathcal{T}} c(t)y(t) + \sum_{p\in\mathcal{P}} c'(p)y(p).$$

Patch costs for each category are given in Table 2.

TABLE 2
Costs of patches.

Category	Number of nodes in Q	Number of nodes in R	Cost
$\mathcal{P}_Q$	n	0	$(n-1)u$
$\mathcal{P}_R$	0	m	$(m-1)a$
$\mathcal{P}_{QR}$	n	m	$nu + (m-1)a$
$\mathcal{P}_{QR}^{big}$	n, of which n_2 have weight 2	m, of which m_2 have weight 2	$(n+n_2)u + (m_2-1)a+$ $\max\{k, ma\}$

When we reduce a channel assignment to a tiling, a patch from $\mathcal{P}_R$ will be used only when the first channel is assigned to a node in R, and a patch from either $\mathcal{P}_Q$ or $\mathcal{P}_{QR}^{big}$ will be used only if the first channel is assigned in Q.

For the rest of this section we will adopt the following terminology. Suppose f is a channel assignment for a constrained graph G with node set V, where $f(V) = \{h_0, h_1, \ldots, h_f\}$, with $h_0 \le h_1 \le \cdots \le h_f$. We say that a tiling y of G *covers* channels h_i to h_j (where $j \ge i$) if y is a tile cover of the subgraph induced by the nodes of G that were assigned channels between h_i and h_j. More precisely, y covers channels $\{h_i, \ldots, h_j\}$ if for each node $v \in V$,

$$\sum_{t\in\mathcal{T}} y(t)t(v) \ge |f(v) \cap \{h_i, \ldots, h_j\}|.$$

Also, when y is a tiling and t is a patch or tile, we use $y + \{t\}$ to mean the tiling where one more copy of t is added, i.e., strictly speaking, the tiling $y + \chi^{\{t\}}$.

We start by stating a lemma that proves that any channel assignment can be reduced to a tile cover for the cliques where there is only one edge constraint and a cosite constraint.

LEMMA 5.2 (see [10]). *Let G be a clique with cosite constraint k and edge constraint u. Let Q be the node set of G, and let the tile set $\mathcal{T}_Q$ and patch set $\mathcal{P}_Q$ be as defined above. Let f be a channel assignment for G, where $f(V) = \{h_0, h_1, \ldots, h_f\}$, $h_0 < h_1 < \cdots < h_f$. Then there exists a tile cover $y \in \mathbb{Z}_+^{\mathcal{T}_Q\cup\mathcal{P}_Q}$ of (G, w) which contains exactly one patch p, covers all channels $\{h_0, \ldots, h_f\}$, and has cost at most $h_f - h_0$. Moreover, the support of p consists of the nodes that are assigned channels $h_{f-n}, \ldots, h_f$, where $n = |V(p)|$ and*

$$c(y - \{p\}) \le h_{f-n} - h_0.$$

The proof of Lemma 5.2 provides the following method of constructing the tile cover y, with patch p. Begin by finding a tile containing the set of nodes that are assigned channels in the range $[h_0, h_0 + k)$. Let t_0 denote that tile. For $j \ge 1$, we recursively find a tile t_j containing the nodes assigned channels in the range $[h_{e_j}, h_{e_j} + k)$, where h_{e_j} is the first channel not covered by the tiling $y_{j-1} = \chi^{\{t_0,t_1,\ldots,t_{j-1}\}}$.

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Tile t_j is chosen so that the cost of $y_j = y_{j-1} + \{t_j\}$ is at most $h_{e_{j+1}} - h_0$, where $h_{e_{j+1}}$ is the first channel not covered by y_j. This process continues until the only channels not covered by the current tiling y_ℓ form a patch p. The cost of this patch is $c'(p) = h_f - h_{e_{\ell+1}}$, where $h_{\ell+1}$ is the first channel not covered by y_ℓ. The required tile cover y is formed by adding p to y_ℓ.

We are now ready to state and prove the technical lemma from which Theorem 5.1 will follow. The proof of this lemma uses a straightforward induction on the number of times the channel assignment "crosses over" from Q to R or vice versa. By invocation of Lemma 5.2, tilings are obtained for the channel assignment up to the first crossover and between the first and second crossovers, respectively. Then induction is used to obtain a tiling of the channel assignment that includes all channels after the second crossover. These tilings are then combined to obtain one new tiling which satisfies the induction hypothesis. The difficulties arise mainly from the fact that three different patches must be combined. As a result, there are a number of cases to be considered. Once the appropriate combinations of tiles and patches are described, verifying the cost of the tiling merely involves finding the appropriate substitutions. This, together with the fact that numerous cases are analogous, compels us to omit the details of the proof in many cases. For a complete treatment of the proof, we refer the reader to [11].

LEMMA 5.3. *Let G be a nested clique with node partition (Q, R) and integer constraints (k, u, a), and let $\mathcal{T}$ and $\mathcal{P}$ be the tile and patch set for G. Let f be a channel assignment for G, where $f(V) = \{h_0, h_1, \dots, h_f\}$, $h_0 < h_1 < \cdots < h_f$. Then there exists a tile cover $y \in \mathbb{Z}_+^{\mathcal{T}\cup\mathcal{P}}$ of (G, w) which contains one patch p, covers all channels $\{h_0, \dots, h_f\}$, and has cost at most $h_f - h_0$. Furthermore, if h_0 is assigned to a node in Q, then $p \notin \mathcal{P}_R$, and if h_0 is assigned to a node in R, then $p \notin \mathcal{P}_Q \cup \mathcal{P}_{QR}^{big}$.*

Proof. Let G be a nested clique and f be a channel assignment, as defined in the statement of the lemma. A crossover is defined to be a pair of channels (h_i, h_{i+1}), where the nodes that receive channels h_i and h_{i+1} are in different parts of the bipartition (Q, R). We now proceed with induction on the number of crossovers.

If f has no crossovers, then the statement follows directly from Lemma 5.2.

Suppose f has exactly one crossover and h_0 is assigned to a node in Q. Let h_ℓ be the first channel in R greater than h_0. By Lemma 5.2, we can cover the channels in $\{h_0, \dots, h_{\ell-1}\}$ with a tiling y_Q, containing one patch $p_Q \in \mathcal{P}_Q$, with cost at most $h_{\ell-1} - h_0$. Likewise, the channels in $\{h_\ell, \dots, h_f\}$ can be covered with a tiling y_R of cost at most $h_f - h_\ell$, containing one patch $p_R \in \mathcal{P}_R$. Combining the two patches into one, we form a new patch $p = p_Q + p_R \in \mathcal{P}_{QR}$ with cost $nu + (m-1)a$, where $n = |V(p_Q)|$ and $m = |V(p_R)|$. So $c'(p) = c'(p_Q) + c'(p_R) + u$. Moreover, $h_\ell - h_{\ell-1} \geq u$ since $h_{\ell-1}$ is assigned to a node in Q, and h_ℓ to a node in R.

Our final tiling is $y = y_Q - \{p_Q\} + y_R - \{p_R\} + \{p\}$ with cost

$$\begin{aligned} c(y) &= c(y_Q) - c'(p_Q) + c(y_R) - c'(p_Q) + c'(p) \\ &\leq (h_{\ell-1} - h_0) + (h_f - h_\ell) + u \\ &\leq h_f - h_0. \end{aligned}$$

When h_0 is assigned to a node in R, the proof is analogous.

For the induction step, assume that f is a channel assignment with g crossovers, where $g \geq 2$, and assume that the lemma holds for any channel assignment with less than g crossovers.

Case 1. Channel h_0 is assigned to a node in Q.

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Let h_ℓ be the first channel assigned to a node in R, and let h_j be the first channel greater than h_ℓ assigned to a node in Q. So $(h_{\ell-1}, h_\ell)$ and (h_{j-1}, h_j) are the first two crossovers of f. Note that $h_\ell \geq h_{\ell-1} + u$ and $h_j \geq h_{j-1} + u$.

By Lemma 5.2, we can find a tiling y_Q (with one patch, $p_Q \in \mathcal{P}_Q$) which covers channels $\{h_0, \dots, h_{\ell-1}\}$ in Q and has cost at most $h_{\ell-1} - h_0$, and a tiling y_R (with one patch, $p_R \in \mathcal{P}_R$) which covers channels $\{h_\ell, \dots h_{j-1}\}$ and has cost at most $h_{j-1} - h_\ell$.

Define n and m to be the number of nodes in $V(p_Q)$ and $V(p_R)$, respectively. By Lemma 5.2, $V(p_Q)$ consists of the nodes that receive channels $\{h_{\ell-n}, \dots, h_{\ell-1}\}$, and $V(p_R)$ consists of the nodes that receive channels $\{h_{j-m}, \dots, h_{j-1}\}$. Note that $c'(p_Q) = (n-1)u$. $c'(p_R) = (m-1)a$.

Case 1A. Tiling y_R contains only the patch p_R (no other tiles).

In this case, patch p_R covers all channels from h_ℓ to h_{j-1}.

(i) Suppose $h_j - h_{\ell-n} \geq k$. Let $y = y_Q - \{p_Q\} + \{t\} + y_{end}$, where $t = p_Q + p_R$, and y_{end} is a tiling covering channels $\{h_j, \dots, h_f\}$ with cost at most $h_f - h_j$ and a patch that is not in $\mathcal{P}_R$. (By induction, such a tiling y_{end} exists.) The new tiling y has the patch of y_{end} as its patch. It is clear that y covers all channels from h_0 to h_f and has a patch of the required type. It now remains to be proved that $c(y) \leq h_f - h_0$.

Since the channels from $h_{\ell-n}$ to h_j cover $n+1$ nodes in Q and m nodes in R, with two crossovers, we have $h_j \geq h_{\ell-n} + (n-1)u + (m-1)a + 2u$. Also, by assumption, $h_j - h_{\ell-n} \geq k$. Therefore, $h_j - h_{\ell-n} \geq \max\{nu + ma + u - a, k\} = c(t)$, and

$$\begin{aligned} c(y) &= c(y_Q - \{p_Q\}) + c(t) + c(y_{end}) \\ &\leq (h_{\ell-n} - h_0) + (h_j - h_{\ell-n}) + (h_f - h_j) \\ &= h_f - h_0. \end{aligned}$$

(ii) Suppose $h_j - h_{\ell-n} < k$. If there is a channel in the range $[h_{\ell-n}+k, h_{\ell-n}+k+u)$ which has been assigned to a node in Q, let h_i denote that channel. (The choice of h_i is unique, since the given range has length less than u.) Otherwise, let h_i be the first channel greater than or equal to $h_{\ell-n} + k + u$. If no such h_i can be chosen, then let $i = f + 1$, so $h_{i-1} = h_f$ and h_i is undefined. Note that it is always the case that $h_{i-1} < h_{l-n} + k + u$.

We form the final tile cover y as follows:

Step 1. Let A be the set of all nodes that receive channels from $\{h_{\ell-n}, \dots, h_{i-1}\}$. Also, let $n_1 = |Q \cap A|$, and let $m_1 = |R \cap A|$.

Step 2. Find a tiling y_{end} which covers channels $\{h_i, \dots, h_f\}$ and has cost at most $h_f - h_i$. Let p_{end} be the patch of y_{end}. (In the case that $h_{i-1} = h_f$, both y_{end} and p are empty.)

Step 3. If $p_{end} \in \mathcal{P}_Q \cup \mathcal{P}_{QR} \cup \mathcal{P}_{QR}^{big}$, form tile $t = \chi^A \in \mathcal{T}_{QR}$ and let $y = y_Q - \{p_Q\} + \{t\} + y_{end}$.

Step 4. If $p_{end} \in \mathcal{P}_R$, then
- 4(a) pick a node $v \in A \cap Q$,
- 4(b) form patch $p = p_{end} + \chi^{\{v\}} \in \mathcal{P}_{QR}$,
- 4(c) form tile $t = \chi^A - \chi^{\{v\}}$,
- 4(d) form tile cover $y = y_Q - \{p_Q\} + \{t\} + y_{end} - \{p_{end}\} + \{p\}$.

Step 5. If p_{end} is empty, then
- 5(a) form patch $p = \chi^A \in \mathcal{P}_{QR}$,
- 5(b) form tile cover $y = y_Q - \{p_Q\} + \{p\}$.

As before, it is easy to see that y covers all channels from h_0 to h_f. Steps 3, 4, and 5 guarantee that the patch of y is not in $\mathcal{P}_R$, as required. We prove that in all cases, $c(y) \leq h_f - h_0$.

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CLAIM 5.4. *No two channels in* $S = \{h_{l-n}, \ldots, h_{i-1}\}$ *are assigned to the same node.*

Proof of claim. Assume two channels h_α and h_β, $l-n \leq \alpha < \beta \leq i-1$, are assigned to the same node. Now, by combining our cosite constraint with a previous remark, we have $k \leq h_\beta - h_\alpha \leq h_{i-1} - h_{l-n} < k+u$.

Since h_{i-1} is in the interval $[h_{l-n}+k, h_{l-n}+k+u)$, it follows from the choice of h_i that no channel in $[h_{l-n}+k, h_{l-n}+k+u)$ is assigned to a node in Q. Hence, all channels from S assigned to Q are in the interval $[h_{l-n}, h_{l-n}+k)$. Since the range of this interval is less than k, it cannot be the case that h_α and h_β are assigned to a node in Q.

Suppose h_α and h_β are both assigned to nodes in R. Since h_{l-n} is assigned to a node in Q, $h_{l-n}+u \leq h_\alpha$ due to our adjacency constraints. Since $h_\beta < h_{l-n}+k+u$, $h_\beta - h_\alpha < k$, which is a contradiction. Hence, no node receives two channels from C. ∎

CLAIM 5.5. *When a channel* h_i *can be chosen,* $h_i - h_{\ell-n} \geq \max\{n_1u + m_1a + u - a, k\}$.

Proof of claim. Suppose h_i is assigned to a node in Q. Since $\{h_{\ell-n}, \ldots, h_i\}$ is covered by n_1+1 nodes in Q and m_1 nodes in R and contains at least two crossovers, we have $h_i - h_{\ell-n} \geq (n_1-1)u + (m_1-1)a + 2u = n_1u + m_1a + u - a$.

If h_i is assigned to a node in R, then $\{h_{\ell-n}, \ldots, h_i\}$ covers n_1 nodes of Q and m_1+1 nodes in R and contains at least three crossovers. Hence, $h_i - h_{\ell-n} \geq (n_1-2)u + (m_1-1)a + 3u = n_1u + m_1a + u - a$.

Whenever h_i is chosen, it is done in such a way that $h_i \geq h_{l-n}+k$. Hence, $h_i - h_{\ell-n} \geq \max\{k+u, n_1u + m_1a + u - a\}$. ∎

In Step 3, we have $t = \chi^A \in \mathcal{T}_{QR}$ and $c(t) = \max\{k, n_1u+m_1a+u-a\} \leq h_i - h_{\ell-n}$. Therefore,

$$c(y) = c(y_Q - \{p_Q\}) + c(t) + c(y_{end}) \leq h_f - h_0.$$

In Step 4, a new patch $p = p_{end} + \chi^{\{v\}} \in \mathcal{P}_{QR}$ is formed, since p_{end} is not of the required type. The cost of this new patch is $c'(p) = c'(p_{end}) + u$. In finding the cost of t there are two possibilities to consider. If $n_1 > 1$, then $t = \chi^A - \chi^{\{v\}} \in \mathcal{T}_{QR}$ and $c(t) = \max\{k, (n_1-1)u + m_1a + u - a\}$. If $n_1 = 1$, then $t \in \mathcal{T}_R$ and $c(t) = \max\{k, m_1a\} \leq \max\{k, (n_1-1)u+m_1a+u-a\}$. Now, since $h_i - h_{\ell-n} \geq \max\{k+u, n_1u+m_1a+u-a\}$, it follows that $c(t) \leq h_i - h_{\ell-n} - u$. Since $c'(p) - c'(p_{end}) \leq u$, it follows that

$$c(y) = c(y_Q - \{p_Q\}) + c(y_{end}) + (c'(p) - c'(p_{end})) + c(t) \leq h_f - h_0.$$

In Step 5, we have $h_{i-1} = h_f$. Since $p \in \mathcal{P}_{QR}$, we have $c'(p) = n_1u + (m_1-1)a$. Furthermore, since $\{h_{\ell-n}, \ldots, h_f\}$ contains n_1 nodes from Q, m_1 nodes from R, and at least two crossovers, $h_f - h_{\ell-n} \geq n_1u + (m_1-1)a = c'(p)$. Therefore,

$$c(y) = c(y_Q - \{p_Q\}) + c'(p) \leq h_f - h_0.$$

Case 1B. y_R contains a tile other than p_R.

By Lemma 5.2, patch p_R covers channels $\{h_{j-m}, \ldots, h_{j-1}\}$, and these channels are all assigned to nodes in R, so $j - m \geq \ell$. Since $(h_{\ell-1}, h_\ell)$ is a crossover, the assignment of channels $\{h_{j-m}, \ldots, h_f\}$ has $g-1$ crossovers. Then, by induction, there exists a tiling y_{end} that covers all channels in $\{h_{j-m}, \ldots, h_f\}$, contains a patch $p_{end} \in \mathcal{P}_{QR} \cup \mathcal{P}_R$, and has cost at most $h_f - h_{j-m}$.

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Let $V_Q = V(p_{end}) \cap Q$, $V_R = V(p_{end}) \cap R$. Also let $n^p = |V_Q|$ and $m^p = |V_R|$. Note that $c'(p_{end}) = n^p u + (m^p - 1)a$ if $p_{end} \in \mathcal{P}_{QR}$, and $c'(p_{end}) = (m^p - 1)a$ if $p_{end} \in \mathcal{P}_R$.

Choose t_R to be any tile from y_R other than p_R. Let $V_t = V(t_R)$ and $m^t = |V_t|$. Note that $t_R \in \mathcal{T}_R$ and $c(t_R) = \max\{k, m^t a\}$. Let $V_{p_Q} = V(p_Q)$. Recall that $|V_{p_Q}| = n$ and $c'(p_Q) = (n-1)u$. In Table 3, we show how to combine p_Q, p_{end}, and t_R into a new tile t and a new patch p.

Table 3
Combining patches.

Case	Condition	Tile t	Patch p
(1)	$p_{end} \in \mathcal{P}_{QR}$		
(1.1)	(1) and $V_Q \cap V_{p_Q} = \emptyset$	t_R	$p_{end} + p_Q$
(1.2)	(1) and $V_Q \cap V_{p_Q} \neq \emptyset$		
(1.2.1)	(1.2) and $V_R \cap V_t = \emptyset$	$p_{end} + t_R$	p_Q
(1.2.2)	(1.2) and $V_R \cap V_t \neq \emptyset$	there is no t	$t_R + p_{end} + p_Q$
(2)	$p_{end} \in \mathcal{P}_R$	t_R	$p_{end} + p_Q$

Case	Cost $c(t)$	$t \in$	Cost $c'(p)$	$p \in$
(1.1)	$c(t_R)$	$\mathcal{T}_R$	$(n+n^p)u + (m^p - 1)a$	$\mathcal{P}_{QR}$
(1.2.1)	$\max\{k, n^p u + m^p a + m^t a + u - a\}$	$\mathcal{T}_{QR}$	$c'(p_Q)$	$\mathcal{P}_Q$
(1.2.2)	–	–	$(n+n^p)u + \lvert V_R \cap V_t\rvert a -$ $a + \max\{k, \lvert V_R \cup V_t\rvert a\}$	$\mathcal{P}_{QR}^{big}$
(2)	$c(t_R)$	$\mathcal{T}_R$	$nu + (m^p - 1)a$	$\mathcal{P}_{QR}$

In Cases (1.1), (1.2.1), and (2), we form the new tiling

$$y = y_Q - \{p_Q\} + y_R - \{p_R\} - \{t_R\} + y_{end} - \{p_{end}\} + \{t\} + \{p\}.$$

In Case (1.2.2), there is no t, so we take the tiling

$$y = y_Q - \{p_Q\} + y_R - \{p_R\} - \{t_R\} + y_{end} - \{p_{end}\} + \{p\}.$$

In all cases, it is straightforward to verify that y covers all channels and has a patch of the required type, and that $c(y) \leq h_f - h_0$.

Case 2. Channel h_0 is assigned to a node in R.

Since this case is very similar to Case 1, we omit the details of the proof. And, unless otherwise stated, the same terminology will apply.

Let h_ℓ be the first channel assigned to a node in Q, and h_j the first channel greater than h_ℓ assigned to a node in R. As in Case 1, by Lemma 5.2, we can find tilings y_R and y_Q of the required cost, which together cover all channels in $\{h_0, \ldots, h_{j-1}\}$. Furthermore, by induction, we can find the appropriate tiling y_{end} (with patch p_{end}) to cover the remaining channels.

We now provide the method for finding a new tiling y that covers all channels in the assignments and has cost at most $h_f - h_0$.

Case 2A. Tiling y_Q contains only the patch p_Q.

(i) If $h_j - h_{l-m} \geq k$, then let $y = y_R - \{p_R\} + \{t\} + y_{end}$, where $t = p_Q + p_R$.

(ii) Suppose $h_j - h_{\ell-m} < k$. Channel h_i is chosen in a manner similar to that in Case 1A. Simply replace "Q" and "n" with "R" and "m," respectively, in the description. Similarly, A denotes the set of nodes receiving channels from $\{h_{\ell-m}, \ldots, h_{i-1}\}$.

If $p_{end} \in \mathcal{P}_R \cup \mathcal{P}_{QR}$, then let $y = y_R - \{p_R\} + \{t\} + y_{end}$, where $t = \chi^A \in \mathcal{T}_{QR}$.

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TABLE 4
Combining patches.

Case	Condition	Tile t	Patch p
(1)	$p_{end} \in \mathcal{P}_{QR}^{big}$		
(1.1)	(1) and $V_t \cap V_Q = \emptyset$	$p_{end} + t_Q$	p_R
(1.2)	(1) and $V_t \cap V_Q \neq \emptyset$	$p_{end} + \chi^{V_t - V_Q}$	$\chi^{V_t \cap V_Q} + p_R$
(2)	$p_{end} \in \mathcal{P}_{QR}$	$t_Q + \chi^{V_R}$	$\chi^{V_Q} + p_R$
(3)	$p_{end} \in \mathcal{P}_Q$	t_Q	$p_{end} + p_R$

Case	Cost $c(t)$	$t \in$	Cost $c'(p)$	$p \in$
(1.1)	$\max\{k, (n^p + n^t)u\} + \max\{k, m^p a\}$ $+n_2^p u + m_2^p a + u - a$	$\mathcal{T}_{QR}^{big}$	$c'(p_R)$	$\mathcal{P}_R$
(1.2)	$\max\{k, \lvert V_t \cup V_Q \rvert u\} + \max\{k, m^p a\}$ $+n_2^p u + m_2^p a + u - a$	$\mathcal{T}_{QR}^{big}$	$\lvert V_t \cap V_Q \rvert u + c'(p_R)$	$\mathcal{P}_{QR}$
(2)	$\max\{k, n^t u + m^p a + u - a\}$	$\mathcal{T}_{QR}$	$n^p u + c'(p_R)$	$\mathcal{P}_{QR}$
(3)	$c(t_Q)$	$\mathcal{T}_Q$	$n^p u + c'(p_R)$	$\mathcal{P}_{QR}$

If $p_{end} \in \mathcal{P}_Q$, $y = y_R - \{p_R\} + \{p\} + y_{end} - \{p_{end}\} + \{t\}$, where $p = p_{end} + \chi^{\{v\}}$, $v \in A \cap R$, and $t = \chi^{A-\{v\}}$.

If $p_{end} \in \mathcal{P}_{QR}^{big}$, then let $t = \chi^{V_Q} + \chi^{V_R}$, $t' = \chi^A$, and $p = \chi^{B_Q} + \chi^{B_R}$. In this case, let $y = y_R - \{p_R\} + y_{end} - \{p_{end}\} + \{t\} + \{t'\} + \{p\}$.

For Cases 2A(i) and 2A(ii), it is straightforward to show that $c(y) \leq h_f - h_0$.

Case 2B. y_Q contains tiles other than p_Q.

Choose t_Q to be any tile from $y_Q - \{p_Q\}$. In Table 4, we show how we will combine p_R, p_{end}, and t_Q into a new tile t and a new patch p. Note that $n^t = |V(t_Q)|$ in this case. In all instances, we form the new tiling

$$y = y_R - \{p_R\} + y_Q - \{p_Q\} - \{t_Q\} + y_{end} - \{p_{end}\} + \{t\} + \{p\}.$$

It is straightforward to verify that y covers all channels and has cost at most $h_f - h_0$. This completes the proof. ∎

6. Conclusions.

We have given a new general method of obtaining lower bounds for the channel assignment problems. When applied to the specific example of nested cliques, this leads to a complete family of lower bounds. This family includes almost all known clique-bounds. The bounds are easy to compute, and give improved results when applied to an example from the literature.

Nested cliques occur naturally in many CAPs. Radio signals decay with distance, so edge constraints between transmitters that are close together are usually stricter than constraints between transmitters that are farther apart. For the same reason, cosite constraints are usually the most restrictive. In this situation, a tight cluster of transmitters in a central area such as a city center, surrounded by a wider ring of more sparsely placed transmitters, will typically form a nested clique. The examples in this paper illustrate this situation.

Further work should address the computational issues related to lower bounds. A computational study comparing the performance of tile cover bounds to the lower bounds from previous work discussed in the introduction on a number of realistic CAP instances would be a valuable addition to this theoretical analysis.

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Another interesting question is whether the tile cover approach can be used to obtain good channel assignments. Knowledge about which lower bound is most restrictive for any particular instance could be used to determine which tiles were most suited to build the best assignment.

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