AIP | Journal of Mathematical Physics

The maximum dimension of the inheriting algebra in perfect fluid space–times

A. A. Coley, G. S. Hall, A. J. Keane, and B. O. J. Tupper

Citation: Journal of Mathematical Physics **43**, 5567 (2002); doi: 10.1063/1.1509087
View online: http://dx.doi.org/10.1063/1.1509087
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/43/11?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 43, NUMBER 11 NOVEMBER 2002

The maximum dimension of the inheriting algebra in perfect fluid space–times

A. A. Coley[a)]
Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia B3H 3J5, Canada

G. S. Hall[b)]
Department of Mathematical Sciences, University of Aberdeen, Dunbar Street, Aberdeen AB24 3UE Scotland, United Kingdom

A. J. Keane[c)]
Department of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, Scotland, United Kingdom

B. O. J. Tupper[d)]
Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick E3B 5A3, Canada

(Received 29 November 2001; accepted 10 June 2002)

We determine the maximum dimension of the Lie algebra of inheriting conformal Killing vectors in perfect fluid space–times. For the case of conformally flat space–times the maximum dimension is eight and for the case of nonconformally flat space–times the maximum dimension is found to be five. We illustrate each case with examples. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1509087]

I. INTRODUCTION

We are interested in space–times which admit conformal Killing vector fields and, in particular, fluid space–times which admit inheriting conformal Killing vector fields. A conformal Killing vector field is said to be an inheriting conformal Killing vector field if fluid flow lines **u** are mapped conformally by the conformal Killing vector field (see Sec. II and Coley and Tupper[1]). The motivation for studying inheriting conformal Killing vector fields was discussed in Ref. 1 and inheriting conformal Killing vector fields in perfect fluid space–times were studied in Refs. 2 and 3. For general space–times, from a kinematical description of matter it has been shown[4] that in order for there to be zero entropy production there must exist a conformal Killing vector field parallel to the fluid four-velocity (which is consequently inheriting).

In this article we determine the maximum dimension of the Lie algebra of inheriting conformal Killing vectors in perfect fluid space–times. For the case of conformally flat space–times the maximum dimension is eight and for the case of nonconformally flat space–times the maximum dimension is found to be five.

In Sec. II we define conformal Killing vector fields and state a number of theorems concerning the maximum dimension of Lie algebras of conformal Killing vector fields. We also address the reducibility of a Lie algebra of conformal Killing vector fields (to a Lie algebra of Killing vector fields) with respect to a conformal scaling of the metric. We consider some cases where the inheriting condition is automatically satisfied. In Sec. III we consider the maximum dimension of the inheriting Lie algebra for conformally flat space–times and present the space–times which admit this maximum number. In Sec. IV we determine the maximum dimension of the inheriting

[a)]Electronic mail: aac@mathstat.dal.ca
[b)]Electronic mail: g.hall@maths.abdn.ac.uk
[c)]Electronic mail: aidan@astro.gla.ac.uk
[d)]Electronic mail: brian@math.unb.ca

Lie algebra for the nonconformally flat space–times. In Sec. V we discuss the results and outline possible future work.

II. CONFORMAL KILLING VECTORS AND INHERITANCE

Let M be a four-dimensional spacetime manifold with metric tensor $\mathbf{g}$ of Lorentz signature. Any vector field $\mathbf{X}$ which satisfies

$$\mathcal{L}_{\xi}\mathbf{g} = 2\psi(x^{\alpha})\mathbf{g} \tag{1}$$

is said to be a *conformal Killing vector* (CKV) of $\mathbf{g}$. If ψ is not constant on M, then ξ is called a *proper conformal Killing vector*; if ψ is constant on M, then ξ is called a *homothetic Killing vector* (HKV); and if ψ is constant and $\psi \neq 0$ on M, then ξ is called *proper homothetic*. If $\psi_{;\alpha\beta}=0$, then ξ is called a *special conformal Killing vector* (SCKV), and if $\psi=0$, then ξ is said to be a Killing vector (KV).

The set of all CKV (respectively, HKV and KV) form a finite dimensional Lie algebra denoted by $\mathcal{C}$ (respectively, $\mathcal{H}$ and $\mathcal{G}$) whose maximum dimension is 15 (respectively, 11 and 10). If $\dim\mathcal{C}=15$, M is conformally flat. If the dimension of $\mathcal{G}$ is 10, then M is of constant curvature. If M is not of constant curvature, then this algebra has dimension at most 7. The algebra of HKV has dimension equal to or at most one greater than that of the KV algebra so that each given space–time admits a basis for $\mathcal{H}$ containing at most one HKV (i.e., all other HKV can be constructed by the addition of a KV). If the algebra $\mathcal{H}$ has its maximum dimension of 11, then M is flat. Any CKV field in a flat space–time is a SCKV and so the maximum dimension of the SCKV algebra is 15. If this occurs, M is flat, while if M is nonflat, its maximum dimension is 8. For details and proofs, see Ref. 5 and references therein (see Ref. 6 for a summary). It will be assumed throughout this article that the space–times considered admit no local (nonglobalizable) conformal Killing vector fields.

We would like to know the maximum dimension of $\mathcal{C}$ in the nonconformally flat case. First, we need to introduce the following terminology. A point $p \in M$ is called a *zero* (or a *fixed point*) of the CKV ξ if $\xi(p)=0$. A zero p of ξ is called *isometric* if $\psi(p)=0$ and *homothetic* if $\psi(p) \neq 0$.

The Petrov type of the Weyl tensor is a statement about the Weyl tensor at a point $p \in M$, and may vary from point to point. If the Petrov type is the same at all points of M, then one can speak of the Petrov type of M.

The following theorem is known,[7–10] but is collected together here for convenience.

Theorem 1: *Let (M,g) be a nonconformally flat space–time and let $\mathcal{C}$ be the conformal algebra of M. Then we have the following.*

(i) If the Petrov type is N at some $p \in M$, $\dim\mathcal{C} \leqslant 7$.
(ii) If the Petrov type is D at some $p \in M$, $\dim\mathcal{C} \leqslant 6$.
(iii) If the Petrov type is III at some $p \in M$, $\dim\mathcal{C} \leqslant 5$.
(iv) If the Petrov type is I or II at some $p \in M$, $\dim\mathcal{C} \leqslant 4$.

In fact it can be shown (using theorem 2 below) that part (iii) of theorem 1 may be strengthened by saying that if the Petrov type is III over some non-empty subset of M then $\dim\mathcal{C}(M) \leqslant 4$.

The following theorem is due to Hall and Steele[8] (see also Ref. 11).

Theorem 2: *Let (M,g) be a space–time that admits an r-dimensional conformal algebra $\mathcal{C}$ and suppose that the Petrov type and the dimension and nature of the orbits associated with $\mathcal{C}$ are the same at each $p \in M$ and that M admits no local (nonglobalizable) conformal vector fields. Then for each $p \in M$ there exists an open neighborhood U of p and a function $\sigma: U \mapsto \mathbf{R}$ such that $\mathcal{C}$ (restricted to U) is a Lie algebra of special conformal vector fields on U with respect to the metric $g' = e^{2\sigma}g$ on U. If the Petrov type is not O, the above local scaling function σ can always be chosen such that $\mathcal{C}$ restricts to a Lie algebra of homothetic Killing vector fields with respect to*

g' on U, and, if (M,g) is not locally conformally related to a generalized plane-wave space–time about any $p \in M$, the above local scaling can always be chosen such that $\mathcal{C}$ restricts to a Lie algebra of Killing vector fields with respect to g' on U.

See Ref. 8 for the definition of a *generalized plane wave*. Minkowski space–time admits 15 (special) CKV fields, (admitting a ten-dimensional subalgebra of KV fields) and it follows that the Lie algebra of CKVs of a conformally flat space–time can, in principle, be locally reduced to a corresponding set of special CKVs with a ten-dimensional subalgebra of KVs. For space–times conformal to the (nonconformally flat) pp-wave space–times, the Lie algebra of CKVs can be locally reduced to a Lie algebra of homotheties,[11] that is, if $\dim\mathcal{C}=7$, then $\mathcal{C}$ can be locally reduced to a seven-dimensional homothety Lie algebra (e.g., 6 KV and 1 HKV); if $\dim\mathcal{C}=6$, then $\mathcal{C}$ can be locally reduced to a six-dimensional homothety Lie algebra (e.g., 5 KV and 1 HKV); and if $\dim\mathcal{C}<6$, then $\mathcal{C}$ can be locally reduced to a Lie algebra of KVs.

The energy momentum tensor for a perfect fluid space–time is given by

$$T_{ab}=(\mu+p)u_a u_b+p g_{ab}, \tag{2}$$

where u^a is the normalized fluid four-velocity and μ and p are, respectively, the energy-density and the pressure. A CKV in a perfect fluid space–time is said to be inheriting if fluid flow lines $\mathbf{u}$ are mapped conformally by the CKV ξ, i.e.,

$$\mathcal{L}_{\xi}\mathbf{u}=-\psi\mathbf{u}. \tag{3}$$

We shall refer to such a CKV as an ICKV. For an HKV or proper CKV which is parallel to the fluid four-velocity vector $\mathbf{u}$, Eq. (3) is automatically true.[1]

We note that the set of ICKVs form a subalgebra $\mathcal{I}$ of the Lie algebra $\mathcal{C}$ and we refer to this as the inheriting algebra. This is proved in Sec. 3.7 of Ref. 12. Thus, the conditions required for a perfect fluid space–time to admit n independent ICKVs are as follows: there must exist n independent vector fields ξ_I, $I=1,\ldots,n$, which satisfy Eqs. (1) and (3). Therefore, in order to determine the maximum dimension of the Lie algebra of ICKV in a space–time, we can either consider the compatibility of these conditions generally or find the answer on a case by case basis.

III. CONFORMALLY FLAT SPACE–TIMES

In the conformally flat (CF) case, in which $\dim\mathcal{C}=15$, it has been shown that the maximum dimension of the inheriting algebra in a perfect fluid space–time is eight (see Sec. 6 in Ref. 13). Since the maximum dimension of the conformal algebra for any non-CF space–time is seven,[7,8] it follows that

$$\mathrm{MAX}(\dim\mathcal{I})=8.$$

In particular, it is known that the Friedmann–Robertson–Walker model with flat spatial geometry admits precisely eight ICKV.[1,2]

Recently, as part of an investigation into the general CKV admitted by CF perfect fluid space–times,[14] all such space–times admitting the maximum eight ICKV and also all admitting seven ICKV have been discovered. Here we present only the results with a brief indication of the calculation that led to the discovery of these space–times.

The CF perfect fluid space–times are all known[15] and fall into two classes, namely the nonexpanding $(\Theta=0)$ generalized Schwarzschild interior solution and the expanding $[\Theta=\Theta(t)\neq 0]$ generalized FRW solution. The space–time metric of each of these classes can be written in the form

$$ds^2=V^{-2}(-F^2dt^2+dx^2+dy^2+dz^2), \tag{4}$$

where $F(t,x,y,z)$ is of the form

$$F(t)=a(t)r^2+b(t)x+c(t)y+d(t)z+e(t), \tag{5}$$

a, b, c, d, e are arbitrary functions of t and $r^2=x^2+y^2+z^2$. For the nonexpanding models, V is given by

$$V=(1+r^2)C/2, \tag{6}$$

where C is a constant and F is of the particular form

$$F=-\tfrac{1}{2}(Cf_4+1)r^2+f_1x+f_2y+f_3z+\tfrac{1}{2}(Cf_4-1), \tag{7}$$

with f_1, f_2, f_3, f_4 arbitrary functions of t. For the expanding models, V is given by

$$V=Hr^2-2Hx_0x-2Hy_0y-2Hz_0z+V_0+Hr_0^2, \tag{8}$$

where H, x_0, y_0, z_0, V_0 are arbitrary functions of t, $r_0^2=x_0^2+y_0^2+z_0^2$ and F is given by

$$F=3\Theta^{-1}\frac{dV}{dt}. \tag{9}$$

In all cases the fluid four-velocity is comoving, i.e., $u^a=VF^{-1}\partial/\partial t$. However, since the condition (3) is conformally invariant, we may consider the ICKV of the underlying space–time

$$d\sigma^2=-F^2dt^2+dx^2+dy^2+dz^2 \tag{10}$$

with four-velocity $u'^a=F^{-1}\partial/\partial t$.

For the space–time (10) the ICKV equations for ξ^a imply that $\xi^0_{,\alpha}=\xi^\alpha_{,0}=0$ $(\alpha=1,2,3)$ and lead to

$$\begin{aligned}
\xi^0&=G(t),\\
\xi^1&=\tfrac{1}{2}A(x^2-y^2-z^2)+Bxy+Cxz+Dx-My+Nz+Q,\\
\xi^2&=\tfrac{1}{2}B(-x^2+y^2-z^2)+Ayx+Cyz+Mx+Dy-Pz+R,\\
\xi^3&=\tfrac{1}{2}C(-x^2-y^2+z^2)+Azx+Bzy-Nx+Py+Dz+S,\\
\psi&=Ax+By+Cz+D,
\end{aligned} \tag{11}$$

where $A, B, C, D, M, N, P, Q, R, S$ are constants, together with the set of equations

$$\begin{aligned}
\frac{d}{dt}(Ga)&=\frac{1}{2}(bA+cB+dC-2aD),\\
\frac{d}{dt}(Gb)&=eA+dN-cM-2aQ,\\
\frac{d}{dt}(Gc)&=eB-dP+bM-2aR,\\
\frac{d}{dt}(Gd)&=eC+cP-bN-2aS,\\
\frac{d}{dt}(Ge)&=eD-bQ-cR-dS.
\end{aligned} \tag{12}$$

A. Space–times admitting eight ICKV

To find those space–times admitting the maximum number of ICKV we must find those functions a, b, c, d, e, G which result in the maximum number of nonzero constants A, $B,\ldots,S$. This is found to occur only when $a=b=c=d=0$, $e\neq 0$ or when $b=c=d=e=0$, $a\neq 0$. However, when the appropriate conformal factor V^{-2} is restored, the second case results only in space–times which are coordinate transformed $(r\mapsto 1/r)$ versions of those occuring in the first case, which is thus the only one we need to consider. A transformation of the coordinate t enables us to put $e=1$, so the underlying space–time (10) is Minkowski space–time with $u'^{a}=\partial/\partial t$. Using the notation of Maartens and Maharaj[16] the eight ICKV are then

$$P_a=\frac{\partial}{\partial x^a},\quad H=x^a\frac{\partial}{\partial x^a},\quad M_{\alpha\beta}=x_\alpha\frac{\partial}{\partial x^\beta}-x_\beta\frac{\partial}{\partial x^\alpha},\tag{13}$$

where $a=0,1,2,3$ and $\alpha,\beta=1,2,3$. From expressions (8) and (9) the corresponding form of V for the expanding case is

$$V=\alpha r^2+\beta x+\gamma y+\delta z+f(t),\tag{14}$$

where α, β, γ, δ are arbitrary constants and $f(t)$ is an arbitrary function of t. Thus the expanding perfect fluid space–times admitting the maximum number of eight ICKV are all of the form

$$ds^2=[\alpha r^2+\beta x+\gamma y+\delta z+f(t)]^{-2}(-dt^2+dx^2+dy^2+dz^2),\tag{15}$$

and the corresponding nonexpanding space–times have V given by (6). There are three cases to consider:

1. Case (i) $\alpha\Theta\neq 0$

A translation of the origin of the form

$$x'=x+\frac{\beta}{2\alpha},\quad y'=y+\frac{\gamma}{2\alpha},\quad z'=z+\frac{\delta}{2\alpha}$$

transforms the metric into

$$ds^2=[f(t)+\alpha r^2]^{-2}(-dt^2+dx^2+dy^2+dz^2),\tag{16}$$

where we have dropped the primes and absorbed the constants into $f(t)$. This is the space–time $S1$ of Ref. 2 with $k=0$ which was shown to admit five proper ICKV, namely P_a and H, together with the three KV of spherical symmetry.

2. Case (ii) $\alpha=0$, $\Theta\neq 0$

A rotation of the spatial axes brings the metric into the form

$$ds^2=[f(t)+kx]^{-2}(-dt^2+dx^2+dy^2+dz^2),\tag{17}$$

where k is an arbitrary constant. This is the plane symmetric model listed under (a) with $k=0$ in Table 1 of in Ref. 3. In addition to the three KV of plane symmetry, this model admits five proper ICKV (a fact not recognized in Ref. 3—see the Appendix), namely P_0, P_1, H, M_{21}, M_{13}. The corresponding conformal scalars ψ are given by

$$(f+kx)\psi=-\frac{df}{dt},\quad -k,\quad f-t\frac{df}{dt},\quad -ky,\quad -kz,\tag{18}$$

respectively. The density of the model is $\mu=3[(df/dt)^2-k^2]$, so we must have $(df/dt)^2>k^2$.

3. Case (iii) $\Theta=0$

For the nonexpanding model, Eq. (6) leads to the metric

$$ds^2=4C^{-2}(1+r^2)^{-2}(-dt^2+dx^2+dy^2+dz^2), \tag{19}$$

i.e., Eq. (7) with $f_1=f_2=f_3=Cf_4+1=0$. This is a special case of the static Schwarzschild interior solution which is known[2] to admit four proper ICKV together with four KV.

The space–times with metrics (16), (17) and (19) are the only perfect fluid space–times admitting the maximum number of eight ICKV.

B. Space–times admitting seven ICKV

Perfect fluid models admitting seven ICKV are found to occur if and only if the functions a, b, c, d, e in Eq. (5) are constant multiples of each other so that, by a redefinition of the time coordinate, we may write the function F in the form

$$F=\alpha r^2+\beta x+\gamma y+\delta z+\epsilon, \tag{20}$$

where α, β, γ, δ, ϵ are constants. For the expanding models, from Eqs. (8) and (9), V is of the form

$$V=K(t)F+\lambda_1 r^2+\lambda_2 x+\lambda_3 y+\lambda_4 z+\lambda_5, \tag{21}$$

where $3dK/dt=\Theta$ and λ_1, λ_2, λ_3, λ_4, λ_5 are constants of integration. For the nonexpanding models V is given by (6). There are three cases to consider:

1. Case (i) $\alpha\Theta\neq 0$

A translation to a new origin, a rotation of the spatial axes and a rescaling of the time coordinate results in the spacetime metric

$$ds^2=[K(t)(1+\alpha r^2)+\omega r^2+\lambda x]^{-2}[-(1+\alpha r^2)^2dt^2+dx^2+dy^2+dz^2], \tag{22}$$

where ω, λ are constants. When $\lambda=0$ these solutions are the spherical symmetric $S1$ models of Ref. 2 with $k\neq 0$. If, in addition, $\omega=0$, these are the $k=\pm 1$ FRW models.

The ICKV of (22) and the corresponding conformal scalars are

$$\begin{aligned}
&\xi_{(1)}=\frac{\partial}{\partial t}, \quad \psi_{(1)}=-(1+\alpha r^2)V^{-1}\frac{dK}{dt},\\
&\xi_{(2)}=[\alpha(x^2-y^2-z^2)+1]\frac{\partial}{\partial x}+2\alpha xy\frac{\partial}{\partial y}+2\alpha zx\frac{\partial}{\partial z}, \quad \psi_{(2)}=(2\alpha\omega r^2x+\lambda\alpha r^2-\lambda)V^{-1},\\
&\xi_{(3)}=2\alpha xy\frac{\partial}{\partial x}+[\alpha(-x^2+y^2-z^2)+1]\frac{\partial}{\partial y}+2\alpha zy\frac{\partial}{\partial z}, \quad \psi_{(3)}=2\alpha\omega r^2yV^{-1},\\
&\xi_{(4)}=2\alpha xz\frac{\partial}{\partial x}+2\alpha yz\frac{\partial}{\partial y}+[\alpha(-x^2-y^2+z^2)+1]\frac{\partial}{\partial z}, \quad \psi_{(4)}=2\alpha\omega r^2zV^{-1},\\
&\xi_{(5)}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}, \quad \psi_{(5)}=\lambda yV^{-1},\\
&\xi_{(6)}=z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}, \quad \psi_{(6)}=-\lambda zV^{-1},\\
&\xi_{(7)}=-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}, \quad \psi_{(7)}=0.
\end{aligned} \tag{23}$$

No linear combination of $\xi_{(1)}$ to $\xi_{(6)}$ will result in a vanishing or constant conformal scalar, so this model admits six proper ICKV and one KV. Putting $\lambda=0$, we see that the $S1$ models admit four proper ICKV and three KV, as shown in Ref. 3, and putting $\lambda=\omega=0$, we see that the $k=\pm 1$ FRW models admit one proper ICKV and six KV, as shown in Ref. 1.

2. Case (ii) $\alpha=0$, $\Theta\neq 0$

A translation to a new spatial origin, a rotation of the spatial axes and a rescaling of the time coordinate results in the space–time metric

$$ds^2=[hr^2+L(t)x+m]^{-2}[-x^2dt^2+dx^2+dy^2+dz^2], \tag{24}$$

where h,m are constants and $L(t)$ is an arbitrary function of time. This space–time admits four proper ICKV, namely,

$$\xi_{(1)}=\frac{\partial}{\partial t},\quad \xi_{(2)}=\frac{\partial}{\partial y},\quad \xi_{(3)}=\frac{\partial}{\partial z},\quad \xi_{(4)}=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}, \tag{25}$$

and three KV, namely,

$$\begin{gathered}\xi_{(5)}=-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z},\\ \xi_{(6)}=2hxy\frac{\partial}{\partial x}+[m+h(-x^2+y^2-z^2)]\frac{\partial}{\partial y}+2hzy\frac{\partial}{\partial z},\\ \xi_{(7)}=2hxz\frac{\partial}{\partial x}+2hyz\frac{\partial}{\partial y}+[m+h(-x^2-y^2+z^2)]\frac{\partial}{\partial z}.\end{gathered} \tag{26}$$

When $h=0$, the space–time is plane symmetric and again admits four proper ICKV and three KV.

3. Case (iii) $\Theta=0$

For the nonexpanding model, using Eqs. (6) and (20) together with a rotation of the spatial axes and a rescaling of the time coordinate, we obtain

$$ds^2=4C^{-2}(1+r^2)^{-2}[-(\alpha r^2+\beta x+1)^2dt^2+dx^2+dy^2+dz^2]. \tag{27}$$

This space–time admits three proper ICKV, namely,

$$\begin{gathered}\xi_{(1)}=[\alpha(x^2-y^2-z^2)+\beta x+1]\frac{\partial}{\partial x}+y(2\alpha x+\beta)\frac{\partial}{\partial y}+z(2\alpha x+\beta)\frac{\partial}{\partial z},\\ \xi_{(2)}=2\alpha xy\frac{\partial}{\partial x}+[\alpha(-x^2+y^2-z^2)+1]\frac{\partial}{\partial y}+2\alpha yz\frac{\partial}{\partial z},\\ \xi_{(3)}=2\alpha xz\frac{\partial}{\partial x}+2\alpha yz\frac{\partial}{\partial y}+[\alpha(-x^2-y^2+z^2)+1]\frac{\partial}{\partial z},\end{gathered} \tag{28}$$

and the four KV,

$$\begin{gathered}\xi_{(4)}=\frac{\partial}{\partial t},\quad \xi_{(5)}=-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z},\\ \xi_{(6)}=2y(1-\alpha+\beta x)\frac{\partial}{\partial x}+[\beta(1-x^2+y^2-z^2)+2(\alpha-1)]\frac{\partial}{\partial y}+2\beta yz\frac{\partial}{\partial z},\\ \xi_{(7)}=2z(1-\alpha+\beta x)\frac{\partial}{\partial x}+2\beta yz\frac{\partial}{\partial y}+[\beta(1-x^2-y^2+z^2)+2(\alpha-1)]\frac{\partial}{\partial z}.\end{gathered} \tag{29}$$

When $\beta=0$, $\xi_{(4)}$ and $\xi_{(5)}$ are KV; the resulting metric is that of the general spherically symmetric Schwarzschild interior solution which is known[2] to admit three proper ICKV and four KV.

The space–times with metrics (22), (24) and (27) are the only CF perfect fluid space–times admitting precisely seven ICKV. Some, but not all, are known admitting precisely six ICKV; none of the known models admit six proper ICKV, thus, so far, the solution (22) is unique in this regard.

IV. NON-CONFORMALLY FLAT SPACE–TIMES

For the case of nonconformally flat space–times, Theorem 1 tells us that $\dim\mathcal{C}\leqslant 7$ and so $\dim\mathcal{I}\leqslant 7$. It is known that the Gödel space–time admits 5 ICKV: This is a perfect fluid homogeneous Petrov type D space–time with 5 KVs. Thus, we ask the question: Does there exist a perfect fluid space–time with $\dim\mathcal{I}=7$ or $\dim\mathcal{I}=6$? From Theorem 1, we only need to consider Petrov types N and D.

First, we give some notation. Let ξ be a CKV. From (1) it follows that $\xi_{a;b}=\psi g_{ab}+F_{ab}$. Now suppose $\xi\not\equiv 0$ but $\xi(p)=0$ for some $p\in M$. Then[9,17] we have the following:

Theorem 3: *(i) If $\psi(p)=0$ (isometric zero), the Petrov type at p is N, D or O. Also, if $F_{ab}(p)=0$, then the Petrov type at p is O. If the Petrov type at p is D, then $F_{ab}(p)\neq 0$ and is a linear combination of the bivectors $l_{[a}n_{b]}$ and $x_{[a}y_{b]}$ where l,n,x,y is a null tetrad ($l_a n^a=x_a x^a=y_a y^a=1$, others zero) at p with l and n repeated principle null directions of the Weyl tensor at p. If the Petrov type at p is N, then $F_{ab}(p)\propto l_{[a}x_{b]}$ where l is the repeated principle null direction of the Weyl tensor at p and $l^a x_a=0$.*

(ii) If $\psi(p)\neq 0$ (homothetic zero), the Petrov type at p is III, N or O. If the Petrov type is III or N, $F_{ab}(p)\neq 0$ and timelike.

Corollary: If ξ is a CKV with $\xi\not\equiv 0$ and $\xi(p)=0$ and the Petrov type at p is not O, then $F_{ab}(p)\neq 0$.

Theorem 4: *If ξ is an ICKV and $\xi\not\equiv 0$ and $\xi(p)=0$, then, if u is the fluid flow velocity at p, $F_{ab}u^b=0$ at p (the fluid flow is assumed nowhere zero).*

Proof: Use $\mathcal{L}_\xi u\propto u$ at p and put $\xi(p)=0$.

□

Theorem 5: *If a perfect fluid space–time is not conformally flat, the dimension of the ICKV algebra is at most 5. If such a space–time admits a maximal ICKV algebra (of dimension 5), it must be of Petrov type D with $F_{ab}\propto x_{[a}y_{b]}$.*

Proof: Suppose this dimension is $\geqslant 6$. Then by taking linear combinations of members of the ICKV algebra one can arrange to have ξ, η as two (i.e., $6-4$) independent ICKV such that

$$\xi_{a;b}=\psi g_{ab}+F_{ab}, \quad \xi(p)=0, \quad \xi\neq 0, \quad \mathcal{L}_\xi u\propto u, \quad F_{ab}u^b=0,$$

$$\eta_{a;b}=\phi g_{ab}+G_{ab}, \quad \eta(p)=0, \quad \eta\neq 0, \quad \mathcal{L}_\eta u\propto u, \quad G_{ab}u^b=0$$

hold at any $p\in M$. Also, by taking linear combinations of ξ and η, we can assume that at least one of ψ and ϕ vanishes at p.

Case (a): $\psi(p)=\phi(p)=0$ and $F_{ab}u^b=G_{ab}u^b=0$. Then from Theorem 3 (i) the (necessarily spacelike) blades of F_{ab} and G_{ab} at p must coincide. Hence $F_{ab}=\mu G_{ab}$ at p ($0\neq\mu\in\mathbf{R}$). Now construct $Z=\xi-\mu\eta$ which is not identically zero. Then Z is an ICKV and $Z_{a;b}=(\psi-\mu\phi)g_{ab}+(F_{ab}-\mu G_{ab})$ with $(\psi-\mu\phi)(p)=0$, $(F_{ab}-\mu G_{ab})(p)=0$ which contradicts nonconformal flatness by Theorem 3 (i).

Case (b): $\psi=0$, $\phi\neq 0$. $F_{ab}u^b=0$, $G_{ab}u^b=0$. Theorem 3 (ii) then shows that $G_{ab}(p)$ is nonzero and timelike (contradicting $G_{ab}u^b=0$ at p), i.e., timelike bivectors cannot have a timelike vector annihilating them.

The second assertion of the theorem follows immediately from this proof and Theorem 3, and is due to the fact that the relevant bivectors are necessarily spacelike.

□

Corollary: The maximum dimension of the ICKV algebra for Petrov type N and type III perfect fluid space–times is at most four.

These results also may be obtained by somewhat tedious direct calculations without recourse to the fixed point theorems of Ref. 17.

A. Examples of Petrov type *D* space–times with five ICKV

(a) The Gödel space–time

$$ds^2=a^2[-(dt+e^x dz)^2+dx^2+dy^2+\tfrac{1}{2}e^{2x}dz^2], \tag{30}$$

when considered as a perfect fluid space–time with zero cosmological constant, has energy density μ and pressure p given by $\mu=p=a^{-2}/2$. It admits five ICKV, all of which are KV.

(b) The plane symmetric Kasner type model

$$ds^2=-dt^2+dx^2+t(dy^2+dz^2) \tag{31}$$

admits four KV and one HKV given by

$$\mathbf{H}=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+\frac{y}{2}\frac{\partial}{\partial y}+\frac{z}{2}\frac{\partial}{\partial z}. \tag{32}$$

The four-velocity is comoving, $\mu=p=t^{-2}/4$.

The space–time with metric

$$d\sigma^2=\omega^{-2}ds^2, \tag{33}$$

where ds^2 is the metric (31) and

$$\omega=a(x^2-2t^2)+b, \tag{34}$$

a and b being nonzero constants, is also a perfect fluid space–time. In this case the KV

$$X=\frac{\partial}{\partial x}$$

becomes a proper ICKV with $\psi_X=-2ax/\omega$ and the HKV H given by (32) also becomes a proper ICKV with $\psi_H=-1+2b/\omega$.

(c) The static spherically symmetric model

$$ds^2=-r^{2m}dt^2+(1+2m-m^2)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{35}$$

has $\mu=(1+2m-m^2)^{-1}m(2-m)r^{-2}$ and $p=m(2-m)^{-1}\mu$ and all energy conditions hold for $0<m\leqslant 1$. The space–time admits four KV and one HKV given by

$$\mathbf{H}=(1-m)t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}. \tag{36}$$

The space–time with metric

$$d\Sigma^2=U^{-2}ds^2, \tag{37}$$

where ds^2 is the metric (35) and $U=a+br^2$, a and b being positive constants, is also a perfect fluid space–time. The four KV of (35) remain as KV but the HKV given by (36) is now a proper ICKV with $\psi=(a-br^2)/(a+br^2)$.

(d) The static spherically symmetric space–time

$$ds^2 = \mathrm{sech}^2\left(\frac{r}{\sqrt{2}}\right)(-dt^2 + dr^2 + d\theta^2 + \sin^2\theta d\phi^2) \tag{38}$$

satisfies the energy conditions. It admits the four KV associated with static spherical symmetry together with the proper ICKV

$$\mathbf{I} = \frac{\partial}{\partial r}.$$

V. DISCUSSION

The results of Secs. III and IV may be summarized in the following theorem:

Theorem 6: *For conformally flat perfect fluid space–times* $\dim\mathcal{I}$ *is at most eight and all such space–times are known. The maximum number of independent proper ICKV is six. For nonconformally flat perfect fluid space–times* $\dim\mathcal{I}$ *is at most five, in which case the space–time is of Petrov type* D.

The example given by (33) and (34) is the only known nonconformally flat perfect fluid solution admitting more than one independent proper ICKV.

It is also of interest to determine the maximum dimension of $\mathcal{I}$ for types N, III, II and I separately. From the analysis of Sec. IV we see that for each of those Petrov types $\dim\mathcal{I} \leqslant 4$. The determination of the exact maximum number in each case may require techniques other than the geometrical approach used here.

ACKNOWLEDGMENTS

We would like to thank Michael Capocci and Chris Clarkson for useful discussions. This work was carried out while one of the authors (A.J.K.) was a postdoctoral fellow in the Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia.

APPENDIX A: ERRATA TO COLEY AND CZAPOR[3]

The result in Sec. III that the plane symmetric CF model given by Eq. (17) admits eight ICKV contradicts Theorem 1 of Ref. 3 which states, in effect, that M_{21} and M_{13} cannot be ICKV (it can be easily checked that they are indeed ICKV). The proof of Theorem 1 in Ref. 3 is correct up to and including Eq. (2.24), but Eq. (2.25) is wrong. In fact, using Eqs. (2.23) and (2.24), Eq. (2.9) becomes (using the notation of Ref. 3)

$$w_x(w_t w_{xt} + w_x w_{tt} - w_{ttx}) = 0,$$

and, since $w_x \neq 0$, it follows that $w_t w_{xt} + w_x w_{tt} - w_{ttx} = 0$; i.e.,

$$(w_t w_x - w_{tx})_x = 0.$$

But Eq. (2.23) states that

$$(w_t w_x - w_{tx})_t = 0,$$

so $w_t w_x - w_{tx} = \text{const}$, which does not contradict the later correct result $w_t w_x = w_{tx}$. [Note Eqs. (2.33a) and (2.33b) are wrong since they are derived from the incorrect Eq. (2.25).] Thus there is no contradiction and solutions do exist for which $\Lambda \neq 0$. Equation (2.37) is correct and substituting this into Eq. (2.9) leads to $w_x^2 w_{xx} + w_{xx}^2 - w_x w_{xxx} = 0$, and putting $e^{-w} = p(t) + s(x)$ we obtain $s_{xx}^2 = s_x s_{xxx}$. If $s_{xx} = 0$, we obtain the solution (18), otherwise we obtain $s = \beta e^{\alpha x}$ which is the first solution in Table I of Ref. 3; i.e., the $k \neq 0$ case. Thus this also admits the "exceptional" ICKV (using the terminology of Ref. 2) but admits only seven ICKV (4 proper ICKV and 3 KV). Thus the plane symmetric case admits models with the ''exceptional'' ICKV corresponding to those in the spherically symmetric case.

[1] A. A. Coley and B. O. J. Tupper, Class. Quantum Grav. **7**, 1961 (1990).
[2] A. A. Coley and B. O. J. Tupper, Class. Quantum Grav. **7**, 2195 (1990).
[3] A. A. Coley and S. R. Czapor, Class. Quantum Grav. **9**, 1787 (1992).
[4] J. Triginer and D. Pavón, Class. Quantum Grav. **12**, 199 (1995).
[5] G. S. Hall, Gen. Relativ. Gravit. **22**, 203 (1990).
[6] G. S. Hall, Conformal symmetries in general relativity, in *Proceedings of the 3rd Hungarian Relativity Workshop*, Tihany (1989).
[7] G. S. Hall, unpublished (1989).
[8] G. S. Hall and J. D. Steele, J. Math. Phys. **32**, 1847 (1991).
[9] J. Ehlers and W. Kundt, *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[10] G. S. Hall, M. S. Capocci, and R. Beig, Class. Quantum Grav. **14**, L49 (1997).
[11] L. Defrise-Carter, Commun. Math. Phys. **40**, 273 (1975).
[12] J. Castejon-Amenedo and A. A. Coley, Class. Quantum Grav. **9**, 2203 (1992).
[13] M. S. Capocci, Class. Quantum Grav. **13**, 1717 (1996).
[14] B. O. J. Tupper, in preparation.
[15] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 1980).
[16] R. Maartens and S. D. Maharaj, Class. Quantum Grav. **3**, 1005 (1986).
[17] G. S. Hall, J. Math. Phys. **31**, 1198 (1990).