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Induced matter theory and embeddings in Riemann flat space–times

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A class of five-dimensional space-times that contain four-dimensional hypersurfaces whose intrinsic metrics are of cosmological interest is investigated. First, the five-dimensional space–time is assumed to be Riemann flat—the problem of determining the intrinsic metrics of the four-dimensional hypersurfaces then becomes a problem of embedding in flat space–time. Second, the Riemann flat solutions are used as a starting point to find solutions to Einstein's vacuum field equations in five dimensions that are not Riemann flat. In particular, a new general class of five-dimensional vacuum solutions is found. © *1996 American Institute of Physics.* [S0022-2488(96)04201-8]

I. INTRODUCTION

Recently, several authors have been interested in Einstein's theory of general relativity in five dimensions[1–7] (higher-dimensional theories have also been considered[8,9]). In these studies the higher-dimensional field equations were taken to be the vacuum Einstein field equations, and the primary goal in several[1–3,6,7] of these studies was to determine whether the four-dimensional properties of matter could be interpreted as being purely geometrical in origin[10]—the embedding of the four-dimensional space–time in the vacuum five-dimensional space-time was interpreted as producing an effective four-dimensional stress–energy tensor.

Curiously, Mc Manus[7] recently observed that a class of five-dimensional vacuum (i.e., Ricci flat) solutions of Ponce de Leon[6] were, in fact, completely (Riemann) flat, that is to say that the five-dimensional Riemann tensor associated with these metrics was identically zero. Thus, one is immediately prompted to ask the following question: are any of the other known five-dimensional Ricci flat solutions also Riemann flat? Of course, it is not very instructive to just simply systematically calculate the Riemann tensor for the known five-dimensional vacuum solutions to determine if the Riemann tensor vanishes. It would be far more beneficial if we could find some general results. Consequently, we now pose the following question: what is the class of Lorentzian four-metrics that can be embedded in five-dimensional Minkowski space-time? In theory, the general solution to this problem is known.[11] For example, consider the following class of Riemann flat five-metrics:

$$ds^2 = g_{\alpha\beta}(x^\gamma,y)dx^\alpha\, dx^\beta + \phi^2(x^\gamma,y)dy^2. \tag{1}$$

where the intrinsic metric, $g_{\alpha\beta}(x^\gamma,y)|_{y=\text{const}}$, of the four-dimensional hypersurfaces $y=$constant is Lorentzian in signature. The fact that the full five-dimensional metric is flat imposes necessary and sufficient conditions on the Riemann tensor, $R_{\alpha\beta\gamma\delta}$, of the intrinsic metric and on the extrinsic curvature, $K_{\alpha\beta}$, of the hypersurface $y=$const, namely that

$$R_{\alpha\beta\gamma\delta} = K_{\alpha\gamma}K_{\beta\delta} - K_{\alpha\delta}K_{\beta\gamma}, \tag{2}$$

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$$K_{\alpha\beta;\gamma}=K_{\alpha\gamma;\beta}, \tag{3}$$

where here ";" denotes covariant differentiation with respect to the intrinsic metric $g_{\alpha\beta}$. (In general,[11] the four-metric ${}^4ds^2=g_{\alpha\beta}(x^\gamma)dx^\alpha\, dx^\beta$ can be embedded in a five-dimensional Riemann flat space–time if and only if there exists a symmetric tensor $\Omega_{\alpha\beta}$ that satisfies ${}^4R_{\alpha\beta\gamma\delta}=2\Omega_{\alpha[\gamma}\Omega_{\delta]\beta}$ and $\Omega_{\alpha[\beta;\gamma]}=0$.) Furthermore, the extrinsic curvature of the surface $y=$const is given by

$$K_{\alpha\beta}=\frac{1}{2\phi}\,\frac{\partial}{\partial y}\,(g_{\alpha\beta}), \tag{4}$$

for the above metric. Unfortunately, the above equations are too complex for an explicit coordinate representation for the metric functions $g_{\alpha\beta}$ and ϕ to be found, in general.

In this paper, we consider the following class of metrics:

$$ds^2=-e^{2F(t,r,y)}\,dt^2+e^{2G(t,r,y)}(dr^2+r^2\,d\Omega^2)+e^{2K(t,r,y)}\,dy^2, \tag{5}$$

where $d\Omega^2\equiv d\theta^2+\sin^2(\theta)d\phi^2$. Various forms of the metric (5) have been extensively investigated in the literature[1,2,6,7]—the vacuum Einstein field equations, $R_{ij}=0$, have been solved for certain subclasses of the metric (5). Although several metrics of the form (5) have been noted to be Riemann flat,[7,13] most notably the Ponce de Leon solutions,[6] the general form of the Riemann flat solutions has not been found.

Therefore, we wish to determine all metrics of the form (5) that are Riemann flat. In other words, we want to find all solutions of the equations $R_{ijkl}=0$ where the metric is given by (5). In principle, the problem is trivial since locally all solutions are Minkowski space-time; however, one has to implement various nontrivial diffeomorphisms to get the solution into this form. However, we shall only permit diffeomorphisms of the form $x^\alpha\rightarrow\bar{x}^\alpha(x^\beta)$ and $y\rightarrow\bar{y}(y)$. Thus, by restricting the permissible diffeomorphisms we ensure that the four-dimensional intrinsic metrics $ds^2 = g_{\alpha\beta}(x^\gamma,y)|_{y=\text{const}}\,dx^\alpha\, dx^\beta$ are not necessarily Riemann flat, even though the five-dimensional metrics are Riemann flat. In technical language, we confine our analysis to four-dimensional metrics of embedding class $p=1$ (see Refs. 8 and 12 for further discussions on the embedding problem).

If we calculate the extrinsic curvature, $K_{\alpha\beta}$, for the metric (5) using (4), then we find that it has the form

$$K_{\alpha\beta}=Av_\alpha v_\beta+Bg_{\alpha\beta}, \tag{6}$$

where $g_{\alpha\beta}v^\alpha v^\beta=-1$. Equation (2) can then be employed to show that the stress–energy tensor associated with the intrinsic metric $g_{\alpha\beta}$ has the form of a perfect fluid, namely $T_{\alpha\beta}=(\mu+P)v_\alpha v_\beta+Pg_{\alpha\beta}$. Now, standard results of embedding theory[12,14,15] tell us about the allowable perfect fluid solutions. In the case $\mu+P\neq 0$, the solution belongs to either the class of generalized interior Schwarzschild solutions if the expansion of the fluid velocity, v_α, is zero, or the class of generalized Friedmann cosmological models if the fluid expansion is nonzero. In the case that $\mu+P=0$, the solutions are de Sitter space–times (that is, space–times of constant curvature). All of these spherically symmetric solutions are conformally flat.

The purpose of our paper is twofold; not only do we wish to find the explicit form for all the Riemann flat solutions of the form (5), but we also wish to find new Ricci flat solutions. In particular, we wish to employ our knowledge of the Riemann flat solutions as an aid to construct new Ricci flat solutions.

The paper is organized as follows: In Sec. II, we analyze the Riemann flat equations for the metric (5), and both classify and find all the five-dimensional solutions explicitly. In Sec. III, we discuss the solutions found in Sec. II, paying particular attention to their interpretation in the context of induced matter theory. Finally, in Sec. IV, we use the Riemann flat solutions as an aid to construct some Ricci flat solutions of the metric (5) that are not Riemann flat.

II. FIELD EQUATIONS

We shall now proceed to solve the equations $R_{ijkl}=0$ for the metric (5). However, we will restrict the allowable diffeomorphisms to diffeomorphisms of the form $x^{\alpha}\rightarrow\bar{x}^{\alpha}(x^{\beta})$, $y\rightarrow\bar{y}(y)$. We start our analysis by noting that the pivotal equations are $R^{\theta}{}_{tr\theta}=0$ and $R^{\theta}{}_{ry\theta}=0$, which yield

$$G_{rt}=G_tF_r. \tag{7}$$

$$G_{ry}=G_yK_r, \tag{8}$$

respectively. (The complete set of the components of the Riemann tensor are listed in the Appendix.) Thus, we immediately observe that we can divide the solutions into four classes: (1) $G=G(r)$; (2) $G=G(r,y)$ with $G_y\neq 0$ and $\exp(K)=G_y/\alpha$; (3) $G=G(r,t)$ with $G_t\neq 0$ and $\exp(F)=G_t/\beta$; and (4) $G=G(t,r,y)$ with $G_tG_y\neq 0$, and $\exp(K)=G_y/\alpha$ and $\exp(F)=G_t/\beta$, where both α and β are nonzero functions of t and y only.

A. $G=G(r)$

The equation $R^{\theta}{}_{rr\theta}=0$ yields the differential equation

$$rG_{rr}+G_r=0, \tag{9}$$

which can easily be solved to get

$$G=c_1\ \ln(r)+c_2, \tag{10}$$

where c_1 and c_2 are arbitrary constants. The field equation $R^{\phi}{}_{\theta\theta\phi}=0$ now reduces to

$$c_1(c_2+2)=0. \tag{11}$$

We can take $c_1=c_2=0$ without loss of generality. (The solutions $c_1=0$ and $c_2=-2$ are related by the transformation $r\rightarrow 1/r$.) Equations $R^{\theta}{}_{tt\theta}=0$ and $R^{\theta}{}_{yy\theta}=0$ imply that the functions F and K are both independent of the variable r. Thus, the metric may be written as

$$ds^2=-A^2(t,y)dt^2+dr^2+r^2\ d\Omega^2+B^2(t,y)dy^2, \tag{12}$$

where the metric functions A and B must satisfy the following equation:

$$\frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial A}{\partial y}\right)=\frac{\partial}{\partial t}\left(\frac{1}{A}\frac{\partial B}{\partial t}\right). \tag{13}$$

Equation (13) follows directly from the equation $R^{t}{}_{yty}=0$. (The remaining equations $R^{i}{}_{jkl}=0$ are trivially satisfied.)

B. $G_t=0$, $G_y\neq 0$

For this class of solutions, we have that the metric function K satisfies

$$e^K=G_y/\alpha(t,y), \tag{14}$$

where $\alpha\neq 0$. The equation $R^{r}{}_{tyr}=0$ implies that $\alpha=(y)$. The equation $R^{\theta}{}_{rr\theta}=0$ then yields the following differential equation for G:

$$G_{rr}+\frac{1}{r}\,G_r+\alpha^2e^{2G}=0, \tag{15}$$

which can easily be solved to obtain the solution

$$G=\frac{1}{2}\,\ln\left[\frac{a^2b}{\alpha^2}\,\frac{r^{a-2}}{(1+br^c)^2}\right], \tag{16}$$

where a and b are arbitrary functions of y only. Furthermore, the equation $R^{\phi}{}_{\theta\theta\phi}=0$ reduces to

$$G_r^2+\frac{2}{r}\,G_r+\alpha^2e^{2G}=0. \tag{17}$$

Inserting the solution (16) for G into (17) implies that $a^2=4$ (we can take $a=2$ without loss of generality). Thus, the solution for G is

$$G=\frac{1}{2}\,\ln\left[\frac{4b(y)}{\alpha^2(y)}\,\frac{1}{(1+b(y)r^2)^2}\right]. \tag{18}$$

Note that we can write the solution for G in the above form, since the equation $R^{\theta}{}_{rr\theta}=0$ implies that $G_r\neq 0$ if $G_t=0$ and $G_y\neq 0$.

For simplicity, we introduce the function A defined by

$$A=e^F. \tag{19}$$

The equations $R^t{}_{rtr}=0$ and $R^{\theta}{}_{tt\theta}=0$ now reduce to

$$\frac{1}{\alpha^2}\,G_y(A_{rr}-G_rA_r)+A_ye^{2G}=0. \tag{20}$$

$$\frac{1}{\alpha^2}\,G_y\left(G_r+\frac{1}{r}\right)A_r+A_ye^{2G}=0. \tag{21}$$

respectively. Subtracting (21) from (20) yields

$$A_{rr}-2A_rG_r-\frac{1}{r}\,A_r=0, \tag{22}$$

which can be integrated to obtain

$$A_r=re^{2G}l(t,y), \tag{23}$$

where $l(t,y)$ is an arbitrary function. Furthermore, Eq. (21) implies that

$$A_y=-\frac{l}{\alpha^2}\,G_y(1+rG_r). \tag{24}$$

With the aid of Eq. (18), Eq. (23) can now be integrated to yield

$$A=-\frac{2l}{\alpha^2}\,\frac{1}{1+br^2}+m(t,y). \tag{25}$$

where $m(t,y)$ is an arbitrary function. In addition, the equation $R^t{}_{rty}=0$ reduces to

$$G_rA_{ry}-G_y^2A_r-A_yG_{ry}=0, \tag{26}$$

which can be integrated to obtain

$$\frac{A_y}{G_y}=A+n(t,y), \tag{27}$$

where $n(t,y)$ is an arbitrary function. If we divide Eq. (24) by G_y and compare the result to (27), then we find that A must be of the form

$$A=-\frac{l}{\alpha^2}\frac{1-br^2}{1+br^2}-n. \tag{28}$$

If we combine Eqs. (25) and (28), then we obtain that the function A may be written as

$$A=\frac{p(t,y)+b(y)q(t,y)r^2}{1+b(y)r^2}. \tag{29}$$

where $p=n-l/\alpha^2$ and $q=n+l/\alpha^2$. In addition, the above solution must also satisfy Eq. (24). Inserting the above solution into Eq. (24), we find that Eq. (24) becomes a power series in r. Thus, equating the various coefficients of r to zero, we find the following equations:

$$(p-q)\left(\frac{b_y}{b}-2\frac{\alpha_y}{\alpha}\right)=4p_y, \tag{30}$$

$$(p-q)b_y=2(q_y+p_y), \tag{31}$$

$$(p-q)\left(\frac{b_y}{b}+2\frac{\alpha_y}{\alpha}\right)=4q_y. \tag{32}$$

Equation (31) is redundant since it can be obtained by adding Eqs. (30) and (32) together. Subtracting Eq. (32) from Eq. (30), we find that

$$2(p_y-q_y)+(p-q)\frac{\alpha_y}{\alpha}=0, \tag{33}$$

which can be integrated to yield the solutions

$$p=u(t)\left[\frac{2}{\alpha}+\int\frac{dy}{\alpha}\frac{b_y}{b}\right]+v(t), \tag{34}$$

$$q=u(t)\left[-\frac{2}{\alpha}+\int\frac{dy}{\alpha}\frac{b_y}{b}\right]+v(t), \tag{35}$$

where u and v are arbitrary functions. The remaining equations $R^i{}_{jkl}=0$ are trivially satisfied.

Thus, if we make the coordinate transformation $Y=\frac{1}{2}\ln b(y)$ and introduce the function $a=-\ln\alpha$, then the metric can be written as

$$ds^2=-A^2\,dt^2+\frac{4e^{2(a+Y)}}{(1+e^{2Y}r^2)^2}(dr^2+r^2\,d\Omega^2)+\frac{e^{2a}}{(1+e^{2Y}r^2)^2}[(a_Y+1)+(a_Y-1)e^{2Y}r^2]^2\,dY^2, \tag{36}$$

where $a=a(Y)$ is an arbitrary function and A can be set to

$$A=\begin{cases} 1 \quad [\text{if } u(t)=0], \\ v(t)+2e^{2a}\,\dfrac{1-e^{2Y}r^2}{1+e^{2Y}r^2}+2\displaystyle\int e^a\, dY \quad [\text{if } u(t)\neq 0], \end{cases} \tag{37}$$

where $v(t)$ is an arbitrary function.

C. $G_y=0$, $G_t\neq 0$

This class is very similar to the previous class. Thus, we forgo writing down the steps of the calculation since they are, in essence, the same as those employed in the derivation of the class (2) solutions. We merely quote the results: there exist coordinates such that the class (3) solutions may be written as

$$ds^2=-\frac{e^{2a}}{(1-e^{2T}r^2)^2}\,[(a_T+1)-(a_T-1)e^{2T}r^2]^2\, dT^2+\frac{4e^{2(a+T)}}{(1-e^{2T}r^2)^2}\,(dr^2+r^2\, d\Omega^2)+B^2\, dy^2, \tag{38}$$

where $a=a(T)$ is an arbitrary function and B can be set equal to

$$B=\begin{cases} 1 \quad (\text{if } B_r=0), \\ v(y)+2e^{2a}\,\dfrac{1+e^{2T}r^2}{1-e^{2T}r^2}+2\displaystyle\int e^a\, dT \quad (\text{if } B_r\neq 0), \end{cases} \tag{39}$$

where $v(y)$ is an arbitrary function.

D. $G_tG_y\neq 0$

In this case, the functions F and K satisfy the relationships

$$e^K=G_y/\alpha(t,y), \tag{40}$$

$$e^F=G_t/\beta(t,y). \tag{41}$$

Now, the equations $R^\theta{}_{rr\theta}=0$ and $R^\phi{}_{\theta\theta\phi}=0$ yield

$$G_{rr}+\frac{1}{r}\,G_r+ae^{2G}=0, \tag{42}$$

$$G_r^2+\frac{2}{r}\,G_r+ae^{2G}=0, \tag{43}$$

respectively, where

$$a\equiv\alpha^2-\beta^2. \tag{44}$$

The above equations can be solved to obtain the solution

$$G=\frac{1}{2}\,\ln\left[\frac{4b^2}{(a+b^2r^2)^2}\right], \tag{45}$$

where $b=b(t,y)$ is an arbitrary function. Furthermore, the equation $R^t{}_{rty}=0$ reduces to

$$G_{yt}=G_tG_y+\frac{\alpha_y}{\alpha}G_t+\frac{\beta_t}{\beta}G_y. \tag{46}$$

Using the solution (45), we can expand the above equation in a power series in r. We find that the following two equations must be satisfied:

$$b_{ty}=(\partial_t \ln \beta)b_y+(\partial_y \ln \alpha)b_t, \tag{47}$$

$$\partial_t\left(\frac{\alpha_y}{\beta}\right)+\frac{1}{\beta}[\alpha(\partial_t \ln b)(\partial_y \ln b)-\alpha_y\partial_t \ln b-\alpha_t\partial_y \ln b]$$
$$=\partial_y\left(\frac{\beta_t}{\alpha}\right)+\frac{1}{\alpha}[\beta(\partial_t \ln b)(\partial_y \ln b)-\beta_y\partial_t \ln b-\beta_t\partial_y \ln b]. \tag{48}$$

(Actually, there is a third equation, but it is automatically satisfied on account of the first two equations.)

The remaining equations $R^i{}_{jkl}=0$ can all be shown to be trivially satisfied. First, using Eq. (46) we find that the equation $R^t{}_{rtr}=0$ can be written as

$$\frac{\partial}{\partial r}[2G_{rr}-G_r^2+ae^{2G}]=0, \tag{49}$$

which is trivially satisfied due to (45). Similarly, the equation $R^r{}_{yyr}=0$ reduces to

$$\frac{\partial}{\partial y}[2G_{rr}-G_r^2+ae^{2G}]=0. \tag{50}$$

Both the equations $R^t{}_{rty}=0$ and $R^y{}_{try}=0$ reduce to

$$\frac{\partial}{\partial r}\left[G_{yt}-G_tG_y-\frac{\alpha_y}{\alpha}G_t-\frac{\beta_t}{\beta}G_y\right]=0 \tag{51}$$

which is trivially satisfied because of Eq. (46). Furthermore, using Eq. (46), the equations $R^\theta{}_{tt\theta}=0$ and $R^y{}_{\theta\theta y}=0$ can be shown to reduce to

$$rG_{rtr}=G_{rt}(1+2rG_r), \tag{52}$$

$$rG_{ryr}=G_{ry}(1+2rG_r), \tag{53}$$

respectively, which again are trivially satisfied on account of Eq. (45). Finally, after a long calculation employing Eqs. (42)–(48), the equation $R^t{}_{yty}=0$ can be shown to be trivially satisfied.

In summary, the metric is given by

$$ds^2=-\left(\frac{G_t}{\beta(t,y)}\right)^2 dt^2+e^{2G}(dr^2+r^2\, d\Omega^2)+\left(\frac{G_y}{\alpha(t,y)}\right)^2 dy^2, \tag{54}$$

where

$$e^{2G}\equiv\frac{4b(t,y)^2}{(\alpha^2-\beta^2+b(t,y)^2r^2)^2}, \tag{55}$$

and α, β and b satisfy (47)–(48).

III. DISCUSSION

We should, of course, determine where the known Riemann flat solutions of the form (5) fit into the above classification scheme. As was noted by Mc Manus,[7] the following three Ponce de Leon metrics[6] are all Riemann flat:

$$ds^2=-y^2\ dt^2+t^{2/\gamma}y^{2/(1-\gamma)}[dr^2+r^2\ d\Omega^2]+\left(\frac{\gamma}{\gamma-1}\right)^2 t^2\ dy^2, \tag{56}$$

$$ds^2=-y^2\ dt^2+y^2e^{2t}[dr^2+r^2\ d\Omega^2]+dy^2, \tag{57}$$

$$ds^2=-dt^2+t^2e^{2y}[dr^2+r^2\ d\Omega^2]+t^2\ dy^2, \tag{58}$$

where $\gamma(\neq 0,1)$ is an arbitrary constant. Clearly, all the above metrics belong to class (4), since $G=G(t,y)$ with $G_tG_y\neq 0$. In particular, the above solutions belong to the special case $a=0$ [that is, $\alpha=\beta$—see Eqs. (40)–(42)]. However, the solution (43) appears to depend on r; this dependency can be removed by the coordinate transformation $r\rightarrow\bar{r}=2/r$. Also note that in this special case Eq. (46) is automatically satisfied. Class (4) solutions with $\alpha\neq\beta$ are known to exist. For instance, the metric

$$ds^2=-dt^2+\frac{1}{4}\ t^2(e^y-\kappa e^{-y})^2\ \frac{dr^2+r^2\ d\Omega^2}{[1+(\kappa/4)r^2]^2}+t^2\ dy^2, \tag{59}$$

given in Ref. 7, is Riemann flat for all values of the constant κ.

In addition, we recall that metrics of the form (5) were originally investigated in the context of induced matter theory.[2] In induced matter theory, the field equations are usually taken to be the the vacuum Einstein field equations in $4+n$-dimensional space–time.[1–3] However, for our analysis we wish to examine the consequences of taking the field equations to be the five-dimensional Riemann flat equations. (Of course, we are immediately neglecting a whole variety of well-known solutions of Einstein's field equations, most notably the Schwarzschild solution that can, at best, be embedded in six-dimensional Minkowski space-time.) Matter is introduced into the theory by considering the embedding of the physical four-dimensional space–time in the full five-dimensional space–time. Basically, the physically relevant metric is taken to be the intrinsic metric on the four-dimensional slices $y=\text{const}$.

The class (1) and (3) solutions, namely (12) and (37)–(38), induce Riemann flat four-metrics on the slices $y=\text{const}$, and are thus physically uninteresting within the context of induced matter theory. The class (2) solutions induce conformally flat four-metrics. The class (2) solutions with $A=1$ [see Eqs. (36)–(37)] represent static Friedman–Robertson–Walker metrics, such that the three-space $t=\text{const}$ has positive constant curvature. The class (2) solutions with $A\neq 1$ can be interpreted as perfect fluid models with constant density,

$$\mu=\frac{3}{e^{2a}}, \tag{60}$$

and non-constant pressure,

$$P=-\frac{1}{e^{2a}}-\frac{4}{e^{2a}}\left[2+e^{-2a}V(t)\ \frac{1+e^{2y}r^2}{1-e^{2y}r^2}\right]^{-1}, \tag{61}$$

where $V(t)\equiv v(t)+2\int^t\exp[a(y)]dy$. The intrinsic metric of these class (2) solutions (on the hypersurface $Y=\text{const}$),

$$ds^2 = -\left\{ v(t) + 2e^{2a} \frac{1-e^{2Y}r^2}{1+e^{2Y}r^2} + 2\int e^a \, dY \right\}^2 dt^2 + \frac{4e^{2(a+Y)}}{(1+e^{2Y}r^2)^2}(dr^2 + r^2 \, d\Omega^2). \tag{62}$$

belongs to the class of generalized interior Schwarzschild solutions.[12,14] The diffeomorphism (with Y=const),

$$R = \frac{2e^{2a}e^Y}{1+e^{2Y}r^2}, \tag{63}$$

reduces the above metric to the standard form,

$$ds^2 = -\{v(t) + \sqrt{1-C^2R^2}\}^2 \, dt^2 + \frac{dR^2}{1-C^2R^2} + R^2 \, d\Omega^2, \tag{64}$$

where $C(= \exp[-a(Y)]_{Y=\text{const}})$ is an arbitrary constant. We note that a specific embedding for the interior Schwarzschild solution, $v(t)$=const, into a six-dimensional Riemann flat solution is known (see Ref. 16). However, to our knowledge, the metric (36) with $A \neq 1$ [see (37)] is the first time that an explicit embedding for the interior Schwarzschild solution into five-dimensional Riemann flat space–time has appeared in the literature.

The class (4) solutions also induce conformally flat four-metrics, and can be interpreted as perfect fluid models whose associated density and pressure are

$$\mu = 3\alpha^2, \tag{65}$$

$$P = -3\alpha^2 + 2\alpha\alpha_t\{\partial_t \ln[b^{-1}(\alpha^2 - \beta^2 + b^2r^2)]\}^{-1}. \tag{66}$$

These solutions belong to the class of generalized Friedmann solutions.[14]

It is clear from (2) that there exist algebraic relationships between $K_{\alpha\beta}$ ($\Omega_{\alpha\beta}$), $R_{\alpha\beta}$ (and hence $T_{\alpha\beta}$), and the Weyl tensor $C_{\alpha\beta\gamma\delta}$. *All the Riemann flat solutions* [classes (1)–(4)] *induce four-metrics that are conformally flat*. Indeed, all the perfect fluid solutions of embedding class one must necessarily be either of Petrov type O (conformally flat) or of Petrov type D.[12] Curiously, results about the embedding of conformally flat four-dimensional metrics into Riemannian flat five-dimensional Lorentzian space-time do not seem to appear in the literature. Results about the embedding of conformally flat four-dimensional metrics into Riemannian flat five-dimensional Euclidean space–times (positive definite metrics) are known (see Refs. 17 and 18). For completeness, we now state the following theorem without proof: If a four-dimensional conformally flat Lorentzian metric is of embedding class one, then its Riemann tensor is given by $R_{\alpha\beta\gamma\delta} = 2\Omega_{\alpha[\gamma}\Omega_{\delta]\beta}$, where $\Omega_{\alpha[\beta;\gamma]}=0$, and furthermore Ω must be of the form

$$\Omega_{\alpha\beta} = An_\alpha n_\beta + Bg_{\alpha\beta}, \tag{67}$$

where n_α is a unit space-like or time-like vector (that is, $n_\alpha n^\alpha = \pm 1$).

IV. RICCI FLAT SOLUTIONS

All the solutions discussed in Secs. II and III are Riemann flat and thus are also automatically Ricci flat. To date, the majority of the known Ricci flat solutions where the metric has the form (5) have been found by examining the special ansatz that the metric functions are separable in the variables t, r, and y. In this section, our aim is to find a class of Ricci flat solutions that contain a subclass of the Riemann flat solutions. Thus, we use the Riemann flat solutions of the previous sections as a springboard to construct new Ricci flat solutions.

We base our first ansatz on the form of the class (2) solutions [we chose the class (2) solutions because (i) they are not as complicated as the class (4) solutions; and (ii) the class (1) and (3) solutions have an uninteresting interpretation in terms of induced matter theory]. Hence, we consider metrics of the following form:

$$e^F=\frac{A(t,y)+B(t,y)r^2}{\alpha(y)+b^2(y)r^2}, \tag{68}$$

$$e^G=\frac{2b(y)+C(y)r^2}{\alpha(y)+b^2(y)r^2}, \tag{69}$$

$$e^K=\frac{D(y)+E(y)r^2}{\alpha(y)+b^2(y)r^2}. \tag{70}$$

Thus, the field equations $R_{ij}=0$ will all reduce to power series in r. The coefficients of each of the power series will, in general, be partial differential equations in t and y only and they must be identically zero. After a lengthy calculation (see Ref. 19 for full details), we find the following equation:

$$C_y=C\,\frac{b_y}{b}. \tag{71}$$

Thus, we find that either (i) $C=0$ or (ii) $C=\pm b\neq 0$. In case (i), the solutions can eventually be shown to reduce to the Riemann flat class (2) solutions. In case (ii), the metric can be shown to be equivalent to

$$ds^2=-y^{-1}\,dt^2+y(dr^2+r^2\,d\Omega^2)+dy^2. \tag{72}$$

The above metric was discussed in Ref. 7 and belongs to the class of generalized Kasner metrics.[9,20,21]

In the process of determining the Riemann flat solutions and the above solutions, we observed that if the metric functions F, G, and K appearing in (5) had a particular form, then some of the Ricci flat field equations could be easily integrated. Thus, based upon our observations, we are led to our second ansatz; we shall now consider metrics of the following form:

$$ds^2=-e^{F(t,r,y)}\,dt^2+e^{2G(r)}[dr^2+r^2\,d\Omega^2]+e^{2K(r,y)}\,dy^2. \tag{73}$$

This ansatz includes both the class (1) Riemann flat solutions and the Davidson–Owen–Gross–Perry solutions.[4,5]

The field equation $R_{\theta\theta}=0$ for metric (73) implies that $(G_r+1/r)F_{tr}=0$. Thus, either (i) $G_r=-1/r$ or (ii) $F_{tr}=0$.

In case (i) we can set $G=0$ without loss of generality by an appropriate diffeomorphism. The field equations $R_{rr}=0$ and $R_{\theta\theta}=0$ can then be employed to show that $F_r=0=K_r$. The remaining field equations can then be used to show that the metric reduces to the class (1) Riemann flat solutions, (12)–(13), with $B=1$.

In case (ii) we find that $F=A(t,y)+B(r,y)$. The equation $R_{\theta\theta}=0$ then yields the additional result $(B+K)_{ry}=0$, and thus $K=-B(r,y)+C(r)$. Furthermore, the equation $R_{rr}=0$ yields $B_{ry}(2B_r-C_r)=0$, which implies that $K=K(r)$ and $F=A(t,y)+B(r)$. Finally, the equation $R_{ry}=0$ implies that $A_y(B_r-K_r)=0$. Thus, either (a) $A_y=0$ or (b) $B_r=K_r$. In case (a), we can take $A=1$ without loss of generality, and the metric then reduces to the Davidson–Owen–Gross–Perry class of soliton solutions,[4,5] namely,

$$ds^2=-\left(\frac{ar-1}{ar+1}\right)^{2\epsilon k}dt^2+\left(\frac{a^2r^2-1}{a^2r^2}\right)^2\left(\frac{ar+1}{ar-1}\right)^{2\epsilon(k-1)}[dr^2+r^2\,d\Omega^2]+\left(\frac{ar+1}{ar-1}\right)^{2\epsilon}dy^2, \tag{74}$$

where ϵ and k are subject to the constraint $\epsilon^2(k^2-k+1)=1$.

In case (b), the general form of the metric may be written as

$$ds^2=-A^2(t,y)e^{2K(r)}\,dt^2+e^{2G(r)}[dr^2+r^2\,d\Omega^2]+e^{2K(r)}\,dy^2. \tag{75}$$

The field equation $R_{tt}=0$ can be employed to show that

$$A_{yy}=\kappa A \tag{76}$$

and

$$e^{2K}[K_{rr}+2K_r^2+K_rG_r+(2/r)K_r]+\kappa e^{2G}=0, \tag{77}$$

where κ is a constant (κ can always be chosen to be equal to either 0 or ±1). Finally, the equations $R_{rr}=0$ and $R_{\theta\theta}=0$ yield

$$K_{rr}+K_r^2-K_rG_r+G_{rr}+\frac{1}{r}\,G_r=0, \tag{78}$$

$$2K_rG_r+\frac{2}{r}\,K_r-G_{rr}+\frac{3}{r}\,G_r+G_r^2=0, \tag{79}$$

respectively. We note that the above system of equations (77)–(79) only has rank 2—Eq. (79) is a first integral of (77) and (78).

If $\kappa=0$, then the above system of equations, (77)–(79), can be solved completely to yield the solution

$$ds^2=-\left(\frac{ar+1}{ar-1}\right)^{2/\sqrt{3}}[\alpha(t)y+\beta(t)]^2\,dt^2+\left(\frac{a^2r^2-1}{a^2r^2}\right)^2\left(\frac{ar-1}{ar+1}\right)^{4/\sqrt{3}}[dr^2+r^2\,d\Omega^2]$$
$$+\left(\frac{ar+1}{ar-1}\right)^{2/\sqrt{3}}dy^2, \tag{80}$$

where a is an arbitrary constant, and α and β are arbitrary functions of t. If $\alpha=0$ then the above metric is a particular solution belonging to the Davidson–Owen–Gross–Perry class of solutions[4,5] [that is, metric (74) with $\epsilon=1/\sqrt{3}$ and $k=-1$]. If $\alpha\neq0$, then we can always set $\alpha=1$: this particular solution was originally found by Ponce de Leon and Wesson[22]—the solution is also very similar in form to the time-dependent soliton solution found by Wesson, Liu, and Lim.[3]

If $\kappa\neq0$, then the equations (77)–(79) are not so simple to solve. For convenience, we introduce the new variable,

$$\rho=\ln r. \tag{81}$$

Equations (77)–(79) can then be reduced to the following system of differential equations:

$$G_{\rho\rho}=-G_\rho^2-2K_\rho G_\rho-2K_\rho-2G_\rho, \tag{82}$$

$$K_{\rho\rho}=-K_\rho^2+3K_\rho G_\rho+G_\rho^2+2G_\rho+3K_\rho, \tag{83}$$

with the first integral

$$4K_\rho G_\rho + G_\rho^2 + K_\rho^2 + 2G_\rho + 4K_\rho + \kappa e^{2(G-K+\rho)} = 0. \tag{84}$$

Equations (82) and (83) form a two-dimensional autonomous system of differential equations. A specific solution can be found by demanding that $K = G + \rho$; Eqs. (82) and (84) then imply that $\kappa = -1$ and that

$$G = \left(-1 \pm \frac{1}{\sqrt{3}} \right) \rho, \tag{85}$$

$$K = \pm \frac{1}{\sqrt{3}}\, \rho. \tag{86}$$

Thus, if we make the transformation $R = r^{\pm 1/\sqrt{3}}$, then the metric, in this special case, may be written as

$$ds^2 = -\cos^2(y + \alpha(t)) R^2\, dt^2 + 3\, dR^2 + R^2\, d\Omega^2 + R^2 dy^2. \tag{87}$$

A. Remarks

Of course, Eqs. (82)–(83) can easily be analyzed as a two-dimensional dynamical systems.[19] It can be shown that the system has four fixed points at finite values and six fixed points at infinity (full details are given in Ref. 19). Analysis of the finite fixed points shows that there are two saddles, an attracting focus and a repelling focus. Both the attracting focus and the repelling focus are described by the metric (87) [for which $\kappa = -1$] and are valid for R tends to infinity and R tends to zero, respectively. The corresponding class of solutions are consequently asymptotically flat in general, in the sense that all components of the Riemann tensor asymptotically vanish (as $r \to \infty$ for the attractor and as $r \to 0$ for the repellor). The corresponding four-dimensional models have the property that asymptotically the stress-energy tensor may be interpreted as an anisotropic fluid, with $p_\parallel = 0$ and $\mu = 2p_\perp$, where $p_\parallel$ and $p_\perp$ are the fluid pressures parallel and perpendicular to the fluid four-velocity, respectively. (At the two saddles the corresponding exact solutions have $\kappa = 0$ and the associated four-dimensional solutions that are also flat.)

Analysis of the singular points at infinity shows that there are two sinks, two sources, and two saddles (appearing in pairs). The asymptotic form of the solution corresponding to the sources at infinity is given by

$$G(r) \approx (\ln r)^{1-2/\sqrt{3}}, \quad K(r) \approx (\ln r)^{1/\sqrt{3}} \tag{88}$$

[note that $K \to \infty$, $G \to 0$, and that $\det(g_{ij}) \to \infty$ as $r \to \infty$], while for the sinks,

$$G(r) \approx (\ln r)^{1+2/\sqrt{3}}, \quad K(r) \approx (\ln r)^{-1/\sqrt{3}}. \tag{89}$$

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APPENDIX: THE RIEMANN TENSOR

The nonzero components of the Riemann tensor, up to the usual symmetries, for the metric (5) are

$$R^t{}_{rtr} = F_{rr} + F_r^2 - F_r G_r - e^{2(G-F)} \{ G_{tt} + G_t^2 - F_t G_t \} + e^{2(G-K)} F_y G_y, \tag{A1}$$

$$R^{t}{}_{rty}=F_{ry}+F_rF_y-F_rG_y-F_yK_r\,, \tag{A2}$$

$$R^{r}{}_{tyr}=G_{ty}+G_tG_y-F_yG_t-K_tG_y \quad (=R^{\theta}{}_{ty\theta}=R^{\phi}{}_{ty\phi}), \tag{A3}$$

$$R^{\theta}{}_{tt\theta}=G_{tt}+G_t^2-F_tG_t-F_r(G_r+1/r)e^{2(F-G)}-e^{2(F-K)}F_yG_y \;\; (=R^{\phi}{}_{tt\phi}), \tag{A4}$$

$$R^{\theta}{}_{tr\theta}=G_{rt}-G_tF_r \quad (=R^{\phi}{}_{tr\phi}), \tag{A5}$$

$$R^{t}{}_{yty}=F_{yy}+F_y^2-F_yK_y+e^{2(K-G)}F_rK_r-e^{2(K-F)}\{K_{tt}+K_t^2-F_tK_t\}, \tag{A6}$$

$$R^{y}{}_{try}=K_{rt}+K_rK_t-K_tF_r-G_tK_r\,, \tag{A7}$$

$$R^{\theta}{}_{rr\theta}=G_{rr}+\frac{1}{r}\,G_r-e^{2(G-F)}G_t^2+e^{2(G-K)}G_y^2 \quad (=R^{\phi}{}_{rr\phi}), \tag{A8}$$

$$R^{\theta}{}_{ry\theta}=G_{ry}-K_rG_y \quad (=R^{\phi}{}_{ry\phi}), \tag{A9}$$

$$R^{r}{}_{yyr}=G_{yy}+G_y^2-G_yK_y+e^{2(K-G)}\{K_{rr}+K_r^2-G_rK_r\}-e^{2(K-F)}G_tK_t\,, \tag{A10}$$

$$R^{\phi}{}_{\theta\theta\phi}=r^2\left\{G_r^2+\frac{2}{r}\,G_r-e^{2(G-F)}G_t^2+e^{2(G-K)}G_y^2\right\}, \tag{A11}$$

$$R^{\theta}{}_{yy\theta}=G_{yy}+G_y^2-G_yK_y-e^{2(K-F)}G_tK_t+e^{2(K-G)}K_r(G_r+1/r) \;\; (=R^{\phi}{}_{yy\phi})\,. \tag{A12}$$

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