PHYSICAL REVIEW D VOLUME 51, NUMBER 2 15 JANUARY 1995

Isotropization of scalar field Bianchi models with an exponential potential

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(Received 22 July 1994)

We study whether homogeneous cosmological models containing a self-interacting scalar field with an exponential potential [of the form $V(\phi) = \Lambda e^{k\phi}$] isotropize. Following Heusler [M. Heusler, Phys. Lett. B **253**, 33 (1991)], we show that Bianchi models, other than possibly those of types I, V, VII, or IX, cannot isotropize if $k^2 > 2$. In this case we note that the solutions of Feinstein and Ibáñez [A. Feinstein and J. Ibáñez, Class. Quantum Grav. **10**, 93 (1993)], which are neither isotropic nor inflationary, act as stable attractors. When $k^2 < 2$ the cosmic no-hair theorem of Kitada and Maeda [Y. Kitada and K. Maeda, Phys. Rev. D **45**, 1416 (1992); *ibid.*, Class. Quantum Grav. **10**, 703 (1993)] applies and the isotropic power-law inflationary FRW solution is the unique attractor for any initially expanding Bianchi model.

PACS number(s): 04.20.Jb, 98.80.Hw

We study homogeneous cosmological models containing a self-interacting scalar field with an exponential potential. Models with an exponential scalar field potential arise naturally in alternative theories of gravity [1,2], in the Salam-Sezgin model of $N = 2$ supergravity coupled to matter [3], and in theories undergoing dimensional reduction to an effective four-dimensional theory [4]. Cosmologies of this type have been studied by a number of authors, including Burd and Barrow [5], Kitada and Maeda [6,7], and Feinstein and Ibáñez [8].

Our aim here is to study qualitatively whether homogeneous scalar field cosmologies with an exponential potential isotropize and/or inflate, thereby determining the applicability of the so-called cosmic no-hair theorem, which essentially asserts that inflation is typical in a wide class of scalar field cosmologies. Also we wish to determine the relevence of the exact homogeneous solutions (of Bianchi types III and IV) of Feinstein and Ibáñez [8] which neither inflate nor isotropize.

Cosmological models with a minimally coupled scalar field have a stress-energy tensor given by

$$T_{ab} = \phi_{;a}\phi_{;b} - g_{ab}\left[\tfrac{1}{2}\phi_{;c}\phi^{;c} + V(\phi)\right], \tag{1}$$

where for a homogeneous scalar field $\phi = \phi(t)$, so that $\phi_{;c}\phi^{;c} = -\dot{\phi}^2$ (where an overdot denotes differentiation with respect to the proper time). In this case we can formally treat the stress-energy tensor as a perfect fluid with a velocity vector $u^a = \phi^{;a}/\sqrt{-\phi_{;a}\phi^{;a}}$, where the energy density and the pressure are given by

$$\rho_\phi \equiv E = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2a}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2b}$$

In the models under consideration, the potential of the scalar field is given by

$$V(\phi) = \Lambda e^{k\phi}, \tag{3}$$

where Λ (> 0) and k are constants.

From the Einstein field equations we have the Raychaudhuri equation governing the evolution of the expansion,

$$\dot{\theta} = -2\sigma^2 - \frac{1}{3}\theta^2 - \dot{\phi}^2 + V(\phi), \tag{4}$$

and the generalized Friedmann equation

$$\theta^2 = 3\sigma^2 + \frac{3}{2}\dot{\phi}^2 + 3V(\phi) - \frac{3}{2}P, \tag{5}$$

where σ is the shear scalar, P is the scalar curvature of the homogeneous hypersurfaces, which is always negative except in the Bianchi type IX case [9], and $V(\phi)$ is given by Eq. (3). The Klein-Gordon equation for the scalar field with an exponential potential is then

$$\ddot{\phi} + \theta\dot{\phi} + kV(\phi) = 0. \tag{6}$$

Defining ψ by

$$\psi = \dot{\phi} + \frac{k}{3}\theta \tag{7}$$

and using Eqs. (4) and (5), the Klein-Gordon equation can be written as

$$\dot{\psi} + \theta\psi + \frac{k}{3}P = 0. \tag{8}$$

We now introduce new expansion-normalized variables and a new time variable as follows:

$$\beta = \sqrt{3}\frac{\sigma}{\theta}, \qquad \frac{dt}{d\Omega} = \frac{3}{\theta},$$
$$\Psi = \frac{\sqrt{6}}{2}\frac{\dot{\phi}}{\theta}, \qquad \Phi = \sqrt{3\Lambda}\frac{e^{k\phi/2}}{\theta}. \tag{9}$$

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With these definitions, Eqs. (4)–(6) can be rewritten as

$$\Psi' = -\Psi(2 - 2\beta^2 - 2\Psi^2 + \Phi^2) - \frac{\sqrt{6}k}{2}\Phi^2, \tag{10a}$$

$$\Phi' = -\Phi\left(-1 - 2\beta^2 - 2\Psi^2 + \Phi^2 - \frac{\sqrt{6}k}{2}\Psi\right), \tag{10b}$$

where a prime denotes differentiation with respect to the new time Ω. The equilibrium points of the system have either $\Phi = \Psi = 0$, which corresponds to the massless scalar field case, or $\beta^2 + \Psi^2 = 1, \Phi = 0$, which represents the Kasner 'initial' (line) singularity, or else (and in all cases of interest here) obey the relation

$$\Phi^2 + \Psi^2 = -\frac{\sqrt{6}}{k}\Psi. \tag{11}$$

In terms of these new expansion-normalized variables the energy density of the scalar field (2a) can be written as

$$\frac{E}{\theta^2} = \frac{1}{3}(\Psi^2 + \Phi^2), \tag{12}$$

and we have that

$$\Psi = -\frac{k}{\sqrt{6}} + \frac{\sqrt{3}}{\sqrt{2}}\frac{\psi}{\theta}. \tag{13}$$

Hence, at the equilibrium points we obtain

$$\frac{E}{\theta^2} = -\frac{\sqrt{6}}{3k}\Psi = \frac{1}{3}\left(1 - \frac{3}{k}\frac{\psi}{\theta}\right), \tag{14a}$$

$$\frac{V}{E} = \frac{\Phi^2}{\Psi^2 + \Phi^2} = 1 - \frac{k^2}{6} + \frac{k}{2}\frac{\psi}{\theta}. \tag{14b}$$

Following Heusler [10], who proved that all Bianchi models with ordinary matter satisfying the usual energy conditions and containing a scalar field with a positive, convex potential [with a local minimum such that $V(\phi_0) = 0$; e.g., $V(\phi) = \frac{1}{2}m\phi^2$], *can only approach isotropy* at infinite times if the underlying Lie group is admitted by a Friedman-Robertson-Walker (FRW) model, the necessary conditions for an anisotropic and homogeneous solution to isotropize are

$$\beta = 0 \tag{15}$$

and (Heusler's proposition 2 [10])

$$\frac{E}{\theta^2} \to \frac{1}{3}, \tag{16a}$$

$$\left\langle \frac{V}{E} \right\rangle \geq \frac{2}{3}, \tag{16b}$$

where angular brackets denote an appropriate time average [Heusler [10], Eq. (20)].

Now, using Eq. (14a), Eq. (16a) implies that

$$\frac{\psi}{\theta} \to 0. \tag{17}$$

Using Eq. (17) we can now compute $\langle V/E \rangle$, viz.,

$$\left\langle \frac{V}{E} \right\rangle = 1 - \frac{k^2}{6} \tag{18}$$

(this replaces Heusler's proposition 3 [10]). Hence Eq. (16b) implies that

$$1 - \frac{k^2}{6} \geq \frac{2}{3} \quad \Rightarrow \quad k^2 \leq 2. \tag{19}$$

Therefore, we have shown that if $k^2 > 2$ and if the model is not of Bianchi types I, V, VII, or IX, then it *cannot* isotropize. Like Heusler [10], we have not completely generalized the Collins-Hawking [11] result that only a subclass of homogeneous models of measure zero can isotropize since we have not explicitly investigated Bianchi models of types VII_h and IX.

The question consequently arises as to what the future asymptotic behavior of the models is when $k^2 > 2$. First, we note that Heusler's proposition 1 [10] is still satisfied (where now $\theta \to 0$ and $V \to 0$ as $t \to \infty$). Second, it can be shown (for $k^2 > 2$) that the Feinstein-Ibáñez solutions [8] play a primary role in that they are stable attractors that are neither isotropic nor inflationary (see [12] for details).

For inflation to occur we must have that

$$2\beta^2 + 2\Psi^2 - \Phi^2 < 0, \tag{20}$$

so that, using Eqs. (11), (13), and (20), at the equilibrium points the solution will inflate if

$$(k^2 - 2) - 3k\frac{\psi}{\theta} < 0. \tag{21}$$

Therefore, from Eqs. (15) and (17), for models to inflate and isotropize k^2 must be less than 2, a well-known result [4,6,7].

We have shown that $k^2 \leq 2$ is a necessary condition for the homogeneous models under consideration to isotropize, and for $k^2 < 2$ these models inflate. What we have *not* shown is that all such models with $k^2 \leq 2$ isotropize. However, the no-hair theorem of Kitada and Maeda [6,7] proves precisely this, namely, that for $k^2 < 2$ the isotropic, power-law inflationary FRW solution is the unique attractor for any initially expanding scalar field Bianchi model with an exponential potential (and with ordinary matter satisfying the usual energy conditions). This result is essentially proven using Wald's [9] approach and reduces to Wald's result in the special case $k = 0$ (corresponding to a cosmological constant), whence the unique attractor is the (exponential) de Sitter solution. In addition, Kitada and Maeda [7] showed that in these models anisotropies generally enhance inflation (over their isotropic counterparts). (As usual in the above the Bianchi type IX case needs to be treated separately — see Kitada and Maeda [7] for details.)

Let us conclude with some brief remarks. First, we note that in our investigation we have not included ordinary matter (satisfying the usual energy conditions). However, matter could easily be included in precisely the same way as in Heusler [10] and in Kitada and Maeda [6,7]. Second, for an exponential potential it can be shown that the equation for the evolution of the expansion (4) decouples from the "reduced dynamical system" in the new expansion-normalized variables (9) [13,14]; consequently the cosmological models corresponding to the equilibrium points of the system are *self-similar*. In particular, the isotropic, power-law inflationary (FRW) attracting solutions (in the case $k^2 < 2$) are self-similar models and the Feinstein-Ibáñez [8] solutions (in the case $k^2 > 2$) are also self-similar.

R.V.D.H. and A.A.C. were supported by the Natural Sciences and Engineering Research Council of Canada and J.I. was supported by the CICYT (Spain) Grant No. PS90-0093. J.I. also thanks the Department of Mathematics, Statistics, and Computing Science of Dalhousie University for its warm hospitality.

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