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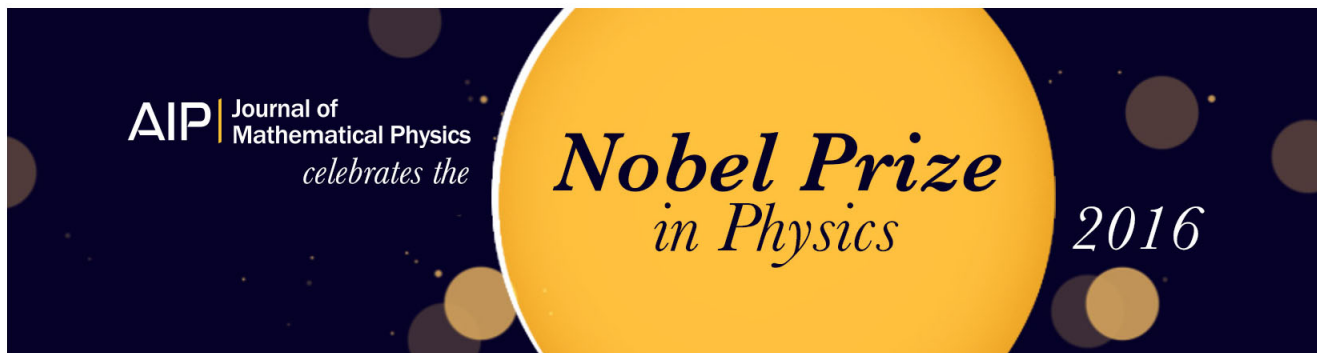Cosmological evolution of a brane Universe in a type 0 string background
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Inhomogeneous M-theory cosmologies

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We study a class of inhomogeneous and anisotropic G_2 string cosmological models. In the case of separable G_2 models we show that the governing equations reduce to a system of ordinary differential equations. We focus on a class of separable G_2 M-theory cosmological models, and study their qualitative behavior (a class of models with time-reversed dynamics is also possible). We find that generically these inhomogeneous M-theory cosmologies evolve from a spatially inhomogeneous and negatively curved model with a nontrivial form field toward spatially flat and spatially homogeneous dilaton-moduli-vacuum solutions with trivial form fields. The late time behavior is the same as that of spatially homogeneous models previously studied. However, the inhomogeneities are not dynamically insignificant at early times in these models. © *2002 American Institute of Physics.*
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I. STRING COSMOLOGY

Nonperturbative M-theory encompasses and unifies all five anomaly free, perturbative superstring theories[1] and corresponds to eleven-dimensional supergravity in the low-energy limit.[2] In particular, the compactification of M-theory on a circle, S^1, leads to the type IIA superstring. A study of the qualitative cosmological effects that can arise in M-theory is therefore of considerable importance. To lowest order (in the inverse string tension), the tree-level effective action for massless fields contains a dilaton, a form field (which in four-dimensions is dynamically dual to pseudoscalar axion field) and (a) stringy cosmological constant(s). Even in this approximation the one-loop string equations of motion for inhomogeneous backgrounds are very difficult to solve, and it is a useful first step to consider models in which the homogeneity is broken only in one spatial direction. Metrics that admit two commuting (orthogonally transitive) space-like Killing vectors are referred to as G_2 space–times.

String models admitting an Abelian group, G_2, of isometries have a number of important physical applications. The spatially homogeneous Bianchi types I–VII$_h$ and locally rotationally symmetric (LRS) types VIII and IX admit a G_2 group of isometries[3] and so the G_2 cosmologies can be considered as inhomogeneous generalizations of these Bianchi models. Nonlinear inhomogeneities in the dilaton and axion fields can be investigated and, in principle, this allows density perturbations in string-inspired inflationary models such as the pre-big bang scenario to be studied.[4,5] Given the potential importance of this scenario it is important to study its generality with respect to inhomogeneities as well with respect to anisotropies. The general effects of small inhomogeneities and anisotropies have been studied by Veneziano.[4]

In general relativity (GR) the generic singularity is neither spatially homogeneous nor isotropic. Hence it is of interest to study more general models. In particular, it has been conjectured that G_2 metrics represent a first approximation to the general solution of Einstein gravity in the vicinity

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of a curvature singularity.[6–9] The high curvature regime is precisely the regime where stringy deviations from GR are expected to be significant. The G_2 models studied here may therefore provide insight into the generic behavior of cosmologies at very early times.

A number of exact inhomogeneous and anisotropic G_2 string cosmologies have been found. Barrow and Kunze studied an inhomogeneous generalization of the Bianchi type I string cosmology[10] and Feinstein, Lazkoz, and Vazquez-Mozo derived a closed, inhomogeneous model by applying duality transformations on the LRS Bianchi type IX cosmology.[11] Clancy et al. have found inhomogeneous generalizations of the Bianchi type VI$_h$ universe and have studied their asymptotic behavior.[12]

In general, the field equations reduce to a system of coupled, partial differential equations in two variables when spatial homogeneity is broken along a single direction. Unfortunately, these equations are still very complicated. However, solutions can be found due to the noncompact global symmetries of the string effective action. When the metric admits two commuting space-like Killing vectors, there exists an infinite-dimensional symmetry on the space of solutions that may be identified infinitesimally with the O(2,2) current algebra.[13–15] This symmetry reduces to the Geroch group, corresponding to the SL(2,R) current algebra, when the dilaton and two-form potential are trivial,[16] and includes the global SL(2,R) S-duality of the action.

New inhomogeneous G_2 string cosmologies containing a nontrivial two-form potential may be generated by an application of both the S- and T-duality symmetries from simpler (dilaton-vacuum) seed solutions. Lidsey et al.[17] discuss the noncompact, global symmetries of the string effective action in a variety of settings and review various methods for solving the Einstein-scalar field equations utilizing generating techniques (from solutions with a minimally coupled, massless scalar field from a vacuum, G_2 cosmology). In particular, Feinstein, Lazkoz, and Vázquez-Mozo[11] present an algorithm which permits the construction of inhomogeneous string solutions by employing a Buscher transformation, inverse scattering techniques,[7,8] followed by the generating technique of Wainwright, Ince, and Marshman.[18] Feinstein et al. employ this algorithm to generate a closed, inhomogeneous string cosmology with S^3 topology from a LRS Bianchi type IX solution.[11,19] However, this algorithm involves a number of nontrivial operations, and an alternative and more straightforward approach is to apply an O(2,2) transformation directly to the seed cosmology.[17]

In this paper we shall consider a class of separable string cosmological models whose governing equations reduce to ordinary differential equations (ODE) which can be studied by qualitative methods. In particular, we shall focus on a class of M-theory cosmological models.

II. STRING ACTION

We consider the general string action in the form[20]

$$S = \int d^4x \sqrt{-g} \left\{ e^{-\Phi} \left[R + (\nabla \Phi)^2 - 6(\nabla \beta)^2 - \frac{1}{2} e^{2\Phi} (\nabla \sigma)^2 - 2\Lambda \right] - \frac{1}{2} Q^2 e^{-6\beta} - \Lambda_M \right\}, \quad (2.1)$$

in terms of the pseudoscalar axion field, σ, the four-dimensional dilaton field Φ, and the modulus field β, where Λ and Λ_M represent cosmological constant terms and Q^2 may be interpreted as a zero-form field strength. This is a phenomenological action representing the bosonic sector of the effective supergravity action for the low-energy limit of M-theory and encompasses other string theories.[20] We are particularly interested in the class of four-dimensional cosmologies derived from the type IIA string and M-theory effective actions and which include a nontrivial Ramond–Ramond (RR) sector.[21] In these models a specific compactification from eleven to four dimensions was considered, where the topology of the internal dimensions was assumed to be a product space consisting of a circle and an isotropic six-torus;[21] this is dynamically equivalent to compactifications on a Calabi–Yau threefold.[22] The FRW models in this class of cosmologies was studied in Ref. 21.

Defining

$$\mathcal{L}_M = -3e^{-\Phi}(\nabla\beta)^2 - \frac{1}{4}e^{\Phi}(\nabla\sigma)^2 - \Lambda e^{-\Phi} - \frac{1}{4}Q^2 e^{-6\beta} - \frac{1}{2}\Lambda_M; \quad T_{\alpha\beta} \equiv g_{\alpha\beta}\mathcal{L}_M - 2\frac{\partial\mathcal{L}_M}{\partial g^{\alpha\beta}},$$

the Euler–Lagrange equations then lead to the field equations (FE)[23]

$$G_{\mu\nu} = -\nabla_\mu\nabla_\nu\Phi + 6\nabla_\mu\beta\nabla_\nu\beta + \tfrac{1}{2}e^{2\Phi}\nabla_\mu\sigma\nabla_\nu\sigma - \tfrac{1}{2}g_{\mu\nu}[(\nabla\Phi)^2 + 6(\nabla\beta)^2 + \tfrac{1}{2}e^{2\Phi}(\nabla\sigma)^2 + 2\Lambda$$

$$+ \tfrac{1}{2}Q^2 e^{\Phi-6\beta} + \Lambda_M e^{\Phi} - 2\square\Phi], \tag{2.2a}$$

$$\square\Phi = \tfrac{1}{2}(\nabla\Phi)^2 + 3(\nabla\beta)^2 - \tfrac{1}{4}e^{2\Phi}(\nabla\sigma)^2 + \Lambda - \tfrac{1}{2}R, \tag{2.2b}$$

$$\square\beta = \nabla_\mu\Phi\nabla^\mu\beta - \tfrac{1}{4}Q^2 e^{\Phi-6\beta}, \tag{2.2c}$$

$$\square\sigma = -\nabla_\mu\Phi\nabla^\mu\sigma. \tag{2.2d}$$

In Eqs. (2.2a)–(2.2d), greek indices take on values 0, 1, 2, 3, and units are chosen so that $16\pi\hat{G} = 1$.

III. G_2 COSMOLOGIES

Let us examine (2.2) within the context of G_2 cosmological models described by the line element

$$ds^2 = e^{2F}(-dt^2 + dz^2) + e^G(e^q\,dx^2 + e^{-q}\,dy^2), \tag{3.1}$$

where the metric functions $\{F,G,q\}$ and the string functions $\{\Phi,\beta,\sigma\}$ are all functions of t and z only. For any $q(t,z)$, we define $\dot{q} \equiv \partial q/\partial t$, $q' \equiv \partial q/\partial z$, and $\Delta^2 q \equiv \ddot{q} - q''$. The local behavior of these models is determined by the gradient $B_\mu \equiv \partial_\mu G$, and cosmological solutions arise if B_μ is globally time-like.

Also, the Ricci scalar is given by

$$R = \tfrac{1}{2}e^{-2F}[4\Delta^2 G + 4\Delta^2 F + 3(\dot{G}^2 - G'^2) + (\dot{q}^2 - q'^2)]. \tag{3.2}$$

Using these expressions, and defining the modified dilaton field,

$$\varphi \equiv \Phi - F - G, \tag{3.3}$$

the field equations become

$$\Delta^2\varphi = \tfrac{1}{2}[(\dot{\varphi}+\dot{F})^2 - (\varphi'+F')^2] + \tfrac{1}{4}(\dot{G}^2 - G'^2) + \tfrac{1}{4}(\dot{q}^2 - q'^2) + 3(\dot{\beta}^2 - \beta'^2)$$

$$- \tfrac{1}{4}e^{2\varphi+2F+2G}(\dot{\sigma}^2 - \sigma'^2) - \Lambda e^{2F}, \tag{3.4a}$$

$$\Delta^2\beta = [(\dot{\varphi}+\dot{F})\dot{\beta} - (\varphi'+F')\beta'] + \tfrac{1}{4}Q^2 e^{\varphi+3F+G-6\beta}, \tag{3.4b}$$

$$\Delta^2\sigma = -[(\dot{\varphi}+\dot{F}+2\dot{G})\dot{\sigma} - (\varphi'+F'+2G')\sigma'], \tag{3.4c}$$

$$\Delta^2 q = [(\dot{\varphi}+\dot{F})\dot{q} - (\varphi'+F')q'], \tag{3.4d}$$

$$\Delta^2 F = \tfrac{1}{2}(\dot{\varphi}+\dot{F})^2 - \tfrac{1}{2}(\varphi'+F')^2 - \tfrac{1}{4}(\dot{G}^2 - G'^2) - \tfrac{1}{4}(\dot{q}^2 - q'^2) - 3(\dot{\beta}^2 - \beta'^2)$$

$$+ \tfrac{1}{4}e^{2\varphi+2F+2G}(\dot{\sigma}^2 - \sigma'^2) - \Lambda e^{2F} - \tfrac{1}{2}Q^2 e^{\varphi+3F+G-6\beta} - \Lambda_M e^{\varphi+3F+G}, \tag{3.4e}$$

$$\Delta^2 G = [(\dot\varphi + \dot F)\dot G - (\varphi' + F')G'] + (\dot\sigma^2 - \sigma'^2)e^{2\varphi + 2F + 2G} - \tfrac{1}{2}Q^2 e^{\varphi + 3F + G - 6\beta} - \Lambda_M e^{\varphi + 3F + G},$$

$$(3.4\text{f})$$

$$(\dot\varphi + \dot F)\dot{} + (\varphi' + F')' = 2(\dot\varphi + \dot F)\dot F + 2(\varphi' + F')F' + \tfrac{1}{2}(\dot G^2 + G'^2) + \tfrac{1}{2}(\dot q^2 + q'^2) + 6(\dot\beta^2 + \beta'^2)$$

$$+ \tfrac{1}{2}(\dot\sigma^2 + \sigma'^2)e^{2\varphi + 2F + 2G}, \tag{3.4g}$$

$$(\dot\varphi + \dot F)' = \tfrac{1}{2}\dot G G' + \tfrac{1}{2}\dot q q' + 6\dot\beta\beta' + F'(\dot\varphi + \dot F) + \dot F(\varphi' + F') + \tfrac{1}{2}e^{2\varphi + 2F + 2G}\dot\sigma\sigma'. \tag{3.4h}$$

These equations reduce to those in Refs. 20 and 21 in the appropriate limits.

IV. SEPARABLE G_2 STRING COSMOLOGIES

A. General case

Let us assume separability of the metric functions of the form

$$F(t,z) \equiv F(t) + f(z), \quad G(t,z) \equiv G(t) + g(z), \quad q(t,z) \equiv q(t) + v(z),$$

and appropriate separability conditions on the matter fields $\Phi(t,z)$, $\beta(t,z)$, $\sigma(t,z)$. Then the Ricci scalar is given by

$$R = \tfrac{1}{2}e^{-2F - 2f}[4\ddot G + 4\ddot F + 3\dot G^2 + \dot q^2 - (4g'' + 4f'' + 3g'^2 + v'^2)].$$

If

$$4g'' + 4f'' + 3g'^2 + v'^2 = C, \tag{4.1}$$

where C is a constant, then we obtain a condition which constrains the spatial dependence of the metric. The Ricci scalar is then given by

$$R = \tfrac{1}{2}e^{-2F - 2f}[4\ddot G + 4\ddot F + 3\dot G^2 + \dot q^2 - C].$$

Putting this expression for the Ricci scalar into the action (2.1), the spatial dependence of the geometrical terms can be eliminated (by integration over the spatial coordinates in the action). After applying any further separability conditions (on the matter fields), the resulting FE will be a system of ODEs. Note that the effect of the spatial dependence is to add a further contribution (C) to the cosmological constant Λ in the action.

B. Specific example: Linear dependence in z

In an attempt to remove the z dependence, let us assume separability of the form

$$F(t,z) \equiv F(t) + \tfrac{1}{2}cz, \quad G(t,z) \equiv G(t), \quad q(t,z) \equiv q(t) + az,$$

$$\Phi(t,z) \equiv \Phi(t) + mz, \quad \beta(t,z) \equiv \beta(t) + nz,$$

$$\sigma(t,z) \equiv \sigma(t) + lz,$$

where a, c, l, m, n are constants [a^2 is equivalent to the constant C in Eq. (4.1)], and therefore

$$\varphi(t,z) = \Phi(t) - F(t) - G(t) + (m - \tfrac{1}{2}c)z \equiv \varphi(t) + (m - \tfrac{1}{2}c)z.$$

With the above-given assumptions, the metric becomes an extension of the inhomogeneous scalar-field G_2 solutions found by Feinstein and Ibanez[24] to M-theoretical models. In addition, for particular values of the parameters, the metric reduces to spatially homogeneous Bianchi I, III, and VI_0 models. Hence,

$$\ddot{\varphi} = \tfrac{1}{2}(\dot{\varphi}+\dot{F})^2 + \tfrac{1}{4}\dot{G}^2 + \tfrac{1}{4}\dot{q}^2 + 3\dot{\beta}^2 - \tfrac{1}{4}e^{2\varphi+2F+2G+2mz}(\dot{\sigma}^2 - l^2) - \Lambda e^{2F+cz} - \tfrac{1}{4}(a^2+2m^2+12n^2),$$
(4.2a)

$$\ddot{\beta} = (\dot{\varphi}+\dot{F})\dot{\beta} + \tfrac{1}{4}Q^2 e^{[\varphi+3F+G-6\beta+(c+m-6n)z]} - mn,$$
(4.2b)

$$\ddot{\sigma} = -(\dot{\varphi}+\dot{F}+2\dot{G})\dot{\sigma} + ml,$$
(4.2c)

$$\ddot{q} = (\dot{\varphi}+\dot{F})\dot{q} - ma,$$
(4.2d)

$$\ddot{F} = \tfrac{1}{2}(\dot{\varphi}+\dot{F})^2 - \tfrac{1}{4}\dot{G}^2 - \tfrac{1}{4}\dot{q}^2 - 3\dot{\beta}^2 + \tfrac{1}{4}e^{2\varphi+2F+2G+2mz}(\dot{\sigma}^2 - l) - \Lambda e^{2F+cz}$$
$$- \tfrac{1}{2}Q^2 e^{[\varphi+3F+G-6\beta+(c+m-6n)z]} - \Lambda_M e^{\varphi+3F+G+(c+m)z} + \tfrac{1}{4}(a^2-2m^2+12n^2),$$
(4.2e)

$$\ddot{G} = (\dot{\varphi}+\dot{F})\dot{G} + (\dot{\sigma}^2 - l^2)e^{2\varphi+2F+2G+2mz} - \tfrac{1}{2}Q^2 e^{[\varphi+3F+G-6\beta+(c+m-6n)z]} - \Lambda_M e^{\varphi+3F+G+(c+m)z},$$
(4.2f)

$$(\ddot{F}+\ddot{\varphi}) = 2(\dot{F}+\dot{\varphi}+m)(\dot{F}+\tfrac{1}{2}c) + \tfrac{1}{2}\dot{G}^2 + \tfrac{1}{2}(\dot{q}+a)^2 + 6(\dot{\beta}+n)^2 + \tfrac{1}{2}(\dot{\sigma}+l)^2 e^{2\varphi+2F+2G+2mz}.$$
(4.2g)

Note that the constraint equation can be rewritten as

$$0 = (\dot{F}+m)^2 - \dot{\varphi}^2 + c(\dot{F}+\dot{\varphi}+m) + \tfrac{1}{2}\dot{G}^2 + \tfrac{1}{2}(\dot{q}+a)^2 + 6(\dot{\beta}+n)^2 + \tfrac{1}{2}(\dot{\sigma}+l)^2 e^{2\varphi+2F+2G+2mz}$$
$$+ \Lambda e^{2F+cz} + \tfrac{1}{2}Q^2 e^{[\varphi+3F+G-6\beta+(c+m-6n)z]} + \Lambda_M e^{\varphi+3F+G+(c+m)z}.$$
(4.3)

In order for the FE to be independent of z, it is necessary that $m=0$ and that either $c=0$ or $c=6n$. Furthermore if $c=0$ then we have that either $n=0$ or $Q=0$. In the $c=6n$ case, we have that $\Lambda = \Lambda_M = 0$. It is the latter case that is of relevance to us here. From here forth we shall assume that $c=6n$ and that $\Lambda = \Lambda_M = 0$. This particular subcase, which is of relevence in M-theory cosmology, is of special physical interest. (The resulting FE in the remaining cases are displayed in Ref. 23.)

V. INHOMOGENEOUS M-THEORY COSMOLOGICAL MODELS

Substituting $c=6n$, $m=0$, $\Lambda=0$, $\Lambda_M=0$ into (3.4), (4.3) (and taking the linear combination $[(4.2g)-(4.2e)-(4.2a)]$) we obtain the following system of ODE with two constraints:

$$\ddot{\varphi} = \tfrac{1}{4}(\dot{q}^2 + \dot{G}^2 - a^2 + 2(\dot{\varphi}+\dot{F})^2 + 12\dot{\beta}^2 - 12n^2 + e^{2\varphi+2F+2G}(l^2 - \dot{\sigma}^2)),$$
(5.1a)

$$\ddot{\beta} = \dot{\beta}(\dot{\varphi}+\dot{F}) + \tfrac{1}{4}Q^2 e^{-6\beta+\varphi+3F+G},$$
(5.1b)

$$\ddot{\sigma} = -\dot{\sigma}(\dot{\varphi}+\dot{F}+2\dot{G}),$$
(5.1c)

$$\ddot{q} = \dot{q}(\dot{\varphi}+\dot{F}),$$
(5.1d)

$$\ddot{F} = \tfrac{1}{4}(3a^2 + \dot{q}^2 + \dot{G}^2 + 36n^2 + 12\dot{\beta}^2 + e^{2\varphi+2F+2G}(l^2 + 3\dot{\sigma}^2) - 2(\dot{\varphi}-\dot{F})^2 + 8\dot{F}^2),$$
(5.1e)

$$\ddot{G} = \dot{G}(\dot{\varphi}+\dot{F}) - e^{2\varphi+2F+2G}(l^2 - \dot{\sigma}^2) - \tfrac{1}{2}Q^2 e^{-6\beta+\varphi+3F+G},$$
(5.1f)

$$0 = 2\dot{\varphi}^2 - \dot{G}^2 - \dot{q}^2 - 12\dot{\beta}^2 - 2\dot{F}^2 - 12n^2 - a^2 - e^{2\varphi + 2F + 2G}(l^2 + \dot{\sigma}^2) - Q^2 e^{-6\beta + \varphi + 3F + G},$$

$$\tag{5.1g}$$

$$0 = a\dot{q} + le^{2\varphi + 2F + 2G}\dot{\sigma} + 6n(\dot{\varphi} + \dot{F} + 2\dot{\beta}). \tag{5.1h}$$

From Eq. (5.1g) we are able solve for and make a global substitution for the quantity $Q^2 e^{-6\beta + \varphi + 3F + G}$. Making this substitution we have the following system of ODEs:

$$\ddot{\varphi} = \tfrac{1}{4}(\dot{q}^2 + \dot{G}^2 - a^2 + 2(\dot{\varphi} + \dot{F})^2 + 12\dot{\beta}^2 - 12n^2 + e^{2\varphi + 2F + 2G}(l^2 - \dot{\sigma}^2)), \tag{5.2a}$$

$$\ddot{\beta} = \tfrac{1}{4}(2\dot{\varphi}^2 - \dot{G}^2 - \dot{q}^2 - 12\dot{\beta}^2 - 2\dot{F}^2 - 12n^2 - a^2 - e^{2\varphi + 2F + 2G}(l^2 + \dot{\sigma}^2)) + \dot{\beta}(\dot{\varphi} + \dot{F}), \tag{5.2b}$$

$$\ddot{\sigma} = -\dot{\sigma}(\dot{\varphi} + \dot{F} + 2\dot{G}), \tag{5.2c}$$

$$\ddot{q} = \dot{q}(\dot{\varphi} + \dot{F}), \tag{5.2d}$$

$$\ddot{F} = \tfrac{1}{4}(3a^2 + \dot{q}^2 + \dot{G}^2 + 36n^2 + 12\dot{\beta}^2 + e^{2\varphi + 2F + 2G}(l^2 + 3\dot{\sigma}^2) - 2(\dot{\varphi} - \dot{F})^2 + 8\dot{F}^2), \tag{5.2e}$$

$$\ddot{G} = \dot{G}(\dot{\varphi} + \dot{F}) - e^{2\varphi + 2F + 2G}(l^2 - \dot{\sigma}^2) - \tfrac{1}{2}(2\dot{\varphi}^2 - \dot{G}^2 - \dot{q}^2 - 12\dot{\beta}^2 - 2\dot{F}^2 - 12n^2 - a^2$$
$$- e^{2\varphi + 2F + 2G}(l^2 + \dot{\sigma}^2)), \tag{5.2f}$$

$$0 = a\dot{q} + le^{2\varphi + 2F + 2G}\dot{\sigma} + 6n(\dot{\varphi} + \dot{F} + 2\dot{\beta}). \tag{5.2g}$$

From the constraint (5.1g) we see that if $\dot{\varphi} = 0$, then all of the other state variables must be simultaneously zero, which can only occur at an equilibrium point of the system. Hence $\dot{\varphi}$ must be positive (or negative) throughout the physical phase space. Here we shall assume $\dot{\varphi} > 0$ (the case $\dot{\varphi} < 0$ can be obtained by a time reversal—see the following).

We define new variables of the form

$$\widetilde{F} = \frac{\dot{F}}{\dot{\varphi}}, \quad \widetilde{G} = \frac{1}{\sqrt{2}}\frac{\dot{G}}{\dot{\varphi}}, \quad \widetilde{q} = \frac{1}{\sqrt{2}}\frac{\dot{q}}{\dot{\varphi}}, \quad \widetilde{\sigma} = \frac{1}{\sqrt{2}}e^{\varphi + F + G}\frac{\dot{\sigma}}{\dot{\varphi}},$$

$$\widetilde{\beta} = \sqrt{6}\frac{\dot{\beta}}{\dot{\varphi}}, \quad \Psi_1 = \sqrt{\frac{a^2 + 12n^2}{2}}\frac{1}{\dot{\varphi}}, \quad \Psi_2 = \frac{1}{\sqrt{2}}e^{\varphi + F + G}\frac{l}{\dot{\varphi}}$$

and a new time variable

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\dot{\varphi}}. \tag{5.3}$$

The variables are chosen so that the transformed dynamical system has a compactified phase space. This property comes from the fact that $Q^2 e^{-6\beta + \varphi + 3F + G} \geqslant 0$ which implies that Eq. (5.1g) yields

$$1 \geqslant \widetilde{F}^2 + \widetilde{G}^2 + \widetilde{q}^2 + \widetilde{\sigma}^2 + \widetilde{\beta}^2 + \Psi_1^2 + \Psi_2^2. \tag{5.4}$$

The dynamical system (5.2) becomes

$$\frac{\mathrm{d}\widetilde{F}}{\mathrm{d}\tau} = \widetilde{F}(2\widetilde{F} - \widetilde{r}) + \frac{1}{2}(\widetilde{q}^2 + \widetilde{G}^2 + \widetilde{\beta}^2 + 3\widetilde{\sigma}^2 + 3\Psi_1^2 + \Psi_2^2 - (1 - \widetilde{F})^2), \tag{5.5a}$$

$$\frac{d\tilde{G}}{d\tau} = \tilde{G}(1+\tilde{F}-\tilde{r}) - \frac{1}{\sqrt{2}}(1-\tilde{F}^2-\tilde{G}^2-\tilde{q}^2-3\tilde{\sigma}^2-\tilde{\beta}^2-\tilde{\Psi}_1^2+\tilde{\Psi}_2^2), \tag{5.5b}$$

$$\frac{d\tilde{q}}{d\tau} = \tilde{q}(1+\tilde{F}-\tilde{r}), \tag{5.5c}$$

$$\frac{d\tilde{\sigma}}{d\tau} = -\tilde{\sigma}(\sqrt{2}\tilde{G}+\tilde{r}), \tag{5.5d}$$

$$\frac{d\tilde{\beta}}{d\tau} = \tilde{\beta}(1+\tilde{F}-\tilde{r}) + \frac{\sqrt{6}}{2}(1-\tilde{F}^2-\tilde{G}^2-\tilde{q}^2-\tilde{\sigma}^2-\tilde{\beta}^2-\tilde{\Psi}_1^2-\tilde{\Psi}_2^2), \tag{5.5e}$$

$$\frac{d\tilde{\Psi}_1}{d\tau} = -\tilde{\Psi}_1\tilde{r}, \tag{5.5f}$$

$$\frac{d\tilde{\Psi}_2}{d\tau} = \tilde{\Psi}_2(1+\tilde{F}+\sqrt{2}\tilde{G}-\tilde{r}), \tag{5.5g}$$

and

$$\tilde{r} = \tfrac{1}{2}[(1+\tilde{F})^2+\tilde{G}^2+\tilde{q}^2+\tilde{\beta}^2+\tilde{\Psi}_2^2-\tilde{\Psi}_1^2-\tilde{\sigma}^2],$$

where the constraint equation becomes

$$0 = \sqrt{\frac{2}{a^2+12n^2}}\tilde{\Psi}_1\left[\sqrt{2}a\tilde{q}+6n\left(1+\tilde{F}+\frac{2}{\sqrt{6}}\tilde{\beta}\right)\right]+2\tilde{\Psi}_2\tilde{\sigma}. \tag{5.6}$$

There exists a first integral in the physical phase space $(\tilde{q}\neq0,\tilde{\Psi}_1\neq0)$ for this system. The function

$$M = \frac{\tilde{\sigma}\tilde{\Psi}_2}{\tilde{q}\tilde{\Psi}_1}$$

is constant, i.e., $M'=0$. This implies a first integral for the original system of ordinary differential equations (5.2)

$$\dot{q} = C\dot{\sigma}e^{2\varphi+2F+2G},$$

where C is a constant.

A. Invariant sets, monotonic functions

We first recall that the phase space for this dynamical system is the interior and boundary of the compact set given by

$$1 \geqslant \tilde{F}^2+\tilde{G}^2+\tilde{q}^2+\tilde{\sigma}^2+\tilde{\beta}^2+\tilde{\Psi}_1^2+\tilde{\Psi}_2^2. \tag{5.7}$$

Various hyperplanes divide the phase space into a number of different regions, they are $\tilde{q}=0$, $\tilde{\sigma}=0$, $\tilde{\Psi}_1=0$, and $\tilde{\Psi}_2=0$ hyperplanes. We note that $\tilde{\Psi}_1=0$ divides the phase space into two distinct regions $\tilde{\Psi}_1<0$ and $\tilde{\Psi}_1>0$. The dynamics in the invariant set $\tilde{\Psi}_1<0$ is the time reversal of the dynamics in the invariant set $\tilde{\Psi}_1>0$ [see (5.3)].

Consider the function

$$M_1 = \frac{\tilde{\sigma}^2 \tilde{\Psi}_2^2}{\tilde{q}^4}$$

and its derivative

$$\frac{dM_1}{d\tau} = -2M_1(1 + \tilde{F}).$$

We easily see that this function is monotonically decreasing in the invariant set $\tilde{q} \neq 0, \tilde{\sigma} \neq 0,$ $\tilde{\Psi}_2 \neq 0$. Therefore, we can conclude that there are no closed or periodic orbits in the seven-dimensional phase space, except possibly on the lower dimensional boundaries of this seven-dimensional invariant set.

We restrict ourselves now to the invariant set $\tilde{\sigma} = 0$. Consider the function

$$M_2 = \frac{\tilde{\Psi}_1^2}{\tilde{q}^2}$$

and its derivative

$$\frac{dM_2}{d\tau} = -2M_2(1 + \tilde{F}).$$

We easily see that this function is monotonically decreasing in the invariant set $\tilde{q} \neq 0, \tilde{\Psi}_1 \neq 0$. Therefore, we can conclude that there are no closed or periodic orbits in this six-dimensional phase space, except possibly on the lower dimensional boundaries of this six-dimensional invariant set.

In the six-dimensional invariant set $\tilde{\Psi}_2 = 0$, the function

$$M_3 = \frac{\tilde{\Psi}_1^2}{\tilde{q}^2}$$

has the derivative

$$\frac{dM_3}{d\tau} = -2M_3(1 + \tilde{F})$$

which is monotonically decreasing in the set $\tilde{\Psi}_2 = 0, \tilde{q} \neq 0, \tilde{\Psi}_1 \neq 0$. Therefore we conclude that there are no closed or periodic orbits in this six-dimensional invariant set.

In the six-dimensional invariant set $\tilde{q} = 0$, the function

$$M_4 = \frac{\tilde{\sigma}^2 \tilde{\Psi}_2^2}{\tilde{\Psi}_1^4}$$

has the derivative

$$\frac{dM_4}{d\tau} = 2M_4(1 + \tilde{F}),$$

which is monotonically increasing in the set $\tilde{q} = 0, \tilde{\sigma} \neq 0, \tilde{\Psi}_2 \neq 0$. Therefore we can conclude that there are no closed or periodic orbits in this six-dimensional set.

   With the existence of these monotonic functions M_1, M_2, M_3, and M_4, we can conclude that there are no closed or periodic orbits in the physical six-dimensional phase space [except possibly on lower dimensional (less than 5) invariant sets].

   The zero-curvature spatially homogeneous and isotropic space–times are contained in the set $\tilde{q} = \tilde{\Psi}_1 = \tilde{\Psi}_2 = \tilde{G} - \sqrt{2}\tilde{F} = 0$ union $\tilde{q} = \tilde{\Psi}_1 = \tilde{\sigma} = \widetilde{G - \sqrt{2}\tilde{F}} = 0$.

   The matter fields in (2.1) satisfy various energy conditions. For example, the positivity of the kinetic energy of the pseudoaxion scalar field, σ, demands that $\dot{\sigma}^2 - l^2 \geq 0$ (i.e., $\tilde{\sigma}^2 - \tilde{\Psi}_2^2 \geq 0$). However, we note that $\tilde{\sigma}^2 - \tilde{\Psi}_2^2 = 0$ is not an invariant set.

B. Equilibrium points and exact solutions

   There are two equilibrium points and one three-dimensional equilibrium set.
   The three-dimensional equilibrium set is given by

$$\{\tilde{F}^2 + \tilde{G}^2 + \tilde{q}^2 + \tilde{\beta}^2 = 1, \tilde{\sigma} = 0, \tilde{\Psi}_1 = 0, \tilde{\Psi}_2 = 0\}.$$

Note that since both $\tilde{\Psi}_1 = 0$ and $\tilde{\Psi}_2 = 0$ we necessarily have that $a = n = l = 0$. At this point the value of $\tilde{r} = 1 + \tilde{F}_0$. The exact solution is then

$$\varphi(t) = h_1 - \frac{1}{\tilde{r}} \ln(\tilde{r}t + h_0),$$

$$F(t) = F_0 \left(h_1 - \frac{1}{\tilde{r}} \ln(\tilde{r}t + h_0) \right) + F_1,$$

$$G(t) = \sqrt{2} G_0 \left(h_1 - \frac{1}{\tilde{r}} \ln(\tilde{r}t + h_0) \right) + G_1,$$

$$q(t) = \sqrt{2} q_0 \left(h_1 - \frac{1}{\tilde{r}} \ln(\tilde{r}t + h_0) \right) + q_1,$$

$$\sigma(t) = \sigma_1,$$

$$\beta(t) = \frac{1}{\sqrt{6}} \beta_0 \left(h_1 - \frac{1}{\tilde{r}} \ln(\tilde{r}t + h_0) \right) + \beta_1,$$

where $F_0^2 + G_0^2 + q_0^2 + \beta_0^2 = 1$ and where F_1, G_1, q_1, σ_1, β_1, h_1, and h_0 are integration constants.

   Since $a = n = l = 0$, this metric is spatially homogeneous (and flat). Since $\tilde{\sigma} = 0$ and $Q = 0$ (which follows from the other conditions), this equilibrium set represents spatially flat solutions where the form-fields (the axion field and the four-form field strength) are trivial and only the dilaton and moduli fields are dynamically important. These solutions are known as the "dilaton-moduli-vacuum" solutions (and their analytical form is given in Ref. 25).

   Recall that the dynamics of these models is restricted by the constraint given by Eq. (5.6). At these equilibrium points we are able to locally solve for the value of $\tilde{\Psi}_1$ and substitute into the remaining equations. The eigenvalues in the six-dimensional constraint surface are

$$0, 0, 0, \sqrt{2}\tilde{G}, -\sqrt{2}\tilde{G} - 1 - \tilde{F}, -1 + \tilde{F} + \sqrt{2}\tilde{G} - \sqrt{6}\tilde{\beta}.$$

The three zero eigenvalues correspond to the fact that this equilibrium set is three-dimensional. It is clear that a subset of this equilibrium set will act as saddles of varying degree of stability, while

another subset will act as sinks. Consequently, a subset of these dilaton-moduli-vacuum solutions with trivial form fields are sinks in the physical phase space, even in the presence of (negative) spatial curvature, and are thus generic attracting solutions.

A second equilibrium point is

$$\left\{\tilde{F}=-\frac{2}{3},\tilde{G}=0,\tilde{q}=0,\tilde{\beta}=-\frac{1}{\sqrt{6}},\tilde{\sigma}=\pm\frac{\sqrt{2}}{6},\tilde{\Psi}_1=\frac{\sqrt{2}}{3},\tilde{\Psi}_2=0\right\}.$$

Note that since $\tilde{\Psi}_2=0$ we necessarily have that $l=0$. At this point the value of $\tilde{r}=0$. [Note, since the dynamical system (5.5) is invariant under the transformation $(\tilde{\Psi}_1,\tilde{\Psi}_2)\rightarrow(-\tilde{\Psi}_1,-\tilde{\Psi}_2)$ there exists a corresponding equilibrium point with a $\tilde{\Psi}_1=-\sqrt{2}/3$.] The exact solution is then

$$\varphi(t)=h_0 t+h_1, \quad F(t)=-\tfrac{2}{3}(h_0 t+h_1)+F_1, \quad G(t)=G_1,$$

$$q(t)=q_1, \quad \sigma(t)=\mp e^{-1/3(h_0 t+h_1)-(F_1+G_1)}+\sigma_1, \quad \beta(t)=-\tfrac{1}{6}(h_0 t+h_1)+\beta_1,$$

where $h_0=\tfrac{3}{2}\sqrt{a^2+12n^2}$, $Q^2=\tfrac{1}{2}(a^2+12n^2)e^{6\beta_1-3F_1-G_1}$, and F_1, G_1, q_1, σ_1, β_1, h_1 are all constants. In this situation the variable $\tilde{q}$ can be eliminated. The eigenvalues restricted to the constraint surface are

$$\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{6}(1\pm\sqrt{15+8\sqrt{2}i}),\tfrac{1}{6}(1\pm\sqrt{15-8\sqrt{2}i}).$$

This point represents a past attractor or a source. This corresponds to a spatially nonvacuum inhomogeneous model with a diagonal Einstein tensor, having negative curvature.

The line element corresponding to this solution (after a few coordinate redefinitions) is

$$ds^2=C^2 e^{-2\sqrt{a^2+12n^2}t+6nz}(-dt^2+dz^2)+(e^{az}dx^2+e^{-az}dy^2). \tag{5.8}$$

The third equilibrium point is

$$\left\{\tilde{F}=-\frac{5}{7},\tilde{G}=\frac{\sqrt{2}}{7},\tilde{q}=0,\tilde{\beta}=-\frac{\sqrt{6}}{7},\tilde{\sigma}=0,\tilde{\Psi}_1=\frac{2\sqrt{3}}{7},\tilde{\Psi}_2=0\right\}.$$

Note that since $\tilde{\Psi}_2=0$ we necessarily have that $l=0$. At this point the value of $\tilde{r}=0$. [Note, since the dynamical system (5.5) is invariant under the transformation $(\tilde{\Psi}_1,\tilde{\Psi}_2)\rightarrow(-\tilde{\Psi}_1,-\tilde{\Psi}_2)$ there exists a corresponding equilibrium point with a $\tilde{\Psi}_1=-2\sqrt{3}/7$.] The exact solution is then

$$\varphi(t)=h_0 t+h_1, \quad F(t)=-\tfrac{5}{7}(h_0 t+h_1)+F_1, \quad G(t)=\tfrac{2}{7}(h_0 t+h_1)+G_1,$$

$$q(t)=q_1, \quad \sigma(t)=\sigma_1, \quad \beta(t)=-\tfrac{1}{2}(h_0 t+h_1)+\beta_1,$$

where $h_0=(7/2\sqrt{3})\sqrt{a^2+12n^2}$, $Q^2=\tfrac{1}{3}(a^2+12n^2)e^{6\beta_1-3F_1-G_1}$, and F_1, G_1, q_1, σ_1, β_1, h_1 are all constants. This solution is a curved inhomogeneous model with a trivial axion field. In this situation the variable $\tilde{q}$ can be eliminated. The eigenvalues restricted to the constraint surface are

$$\tfrac{2}{7},\tfrac{2}{7},\tfrac{4}{7},-\tfrac{2}{7},\tfrac{1}{7}(1\pm\sqrt{23}i).$$

This point is always a saddle.

VI. DISCUSSION

We have studied several classes of inhomogeneous string models whose governing equations reduce to ODE. In particular, we have found that generically solutions of the class of separable G_2

inhomogeneous M-theory cosmologies studied evolve from a spatially inhomogeneous and negatively curved model with a nontrivial form fields toward (a subset) of spatially flat and spatially homogeneous dilaton-moduli-vacuum solutions where the form-fields (the axion field and the four-form field strength) are trivial and only the dilaton and moduli fields are dynamically important. This late time behavior is the same as that of the spatially homogeneous models studied previously. However, in these models the inhomogeneities are not dynamically insignificant at early times, and the models asymptote (in the past) toward a new class of inhomogeneous cosmological models.

As noted earlier, the time-reversed dynamics of the $\dot{\varphi}>0$ models we have considered thus far is equivalent to the dynamics of the case where $\dot{\varphi}<0$. This follows by redefining the time variable according to $dt/d\tau = -1/\dot{\varphi}$ and appropriate definitions of the other state variables. The evolution equations will have an ''overall'' change in sign, and hence the equilibrium points are identical in both cases, but the eigenvalues have opposite signs. Consequently, the dynamics of the $\dot{\varphi}<0$ models is the time reversal of the $\dot{\varphi}>0$ models and the time-reversed dynamics of the above class of models is deduced by interchanging the sources and sinks and reinterpreting expanding solutions in terms of contracting ones, and vice versa.

Although at late times (in the $\dot{\varphi}>0$ models) the inhomogeneities decay, the inhomogeneities are important at intermediate times and, in particular, at early times. Thus the qualitative features of the models are quite different to those of spatially homogeneous models studied previously. For example, in a study of FRW models[21] it was found that all negatively curved FRW models evolve from the solution corresponding to a global source in which the curvature is (negative and) dynamically important (but with a trivial axion field) toward the dilaton-moduli-vacuum solutions,[25] even in the presence of spatial curvature. The physical interpretation of these models, where both the NS–NS two-form potential and RR three-form potential are dynamically significant, was discussed in Ref. 21, with particular emphasis on the fact that the RR field causes the universe to collapse, but the NS–NS field has the opposite effect, whereby the interplay between these two fields leads to the models undergoing bounces. In the models under investigation here, orbits in the full phase space (with $\dot{\varphi}$ monotone) approach the dilaton-moduli-vacuum solution on the zero-curvature boundary (at late times) and again exhibit a ''bouncing'' behavior; this bouncing behavior is the result of the orbits shadowing orbits in the boundary that are constantly being redirected to saddle points of the same or higher stability until it reaches a stable equilibrium.

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