

**Two-component scaling near the metal-insulator bifurcation in two dimensions**D. J. W. Geldart<sup>1,2,3</sup> and D. Neilson<sup>2,4</sup><sup>1</sup>*Department of Physics, Dalhousie University, Halifax, NS B3H3J5, Canada*<sup>2</sup>*School of Physics, University of New South Wales, Sydney 2052, Australia*<sup>3</sup>*Laboratoire des Verres, Université de Montpellier II, Montpellier 34095, France*<sup>4</sup>*Dipartimento di Fisica, Università di Camerino, 62032 Camerino, Italy*

(Received 11 December 2002; published 12 May 2003)

We consider a two-component scaling picture for the resistivity of two-dimensional (2D) weakly disordered interacting electron systems at low temperature with the aim of describing both the vicinity of the bifurcation and the low resistance metallic regime in the same framework. We contrast the essential features of one-component and two-component scaling theories. We discuss why the conventional lowest order renormalization group equations do not show a bifurcation in 2D, and a semiempirical extension is proposed which does lead to bifurcation. Parameters, including the product  $z\nu$ , are determined by least squares fitting to experimental data. An excellent description is obtained for the temperature and density dependence of the resistance of silicon close to the separatrix. Implications of this two-component scaling picture for a quantum critical point are discussed.

DOI: 10.1103/PhysRevB.67.205309

PACS number(s): 71.10.Ca, 71.30.+h, 73.20.Qt

**I. INTRODUCTION**

The determination of the low-temperature properties, and ultimately the ground-state phase diagram, of two-dimensional (2D) systems of strongly interacting electrons at low carrier density in the presence of low levels of disorder remains a very challenging problem in spite of a great deal of experimental and theoretical effort. The longstanding view that all charge carrier states become localized at zero temperature ( $T=0$ ) in the limit of a large 2D system has been called into question by experimental observations of a finite temperature “transition” from insulatorlike ( $\partial\rho/\partial T < 0$ ) to metal-like ( $\partial\rho/\partial T > 0$ ) behavior as the electron density is increased in very high purity metal-oxide semiconductor field-effect transistors (MOSFET’s) and heterostructures.<sup>1–4</sup> This bifurcation of the resistivity  $\rho(T)$  into two families of curves occurs at a critical carrier density of the 2D electron (or hole) system. The critical carrier density depends on the specific disorder characteristics of the given sample. In the vicinity of the bifurcation,  $\rho(T)$  has intriguing scaling properties as a function of density and temperature. The lower density insulating family of curves having  $d\rho/dT < 0$  can be collapsed onto a single curve when plotted as a function of a scaling variable  $T_0(\delta)/T$ . The new density dependent temperature scale  $T_0(\delta)$  has a power law dependence on the magnitude of  $\delta = (n - n^*)/n^*$ , where  $n^*$  is the critical density at the bifurcation. Similarly, the metallic family of curves at higher density with  $d\rho/dT > 0$  collapses onto another unique curve with a power law dependence on  $T_0(\delta)/T$  with precisely the same critical exponent. There is also scaling behavior with respect to density via  $|\delta|$  and applied electric field, with a different characteristic critical exponent. A review has been given by Abrahams *et al.*<sup>5</sup>

A bifurcation with scaling behavior at finite  $T$  is expected to be a generic feature associated with a  $T=0$  quantum critical point (QCP).<sup>6</sup> Consequently, the demonstrated scaling for the resistivity can be taken as evidence for a metal-insulator transition (MIT). With this assumption, Dobrosavljević *et al.*

have given a phenomenological description of the observed scaling properties and power laws in terms of a one-parameter scaling theory.<sup>7</sup> In this picture, the bifurcation point lies on a separatrix which terminates at  $T=0$  at a QCP which is a fixed point of a set of scaling equations. The fixed point is repulsive with respect to the temperature variable and separates the insulating and metallic regions of the low- $T$  phase diagram. There is a unique correspondence between the QCP and the bifurcation point and a two-phase ground state is implied.

However, this correspondence provides only indirect evidence for an MIT. Of course the power law fits near the bifurcation point are based on linearization of a scaling flow equation and the linearization must eventually fail at sufficiently low temperature, for which  $T_0(\delta)/T$  is no longer small, but this is not evidence against a QCP. A more crucial point is that a physical effect which is negligible for temperatures  $T \sim 1$  K near the bifurcation, may become dominant in the low- $T$  limit. Scaling properties characteristic of an imminent QCP would then still be observed near the bifurcation but the suggested QCP itself could be totally removed in the zero-temperature limit.

A possible example of this situation is the proposal by Simmons *et al.*<sup>8</sup> that in  $p$ -type GaAs the  $\rho(T)$  curves of the metallic family will eventually turn upward to exhibit insulatorlike behavior with  $\partial\rho/\partial T < 0$  if the electron temperature  $T$  could be made sufficiently low. A similar conclusion has recently been reached for Si MOSFET’s.<sup>9</sup> However, the turn up is actually only directly observed for metallic curves just above the critical density and, as we discuss below, the magnetoresistance data cited as evidence of a universal weak localization in 2D in Refs. 8 and 9 may alternatively reflect different magnetic field dependencies of the competing localizing and delocalizing triplet spin state interactions. Irrespective of the mechanism responsible for the upturn, properties of the system at temperatures well below the bifurcation temperature are not correctly described by a one-component scaling theory. We will outline a semiempirical two-

component scaling theory that describes within a single framework the temperature-dependent resistivity both in the vicinity of the bifurcation and in the low resistance metallic regime.

Following early work by Altshuler *et al.*<sup>10</sup> further progress toward an understanding of disordered 2D electron systems in the strong coupling regime was made by Finkelstein<sup>11,12</sup> and by Castellani *et al.*<sup>13</sup> who showed that electron-electron interactions can lead to  $d\rho/dT > 0$  (metallic behavior) in the disordered 2D system at finite  $T$ . This is in contrast to the localizing insulating behavior found in the absence of electron-electron interactions.<sup>14</sup> The physical properties of the system at low  $T$  are determined by nonlinear interactions of low-energy diffusive modes. A set of renormalization group (RG) equations was generated by means of a formal perturbation expansion in powers of a dimensionless temperature-dependent resistivity  $\mathcal{R}$  which is related to the physical resistance per square by  $\mathcal{R} = (e^2/\pi h)R_{\square}$ . These scaling results include electron-electron interactions to all orders and are valid in the diffusive regime  $k_B T < \hbar/\tau$  provided  $\mathcal{R}$  is small (in principle,  $\mathcal{R} \ll 1$ ). Additional progress was made by Zala *et al.*<sup>15</sup> who considered interaction corrections to transport properties at intermediate temperatures in the ballistic regime  $k_B T > \hbar/\tau$  to all orders in the electron-electron interaction using a Fermi-liquid approach. These results are also restricted to the perturbative regime  $\mathcal{R} \ll 1$ . Comparisons with experiment for the temperature dependent resistivity of Si MOSFET's have been made for both the scaling theory<sup>16</sup> and the intermediate temperature theory.<sup>17,18</sup> The intermediate temperature theory gives consistent semiquantitative agreement with experiment in the range of densities for which  $\mathcal{R}$  is small.

We emphasize that both these theories are perturbative in  $\mathcal{R}$  and are limited to the regime of small  $\mathcal{R}$ . Hence they must fail to describe the regime near bifurcation where  $\mathcal{R} \sim 1$  and they contain no bifurcation point. In addition, in the RG procedure the equations develop singularities at a nonzero temperature so that the approach to the ground state cannot be described.

In contrast to the 2D case there is a qualitative understanding of the metal-insulator transition in 3D electron systems. A consistent qualitative picture of the transition has been obtained by RG methods in a space of dimension  $2 + \epsilon$  with  $\epsilon > 0$  taken as an expansion parameter. Physical quantities such as critical exponents are expressed as power series in  $\epsilon$ . While only the leading terms in  $\epsilon$  have been obtained it is presumed that resummation methods would give good results in 3D if sufficient correction terms could be calculated. In the case of  $\epsilon > 0$  it is clear that the interplay of strong electron-electron interactions with disorder is an essential aspect of the bifurcation.

It is remarkable that in spite of great effort a corresponding theory does not exist for 2D systems, that is for  $\epsilon = 0$ . This lack casts doubt on the physical significance and relevance of the ‘‘universal’’ aspects of electron-electron interactions in disordered systems. In addition the lack of even an approximate scaling theory has prevented quantitative discussion of the low-temperature limit and the approach to the ground state. We expect that the interplay of strong electron-

electron interactions with disorder must be an essential aspect of the bifurcation in 2D irrespective of whether or not a true metal-insulator transition occurs. Of course in some systems material dependent effects also contribute.

In this paper we discuss why the lowest order RG equations in 2D do not show a bifurcation and we give a semi-empirical extension which does describe a bifurcation region. We focus on the immediate vicinity of the bifurcation and on the metallic regime. The strong insulator limit will be discussed elsewhere. The proposed scaling equations have a physical low-temperature limit and have no singularities at finite length or temperature scales. This scaling picture provides a theoretical framework for the interpretation of experimental results, more specifically the ‘‘universal’’ contributions due to electron-electron interactions and disorder.

## II. RENORMALIZATION GROUP EQUATIONS

The RG equations established at one-loop level for a 2D system of electrons in the presence of disorder are based on four dimensionless scaling parameters  $\mathcal{R}$ ,  $\gamma_2$ ,  $Z$ , and  $\gamma_c$ ,<sup>11-13</sup> where  $\Gamma_2 = Z\gamma_2$  is the electron-hole scattering amplitude for the triplet spin state,  $\Gamma_c = Z\gamma_c$  is the singlet state particle-particle scattering amplitude and  $Z$  is the dynamical energy rescaling function. Together with  $\mathcal{R}$ , these quantities are all functions of the variable  $y = \ln \lambda^{-1}$  which describes rescaling of the momenta after integrating over the momentum shell specified by  $\lambda k_0^2 < k^2 < k_0^2$ .<sup>19</sup> We consider only the case when the disorder is due to purely potential scattering. When a connection between length and temperature scaling is needed, we relate a thermal length  $l_{th}$  to temperature by  $l_{th}/l_{el} = (T_{el}/T)^{1/z}$ , where  $l_{el}$  and  $T_{el}$  are the length and temperature scales for elastic scattering and  $z$  is the dynamical critical exponent.<sup>6</sup>

Particle-particle scattering is not considered to play an essential role relative to the electron-hole scattering represented by  $\gamma_2$  so we omit consideration of  $\gamma_c$ . The energy scaling function  $Z(y)$  is not followed explicitly but its effect is taken into account at the bifurcation fixed point by  $y = \ln[(T_{el}/T)^{1/z}]$ . These approximations permit discussion of a two-component scaling theory based on the scaling parameters  $\mathcal{R}$  and  $\gamma_2$ . With these simplifications the RG equations are

$$d\mathcal{R}/dy = \alpha(\gamma_2)\mathcal{R}^2, \quad (1)$$

$$d\gamma_2/dy = \frac{(1 + \gamma_2)^2}{2}\mathcal{R}, \quad (2)$$

where

$$\alpha(\gamma_2) = 1 + \left[ n_v + [(2n_v)^2 - 1] \left\{ 1 - \frac{1 + \gamma_2}{\gamma_2} \ln(1 + \gamma_2) \right\} \right]. \quad (3)$$

$n_v$  is the number of valleys, so for Si  $n_v = 2$ . Equations (1) and (2) are to be integrated upward in  $y$ , starting from initial bare values for the dependent variables at  $y = y(0)$  for which  $T_{el}/T$  is of order unity. Increasing  $y$  corresponds to integrating out shorter wavelength and higher-energy excitations.

This amounts to increasing the length scale and decreasing the temperature scale. Our Eqs. (1)–(3) are essentially the same as Eqs. (1) and (2) of Ref. 16. The notational difference is that we use a running variable  $y = -\ln(\lambda)$  as in Ref. 19, while Ref. 16 uses  $\xi = -\ln(k_B T \tau / \hbar)$ . Prior to renormalization due to interacting diffusive modes, we can set  $Z(y) = 1$ . After the RG flow has proceeded to the bifurcation fixed point, we set  $y = \ln[(T_{c1}/T)^{1/z}]$ , where  $z$  is the dynamical critical exponent of the fixed point.

The first term (unity) on the right-hand side of Eq. (3) contains no electron-electron interactions and arises from weak localization. On scaling this term enhances the resistivity. The other term, in the square brackets, is due to electron-electron interactions in the singlet and triplet particle-hole spin states and can be negative when  $\gamma_2$  is large enough. When it is negative this term has the opposite trend of reducing the resistivity upon scaling. The bare value of  $\gamma_2$  is given by  $\gamma_2^{\text{bare}} = -F_0^a / (1 + F_0^a)$ , where  $F_0^a < 0$  is the spin-antisymmetric Landau parameter. From Eq. (2),  $\gamma_2$  increases with rescaling. With  $n_p = 2$ , the expression for  $\alpha(\gamma_2)$  changes sign when  $\gamma_2$  reaches 0.46.

A change in sign of  $\alpha(\gamma_2)$ , corresponding to a net delocalizing effect in zero external magnetic field, has no particular *a priori* implication for magnetoresistance. The magnetoresistance data of Refs. 8,9 can be consistent with our scaling conclusions if, for example, the localizing contribution in Eq. (3) has a stronger magnetic field dependence (at small fields) than the delocalizing contribution from the electron-electron interactions. Thus the appearance of a weak localization precursor signature in magnetoresistance data does not necessarily imply there will be an eventual turn up in the zero field resistivity at very low temperatures.

As we integrate Eq. (2),  $\gamma_2$  increases from its initial bare value but diverges at a finite value of  $y = y_{\text{max}}$  provided  $\mathcal{R}$  remains finite at  $y = y_{\text{max}}$ .<sup>11–13</sup> It is easy to confirm that a solution of Eqs. (1) and (2) with finite  $\mathcal{R} = \mathcal{R}_{\text{min}}$  and  $\gamma_2$  arbitrarily large is consistent. In this limit Eq. (2) can be written

$$d\gamma_2/dy = (\mathcal{R}_{\text{min}}/2)\gamma_2^2. \quad (4)$$

It follows that  $\gamma_2$  diverges at a finite  $y_{\text{max}}$ . To verify the consistency of a finite  $\mathcal{R}_{\text{min}}$  when  $\gamma_2$  diverges, we can divide Eq. (1) by (2) and rearrange obtaining

$$d\mathcal{R}/\mathcal{R} = -d\gamma_2(6 \ln \gamma_2)/\gamma_2^2. \quad (5)$$

The integral of the left hand side is finite so it follows that  $\mathcal{R}$  indeed reaches a finite lower limit  $\mathcal{R}_{\text{min}}$  as  $\gamma_2 \rightarrow \infty$ . The rescaling cannot be continued beyond the singularity at  $y = y_{\text{max}}$  which means that the zero-temperature limit cannot be reached.

In principle, this singularity in the triplet state scattering amplitude might signal the onset of a magnetic instability in the system. On the other hand, the divergence might simply be an artifact of a low order perturbation expansion. The ultimate fate of this singularity at very low  $T$  is not completely clear. Castellani *et al.*<sup>19</sup> showed that a similar divergence in the energy rescaling function  $Z(y)$  can cause the singular point to shift to extremely low  $T$ . The paramagnetic

metallic regime would then extend over a wide temperature. More recently, Kirkpatrick and Belitz<sup>20</sup> and Chamon and Mucciolo<sup>21</sup> have indeed found a solution of the RG equations corresponding to a disordered ferromagnet. The question of how low the temperatures would be where such transitions might occur is open. In the absence of theoretical guidance on this point we turn to experiment for information on the triplet spin state scattering amplitude  $\gamma_2$  in the temperature range of the bifurcation.

Reference 22 concluded from an analysis of magnetoresistance data that there would be a ferromagnetic instability very near the density of the bifurcation. However, measurements in Refs. 23,24 found an enhancement of the effective  $g^*$ , but no singularity for densities down to  $r_s = 8.4$ , a range which includes the bifurcation. This is interpreted in terms of the spin-antisymmetric Landau parameter as  $g^* = 2/(1 + F_0^a)$ .  $F_0^a$  would be  $-1$  at the onset of a ferromagnetic instability. For  $r_s = 8$  Ref. 24 gives  $F_0^a \approx -0.5$ , corresponding to a spin susceptibility enhancement factor of 2. At temperatures well above the bifurcation the bare values of the electron-electron interaction amplitudes such as  $\gamma_2^{\text{bare}}$  have negligible diffusion corrections.  $\gamma_2^{\text{bare}}$  is then related to  $F_0^a$  by  $\gamma_2^{\text{bare}} = -F_0^a / (1 + F_0^a)$ . A value  $F_0^a = -0.5$  corresponds to  $\gamma_2^{\text{bare}} = 1$ . On the basis of the direct measurement of  $g^*$  we will assume that the triplet spin state scattering amplitude  $\gamma_2$  is finite and well behaved throughout the density and temperature range of the experiments we consider. With this assumption, an explicit RG equation is not needed for  $\gamma_2$ . Nonmagnetic Fermi liquid behavior of the system is a sufficient condition for a smooth  $\gamma_2$  but may not be necessary.

### III. BIFURCATION IN 2D

To discuss why Eqs. (1) and (2) fail to describe a bifurcation in 2D electron systems, we first recall the corresponding RG results in  $d = 2 + \epsilon$  dimensions with  $\epsilon$  small and positive. Making the same physical assumptions regarding the interacting diffusive modes and the same one-loop approximation, Eq. (1) becomes

$$d\mathcal{R}/dy = -(\epsilon/2)\mathcal{R} + \alpha(\gamma_2)\mathcal{R}^2. \quad (6)$$

The first term on the right hand side of Eq. (6) is the consequence of the  $L^{2-d}$  form factor when converting resistivity to resistance in a space of  $d$  dimensions, and the coefficient  $\alpha(\gamma_2)$  in the second term is the same as in Eq. (3).

We identify the bifurcation point as the point at which  $d\mathcal{R}/dy = 0$ . The zero of Eq. (6) occurs at the critical value  $\mathcal{R}^* = \epsilon / (2\alpha)$ , provided  $\alpha > 0$ , where  $\alpha$  is the value of  $\alpha(\gamma_2)$  when the bifurcation occurs. In order for the bifurcation to be a precursor for a quantum critical point, the temperature must be a relevant variable. This requires that  $\tau^{-1} > 0$  in the linearized flow equation

$$d(\mathcal{R} - \mathcal{R}^*)/dy = \tau^{-1}(\mathcal{R} - \mathcal{R}^*). \quad (7)$$

This procedure for identifying a critical point by a linearized flow equation is standard. From Eq. (6),  $\tau^{-1} = \epsilon/2 > 0$ . The



condition that  $\tau^{-1} > 0$  implies that the resistance has positive (negative) slope with respect to  $T$  in the metallic (insulating) regime.

It is clear that a nontrivial root of  $d\mathcal{R}/dy = 0$  requires at least two terms and that for the 2D case Eq. (1) as presented [that is, Eq. (6) with  $\epsilon = 0$ ] will not be sufficient. Attempts have been made to discuss possible metal-insulator transitions in 2D using Eq. (1) by tuning the parameter  $\alpha(\gamma_2)$  to zero. However, a critical point is a robust property of the entire  $d\mathcal{R}/dy$  and cannot be described by the properties of only a single term  $\alpha(\gamma_2)\mathcal{R}^2$  in a series. Such a procedure is not stable to the addition of higher order terms. The simplest modification of Eq. (1) that can show a bifurcation and is consistent with two-component scaling is

$$d\mathcal{R}/dy = \alpha(\gamma_2)\mathcal{R}^2 + \beta(\gamma_2)\mathcal{R}^3 + \dots \quad (8)$$

The function  $\beta(\gamma_2)$  is not known explicitly but its sign can be determined by the conditions that  $\mathcal{R}^*$  and  $\tau^{-1}$  are both positive. Linearizing Eq. (8) about the zero of its right hand side gives the linearized flow equation, Eq. (7), with  $\mathcal{R}^* = -\alpha/\beta$  and exponent  $\tau = \beta/\alpha^2$ . Here  $\alpha$  and  $\beta$  are the values of  $\alpha(\gamma_2)$  and  $\beta(\gamma_2)$  when the bifurcation occurs.  $\mathcal{R}^*$  and  $\tau^{-1}$  are both positive provided  $\alpha < 0$  and  $\beta > 0$ . There is no bifurcation in 2D if  $\alpha(\gamma_2)$  is always positive. Integrating Eq. (7) starting from an initial  $\mathcal{R}_0$  at  $y = y_0$  gives two families of curves. A metallic regime and a bifurcation can thus be described.

We conclude that the bifurcations in the 2D system and the  $2 + \epsilon$  system are controlled by different fixed points. The fixed point  $\mathcal{R}^* = \epsilon/\alpha$  for  $2 + \epsilon$  requires  $\alpha > 0$  which is in the range of weak electron-electron interactions. This fixed point becomes trivial ( $\mathcal{R}^* = 0$ ) in the  $\epsilon \rightarrow 0$  limit and plays no role in 2D. This allows a new fixed point  $\mathcal{R}^* = -\alpha/\beta$  to become physical in 2D at a scale where electron-electron interactions have become strong enough to change the sign of  $\alpha$ . In each case there is only one fixed point and the physical picture of interacting diffusive modes is correct for the determination of the universal contributions.

The strength of the electron-electron interactions in 2D is crucial for generating a bifurcation. At high densities where the electron-electron interactions are weak,  $\gamma_2^{\text{bare}}$  is small and  $\alpha(\gamma_2^{\text{bare}})$  is positive. As the density is lowered the initial  $\gamma_2^{\text{bare}}$  increases. Using the data of Ref. 24 for  $g^*$  in Si, the sign change in  $\alpha(\gamma_2^{\text{bare}})$  occurs for a density corresponding to  $r_s \approx 3$ . This provides an upper limit to the density at which a bifurcation can occur in Si. For  $r_s < 3$  the electron-electron interactions are too weak.

In order to describe quantitatively the resistivity near the bifurcation where  $R \sim 1$ , as well as in the insulating regime where  $R \gg 1$ , the sum of the series implied in Eq. (8) must be adequately represented. If the series is truncated at an arbitrary finite order a spurious divergence at a finite  $y_{\text{max}}$  can occur in the insulating region. While in the insulating limit  $\mathcal{R}$  and  $d\mathcal{R}/dy$  are expected to diverge as  $y \rightarrow \infty$  (that is, at  $T \rightarrow 0$ ), the divergence in  $d\mathcal{R}/dy$  must be sufficiently weak that a spurious divergence in  $\mathcal{R}$  at a finite  $y_{\text{max}}$  does not occur. A linear power law in  $\mathcal{R}$  is the strongest growth of  $d\mathcal{R}/dy$  at large  $\mathcal{R}$  that permits this (with possible logarithmic

corrections). For simplicity, we maintain the fixed point structure of the low order terms and introduce a denominator into Eq. (8) to represent the net effect of higher order terms including the linear growth at large  $\mathcal{R}$ ,

$$\frac{d\mathcal{R}}{dy} = \frac{\alpha(\gamma_2)\mathcal{R}^2 + \beta(\gamma_2)\mathcal{R}^3}{1 + \kappa(\gamma_2)\mathcal{R}^2}. \quad (9)$$

Equation (9) satisfies the minimal conditions of having a bifurcation with two classes of well defined solutions (metallic and insulating) depending on the choice of initial conditions at  $y = y(0)$ . It may be regarded as a semi-empirical Padé approximation to the full series. Of course, these minimal conditions do not uniquely determine the functional form (see also Sec. IV B).

We focus in this paper on the metallic regime and the close vicinity of the separatrix. The known perturbative results are contained explicitly in Eq. (9) so the solution for  $y \rightarrow \infty$  and  $\mathcal{R} \rightarrow 0$  is exact. The strongly insulating limit  $\mathcal{R} \rightarrow \infty$  contains additional logarithmic corrections so Eq. (9) is incomplete in this limit, and a detailed discussion of the deep insulating regime will be given elsewhere.

Close to the separatrix the functions  $\alpha(\gamma_2)$ ,  $\beta(\gamma_2)$ , and  $\kappa(\gamma_2)$  are taken to be slowly varying and so are replaced by their constant values  $\alpha$ ,  $\beta$ , and  $\kappa$  for  $\gamma_2$  near the start of the bifurcation. Information on these parameters is given by fitting to experimental data in the following section. Equation (9) can then be rewritten in the form

$$\frac{1}{\mathcal{R}} \frac{d\mathcal{R}}{dy} = \frac{1}{\tau} \frac{\Delta + \Delta^2}{1 + \phi(2\Delta + \Delta^2)}, \quad (10)$$

where

$$\mathcal{R}^* = -\alpha/\beta,$$

$$\mathcal{R}/\mathcal{R}^* = 1 + \Delta,$$

$$\tau^{-1} = \beta\mathcal{R}^{*2}/(1 + \kappa\mathcal{R}^{*2}),$$

$$\phi = \kappa\mathcal{R}^{*2}/(1 + \kappa\mathcal{R}^{*2}). \quad (11)$$

Linearizing Eq. (10) in  $\Delta$  we recover Eq. (7), with the solution

$$\ln\{|\mathcal{R} - \mathcal{R}^*|/|\mathcal{R}(0) - \mathcal{R}^*|\} = \tau^{-1}[y - y(0)]. \quad (12)$$

Since  $\tau$  describes the rescaling of an inverse length squared, it is related to the critical exponent  $\nu$  of the correlation length by  $\tau = 2\nu$ . The temperature is introduced by the thermal length  $l_{\text{th}} \sim 1/T^{1/z}$  giving

$$|\mathcal{R} - \mathcal{R}^*| = |\mathcal{R}(0) - \mathcal{R}^*|(T_0/T)^{1/z\nu}, \quad (13)$$

with the same exponent for both the metallic and the insulating branches. The prefactor defines a temperature scale  $T_0(\delta) \sim |\delta|^{z\nu} T_0$ . Both of these features agree with the observed scaling and with the phenomenological scaling of Ref. 7.

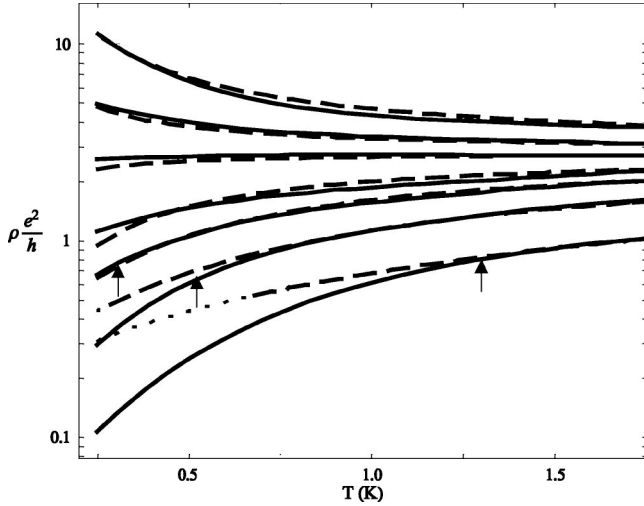


FIG. 1. Solid lines: measured resistivity  $\rho(T)$  in units of  $h/e^2$  from Ref. 25 in a Si-MOSFET close to separatrix for electron densities (from the top)  $n=8.6, 8.8, 9.0, 9.3, 9.5, 9.9, 11.0 \times 10^{10} \text{ cm}^{-2}$ . Dashed lines: our calculations using Eq. (10). The values of the parameters are given in the text. Dotted lines are in the small  $T$  region  $T_0(\delta)/T > 0.5$  that is excluded from the fit. Small vertical arrows: see text.

## IV. RESULTS

### A. Vicinity of separatrix

We now compare our results based on Eq. (10) with experimental values of the resistivity  $\rho(T)$  close to the separatrix obtained for a Si-MOSFET taken from Fig. 1(b) of Ref. 25 (see also Ref. 5). Equation (10) can be integrated analytically. The parameters  $\mathcal{R}^*$ ,  $z$ ,  $\nu$ ,  $\alpha$ ,  $\beta$ , and  $\kappa$  in  $\mathcal{R}(y)$  are chosen to give a best fit to the experimental data. Not all these parameters are independent of each other. The experimental data fix  $\mathcal{R}^*=2.8$ . Since  $\mathcal{R}^*=-\alpha/\beta$ , we can then consider  $\alpha$  to be fixed with  $\beta$  the free variable. The  $z$  and  $\nu=\tau/2$  enter together as a product via  $\tau^{-1}[y-y(0)]=\ln(T_0/T)^{1/z\nu}$ . The temperature  $T_0$  just prior to the bifurcation was taken to correspond to the temperature scale of elastic scattering  $T_{el}=1.75$  K. From the definition of  $\tau$ , we have  $\kappa=\tau\beta-1/\mathcal{R}^{*2}$ , which relates  $\kappa$  to  $\beta$ . There are therefore three independent variables  $z$ ,  $\nu=\tau/2$ , and  $\beta$ .

From the combination of electric field scaling and temperature scaling of the resistivity the dynamical critical exponent  $z$  is believed to be in the range 0.8 to 1.2.<sup>5</sup> We have therefore made least squares fits to the experimental data with  $b=z\nu$  and  $\beta$  as free parameters for the fixed values of  $z$  in the range  $0.8 \leq z \leq 1.2$ . The optimum values of the fitting parameters are determined by minimizing the root mean square relative deviation between theory and experiment

$$D = \frac{1}{N} \sum_{j=1}^N \frac{\sqrt{(\mathcal{R}_j^{\text{theory}} - \mathcal{R}_j^{\text{expt}})^2}}{\mathcal{R}_j^{\text{expt}}}, \quad (14)$$

where  $N$  is the total number of points included in the fit.

We have also examined the sensitivity of the fitted parameters to the temperature and density range of the fits. This is essential because the values of the least squares fitted param-

eters can vary with the range of temperature and density considered. Equation (10) has been established on the assumption of constant parameters for sufficiently small  $T_0(\delta)/T$ . Fits to data will be valid only if the derived parameters are stable with respect to reducing the maximum allowed  $T_0(\delta)/T$ , that is by restricting the temperature and density range of the fits.

We first carried out fits including all of the data points in Fig. 1(b) of Ref. 25. To test sensitivity to the range of fit we successively restricted the allowed data points by the conditions  $T_0(\delta)/T < 1.0, 0.5$ , and  $0.25$ . For  $T_0(\delta)/T < 0.5$  and fixed  $z$  the  $D$  and the values of the fitted parameters become stable. The coefficients  $\beta$  and  $\kappa$  vary slowly with  $z$ . However, the essential parameters from the point of view of fitting to a universal scaling form are constant throughout the range of  $z$ . These values are  $D=0.036 \pm 0.0005$ ,  $b=z\nu=1.09 \pm 0.005$  and  $\phi=0.80 \pm 0.005$ , where the uncertainties reflect the small variations due to  $z$ . A constant  $b$  and  $\phi$  is consistent, since  $(2\phi-1)/b$  is the universal coefficient of the first nonlinear correction  $\Delta^2$  in Eq. (10).

Figure 1 compares our results (dashed lines) from Eq. (10) for the dimensionless resistivity in the form  $\rho e^2/h = \mathcal{R}\pi$  with the experimental measurements (solid lines). The agreement is excellent. The dotted lines indicate the data points excluded from the fit by the condition  $T_0(\delta)/T < 0.5$ , and the small arrows show the edge of the corresponding excluded data range for  $T_0(\delta)/T < 0.25$ .

### B. Exponential form in metallic region

Previous fits to experimental data have shown that the temperature dependence of the resistivity is approximately exponential.<sup>26,27</sup> This result was accounted for by Ref. 7 by arguing that the beta function of the conductivity was a logarithmic function of conductivity, even in the metallic range near the separatrix. This is equivalent to assuming that the sum of the full series in Eq. (8), again with the coefficients evaluated at the bifurcation, is approximately logarithmic. Then Eq. (10) is replaced by

$$\frac{d \ln \mathcal{R}/\mathcal{R}^*}{dy} \approx \tau^{-1} \ln \mathcal{R}/\mathcal{R}^*, \quad (15)$$

giving

$$\mathcal{R}(T) \approx \mathcal{R}^* \exp \left\{ -C \left[ \frac{T_0(\delta)}{T} \right]^{1/z\nu} \right\} \quad (16)$$

with  $C$  a positive constant of order unity.

For the purposes of least squares fitting to data for small  $|\Delta|$ , it is important to note that Eqs. (10) and (15) are not inconsistent. Standard procedures for Padé approximations give the identity

$$\ln(1+\Delta) = \frac{\Delta + \frac{1}{2}\Delta^2}{1 + \Delta + \frac{1}{6}\Delta^2} + \mathcal{O}(\Delta^5). \quad (17)$$

The exact sum of the perturbation series near the bifurcation has the same form

$$\frac{d \ln \mathcal{R}/\mathcal{R}^*}{dy} = \frac{1}{\tau} \frac{\Delta + a\Delta^2}{1 + b\Delta + c\Delta^2} + \mathcal{O}(\Delta^5), \quad (18)$$

with the coefficients being of order unity. Equation (10) and Eq. (15) with (17) can thus be regarded as different approximations to Eq. (18) and are therefore equivalent from the point of view of least squares fitting to data for small  $|\Delta|$ . It is interesting to note that the low order expansion of Eq. (17),  $\ln(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \mathcal{O}(\Delta^3)$ , gives a value  $\phi = \frac{3}{4}$  which is comparable to our fitted value of  $\phi = 0.80$ .

### C. Deep metallic limit

Due to its  $y$  dependence  $\gamma_2$  continues to rescale as the low temperature limit of the metallic regime is approached. If we assume that  $\gamma_2$  approaches a finite limiting plateau value  $\gamma_2^0$  as  $T$  approaches 0, then  $\alpha(\gamma_2)$  and  $\beta(\gamma_2)$  also approach finite limiting values  $\alpha(\gamma_2^0)$  and  $\beta(\gamma_2^0)$ . Similarly, again due to the rescaling, the low-temperature value  $z^0 \nu^0$  of the exponent  $z\nu$  is expected to be different from the bare value and also from the value determined at the bifurcation. In this case for  $T$  sufficiently small we can obtain a consistent solution of Eq. (8) with  $\mathcal{R} \ll 1$  so that the perturbation expansion is well represented by the leading term

$$d\mathcal{R}/dy = \alpha(\gamma_2^0)\mathcal{R}^2 + \mathcal{O}(\mathcal{R}^3), \quad (19)$$

with the solution

$$\mathcal{R}/\mathcal{R}_0 = [1 - \alpha(\gamma_2^0)(y - y_0)]^{-1}. \quad (20)$$

Using  $(y - y_0) = \ln(T_0/T)^{1/z}$  and with the dimensionless conductivity  $g = \mathcal{R}^{-1}$ , we find

$$\frac{dg}{d \ln T} = z^0 \alpha(\gamma_2^0). \quad (21)$$

The dominant temperature dependence is then logarithmic as observed experimentally. References 27 and 28 express this logarithmic contribution to the conductivity  $G$  as  $\Delta G = (e^2/h)C(n)\ln T$ . Using Eq. (21) we identify  $C(n) = \alpha(\gamma_2^0)/(\pi z^0)$ . As the density increases  $\gamma_2^0$  and  $\alpha(\gamma_2^0)$  decrease in magnitude and  $\alpha(\gamma_2^0)$  may even change sign. This dependence of  $C(n)$  on density is in agreement with that observed experimentally.

### V. CONCLUSIONS

The low order perturbative RG equations of Refs. 11–13 do not describe a bifurcation in 2D, but a proper description

of a fixed point and a bifurcation in 2D can be obtained when higher order terms in the perturbation expansion for  $\mathcal{R}$  are retained [see Eqs. (8), (9)]. The resulting 2D fixed point exists only for  $\alpha(\gamma_2)$  negative and so is unrelated to the fixed point in dimensions  $2 + \epsilon$  for which  $\alpha(\gamma_2)$  is positive.

Our results provide a coherent semiempirical two-component scaling description of the density and temperature regime near the observed bifurcation and throughout the metallic regime. Near the separatrix a least squares fit to experimental data using Eq. (10) gives an excellent description of the observed density and temperature dependence of the resistivity. A full discussion for the insulating range will be given elsewhere. The scaling results apply only to the “universal” contributions to the resistivity which are a generic consequences of the interplay between electron-electron interactions and disorder. There are also “nonuniversal” contributions to the resistivity which will be material dependent.

The present picture is based on a  $\mathcal{R}$  which shows a bifurcation and on a  $\gamma_2$  which scales smoothly with temperature. The bifurcation “point”  $\mathcal{R}^* = -\alpha/\beta$  therefore varies smoothly with temperature and the separatrix, at which  $d\mathcal{R}/dT = 0$ , is “tilted” upwards. A tilted separatrix can lead in the metallic regime to a turn up of the resistance at low temperatures. This cannot occur in a one-component scaling theory where the separatrix is flat and the bifurcation point is uniquely determined as a function of density.

An exact solution of Eq. (19) in the very low-temperature limit shows a logarithmic dependence of the conductance on temperature with a coefficient which is negative and decreases in magnitude as the density increases. The low-temperature behavior is in agreement with experimental results.<sup>27,28</sup> The physical origin of this logarithmic electron-electron contribution to the conductivity is the same as that observed at high temperature. However, the numerical value of the low- $T$  coefficient can differ from the bare value at higher temperature.

### ACKNOWLEDGMENTS

This work is supported by the Natural Sciences and Engineering Research Council of Canada and an Australian Research Council Grant. We thank Alex Hamilton and Michelle Simmons for very useful discussions. DJWG thanks Ian Campbell for hospitality at the Université de Montpellier II.

<sup>1</sup>S.V. Kravchenko, W.E. Mason, G.E. Bowker, J.E. Furneaux, V.M. Pudalov, and M. D'Iorio, Phys. Rev. B **51**, 7038 (1995); S.V. Kravchenko, D. Simonian, M.P. Sarachik, W. Mason, and J.E. Furneaux, Phys. Rev. Lett. **77**, 4938 (1996).

<sup>2</sup>D. Simonian, S.V. Kravchenko, M.P. Sarachik, and V.M. Pudalov, Phys. Rev. Lett. **79**, 2304 (1997).

<sup>3</sup>P.T. Coleridge, R.L. Williams, Y. Feng, and P. Zawadzki, Phys.

Rev. B **56**, R12 764 (1997).

<sup>4</sup>M.Y. Simmons, A.R. Hamilton, M. Pepper, E.H. Linfield, P.D. Rose, D.A. Ritchie, A.K. Savchenko, and T.G. Griffiths, Phys. Rev. Lett. **80**, 1292 (1998).

<sup>5</sup>E. Abrahams, S.V. Kravchenko, and M.P. Sarachik, Rev. Mod. Phys. **73**, 251 (2001).

<sup>6</sup>S.L. Sondhi, S.M. Girvin, J.P. Carini, and D. Shahar, Rev. Mod.

- Phys. **69**, 315 (1997).
- <sup>7</sup>V. Dobrosavljević, E. Abrahams, E. Miranda, and S. Chakravarty, Phys. Rev. Lett. **79**, 455 (1997).
- <sup>8</sup>M.Y. Simmons, A.R. Hamilton, M. Pepper, E.H. Linfield, P.D. Rose, and D.A. Ritchie, Phys. Rev. Lett. **84**, 2489 (2000).
- <sup>9</sup>O. Prus, M. Reznikov, U. Sivan, and V. Pudalov, cond-mat/0108018 v2 (unpublished).
- <sup>10</sup>B.L. Altshuler, A.G. Aronov, and P.A. Lee, Phys. Rev. Lett. **44**, 1288 (1980).
- <sup>11</sup>A.M. Finkelstein, Zh. Éksp. Teor. Fiz. **84c**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)].
- <sup>12</sup>A.M. Finkelstein, Z. Phys. B **56**, 189 (1984).
- <sup>13</sup>C. Castellani, C. Di Castro, P.A. Lee, and M. Ma, Phys. Rev. B **30**, 527 (1984).
- <sup>14</sup>E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
- <sup>15</sup>G. Zala, B.N. Narozhny, and I.L. Aleiner, Phys. Rev. B **64**, 214204 (2001).
- <sup>16</sup>A. Punnoose and A.M. Finkelstein, Phys. Rev. Lett. **88**, 016802 (2001).
- <sup>17</sup>S.A. Vitkalov, K. James, B.N. Narozhny, M.P. Sarachik, and T.M. Klapwijk, cond-mat/0204566 v1 (unpublished).
- <sup>18</sup>V.M. Pudalov, M.E. Gershenson, H. Kojima, G. Brunthaler, A. Prinz, and G. Bauer, cond-mat/0205449 v1 (unpublished).
- <sup>19</sup>C. Castellani, C. Di Castro, and P.A. Lee, Phys. Rev. B **57**, R9381 (1988).
- <sup>20</sup>T.R. Kirkpatrick and D. Belitz, Phys. Rev. B **62**, 952 (2000).
- <sup>21</sup>C. Chamon and E.R. Mucciolo Phys. Rev. Lett. **85**, 5607 (2000).
- <sup>22</sup>S.A. Vitkalov, H. Zheng, K.M. Mertes, M.P. Sarachik, and T.M. Klapwijk, Phys. Rev. Lett. **87**, 086401 (2001); A.A. Shashkin, S.V. Kravchenko, V.T. Dolgoplov and T.M. Klapwijk, *ibid.* **87**, 086801 (2001).
- <sup>23</sup>V.M. Pudalov, M.E. Gershenson, and H. Kojima, cond-mat/0201001 v1 (unpublished); V.M. Pudalov, M.E. Gershenson, and H. Kojima, cond-mat/0110160 v1 (unpublished); V.M. Pudalov, G. Brunthaler, A. Prinz, and G. Bauer, Phys. Rev. Lett. **88**, 076401 (2002).
- <sup>24</sup>V.M. Pudalov, M.E. Gershenson, H. Kojima, N. Butch, E.M. Dizhur, G. Brunthaler, A. Prinz, and G. Bauer, cond-mat/0105081 v2 (unpublished).
- <sup>25</sup>M.P. Sarachik and S.V. Kravchenko, Proc. Natl. Acad. Sci. U.S.A. **96**, 5900 (1999).
- <sup>26</sup>V.M. Pudalov, JETP Lett. **66**, 175 (1997).
- <sup>27</sup>V.M. Pudalov, G. Brunthaler, A. Prinz, and G. Bauer, JETP Lett. **68**, 442 (1998).
- <sup>28</sup>V.M. Pudalov, G. Brunthaler, A. Prinz, and G. Bauer, JETP Lett. **68**, 534 (1998).