

ART. III.—ON THE ELEMENTARY TREATMENT OF THE PROPAGATION OF LONGITUDINAL WAVES.—BY PROF. J. G. MACGREGOR, D. SC.

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The best elementary treatment of the propagation of longitudinal waves known to me is that contained in Maxwell's Theory of Heat, chapter XV. It is based on Rankine's more difficult treatment of the same subject in his paper "On the thermodynamic theory of waves of finite longitudinal disturbance," published in the Philosophical Transactions of 1870, and in Rankine's Miscellaneous Scientific Papers, page 530. "The kind of waves to which the investigation applies are those in which the motion of the parts of the substance is along straight lines parallel to the direction in which the wave is propagated, and the wave is defined to be one which is propagated with constant velocity, and the type of which does not alter during its propagation." Maxwell's investigation involves an error which vitiates his result, and it is the object of this paper to point out the error and to obtain the same result in a legitimate manner.

Maxwell imagines a plane of unit area, which he calls the plane A, perpendicular to the direction of propagation of the wave, to move with the velocity of the wave. Behind it another plane, B, of the same area, moves with the same velocity, thus maintaining its distance from A. As both these planes move with the velocity of the wave, the velocity, the specific volume (*i. e.*, volume of unit of mass), and the pressure, of the substance, at each of them, must remain the same. The same is true of any plane normal to the direction of propagation of the wave and moving with its velocity. During the passage of the wave portions of the substance are continually passing through the planes A and B.

Let V be the velocity of the wave, $u_1, v_1,$ and $p_1,$ the velocity, specific volume, and pressure, of the substance at P, $u_2, v_2,$ and $p_2,$ the velocity, specific volume and pressure of the substance at

B, and Q and Q_2 the masses of the substance passing through A and B respectively, in a direction opposite to that of the wave, in unit of time.

Maxwell first shows, in an unexceptionable manner, that

$$Q_1 = Q_2 = Q \text{ (say) } \dots\dots\dots (4)$$

(we retain Maxwell's numbering of his equations), and that

$$u_1 = U - Qv_1 \text{ and } u_2 = U - Qv_2 \dots\dots (5)$$

He then points out (1) that the substance between the planes A and B is continually acted upon by a resultant force equal to $p_2 - p_1$ in the direction of the wave, and (2) that the momentum of the substance between these planes, in the direction of the wave, increases at a rate equal to $Q(u_1 - u_2)$ per unit of time. "The only way in which this momentum can be produced," he says, "is by the action of the external pressures p_1 and p_2 ," and hence in the earlier editions of the Theory of Heat, he puts

$$p_2 - p_1 = Q(u_1 - u_2) \dots\dots (6),$$

thus applying Newton's Second Law of Motion. He then substitutes in this equation the above values of u_1 and u_2 , and by a slip finds that

$$p_2 - p_1 = Q^2(v_1 - v_2) \dots\dots (7),$$

and consequently

$$p + Q^2 v_1 = p + Q^2 v_2 \dots\dots (8).$$

It follows that, as A and B are any planes whatever, normal to the direction of the wave, and moving with its velocity, if p and v are the pressure and specific volume of the substance at any such plane, we have

$$p + Q^2 v = \text{const.}$$

Hence, since for small changes of volume of actual substances the increase of pressure is proportional to the decrease of volume, the kind of wave under consideration is proved to be possible for actual substances, provided the changes of pressure and volume involved be small. Expressing, therefore, the elasticity (E) in terms of v and Q , by the aid of equation (7), he obtains

$$E = v Q^2,$$

and deduces at once the important result,

$$U^2 = E v.$$

If we correct the slip in substitution referred to above, it will be obvious that Maxwell should have deduced from equation (6), not equation (7), but the equation

$$p_2 - p_1 = Q^2 (v_2 - v_1),$$

from which it would have followed that

$$p - Q^2 v = \text{const.},$$

and consequently, that as an increase of pressure produces an increase of volume in no known substance, the form of wave under consideration was not possible in actual substances.

In later editions of the Theory of Heat there is substituted for equation (6) the equation

$$p_1 - p_2 = Q (u_1 - u_2),$$

and from this equation it follows that

$$p + Q^2 v = \text{const.};$$

but no reason is given for this modification of equation (6). Now, if the Second Law of Motion is applicable in the way in which Maxwell has applied it, equation (6) is correct in the earlier editions and incorrect as given in the later editions. For $p_2 - p_1$ is the value of the resultant force, in the direction of the propagation of the wave, acting on the portion of the substance between A and B, and $Q (u_1 - u_2)$ is the rate of increase of the momentum in the same direction of the substance between A and B. Hence, if the Second Law is applicable in this way the conclusion should be drawn that waves of this kind cannot be propagated by actual substances.

But the Second Law of Motion seems to be inapplicable in the present case. That law asserts that the resultant force acting on a *body* is equal to the rate of increase of the momentum of the body in the direction of the force, but not that the resultant force on the portion of a moving substance enclosed by certain bounding planes is equal to the rate of increase of the momentum of the substance thus enclosed. And hence the earlier form of equation (6) is not legitimately established.

Some other method must therefore be adopted of obtaining a relation between p , v and Q . The following method gives us the required relation: Let A and B be indefinitely near one another. Then the mass entering the space between A and B

at A during any time may be supposed to be identical with the mass issuing at B; and during its motion it may be supposed to be acted upon by a resultant force equal to $p_2 - p_1$ in the direction of the propagation of the wave. The mass Q , which enters in unit of time has initially, on entering, the momentum Qu_1 in the same direction, and finally, on issuing, the momentum Qu_2 also in the same direction. Hence the rate of increase of the momentum in the direction of the resultant force $p_2 - p_1$ is $Q(u_2 - u_1)$, and therefore, by the Second Law of Motion,

$$p_2 - p_1 = Q(u_2 - u_1).$$

When, now, we substitute in this equation the values of u_1 and u_2 given by equations (5), we obtain

$$p_2 - p_1 = Q^2(v_1 - v_2).$$

Consequently

$$p + Q^2 v = \text{const.},$$

for planes indefinitely near, and therefore also for planes at a finite distance from one another. Waves of the kind under consideration are thus seen to be possible of transmission through actual substances, provided the changes of pressure and volume are small, and the result,

$$U^2 = E v,$$

may therefore be deduced.