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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS RÉCU.

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INDICIAL METHODS
FOR
RELATIVE CATEGORIES

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Approved by

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ABSTRACT

Let \( V \) be a monoidal category. \( V \)-categories can be effectively studied as categories equipped with \( I \)-indexed families of morphisms, for all \( I \in \mathbb{V} \), and substitution operations. Abstraction of this structure leads to the notion of large \( V \)-category. The 2-category of large \( V \)-categories is well behaved, regardless of \( V \), and is, in a sense, to \( \mathbf{V-Cat} \) what the 2-category of large categories is to the 2-category of locally small categories.

A large \( V \)-category is a category object of a certain type in the total category, \( \mathbf{V-ind-SET} \), of a fibration over the category of finitely presented, monoids. A \( V \)-indexed category is a category object of arbitrary type in \( \mathbf{V-ind-SET} \). \( V \)-indexed categories have \( \hat{I} \)-indexed families of objects and \( \hat{J} \)-indexed families of morphisms, where \( \hat{I} \) and \( \hat{J} \) are objects of categories that can be constructed from the monoidal data of \( V \) using essentially only finite limits. \( V \)-indexed categories permit constructions of ordinary category theory which are generally not appropriate for \( V \)-categories. In particular, "discrete" fibrations, limits, cotensor products and mean cotensor products in \( V \)-categories can be studied using \( V \)-indexed categories.
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INTRODUCTION

Let I be a set. The concept of an I-indexed family of mathematical objects, \((X_i)_{i \in I}\), is one that appears throughout mathematics and can often be taken for granted. If the \(X_i\) are objects of a category \(X\), we can regard the family simply as a functor, \(I \xrightarrow{X} X\), where I is viewed as a discrete category.

Now, if one is concerned with mathematics relative to an elementary topos, \(E\), it is desirable, for reasons that we will not dwell on here, to avoid recourse to set-based mathematics wherever possible. Thus one tries, for example, to speak of I-indexed families of objects of \(E\), where I is itself an object of \(E\). The notion of a functor \(I \xrightarrow{X} E\) no longer makes sense à priori, for in general there is not a useful functor \(E \xrightarrow{D} \text{CAT}\), that allows one to define \(X\) as an \(E\)-valued functor on \(\mathcal{A}_I\). The equivalence, \(\text{set}^I \simeq \text{set}/I\), suggests another possibility. An object of \(E\) over I, \(X \xrightarrow{P} I\), may be thought of as defining an I-indexed family of objects of \(E\), since an "element" \(\iota \xrightarrow{i} I\) of I determines an object, \(X_i = X_i^*\), of \(E\) by pulling back \(p\) along \(i\). Significantly though, this procedure applies equally well to "elements" of \(I\), \(U \xrightarrow{U} I\) defined over arbitrary objects of \(E\).
Defining the category of $I$-indexed families of objects of $E$, $E^I$, to be $E/I$, enables one to speak of $I$-families of many kinds of mathematical objects defined relative to $E$. For $E/I$ is again a poset and a construction possible in $E$ will usually also be possible in $E/I$. For example, if $E$-gr is the category of group objects in $E$, it makes sense to define $(E$-gr)$^I$ to be $(E/I$)-gr. That is, an $I$-indexed family of groups in $E$ is defined to be a group in $E/I$. The adjunction, $\xymatrix{ E \ar[r]^-A & E/I }$, lifts to, $\xymatrix{ E$-gr \ar[r]^-A & (E$-gr)$^I }$, so the notion of $E$-families allows one to extend concepts such as $E$-completeness to categories defined relative to $E$.

An important feature of the examples $E$ and $E$-gr is that for each $I \in E$ we have a category, $A^I$, and for each $J \xrightarrow{u} I$ in $E$, a functor, $A^J \xleftarrow{u^*} A^I$, with the assignments functorial in $u$, up to coherent natural isomorphism. Observing that the "indexing" for $E$ itself depends only on finite limits leads to the notion of an $S$-indexed category in the sense of Awodey, Penon, Bénabou, Paré and Schumacher, where $S$ is any category with finite limits. If we ignore the isomorphisms mentioned above, an $S$-indexed category is just a functor, $S^{\text{op}} \xrightarrow{A^*} \text{CAT}$. Thus, roughly speaking, $S$-indexed category theory is category theory in $\text{SET}^{S^{\text{op}}}$. Indeed the "useful" $\xymatrix{ E \ar[r]^-D & \text{CAT} }$ that we want is replaced by
$S \overset{\gamma}{\rightarrow} \text{SET}^{\text{op}} \overset{D^{\text{op}}}{\longrightarrow} \text{CAT}^{\text{op}}$, where $\gamma$ is the Yoneda embedding and $D$ is the usual embedding of $\text{SET}$ in $\text{CAT}$.

Now while a great deal of indicial theory can be carried out as above, assuming only finite limits in $S$, the theory is mainly applicable to $\text{top}$-like and categories for which sums are disjoint and universal, like $\text{top}$. A morphism $X \overset{P}{\rightarrow} I$ in $\text{ab}$ is not a good notion of $I$-indexed family of abelian groups and this is usually the case for categories with an important monoidal structure which is distinct from the cartesian one. In fact, the theory may also be lacking for cartesian categories as the example $\text{cat}$ illustrates. A functor $X \overset{P}{\rightarrow} I$ should be a cofibration if one wishes to regard it as an $I$-indexed family of categories in a fairly rigid sense.

There are strong connections however between the theory of $S$-indexed categories and the theory of $S$-categories, [E&K], where $S$ is regarded as a monoidal category via its cartesian structure. If an $S$-indexed category $A$ is viewed as a category object in $\text{SET}^{\text{op}}$:

$$
\begin{array}{ccc}
M \times M & \overset{\delta_0}{\longrightarrow} & M \\
\downarrow \phantom{M \times M} & & \downarrow \phantom{M} \\
O & \overset{\delta_1}{\longleftarrow} & O
\end{array}
$$

we may ask whether for each $I$ in $S$, and each pair of $I$-indexed families of objects of $A$, $\langle A, B \rangle$, the pullback of $M \overset{\langle \delta_0, \delta_1 \rangle}{\longrightarrow} O$ along $I \overset{\langle A, B \rangle}{\rightarrow} O$.


lies in $S$. Such $S$-indexed categories are said to be *locally small* and for $A$ with this property the categories $A^I$ are easily seen to be $S/I$-categories in a compatible way.

In the other direction assume that $V$ is a monoidal category and that $A$ is a $V$-category. For $I \in V$ and $A,B \in A$, a morphism, $I \rightarrow [AB]$, in $V$ provides a reasonable notion of $I$-indexed family of morphisms from $A$ to $B$. Furthermore, a cotensor product, $[IB]$ in $A$, has long been regarded as a sort of "product" of $I$ copies of $B$.

Guided by these observations, this thesis represents an initial attempt to develop a general theory of $V$-indexed categories for monoidal $V$. We believe that this is an important program, for while $S$-indexing seems to perfectly capture the essence of set-like indexing, many of the parametrizing objects encountered repeatedly in mathematics are far from being arbitrary sets. The objects $N, Z, R, [0,1], \{z \in C | |z| = 1\}$, etc., come readily to mind and the uses of order, arithmetic, topology, etc. in parametrizations involving these
objects are too familiar to require further comment.

V-indexing, as presented here, while borrowing the ideas and terminology of S-indexing, differs considerably from the latter in the techniques involved. On the one hand it is more simple minded in that "substitution" is, in practice, often accomplished by mere composition in V; the analogous S-technique might be said to be "pulling back". On the other hand V-indexing typically requires more indexing variables and these may be subject to "constraints". Thus, while S-indexing involves I-families for all I in S, we will have to come to terms with I,J,K-families and (I,J,K,L such that I@J = K@L) - families where the variables are objects of V.

In chapter 1 we discuss V-categories in some detail from the point of view of categories equipped with V-indexed families of morphisms (assuming only that V is monoidal). The natural objects to study here are what we call large V-categories. These are equivalent to SET\textsuperscript{op} - categories, but we work with them like graded categories (which are also SET\textsuperscript{N} - categories). The approach makes for a very simple and conceptual treatment of ordinary V-categories. They are locally small large V-categories.
It is well known that the comma theory of $V$-$\mathcal{C}at$ is not rich enough to be used in the same way as that of $\mathcal{C}at$, and the considerations of chapter 1 amplify where the difficulties lie. After some preliminary motivation along these lines in chapter 2, we proceed with the constructions that we use to define $V$-indexed categories. Large $V$-categories arise as a special case of these generalized $V$-categories which typically have indexed families of objects as well as indexed families of morphisms. Various examples and applications of the general theory are given.

In chapter 3 we define $V$-discrete, $V$-fibrations using $V$-indexed categories. We show that these are related to the comma theory of $V$-indexed categories and allow theorems of ordinary category theory to be proved in the enlarged $V$-world. The mean cotensor products of Borceux and Kelly [B&K] are examined in considerable detail and shown to behave just like ordinary limits provided that $V$-indexed categories are included in the discussion.

We conclude with an appendix which indicates a further generalization of $V$-indexing that allows for a cartesian closed 2-category of $V$-indexed categories.

Much remains to be done with indexing in general and $V$-indexing in particular. No claim is made here that the ideas of this paper are particularly appropriate for cartesian monoidal structures and the collapsing of $V$-indexing to $S$-indexing awaits development. The case
\[ V = \text{cat} \] seems to require still another perspective.

Our formulation in terms of "types" should be studied further. It is clear that some "types" are more important than others. This is particularly true for "types" of \( V \)-indexed functors and \( V \)-indexed natural transformations. If the "type" notion is carried over to \( V \)-indexing for monoidal closed \( V \), it is reasonable to conjecture that finitely presented groups may replace the finitely presented monoids that we have used here. In this case one might have to deal with \( (I, J, K \) such that \( I \otimes J \to K \)-families where the morphism is \textit{compatible}, as in our definition in chapter 2, but is not necessarily an isomorphism.

Smallness in general, particularly local smallness of \( V \)-indexed categories, also remains to be studied.
CHAPTER 1
LARGE \( \mathbf{V} \)-CATEGORIES

1. PRELIMINARIES

Let \( U_1, U_2, U_3, \ldots \) denote universes in the sense of [Gk], with \( U_i \subseteq U_{i+1} \). Throughout most of this paper the reader may think of \( U_1 \) as the smallest universe containing \( \mathbb{N} \) as an element, so that \( U_1 \) sets are those usually encountered in mathematics. Similarly \( U_2 \) may be regarded as the smallest universe such that \( U_1 \subseteq U_2 \).

We make no essential use of the \( U_i \) with \( i \geq 3 \). set (respectively \( \text{SET}, \text{SET} \)) denotes the category of \( U_1 \) (respectively \( U_2, U_3 \)) sets and functions.

We write \( \text{cat} \) (respectively \( \text{CAT}, \text{CAT} \)) for the 2-category of category objects in \( \text{SET} \) (respectively \( \text{SET}, \text{SET} \)). Thus \( \text{SET} \) is an object of \( \text{CAT} \) as is \( \text{ab} \), (respectively \( \text{top} \), \( \text{top} \). etc.), the category of all abelian groups (respectively topological spaces etc.) whose underlying set is an element of \( U_1 \). Among the objects of \( \text{CAT} \) are all the usual "large" categories. However, as a consequence of our set theory, \( \text{CAT} \) is cartesian closed so that functor categories like \( \text{SET}^{\text{ab}} \) are objects of \( \text{CAT} \) also. Note also that \( \text{SET}^{\text{ab}} \) etc. is comparable to \( \text{SET} \); both are objects of \( \text{CAT} \).

In general, if \( S \) is a category with finite limits, \((S)\text{cat}\) denotes the 2-category of category objects in \( S \).
We avoid use of the words "large" and "small" etc. in any set theoretic sense, reserving these for technical terms to be introduced in the sequel. Thus, for example, we say \( X \) is \( U_1 \)-complete when \( X \) has limits of all diagrams with domain an object of \( \text{cat} \). If \( X \) is an object of \( \text{CAT} \) its hom functor has codomain \( \text{SET} \). We say that \( X \) is locally \( U_1 \) if its hom functor factors over \( \text{SET} \). \( \text{Cat} \) denotes the 2-full sub 2-category of \( \text{CAT} \) determined by the locally \( U_1 \) categories. Thus set, ab, top etc. are objects of \( \text{Cat} \), but note that it is neither cartesian closed nor \( U_1 \)-cocomplete. (cf. §4).

\( \mathcal{V} = \langle \mathcal{V}, U, \theta, \lambda, \rho, \alpha \rangle \) is to be a fixed monoidal category with underlying category an object of \( \text{Cat} \). \( U \) is the unit object; other symbols have their usual meaning. The functor \( \mathcal{V} (U, -) \to \text{set} \) is known as the base functor.

If for each \( V \in \mathcal{V} \) the functors \( \mathcal{V} \theta \) have right adjoints \( [V-] \) we say that \( \mathcal{V} \) is closed, and if the functors \( - \theta V \) have right adjoints \( [V-]^{rev} \) we say that \( \mathcal{V} \) is rev-closed. We write \( \mathcal{V}^{rev} \) (read \( \mathcal{V} \) reverse) for the monoidal category \( \langle \mathcal{V}, U, \theta^{rev}, \rho, \lambda, \alpha^{rev} \rangle \), where \( \mathcal{V}^{rev} \mathcal{W} = \mathcal{Wv} \mathcal{V} \) and \( \alpha^{rev}_{W} = \alpha^{-1}_{XWV} \). Thus \( \mathcal{V} \) is rev-closed if and only if \( \mathcal{V}^{rev} \) is closed.

It will be understood that the set of objects of a \( \mathcal{V} \)-category is an element of \( U_2 \), and we write \( \mathcal{V} \)-Cat for the 2-category of \( \mathcal{V} \)-categories. A \( U_1 \) \( \mathcal{V} \)-category is one whose set of objects is an element of \( U_1 \).
\textit{V-cat} denotes the 2-category of \( U_1 \) \textit{V-categories}.

Identity morphisms (functors etc.) will be notationally confused with the corresponding object (category etc.) or denoted by a \( 1 \). Composites will be written in diagrammatic order except when tradition absolutely dictates otherwise.

\section{Large \textit{V}-Categories}

A \textit{large \textit{V}-category} \( A \) consists of a \( U_2 \) set of objects \( O \) together with, for each pair of objects \( A, B \in O \), and for each object \( I \in V \), a \( U_2 \) set, \( (I)[[AB]] \), formally declared to be the set of \( I \)-indexed families of morphisms from \( A \) to \( B \). A typical such family will be written

\[ A \xrightarrow{\langle I; f \rangle} B \]

If \( J \xrightarrow{u} I \) is a morphism in \( V \) we are to have a function \( (I)[[AB]] \to (J)[[AB]] \), known as substitution along \( u \), whose values will be written as:

\[ (A \xrightarrow{\langle I; f \rangle} B) \xrightarrow{u} (A \xrightarrow{\langle J; f u^* \rangle} B) \].

These assignments are to be functorial in \( u \). We further postulate that families compose as in:

\[ A \xrightarrow{\langle I; f \rangle} B \xrightarrow{\langle J; g \rangle} C \]

\[ \langle I \circ J; fg \rangle \]
and composition is \textit{stable under substitution}. Thus if $I' \xrightarrow{u} I$ and $J' \xrightarrow{v} J$ are morphisms of $\mathcal{V}$, $(fu^*)(gv^*) = (fg)(u \circ v)^*$. For each object $A \in \mathcal{O}$ there is a distinguished $\mathcal{U}$-\textit{indexed family} of morphisms:

$$A \xrightarrow{<U; A>} A$$

Finally we require that composition of composable families be associative and unitary as the $\theta$ of $\mathcal{V}$.

More formally:

1.1 \textbf{DEFINITION} A \textit{large $\mathcal{V}$-category} $\mathcal{A}$ consists of the following data:

1. A $\mathcal{U}$ set $\mathcal{O}$

2. A functor $\mathcal{V}^{\text{op}} \xrightarrow{\mathcal{M}} \text{SET}$

3. Functions $\mathcal{I} \xrightarrow{\mathcal{M}} \mathcal{O} \times \mathcal{O}$

natural in $\mathcal{I}$

4. A function $\mathcal{O} \xrightarrow{1} \mathcal{U} \mathcal{M}$ satisfying

$$\mathcal{O} \xrightarrow{1} \mathcal{U} \mathcal{M}$$

$$\Delta \xrightarrow{\mathcal{U} \theta = <\mathcal{U} \theta_0, \mathcal{U} \theta_1>} \mathcal{O} \times \mathcal{O}$$

5. Functions $\mathcal{I} \times \mathcal{J} \xrightarrow{\mathcal{Y}} (\mathcal{I} \otimes \mathcal{J}) \mathcal{M}$

natural in $\mathcal{I}$ and $\mathcal{J}$, satisfying

$$\mathcal{I} \times \mathcal{J} \xrightarrow{\mathcal{Y}} (\mathcal{I} \otimes \mathcal{J}) \mathcal{M}$$

$$\sigma \xrightarrow{\theta} \mathcal{O} \times \mathcal{O}$$

$$\mathcal{O} \times \mathcal{O} \times \mathcal{O} \xrightarrow{<\mathcal{P}_0, \mathcal{P}_2>} \mathcal{O} \times \mathcal{O}$$
The vertical arrow on the left is defined by the following pullback in SET:

\[ \begin{array}{ccc}
   IM \times JM & \xrightarrow{\pi_0, \pi_1} & IM \\
   \downarrow{\sigma_0} & & \downarrow{\sigma_1} \\
   O & & O \\
   \downarrow{\sigma_2} & & \downarrow{\sigma_2} \\
   O & & O \\
\end{array} \]

The data is subject to the axioms:

\[ \begin{array}{ccc}
   IM & \xrightarrow{<\sigma_0, \lambda_1>} & UM \times IM \\
   \downarrow{\gamma} & & \downarrow{\gamma} \\
   IM & & IM \\
   \downarrow{\lambda_M} & & \downarrow{\rho_M} \\
   IM & & IM \\
   \downarrow{1} & & \downarrow{1} \\
   (I \otimes I)M & & (I \otimes U)M \\
   \end{array} \]

and

\[ \begin{array}{ccc}
   IM \times (JM \times KM) & \xrightarrow{1 \times \gamma} & IM \times (J \otimes K)M \\
   \downarrow{\delta_0} & & \downarrow{\gamma} \\
   (IM \times JM) \times KM & & (I \otimes J)M \times KM \\
   \downarrow{\gamma \times 1} & & \downarrow{\gamma} \\
   O & & O \\
\end{array} \]
We can recover the \textit{hom set} definition, given informally at the beginning of this section, by regarding 0 as a constant functor on $\mathcal{V}^{\text{op}}$ and defining for each pair of objects $A, B \in \mathcal{O}$ a functor $\mathcal{V}^{\text{op}} \to \text{SET}$ by the following pullback:

\[
\begin{array}{c}
\text{[[AB]]} \\
\downarrow \\
A \\
\downarrow \\
<_{A,B} \\
\end{array}
\quad \begin{array}{c}
\Rightarrow \\
M \\
\downarrow \\
\emptyset \\
\downarrow \\
0 \times 0
\end{array}
\]

Then an \textit{I-indexed family of morphisms from $A$ to $B$}, $A \xrightarrow{\langle I; f \rangle} B$, is just an element of $(I) \text{[[AB]]}$. In the sequel we will often simply say that $\langle I; f \rangle$ is an \textit{I-morphism from $A$ to $B$}.

By considering only $U$-morphisms and using instances of the canonical isomorphism $U \xrightarrow{\cong} U \otimes U$ in $\mathcal{V}$ we can associate to any large $\mathcal{V}$-category $\mathcal{A}$ an ordinary underlying category $\underline{\mathcal{A}}$, with the same set of objects $\mathfrak{A}$.

For any $I \in \mathcal{V}$ define $I(0) = U$, $I(1) = I$, and $I(n) = (I(n-1)) \otimes I$ for $n > 1$. Then to any large $\mathcal{V}$-category $\mathcal{A}$ and any object $I \in \mathcal{V}$ we can associate a graded category, $(I)\underline{\mathcal{A}}$, having the same set of objects as $\mathfrak{A}$, with morphisms of degree $n$, between objects $A$ and $B$, given by $(I(n))\text{[[AB]]}$. Composition of morphisms, $A \xrightarrow{\langle I(n)f \rangle} B \xrightarrow{\langle I(n)g \rangle} C$, is given by composition in $\underline{\mathcal{A}}$ followed by $\tilde{\otimes}$. Here
\[ \alpha \] is the isomorphism constructed from instances of \( \alpha, \lambda \) and \( \rho \).

With notation as above we make the following:

1.2 DEFINITION A large \( \mathcal{V} \)-category \( \mathcal{A} \) is said to be locally small if for each pair of objects \( A, B \in \mathcal{A}, [AB] \) is representable. We will denote the representing objects by \([AB]\) when this is the case.

Thus for locally small large \( \mathcal{V} \)-categories we have a bijection:

\[
\begin{array}{ccc}
A \overset{\mathcal{V}}{\longrightarrow} & B & \text{in } \mathcal{A} \\
I & \longrightarrow & [AB] \text{, in } \mathcal{V}
\end{array}
\]

In any event for \( \mathcal{A} = \langle O, M, \partial, i, \gamma \rangle \), we have \( M = \sum_{A, B \in O} [AB] \). As one would expect \( i \) and \( \gamma \) can also be recovered from local data and the following two propositions bear this out.

1.3 PROPOSITION To any locally small large \( \mathcal{V} \)-category we can associate an ordinary \( \mathcal{V} \)-category and conversely.

**Proof:** (⇒) Given a \( \mathcal{V} \)-category \( \mathcal{A} \) with objects \( O \) and \( \mathcal{V} \)-structure \( U \xrightarrow{1} [AA], [AB] \circ [BC] \xrightarrow{\gamma} [AC] \) we define:

\[
M = \sum_{A, B \in O} (-, [AB])
\]
Certainly the resulting large \( \text{V} \)-category with objects \( O \) is locally small by construction.

(+) Given a locally small large \( \text{V} \)-category \( A \) we construct an ordinary \( \text{V} \)-category \( A' \) with the same set of objects and \( \text{V} \)-valued homs given by the objects \( \text{[AB]} \) representing the \( \text{[AB]} \) of \( A \). For each \( A \in O \) we have

\[
\begin{array}{ccc}
1 & \xrightarrow{A} & O \\
\downarrow{A} & & \downarrow{\emptyset} \\
\langle A, A \rangle & \xrightarrow{\emptyset} & O \times O
\end{array}
\]

and hence

\[
\begin{array}{ccc}
1 & \xrightarrow{A_1} & O \\
\downarrow{(U, [AA])} & & \downarrow{\emptyset} \\
\langle A, A \rangle & \xrightarrow{\emptyset} & O \times O
\end{array}
\]

i.e. \( U \xrightarrow{1} [AA] \) for each \( A \in O \). For each pair of objects \( A, B \in O \) there is a generic \( [AB] \)-morphism from \( A \) to \( B \), namely \( 1 \in ([AB], [AB]) \). Composing this with the generic \( [BC] \)-morphism from \( B \) to \( C \) yields an \( [AB] \times [BC] \)-morphism from \( A \) to \( C \) and this latter is
simply a morphism $[AB] \circ [BC] \to [AC]$ in $V$ which provides the desired internal composition for $A'$. That internal composition is associative and unitary follows by Yoneda type arguments.

Henceforth we will also denote ordinary $V$-categories with notation such as $A$ and write $A$ for the associated underlying category.

For $V$ monoidal $\mathbf{SET}^{V^{\text{op}}}$ becomes a closed and rev-closed monoidal category in a canonical way. Furthermore, a symmetry for $V$ induces a symmetry on $\mathbf{SET}^{V^{\text{op}}}$. The details may be found in Day, [Dy]. Here we just record the formulas for the $\circ$ and the two internal homs and establish one of the adjointness relations. $\circ$ is defined as a left Kan extension:

$$
\begin{align*}
\mathbf{Y}^\times\mathbf{Y} & \to \mathbf{SET}^{V^{\text{op}}} \times \mathbf{SET}^{V^{\text{op}}} \\
\circ & \quad \downarrow \quad \circ = \text{Lan}_{Y^\times Y}(\theta) \\
\mathbf{Y} & \quad \downarrow \\
\mathbf{SET}^{V^{\text{op}}} & 
\end{align*}
$$

where $Y$ is the Yoneda embedding. Thus for $F, G \in \mathbf{SET}^{V^{\text{op}}}$ we have:

$$
F \circ G = \int_{J, K} J^\times K \times (\cdot, J \circ K).
$$

For $H \in \mathbf{SET}^{V^{\text{op}}}$ define:

$$
[FH] = \int_J (J^\circ, (J \circ H))
$$
Then \((F \circ G, H) \cong \int_\mathcal{I} (\int_\mathcal{J} (J \circ \mathcal{F} \circ \mathcal{K} \circ \mathcal{I}, \mathcal{I} \circ \mathcal{K}), \mathcal{I}H)\)
\[= \int_\mathcal{I} \int_\mathcal{J} \int_\mathcal{K} (J \circ \mathcal{F} \circ \mathcal{K} \circ \mathcal{I}, \mathcal{I} \circ \mathcal{K}, \mathcal{I}H)\]
\[= \int_\mathcal{K} (\mathcal{K} \circ \mathcal{J} \circ \int_\mathcal{I} (J \circ \mathcal{F} \circ (\mathcal{I} \circ \mathcal{K}), \mathcal{I}H))\]
\[= \int_\mathcal{K} (\mathcal{K} \circ \mathcal{J} \circ (\int_\mathcal{I} \mathcal{I} \circ \mathcal{K}, J \circ \mathcal{F}, \mathcal{I}H))\]
\[= \int_\mathcal{K} (\mathcal{K}, \mathcal{J} \circ (\mathcal{I} \circ \mathcal{K}, J \circ \mathcal{F}) \mathcal{H})\]
\[= (G, [F \circ H])\],
so that \(F \circ \emptyset = - [F -]\). Similarly by defining
\[\mathcal{G} \circ \mathcal{H}^{\text{rev}} = \int_\mathcal{K} (\mathcal{K} \circ \emptyset, \mathcal{H})\] we have \((F \circ G, H) \cong (F, [G \circ \mathcal{H}^{\text{rev}}])\).
In other words \(- \mathcal{G} \circ [- G -]\).

The unit object for \(\text{SET}^{\mathcal{V}}\) is \((- , \mathcal{U})\) and thus the base functor \(\text{SET}^{\mathcal{V}} \Rightarrow \text{SET}\) is evaluation at \(\mathcal{U}\).

We will consider \(\text{SET}^{\mathcal{V}}\)-categories but only those with a \(\mathcal{U}_2\) set of objects. Since \(\text{SET}^{\mathcal{V}}\) is \(\mathcal{U}_2\) based while \(\mathcal{V}\) is \(\mathcal{U}_1\) based, such \(\text{SET}^{\mathcal{V}}\)-categories are analogous to the \(\mathcal{U}_1\ \mathcal{V}\)-categories mentioned in §1. \(\text{SET}^{\mathcal{V}}\) is \(\mathcal{U}_2\) complete and \(\mathcal{U}_2\) co-complete, so if \(\mathcal{V}\) is symmetric all of the theory developed in [Dc]or, [D\&K] for "small" "\(\mathcal{V}\)"-categories is applicable to these \(\text{SET}^{\mathcal{V}}\)-categories.

The 2-category of \(\mathcal{U}_2\ \text{SET}^{\mathcal{V}}\) categories will be denoted \(\text{SET}^{\mathcal{V}}\)-\(\text{cat}\).

1.4 PROPOSITION To any large \(\mathcal{V}\)-category we can associate an object of \(\text{SET}^{\mathcal{V}}\)-\(\text{cat}\) and conversely.

Proof: (+) Given \(\mathcal{A} = \langle \mathcal{O}, \mathcal{M}, \emptyset, \mathcal{I}, \gamma \rangle\) we define a \(\text{SET}^{\mathcal{V}}\) category with objects \(\mathcal{O}\) by setting the internal hom
between each $A, B \in O$ equal to $[[AB]]$. Strong identities are obvious. To define a strong composition,

$[[AB]] \otimes [[BC]] \xrightarrow{\gamma'} [[AC]]$, note that:

$([[AB]] \otimes [[BC]], [[AC]]) = \int_I (f_{j, k}^2 (J) [[AB]] \times (K)[[BC]] \times (I, J \otimes K), (I) [[AC]])$

$= \int_{I, J, K} (J) [[AB]] \times (K)[[BC]] \times (I, J \otimes K), (I) [[AC]])$

The components of $\gamma'$ are then constructed as follows. Form the indicated map below. (The front and back faces of the cube are pullbacks.)

To each $I \xrightarrow{u} J \otimes K$ associate:

$(J) [[AB]] \times (K)[[BC]] \xrightarrow{(u) [[AC]]} (J \otimes K)[[AC]] \xrightarrow{(I) [[AC]]} (I) [[AC]]$, thus defining a function $(I, J \otimes K) \rightarrow ((J) [[AB]] \times (K)[[BC]], (I) [[AC]])$. The exponential transpose of this function is the $I, J, K'$th component of $\gamma'$. 
Given a \( \mathbf{SET}^{\mathbf{OP}} \) category with objects \( 0 \) and internal homs \( [[AB]] \) define \( M = \sum_{A,B \in \mathcal{O}} [[AB]] \). Then

\[
\begin{align*}
J^M \times K^M & \cong \sum_{A,B,C \in \mathcal{O}} (J)[[AB]] \times (K)[[BC]] \quad \text{To define} \\
0 & \quad A,B,C \in \mathcal{O}
\end{align*}
\]

\[
J^M \times K^M \rightarrow (J \circ K) M \quad \text{we require a family:}
\]

\[
\begin{align*}
\langle (J)[[AB]] \times (K)[[BC]] & \rightarrow \sum_{A,C \in \mathcal{O}} (J \circ K)[[AC]] \rangle \\
A,C \in \mathcal{O} & \quad A,B,C \in \mathcal{O}
\end{align*}
\]

We have, from the data for a \( \mathbf{SET}^{\mathbf{OP}} \) category, compatible families:

\[
\langle (J)[[AB]] \times (K)[[BC]] \times (I,J \circ K) \rightarrow (I)[[AC]] \rangle_{J,K}
\]

for each \( A,B,C \in \mathcal{O} \). By taking \( I = J \circ K \) above and evaluating at \( J \circ K \rightarrow J \circ K \) we obtain a function which when followed by the \( A,C \)'th injection into the coproduct provides the \( A,B,C \)'th component of the family of functions that we want.

\[\]
subject to the axioms:

\[
\begin{array}{ccc}
M & \xrightarrow{F} & M' \\
\downarrow \beta & & \downarrow \beta' \\
0 \times 0 & \xrightarrow{F \times F} & 0' \times 0'
\end{array}
\]

\[
\begin{array}{ccc}
UM & \xrightarrow{UF} & UM' \\
\downarrow 1 & & \downarrow 1' \\
O & \xrightarrow{F} & O'
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
IM \times JM & \xrightarrow{F} & IM' \times JM' \\
\downarrow \gamma & & \downarrow \gamma' \\
(I\otimes J)M & \xrightarrow{(I\otimes J)F} & (I\otimes J)M'
\end{array}
\]

A large \( Y \)-natural transformation

\[ A = \langle O, M, \beta, 1, \gamma \rangle \xrightarrow{F \oplus t} \langle O', M', \beta', 1', \gamma' \rangle = A' \]

consists of a function \( O \xrightarrow{t} UM' \) satisfying:

\[
\begin{array}{ccc}
O & \xrightarrow{t} & UM' \\
\downarrow <F, G> & & \downarrow U\Theta' \\
0' \times 0' & \xrightarrow{} & 0'
\end{array}
\]

subject to the axiom:

\[
\begin{array}{ccc}
IM & \xrightarrow{IF, \beta, t} & IM' \times UM' \\
\downarrow \gamma & & \downarrow \gamma' \\
\langle IF, \beta, t, IG \rangle & & \langle (I\otimes U)M', \beta, t \rangle \\
\downarrow \gamma & & \downarrow \gamma' \\
UM' \times IM' & \xrightarrow{} & O'
\end{array}
\]

//
1.6 THEOREM  Large $\mathcal{V}$-categories, large $\mathcal{V}$-functors and large $\mathcal{V}$-natural transformations constitute a 2-category which is equivalent, as a 2-category, to $\mathcal{V}^{\text{op}}$-cat.

Moreover, the full and locally full sub 2-category of all large $\mathcal{V}$-categories determined by the locally small large $\mathcal{V}$-categories is equivalent, as a 2-category, to $\mathcal{V}$-Cat.

**Proof:** The verification, an elaboration of the correspondences given in the proofs of the preceding propositions, is straightforward. Here we just give two details. First, a large $\mathcal{V}$-functor $A \xrightarrow{F} A'$ between locally small large $\mathcal{V}$-categories is just an ordinary $\mathcal{V}$-functor between the corresponding ordinary $\mathcal{V}$-categories. For $M \xrightarrow{F} M'$ induces natural transformations $\langle AB \rangle \xrightarrow{\langle A, B \rangle \times \langle A', B' \rangle} \langle [AF, BF] \rangle$, which for $A$ and $A'$ locally small are just morphisms $\langle AB \rangle \xrightarrow{\langle A, B \rangle \times \langle A', B' \rangle} \langle [AF, BF] \rangle$ in $\mathcal{V}$ and conversely. Second, a large $\mathcal{V}$-natural transformation $A \xrightarrow{\gamma} A'$ where $A$ and $A'$ are locally small is just an ordinary $\mathcal{V}$-natural transformation. For the description of $\gamma$ gives us morphisms in $\mathcal{V}$: $\langle U \xrightarrow{\gamma} [AF, AG] \rangle$ and for these to constitute an ordinary $\mathcal{V}$-natural transformation we require that

$$
\begin{array}{c}
\text{[AB]} \circ \mathcal{U} \xrightarrow{\text{FOBT}'} \mathcal{V} \circ \mathcal{U} \circ \mathcal{F}
\end{array}
$$
commutes for all $A, B$. The description of $t$ however says that for any $I \xrightarrow{f} [AB]$ (i.e., $I$-morphism $f$ from $A$ to $B$)

\[ \begin{array}{ccc}
\beta^{-1} & \xrightarrow{I\otimes U} & [AF, BF] \otimes [BF, BG] \\
\downarrow & \downarrow & \downarrow \\
I \otimes U & \xrightarrow{f \otimes \text{refl}_I} & [AF, BF] \otimes [BF, BG] \\
\downarrow & \downarrow & \downarrow \\
\lambda^{-1} & \xrightarrow{U \otimes I} & [AF, AG] \otimes [AG, BG] \\
\end{array} \]

\text{commutes, where } fF \text{ is } I \xrightarrow{f} [AB] \xrightarrow{F} [AF, BF] \text{ and } fG \text{ is } I \xrightarrow{f} [AB] \xrightarrow{G} [AG, BG]. \text{ Applying the above to } [AB] \xrightarrow{g} [AB], \text{ (the generic } [AB]-\text{morphism from } A \text{ to } B) \text{ we have what we want. The converse is immediate.}

1.7 REMARK One of our main tools is the biclosed monoidal structure of $\text{SET}^{\mathcal{V}^{\text{op}}}$. In [Dy] the minimal structure on $\mathcal{V}$ required to make $\text{SET}^{\mathcal{V}}$ biclosed monoidal is abstracted and called a promonoidal structure. The main component of such a structure is a functor $\mathcal{V}^{\text{op}} \times \mathcal{V}^{\text{op}} \times \mathcal{V} \xrightarrow{P} \text{SET}$. Monoidal structure on $\mathcal{V}$ gives rise to promonoidal structure on $\mathcal{V}$ by setting $(I, J, K)P = (I \otimes J, K)$. However what we have really used thus far is the pro-monoidal structure on $\mathcal{V}^{\text{op}}$ (induced by the monoidal structure on $\mathcal{V}^{\text{op}}$, induced by the monoidal structure on $\mathcal{V}$!) Thus, suppose temporarily that we are just given a promonoidal structure on $\mathcal{V}^{\text{op}}$ which we will write as $\mathcal{V}^{\text{op}} \times \mathcal{V} \times \mathcal{V} \xrightarrow{P} \text{SET}$.
(thinking of \((K, I, J)P\) as a generalization of \((K, I\&J)\)).

Then we could define a large (or pro-) \(V\)-category to be an object of \(\text{CAT}_A\), \(A\), equipped with (amongst further data) a functor \(A^{\text{OP}} \times V^{\text{OP}} \times A \rightarrow \text{SET}_{\text{PA}}\), suitably compatible with the hom functor for \(A\). \((A, I, B)\) \(\text{PA}\) would then be the set of \(I\)-indexed families of morphisms from \(A\) to \(B\). Unable to specify composition in the form,

\[
(A, I, B)_{\text{PA}} \times (B, J, C)_{\text{PA}} \rightarrow (A, I\&J, C)_{\text{PA}}
\]

we would give instead,

\[
\int_I^J (A, I, B)_{\text{PA}} \times (B, J, C)_{\text{PA}} \times (K, I, J)_{\text{PA}} \rightarrow (A, K, C)_{\text{PA}}
\]

The domain of the above function clearly provides a definition for a \(K\)-indexed family of composable pairs for the 3-tuple \(<A, B, C>\).

Henceforth we will denote the 2-category of large \(V\)-categories by \(V\)-\text{CAT} and drop uses of the word large except when needed for emphasis. We will refer to ordinary \(V\)-categories as locally small \(V\)-categories.

In passing we note that the inclusion:

\[
\text{V-Cat} \rightarrow \text{V-CAT}
\]

is induced by the strong monoidal functor \(V \rightarrow \text{SET}_{V^{\text{OP}}}\). \(Y\) has a partially defined left adjoint \(L\) whose value at \(F \in \text{SET}_{V^{\text{OP}}}\) is given by

\[
\lim(\text{Y/F} \xrightarrow{P} V)
\]
when the colimit exists. \((P_F \text{ is the discrete fibration associated to } F.)\)  The domain of \(L\) is the full subcategory of \(\text{SET}^{\text{op}}_\mathcal{V}, \mathcal{V}\), determined by those \(F\) for which \(\lim_{P_F} \rightarrow\) exists. In particular \(\mathcal{V}\) contains all the representables and \(YL = \mathcal{V}\).

1.8 **PROPOSITION** IF the functors \(I\theta-\) and \(-\Theta I\) preserve colimits for all \(I\in \mathcal{V}\), \(\mathcal{V}\) is a full sub-monoidal category of \(\text{SET}^{\text{op}}_\mathcal{V}\) and \(L\) is a strong monoidal functor.

**Proof:** We have only to show that if \(F\) and \(G\) are in \(\mathcal{V}\) then \(F\circ G\) is in \(\mathcal{V}\) and \((F\circ G)L \Rightarrow F\circ L\circ G\).

\(FL \simeq \int^J JF \cdot J\) and \(GL \simeq \int^K KG \cdot K\) since the coends exist if and only if the assumed colimits exist, in which case they are isomorphic. \((JF \cdot J\) denotes the \(JF\)'th copower of \(J\).) For any \(J, K\) in \(\mathcal{V}\) we have \(J\circ K \simeq \int^i (I, J\circ K).I\)

\[
\begin{align*}
FL \circ GL & \simeq (\int^J JF \cdot J) \circ (\int^K KG \cdot K) \\
& \simeq \int^J JF \cdot (I \circ \int^K KG \cdot K) \\
& \simeq \int^J JF \cdot \int^K KG \cdot J \circ K \\
& \simeq \int^J JF \cdot KG \cdot J \circ K \\
& \simeq \int^J JF \cdot KG \cdot \int^i (I, J \circ K).I \\
& \simeq \int^i (JF \cdot KG \cdot (I, J \circ K)).I \\
& \simeq \int^i (I \circ F \circ G).I \\
& \simeq (F \circ G)L
\end{align*}
\]
1.9 **COROLLARY** With hypotheses as above, $\mathcal{V}$-$\text{Cat} \to \mathcal{V}$-$\text{Cat}$ has a left adjoint, $L$.

**Proof:** Obvious.

In the next section we show that $\mathcal{V}$-$\text{CAT}$ is $U_2$ cocomplete, however a colimit (or tensor with a category) of locally small $\mathcal{V}$-categories, when taken in $\mathcal{V}$-$\text{CAT}$, is usually not locally small. (Recall that the Yoneda embedding preserves only absolute colimits [P].) If the colimit can be identified as an object of $\mathcal{V}$-$\text{Cat}$, however, the above corollary enables us to reflect it back to $\mathcal{V}$-$\text{Cat}$ and thus obtain the corresponding colimit as a locally small $\mathcal{V}$-category. This sometimes provides a simplification of the usual constructions.

It should be pointed out here that large $\mathcal{V}$-categories are not intended as a disguise for the well understood notion of $\mathcal{V}$-$\text{OP}$-categories. Definitions 1.1 and 1.5 were adopted because they are amenable to generalizations that we pursue in subsequent chapters. At the same time however the family approach provides an efficient method of externalizing $\mathcal{V}$-constructions in the same way that the $\mathcal{S}$-indexed categories of Paré and Schumacher [P&S] provide externalizations of $\mathcal{S}$-constructions such as category objects. The remainder of this chapter briefly explores this possibility.
The following convention will be useful. By

\[
\begin{array}{c}
\beta
\end{array}
\]

we will understand that \(<J;g> = <J;f\chi^*>, for some isomorphism \(J \overset{\chi}{\rightarrow} I\) in \(\mathcal{V}\). When \(\chi\) is obvious we will not explicitly mention it. Then if \(A \xrightarrow{F \downarrow t} A'\)

is a \(\mathcal{V}\)-natural transformation we can associate to an \(I\)-morphism \(A \xrightarrow{<I;f>} B\) in \(\mathcal{A}\) an \(I\)-square,

\[
\begin{array}{c}
AF \\
<U;At> \\
AG \\
<U;Bt> \\
BGF
\end{array}
\]

and all of the \(\mathcal{V}\)-information in \(t\) can be recovered from such squares. Thus the notion of \(I\)-morphism allows us to work with \(\mathcal{V}\)-categories as if they were ordinary categories.

\section{Completeness of \(\mathcal{V}\)-CAT}

\subsection*{1.10 Proposition \(\mathcal{V}\)-CAT admits cotensoring with \(2\).}

\textbf{Proof:} If \(A \in \mathcal{V}\)-CAT, then \(A^2\) has as objects all \(U\)-morphisms \(A \xrightarrow{<U;f>} B\) and an \(I\)-morphism from \(A \xrightarrow{<U;f>} B\) to \(A' \xrightarrow{<U;f'>} B'\) is an \(I\)-square.
\[ \begin{array}{c}
A \xymatrix{<I;a> \ar[r] & A' \\
< U;f > \ar[r] \ar[d] & < U;f' > \\
B \xymatrix{< I;b > \ar[r] & B'}
\end{array} \]

These are composed by horizontal pasting (cf. [K&S]).

From the discussion at the end of the last section it follows that we get an isomorphism of (ordinary) categories:
\[ (B,A^2) \cong (B,A)^2 \text{ for all } B \in \mathbf{V-CAT}. \]

From the above elemental description it is obvious that \([ff']\) , for objects \(f\) and \(f'\) as above, is given by the pullback

\[ \begin{array}{c}
[[ff']] \ar[r] & [[AA']] \\
\downarrow & \downarrow \\
[[BB']] \ar[r] & [[AB']] \\
\end{array} \]

in \(\mathbf{SET}^{\mathbf{V}^{\text{op}}}\). If \(A\) is locally small, so that \([AA']\) etc. are representable, and \(V\) has pullbacks, then \([ff']\) is representable and \(A^2\) is locally small. If \(V\) does not have pullbacks the \(V\)-categories \(A^2\), for \(A\) locally small, provide examples of \(V\)-categories which are not locally small.
1.11 PROPOSITION \( \text{V-CAT} \) admits tensoring with 2.

Proof: If \( B \) is a \( \text{V} \)-category with objects 0, \( B \times 2 \) has objects \( 0 \times 2 \) with

\[
(I)[(A,i)(B,j)] = \begin{cases} (I)[AB] & \text{for } i \leq j \\ \emptyset & \text{for } i > j \end{cases}, \quad i,j \in \{0,1\}
\]

Composition is given by the rules:

\[
\begin{align*}
<I;f,0><J;g,0> &= <I\otimes J;fg,0> \\
<I;f,0><J;g,> &= <I\otimes J;fg,> \\
<I;f,><J;g,1> &= <I\otimes J;fg,> \\
<I;f,1><J;g,1> &= <I\otimes J;fg,1>
\end{align*}
\]

where in each case the middle component of the composite is given by composition in \( B \) and the labels in the third component reflect the structure of 2. Then any \( I \)-morphism \( (A,0) \xrightarrow{<I;f,>} (B,1) \) factors canonically as:

\[
\begin{array}{ccc}
(A,0) & \xrightarrow{<I;f,0>} & (B,0) \\
\downarrow U;A,> & & \downarrow U;B,> \\
(A,1) & \xrightarrow{<I;f,1>} & (B,1)
\end{array}
\]

whence we establish an isomorphism of categories:

\[
(B \times 2, A) \cong (B, A)^2 \quad \text{for all } A \in \text{V-CAT}.
\]
B × 2 is never locally small (since \([(A, 1)(B, 0)] = \emptyset\)
is not representable) for non-empty B, however we do have the following:

1.12 COROLLARY If V has an initial object 0 such that
I00 \sim 0 \sim 00I for all I in V, and B \in V-Cat then
B × 2 \in V-Cat. Hence (B × 2)1 is the cotensor of B
with 2 in V-Cat.

Proof: See corollary 1.9 and subsequent remarks.

1.13 THEOREM V-CAT is \textit{U}_2 complete and \textit{U}_2 co-complete
as a 2-category.

Proof: In view of propositions 1.10 and 1.11 it suffices
to show that V-CAT is complete and cocomplete as a
category. Limits and sums are as expected so we will
just consider coequalizers.

First, for any (large) V-category C, a relation
on C, R, consists of a collection of binary relations
R<\(I; A, B\)> on the sets \(I[[AB]]\) where \(I \in V\) and \(A, B \in C\).
A relation E on C is a congruence relation on C if
it is an equivalence relation, stable under substitution
and composition. That is:

1) Each \(E_{<I; A, B>}\) is an equivalence relation.

2) If \((A \xrightarrow{<I; f>} B)E_{<I; A, B>} (A \xrightarrow{<I; f'>} B)\)

and \(J \xrightarrow{u} I\) is a morphism of V,
then \((A \xrightarrow{J;fu} B_E \times_{J;A,B} (A \xrightarrow{J;f'u} B))\)

3) If \((A \xrightarrow{I;f} B) E_{<I;A,B>} (A \xrightarrow{I;f'} B)\)

and \((B \xrightarrow{J;g} C) E_{<J;B,C>} (B \xrightarrow{J;g'} C)\)

then \((A \xrightarrow{I\otimes J;fg} C) E_{<I\otimes J;A,C>} (A \xrightarrow{I\otimes J;f'g'} C)\).

It is clear that the intersection of a collection of congruence relations is again a congruence relation. Thus if \(R\) is any relation on \(C\) there is a smallest congruence relation \(E(R)\) on \(C\) which contains \(R\).

Define \(C/R\) to be the \(V\)-category with objects those of \(C\), \(I\)-indexed families of morphisms from \(A\) to \(B\) given by \((I)[[AB]] / E(R)_{<I;A,B>}\), and structure inherited from that of \(C\). The functions \((I)[[AB]] \rightarrow (I)[[AB]] / E(R)_{<I;A,B>}\), jointly constitute a \(V\)-functor from \(C\) to \(C/R\) with the following universal property: If \(C \rightarrow H \rightarrow D\) is any \(V\)-functor for which 

\((A \xrightarrow{I;f} B) R_{<I;A,B>} (A \xrightarrow{I;f'} B)\) implies \(fH = f'H\), for all related \(f,f'\), there exists a unique \(V\)-functor \(C/R \rightarrow H \rightarrow D\) such that

\[
\begin{array}{ccc}
C & \xrightarrow{H} & C/R \\
\downarrow{H} & & \downarrow{H} \\
D & & D
\end{array}
\]

commutes.
Now given \( V \)-functors \( A \xrightarrow{F} B \) construct a \( V \)-category \( C \) as follows: The set of objects of \( C \) is the coequalizer of the underlying object functions of \( F \) and \( G \),

\[
\begin{array}{c}
\xrightarrow{F} \\
|A| \\
G
\end{array}
\]

If \( B \) and \( B' \) are objects of \( C \) an \( I \)-morphism from \( B \) to \( B' \) is a finite list:

\[
< I_1; b_1 > \xrightarrow{I_1; b_1} B_0 \xrightarrow{I_2; b_2} B_1 \xrightarrow{I_2; b_2} B_2 \xrightarrow{\ldots} B_{n-1} \xrightarrow{I_n; b_n} B'
\]

where \( i \) is a morphism of \( V \), the \( b_j \)'s for \( j = 1, \ldots, n \) are morphisms of \( B \), and \( B_0P = B', B_1P = B', \ldots, B_{n-1}P = B' \) for \( j = 1, \ldots, n-1 \). When \( B' = B \) we also have a distinguished \( U \)-morphism (the identity): \( < U \xrightarrow{id} U; > \).

If \( J \xrightarrow{U} I \) is a morphism of \( V \), composition with \( i \) defines substitution along \( u \). Composition of lists is given by tensoring the morphisms of \( V \) and concatenating the strings of morphisms of \( B \).

If \( B \xrightarrow{I; b} B' \) is an \( I \)-morphism of \( B \) define \( b \) to be the \( I \)-morphism \( I \xrightarrow{I; b} B \xrightarrow{I; b} B' \) from \( BP \) to \( B'P \) in \( C \). These assignments do not define a \( V \)-functor from \( B \) to \( C \). Call such an assignment a pre-\( V \)-functor. It is clear that if \( C \xrightarrow{H} D \) is any \( V \)-functor component-wise composition of \( P \) and \( H \) yields a pre-\( V \)-functor from \( B \) to \( D \) as suggested by:
Define $Q = C/E$ where $E$ is the smallest congruence relation on $C$ such that the composite

$(\sim P \to C \to C/E) = K$ is a $V$-functor which coequalizes $F$ and $G$. $B \xrightarrow{K} Q$ is then the coequalizer of $F$ and $G$ in $V$-$\text{CAT}$ . It follows that $E$ is $E(R)$ where $R$ is specified by:

1) $<J; B \xrightarrow{J; bu*} B'> R <J; BP, B'P> <J; \xrightarrow{u} I; B \xrightarrow{I; b} B'>$ for all $J \xrightarrow{u} I$ in $V$ and $B \xrightarrow{I; b} B'$ in $\sim$.

2) $<U; B \xrightarrow{U; B} B' > R <U; BP, BP> <U; >$ for all $B$ in $\sim$.

3) $<\text{IOJ}; B \xrightarrow{\text{IOJ}; bb'} B'' > R <\text{IOJ}; BP, B''P> <\text{IOJ}; \xrightarrow{I; b} B', B' \xrightarrow{J; b'a} B'' >$ for all composable $b, b'$ in $\sim$.

4) $<I; AF \xrightarrow{I; aF} A'F > R <I; A FP, A' FP > <I; AG \xrightarrow{I; aG} A'G >$ for all $A \xrightarrow{I; a} A'$ in $A$.

In view of theorem 1.6 the construction of $[W]$ (also outlined in $[Gy_1]$) is applicable for the construction of coequalizers in $V$-$\text{CAT}$ . (Symmetry, while assumed in $[W]$ , is unnecessary. The construction only requires that $V\theta$- and $\sim V$ preserve colimits which is certainly the case for $\text{SET}^{\text{op}}$ .) If $A$ and $B$ , as above, are $U_1$ locally
small \( \mathcal{V} \)-categories, the construction of \([\mathcal{W}]\) exhibits
the \([\text{Set}']\) of \( Q \) as \( U_1 \) colimits of representables.
Thus if \( \mathcal{V} \) is \( U_1 \) cocomplete the \([\text{Set}']\) lie in \( \mathcal{V} \)
and \( Q \mathbb{L} \), where \( \mathbb{L} \) is the reflector, is the coequalizer
of \( F \) and \( G \) in \( \mathcal{V} \)-Cat. It is a fortiori the coequalizer
in \( \mathcal{V} \)-cat which is \( U_1 \) cocomplete when \( \mathcal{V} \) is. However
\( \mathcal{V} \)-Cat is unlikely to be \( U_1 \) cocomplete even when \( \mathcal{V} \) is.
The reason is that the hom objects for a general coequalizer
of locally small \( \mathcal{V} \)-categories are typically \( U_2 \) colimits,
which \( \mathcal{V} \) fails to have in general, unless it is a
partially ordered set. (cf. §7). \( \text{Cat} \) (which is set-Cat)
provides a simple counterexample. The diagram
\[
\begin{array}{ccc}
|\text{set}| \times |\text{set}| & \longrightarrow & \text{set} \\
\downarrow & & \downarrow \\
|\text{set}| & \longrightarrow & \text{set}
\end{array}
\]
where \( |\text{set}| \) denotes the discrete category whose objects
are all \( U_1 \) sets, has no coequalizer in \( \text{Cat} \). Its
coequalizer in \( \text{CAT} \) has one object and a \( U_2 \) set of
morphisms.

An immediate consequence of theorem 1.13 is that \( \mathcal{V} \)-CAT
has comma and op-comma objects. Once again; op-comma
\( \mathcal{V} \)-categories are rarely locally small and \( \mathcal{V} \)-Cat is
closed with respect to formation of comma \( \mathcal{V} \)-categories when
\( \mathcal{V} \) has pullbacks.

The most important example of a \( \mathcal{V} \)-category which is
not necessarily locally small is \( \mathcal{V} \) itself. We define
a \mathcal{V}\text{-category } \mathcal{V} \text{ with objects those of } \mathcal{V} \text{ and }
(\mathcal{I})[[\mathcal{VW}]] = (\mathcal{VW}, W) \text{ for } \mathcal{V}, W \in \mathcal{V}. \text{ Thus for } \mathcal{V}, \mathcal{W} \in \mathcal{V}, \mathcal{V} \xrightarrow{\mathcal{I}; f} \mathcal{W} \text{ denotes a morphism } \mathcal{VW} \xrightarrow{f} \mathcal{W} \text{ in } \mathcal{V}
\text{ and composition is given by:}

\begin{align*}
\langle \mathcal{V} \xrightarrow{\mathcal{I}; f} \mathcal{W}, \mathcal{W} \xrightarrow{\mathcal{J}; g} \mathcal{X} \rangle &= \langle \mathcal{VW} \xrightarrow{f} \mathcal{W}, \mathcal{WJ} \xrightarrow{g} \mathcal{X} \rangle \\
&\xrightarrow{-1} (\mathcal{V}(\mathcal{I}\mathcal{J}) \xrightarrow{\alpha} (\mathcal{V}\mathcal{I})\mathcal{J} \xrightarrow{\mathcal{I}g} \mathcal{WJ} \xrightarrow{g} \mathcal{X}) \\
&= (\mathcal{V} \xrightarrow{\mathcal{I}\mathcal{J}; \alpha^{-1}(\mathcal{I}g)} \mathcal{X})
\end{align*}

1.14 REMARK Further to remark 1.7 we note that the above \mathcal{V}\text{-structure for } \mathcal{V} \text{ itself is induced by the promonoidal structure on } \mathcal{V}, \mathcal{V}^{\text{op}} \times \mathcal{V}^{\text{op}} \times \mathcal{V} \xrightarrow{P_{\mathcal{V}}} \text{SET}, \text{ where } \mathcal{P}_{V} = (\mathcal{VW}, W). \text{ Thus the } \mathcal{O} \text{ on } \mathcal{V} \text{ plays essentially two different roles in } \mathcal{V}\text{-theory. On the one hand it provides the syntactic data } P \text{ and on the other the semantic data } P_{\mathcal{V}}. \text{ Regarding } P \text{ as a profunctor, } \mathcal{V} \times \mathcal{V} \xrightarrow{P} \mathcal{V} \text{ and } P_{\mathcal{V}} \text{ similarly we have } P \vdash P_{\mathcal{V}} \text{ as profunctors.}

\mathcal{V} \text{ is locally small if and only if } \mathcal{V} \text{ is closed, in which case the above structure is isomorphic to the usual one. Moreover, if } \mathcal{A} \text{ is locally small a } \mathcal{V}\text{-functor } \mathcal{A} \xrightarrow{F} \mathcal{V} \text{ is precisely what is usually meant by a } \mathcal{V}\text{-valued } \mathcal{V}\text{-functor when } \mathcal{V} \text{ is not necessarily closed, [In]. For } F \text{ must take } \mathcal{I}\text{-morphisms in } \mathcal{A} \text{ to } \mathcal{I}\text{-morphisms in } \mathcal{V}, \text{ that is } (\mathcal{I} \xrightarrow{F} [\mathcal{AB}]) \xrightarrow{F} (\mathcal{AFI} \xrightarrow{FP} BF).
In particular to the generic \([AB]\)-morphism from \(A\) to \(B\) we associate \(\text{A} \otimes [AB] \xrightarrow{<A,B>F} \text{BF}\), which is the usual strength of \(F\). Conversely given the usual data with strength as above we obtain a \(\text{V}\)-functor by composition. I.e. \((\text{I} \xrightarrow{f} [AB]) \sqcup \xrightarrow{<A,B>F} \text{BF}\), \(\text{V-CAT}\) thus eliminates the need to deal with a special case.

Actually we can do somewhat more. To define a \(\text{V}\)-functor \(\text{V} \xrightarrow{F} \text{V}\), we require an ordinary functor \(\text{V} \xrightarrow{F} \text{V}\) together with a specification of \(F\) on \(I\)-indexed families of morphisms, \(F_I\), for each \(I\) in \(\text{V}\). Thus if \(\text{V} \otimes I \xrightarrow{f} \text{W}\) is regarded as an \(I\)-morphism from \(\text{V}\) to \(\text{W}\) we require a morphism \(\text{V} \otimes I \xrightarrow{ff} \text{W}\).

Since \(f\) is the composite of the "distinguished" \(I\)-morphism \(\text{V} \otimes I \longrightarrow \text{V} \otimes I\) and the "ordinary" morphism \(\text{V} \otimes I \xrightarrow{f} \text{W}\), ("ordinary" morphisms are in bijective correspondence with \(U\)-morphisms), we must have

\[
\begin{array}{c}
\text{V} \otimes I \xrightarrow{f} \text{W} \\
\text{V} \otimes I \xrightarrow{ff} \text{W} \\
(V \otimes I)^F
\end{array}
\]

where \(V_F^I\) is \(F_I\) of the "distinguished" \(I\)-morphism. Thus it suffices to specify the \(VF^I\), naturally, for all \(V\) and \(I\) in \(\text{V}\), subject to the requirement that the diagrams
If \( V \) is closed, specification of the \( VF_\lambda \) is equivalent to specification of a strength \( [V,W] \xrightarrow{F} [VF,WF] \). For given the latter we define \( VF_\lambda \) to be the transpose of \( \lambda: [V,V \Theta I] \xrightarrow{F} [VF,(V \Theta I)F] \), where \( \lambda \) is the unit of the \( V \Theta - I \) \( [V,-] \) adjunction. Conversely, given the \( V \Theta I \xrightarrow{VF} (V \Theta I)F \), the transpose of \( \text{VF}[V,W] \xrightarrow{VF} (V \Theta [V,W])F \xrightarrow{EF} WF \) defines a strength for \( F \), and the assignments are seen to be mutually inverse.

The above discussion also enables us to classify \( V \)-functors \( \xrightarrow{\text{VF}} A \), for \( A \) locally small. In addition to an ordinary functor \( \xrightarrow{\text{VF}} A \) one has also to give morphisms \( I \xrightarrow{\text{VF}} [VF,(V \Theta I)A] \) in \( V \) for all \( V \) and \( I \) in \( V \), subject to equations which are analogous to those in the previous case.
The "distinguished" I-morphism \( V \otimes I \rightarrow V \otimes I \) has an intuitive interpretation. If we think of \( V \otimes I \) as being an internal coproduct of \( I \) copies of \( V \) with itself, then the "distinguished" I-morphism may be regarded as giving the I-indexed family of injections of \( V \) into \( V \otimes I \). Thus any "ordinary" morphism \( V \otimes I \xrightarrow{f} W \) out of the coproduct should correspond to an I-indexed family of morphisms from \( V \) to \( W \), and this correspondence should be mediated by the family of injections. This is the basis for the trivial factorization given above on which the discussion turns.

In ordinary category theory one has the following fact: If \( E \xrightarrow{p} B \) is a discrete cofibration with \( U_1 \) fibres there exists a functor, unique up to isomorphism, \( B \xrightarrow{F} \text{set} \), such that the following diagram is a pullback in \( \text{CAT} \):

\[
\begin{array}{ccc}
E & \rightarrow & \text{set}_* \\
\downarrow \quad \quad & & \downarrow \\
B & \xrightarrow{F} & \text{set}
\end{array}
\]

Here \( \text{set}_* \rightarrow \text{set} \) denotes the category of pointed \( U_1 \) sets equipped with the functor which forgets distinguished points. \( \text{set}_* \simeq 1/\text{set} \) where \( 1 \) denotes a chosen one-element set. Consequently \( E \xrightarrow{p} B \) is isomorphic to \( 1/F \rightarrow B \), that is the comma category, with one of its projections, given by:
It should be noted here that by replacing $1 \rightarrow \text{set}$ by $U \rightarrow \mathcal{Y}$, where $U$ denotes the usual unit $\mathcal{Y}$-category, and consequently replacing $\text{set} \rightarrow \text{set}$ by $U/\mathcal{Y} \rightarrow \mathcal{Y}$, we have the necessary ingredients to seek a $\mathcal{Y}$-analogue of the above theorem of ordinary category theory. However, the comma theory of $\mathcal{Y}$-CAT is simply not rich enough to allow this. For pulling back $U/\mathcal{Y} \rightarrow \mathcal{Y}$ along distinct $B \rightarrow \mathcal{Y}$ may yield the same $\mathcal{Y}$-category over $B$. That is, information is lost in forming $(U/\mathcal{Y} \rightarrow \mathcal{Y})F^*$. The difficulty is best illustrated by the construction at the heart of comma theory: cotensoring with $I$. A square in $\mathcal{A}$ is a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\langle I; a \rangle} & A' \\
\downarrow^{\langle J; f \rangle} & & \downarrow^{\langle J'; f' \rangle} \\
B& \xleftarrow{\langle I'; b \rangle} & B'
\end{array}
$$

($I \circ J' \rightarrow J \circ I'$ in $\mathcal{Y}$), but morphisms in $\mathcal{A}^2$ involve only $I$-squares,
It is this difficulty which leads us to the further generalization of the notion of \( V \)-category in chapter 2.

A further difficulty with \( \mathbf{V-CAT} \) is that it does not admit opposites in general. As usual, it does of course when \( V \) is symmetric, but note that in any event we can define \( V^{(op)} \) by \((I)(VW)^{(op)} = (I\circ W, V)\) and further that \( V \)-functors \( A \rightarrow V^{(op)} \) for \( A \) locally small, are precisely what are usually meant by contravariant \( V \)-valued \( V \)-functors when \( V \) is not necessarily closed, [Ln]. If \( A \in \mathbf{V-CAT} \) we may consider \( A^{op} \) to be an object of \( V^{rev}\mathbf{-CAT} \) by defining \((I)(AB)^{op} = (I)(BA)A \)
(cf. [R]). Our construction in chapter 2 will "absorb" this difficulty, however, so we do not pursue the interplay between \( \mathbf{V-CAT} \) and \( V^{rev}\mathbf{-CAT} \).

\section{COTENSORS AND LIMITS}

This section is intended to show briefly some further uses of \( I \)-morphisms.

\begin{definition}
For \( B \in \mathbf{V-CAT} \) and \( I \in V \), the weak cotensor of \( B \) with \( I \) is an object \([IB]\) of \( A \)
\end{definition}
together with an I-morphism \([IB] \overset{<I;p>}{\rightarrow} B\) such that for any I-morphism \(A \overset{<I;f>}{\rightarrow} B\) there is a unique U-morphism \(A \overset{<U;\hat{f}>}{\rightarrow} [IB]\) such that:

\[
\begin{array}{c}
A \\
\downarrow_{<U;\hat{f}>} \\
[IB] \\
\downarrow_{<I;p>} \\
B
\end{array}
\]

1.16 PROPOSITION If \(A \in \mathcal{V}_{\text{cat}}\), for \(\mathcal{V}\) as in [Dc], the above agrees with the implicit definition in [Dc] (page 26) which says that there is a bijection:

\[
\begin{array}{c}
I \\
\rightarrow [AB] \\
A \\
\rightarrow [IB]
\end{array}
\]

natural in \(A\).

Weak cotensors are of no real interest in \(\mathcal{V}\)-theory and we mention them only to make a comparison.

1.17 DEFINITION For \(B \in \mathcal{V}_{\text{cat}}\) and \(I \in \mathcal{V}\) the cotensor of \(B\) with \(I\) is an object \([IB]\) of \(A\) together with an I-morphism \([IB] \overset{<I;p>}{\rightarrow} B\) such that for any J\(\mathcal{O}I\)-morphism, (for any \(J\) in \(\mathcal{V}\)), \(A \overset{<J;\hat{f}>}{\rightarrow} B\) there is a unique J-morphism \(A \overset{<J;\hat{f}>}{\rightarrow} [IB]\), such that:

\[
\begin{array}{c}
A \\
\downarrow_{<J;\hat{f}>} \\
[IB] \\
\downarrow_{<I;p>} \\
B
\end{array}
\]
1.18 PROPOSITION If \( A \in V\text{-Cat} \), for \( V \) as in [Dc],
the above agrees with definition 1.2.1 in [Dc] which
says that: \([A, [IB]]_A \cong [I, [AB]_A)_V \) in \( \mathcal{V} \), \( \mathcal{V} \)-natural
in \( A \).

In the classical setting 'cotensor' means weak cotensor
preserved by all the \( \mathcal{V} \)-valued representables. Similarly,
weak limit (respectively end) in \( A \) means limit (respectively end) in \( A \) preserved by all the \( \mathcal{V} \)-valued
representables. In our setting a weak \( \text{lim} \) gadget survives
testing by \( U \)-morphisms into it while a \( \text{lim} \) gadget
must survive testing by arbitrary \( J \)-morphisms into it.
In the first case we require that the \( U \)-morphisms correspond
to 'cones' and in the second that the \( J \)-morphisms correspond
to \( J \)-indexed families of cones. We will show later that
an \( I \)-morphism \( A \xleftarrow{\langle I;f \rangle} B \) is a cone from \( A \) to a
generalized \( \mathcal{V} \)-functor and that a \( J \Omega I \)-morphism
\( A \xleftarrow{\langle J\Omega I;f \rangle} B \), which may be regarded as a \( J \)-indexed
family of \( I \)-indexed families of morphisms, is a
\( J \)-indexed family of cones from \( A \) to the same generalized
\( \mathcal{V} \)-functor. For now we can simply say: \( [IB] \xleftarrow{\langle I;p \rangle} B \)
is a weakly universal \( I \)-morphism with codomain \( B \) and
\( [IB] \xrightarrow{\langle I;p \rangle} B \) is a universal \( I \)-morphism with codomain
\( B \). If we regard \([IB] \) as an internal \( I \)-fold product of
\( B \) with itself then \( \langle I;p \rangle \) can be interpreted as the
\( I \)-indexed family of projections.
We illustrate the above ideas for ordinary binary products. The weak product of objects \( A \) and \( B \) in \( A \) is an object \( A \times B \) together with \( U \)-morphisms, \( A \xrightarrow{U;p} A \times B \xleftarrow{U;q} B \) such that for any pair of \( U \)-morphisms \( X \xrightarrow{U;f} A, X \xrightarrow{U;g} B \), there is a unique \( U \)-morphism \( X \xrightarrow{U;<f,g>} A \times B \) such that it commutes.

The product of \( A \) and \( B \) is an object \( A \times B \) together with \( U \)-morphisms \( U;p \) and \( U;q \) such that for any pair of \( J \)-morphisms (for any \( J \)), \( X \xrightarrow{U;f} A, X \xrightarrow{U;j} B \), that is a \( J \)-indexed family of cones, there is a unique \( J \)-morphism \( J;<f,g> \) such that the obvious diagram commutes.

The distinction between ordinary limits and cotensors becomes very clear in this setting. In the former case all universals are \( U \)-morphisms while for cotensors the projection itself is an \( I \)-morphism. We will generalize these ideas considerably in chapter 3.
§ 6 \( \Sigma \)-CELLS AND \( \Sigma \)-FUNCTOR CATEGORIES

In this section we assume that \( \Sigma \) is symmetric with symmetry \( \sigma \).

1.19 DEFINITION For \( K \in \Sigma \) a \( K \)-cell

\[
F \xrightarrow{t} G
\]

\( \Lambda = \langle O, M, \beta, 1, \gamma \rangle \xrightarrow{t} \langle O', M', \beta', 1', \gamma' \rangle = \Lambda' \)

consists of a function \( O \xrightarrow{t} KM' \) satisfying:

subject to the commutativity of

\[
\begin{array}{ccc}
IM' \times KM' & \xrightarrow{\gamma'} & O' \\
(F, G) \downarrow & & \downarrow (K, \sigma') \\
O' \times O' & \xrightarrow{(I \otimes K)M'} & (K \otimes I)M'
\end{array}
\]

naturally in \( I \).

In terms of our simplified terminology this means that to each \( I \)-morphism \( A \xrightarrow{t, f} B \) in \( A \), \( \langle K, t \rangle \) associates an \( I, K \)-square in \( A' \).
A U-cell is simply a V-natural transformation, and a K-cell is a K-indexed family of V-natural transformations.

A V-cell is a K-cell for some KcV : V-cells compose vertically and horizontally and satisfy analogues of the Godement rules.

\[
\begin{align*}
\text{If } A &\xrightarrow{F} A' \quad \text{then } A &\xrightarrow{F_{\mathfrak{L}}} A' \text{ is formally defined by } O \xrightarrow{S} \text{KM} \times \text{LM} \quad \nu \quad O' \rightarrow (K\Omega L)M' \\
\text{If } A &\xrightarrow{F} A' \quad \text{then } A &\xrightarrow{F_{\mathfrak{L}}} A' \quad \text{we first define } <K;\#>F' = <K;S'>FF' \rightarrow GF' \quad \text{by } O \xrightarrow{S} \text{KM} \quad \text{SF}' \quad \text{KM}'' \\
&\quad \quad \quad \quad \text{and } G_{\mathfrak{L}}t = <L;Gt>: GF' \rightarrow GG' \quad \text{by } O \xrightarrow{G} O' \quad t \rightarrow \text{LM}'' \\
\end{align*}
\]
Then since \( \langle K; s' \rangle \langle L; Gt \rangle \)
\[ = \langle K; (s') (Gt) \rangle \]
\[ = \langle L; (K; (Ft)) (G' \rangle \]
\[ = \langle L; Ft \rangle \langle K; s' \rangle \]

we may choose either value as \( \langle K; s \rangle \circ \langle L; t \rangle \) which we
denote by \( \langle K; (s') \circ (Ft) \rangle \). These definitions can of course
be easily translated into the simplified terminology.

The pleasing feature of \( V \)-cells is that they
provide a simple minded description of the enriched
structure of \( V \)-functor categories. For given \( V \)-
categories \( A \) and \( B \) we define \( [AB] \) to have as
objects \( V \)-functors \( A \rightarrow B \) and as \( I \)-morphisms,
\( I \)-cells between them. Thus we obviate the necessity of
requiring ends in \( V \). Of course we are tacitly using
ends in \( \text{SET}^{\text{top}} \) but the elemental nature of \( V \)-cells
allows us to manipulate them like ordinary natural
transformations and forget the more complicated
structure when it is not required. It is a consequence
of our set theory that \( [AB] \) exists for all \( \overset{\sim}{A}, \overset{\sim}{B} \in \overset{\sim}{V}\text{-CAT} \). The usual approach requires "size"
considerations as in the following:

1.20 Proposition If \( A \in \overset{\sim}{V}\text{-Cat} \), \( B \in \overset{\sim}{V}\text{-Cat} \), and \( V \)
is closed and complete, \( [AB] \) agrees with the usual
definition (as in say \( [D\&K] \)) and hence is an object of
\( \overset{\sim}{V}\text{-Cat} \).
Proof: Certainly by theorem 1.6 the objects of $[AB]$ are not in dispute so let $F$ and $G$ be $\mathcal{V}$-functors from $A$ to $B$. Then the usual definition of $[FG]$ says that it is the end of:

$$A^{\mathcal{V}} \xrightarrow{F^{\mathcal{V}}} B^{\mathcal{V}} \rightarrow \mathcal{V}$$

That is, we have:

$$<[FG] \xrightarrow{\text{As}} [AF,AG] \rangle_{A \triangleleft A}$$

$\mathcal{V}$-extra natural and terminal with that property.

Thus

$$K \xrightarrow{t} [FG]$$

which means explicitly that:


$$\downarrow [AF,G] \quad \quad \downarrow [At,1] \mathcal{V}$$


$$\downarrow [A',1] \mathcal{V}$$

commutes in $\mathcal{V}$ for all $A,A' \triangleleft A$. Homming into these diagrams by $I_{\mathcal{V}}$ and transposing yields diagrams which are collectively equivalent to the diagram in 1.19.

Conversely that diagram is to hold naturally in $I$.
so the steps are reversible and we have what we want.

In the above we have used the familiar tensor product of ordinary \( \mathcal{V} \)-categories. As expected this is also available for \( \mathcal{V} \)-\textsc{cat} and may be described in the usual way using the \( \oplus \) on \( \text{SET}^{\mathcal{V}^{\mathcal{O}p}} \). We can also form the \( \oplus \) product of \( \mathcal{V} \)-cells. If \( A \xrightarrow{\langle K,s \rangle} C \) and \( B \xrightarrow{\langle L,t \rangle} D \), that is we have \( O_A \xrightarrow{S} (K)M_C \)

and \( O_B \xrightarrow{t} (L)M_D \) ; \( A \otimes B \xrightarrow{\langle \mathcal{K}\mathcal{O}\mathcal{L},s\otimes t \rangle} C \otimes D \) is given by

\[
O_A \times O_B \xrightarrow{\text{sxt, } I \mathcal{K}\mathcal{O}\mathcal{L}} (K)M_C \times (L)M_D \times (\mathcal{K}\mathcal{O}\mathcal{L}, \mathcal{K}\mathcal{O}\mathcal{L})
\]

\[
(K,L) \mapsto \int_{I,J} (I)_M C \times (J)_M D \times (K\mathcal{O}\mathcal{L},I\mathcal{O}\mathcal{J})
\]

\[
= (K\mathcal{O}\mathcal{L})(M_{\mathcal{O}\mathcal{M}}) = (K\mathcal{O}\mathcal{L})M_{C \otimes D}
\]

One can establish that

\[ [A \otimes B, C] \cong [B, [AC]] \] as \( \mathcal{V} \)-categories.

Indeed:

1.21 \textsc{Theorem} For \( \mathcal{V} \) symmetric, \( \mathcal{V} \)-\textsc{cat} is a symmetric monoidal closed 2-category which admits an \( \mathcal{V} \otimes_{\mathcal{V}} \).
§7 EXAMPLES

If \(A\) is any \(U_2\) category equipped with a left action of \(V^{\text{op}}\), \(V^{\text{op}} \times A \rightarrow A\), written \(<I,B> \mapsto [IB]\), with the obvious structural isomorphisms and equations, \(A\) becomes a \(V\)-category \(A\) in a canonical way. This is accomplished by defining an \(I\)-morphism from \(A\) to \(B\) in \(A\) to be a morphism from \(A\) to \([IB]\) in \(A\). Thus

\[
\begin{array}{c}
A \xrightarrow{<I,B>} B \quad \text{in } A \\
\downarrow \\
A \rightarrow [IB] \quad \text{in } A
\end{array}
\]

The identity \([IB] \rightarrow [IB]\) in \(A\), for any \(I \in V\) and \(B \in A\), provides a distinguished \(I\)-morphism \([IB] \xrightarrow{<I,I>} B\) in \(A\) which is easily seen to be the cotensor product of \(B\) with \(I\) in \(A\).

A similar observation holds for categories equipped with a right action of \(V\), \(A \times V \rightarrow A\), written \(<A,I> \mapsto A \otimes I\). In this case of course we have

\[
\begin{array}{c}
A \xrightarrow{<I,B>} B \quad \text{in } A \\
\downarrow \\
A \otimes I \rightarrow B \quad \text{in } A
\end{array}
\]

Both situations amount to a simple way of giving the data mentioned in remark 1.7. Thus the cotensored and tensored categories of \([R]\) (chapter 5) are special
cases of large $\mathcal{V}$-categories; and our general definitions of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations cover the cases of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations between (co)tensored categories and locally small $\mathcal{V}$-categories etc.

In terms of the $\mathcal{A}^{\text{op}} \times \mathcal{V} \times \mathcal{A}^{\text{op}} \times \mathcal{A} \xrightarrow{\mathcal{P}_A} \text{SET}$ description of $\mathcal{V}$-categories in 1.7 we can summarize the above observations by:

1) Cotensored categories are those for which $(-,I,B)_A^\mathcal{P}$ is representable for all $I \in \mathcal{V}, B \in \mathcal{A}$.

2) Locally small $\mathcal{V}$-categories are those for which $(A,-,B)_A^\mathcal{P}$ is representable for all $A, B \in \mathcal{A}$.

3) Tensored categories are those for which $(A, I, -)_A^\mathcal{P}$ is representable for all $A \in \mathcal{A}, I \in \mathcal{V}$.

Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a $\mathcal{V}$-functor with $\mathcal{A}$ of the form 1) and $\mathcal{B}$ of the form 3) above. We can now tabulate the nine specific forms that the strength of $F$ takes for these special $\mathcal{V}$-categories.
In the above table \( A \) and \( B \) are objects of \( \mathcal{V} \) and \( I \) is an object of \( \mathcal{V} \). The entries are determined by considerations similar to those in §4 where \( \mathcal{V} \)-functors \( \mathcal{V} \overset{F}{\to} \mathcal{V} \) (a special case of (3)3)), were classified. The reader should have no difficulty in supplying the relevant equations.

Our treatment of \( \mathcal{V} \)-categories is reminiscent of graded categories. This is not surprising for graded categories are just locally small \( \mathbf{SET}^\mathbb{N} \)-categories or, in our terminology, (large)\( \mathbb{N} \)-categories where \( \mathbb{N} \) is regarded as a discrete monoidal category. For \( F, G \in \mathbf{SET}^\mathbb{N} \), \((k)F \circ G = \bigoplus_{m+n=k} mF \times nG \times (k, m+n)\)

Thus the \( \emptyset \) on \( \mathbf{SET}^{\mathbb{V}^{\text{op}}} \) for arbitrary \( \mathcal{V} \) should be regarded as a categorical generalization of power series multiplication.

In this example our suggestion in §1 regarding the interpretation of \( U_1 \) and \( U_2 \) is applicable but not particularly appropriate. Here we could take \( U_1 \) to be the set of all finite sets and \( U_2 \) to be the smallest
universe containing \( \mathbb{N} \) as an element. In view of naturally occurring examples one would then also want to consider \( \mathcal{U}_3 \mathbb{N} \)-categories, but this necessitates no essential changes in the theory. Similar remarks apply to the following example.

Let \( \mathbb{R} \) denote the set of non-negative real numbers together with \( \infty \), viewed as a partially ordered set, \( \langle \mathbb{R}, \geq \rangle \), in the usual way with \( \infty \geq a \) for all \( a \in \mathbb{R} \). \( \mathbb{R} \) is bicomplete with \( \lim \) given by sup and \( \lim \) by inf. Moreover, \(+, 0\) and truncated subtraction equip \( \mathbb{R} \) with the structure of a strict symmetric monoidal closed category. Setting \( \mathcal{V} = \mathbb{R} \) it is shown in [Le1] that a locally small \( \mathcal{V}\)-category is a generalized metric space. That is, a locally small \( \mathbb{R}\)-category is a set \( X \) together with a function, \( X \times X \xrightarrow{[\cdot - \cdot]} \mathbb{R} \), satisfying

\[
[xy] + [yz] \geq [xz]
\]

and

\[
0 \geq [xx]
\]

for all \( x, y, z \in X \). Now a large \( \mathbb{R}\)-category is a set \( X \) together with the following extra structure. For each pair of elements \( x, y \in X \) and each "real number" \( a \in \mathbb{R} \) there is given a set (a) \( [xy] \) which may be considered as defining a set of paths from \( x \) to \( y \) with "length" less than or equal to \( a \). This is to be functorial in \( a \) so that if \( b > a \) we have a function
(a)\([xy]\) \rightarrow (b)\([xy]\) \text{ etc. For each element } x \text{ in } X \text{ we require a distinguished path of length (less than or equal to) } 0 \text{ from } x \text{ to } x \text{. Finally, if } \gamma \text{ is a path of length less than or equal to } a \text{ from } x \text{ to } y \text{ and } \delta \text{ is a path of length less than or equal to } b \text{ from } y \text{ to } z \text{ then there is to be a composite path } \gamma \delta \text{ of length less than or equal to } a + b \text{ from } x \text{ to } z \text{.}

The data must of course further satisfy the usual axioms.

We note that in this example the left adjoint to the Yoneda embedding, \( \mathbb{R} \xrightarrow{Y} \text{SET}^{\text{op}} \), is fully defined. Indeed for \( \text{SET}^{\text{op}} \xrightarrow{F} \text{SET} \), \( FL \) (see §3) is given by

\[
\int^a aF \cdot a = \inf\{a \mid aF \neq \phi\}
\]

Thus we have \( \mathbb{R}\text{-Cat} \xrightarrow{T} \mathbb{R}\text{-CAT} \) and the reflection of a large \( \mathbb{R} \)-category \( X \) is seen to be a generalized metric space structure on the set \( X \) with \( [xy] \) the infimum of the set of real numbers \( a \) such that there is a path of length \( \leq a \) from \( x \) to \( y \).
CHAPTER 2
GENERALIZED $\mathcal{V}$-CATEGORIES

§1 SET VALUED FUNCTORS

We remarked in the introduction that the theory of $S$-indexed categories, as developed by Paré and Schumacher et al., is (roughly) category theory in $\text{SET}^{S^{\text{op}}}$. In chapter 1 we used $\text{SET}^{\mathcal{V}^{\text{op}}}$, in much the same spirit as the above authors, to speak of $\mathcal{V}$-indexed families of morphisms in a category. However, we have also used $\text{SET}^{(\mathcal{V} \times \mathcal{V})^{\text{op}}}$ to speak of $\mathcal{V} \times \mathcal{V}$-indexed families of composable pairs of morphisms and (somewhat pedantically) $\text{SET}^{1^{\text{op}}}$ to speak of objects. This state of affairs is dictated by the existing theory and further examples that we wished to cover. We cannot say all that we want to (or indeed very much) if we insist that every concept be expressible in $\text{SET}^{\mathcal{V}^{\text{op}}}$. The difficulty, for what we have considered thus far, can be said to reside in the fact that $\mathcal{V} \xrightarrow{\Theta I} \mathcal{V}$, for $I \in \mathcal{V}$, does not carry a cotriple structure.

The above considerations suggest that we need at least $\mathcal{V}^n$-indexed families for $n \in \mathbb{N}$—a somewhat multilinear approach. (cf. [Ln] and [R]). We will require more though. For $A \in \mathcal{V}$-CAT recall the notion of a square in $A$ as given in §4 of chapter 1. It is a $\mathcal{V}^I$-indexed family, where $\mathcal{V}^I$ has as objects lists of the form $<I,J,J',I' : IOJ' \xrightarrow{\lambda} J\Theta I'>$ Roughly speaking we
are going to consider $A(V)$-indexed families, where $A(V)$ is any category constructed from $V$ and its monoidal data using "finite limits". Thus we will do category theory in $\text{SET}^{A(V)\text{op}}$ with $A(V)$ allowed to vary.

In this section we briefly study the (2-)category of all functors $A^{\text{op}} \to \text{SET}$ for all categories $A$ since it provides an ambient setting for the above and allows us in the appendix to consider still more general families. It also permits consideration of change of base questions in the presence of a monoidal functor $V \xrightarrow{F} W$, but only an indication of this will be given in this paper.

$\text{SET}^\text{op}$ denotes the 2-category whose objects are pairs $<A,\Gamma>$ with $A \in \text{CAT}$ and $A^{\text{op}} \xrightarrow{\Gamma} \text{SET}$. A morphism from $<A,\Gamma>$ to $<B,\Phi>$ is a pair $<F,\alpha>$ where $A \xrightarrow{F} B$ and

```
\begin{array}{c}
A^{\text{op}} \xrightarrow{F^{\text{op}}} B^{\text{op}} \\
\Gamma \downarrow \alpha \downarrow \Phi \\
\text{SET}
\end{array}
```

If $<A,\Gamma> \xrightarrow{<F,\alpha>} <B,\Phi>$, a 2-cell from $<F,\alpha>$ to $<G,\beta>$ is a natural transformation $F \xrightarrow{t} G$ such that:
The forgetful 2-functor \( \text{SET}^{\text{op}} \) \( \xrightarrow{F} \) \( \text{CAT} \)

given by: \( \langle A, \Gamma \rangle \xrightarrow{F, \alpha} \langle B, \phi \rangle \xrightarrow{t} A \xrightarrow{F} B \)

is a split normal fibration. The fibre over \( A \) is \( \text{SET}^{A^{\text{op}}} \).

As with other fibrations we will write

\[
\begin{array}{c}
\Gamma \xrightarrow{\alpha} \phi \\
\vdots \\
\vdots \\
A \xrightarrow{F} B
\end{array}
\]

e tc. for \( \langle A, \Gamma \rangle \xrightarrow{F, \alpha} \langle B, \phi \rangle \) etc. when it serves to clarify the discussion. In this case the notation is particularly justified for \( \text{SET}^{\text{op}} \) is equivalent to the 2-full sub 2-category of \( \text{CAT}^2 \) determined by the discrete fibrations. Thus if \( \Gamma \rightarrow A \) etc. is the discrete fibration associated to \( A^{\text{op}} \rightarrow \text{SET} \) etc. \( F, \alpha \) becomes simply a commutative square in \( \text{CAT} \).
2.1 PROPOSITION \( \text{SET}(\_)^{\text{op}} \) admits cotensoring and tensoring with \( \mathbb{2} \) and \( \mathbb{F} \) preserves these constructions.

Proof: A 2-cell \( \langle A, G, \alpha \rangle \xrightarrow{\Phi, \beta} \langle B, \phi \rangle \) is determined by \( \beta \) and \( \Phi \), since \( \alpha = \beta(t^{\text{op}}) \). Thus \( \langle B, \phi \rangle^{\mathbb{2}} \) is seen to be \( \langle B^{\mathbb{2}}, B^{\text{op}} \rangle \xrightarrow{\text{op}} B^{\text{op}} \xrightarrow{\phi} \text{SET} \), and \( \langle A, G \rangle \times \mathbb{2} = \langle A \times \mathbb{2}, (A \times \mathbb{2})^{\text{op}} \rangle \xrightarrow{\text{op}} A^{\text{op}} \xrightarrow{\Gamma} \text{SET} \).

2.2 THEOREM \( \text{SET}(\_)^{\text{op}} \) is \( \mathbb{U}_2 \) complete and \( \mathbb{U}_2 \) cocomplete as a 2-category and \( \mathbb{F} \) is continuous and cocontinuous as a 2-functor.

Proof: Given \( \mathbb{I} \xrightarrow{D} \text{SET}(\_)^{\text{op}} \), \( (\mathbb{I} \in \text{CAT}) \),

\[
\lim D = \langle L = \lim (D \mathbb{F}) \rangle, \quad \text{OP} \xrightarrow{\text{SET}} \mathbb{I} \xrightarrow{\lim} \text{SET} \]

where \( \text{OP} \xrightarrow{\text{SET}} \mathbb{I} \) is determined as follows: For each \( \mathbb{I} \xrightarrow{a} j \) in \( \mathbb{I} \) we have...
where the $P_i$ denote the projections. This determines a functor, $I \longrightarrow \text{SET}^{\text{op}}$, the double transpose of which is $L^{\text{op}} \longrightarrow \text{SET}^I$ above. Thus, if we denote the composite $L^{\text{op}} \longrightarrow \text{SET}^I \overset{\text{lim}}{\longrightarrow} \text{SET}$ by $\Lambda$, we have

$\Lambda \alpha = \underset{\longleftarrow}{\text{lim}} \quad L^{\text{op}}_i P_{D_i}$ for $\leq \Lambda$.

Hence

\[
\begin{array}{ccc}
\langle A, \Gamma \rangle & \xrightarrow{\langle F, \alpha \rangle} & \langle L, \Lambda \rangle \\
A & \xrightarrow{F} & L, \quad \Gamma & \xrightarrow{\alpha} & F^{\text{op}} \Lambda \\
\hline
\langle A, F_i \rangle & \xrightarrow{D_i \gg} & \langle i, \gg \rangle \\
A & \xrightarrow{F} & L, \quad \Lambda & \xrightarrow{\alpha} & \langle D_i, \gg \rangle \\
\hline
\end{array}
\]

as required. By proposition 2.1 all limits are 2-enriched.

Clearly $\gg$ is 2-continuous.
We implicitly mentioned the 2-functor $\mathbf{SET}^\text{op} \xrightarrow{\Pi} \mathbf{CAT}^2$ given by $\langle \mathbb{A}, \Pi \rangle \Pi = (\mathbb{A}/\mathbb{T} = \Pi \xrightarrow{P_T} \mathbb{A})$.

$\Pi$ is full and faithful and locally full and faithful. It has a left adjoint over $\mathbf{CAT}$ as in the following diagram:

$\mathbf{SET}^\text{op} \xrightarrow{\Pi} \mathbf{CAT}^2$

$\downarrow \Pi$  \hfill $\downarrow \Pi$

$\mathbf{CAT}$ \hfill $\mathbf{CAT}$

$\mathbb{P}$

Claim: $(\mathbb{B}, \mathbb{P} \rightarrow \mathbb{B}) \mathbb{L} = \mathbb{P}^\text{op} \rightarrow \mathbb{P} \to \mathbb{B} \xrightarrow{\Pi} \mathbb{A}$

where $\mathbb{P} \to \mathbb{B}$ denotes the connected components functor from $\mathbf{CAT}$ to $\mathbf{SET}$.

Fix $\mathbb{B} \xrightarrow{\mathbb{F}} \mathbb{A}$, then

$\mathbb{E} \xrightarrow{\mathbb{P}} \mathbb{B} \xrightarrow{\mathbb{F}} \mathbb{A}$

in $\mathbf{CAT}^2$

$\mathbb{P} \xrightarrow{\mathbb{P}_T} \mathbb{A}$

$\mathbb{B} \xrightarrow{\mathbb{F}} \mathbb{A}$

$\mathbb{E} \xrightarrow{\mathbb{P} \mathbb{F}^*} \mathbb{A}$

$\mathbb{E} \xrightarrow{\mathbb{P} \mathbb{F}^*} \mathbb{A}$

$\mathbb{B}$

$\mathbb{F} \xrightarrow{\mathbb{F} \mathbb{P}^*} \mathbb{A}$

$\mathbb{P} \mathbb{F}^* = \mathbb{P} \mathbb{F}^* \xrightarrow{\mathbb{P} \mathbb{F}^*} \mathbb{A}$
The second bijection results from the adjunction

\[ \text{CAT} \leftarrow \text{SET}^{\text{op}} \]

Thus from the diagram (*) we see that \text{SET}(\text{op}) is cocomplete, 2-cocomplete in virtue of 2.1, and that \text{P} is 2-cocontinuous. If \text{SET}(\text{op}) is regarded as a sub 2-category of \text{CAT}^2, then limits are calculated as in \text{CAT}^2.

2.3 THEOREM \text{SET}(\text{op}) is cartesian closed and \text{P} preserves exponentiation. \text{<C,Ψ><A,Γ> is given by <C,A,Ψ> where for } G \in (\text{C})^{\text{op}}, (G)^{\Gamma} = \int_A (A^\Gamma, AG^\Psi) \text{, the end taken over } A.

Proof: Let \Gamma, \Phi, and \Psi.

\[
\begin{align*}
&\vdots \\
&\vdots \\
&\vdots \\
&A & B & C \\
\end{align*}
\]

Then we have:

\[
\begin{align*}
\Gamma \times \Phi \\
\vdots \\
A \times B \\
\end{align*}
\]

where \text{<A,B>Γ×Φ = AΓ × BΦ}, by theorem 2.2. Let
\[ A \times B \xrightarrow{F} C \quad \text{and} \quad B \xrightarrow{\hat{F}} C^A \quad \text{its exponential transpose. We show that morphisms over } F \text{ are in bijective correspondence with morphisms over } \hat{F} \text{ as suggested by the diagrams:} \]

\[ \begin{array}{c}
\Phi \times \Phi \xrightarrow{\gamma} \gamma' \\
\begin{array}{c}
A \times B \xrightarrow{F} C
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\Phi \xrightarrow{\gamma'} \gamma' \\
B \xrightarrow{\hat{F}} C^A
\end{array} \]

\[ 2 = (\Phi, \hat{F}^{\text{OP}} \gamma') \left( \text{SET}^{\text{OP}} \right) \]

\[ \sim \int_{B} (B \Phi, (BF^{\text{OP}}) \gamma') \]

\[ \sim \int_{B} (B \Phi, \int_{A} (A \Gamma, A(BF^{\text{OP}}) \gamma')) \]

\[ \sim \int_{A, B} (B \Phi, (A \Gamma, <A, B>F^{\text{OP}} \gamma')) \]

\[ \sim \int_{A, B} (A \Gamma \times B \Phi, <A, B>F^{\text{OP}} \gamma') \]

\[ \sim (\Gamma \times \Phi, F^{\text{OP}} \gamma') \left( \text{SET}^{(A \times B)^{\text{OP}}} \right) \]

\[ = 1 \]

2.4 PROPOSITION \[ \text{SET}^{(\cdot)^{\text{OP}}} \xrightarrow{\text{L}} \text{CAT}^{2 \times 2} \]

preserves finite products.

Proof: Let \( E \xrightarrow{P} B \) and \( E' \xrightarrow{P'} B' \) be objects of \( \text{CAT}^{2 \times 2} \). To show that \( (P \times P') \Pi_0 = PL \times P'L \) we have to show that \( <B, B'>/(P \times P') \Pi_0 \subseteq (B/P) \Pi_0 \times (B'/P') \Pi_0 \) for all \( <B, B'> \in B \times B' \). \( <B, B'>/(P \times P') \subseteq B/P \times B'/P' \) so it suffices
to show that \( \text{CAT} \xrightarrow{\pi_0} \text{SET} \) preserves binary products.

But if \( A \in \text{CAT} \) is partially described by

\[
\begin{array}{c c c}
\partial_0 & \rightarrow & A_0 \\
\downarrow & & \downarrow \\
\text{A}_1 & \leftarrow & A_1
\end{array}
\]

\( A_{\pi_0} \) is the coequalizer of the reflexive pair \( \langle \partial_0, \partial_1 \rangle \). Since, in \( \text{SET} \), a product of coequalizers of reflexive pairs is a coequalizer of the product of the reflexive pairs we have what we want. Trivially

\( (1) \pi_0 = 1 \), from which we have \( (1 \rightarrow 1) \mathbb{L} \xrightarrow{1} \text{I}^{\text{op}} \rightarrow \text{SET} \).

If we regard \( \text{SET}^{(\cdot)^{\text{op}}} \), via \( \mathbb{I} \), as the 2-full subcategory of \( \text{CAT}^{2} \) determined by the discrete fibrations, proposition 2.4 allows us to extend theorem 2.3 as follows.

2.5 PROPOSITION If \( F \) and \( P \) are objects of \( \text{CAT}^{2} \) and \( F \) is a discrete fibration then \( F^{P} \) is a discrete fibration.

Proof: \( \text{CAT}^{2} = (\text{SET}^{2})_{\text{cat}} \) which is certainly cartesian closed. We have shown that \( \text{SET}^{(\cdot)^{\text{op}}} \) is a full, reflective subcategory of \( \text{CAT}^{2} \) and that the reflector preserves finite products. The proposition now follows from [F], proposition 1.31.
§2 "CATEGORIES" IN $\text{SET}(\cdot)^{\text{op}}$

A concept defined by commutative diagrams in a category admits various generalizations in a 2-category. For some, or all, of the commutativity conditions can be replaced by isomorphisms, as in

\[
\begin{array}{c}
\downarrow \\
\triangle
\end{array}
\]

to obtain a corresponding \textit{pseudo} concept. Such isomorphisms are often required to satisfy coherence axioms. Just which equalities are to be "relaxed", and the specific nature of the coherence conditions is usually determined by examples which occur in practice.

Now a pseudo diagrammatic concept in the 2-category $\text{SET}(\cdot)^{\text{op}}$ involves, via $\text{FP}$, an underlying pseudo diagrammatic concept in $\text{CAT}$. Moreover, all the "pseudoness" is contained in the underlying concept, since the fibres of $\text{FP}$ are just ordinary categories. We now consider briefly some "pseudo category objects" in $\text{SET}(\cdot)^{\text{op}}$. Since it is obvious in each case which equalities have been relaxed in $\text{CAT}$, we do not give a formal definition and speak just of "categories" in $\text{SET}(\cdot)^{\text{op}}$.

2.6 PROPOSITION. A large $\mathcal{V}$-category, as in definition 1.1, is precisely a "category" in $\text{SET}(\cdot)^{\text{op}}$, over the
pseudo monoid \( V : V \times V \xrightarrow{\theta} V \xleftarrow{U} 1 \) in \( \text{CAT} \).

**Proof:** Transcription of the data for \( V \) in 1.1 readily shows that \( M \times M \) is a pullback in \( \text{SET}^{(\_)^{\text{op}}} \) where \( \circ \) is considered as \( \text{Id}^{\text{op}} \xrightarrow{\circ} \text{SET} \). The relevant diagram is

\[
\begin{array}{ccc}
A & \xrightarrow{V_0} & M \\
\downarrow \circ & & \downarrow \circ \\
\text{Id} & \xrightarrow{\text{Id}_1} & M & \xleftarrow{\text{Id}_1} & \text{Id} \\
\downarrow & & \downarrow & & \downarrow \\
V & \xrightarrow{V_0} & V & \xleftarrow{U} & 1
\end{array}
\]

with notation exactly as in definition 1.1.

For \( \text{Id}V \) let \( I \) denote the "category":

\[
\begin{array}{cccc}
&&& \\
\text{I} & \xrightarrow{=} & \text{I} & \xleftarrow{=} & \text{I} \\
\downarrow & & \downarrow & & \downarrow \\
\text{V} & \xrightarrow{=} & \text{V} & \xleftarrow{=} & \text{V}
\end{array}
\]

where \( I \) "over" \( V \) means \( \text{Id}^{\text{op}} \xrightarrow{(-, I)} \text{SET} \) and all morphisms are identities.

\( I \) is a discrete "category" but the important feature to note at this stage is that it has \( V \)-indexed families of objects. For \( \text{Je}V \), a \( J \)-indexed family of objects of \( I \) or \( J \)-object is simply a morphism \( \text{J} \xrightarrow{u} \text{I} \) in \( V \).

For any \( A \in \text{V-CAT} \), regarded as a "category", we will
presently show that a "functor" $I \rightarrow \mathcal{A}$ is determined by an object, $B$ say, of $\mathcal{A}$ and the "limit" of such a functor exists if and only if the cotensor $[IB]$ exists, in which case they are isomorphic.

For our second example let $A_0 \in \text{CAT}_V$. Consider first the comma category $A_0/A$ in $\text{CAT}_V$. From the description of $A_0^2$ in chapter 1, it follows that $A_0/A$ has as objects $U$-morphisms with domain $A_0$ and as $I$-morphisms, $I$-squares of the form:

$$
\begin{array}{c}
A_0 \\
\downarrow \langle U; a \rangle \\
A \\
\downarrow \langle I; f \rangle \\
B
\end{array}
\quad
\begin{array}{c}
\langle I; A_0 u \rangle \\
\downarrow \langle U; b \rangle \\
A_0
\end{array}

in $\mathcal{A}$, where $I \xrightarrow{u} U$ is in $V$.

Incidentally we have immediately, by inspection of the above, that $[ab]$ is given by the following pullback in $\text{SET}^{U^\text{OP}}$:

$$
\begin{array}{c}
[ab] \\
\downarrow \\
U
\end{array}
\quad
\begin{array}{c}
\downarrow \\
\downarrow \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
[ab] \\
\downarrow \\
\downarrow \\
\rightarrow
\end{array}

\begin{array}{c}
[AB] \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
[[AB]] \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
[[aB]] \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
[[A_0 B]]
\end{array}
This shows explicitly that for \( A \) locally small and \( V \)
with pullbacks that \( A_0/A \) is locally small. \( A_0 \overset{i \in I}{\longrightarrow} A \)
may be regarded as an \( I \)-indexed family of "scalar
multiplications". For if \( V \) is the category of \( (U_I) \)
k-vector spaces over a field \( k \), \( i \in I \), \( i \in I \), \( \langle i \rangle_j \cdot A_0 \)
where \( (i) u \cdot A_0 \) denotes multiplication of the identity
by the scalar \( (i) u \). For a general \( V \) of course we
can only say that \( U \) is a commutative monoid object in
\( V \) which acts canonically on both sides of all objects of
\( V \). The "scalars" are then just morphisms \( U \longrightarrow U \) and
\( I \)-indexed families of scalars are morphisms \( I \longrightarrow U \).

Returning to general consideration of the \( V \)-category
\( A_0/A \), we note that it sacrifices information that is
canonically available. For the objects of \( A_0/A \) can be
"clumped", that is written as \( \sum_{A \in A} (U)(A_0, A) \); and each of
the "clumps" \( \sum_{A \in A} (U)(A_0, A) \) can be enriched to \( (A_0, A) \). To
paraphrase two well known \( V \)-authors: "the \( (U)(A_0, A) \),
of themselves, are clearly not the proper objects of
interest for \( V \)-categories. For instance, when \( V \) is graded
abelian groups, \( (U)(A_0, A) \) contains only an infinitesimal
part of the information in \( (A_0, A) \)." We are led to define
an \( I \)-indexed family of objects of \( A_0/A \) as an \( I \)-morphism
\( A_0 \overset{i \in I}{\longrightarrow} A \) for some \( A \) in \( A \).

We can give a heuristic justification for the above
as follows. An \( I \)-indexed family of objects in a \( V \)-indexed
category \( X \), whatever that is to be, should be determined by a functor from \( I \) regarded as a discrete \( V \)-indexed category to \( X \). Thus if we construct a more enriched "category", from the data used to form \( A_0/A \), with "objects" given by \( \underset{A \in A}{\varinom{op}} A \), a functor from \( I \) to such a category should be determined by a natural transformation:

\[
(\gamma I) \quad \rightarrow \quad \Sigma \underset{A \in A}{[A_0 A]}
\]

Such a natural transformation is, by the Yoneda lemma, what we call an \( I \)-morphism \( A_0 \xrightarrow{I:a} A \) for some \( A \in A \).

We now express the above ideas in our second new example of a "category", which we provisionally denote by \( A_0//A \):

\[
\begin{array}{ccc}
A_0//A : & \Sigma \underset{A,B,C \in A}{[A_0 A] \times [A B] \times [B C]} & \rightarrow & \Sigma \underset{A,B \in A}{[A_0 A] \times [A B]} & \rightarrow & \Sigma \underset{A \in A}{[A_0 A]} \\
& \rightarrow & A,B \in A & \rightarrow & A \in A
\end{array}
\]

\[
\begin{array}{ccc}
V \underset{\sim}{\times V \times V} & \xrightarrow{<P_0,P_1>} & V \times V & \xleftarrow{<V,U>} & V
\end{array}
\]

\[A_0//A\] has for each \( I \) and \( J \) in \( V \), \( I,J \)-indexed families of morphisms or \( I,J \)-morphisms, a typical one being
where the components are $V$-morphisms of $A$. The domain of such an $I,J$-morphism is the $I$-object $A_0 \xrightarrow{<I;a>} A$ and the codomain is the $I \circ J$-object $A_0 \xrightarrow{<I \circ J; af>} B$. Given an $I$-object $A_0 \xrightarrow{<I;a>} A$, we can associate to it an $I,U$-morphism:

the identity; and given an $I,J,K$-composable pair

its composite is the $I, J \circ K$-morphism.
Lest this "category" appear bizarre, the reader should note that the indexing "category" in $\text{CAT}_{\frac{0}{\mathcal{V}}}$, is itself the comma "category" $0/\mathcal{V}$, for "categories" in $\text{CAT}$, where 0 denotes the unique "object" of $\mathcal{V}$. Noting that the indexing "category" of the discrete "category" $\mathbb{I}$ is discrete as a "category" in $\text{CAT}$ we are led to the following general principle: When categorical constructions are applied to "categories" the types of families involved should be subjected to the same constructions.

§3. $\mathcal{V}$-INDEXED SETS AND $\mathcal{V}$-INDEXED CATEGORIES

In this section we pursue the discussion introduced at the beginning of §1 and extract from $\text{SET}^{(\cdot)^{\text{op}}}$ that which we require to speak of a general $\mathcal{V}$-indexed category.

Let $\mathcal{T}$ denote the category of finitely presented monoids. An object $T$ of $\mathcal{T}$ is a diagram

\[
\begin{array}{c}
\bar{m} \\
\downarrow f \\
\bar{n}
\end{array}
\leftarrow
\begin{array}{c}
\bar{r} \\
\downarrow g
\end{array}
\rightarrow
\bar{n}
\]

where $\bar{m}$ (respectively $\bar{n}$) denotes the finitely generated free monoid on $m$ (respectively $n$) generators, and $f$, $g$, and $r$ are monoid homomorphisms satisfying $rf = \bar{n} = rg$. We will suppress mention of $r$ (the reflexivity) when it is convenient to do so. If $T' = (\bar{m}' \begin{array}{c} f'_1 \\ \downarrow g'_1 \end{array} \bar{n}')$ is also an
object of \( T \), a morphism from \( T' \) to \( T \) is an
equivalence class, \( \bar{h} \), of monoid homomorphisms
\( \bar{n}' \xrightarrow{h} \bar{n} \) such that \( f'\bar{h} = g'\bar{h} \). The equivalence
relation is generated by the following relation: \( h \) is
related to \( h' \) if there exists a homomorphism \( \bar{n}' \xrightarrow{k} \bar{m} \)
such that \( kf = h \) and \( kg = h' \) as suggested by the
diagram below.

\[
\begin{align*}
T' & \xrightarrow{f'} \bar{m}' \xrightarrow{g'} \bar{n}' \\
\bar{n} & \xrightarrow{h} \bar{h}' \\
T & \xrightarrow{f} \bar{m} \xrightarrow{g} \bar{n}
\end{align*}
\]

\( f'\bar{h} \) (respectively \( g'\bar{h} \)) denotes the corresponding
equivalence class of morphisms \( \bar{m}' \longrightarrow \bar{n} \) obtained by
composing with \( f \) (respectively \( g \)). The situation
is best described by:

\[
\begin{align*}
(T', T) & \\
(\bar{n}', \bar{m}) & \xrightarrow{(\bar{n}', f)} (\bar{n}', \bar{n}) \xrightarrow{(\bar{n}', T)} (\bar{n}', T) \\
(f', \bar{m}) & \xrightarrow{(g', \bar{m}) (f', \bar{n}) (f', T) (g', T)} (\bar{m}', \bar{n}) \xrightarrow{(\bar{m}', T)} (\bar{m}', T)
\end{align*}
\]
where the two rows are coequalizers and the third column is an equalizer. A routine calculation shows that composition of equivalence classes via representatives is well defined. (Note that the suppressed morphisms, \( r \), occurring in the definition of objects of \( T \), are not redundant when morphisms are presented as above.)

2.7 **LEMMA** \( T \) is finitely cocomplete.

**Proof:** \( \vec{0} \xrightarrow{0} \vec{0} \) is initial and the coproduct of objects \( T \) and \( T' \), as above, is given by

\[
\vec{m} + \vec{m}' = \vec{m} + \vec{m}' \xrightarrow{f + f'} \vec{n} + \vec{n}'.
\]

If \( h \) and \( h' \) are representatives of morphisms \( \vec{h} \) and \( \vec{h}' \) defining a reflexive pair from \( T' \) to \( T \) as in:

\[
\begin{array}{ccc}
T' & \xrightarrow{\vec{m}'} & \vec{n}' \\
\downarrow{\vec{h}} & & \downarrow{\vec{h}'} \\
T & \xrightarrow{\vec{m}} & \vec{n}
\end{array}
\]

their coequalizer may be given as:

\[
\begin{array}{ccc}
\frac{f}{g} & \xrightarrow{(f'h)} & \vec{n} \\
\frac{f}{g} & \xrightarrow{(g'h')} & \vec{n}
\end{array}
\]
with coequalizer morphism the equivalence class of the identity.

If we denote the category of finitely generated free monoids (that is the theory of monoids) by \( F \), the functor \( F \to T \), given by \( \bar{m} \mapsto (\bar{m} \to \bar{m}) \), is full and faithful and preserves finite coproducts. Moreover, the object \( \bar{m} \xrightarrow{f} \bar{n} \) in \( T \) is the coequalizer of \( (\bar{m} \to \bar{m}) \xrightarrow{f} (\bar{n} \to \bar{n}) \). In fact:

2.8 LEMMA \( T \) is the finite colimit completion of \( F \).

That is, if \( F \xrightarrow{G} C \) is any finite coproduct preserving functor into a category with finite colimits, there is a finite colimit preserving functor, \( T \xrightarrow{\xi} C \), unique up to isomorphism, such that

\[
\begin{array}{ccc}
F & \to & T \\
G & \downarrow \ & \downarrow \xi \\
& \to & C
\end{array}
\]

commutes.

The category \( T \) is equivalent to the full subcategory of monoids determined by the finitely presentable monoids. The equivalence functor sends an object \( \bar{m} \xrightarrow{f} \bar{n} \) in \( T \) to the coequalizer of the corresponding pair in the
category of monoids. The distinction between these categories is important for us since we will define a \( \text{CAT} \)-valued pseudo functor on \( T^{\text{op}} \), and pseudo functors do not preserve isomorphisms.

We will write \( \vec{n} \) or \( \underbrace{\vec{N} + \ldots + \vec{N}}_{\vec{n}} \) for the objects \( \vec{n} \) in \( T \).

Next, consider the assignments \( \vec{n} \to \underbrace{\vec{V}^n}_{\vec{x}} \),

\[
\begin{align*}
(\overline{1} & \xrightarrow{x y} \overline{2}) \quad & \mapsto & \quad (\overline{V^2} & \xrightarrow{a} \overline{V}) \quad , \quad \text{and} \quad (\overline{1} & \xrightarrow{1} \overline{0}) \quad \mapsto \quad \overline{V^0} \quad , \\
(\overline{V^0} & \xrightarrow{u} \overline{V}) \quad \text{where} \quad \underbrace{\overline{V^n \times \ldots \times \overline{V}_n}}_{\overline{V}^n}, \quad \overline{V^0} = \overline{I},
\end{align*}
\]

and \( x y \) and \( 1 \) are the defining operations for the theory of monoids. We extend the above assignments to all of \( F \) as follows: If \( \overline{I} \xrightarrow{w} \overline{n} \) is the morphism \( \overline{I} \xrightarrow{1} \overline{0} \xrightarrow{1} \overline{n} \) we assign to it the functor \( \overline{V^n} \xrightarrow{1} \overline{I} \xrightarrow{u} \overline{V} \), otherwise we apply the leftmost association to the letters of \( w \), (with all 1's deleted), and assign to it the corresponding functor built from instances of \( \emptyset \) and the functors involved in the product structure of the \( \overline{V^k} \). Thus, for example, if \( \langle x, y, z, u, v \rangle \) denotes the (ordered) set of generators of \( \overline{5} \), we assign to the morphism \( \overline{I} \xrightarrow{vvyyzu} \overline{5} \) the functor \( \overline{V^5} \xrightarrow{\overline{V}} \overline{V} \) which sends \( \overline{I, J, K, L, M} \) to \( \overline{((M@J)@J)@K}@L \). Finally, if to \( \overline{I} \xrightarrow{w_i} \overline{n} \), for \( i = 1, \ldots, m \), are assigned
the functors $\mathbb{V}^n \xrightarrow{W_i} \mathbb{V}$, we assign to $\bar{m} \xrightarrow{w_1 \ldots w_m} \bar{n}$ the functor $\mathbb{V}^n \xrightarrow{\langle w_1, \ldots, w_m \rangle} \mathbb{V}^m$. These assignments do not define a functor $\mathcal{P}^{\text{op}} \rightarrow \text{CAT}$ since composition is preserved only up to isomorphism. However, the isomorphisms involved are built from instances of

\begin{align*}
\begin{tikzcd}
\mathbb{V} & \mathbb{V}^2 \\
\mathbb{V} \\
\end{tikzcd}
\end{align*}

\begin{align*}
\begin{tikzcd}
\mathbb{V} & \mathbb{V}^2 \\
\mathbb{V} \\
\end{tikzcd}
\end{align*}

and

\begin{align*}
\begin{tikzcd}
\mathbb{V}^3 \\
\mathbb{V} \\
\end{tikzcd}
\end{align*}

which are assumed coherent. Thus we have:

\[ 2.9 \text{ LEMMA } \] The above assignments, which depend only on $\mathbb{V}$, determine a (normalized) pseudo functor, (in the sense of say [K&S]), $\mathcal{P}^{\text{op}} \xrightarrow{\mathbb{V}^-} \text{CAT}$, which preserves finite products

If $\bar{m} \xrightarrow{f} \bar{n}$ is an arbitrary morphism in $\mathcal{F}$ we will denote $\mathbb{V}^-$ of it by $\mathbb{V}^n \xrightarrow{f^\#} \mathbb{V}^m$. 
In the spirit of lemma 2.8 we wish to extend \( V^- \) to \( T^{OP} \). However, if we naively define \( V^- \) at

\[
T = (\bar{m} \xrightarrow{f} \bar{n}) \text{ to be the equalizer of } \begin{array}{ccc} V^n & \xrightarrow{f^#} & V^m \\ \downarrow g & & \downarrow g^* \end{array}
\]

we do not obtain categories like \( V^{[n]} \) (as in §1) in the image of \( V^- \). We want to interpret identities between "words" via specified isomorphisms. This requires some preliminaries.

Let \( f \) and \( g \) be as above and introduce the formal expression, \( f \sim g \). If \( \bar{I} \xrightarrow{k} \bar{m} \) we write \( kf \sim kg \) and \( kg \sim kf \) and call these simple consequences of \( f \sim g \). If \( h = h_1 \sim h_2 \), \( h_2 \sim h_3 \), \ldots, \( h_{k-1} \sim h_k = h' \), for some finite cardinal \( k \geq 1 \), are simple consequences of \( f \sim g \), or \( h = h' \); we write \( h \sim h' \) and call it a consequence of \( f \sim g \). A finite list of simple consequences as above involves a finite, well formed list of morphisms \( \bar{I} \xrightarrow{k} \bar{m} \), and we call such a list a proof of \( h \sim h' \).

Intuitively we think of the pair \( \langle f, g \rangle \) as defining \( m \) "equations" in the free monoid \( \bar{n} \). Using various instances of some or all of these equations, together with identities that hold in the theory of monoids, we can generate more equations: the "consequences" of \( f \sim g \). The form of a formal proof records where monoid identities and assumed equations are used in deriving a consequence.

We might have for example:
\[ h = k_1 f \sim k_1 g = k_2 f \sim k_2 g = k_3 g \sim k_3 f = h' . \]

We now interpret these formal concepts "in \( V \). Suppose that we have an object \( \hat{I} \in V^h \) and an isomorphism \( \hat{I}f^\# \rightarrow \hat{I}g^\# \) in \( V^m \). Thus \( \gamma = \langle \gamma_1, \ldots, \gamma_m \rangle \) consists of \( m \) isomorphisms in \( V \) and we regard it as an interpretation of \( f \sim g \) in \( V \). If \( \hat{I} \rightarrow \hat{m} \), as before, we say that
\[
\hat{I}f^\# \rightarrow \hat{I}g^\# \text{ and } \hat{I}g^\# \rightarrow \hat{I}f^\#
\]
are simple consequences of \( \gamma \) in \( V \). For any morphisms \( a, b, c, \) and \( d \) in \( F \) with \( ab = cd \), we write
\[
b^\# \rightarrow d^\# \text{ for the isomorphisms, } b^\# \rightarrow (ab)^\# = (cd)^\#
\]
are constructed from the pseudo functor data of \( V \). Now for \( \hat{I} \rightarrow \hat{n} \) in \( F \) we say that an isomorphism
\[
\hat{I}h^\# \xrightarrow{\hat{I}, \gamma} \hat{I}h'^\#
\]
is a consequence of \( \gamma \) in \( V \) if it is either a composite of simple consequences in \( V \) and \( \beta \)'s or, if \( h = h' \), an identity.

If \( h \sim h' \) is a consequence of \( f \sim g \) and
\[
\hat{I}f^\# \rightarrow \hat{I}g^\#
\]
is given we obtain for each proof of \( h \sim h' \) a consequence, \( \hat{I}h^\# \xrightarrow{\hat{I}, \gamma} \hat{I}h'^\# \), of \( \gamma \) in \( V \) by interpreting simple consequences as simple consequences in \( V \) and monoid identities as \( \beta \)'s.
In general distinct proofs of $h \simeq h'$ will yield distinct consequences of $\gamma$ in $V$. We wish to single out those $\gamma$ for which this does not happen. Hence we say: An isomorphism $\frac{f}{g}$ is compatible if for every consequence $h \simeq h'$ of $f \simeq g$ there is a unique consequence of the form $\frac{h}{h'}$ of $\gamma$ in $V$.

"Compatibility" is a precise way of saying that any "well formed" diagram in $V$ involving the $\gamma_i$, $1 \leq i \leq m$, and the monoidal data for $V$ commutes. As such it involves an infinite set of diagrams in $V$. Whether, or when, compatibility can be expressed finitely is a coherence problem that we have not investigated. For the purpose of this paper it is not a question of immediate relevance. We give two examples.

Let $\langle f, g \rangle$ be given by morphisms $\frac{\mathbb{A}}{} \longrightarrow \frac{\mathbb{B}}{}$ defining $x = x$, $y = y$, $xy = yx$ and $x^2y = yx^2$. Let $\gamma = \langle I, J, I \circ J, \gamma_3, (I \circ I) \circ J, \gamma_4, (I \circ I) \circ I \rangle$ be an interpretation of $f \simeq g$. For $\gamma$ to be compatible we require that $\gamma_1$ and $\gamma_2$ be identities, (which is the case in general for a compatible interpretation of a reflexivity equation), and that
commutes.

If \( f \xrightarrow{f} g \) defines \( x = x, y = y, u = u, v = v \) and \( xy = uv \), and \( I \circ J \xrightarrow{I \circ J} K \circ L \) is any isomorphism in \( V \) then

\[
\langle I \mapsto I, J \mapsto J, K \mapsto K, L \mapsto L, I \circ J \xrightarrow{I \circ J} K \circ L \rangle
\]

is a compatible interpretation of \( f \sim g \).

Now for \( T = (m \xrightarrow{f} n) \) in \( T \) we define a category \( [TV] \) as follows. The objects are pairs \( \langle \tilde{I}, \gamma \rangle \) where \( \tilde{m} \xrightarrow{\gamma} \tilde{n} \) and \( \tilde{f} \xrightarrow{\gamma} \tilde{g} \) is a compatible isomorphism.

A morphism from \( \langle \tilde{I}, \gamma \rangle \) to \( \langle \tilde{I}', \gamma' \rangle \) is a triple \( \langle \gamma, \tilde{u}, \gamma' \rangle \) where \( \tilde{u} \xrightarrow{\gamma} \tilde{u} \) is in \( V \) and

\[
\begin{align*}
\tilde{I}_f & \xrightarrow{\tilde{u}_f} \tilde{I}'_f \\
\gamma & \downarrow \\
\tilde{I}_g & \xrightarrow{\tilde{u}_g} \tilde{I}'_g
\end{align*}
\]

commutes. There is a functor \( [TV] \xrightarrow{D} V^n \) and a natural isomorphism \( D_i \xrightarrow{t} D_{i'} \) where

\( (\langle \tilde{I}, \gamma \rangle, \langle \gamma, \tilde{u}, \gamma' \rangle, \langle \tilde{I}', \gamma' \rangle) D = \tilde{I} \xrightarrow{\tilde{u}} \tilde{I}' \), and

\( \langle \tilde{I}, \gamma \rangle t = \gamma \). The data \( \langle [TV], D, t \rangle \) then constitutes a notion intermediate between that of ordinary equalizer and subequalizer in the sense of Lambek [1k]. For objects
of the form \( \tilde{n} \) in \( T \), the isomorphisms involved in
the objects of \([\tilde{n}V]\) are interpretations of reflexivity
so they are forced, by the compatibility requirement, to
be identities. It follows that \([\tilde{n}V]\) and \(V^n\) are
isomorphic and can be safely identified, so that the
assignments \([TV]\) extend \(V^-\) to \(T^{op}\) on objects.
Note also that \(\tilde{V}^j\) arises from the \(T = \langle f, g \rangle\) given
in our second example of compatibility.

To define \([-V]\) on morphisms of \(T^{op}\) we first
consider the diagram

\[
\begin{array}{c}
\tilde{n}
\downarrow h
\\hline
\tilde{n}'
\end{array}
\]

\[
(T; \tilde{m} \xrightarrow{f} \tilde{n} \xrightarrow{g} \tilde{n'})
\]

in \( F \), and define \([TV] \tilde{h} \rightarrow V^{n'}\) to be the indicated
composite.

\[
\begin{array}{c}
[TV] \xrightarrow{D} V^{n} \xrightarrow{f \#} V^{m}
\downarrow h^*
\\hline
V^{n'}
\end{array}
\]

2.10 LEMMA If in the situation above \( \tilde{n}' \xrightarrow{h'} \tilde{n} \) is also
given, with \( h \) equivalent to \( h' \) via \( f \) and \( g \), then
h is coherently and naturally isomorphic to h'. Here by coherent we mean the following: If h \overset{s}{\longrightarrow} h', h' \overset{s'}{\longrightarrow} h'' and h \overset{s''}{\longrightarrow} h'' are the isomorphisms resulting from the equivalences h \simeq h', h' \simeq h'' and h \simeq h'', then

\[
\begin{array}{ccc}
h & \overset{s}{\longrightarrow} & h' \\
\downarrow{s''} & & \downarrow{s'} \\
h'' & \overset{?}{\longrightarrow} & h''
\end{array}
\]

commutes.

Proof: Assume first that \( \tilde{n}' = \tilde{I} \). In the vocabulary of the preceding paragraphs, h is equivalent to h' via f and g, means that h \simeq h' is a consequence of f \simeq g. Thus for any isomorphism, if \( \overset{\sim}{\longrightarrow} \), we have a consequence \( \tilde{h}' \overset{<I,\gamma>}{\longrightarrow} \tilde{h}'' \) in V, and the \( <I,\gamma> \)s constitute a natural isomorphism h = Dh' \overset{s}{\longrightarrow} Dh'' = h''. The coherence assertion is equivalent to the statement that the \( <I,\gamma> \)s are independent of the "proof" of h \simeq h', but this is trivially the case since the \( \gamma \)'s are compatible by definition.

For general \( \tilde{n}' \) the result follows from the above since

\[
\begin{array}{ccc}
\tilde{n}' & \overset{\lambda}{\longrightarrow} & \tilde{x} \\
\overset{<I,\gamma>}{\longrightarrow} & & \\
1 \leq i \leq n'
\end{array}
\]
and we can speak of \( n' \)-tuples of "consequences", etc.

Now suppose that \( T' \xrightarrow{f'} T \) is a morphism in \( T \),
a representative of which is given by:

\[
\begin{array}{c}
m' \\
| \\
\downarrow g' \\
| \\
\downarrow h \\
| \\
\downarrow g \\
\end{array}
\quad \begin{array}{c}
m' \xrightarrow{f'} n' \\
| \\
\downarrow h \\
\end{array}
\]

Thus \( f' \circ h \) is equivalent to \( g' \circ h \) and by lemma 2.10
we have a coherent natural isomorphism \( D(f' \circ h) \xrightarrow{s} D(g' \circ h) \).
From this we obtain by application of "\( \beta \)'s" a natural
isomorphism \( D h' \xrightarrow{s} D h' \), the components of which
are easily seen to be compatible. This induces a functor,
\( h' \), as in the diagram below.

\[
\begin{array}{ccc}
[TV] & \xrightarrow{D} & V^n \\
\downarrow h' & & \downarrow h' \\
[T'V] & \xrightarrow{D'} & V'^n
\end{array}
\quad \begin{array}{ccc}
V^n & \xrightarrow{f^*} & V^m \\
\downarrow g^* & & \downarrow g^* \\
V^m & \xrightarrow{f'^*} & V'^m
\end{array}
\]

Here the above square commutes and also \( h'^* t' = s \), where
\( t' \) is the "equalizer" isomorphism.

We have made a choice of representatives to define
but any choices $h$ and $h'$ define isomorphic $\tilde{h}^\#$ and $\tilde{h}'^\#$. This follows easily by lemma 2.10 and the definition of compatibility. If also $T \xrightarrow{\tilde{h}} T'$ lemma 2.10 further provides a coherent isomorphism $\tilde{h}^\# \tilde{h}'^\# \cong (\tilde{h}'\tilde{h})^\#$, regardless of the choices of representatives for $\tilde{h}$, $\tilde{h}'$, and $\tilde{h}'\tilde{h}$. For an identity morphism $T \cong T$ in $T$ we may choose an identity morphism in $\mathcal{F}$ as the representative. Summarizing, we have:

2.11 PROPOSITION The assignments above define a normalized pseudo functor $T^{\text{op}} \xrightarrow{[-\mathcal{V}]} \text{CAT}$ which renders

\[
\begin{array}{ccc}
\mathcal{F}^{\text{op}} & \xrightarrow{[-\mathcal{V}]} & T^{\text{op}} \\
\downarrow & & \downarrow \\
\text{CAT} & \xrightarrow{[-\mathcal{V}]} & \text{CAT}
\end{array}
\]

commutative.

We will continue to write $\mathcal{V}^n$, rather than $[\tilde{h}\mathcal{V}]$, for the value of $[-\mathcal{V}]$ on frees, and also $\tilde{h}^\#$ etc. for the values of $[-\mathcal{V}]$ on arbitrary morphisms of $T$. The essential feature of $[-\mathcal{V}]$ for most of our discussion is that it is a pseudo functor from $T^{\text{op}}$ to $\text{CAT}$ which depends only on $\mathcal{V}$. 
Now the split normal fibration \( \text{SET} \rightarrow \text{CAT} \)
of §1 corresponds to a functor, \( \text{CAT} \rightarrow \text{CAT} \),
given by:

\[
\begin{array}{c}
\text{A} \xrightarrow{F} \text{B} \\
\downarrow^G \\
\text{G}
\end{array} \rightarrow 
\begin{array}{c}
\text{SET} \leftarrow \text{SET}^\text{op} \rightarrow \text{SET}^\text{op} \leftarrow \text{SET}^\text{op}
\end{array}
\]

Upon composing this with the opposite of \( [-V] \) we obtain
a normalized pseudo functor: \( T \rightarrow \text{CAT} \rightarrow \text{CAT} \).

The corresponding normal fibration over \( T^\text{op} \) we denote
by \( V\text{-ind-SET} \rightarrow T^\text{op} \). Rather heuristically,
\( P_V \) is the "pullback" of \( T \) along the pseudo functor
\( [-V] \).

The somewhat complicated construction of \( P_V \) is
only imposed on us by the need to keep track of the coherence
data for \( V \) and the need to interpret certain equalities
by specified isomorphisms. An object of \( V\text{-ind-SET} \) is
simply a pair \( <T, \Gamma> \) where \( T \) is a finitely presented
monoid and \( \Gamma \) is a functor \( [TV]^\text{op} \rightarrow \text{SET} \). The categories
\( [TV] \) that one encounters in the study of ordinary (or
even large) \( V \)-categories are, like \( V \), for the most part quite manageable. We will in general refer to the objects of \( V \)-\textit{ind-SET} as \( V \)-\textit{indexed sets}, and specifically we will say that \( T \) is a \( V \)-\textit{indexed set of type} \( T \) when \( <T,T> \) is an object of \( V \)-\textit{ind-SET}. A morphism from \( <T,T> \) to \( <S,S> \) is a pair \( <f,\gamma> \) where \( S \xrightarrow{f} T \) is a monoid homomorphism and \( \gamma \) is a natural transformation as indicated below.

\[
\begin{array}{ccc}
[TV]^{\text{op}} & \xrightarrow{f^\#^\text{op}} & [SV]^{\text{op}} \\
\gamma \downarrow & & \phi \\
\text{SET} & & 
\end{array}
\]

\( V \)-\textit{ind-SET} is just a category. The 2-structure of \( \text{SET}(V)^{\text{op}} \) is forgotten, save for the coherent isomorphisms which make \([-V] \) a pseudo functor, and these are absorbed in the composition rules of \( V \)-\textit{ind-SET}.

One of the main contentions of this thesis is that \( V \)-indexed sets are the correct "fundamental" objects for studying those aspects of \( V \)-theory which involve conjunction, truth and equality. Our earlier remarks about the ambient nature of \( \text{SET}(V)^{\text{op}} \) concern the "pullback" diagram above. By enlarging the category of "types", \( T^{\text{op}} \), to include types corresponding to disjunction and implication etc., we can obtain higher order \( V \)-indexed sets.
This will be briefly considered in the appendix. On the other hand by restricting $\mathcal{T}^{op}$ to $\mathcal{F}^{op}$ we obtain those $\mathcal{V}$-indexed sets which are necessary for a theory involving only truth and conjunction. In any event, given "change of base" data $\mathcal{V} \xrightarrow{F} \mathcal{W}$, where $\mathcal{W}$ is also a monoidal category, we obtain a corresponding natural transformation $[-\mathcal{V}] \xrightarrow{[-F]} [-\mathcal{W}]$ which induces a functor over $\mathcal{T}^{op}$, $\mathcal{F}^{op}$, or "higher types" as indicated.

\[
\begin{array}{ccc}
\mathcal{V}-\text{ind-SET} & \xrightarrow{F} & \mathcal{W}-\text{ind-SET} \\
\mathcal{P}_\mathcal{V} & \xrightarrow{\mathcal{T}^{op}} & \mathcal{P}_\mathcal{W}
\end{array}
\]

Returning to $\mathcal{V}$-indexed sets as presently defined we note first that $\mathcal{V}$-ind-SET is suitable for discussing category objects.

2.12 Lemma $\mathcal{E} \xrightarrow{F} \mathcal{B}$ is a fibration and for some class of limits, $\mathcal{C}$, $\mathcal{B}$ has $\mathcal{C}$-limits, if $\mathcal{B}$ has $\mathcal{C}$-limits for all $I$ in $\mathcal{B}$ and $u^P$ preserves $\mathcal{C}$-limits for all $J \xrightarrow{u} I$ in $\mathcal{B}$, then $\mathcal{E}$ has $\mathcal{C}$-limits and $P$ preserves them.
2.13 PROPOSITION \( V \)-ind-SET has finite limits and \( P_V \) preserves them.

Proof: \( T^\text{op} \) has finite limits while \( T^V_P = \text{SET}^{[TV]}_\text{op} \) is \( U_2 \)-complete and \( fP^V_P = \text{SET}^{f_\text{op}}_\text{op} \) is \( U_2 \)-continuous for all \( T \to S \) in \( T \).

2.14 DEFINITION A \( V \)-indexed category is a category object in \( \text{V-ind-SET} \). Other terminology is similarly determined by:

\[
\text{V-ind-CAT} = (\text{V-ind-SET})\text{cat}
\]

By proposition 2.13 a \( V \)-indexed category projects to a category object in \( T^\text{op} \), that is to a cocategory object in \( T \). If

\[
\begin{array}{cccccc}
A & \xleftarrows{M \times M} & M & \xleftarrow{M} & O \\
& & \downarrow & & \\
& & & \vdots & & \\
& & & \vdots & & \\
& & & \vdots & & \\
& & & \vdots & & \\
T & \xleftarrow{T_0} & T_1 & \xleftarrow{T_1} & T_0
\end{array}
\]

is a \( V \)-indexed category we will say that \( A \) is a \( V \)-indexed category of type \( T \), where \( T \) is the displayed cocategory object in \( T \). It will frequently be convenient to write
just $A$ for a $V$-indexed category $\langle T, A \rangle$. If when this is done we later have to refer to the type of $A$, we will denote it by $\overline{A}$ if $\overline{T}$ is not available from the context. An analogous convention will apply to $V$-indexed sets, $V$-indexed functors etc.

From definition 2.14 and proposition 2.13 we immediately have:

2.15 PROPOSITION $\mathbf{V}$-ind-CAT is finitely complete as a 2-category and $(P_V)_{\text{cat}} = P_V$ preserves this property.

§4 EXAMPLES

Every $V$-category is a $V$-indexed category of type $\overline{N}$:

\[
\begin{array}{ccc}
N + N & \xleftarrow{X} & \overline{N} \\
\downarrow & & \downarrow \overset{Y}{\longrightarrow} \\
\overline{N} & \xleftarrow{1} & \overline{0}
\end{array}
\]

, and conversely every $V$-indexed category of type $\overline{N}$ is a $V$-category. If $A$ and $B$ are $V$-categories a $V$-indexed functor $A \xrightarrow{F} B$ has a type $\overline{N} \xrightarrow{t} \overline{N}$. This is determined by a morphism $N \xleftarrow{X^n} N$, between the middle components, and from the requirement that $t$ be a cofunctor we conclude that $n$ must be either 0 or 1. For $n = 1$ $F$ is a $V$-functor in the usual sense, while for $n = 0$ $F$ is an assignment that takes objects to objects and $I$-morphisms to $U$-morphisms for all $I$ in $V$, (preserving domain, codomain,
identities and composites as do all \( \mathcal{V} \)-indexed functors.

If \( F \) and \( G \) are \( \mathcal{V} \)-functors, between \( \mathcal{V} \)-categories \( A \) and \( B \), a \( \mathcal{V} \)-indexed natural transformation between them is precisely a \( \mathcal{V} \)-natural transformation. If \( F \) and \( G \) are of type \( 0 \) a \( \mathcal{V} \)-indexed natural transformation is a collection of \( U \)-morphisms in \( B \) with the usual property. If \( F \) and \( G \) are of different types there are no \( \mathcal{V} \)-natural transformations between them.

There is an obvious underlying \( \mathcal{V} \)-indexed set functor,\[
\mathcal{V}\text{-ind-CAT} \xrightarrow{\sim} \mathcal{V}\text{-ind-SET}
\]
which has a left adjoint. The latter assigns to a \( \mathcal{V} \)-indexed set \( \langle T, \Gamma \rangle \) the \( \mathcal{V} \)-indexed category:

\[
\begin{array}{ccc}
\Gamma & : & \Gamma \\
\downarrow & & \downarrow \\
\vdots & : & \vdots \\
\downarrow & & \downarrow \\
T & \cong & T \\
\end{array}
\]

The "category" \( I \) introduced in \( \S 2 \) becomes the discrete \( \mathcal{V} \)-indexed category on \( \langle \mathbb{N}, (-, I) \rangle \), and we will refer to it simply as \( I \). To classify functors from \( I \) to a \( \mathcal{V} \)-category \( A \) it suffices, since \( I \) is discrete, to consider just morphisms,
in $\mathbf{V}$-ind-$\mathbf{SET}$ where $0$ is the set of objects of $A$.

Such a morphism is, by the Yoneda lemma, determined by an object, $B$, say, of $A$. Thus all functors $I \to A$ are constant or in other words; all $I$-indexed families of objects of $A$ are determined by a single object of $A$.

To take the limit of a functor $I \to A$ we can proceed as in ordinary category theory. For $\mathbf{V}$-ind-$\mathbf{CAT}$ has a terminal object, $1$, which is the discrete $\mathbf{V}$-indexed category on $\langle 0,1 \rangle$, and a $\mathbf{V}$-indexed functor $1 \to A$ is also determined by an object, $A$, say, of $A$. Thus we can define a cone from $A$ to $B$ to be a $\mathbf{V}$-indexed natural transformation,

$$
\begin{array}{ccc}
I & \to & \mathbb{0} \\
\downarrow & \downarrow & \downarrow \\
B & \to & A
\end{array}
$$

Such an $f$ itself has a type, indicated by the dotted arrow below.
The commutativity requirement for naturality imposes no further restrictions on $x^n$ so that the cone $f$ may have "type $n$" for each $n \in \mathbb{N}$. The limit of $B$ should be a universal cone to $B$ and this should necessarily have a universal type. Direct calculation shows that there is no universal type, so $B$ does not have a limit in the usual sense. However for a fixed $n$ we can ask for a universal cone of type $n$, which we denote by

$$\lim B \xrightarrow{\overline{P_n}} B$$

A cone of type $n$ from $\overline{A}$ to $B$ is easily seen to be an $I^{(n)}$-morphism, $\overline{A} \xrightarrow{\langle I^{(n)}; f \rangle} B$, in $A$.

The universal such cone, $\lim B \xrightarrow{\langle I^{(n)}; \overline{P_n} \rangle} B$, has the property that for any $\overline{A} \xrightarrow{\langle I^{(n)}; f \rangle} B$ there exists a unique $\overline{A} \xrightarrow{\langle U; f' \rangle} \lim B$ such that
It follows from the considerations of §5 in chapter 1 that \( \lim B \) exists if and only if the weak cotensor \( [I(n)B]' \) exists, in which case they are isomorphic.

Since this answer is not as tidy as one might like we pursue the discussion further. On the one hand we have somewhat more than we want: \( [I(n)B]' \) for all \( n \in \mathbb{N} \), and on the other hand somewhat less: \( [IB]' \) rather than \( [IB] \). The first difficulty seems unavoidable, for even if \( [IB]' \) exists for all \( B \in \mathcal{A} \), \( A \xrightarrow{[I-]'} \mathcal{A} \) does not carry a canonical triple structure and we are forced to consider iterations of it separately. The second difficulty is less specific in nature. In any 2-category one can define Kan extensions, and if a terminal object is present limits and colimits can be discussed as special cases. However 1-cells \( \mathbb{I} \xrightarrow{A} \mathcal{A} \) are not necessarily particularly informative about the "objects" of \( \mathcal{A} \) in a general 2-category, and 2-cells \( \mathbb{I} \xrightarrow{A \xrightarrow{f} B} \mathcal{A} \) may be similarly lacking of information
about "morphisms" in $A$. Thus limits in a 2-category are generally not, of themselves, as useful as those in ordinary category theory. In our present situation the problem is with the morphisms. $V$-indexed natural transformations $\begin{array}{c} A \\ B \end{array} \rightarrow A$, for $A \in V\text{-CAT}$ are in bijective correspondence with $U$-morphisms from $A$ to $B$ in $A$. For $J \in V$, $J$-morphisms from $A$ to $B$ in $A$ are in bijective correspondence with $V$-indexed natural transformations, $\begin{array}{c} A \\ B \end{array} \rightarrow A$ of "type 1".

We now show that the universality of $[IB]$ can be expressed in terms of "Kan, extensions" in $V\text{-ind-CAT}$. True Kan extensions in $V\text{-ind-CAT}$ are too rare to be useful so we make a definition that takes account of types, as we did for the specific limits just described.

2.16 DEFINITION For $\varepsilon$ as shown:

$$
\begin{array}{ccc}
B & \xrightarrow{H} & C \\
& \searrow & \\
G & \xleftarrow{\varepsilon} & F \\
& \nearrow & \\
A & & 
\end{array}
$$

of type $t$, we say that $\varepsilon$ exhibits $F$ as a right $t,s$-extension of $G$ along $H$ if for any $\begin{array}{c} C \\ F' \end{array} \rightarrow A$ and any $\begin{array}{c} HF' \\ t \end{array} \rightarrow G$ of type $t$, there exists a unique $\begin{array}{c} F' \\ t' \end{array} \Rightarrow F$ of type $s$ such that
The product in $\mathcal{V}$-ind-SET of $\langle N, J \rangle$ and $\langle N, I \rangle$ is $\langle NN, J \times I \rangle$ where $\langle K, L \rangle J \times I = (K, J) \times (L, I)$ for all $\langle K, L \rangle$ in $\mathcal{V} \times \mathcal{V}$ $^{\text{op}}$. We denote the discrete $\mathcal{V}$-indexed category on $\langle NN, J \times I \rangle$ by $J \times I$.

2.17 PROPOSITION A weak cotensor $[IB]' \xrightarrow{[IP']^r} B$ in a $\mathcal{V}$-category $A$, is a cotensor if and only if for all $Jc\mathcal{V}$, $P_1P$ exhibits $[IB]' = [IB]'$ as a right $t,s$-extension of $P_1B$ along $P_0$; where the $P_i$ are the projections from the product as indicated below:

\[ \begin{array}{ccc}
 J \times I & \xrightarrow{P_0} & J \\
 \downarrow P_1 & & \downarrow I \\
 I & \xrightarrow{I} & \mathrm{II} \\
 \downarrow I & & \downarrow \mathrm{II} \\
 B & \xleftarrow{P} & [IB]' \\
 \end{array} \]

$t$ (the type of $P_1P$) is, in abbreviated form, $N+N \xleftarrow{V} N$, and $s$ abbreviated is $N \xleftarrow{X} N$. 
Proof: (The type $s$ formally includes the type of its domain. In this case there is only one possibility for the type of a $\mathcal{V}$-indexed functor $J \rightarrow A$ that is the domain of a $\mathcal{V}$-indexed natural transformation of "type $s$".) Assume that $J \xrightarrow{A} A$ is given. The type $\bar{F}_0$ is easily seen to be $\mathbb{N} \xleftarrow{\times} \mathbb{N}$, thus the type $\bar{F}_0$ is $\mathbb{N} \xleftarrow{\times V} \mathbb{N}$. Consider a $\mathcal{V}$-indexed natural transformation, $f$, of type $xy$ from $P_0A$ to $P_1B$. This is the same thing as an ordinary natural transformation,

\[
\begin{array}{ccc}
(y \times y)^{op} & \xrightarrow{\phi^{op}} & \mathcal{V}^{op} \\
J \times I & \xrightarrow{f} & M \\
\downarrow & \downarrow & \downarrow \\
\text{SET} & & \text{SET}
\end{array}
\]

where $M$ is the $\mathcal{V}$-indexed set of morphisms of $A$, subject to the requirement that $f_{\partial_0} = A$ and $f_{\partial_1} = B$, $\partial_0$ and $\partial_1$ being the domain and codomain assignments of $A$. Such an $f$ however is the same thing as a natural transformation from $\text{Lan}_{\mathcal{V}}^{J \times I}$ to $M$, and $\text{Lan}_{\mathcal{V}}^{J \times I} = (-, J \Omega I)$. Thus by the Yoneda lemma $f$ is completely determined by a $J \Omega I$-morphism from $A$ to $B$ in $A$: $A \xrightarrow{\langle J \Omega I; f \rangle} B$. If the diagram in the statement of the proposition is a right $t,s$-extension then for any $J \Omega I$-morphism $f$ as above there exists a unique $A \xrightarrow{\langle J; f' \rangle} [IB]$ such that
commutes. The proposition now follows immediately.

Just as an I-morphism, \( A \xrightarrow{I;f} B \), is a cone of type \( x \) from \( A \) to the functor \( I \xrightarrow{B} A \), so the above proposition provides us with a theoretical justification for considering a \( J@I \)-morphism \( A \xrightarrow{J@I;f} B \) as a \( J \)-indexed family of such cones from \( A \) to \( I \xrightarrow{B} A \).

Noting that
\[
\begin{array}{ccc}
J \times I & \xrightarrow{P_0} & J \\
\downarrow P_1 & & \downarrow 1 \\
I & \xrightarrow{!} & I
\end{array}
\]

is a trivial comma diagram in \( V\text{-ind-CAT} \), we see that the above proposition is an application of the "pointwise Kan-extension" concept introduced by Street, [S]. For a general \( V \) this concept is at variance in \( V\text{-Cat} \) with the notion of "pointwise" as introduced by Dubuc, [Dc]. The latter means that extensions are respected by the \( V \)-valued hom functors. Initial observations indicate that the two concepts are more compatible in \( V\text{-ind-CAT} \).
2.18 PROPOSITION A weak cotensor \([IB]^\prime \downarrow [\Lambda] \rightarrow B\) in a locally small \(\mathcal{V}\)-category \(\mathcal{A}\) is a cotensor if and only if for all \(A\) in \(\mathcal{A}\) it is preserved by \(\mathcal{A} \rightarrow [\Lambda] \rightarrow \mathcal{V}\).

Proof: Consider

\[
\begin{array}{ccc}
I & \rightarrow & \Pi \\
\downarrow & & \downarrow \\
B & \rightarrow & [IB]^\prime \\
\downarrow & & \downarrow \\
A & \rightarrow & [\Lambda] \\
\downarrow & & \downarrow \\
& V & \\
\end{array}
\]

where \(p\) is of type \(x\). We have \([A, [IB]^\prime] \downarrow [\Lambda] \rightarrow [AB]\) in \(\mathcal{V}\), that is \([A, [IB]^\prime] \otimes I \rightarrow [AB]\) in \(\mathcal{V}\). For it to constitute \(\operatorname{lim}_{\mathcal{V}} [AB] \rightarrow [AB]\) we require that for any \(J \in \mathcal{V}\) and any \(I\)-morphism from \(J\) to \([AB]\), \(J \otimes I \rightarrow [AB]\), there exists a unique ordinary morphism, \(J \rightarrow [A, [IB]^\prime]\), such that

\[
\begin{array}{ccc}
[A, [IB]^\prime] \otimes I & \rightarrow & [AB] \\
\downarrow f \otimes I & & \downarrow f \\
J \otimes I & \rightarrow & \end{array}
\]

commutes. But this is just the specific form that the definition of \([IB]\) takes in a locally small \(\mathcal{V}\)-category.
(From the bijection

$$
\begin{align*}
J \& I & \longrightarrow [AB] \\
J & \longrightarrow [A, [IB]]'
\end{align*}
$$

it follows that \([A, [IB]]' = [I, [AB]]^{rev}\).

Let \(A\) be a \(V\)-category considered as a \(V\)-indexed category. Thus we have a diagram:

$$
\begin{array}{c}
A : M \times M \rightarrow M \leftarrow I \rightarrow \emptyset \\
\vdots \\
\bar{N} : N + N \leftarrow XY \rightarrow N \rightarrow \emptyset
\end{array}
$$

in \(V\text{-ind-SET}\). To compute \(A^2\) in \(V\text{-ind-CAT}\) we can proceed as in any 2-category of the form \((S)\text{cat}\) where \(S\) is a category with finite limits. Hence the object of objects of \(A^2\) is given by the \(V\)-indexed set \(\langle N, M \rangle\). The object of morphisms is the pullback:

$$
\begin{array}{c}
\langle P, S \rangle \longrightarrow \langle N + N, M \times M \rangle \\
\downarrow \\
\langle N + N, M \times M \rangle \leftarrow \langle xy, \gamma \rangle \\
\downarrow \\
\langle N, M \rangle \leftarrow \langle xy, \gamma \rangle \\
\end{array}
$$
in $\text{V-ind-SET}$. Thus, $P$ is the pushout:

\[
\begin{array}{ccc}
N & \xrightarrow{xy} & N+N \\
\downarrow{xy} & & \downarrow{xy} \\
N+N & \rightarrow & P
\end{array}
\]

in $\text{T}$ which can be described as a monoid with four generators, $\langle x, y, u, v \rangle$, subject to the relation $xy = uv$. It follows that $[\text{PV}]$ is $\forall \chi$ and has as objects lists of the form: $\langle I, J, K, L; I\Theta J \leftarrow \chi K\Theta L \rangle$, with entries in $\text{V}$. Then $[\text{PV}]^\text{op} \xrightarrow{S} \text{SET}$ is given by the pullbacks:

\[
\begin{array}{ccc}
<I, J, K, L; \chi> & \rightarrow & <K, L> M < M \\
\downarrow & & \downarrow \chi M \\
<K, L> M < M & \rightarrow & (I\Theta J) M \leftarrow \chi M (K\Theta L) M
\end{array}
\]

in $\text{SET}$. Thus the $[\text{PV}]$-indexed families of morphisms of $A^2$ are indeed squares in $A$,

\[
\begin{array}{ccc}
A & \xrightarrow{<K; f>} & B \\
\downarrow{<I; a>} & & \downarrow{<L; b>} \\
A & \xrightarrow{<J; g>} & B
\end{array}
\]
as mentioned earlier. Of course in practice such calculations are generally unnecessary since one can be safely guided by the types of families involved.

If $A_0/A$ as above we can now compute the comma category $A_0/\mathcal{A}$ in $\mathcal{V}$-$\text{ind-CAT}$ via the usual pullback construction. However the type of the object of morphisms is clearly isomorphic in $\mathcal{T}$ to $\mathcal{N} \times \mathcal{N}$, so we can obtain a simpler description via the simpler presentation. In fact the "category" $A_0//\mathcal{A}$ given in §2 "is" $A_0/A$.

once $\mathcal{V}$'s are rewritten as $\mathcal{N}$'s and arrows "below the dots" are reversed and suitably relabelled.

§5 FURTHER CONSTRUCTIONS FOR $\mathcal{V}$-INDEXED CATEGORIES

A $\mathcal{V}$-indexed category of trivial type,

$$\overline{\mathcal{O}} : \overline{\mathcal{O}} \xrightarrow{\overline{\mathcal{O}}} \overline{\mathcal{O}} \xleftarrow{\overline{\mathcal{O}}} \overline{\mathcal{O}}$$

is just an ordinary category.

Indeed:

2.19 PROPOSITION. The 2-functor $I$ defined by the following pullback is 2-full and faithful.
When studying \( \mathbf{V}\text{-Cat} \) it is frequently convenient to be able to discuss ordinary categories in the \( \mathbf{V}\)-world. If \( \mathbf{V} \) admits \( \Pi_1 \) copowers which are preserved by \( \Pi \) and \( \Theta \) for all \( I \) in \( \mathbf{V} \), the forgetful functor \( \mathbf{V}\text{-Cat} \to \mathbf{Cat} \) has a left adjoint which associates to a category \( \mathbf{A} \) an ordinary \( \mathbf{V}\)-category \( \mathbf{AF} \) with the same objects and \( [AB] = \mathbf{U} \cdot (A, B) \) for all \( A, B \) in \( \mathbf{A} \). With no extra assumptions on \( \mathbf{V} \) we have a left adjoint to the forgetful functor \( \mathbf{V}\text{-CAT} \to \mathbf{CAT} \), since \( \mathbf{SET}^{\mathbf{V}^{\mathbf{op}}} \) is \( \Pi_2 \)-cocomplete and \( \Pi \) and \( \Theta \) have right adjoints for all \( F \in \mathbf{SET}^{\mathbf{V}^{\mathbf{op}}} \).

In the \( \mathbf{V}\)-indexed setting this can be expressed in a way that admits convenient generalization. We have for the situation above,

\[
\begin{array}{ccc}
A & \rightarrow & \mathbf{AF} \\
\downarrow & & \downarrow \\
\mathbf{N} & \leftarrow & \mathbf{N}
\end{array}
\]

which is a co-cartesian square relative to \( \mathbf{P}_\mathbf{V} \). That is to say, for any \( \langle T, B \rangle \) in \( \mathbf{V}\text{-ind} T \), for any

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
\mathbf{T} & \leftarrow & \mathbf{T}
\end{array}
\]

and any \( T \rightarrow \mathbf{N} \)
(such that \( \tilde{T} \) commutes) there exists a unique \( \alpha_{\tilde{T}} : \tilde{T} \to \tilde{N} \) such that

\[
\begin{array}{ccc}
\tilde{T} & \xrightarrow{} & \tilde{N} \\
\alpha_{\tilde{T}} \downarrow & & \downarrow \\
\tilde{F} & \xrightarrow{} & \tilde{B}
\end{array}
\]

commutes. (In the present case, the parenthesized condition is of course redundant.) Setting \( \tilde{T} = \tilde{N} \) we see that this extends the usual universal property of \( \tilde{F} \). Note that the terminal object of \( \mathcal{V}-\text{ind-CAT} \), \( \tilde{U} \), is of type \( \tilde{0} \) and that \( \tilde{F} = \tilde{U} \), the usual unit \( \mathcal{V} \)-category.

For a general \( \mathcal{V} \)-indexed category \( \tilde{A} \) of type \( \tilde{T} \) and a cofunctor \( \tilde{T} \xleftarrow{f} \tilde{S} \), the problem of determining a \( \mathcal{V} \)-indexed category \( \tilde{A}_{f^!} \) of type \( \tilde{S} \), together with a functor \( \tilde{A} \xrightarrow{f^!} \tilde{A}_{f^!} \) of type \( f^! \), satisfying the co-cartesian criterion is quite complicated. Consider though the objects of \( \mathcal{V} \)-ind-\text{SET} \) defined by left Kan extension as follows:
for \( i = 0,1,2 \). Here \( (A)_0, (A)_1, (A)_2 \) respectively is the object of objects (morphisms, composable pairs respectively) of \( A \). Similarly \( T_i, f_i \), and \( S_i \) are the components of \( T, f \), and \( S \). From the universal property of left Kan extension we have morphisms,

\[
(\mathbf{Af})_1 \xrightarrow{\partial_1} (\mathbf{Af})_0,
\]

in \( \mathbf{V}-\text{ind-SET} \) induced by the morphisms \( (A)_1 \xrightarrow{\partial_1} (A)_0 \).

2.20 PROPOSITION If, with notation as above,

\[
\begin{array}{ccc}
(\mathbf{Af})_2 & \xrightarrow{\pi_1} & (\mathbf{Af})_1 \\
\downarrow \pi_0 & & \downarrow \partial_0 \\
(\mathbf{Af})_1 & \xrightarrow{\partial_1} & (\mathbf{Af})_0
\end{array}
\]

is a pullback in \( \mathbf{V}-\text{ind-SET} \), we can extend the constructed data canonically so that

\[
\begin{array}{ccc}
\mathbf{Af}_1 : (\mathbf{Af})_2 & \xrightarrow{\pi_0} & (\mathbf{Af})_1 \\
\downarrow \pi_1 & & \downarrow \partial_1 \\
(\mathbf{Af})_1 & \xrightarrow{\partial_0} & (\mathbf{Af})_0
\end{array}
\]

becomes a \( \mathbf{V} \)-indexed category of type \( S \) and \( f' \) becomes a
functor of type $f$, cocartesian over $f$.

A sufficient condition that the above construction be applicable is that $\overline{T} \leftarrow \overline{S}$ be in $(\mathcal{F}^{\text{op}})_{\text{cat}}$ (e.g. $\overline{0} \leftarrow \overline{N}$ or $\overline{N} \leftarrow \overline{N}$ as in §1 of chapter 3).

2.21 REMARK An interesting application of the above is provided when $\overline{V}$ is symmetric and $\overline{T}$ is taken to be the category of finitely presented commutative monoids. Adapting $\overline{V}\text{-}\text{ind-CAT}$ accordingly, a (large) $\overline{V}$-category $\overline{A}$ becomes a $\overline{V}$-indexed category of type $\overline{N} : \overline{N} \overline{G} \leftarrow \triangle \overline{N} \overline{G} \rightarrow \overline{0}$, and the product of $\overline{V}$-categories $\overline{A}$ and $\overline{B}$, as $\overline{V}$-indexed categories, $\overline{A} \times \overline{B}$, is of type,

$$\overline{N} \overline{G} \overline{N} : (\overline{N} \overline{G}) \circ (\overline{N} \overline{G}) \leftarrow (\triangle \triangle \triangle) \overline{m} \rightarrow \overline{0}$$

where $\overline{m}$ is the "middle four interchange" isomorphism. We have a cofunctor $\overline{N} \overline{G} \overline{N} \leftarrow \triangle \overline{N}$ and the considerations of the previous proposition apply.

$$\overline{A} \times \overline{B} \leftarrow \overline{A} \otimes \overline{B}$$

is cocartesian where $\overline{A} \otimes \overline{B}$ denotes the usual tensor
product of $\mathcal{V}$-categories. To see this assume, for convenience that $A$ and $B$ are locally small. Then consider:

\[
(\mathcal{V} \times \mathcal{V})^{op} \xrightarrow{\Theta^{op}} \mathcal{V}^{op} \xrightarrow{\text{Lan}_{\Theta^{op}}}(A)_{0} \times (B)_{0} \xrightarrow{\text{SET}} \text{Lan}_{\Theta^{op}}(A)_{0} \times (B)_{0}
\]

\[
(I, J)(A)_{0} \times (B)_{0} = \sum_{A, A' \in (A)_{0}} (I, [AA']) \times \sum_{B, B' \in (B)_{0}} (J, [BB'])
\]

so \(\text{Lan}_{\Theta^{op}}(A)_{0} \times (B)_{0}\)

\[
= \int_{I, J}(K, I \Theta J) \times (I, [AA']) \times (J, [BB'])
\]

\[
= \sum_{A, A' \in (A)_{0}} \sum_{B, B' \in (B)_{0}} \int_{I, J}(K, I \Theta J) \times (I, [AA']) \times (J, [BB'])
\]

\[
= \{A, B, A', B' \in (A)_{0} \times (B)_{0} \}
\]

\[
= (K)(A \Theta B)_{0}
\]

as required. This construction illustrates an important aspect of $\mathcal{V}$-\text{ind-CAT}. When $\mathcal{V}$ is not symmetric a $\Theta$ is not available for $\mathcal{V}$-\text{CAT} (or $\mathcal{V}$-\text{Cat}). Yet the product of $\mathcal{V}$-categories, as $\mathcal{V}$-indexed categories, contains the information.
necessary for the $\Theta$ construction were $V$ symmetric. It is clear that the product in $V$-$\text{CAT}$ does not.

As remarked earlier $V$-$\text{CAT}$ does not admit opposites when $V$ is not symmetric. For $A$ of type $T$ in $V$-$\text{ind-CAT}$ though we can define $A^{\text{op}}$ of type $\tilde{T}$

\[
\begin{array}{rcl}
T_0 & \xrightarrow{d_0} & T_1 \xleftarrow{d_1} \xrightarrow{c} T_2 \xrightarrow{p_1} \tilde{T} \xleftarrow{\tilde{p}_0} \tilde{T}_0
\end{array}
\]

where we have interchanged the roles of $d_0$ and $d_1$. In particular $V^{\text{rev-CAT}}$ is the subcategory of $V$-$\text{ind-CAT}$ determined by the $V$-indexed categories of type $\tilde{N}$

\[
\begin{array}{rcl}
\tilde{N} & \xrightarrow{x} & N \xleftarrow{\epsilon} \tilde{N} \xrightarrow{x} \tilde{N} \xleftarrow{\epsilon} \tilde{N}
\end{array}
\]

and $V$-indexed functors of type "1" between them. There are no non-trivial cofunctors between $\tilde{N}$ and $\tilde{N}^{\text{op}}$ however so a "cotravariant" functor between $V$-categories $A$ and $B$ requires that one of them possesses additional structure.

In chapter 1 we mentioned the $V$-category $V^{(\text{op})}$ $V^{(\text{op})}$ is itself the opposite, in $V$-$\text{ind-CAT}$, of a $V^{\text{rev}}$-category that we denote by $V^{\text{rev}}$. $V^{\text{rev}}$ has the same objects as $V$ and for $V,W \in V$ the set of $I$-indexed families of morphisms from $V$ to $W$, $(I[[V,W]])$ is given by $(I\Omega V,W)$. If we think of a $V$-category as a "right $V$-module" and a $V^{\text{rev}}$-category as a "left $V$-module" we see that "$W" is both and it makes sense to ask if it
is a "bimodule". This is the case. Consider the type

\[ \mathbb{N} \otimes \mathbb{N} \]

where \( <x, y, u, v> \) are the generators of \( A \). We denote by \( V^b_i \) the \( V \)-indexed category of type \( \mathbb{N} \otimes \mathbb{N} \) whose objects are those of \( V \) and for which \( I, J \)-morphisms from \( V \) to \( W \) are given by the set \( (I \otimes V \otimes J, W) \) with the obvious composition. If \( A \) and \( B \) are \( V \)-categories, \( A^\text{op} \times B \) is also a \( V \)-indexed category of type \( \mathbb{N} \otimes \mathbb{N} \) and we can speak of a \( V \)-indexed functor, \( F \), of identity type from \( A^\text{op} \times B \) to \( V^b_i \). It is easy to see that when \( A \) and \( B \) are locally small \( V \)-categories such an \( F \) is the same thing as a "\( V \)-valued \( V \)-bifunctor, of mixed variance" as defined in [Ln].
CHAPTER 3

V-DISCRETE V-FIBRATIONS AND V-LIMITS

§1 TYPE FIBRATIONS

In this chapter we provide our main reasons for the introduction of V-indexed categories. We define V-discrete V-fibrations and prove V-analogues of propositions of ordinary category theory such as that mentioned in §4 of chapter 1. Using V-discrete V-cofibrations we analyse the mean cotensor products of Borceux and Kelly, [B&K], in terms of V-limits as in chapter 2. This considerably extends the scope of the Borceux and Kelly V-limits. In particular no assumptions about V are required and the V-categories under consideration do not have to be locally small.

We begin by discussing the "types" that we require. Our discussion could be phrased in terms of comma objects in (TOP)cat, however, in practice it is desirable to have elemental descriptions on hand, so we follow this approach.

The cofunctor, y, in (T)cocat, given by:
will be called the universal type cofibration. If $\mathcal{A}$ is a $\mathcal{V}$-category and $A_0 \in A$, it is easy to see that $\bar{N}^*$ is just the type of the comma category $A_0/A$ in $\mathcal{V}$-$\text{ind-CAT}$, and $y$ is the type of the projection to $\bar{A}$.

(y and the trivial cofunctor, $\bar{N} \to \bar{0} \to N^*$, are the only cofunctors from $\bar{N}$ to $\bar{N}^*$. The only cofunctor from $\bar{N}^*$ to $\bar{N}$ is the trivial one.)

An arbitrary cofunctor with domain $\bar{N}$:
is completely determined by an element \( mcM_1 \) satisfying \( mi = 1 \) (in \( M_0 \)) and \( mc = (mp_0) \cdot (mp_1) \) (in \( M_2 \)). We denote the pushout of \( y \) along such an \( m \), in \( (T)cocat \), by \( \hat{m} \),

\[
\begin{array}{c}
1/m \\
\downarrow \hat{m} \\
M \\
\downarrow y \\
N
\end{array}
\]

and call such a cofunctor a *type cofibration* under \( M \).

Explicitly \( \hat{m} \) is given by:

\[
\begin{array}{c}
N+p_0 & N+d_0 \\
\downarrow x(mP_0) & \downarrow x(mP_1) \\
P_0J_1 & P_1J_1 \\
\downarrow \hat{m} & \downarrow j_1 \\
M_2 & M_1 \\
\downarrow c & \downarrow i \\
P_1 & M_0
\end{array}
\]

where the \( j_1 \)'s are coproduct injections.

\[ \text{diagram} \]
3.1 REMARK The square,

\[
\begin{array}{c}
\begin{array}{c}
\text{N} \\
\downarrow j_1 \\
\text{M}_1 \\
\downarrow d_0 \\
\text{M}_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{N+M}_0 \\
\downarrow N+d_0 \\
\text{N+M}_1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{N} \\
\downarrow j_1 \\
\text{M}_1 \\
\downarrow d_0 \\
\text{M}_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{N+M}_0 \\
\downarrow N+d_0 \\
\text{N+M}_1
\end{array}
\end{array}
\]

above is a pushout, paralleling the situation for ordinary discrete cofibrations in \text{CAT}. Indeed if \( \tilde{M} \to \tilde{F} \) is any cofunctor whose "domain" data satisfies the above specific pushout condition it is either of the form \( \tilde{m} \), or else has \( (\xymatrix{d_1j_1}) \) replaced by \( (\xymatrix{m_1j_1}) \), with a corresponding change in \( (\xymatrix{p_1j_1}) \), for some \( m \in M_1 \) satisfying \( m_i = 1 \) and \( m_c = (mp_1)(mp_0) \). (The latter case corresponds to a cofunctor from \( \tilde{N}_{\text{op}} \) to \( \tilde{M} \).)

Thus to check whether a cofunctor, \( \tilde{M} \to \tilde{F} \) is a type cofibration it suffices to show that the "domain" data forms a pushout of the specific form above and that the "codomain" operation of \( \tilde{F} \) is of the form, \( (\xymatrix{a_1j_1}) \), for some \( m \in M_1 \).

Next suppose that we have a conatural transformation,

\[
\tilde{N} \to \tilde{N}
\]

between cofunctors of domain \( \tilde{N} \). It is
straightforward to verify that such a \( t \) is determined by an element \( t \in M_0 \) satisfying \( m_0 \cdot (td_1) = (td_0) \cdot m_1 \) (in \( M_1 \)), where \( d_0 \) and \( d_1 \) are, as before, the "domain" and "codomain" operations of \( \tilde{M} \). Such a \( t \) induces a cofunctor, \( \tilde{\epsilon} \), from \( \tilde{M}_1 \) to \( \tilde{M}_0 \) under \( \tilde{M} \):

\[
\begin{array}{c}
\tilde{M}_0 \\
\tilde{M}_1 \\
\tilde{M}
\end{array}
\xymatrix{
1/m_0 \ar[rd]_{\tilde{\epsilon}} \ar@{.>}[r] & 1/m_1 \\
\tilde{M}_0 & \tilde{M}_1 \ar[lu]_{\tilde{\epsilon}}
}
\]

which is explicitly given by:

\[
\begin{array}{c}
1/m_0 \ar[rd]_{\tilde{\epsilon}} \ar[rr] & & 1/m_1 \\
\cdot \cdot \cdot \ar[uu]_{\left( \begin{smallmatrix} x \cdot t \\ j_1 \end{smallmatrix} \right)} & & \cdot \cdot \cdot \\
N+M_0 \ar[uu]_{N+i} \ar[uu]_{N+d_0} & & N+M_0 \ar[uu]_{N+i} \ar[uu]_{N+d_0}
\end{array}
\]

\[
\begin{array}{c}
N+d_0 \ar[rr] & & N+M_1 \\
\downarrow \downarrow & & \downarrow \downarrow \\
N+M_1 \ar[rr] & & N+M_1
\end{array}
\]

\[
\begin{array}{c}
N+P_0 \ar[rr] & & N+M_2 \\
\downarrow \downarrow & & \downarrow \downarrow \\
N+M_2 \ar[rr] & & N+M_2
\end{array}
\]

\[
\begin{array}{c}
N+P_0 \ar[rr] \ar[rr] & & N+M_2 \\
\downarrow \downarrow & & \downarrow \downarrow \\
N+M_2 \ar[rr] \ar[rr] & & N+M_2
\end{array}
\]
Cofunctors of the form $t$ between type cofibrations will be known as morphisms of type cofibrations. If $m_0 \xrightarrow{t} m_1 \xrightarrow{s} m_2$ are conatural transformations between cofunctors defined on $\mathcal{N}$ it follows that $\hat{ts} = \hat{st}$, so that morphisms of type cofibrations form a subcategory of the category of cocategories under $\mathcal{N}$. Note however that it is not full.

The $\mathcal{V}$-discrete $\mathcal{V}$-cofibrations that we will presently define have as their types, type cofibrations. We will also define $\mathcal{V}$-discrete $\mathcal{V}$-fibrations. Since the "duality" between cofibrations and fibrations involves that between $\mathcal{V}$ and $\mathcal{V}^{(op)}$ rather than that between $\mathcal{V}$ and $\mathcal{V}^{op}$ we define type fibrations separately, but in somewhat less detail.

The universal type fibration is:

\[
\begin{array}{c}
\begin{array}{c}
\mathcal{N} \to \mathcal{N} \\
\mathcal{N} \to \mathcal{N}
\end{array}
\end{array}
\]
The pushout of $x$ along a cofunctor $\overline{m} \leftarrow m \rightarrow \overline{N}$ will be written as $m/! \leftarrow \overline{m} \rightarrow \overline{N}$ and such cofunctors will be called type fibrations. $m/!$ is given by:

$$
\begin{array}{ccc}
\left( \begin{array}{c}
P_0 \\
\overline{m}
\end{array} \right) & \left( \begin{array}{c}
P_0 \\
\overline{m}
\end{array} \right) \\
\downarrow & \downarrow \\
\left( \begin{array}{c}
P_1 \\
\overline{m}
\end{array} \right) & \left( \begin{array}{c}
P_1 \\
\overline{m}
\end{array} \right)
\end{array}
\left( \begin{array}{c}
P_0 \\
\overline{m}
\end{array} \right)

m/!: M_2 \times N \xleftarrow{c \in N} M_1 \times N \xrightarrow{i \in N} M_0 \times N

\end{array}
$$

and the components of $\overline{m}$ are the obvious coproduct injections. Morphisms of type fibrations are as expected.

If $m_0 \rightarrow m_1$ is a conatural transformation between cofunctors defined on $\overline{N}$, the first component of $\overline{m}_0 \rightarrow \overline{m}_1$ is given by $(j_0)_{tx}$. It is convenient to postpone remarks about the "duality" between type fibrations and type cofibrations until we are actually discussing $V$-indexed categories.

§2 $V$-DISCRETE $V$-FIBRATIONS

In analogy with pointed sets we write $V_*$ for the comma category $U/V$ in $\text{V-ind-CAT}$. As remarked in the previous section $V_*$ is of type $\overline{N}$. Specifically, $V_*$ has $I$-indexed families of objects, or simply $I$-objects, for all $I \in V$ which are given by $I$-morphisms of domain $U$, $U \xrightarrow{<I;x>} V$, in $V$. We regard $<I;x>$ as an object of $V$, $V$, together with an $I$-indexed family of its "elements". $V_*$ has $I,J$-morphisms for all $I$, $J \in V$, where such morphisms consist of pairs.
\(<U <I;x> \rightarrow V, V <J;f> \rightarrow W>\). That is, we are to have an I-object together with a J-indexed family of morphisms out of its underlying object in V. The domain of such an I, J-morphism is \(<I;x>\) while the codomain is the composite of the pair in V, \(<\bowtie J;xf>\). An I,J,K-composable pair is a 3-tuple, \(<U <I;x> \rightarrow V, V <J;f> \rightarrow W, W <K;g> \rightarrow X>\) and its composite is the I,J,K-morphism \(<U <I;x> \rightarrow V, V <J;f> \rightarrow X>\). As a comma category \(V_*\) comes equipped with a projection, \(\pi Y \rightarrow V\), which associates to an I,J-morphism \(<I;x>,<J;f>\) in \(V_*\) the J-morphism \(<J;f>\) in V. \(Q\) is of type \(V\) and we will refer to it as the universal V-discrete V-cofibration.

Now let \(B\) be a V-indexed category of type \(\bar{M}\) and \(B \rightarrow V\) a \(V\)-indexed functor of type \(m\), and form the indicated pullback in \(V\text{-ind-CAT}\).

The diagram is as follows:

- \(U/F \rightarrow V_*\)
- \(Q_F \rightarrow Q\)
- \(\pi Y \rightarrow V\)

The arrows and objects are connected as shown in the diagram.
(The pullback is $U/F$ since $V^*_F = U/V$.) We seek to characterize those $B$-valued $V$-indexed functors of the form $Q_F$ and show that the $F$ is unique up to isomorphism. The explicit description of $i/m$, $\hat{m}$ etc. given in the previous section settles the question at the level of "types", so the kinds of families involved are completely determined and we freely use the notation established in §1. Generalizing the kind of terminology that we have used throughout this paper, we will speak of $I_0$-objects of $B$ etc., for $I_0 \in [M_0, V]$ , and write $<I_0; A>$ etc.

Suppose that $B \xrightarrow{t} V$ of type $\bar{M} \xleftarrow{m_0} \bar{M}_1$ is given. Thus for any $I_1 \in [M_1, V]$ and any $I_1$-morphism $<I_1; f> \xrightarrow{<I_1; f>} <J_0; B>$ in $B$ (tacitly $I_0 = I_1d_0^\#$, $J_0 = I_1d_1^\#$) we have a "commutative" square,

\[
\begin{array}{ccc}
AF_0 & \xrightarrow{<I_1; m_0^\#fF_0>} & BF_0 \\
| & & | \\
<I_0; t^\#; A> & \downarrow & <J_0; t^\#; B> \\
| & & | \\
AF_1 & \xrightarrow{<I_1; m_1^\#fF_1>} & BF_1
\end{array}
\]

in $V$. (The isomorphism $-m_0^\# \rightarrow d_1^\# t^\# \rightarrow d_0^\# t^\# \circ \circ \rightarrow m_1^\#$ is that arising from the pseudo functor structure of $^{\#}$'ing applied to the equation $m_0 \cdot (t d_1) = (t d_0) \cdot m_1$ in $M_1$.)
We can now describe the $V$-indexed categories $U/F_0$ and $U/F_1$, over $B$, and also the $V$-indexed functor,

\[ U/F_0 \xrightarrow{U/I} U/F_1 \]

\[ Q_{F_0} \xrightarrow{B} Q_{F_1} \]

over $B$, induced by $F_0 \xrightarrow{I} F_1$. For, $<I,I_0> \in V \times [M_0,V]$, an $I,I_0$-object of $U/F_0$ is a pair, $<U \xrightarrow{I;X} A_{F_0}, <I_0;A>>$, where $A$ is an $I_0$-object of $B$. For $<I,I_1> \in V \times [M_1,V]$ an $I,I_1$-morphism of $U/F_0$, is a pair $<U \xrightarrow{I;X} A_{F_0}, <I_0;A> \xrightarrow{I_1;f} J_0;B>>$ where $f$ is an $I_1$-morphism of $B$. The domain of such a morphism is $<U \xrightarrow{I;X} A_{F_0}, <I_0;A>>$ and the codomain is $<U \xrightarrow{I_0;M_{I_0}X(f_0)} B_{F_0}, J_0;B>>$.

The functor $Q_{F_0}$ is given by forgetting the first components. $U/I$ sends $<U \xrightarrow{I;X} A_{F_0}, <I_0;A>>$ to $<U \xrightarrow{I;X} A_{F_0} \xrightarrow{I_0;A} A_{F_1}, <I_0;A>>$, (an $I_0;I_0^\#$, $I_0$-object of $U/F_1$), and $<I;X, <I_1;f>>$ to $<I_0;X(A_1), <I_1;f>>$, (which is an $I_0;I_1^d, I_1$-morphism of $U/F_1$).
3.2 PROPOSITION If \( B \xrightarrow{F_0} V \) are of types \( m_0 \)
and \( m_1 \) respectively, \( m_0 \xrightarrow{t} m_1 \) is a conatural
transformation, and \( G \) as below.

\[
\begin{array}{ccc}
U/F_0 & \xrightarrow{G} & U/F_1 \\
\downarrow & & \downarrow \\
B & \xrightarrow{} & \\
\end{array}
\]

is of type \( t \), then there exists a unique \( F_0 \xrightarrow{t} F_1 \)
of type \( t \), such that \( G = U/t \).

Proof: If \( \langle U \xrightarrow{<I; x>} AF_0, <I_0; A> \rangle \) is an \( I, I_0 \)-object
of \( U/F_0 \) and \( G \) is as above, \( G \) sends such an object
to an \( I\otimes I_0 t^\# , I_0 \)-object, of \( U/F_1 \), of the form
\( \langle I\otimes I_0 t^\# ; y \rangle \xrightarrow{<U \xrightarrow{<I_0; A}>} AF_1, <I_0; A> \). We will just show that
y necessarily factors over \( x \) and leave the remaining
details to the reader.

Observe that \( \langle <I; x>, <I_0; A> \rangle \) is obtained from the
generic \( AF_0, I_0 \)-object, \( \langle U \xrightarrow{<AF_0; (AF_0)^\lambda>} AF_0, <I_0; A> \rangle \),
via substitution along the morphism \( \langle I \xrightarrow{x} AF_0, I_0 \xrightarrow{1} I_0 \rangle \)
in \( V \times [M_0 V] \). As a \( V \)-indexed functor \( G \) preserves
substitution so it suffices to consider \( G \) of the generic
\( AF_0, I_0 \)-object when determining \( G \) of \( \langle <I; x>, <I_0; A> \rangle \).
But \( G \) of the generic object is a pair,
<AF_0\theta I_0t^#;z> 
<U \rightarrow AF_1; \langle I_0; A \rangle >

and from the description of the V-indexed structure of V we see that z is the same thing as an I_0t^#-morphism, A_1, from AF_0 to AF_1 in V. It follows that y = x(A_1).

If \[ E \rightarrow \rightarrow B \] is any V-indexed functor whose type is a type cofibration, say \[ !/m \leftarrow \hat{m} \leftarrow \bar{m} \], we can ask whether it satisfies the following:

3.3 DEFINITION Lifting condition. For any \[ \langle I, I_0 \rangle \in V \times [M_0; V] \], any \( I, I_0 \)-object of E, \( \langle I, I_0; X \rangle \), any \( I_1 \in [M_1; V] \) with \( I_1d_0^# = I_0 \), and any \( I_1 \)-morphism of B, \( \langle I_1; f \rangle \) whose domain is \( \langle I, I_0; X \rangle \) and G of which is \( \langle I_1; f \rangle \).

Diagramatically:

\[ \langle I, I_0; X \rangle \xrightarrow{\raisebox{-2.5em}{\begin{array}{c} \text{Diagrammatically:} \\
\end{array}}} \langle I0; Y \rangle \]

\[ \langle I, I_0; X \rangle \xrightarrow{\langle I_1; f \rangle} \langle I_1; B \rangle \]

(that the codomain of the dotted postulated morphism takes the indicated form follows from the description of
3.4 Lemma The lifting condition is "stable under substitution". That is, if we denote the lifted morphism in the previous diagram by \( \langle I', I_1; \tilde{f} \rangle \) and
\[
\langle I', I_0; f \rangle \xrightarrow{\langle u, u_0 \rangle} \langle f, I_0 \rangle \text{ in } V \times [M_0 V] \text{ and } I_1 \xrightarrow{u_1} I_1
\]
in \([M_1 V]\) are given with \( u_1 d_0^\# = u_0 \); then
\[
\langle I', I_1; \tilde{f} < u, u_1 > \rangle \text{ is the lift of } \langle I_1; f u_1 \rangle \text{ to a morphism out of } \langle I', I_0; X < u, u_0 > \rangle.
\]

Proof: The domain of \( \langle I', I_1; \tilde{f} < u, u_1 > \rangle \) is \( \langle I', I_0; X < u, u_0 > \rangle \) and \( G \) applied to \( \tilde{f} < u, u_1 > \) is \( f u_1 \); since \( V \)-indexed functors by definition preserve "substitution". The conclusion follows by uniqueness of "lifts".

3.5 Lemma If \( B \xrightarrow{F_0} V \), then \( U/F_0 \xrightarrow{QF_0} B \) satisfies the lifting condition.

Proof: Consider the diagram:

\[
\begin{array}{ccc}
\langle I; x \rangle, \langle I_1; f \rangle & \xrightarrow{\langle I; x \rangle, \langle I_0; A \rangle} & \langle \langle I \otimes I_1 \rangle^\#; x (f F_0) \rangle, \langle J_0; B \rangle \\
\langle I_0; A \rangle & \xrightarrow{I_1; f} & \langle J_0; B \rangle
\end{array}
\]
and the descriptions of $U/F_0$ and $Q_{F_0}$ that precede proposition 3.2.

For $E \xrightarrow{G} B$ of type $\lambda/m \xleftarrow{\bar{m}} M$ as before, we can define, for each $I_0 \in [M_0,V]$ and each $I_0$-object, $A$, of $B$, a contravariant $\text{SET}$-valued functor on $V$:

$$V^{\text{op}} \xrightarrow{I_0;A} \text{SET}$$

whose value at $IcV$ is the set of all $I,I_0$-objects of $E$, $\langle I,I_0;X \rangle$, such that $XG = A$. If $I \xrightarrow{u} I$ is in $V$, $\forall (u) \langle I_0;A \rangle^{-1}$ is given by substitution along $\langle u,I_0 \rangle$. We can ask whether $G$ satisfies the following:

3.6 DEFINITION Smallness condition. For each $I_0 \in [M_0,V]$ and each $I_0$-object, $A$, of $B$, $\langle I_0;A \rangle^{-1}$ is representable and further, for any $I_0 \xrightarrow{u_0} I_0$ in $[M_0,V]$, $\langle I_0;Au_0 \rangle^{-1} \simeq \langle I_0;A \rangle^{-1}$.

3.7 LEMMA If $B \xrightarrow{F_0} V$, then $U/F_0 \xrightarrow{Q_{F_0}} B$ satisfies the smallness condition.
Proof: Referring to the description of $U/F_0$ and $Q_{F_0}$ preceding proposition 3.2, we see that $(I)<I_0,A>Q_{F_0}^{-1}$ is the set of $I,I_0$-objects of $U/F_0$ of the form $<U< I_0 ; x > A F_0 ,< I_0 ; A > F_0$ and this is isomorphic to $(I)[(U,AF_0)] F_0$. But the latter is, by definition, $(U @ I, AF_0) F_0 \approx (I, AF_0) V$. Thus $AF_0$ represents $<I_0 ; A > Q_{F_0}^{-1}$. If $I_0 \rightarrow I_0$ is in $[M_0 V]$, $(A u) F_0 \approx AF_0$ since $F_0$ is $V$-indexed; specifically, on objects $F_0$ is a $V$-indexed function of type $N \leftarrow 0$.

3.8 Definition A $V$-indexed functor $E \rightarrow Q \rightarrow B$, whose type is a type cofibration, is a $V$-discrete $V$-cofibration if it satisfies:

1) the lifting condition
2) the smallness condition.

3.9 Remark We could have arranged definitions 3.3 and 3.6 in such a way that the type requirement above is a consequence of 1) and 2). For if $\hat{m}$ is a type cofibration its form on "objects", $N \times M_0 \leftarrow M_0$, is the smallness condition at the level of types, and the pushout condition on the "domain" operations is the lifting condition for types. However such an approach is clumsy. The type
cofibration requirement is, in any event, necessary and, as we remarked in 3.1, it is easy to verify.

If \( E_0 \xrightarrow{Q_0} B \) and \( E_1 \xrightarrow{Q_1} B \) are \( \tilde{V} \)-discrete \( \tilde{V} \)-cofibrations over \( B \), and \( G \) is any \( \tilde{V} \)-indexed functor from \( Q_0 \) to \( Q_1 \) over \( B \),

\[
\begin{array}{ccc}
E_0 & \xrightarrow{G} & E_1 \\
Q_0 & \searrow & Q_1 \\
& B & \\
\end{array}
\]

whose type is a morphism of type cofibrations, we call \( G \) a morphism of \( \tilde{V} \)-discrete \( \tilde{V} \)-cofibrations. By lemmas 3.5 and 3.7, \( \tilde{V} \)-indexed functors of the form \( Q_{\tilde{F}} \) are \( \tilde{V} \)-discrete \( \tilde{V} \)-cofibrations and proposition 3.2 suggests that the above definition makes sense. Indeed, the above definition allows us, for each \( B \in \tilde{V} \text{-ind-CAT} \) to speak of the (augmented) category of \( \tilde{V} \)-discrete \( \tilde{V} \)-cofibrations over \( B \), \( \tilde{V} \text{-}d_0 \text{-fib}_B \). "Modulo types", \( \tilde{V} \text{-}d_0 \text{-fib}_B \), is a full subcategory of \( \tilde{V} \text{-ind-CAT}/B \), paralleling the situation for set-based categories.

We write \((\tilde{M}, \tilde{N})\) for the category whose objects are cofunctors, \( \tilde{M} \leftarrow \tilde{N} \), and whose morphisms are conatural transformations between them. From the discussion in §1 we have that \((\tilde{M}, \tilde{N})\) is isomorphic to the opposite of the
category of type cofibrations under $\vec{M}$. Thus $V$-d_0 fib_{\vec{B}}$
for $B$ of type $\vec{M}$, is augmented by (or defined over)
$(\vec{M}, \vec{N})$, as is $(B, V)$, the category of $V$-valued functors
on $B$.

3.10 THEOREM $V$-d_0 fib_{\vec{B}}$ and $(B, V)$ are equivalent
augmented categories.

Proof: The assignments, $(F_0 \xrightarrow{f} F_1) \mapsto (Q_{F_0} \xrightarrow{U/T} Q_{F_1})$,
given by pulling back the universal $V$-discrete $V$-cofibration,
clearly define a functor, $(B, V) \xrightarrow{Q} V$-d_0 fib_{\vec{B}}$, over
$(\vec{M}, \vec{N})$.

In the other direction, given $E \xrightarrow{Q} B \in V$-d_0 fib_{\vec{B}}$
of type $\vec{M}$, we proceed as follows. For each $I_0 \in [M_0 V]$ and
each $I_0$-object, $A$ of $B$, choose a representing object, $AF_Q \in V$,
for $\langle I_0; A, Q^{-1} \rangle$, in such a way
that if $\xrightarrow{u_0} I_0$ is in $[M_0 V]$, $Au_0^* Q = AF_Q$. This
we can do since for $u_0$ as above we have
$\langle I_0; Au_0^* Q, A, Q^{-1} \rangle$. Let $(-, AF_Q) \xrightarrow{\phi} \langle I_0; A, Q^{-1} \rangle$
denote such a representation, then $g = (1)(AF_Q)^\phi$ is a
generic $AF_Q, I_0$-object in $E$ over $\langle I_0; A \rangle$. Now for an
$I_1$-morphism, $\xrightarrow{I_1; f} \langle J_0; B \rangle$, in $B$ consider its
lift to a morphism in $E$ out of $\langle AF_Q, I_0; g \rangle$.
\[ <AF_Q, I_0; g> \quad \rightarrow \quad <AF_Q \otimes I_1 m^\#; J_0; Y> \]

\[ \vdots \quad \vdots \]

\[ <I_0; A> \xrightarrow{<I_1; f>\quad <J_0; B> \]

The codomain of the lift is an element of \((AF_Q \otimes I_1 m^\#) <J_0; B> Q^{-1}\) and if we denote the representation of \(<J_0; B> Q^{-1}\) by \((-; BF_Q) \xrightarrow{\phi} <J_0; B> Q^{-1}\), then \(\phi = (Y)(AF_Q \otimes I_1 m^\#) Q^{-1}\) is a morphism, \(AF_Q \otimes I_1 m^\# \xrightarrow{\phi} BF_Q\) in \(V\), that is an \(I_1 m^\#\)-morphism from \(AF_Q\) to \(BF_Q\) in \(V\). \(F_Q\) as defined is a \(V\)-indexed functor of type \(m\).

Next consider a morphism:

\[
\begin{array}{ccc}
E_0 & \xrightarrow{G} & E_1 \\
\downarrow{Q_0} & & \downarrow{Q_1} \\
B & \sim & \sim
\end{array}
\]

of type \(\tilde{\mathcal{C}}\), in \(\text{\textunderscore\textunderscore d}_0 \text{\textunderscore\textunderscore fib}_B\). From the type considerations of \(\S 1\), it follows that if \(X\) is an \(I, I_0\)-object of \(E_0\), then \(XG\) is an \(I_0 I_0 t^\#, I_0\)-object of \(E_1\). So consider an \(I_0\)-object, \(A\), in \(B\) and the generic \(AF_{Q_0}, I_0\)-object, \(g\), in \(E_0\), over \(A\). \(gG\) is an \(AF_{Q_0} \otimes I_0 t^\#, I_0\)-object of \(E_1\) over \(A\) since the above diagram commutes. Thus by representability of \(<I_0; A> Q^{-1}\), \(gG\) is the isomorph of a morphism \(AF_{Q_0} \otimes I_0 t^\# \xrightarrow{A \sim G} AF_{Q_1}\).
in \( V \), that is
\[
\begin{array}{ccc}
\mathcal{F}_0 & \xrightarrow{t^*} & \mathcal{A}_G \\
\downarrow \mathcal{F}_0 & & \downarrow \mathcal{A}_G \\
\mathcal{F}_1 & \xrightarrow{t} & \mathcal{A}_G
\end{array}
\]
\( \tau_G \) is a \( V \)-indexed natural transformation of type \( t \) from \( \mathcal{F}_0 \) to \( \mathcal{F}_1 \), and the assignments \( (Q_0 \xrightarrow{G} Q_1) \mapsto (\mathcal{F}_0 \xrightarrow{\tau_G} \mathcal{F}_1) \)

define a functor \( V\text{-}d_0\text{-}\text{fib}_B \xrightarrow{F} (B, V) \) over \((\bar{M}, \bar{N})\). We leave it to the reader to show that \( F \Rightarrow Q \) is an equivalence.

Thus, up to isomorphism, \( V \)-discrete \( V \)-cofibrations are pullbacks of the universal \( V \)-discrete \( V \)-cofibration.

Let \( A \) be a \( V \)-indexed category of type \( \bar{M} \). While \( V \)-indexed functors of the form, \( \xrightarrow{A_{\text{op}}} \rightarrow \mathcal{V} \), make sense, \( V \)-indexed functors, \( \xrightarrow{A_{\text{op}}} \rightarrow \mathcal{V}_{\text{rev}} \), occur more often in practice. For example if \( A \) is a locally small \( V \)-category the functors \( [-A] \), for \( A \in \bar{A} \), are of this form. To \( \xrightarrow{A_{\text{op}}} \rightarrow \mathcal{V}_{\text{rev}} \) we wish to associate a \( V \)-indexed functor with codomain \( A \), \( \mathcal{P}_F \), the corresponding \( V \)-discrete \( V \)-fibration. To do this it is slightly more convenient to regard \( F \) as a \( V \)-indexed functor from \( A \) to \( \mathcal{V}_{\text{op}} \).

For there is a \( V \)-indexed functor \( \xrightarrow{V \mathcal{P}} \rightarrow \mathcal{V}_{\text{op}} \) of type \( x \), (as in \( \S1 \)), which we can refer to as the universal \( V \)-discrete \( V \)-fibration. This approach allows us to define \( \mathcal{P}_F \) as simply the pullback of \( \mathcal{P} \) along \( F \).
$V$, of type $\overrightarrow{\mathbb{N}}$, has $I$-objects, for all $I \in V$, given by $I$-morphisms of domain $U$, $U \xrightarrow{<x;I>} V$, in $v^{\text{rev}}$. (Thus we have $I(U) \xrightarrow{x} V$.) A $J, I$-morphism in $V$ is a pair, $<W \xleftarrow{f;J} V, V \xleftarrow{<x;I> U}>$, in $v^{\text{rev}}$. The domain of such a morphism is the $J \otimes I$-object, $U \xrightarrow{xI} W$; its codomain is $<x;I>$. A typical $K, J, I$-composable pair is given by $<X \xleftarrow{g;K} W, W \xleftarrow{f;J} V, V \xleftarrow{<x;I> U}>$. Its composite is $<X \xleftarrow{fg;KJ} V, V \xleftarrow{<x;I> U}>$. The $V$-indexed functor $P$ sends $<x;I>$ to $V$ and $<f;J>, <x;I> >= V \xleftarrow{f;J} V$, (which is a $J$-morphism from $W$ to $V$ in $V^{(op)}$).

A general $V$-discrete $V$-fibration, $E \xrightarrow{P} A$, may now be defined as a $V$-indexed functor whose type is a type fibration and which satisfies an obvious lifting and smallness condition. An analogue of theorem 3.10 is easy to state and prove.

§3 EXAMPLES

If $B \xrightarrow{F} V$ is a $V$-functor and $B$ is locally small, $U/F \xrightarrow{QF} B$ is of type $\overrightarrow{\mathbb{N}} \xleftarrow{V} \overrightarrow{\mathbb{N}}$. Specializing the previous section we see that an $I$-object of $U/F$ is a pair $<I \xrightarrow{x} AF, A>$, where $A \in B$ and $x$ is in $V$. An $I, J$-morphism is a pair $<I \xrightarrow{x} A F, J \xrightarrow{f} [AB]>$ where $f$ is in $V$. Its codomain is
\[ \langle \Theta \xrightarrow{\phi f} \alpha \Theta [\alpha \beta] \xrightarrow{\Phi} \beta \Gamma, \beta \rangle \]

If \( F \) above is \([A_0-] \), then \( U/[A_0-] \rightarrow B \) is \( A_0/B \rightarrow B \). The latter always makes sense. In particular if \( B \) is a large \( \mathcal{V} \)-category which is not locally small, \([A_0-]\) may fail to exist, but \( A_0/B \rightarrow B \) contains all the information that \([A_0-]\) would were it available. Furthermore, it is of type \( \gamma \) and satisfies the lifting condition.

If \( \mathcal{V} = ab \) and \( B = R \) is a ring, regarded as an \( ab \)-category with one object, then \( F \) is just a right \( R \)-module, \( M \). The associated total category \( \mathbb{Z}/M \) has group homomorphisms \( I \xrightarrow{\chi} M \) as \( I \)-objects, and pairs of homomorphisms \( \langle I \xrightarrow{\chi} M, J \xrightarrow{\phi} R \rangle \) as \( I,J \)-morphisms. The codomain of such a morphism is \( \Theta \xrightarrow{\phi f} M \Theta R \xrightarrow{a} M \)

where \( a \) is the action of \( R \) on \( M \). \( \mathbb{Z}/M \) contains no more (or less) information than the module \( M \) but it now makes sense to speak of \( ab \)-indexed functors into and out of "\( M \)."

Let \( \mathcal{V} = R \) as in §7 of chapter 1. Let \( B \xrightarrow{f} R \) be a distance decreasing map between generalized metric spaces (i.e. an \( R \)-functor). For \( i \in R \) the \( i \)-objects of \( 0/f \) are given by \( \{ b \in B | i > bf \} \). For \( i,j \in R \) an \( i,j \)-morphism is a pair \( \langle b,c \rangle \in B \times B \) with \( i > bf \) and \( j > [bc] \). Its codomain is the \( i+j \) object \( c \). \( (i+j > bf + [bc] > cf) \). If \( bf \) is the distance from \( b \) to some closed subset \( B' \) of \( B \), then \( 0/f \) can be viewed as \( B \) equipped
with an $\mathbb{R}$-indexed family (in the usual sense) of concentric subsets, the innermost being $B'$ (which determines the others).

§4 MEAN COTENSOR PRODUCTS

In §6 of chapter 1 we introduced, for symmetric $\mathcal{V}$, the notion of $\mathcal{V}$-cells between $\mathcal{V}$-functors and used it to give a simplified description of $\mathcal{V}$-functor $\mathcal{V}$-categories $[\mathcal{B} \mathcal{A}]$. For $A = V$ we can modify the construction so as to be able to construct $[B, V]_{\mathcal{V}}^{rev}$-$CAT$ when $\mathcal{V}$ is not necessarily symmetric.

3.11 DEFINITION Given $\begin{array}{c} F \\ \downarrow G \end{array} \xrightarrow{\sim} \mathcal{V}$ and $K_{\mathcal{V}}$, a $K$-cell $\begin{array}{c} F \\ \downarrow K \end{array} \xrightarrow{\sim} G$ consists of a set theoretic family of morphisms in $\mathcal{V}$, parametrized by the objects of $\mathcal{B}$,

$$\langle K@BF \ xrightarrow{B_t} BG \rangle_{B \in \mathcal{B}}$$

subject to the requirement that for all $\begin{array}{c} I \\ \downarrow b \end{array} \xrightarrow{\sim} B'$ in $\mathcal{B}$,

$$\begin{array}{ccc} K@((BFOI)) & \xrightarrow{K@BF} & K@B'F \\ \downarrow (K@BF)@I & & \downarrow B't \end{array}$$

$$\begin{array}{ccc} Bt@I & \downarrow Bt@I & \downarrow B'G \\ \downarrow BG @I & \end{array}$$

commutes.
It should be clear that the above is equivalent to definition 1.19 when $V$ is symmetric and that for $B \xrightarrow{H} V$, $G \xleftarrow{G;L} H$ we can define a composite, $F \xleftarrow{G;L@K} H$, thus endowing the set of $V\text{-}\text{functors}$ from $B$ to $V$ with a $V^{\text{rev}}\text{-category structure}$, $[B, V]$.

3.12 REMARK The essential feature of "$V$" that the above definition uses is that it comes equipped with a "bimodule" structure. For any $A \in V\text{-CAT}$ so endowed we could define $[B, A]_{\in V^{\text{rev}}\text{-CAT}}$.

Kelly and Borceux defined in [B&K] a notion of limit for locally small $V\text{-categories}$ which subsumes all the limit-like notions usually discussed in $V\text{-theory}$. Given $V\text{-functors}$,

$$
\begin{array}{ccc}
B & \xrightarrow{G} & A \\
\downarrow & & \downarrow \\
F & \xrightarrow{V} & V
\end{array}
$$

they define the mean cotensor product of $F$ and $G$, $\{F, G\}$, to be an object of $A$ which when $A$ admits cotensor products is given by,

$$\{F, G\} = \int_B [BF, BG]$$
existing whenever the end does. In the absence of the cotensors \([BF, BG]\), for all \(B\) in \(\mathcal{B}\), \([F, G]\) may still exist; in which case it is the solution to the \(V\)-universal problem that \(\int_B [BF, BG]\) solves when the requisite cotensors do exist. Following \([B&K]\) we can state the universal property as follows. The mean cotensor product of \(F\) and \(G\) is an object \([F, G]\) of \(\mathcal{A}\), together with a \(V\)-natural transformation

\[
F \xrightarrow{\pi} \{[F, G], \cdot G\}_A,
\]

which for every \(A \in \mathcal{A}\) and every \(J \in V\) mediates a bijection,

\[
(J, [A, \{F, G\}]_A) \xrightarrow{\pi} V\text{-nat}(F, [J, [A, -]_A])
\]

where for \(J \xrightarrow{f} [A, \{FG\}]_A\) in \(V\), \((f)\pi\) is the composite

\[
F \xrightarrow{\pi} \{[F, G], -G\}_A \xrightarrow{[A, -]} \{[A, \{F, G\}]_A, [A, -G]_A\} \xrightarrow{[f, 1]} [J, [A, -G]_A].
\]

It should be pointed out that the authors of \([B&K]\) assume that \(V\) is symmetric and closed, and all their \(V\)-categories are locally small in our sense. We assume, for now, the latter property for \(\mathcal{A}\) and \(\mathcal{B}\), but maintain our usual weaker assumptions about \(V\).

3.13 LEMMA Replacing \(\pi\) above by

\[
(J, [A, \{F, G\}]_A) \xrightarrow{\pi'} V\text{-nat}(J\Theta - F, [A, -G]_A)
\]

where \((f)\pi'\) is the composite
provides a definition for \( \{F,G\} \) which makes sense when \( \mathcal{V} \) is not necessarily (symmetric and) closed and agrees with the above when \( \mathcal{V} \) is.

**Proof:** Observe that for \( \mathcal{V} \) closed the adjointness
\[ J^\mathbf{0} - \rightarrow [J,-] \]
induces an isomorphism
\[ \mathcal{V}\text{-nat}(J^\mathbf{0}-F,[A,-]A) \cong \mathcal{V}\text{-nat}(F,[J,[A,-]A]) \]
for which \( \pi' \Phi = \pi \)

3.14 **PROPOSITION**

\[ [A,\{F,G\}]A \longrightarrow [[F,[A,-]A],[B,\mathcal{V}]], \mathcal{V}\text{-natural in } A. \]

**Proof:** From lemma 3.13 and the definitions we have

\[
\begin{align*}
F & \longrightarrow [A,-]A \\
J^\mathbf{0}-F & \longrightarrow [A,-]A \\
J & \longrightarrow [A,\{F,G\}] & \text{in } \mathcal{V} \\
A & \longrightarrow \{F,G\} & \text{J-morphisms in } A
\end{align*}
\]
natural in \( J \).
In [B&K] the above is pointed out to be valid when [B,Y] "exists". [B,Y] "exists" in the setting of {B&K} precisely when it is locally small in our setting.

We now make {F,G} look more like an ordinary limit by considerations generalizing those in §5 of chapter 1.

3.15 LEMMA V-naturality of \( F \xrightarrow{P} \{[F,G],-\} \)A is equivalent to the commutativity of

\[
\begin{array}{c}
\{F,G\} \xrightarrow{\langle BF;Bp\rangle} BG \\
\downarrow \langle BF\circ I; (B'p)(bF)^* \rangle \\
B'G
\end{array}
\]

for every \( B \xrightarrow{\langle I;b \rangle} B' \) in B.

Proof: Since B and A were assumed to be locally small \( B \xrightarrow{\langle I;b \rangle} B' \) is a morphism \( I \xrightarrow{b} [BB'] \) in V, bG is the composite \( I \xrightarrow{b} [BB'] \xrightarrow{G} [BG,B'G] \) and bF is \( BF \circ I \xrightarrow{\langle I;b \rangle} BF\circ [BB'] \xrightarrow{BF} B'F \). Bp is the B'th component of p, \( BF \xrightarrow{Bp} \{[F,G],BG\} \), hence \( (B'p)(bF)^* \) is the composite \( (\langle I;b \rangle)(F)(B'p) \). Thus commutativity of the triangles for all \( B \xrightarrow{\langle I;b \rangle} B' \) means precisely commutativity of the outer diagram.
for all $I \xrightarrow{b} [BB']$. Specializing to the generic $[BB']$ morphism from $B$ to $B'$, $[BB'] \Rightarrow [BB']$, gives us commutativity of the lower inner diagram, which is the (transpose of the) usual requirement for $V$-naturality of $F \xrightarrow{p} [[F,G]-G]$. The converse follows immediately from the diagram.

The lemma is clearly just a reshuffling of the data for $p$ via the Yoneda principle, however it has at least two important aspects. On the one hand commutativity of the triangles for every $B \xrightarrow{I;b} B'$ in $\sim$ makes sense for arbitrary $A$ and $B$ in $V\text{-CAT}$. On the other the reformulation exhibits $\langle F,G,p \rangle$ in the familiar way as an object of $A$ equipped with a "family" of projections. Here the "family" is a set theoretic family of our technical families, (the $\langle BF;Bp \rangle$), with the set theoretic aspect parametrized by the objects of $B$. We simplify this further. For $A\in A$, let us call a "family" $c$. 
of \( \mathcal{V} \)-morphisms in \( A \), \( \langle A ; \langle BF; Bc \rangle \rightarrow BG \rangle \) an \( \mathcal{F} \)-cone

from \( A \) to \( G \) if

\[
\begin{align*}
A \xrightarrow{\langle BF; Bc \rangle} BG \\
\langle BF\Theta I; (B'c) (bF)^* \rangle & \downarrow \langle I; bG \rangle \\
& \quad \downarrow \langle I; bG \rangle \\
& \quad \downarrow \langle I; bG \rangle \\
& \quad \downarrow \langle I; bG \rangle \\
B'G
\end{align*}
\]

commutes for every \( B \xrightarrow{\langle I; b \rangle} B' \) in \( B \). We will write

\[
A \xrightarrow{\langle F; c \rangle} G
\]

For \( J \in \mathcal{V} \) we may further define a \( J \)-indexed family of \( \mathcal{F} \)-cones from \( A \) to \( G \) to be a "family", \( c \), of \( \mathcal{V} \)-morphisms in \( A \), \( \langle A ; \langle J\Theta BF; Bc \rangle \rightarrow BG \rangle \) such that

\[
\begin{align*}
A \xrightarrow{\langle J\Theta BF; Bc \rangle} GB \\
\langle J\Theta (BF\Theta I); (B'c) (J\Theta bF)^* \rangle & \downarrow \langle I; bG \rangle \\
& \quad \downarrow \langle I; bG \rangle \\
& \quad \downarrow \langle I; bG \rangle \\
& \quad \downarrow \langle I; bG \rangle \\
B'G
\end{align*}
\]

commutes for every \( B \xrightarrow{\langle I'; b \rangle} B' \) in \( B \). For brevity we will refer to these as \( \langle J\Theta F; c \rangle \)-cones from \( A \) to \( G \), and write \( A \xrightarrow{\langle J\Theta F; c \rangle} G \). An easy calculation shows that if \( A' \xrightarrow{\langle K; a \rangle} A \) is a \( K \)-morphism in \( A \), pointwise composition with \( \langle J\Theta F; c \rangle \) yields a \( (K \Theta J)\Theta F \)-cone from \( A' \) to \( G \), \( A' \xrightarrow{\langle (K \Theta J)\Theta F; ac \rangle} G \) (after an application of the associativity isomorphism for \( \mathcal{V} \)). Thus diagrams of the
form:

\[
\begin{array}{ccc}
  & A <J\otimes F; c> & \rightarrow G \\
  & <K;a> & \downarrow \\
 A & \rightarrow & \rightarrow (K\otimes J) \otimes F; ac
\end{array}
\]

make sense.

3.16 PROPOSITION The mean cotensor product of \( V \)-functors

\[
\begin{array}{ccc}
  B & \rightarrow G & \rightarrow A \\
  F & \downarrow V & \\
 & 
\end{array}
\]

is an object \( \{F,G\} \) of \( A \) together with an \( F \)-cone, \( \{F,G\}<F;P> \rightarrow G \), such that for any \( J \) in \( V \) and any \( J\otimes F \)-cone, \( A <J\otimes F; c> \rightarrow G \), there exists a unique \( J \)-morphism, \( A <J;\overline{c}> \rightarrow \{F,G\} \), in \( A \) rendering the diagram,

\[
\begin{array}{ccc}
\{F,G\}<F;P> & \rightarrow G \\
<J;\overline{c}> & \downarrow & <J\otimes F; c> \\
A & \rightarrow & A
\end{array}
\]

commutative.

Proof: Comparing the above statement with that of lemma 3.13 the proposition becomes tautologous, since \( J\otimes F \)-cones from \( A \) to \( G \) "are" just \( V \)-natural transformations from \( J\otimes F \) to \([A,-G]A \).
The statement of the proposition makes sense for arbitrary $A$ and $B$ in $\mathbf{V}$-$\mathbf{CAT}$ so we henceforth take it as our definition.

The authors of [B&K] also say that $\{F,G\}$ exists weakly if $\pi$ in their definition is an isomorphism for $J = U$ but is not necessarily so for arbitrary values of $J$. The reader will have no difficulty in verifying that this amounts to universality of $\langle F;p \rangle$ with respect to $F$-cones.

Thus far everything in this section concerns only (large) $\mathcal{V}$-categories. Consider now a $\mathcal{V}$-indexed functor $E \stackrel{H}{\to} A$ of type $N^* \longleftarrow \mathcal{V} \to N$. Thus $A$ is just a $\mathcal{V}$-category and $E$ has $I$-indexed families of objects for all $I \in \mathcal{V}$ etc. For $A \in A$ we define a cone, $c$, from $A$ to $H$ to consist of the following data: for each $I \in \mathcal{V}$ and each $I$-object $X$ of $E$ an $I$-morphism $A \xrightarrow{I;Xc} XH$ in $A$, subject to the requirement that for every $I,J$-morphism, $X \xrightarrow{I,J;x} X'$ in $E$,

$$\begin{array}{c}
A \xrightarrow{I;Xc} XH \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\xrightarrow{J;XH} \\
\xrightarrow{I;\theta J;X'c} X'H
\end{array}$$

commutes. $c$ must further respect substitution. That is $\langle I';(Xc)u^* \rangle = \langle I';(Xu^*)c \rangle$ for all $I' \xrightarrow{u} I$ in $\mathcal{V}$.

We write $A \xleftarrow{i_0^*} H$. For $K \in \mathcal{V}$ we say that $c$ is a $K$-indexed family of cones from $A$ to $H$, or simply
that \( c \) is a \( K \)-cone from \( A \) to \( H \), if \( c \) consists of a \( K \Theta I \)-morphism, \( A \xrightarrow{\langle K \Theta I; Xc \rangle} XH \), in \( A \) for each \( I \)-object \( X \) in \( E \) with

\[
\begin{align*}
A & \xrightarrow{\langle K \Theta I; Xc \rangle} XH \\
& \downarrow \quad \downarrow J; xH \\
\langle K \Theta (I \Theta J); X'; c \rangle & \rightarrow X'; H
\end{align*}
\]

for every \( \langle I, J; x \rangle \) in \( E \), and substitution respected as above. As expected, we denote such \( c \) by \( A \xrightarrow{\langle K; c \rangle} H \) and clearly expressions such as

\[
\begin{align*}
A & \xrightarrow{\langle K; c \rangle} H \\
& \downarrow \quad \downarrow J; xH \\
\langle L; a \rangle & \rightarrow \langle L; ac \rangle
\end{align*}
\]

and

\[
\begin{align*}
A & \xrightarrow{\langle K; c \rangle} H \\
& \downarrow \quad \downarrow J; xH \\
\langle L; a \rangle & \rightarrow \langle L \Theta K; ac \rangle
\end{align*}
\]

make sense, for any \( L \)-morphism \( a \) in \( A \).

3.17 DEFINITION For \( E \xrightarrow{H} A \) a \( V \)-indexed functor of type \( \mathbf{N}^* \leftarrow \mathbf{N} \), the limit of \( H \) (if it exists) is an object \( \lim H \) of \( A \) together with a cone,

\[
\lim H \xrightarrow{\langle \mathbf{P} \rangle} H
\]

such that for any \( K \)-cone, \( A \xrightarrow{\langle K; c \rangle} H \),
there exists a unique $K$-morphism, $A \xrightarrow{\langle K; \bar{c} \rangle} \lim H$, such that

\[
\begin{array}{ccc}
\text{lim } H & \xleftarrow{iP} & H \\
\downarrow & & \downarrow \\
\langle K; \bar{c} \rangle & \xrightarrow{\langle K; c \rangle} & A
\end{array}
\]

commutes.

3.18 THEOREM Given $V$-functors

\[
\begin{array}{ccc}
B & \xrightarrow{G} & A \\
\downarrow & & \downarrow \\
F & \xrightarrow{V} & V
\end{array}
\]

let $U/F \xrightarrow{Q_F} B$ denote the $V$-discrete $V$-cofibration associated to $F$. $\{F, G\}$ exists if and only if $\lim (Q_F G)$ exists and when they do exist they are isomorphic.

Proof: Comparing proposition 3.16 and definition 3.17 we see that it suffices to show

\[
\begin{array}{ccc}
A & \rightarrow & G & \text{K\&F-cones} \\
\text{K-cones}
\end{array}
\]

\[
\begin{array}{ccc}
A & \rightarrow & Q_F G
\end{array}
\]
for all \( K \) in \( V \), (\( V \)-natural in \( A \)). Thus suppose that we are given \( A \xrightarrow{K\theta F;m} G \); that is we have compatible \( K\theta BF \)-morphisms, \( A \xrightarrow{K\theta BF;Bc} BG \). To define a \( K \)-cone \( \hat{c} \) from \( A \) to \( Q_F G \) we require for each \( I \)-object \( <I;x,B> \) (ie \( I \xrightarrow{x} BF \)) of \( U/F \) a \( K\theta I \)-morphism

\[
A \xrightarrow{K\theta I;(x,B)\hat{c}} (x,B)Q_F G = BG.
\]

For each \( B \in B \) we have a generic \( BF \)-object, namely \( <BF;BF,B> = BF \xrightarrow{BF} BF \), in \( U/F \), so define \( (BF,B)\hat{c} = Bc \). Since \( <I;x,B> = <BF;BF,B;x> \) this obliges us to set \( (x,B)\hat{c} = (Bc)(K\theta x) \). Let \( <I;x,B> \xrightarrow{I;J;b} <I\theta J;x\theta J;bF,B'> \) denote an \( I,J \) morphism in \( U/F \) (thus \( B \xrightarrow{J;b} B' \) is a \( J \)-morphism in \( B \)). Then we have

\[
((x,B)\hat{c})(bG) = (Bc)(K\theta x)(bG)
\]

\[
= ((Bc)(bG))(K\theta x)\theta J
\]

\[
= (B'c)(K\theta bF)(K\theta (x\theta J))
\]

\[
= (B'c)(K\theta (x\theta J)bF)
\]

\[
= ((x\theta J)bF,B')\hat{c}
\]

which shows that \( \hat{c} \) is indeed a \( K \)-cone. Conversely given \( A \xrightarrow{K\phi c} Q_F G \), a \( K \)-cone, we set \( Bc = (BF,B)\hat{c} \) and since

\[
(Bc)(bG) = ((BF,B)\hat{c})(bG)
\]

\[
= (bF,B')\hat{c}
\]

\[
= (B'c)(K\theta bF).
\]
this assignment determines a $\mathcal{K}_F$-cone from $A$ to $G$.

The correspondence $c \leftrightarrow \hat{c}$ is clearly bijective and $\mathcal{V}$-natural.
APPENDIX

We cannot discuss true functor categories in \( \text{V-Ind-CAT} \) because \( \text{V-Ind-SET} \) is not cartesian closed. We showed, however, in theorem 2.3 that the ambient \((2-)\)category \( \text{SET}^{(\cdot)^{\text{op}}} \) is, and that \( \text{SET}^{(\cdot)^{\text{op}}} \xrightarrow{\mathcal{F}} \text{CAT} \) preserves exponentiation. Here we indicate a way of making use of this structure by introducing "higher order types" as suggested in §3 of chapter 2.

We will assume that \( \text{V} \) is a strict monoidal category. This allows us to simplify the pseudo functor \( \text{TOP}^{\text{op}} \xrightarrow{[-\text{V}]} \text{CAT} \). Assume that \( \text{Mon}^{\text{op}} \) and redefine \( \text{T} \) to be the full subcategory of \( \text{Mon} \) monoids determined by the finitely presentable ones. For \( \text{T} \text{C} \text{T} \) we now set \([\text{T} \text{V} \text{C} \text{V}]\) equal to the category of strict monoidal functors from \( \text{T} \), regarded as a discrete monoidal category, to \( \text{V} \). This makes \([[-\text{V}]]\) an ordinary functor.

We define a functor, \( \text{set}^{\text{T}} \xrightarrow{\text{V}} \text{CAT} \), by left Kan extension.

\[
\begin{array}{ccc}
\text{TOP}^{\text{op}} & \xrightarrow{\text{V}} & \text{set}^{\text{T}} \\
[-\text{V}] & \xrightarrow{\eta} & \text{CAT} \\
& \downarrow & \text{V} \text{C} \text{V} = \text{Lan}_{\text{T}} [-\text{V}] \\
\end{array}
\]

Thus for \( \text{F} \text{E} \text{set}^{\text{T}} \), \( \text{V} \text{C} \text{F} = \int^{\text{T} \text{F} \cdot \text{C} \text{V}} \text{C} \text{V} \), the coend taken over \( \text{T} \text{C} \text{T} \).
A.1 **PROPOSITION** $\mathcal{V}_\varepsilon$ preserves colimits.

**Proof:** This can be seen directly but we can get slightly more. $\mathcal{V}_\varepsilon$ can be defined in the same way on $\text{SET}^T$. Then the original $\mathcal{V}_\varepsilon$ factors as:

$$\xymatrix{\text{SET}^T \ar[r] & \text{SET}^T \ar[r]^{\mathcal{V}_\varepsilon} & \text{CAT}}$$

where the first functor, an inclusion, preserves all colimits and the second functor has a right adjoint, $R$. For given $\mathcal{C} \in \text{CAT}$ let $\mathcal{C}_R = |\mathcal{C}^{[-\mathcal{V}]}|$. Then,

$$\xymatrix{\mathcal{V}_\varepsilon \mathcal{F} \ar[r] & \mathcal{C} \ar[rr] & & \mathcal{T}^T \mathcal{F} \cdot [\mathcal{V}] \ar[r] & \mathcal{C} \ar[rr] & & \mathcal{C}_R \ar[r] & \mathcal{C}_R^{T \varepsilon T} \ar[r] & \mathcal{F} \ar[r] & |\mathcal{C}^{[-\mathcal{V}]}|}$$

Moreover

$$\xymatrix{\text{SET}^T \ar[r]^{\mathcal{V}_\varepsilon} & \text{CAT} \ar[rr] & & \Gamma \ar[r]^{R} & \text{SET} \ar[r]^{D} & |\text{SET}|}$$
commutes, for \( F \Gamma = 0F \) for all \( F \), hence
\[
|c^{-Y}| \Gamma = |c^{0Y}| = |c^N| = |c|.
\]

The proof of the above proposition suggests that we are adapting the "model classifier" idea of [J&W]. It seems unlikely that \( \mathcal{V} \theta \) is left exact, but we require only the following.

A.2 PROPOSITION \( \mathcal{V} \theta \) preserves finite products.

Proof: It is clear that \( \mathcal{V} \theta \) preserves the terminal object so we just require that \( \mathcal{V} \theta (F \times G) = (\mathcal{V} \theta F) \times (\mathcal{V} \theta G) \) for \( F, G \in \text{set}^T \). Consider first the case where \( F = \text{Set}^{op} \) (i.e. \( F = (S, -) \)). Then,

\[
\mathcal{V} \theta (S \times G) = f^T(S, T) \times \mu G \cdot [TV]
\]

\[
= f^T(S, T) \times (f^R(R, T) \times RG) \cdot [TV]
\]

\[
= f^R f^T(S, T) \times (R, T) \times RG \cdot [TV]
\]

\[
= f^R f^T(S, T) \times (R, T) \cdot [TV]
\]

\[
= f^R f^T(S + R, T) \cdot [TV]
\]

\[
= f^R f^R[G(S + R, V)]
\]

\[
= f^R f^R[G(S, V) \times (R, V)]
\]

\[
= [S, V] \times (\mathcal{V} \theta G)
\]

\[
= (\mathcal{V} \theta S) \times (\mathcal{V} \theta G)
\]
Next,
\[ \psi_{F \times G} = \int^T_{S,F \times T,G \cdot TV} \]
\[ = \int^T (\int^S_{(S,T) \times SF \times TG \cdot TV}) \]
\[ = \int^S_{SF \cdot TV} \int^T_{(S,T) \times TG \cdot TV} \]
\[ = \int^S_{SF \cdot \psi(S \times G)} \]
\[ = (\int^S_{SF \cdot [SV]} \times (\psi \circ G)) \]
\[ = (\psi \circ F) \times (\psi \circ G) \]

A.3 LEMMA Let the following diagram be a pullback in \textbf{CAT}:

\[
\begin{array}{ccc}
\mathcal{E} \mathcal{F}* & \xrightarrow{F} & \mathcal{E} \\
\mathcal{P} \mathcal{F}* & \downarrow{P} & \\
\mathcal{A} & \xrightarrow{F} & \mathcal{B}
\end{array}
\]

If 1/ \( P \) is a fibration
2/ \( A, E, \) and \( B \) are cartesian closed
3/ \( P \) is a cartesian closed functor
4/ \( F \) is a cartesian functor
then
1/ \( \mathcal{P} \mathcal{F}* \) is a fibration
2/ \( \mathcal{E} \mathcal{F}* \) is cartesian closed
3/ \( \mathcal{P} \mathcal{F}* \) is a cartesian closed functor
4/ \( \bar{F} \) is a cartesian functor.
Proof: We have only to check those parts of 2/ and 3/ involving exponentiation, since everything else is well known (and does not require the full set of hypotheses).

Write $A, B, C$ etc. for objects in $A$ and $I, \psi, \psi'$ etc. for objects in $E$. $^\psi A$ will be an abbreviation for $A$.

$\langle A, \Gamma \rangle$ $\Gamma$ $\Gamma$
$\vdots$ $\vdots$
Thus $\psi$ is "in PF*" if and only if $A \rightarrow A^\psi A$

is "in $P$". Since $F$ is cartesian we have for each $A$ and $C$ in $A$ a comparison morphism $(C^A)_F \xrightarrow{k} C_F A^F$
in $B$, namely the exponential transpose of

$A_F \times (C^A)_F \xrightarrow{\varepsilon} (A \times C^A)_F \xrightarrow{\varepsilon F} C_F$, where $\varepsilon$ is the

evaluation. For $A$ and $C$ in PF* we define $A \rightarrow C$
in PF* to be $(\psi^T)_k^*$, as in the following diagram:

$(C^A)_F$

$\psi^T$ $\psi^T$
$\vdots$ $\vdots$
$(\psi^T)_k^* \xrightarrow{k} \psi^T$

in $P$. Now,
In the above $\hat{h}$, respectively $\hat{hF}$, denotes the exponential transpose of $h$, respectively $hF$, and we have (*) in virtue of the commutativity of

\[
\begin{array}{c}
\text{BF} \\
\hat{hF} \\
(C^A)_F \\
\end{array}
\quad
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
k \\
\end{array}
\quad
\begin{array}{c}
\text{AF} \\
\text{CP} \\
\end{array}
\]

We define a category of "higher order" $V$-indexed sets by the following pullback.

\[
\begin{array}{ccc}
V\text{-ind-SET} & \longrightarrow & \text{SET}^{(\cdot)^{\text{op}}} \\
P_V^1 & \quad & P \\
\downarrow \quad & \quad & \downarrow \\
\text{set}^T & \quad & \text{CAT} \\
\end{array}
\]

A.4 Theorem $V$-ind-SET is $\mathcal{U}_1$-complete and cartesian closed, and $P_V^1$ preserves $\mathcal{U}_1$-limits and exponentiation.

Proof: Existence of $\mathcal{U}_1$-limits and their preservation by $P_V^1$ follows from lemma 2.12. The corresponding statement about exponentials is a consequence of lemma A.3, after application of theorem 2.3 and proposition A.2.
As before we can define
\[ V \operatorname{-ind-CAT} \xrightarrow{P V'} \operatorname{cat} T = (V \operatorname{-ind-SET}^t) \operatorname{cat} \xrightarrow{(P V') \operatorname{cat}} \operatorname{set} T \operatorname{-cat}, \]
which is clearly \( U_1 \)-complete as a 2-category and cartesian closed. \( V \operatorname{-ind-SET}^t \) contains \( V \operatorname{-ind-SET} \) as a full-subcategory so we have
\[ V \operatorname{-ind-CAT} \xrightarrow{\_} V \operatorname{-ind-CAT}^t, \]
2-full and faithful.

We also have, from theorem 2.2 and proposition A.1, that \( V \operatorname{-ind-SET}^t \) is \( U_1 \)-cocomplete and \( P \) preserves \( U_1 \)-colimits.

To briefly illustrate the ideas in this appendix, assume that \( V \) is symmetric and that \( T \) is the category of finitely presentable commutative monoids. We calculate \( A^I \) in \( V \operatorname{-ind-CAT}^t \) when \( A \) is a \( V \)-category.

\( A^I \) is of type:

\[ (\mathbf N, -) \xrightarrow{\otimes} (\mathbf N, -) \xrightarrow{(\mathbf N, -)} (\mathbf N, -) \xrightarrow{(\mathbf N, -)} (\mathbf N, -) \xrightarrow{\otimes} (\mathbf N, -) \]

which we write as:

\[ \mathbf N^N \times \mathbf N^N \xrightarrow{} \mathbf N^N \xrightarrow{} \emptyset \]

\( T \mathbf N^N \) is the underlying set of \( \mathbf N \otimes T \).

Let \( <\emptyset, 0> \), respectively \( <\mathbf N, \mathbf M> \), denote the \( V \)-indexed set of objects, respectively morphisms, of \( A \).

The objects of \( A^I \) are easily seen to be given by \( <\emptyset, 0> \). To calculate the morphisms, \( <\mathbf N, \mathbf M> \xrightarrow{<\mathbf N, I>} \), of \( A^I \), we proceed as follows. \( V \otimes (\mathbf N, -) \xrightarrow{} V \), so first consider
<V,M> in \text{SET}(). By theorem 2.3 the exponential is given by \( <V,M> \), where
\[
G \in (V^\text{op})^\text{op}, \quad (G)^I = \int_J ((J,I), JGM) = IG_M.
\]

Now,
\[
\varnothing^N \Rightarrow \int^T (N,N) \times (N,T) \cdot [TV]
\]
\[
\Rightarrow \int^T (N \times N) \times (N,T) \cdot [TV]
\]
\[
\Rightarrow N \times \int^T (N,T) \cdot [TV]
\]
\[
\Rightarrow \int^T (N \times V)
\]

and the comparison morphism, \( N \times V \xrightarrow{k} V \), can be shown to be the functor which sends \((n,J) \in N \times V\) to the functor defined by \((K \xrightarrow{\sim} J \otimes \text{K}(n))\). Thus by the proof of lemma A.3,
\[
<N,M> \xrightarrow{k} V^\text{op} \xrightarrow{(V)^\text{op}} \Rightarrow \int^T (n \times V) \Rightarrow \int^T (N,N) \times (N,T) \cdot [TV]
\]

in \text{V-ind-SET} ; and \((n,J)k^\text{op}M^I = (J \otimes \text{K}(n))M\) for \((n,J) \in N \times V\). In short the objects of \( \text{A}^I \) are those of \( \text{A} \) and for \( \text{A}, \text{B} \in \text{A}^I \), an \( n,J \)-morphism from \( \text{A} \) to \( \text{B} \) in \( \text{A}^I \) is a \( J \otimes \text{I}(n) \)-morphism from \( \text{A} \) to \( \text{B} \) in \( \text{A} \). Morphisms are composed as in \( \text{A} \) (this making sense since \( V \) was assumed symmetric).

If we consider only the \( n,J \)-morphisms of \( \text{A}^I \), for variable \( n \in \text{N} \), we recover the graded category \((I)\text{A}\) introduced in §2 of chapter 1. The diagonal \( \text{V}-\text{indexed functor}, \text{A} \xrightarrow{\text{A}^I} \), is the identity on objects and sends \( J \)-morphisms in \( \text{A} \) to the corresponding
0,J-morphisms in $\tilde{A}^I$. It clearly restricts to an ordinary functor, $\tilde{A} \xrightarrow{\Delta} (I)\tilde{A}$, where $\tilde{A}$ is the underlying ordinary category of $\tilde{A}$. For $\tilde{A} = \tilde{V}$ this functor has a universal property.

Note first that $(I)\tilde{V}$ is again a (strict) symmetric monoidal category, that $\tilde{V} \xrightarrow{\Delta} (I)\tilde{V}$ is a (strict) monoidal functor, and that there is a distinguished morphism $U \xrightarrow{\kappa} I$ of degree 1 in $(I)\tilde{V}$, namely that corresponding to the identity $I \xrightarrow{\kappa} I$ in $\tilde{V}$.

Now for $\tilde{W}$ any (strict) symmetric monoidal category, $\tilde{V} \xrightarrow{F} \tilde{W}$ any (strict) monoidal functor, and $U \xrightarrow{W} IF$ any "U-element" of $IF$ in $\tilde{W}$, there exists a unique (strict)monoidal functor $(I)\tilde{V} \xrightarrow{F} \tilde{W}$ such that

$$ \tilde{V} \xrightarrow{\Delta} (I)\tilde{V} \xrightarrow{F} \tilde{W} $$

commutes and $xF = w$.

It may be interesting to observe that for $S$ with finite limits, $S \xrightarrow{A} S/I$ satisfies the same "polynomial" property with respect to left exact functors, (cf. also with [Lk₂]). At the same time $S/I$ is equivalent to the $S$-indexed category (in the sense of [F&S]), $S^I$, at 1.
REFERENCES


[Lk2] J. Lambek, From types to sets, Preprint from McGill University, 1974.


