Reliability-Based Bearing Capacity Design of Shallow Foundations

by

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List of Abbreviations

ASCE  American Society of Civil Engineers
CLT  Central Limit Theorem
CPT  cone penetration test for in-situ soil testing
FEM  Finite Element Model
FHWA  Federal Highway Administration
LAS  Local Average Subdivision
LCG  linear congruential generators
LRFD  Load and Resistance Factor Design
LSD  Limit States Design
RFEM  Random Finite Element Method
RBD  Reliability-Based Design
SLS  Serviceability Limit State
SPT  standard penetration test for in-situ soil testing
ULS  Ultimate Limit State
WSD  Working Stress Design
List of Symbols

<table>
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<tr>
<td>$c$</td>
<td>load factor</td>
</tr>
<tr>
<td>$c_D$</td>
<td>dead load factor</td>
</tr>
<tr>
<td>$c_L$</td>
<td>live load factor</td>
</tr>
<tr>
<td>$\beta$</td>
<td>reliability index</td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>test statistic of Chi-Square</td>
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<td>$\phi$</td>
<td>internal friction angle of soil</td>
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<tr>
<td>$\phi_A$</td>
<td>arithmetic average of friction angle</td>
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<tr>
<td>$\phi_G$</td>
<td>geometric average of friction angle</td>
</tr>
<tr>
<td>$\phi_{\min}$</td>
<td>lower bound of $\phi$</td>
</tr>
<tr>
<td>$\phi_{\max}$</td>
<td>upper bound of $\phi$</td>
</tr>
<tr>
<td>$\phi_{eff}$</td>
<td>effective friction angle</td>
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<td>$\phi(x_i)$</td>
<td>one of the $n$ friction angle ‘samples’ forming a partition in the region under the footing</td>
</tr>
<tr>
<td>$\phi^o(x_j)$</td>
<td>one of $m$ friction angle soil samples actually observed</td>
</tr>
<tr>
<td>$\hat{\phi}$</td>
<td>characteristic friction angle</td>
</tr>
<tr>
<td>$\phi_g$</td>
<td>resistance factor</td>
</tr>
<tr>
<td>$\phi_n$</td>
<td>normalized form of $\phi$ on interval (0,1)</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>standard normal cumulative distribution function</td>
</tr>
<tr>
<td>$\psi$</td>
<td>dilation angle</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>unit weight of the soil</td>
</tr>
<tr>
<td>$\gamma_c$</td>
<td>variance reduction function of $c$</td>
</tr>
<tr>
<td>$\gamma_\phi$</td>
<td>variance reduction function of $\phi$</td>
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<tr>
<td>$\mu$</td>
<td>mean</td>
</tr>
<tr>
<td>$\mu_c$</td>
<td>mean cohesion</td>
</tr>
<tr>
<td>$\mu_{\ln c}$</td>
<td>mean of the log-cohesion</td>
</tr>
<tr>
<td>$\mu_{\ln \hat{c}}$</td>
<td>mean of the logarithm of the estimated cohesion</td>
</tr>
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</table>
\( \mu_{\ln e_{\text{eff}}} \) mean of the logarithm of the effective cohesion underlying the footing

\( \mu_\phi \) mean friction angle

\( \mu_{\hat{\phi}} \) mean of the estimated friction angle

\( \mu_{\phi_{\text{eff}}} \) mean of the effective friction angle

\( \mu_{\ln N_c} \) mean of the logarithm of \( N_c \)

\( \mu_{\ln \hat{N}_c} \) mean of the logarithm of the estimated \( N_c \)

\( \mu_{\ln N_{c, \text{eff}}} \) mean of the logarithm of the effective \( N_c \) underlying the footing

\( \mu_L \) mean footing load

\( \mu_{L_D} \) mean dead load

\( \mu_{L_L} \) mean live load

\( \mu_{\ln L} \) mean of the log-footing load

\( \mu_R \) mean resistance

\( \mu_{\ln R} \) mean of the log-resistance

\( \nu \) Poisson’s ratio

\( \rho \) correlation coefficient

\( \rho_{i,j}^{c,\phi} \) correlation coefficient between \( \ln c^\phi(x_i) \) and \( \ln c^\phi(x_j) \)

\( \rho_{i,j}^{\phi,\phi} \) correlation coefficient between \( \phi^\phi(x_i) \) and \( \phi^\phi(x_j) \)

\( \rho_{i,j}^{c,c} \) correlation coefficient between \( \ln c(x_i) \) and \( \ln c(x_j) \)

\( \rho_{i,j}^\phi \) correlation coefficient between \( \phi(x_i) \) and \( \phi(x_j) \)

\( \rho_{c,i,j} \) correlation coefficient between \( \ln c^\phi(x_i) \) and \( \ln c(x_j) \)

\( \rho_{n,i,j} \) correlation coefficient between \( \ln N_{c,\text{eff}}(x_i) \) and \( \ln N_c(x_j) \)

\( \sigma \) standard deviation

\( \sigma_{\ln c} \) standard deviation of the log-cohesion

\( \sigma_{\ln \hat{c}} \) standard deviation of the logarithm of the estimated cohesion

\( \sigma_{\ln e_{\text{eff}}} \) standard deviation of the logarithm of the effective cohesion underlying the footing

\( \sigma_{\ln \hat{N}_c} \) standard deviation of the logarithm of the estimated \( N_c \)

\( \sigma_{\ln N_{c,\text{eff}}} \) standard deviation of the logarithm of the effective \( N_c \) underlying the footing

\( \sigma_{\ln L_D} \) standard deviation of the logarithm of dead load
\( \sigma_{\ln L} \) standard deviation of the logarithm of live load
\( \sigma_{\hat{p}_f} \) standard deviation of \( \hat{p}_f \)
\( \sigma_{n_f} \) standard deviation of \( n_f \)
\( \tau \) distance between points or centers of elements
\( \theta \) correlation length
\( \theta_{mc} \) isotropic correlation length of the log-cohesion field
\( \theta_{\phi^{-1}} \) isotropic correlation length of \( G_\phi(x) \)
\( A \) footing area
\( B \) footing width
\( \bar{B} \) mean footing width estimated from the mean soil parameters
\( B_{req} \) required footing width
\( c \) cohesion of soil
\( c_{eff} \) effective cohesion
\( \hat{c} \) characteristic cohesion
\( c_a \) geometric average of cohesion
\( c_A \) arithmetic average of cohesion
\( c(x_i) \) one of \( n \) cohesion ‘samples’ forming a partition in the region under the footing
\( c^o(x_j) \) one of \( m \) cohesion soil samples actually observed
\( C \) footing length
\( dx \times dy \) element sizes
\( D \) effective domain \( D = 2W \times W \)
\( D_f \) depth of the foundation
\( E \) elastic modulus
\( F_s \) factor of safety
\( G_{mc}(x) \) standard normal (Gaussian) random field
\( G_\phi(x) \) standard normal (Gaussian) random field
\( H \) the width of the soil model in the horizontal direction
\( k_{D} \) bias factor of dead load
$k_L$ bias factor of live load
$L$ load
$L_c$ characteristic load
$L_d$ dead load
$L_{dc}$ characteristic dead load
$L_l$ live load
$L_{lc}$ characteristic live load
$M$ safety margin
$n$ number of simulations
$n_b$ number of elements under the footing
$n_f$ number of footing design failure
$n_x$ number of elements in $x$ direction
$n_y$ number of elements in $y$ direction
$N_{\gamma}$ bearing capacity factor that accounts for the influence of the weight of the soil
$N_c$ bearing capacity factor that concerns with the cohesion, $c$
$N_{c, eff}$ effective $N_c$
$N_{c, eff}$
$N_{c, eff}$ characteristic $N_c$
$N_q$ bearing capacity factor that concerns with the embedment depth $D$
$p_f$ the probability of failure of foundations
$\hat{p}_f$ estimated probability of failure
$\rho_{max}$ maximum acceptable risk of design failure
$\bar{q}$ overburden stress
$q_a$ allowable bearing capacity
$q_u$ ultimate (actual) bearing capacity
$\hat{q}_u$ characteristic ultimate bearing capacity
$R$ resistance
$R_c$ characteristic resistance
$R_{D/L}$ dead to live load ratio $R_{D/L} = \mu_{D/L} / \mu_{L/L}$

xx
$s_\Phi$ the factor that controls the distribution of friction angle
$s_u$ undrained shear strength
$T$ local averaging width
$V_c$ coefficient of variation of cohesion
$V_\phi$ coefficient of variation of friction angle
$V_L$ coefficient of variation of load $L$
$V_{LD}$ coefficient of variation of dead load
$V_{LL}$ coefficient of variation of live load
$V_E$ error factor considering the measurement and model errors
$W$ mean wedge zone depth
$g$ spatial coordinate or position
$X$ random viable or field
$X_A$ arithmetic average of $X$
$X_G$ geometric average of $X$
$X_H$ harmonic average of $X$
$X_T$ local average of random fields over length $T$
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Abstract

Geotechnical engineers have primarily used a traditional approach, Working Stress Design (WSD), for the bearing capacity design of shallow foundations, in which a single factor of safety is applied to capture all uncertainty. One problem with WSD is that there is no rational consideration of different modes of failure and their consequences. The structural community largely abandoned WSD several decades ago in favor of Reliability-Based Design (RBD) methods which have been implemented using Load and Resistance Factor Design (LRFD).

The Load and Resistance Factor Design (LRFD) is an evolution of WSD, with the consideration of different possible failure modes and separate load and resistance factors. LRFD is a convenient and rational way of accounting for the sources of uncertainty in design. The application of LRFD in geotechnical engineering helps harmonize with the structural community and should reduce expense. The geotechnical community is currently working on the transition from Working Stress Design (WSD) to Load and Resistance Factor Design (LRFD).

In this research, LRFD has been considered in some detail for bearing capacity design of shallow foundations. The actual performance of any geotechnical footing can be quite different than expected due to the soil's spatial variability. A novel mathematical theory was developed to analytically estimate the probability of bearing capacity failure. The spatial random soil field was modeled using random field theory, in which three parameters were considered: the mean, the standard deviation of soil properties, and the correlation length of the soil field. Monte Carlo simulation was then used to simulate the soil field, estimate the probability of bearing capacity failure, and validate the theory. Specifically, Random Finite Element Method (RFEM) was used to simulate the soil's spatial variability and estimate failure probabilities. Once the theory has been validated, the ‘optimal’ resistance factors required to achieve certain levels of reliability were suggested using the proposed theory.
Chapter 1

Introduction

1.1 General

The function of a footing is to transmit loads from the structure to the supporting subsurface soil or rock without collapse or excessive deformation. The primary objectives of engineering design, including foundation design, are to achieve safety, serviceability, and economy. In order to achieve these objectives we commonly consider two limit states or failure modes: Ultimate Limit State (ULS) and Serviceability Limit State (SLS). For shallow foundations the ULS corresponds to bearing capacity failure, and the SLS to settlement failure. At both these limit states, the structure no longer performs its intended function.

Traditionally, the geotechnical engineer has primarily used a design approach commonly referred to as Working Stress Design (WSD), in which a single factor of safety is used to capture all uncertainty. The factor of safety is developed from previous experience with similar structures in similar environments, or under similar conditions, and is applied to the resistance at the design stage to reduce the risk of potential adverse performance (collapse, excessive deformations, etc.).

With increasing public awareness of the failure consequences of engineered geotechnical projects, rational risk assessments as part of the design are becoming necessary. As mentioned by Becker (1996 a), “There is an ever-increasing demand on the geotechnical engineering community to adopt limit states design.” This demand can best be accommodated by the implementation of a rational Reliability-Based geotechnical Design approach (RBD), such as Load and Resistance Factor Design (LRFD) method. The main advantage of using RBD is that a known level of reliability can be achieved consistently over a wide range of design conditions.

Unlike concrete and steel, however, where fixed strength/resistance factors and characteristic strengths can be used, soil is an in-situ material over which little or no ‘quality control’
can be exercised. In the case of natural soils, the optimum approach is then to assess the variability of the in-situ 'material' through site investigation and then estimate probability of failure for a given design problem. This can be achieved using RBD since it relates the spatial variability of soils to probability of failure, allowing the estimation of load and resistance factors as a function of soil variability.

Structural codes largely abandoned the Working Stress Design approach several decades ago in favor of probabilistic methods which have been implemented using Load and Resistance Factor Design (LRFD) since 1970's in Canada. One advantage to LRFD is that it separately considers the uncertainties due to loading and resistance. The load and resistance factors are selected so as to achieve a desired structural reliability over a wide range of design conditions. Even though LRFD is deemed superior to WSD and the application of LRFD would harmonize with structural codes, while optimizing economy, it has yet to be fully adopted in geotechnical engineering.

This thesis considers in some detail a Load and Resistance Factor Design (LRFD) methodology for shallow foundations against bearing failure. This is accomplished by investigating bearing capacity resistance of shallow foundations using the Random Finite Element Method (RFEM). The intent is to derive a reliability-based bearing capacity design approach for shallow foundations, using load factors as specified by structural codes and resistance factors determined in this research to achieve an acceptable reliability level.

1.2 Background to Design Methodologies

Working Stress Design (WSD) (also referred to as allowable stress or permissible stress design (ASD)) was one of the first rational design approaches developed in civil engineering and it became the traditional design basis after its introduction in the early 1800's. When applied to foundation design, the objective of WSD is to ensure that when the foundation is subjected to the working (or service) load transmitted from the superstructure, the induced stresses are less than the allowable stresses. In order to safely fulfill this objective, a single, sometimes called "global", factor of safety, \( F_s \), is used to characterize all uncertainties associated with the design process, such as uncertainties associate with loads, soil strength, model accuracy, and so on. Typical ranges of factors of safety (Terzaghi and Peck, 1967) are shown in Table 1.1.
Table 1.1  Typical ranges of factors of safety (Terzaghi and Peck, 1967).

<table>
<thead>
<tr>
<th>Failure Type</th>
<th>Item</th>
<th>Factor of Safety</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shearing</td>
<td>Earthworks</td>
<td>1.3 – 1.5</td>
</tr>
<tr>
<td></td>
<td>Earth-retaining structures, excavations</td>
<td>1.5 – 2</td>
</tr>
<tr>
<td></td>
<td>Foundations</td>
<td>2 – 3</td>
</tr>
<tr>
<td>Seepage</td>
<td>Uplift, heave</td>
<td>1.5 – 2</td>
</tr>
<tr>
<td></td>
<td>Exit gradient, piping</td>
<td>2 – 3</td>
</tr>
</tbody>
</table>

As shown in Fig. 1.1, the load applied to a foundation is not known for certain – it has some distribution. Similarly, the soil resistance cannot be predicted with certainty. It also has a distribution. These distributions express the relative likelihoods that \( L \) or \( R \) take on any particular value. The first step in WSD is to express the loads in terms of characteristic, also called the nominal, values. Similarly, the soil resistance is computed using characteristic soil properties. The characteristic load, \( L_c \), and characteristic resistance, \( R_c \), are shown in Fig. 1.2. Essentially, WSD ensures that the characteristic load \( L_c \) acting on a foundation does not exceed some allowable limit.

![Probability Density of R and L](image)

**Figure 1.1**  Randomness of load and resistance.
As shown in Fig. 1.2, there are two definitions of the factor of safety, the mean factor of safety and the nominal factor of safety.

\[ F_s = \frac{\text{mean of resistance}}{\text{mean of load}} = \frac{\mu_R}{\mu_L} \]  

(1.1)  

or

\[ F_s = \frac{\text{characteristic resistance}}{\text{characteristic load}} = \frac{R_c}{L_c} \]  

(1.2)

where the mean values can be estimated from the sampled data of soil and loads, the characteristic values can be taken as percentiles of the load and resistance distributions. For example, in Fig. 1.2, \( L_c \) might be the value that only 10% of the loads exceed, while \( R_c \) is the value that 90% of the resistances exceed. In geotechnical design, \( F_s \) is typically selected using engineering judgement and experience and then can be used to determine the allowable stress for the estimation of the required footing size.

The mean factor of safety is probably the more commonly used of the two. The lack of a clear definition for the factor of safety has led to significant confusion in the past.

Although WSD is simple and useful, it is also accompanied by difficulties and ambiguities. First, the mean factor of safety makes no distinction between loads and resistances having different variabilities. Foundations with the same mean factor of safety can have vastly
different probabilities of failure. For example, Fig. 1.3 shows three different load and resistance distributions, all having identical factors of safety. The uppermost plot has the least uncertainty/variability associated with the load and resistance and so has a failure probability (where load exceeds resistance, \( L > R \)) which is quite small. The lowermost plot, on the other hand, has a higher probability of failure simply because the load and resistance have higher uncertainty/variability.

The probability of failure, \( p_f \), is computed as the probability that load exceeds resistance,

\[
p_f = P \left[ L > R \right] = P \left[ R/L < 1 \right] = P \left[ \ln R - \ln L < 0 \right]
\]

where, if it is assumed that \( R \) and \( L \) are lognormally distributed, then \( (\ln R - \ln L) \) is normally distributed. If we let \( M = R/L \), where \( M \) is called the safety margin (\( M \) is less than one if the load exceeds the resistance so that failure occurs), then

\[
\ln M = \ln R - \ln L
\]

which is normally distributed with parameters

\[
\begin{align*}
\mu_{\ln M} &= \mu_{\ln R} - \mu_{\ln L} \\
\sigma^2_{\ln M} &= \sigma^2_{\ln R} + \sigma^2_{\ln L}
\end{align*}
\]

where it is assumed \( R \) and \( L \) are independent. Now we find that

\[
P \left[ L > R \right] = P \left[ \ln M < 0 \right] = P \left[ Z < \frac{0 - \mu_{\ln M}}{\sigma_{\ln M}} \right]
\]

\[
= P \left[ Z < \frac{0 - (\mu_{\ln R} - \mu_{\ln L})}{\sqrt{\sigma^2_{\ln R} + \sigma^2_{\ln L}}} \right]
\]

\[
= \Phi \left( -\frac{\mu_{\ln R} - \mu_{\ln L}}{\sqrt{\sigma^2_{\ln R} + \sigma^2_{\ln L}}} \right)
\]

where \( Z = (\ln M - \mu_{\ln M})/\sigma_{\ln M} \) has a standard normal distribution, and \( \Phi \) is the standard normal cumulative distribution function. We can now define the reliability index, \( \beta \), to be

\[
\beta = \frac{\mu_{\ln R} - \mu_{\ln L}}{\sqrt{\sigma^2_{\ln R} + \sigma^2_{\ln L}}}
\]

which represents the number of standard deviations that \( (\ln R - \ln L) \) is away from the "failure" region (i.e. 0). As \( \beta \) becomes smaller, the probability of failure increases. Typically, a \( \beta \) value of 3.0 to 3.5 is aimed for in structural engineering.
Figure 1.3  Three geotechnical problems can have the same mean factor of safety, $F_s$, and yet quite different probabilities of failure, $p_f$. 
Even though the factors of safety are the same in Fig. 1.3, the footing reliabilities are quite different, as shown by the different failure probability and reliability indices. Hence, in most cases, there is no way of quantifying the probability of failure from the factor of safety. In other words, Working Stress Design (WSD) generally fails to reflect the true reliability. Note that as the area of the overlap by load and resistance distribution, as shown in Fig. 1.3, increases, the probability of failure increases, although the area is not equal to the probability of failure.

Another key problem with WSD is that it does not explicitly differentiate between the behaviour of the structure under ultimate and serviceability limit states. In other words, there is no rational consideration of different modes of failure and their consequences. However, as will be discussed follows, when we combine Limit States Design (LSD) with Load and Resistance Factor Design (LRFD), we can both consider different modes of failure and achieve appropriate levels of safety for each failure mode.

1.3 Reliability-Based Design

As Lacasse (1994) said, "probabilistic solutions are not a panacea but they provide a rational framework to include in a consistent fashion the relevant uncertainties of the predicted behavior." In a Reliability-Based Design (RBD) the basic objective is to ensure that the probability of failure of a system does not exceed a certain target reliability or a certain acceptable level. This is achieved by taking the load and resistance parameters as being random variables or random fields instead of constant deterministic values.

Although RBD methods are widely accepted in the structural community, geotechnical engineers are reluctant to accept these methods. The first reason for this lack of acceptance is that reliability involves concepts and theories that many geotechnical engineers are not familiar with. Secondly, it is commonly perceived that using reliability theory would require more data, time, and effort than available in most circumstances. This requirement would complicate the design and increase its expense. Most importantly, the inherent variabilities and uncertainties involved in geotechnical problems are quite difficult to quantify, which also hampers the implementation of RBD in foundation engineering.

However, much current research, including this thesis, is aimed at remedying these problems.
1.3.1 Limit States Design

Limit states are conditions which, when reached, render the structure no longer able to perform its intended function. For example, the ultimate limit state of collapse renders the building unusable. In order to achieve the design objectives of engineering, i.e., safety, serviceability, and economy, two limit states are usually considered in foundation engineering: Ultimate limit State (ULS) and Serviceability Limit State (SLS). In 1943 Terzaghi pointed out two principal groups or classes of problems in geotechnical analysis, namely stability problems, defined as “the conditions for the equilibrium of ideal soils immediately preceding ultimate failure”, and elasticity problems, dealing with “the deformation of the soil”. These coincide with ULS and SLS and so geotechnical engineers have actually been practicing limit state design for more than sixty years now.

Allen (1975) defines SLS as “excessive deflection, vibration, cracking or permanent deformation”, which in foundation engineering mostly translates into a excessive settlement or differential settlement. He also defines ULS as “collapse due to crushing, fracture, buckling, etc.; overturning, sliding; large deformation, flutter”, in which the foundation and/or the structure it supports are deemed to be unsafe. ULS is generally concerned with failure modes that are dangerous or life threatening. For foundations at ULS one is typically concerned with bearing capacity failure.

Limit States Design (LSD) is an evolution of WSD, with the consideration of the possible failure modes or critical limit states. Over the past 30 years much effort has been made to investigate LSD in structural and geotechnical engineering by many researchers and practitioners, such as Meyerhof (1970, 1982, 1984, 1993, 1994, 1995), Allen D.E.(1975, 1991), MacGregor (1976), Becker (1996a, 1996b, 2003), and Green and Becker (2000).

1.3.2 Load and Resistance Factor Design

The concept of Load and Resistance Factor Design can be traced back to the late 1940’s and early 1950’s to the work of Taylor (1948), Hansen (1953), and Freudenthal (1956). They suggested separate safety factors for loads and resistance instead of the single factor of safety, $F_s$, used in Working Stress Design (WSD).

As a reliability-based design method, Load and Resistance Factor Design (LRFD) can at least approximately accomplish the basic objective of RBD, i.e., ensure the structure has a certain target reliability. This is achieved by employing partial factors of safety separately on
the individual components of load and resistance. Since the variability of load differs from that of resistance, it makes sense that they have different partial factors of safety applied to them. Using LRFD provides the opportunity for the design to be more responsive to the differences among types of loads and resistances, fundamental behaviour of the foundation, and consequence of different failure modes (i.e. limit states) leading to unsatisfactorily performance.

One version of LRFD design of geotechnical components, valid for any limit state, involves satisfying the following equation:

$$\phi_g R \geq \sum \alpha_i L_i$$  \hspace{1cm} (1.8)

where $\phi_g$ is the resistance factor (normally less than one), $\alpha_i$ is the load factor acting on the load effect $L_i$ (normally greater than one), and $R$ is the resistance. The function of the resistance factor, $\phi_g$, is to reduce the resistance to a level very likely to be less than true resistance, which is called the factored resistance. Similarly, the load factor, $\alpha$ increases the load to a factored load, having a small probability of exceedance.

The random characteristics of loads and strength (resistance) in structural engineering are fairly well known and reasonably well established (Allen, 1975, MacGregor, 1976, Scott et al., 2003). This is because, for structural material such as concrete, steel, and wood, representative testing can easily be performed so that distributions are relatively easily estimated. Because structural materials are typically quality controlled, their distributions remain relatively constant at any building site. Thus, it is typically only necessary to take a few samples of the building material at the site to ensure that design criteria are met.

Unfortunately, the same cannot be said about soil properties. The difficulty with geotechnical engineering in terms of LRFD is that geotechnical materials, e.g. soil or rocks, are not manufactured to specified criteria, as is the case for most structural materials. In order to have a reasonably accurate estimation of soil variability we must conduct intensive site investigations. It is also hard to deliver undisturbed soil samples to test facilities to accurately determine their properties. In other words, to capture the variability of soil accurately many carefully taken samples would be required to estimate both mean and variance values, which adds expense and difficulties to foundation design.

Another difficulty with geotechnical engineering is the determination of spatial randomness of soil and its effect on the design reliability. Soil properties often vary dramatically from point to point within the same site and a thorough awareness of this inherent variability can
be vital to the success of the design. Because soil properties vary from point to point soils should be modeled using random field theory (Vanmarcke, 1984).

In spite of the difficulties stated above, there are important technical benefits associated with the use of LRFD for geotechnical aspects of foundation design, as discussed below.

First of all, the use of separate load and resistance factors is logical and realistic because loads and resistance have separate and unrelated sources of uncertainty. Using separate factors is a convenient and rational way of accounting for the sources of uncertainty in design. In addition, soil can act either as a load or as a resistance or both. For example, the soil behind a retaining wall acts as a load while soil in front of the retaining wall may act as a resistance. It would be better to factor these actions separately.

Secondly, the application of LRFD to geotechnical design helps harmonize with the structural community and minimize any incompatibility between structural and foundation engineers. This leads to a consistent design approach/philosophy orchestrated by the structural and geotechnical engineers.

Finally, the fact that all components of the structural system, including the foundation, are designed to a consistent and appropriate level of safety or reliability leads to a more economical design.

The geotechnical community is currently mainly preoccupied with the transition from Working Stress Design (WSD) to Load and Resistance Factor Design (LRFD). Many researchers are overcoming the difficulties noted above to encourage this process. For example, Fenton et al. (2005) applies LRFD principles to the serviceability limit state of shallow foundation settlement.

Descriptions of the application of LRFD in geotechnical engineering can be found in several design codes and reports, such as the American Association of State Highway and Transportation Officials (AASHTO, 2004), the National Cooperative Highway Research Program (NCHRP) Report 343 (Barker et al., 1991), the Canadian Foundation Engineering Manual (CFEM, 1992, 2006), Eurocode 7 (1997, 2003), the Federal Highway Administration (FHWA) Load and Resistance Factor Design (LRFD) for Highway Bridge Substructures (2001), the NCHRP Report 507 (2004), the NCHRP 12-55 Load and Resistance Factors for Earth Pressures on Bridge Substructures and Retaining Walls (2004), and the FHWA Development of Geotechnical Resistance Factors and Downdrag Load Factors for LRFD Foundation Strength Limit State Design Reference Manual (Allen, 2005).
Even though the evolution from WSD to Load and Resistance Factor Design (LRFD) in geotechnical engineering community is entirely natural with the development of the public awareness of the benefits, there is some concerns about the position of engineering judgement and experience. It should be pointed out that engineering judgement and experience are, and always will be, an essential part of geotechnical engineering, especially for the design aspects that are beyond the scope of mathematical analysis. For example, the selection of characteristic values for any given limit state will involve engineering judgement and experience.

The incorporation of LSD and LRFD in foundation engineering will provide a consistent design philosophy between geotechnical and structural engineers and result in a more consistent and rational risk management in foundation engineering, and will be adopted in the present study. The basic design approach is to satisfy the following equation at each possible limit state,

\[
\text{Factored resistance} \geq \text{Factored load effects}
\]

1.3.3 Load and Resistance Factor Design Implementation

During the development of LRFD in geotechnical engineering, European countries and North America went two different ways with regards to applying the resistance factors. As Becker indicated (1996a) "there are two different directions developed for the concepts of LSD with the use of partial factors of safety in the United States and in Europe, and the main difference between them is how to calculate the factored resistance for ULS".

The factored strength approach is used in European countries. This approach applies specified partial factors directly to the soil strength parameters appearing in prediction equations, such as cohesion, \(c\), and friction angle, \(\phi\), to ensure the safety of structures for ULS and SLS. For example, one model given in Eurocode 7 (2003) specifies that undrained shear strength, \(s_u\), is to be divided by a partial factor equal to 1.4 while the effective friction angle is to be divided by a partial factor equal to 1.25.

The factored total resistance approach, used in North America, applies a single resistance factor to the soil resistance computed by traditional means using unfactored characteristic soil properties. More and more manuals and codes in Canada are incorporating the total resistance factor design approach, such as the Canadian Highway Bridge Design Code (CHBDC, 2000), the National Building Code of Canada (NBCC, 2005), and the Canadian
Foundation Engineering Manual (CFEM, 2006). For example, the bearing capacity under a shallow foundation could be computed using the bearing capacity formula in which soil properties, \( c \) and \( \phi \), are characteristic values. The final predicted ‘resistance’ is then decreased by a total resistance factor for use in design.

In the opinion of author, the factored strength approach has some shortcomings. For example, Been (1989) stated that the factored strength approach does not capture the true mechanism of failure when failure is influenced by soil behaviour. In other words, factoring the soil properties prior to their use in predictive formula sometimes results in changes in the critical failure mechanism and leads to loss of the understanding of the real soil behaviour. In addition, the factored strength approach requires a multitude of resistance factors all of which have to be estimated separately. This is a difficult task, requiring significant amounts of experimental data, and prone to significant error. Finally, the use of myriad factors can lead to confusion in the design process.

Compared to the factored strength approach, the factored total resistance approach is simpler in that one total resistance factor is used to capture all sources of uncertainty, including the parameter (soil properties) uncertainty, estimation approach uncertainty, analytical model uncertainty, site investigation uncertainty, construction uncertainty, limit state uncertainty and so on. Not only is the factored total resistance approach easier to use, but a better representation of the actual failure mechanism can be obtained, and the failure mechanism is preserved (the computed resistance prior to factoring is the ‘best’ guess at the soil behavior).

In addition, the using a single soil resistance factor in geotechnical design is consistent with the structure codes, since soil is taken as an engineering material having its own single resistance factor, like concrete and steel. For example, in the National Building Code of Canada (NBCC, 2005) the resistance factor, \( \phi \), is defined as “a resistance factor applied to the resistance or specified material property which takes into account variability of material properties and dimensions, workmanship, type of failure (i.e. brittle verses ductile) and uncertainty in the prediction of resistance.”

Finally, the single resistance factor is much easier to estimate from observed data, and it allows for a smoother transition from Working Stress Design (WSD) to Load and Resistance Factor Design (LRFD). The using of a single resistance factor in LRFD is similar to the global factor of safety, \( F_s \), in WSD, except that the single resistance factor is specifically applied to the resistance and the loads are now separately factored.
This research will use the factored total resistance approach, employing a single total resistance factor, $\phi_g$, which will be consistent with many existing codes of practice currently in use in Canada and the United States. This approach will be referred to as Load and Resistance Factor Design (LRFD) in the following chapters.

1.4 Basic Properties of Random Fields and Local Averages

1.4.1 Random Fields

1.4.1.1 General

A random variable is a function that assigns a real number, $X(\omega)$, to every outcome, $\omega \in \Omega$. Although there is exactly one value $X(\omega)$ for each $\omega$, different $\omega$ could lead to the same $X(\omega)$. A random field $X(t)$ is a collection of random variables, $X_1 = X(\xi_1), X_2 = X(\xi_2), \ldots$, whose values are mapped onto a space (of n dimensions), one for each point in the field. Values in a random field are usually spatially correlated in one way or another.

In the present study three parameters are considered to describe the random soil model, the mean, $\mu$, the standard deviation, $\sigma$, and the correlation length, $\theta$. Frequently, it is more convenient to express standard deviation (or variance) as a coefficient of variation, $V$, defined as the ratio of standard deviation, $\sigma$, to the mean, $\mu$, $V = \sigma / \mu$.

Spatial correlation length $\theta$, also called scale of fluctuation, is the area under the correlation function, $\rho(\tau)$ (see next),

$$\theta = \int_{-\infty}^{\infty} \rho(\tau)d(\tau)$$  \hspace{1cm} (1.9)

Loosely speaking $\theta$ is the distance beyond which points are deemed to be effectively independent. Conversely, two points separated by a distance ($\tau$) less than $\theta$ will be significantly correlated. Fields with small $\theta$ tend to be ‘rough’, while fields with larger $\theta$ are usually smoother. The comparison of ‘rough’ and ‘smooth’ fields are shown in Fig. 1.4 and Fig. 1.5, where $X(t)$ is the random field, a collection of random variables $X$. Unfortunately, the correlation length is difficult to estimate even with large amounts of data. Fortunately, a “worst” case correlation length, leading to the highest probability of failure, often exists for geotechnical problems (Fenton, 2003).
Figure 1.4  Rough field due to small scale of fluctuation, $\theta = 0.2$.

Figure 1.5  Smooth field due to large scale of fluctuation, $\theta = 10.0$. 
In this research, the correlation length, $\theta$, corresponds to the underlying normally distributed random field. In other words, if the random field $X$ is lognormally distributed, then $\theta$ corresponds to correlations between $\ln X$ at two points.

1.4.1.2 Common Types of Correlation Functions

In random fields a correlation coefficient can be used to characterize the spatial dependence of the fields. The correlation coefficient will be given by correlation function parameterized by the correlation length, $\theta$.

There are several commonly used correlation functions, for example,

1) Markov correlation function

$$\rho(\tau) = \exp \left\{ -\frac{2|\tau|}{\theta} \right\}$$ (1.10)

2) Gaussian type correlation function

$$\rho(\tau) = \exp \left\{ -\pi \left( \frac{|\tau|}{\theta} \right)^2 \right\}$$ (1.11)

3) Polynomial decaying correlation function

$$\rho(\tau) = \frac{\theta^3}{(\theta + |\tau|)^3}$$ (1.12)

These functions all imply that the magnitude of correlation between two points depends on the correlation length, $\theta$, and the absolute distance between any each points in the domain, $|\tau|$. Of all the functions, Markov is most commonly used because of its simplicity and because, in one-dimension, it is a memoryless process. This means for a stochastic process the future of the random process only directly depends on the present and not on the past.

1.4.2 Local Averages

Measured engineering properties are usually some sort of a local average. Properties are rarely measured at a point. They are usually obtained by sampling over some size. For example, with our eyes we don’t see atomic structure, we see averages.

We will define an arithmetic local average as

$$X_\tau(t) = \frac{1}{T} \int_0^T X(\xi)d\xi$$ (1.13)
where \( X_T(t) \) is the local average of \( X(t) \) over a length \( T \). The moments of the local average \( X_T(t) \) are

\[
\begin{align*}
\mathbb{E} [X_T(t)] &= \mu_X \quad (1.14a) \\
\text{Var} [X_T(t)] &= \sigma_X^2 \gamma(T) \quad (1.14b)
\end{align*}
\]

where \( \gamma(T) \) is the variance reduction function, as shown below, which measure the reduction in variance due to local averaging. The variance reduction function, \( \gamma(T) \), lies between zero and one. \( \gamma(T) \) is one (no reduction in variance) when \( T = 0 \) (no local averaging) or when \( \theta = \infty \); conversely as \( T \to \infty \), \( \gamma(T) \to 0 \).

\[
\gamma(T) = \frac{1}{T^2} \int_0^T \int_0^T \rho_X(\tau) d\xi d\eta \quad (1.15)
\]

where \( \tau = \xi - \eta \), is the distance between two points \( X(\xi) \) and \( X(\eta) \), and \( \rho_X(\tau) \) is the correlation function of \( X(t) \). As we can see from Eq. (1.15), \( \gamma(T) \) is actually an average of \( \rho_X(\tau) \) between every pair of points in the field, and \( \gamma(T) \) varies positively with \( \rho_X(\tau) \). If all points in the field are perfectly correlated \( (\rho_X(\tau) = 1) \), \( \gamma(T) \) will be one.

Local averaging is a low pass filter. To illustrate this, consider a ‘boat in the water’ example: the motion of a piece of sawdust on the surface of the ocean will certainly have as much variability as the waves themselves. But an oil tanker no longer moves with the little (high frequency) waves – in fact it only moves very slowly up and down with the very largest of waves (low frequency). So the local averages damp out the high frequency components first and pretty well leave untouched low frequency components (those with the wavelength larger than the correlation length).

One of the main effects of local averaging is to reduce variance. The amount of variance reduction increases with increasing high-frequency content in the random field. In other words, variance reduction increases when the random field consists of more “independence”.

1.4.2.1 Local Averages of White Noise

Assume \( X_T \) is a local average of the 1-D random field, \( X \), assumed to have finite variance, over length \( T \), as shown in Eq. (1.13). When the scale of fluctuation, \( \theta \to 0 \), the field becomes white noise. The white noise is a ‘rough’ process because all the points are independent, which is physically unrealizable. As \( \theta \) increases, the correlation becomes
stronger and the random process becomes smoother, as illustrated in Figures 1.4 and 1.5. The correlation coefficient \( \rho(\tau) \) of white noise is

\[
\rho(\tau) = \begin{cases} 
1 & \tau = 0 \\
0 & \text{otherwise}
\end{cases}
\]

(1.16)

where \( \tau \) is the distance between two points.

Accordingly, the variance of \( X_\tau \), \( \text{Var}[X_\tau] \) can be computed as

\[
\text{Var}[X_\tau] = \mathbb{E} \left[ \frac{1}{T^2} \int_0^T \int_0^T X(\xi)X(\eta)d\xi d\eta \right] \\
= \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} [X(\xi)X(\eta)] d\xi d\eta \\
= \frac{\sigma^2}{T^2} \int_0^T \int_0^T \rho(\xi - \eta)d\xi d\eta \\
= \frac{\sigma^2}{T^2} \int_0^T (1d\xi) d\eta \\
= 0
\]

(1.17)

so long as the variance of \( X(t) \) is finite, where \( \rho(\xi - \eta) \) is the correlation coefficient between points \( X(\xi) \) and \( X(\eta) \), and \( \sigma^2 \) is the variance of \( X \). Note that this is not a true white noise process because the variance, \( \sigma^2 \), has been kept finite. We do this because all soils have finite variance and yet we want to investigate what happens as \( \theta \to 0 \).

The mean of \( X_\tau \) is

\[
\mathbb{E} [X_\tau(t)] = \mu_{X_\tau} = \mathbb{E} \left[ \frac{1}{T} \int_0^T X(\tau)d\tau \right] = \frac{1}{T} \int_0^T \mathbb{E} [X(\tau)] d\tau = \mu_X
\]

(1.18)

Hence, for zero scale of fluctuation, \( X_\tau \) has zero variance and the same mean as \( X \). The above process holds true in two-dimensions as well. We can conclude that any local average of white noise with finite variance is no longer random since its variance is zero. The local average becomes deterministic with value \( \mu_X \).

1.4.2.2 Discussion of Three Local Averages

Three commonly used local averages over some domain \( A \) can be defined as

1) Arithmetic average

\[
X_A = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{A} \int_A X(\xi) d\xi
\]

(1.19)
for the discrete and continuous cases. The arithmetic average weights high and low values equally. The concept is simple and commonly used. More importantly, the arithmetic average tends to a normal distribution by the Central Limit Theorem (CLT). The Central Limit Theorem states that the sum of a sufficient number of independent and identically-distributed random variables will be approximately normally distributed.

2) Geometric Average

\[
X_G = \left( \prod_{i=1}^{n} X_i \right)^{1/n} = \exp \left\{ \frac{1}{n} \sum_{i=1}^{n} \ln X_i \right\} = \exp \left\{ \frac{1}{A} \int_{A} \ln X(\xi)d\xi \right\} 
\]  
(1.20)

The geometric average is dominated by low values of \( X \) and will always be less than the arithmetic average. If the random variable \( X \) is lognormally distributed, the geometric average of \( X \), \( X_G \), also has a lognormal distribution with the median \( e^{\mu \ln X} \) preserved. The CLT states that the geometric average will tend to a lognormal distribution for positively distributed \( X \).

3) Harmonic Average

\[
X_H = \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{X_i} \right]^{-1} = \left[ \frac{1}{A} \int_{A} \frac{1}{X(\xi)} d\xi \right]^{-1} 
\]  
(1.21)

The harmonic average is what is used to predict the settlement of strongly horizontally layered soils. It is more strongly low value dominated and is smaller than either the geometric or arithmetic averages. Unfortunately, it is difficult to use in a probabilistic framework due to its unknown distribution.

1.4.2.3 Geometric Average Mean

As will be shown in Chapter two, the effective cohesion, \( c_{eff} \), as seen by the footing is taken to be a geometric average of the cohesion over some domain, \( D \), under the footing. The objective of this section is to study how the geometric average mean varies with the scale of fluctuation.

The discussion considers a lognormally distributed isotropic random field \( c \) with mean, \( \mu_c \), and variance, \( \sigma_c^2 \). Then the normally distributed \( \ln c \) has mean, \( \mu_{\ln c} \), and variance, \( \sigma_{\ln c}^2 \). The geometric average of \( c \) over a 2-D domain of size \( T_1 \times T_2 \) is

\[
c_G = \exp \left\{ \frac{1}{T_1 T_2} \int_{0}^{T_1} \int_{0}^{T_2} \ln c(\xi)d\xi \right\} 
\]  
(1.22)
$c_o$ is also lognormally distributed with parameters

$$
\mu_{\ln c_o} = \mathbb{E} [\ln c_o] = \mathbb{E} \left[ \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \ln c(x) dx \right]
$$

$$
= \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \mathbb{E} [\ln c(x)] dx
$$

$$
= \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \mu_{\ln c}
$$

$$
= \mu_{\ln c}
$$

(1.23a)

$$
\sigma_{\ln c_o}^2 = \mathbb{E} [(\ln c_o - \mu_{\ln c_o})^2] = \mathbb{E} \left[ \left( \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \ln c(x) dx - \mu_{\ln c_o} \right)^2 \right]
$$

where, since $\mu_{\ln c_o} = \mu_{\ln c}$

$$
\sigma_{\ln c_o}^2 = \mathbb{E} \left[ \left( \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} (\ln c(x) dx - \mu_{\ln c}) dx \right)^2 \right]
$$

$$
= \left( \frac{1}{T_1 T_2} \right)^2 \int_0^{T_1} \int_0^{T_2} \int_0^{T_1} \int_0^{T_2} \mathbb{E} \left[ (\ln c(\xi) - \mu_{\ln c}) (\ln c(\eta) - \mu_{\ln c}) \right] d\xi d\eta
$$

$$
= \frac{\sigma_{\ln c}^2}{(T_1 T_2)^2} \int_{-T_2}^{T_2} \int_{-T_1}^{T_1} (T_1 - |\tau_1|)(T_2 - |\tau_2|) \rho(\tau_1, \tau_2) d\tau_1 d\tau_2
$$

$$
= \sigma_{\ln c}^2 \gamma_c(T_1, T_2)
$$

(1.23b)

where $\rho(\tau_1, \tau_2)$ is the correlation coefficient between points $c(\xi_1, \eta_1)$ and $c(\xi_2, \eta_2)$ with $\tau_1 = \xi_1 - \xi_2$ and $\tau_2 = \eta_1 - \eta_2$, the variance reduction function, $\gamma_c(T_1, T_2)$, is (Vanmarcke, 1984)

$$
\gamma_c(T_1, T_2) = \frac{2^2}{(T_1 T_2)^2} \int_0^{T_1} \int_0^{T_2} (T_1 - \tau_1)(T_2 - \tau_2) \rho(\tau_1, \tau_2) d\tau_1 d\tau_2
$$

(1.24)

The mean of the geometric average is thus

$$
\mu_{c_o} = \exp \left\{ \mu_{\ln c} + \frac{1}{2} \sigma_{\ln c}^2 \gamma_c(T_1, T_2) \right\}
$$

(1.25)

When the correlation length $\theta \to 0$, $\gamma_c \to 0$ for non-zero $T_1$ and $T_2$. In this case, the points in the field are independent – the field is now a white noise. As mentioned earlier, any local average of white noise with finite variance has zero variance, so that

$$
\mu_{c_o} = \exp \{ \mu_{\ln c} \}
$$

(1.26)

which is the median of $c$. 
Alternatively, when the correlation length $\theta \to \infty$, the points in the field are perfectly correlated, the variance reduction function, $\gamma_c \to 1$, so that

$$\mu_{cg} = \exp\{\mu_{ln c} + \frac{1}{2} \sigma_{ln c}^2\} = \mu_c$$ \hspace{1cm} (1.27)

Equations (1.26) and (1.27) imply that, for an isotropic random field $c$ with varying of scale of fluctuation, the geometric average mean will lie between the median and the mean of $c$.

For example, assume $c$ is lognormally distributed with $\mu_c = 105.0$ and $\sigma_c = 52.5$. Then the median of $c$ is 93.9. The change of $\mu_{cg}$ with scale of fluctuation over area $(0.1 \times 0.1)$ is shown in Fig. 1.6. Clearly, the geometric average mean of lognormally distributed $c$ is between the median (93.9) and the mean (105).

**Figure 1.6** Geometric average mean vs. scale of fluctuation for $V_c = 0.5$. 
1.5 Research Objectives

According to Terzaghi's (1943) bearing capacity theory, shallow foundations are defined as those having depth less than the footing width. Usually, shallow foundation design is governed by one of two possible modes of failure; settlement failure, a Serviceability Limit State (SLS), and bearing capacity failure, an Ultimate Limit State (ULS). Bearing capacity is defined as the ability of a soil to support the imposed loads without experiencing a shearing failure. Even though shallow foundation design is often governed by the SLS, the ULS may lead to loss of human life and/or major financial losses. As a result, lower probabilities of failure are expected for ULS.

Furthermore it has been observed that failure in soils carrying a foundation load will be due primarily to shear rather than tension or compression. In this research the definition of bearing capacity failure will be ultimate shear failure in the soil which occurs when the load stress equals the ultimate bearing capacity, $q_u$. The thesis, then, will concentrate on ultimate limit state design of shallow foundations against bearing capacity failure.

Most current geotechnical LRFD implementations are determined by direct calibration to the global factor of safety used in traditional WSD codes or by engineering judgement, in other words, the resistance factors are calibrated to satisfy the safety established by the old code, which presumably was deemed acceptable. Whether this calibration can reflect the true reliability is questionable due to the fact that the factor of safety only reflects variability and uncertainty through the engineer's judgement. As Phoon (2004) mentioned, "there are strong practical reasons to consider geotechnical LRFD as a simplified reliability-based design procedure, rather than an exercise in rearranging the original global factor of safety."

It is important, therefore, to critically review those resistance factors and select appropriate values that reflect the desired reliability. This must be done by considering all sources of uncertainty and variability, including spatial randomness of soil properties.

The development of resistance factors for settlement design of shallow foundation using LRFD (Fenton et al., 2005) makes it necessary and reasonable for this study to continue on into the ultimate limit state design. In this research we consider a strip footing in two dimensions placed on a spatially random soil. It is assumed, therefore, that the correlation length in the out-of-plane (i.e., in the third dimension which is in the direction of the strip) is infinite. This is a reasonable assumption in that averaging of soil properties by the strip footing in the out-of-plane direction largely eliminates the effect of spatial variability in this direction in any case.
This study concentrates on proposing a reliability-based bearing capacity design approach for the practicing engineer with the aim of reducing cost without compromising safety. For shallow foundation design, this will be accomplished by risk assessment of bearing capacity failure. The final goal is to investigate the effect of a soil’s spatial variability and site investigation intensity on the resistance factors via theory and via simulation, the latter using the Random Finite Element Method (RFEM) (Fenton et al., 2005).

1.6 Scope of Work

The initial research effort was directed at reviewing the classic bearing capacity theory background in terms of the fundamental assumptions and limitations. The most popular theory was extended to handle spatial variability in soils, as discussed in Chapter 2. It will be assumed that the strip footing being considered is founded on a $c - \phi$ soil, where the cohesion, $c$, is lognormally distributed and the friction angle, $\phi$, has a symmetrically bounded distribution. Chapter 2 also presents a novel mathematical theory, developed by the author, which allows the probability of bearing capacity failure to be estimated analytically. This theory is later validated by simulation and will be used to deduce the required resistance factors, $\phi_g$.

Chapter 3 describes the Finite Element Model (FEM) which is used to represent the spatially variable soil being loaded by the footing. The model accuracy was verified by comparison to Prandtl’s theory in the deterministic case over a variety of footing widths and number of elements. Chapter 3 also describes the Random Finite Element Method (RFEM) used in this thesis to investigate required resistance factors. Essentially, a random soil field is first simulated having certain statistical characteristics (e.g. mean, variance and correlation structure). The simulated field is then ‘sampled’ at a series of locations to yield an estimate of the soil properties, as would be done in the field. The estimated soil properties, along with a hypothesized resistance factor, are used to design the required footing width. The designed footing is then ‘placed’ on the simulated soil and loaded randomly. The response of the soil-foothing system is evaluated by the FEM and whether bearing capacity failure occurs is determined. Repeating this process 2000 times allows the estimation of the probability of bearing capacity failure at this specific value of the resistance factor. Repeating the entire process for a variety of different resistance factors allows the determination of the relationship between probability of failure and resistance factor for different soil field statistics. The simulation methodology presented in this Chapter by the author is a new way of assessing shallow foundation design reliability.
Chapter 4 presents the results of the simulation and compares these results to the theory developed in Chapter 3. The agreement between simulation and theory was quite good.

Chapter 5 analyzes the results of Chapter 4 and presents recommendations regarding resistance factors to be used for target probabilities of failure. Plots and tables are given which would allow a designer to easily select the resistance factor required for specific situation. Comparisons are made to resistance factors recommended by various other sources (e.g. other codes and the literature). The Chapter concludes with a design example.

Finally, the limitations of the overall results and recommendations are discussed, and conclusions drawn, in Chapter 6. Directions for future research are also suggested.
Chapter 2

Theory

Geotechnical designs are beginning to migrate from Working Stress Design (WSD) towards Reliability-Based Design (RBD). The Load and Resistance Factor Design (LRFD) methodology has been chosen for shallow foundation design against bearing capacity failure in the present study. As we all know, the development of resistance factors for use in geotechnical engineering is much more difficult than for structural engineering due to the inherent variability of soil properties. In this chapter a theoretical approach will be developed to estimate ultimate bearing capacity failure probability in shallow foundations. This theory will be used for the recommendation of resistance factors.

2.1 Classical Theory

2.1.1 General

The ultimate bearing capacity of a shallow foundation, $q_u$, is defined as the average load per unit area required to produce bearing capacity failure in the soil under the foundation. $q_u$ not only depends on the properties of the soil but also on the size of the loading area, loading shape, and loading location and inclination.

In Working Stress Design (WSD), the allowable bearing capacity, $q_a$, is used to ensure the safety of foundations. $q_a$ is defined as the maximum stress that can be permitted on a foundation soil, considering all pertinent factors. It is also the value that allows the footing to maintain adequate safety against rupture of the soil mass or movement of such magnitude that the structure needs to be repaired. In other words, $q_a$ can be obtained by reducing $q_u$ to a certain safe level.

A shallow foundation is normally defined as having depth of foundation embedment less than the foundation width. As mentioned in the Canadian Foundation Engineering Manual (CEFM, 1992), “A shallow foundation generally derives its support from the soil or rock
close to the lowest part of the building that it supports. A deep foundation is a foundation unit that provides support for a structure by toe resistance in a competent soil or rock at some depth below the structure, and/or by shaft resistance in the soil or rock in which it is placed.” Compared to deep foundations, loads or resistances on the vertical sides of the shallow foundations, due to cohesion and friction angle, are normally neglected. The common types of shallow foundations are spread footings, strip footings, mats, and so on. A typical strip footing is considered in the present study.

The design of a footing against bearing capacity failure for ultimate limit state may be based on empirical bearing capacity method, i.e., field tests on the soil, or upon traditional equations using characteristic soil values obtained from laboratory tests. The latter methods will be discussed next.

### 2.1.2 Traditional Equations of Bearing Capacity Analysis

One type of bearing capacity shear failure is general shear failure. It normally occurs in a dense sand or a stiff cohesive soil and is sudden and catastrophic. This form of failure can be characterized by the existence of a well-defined failure surface, i.e., a logarithmic spiral shape. The failure surface in the soil will extend to the ground surface.

A number of bearing capacity equations are reported in the literature which provide explicit solutions for the ultimate bearing capacity. One of the early sets of the ultimate bearing capacity equations for the mode of the bearing capacity failure was proposed by Terzaghi (1943), as shown below. This formula, referred to as bearing capacity formula in this research, is involved in most modern bearing capacity predictions. It considers several bearing capacity factors, such as depth and shape factors, and load inclination and eccentricity factors,

\[
q_u = c N_c + \bar{q} N_q + \frac{1}{2} \gamma B N_\gamma
\]  

(2.1)

where \( q_u \) is the ultimate bearing capacity stress; \( c \) is the cohesion; \( \bar{q} \) is the overburden stress, \( \bar{q} = \gamma D_f \), where \( D_f \) is the depth of the foundation; \( \gamma \) is the unit weight of the soil under the foundation; \( B \) is the foundation width. \( N_\gamma \) is the bearing capacity factor of a cohesionless soil (having internal friction angle, \( \phi \)) which accounts for the influence of the weight of the soil. \( N_q \) is the bearing capacity factor concerned with the embedment depth \( D_f \), and \( N_c \) is the bearing capacity factor concerned with the cohesion, \( c \). All the bearing capacity factors are dimensionless quantities depending only on the friction angle, \( \phi \).
The bearing capacity formula has been validated by many empirical bearing capacity studies, i.e., through the correlation of the site tests results and the ultimate bearing capacity of the soil. The typical site tests include Standard Penetration Test (SPT), Cone Penetration Test (CPT), and Pressuremeter Test.

For more complicated foundations (rectangular footings, eccentric loads, etc), each bearing capacity factor is multiplied by a series of correction factors, i.e., shape correction factors, inclination correction factors, and so on.

Most of the bearing capacity predictions, involving the specification of N factors, are summarized by Bowles (1996), as shown in Table 2.1. The formulas by Hansen (1970) and Vesic (1975) were used in the Canadian Foundation Engineering Manual (CFEM, 1992).

**Table 2.1** Classical expressions of bearing capacity factors.

<table>
<thead>
<tr>
<th>Author</th>
<th>$N_q$</th>
<th>$N_c$</th>
<th>$N_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prandtl (1921)</td>
<td>$\frac{\tan \phi}{2} \left( K_{qf} \frac{\cos^2 \phi}{\cos^2 \phi} - 1 \right)$</td>
<td>$(N_q - 1)/\tan \phi$</td>
<td>$\frac{a^2}{2 \cos^2(\pi/4 + \phi/2)}$</td>
</tr>
<tr>
<td>Meyerhof (1951, 1963)</td>
<td>$(N_q - 1)\tan(1.4\phi)$</td>
<td>$(N_q - 1)/\tan \phi$</td>
<td>$\tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \exp(\pi \tan \phi)$</td>
</tr>
<tr>
<td>Hansen (1970)</td>
<td>$1.5(N_q - 1)\tan(\phi)$</td>
<td>$(N_q - 1)/\tan \phi$</td>
<td>$\tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \exp(\pi \tan \phi)$</td>
</tr>
<tr>
<td>Vesci (1973, 1975)</td>
<td>$2(N_q + 1)\tan(\phi)$</td>
<td>$(N_q - 1)/\tan \phi$</td>
<td>$\tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \exp(\pi \tan \phi)$</td>
</tr>
<tr>
<td>Eurocode 7 (1996)</td>
<td>$(N_q - 1)\tan(\phi)$</td>
<td>$(N_q - 1)\cot \phi$</td>
<td>$\tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \exp(\pi \tan \phi)$</td>
</tr>
</tbody>
</table>

where $K_{qf}$ is given in tables and $a = \exp((3\pi/4 - \phi/2)\tan \phi)$.

The fundamental assumption of bearing capacity formula is that there is a uniform soil underlying the footing, i.e., the soil properties are spatially constant. In reality, however, soil properties are inherently variable from point to point through the field. The range of the variability depends on the heterogeneity of constituent materials forming the soil matrix, the geological, environmental and physical-chemical history of soil formation and modification by nature.

The question now is if this uniform soil base theory can be used to solve a randomly varying non-uniform soil problem. In the following discussion the sources of the spatial variability of the soil will be generalized along with the application of this theory to a spatially variable soil field.
2.2 Spatial Variability of Soil

The sources of uncertainty that cause the discrepancy between the estimated and the actual performance of any geotechnical system are summarized as (Griffiths et al., 2002, Phoon and Kulhawy, 1999, 2003):

1) Inherent variability of the soil conditions at the site;
2) Errors in laboratory testing or site investigation;
3) Errors in evaluation of the engineering properties of the geotechnical materials;
4) Errors in selections of design parameters and assumptions of analytical method;
5) Precision of the analytical model;
6) Variability of loadings (e.g. dead, live, wind) transferred from superstructures.

Of all that stated above, the uncertainties associated with the natural variability of soil profile is considered in this research along with the variability of loadings. This is more realistic than the traditional bearing capacity analysis of shallow foundations as normally the significant source of error – the spatial variability of soil – is ignored in the traditional model.

A series of studies on geotechnical variability (Vanmarcke, 1977, Kulhawy et al., 1992, Phoon et al., 1995, Fenton, 1999, Phoon et al., 1999, Fenton and Griffiths, 2003) have been conducted to establish realistic statistical estimates of the variability of design soil properties. These research efforts provide guidelines for the calibration of geotechnical probabilistic analysis. For example, the spatial variability of soil under a strip footing was modeled using random field theory and elasto-plastic finite element analysis to evaluate the effect of spatial variability on bearing capacity by Fenton and Griffiths (2003). With the achievement of these efforts the spatial variability of soil can be included in the estimation of the soil parameters for design of shallow foundations using Load and Resistance Factor Design (LRFD) method.

The bearing capacity formula for the ultimate bearing capacity of soil was based on the failure surface under the foundation following a logarithmic spiral shape. Accordingly, the bearing capacity factors listed in table 2.1 also assume the same failure surface. However, in a spatially random soil the failure surface will follow the weakest path through the soil (on average the path might follow a logarithmic spiral shape). One possible way to estimate the ultimate bearing capacity on spatial variable soil can be realized by selecting the “effective” soil parameters of the random soil in the bearing capacity formula, as stated below.
It is hypothesized here that spatial variability can be handled by using appropriately selected "effective" soil properties in traditional formula, such as the effective cohesion, $c_{eff}$, and the effective friction angle, $\phi_{eff}$. The "effective" soil properties are defined as those values that have the same bearing capacity stability as the actual random soil. In other words, they are the uniform values which would produce a predicted bearing capacity identical to the actual bearing capacity.

2.3 Development of Theory for Load and Resistance Factor Design

This thesis considers a strip footing, assumed to be of infinite length in the out-of-plane direction, founded on the surface of a two-dimensional spatially random soil mass. To simplify the analysis and concentrate on the stochastic behavior of soil parameters, the soil mass is assumed to be weightless. This means the density of soil, $\gamma$, and the parameter related with $\gamma$, overburden stress, $\bar{q}$, are assumed to be zero in the bearing capacity formula. This is a conservative assumption since including the soil weight leads to an increased bearing capacity. Thus, the predicted failure probabilities will be higher than in reality.

Furthermore, the weightless soil is a $c - \phi$ soil with both cohesion $c$ and friction angle $\phi$ greater zero. Nevertheless, we expect that the qualitative behavior of $c = 0$ or $\phi = 0$ soils to be similar.

The object of this section is to propose a reliability-based design methodology for the ultimate limit state of bearing capacity. The uncertainty in both load and resistance will be considered by applying separate factors to each instead of a single global factor of safety, as in the Working Stress Design (WSD) approach. This is consistent with the desired or specified level of safety and reliability since the variability of load differs from that of resistance in nature. Due to the consequence of failure, Ultimate Limit State (ULS) conditions are usually designed for a low probability of occurrence.

For reliability-based design the goal is to reduce to an acceptable level the probability that the footing will experience a bearing capacity failure

$$ p_f = P \left[ L > q_u A \right] $$

(2.2)

where $p_f$ is the probability of failure of the foundation, $L$ is the true load applied to the footing, $q_u$ is the actual bearing stress capacity of the random soil mass, and $A = B \times C$ is the footing area, of width $B$ and length $C$. 
In reality, load $L$, soil properties, such as $c$ and $\phi$, and actual bearing stress capacity $q_u$ are all random variables. Since the designed footing area, $A$, is based on the soil properties and loads, it is also a random variable. For the two horizontal dimensions $B \times C$ of a strip footing (2-D), only footing width $B$ is random. Length $C$ is infinite, which is handled by considering the load to act per unit length. A mean dead load of $\mu_{L_D} = 600$ kN/m and a mean of live load of $\mu_{L_L} = 200$ kN/m are used in this research.

The bearing capacity is predicted using the bearing capacity formula, in Eq. (2.1). Under the assumption that the soil is weightless the bearing capacity equation simplifies to

$$q_u = c_{\text{eff}} N_{c,\text{eff}}$$

(2.3)

in which $c_{\text{eff}}$ is the “effective” cohesion of a uniform soil having the same bearing capacity as the actual random soil, $N_{c,\text{eff}}$ is the “effective” $N_c$ computed from the “effective” friction angle $\phi_{\text{eff}}$.

As suggested by Fenton and Griffiths (2003) the effective cohesion, $c_{\text{eff}}$, as seen by the footing, is a geometric average of the soil properties over some domain $D$ under the footing. The geometric average is defined as

$$c_{\text{eff}} = \left( \prod_{i=1}^{n} c(x_i) \right)^{1/n} = \exp \left\{ \frac{1}{n} \sum_{i=1}^{n} \ln c(x_i) \right\} = \exp \left\{ \frac{1}{D} \int_D \ln c(x) dx \right\}$$

(2.4)

where the right-hand term is obtained when $c(x)$ varies continuously in space. In $c(x_i)$ is defined to be the geometric average of $\ln c(x)$ over the $i^{th}$ element, for $i = 1, \ldots, n$.

In this study cohesion, $c$, is assumed to be lognormally distributed. One reason to use the lognormal distribution (instead of the normal distribution) is because it avoids negative soil properties. If $c$ is assumed to be lognormally distributed, then $c_{\text{eff}}$ is also lognormally distributed.

The size of the averaging area, $D$, can be estimated based on the mean wedge zone depth $W$,

$$W = \frac{1}{2} B \tan \left( \frac{\pi}{4} + \frac{\mu_{\phi}}{2} \right)$$

(2.5)

The area surrounded by the failure surfaces and the bottom of the footing is called the wedge zone. $W$ is the depth of the wedge zone, which is the vertical distance from the bottom of the footing to the log spiral failure curves. In this research $D$ is taken as $2W \times W$, which is a rough approximation of the failure area.
The friction angle, $\phi$, is assumed to have a distribution which is symmetrically bounded between $\phi_{\text{max}}$ and $\phi_{\text{min}}$. The use of the arithmetic average here is motivated by the fact that for a symmetric distribution the mean is preserved under arithmetic averaging but not under geometric averaging (note the geometric averaging preserves the median of a lognormal distribution). For such a symmetrically bounded distribution, the mean equals the median and the mode. The effective friction angle $\phi_{\text{eff}}$ is assumed to be the arithmetic average of $\phi$ over the domain $D$

$$\phi_{\text{eff}} = \frac{1}{D} \int_D \phi(x) dx = \frac{1}{n} \sum_{i=1}^{n} \phi(x_i)$$ (2.6)

where $\phi(x_i)$ is defined to be the local average of $\phi(x)$ over the $i^{th}$ element, for $i = 1, \ldots, n$.

As recommended by Canadian Foundation Engineering Manual (1992), the equation originally by Prandtl (1921), as shown in Table 2.1, was applied to estimate the effective $N_{c,\text{eff}}$, as follows

$$N_{c,\text{eff}} = \frac{\tan^2\left(\frac{\pi}{2} + \frac{\phi_{\text{eff}}}{2}\right) \exp(\pi \tan \phi_{\text{eff}}) - 1}{\tan \phi_{\text{eff}}}$$ (2.7)

In this study $N_{c,\text{eff}}$ is assumed to be lognormally distributed.

Combining Eq. (2.3) into Eq. (2.2) gives the failure probability

$$p_f = P \left[ L > c_{\text{eff}}N_{c,\text{eff}}B \times 1 \right]$$ (2.8)

In a strip footing design, geotechnical engineers try to find the required footing width $B$, to avoid a bearing capacity failure, based on the soil properties and the applied loads. Ultimate Limit State (ULS) conditions are then checked using separate factors on loads and on characteristic geotechnical resistance. This leads to the Load and Resistance Factor Design (LRFD) approach, which requires that the factored resistance exceed the factored load,

$$\phi_g \hat{q}_u B \geq \alpha L_c$$ (2.9)

where $\phi_g$, the resistance factor, is typically less than one and accounts for variabilities in geotechnical parameters as well as model uncertainty; For a specific site, the resistance factor depends on intensity of the site investigation, the design sophistication, level of construction control, and the failure consequences; $\hat{q}_u$ is the characteristic ultimate bearing
capacity predicted from sampled characteristic soil properties. \( \alpha \), the load factor, is usually greater than one and accounts for uncertainty in loads; and \( L_c \) is the characteristic load, also called the design load, in this case expressed per unit length of the footing. The selection of design load involves engineering judgement and experience.

In this research only dead and live loads will be considered. Hence,

\[
\alpha L_c = \alpha_D L_{DC} + \alpha_L L_{LC}
\]

(2.10)

where the dead load factor \( \alpha_D = 1.25 \) and the live load factor \( \alpha_L = 1.5 \) (NBCC, 2005) are applied in the present study. \( L_{DC}, L_{LC} \) are the non-random characteristic dead and live loads. The characteristic load is defined as the mean load times the bias. In this research the bias were chosen as 1.18 for dead load and 1.43 for live load (Becker, 1996, Allen, 1975).

The characteristic ultimate bearing stress capacity, \( \dot{q}_u \), can be estimated as

\[
\dot{q}_u = \hat{c} \hat{N}_c
\]

(2.11)

in which \( \hat{c} \) is the estimated characteristic cohesion and \( \hat{N}_c \) is the estimated characteristic bearing capacity factor, derived from the characteristic internal friction angle, \( \hat{\phi} \). The two soil properties \( \hat{c} \) and \( \hat{\phi} \) can be estimated directly from sampled data as discussed next. Note that the choice of characteristic ultimate bearing capacity is just one possibility. As can be seen in Eq. (2.9), resistance factor, \( \phi_r \), varies negatively with characteristic ultimate bearing stress capacity, \( \dot{q}_u \). In other words, higher resistance factors, corresponding to larger footing, would be expected by using a lower characteristic resistance.

Assume there are \( m \) soil samples, \( c^o(x_i), i = 1, 2, \ldots, m \), available, where \( c^o(x_i) \) is the cohesion estimated from each observed sample. Since the effective cohesion value, \( c_{eff} \), is assumed to be a geometric average (see in Eq. (2.4)), then if \( \hat{c} \) is a good estimate of \( c_{eff} \), it should also be determined as a geometric average of the observed samples \( c^o(x_i) \).

\[
\hat{c} = \left( \prod_{i=1}^{m} c^o(x_i) \right)^{1/m} = \exp \left\{ \frac{1}{m} \sum_{i=1}^{m} \ln c^o(x_i) \right\}
\]

(2.12)

Note that the geometric average preserves the lognormal distribution, so that \( \hat{c} \) is also lognormal.

In a similar way, \( \hat{\phi} \) can be evaluated using the arithmetic average of the observed samples \( \phi^o(x_i) \)

\[
\hat{\phi} = \frac{1}{m} \sum_{i=1}^{m} \phi^o(x_i)
\]

(2.13)
The characteristic value \( \hat{N}_c \) is then
\[
\hat{N}_c = \frac{\tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \exp(\pi \tan \hat{\phi}) - 1}{\tan \hat{\phi}}
\]  
(2.14)

where \( \hat{N}_c \) is also assumed to have a lognormal distribution.

Substituting Eq. (2.11) into Eq. (2.9) gives the design requirement
\[
\phi_g \hat{c} \hat{N}_c B \geq \alpha L_c
\]  
(2.15)

To avoid a bearing capacity failure, the footing width required by the Load Resistance Factor Design (LRFD) is
\[
B = \frac{\alpha L_c}{\phi_g \hat{c} \hat{N}_c}
\]  
(2.16)

Combining Eq. (2.16) into Eq. (2.8), the failure probability becomes
\[
p_f = p_{max} = P \left[ \frac{L}{c_{eff} N_{c,eff}} > \frac{\alpha L_c}{\phi_g} \right]
\]  
(2.17)

where \( p_{max} \) is the maximum acceptable risk of design failure. The objective now is to evaluate the resistance factor, \( \phi_g \), for bearing capacity failure of shallow foundations given the acceptable probability level \( p_{max} \).

Defining
\[
Y = L \frac{\hat{c} \hat{N}_c}{c_{eff} N_{c,eff}}
\]  
(2.18)

it can be seen that, since \( c_{eff}, N_{c,eff}, \hat{c}, \hat{N}_c \) and \( L \) are all lognormally distributed, \( Y \) is also lognormally distributed. In other words
\[
\ln Y = \ln L + \ln \hat{c} + \ln \hat{N}_c - \ln c_{eff} - \ln N_{c,eff}
\]  
(2.19)

is normally distributed. In this research \( c \) and \( \phi \) are assumed to be independent, so that, \( c \) and \( N_c \) are independent. The mean and variance of \( \ln Y \) are thus
\[
\mu_{\ln Y} = \mu_{\ln L} + \mu_{\ln \hat{c}} + \mu_{\ln \hat{N}_c} - \mu_{\ln c_{eff}} - \mu_{\ln N_{c,eff}}
\]  
(2.20a)

\[
\sigma^2_{\ln Y} = \sigma^2_{\ln L} + \sigma^2_{\ln \hat{c}} + \sigma^2_{\ln \hat{N}_c} + \sigma^2_{\ln c_{eff}} + \sigma^2_{\ln N_{c,eff}} - 2 \text{Cov} \left[ \ln c_{eff}, \ln \hat{c} \right] - 2 \text{Cov} \left[ \ln N_{c,eff}, \ln \hat{N}_c \right]
\]  
(2.20b)

The load \( L \) is defined as
\[
L = L_d + L_L
\]  
(2.21)
where the dead load, $L_D$, and live load, $L_L$, are the actual (random) loads. We first estimate the nature of resistance terms in Eq. (2.19). Our first effort is to determine the resistance means in Eq. (2.20a).

From Eq. (2.4) and Eq. (2.12) the means, $\mu_{\ln c_{\text{eff}}}$ and $\mu_{\ln \hat{c}}$, are

$$
\mu_{\ln c_{\text{eff}}} = \mathbb{E} \left[ \ln c_{\text{eff}} \right] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \ln(c(x_i)) \right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \ln c(x_i) \right] = \frac{1}{n} \sum_{i=1}^{n} \mu_{\ln c} = \mu_{\ln c}
$$

(2.22)

$$
\mu_{\ln \hat{c}} = \mathbb{E} \left[ \ln \hat{c} \right] = \mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^{m} \ln c^0(x_i) \right] = \frac{1}{m} \sum_{i=1}^{m} \mathbb{E} \left[ \ln c^0(x_i) \right] = \frac{1}{m} \sum_{i=1}^{m} \mu_{\ln c} = \mu_{\ln c}
$$

(2.23)

From Eq. (2.6) and Eq. (2.13) we similarly conclude that

$$
\mu_{\hat{\phi}} = \mu_{\phi_{\text{eff}}} = \mu_{\phi}
$$

(2.24)

$N_c$ is a function of the random variable $\phi$. According to Fenton and Griffiths (2003) a first order approximation to $\mu_{\ln N_c}$ is reasonably accurate. In the present study the first order approximation to the mean using the Taylor’s series expansion is used, such that

$$
\mu_{\ln N_c} \simeq \ln \hat{N}_c(\mu_{\hat{\phi}}) = \ln \frac{e^{\pi \tan \mu_{\hat{\phi}} \tan^2(\pi / 4 + \mu_{\hat{\phi}}/2) - 1}}{\tan \mu_{\hat{\phi}}}
$$

(2.25)

Similarly

$$
\mu_{\ln N_{c,\text{eff}}} \simeq \ln N_{c,\text{eff}}(\mu_{\phi_{\text{eff}}}) = \ln \frac{e^{\pi \tan \mu_{\phi_{\text{eff}}} \tan^2(\pi / 4 + \mu_{\phi_{\text{eff}}}/2) - 1}}{\tan \mu_{\phi_{\text{eff}}}}
$$

(2.26)

According to Eq. (2.24)

$$
\mu_{\ln N_c} = \mu_{\ln N_{c,\text{eff}}} \simeq \ln \frac{e^{\pi \tan \mu_{\phi} \tan^2(\pi / 4 + \mu_{\phi}/2) - 1}}{\tan \mu_{\phi}}
$$

(2.27)

With these results the mean of $\ln Y$ simplifies to

$$
\mu_{\ln Y} = \mu_{\ln L} + \mu_{\ln N_c} - \mu_{\ln N_{c,\text{eff}}} = \mu_{\ln L}
$$

(2.28)

We now turn our attention to estimating the variance of $\ln Y$. The variance is effected by local averaging. The main effect of local averaging is to reduce the variance.

The variances of $\ln \hat{c}$ and $\hat{\phi}$ can be expressed individually in terms of variances of $\ln c$ and $\phi$, using the variance reduction function $\gamma$

$$
\sigma_{\ln \hat{c}}^2 = \gamma_c(m)\sigma_{\ln c}^2
$$

(2.29a)
\[ \sigma^2_{\phi} = \gamma_{\phi}(m)\sigma^2_{\phi} \]  

(2.29b)

where \( \sigma_{\phi} \) is defined in Table 3.5, \( \sigma_{in\,c} \) is determined from Eq. (3.1) and

\[ \gamma_{c}(m) = \frac{1}{Q^2} \int_Q \int_Q \rho^{\phi,\xi}(\xi - \eta)d\xi\,d\eta \approx \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \rho^{\phi,\xi}_{ij} \]  

(2.30a)

\[ \gamma_{\phi}(m) = \frac{1}{Q^2} \int_Q \int_Q \rho^{\phi,\phi}(\xi - \eta)d\xi\,d\eta \approx \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \rho^{\phi,\phi}_{ij} \]  

(2.30b)

In the above, \( \gamma_{c}(m) \) and \( \gamma_{\phi}(m) \) are the variance reduction functions that measure the reduction in variance due to local averaging over the sampling domain \( Q \). \( m \) is the number of sampling elements in \( Q \). \( \rho^{\phi,\xi}_{ij} \) is the correlation coefficient between \( \ln c^{\phi}(x_i) \) and \( \ln c^{\phi}(x_j) \), \( \rho^{\phi,\phi}_{ij} \) is the correlation coefficient between \( \phi^{\phi}(x_i) \) and \( \phi^{\phi}(x_j) \).

The variances of \( \ln c_{eff} \) and \( \phi_{eff} \) can be estimated over the domain \( D = 2W \times W \) in a similar way,

\[ \sigma^2_{ln\,c_{eff}} = \gamma_{c}(n)\sigma^2_{ln\,c} \]  

(2.31a)

\[ \sigma^2_{\phi_{eff}} = \gamma_{\phi}(n)\sigma^2_{\phi} \]  

(2.31b)

where \( \sigma_{\phi} \) is defined in Table 3.5, \( \sigma_{in\,c} \) is determined from Eq. (3.1), and

\[ \gamma_{c}(n) = \frac{1}{D^2} \int_D \int_D \rho^{\phi}(\xi - \eta)d\xi\,d\eta \approx \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho^{\phi}_{ij} \]  

(2.32a)

\[ \gamma_{\phi}(n) = \frac{1}{D^2} \int_D \int_D \rho^{\phi}(\xi - \eta)d\xi\,d\eta \approx \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho^{\phi}_{ij} \]  

(2.32b)

and where \( \rho^{\phi}_{ij} \) is the correlation coefficient between \( \ln c(x_i) \) and \( \ln c(x_j) \); \( \rho^{\phi}_{ij} \) is the correlation coefficient between \( \phi(x_i) \) and \( \phi(x_j) \), as illustrated in Fig. 2.1.
As mentioned earlier the effective parameters as seen by the footings can be estimated as averages over domain $D = 2W \times W$. So the 2-D variance reduction functions, $\gamma_c(n), \gamma_s(n)$, are computed over $D$. As shown in Eq. (2.5), domain $D$ depends on the footing width $B$. But $B$ is a random variable and having a random variance function is inconvenient. One possible solution is to assume that the random deviations of $B$ from its mean do not have a great influence on the variance reduction, and take $B = \bar{B}$, where $\bar{B}$ is the footing width obtained from the mean soil parameters.

The soil field is meshed into $n_x$ elements in the $x$ direction and $n_y$ elements in the $y$ direction. The correlation coefficient between each pair of elements is approximated by the correlation coefficient, $\rho(\tau)$, where $\tau$ is the distance between their centers. This is shown in Fig. 2.1, where elements $x_i$ and $x_j$ have the same size $(dx \times dy)$. If $\tau$ is larger than $\theta$, then the elements are effectively independent. Conversely, two elements separated by a distance $\tau$ less than $\theta$ will be significantly correlated.
First order estimates of $\sigma_{\ln N_c}$, $\sigma_{\ln N_{c,eff}}$, and $\sigma_{\ln N_c}^2$ using Taylor’s series, are

$$\sigma_{\ln N_c}^2 \simeq \sigma_\phi^2 \left( \frac{d \ln \hat{N}_c}{d \phi} \bigg|_{\mu_\phi} \right)^2$$  (2.33a)

$$\sigma_{\ln N_{c,eff}}^2 \simeq \sigma_{\phi_{eff}}^2 \left( \frac{d \ln N_{c,eff}}{d \phi_{eff}} \bigg|_{\mu_{\phi_{eff}}} \right)^2$$  (2.33b)

$$\sigma_{\ln N_c}^2 \simeq \sigma_\phi^2 \left( \frac{d \ln N_c}{d \phi} \bigg|_{\mu_\phi} \right)^2$$  (2.33c)

where $\sigma_\phi$ is defined in Table 3.5.

From Eq. (2.7) the derivative in Eq. (2.33), which will be denoted as $\beta(\phi)$, is

$$\beta(\phi) = \frac{d \ln N_c}{d \phi} = \frac{bd}{bd^2 - 1} \left[ \pi (1 + a^2) d + 1 + a^2 \right] - \frac{1 + a^2}{a}$$  (2.34)

where $a = \tan(\phi)$, $b = e^{\pi \alpha}$, $d = \tan(\pi/4 + \phi/2)$.

The covariance terms in Eq. (2.20b) are computed from

$$\text{Cov}[\ln \hat{N}_c, \ln \epsilon_{eff}] \simeq \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \text{Cov} [\ln \epsilon^o(x_i), \ln c(x_j)] = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_{\ln \epsilon \rho c,ij}$$  (2.35a)

$$\text{Cov}[\ln \hat{N}_c, \ln N_{c,eff}] \simeq \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \text{Cov} [\ln N_c^o(x_i), \ln N_c(x_j)] = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_{\ln N_c,\rho n,ij}$$  (2.35b)

where $\rho_{c,ij}$ is the correlation coefficient between $\ln \epsilon^o(x_i)$ and $\ln c(x_j)$; $\rho_{n,ij}$ is the correlation coefficient between $\ln N_c^o(x_i)$ and $\ln N_c(x_j)$. Where $N^o_c(x_i)$ and $N_c(x_j)$ are individually defined as the local average of $\ln N_c(x)$ over the element $x_i$ and $x_j$. In this research $\ln N_c$ is assumed to have the same scale of fluctuation as $\phi$ and $\ln c$.

Using the above definitions and equations the estimated variance of $\ln Y$ is

$$\sigma_{\ln Y}^2 = \sigma_{\ln L}^2 + \sigma_{\ln \epsilon}^2 + \sigma_{\ln c_{eff}}^2 + \sigma_{\ln N_{c,eff}}^2 + \sigma_{\ln \hat{N}_c}^2 - \frac{2}{mn} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_{\ln \epsilon \rho c,ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_{\ln N_c,\rho n,ij} \right)$$  (2.36)

Now we turn our attention to estimate the load terms in Eq. (2.19). By assuming dead load and live are independent of each other, the mean and variance of $L$ becomes

$$\mu_L = \mu_{L_D} + \mu_{L_L}$$  (2.37a)
\[
\sigma_L^2 = \sigma_{LD}^2 + \sigma_{LL}^2
\]  
(2.37b)

We will assume that \( L \) is lognormally distributed (see section 2.4 for verification of this assumption) with parameters,

\[
\sigma_{\ln L}^2 = \ln (1 + V_L^2)
\]  
(2.38a)

\[
\mu_{\ln L} = \ln(\mu_L) - \frac{1}{2} \sigma_{\ln L}^2
\]  
(2.38b)

where \( V_L = \sigma_L / \mu_L \) is the coefficient of variation of load \( L \). The non-negative contribution of lognormal distribution for load avoids footings have chances to support tension instead of compression, which is rarely the case.

With all these results the design probability becomes

\[
P \left[ \frac{Y}{\phi_g} > \frac{\alpha L_c}{\phi_g} \right] = P \left[ \ln Y > \ln(\alpha L_c) - \ln \phi_g \right] = p_{max}
\]  
(2.39)

So

\[
p_f = p_{max} = 1 - \Phi \left( \frac{\ln(\alpha L_c) - \ln \phi_g - \mu_{\ln Y}}{\sigma_{\ln Y}} \right)
\]  
(2.40)

From normal distribution tables we can get

\[
z_{1-p_{max}} = \frac{\ln(\alpha L_c) - \ln \phi_g - \mu_{\ln Y}}{\sigma_{\ln Y}}
\]  
(2.41)

Finally the required resistance factor, \( \phi_g \), can be found as

\[
\phi_g = \exp \{ \ln(\alpha L_c) - \mu_{\ln Y} - z_{1-p_{max}} \sigma_{\ln Y} \}
\]  
(2.42)

It has been noticed that in Eq. (2.42), the load and resistance factors are related to each other. This is as expected (see, for example, Becker, 1996b) – as the load factor(s) increase, the resistance factor increases towards one.

### 2.4 Load Distribution

As will be shown in Chapter 3, the lognormally distributed dead load, \( L_D \), and live load, \( L_L \), are modeled separately in the simulation. However, the total load, \( L = L_D + L_L \), is considered in the theory. The objective in this section is to establish a reasonable model (distribution) for the variability of \( L \) based on lognormally distributed \( L_D \) and \( L_L \).

Note that a load ratio, \( R_{DL} = L_D / L_L \), is applied in the present study to include the long developed database of load ratios in Working Stress Design (WSD). As will be discussed in Chapter 3, dead load and live load are simulated given the mean and the standard deviation in this research, so \( R_{DL} \) is assumed to be \( \mu_{L_D} / \mu_{L_L} \).
2.4.1 Methodology

To find the distribution of the total load, \( L = L_D + L_L \), where dead load \( L_D \) and live load \( L_L \) are lognormally distributed, a simulation was performed, as follows;

1) estimate the parameters of the lognormal distributions of \( L_D \) and \( L_L \), \( \mu_{lnL_D} \), \( \sigma_{lnL_D} \), \( \mu_{lnL_L} \) and \( \sigma_{lnL_L} \), using the transformations in Eq. (2.38a) and Eq. (2.38b);

2) simulate realizations of \( L_D \) and \( L_L \) from their distributions;

3) form a realization of the total load, \( L = L_D + L_L \);

4) repeat from step (2) one thousand times;

5) plot a histogram and superimpose the fitted distribution.

In this research a lognormal distribution is selected to fit to the total load. This is physically appropriate since the lognormal distribution is non-negative as are the true loads (assumed).

Once a distribution has been selected, the fit must be assessed. Note that no fitted distributions will be an exact fit, but what we ask here is if the fitted distribution is reasonable enough for its intended purpose? Here we consider two approaches to answer this question, heuristic procedures and goodness-of-fit tests. In heuristic procedures there are frequency comparisons and probability plots.

Frequency comparisons are made by plotting the observed frequency of occurrence against that predicted by the fitted theoretical distribution. When the data come from a continuous distribution, the histogram can be obtained by breaking up the range of values covered by the data into \( k \) intervals \([a_0, a_1), \ldots, [a_{k-1}, a_k)\) and counting the frequency of occurrence of the data within each interval. As suggested by Fenton (2005) a 'good' number of intervals, \( k \), commonly is taken to be about 5\% to 20\% of the number of observations, \( n \), and at least 5 observations occurring in each interval. The histogram is normalized by dividing each frequency value by \( n\Delta x \), where \( \Delta x \) is the interval width associated with each frequency, in order to produce a plot having area 1.0.

There are a number of quantitative goodness-of-fit tests available and one of the easiest of these to implement is the Chi-Square test, as applied in this study. The Chi-square test is defined for the hypotheses,

\[ H_0: \text{The data follow a specified distribution.} \]

\[ H_a: \text{The data do not follow the specified distribution.} \]
For the Chi-Square goodness-of-fit computation, the data are divided into \( k \) buckets and the test statistic is defined as

\[
\chi^2 = \sum_{i=1}^{k} \frac{(V_i - E_i)^2}{E_i}
\]  
(2.43)

where \( V_i \) is the observed frequency for bucket \( i \) and \( E_i \) is the expected frequency for bucket \( i \). The expected frequency is calculated by

\[
E_i = n[F(a_i) - F(a_{i-1})]
\]  
(2.44)

where \( F \) is the cumulative distribution function for the distribution being fit to the data, and \( n \) is the sample size.

It should be pointed out that since the distribution of total load, \( L \), cannot be derived analytically, we must check the form of the distribution by simulation.

### 2.4.2 Examples

There are two examples used to illustrate the distribution of total load, \( L \). The first example takes the dead to live load ratio to be 3:1, as also assumed in the theory and simulation. This choice follows Allen (2005)’s suggestion that a typical ratio of dead load to live load is 3.0. This value was also used by Barker et al. (1991). Note that neither Allen (2005) nor Barker et al. (1991) advocated a resistance factor that varies with the load ratio when design for the same limit state using reliability theory. The possible reason might because the resistance factor is not all that sensitive to the load ratio.

In particular, the lognormally distributed dead load \( L_D \) is taken to have mean \( \mu_{L_D} = 900 \) kN/m and coefficient of variation \( V_{L_D} = 0.15 \) (Becker, 1996), while the lognormally distributed live load \( L_L \) has mean \( \mu_{L_L} = 300 \) kN/m and coefficient of variation \( V_{L_L} = 0.3 \) (Allen, 1975).

The distribution of \( L = L_D + L_L \) is estimated via simulation. The results are shown in Fig. 2.2. Ten buckets are used for the histograms. A lognormal distribution is fit to the histogram, and the p-value of the Chi-Square goodness-of-fit is reported. The p-value is the probability of obtaining a result at least as extreme as that observed. It is the smallest level of significance at which null hypothesis \( H_0 \) would be rejected. Large p-values support \( H_0 \).

In general, one rejects \( H_0 \) if the p-value is smaller than or equal to the significance level.
As illustrated in Fig. 2.2, the p-value (0.27) indicates good agreement between the hypothesized distribution (lognormal) and the simulated data. Note that in Fig. 2.2 the frequency intervals are calculated in log-space and the mean, $\mu_{\ln L} = 7.077$, and the standard deviation, $\sigma_{\ln L} = 0.117$, are estimated from the log-data.

A second example was considered where the dead load to live load ratio was selected to be 1:1. In this case, both dead load and live load are lognormally distributed with the means 600 kN/m. The coefficient of variation is 0.15 (Becker, 1996) for dead load and 0.3 (Allen, 1975) for live load. The simulation results are shown in Fig. 2.3.

The Chi-Square test is encouraging since the p-value (0.17) indicates that the lognormal distribution is quite reasonable.

It can be seen that in both cases, where the load ratios are quite different, the total load is well approximated by a lognormal distribution.
2.5 Probability of Failure for $\theta \to 0$ and $\theta \to \infty$

It is often instructive to consider the analysis over a wide range of correlation lengths. In this section we analyze the accuracy of the theory developed previously by estimation of the probability of bearing capacity failure of foundations for zero and infinity scale of fluctuations.

When the scale of fluctuation $\theta \to 0$, the variance of any local average goes to zero as shown in Chapter 1. The covariance terms also become zero (since the covariance is always less than the multiplication of variances, i.e., $\text{Cov}[X, Y] \leq \sigma_X \sigma_Y$, where $X$ and $Y$ are random fields or variables). In other words, as $\theta \to 0$,

$$\sigma_{\ln \bar{d}} = \sigma_{\ln c_{eff}} = 0$$

$$\sigma_\phi = \sigma_{\phi_{eff}} = 0$$

$$\sigma_{\ln N_c} = \sigma_{\ln N_{c,eff}} = 0$$

and
\[ \text{Cov} \left[ \ln \hat{\epsilon}, \ln c_{eff} \right] = 0 \]
\[ \text{Cov} \left[ \ln \hat{N}_e, \ln N_{e,eff} \right] = 0 \]

Eq. (2.20a) and Eq. (2.20b) now turn out to be

\[ \mu_{\ln Y} = \mu_{\ln L} \quad (2.45a) \]
\[ \sigma_{\ln Y} = \sigma_{\ln L} \quad (2.45b) \]

and the probability of failure of Eq. (2.40) for zero correlation length becomes

\[ p_f = 1 - \Phi \left( \frac{\ln(\alpha L_c) - \ln \phi_g - \mu_{\ln L}}{\sigma_{\ln L}} \right) \quad (2.46) \]

This equation implies that for zero correlation length the probability of bearing capacity failure of foundations only depends on the load and resistance factors and the load distribution. In other words, for a finite variance white noise, the failure probability \( p_f \) becomes

\[ p_f = P \left[ L > \frac{\alpha L_c}{\phi_g} \right] \quad (2.47) \]

This is obviously true because of the fact that if there is no effective variability in the averaged soil properties, the failure of the footing involves only load variability.

When the scale of fluctuation \( \theta \to \infty \), all soil points in the field become perfectly correlated. The field becomes a uniform field. The analysis is expected to approach a single random field method. This means all correlation coefficient terms and variance reduction functions are one, i.e., there is no variance reduction, and since \( \hat{\epsilon} = c_{eff} = c \), and \( \phi = \phi_{eff} = \phi \),

\[ \sigma_{\ln \hat{\epsilon}} = \sigma_{\ln c_{eff}} = \sigma_{\ln c} \]
\[ \sigma_{\phi} = \sigma_{\phi_{eff}} = \sigma_{\phi} \]
\[ \sigma_{\ln N_e} = \sigma_{\ln N_{e,eff}} = \sigma_{\ln N_e} \]

and the covariance terms in Eq. (2.35a) and Eq. (2.35b) become

\[ \text{Cov} \left[ \ln \hat{\epsilon}, \ln c_{eff} \right] = \sigma_{\ln c}^2 \]
\[ \text{Cov} \left[ \ln \hat{N}_e, \ln N_{e,eff} \right] = \sigma_{\ln N_e}^2 \]
In this case, the parameters of the distribution $Y$ become the same as Eq. (2.45a) and (2.45b),

$$
\mu_{\ln Y} = \mu_{\ln L}
$$

$$
\sigma_{\ln Y} = \sigma_{\ln L}
$$

and $p_f$ can also be estimated according to Eq. (2.47). This means the same probability of failure can be estimated for zero and infinity correlation length, involving only load and resistance factors and randomness of loading.
Chapter 3

Simulation

3.1 General

As mentioned by Fenton (2006) "simulation is the process of producing reasonable replications of the real world in order to study the probabilistic nature of the response to the real world."

More recently an advanced method, Monte Carlo simulation, as a powerful means of obtaining probability distribution estimates for very complicated problems, has been used for performing the reliability analysis for bearing capacity of foundation. It randomly generates numbers for uncertain variables and fields repeatedly for the investigation of stochastic problems. The procedure of Monte Carlo simulation involves the following steps

1) generate a uniformly distributed random number using a random number generator.
   The generator used in this study is Numerical Recipes' (Press et al., 1992) RAN2 generator which is based on linear congruential generators (LCGs) first introduced by Lehmer (1951).

2) the realizations of a random variable or field with defined distribution can be obtained using an appropriate transformation method.

In practice, the accuracy of the Monte Carlo method depends on how well the probability distribution fits the real stochastic process. If the fit is reasonable, the accuracy increases with the number of simulation runs, i.e., better results will be obtained as the number of simulation realizations increases. The new generation of computers allows for tens of thousands to millions of realizations in seconds.

In this chapter probabilistic analysis of bearing capacity using Monte Carlo simulation was conducted by the writer. The objective is to investigate the failure probability of a strip footing on a weightless soil with spatially varying random fields, c and φ, via simulation in order to validate the theory developed in the previous chapter.
3.2 Design Approach

The Random Finite Element Method (RFEM), combining random field theory with finite element analysis together, will be used to simulate the soil fields and analyze their behavior under a strip footing. The details of RFEM will be discussed in Section 3.3. RFEM was developed by Fenton and Griffiths and has been applied to a number of geotechnical problems (Griffiths and Fenton, 2001, Griffiths et al., 2002, Fenton and Griffiths, 2003, Fenton et al., 2005). It is modified in this study to estimate the probability of failure of foundations founded on spatially random soil against bearing capacity failure.

The approach applied in the simulation is described as follows:

1) compute the factored design load, \( \alpha L_c = \alpha_D L_{DC} + \alpha_L L_{LC} \), using the mean loads, \( \mu_{L_D} \) and \( \mu_{L_L} \), the load factors, \( \alpha_D \) and \( \alpha_L \), and the bias factors, \( k_D \) and \( k_L \);

2) simulate a random soil mass with a lognormally distributed cohesion field, \( c(x) \), and a bounded distributed friction angle field, \( \phi(x) \), by using the specified mean of cohesion, \( \mu_c \), the coefficient of variation, \( \nu_c \), bounds \( \phi_{max} \), \( \phi_{min} \), the factor that controls the distribution of friction angle, \( s_\phi \), and the correlation length, \( \theta \). In particular, the 2-D Local Average Subdivision (LAS) method (Fenton and Vanmarcke, 1990) is applied to simulate both fields. The LAS cells are mapped to the finite elements. Cross-correlation between \( c \) and \( \phi \) is ignored due to its slight influence on bearing capacity (Fenton, 2003);

3) sample the soil in various locations to estimate the design soil parameters, \( \hat{c} \) and \( \hat{\phi} \), to determine the ultimate bearing capacity \( q_u \);

4) design a strip footing, given the estimated cohesion, \( \hat{c} \), the estimated friction angle, \( \hat{\phi} \), the design load, \( L_c \), and the resistance factor, \( \phi_g \). The footing width \( B \) is calculated according to Eq. (2.16);

5) place the designed footing on the same random mass from step 2) and compute the ‘actual’ bearing capacity, \( q_u \), using the finite element analysis;

6) simulate lognormally distributed dead load, \( L_D \), and live load, \( L_L \), having means, \( \mu_{L_D} \), \( \mu_{L_L} \), and standard deviations, \( \sigma_{L_D}, \sigma_{L_L} \). The random total load \( L \) is obtained by adding \( L_D \) and \( L_L \) together;

7) compare \( L \) with \( q_u \times B \). If \( L \geq q_u \times B \), the footing design is assumed to have failed;

8) repeat from step 2) a large number of times (\( n = 2000 \) in the present study), counting the number of footing design failure, \( n_f \). The probability of failure is estimated to be \( p_f = n_f / n \).
9) repeat the entire process, varying the random field parameters (coefficient of variation and the correlation length), and the resistance factor;

10) extract the estimated probabilities of failure obtained from simulation for each correlation length, $\theta$, coefficient of variation of cohesion, $V_c$ (and $s_\theta$), and resistance factor, $\phi_y$;

11) compare the probability of failure obtained from simulation with those from theory;

12) decide on the maximum acceptable risks of design failure, $P_{max}$;

13) recommend the 'optimal' resistance factors for bearing capacity design of shallow foundations.

The Monte Carlo simulations along with the finite element analysis were realized by a computer program originally written by Dr. G.A. Fenton and Dr. D.V. Griffiths but modified by the author to design the footings, simulate the loads, and estimate the failure probability of footings. The accuracy of these computations depends on the accuracy of the random finite element model, the accuracy of the assumed limit state design methodology, and the number of simulations (computer runs). The effects of these factors will be discussed in section 3.4.

### 3.3 Random Finite Element Method

Due to the inherent randomness of soil properties, the uncertainty of predictions is quantified using random field theory (Vanmarcke, 1984, Fenton and Vanmarcke, 1990). This is combined with elasto-plastic finite element analysis (Smith and Griffiths, 2004) in the analysis.

Random field theory is more realistic for the representation of the soil properties since it can better capture the variability of soil. It is an efficient and scientific way to include the effects of soil randomness in a probabilistic study. It does so by expressing the soil properties in the form of a probability density function.

For example, Fig. 3.1 illustrates a bearing capacity failure in a realistic soil with spatially varying properties. In this plot darker regions represent stronger soil and lighter regions indicate weaker soil. The failure surface no longer is a nice smooth symmetric log-spiral, but tends to follow the weakest path through the soil.
Figure 3.1  Typical bearing capacity failure of spatially random soil.

The soil properties that relate to bearing capacity failure of a shallow foundation include:

1) Cohesion $c$;
2) Friction angle $\phi$;
3) Elastic modulus $E$;
4) Dilation angle $\psi$;
5) Poisson’s ratio $\nu$.

Of these, the parameters of primary interest to bearing capacity are cohesion, $c$, and the friction angle, $\phi$. They are modeled as random fields, while all the other parameters are assumed to be deterministic.

To characterize the variability of cohesion, there are three parameters to be considered, the mean, $\mu_c$, the standard deviation, $\sigma_c$, and the spatial correlation length, $\theta_{lnc}$.

If the cohesion $c$ is assumed to be lognormally distributed, then $\ln c$ is normally distributed. The parameters of the normal distribution, $\sigma_{lnc}^2$ and $\mu_{lnc}$, can be obtained from the parameters of the lognormal distribution, $\sigma_c^2$ and $\mu_c$, using following transformations:

$$\sigma_{lnc}^2 = \ln \left(1 + V_c^2\right)$$  
$$\mu_{lnc} = \ln \mu_c - \frac{1}{2} \sigma_{lnc}^2$$

where $V_c = \sigma_c / \mu_c$ is the coefficient of variation of cohesion $c$. We can also invert these to obtain

$$\mu_c = \exp(\mu_{lnc} + \frac{1}{2} \sigma_{lnc}^2)$$
$$\sigma_c^2 = \mu_c^2(\exp^2 \sigma_{lnc}^2 - 1)$$
Aside from the mean and the variance, the scale of fluctuation of cohesion also needs to be established. To date the correlation structure of soil properties remains largely unknown. Establishing the correlation structure for an individual site would involve extensive investigation which would be very expensive. In this study we will look for a worst case scale of fluctuation and use that in our design recommendations.

According to Fenton (1999) the correlation structure is largely insensitive to the lognormal transformation. For simplicity, an exponentially decaying Markovian spatial correlation function (also called exponential correlation function) is selected in the present study, as shown below.

$$\rho(\tau) = \exp \left\{ -\frac{2|\tau|}{\theta} \right\}$$

(3.3)

where $\rho$ is the correlation coefficient between log cohesion values at any two spatial points $x_i$ and $x_j$ in the soil field with a distance $\tau = x_i - x_j$, and $\theta$ is $\theta_{\ln c}$ for cohesion. When the absolute length $|\tau|$ is greater than the correlation length $\theta$, the points are largely uncorrelated.

The lognormally distributed cohesion field $c(x_i)$ can be obtained from a normally distributed random field $G_{\ln c}(x)$, whose point distribution has zero mean, unit variance, and spatial correlation length $\theta_{\ln c}$, using 2-D Local Average Subdivision (LAS) algorithm (Fenton and Vanmarcke, 1990)

$$c(x_i) = \exp \{ \mu_{\ln c} + \sigma_{\ln c} G_{\ln c}(x_i) \}$$

(3.4)

where $x_i$ is the vector containing the coordinates of the center of the $i^{th}$ finite element. Note that if $G_{\ln c}(x_i)$ is produced using LAS, then the cohesion $c(x_i)$ is the geometric average of $c(x)$ over the finite element because in LAS $G_{\ln c}(x)$ is obtained by local averaging $\ln c(x)$ over the finite element.

Local Average Subdivision (LAS) is a method of producing realizations of a discrete “local average” random field. One-dimensional LAS involves a top-down recursion (Fenton, 1990, 1994), as illustrated in Fig. 3.2. The parent domain is subdivided into two child regions whose “local” averages must in turn average to the parent value. The global average remains constant through the subdivision by keeping the average of child regions equal to the parent value.

In two dimensions, a rectangular domain is defined. LAS then involves a subdivision process in which a ‘parent’ cell is divided into 4 equal sized cells ($2 \times 2$), in a series of stages. For example, in Fig. 3.3, the parent cell $P_3$ is divided to child cells $Q_j, j = 1, 2, 3, 4$. The similar subdivision can be obtained for other parent cells $P_i, i = 1, 2, ..$. In each subdivision stage new soil property values are generated, preserving upwards averaging,
while keeping variance and covariance relationships between each cell approximate correct. The preservation of upwards averaging means that the average of the four new values is the same as the parent cell value, i.e., \( P_3 = \frac{1}{4}(Q_1 + Q_2 + Q_3 + Q_4) \).

\[\begin{array}{ccc}
\text{Stage 0} & \text{Z}_1^0 & \text{Z}_2^1 \\
\text{Stage 1} & \text{Z}_1^1 & \text{Z}_2^2 & \text{Z}_3^2 & \text{Z}_4^2 \\
\text{Stage 2} & \text{Z}_1^2 & \text{Z}_2^3 & \text{Z}_3^3 & \text{Z}_4^3 \\
\text{Stage 3} & \text{Z}_1^3 & \text{Z}_2^4 & \text{Z}_3^4 & \text{Z}_4^4 & \text{Z}_5^4 & \text{Z}_6^4 & \text{Z}_7^4 & \text{Z}_8^4 \\
\text{Stage 4} & & & & & & & & & &
\end{array}\]

**Figure 3.2** Local Average Subdivision in one-dimension.

\[\begin{array}{ccc}
P_1 & P_2 & P_3 \\
& & \\
P_4 & Q_1 & Q_2 \\
& P_5 & & P_6 \\
& Q_3 & Q_4 & \\
P_7 & P_8 & P_9 & \\
& & & & & & & & & &
\end{array}\]

**Figure 3.3** Local Average Subdivision in two-dimensions.

A particular gray image of variance function in a two-dimensional field is shown in Fig. 3.4. Although LAS is superior to other approximate generators (e.g. Turning Bands and Fast
Fourier Transform), it is not perfect (Fenton, 1994). In particular, the variance at any point in the field tends to fluctuate in a regular fashion. This is illustrated in Fig. 3.4.

![Figure 3.4](image)

**Figure 3.4** Two-dimensional LAS variance field estimated over 2000 realizations for target point variance $\sigma^2 = 1$, correlation length $\theta = 2$, and random seed kseed = 1871.

Fig. 3.4 was produced by simulating a random field 2000 times and estimating the variance at each point. The regular pattern is clearly evident. For a target point variance one there are $95\%$ of variances within interval $(0.919,1.068)$, $90\%$ within $(0.931,1.057)$ and $80\%$ within $(0.945,1.044)$, so the magnitude of the error is not a problem. It is just that the lows and highs tend to always occur in the same locations. Fig. 3.5 illustrates one realization of variance values across the middle of this two-dimensional field ($y = 2.5$ in Fig. 3.4). It can
be seen in Fig. 3.5 that the minimum variances tend to occur every certain distance. We shall see later that this leads to small errors in the estimated failure probabilities.

![Graph showing LAS variance along the middle line of a two-dimensional field for target point variance $\sigma^2 = 1$, correlation length $\theta = 2$, and random seed kseed = 1871.](image)

**Figure 3.5** LAS variance along the middle line of a two-dimensional field for target point variance $\sigma^2 = 1$, correlation length $\theta = 2$, and random seed kseed = 1871.

Since all measured engineering properties are based on local averages over some finite domain, an appropriate "local average" random field is, however, more related to the actual engineering property. The major advantage of LAS is that statistical consistency can be achieved in the random field regardless of the mesh size. In other words, the variance and covariance structures are properly adjusted when the mesh is changed (unlike "point" generators, such as Turning Bands and Fast Fourier Transform). As such, it is well suited to problems with average properties over a local element.

Specifically, in the present study, a 2-D symmetric homogeneous (stationary) Gaussian local average random field $G_{lae}(x)$ with Markovian spatial correlation structure is produced using LAS.
Now attention is turned to the second random soil parameter, the friction angle $\phi$, which has symmetrically bounded distribution between $\phi_{\text{max}}$ and $\phi_{\text{min}}$. A transformation $T(\phi)$ of a standard normal random field, $G_{\phi}(\vec{x})$, having point distribution with zero mean and unit variance, is chosen to produce the friction angle field, as shown below

$$
\phi(\vec{x}) = T(G_{\phi}(\vec{x})) = \phi_{\text{min}} + \frac{1}{2}(\phi_{\text{max}} - \phi_{\text{min}}) \left\{ 1 + \tanh \left( \frac{s_{\phi} G_{\phi}(\vec{x})}{2\pi} \right) \right\} \quad (3.5)
$$

where $s_{\phi}$ is a factor which controls the variability of the friction angle $\phi$ between $\phi_{\text{max}}$ and $\phi_{\text{min}}$. The inverse function of $T(G_{\phi}(\vec{x}))$ is

$$
G_{\phi}(\vec{x}) = T^{-1}(\phi(\vec{x})) = \frac{\pi}{s_{\phi}} \ln \left( \frac{\phi(\vec{x}) - \phi_{\text{min}}}{\phi_{\text{max}} - \phi(\vec{x})} \right) \quad (3.6)
$$

The same methodology (LAS) is used to produce $G_{\phi}(\vec{x})$ as $G_{\text{ln.c}}(\vec{x})$. As $\theta$ has a symmetrically bounded distributed, its shape looks a lot like normal distribution $N(\mu_{\phi}, \sigma_{\phi})$ (see Fig. 3.10).

Note that the scale of fluctuation, $\theta$, is measured relative to $T^{-1}(\phi)$. In other words, to estimate the scale of fluctuation we first transform the observed friction angle, $(\phi_1, \phi_2, \cdots)$, into a series of normally distributed observations $x_i = T^{-1}(\phi_i)$ and then estimate $\theta$ from $x_i$ values.

It should be pointed out that the correlation length $\theta_{\text{ln.c}}$ may be different from $\theta_{T^{-1}(\phi)}$. However, for the same site it is probably reasonable to assume they are similar to each other as a result of common geologic processes. In the present study they are assumed to be the same. The consideration of differing scales of fluctuation will be left for future refinements.

The cross correlation coefficient, $\rho$, between cohesion, $c$, and friction angle, $\phi$, is difficult to determine and no real consensus is found in the literature. In this study $c$ and $\phi$ are assumed to be independent following the conclusion by Fenton (2003), that “varying the cross-correlation $\rho$ from -1 to +1 was found to have only a minor influence on the stochastic behaviour of the bearing capacity". It was also noticed by Fenton (2003) that mean bearing capacity for independent $c$ and $\phi$ ($\rho = 0$) is smaller than the negatively correlated case ($\rho = -1$). If the negative cross correlation between $c$ and $\phi$ (Cherubini, 2000, Wolff, 1985) is more reasonable, it is believed that assuming independence is conservative since a larger probability of failure, $p_f$, occurs for the same footing since the mean bearing capacity is smaller.

LAS can be used to produce both isotropic random fields and anisotropic random fields. In the present study, an isotropic soil field in two-dimensions is assumed. This implies
that the joint probability density function of log-cohesion or friction angle is invariant under rotation. It also implies this two-dimensional random field is homogeneous, in other words, the mean, variance and higher order moments are constant through soil space. For our purposes isotropy means the correlation between any two points only depends on the distance between them, instead of their orientation relative to one another. Hence, the same correlation length was assumed in any direction through the soil. Even though the properties of soil are often more correlated horizontally than vertically, because of soil layering, in this research we concentrate on the isotropic case and leave the anisotropic case for site specific studies.

To better represent the failure surface and its tendency to seek out the weakest path, a non-linear finite element model of the soil subjected to an overlying load transmitted through a rigid footing is considered in this research. The rigid footing is defined as one that has no rotation and is inflexible. The soil model is 18 m wide by 4.8 m deep meshing with $120 \times 32$ square elements, as shown in Fig. 3.6, each having side length of 0.15 m. The soil is treated as an elastic-plastic Von-Mises solid, and plain strain deformation is computed using 8-node quadrilateral elements, within the viscoplastic strain method (Smith and Griffiths, 2004). The displacement increment of finite element analysis is 0.01 m. This means after the designed footing is placed on the soil model, it is pushed down 0.01 m in steps until the soil fails. The failure of soil occurs when the soil ‘slips’ out from under the footing(s) by shearing along some failure surface.

![Figure 3.6 Meshing of finite element analysis applied in the present study.](image)

Note that each grid cell in the Local Average Subdivision process corresponds to an element in the finite element analysis. In other words local average soil properties in each LAS cell are mapped directly to the finite element properties.
3.4 Accuracy of Finite Element Analysis (Deterministic Analysis)

This section investigates the accuracy of the finite element analysis. A two-dimensional deterministic soil mass subjected to a rigid strip footing is analyzed by a non-linear finite element model under plane strain conditions.

In finite element analysis an estimate that is based on only a few elements will tend to have a large error. In order to achieve accurate analysis, an appropriate mesh will be required.

The footing size, relative to the size of the soil model, will also affect the finite element accuracy. Here, we test the influence of number of elements under footing using different meshing schemes on the same soil model and different soil models with the same meshing scheme, while the footing width remains the same (1.5 m). Specifically, the cases considered are listed in Table 3.1 and Table 3.2, where \( n_b \) is the number of elements under the footing and \( \hat{q}_u \) is the estimated bearing capacity by finite element analysis.

### Table 3.1 Study of effects number of elements under footing on finite element analysis of bearing capacity, \( q_u \), for different soil mass, \( n_b \) is the number of elements under the footing.

<table>
<thead>
<tr>
<th>FEM mesh</th>
<th>Element size ( (dx \times dy) )</th>
<th>( n_b )</th>
<th>( \hat{q}_u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>120 ( \times 60 )</td>
<td>0.1 m( \times )0.1 m</td>
<td>15 elements</td>
<td>1465.8 kPa</td>
</tr>
<tr>
<td>120 ( \times 60 )</td>
<td>0.3 m( \times )0.3 m</td>
<td>5 elements</td>
<td>1458.9 kPa</td>
</tr>
<tr>
<td>120 ( \times 60 )</td>
<td>0.5 m( \times )0.5 m</td>
<td>3 elements</td>
<td>1432.7 kPa</td>
</tr>
<tr>
<td>120 ( \times 32 )</td>
<td>0.1 m( \times )0.1 m</td>
<td>15 elements</td>
<td>1463.5 kPa</td>
</tr>
</tbody>
</table>

### Table 3.2 Study of effects of discretization resolution on finite element analysis of bearing capacity, \( q_u \), \( n_b \) is the number of elements under footing.

<table>
<thead>
<tr>
<th>FEM mesh</th>
<th>Element size ( (dx \times dy) )</th>
<th>( n_b )</th>
<th>( \hat{q}_u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>120 ( \times 60 )</td>
<td>0.1 m( \times )0.1 m</td>
<td>15 elements</td>
<td>1465.8 kPa</td>
</tr>
<tr>
<td>40 ( \times 20 )</td>
<td>0.3 m( \times )0.3 m</td>
<td>5 elements</td>
<td>1458.5 kPa</td>
</tr>
<tr>
<td>24 ( \times 12 )</td>
<td>0.5 m( \times )0.5 m</td>
<td>3 elements</td>
<td>1450.9 kPa</td>
</tr>
</tbody>
</table>

The deterministic finite element analysis results, uses cohesion and friction angle, values of 100 kN/m\(^2\) and 20°, respectively. The resulting computed bearing capacity, \( q_u \), are given in Tables 3.1 and 3.2. The exact solution, given by Prandtl’s (1921) formula, is 1483 kPa. Figures 3.6 and 3.7 illustrate the relative error between Prandtl's solution and the finite element analysis.
Figure 3.7  Effects of number of elements under footing on the finite element analysis of bearing capacity for different soil mass.

Figure 3.8  Effects of discretization resolution on finite element analysis of bearing capacity.
It can be observed the estimated values by finite element analysis are slightly less than the Prandtl solution. Since the 120 × 32 results are very similar to the 120 × 60 results in Table 3.1, we will use a mesh size of 120 × 32. This results in considerable savings in computer time.

As we can see the discrepancy of bearing capacity between Prandtl's formula and finite element analysis is decreasing with the increasing number of elements. The fifteen elements footing width \( (n_b = 15) \) is the most accurate case considered, in which case the error is 1.16%. When the number of elements under the footing \( n_b = 3 \), the error is 3.4%.

If the footing width is large compared to the soil model, edge effects become important. These may also decrease the accuracy of the finite element analysis of bearing capacity. The influence of edge effects were tested by keeping the same footing width \( (B = 6 \text{ m}) \) and decreasing the width, \( H \), of the soil model. Details are shown in the following table.

**Table 3.3** Study of edge effects on finite element analysis of bearing capacity \( q_u \). \( B \) is the footing width and \( H \) is the soil model width.

<table>
<thead>
<tr>
<th>FEM mesh</th>
<th>Element size ( (dx \times dy) )</th>
<th>( B/H )</th>
<th>Estimate bearing capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>120 × 60</td>
<td>0.2 × 0.1</td>
<td>6/24</td>
<td>1411.7 kPa</td>
</tr>
<tr>
<td>120 × 60</td>
<td>0.15 × 0.1</td>
<td>6/18</td>
<td>1439.9 kPa</td>
</tr>
<tr>
<td>120 × 60</td>
<td>0.1 × 0.1</td>
<td>6/12</td>
<td>1435.0 kPa</td>
</tr>
<tr>
<td>120 × 60</td>
<td>0.075 × 0.1</td>
<td>6/9</td>
<td>1739.9 kPa</td>
</tr>
</tbody>
</table>

The results of the edge effect analysis are shown in Fig. 3.9, where \( B \) is footing width and \( H \) is the width of the soil model in the horizontal direction. As can be seen from Fig. 3.9, when the footing width approaches 2/3 of the soil model width, the finite element analysis considerably overestimate the bearing capacity. The error is 17.32%.

The edge effects begin to become significant when the footing is over about half of soil model width. Accordingly, in this research shallow foundations having footing widths over one-third of the soil model width will be defined as being too big. It is assumed, in this case, that the soil is too weak to support a normal shallow foundation and that the designer would choose some alternative. The specified details will be discussed in the following section.
Figure 3.9 Influence of edge effects on finite element analysis of bearing capacity.

3.5 Footing Design Simulation

A specific design problem will be considered here in order to investigate the probability of bearing capacity failure for the recommendation of the resistance factors for shallow foundations.

The simulation computer program allows for five random parameters (cohesion, $c$, friction angle, $\phi$, elastic modulus, $E$, dilation angle, $\psi$, and Poisson’s ratio, $\nu$). However, the primary parameters of bearing capacity problem are cohesion, $c$, and friction angle, $\phi$. Hence, in the present study, to simplify the analysis, $E$, $\psi$ and $\nu$ are held constant at $10^5$ kN/m$^2$, 0.3, and 0 respectively. Note that zero dilation angle means there is no plastic dilation during yield of the soil.

In the following parametric study, the mean of the lognormally distributed cohesion, $c$, is held constant at 100 kN/m$^2$. The symmetrically bounded distributed friction angle, $\phi$, lies between (10°, 30°). The reason of taking 30° as the maximum value of friction angle is that, footings design based on soils with higher friction angles will usually be governed by settlement instead of bearing capacity, which is beyond our interest.

According to Becker (1996a), “the values of resistance factor, $\phi_y$, lie in the range of about 0.3 to about 0.9. In general, values between 0.5 and 0.85 are normally used or recommended
in foundation design." Barker R.M. et al. (1991) suggested that the typical range of φ_g is from 0.35 to 0.9. φ_g can be effected by the classification of material (such as clay and sand), the types of the geotechnical structures, such as shallow foundations, deep foundations, and retaining walls, the estimated method of resistance. The values of φ_g selected for study in the present research are listed in Table 3.4. In addition, the coefficient of variation of cohesion, V_c, and spatial correlation length, θ, are varied systematically according to the following table:

**Table 3.4** Random field parameters applied in the study.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values considered</th>
</tr>
</thead>
<tbody>
<tr>
<td>V_c</td>
<td>0.1 0.2 0.3 0.5</td>
</tr>
<tr>
<td>θ</td>
<td>0.1 1.0 2.0 3.0 6.0 10.0 50.0</td>
</tr>
<tr>
<td>φ_g</td>
<td>0.4 0.5 0.6 0.7 0.8 0.9</td>
</tr>
</tbody>
</table>

The definitive levels of coefficient of variation V in geotechnical characterization is still unclear. The appropriate values in a specific field mainly depends on the site investigation and engineering properties of the soil. For example, poor site investigation corresponds to a higher V. In this research typical values of V are chosen. In that the correlation between c and φ is assumed to be independent, the correlation coefficient ρ between G_{loc} and G_{φ} is zero. The correlation structure of each field is Markovian (Eq. (3.3)) with scale of fluctuation shown in Table 3.4.

Following 2000 realizations, the estimated probability of failure, \( \hat{p}_f = n_f/n \), of a rigid strip footing founded on random soil mass are estimated, where \( n_f \) is the number of design failures, and n is the number of simulation. The standard deviation of \( \hat{p}_f \) can be estimated as

\[
\sigma_{\hat{p}_f} = \sqrt{\frac{\hat{p}_f(1 - \hat{p}_f)}{n}}
\]  

(3.7)

In other words, in the present study the estimated failure probability, \( \hat{p}_f \), from simulation has standard error \( \sqrt{\hat{p}_f(1 - \hat{p}_f)/2000} \).

Note that with the chosen parameters stated earlier one run of 2000 realizations takes about 22 hours on a 36HZ CPU. 168 cases considered in Table 3.4 took about 140 CPU-days to run.

In practice, the footing width dimension is always increased to an easily measured quantity for the purpose of construction. Similarly, in the simulation the design width B is rounded up to the next element boundary. This means once the required footing width, B_{req}, has
been found in step (4), the design footing width $B$ must be increased to align with an element boundary. For example, if the required footing width $B_{req} = 2.34$ m, and the element length $dx = 0.15$ m, the design footing width $B$ must be increased to 2.4 m, which corresponds to 16 elements. Note that if the required footing width is sufficiently close to a lower element boundary (within 1%), then the design footing width is decreased. For example, if $B_{req} = 2.2507$ m, then it would be rounded down to 2.25 m, rather than being increased to 2.4 m.

Another issue should be pointed out is that, in practice if a spread footing covers more than about one-third of the building footprint area, a mat or some type of deep foundation would probably be more economical, not to mention that a strip footing being over 6 m in width is unlikely to happen. In this research the maximum footing size is taken to be equal to one-third of the soil mass length, i.e., 6 m. The one-third rule also ensures that the edge effects discussed earlier do not become significant in the finite element analysis.

From the viewpoint of assessing the reliability of the ‘designed’ strip footing, it is necessary to decide if footing widths over one-third of soil model width would correspond to a success, or to a failure. In this research we shall assume that the subsequent design of the alternative foundation would be a success, since it would have its own (high) reliability. For the estimation of probability of failure, $\hat{\beta}_f$, the total number of realizations $n$ would still remain the same ($n = 2000$ in this research). Footings found to be too large are simply assumed to not fail. Regardless, the fraction of footings found larger than 1/3 the soil width always remained under 6% in the present study.

To avoid footings which are too small to construct, there will also be a minimum footing width. In the simulation the footing width $B$ will be taken as not less than four elements. According to a similar deterministic study as in Table 3.1, the four elements footing width ($n_b = 4$) has error 2.32%. Loaded areas smaller than this tend to have significant finite element errors. In the present study the minimum footing width of 0.6 m, corresponding to element side length 0.15 m, is also consistent with the recommendation by S.E. French (1999) that the minimum size of a footing be 0.6 m.
3.6 Mapping of the Coefficient of Variation of Friction Angle, $V_\phi$

Friction angle, $\phi$, was assumed to be a symmetrically bounded distribution between 10° and 30°. A transformation originally from Fenton's Ph.D. thesis (1990) was applied to transform the Gaussian process $G \sim N(0, 1)$ into the bounded distribution of $\phi_n$, as shown below

$$\phi_n = \frac{1}{2} \left\{ 1 + \tanh \left( \frac{Y}{2\pi} \right) \right\}$$  

(3.8)

where $Y = s_\phi G$, $s_\phi$ is a factor that controls the distribution of $\phi_n$, and $s_\phi$ is also the standard deviation of the normally distributed $Y$. $\phi_n$ is bounded on the interval (0, 1), and so is a normalized form of $\phi$. The objective of this section is to determine how the standard deviation of $\phi$ varies with factor $s_\phi$. The probability distribution of $\phi_n$ is

$$f_{\phi_n}(x) = \frac{\sqrt{\pi}}{x(1-x)s_\phi} \exp \left\{ -\frac{1}{2} \left( \frac{\pi \ln \left( \frac{x}{1-x} \right)}{s_\phi} \right)^2 \right\}, x \in (0, 1)$$  

(3.9)

which is symmetric about $x = \frac{1}{2}$.

Fig. 3.10 illustrates the bounded distribution for a variety of values of $s_\phi$. For $s_\phi = 5$, the distribution is almost uniform between zero and one. When $s_\phi$ is greater than five, the distribution becomes u-shaped with higher densities at $\phi_n = 0$ or 1 than at $\phi_n = 0.5$, which is not particularly meaningful. Physically reasonable distributions are obtained for $s_\phi \leq 5$.

The random friction angle, $\phi$, is obtained from $\phi_n$ by stretching and shifting Eq. (3.8)

$$\phi = \phi_{\text{min}} + \frac{1}{2}(\phi_{\text{max}} - \phi_{\text{min}}) \left\{ 1 + \tanh \left( \frac{s_\phi G}{2\pi} \right) \right\}$$  

(3.10)

A program was written to simulate $\phi$ from the symmetrically bounded distribution shown in Fig. 3.10 using $\phi_{\text{min}} = 10^\circ$, $\phi_{\text{max}} = 30^\circ$ and $n = 100,000$ realizations. Then the sample standard deviation of $\phi$, was estimated from the simulation. Fig. 3.11 shows how the sample standard deviation of $\phi$ varies with $s_\phi$. 

Figure 3.10  Bounded distribution of normalized friction angle for different values of $s_\phi$.

Figure 3.11  The standard deviation of friction angle $\phi$, as a function of factor $s_\phi$ (estimated by simulation).
Note that when \( s_\phi \) goes to infinity the bounded distribution of Eq. (3.9) becomes a Bernoulli distribution on \((0, 1)\) with \( p = 0.5 \). The standard deviation of a Bernoulli distribution is \( \sigma = \sqrt{pq} = \sqrt{0.5 \times 0.5} = 0.5 \), where \( q = 1 - p \). An expanded Bernoulli distribution on \((10, 30)\) has standard deviation 10, and it can be seen in Fig. 3.11 that the sample standard deviation tends towards this limiting value. This suggests that

\[
\sigma_\phi \approx \frac{1}{4} (\phi_{max} - \phi_{min}) \left( 1 + \frac{s_\phi}{a + s_\phi} - e^{-bs_\phi} \right)
\]

where \( a = 3.8 \) and \( b = 0.15 \) in the present study.

In this research the coefficient of variation of cohesion, \( V_c \), was varied over values 0.1, 0.2, 0.3, 0.5. Correspondingly, \( s_\phi \) values were taken as \( s_\phi = 1, 2, 3, 5 \) (Table 3.5 gives the corresponding coefficient of variation). In other words, the variability of \( c \) and \( \phi \) were tied together in this research: when \( V_c = 0.1 \), \( s_\phi \) was set to 1, when \( V_c = 0.2 \), \( s_\phi \) was set to 2, and so on. The choice of \( s_\phi \) is arbitrary, but it is reasonable that the variability of cohesion is consistent with the variability of friction angle for the same site.

<table>
<thead>
<tr>
<th>( V_c )</th>
<th>( s_\phi )</th>
<th>( \sigma_\phi )</th>
<th>( V_\phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1</td>
<td>1.5528</td>
<td>0.0776</td>
</tr>
<tr>
<td>0.2</td>
<td>2</td>
<td>2.9184</td>
<td>0.1459</td>
</tr>
<tr>
<td>0.3</td>
<td>3</td>
<td>4.0307</td>
<td>0.2015</td>
</tr>
<tr>
<td>0.5</td>
<td>5</td>
<td>5.5976</td>
<td>0.2799</td>
</tr>
</tbody>
</table>

### 3.7 Simulation of the Footing Loads

The load \( L \) is also a random variable involving inherent uncertainties. The sources of uncertainties in dead load include unit weight and volume of material while the uncertainties of live load include the weight of people, equipment, and furnishing material. The objective of this section is to estimate the design load and simulate lognormally distributed dead and live loads. The total load can be obtained by adding dead and live loads for each realization. Whether the footing fails or not is determined by comparing the total load with the actual bearing capacity obtained from finite element analysis.

Both dead and live loads are assumed to be lognormally distributed random variables. They are also assumed to be independent with means and the coefficient of variations, \( \mu_{\nu D}, V_{\nu D} \)
and \( \mu_{LL}, V_{LL} \). A dead load to live load ratio of 3.0 was applied in the present study to be consistent with previous work (Barker et al., 1991, Allen, 2005). The mean dead load was chosen to be 600 kN/m, with mean live load of 200 kN/m.

Load and Resistance Factor Design (LRFD), particularly load and resistance factors, have been calibrated and adjusted in structural engineering over many years. It would be appropriate to use the same loads, load factors and load combinations in foundation design as used in current structural practice. Doing this will not only attain a consistent design between superstructure and substructure and harmonize with structural community, but also significantly simplify the design process. According to the National Building Code of Canadian (NBCC, 2005), Section 4.1 “Structural Loads and Procedures”, load factors, \( \alpha \), and load combinations shall follow combinations in Table 3.6.

**Table 3.6** Load combinations for ultimate limit states.

<table>
<thead>
<tr>
<th>Case</th>
<th>Load Combination</th>
<th>Companion Loads</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.4( L_{DC} )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(1.25( L_{DC}^{(1)} ) or 0.9( L_{DC}^{(2)} )) + 1.5( L_{LC}^{(3)} )</td>
<td>0.5( S ) or 0.4( W )</td>
</tr>
<tr>
<td>3</td>
<td>(1.25( L_{DC}^{(1)} ) or 0.9( L_{DC}^{(2)} )) + 1.5( S )</td>
<td>0.5( L_{LE}^{(4)} ) or 0.4( W )</td>
</tr>
<tr>
<td>4</td>
<td>(1.25( L_{DC}^{(1)} ) or 0.9( L_{DC}^{(2)} )) + 1.4( W )</td>
<td>0.5( L_{LE}^{(4)} ) or 0.5( S )</td>
</tr>
<tr>
<td>5</td>
<td>1.0( L_{DC}^{(2)} ) + 1.0( S^{(5)} )</td>
<td>0.5( L_{LE}^{(4)} ) + 0.25( S )</td>
</tr>
</tbody>
</table>

Footnotes to Table 3.6:

1. The load factor 1.25 for dead load, \( L_{DC} \), for soil, superimposed earth, plants and trees shall be increased to 1.5, except that when the soil depth exceeds 1.2 m, the factor may be reduced to \( (1 + 0.6/h_s) \) but not less than 1.25, where \( h_s \) is the depth of soil in meters supported by the structure.

2. Except as provided in Sentence 4.1.8.16.(1) (NBCC, 2005), the counteracting, factored dead load, 0.9\( L_{DC} \) in load combination cases 2, 3 and 4 and 1.0\( L_{DC} \) in load combination case 5, shall be used when the dead load acts to resist overturning, uplift, sliding, failure due to stress reversal, and to determine anchorage requirements and the factored resistance of members.

3. The principal-load factor 1.5 for live load, \( L_{LC} \), may be reduced to 1.25 for liquids in tanks.

4. The companion-load factor 0.5 for live load, \( L_{LE} \), shall be increased to 1.0 for storage areas, and equipment areas and service rooms referred to in Table 4.1.5.3 (NBCC, 2005).
(5) Earthquake load, $E$, in load combination case 5 includes horizontal earth pressure due to earthquake determined in accordance with Sentence 4.1.8.16.(4) (NBCC, 2005).

In Table 3.6, $L_{DC}$ is the (characteristic) dead load, $E$ is earthquake load and effects, $H$ is a permanent load due to lateral earth pressure, including groundwater, $L_{LC}$ is the (characteristic) live load, $S$ is variable load due to snow, including ice and associated rain, and $W$ is wind load.

A literature review of dead load and live load factors used around the world are shown in Table 3.7.

<table>
<thead>
<tr>
<th>Source</th>
<th>dead load</th>
<th>live load</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hansen, J.B. (1956)</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td>Denmark (1965)</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td>ANSI A58 (1980)</td>
<td>1.2–1.4</td>
<td>1.6</td>
</tr>
<tr>
<td>NCHRP 343 (Barker et al., 1991)</td>
<td>1.3</td>
<td>2.17</td>
</tr>
<tr>
<td>NCHRP 12–55 (2004)</td>
<td>1.25</td>
<td>1.75</td>
</tr>
<tr>
<td>CFEM (1992)</td>
<td>1.25</td>
<td>1.5</td>
</tr>
<tr>
<td>CHBDC (2000)</td>
<td>1.25</td>
<td>1.5</td>
</tr>
<tr>
<td>AS 4678 (2002)</td>
<td>1.25</td>
<td>1.5</td>
</tr>
<tr>
<td>AS 5100 (2004)</td>
<td>1.2</td>
<td>1.5</td>
</tr>
<tr>
<td>NBCC (2005)</td>
<td>1.25</td>
<td>1.5</td>
</tr>
<tr>
<td>Eurocode 7 (Model 1)</td>
<td>1.0</td>
<td>1.3</td>
</tr>
<tr>
<td>Eurocode 7 (Model 2)</td>
<td>1.35</td>
<td>1.5</td>
</tr>
</tbody>
</table>

The dead and live load factor, $\alpha_d$, $\alpha_L$, respectively, are chosen to be 1.25 and 1.5 (e.g. Bartlett, 2003) in the present study, as specified by NBCC (2005).

In foundation design a nominal load value frequently characterizes the load. Estimation of the characteristic values of load is the first step in engineering design. The characteristic or nominal load does not necessarily need to be the mean value. Frequently, a value higher than the mean is selected by geotechnical and structural engineers as the characteristic value for design purposes. The characteristic load can be defined as follows:

$$L_{DC} = k_D \mu_D$$

Similarly,

$$L_{LC} = k_L \mu_L$$

(3.12)

(3.13)
where $k_D$ and $k_L$ are respectively the bias factor representing the ratio of the specified (characteristic) value to the mean value for dead and live load effects. Particularly, $k_D$ and $k_L$ are usually less than or equal to one. It would be one when the mean value is taken to be the characteristic load.

The choices of a characteristic load or bias factor for any given limit state will necessarily involve engineering judgment and experience. Different engineers with varying background and experience would use different values. It is necessary to base the design on characteristic values that are consistent within the society. A typical range of bias factor for load is between 1.0 and 1.43 (Allen, 1975, Becker, 1996).

To characterize the load distribution, the coefficient of variation, $V$, needs to be considered along with the mean value. Live loads are generally more variable than dead loads, i.e., usually the coefficient of variation of live load is much higher than that for dead load. Table 3.8 shows typical coefficients of variation as determined by various researchers.

### Table 3.8  Literature review of coefficient of variation for dead and live load.

<table>
<thead>
<tr>
<th>Source</th>
<th>$V_D$</th>
<th>$V_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allen (1975)</td>
<td>0.07</td>
<td>0.3</td>
</tr>
<tr>
<td>Meyerhof (1993, 1995)</td>
<td>0.05 – 0.15</td>
<td>0.2 – 0.6</td>
</tr>
<tr>
<td>Ellingwood and Tekei (1999)</td>
<td>0.08 – 0.15</td>
<td>0.25</td>
</tr>
<tr>
<td>Bartlett et al. (2003)</td>
<td>0.05 – 0.075</td>
<td>0.17</td>
</tr>
<tr>
<td>Scott et al. (2003)</td>
<td>0.07 – 0.16</td>
<td>0.2 – 0.3</td>
</tr>
</tbody>
</table>

In this research, a bias factors of $k_D = 1.18$ (Becker, 1996) was conservatively selected, along with a coefficient of variation of $V_D = 0.15$ for dead load. For live load, a bias factor of $k_L = 1.43$ (Allen, 1975) was selected, along with a coefficient of variation $V_L = 0.3$. Accordingly, the indicated load parameters applied in this research are shown in the following table:

### Table 3.9  Load characters applied in this research.

<table>
<thead>
<tr>
<th>Characteristic value</th>
<th>Dead load (kN/m)</th>
<th>Live load (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{DC}$ = 705.88</td>
<td>600</td>
<td>200</td>
</tr>
<tr>
<td>$L_{LC}$ = 285.71</td>
<td>90</td>
<td>60</td>
</tr>
</tbody>
</table>

In Table 3.9, $L_{DC}$ and $L_{LC}$ are, respectively, the characteristic dead and live load. Hence, the factored design load in the present study, $\alpha L_c = \alpha_D L_{DC} + \alpha_L L_{LC} = 1.25(705.88) + 1.5(285.71) = 1310.9$ kN/m.
The lognormally distributed dead and live load can be simulated using the following formula

\[ L = \exp(\mu_{\ln L} + \sigma_{\ln L} G) \]  

(3.14)

where \( L \) is either \( L_D \) or \( L_L \) and \( \mu_{\ln L}, \sigma_{\ln L} \) are the mean and standard deviation of the normally distributed \( \ln L \). They can be computed from \( \mu_L \) and \( \sigma_L \) using the transformation given by Eq. (2.38). \( G \) is a standard normally distributed random variable having zero mean and unit variance. \( G \) is obtained from uniformly distributed random numbers \( u_1 \sim U(0, 1) \), \( u_2 \sim U(0, 1) \) through the following transformations

\[ G_i = \sqrt{-2 \ln u_1} \cos(2\pi u_2) \]  

(3.15)

and

\[ G_{i+1} = \sqrt{-2 \ln u_1} \sin(2\pi u_2) \]  

(3.16)

where \( G_i \) is the \( i^{th} \) realization of \( G \).

The total load is then set to the sum of the simulated dead and live loads, i.e., \( L = L_D + L_L \). \( L \) is then compared to the “actual” bearing capacity, \( q_u \times B \), estimated from the finite element analysis. If the total load is greater than the “actual” bearing capacity, then the designed footing cannot support the loads and a bearing capacity failure occurs for this individual realization. Otherwise, the footing is assumed to perform well during its design life.

### 3.8 Sampling

In this research sampling one element at a time was selected to yield the soil samples. Elements over the entire depth (4.8 m, corresponding to 32 elements, in the present study) in one column are inspected, as shown in Fig. 3.12, and their average is computed. This sampling approach is comparable to a cone penetration test (CPT), where the soil is sampled in a continuous mode over a significant depth.

There are three sampling locations considered, sampling directly under the footing, sampling 4.5 m away from the footing center, and sampling 9 m away from the footing center, which are illustrated in Fig. 3.12.

Once a soil column has been sampled, the characteristic cohesion, \( \hat{c} \), used in the design, is computed as the geometric average

\[ \hat{c} = \left( \prod_{i=1}^{32} c^e(g_i^c) \right)^{1/32} = \exp \left\{ \frac{1}{32} \sum_{i=1}^{32} \ln c^e g_i^c \right\} \]  

(3.17)
where $c^o(x_i)$ is the $i^{th}$ element in the sampled column.

Similarly, the characteristic friction angle, $\hat{\phi}$, is computed using an arithmetic average, i.e.,

$$\hat{\phi} = \frac{1}{32} \sum_{i=1}^{32} \phi^o(x_i)$$

(3.18)

where $\phi^o(x_i)$ is the $i^{th}$ element in the sampled column.

**Figure 3.12** Sampling locations, (a) corresponds to sampling at a distance 9 m from the footing center, (b) corresponds to sampling at a distance 4.5 m from the footing center, and (c) is directly under the footing.

Note that measurement error is not considered in this study.
Chapter 4

Simulated Results and Comparison with Predicted Results

The simulation results are presented in this chapter and compared with theory. Specifically, the probability of failure, $p_f$, as a function of correlation length, $\theta$, and coefficient of variation, $V_c$, are discussed. The simulated $\hat{p}_f$ and the predicted $p_f$ using the theory presented in Chapter 2 are compared for various $\theta$ and $V_c$.

4.1 Simulation-based Estimates of Failure Probability

We will start by discussing the effects of the correlation length, $\theta$, on the probability of failure for specific values of $V_c$ and $\phi_0$, and for the various sampling locations considered (see section 3.8).

Recalling that as $\theta \to 0$, the random field becomes a white noise with independent values at any two separate points and that when $\theta \to \infty$, the random field is a uniform field. This suggests that fields with smaller correlation lengths tend to be rough and fields with larger correlation lengths tend to be smooth.

Fig. 4.1 illustrates a random cohesion field, $c$, corresponding to a small correlation length, $\theta = 0.1$, in which darker regions indicate stronger soils.

![Image](image.jpg)

**Figure 4.1** Generated cohesion field for correlation length $\theta = 0.1$ when coefficient of variation $V_c = 0.3$, darker regions indicate strong soils.
Similarly, Fig. 4.2 shows another random cohesion field having a large scale of fluctuation, $\theta = 10.0$. As expected, the field is ragged for the small scale of fluctuation, $\theta = 0.1$, and smoother for the large scale of fluctuation, $\theta = 10.0$.

**Figure 4.2** Generated cohesion field for correlation length $\theta = 10.0$ when coefficient of variation $V_c = 0.3$, darker regions indicate strong soils.

The effect of the correlation length on the probability of failure for sampling directly under the footing, 4.5 m away from the footing center, and 9 m away from the footing center with coefficient of variation of cohesion $V_c = 0.5$, and the factor that describes the distribution of friction angle $s_\phi = 5$ are illustrated in Figures 4.3, 4.4 and 4.5. The other cases, with different $V_c$ values and sampling locations, displayed similarly shaped curves.

Note that the correlation length, $\theta$, in Figures 4.3, 4.4 and 4.5, is one of $\theta_{\ln c}$ or $\theta_{T-1(\phi)}$ (the same correlation length has been assumed for normally distributed $G_{\ln c}$ and $G_\phi$), where $c$ is cohesion and $\phi$ is friction angle. The numerical tables of probability of failure estimated from simulation can be found in Appendix A.

It can be seen from Figures 4.3, 4.4 and 4.5 that the probability of failure tends to be higher for all three sampling locations when the value of $\theta$ is in an intermediate range, i.e., somewhere between 1 and 10. This is as expected, since for homogeneous (stationary) random fields the probability of failure depends on the quality of the estimation of the effective soil properties, i.e. on the discrepancy between the estimated and actual effective soil parameters. Better estimated soil properties can be obtained for small correlation length, since samples are more independent, and large correlation length, since samples are more correlated to properties under the footing. So the largest difference between the estimated and effective values can be expected in the intermediate range of correlation length. This leads to higher values of probability of failure in this range.
**Figure 4.3** Effect of correlation length, $\theta$, on probability of failure, $\hat{p}_f$, estimated from simulation, for $V_c = 0.5$, $s_\phi = 5$ and sampling directly under the footing.

**Figure 4.4** Effect of correlation length, $\theta$, on probability of failure, $\hat{p}_f$, estimated from simulation, for $V_c = 0.5$, $s_\phi = 5$ and sampling 4.5 m away from the footing center.
Figure 4.5  Effect of correlation length, θ, on probability of failure, \( \hat{P}_f \), estimated from simulation, for \( V_c = 0.5, s_φ = 5 \) and sampling 9 m away from the footing center.

It should be pointed out that the probabilities of failure for both small correlation lengths and large correlation lengths decreases, as shown in Figures 4.3 – 4.5. This is discussed further in Section 4.2.1.

Therefore, for design, the worst case corresponding to the highest probability of failure is when the correlation length is between 1 and 10. In the following, \( \theta = 1, 3, \) and 10 will be concentrated on since this is conservative.

We turn our attention now to how the soil variability (i.e. \( V_c \) and \( s_φ \)) affects the probability of failure. Three representative worse cases of estimated probability of failure, \( \hat{P}_f \), for \( \theta = 1, 3, 10 \) when sampling at three locations are discussed in the following. Uncertainty regarding conditions under the footing is a function of both distance to sampling location and correlation length. Three representative figures are shown in Figures 4.6, 4.7 and 4.8. The other plots can be seen in Appendix B.

Fig. 4.6 to Fig. 4.8 display how the probability of failure varies with \( V_c \) (and \( s_φ \)) and \( φ_θ \) when sampling in different locations. As can be seen from these figures, the probability of failure increases with increasing soil variability, as expected.
Figure 4.6 Effect of coefficient of variation, $V_c$, on probability of failure, $\hat{p}_f$, for $\theta = 3.0$, sampling directly under the footing.

Figure 4.7 Effect of coefficient of variation, $V_c$, on probability of failure, $\hat{p}_f$, for $\theta = 3.0$, sampling 4.5 m away from the footing center.
Figure 4.8  Effect of coefficient of variation, $V_c$, on probability of failure, $\hat{p}_f$, for $\theta = 3.0$, sampling 9 m away from the footing center.

Fig. 4.9 and Fig. 4.10 illustrates how the probability of failure varies with resistance factor, $\phi_g$. Only the worst case scale of fluctuation, $\theta = 3.0$, and two levels of soil variability, $V_c = 0.2$ ($s_\phi = 2$) and $V_c = 0.5$ ($s_\phi = 5$), are shown. The plots are representative.

Fig. 4.9 and 4.10 demonstrates that the influence of resistance factor, $\phi_g$, on probability of failure, $p_f$, is significant. The probability of failure $p_f$ increases with the increasing $\phi_g$. This is because smaller footings would be designed when using higher values of $\phi_g$. Therefore, there is a higher failure risk for footings designed using larger resistance factors than those designed using small resistance factors.

It is also observed that there is lower probability of failure when sampling directly under the footing (in the middle) comparing to sampling away from the footing (at the quarter and at the edge). This is as expected since better estimation of soil parameters can be obtained when the sampling location is close to the footing location.

By comparing Fig. 4.9 and Fig. 4.10 we can conclude that better sampling scheme can make a significant difference to the resistance factor when soil is highly variable. Hence, a smaller footing, having reduced expense, can be designed and constructed by improving the sampling scheme if significantly variable soil properties are expected.
Figure 4.9  Effect of resistance factor, $\phi_g$, on probability of failure, $\hat{p}_f$, for $V_c = 0.2, \theta = 3.0$.

Figure 4.10  Effect of resistance factor, $\phi_g$, on probability of failure, $\hat{p}_f$, for $V_c = 0.5, \theta = 3.0$. 
Figure 4.11  Effect of resistance factor, $\phi_g$, on probability of failure, $\hat{p}_f$, for $V_c = 0.5$, $\theta = 1.0$.

Note that in Fig. 4.11 there are a few cases where the probability of failure is slightly higher when sampling at 4.5 m than when sampling at 9 m, particularly for smaller scales of fluctuation. This is caused by the systematic bias of the variance field in two dimensions discussed in Section 3.3 that a regular pattern is clearly evident in two-dimensional Local Average Subdivision (LAS) variance field. The pattern in the variance field means that the variance at 4.5 m is consistently higher (by about 10-14\%) than at 9.0 m resulting in a slightly higher probability of failure at 4.5 m than at 9.0 m. This is verified theoretically shortly.

4.2  Comparison between Simulation and Theory

4.2.1  Cases with $\theta \to 0$ and $\theta \to \infty$

First the comparison of probability of failure for cases with $\theta \to 0$ and $\theta \to \infty$, between simulation and theory, are discussed here.
As mentioned earlier, the probabilities of failure estimated by theory when \( \theta \to 0 \) and \( \theta \to \infty \) are the same and depend only on the loading. These probabilities can be calculated as follows

\[
 p_f = P \left[ L > \frac{\alpha L_c}{\phi_g} \right] \tag{4.1}
\]

where \( L_c \) is the non-random characteristic load. If total load \( L = L_D + L_L \) is assumed to be lognormally distributed, \( \ln L \) is normally distributed. Then \( p_f \) can be calculated according to

\[
 p_f = 1 - P \left[ \ln L < \ln \left( \frac{\alpha L_c}{\phi_g} \right) \right] \tag{4.2}
\]

The normal distribution \( \ln L \sim N(\mu_{\ln L}, \sigma_{\ln L}^2) \) can be transformed to a standard normal distribution \( Z \sim N(0, 1) \) using \( Z = (\ln L - \mu_{\ln L})/\sigma_{\ln L} \). Then the probability of failure, \( p_f \), is

\[
 p_f = P \left[ Z < z \right] \tag{4.3}
\]

where \( z = \left( \ln \left( \frac{\alpha L_c}{\phi_g} \right) - \mu_{\ln L} \right)/\sigma_{\ln L} \) is called the standardized normal variate.

In the present study \( \mu_{\ln L} = 6.676 \text{ kN}, \sigma_{\ln L} = 0.135 \text{ kN} \), and the factored characteristic load \( \alpha L_c = 1310.924 \text{ kN} \). Then the standardized normal variate \( z \) can be computed for different resistance factors using these values. The results are listed in Table 4.1.

<table>
<thead>
<tr>
<th>( \phi_g )</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>10.51</td>
<td>8.86</td>
<td>7.51</td>
<td>6.36</td>
<td>5.38</td>
<td>4.50</td>
</tr>
</tbody>
</table>

Hence, using the normal distribution tables we can conclude that the probability of failure, \( p_f \), estimated from theory, when \( \theta \to 0 \) and \( \theta \to \infty \), are close to zero. Note that these results agree qualitatively with what is seen in Figures 4.3 to 4.5, in that these figures show \( p_f \) becoming very small as \( \theta \to 0 \) or \( \theta \to \infty \).

In simulation the dead load, \( L_D \), and live load, \( L_L \), are assumed to be individually lognormally distributed. Therefore, when \( \theta \to 0 \) and \( \theta \to \infty \), the probability of failure, \( p_f \), estimated from simulation is

\[
 p_f = P \left[ L_D + L_L > \frac{\alpha L_c}{\phi_g} \right] \tag{4.4}
\]

Although the sum of lognormals is not lognormal, we saw in Section 2.4 that the sum, \( L_D + L_L \), is approximately lognormal. Nevertheless, a simulation program was written
to estimate Eq. (4.4). In this program, lognormally distributed dead and live loads are simulated individually. 2000 realizations were generated for each of six resistance factors, $\phi_g = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$, and failure probabilities were estimated.

The estimated probabilities of failure obtained from this simulation were all zero. We conclude that Eq. (4.1) and Eq. (4.4) give similar results. This is consistent with the previous conclusion (see section 2.4) that the total load, as a sum of lognormally distributed dead and live loads, approximately follows a lognormal distribution.

When the correlation length, $\theta \to \infty$, the soil field becomes uniform field. Further analysis of $p_f$ estimated from simulation has been conducted by assuming cohesion, $c$, and friction angle, $\phi$, to be uniform fields ($\theta \to \infty$). In other words, single random variables are produced in the RFEM simulation instead of random fields. Since the effective (actual) and estimated (sampled) soil properties will be identical, no averaging is needed to estimate characteristic soil properties. The probability of failure again only depends on load variability. For the parameters considered in the present study and 2000 realizations, the estimated failure probabilities were all zero when uniform fields were used in the simulation.

Based on the uniform field simulation, and from the trends evident in Figures 4.3 to 4.5 we can conclude that there is good agreement between simulation and theory when $\theta \to 0$ and $\theta \to \infty$.

### 4.2.2 Cases with $\theta \in (0.1, 50.0)$

The simulated probability of failure and the predicted values from theory for different sampling locations are discussed as follows. Note that when sampling in the middle (directly under the footing), the estimated failure probabilities obtained from simulation are all zero for $V_c = 0.1, 0.2$ and $\phi_g = 0.4, 0.5$. Theory suggests that the probabilities of failure for these cases is very small, $p_f < 1 \times 10^{-4}$ or so. For such small probabilities, 2000 realizations is insufficient to achieve accurate probability estimates. Large relative errors would be expected from such a simulation for small probabilities of failure. For example, if the probability of failure is $p_f = 1 \times 10^{-4}$, then the standard error of its estimate, $\hat{p}_f$, is $\sigma_{\hat{p}_f} = \sqrt{p_f(1 - p_f)/n} = \sqrt{10^{-4} \times (1 - 10^{-4})/2000} = 2.2 \times 10^{-4}$, which is larger than $p_f$. Hence, the comparison of $p_f$ between simulation and theory will mostly concentrate on larger values of probability of failures, specifically, $p_f > 10^{-4}$ in the present study.

To check how the systematic errors in the variance field of LAS simulations (see Section 3.3) affects the probability of failure, $p_f$, the theory has been run with $\sigma_{in_c}^2 \pm 7\%$ (on average
the relative error on variance due to 2-D LAS is 7%) to estimate \( p_f \) when sampling at 4.5 m and 9.0 m. For \( \phi_g = 0.9, \ V_c = 0.3 \ (s_\phi = 3), \ \theta = 1.0 \), the probability of failure when using \( 1.07 \sigma_{in}^2 \) is 0.0402 when sampling at 4.5 m. Alternatively, the probability of failure when using \( 0.93 \sigma_{in}^2 \) drops to 0.0257. In other words, small changes in variance can make significant changes to these small probabilities, so small discrepancies are not surprising. Overall, however, the simulation results show behaviour that is as expected. More importantly the simulation results substantiate the theory, as we show next.

Representative comparison between the predicted probability of failure obtained from theory and the estimated probability of failure obtained via simulation are illustrated in Fig. 4.12 to 4.17. All the other figures can be found in Appendix B. It can be seen that the agreement in general is remarkably good, considering the fact that the averaging domain \( 2W \times W \) was rather arbitrarily selected. Note that different scales on the \( p_f \) axis are used in Fig. 4.12 to 4.17 since the probability magnitudes varied considerably.

In general the predicted results capture the shape of the probability of failure. The peak probabilities of failure, corresponding to the worst (intermediate) range of correlation length, agrees pretty well between simulation and theory. Both simulated results and predicted results converge consistently for small and large scale of fluctuation. The predicted probability of failure is seen to be conservative at small coefficient of variation by slightly overestimating the probability of failure.

Note that there are a few outliers in Figures 4.12, 4.13, and 4.15. This is due to several reasons. First, only a first order approximation to the mean using Taylor's series was used in the theory. For example, the characteristic value of the log-bearing capacity factor, \( \ln \hat{N}_c \), is calculated by

\[
\ln \hat{N}_c = g(\hat{\phi}) = \ln \frac{\tan^2(\frac{\pi}{4} + \frac{\hat{\phi}}{2}) \exp(\pi \tan \hat{\phi}) - 1}{\tan \hat{\phi}}
\]

Using Taylor's series \( \hat{N}_c \) can be written as

\[
\ln \hat{N}_c = g(\mu_\hat{\phi}) + (\hat{\phi} - \mu_\hat{\phi}) \frac{dg}{d\hat{\phi}} \bigg|_{\mu_\hat{\phi}} + \frac{1}{2}(\hat{\phi} - \mu_\hat{\phi})^2 \frac{d^2g}{d\hat{\phi}^2} \bigg|_{\mu_\hat{\phi}} + \cdots
\] (4.5)
Figure 4.12  Comparison of probability of failure, \( p_f \), between simulation and theory for \( \phi_g = 0.5 \), and sampling directly under the footing.

Figure 4.13  Comparison of probability of failure, \( p_f \), between simulation and theory for \( \phi_g = 0.7 \), and sampling directly under the footing.
Figure 4.14  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.9$, and sampling directly under the footing.

Figure 4.15  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.4$, and sampling 4.5 m away from the footing center.
Figure 4.16  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.7$, and sampling 4.5 m away from the footing center.

Figure 4.17  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.8$, and sampling 9 m away from the footing center.
The mean and variance of $\ln \hat{N}_c$ using a first order approximation are then

$$E[\ln \hat{N}_c] \approx g(\mu_\phi)$$  \hspace{1cm} (4.6a)

$$\text{Var}[\ln \hat{N}_c] \approx \text{Var}\left[\frac{dg}{d\phi}\right]^2$$  \hspace{1cm} (4.6b)

which leads to Eq. (2.25) and (2.33a).

Secondly, the very small probabilities, such as $p_f < 10^{-4}$, are very sensitive to things that 
not adequately modeled in the simplified theory. For example, the actual failure probability
(and thus the required resistance factor) depends on the distribution of the footing width,
$B$, which is unknown. We used a fixed width $B$, based on the mean soil properties. The 
simulation does not require such an assumption, so difference are expected.

Finally, 2000 realizations in simulation is insufficient to achieve accurate estimates of very 
small probabilities.

### 4.3 Summary

This chapter presented the results of probability of failure obtained from simulation and
compared these with the predicted values estimated from theory. The following summary

1) The worst case of scales of fluctuation, corresponding to higher probabilities of failure,
   are in the intermediate range of $\theta$, particularly, $1 < \theta < 10$.

2) The probability of failure, $p_f$, increases with soil variability, $V_c$ and $s_\phi$. It also increases
   significantly with increasing resistance factor, $\phi$.

3) As expected, the probabilities of failure are smaller when the soil is sampled under the
   footing than when sampled some distance away. This means that considerable savings
   can be achieved by improving the sampling scheme, especially, when significant soil
   variability exists.

4) The agreement between simulation and theory is very good. This implies that the
   first-order theory can be used to reliably estimate bearing capacity failure probabilities.

The last conclusion will be used in the following chapter to provide recommendations
regarding the ‘optimal’ resistance factors for certain target probabilities of failure.
Chapter 5

Recommendations for Load and Resistance Factor Design

As mentioned earlier, load factors and resistance factors are applied separately in Load and Resistance Factor Design (LRFD) to accommodate the individual uncertainties associated with loads and resistance. In this research, the load factors and load combinations are selected from the structural codes. The resistance factors will be estimated in this chapter to provide an acceptable probability of failure.

The comparison of the probability of failure between simulation and theory was conducted in the previous chapter. The agreement between simulation and theory is quite good. The theory will be used here to develop a set of resistance factors for bearing capacity design of shallow foundations.

The process considered for the development of resistance factors is to:

1) specify an acceptable reliability level. This specification is in the form of a target probability of failure, $p_{\text{max}}$;

2) plot the probability of failure, $p_f$, obtained from theory, against resistance factor, $\phi_g$, for various sampling locations and scales of fluctuation;

3) recommend the ‘optimal’ resistance factors for different sets of parameters by comparing $p_{\text{max}}$ to the plots of $p_f$;

4) compare the recommended resistance factors with values currently used in practice;

5) illustrate the recommendations via an example design of a shallow foundation using an ‘optimal’ resistance factor.
5.1 Target Probability of Failure

The utilization of Load and Resistance Factor Design (LRFD) involves one or more target probability failure levels, $p_{max}$. Different levels of $p_{max}$ may be selected to consider the “importance” of the structure. In turn, $p_{max}$ determines the magnitude of the resistance factor, $\phi_g$. The correct choice resistance factor ensures that the probability of failure of a footing does not exceed an acceptable level so that safety is maintained. This acceptable level is called acceptable risk which change with the severity of the consequences of the ‘failure’.

The choice of a target probability of failure, $p_{max}$, should consider the margin of safety implicit in current foundation designs and the levels of reliability for geotechnical design as reported in the literature. The values of target probability of failure for foundation designs are nearly the same or less than the requirement of concrete and steel structure designs because of the difficulties and high expense of foundation repairs. A literature review of the probability of failure for foundations is listed in Table 5.1.

**Table 5.1** Literature review of probability of failure for foundation design.

<table>
<thead>
<tr>
<th>Source</th>
<th>$p_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simpson et al. (1981)</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Barker et al. (1991)</td>
<td>$10^{-2} - 10^{-4}$</td>
</tr>
<tr>
<td>Becker (1996 a)</td>
<td>$10^{-3} - 10^{-4}$</td>
</tr>
</tbody>
</table>

As stated by Meyerhof (1995), “The order of magnitude of lifetime probabilities of stability failure is about $10^{-2}$ for offshore foundation, about $10^{-3}$ for earthworks and earth retaining structures, and about $10^{-4}$ for foundations on land.”

In the present study three target failure probabilities, $10^{-2}$, $10^{-3}$, and $10^{-4}$ are considered. These are appropriate for designs involving low, medium and high failure consequence structures, respectively. Resistance factors to achieve these target probabilities will be recommended.
5.2 Predicted Probability of Failure $p_f$ vs. Resistance Factor $\phi_g$

The theoretically predicted probabilities of failure, $p_f$, against resistance factors, $\phi_g$, are discussed in this section. The objective here is to show how predicted probabilities of failure varies with resistance factors. Three representative correlation length values, $\theta = 1.0, 3.0, 10.0$, and four coefficient of variation, $V_c = 0.1, 0.2, 0.3, 0.5$ and $s_\phi = 1, 2, 3, 5$, are illustrated for the three sampling locations, i.e., sampling directly under the footing, sampling 4.5 m away from the footing center, and sampling 9 m away from the footing center.

Three representative plots of $p_f$ vs. $\phi_g$ are illustrated in Figures 5.1 – 5.3. More figures of $p_f$ vs. $\phi_g$ can be seen in Appendix B.

Figures of $p_f$ vs. $\phi_g$, such as Fig. 5.1 – 5.3, can be used to determine ‘optimal’ resistance factors for use in the design of shallow foundations against bearing capacity failure. To do so, a horizontal line can be drawn across at the target probability of failure, $p_{max}$, as shown in Fig. 5.1, then a required resistance factor can be read for specific coefficient of variation and scale of fluctuation and a given sampling location. For example, assume the correlation length of soil parameters, $\theta$, is 3.0 and the coefficient of variation of cohesion, $V_c$, is 0.3. When the soil is sampled directly under the footing, a target failure probability, $p_{max} = 0.01$, yields a resistance factor, in Fig. 5.1, of 0.82.
Figure 5.1  Probability of failure, $p_f$, vs. resistance factor, $\phi_g$, for sampling directly under the footing, $\theta = 3.0$. 
Figure 5.2  Probability of failure, $p_f$, vs. resistance factor, $\phi_g$, for sampling 4.5 m away from the footing center, $\theta = 3.0$. 

![Graph showing the relationship between $p_f$ and $\phi_g$ for different $V_c$ values.](image-url)
Figure 5.3  Probability of failure, $p_f$, vs. resistance factor, $\phi_g$, for sampling 9 m away from the footing, $\theta = 3.0$. 
5.2.1 **Recommendations of Resistance Factors**

The recommended resistance factors corresponding to target probabilities of failure for the design of shallow foundations against bearing capacity failure are presented in Tables 5.3 to 5.5, where \( \theta \) is the correlation length of soil fields, \( V_c \) is the coefficient of variation of cohesion \( c \), and \( \phi_g \) is the resistance factor. In this research three values of \( p_{max} \) have been considered, i.e., \( 10^{-2}, 10^{-3}, 10^{-4} \), corresponding to low, medium, and high failure consequence foundations, respectively.

The resistance factors shown in tables 5.3 – 5.5 are obtained by interpolation. For example, for correlation length \( \theta = 3.0 \) and coefficient of variation \( V_c = 0.3 \) (\( s_\phi = 3 \)), the predicted theoretical probabilities of failure, when sampling directly under the footing, are listed in Table 5.2.

**Table 5.2** Listing of predicted probability of failure, \( p_f \), for \( \theta = 3.0 \), \( V_c = 0.3 \) (\( s_\phi = 3 \)), and when sampling under the footing.

<table>
<thead>
<tr>
<th>( \phi_g )</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_f )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.00010</td>
<td>0.00125</td>
<td>0.00720</td>
<td>0.02393</td>
</tr>
</tbody>
</table>

For a target probability of failure \( p_{max} = 0.001 \), the resistance factor can be computed by interpolating linearly between \( \phi_g = 0.6 \) and \( \phi_g = 0.7 \).

\[
\phi_g = \frac{0.001 - 0.00010}{0.00125 - 0.00010} \times 0.1 + 0.6 = 0.68
\]

As can be seen in Tables 5.3 – 5.5, the resistance factors when sampling directly under the footing are larger than when sampling further from the footing. This is as expected since there are more uncertainties when sampling at a distance from the footing. Accordingly, larger footings will be designed and constructed when sampling at a distance from the footing.

It also can be noticed that for smaller correlation lengths, such as \( \theta = 1.0 \), the same resistance factor is suggested regardless of whether sampling at 4.5 m or 9 m from the footing. This is because fields with smaller correlation length tend to be rough and are less correlated. For these fields sampling at distances greater than \( \theta \) away from the footing center makes little difference to the quality of the estimate of soil probabilities used in the bearing capacity prediction. In other words, if our sample is more than \( \theta \) away from the footing, our failure probabilities is not as much affected.
Table 5.3  Recommended resistance factors from theory when sampling under the footing.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$V_c$</th>
<th>$\phi_g$ $P_{max} = 0.01$</th>
<th>$\phi_g$ $P_{max} = 0.001$</th>
<th>$\phi_g$ $P_{max} = 0.0001$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>&gt; 0.90</td>
<td>&gt; 0.90</td>
<td>&gt; 0.90</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2</td>
<td>&gt; 0.90</td>
<td>0.84</td>
<td>0.74</td>
</tr>
<tr>
<td>1.0</td>
<td>0.3</td>
<td>0.87</td>
<td>0.73</td>
<td>0.63</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5</td>
<td>0.72</td>
<td>0.60</td>
<td>0.50</td>
</tr>
<tr>
<td>3.0</td>
<td>0.1</td>
<td>&gt; 0.90</td>
<td>&gt; 0.90</td>
<td>0.87</td>
</tr>
<tr>
<td>3.0</td>
<td>0.2</td>
<td>&gt; 0.90</td>
<td>0.81</td>
<td>0.71</td>
</tr>
<tr>
<td>3.0</td>
<td>0.3</td>
<td>0.82</td>
<td>0.68</td>
<td>0.60</td>
</tr>
<tr>
<td>3.0</td>
<td>0.5</td>
<td>0.65</td>
<td>0.51</td>
<td>0.41</td>
</tr>
<tr>
<td>10.0</td>
<td>0.1</td>
<td>&gt; 0.90</td>
<td>&gt; 0.90</td>
<td>&gt; 0.90</td>
</tr>
<tr>
<td>10.0</td>
<td>0.2</td>
<td>&gt; 0.90</td>
<td>0.87</td>
<td>0.78</td>
</tr>
<tr>
<td>10.0</td>
<td>0.3</td>
<td>&gt; 0.90</td>
<td>0.75</td>
<td>0.65</td>
</tr>
<tr>
<td>10.0</td>
<td>0.5</td>
<td>0.74</td>
<td>0.61</td>
<td>0.51</td>
</tr>
</tbody>
</table>

Table 5.4  Recommended resistance factors from theory when sampling 4.5 m away from the footing center.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$V_c$</th>
<th>$\phi_g$ $P_{max} = 0.01$</th>
<th>$\phi_g$ $P_{max} = 0.001$</th>
<th>$\phi_g$ $P_{max} = 0.0001$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>&gt; 0.90</td>
<td>&gt; 0.90</td>
<td>0.84</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2</td>
<td>&gt; 0.90</td>
<td>0.77</td>
<td>0.69</td>
</tr>
<tr>
<td>1.0</td>
<td>0.3</td>
<td>0.79</td>
<td>0.63</td>
<td>0.53</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5</td>
<td>0.61</td>
<td>0.47</td>
<td>0.40</td>
</tr>
<tr>
<td>3.0</td>
<td>0.1</td>
<td>&gt; 0.90</td>
<td>0.89</td>
<td>0.79</td>
</tr>
<tr>
<td>3.0</td>
<td>0.2</td>
<td>0.80</td>
<td>0.63</td>
<td>0.53</td>
</tr>
<tr>
<td>3.0</td>
<td>0.3</td>
<td>0.63</td>
<td>0.48</td>
<td>0.40</td>
</tr>
<tr>
<td>3.0</td>
<td>0.5</td>
<td>0.43</td>
<td>&lt; 0.40</td>
<td>&lt; 0.40</td>
</tr>
<tr>
<td>10.0</td>
<td>0.1</td>
<td>&gt; 0.90</td>
<td>&gt; 0.90</td>
<td>0.81</td>
</tr>
<tr>
<td>10.0</td>
<td>0.2</td>
<td>0.83</td>
<td>0.66</td>
<td>0.55</td>
</tr>
<tr>
<td>10.0</td>
<td>0.3</td>
<td>0.66</td>
<td>0.50</td>
<td>0.41</td>
</tr>
<tr>
<td>10.0</td>
<td>0.5</td>
<td>0.45</td>
<td>&lt; 0.40</td>
<td>&lt; 0.40</td>
</tr>
</tbody>
</table>
Table 5.5  Recommended resistance factors from theory when sampling 9 m away from the footing center.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$V_c$</th>
<th>$\phi_g$</th>
<th>$\phi_g$</th>
<th>$\phi_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$p_{max} = 0.01$</td>
<td>$p_{max} = 0.001$</td>
<td>$p_{max} = 0.0001$</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>$&gt; 0.90$</td>
<td>$&gt; 0.90$</td>
<td>0.84</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2</td>
<td>$&gt; 0.90$</td>
<td>0.77</td>
<td>0.69</td>
</tr>
<tr>
<td>1.0</td>
<td>0.3</td>
<td>0.79</td>
<td>0.63</td>
<td>0.53</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5</td>
<td>0.61</td>
<td>0.47</td>
<td>0.40</td>
</tr>
<tr>
<td>3.0</td>
<td>0.1</td>
<td>$&gt; 0.90$</td>
<td>0.87</td>
<td>0.77</td>
</tr>
<tr>
<td>3.0</td>
<td>0.2</td>
<td>0.79</td>
<td>0.62</td>
<td>0.52</td>
</tr>
<tr>
<td>3.0</td>
<td>0.3</td>
<td>0.62</td>
<td>0.45</td>
<td>$&lt; 0.40$</td>
</tr>
<tr>
<td>3.0</td>
<td>0.5</td>
<td>0.42</td>
<td>$&lt; 0.40$</td>
<td>$&lt; 0.40$</td>
</tr>
<tr>
<td>10.0</td>
<td>0.1</td>
<td>$&gt; 0.90$</td>
<td>0.85</td>
<td>0.73</td>
</tr>
<tr>
<td>10.0</td>
<td>0.2</td>
<td>0.74</td>
<td>0.58</td>
<td>0.46</td>
</tr>
<tr>
<td>10.0</td>
<td>0.3</td>
<td>0.56</td>
<td>0.41</td>
<td>$&lt; 0.40$</td>
</tr>
<tr>
<td>10.0</td>
<td>0.5</td>
<td>$&lt; 0.40$</td>
<td>$&lt; 0.40$</td>
<td>$&lt; 0.40$</td>
</tr>
</tbody>
</table>

However, for other larger correlation lengths, the resistance factors are larger when sampling 4.5 m away from the footing center than when sampling 9 m away from the footing center. This means that since the soil is highly correlated, the closer the sampling locations are to the footing, the better the prediction of soil properties and the larger the resistance factor. This is as expected since there is a better understanding of the soil by sampling closer to the footing. Hence, for highly correlated soil fields smaller footings are required when sampling locations are closer to the footing, corresponding to larger resistance factors. Accordingly, significant savings might be achieved by sampling closer to the footing.

Note that the resistance factors in Table 5.3 – 5.5 should be considered to be *upper bounds* due to the fact that measurement and model errors are not included. In other words, lower values of resistance factors would be expected by including the measurement and model error.

5.3 Comparison with the Current Resistance Factors

As mentioned in Chapter 2, the load and resistance factors are interrelated. To compare the recommended resistance factors, $\phi_g$, with values in the current codes and literature, the ratio of the resistance factor to the load factor will be developed and compared in this section.
The dead load factor, $\alpha_D = 1.25$, and live load factor, $\alpha_L = 1.5$, were selected according to National Building Code of Canadian (2005). The bias factors of $k_D = 1.18$ (Becker, 1996) and $k_L = 1.43$ (Allen, 1975) were conservatively chosen in this research. Using a dead to live load ratio $R_{D/L} = 3.0$, the combined load factor, $\alpha$, according to Eq. (2.10), is

$$\alpha = \frac{\alpha_D L_{de} + \alpha_L L_{Lc}}{L_e} = \frac{\alpha_D L_{de} + \alpha_L L_{Lc}}{L_{de} + L_{Lc}}$$

$$= \frac{\alpha_D k_D \mu_{Ld} + \alpha_L k_L \mu_{Ll}}{k_D \mu_{Ld} + k_L \mu_{Ll}} = \frac{\alpha_D k_D (3 \mu_{Ll}) + \alpha_L k_L \mu_{Ll}}{k_D (3 \mu_{Ll}) + k_L \mu_{Ll}}$$

$$= \frac{1.25(1.18)(3) + 1.5(1.43)}{1.18(3) + 1.43} = 1.32$$

(5.1)

where $L_{de}$ and $L_{Lc}$ are characteristic dead and live loads, and $\mu_{Ld}$ and $\mu_{Ll}$ are mean dead and live loads. For example, when sampling directly under the footing, the resistance factor recommended by the author is 0.68 when $\theta = 3.0$, $V_c = 0.3$, and $p_{max} = 0.001$. The ratio of resistance factor to load factor, $\phi_g/\alpha = 0.68/1.32 = 0.52$, as shown in Table 5.6.

**Table 5.6** Comparison of the resistance factors recommended in this study with the current code or literature values for shallow foundations, where $R_{D/L}$ is the dead to live load ratio.

<table>
<thead>
<tr>
<th>Source</th>
<th>Applied parameters</th>
<th>$\phi_g$</th>
<th>$\phi_g/\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling under the footing</td>
<td>$R_{D/L} = 3.0$, $\alpha_D = 1.25$, $\alpha_L = 1.5$</td>
<td>0.68$^1$</td>
<td>0.52</td>
</tr>
<tr>
<td>Sampling 4.5 m from the footing</td>
<td>$R_{D/L} = 3.0$, $\alpha_D = 1.25$, $\alpha_L = 1.5$</td>
<td>0.48$^1$</td>
<td>0.36</td>
</tr>
<tr>
<td>Sampling 9 m from the footing</td>
<td>$R_{D/L} = 3.0$, $\alpha_D = 1.25$, $\alpha_L = 1.5$</td>
<td>0.45$^1$</td>
<td>0.34</td>
</tr>
<tr>
<td>Foye et al. (2006)</td>
<td>$R_{D/L} = 1.0$, $\alpha_D = 1.2$, $\alpha_L = 1.6$</td>
<td>0.70</td>
<td>0.49</td>
</tr>
<tr>
<td>Canadian Foundation Engineering Manual (2006)</td>
<td>$R_{D/L} = 3.0$, $\alpha_D = 1.25$, $\alpha_L = 1.5$</td>
<td>0.5</td>
<td>0.38</td>
</tr>
<tr>
<td>Australian Standard Bridge Design (2004)</td>
<td>$R_{D/L} = 3.0^2$, $\alpha_D = 1.2$, $\alpha_L = 1.8$</td>
<td>0.45$^3$</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Footnotes to Table 5.6:

1. The resistance factors recommended in Table 5.6 are for $\theta = 3.0$, $V_c = 0.3$ ($s_\phi = 3$), and $p_{max} = 0.001$;
2. The mean dead and live load ratio is assumed to be 3.0 in Australian Standard for the analysis of the ratio of the load factor to the resistance factor in the present study;
3. The resistance factor is assumed to be based on CPT tests.
It can be seen from Table 5.6 that the ratio of the resistance factor to the combined load factor, $\phi_g/\alpha$, recommended in this research when sampling directly under the footing is close to the one given by Foye et al. (2006) and higher than the values from the current codes. The reason that the resistance factors proposed in the present study are higher than the values from the current codes might because measurement and model error have been included in the codes when estimating the resistance factors. For example, CPT tests were used for the estimation of the resistance factors suggested by Australian Standard Bridge Design (2004). In the present study these errors can be considered by using an error factor, $V_E$.

However, the code values correspond very well to those recommended in this research when samples are taken some distance away from the footing, which may also be what the codes are assuming.

From above comparison it can be concluded that even though the dead to live load ratio will be different for each structure, depending on the size of superstructures and the other factors, the resulting resistance factor is not all that sensitive to the load ratio. As also noticed by Foye et al. (2006), “the variations in dead load and live load ratio have little effect on the required value of resistance factor.”

### 5.4 Trial Design

Optimal resistance factors were suggested in the previous section. An example of the preliminary design of a strip footing, as shown in Fig. 5.4, founded on a weightless soil against bearing capacity failure (ultimate limit state) will be illustrated here. A comparison between the traditional approach – Working Stress Design (WSD) and Load and Resistance Factor Design (LRFD) using the ‘optimal’ resistance factors is as follows.

![Figure 5.4](image)

**Figure 5.4** Strip footing of width $B$ founded on a weightless soil

We assume that loads are transferred from the superstructure, having mean of dead load, $\mu_D = 600 \text{kN/m}$, and the mean of live load, $\mu_L = 200 \text{kN/m}$. Assume that the soil is sampled in three locations with results shown in Table 5.7.
Table 5.7 Sampled cohesion and internal friction angle.

<table>
<thead>
<tr>
<th>Soil Sample</th>
<th>Cohesion $c$ (kN/m$^2$)</th>
<th>Friction angle $\phi$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>91.3</td>
<td>18.8</td>
</tr>
<tr>
<td>2</td>
<td>101.7</td>
<td>19.3</td>
</tr>
<tr>
<td>3</td>
<td>113.2</td>
<td>22.1</td>
</tr>
<tr>
<td>Arithmetic Average</td>
<td>102.1</td>
<td>20.1</td>
</tr>
<tr>
<td>Geometric Average</td>
<td>101.7</td>
<td>20.0</td>
</tr>
<tr>
<td>Characteristic Value</td>
<td>100.0</td>
<td>20.0</td>
</tr>
</tbody>
</table>

Determine the design footing width, $B$, using a traditional working stress design (WSD) approach as well as a load and resistance factor design (LRFD) approach based on a total resistance factor, $\phi_g$.

First determine the characteristic values of cohesion and internal friction angle to be used in the design. The arithmetic average of the soil samples are:

$$c_A = \frac{1}{n} \sum_{i=1}^{n} c_i = \frac{1}{3}(91.3 + 101.7 + 113.2) = 102.1 \text{ kN/m}^2$$

$$\phi_A = \frac{1}{n} \sum_{i=1}^{n} \phi_i = \frac{1}{3}(18.8 + 19.3 + 22.1) = 20.1 \text{ degrees}$$

The geometric averages of the soil samples are

$$c_G = \left( \prod_{i=1}^{n} c_i \right)^{1/n} = (91.3 \times 101.7 \times 113.2)^{1/3} = 101.7 \text{ kN/m}^2$$

$$\phi_G = \left( \prod_{i=1}^{n} \phi_i \right)^{1/n} = (18.8 \times 19.3 \times 22.1)^{1/3} = 20.0 \text{ degrees}$$

The average values are shown in Table 5.7. Based on the limited soil samples, 100 kN/m$^2$ and 20° are chosen to be our characteristic design values, $c$ and $\phi$, respectively.

Using the characteristic values suggested in Table 5.7 in Prandtl’s formula (1921) the ultimate bearing stress capacity, $\hat{q}_u$, is

$$\hat{q}_u = \hat{c}\hat{N}_c = \hat{c} \left( \tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \exp \left( \pi \tan \hat{\phi} \right) - 1 \right) \tan \hat{\phi} = 100 \times 14.83 = 1483 \text{ kN/m}^2$$ \hspace{1cm} (5.2)

where $\hat{c}$ is the cohesion, $\hat{N}_c$ is the bearing capacity factor, and friction angle $\hat{\phi}$ was first converted to radians, $\phi = 20° \times \pi/180° = 0.349 \text{ rad}$. 
First the strip footing will be designed using the traditional Working Stress Design approach by applying a factor of safety, $F_s$, as follows.

It is probably reasonable to assume the characteristic dead load equal to the mean dead load since dead loads are relatively well defined and can be computed by multiplying volumes by mean densities, particularly, 600 kN/m. Live loads, on the other hand, are more difficult to characterize since there are higher variabilities and uncertainties. In this case we will assume that the mean of live load is the maximum value over the designed life-time of the structure and take it as the characteristic live load, specifically, 200 kN/m.

Assuming $F_s = 3$ (recommended by Canadian Foundation Engineering Manual, 1992), the allowable bearing capacity, $q_a$, is

$$q_a = \frac{q_u}{F_s} = \frac{1483}{3} = 494.33 \text{ kN/m}^2$$

The required footing width, $B$, is

$$B = \frac{\mu_D + \mu_L}{q_a} = \frac{600 + 200}{494.33} = 1.62 \text{ m}$$

Similarly, when $F_s = 2.5$ and 3.5, the required footing width are

$$q_a = \frac{q_u}{F_s} = \frac{1483}{2.5} = 593.20 \text{ kN/m}^2$$

$$B = \frac{\mu_D + \mu_L}{q_a} = \frac{600 + 200}{593.20} = 1.35 \text{ m}$$

and

$$q_a = \frac{q_u}{F_s} = \frac{1483}{3.5} = 423.71 \text{ kN/m}^2$$

$$B = \frac{\mu_D + \mu_L}{q_a} = \frac{600 + 200}{423.71} = 1.89 \text{ m}$$

respectively.

Next design a strip footing against bearing failure using the proposed Load and Resistance Factor Design method. The resistance factors, $\phi_g$, for $V_c = 0.3$, $\theta = 3.0$, and target probability of failure, $p_{max} = 0.001$, are listed as follows.

Table 5.8 Resistance factor, $\phi_g$, for $V_c = 0.3$ ($s_\phi = 3$), $\theta = 3.0$, and $p_{max} = 0.001$, as suggested in Section 5.2.1.

<table>
<thead>
<tr>
<th>Sampling Location</th>
<th>$\phi_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling directly under footing</td>
<td>0.68</td>
</tr>
<tr>
<td>&lt; 4.5 m away</td>
<td>0.48</td>
</tr>
<tr>
<td>&lt; 9.0 m away</td>
<td>0.45</td>
</tr>
</tbody>
</table>
The characteristic loads are chosen using bias factors of 1.18 and 1.43 for dead and live loads (Becker, 1996, Allen, 1975). The characteristic loads are thus

\[ L_{DC} = k_d \mu_d = 1.18(600) = 705.88 \text{ kN/m} \]
\[ L_{LC} = k_L \mu_L = 1.43(200) = 285.71 \text{ kN/m} \]

The LRFD design equation is

\[ \phi_g R = \alpha_D L_{DC} + \alpha_L L_{LC} \quad (5.3) \]

where the resistance \( R \) is

\[ R = B q_u \quad (5.4) \]

Substitute Eq. (5.4) to Eq. (5.3) and solve for the required design footing width, \( B \);

\[ B = \frac{\alpha_D L_{DC} + \alpha_L L_{LC}}{\phi_g q_u} \quad (5.5) \]

When soil samples are taken directly under the footing (\( \phi_g = 0.68 \)), so that we have a high understanding of the soil supporting the footing, our required footing width is

\[ B = \frac{\alpha_D L_{DC} + \alpha_L L_{LC}}{\phi_g q_u} = \frac{1.25 \times 705.88 + 1.5 \times 285.71}{0.68 \times 1483} = 1.30 \text{ m} \ (F_s \simeq 2.4) \]

When soil samples are taken approximately 4.5 m away from the footing center, so that we have an 'average' understanding of the soil, the resistance factor becomes \( \phi_g = 0.48 \) and

\[ B = \frac{\alpha_D L_{DC} + \alpha_L L_{LC}}{\phi_g q_u} = \frac{1.25 \times 705.88 + 1.5 \times 285.71}{0.48 \times 1483} = 1.84 \text{ m} \ (F_s \simeq 3.4) \]

When soil samples are taken approximately 9 m away from the footing center, so that we have a low understanding of the soil, then \( \phi_g = 0.45 \) and

\[ B = \frac{\alpha_D L_{DC} + \alpha_L L_{LC}}{\phi_g q_u} = \frac{1.25 \times 705.88 + 1.5 \times 285.71}{0.45 \times 1483} = 1.96 \text{ m} \ (F_s \simeq 3.6) \]

Note that if \( \phi_g = 0.5 \) (CFEM, 2006) is chosen, the required footing width \( B = 1.77 \text{ m} \ (F_s \simeq 3.3) \).
Chapter 6
Summary and Conclusions

6.1 Summary and Conclusions

This research studied reliability-based design, specifically, the Load and Resistance Factor Design (LRFD) of shallow foundations against bearing capacity failure. The load factors and load combinations are as used in structural codes. A probabilistic analysis of resistance factors of a strip footing subjected to ultimate limit state failure and founded on a spatially random soil field has been carried out to assess foundation reliability. The Random Finite Element Method (RFEM) was used in the analysis, in which random field theory is combined with a nonlinear finite element algorithm, and Monte Carlo simulation are employed to assess the probability of bearing capacity failure and compare to theoretical model.

A novel mathematical theory was developed to analytically estimate the probability of bearing capacity failure. The advantages offered by this theory are discussed as follows. First, a more realistic soil can be modeled by assuming a spatial random soil field. Second, more consistent reliability levels in shallow foundation design can be reached by probabilistic analysis of bearing capacity of soil. Finally, a consistent design philosophy between geotechnical and structural communities is obtained.

The theoretical model assumes a homogeneous random soil field with lognormally distributed cohesion $c$ and a bounded symmetrically distributed friction angle $\phi$. The weightless $c - \phi$ soil with both cohesion $c$ and friction angle $\phi$ greater than zero is assumed in this research. Nevertheless, we expect that the qualitative behaviour of $c = 0$ or $\phi = 0$ soils to be similar.

The effect of the soil’s spatial variability and site investigation intensity on the resistance factor has been investigated via simulation and theory by considering various soil statistics and sampling locations. The study carried out in simulation involves 2000 realizations for each set of parameters. The results from the Monte-Carlo simulations are compared to
the proposed theory. The ‘optimal’ resistance factors are recommended for the design of shallow foundations against bearing capacity failure for three target probabilities of failure.

The suggested design procedure using the proposed Load and Resistance Factor Design (LRFD) method are summarized as follows:

1) decide on a target probability of failure, \( p_{\text{max}} \), considering the consequence of failure of a single footing. Due to the higher risk of ultimate limit state failures, a lower target probability of failure would normally be considered when compared to serviceability problems;

2) estimate the characteristic soil properties using Eq. (2.12) and (2.13). The ultimate bearing capacity is calculated using the bearing capacity formula, as shown in Eq. (2.11);

3) determine load factors, \( \alpha \), and the load combinations from structural design codes as described previously in Chapter 3. The characteristic dead and live loads, \( L_{dc} \) and \( L_{le} \), are estimated according to Eq. (3.12) and (3.13);

4) select the resistance factor from tables (5.3) – (5.5) for given correlation length, \( \theta \), coefficient of variation, \( V_c \), target probability of failure, \( p_{\text{max}} \), and sampling locations;

5) estimate the footing size given \( \alpha, L_c, \phi_y, \) and \( q_u \) using LRFD, as described in this research.

This design process can be used in practice for the design of shallow foundations against bearing capacity failure with the recommended resistance factors in the present study. Note that the recommended resistance factors are used in a comparative fashion, i.e., relative to the existing resistance factors

The following specific conclusions arising from this study can be made:

1) probabilistic methods offer a rational way of approaching geotechnical analysis (through which probabilities of design failure can be assessed). Stochastic modeling of the soil allows traditional geotechnical engineering problems to be reassessed in a probabilistic light and reliability estimates to be made. The theory developed in this research is more realistic than the traditional design method, Working Stress Design (WSD), because three parameters, the mean, \( \mu \), the coefficient of variation, \( V \), and the correlation length, \( \theta \), have been considered for the modeling of random soil field. Comparatively, two statistical parameters, the mean and the coefficient of variation, are considered to describe the soil properties in the second moment probabilistic method (MacGregor, 1976, Cherubini, 2000).
2) recommended resistance factors for ultimate limit state design of shallow foundations against bearing capacity failure were given in Chapter 5 and also compared with the values currently used in practice. The ‘optimal’ resistance factors recommended in the present study should be considered to be upper bounds due to the fact that measurement and model errors have not been included. These errors can be considered by using an error factor, $V_E$, which will be left for additional study. The proposed resistance factors in Chapter 5 are based on the assumption of weightless soil ($c-\phi$ soil). This assumption is conservative since a higher bearing capacity would be expected by including the soil weight, which leads to lower resistance factors. To some extent, the influence of measurement and model errors and assumption of weightless soil can balance each other for the estimation of resistance factors.

3) it has been shown in Chapter 2 that the resistance factors are related to the load factors. This is as expected since with the increase of the load factor(s), the resistance factors increase towards one. The evaluation of resistance factor for foundation design involves the soil field’s uncertainty level (e.g. coefficient of variation, $V$) and correlation level (e.g. correlation length, $\theta$), and sampling locations. This knowledge suggests the requirement of better sampling schemes and appropriate determination of the soil field’s parameters and the correlation parameters for a given site.

4) since coefficients of variation, $V$, and correlation lengths, $\theta$, are usually unknown for a given site, various $V$ are considered in this study for shallow foundation design, along with a worse case range of $\theta$, i.e., the intermediate range corresponding to the higher probabilities of failure.

5) several sampling schemes have been considered in the present study. Better estimates of the soil field can be obtained when sampling directly under the footing than when sampling away from the footing center. Specifically, lower probabilities of failure and larger resistance factor values are obtained by sampling directly under the footing. Hence, preference should always be given to the soil properties sampled directly under the footing than elsewhere.
6.2 Future Work

There are several areas left for future work: This research concentrates on single sampling locations. Multiple sampling locations scheme will be left for further analysis. At each sampling location, the soil were sampled over the entire depth in one column, topics about numbers of sampled elements in each column can also be studied in the future.

In this research the soil properties are assumed to be isotropic fields with the same correlation length in both horizontal and vertical directions. The anisotropic case is left for future analysis. Anisotropy means the correlation between two points not only depends on their distance, but also depends on their orientation relative to each other. This would require different correlation length for different directions.

A similar research can also be processed to rectangular footings or deep foundations in the future. Then a three dimensional random field need to be analyzed instead of a two dimensional field in the current study.

There are two soil parameters considered random in this research, $c$ and $\phi$. The correlation between them was assumed to be independent. Additional study are required on the correlation of $c$ and $\phi$ in the future. Elastic modulus, $E$, dilation angle, $\psi$, and Poisson’s ratio, $\nu$, also needed to be considered as random fields for future work.
References


Appendix A

Estimated Probability of Failure, $\hat{p}_f$, from Simulation
A.1 Sampling Directly under the Footing

Note that in the following tables $\theta$ is the correlation length and $V_c$ is the coefficient of variation.

Table A.1 Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.4$ and sampling directly under the footing.

<table>
<thead>
<tr>
<th>$\hat{p}_f$</th>
<th>$\theta = 0.1$</th>
<th>$\theta = 1.0$</th>
<th>$\theta = 2.0$</th>
<th>$\theta = 3.0$</th>
<th>$\theta = 6.0$</th>
<th>$\theta = 10.0$</th>
<th>$\theta = 50.0$</th>
</tr>
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<tbody>
<tr>
<td>$V_c = 0.1$</td>
<td>0.0</td>
<td>0.0</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.2$</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.3$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.5$</td>
<td>0.0</td>
<td>0.0</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table A.2 Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.5$ and sampling directly under the footing.

<table>
<thead>
<tr>
<th>$\hat{p}_f$</th>
<th>$\theta = 0.1$</th>
<th>$\theta = 1.0$</th>
<th>$\theta = 2.0$</th>
<th>$\theta = 3.0$</th>
<th>$\theta = 6.0$</th>
<th>$\theta = 10.0$</th>
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<td>0.0</td>
<td>0.0</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.3$</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.5$</td>
<td>0.0</td>
<td>0.0015</td>
<td>0.0005</td>
<td>0.0005</td>
<td>0.0005</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table A.3 Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.6$ and sampling directly under the footing.

<table>
<thead>
<tr>
<th>$\hat{p}_f$</th>
<th>$\theta = 0.1$</th>
<th>$\theta = 1.0$</th>
<th>$\theta = 2.0$</th>
<th>$\theta = 3.0$</th>
<th>$\theta = 6.0$</th>
<th>$\theta = 10.0$</th>
<th>$\theta = 50.0$</th>
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<tr>
<td>$V_c = 0.1$</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.2$</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.3$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.001</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.5$</td>
<td>0.0</td>
<td>0.0055</td>
<td>0.0065</td>
<td>0.01</td>
<td>0.0015</td>
<td>0.0025</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table A.4 Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.7$ and sampling directly under the footing.

<table>
<thead>
<tr>
<th>$\hat{p}_f$</th>
<th>$\theta = 0.1$</th>
<th>$\theta = 1.0$</th>
<th>$\theta = 2.0$</th>
<th>$\theta = 3.0$</th>
<th>$\theta = 6.0$</th>
<th>$\theta = 10.0$</th>
<th>$\theta = 50.0$</th>
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<tbody>
<tr>
<td>$V_c = 0.1$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.2$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.3$</td>
<td>0.0</td>
<td>0.0005</td>
<td>0.0025</td>
<td>0.001</td>
<td>0.0005</td>
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<td>0.0</td>
</tr>
<tr>
<td>$V_c = 0.5$</td>
<td>0.0</td>
<td>0.0135</td>
<td>0.0295</td>
<td>0.031</td>
<td>0.0115</td>
<td>0.005</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Table A.5  Failure probabilities, \( \hat{p}_f \), for resistance factor \( \phi_y = 0.8 \) and sampling directly under the footing.

<table>
<thead>
<tr>
<th>( \hat{p}_f )</th>
<th>( \theta = 0.1 )</th>
<th>( \theta = 1.0 )</th>
<th>( \theta = 2.0 )</th>
<th>( \theta = 3.0 )</th>
<th>( \theta = 6.0 )</th>
<th>( \theta = 10.0 )</th>
<th>( \theta = 50.0 )</th>
</tr>
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<tr>
<td>( V_c = 0.1 )</td>
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<td>0.0</td>
<td>0.0</td>
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</tr>
<tr>
<td>( V_c = 0.2 )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0005</td>
<td>0.0</td>
<td>0.0005</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( V_c = 0.3 )</td>
<td>0.0</td>
<td>0.003</td>
<td>0.006</td>
<td>0.007</td>
<td>0.0025</td>
<td>0.003</td>
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</tr>
<tr>
<td>( V_c = 0.5 )</td>
<td>0.0</td>
<td>0.0305</td>
<td>0.0545</td>
<td>0.057</td>
<td>0.033</td>
<td>0.0245</td>
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</tr>
</tbody>
</table>

Table A.6  Failure probabilities, \( \hat{p}_f \), for resistance factor \( \phi_y = 0.9 \) and sampling directly under the footing.

<table>
<thead>
<tr>
<th>( \hat{p}_f )</th>
<th>( \theta = 0.1 )</th>
<th>( \theta = 1.0 )</th>
<th>( \theta = 2.0 )</th>
<th>( \theta = 3.0 )</th>
<th>( \theta = 6.0 )</th>
<th>( \theta = 10.0 )</th>
<th>( \theta = 50.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_c = 0.1 )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( V_c = 0.2 )</td>
<td>0.0</td>
<td>0.001</td>
<td>0.004</td>
<td>0.0035</td>
<td>0.003</td>
<td>0.0005</td>
<td>0.0</td>
</tr>
<tr>
<td>( V_c = 0.3 )</td>
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<td>0.016</td>
<td>0.018</td>
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<td>0.0125</td>
<td>0.006</td>
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<tr>
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<td>0.0765</td>
<td>0.0395</td>
<td>0.0015</td>
</tr>
</tbody>
</table>

A.2  Sampling 4.5 m away from the Footing Center

Table A.7  Failure probabilities, \( \hat{p}_f \), for resistance factor \( \phi_y = 0.4 \) and sampling 4.5 m away from the footing center.

<table>
<thead>
<tr>
<th>( \hat{p}_f )</th>
<th>( \theta = 0.1 )</th>
<th>( \theta = 1.0 )</th>
<th>( \theta = 2.0 )</th>
<th>( \theta = 3.0 )</th>
<th>( \theta = 6.0 )</th>
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<th>( \theta = 50.0 )</th>
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<tr>
<td>( V_c = 0.2 )</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( V_c = 0.3 )</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
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<td>0.0</td>
<td>0.0045</td>
<td>0.013</td>
<td>0.0155</td>
<td>0.0075</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table A.8  Failure probabilities, \( \hat{p}_f \), for resistance factor \( \phi_y = 0.5 \) and sampling 4.5 m away from the footing center.

<table>
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<tr>
<th>( \hat{p}_f )</th>
<th>( \theta = 0.1 )</th>
<th>( \theta = 1.0 )</th>
<th>( \theta = 2.0 )</th>
<th>( \theta = 3.0 )</th>
<th>( \theta = 6.0 )</th>
<th>( \theta = 10.0 )</th>
<th>( \theta = 50.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_c = 0.1 )</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<tr>
<td>( V_c = 0.2 )</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( V_c = 0.3 )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<td>0.001</td>
<td>0.0015</td>
<td>0.0</td>
</tr>
<tr>
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<td>0.026</td>
<td>0.028</td>
<td>0.017</td>
<td>0.001</td>
</tr>
</tbody>
</table>
Table A.9  Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.6$ and sampling 4.5 m away from the footing center.

<table>
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<tr>
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<tr>
<td>$V_e = 0.5$</td>
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<td>0.036</td>
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Table A.10  Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.7$ and sampling 4.5 m away from the footing center.

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<td>0.0715</td>
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Table A.11  Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.8$ and sampling 4.5 m away from the footing center.

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<td>0.102</td>
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Table A.12  Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.9$ and sampling 4.5 m away from the footing center.

<table>
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<td>0.0665</td>
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#### A.3 Sampling 9 m away from the Footing Center

**Table A.13** Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.4$ and sampling 9 m away from the footing center.

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<td>0.017</td>
<td>0.022</td>
<td>0.024</td>
<td>0.0005</td>
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</table>

**Table A.14** Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.5$ and sampling 9 m away from the footing center.

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<td>0.032</td>
<td>0.0395</td>
<td>0.0475</td>
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**Table A.15** Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.6$ and sampling 9 m away from the footing center.

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<td>0.0895</td>
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**Table A.16** Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.7$ and sampling 9 m away from the footing center.

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Table A.17  Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.8$ and sampling 9 m away from the footing center.

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Table A.18  Failure probabilities, $\hat{p}_f$, for resistance factor $\phi_g = 0.9$ and sampling 9 m away from the footing center.

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Appendix B

Failure Probability Plots
Figure B.1  Effect of coefficient of variation, $V_c$, on probability of failure, $\hat{p}_f$, for $\theta = 1.0$, sampling directly under the footing.

Figure B.2  Effect of coefficient of variation, $V_c$, on probability of failure, $\hat{p}_f$, for $\theta = 1.0$, sampling 4.5 m away from the footing center.
Figure B.3  Effect of coefficient of variation, $V_c$, on probability of failure, $\hat{p}_f$, for $\theta = 1.0$, sampling 9 m away from the footing center.

Figure B.4  Effect of coefficient of variation, $V_c$, on probability of failure, $\hat{p}_f$, for $\theta = 10.0$, sampling directly under the footing center.
**Figure B.5** Effect of coefficient of variation, $V_c$, on probability of failure, $\hat{p}_f$, for $\theta = 10.0$, sampling 4.5 m away from the footing center

**Figure B.6** Effect of coefficient of variation, $V_c$, on probability of failure, $\hat{p}_f$, for $\theta = 10.0$, sampling 9 m away from the footing center
Figure B.7 Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.6$, and sampling directly under the footing.

Figure B.8 Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.8$, and sampling directly under the footing center.
Figure B.9  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_y = 0.9$, and sampling directly under the footing center.

Figure B.10  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_y = 0.5$, and sampling 4.5 m away from the footing center.
Figure B.11  Comparison of probability of failure, \( p_f \), between simulation and theory for \( \phi_g = 0.6 \), and sampling 4.5 m away from the footing center.

Figure B.12  Comparison of probability of failure, \( p_f \), between simulation and theory for \( \phi_g = 0.7 \), and sampling 4.5 m away from the footing.
Figure B.13  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.8$, and sampling 4.5 m away from the footing center
Figure B.14  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.9$, and sampling 4.5 m away from the footing center.
Figure B.15  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.4$, and sampling 9 m away from the footing center.

Figure B.16  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.5$, and sampling 9 m away from the footing center.
Figure B.17  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_y = 0.6$, and sampling 9 m away from the footing center.
Figure B.18 Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.7$, and sampling 9 m away from the footing center.
Figure B.19  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.8$, and sampling 9 m away from the footing center.
Figure B.20  Comparison of probability of failure, $p_f$, between simulation and theory for $\phi_g = 0.9$, and sampling 9 m away from the footing center
Figure B.21  Probability of failure, $p_f$, vs. resistance factor, $\phi_g$, for sampling directly under the footing, $\theta = 1.0$
Figure B.22  Probability of failure, $p_f$, vs. resistance factor, $\phi_g$, for sampling directly under the footing, $\theta = 10.0$
Figure B.23  Probability of failure, $p_f$, vs. resistance factor, $\phi_g$, for sampling 4.5 m away from the footing center, $\theta = 1.0$
Figure B.24  Probability of failure, $p_f$, vs. resistance factor, $\phi_g$, for sampling 4.5 m away from the footing center, $\theta = 10.0$
Figure B.25  Probability of failure, $p_f$, vs. resistance factor, $\phi_g$, for sampling 9 m away from the footing center, $\theta = 1.0$
Figure B.26  Probability of failure, $p_f$, vs. resistance factor, $\phi_g$, for sampling 9 m away from the footing center, $\theta = 10.0$