GENERALIZED SUPREMUM-ENRICHED CATEGORIES AND THEIR SHEAVES

By
Dale Garraway

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "Generalized Supremum Enriched Categories and Their Sheaves" by Dale Garraway, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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External Examiner: [redacted]

Research Supervisor: [redacted]

Examining Committee: [redacted]

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Author: **Dale Garraway**

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Abstract

This work is an exploration of Supremum-enriched semicategory theory (quantaloids) and the relationship with sheaves. We begin with a review of some basic constructions and structures then introduce enriched semicategories and taxons. Next we define the category of sheaves for an involutive quantaloid $\mathcal{Q}$ and give an equivalence with $\mathcal{Q}$-valued sets. We close by showing that a sheaf is an infimum preserving semifunctor, $F : \mathcal{Q}^{co} \to \mathbf{REL}$.

Introduction

A sheaf on a topological space $(X, \mathcal{T})$ is a contravariant functor from the lattice of open sets to the category of sets ($F : \mathcal{T}^{op} \to \mathbf{SET}$) that satisfies a *patching* condition. The lattice of opens for a topological space is a complete Heyting algebra and the theory of sheaves for complete Heyting algebras is well understood. From a complete Heyting algebra $\mathcal{H}$, Higgs [10] constructed the category of $\mathcal{H}$-valued sets and showed that this category is equivalent to the category of sheaves on $\mathcal{H}$. For Higgs an $\mathcal{H}$-valued set is a *matrix* that takes its values in $\mathcal{H}$.

The Gelfand-Naimark theorem tells us that the category of locally compact Hausdorff spaces is equivalent to the category of commutative C^*-algebras. So in some sense the study of C^*-algebras is the study of noncommutative topologies. From a C^*-algebra $\mathcal{A}$ Mulvey[16] studied the lattice of closed linear subspaces of $\mathcal{A}$. This lattice is a one object supreiched *semicategory* (a *quantale*) which is a noncommutative generalization of complete Heyting algebras.

For a Grothendieck topos $\mathcal{E}$ the category of relations on $\mathcal{E}$ ($\mathbf{REL}(\mathcal{E})$) is a supremum-enriched category with extra structure. Pitts[20] called these *bounded complete distributive categories of relations* (*bcDCR*), and showed that the category with objects *bcDCR*'s and morphisms supremum preserving functors, is equivalent to the category of Grothendieck toposes. A DCR is said by Pitts to be complete if it has all coproducts and all *symmetric* idempotents split. A complete Heyting algebra $\mathcal{H}$ is a bounded DCR, and if we take the category of maps in the completion of $\mathcal{H}$ then what we have is the category of $\mathcal{H}$-valued sets.

In the past Borceux, Mulvey, Gylys et al, have studied $\mathcal{Q}$-valued sets for $\mathcal{Q}$ a quantale and more generally supremum-enriched semicategories (quantaloids). Using the work of Higgs as a guide these authors have defined a $\mathcal{Q}$-valued set to be a matrix, taking its values in $\mathcal{Q}$, satisfying a set of axioms.

In the present work we set out to answer two questions. What is the correct notion of presheaf and sheaf for a quantaloid? What is the relationship between $\mathcal{Q}$-valued sets, sheaves and semicategories?

To answer the first question we start with a short analysis of the work of Higgs. To show the equivalence between $\mathcal{H}$-valued sets and sheaves on $\mathcal{H}$, for $\mathcal{H}$ a complete Heyting algebra, Higgs used *singleton* $\mathcal{H}$-valued sets and their associated morphisms. Building on this we define the category of $\mathcal{Q}$-valued sets ($\mathcal{Q}$-SET) for $\mathcal{Q}$ a supremum-enriched semicategory, to be the completion (in the sense of Pitts) of $\mathcal{Q}$. The objects are idempotent matrices and the arrows are the left adjoint matrices in the idempotent splitting completion of the category of matrices of $\mathcal{Q}$. Of particular interest are the $\mathcal{Q}$-valued sets (X, ρ, δ) that are completely determined by the morphisms with codomain (X, ρ, δ) and domain a singleton $\mathcal{Q}$-valued set.

For $\mathcal{H}$ a complete Heyting algebra, thought of as a one object quantaloid, if we take the category whose objects are the idempotent arrows in $\mathcal{H}$ and arrows the left adjoints, we then recover $\mathcal{H}$ as a lattice. Applying this process to an *involutive* quantaloid $\mathcal{Q}$ we construct a category $\bar{\mathcal{Q}}$ and define the category of presheaves to be the category $SET^{\bar{\mathcal{Q}}^{op}}$. We then define a sheaf in this category to be a functor that has a unique *amalgamation* for every *matching family*. Generalising the work of Higgs we construct an associated sheaf functor that factors through the category $\mathcal{Q}$-SET. With this we can show that the category of sheaves on $\mathcal{Q}$ is equivalent to the category of $\mathcal{Q}$-valued sets for $\mathcal{Q}$ *pseudo-rightsided.*

To answer the second question we must begin with an exploration of enriched semicategory theory. For semicategories the notion of transformation is problematical since the usual notion of transformation is tied to the existence of identities. This leads to the idea that we should define transformations using the arrows. Unfortunately in the most general setting there is no natural way to define the composition of transformations. If we ask that the composition in a semicategory be a coequalizer then we can overcome the difficulties. Such a semicategory is called a *taxon* (Koslowski[12]). Our main interest is with taxons enriched in the monoidal categories **ORD**, **SUP** and **INF**. In these settings we can define lax-semifunctors and lax-transformations, the latter comes in two flavours: modular and strong.

A *relational presheaf* on a quantaloid $\mathcal{Q}$ is an infimum preserving lax-semifunctor, $F : \mathcal{Q}^{co} \to \mathbf{REL}$. We show that the category of relational presheaves and *modular* lax-transformations between them is equivalent to the semicategory of *modules* on the matrices of $\mathcal{Q}$. This result helps us show that for pseudo-rightsided supremum-enriched taxons, a sheaf is an infimum preserving semifunctor $F : \mathcal{Q}^{co} \to \mathbf{REL}$.

Since a **SUP**-enriched semicategory $\mathcal{Q}$ is *like* a bicategory we can talk about the category $\mathcal{Q}$-TAX. The objects are exactly the $\mathcal{Q}$-valued sets and the arrows are *$\mathcal{Q}$-semifunctors* defined in the obvious way. With a little work we can show that $\mathcal{Q}$-TAX is equivalent to the category $\mathcal{Q}$-SET for $\mathcal{Q}$ a pseudo-rightsided quantaloid. Thus $\mathcal{Q}$-TAX is equivalent to the category of sheaves on $\mathcal{Q}$.

We finish with a look at the Grothendieck construction of the category of elements for a relational presheaf. For $\mathcal{Q}$ a quantaloid we show that the category of relational presheaves and the modular lax-transformations between them is equivalent to the category whose objects are faithful supremum preserving semifunctors $L_1 : \mathcal{Q}_1 \to \mathcal{Q}$ ($\mathcal{Q}_1$ a quantaloid) satisfying Frobenius reciprocity, and a morphism $\tau : (\mathcal{Q}_1, L_1) \to (\mathcal{Q}_2, L_2)$ is a subprofunctor of $\mathcal{Q}(L_1-, L_2-)$. If a **SUP**-taxon $\mathcal{Q}$ is pseudo-rightsided, then if we restrict to the faithful semifunctors $F : \mathcal{Q}_1 \to \mathcal{Q}_2$ that satisfy $L_2F = L_1$. The resulting category then is equivalent to sheaves on $\mathcal{Q}$.

Chapter 1

Constructions and Structures

1.1 ∇-Categories and Matrices

1.1.1 ∇-Categories

It is well known that the free completion with respect to coproducts of a category $\mathcal{C}$ is the category $FAM(\mathcal{C})$. In many instances some of the structure that $\mathcal{C}$ may have is lost in $FAM(\mathcal{C})$.

Example 1.1.1 A category is said to be supremum-enriched if every hom set is a complete lattice with all suprema preserved by the composition. The category $\mathbf{1}$ admits a unique supremum-enrichment and $FAM(1) = \mathbf{SET}$ admits none. For if X is a non-empty set then $\mathbf{SET}(X, \emptyset) = \emptyset$, which underlies no complete lattice.

In this section we investigate a particular class of examples, which we call ∇-categories, and the construction of *matrices* on a ∇-category, which completes the category with respect to coproducts while maintaining the structure.

Definition 1.1.2 A ∇*-category* is a category $\mathcal{C}$ together with an operation

$$\overset{\alpha}{\nabla}: \mathcal{C}(A,B)^{\alpha} \longrightarrow \mathcal{C}(A,B)$$

for each set α, such that, for every $b : B \to B'$, $a : A' \to A$, and $(h_i : A \to B)_{i \in \alpha}$.

$$\overset{1}{\nabla} = 1_{C(A,B)} \tag{1.1}$$

$$b \circ \overset{\alpha}{\nabla}(h_i) = \overset{\alpha}{\nabla}(b \circ (h_i)) \tag{1.2}$$

$$\overset{\alpha}{\nabla}(h_i) \circ a = \overset{\alpha}{\nabla}((h_i) \circ a) \tag{1.3}$$

$$\overset{\beta}{\nabla} \circ \prod_{i\in\beta} \overset{\gamma_i}{\nabla} = \overset{\alpha}{\nabla} \text{ for } \sum_{i\in\beta} \gamma_i = \alpha \tag{1.4}$$

$$\overset{\alpha}{\nabla} = \overset{\alpha'}{\nabla} \text{ for any isomorphic sets } \alpha, \alpha'. \tag{1.5}$$

Such a category then has a distinguished element in every hom set $C(A, B)$ given by $\overset{\emptyset}{\nabla}$. We will refer to this element as $\perp$. Notice that for any arrow f in $C(A, B)$ we have $f\perp = \perp$, (since $\overset{0}{\nabla}(f \circ \perp) = \perp$) and $\overset{2}{\nabla}(f, \perp) = f$ (since $\overset{1}{\nabla} = \overset{2}{\nabla} \circ (\overset{0}{\nabla} \times \overset{1}{\nabla})$). A ∇-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, between ∇-categories is a functor that preserves ∇.

Example 1.1.3 Any supremum-enriched category is a ∇-category with $\nabla = \bigvee$. So as a special case a Heyting algebra with composition given by $\wedge$ is a ∇-category.

In a ∇-category if the coproduct of a family of objects exists then the resulting object is the product object as well, the proof of which follows the lines of Mac Lane [13].

Definition 1.1.4 For a family $\langle A_j \rangle_{j\in J}$ of objects in a ∇-category $\mathcal{C}$, a *J-biproduct* consists of a diagram

$$A_{j\in J} \underset{\rho_j}{\overset{\iota_j}{\rightleftarrows}} C$$

such that the following equations hold

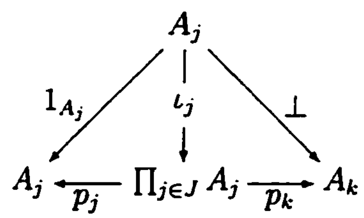
$$\rho_j \iota_k = \begin{cases} 1_j & \text{if } j = k \\ \perp & \text{otherwise} \end{cases} \tag{1.6}$$

$$\nabla(\iota_j \rho_j) = 1_C.$$

Theorem 1.1.5 For a family of objects $\langle A_j \rangle_{j \in J}$, in a ∇ category C, the following are equivalent.

1. The product of $\langle A_j \rangle_{j \in J}$ exists,
2. $\langle A_j \rangle_{j \in J}$ has a J-biproduct diagram,
3. The coproduct of $\langle A_j \rangle_{j \in J}$ exists.

Proof: Assume the product of $\langle A_j \rangle_{j \in J}$ exists and examine the following diagram.



where $j \neq k$. Clearly the first set of equations for a biproduct diagram holds when we set $\rho_j = p_j$.

$$\begin{aligned} p_k \overset{J}{\nabla} (\iota_j p_j) &= \overset{J}{\nabla} (p_k \iota_j p_j) \\ &= \overset{J}{\nabla} (p_k, \perp_j) && \text{from 1.6 above} \\ &= \overset{2}{\nabla} (p_k, \perp) && \text{since } \overset{\alpha}{\nabla} (\perp_{i \in \alpha}) = \perp \\ &= p_k. \end{aligned}$$

This implies that $\overset{J}{\nabla} (\iota_j p_j) = 1_{\prod A_j}$, thus 1 implies 2. Now assume we have a J-biproduct diagram and arrows $f_j : D \to A_j$. We have the arrow $\overset{J}{\nabla} (\iota_j f_j) : D \to C$ which satisfies $\rho_k \overset{J}{\nabla} (\iota_j f_j) = \overset{J}{\nabla} (\rho_k \iota_j f_j) = f_k$. Now if there is another arrow $h : D \to C$ such that $\rho_j h = f_j$ then we have

$$\begin{aligned} h &= \overset{J}{\nabla} (\iota_j \rho_j) h \\ &= \overset{J}{\nabla} (\iota_j \rho_j h) \\ &= \overset{J}{\nabla} (\iota_j f_j). \end{aligned}$$

Thus the arrows ρ_j together with the object C form a product for the family $\langle A_j \rangle_{j \in J}$. Thus 2 implies 1. By duality we have 2 if and only if 3. ∎

This tells us that if we complete a ∇-category $\mathcal{C}$, with respect to coproducts *as a ∇-category* then we will also be completing $\mathcal{C}$ with respect to products.

1.1.2 Matrices

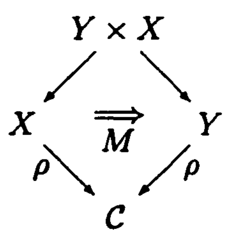
If we wish the completion, with respect to coproducts, of a ∇-category as a ∇-category then we need more than what $Fam(\mathcal{C})$ gives us. What turns out to be the correct notion is the category of matrices of $\mathcal{C}$, $Mat(\mathcal{C})$ (Pitts [20]).

Definition 1.1.6 For a ∇-category $\mathcal{C}$, define the category of matrices of $\mathcal{C}$, $MAT(\mathcal{C})$, as follows.

Objects: pairs (X, ρ) where X is a set and $\rho : X \longrightarrow \mathcal{C}$ is a functor.

Arrows: $M : (X, \rho) \longrightarrow (Y, \rho)$ is a function $M : Y \times X \longrightarrow arrows(\mathcal{C})$ such that $M(y, x) : \rho(x) \longrightarrow \rho(y)$.

In other words



is a natural transformation.

The composition of arrows $M : (X, \rho) \longrightarrow (Y, \rho)$ and $N : (Y, \rho) \longrightarrow (Z, \rho)$ is given by

$$NM(z, x) = \overset{Y}{\nabla}(N(z, y) \circ M(y, x)).$$

Theorem 1.1.7 $MAT(\mathcal{C})$ is a ∇-category with the identities given by the Kronecker delta and ∇ defined pointwise.

Proof: Clearly $\overset{1}{\nabla} M = M$. Since ∇ is defined pointwise $\overset{\alpha}{\nabla} = \overset{\alpha'}{\nabla}$ for any pair of isomorphic sets. So axiom (1.5) holds.

$$\begin{aligned}(\overset{\beta}{\nabla} \circ \coprod_{i\in\beta} \overset{\gamma_i}{\nabla} M_{\gamma_i})(x,y) &= \overset{\beta}{\nabla} \circ \coprod_{i\in\beta}(\overset{\gamma_i}{\nabla} M_{\gamma_i}(x,y)) \\ &= \overset{\alpha}{\nabla} M_j(x,y) \text{ for } j \in \alpha \text{ and } \coprod_i \gamma = \beta.\end{aligned}$$

So (1.4) holds. Now for (1.2) and (1.3)

$$\begin{aligned}(\overset{\alpha}{\nabla} M_i)N(z,x) &= \overset{Y}{\nabla}\overset{\alpha}{\nabla} \{M_i(z,y) \circ N(z,x)\} \\ &= \overset{\alpha}{\nabla}\overset{Y}{\nabla} \{M_i(z,y) \circ N(y,x)\} \text{ by 1.5} \\ &= \overset{\alpha}{\nabla} \{M_i N(z,x)\}.\end{aligned}$$

∎

Theorem 1.1.8 For C a ∇-category $MAT(C)$ has all coproducts.

Proof: For $(X_i, \rho_i)_{i\in I}$ the coproduct is given by

$$\coprod(X_i,\rho_i) \xleftarrow{\;\iota_i\;} (X_i,\rho_i)$$

where

$$\coprod(X_i,\rho_i) = (\coprod X_i, \coprod \rho_i)$$

and

$$\iota_i(a,b) = \begin{cases} 1_{X_i} & \text{if } a = b \\ \bot & \text{otherwise.}\end{cases} \tag{1.7}$$

∎

We will frequently be working in the context of *semicategories*. In that setting we will make use of the $Mat(C)$ construction, but without the existence of identities this need not have all coproducts. Here we have defined coproduct in terms of a universal cone.

There is a natural embedding, κ, of $\mathcal{C}$ into $MAT(\mathcal{C})$ which sends an object $C \in \mathcal{C}$ to the pair $(\{C\}, \rho)$, where $\rho(C) = C$. The embedding then sends an arrow $f : C \to C'$ to the matrix $(\{C\}, \rho) \stackrel{M_f}{\to} (\{C'\}, \rho)$, where $M_f(C', C) = f$. Since ∇ is defined pointwise in $MAT(\mathcal{C})$, the embedding, κ, preserves ∇.

Theorem 1.1.9 If there is a ∇-functor L from a ∇-category $\mathcal{C}$ to a ∇-category $\mathcal{D}$ with all coproducts, then there is an essentially unique coproduct preserving ∇-functor $L' : MAT(\mathcal{C}) \to \mathcal{D}$, such that $L'\kappa = L$.

Proof: L' sends an object (X, ρ) to the object $\coprod L(\rho(x))$ and a morphism M : $(X, \rho) \to (Y, \rho)$ to the unique morphism given by

$$\begin{array}{ccc} L(\rho(x)) & \xrightarrow{L(M(y,x))} & L(\rho(y)) \\ \uparrow & & \uparrow \\ \downarrow & & \\ \coprod L(\rho(x)) & \xrightarrow[L'(M)]{} & \coprod L(\rho(y)) \end{array}$$

The arrow $L'(M)$ is unique since $\coprod L(\rho(y))$ is also the product. In the same manner M is the unique arrow that we get from the trivial matrices $(\kappa(\rho(x)) \stackrel{\kappa(M(y,x))}{\longrightarrow} \kappa(\rho(y))$ in $MAT(\mathcal{C})$. So it follows that L' is essentially unique. ∎

1.2 Complete Heyting Algebras and Quantales

In this section we will explore the basic structure of *complete Heyting algebras* which generalise the lattice of opens for a topological space. We begin with an exploration of partial orders and lattices.

1.2.1 Orders and Lattice Theory

Definition 1.2.1 A *partial order* $\mathcal{O}$ is a set together with a transitive, reflexive and antisymmetric relation.

An equivalent way to define a partial order is as a category for which each *hom* set has at most one element and the only isomorphisms are the identity arrows.

Example 1.2.2 The power set of a set X is a partial order with the order given by subset inclusion.

Example 1.2.3 For a partial order $\mathcal{O}$ we create another partial order $D(\mathcal{O})$; where an element A of $D(\mathcal{O})$ is a subset of the elements of $\mathcal{O}$ that satisfies the condition that if $a \leq b \in A$ then $a \in A$. We call such sets down sets. There is an embedding of $\mathcal{O}$ into $D(\mathcal{O})$ given by sending an element a of $\mathcal{O}$ to the down set $\downarrow (a) = \{b \in \mathcal{O} \mid b \leq a\}$.

We will later see that this construction gives us the free *sup-lattice* on a partial order. A morphism of partial orders is an order preserving function (categorically a morphism is a functor). We denote the category of partial orders by **ORD**.

Definition 1.2.4 A *lattice* is a partial order with all finite suprema (also known as finite sups or joins) and finite infima (also known as finite infs or meets). The lattice is *complete* if it has all suprema. The category of *sup-lattices* has as its objects complete lattices and morphisms supremum-preserving functions. We denote this category by **SUP**.

When we think of a partial order as a category then it is a lattice if and only if it has all finite products and all finite coproducts. It is complete if it has all colimits. A morphism of sup-lattices is a functor that preserves colimits (a left adjoint order preserving functor).

Proposition 1.2.5 A partial order $\mathcal{O}$ is a sup-lattice if and only if the embedding $\downarrow: \mathcal{O} \to D(\mathcal{O})$ has a left adjoint.

Example 1.2.6 The open sets of a topological space form a sup-lattice with the order given by subset inclusion and supremum by union. Here, binary meet (intersection) preserves suprema in each variable separately (this will be our defining property of a *complete Heyting algebra*).

1.2.2 Complete Heyting Algebras

One of the properties that the category of sets has is that the union of sets distributes over the intersection $(\bigcup_i (A_i \cap B) = (\bigcup_i A_i) \cap B)$. This property also holds for the lattice of opens of a topological space. This naturally leads us to complete lattices with the added property that binary infima distribute over suprema and more generally to complete lattices with an associative binary operation that distributes over suprema.

Definition 1.2.7 A *complete Heyting algebra* $\mathcal{H}$ is a sup-lattice for which the map $- \wedge a : \mathcal{H} \to \mathcal{H}$, for each $a \in \mathcal{H}$, has a right adjoint $a \Rightarrow - : \mathcal{H} \to \mathcal{H}$.

Theorem 1.2.8 A sup-lattice $\mathcal{L}$ is a complete Heyting algebra if and only if for every $h, < h_i >_{i \in I}$ elements of $\mathcal{L}$ the following equation holds

$$\bigvee_{i \in I}(h \wedge h_i) = h \wedge (\bigvee_{i \in I} h_i).$$

Proof: For complete lattices X, A, $X \xrightarrow{f} A$ has a right adjoint if and only if it preserves suprema. So in particular this is true for $- \wedge a : \mathcal{H} \to \mathcal{H}$. ∎

A complete Heyting algebra is a one-object ∇-category with ∇ given by the supremum and composition given by meet.

Example 1.2.9 A complete boolean algebra is a complete Heyting algebra.

Example 1.2.10 The lattice of open sets of a topological space forms a complete Heyting algebra.

Example 1.2.11 For $\mathcal{O}$ a partial order, $D(\mathcal{O})$ is a complete Heyting algebra. We have binary meets given by intersection and suprema given by union. To show that this is Heyting we need that $A \cap (\bigcup_{i\in I} B_i)$ is equal to $\bigcup_{i\in I}(A \cap B_i)$, for down closed subsets A and $B_{i\in I}$ of $\mathcal{O}$.

$$\begin{aligned} x \in A \cap (\bigcup_{i\in I} B_i) &\Leftrightarrow x \in A \text{ and } x \in \bigcup_{i\in I} B_i \\ &\Leftrightarrow x \in A \text{ and } \exists_i \quad x \in B_i \\ &\Leftrightarrow x \in \bigcup_i (A \cap B_i). \end{aligned}$$

1.2.3 Quantales

The Gelfand-Naimark theorem tells us that the category of commutative C^*-algebras is equivalent to the category of locally compact Hausdorff spaces. This naturally leads to the question of what is the appropriate notion of a noncommutative topology and the associated lattice of opens. As an answer to this question Mulvey [16] introduced *quantales* as a non-commutative generalisation of complete Heyting algebras. These abandon the meet operation in favour of an arbitrary binary operation.

Definition 1.2.12 A *quantale* Q is a complete lattice together with an associative binary operation $\& : Q \times Q \to Q$ such that the maps $-\&a : Q \to Q$ and $b\&- : Q \to Q$ have right adjoints $- \Rightarrow a : Q \to Q$ and $b \Leftarrow - : Q \to Q$ respectively. Frequently we will write $q\&p$ as qp. If the quantale contains an element e that satisfies $e\&q = q = q\&e$ for every q in Q then we say that the quantale is *unital*.

Theorem 1.2.13 A complete lattice $\mathcal{L}$ with an associative binary operation is a quantale if and only if for q and $< q_i >_{i \in I}$ elements of $\mathcal{L}$ the following equations hold

$$\begin{aligned} \bigvee_{i \in I} (q \& q_i) &= q \& (\bigvee_{i \in I} q_i) \\ \bigvee_{i \in I} (q_i \& q) &= (\bigvee_{i \in I} q_i) \& q \end{aligned}$$

Proof: A morphism has a right adjoint if and only if it preserves the suprema. ∎

Of particular interest to us are quantales that come equipped with an *involution*. Many of the interesting examples are *involutive* and the involution will help us later in our analysis of sheaves.

Definition 1.2.14 An involutive quantale is a quantale $\mathcal{Q}$ together with a supremum preserving operation $()^* : \mathcal{Q}^{op} \rightarrow \mathcal{Q}$ (called the involution) such that for every q in $\mathcal{Q}$ we have $q^{**} = q$

Example 1.2.15 For a $C^* - algebra$ $\mathcal{A}$ we construct the quantale of closed linear subspaces $MAX(\mathcal{A})$ by taking the composite of subspaces A and B to be the subspace $A \& B = Closure(Span\{ab | a \epsilon A \text{ and } b \epsilon B\})$. $MAX(\mathcal{A})$ is unital if $\mathcal{A}$ is unital, and in that case the unit is given by $span\{1\}$. $MAX(\mathcal{A})$ is also involutive with the involution on a closed linear subspace X given by $X^* = \{x^* \mid x \in X\}$

1.3 Enriched Categories

The definition of *semicategory* is obtained by deleting from the definition of category the existence of identity arrows (and the equations they are required to satisfy). Thus semicategories with one object are precisely semigroups in the same way that categories with one object are monoids. For categories the notion of enrichment is well known, as also the notions of enriched functor and enriched natural transformation.

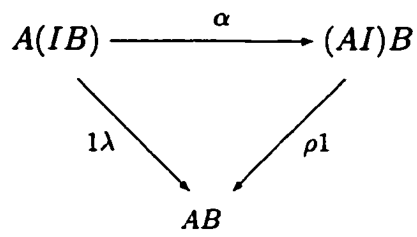
In this work we will need enriched semicategories. In particular we will need **SUP**-enriched semicategories. To motivate this recall that a quantale can be described as a **SUP**-enriched semigroup. To say that a quantale is unital is to say that it, together with its unit, is a **SUP**-enriched monoid.

It is not useful to merely adjoin a unit to a quantale that is not unital. Consider the quantale $MAX(\mathcal{A})$ for $\mathcal{A}$ a C^*-algebra. For commutative C^*-algebras the existence of the identity element encodes compactness, along the Gelfand-Naimark equivalence.

In considering enriched semicategories we are led to the counterparts of functor and of natural transformation. The latter requires some care. Accordingly we begin with a review of monoidal categories and categories enriched in them and focus our attention on two equivalent definitions of natural transformations.

Definition 1.3.1 A *monoidal category* $\mathcal{C}$ is a category together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (for which $A \otimes B$ may be abbreviated by AB), an object I, and natural isomorphisms $\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $\rho : A \otimes I \to A$ and $\lambda : I \otimes A \to A$ satisfying

$$\begin{array}{ccccc}
A(B(CD)) & \xleftarrow{\alpha} & (AB)(CD) & \xleftarrow{\alpha} & ((AB)C)D \\
{\scriptstyle 1\alpha}\uparrow & & & & \downarrow{\scriptstyle \alpha 1} \\
A((BC)D) & & \xleftarrow{\qquad\alpha\qquad} & & (A(BC))D
\end{array}$$



Example 1.3.2 The category of partial orders and order preserving functions, **ORD**, is a monoidal category. The tensor product is given by the product of sets with the partial order on the tensor defined piecewise.

Example 1.3.3 The category of complete lattices and supremum preserving functors, **SUP**, is a monoidal category. For complete lattices $\mathcal{Q}_1$ and $\mathcal{Q}_2$ their tensor product $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ is the set
$\{W \in D(\mathcal{Q}_1 \times \mathcal{Q}_2) \mid (\forall S \times T \in D(\mathcal{Q}_1 \times \mathcal{Q}_2))\ S \times T \subseteq W \text{ implies } (\bigvee S, \bigvee T) \in W\}$
[26].

Example 1.3.4 The category of **INF**-lattices and infimum preserving functors is a monoidal category. This category is denoted by **INF**. The functor $()^{op} : \mathbf{SUP} \to \mathbf{INF}$ is an involutive isomorphism of categories [19].

Definition 1.3.5 For $\mathcal{V}$ a monoidal category, a $\mathcal{V}$*-category* $\mathcal{C}$ consists of the following

1. a class of objects, $|\mathcal{C}|$
2. for every pair A, B of objects, an object $\mathcal{C}(A, B)$ of $\mathcal{V}$
3. for every triple A, B, C of objects, a morphism in $\mathcal{V}$, known as 'composition', $c_{ABC} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)$.
4. for every object A, a 'unit' morphism in $\mathcal{V}$, $c_A : I \to \mathcal{C}(A, A)$.

These morphisms must satisfy

$$\begin{array}{ccc}
(\mathcal{C}(A,B)\otimes\mathcal{C}(B,C))\otimes\mathcal{C}(C,D) & \xrightarrow{\mathcal{C}_{ABC}\otimes 1} & \mathcal{C}(A,C)\otimes\mathcal{C}(C,D) \\
\downarrow{\scriptstyle \alpha} & & \\
\mathcal{C}(A,B)\otimes(\mathcal{C}(B,C)\otimes\mathcal{C}(C,D)) & & \downarrow{\scriptstyle \mathcal{C}_{ACD}} \\
\downarrow{\scriptstyle 1\otimes\mathcal{C}_{BCD}} & & \\
\mathcal{C}(A,B)\otimes\mathcal{C}(B,D) & \xrightarrow[\mathcal{C}_{ABD}]{} & \mathcal{C}(A,D)
\end{array}$$

$$\begin{array}{ccccc}
I\otimes\mathcal{C}(A,B) & \xrightarrow{\lambda} & \mathcal{C}(A,B) & \xleftarrow{\rho} & \mathcal{C}(A,B)\otimes I \\
\downarrow{\scriptstyle \mathcal{C}_A\otimes 1} & & \downarrow{\scriptstyle 1_{\mathcal{C}(A,B)}} & & \downarrow{\scriptstyle 1\otimes\mathcal{C}} \\
\mathcal{C}(A,A)\otimes\mathcal{C}(A,B) & \xrightarrow[\mathcal{C}_{AAB}]{} & \mathcal{C}(A,B) & \xleftarrow[\mathcal{C}_{ABB}]{} & \mathcal{C}(A,B)\otimes\mathcal{C}(B,B)
\end{array}$$

Definition 1.3.6 For $\mathcal{V}$ a monoidal category, and $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$ a *$\mathcal{V}$-functor* $F:\mathcal{A}\to\mathcal{B}$ consists of the following

1. A function $F:|\mathcal{A}|\to|\mathcal{B}|$.

2. for every pair of objects A_1, A_2 in $\mathcal{A}$, a morphism $F_{A_1,A_2}:\mathcal{A}(A_1,A_2)\to\mathcal{B}(FA_1,FA_2)$ in $\mathcal{V}$, such that the following diagrams commute.

$$\begin{array}{ccc}
\mathcal{A}(A_1,A_2)\otimes\mathcal{A}(A_2,A_3) & \xrightarrow{\mathcal{A}_{A_1A_2A_3}} & \mathcal{A}(A_1,A_3) \\
\downarrow{\scriptstyle F_{A_1A_2}\otimes F_{A_2A_3}} & & \downarrow{\scriptstyle F_{A_1A_3}} \\
\mathcal{B}(FA_1,FA_2)\otimes\mathcal{B}(FA_2,FA_3) & \xrightarrow[\mathcal{B}_{FA_1FA_2FA_3}]{} & \mathcal{B}(FA_1,FA_3)
\end{array}$$

$$\begin{array}{ccc}
I & \xrightarrow{\mathcal{A}_A} & \mathcal{A}(A,A) \\
& {\scriptstyle \mathcal{B}_{FA}}\searrow & \downarrow{\scriptstyle F_{AA}} \\
& & \mathcal{B}(FA,FA)
\end{array}$$

Definition 1.3.7 For $\mathcal{V}$ a monoidal category, $\mathcal{A}$, $\mathcal{B}$ two $\mathcal{V}$-categories, and $F, G : \mathcal{A} \to \mathcal{B}$ two $\mathcal{V}$-functors an object-based *$\mathcal{V}$-natural transformation* $\gamma : F \to G$ consists of an $|\mathcal{A}|$ indexed family of arrows $\gamma_A : I \to \mathcal{B}(FA, GA)$ in $\mathcal{V}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{A}(A_1, A_2) \otimes I & \xrightarrow{F \otimes \gamma} & \mathcal{B}(FA_1, FA_2) \otimes \mathcal{B}(FA_2, GA_2) \\
\uparrow{\scriptstyle \rho^{-1}} & & \downarrow{\scriptstyle \mathcal{B}_{FA_1GA_2GA_3}} \\
\mathcal{A}(A_1, A_2) & & \mathcal{B}(FA_1, GA_2) \\
\downarrow{\scriptstyle \lambda^{-1}} & & \uparrow{\scriptstyle \mathcal{B}_{FA_1FA_2GA_3}} \\
I \otimes \mathcal{A}(A_1, A_2) & \xrightarrow[\gamma \otimes G]{} & \mathcal{B}(FA_1, GA_2) \otimes \mathcal{B}(GA_2, GA_2)
\end{array}
$$

Recall that for a category, there is an equivalent *arrows-based* definition of a natural transformation (Mac Lane[13] pg 19). This is also true in the more general setting of $\mathcal{V}$-enriched categories.

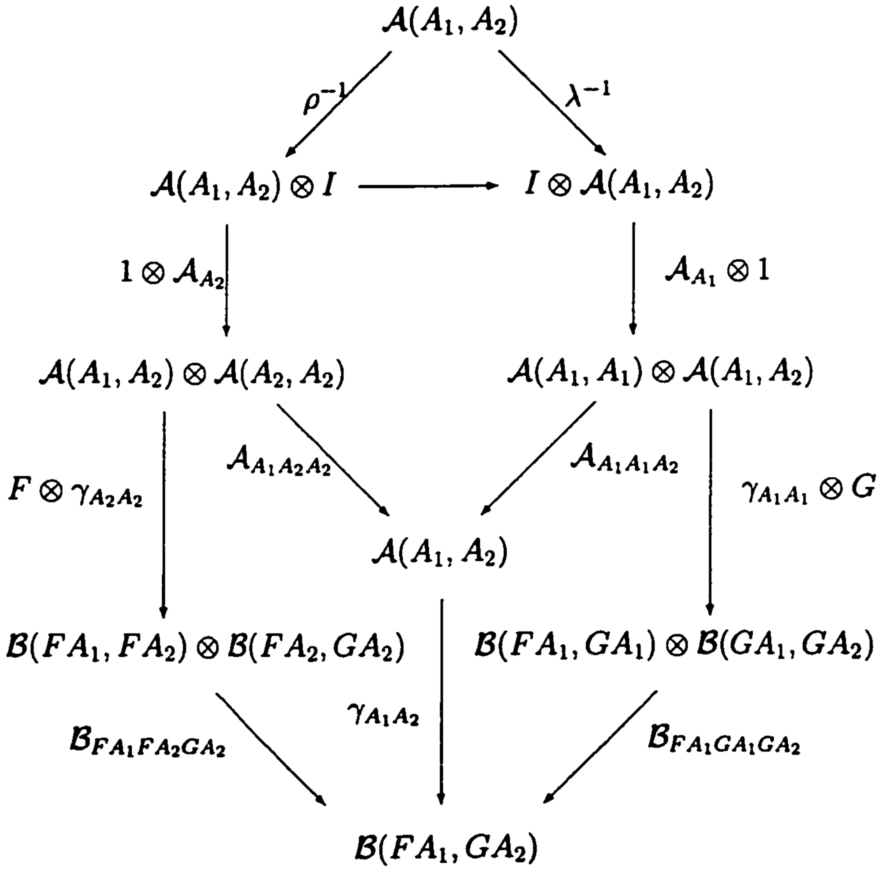
Definition 1.3.8 For $\mathcal{V}$ a monoidal category, $\mathcal{A}$, $\mathcal{B}$ two $\mathcal{V}$-categories, and $F, G : \mathcal{A} \to \mathcal{B}$ two $\mathcal{V}$-functors an *arrows-based $\mathcal{V}$-natural transformation* $\gamma : F \to G$ consists of an $|\mathcal{A}| \times |\mathcal{A}|$ indexed family of arrows $< \gamma_{A_1A_2} : \mathcal{A}(A_1, A_2) \to \mathcal{B}(FA_1, GA_2) >$ such that the following diagrams commute

$$
\begin{array}{ccc}
\mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_2, A_3) & \xrightarrow[F \otimes \gamma_{A_2A_3}]{} & \mathcal{B}(FA_1, FA_2) \otimes \mathcal{B}(FA_2, GA_3) \\
{\scriptstyle \mathcal{A}_{A_1A_2A_3}}\downarrow & & \downarrow{\scriptstyle \mathcal{B}_{FA_1FA_2GA_3}} \\
\mathcal{A}(A_1, A_3) & \xrightarrow{\gamma_{A_1A_3}} & \mathcal{B}(FA_1, GA_3)
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_2, A_3) & \xrightarrow[\gamma_{A_1A_2} \otimes G]{} & \mathcal{B}(FA_1, GA_2) \otimes \mathcal{B}(GA_2, GA_3) \\
{\scriptstyle \mathcal{A}_{A_1A_2A_3}}\downarrow & & \downarrow{\scriptstyle \mathcal{B}_{FA_1GA_2GA_3}} \\
\mathcal{A}(A_1, A_3) & \xrightarrow{\gamma_{A_1A_3}} & \mathcal{B}(FA_1, GA_3)
\end{array}
$$

Theorem 1.3.9 The two definitions are equivalent.

Proof: We start by describing how to construct an object-based $\mathcal{V}$-natural transformation from a given arrows-based $\mathcal{V}$-natural transformation $< \gamma_{A_1A_2} : \mathcal{A}(A_1, A_2) \to \mathcal{B}(FA_1, GA_2) >$. For an object $A \in \mathcal{A}$ define $\bar{\gamma}$ to be the family of arrows $< I \xrightarrow{\mu} \mathcal{A}(A, A) \xrightarrow{\gamma_{AA}} \mathcal{B}(FA, GA) >$. The following diagram chase shows us that this family is an object-based $\mathcal{V}$-natural transformation since the boundaries of the diagram represent the boundaries of the defining square.



Now given an object-based $\mathcal{V}$-natural transformation $< \gamma_A : I \to \mathcal{A}(A, A) >$ we define an arrows-based $\mathcal{V}$-natural transformation by

$$< \mathcal{A}(A_1, A_2) \xrightarrow{\rho} \mathcal{A}(A_1, A_2) \otimes I \xrightarrow{F \otimes \gamma_{A_2}}$$
$$\mathcal{B}(FA_1, FA_2) \otimes \mathcal{B}(FA_2, GA_2) \xrightarrow{\mathcal{B}} \mathcal{B}(FA_1 GA_2) >.$$

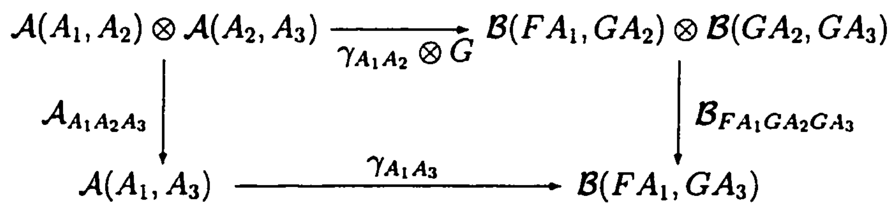
The following diagram chases give the defining squares for the arrows-based definition.

$$\begin{array}{ccc}
 & \mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_2, A_3) & \\
\swarrow{\scriptstyle \mathcal{A}_{A_1 A_2 A_3}} & & \searrow{\scriptstyle 1 \otimes \rho^{-1}} \\
\mathcal{A}(A_1, A_3) & & \\
\downarrow{\scriptstyle \rho^{-1}} & & \\
\mathcal{A}(A_1, A_3) \otimes I & \xleftarrow{\mathcal{A}_{A_1 A_2 A_3} \otimes 1} & \mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_2, A_3) \otimes I \\
\downarrow{\scriptstyle F \otimes \gamma_{A_3}} & & \downarrow{\scriptstyle F \otimes F \otimes \gamma_{A_3}} \\
\mathcal{B}(FA_1, FA_3) \otimes \mathcal{B}(FA_3, GA_3) & \xleftarrow{\mathcal{B}_{FA_1 FA_2 FA_3} \otimes 1} & \mathcal{B}(FA_1, FA_2) \otimes \mathcal{B}(FA_2, FA_3) \otimes \mathcal{B}(FA_3, GA_3) \\
\downarrow{\scriptstyle \mathcal{B}_{FA_1 FA_3 GA_3}} & & \downarrow{\scriptstyle 1 \otimes \mathcal{B}_{FA_2 FA_3 GA_3}} \\
\mathcal{B}(FA_1, GA_3) & \xleftarrow[\mathcal{B}_{FA_1 FA_3 GA_3}]{} & \mathcal{B}(FA_1, FA_3) \otimes \mathcal{B}(FA_3, GA_3)
\end{array}$$

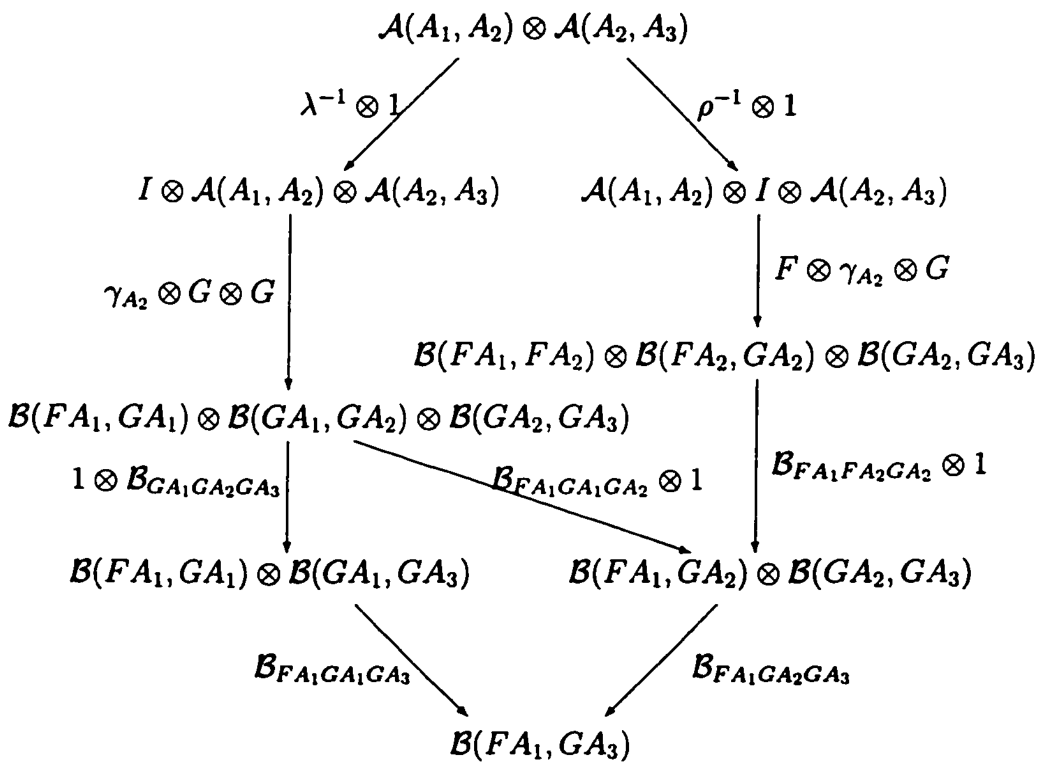
The boundaries of this diagram give us the following square which is the first of the two required in the arrows-based definition.

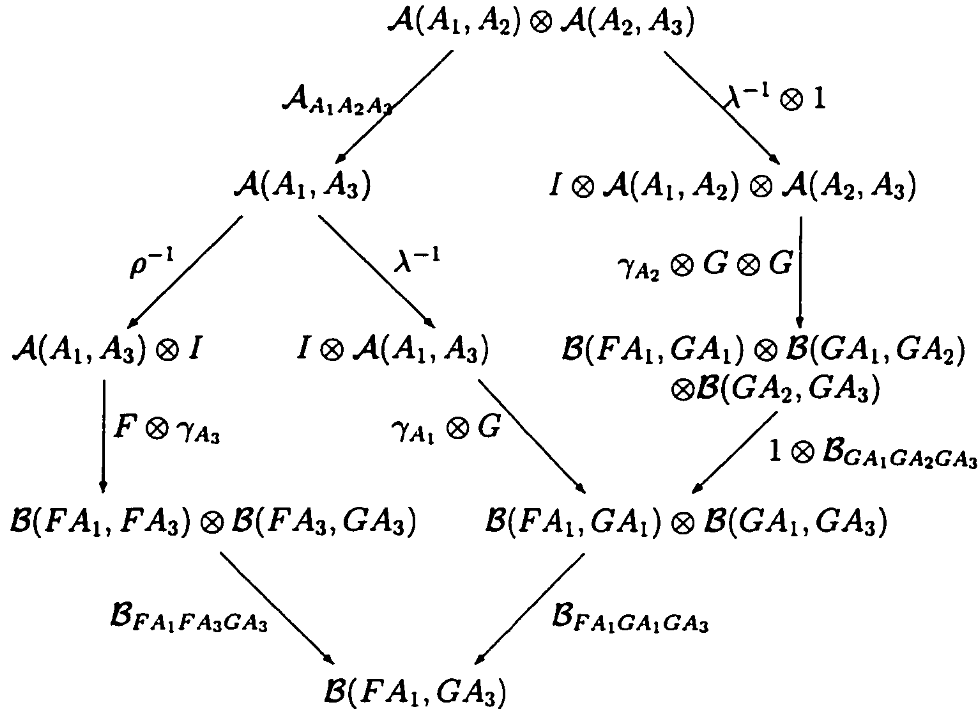
$$\begin{array}{ccc}
\mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_2, A_3) & \xrightarrow[F \otimes \gamma_{A_2 A_3}]{} & \mathcal{B}(FA_1, FA_2) \otimes \mathcal{B}(FA_2, GA_3) \\
\downarrow{\scriptstyle \mathcal{A}_{A_1 A_2 A_3}} & & \downarrow{\scriptstyle \mathcal{B}_{FA_1 FA_2 GA_3}} \\
\mathcal{A}(A_1, A_3) & \xrightarrow{\gamma_{A_1 A_3}} & \mathcal{B}(FA_1, GA_3)
\end{array}$$

The diagrams on the next two pages show that the second of the two required diagrams (below) in the arrows-based definition holds.



The left boundary of the first diagram (below) matches up with the right boundary of the second diagram (next page), and putting the two together gives us the required equality.

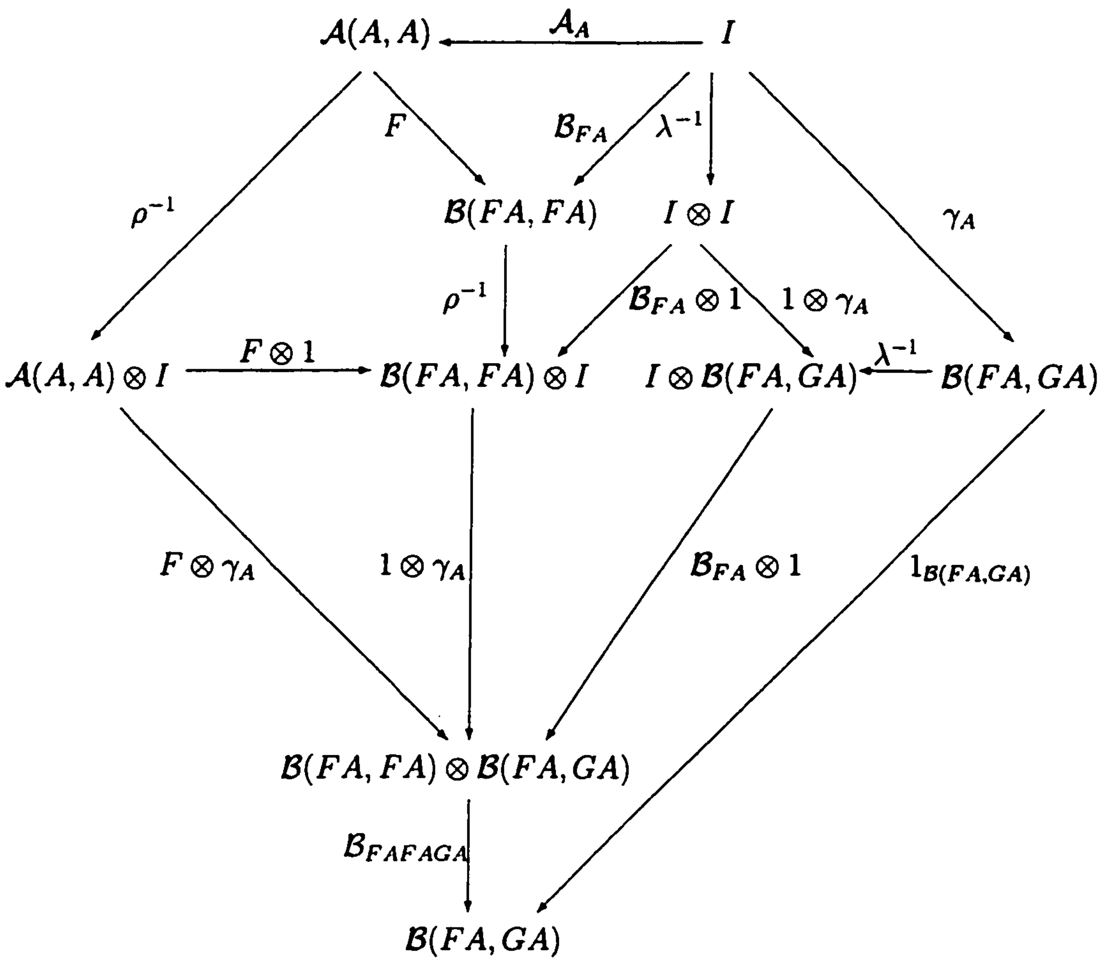




This shows us that we have an arrows-based $\mathcal{V}$-natural transformation. We now want to show that by applying the constructions twice we return to the original transformation. Assume that $< I \xrightarrow{\gamma_A} \mathcal{B}(FA, GA) >$ is an object-based $\mathcal{V}$-natural transformation. Converting into an arrows-based natural transformation and back again yields the object-based $\mathcal{V}$-natural transformation given by

$$< I \xrightarrow{A_A} \mathcal{A}(A, A) \xrightarrow{\rho^{-1}} \mathcal{A}(A, A) \otimes I \xrightarrow{F \otimes \gamma_A}$$
$$\mathcal{B}(FA, FA) \otimes \mathcal{B}(FA, GA) \xrightarrow{\mathcal{B}} \mathcal{B}(FA, GA) >.$$

The following diagram shows us that this family is the original family.



Finally given an arrows-based $\mathcal{V}$-natural transformation we construct a new arrows-based $\mathcal{V}$-natural transformation by first converting it into an object-based one and then converting back. The final diagram below shows us that the new arrows-based $\mathcal{V}$-natural transformation

$$< \mathcal{A}(A_1, A_2) \xrightarrow{\rho^{-1}} \mathcal{A}(A_1, A_2) \otimes I \xrightarrow{1 \otimes \mathcal{A}} \mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_2, A_2) \xrightarrow{F \otimes \gamma_{A_2 A_2}} \mathcal{B}(FA_1, FA_2) \otimes \mathcal{B}(FA_2, GA_2) \xrightarrow{\mathcal{B}} \mathcal{B}(FA_1, GA_2) >$$

is the same as the original one.

$$
\begin{array}{ccc}
 & \mathcal{A}(A_1, A_2) \otimes I & \\
 \swarrow^{\rho} & & \searrow^{1 \otimes \mathcal{A}_{A_2}} \\
\mathcal{A}(A_1, A_2) & \xleftarrow{\mathcal{A}_{A_1 A_2 A_2}} & \mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_2, A_2) \\
\Big\downarrow {\scriptstyle \gamma_{A_1 A_2}} & & \Big\downarrow {\scriptstyle F \otimes \gamma_{A_2 A_2}} \\
\mathcal{B}(FA_1, GA_2) & \xleftarrow[\mathcal{B}_{FA_1 FA_2 GA_2}]{} & \mathcal{B}(FA_1, FA_2) \otimes \mathcal{B}(FA_2, GA_2)
\end{array}
$$

So this gives us that the arrows-based and object-based definitions of $\mathcal{V}$-natural transformation are equivalent ∎

1.4 Enriched Semicategories and Taxons

The Gelfand-Naimark theorem tells us that the category of commutative C^*-algebras is equivalent to the category of locally compact Hausdorff spaces and that the category of compact Hausdorff spaces is equivalent to the category of unital C^*-algebra. This has led many to refer to non-commutative C^*-algebras as non-commutative topologies. Mulvey[16] extended the Gelfand-Naimark theorem to non-commutative C^*-algebras $\mathcal{A}$. For his work he used the associated quantale $MAX(\mathcal{A})$. This result indicates that we should work with structures that need not have a unit. This is problematical though, because the simple idea of dropping the unit from a category is insufficient when we wish to work with natural transformations. When we choose to work with an arrows-based definition of natural transformations there is no natural way to define their composite. In this section we will explore a class of structures, called *Taxons*, which overcomes this difficulty. These were introduced by Koslowski [12] in his work on *interpolads*.

Definition 1.4.1 For $\mathcal{V}$ a monoidal category a $\mathcal{V}$-*semicategory* consists of

1. a class of objects, $|\mathcal{C}|$
2. for every pair A, B of objects, an object $\mathcal{C}(A, B)$ of $\mathcal{V}$
3. for every triple A, B, C of objects, a morphism in $\mathcal{V}$, known as 'composition', $c_{ABC} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)$, such that for objects A, B, C, D the following diagram commutes

$$
\begin{array}{ccc}
(\mathcal{C}(A,B)\otimes\mathcal{C}(B,C))\otimes\mathcal{C}(C,D) & \xrightarrow{c_{ABC}\otimes 1} & \mathcal{C}(A,C)\otimes\mathcal{C}(C,D) \\
\downarrow{\scriptstyle \alpha} & & \\
\mathcal{C}(A,B)\otimes(\mathcal{C}(B,C)\otimes\mathcal{C}(C,D)) & & \downarrow{\scriptstyle c_{ACD}} \\
\downarrow{\scriptstyle 1\otimes c_{BCD}} & & \\
\mathcal{C}(A,B)\otimes\mathcal{C}(B,D) & \xrightarrow[c_{ABD}]{} & \mathcal{C}(A,D)
\end{array}
$$

Definition 1.4.2 For $\mathcal{V}$ a monoidal category, and $\mathcal{V}$-semicategories $\mathcal{A}$, $\mathcal{B}$ a $\mathcal{V}$-*semifunctor* $F : \mathcal{A} \to \mathcal{B}$ consists of the following

1. for every object A in $\mathcal{A}$ an object FA in $\mathcal{B}$.
2. for every pair of objects A_1, A_2 in $\mathcal{A}$, a morphism $F_{A_1A_2} : \mathcal{A}(A_1, A_2) \to \mathcal{B}(FA_1, FA_2)$ in $\mathcal{V}$, such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_2, A_3) & \xrightarrow{\mathcal{A}_{A_1A_2A_3}} & \mathcal{A}(A_1, A_3) \\ {\scriptstyle F_{A_1A_2} \otimes F_{A_2A_3}} \downarrow & & \downarrow {\scriptstyle F_{A_1A_3}} \\ \mathcal{B}(FA_1, FA_2) \otimes \mathcal{B}(FA_2, FA_3) & \xrightarrow[\mathcal{B}_{FA_1FA_2FA_3}]{} & \mathcal{B}(FA_1, FA_3) \end{array}$$

When $\mathcal{V}$ is the monoidal category **SET** we will refer to the $\mathcal{V}$-semicategories and $\mathcal{V}$-semifunctors simply as semicategories and semifunctors and denote the resulting category by **SCAT**. We saw that for $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$, and $\mathcal{V}$-functors $F, G : \mathcal{C} \to \mathcal{D}$ there are two equivalent definitions of natural transformations between F and G to choose from, the object-based and arrows-based definitions. The equivalence between the two was determined by how the arrows-based definition acts on the identity arrows. This indicates that the object-based definition is closely tied to the existence of identity arrows. So with their absence we should use the arrows-based definition. Unfortunately for semicategories composing natural transformations using the arrows-based definition is a problem. This difficulty is overcome using *taxons* (Koslowski[12]). This though does require that $\mathcal{V}$ have all coproducts.

Definition 1.4.3 *:* If the monoidal category $\mathcal{V}$ has all coproducts a $\mathcal{V}$-*taxon* is a $\mathcal{V}$-semicategory $\mathcal{C}$ for which the unique arrow $m : \coprod_X \mathcal{C}(A, X) \otimes \mathcal{C}(X, B) \longrightarrow \mathcal{C}(A, B)$, obtained from the universal property of coproducts and the arrows $\mathcal{C}_{AXB}$, is the coequalizer of

$$\coprod_{U,V} \mathcal{C}(A,U)\otimes\mathcal{C}(U,V)\otimes\mathcal{C}(V,B) \underset{1\circ m}{\overset{m\circ 1}{\rightrightarrows}} \coprod_X \mathcal{C}(A,X)\otimes\mathcal{C}(X,B)$$

where $1\circ m$ and $m\circ 1$ are also obtained from the universal property of coproducts in the obvious way.

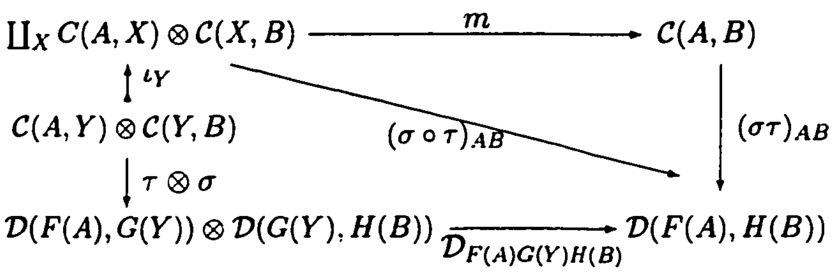
This is enough to allow us to define the composite of natural transformations for the arrows-based definition. For a monoidal category $\mathcal{V}$ we will denote the category of $\mathcal{V}$-taxons and $\mathcal{V}$-semifunctors by $\mathbf{TAX}_{\mathcal{V}}$ (The standard notation is $\mathcal{V}$-**TAX**, but for simplicity we prefer $\mathbf{TAX}_{\mathcal{V}}$). For the monoidal category **SET** we will denote the resulting category $\mathbf{TAX}_{\mathbf{SET}}$ by **TAX** and an object will be called a *taxon*. One of the consequences of the definition is that every arrow in a taxon $\mathcal{C}$ can be written as a composite.

Definition 1.4.4 For $\mathcal{V}$ a monoidal category, $\mathcal{A}$, $\mathcal{B}$ two $\mathcal{V}$-taxons, and $F,G:\mathcal{A}\to\mathcal{B}$ two $\mathcal{V}$-semifunctors a $\mathcal{V}$*-natural transformation* $\gamma : F \to G$ consists of an $|\mathcal{A}|\times|\mathcal{A}|$ indexed family of arrows $\langle \gamma_{A_1A_2} : \mathcal{A}(A_1,A_2)\to\mathcal{B}(FA_1,GA_2)\rangle$ such that the following diagrams commute

$$\begin{array}{ccc}
\mathcal{A}(A_1,A_2)\otimes\mathcal{A}(A_2,A_3) & \xrightarrow[F\otimes\gamma_{A_2A_3}]{} & \mathcal{B}(FA_1,FA_2)\otimes\mathcal{B}(FA_2,GA_3) \\
\Big\downarrow{\scriptstyle \mathcal{A}_{A_1A_2A_3}} & & \Big\downarrow{\scriptstyle \mathcal{B}_{FA_1FA_2GA_3}} \\
\mathcal{A}(A_1,A_3) & \xrightarrow{\gamma_{A_1A_3}} & \mathcal{B}(FA_1,GA_3)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{A}(A_1,A_2)\otimes\mathcal{A}(A_2,A_3) & \xrightarrow[\gamma_{A_1A_2}\otimes G]{} & \mathcal{B}(FA_1,GA_2)\otimes\mathcal{B}(GA_2,GA_3) \\
\Big\downarrow{\scriptstyle \mathcal{A}_{A_1A_2A_3}} & & \Big\downarrow{\scriptstyle \mathcal{B}_{FA_1GA_2GA_3}} \\
\mathcal{A}(A_1,A_3) & \xrightarrow{\gamma_{A_1A_3}} & \mathcal{B}(FA_1,GA_3)
\end{array}$$

For two $\mathcal{V}$-natural transformations $F \xrightarrow{\tau} G \xrightarrow{\sigma} H$, their composite $(\sigma\tau)_{AB}$ is defined to be the unique arrow, in the following diagram, satisfying $(\sigma\tau)_{AB} = (\sigma\circ\tau)_{AB}$.



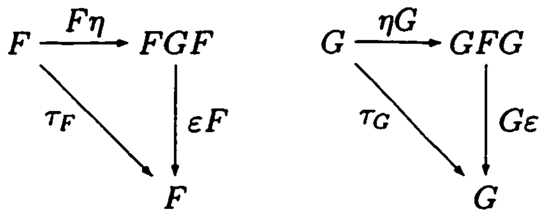
Here $(\sigma\circ\tau)_{AB}$, derived from the universal property of coproducts, coequalizes $1\circ m$ and $m\circ 1$. Observe that for a $\mathcal{V}$-semifunctor $F:\mathcal{C}\to\mathcal{D}$ there is an associated $\mathcal{V}$-natural transformation $\tau_F : F\to F$, defined by $\tau_{F_{A_1A_2}} = F_{A_1A_2}$ which is the identity $\mathcal{V}$-natural transformation on F.

In addition to the natural transformations τ and σ assume we have semifunctors $J,K,L:\mathcal{D}\to\mathcal{E}$, and natural transformations $J\xrightarrow{\psi}K\xrightarrow{\phi}L$. If we define the horizontal composite $(\phi\diamond\tau)_f$ to be the natural transformation given by the family $<\phi_{\tau_f}>$, then the interchange law holds.

$$\begin{aligned}
[(\phi\psi)\diamond(\sigma\tau)]_f &= (\phi\psi)_{\sigma\tau_f}\\
&= (\phi\psi)_{\sigma_g\tau_h} \text{ for every } gh=f\\
&= \phi_{\sigma_g}\psi_{\tau_h}\\
&= (\phi\diamond\sigma)_g(\psi\diamond\tau)_h\\
&= [(\phi\diamond\sigma)(\psi\diamond\tau)]_f
\end{aligned}$$

So we have that **TAX** together with $\mathbf{TAX}(\mathcal{A},\mathcal{B})$ and $\diamond$ is a 2-category.

Definition 1.4.5 For $\mathcal{C}$ and $\mathcal{D}$ taxons, an *adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is a quadruple $<F,G,\eta,\varepsilon>$ such that $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{C}$ are semifunctors and $\eta:\mathbf{1}\to GF$ and $\varepsilon:FG\to\mathbf{1}$ are natural transformations that satisfy the following:



These triangles say that for arrows $C \xrightarrow{f} C' \xrightarrow{g} C''$ in $\mathcal{C}$ and arrows $D \xrightarrow{r} D' \xrightarrow{s} D''$ in $\mathcal{D}$, the following equations hold

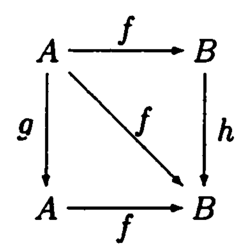
$$\varepsilon_{F(g)}F(\eta_h) = F(gh) \text{ and } G(\varepsilon_r)\eta_{G(s)} = G(rs)$$

For such an adjunction we say that F is left adjoint to G and G is right adjoint to F, and denote the adjunction by $F \dashv G$.

For $\mathcal{C}$ and $\mathcal{D}$ taxons and $F, G : \mathcal{C} \to \mathcal{D}$ semifunctors, a *natural isomorphism* is a transformation $\tau : F \to G$ such that there exists a transformation $\sigma : G \to F$ satisfying $\tau\sigma = \tau_G$ and $\sigma\tau = \tau_F$. We will usually denote σ by τ^{-1}.

Definition 1.4.6 Two taxons $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* if there exist semifunctors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathbf{1}_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to \mathbf{1}_{\mathcal{D}}$.

For a taxon $\mathcal{C}$ the idempotent splitting completion (also known as the Karoubian envelope) has objects idempotent arrows in $\mathcal{C}$ and arrows $f : g \to h$ satisfying the following:

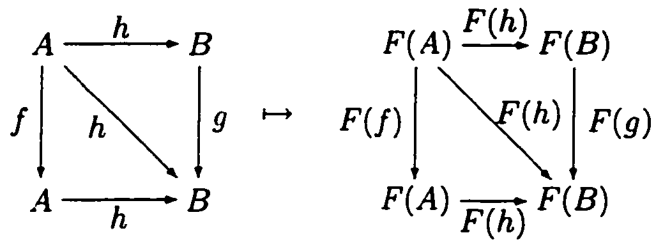


We can now show that the idempotent splitting completion, $KAR : \mathbf{TAX} \longrightarrow \mathbf{CAT}$ is a 2-functor and right adjoint to the inclusion of **CAT** into **TAX**.

Theorem 1.4.7

$$\mathbf{CAT} \underset{\iota}{\overset{KAR}{\underset{\top}{\leftrightarrows}}} \mathbf{TAX}$$

Proof: First we wish to show that KAR is functorial. For $\mathcal{C}$ and $\mathcal{D}$ taxons and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a semifunctor $KAR(F) : KAR(\mathcal{C}) \longrightarrow KAR(\mathcal{D})$ is defined by the mapping on the following square, which is an arrow in $KAR(\mathcal{C})$,



Clearly we have $KAR(GF)$ equal to $KAR(G)KAR(F)$. For $G : \mathcal{C} \longrightarrow \mathcal{D}$ another semifunctor and $\tau : F \longrightarrow G$ a natural transformation define $KAR(\tau) : KAR(F) \longrightarrow KAR(G)$ to be the natural transformation defined on an arrow $h : f \longrightarrow g$ in $KAR(\mathcal{C})$, by $KAR(\tau)_h = \tau_h$. For arrows h and k in $KAR(\mathcal{C})$ the following shows that $KAR(\tau)$ is a natural transformation from $KAR(F)$ to $KAR(G)$.

$$\begin{aligned} KAR(\tau)_k KAR(F)(h) &= \tau_k F(h) \\ &= \tau_{kh} \\ &= KAR(\tau)_{kh}. \end{aligned}$$

Similarly we have $KAR(G)(k)KAR(\tau)_k = KAR(\tau)_{kh}$. If we have a third semifunctor $H : \mathcal{D} \rightarrow \mathcal{E}$ and a natural transformation $\sigma : G \rightarrow H$ then clearly we have $KAR(\sigma\tau)_{fg}$ equal to $KAR(\sigma)_f KAR(\tau)_g$ so KAR is a 2-functor. It is well know that for a category there is an embedding of $\mathcal{C}$ into $KAR(\mathcal{C})$ given by taking $C \in \mathcal{C}$ to 1_C and an arrow f to f. We take this embedding to be the unit for our adjunction. For a taxon $\mathcal{C}$, the counit is defined to be the forgetfull arrow $\varepsilon_{\mathcal{C}} : KAR(\mathcal{C}) \rightarrow \mathcal{C}$. Note that $\varepsilon_{\mathcal{C}}$ is a genuine semifunctor, even for $\mathcal{C}$ a category, since it does not preserve identities. It is now clear that $(KAR\varepsilon)(\eta KAR)$ is the identity transformation from KAR to KAR while $(\varepsilon\iota)(\iota\eta)$ is the identity transformation on the inclusion ι. ∎

Observe that if $\mathcal{A}$ is a taxon the category $\mathbf{TAX}(\mathbf{1}, \mathcal{A})$, which has as its objects semifunctors and arrows transformations, is equivalent to the category $KAR(\mathcal{A})$. This follows since each semifunctor picks out an idempotent arrow in $\mathcal{A}$ while a transformation picks out an arrow in $\mathcal{A}$ that satisfies the two triangles. That is $\tau_1 F(1) = \tau_1$ and $G(1)\tau_1 = \tau_1$.

1.5 Order and Supremum-enriched Taxons

Recall that complete Heyting algebras are one-object **SUP**-enriched categories with composition given by meet while a quantale is a one-object **SUP**-enriched semicategory. A quantale is said to be unital if it has an identity morphism. Our particular interest is with the more general semicategories and taxons enriched in the monoidal categories, **ORD** and **SUP**. We will call a semicategory enriched in the category **SUP** a *quantaloid.*

Example 1.5.1 The category of relations is sup-enriched. This category has as its objects sets, morphisms relations and the supremum of relations is the union of sets. For any Grothendieck topos $\mathcal{E}$ the category of relations on $\mathcal{E}$ is **SUP**-enriched.

Example 1.5.2 For a semicategory $\mathcal{C}$ we can construct a **SUP**-enriched semicategory $\mathcal{PC}$, where the objects of $\mathcal{PC}$ are the same as in $\mathcal{C}$, and the arrows from A to B are the subsets of $\mathcal{C}(A, B)$. The composite of two arrows, $X \subseteq \mathcal{C}(A, B)$ and $Y \subseteq \mathcal{C}(B, C)$, is the subset $\{gf \mid f \in X,\ g \in Y\}$.

Rosenthal[21] showed that $\mathcal{P}$ is a functor from the category of locally small categories into the category of locally small **SUP**-enriched categories and that it is left adjoint to the forgetful functor. It turns out that the category of algebras for this adjunction is the category of locally small supremum-enriched categories.

Example 1.5.3 For an **ORD**-semicategory $\mathcal{O}$, we can construct a quantaloid $D(\mathcal{O})$, where the objects of $D(\mathcal{O})$ are the same as in $\mathcal{O}$, and the arrows from objects A to B are the downsets of $\mathcal{O}(A, B)$. For down sets $X \subseteq \mathcal{O}(A, B)$ and $Y \subseteq \mathcal{O}(B, C)$ the composition of X and Y is given by, $YX = \{yx | y \in Y,\ x \in X\}^{\downarrow}$ and the supremum is the union of sets. Here we write $S^{\downarrow} = \{x \mid \exists_y\ x \leq y \in S\}$ for the downset associated to an arbitrary subset of an ordered set. For x an element of an ordered set we will denote $\{x\}^{\downarrow}$ by $\downarrow x$. If $\mathcal{O}$ is involutive then $D(\mathcal{O})$ is also involutive. For X a downset the involution is given by setting $X^{\bullet} = \{x^{\bullet} \mid x \in X\}$.

The free **SUP**-enriched category $\mathcal{PC}$ of a category $\mathcal{C}$ is a special case of the following theorem, where each hom set of $\mathcal{C}$ is endowed with the trivial order.

Theorem 1.5.4 For $\mathcal{O}$ an **ORD**-taxon, $D(\mathcal{O})$ is the free **SUP**-enriched taxon of $\mathcal{O}$.

Proof: We have $D : \mathbf{TAX_{ORD}} \to \mathbf{TAX_{SUP}}$ given by

- • On Objects • For $\mathcal{O}$ an **ORD**-taxon, $D(\mathcal{O})$ has
  $|D(\mathcal{O})| = |\mathcal{O}|$
  $D(\mathcal{O})(A,B) = \{X \subseteq \mathcal{O}(A,B) \mid X \text{ is a down set }\}$
  with composition $XY = \{xy \mid x \in X, y \in Y\}^{\downarrow}$

- • On Arrows • For $\mathcal{O}_1 \xrightarrow{f} \mathcal{O}_2$, order preserving
  $Df(A) = \{f(a) \mid a \in A\}^{\downarrow}$.

We have that $\coprod_X \mathcal{O}(A,X) \times \mathcal{O}(X,B) \xrightarrow{m} \mathcal{O}(A,B)$ is a coequalizer. Now the arrow D : **ORD** $\to$ **SUP** is left adjoint to the inclusion, and thus preserves coequalizers and coproducts. Also it is known that for partial orders $\mathcal{O}_1$ and $\mathcal{O}_2$, $D(\mathcal{O}_1 \times \mathcal{O}_2) = D\mathcal{O}_1 \otimes D\mathcal{O}_2$. Thus it follows that $D\mathcal{O}$ is a **SUP**-taxon. We now show that $D \dashv U$, where U is the 2-functor that forgets suprema. The unit of the adjunction $\eta : \mathbf{1} \to UD$ is the natural transformation defined by:

$$\text{For } \mathcal{O}_1 \xrightarrow{f} \mathcal{O}_2 \text{ an order arrow}$$
$$\eta_f(x) = \downarrow f(x).$$

The counit of our adjunction $\varepsilon : DU \to \mathbf{1}$ is the natural transformation defined by:

$$\text{For } \mathcal{Q}_1 \xrightarrow{h} \mathcal{Q}_2 \text{ a supremum arrow}$$
$$\varepsilon_h(X) = \bigvee\{h(x) \mid x \in X\}.$$

Given $\mathcal{O}_1 \xrightarrow{f} \mathcal{O}_2 \xrightarrow{g} \mathcal{O}_3$ we have for X a down set of $\mathcal{O}_1$

$$\begin{aligned} \varepsilon_{Dg} D\eta_f(X) &= \varepsilon_{Dg}\{f(x) \mid x \in X\} \\ &= \bigcup_x \downarrow (gf(x)) \\ &= D_{gf}(X). \end{aligned}$$

Now for $\mathcal{Q}_1 \xrightarrow{h} \mathcal{Q}_2 \xrightarrow{k} \mathcal{Q}_3$ we have

$$\begin{aligned} U\varepsilon_k \eta_{Uh}(q) &= U\varepsilon_k(\downarrow h(q)) \\ &= \bigvee\{k(x) \mid x \leq h(q)\} \\ &= kh(q). \end{aligned}$$

And so we have that $D \dashv U$. ∎

1.5.1 Constructing ORD-Semicategories

In this section we will explore some ways to construct new **ORD**-enriched semicategories or **SUP**-enriched semicategories from given ones.

Earlier we defined ∇-categories to be categories together with a ∇ structure. In the obvious way we define ∇-semicategories to be semicategories together with a ∇ structure.

Example 1.5.5 If $\mathcal{Q}$ is a quantaloid then $MAT(\mathcal{Q})$ is a quantaloid. For matrices $X \xrightarrow{M,N} Y$ we say that $M \leq N$ if and only if $M(y,x) \leq N(y,x)$ for every $y \in Y, x \in X$. For a family of matrices $Y \xrightarrow{R_i} Z$, we take $\overset{I}{\nabla} R_{i \in I}$ to be the pointwise supremum. It is easy to show that this is the supremum.

Definition 1.5.6 For $\mathcal{O}$, an **ORD**-semicategory, the semicategory of *modules* on $\mathcal{O}$, $MOD(\mathcal{O})$ has

- Objects • f an endo arrow in $\mathcal{O}$
  such that
  $ff \leq f$

- Arrows • For $f : A \to A$ and $g : B \to B$ objects
  $r : f \to g$ is an arrow $r : A \to B$
  such that
  $rf \leq r$ and $gr \leq r$.

Rosenthal[22], et al, studied the category of *bimodules* on a **SUP**-enriched category $\mathcal{Q}$. This category of bimodules is the full subcategory of $MOD(MAT(\mathcal{Q}))$ determined by those matrices $\delta : (X, \rho) \to (X, \rho)$, for which $1_{(X,\rho)} \leq \delta$ in $MAT(\mathcal{Q})((X, \rho), (X, \rho))$. (These objects are what Walters[24] called $\mathcal{Q}$-categories, where $\mathcal{Q}$ is regarded as a bicategory). In our case we are interested in quantaloids that may not have identities. We will call an object of $MOD(MAT(\mathcal{Q}))$ a *$\mathcal{Q}$-semicategory* and denote it by (X, ρ, δ).

Definition 1.5.7 For $\mathcal{O}$ an **ORD**-semicategory, the semicategory of *right modules* consists of

- Objects • f an endo arrow in $\mathcal{O}$
  such that
  $ff \leq f$

- Arrows • For $f : A \to A$ and $g : B \to B$ objects
  $r : f \to g$ is an arrow $r : A \to B$
  such that
  $rf = r$ and $gr \leq r$.

It is easy to see that the semicategory of right modules $(RMOD(\mathcal{O}))$, for $\mathcal{O}$ an **ORD**-semicategory, is also an **ORD**-semicategory.

Recall that for $\mathcal{O}$ an **ORD**-semicategory the idempotent splitting completion (Karoubian envelope) $KAR(\mathcal{O})$ has

- Objects • f an endo arrow in $\mathcal{O}$
  such that
  $ff = f$

- Arrows • For $f : A \to A$ and $g : B \to B$ idempotents
  $r : f \to g$ is an arrow $r : A \to B$
  such that
  $rf = r = gr$.

Observe, for $\mathcal{Q}$ a **SUP**-semicategory, the arrow $\delta\delta \leq \delta$ in $MAT(\mathcal{Q})((X,\rho),(X,\rho))$ coequalizes the arrows $\leq \circ\delta, \delta\circ \leq : \delta\delta\delta \to \delta\delta$ (both of which are the arrow $\delta\delta\delta \leq \delta\delta$ in $MAT(\mathcal{Q})((X,\rho),(X,\rho)))$ if and only if $\delta\delta = \delta$. We will thus call an object of $KAR(MAT(\mathcal{Q}))$ a *$\mathcal{Q}$-taxon*.

Clearly for $\mathcal{O}$ an **ORD**-semicategory $KAR(\mathcal{O})$ is an **ORD**-semicategory. Similarly for $\mathcal{Q}$ a quantaloid, $MOD(\mathcal{Q}), RMOD(\mathcal{Q})$ and $KAR(\mathcal{Q})$ are quantaloids.

For a **SUP**-category $\mathcal{Q}$ we will frequently be working with the *left adjoints* in $\mathcal{Q}$. That is the elements $A \xrightarrow{p} B$ in $\mathcal{Q}$, such that there exists an arrow $B \xrightarrow{p^\#} A$ satisfying

$$\begin{aligned} 1_A &\leq p^\# p \\ pp^\# &\leq 1_B. \end{aligned}$$

In this case we denote p left adjoint to $p^\#$ by $p \dashv p^\#$. We will call a left adjoint p a *map* and write $MAP(\mathcal{Q})$ for the **SUP**-category with $|MAP(\mathcal{Q})| = |\mathcal{Q}|$ and the arrows the maps of $\mathcal{Q}$. We can expand this slightly to **ORD**-semicategories of the form $MOD(\mathcal{O})$ since each object comes equipped with a distinguished arrow although there may be no identities. For $\mathcal{O}$, an **ORD**-semicategory, a map in $MOD(\mathcal{O})$ is an arrow $p : f \to g$, together with an arrow $p^\# : g \to f$ such that

$$\begin{aligned} f &\leq p^{\#}p \\ pp^{\#} &\leq g. \end{aligned}$$

Denote the $\mathcal{O}$-semicategory of maps on $MOD(\mathcal{O})$ by $MAP(MOD(\mathcal{O}))$.

Lemma 1.5.8 For $\mathcal{Q}$ a **SUP**-category and $p : A \to B$ a map we have the following

1. $p = pp^{\#}p$
2. p is monomorphic if and only if $p^{\#}p = 1_A$
3. p is epimorphic if and only if $pp^{\#} = 1_B$
4. p is isomorphic if and only if it is both a monomorphism and an epimorphism.

Proof: For the first result we have

$$p = p1_A \leq pp^{\#}p \leq 1_B p = p.$$

Secondly, if p is a monomorphism then we have $p1_A = pp^{\#}p$ implying that $1_A = p^{\#}p$. The opposite direction is clear. By duality we have the third and fourth properties. ∎

1.5.2 Involution

Definition 1.5.9 For $\mathcal{V}$ a monoidal category a $\mathcal{V}$-semicategory $\mathcal{C}$ together with a $\mathcal{V}$-semifunctor $()^* : \mathcal{C}^{op} \to \mathcal{C}$ that is the identity on the objects and satisfies $()^*()^{*^{op}} \simeq 1_{\mathcal{C}}$ is an *involutive* $\mathcal{V}$-semicategory. A semifunctor $F : \mathcal{C} \to \mathcal{D}$ between two involutive $\mathcal{V}$-semicategories is involutive if it preserves the involution.

Denote the category of involutive $\mathcal{V}$-taxons and involutive semifunctors by ${}^*\mathbf{TAX}_{\mathcal{V}}$ (similarly for $\mathcal{V}$-semicategories). Unless otherwise stated we will assume that a semifunctor between involutive $\mathcal{V}$-semicategories preserves the involution.

For $\mathcal{C}$ a semicategory an endo arrow f is said to be *symmetric (self adjoint)* if $f = f^*$. Similarly for **ORD** and **SUP** enriched semicategories. Denote by ${}^*KAR(\mathcal{C})$ the full subcategory of $KAR(\mathcal{C})$ generated by the symmetric idempotents. Clearly ${}^*KAR(\mathcal{C})$ is involutive. This is because the objects are symmetric. This together with Theorem(1.4.7) gives us the following result.

Theorem 1.5.10 ${}^*KAR$: ***TAX** $\rightarrow$ ***CAT** is right adjoint to the inclusion of ***CAT** into ***TAX** (where ***CAT** is the category of involutive categories and involution preserving functors).

For $\mathcal{O}$ an involutive **ORD**-semicategory ${}^*MOD(\mathcal{O})$ is the full sub **ORD**-semicategory of $MOD(\mathcal{O})$ generated by the symmetric modules.

For $\mathcal{Q}$ an involutive **SUP**-semicategory, $MAT(\mathcal{Q})$ is involutive with the involution defined pointwise. For a matrix $M : (X, \rho) \rightarrow (Y, \rho)$ define $M^\circ : (Y, \rho) \rightarrow (X, \rho)$ by setting $M^\circ(x, y) = M(y, x)^*$. A *symmetric map* in a **SUP**-category is a map p such that $p^\# = p^*$. The category ${}^*MAP(\mathcal{Q})$ has objects $|{}^*MAP(\mathcal{Q})| = |\mathcal{Q}|$ and arrows the symmetric maps. In particular our interest is drawn towards those matrices which satisfy $M \dashv M^\circ$. If we interpret a matrix as a type of relation, then $M \dashv M^\circ$, tells us that M is 'functional'.

Theorem 1.5.11 For $\mathcal{Q}$ an involutive **SUP**-semicategory ${}^*MAP(\mathcal{Q})$ is a category.

Proof: For maps p and q, if $p \leq q$ then by adjunction we have $q^* \leq p^*$. But the involution preserves the order so we have $p^* \leq q^*$. Thus $p^* = q^*$ and so $p = q$. ∎

Example 1.5.12 The category of relations **REL** is involutive with the involution given by taking the opposite relation. This is the category ${}^*KAR(MAT(\mathcal{Q}))$, where $\mathcal{Q}$ is the Heyting algebra $\bot \leq \top$ and the involution is the identity on $\mathcal{Q}$. It is easy to see that $MAT(\mathcal{Q})$ is equivalent to **REL** and since all symmetric idempotents split in **REL** it follows that ${}^*KAR(MAT(\mathcal{Q}))$ is equivalent to **REL**.

Example 1.5.13 For a C^*-algebra $\mathcal{A}$ we can construct a quantaloid $\mathcal{Q}_{\mathcal{A}}$ by taking as our objects the sub C^*-algebras of $\mathcal{A}$. For two sub C^*-algebras A, B an arrow $M: A \to B$ is a closed linear subspace M of $\mathcal{A}$ such that

$$MA = Closure(span\{ma \mid m \in M, a \in A\}) = M$$

and similarly $BM = M$. Note that $\mathcal{Q}_{\mathcal{A}}$ is equivalent to the category $KAR(MAX(\mathcal{A}))$. The involution on $\mathcal{A}$ makes this an involutive quantaloid. That is for X, a closed linear subspace of $\mathcal{A}$, X^* is the set $\{x^* \mid x \in X\}$.

Definition 1.5.14 An involutive quantaloid $\mathcal{Q}$ satisfies *Freyd's law of modularity* if for every triple of arrows $f : A \to B$, $g : B \to C$, and $h : A \to C$ we have $gf \wedge h \leq g(f \wedge g^*h)$.

Example 1.5.15 For $\mathcal{H}$ a Heyting algebra, $\mathcal{H}$ taken as a one object involutive quantaloid satisfies Freyd's modular law. Since composition is the meet and involution is the identity this is just the inequality $g \wedge f \wedge h \leq g \wedge (f \wedge g \wedge h)$.

Example 1.5.16 For a group G if we take as objects the sub groups of G and as arrows $F : A \to B$ the subsets of G that satisfy $FA = \{fa | f \epsilon F \text{ and } a \epsilon A\} = F$ and $BF = F$ then we have a quantaloid $\mathcal{Q}_G$. Note that $\mathcal{Q}_G$ is ${}^*KAR(\mathcal{P}(G))$, where $\mathcal{P}(G)$ is the quantale of subsets of G. This is an involutive quantaloid with the involution given by inverse. That is for $X \subseteq G$ we have $X^* = \{x^{-1} \mid x \in X\}$. This satisfies Freyd's law of modularity.

Example 1.5.17 The category of relations satisfies Freyd's modular law. Given relations $R : X \to Y$; $S : Y \to Z$ and $T : X \to Z$ assume that $z(SR \wedge T)x$. This means that zTx and there is a y in Y such that $zSyRx$. So we have $y(R \wedge S^{\circ}T)x$ and zSy which gives us $zS(R \wedge S^{\circ}T)x$.

Definition 1.5.18 An involutive quantaloid $\mathcal{Q}$ is said to be *pseudo-rightsided* if for every $q \in \mathcal{Q}$ we have $qq^*q \leq q$ implies that $qq^*q = q$.

Lemma 1.5.19 For $\mathcal{Q}$ an involutive quantaloid, if for every $q \in \mathcal{Q}$, $q \leq q\top$ and if $\mathcal{Q}$ satisfies Freyd's modular law then $\mathcal{Q}$ is pseudo-rightsided.

Proof: Let $q : A \to B$ be an arrow of $\mathcal{Q}$, and assume $\mathcal{Q}$ satisfies Freyd's law of modularity. Then we have $q\top \wedge q \leq q(\top \wedge q^*q)$. This is just $q \leq qq^*q$, so if $qq^*q \leq q$ then equality immediately follows. ∎

We would like to say that for a C^*-algebra $\mathcal{A}$, the quantale $MAX(\mathcal{A})$ is pseudo-rightsided. Unfortunately we have not been able to show that this is true though we strongly suspect that it is the case. The reason for this is that for A a closed linear subspace of $\mathcal{A}$, if $AA^*A \leq A$ then $AA^*AA^* \leq AA^*$. This implies that AA^* is a sub C^*-algebra of $\mathcal{A}$ and thus has an aproximate unit constructed out of elements of A and A^*. So it seems reasonable to expect that if $AA^*A \leq A$ then we can recover all of A from the aproximate unit.

Lemma 1.5.20 If $\mathcal{Q}$ is a pseudo-rightsided quantaloid then $^*MAP(^*RMOD(\mathcal{Q})$ is equivalent to $^*MAP(^*KAR(\mathcal{Q}))$.

Proof: Assume q is an object in $^*RMOD(\mathcal{Q})$. Then we have $qqq \leq qq \leq q$, hence q is a symmetric idempotent. Now if $p : q_1 \to q_2$ is an arrow in $^*MAP(^*RMOD(\mathcal{Q})$ then we have $pp^*p = p$ and so $p = pp^*p \leq q_2p \leq p$ implying that p is an arrow in $^*MAP(^*KAR(\mathcal{Q}))$. The result now follows. ∎

Lemma 1.5.21 If q is an involutive **SUP**-category satisfying Freyd's law of modularity then $p \dashv q$ implies that $q = p^*$.

Proof: First

$$pq = pqq^*q \leq q^*q \quad \text{and} \quad q^*p^* = q^*p^*pp^* \leq pp^*.$$

Using these we now apply Freyd twice. We first show that $q \leq p^*$.

$$\begin{aligned} q &= 1q \\ &= p^*q^*q \wedge q \\ &\leq p^*(q^*q \wedge pq) \\ &= p^*pq \\ &\leq p^*. \end{aligned}$$

Now to show that $p^* \leq q$

$$\begin{aligned} p^* &= 1p^* \\ &= qpp^* \wedge p^* \\ &\leq q(pp^* \wedge q^*p^*) \\ &= qq^*p^* \\ &\leq q. \end{aligned}$$

Thus we have $q = p^*$. ∎

Example 1.5.22 Pitts[20] studied a class of **SUP**-enriched categories called distributive categories of relations(DCR). For a DCR there is an involution which is derived from other data and satisfies Freyd's law of modularity. For $\mathcal{Q}$ a DCR, $\mathcal{Q}$ is said by Pitts to be complete if all symmetric idempotents split and if it has all coproducts. The completion of $\mathcal{Q}$, in this sense, is given by $KAR(MAT(\mathcal{Q}))$. $\mathcal{Q}$ is bounded if it has a set of generating objects. Denote the category of bounded complete DCRs by $bcDCR$. Expanding on the work of Carboni-Walters[6], Pitts showed that if $\mathcal{Q}$ was bounded and complete then $MAP(\mathcal{Q})$ is a Grothendieck topos and if $\mathcal{E}$ is a Grothendieck topos then $REL(\mathcal{E})$ (the category of relations on $\mathcal{E}$) is a bcDCR. Pitts also showed that $MAP : bcDCR \to GTOP^{op}$ and $REL : GTOP^{op} \to bcDCR$ are functors such that $MAP \dashv REL$ is an equivalence.

1.5.3 Lax-Semifunctors and Transformations

One of our main goals is to explore the relationship between the constructions of the previous section and laxity. In this section we will explore what this means in the world of **ORD** and **SUP** enriched semicategories. (We will also be interested in laxity for semicategories enriched in the monoidal category of infimum lattices. **SUP**-lattices and **INF**-lattices are related via $()^{op}$: **SUP** $\to$ **INF** which is an involutive isomorphism [19].

Recall that **ORD** and **SUP** are monoidal. Also recall that for $\mathcal{O}$ an **ORD**-taxon the unique arrow $m : \coprod_X \mathcal{O}(A, X) \otimes \mathcal{O}(X, B) \longrightarrow \mathcal{O}(A, B)$, constructed from the composition arrows via the universal property of coproducts is a coequalizer. For the monoidal category **ORD** the tensor product $- \otimes -$ is given by the product $- \times -$.

Definition 1.5.23 For $\mathcal{O}_1$ and $\mathcal{O}_2$, **ORD**-taxons [**SUP**-taxons], a *lax-semifunctor* $F : \mathcal{O}_1 \to \mathcal{O}_2$ consists of the following:

1. a function $F : |\mathcal{O}_1| \to |\mathcal{O}_2|$
2. For every pair of objects A_1, A_2 in $|\mathcal{O}_1|$ a morphism $F_{A_1A_2} : \mathcal{O}_1(A_1, A_2) \to \mathcal{O}_2(FA_1, FA_2)$ in **ORD** [**SUP**], such that

$$\begin{array}{ccc}
\mathcal{O}_1(A_1, A_2) \otimes \mathcal{O}_1(A_2, A_3) & \xrightarrow{\mathcal{O}_{1A_1A_2A_3}} & \mathcal{O}_1(A_1, A_3) \\
{\scriptstyle F_{A_1A_2} \otimes F_{A_2A_3}} \downarrow & \leq & \downarrow {\scriptstyle F_{A_1A_3}} \\
\mathcal{O}_2(FA_1, FA_2) \otimes \mathcal{O}_2(FA_2, FA_3) & \xrightarrow[\mathcal{O}_{2FA_1FA_2FA_3}]{} & \mathcal{O}_2(FA_1, FA_3)
\end{array}$$

Since the universal property of natural transformations is given in terms of two commuting diagrams we have some choice in how we define laxity.

Definition 1.5.24 For $\mathcal{O}_1$ and $\mathcal{O}_2$, **ORD**-taxons [**SUP**-taxons] and $F, G : \mathcal{O}_1 \to \mathcal{O}_2$ lax-semifunctors a *strong lax-transformation* $\gamma : F \to G$ is a family of **ORD** [**SUP**] morphisms $\langle \gamma_{A_1A_2} : \mathcal{O}_1(A_1, A_2) \to \mathcal{O}_2(FA_1, GA_2)\rangle_{A_1, A_2 \in |\mathcal{O}_1|}$ such that

$$\begin{array}{ccc}
\mathcal{O}_1(A_1,A_2)\otimes\mathcal{O}_1(A_2,A_3) & \xrightarrow{\mathcal{O}_{1A_1A_2A_3}} & \mathcal{O}_1(A_1,A_3) \\
{\scriptstyle F_{A_1A_2}\otimes\gamma_{A_2A_3}}\downarrow & = & \downarrow{\scriptstyle \gamma_{A_1A_3}} \\
\mathcal{O}_2(FA_1,FA_2)\otimes\mathcal{O}_2(FA_2,GA_3) & \xrightarrow[\mathcal{O}_{2FA_1FA_2FA_3}]{} & \mathcal{O}_2(FA_1,GA_3)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{O}_1(A_1,A_2)\otimes\mathcal{O}_1(A_2,A_3) & \xrightarrow{\mathcal{O}_{1A_1A_2A_3}} & \mathcal{O}_1(A_1,A_3) \\
{\scriptstyle \gamma_{A_1A_2}\otimes G_{A_2A_3}}\downarrow & \leq & \downarrow{\scriptstyle \gamma_{A_1A_3}} \\
\mathcal{O}_2(FA_1,GA_2)\otimes\mathcal{O}_2(GA_2,GA_3) & \xrightarrow[\mathcal{O}_{2FA_1GA_2GA_3}]{} & \mathcal{O}_2(FA_1,GA_3)
\end{array}$$

The composition of strong lax-transformations is given similar to the composition of natural transformations. So for $\tau : F \to G$ and $\sigma : G \to H$, strong lax-transformations between **ORD**-semifuntors, we define $(\sigma\tau)_f = \sigma_h\tau_g$ for some $f = hg$. Between **SUP**-semifunctors the composition is $(\sigma\tau)_f = \vee\sigma_h\tau_g$ where $hg \leq f$.

Recall that for $\mathcal{O}_1$ and $\mathcal{O}_2$, **ORD**-categories a lax-transformation (Pitts[20]) $\tau : F \to G$ between **ORD**-functors $F, G : \mathcal{O}_1 \to \mathcal{O}_2$, satisfies for every arrow $f : A \to B$

$$G(f)\tau_A \leq \tau_B F(f).$$

Theorem 1.5.25 For $\mathcal{O}_1$ and $\mathcal{O}_2$, **ORD**-categories, $F, G : \mathcal{O}_1 \to \mathcal{O}_2$ functors, the concepts of strong lax-transformation and lax-transformation (in the sense of Pitts) coincide.

Proof: Given a strong lax-transformation τ the family of arrows τ_{1_A} clearly gives a lax-transformation. For a lax-transformation ω define τ_f to be the arrow $\omega_{1_A}F(f)$. We have

$$\tau_f F(g) = \omega_{1_A}F(f)F(g) = \omega_{1_A}F(fg) = \tau_{fg}$$

$$G(f)\tau_g = G(f)\omega_{1_B}F(g) \leq \omega_{1_A}F(f)F(g) = \tau_{fg}.$$

Thus this is a strong lax-transformation. It is easy to see that going back and forth gives back the original transformations. ∎

Definition 1.5.26 For $\mathcal{O}_1$ and $\mathcal{O}_2$, **ORD**-taxons [**SUP**-taxons] and $F, G : \mathcal{O}_1 \to \mathcal{O}_2$ lax-semifunctors a *modular lax-transformation* $\gamma : F \to G$ is a family of **ORD** [**SUP**] morphisms $\langle \gamma_{A_1A_2} : \mathcal{O}_1(A_1, A_2) \to \mathcal{O}_2(FA_1, GA_2)\rangle_{A_1,A_2 \in |\mathcal{O}_1|}$ such that

$$\begin{array}{ccc} \mathcal{O}_1(A_1, A_2) \otimes \mathcal{O}_1(A_2, A_3) & \xrightarrow{\mathcal{O}_{1A_1A_2A_3}} & \mathcal{O}_1(A_1, A_3) \\ {\scriptstyle F_{A_1A_2} \otimes \gamma_{A_2A_3}} \downarrow & \leq & \downarrow {\scriptstyle \gamma_{A_1A_3}} \\ \mathcal{O}_2(FA_1, FA_2) \otimes \mathcal{O}_2(FA_2, GA_3) & \xrightarrow[\mathcal{O}_{2FA_1FA_2FA_3}]{} & \mathcal{O}_2(FA_1, GA_3) \end{array}$$

$$\begin{array}{ccc} \mathcal{O}_1(A_1, A_2) \otimes \mathcal{O}_1(A_2, A_3) & \xrightarrow{\mathcal{O}_{1A_1A_2A_3}} & \mathcal{O}_1(A_1, A_3) \\ {\scriptstyle \gamma_{A_1A_2} \otimes G_{A_2A_3}} \downarrow & \leq & \downarrow {\scriptstyle \gamma_{A_1A_3}} \\ \mathcal{O}_2(FA_1, GA_2) \otimes \mathcal{O}_2(GA_2, GA_3) & \xrightarrow[\mathcal{O}_{2FA_1GA_2GA_3}]{} & \mathcal{O}_2(FA_1, GA_3) \end{array}$$

The composition of modular lax-transformations is identical to the composition of lax-transformations. A lax-semifunctor $F : \mathcal{O}_1 \to \mathcal{O}_2$, gives rise to a modular lax-transformation τ_F where $\tau_{F_{AB}} = F_{AB}$. Call a lax-semifunctor F an idempotent if $\tau_F \tau_F = \tau_F$. We will denote the semicategory of order preserving lax-semifunctors and strong lax-transformations by $SLAX_{ord}(\mathcal{O}_1, \mathcal{O}_2)$ and that of lax-semifunctors and modular lax-transformations by $MLAX_{ord}(\mathcal{O}_1, \mathcal{O}_2)$. Similarly for supremum and infimum preserving lax-semifunctors and modular lax-transformations we denote the respective semicategories by $SLAX_{sup}(\mathcal{O}_1, \mathcal{O}_2)$ and $MLAX_{sup}(\mathcal{O}_1, \mathcal{O}_2)$.

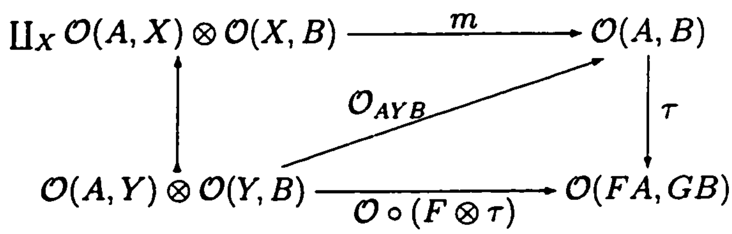
Observe that for the category $\mathbf{1}$, the semicategory $SLAX_{ord}(\mathbf{1}, \mathcal{O})$ is equivalent to $RMOD(\mathcal{O})$ and $MLAX_{ord}(\mathbf{1}, \mathcal{O})$ is equivalent to $MOD(\mathcal{O})$.

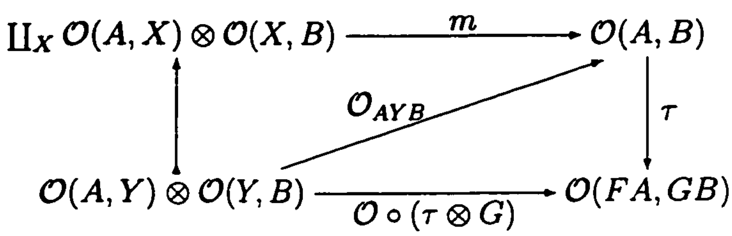
We will say that a modular lax-transformation, τ, between lax-semifunctors is a natural transformation if both defining diagrams commute.

Lemma 1.5.27 For $\mathcal{O}_1$ and $\mathcal{O}_2$, **ORD**-taxons (**SUP**-taxons) and $F, G : \mathcal{O}_1 \to \mathcal{O}_2$ lax-semifunctors, a modular lax-transformation $\tau : F \to G$ is a natural transformation if and only if

$$\tau_G \tau = \tau = \tau \tau_F.$$

Proof: The following diagrams give the result





Observe that in the first diagram the arrow τ is the arrow $\tau\tau_F$ by definition of the composite. And in the second diagram the arrow τ is the arrow $\tau_G\tau$. ∎

The following is an immediate consequence of this

Corollary 1.5.28 For $\mathcal{O}_1$ and $\mathcal{O}_2$ **ORD**-taxons (**SUP**-taxons), a lax-semifunctor $F : \mathcal{O}_1 \to \mathcal{O}_2$ is a semifunctor if and only if the induced modular lax-transformation is idempotent.

Corollary 1.5.29 For $\mathcal{Q}_1$ and $\mathcal{Q}_2$ **SUP**-taxons, $KAR(MLAX_{sup}(\mathcal{Q}_1, \mathcal{Q}_2))$ is equivalent to $TAX_{sup}(\mathcal{Q}_1, \mathcal{Q}_2)$.

Proof: For $\tau : F \to F$ an idempotent modular lax-transformation define a lax-semifunctor F_τ by

$$\text{For an object } A, \quad F_\tau(A) = F(A)$$
$$\text{For a morphism } q, \quad F_{\tau_q}(x, y) = \tau_q(x, y).$$

Since τ is idempotent the induced lax-semifunctor F_τ is idempotent and therefore is a semifunctor. Any modular lax-transformation $\omega : \tau \to \sigma$ in $KAR(MLAX_{sup}(\mathcal{Q}_1, \mathcal{Q}_2))$, by lemma(1.5.27) is a transformation from F_τ to F_σ. It is easy to see that we have a functor Ψ from $KAR(MLAX_{sup}(\mathcal{Q}_1, \mathcal{Q}_2))$ to $TAX_{sup}(\mathcal{Q}_1, \mathcal{Q}_2)$. This together with the inclusion ι of $TAX_{sup}(\mathcal{Q}_1, \mathcal{Q}_2)$ into $KAR(MLAX_{sup}(\mathcal{Q}_1, \mathcal{Q}_2))$ gives us the equivalence. Clearly $\Psi\iota$ is the identity on $TAX_{sup}(\mathcal{Q}_1, \mathcal{Q}_2)$. For τ, an idempotent modular lax-transformation, $\Psi\iota(\tau)$ is the idempotent $\tau : F_\tau \to F_\tau$. Now τ is an isomorphism between the objects $\tau : F \to F$ and $\tau : F_\tau \to F_\tau$ telling us that $\Psi\iota$ is isomorphic to the identity functor. ∎

Chapter 2

Sheaves and $\mathcal{Q}$-valued Sets

2.1 Sheaves on a Heyting Algebra

A sheaf on a Heyting algebra $\mathcal{H}$ is a functor $F : \mathcal{H}^{op} \to SET$ satisfying a *patching condition.* The following example is a canonical example of a sheaf.

Example 2.1.1 For $(X, \mathcal{T})$ a topology, there is a functor $F : \mathcal{T}^{op} \to SET$, defined for $U \subseteq V$ in $\mathcal{T}$ by

$$F(U) = \{f : U \to \mathbf{R}, \text{ continuous}\}$$
$$\text{with}$$
$$F(U \subseteq V)(f) = f_{|_U}.$$

Notice that for a family of functions $f_i : U_i \to \mathbf{R}$, if each pair of functions (f_i, f_j) agree on $U_i \cap U_j$ then this tells us there is a unique function on the union of the U_i which when restricted to U_j, for some j, is equal to f_j. This is the property that is used to define the notion of sheaf. In the following if $k \leq h$ in a Heyting algebra $\mathcal{H}$, then for $x \in F(h)$ we will denote the element $F(k \leq h)(x)$ by $x_{|_k}$ and we will say that x is restricted to k.

Definition 2.1.2 For $\mathcal{H}$ a complete Heyting algebra and h an element of $\mathcal{H}$, a *cover of* h is a family of elements $< h_i >_{i \in I}$ in $\mathcal{H}$ such that $\bigvee_i h_i = h$.

Definition 2.1.3 For $< h_i >_{i \in I}$ a cover of h an element in a complete Heyting algebra $\mathcal{H}$ and $F : \mathcal{H}^{op} \to SET$, a *matching family* consists of a family $< x_i \in F(h_i) >_{i \in I}$ such that $x_{i|_{h_i \wedge h_j}} = x_{j|_{h_i \wedge h_j}}$ for all $i, j \in I$.

Definition 2.1.4 For a Heyting algebra $\mathcal{H}$, $F : \mathcal{H}^{op} \to SET$ and a matching family $< x_i \in F(h_i) >_{i \in I}$, an *amalgamation* is an element $x \in F(h)$ such that $x_{|_{h_i}} = x_i$ for all $i \in I$.

Definition 2.1.5 For a Heyting algebra $\mathcal{H}$, $F : \mathcal{H}^{op} \to SET$ is a *sheaf* if every matching family has a unique amalgamation.

Example 2.1.6 For $X \in D\mathcal{H}$, we have the following functor

$$F^X(h) = \begin{cases} \{*\} & \text{if } h \in X \\ \emptyset & \text{otherwise.} \end{cases} \tag{2.1}$$

Theorem 2.1.7 For $X \in D\mathcal{H}$, F^X is a sheaf if and only if X is a principal down set.

Proof: If F^X is a sheaf then for each $x \in X$ there is an associated singleton $\{*\}$ in $F(x)$. The elements of X form a cover of $\bigvee X$ and the singletons $\{*\}$ are a matching family for this cover. Since F^X is a sheaf there is an element in the set $F(\bigvee X)$, which implies that $\bigvee X \in X$. Thus X is a principal downset.

Now assume that X is a principal downset. Let $\langle h_i \rangle_{i \in I}$ be a cover of h and let $\langle x_i \rangle_{i \in I}$ be a matching family. Since there is a matching family for the cover that implies that each set $F(h_i)$ is non-empty. Thus each h_i is an element of X. So $\bigvee h_i$ is an element of X since X is a principal downset. This tells us the there is an unique element in $F(\bigvee h_i)$ which is an amalgamation of the matching family. ∎

We will later see that all sheaves share a more general version of this.

The full subcategory of $SET^{\mathcal{H}^{op}}$ generated by sheaves will be denoted by $SHV(\mathcal{H})$.

The following results are well known and will be a consequence of the work to come. The first proposition below will follow from the more general work that will be done in the last section of this chapter.

Proposition 2.1.8 The inclusion functor from $SHV(\mathcal{H})$ to $SET^{\mathcal{H}^{op}}$ has a left adjoint, the associated sheaf functor.

In the next section we define the category of $\mathcal{H}$-valued sets for $\mathcal{H}$ a complete Heyting algebra and then prove that for $\mathcal{O}$ a partial order that the category of $D\mathcal{O}$-valued sets is equivalent to $SET^{D\mathcal{O}^{op}}$. In the last section we also show that the category $SHV(\mathcal{H})$ is equivalent to the category of $\mathcal{H}$-valued sets. Putting these results together gives the following results.

Proposition 2.1.9 For $\mathcal{O}$ a partial order, the category of sheaves on $D\mathcal{O}$ is equivalent to $SET^{\mathcal{O}^{op}}$.

Corollary 2.1.10 For $\mathcal{H}$ a complete Heyting algebra, $SHV(D\mathcal{H})$ is equivalent to $SET^{\mathcal{H}^{op}}$.

2.2 $\mathcal{H}$-Sets

2.2.1 $\mathcal{H}$ Valued Sets

In 1984 Dennis Higgs[10] defined the category of *$\mathcal{H}$-Valued Sets*, for $\mathcal{H}$ a complete Heyting algebra, and showed that this category is equivalent to the category of sheaves on $\mathcal{H}$. This new way of looking at the category of sheaves has been the main area of study for exploring *sheaves* on a quantale and more generally a quantaloid. We will explore the category of $\mathcal{H}$-Valued Sets as Higgs defined them and show their relationship to some of the constructions in preceding sections.

Definition 2.2.1 For $\mathcal{H}$ a complete Heyting algebra, an $\mathcal{H}$-valued set is a pair (X, δ), where X is a set and δ is a function $\delta : X \times X \longrightarrow \mathcal{H}$ satisfying the following two conditions

$$\delta(x,y) = \delta(y,x) \qquad \text{symmetry} \tag{2.2}$$

$$\delta(x,y) \wedge \delta(y,z) \leq \delta(x,z) \qquad \text{transitivity.} \tag{2.3}$$

The $\mathcal{H}$-valued set (X, δ) then is a symmetric module on the category of matrices on $\mathcal{H}$. Recall that $MAT(\mathcal{H})$ satisfies Freyd's law of modularity so it immediately follows that δ is a symmetric idempotent. A morphism $R : (X, \delta) \longrightarrow (Y, \delta)$ of $\mathcal{H}$-valued sets consists of a function $R : Y \times X \longrightarrow \mathcal{H}$ satisfying the following conditions.

$$R(y, x_1) \wedge \delta_X(x_1, x) \leq R(y, x) \tag{2.4}$$

$$\delta_Y(y, y_1) \wedge R(y_1, x) \leq R(y, x) \tag{2.5}$$

$$R(y_1, x) \wedge R(y_2, x) \leq \delta_Y(y_1, y_2) \tag{2.6}$$

$$\bigvee_y R(y, x) = \delta_X(x, x). \tag{2.7}$$

For another morphism $S : (Y, \delta) \to (Z, \delta)$ the composite morphism SR is given by $SR(z, x) = \bigvee_y \{S(z, y) \wedge R(y, x)\}$. We will denote the category of $\mathcal{H}$-valued sets by $\mathcal{H}$-SET.

Note that for $R : (X,\delta) \to (Y,\delta)$, δ_X, δ_Y and R are $\mathcal{H}$-valued matrices. We next show that the category of $\mathcal{H}$-valued sets can be characterized in terms of certain constructions on $\mathcal{H}$.

Theorem 2.2.2 The category $\mathcal{H}$-SET is $MAP({}^*KAR(MAT(\mathcal{H})))$.

Proof: We have that for the $\mathcal{H}$-valued set (X,δ), δ is a symmetric idempotent. Let $R : (X,\delta) \to (Y,\delta)$ be an $\mathcal{H}$-SET morphism. We need to show that $R\delta = R = \delta R$ and $R \dashv R^{\circ}$. By (2.4) and (2.5) we have that $R\delta_X \leq R$ and $\delta_Y R \leq R$. (2.7) tells us that $R(y,x) \leq \delta_X(x,x)$ so $R(y,x) \wedge \delta_X(x,x) = R(y,x)$ giving us $R \leq R\delta_X$ and so $R = R\delta_X$. By (2.6) we have

$$
\begin{aligned}
R(y,x) &= R(y,x) \wedge R(y,x) \wedge R(y,x) \\
&\leq \delta_Y(y,y) \wedge R(y,x)
\end{aligned}
$$

and so $R = \delta_Y R$.

(2.6) tells us that $R^{\circ}R \leq \delta_Y$. All we have left to show is that $\delta_X \leq RR^{\circ}$. By (2.7) we have

$$
\begin{aligned}
\bigvee_y \{R(y,x)\} &= \bigvee_y \{R^{\circ}(x,y) \wedge R(y,x)\} \\
&= R^{\circ}R(x,x).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\delta_X(x_1,x_2) &= \delta_X(x_1,x_2) \ \wedge \ \delta(x_2,x_2) \\
&= \delta_X(x_1,x_2) \ \wedge \ R^{\circ}R(x_2,x_2) \\
&\leq R^{\circ}R(x_1,x_2).
\end{aligned}
$$

So we have that every $\mathcal{H}$-valued set morphism is a left adjoint in the category $KAR(MAT(\mathcal{H}))$. Clearly a symmetric idempotent matrix is an $\mathcal{H}$-valued set. Now let us assume that $R : (X,\delta) \to (Y,\delta)$ is a left adjoint in $KAR(MAT(\mathcal{H}))$. (2.2), (2.3) and (2.4) are automatic so all we need to show is that $\bigvee_y\{R(y,x)\}$ is equal to

$\delta(x,x)$. By the adjunction we have $\delta(x,x) \leq R^{\circ}R(x,x)$ so $\delta(x,x) \leq \bigvee_y\{R(y,x)\}$. Now $R(y,x_1) \wedge \delta(x_1,x) \leq \delta(x_1,x) \leq \delta(x,x)$. Thus $R(y,x) \leq \delta(x,x)$ which now gives the required equality. ∎

Recall that there are two different ways to view a complete Heyting Algebra: as a partial order and as an involutive **SUP**-enriched category.

Lemma 2.2.3 For a complete Heyting algebra $\mathcal{H}$ interpreted as a one object **SUP**-enriched category the category $MAP(KAR(\mathcal{H}))$ is $\mathcal{H}$ interpreted as a partial order.

Proof: The objects of $MAP(KAR(\mathcal{H}))$ are the idempotents of $\mathcal{H}$, which are all of the elements of $\mathcal{H}$. An arrow $h_1 \xrightarrow{k} h_2$ in $KAR(\mathcal{H})$ is just an element that satisfies $k \wedge h_1 = k = k \wedge h_2$. This implies that $k \leq h_1$ and $k \leq h_2$. Now if $k \dashv k^{\#}$ is a map then $h_1 \leq k^{\#} \wedge k$. Which implies that $h_1 \leq k$. So we have k is a map if and only if $h_1 = k \leq h_2$. ∎

This relationship is key to allowing us to define the category of presheaves on a quantaloid. Recall that for $\mathcal{O}$ a partial order the partial order $D\mathcal{O}$ is a complete Heyting algebra.

Theorem 2.2.4 For $\mathcal{O}$ a partial order the category *DO-Set* is equivalent to $SET^{\mathcal{O}^{op}}$.

Proof: We define our functors as follows. For Ψ : *DO-Set* $\rightarrow SET^{\mathcal{O}^{op}}$,

- On objects (X, δ) • $\Psi((X, \delta)) = \Psi_\delta$ is given by
  For $x \in \mathcal{O}$; $\Psi_\delta(h) = \{x \in X | h \in \delta(x, x)\}/ \sim_h$
  where $x \sim_h y$ if and only if $h \in \delta(x, y)$
  we will denote the elements of $\Psi_\delta(h)$ by $[x]_h$

  For $h_1 \leq h_2$, $\Psi_\delta([x]_{h_2}) = [x]_{h_1}$.

- On arrows R • $\Psi(R) = \Psi_R$ where
  $\Psi_R([x]_h) = [y]_h$ where $h \in R(y, x)$.

We need first to show that $\sim_h$ is an equivalence relation. By the definition of δ, reflexivity and symmetry are clear. If $h \in \delta_X(x, y)$ and $h \in \delta_X(y, z)$ then $h \in \delta_X(x, z)$ by definition of the composition of matrices. Now we need to show that Ψ_R is well defined. Since R is a left adjoint we have that if $h \in \delta_X(x, x)$ then $h \in R^\circ R(x, x)$, so there exists y such that $h \in R(y, x)$. If $h \in \delta_X(x, \bar{x})$ then we have $h \in R(y, \bar{x})$ since $R\delta = R$ so the set of y such that $h \in R(y, x)$ is independent of the choice of representative of the class $[x]_h$. We will now show that this set is the class of y. If $h \in R(\bar{y}, x)$ as well then $h \in RR^\circ(\bar{y}, y)$ which by the adjunction property implies that $h \in \delta_Y(\bar{y}, y)$. Now $h \in \delta_Y(\bar{y}, y)$ implies that $h \in R(\bar{y}, x)$ since $\delta_Y R = R$. Thus Ψ_R is well defined. If S is an arrow composable with R then we have

$$\begin{aligned}
\Psi_{SR}([x]_h) = [z]_h &\Leftrightarrow h \in SR(z, x) \\
&\Leftrightarrow \exists y : h \in S(z, y) \text{ and } h \in R(y, x) \\
&\Leftrightarrow \exists y : \Psi_S([y]_h) = [z]_h \text{ and } \Psi_R([x]_h) = [y]_h \\
&\Leftrightarrow \Psi_S \Psi_R([x]_h) = [z]_h.
\end{aligned}$$

For $\Phi : SET^{\mathcal{O}^{op}} \longrightarrow D(\mathcal{O})\text{-}Set$.

- On objects F • $\Phi(F) = (X_F, \delta_F)$ is given by
  $X_F = \coprod_{a \in \mathcal{O}} F(a)$
  For $x \in F(a_1), y \in F(a_2)$; $\delta_F(x, y) = \{a \in \mathcal{O} | x_{|a} = y_{|a}\}$.

- On arrows τ • $\Phi(\tau) = \Phi_\tau$ where
  $\Phi_\tau(y, x) = \{a \in \mathcal{O} | \tau(x)_{|a} = y_{|a}\}$.

We wish to show that δ_F is an $\mathcal{H}$-valued set. Clearly δ_F is symmetric. To show that it is idempotent we have

$$\begin{aligned}
\delta_F\delta_F(z, x) &= \{a \in \mathcal{O} \mid \exists_y \ a \in \delta_F(z, y) \text{ and } a \in \delta_F(y, x)\} \\
&= \{a \in \mathcal{O} \mid \exists_y \ z_{|a} = y_{|a} = x_{|a}\} \\
&= \{a \in \mathcal{O} \mid z_{|a} = x_{|a}\} \\
&= \delta_F(z, x).
\end{aligned}$$

To see that Φ_τ is a morphism we have

$$\begin{aligned}
\Phi_\tau\delta(y, x) &= \{a \in \mathcal{O} \mid \exists_{\bar{x}} \ a \in \Phi_\tau(y, \bar{x}) \text{ and } a \in \delta(\bar{x}, x)\} \\
&= \{a \in \mathcal{O} \mid \exists_{\bar{x}} \ y_{|a} = \tau(\bar{x})_{|a} \text{ and } \bar{x}_{|a} = x_{|a}\} \\
&= \{a \in \mathcal{O} \mid y_{|a} = \tau(x)_{|a}\} \\
&= \Phi_\tau(y, x).
\end{aligned}$$

Similarly $\delta\Phi_\tau = \Phi_\tau$. To show that Φ_τ is left adjoint to Φ°_τ.

$$\begin{aligned}
a \in \delta_F(x, \bar{x}) &\Leftrightarrow x_{|a} = \bar{x}_{|a} \\
&\Rightarrow \tau(x)_{|a} = \tau(\bar{x})_{|a} \\
&\Leftrightarrow a \in \Phi_\tau(\tau(x), \bar{x}) \\
&\Rightarrow a \in \Phi^\circ_\tau\Phi_\tau(x, \bar{x}).
\end{aligned}$$

The last implication follows since if $a \in \delta_F(x, \bar{x})$ then $a \in \delta_F(x, x)$ so $a \in \Phi_\tau(\tau(x), x)$.

$$a \in \Phi_\tau\Phi^\circ_\tau(y_1, y_2) \Leftrightarrow \exists_x \ a \in \Phi_\tau(x, y_1) \text{ and } a \in \Phi_\tau(x, y_2)$$

$$\Leftrightarrow \exists_x\ \tau(x)_{|a} = y_{1|a} \text{ and } \tau(x)_{|a} = y_{2|a}$$
$$\Rightarrow y_{1|a} = y_{2|a}$$
$$\Leftrightarrow a \in \delta_G(y_1, y_2).$$

Before we define the unit and counit for our adjunction let us examine the composite functors $\Psi\Phi$ and $\Phi\Psi$. For $F : \mathcal{O}^{op} \to SET$ we have

$$\begin{aligned} \Psi\Phi(F)(a) &= \{x \in X_F \mid a \in \delta_F(x,x)\}/\sim_a \\ &= \{x \mid x_{|a} = x_{|a}\}/\sim_a \\ &= \{x \mid x \in F(a)\} \\ &= \text{F(a)}. \end{aligned}$$

For $a_1 \leq a_2$ $\quad \Psi\Phi(F)(a_1 \leq a_2)(x)$

$$\begin{aligned} &= [x]_{a_1} \\ &= x_{|a_1}. \end{aligned}$$

This tells us that the composite $\Psi\Phi$ is the identity. And so we can take as the counit the identity transformation. Now for (X, δ) a $D(\mathcal{O})$-*Set* we have

$$\Phi\Psi_X = \{[x]_a \mid a \in \delta(x,x)\}.$$

$$\begin{aligned} \Phi\Psi_\delta([x]_{a_1}, [y]_{a_2}) &= \{a \in \mathcal{O} \mid [x]_{a_1|a} = [y]_{a_2|a}\} \\ &= \delta(x,y) \cap \downarrow a_1 \cap \downarrow a_2. \end{aligned}$$

Now we define the unit for the adjunction $\Psi \dashv \Phi$. The unit $\mathbf{1} \xrightarrow{\eta} \Phi\Psi$ is defined on an object (X, δ) as follows

$$\eta_\delta([x]_a, y) = \delta(x,y) \cap \downarrow a.$$

We need to show that η_δ is a morphism.

$$\begin{aligned} \eta_\delta\delta([x]_a, z) &= \cup_y\{\eta([x]_a, y) \cap \delta(y,z)\} \\ &= \cup_y\{\downarrow a \cap \delta(x,y) \cap \delta(y,z)\} \end{aligned}$$

$$
\begin{aligned}
&= \ \downarrow a \cap \delta(x,z) \\
&= \ \eta_\delta([x]_a, z).
\end{aligned}
$$

$$
\begin{aligned}
\Psi\Phi_\delta\eta_\delta([x]_a, z) &= \bigcup_{[y]_{a'}}\{\Psi\Phi_\delta([x]_a,[y]_{a'}) \cap \eta_\delta([y]_{a'}, z)\} \\
&= \bigcup_{[y]_{a'}}\{\downarrow a \cap \downarrow a' \cap \delta(x,y) \cap \delta(y,z)\} \\
&= \ \downarrow a \cap \delta(x,z)\} \\
&= \eta([x]_a, z).
\end{aligned}
$$

For η° defined in the obvious way the preceeding equalities follow. Now we show that η_δ is a left adjoint. We will in fact show that η is an isomorphism.

$$
\begin{aligned}
\eta^\circ\eta(x,z) &= \bigcup_{[y]_{a'}}\{\eta^\circ(x,[y]_{a'}) \cap \eta([y]_{a'}, z)\} \\
&= \bigcup_{[y]_{a'}}\{\downarrow a' \cap \delta(x,y) \cap \delta(y,z)\} \\
&= \delta(x,z).
\end{aligned}
$$

$$
\begin{aligned}
\eta\eta^\circ([x]_{a_1},[z]_{a_2}) &= \bigcup_y\{\eta([x]_{a_1}, y) \cap \eta^\circ(y,[z]_{a_2})\} \\
&= \bigcup_y\{\downarrow a_1 \cap \downarrow a_2 \cap \delta(x,y) \cap \delta(y,z)\} \\
&= \ \downarrow a_1 \cap \downarrow a_2 \cap \delta(x,z) \\
&= \Psi\Phi_\delta([x]_{a_1},[z]_{a_2}).
\end{aligned}
$$

So η_δ is an isomorphism. The counit $\varepsilon : \Psi\Phi \to 1$ is defined on a functor $F : \mathcal{O}^{op} \to Set$ to be the identity transformation. Thus we have an equivalence. ∎

Corollary 2.2.5 For $\mathcal{H}$ a complete Heyting algebra we have $SET^{\mathcal{H}^{op}}$ is equivalent to $D\mathcal{H}$-*Set*.

2.2.2 Singletons

We turn our attention to the concept of a *singleton* $\mathcal{H}$-valued set for $\mathcal{H}$ a complete Heyting algebra. Higgs used the singleton $\mathcal{H}$-valued sets and associated morphisms to show that the category of $\mathcal{H}$-valued sets and their morphisms is equivalent to the category of sheaves on $\mathcal{H}$. We will introduce the ideas here but we will save the proof of the equivalence until we study $\mathcal{Q}$-valued sets and sheaves on $\mathcal{Q}$ for $\mathcal{Q}$ a quantaloid.

Definition 2.2.6 A *singleton* $\mathcal{H}$-valued set is an $\mathcal{H}$-valued set (X, δ), for which $X = \{*\}$ is a one element set.

Example 2.2.7 Given $h \in \mathcal{H}$ there is a singleton $(\{*\}, \delta^h)$ given by $\delta(*,*) = h$. In fact all singletons are of this form.

The following lemma turns out to be very important when we want to define sheaves on a quantaloid.

Lemma 2.2.8 The full subcategory of $\mathcal{H}$-*Set* whose objects are the singletons is equivalent to the partial order $\mathcal{H}$.

Proof: Let $Sing(\mathcal{H})$ denote the full subcategory of $\mathcal{H}$-*Set* generated by the singleton objects. We define the functors $\Phi : Sing(\mathcal{H}) \to \mathcal{H}$ and $\Psi : \mathcal{H} \to Sing(\mathcal{H})$ as follows. Given $\alpha : (\{*\}, \delta) \to (\{*\}, \gamma)$, an $\mathcal{H}$-valued set morphism, Φ sends $(\{*\}, \delta)$ to the element $\delta(*,*)$ and $(\{*\}, \gamma)$ to $\gamma(*,*)$. We have $\delta(*,*) \leq \alpha^{\circ}\alpha(*,*)$ and $\alpha\alpha^{\circ}(*,*) \leq \gamma(*,*)$. This is if and only if $\delta(*,*) \leq \alpha(*,*)$ and $\alpha(*,*) \leq \gamma(*,*)$. Notice though that we also have $\alpha\delta(*,*) = \alpha(*,*)$ which tells us that $\alpha(*,*) \leq \delta(*,*)$. Hence α is a morphism from $(\{*\}, \delta)$ to $(\{*\}, \gamma)$ if and only if $\delta(*,*) \leq \gamma(*,*)$. On the other hand if $h \leq k$ then Ψ sends this to the $\mathcal{H}$-valued sets $(\{*\}, \delta^h)$ and $(\{*\}, \delta^k)$ where $\delta^h(*,*) = h$ and $\delta^k(*,*) = k$ and the unique morphism between them. ∎

In future we will denote a singleton $(\{*\}, \delta)$ by $[h]$, where $h \in \mathcal{H}$ and $h = \delta(*,*)$. Given an $\mathcal{H}$-valued set (X, δ), for each $x \in X$ denote the singleton $[\delta(x,x)]$ by $[x]$. We will denote the singleton matrix associated to $[x]$ by δ_x.

Lemma 2.2.9 Given an $\mathcal{H}$-valued set (X, δ), for each x in X there is a monomorphism $\alpha_x : [x] \hookrightarrow (X, \delta)$ where $\alpha_x(x_1, *) = \delta(x_1, x)$.

Proof: : First we show that α_x is a morphism in $KAR(MAT(\mathcal{H}))$.

$$\begin{aligned} \alpha_x\delta_X(y,*) &= \alpha_x(y,*) \;\&\; \delta_x(*,*) \\ &= \delta(y,x) \;\&\; \delta(x,x) \\ &= \delta(y,x) \qquad (\text{since } \delta(y,x) \le \delta(x,x)) \\ &= \alpha_x(y,*). \end{aligned}$$

$$\begin{aligned} \delta_Y\alpha_x(y,*) &= \bigvee_{x'}\{\delta(y,x') \;\&\; \alpha_x(x',*)\} \\ &= \bigvee_{x'}\{\delta(y,x') \;\&\; \delta(x',x)\} \\ &= \delta(y,x) \\ &= \alpha_x(y,*). \end{aligned}$$

With $\alpha_x^\circ(*,x') = \delta(x,x')$ we have

$$\begin{aligned} \alpha_x\alpha_x^\circ(x_1,x_2) &= \alpha_x(x_1,*) \;\&\; \alpha_x^\circ(*,x_2)\} \\ &= \delta(x_1,x) \;\&\; \delta(x,x_2) \\ &\le \delta(x_1,x_2). \end{aligned}$$

$$\begin{aligned} \alpha_x^\circ\alpha_x(*,*) &= \bigvee_{x'}\{\alpha_x^\circ(*,x') \;\&\; \alpha_x(*,x')\} \\ &= \bigvee_{x'}\{\delta(x,x') \;\&\; \delta(x',x)\} \\ &= \delta(x,x). \end{aligned}$$

Thus α_x is a monomorphism of $\mathcal{H}$-valued sets. ∎

Observe that for a singleton morphism $\alpha : [h] \hookrightarrow (X,\delta)$ the family of arrows $\langle\alpha(x)\rangle_{x\in X}$ is a cover of h. We will see later that we can interpret the elements of X to be a matching family for this cover and that α can then be thought of as the unique amalgamation for this matching family.

We finish off this section with two propositions. The first tells us of an adjunction between the category of pre-sheaves on $\mathcal{H}$ and the category of $\mathcal{H}$-valued sets. The second by Higgs[10] shows that the category of sheaves on a Heyting algebra $\mathcal{H}$ is equivalent to the category of $\mathcal{H}$-valued sets. Combining these gives us the associated sheaf functor. The proofs of these will follow automatically from the work we will do when we explore the relationship between sheaves on a quantaloid and $\mathcal{Q}$-valued sets.

Proposition 2.2.10 There is an adjuction $\Psi \dashv \Phi : SET^{\mathcal{H}^{op}} \to \mathcal{H}\text{-}Set$.

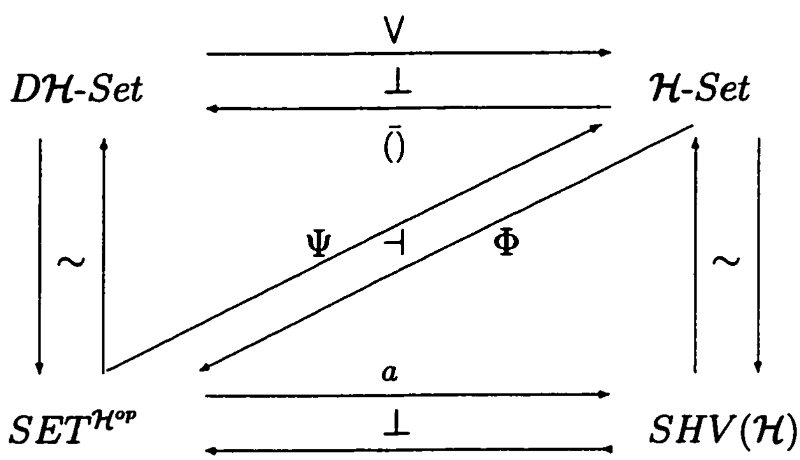
Φ takes a functor to the $\mathcal{H}$-valued set (X_F, δ_F) where $X_F = \coprod_h F(h)$ and $\delta(x,y) = \bigvee\{h \mid x_{|_h} = y_{|_h}\}$. Ψ takes an $\mathcal{H}$-valued set to the functor F_δ where $F_\delta(h) = \{[h] \hookrightarrow (X,\delta)\}$, the set of monomorphisms from $[h]$ to (X,δ). The restriction for $k \leq h$ takes a singleton and precomposes with the unique arrow $[k] \xrightarrow{!} [h]$.

Proposition 2.2.11 *(Higgs)* The category of $\mathcal{H}$-valued sets is equivalent to the category of sheaves on $\mathcal{H}$.

Corollary 2.2.12 For $\mathcal{O}$ a partial order, $SET^{\mathcal{O}^{op}}$ is equivalent to $SHV(D\mathcal{O})$.

Corollary 2.2.13 *(The associated sheaf functor)* There is an adjuction $a \dashv \iota : SET^{\mathcal{H}^{op}} \to SHV(\mathcal{H})$.

The results of this section can be encapsulated by the following diagram:



2.3 $\mathcal{Q}$-Sets

After Higgs constructed the category of $\mathcal{H}$-valued sets for a Heyting algebra $\mathcal{H}$ and showed how they are related to the category of sheaves on $\mathcal{H}$, Nawaz, Borceux, Gylys et al [4, 5, 8, 9, 18], used this template to define sheaves for quantaloids. Starting from matrices on a quantaloid $\mathcal{Q}$, $\mathcal{Q}$-Sets have been described by a set of axioms that these matrices and their morphisms must satisfy. The most interesting work has resulted when the category of $\mathcal{Q}$-Sets on a quantaloid $\mathcal{Q}$ is defined to be the category $MAP(KAR(MAT(\mathcal{Q})))$. An early result of Pitts[10] showed that when $\mathcal{Q}$ is a bounded distributive category of relations ($bDCR$) then the category of sheaves on $\mathcal{Q}$ in this sense is a Grothendieck topos. Moreover he showed that the category of complete $bDCR$'s is equivalent to the category of Grothendieck toposes, where completeness means that all symmetric idempotents split and all coproducts exist. Van den Bosche[23] explored the category of sheaves on a quantaloid constructed out of a ring R. The quantaloid is a two object $(0, 1)$ quantaloid where the hom sets are

$\mathcal{Q}(0,0)$	2 sided ideals
$\mathcal{Q}(0,1)$	Left sided ideals
$\mathcal{Q}(1,0)$	Right sided ideals
$\mathcal{Q}(1,1)$	Additive sub groups.

Using this Van den Bosche was able to show some nice relationships between the ring R and $\mathcal{Q}$-Sets on the above quantaloid. Recently Gylys[9] has also studied this category. He defines sheaves to be a subcategory of $MAP(KAR(MAT(\mathcal{Q})))$. We will explore Gylys' notion of *strict* $\mathcal{Q}$-valued sets and its relationship to sheaves. In particular a sheaf for Gylys is a *strict* and *separated* $\mathcal{Q}$-valued set that is *complete.* For this work we will only use the strictness property.

2.3.1 $\mathcal{Q}$-Valued Sets

We begin by defining the category of $\mathcal{Q}$-valued sets for $\mathcal{Q}$ a quantaloid and show some simple consequences of the definition that will be useful later.

Definition 2.3.1 For $\mathcal{Q}$ a quantaloid the category of *$\mathcal{Q}$-valued sets* is the category $MAP(KAR(MAT(\mathcal{Q})))$.

So a $\mathcal{Q}$-valued set is a triple (X, ρ, δ) where δ is an idempotent matrix on (X, ρ). If $\mathcal{Q}$ is involutive then the category of *$\mathcal{Q}$-valued sets* is the category $^*MAP(^*KAR(MAT(\mathcal{Q})))$. So a $\mathcal{Q}$-valued set is then a symmetric idempotent matrix.

A morphism of $\mathcal{Q}$-valued sets then is a matrix, $R : (X, \rho, \delta) \to (Y, \rho, \delta)$, that is a map. That is a pair of matrices $R : (X, \rho) \to (Y, \rho)$ and $R^\# : (Y, \rho) \to (X, \rho)$ (if $\mathcal{Q}$ is involutive then we ask that $R^\#$ be equal to R°), such that as matrices the following equations hold

$$
\begin{aligned}
R\delta_X = R \quad &\text{and} \quad \delta_X R^\# = R^\# \\
\delta_Y R = R \quad &\text{and} \quad R^\# \delta_Y = R^\# \\
\delta_X &\leq R^\# R \\
RR^\# &\leq \delta_Y .
\end{aligned}
$$

We will denote the category of $\mathcal{Q}$-valued sets by $\mathcal{Q}$-SET.

Recall that for $\mathcal{H}$ a Heyting algebra the category $\mathcal{H}$-*Set* is equivalent to the category $\mathcal{Q}$-SET when we interpret $\mathcal{H}$ as a one-object involutive quantaloid.

Define $\widehat{()} : D\mathcal{Q}\text{-}Set \to \mathcal{Q}\text{-}Set$ as follows

- On objects (X, ρ, δ) • $\widehat{(X, \rho, \delta)}$ equals $(\widehat{X}, \widehat{\rho}, \widehat{\delta})$ where
$$
\begin{aligned}
\widehat{X} &= X \\
\widehat{\rho} &= \rho \\
\widehat{\delta}(x, y) &= \bigvee\{q : q \in \delta(x, y)\}.
\end{aligned}
$$

- On Arrows R •
$$
\begin{aligned}
\widehat{R}(x, y) &= \bigvee\{q : q \in R(x, y)\} \\
\widehat{R}^\#(y, x) &= \bigvee\{q : q \in R^\#(y, x)\}.
\end{aligned}
$$

Lemma 2.3.2 For $\mathcal{Q}$ a quantaloid $\widehat{()}$ is a functor.

Proof:

If $\mathcal{Q}$ is involutive then $\widehat{\delta}$ is symmetric

$$\begin{aligned}\widehat{\delta}^{\circ}(x,y) &= \widehat{\delta}(y,x)^{*}\\ &= \bigvee\{f \mid f \in \delta(y,x)\}^{*}\\ &= \bigvee\{f^{*} \mid f \in \delta(y,x)\}\\ &= \bigvee\{f^{*} \mid f^{*} \in \delta(x,y)\}\\ &= \widehat{\delta}(x,y).\end{aligned}$$

Now to show that it is an idempotent matrix we have

$$\begin{aligned}\widehat{\delta}\widehat{\delta}(x,y) &= \bigvee_{z}\{\widehat{\delta}(x,z) \,\&\, \widehat{\delta}(z,y)\}\\ &= \bigvee_{z}\{\bigvee\{q_1 : q_1 \in \delta(x,z)\} \,\&\, \bigvee\{q_2 : q_2 \in \delta(z,y)\}\}\\ &= \bigvee_{z}\{q_1q_2 : q_1 \in \delta(x,z) \text{ and } q_2 \in \delta(z,y)\}\\ &= \bigvee\{q : q \in \delta(x,y)\}\\ &= \widehat{\delta}(x,y).\end{aligned}$$

Now to show that the matrix $\widehat{R}$ is a $\mathcal{Q}$-valued set morphism we have

$$\begin{aligned}\widehat{\delta}\widehat{R}(x,y) &= \bigvee_{z}\{\widehat{\delta}(x,z) \,\&\, \widehat{R}(z,y)\}\\ &= \bigvee_{z}\{\{q_1 : q_1 \in \delta(x,z)\} \,\&\, \{q_2 : q_2 \in R(z,y)\}\}\\ &= \bigvee_{z}\{q_1q_2 : q_1 \in \delta(x,z) \text{ and } q_2 \in R(z,y)\}\\ &= \bigvee\{q : q \in R(x,y)\}\\ &= \widehat{R}(x,y).\end{aligned}$$

In a similar way we have $\widehat{R}\widehat{\delta}$ equal to $\widehat{R}$ and that $\widehat{R} \dashv \widehat{R}^{\#}$. If $S : (Y, \rho, \delta) \to (Z, \rho, \delta)$, is another morphism then $\widehat{S}\widehat{R} = \widehat{SR}$. ∎

Lemma 2.3.3 If a quantaloid $\mathcal{Q}$ is pseudo-rightsided then there is a lax-functor $\bar{\downarrow} : KAR(MAT(\mathcal{Q})) \to KAR(MAT(D\mathcal{Q}))$.

Proof: For (X, ρ, δ) with δ a symmetric idempotent, let $\bar{\downarrow}(X, \rho, \delta) = (X, \rho, \bar{\delta})$ where $\bar{\delta}(x, y) = \downarrow \delta(x, y)$

$$\begin{aligned} \bar{\delta}\bar{\delta}(x, z) &= \bigcup_y \{\bar{\delta}(x, y) \ \& \ \bar{\delta}(y, z)\}^{\downarrow} \\ &= \bigcup_y \{\downarrow \delta(x, y) \ \& \ \downarrow \delta(y, z)\}^{\downarrow} \\ &= \bigcup_y \{\delta(x, y) \ \& \ \delta(y, z)\}^{\downarrow} \\ &\leq \ \downarrow \delta(x, z). \end{aligned}$$

We have $\delta(x, z) \ \& \ \delta(x, z)^* \ \& \ \delta(x, z) \leq \delta(x, z)$, and since $\mathcal{Q}$ is pseudo-rightsided this is an equality. This gives us $\downarrow \delta(x, z) = \downarrow \delta(x, x) \ \& \ \downarrow \delta(x, z)$ and so $\bar{\delta} = \bar{\delta}\bar{\delta}$. In a similar manner we define $\bar{\downarrow} R$ for a morphism $R : (X, \rho, \delta) \to (Y, \rho, \delta)$. Using $\downarrow \delta(x, z) = \downarrow \delta(x, x) \ \& \ \downarrow \delta(x, z)$ it is easy to see that this is a morphism in $KAR(MAT(\mathcal{Q}))$. For $S : (Y, \rho, \delta) \to (Z, \rho, \delta)$ the composite (SR) gives

$$\begin{aligned} \overline{SR}(z, x) &= \ \downarrow (SR(z, x)) \\ &= (\bigvee_y \{S(z, y) \ \& \ R(y, x)\})^{\downarrow} \\ &\geq \{f \mid \exists_y \, f \leq S(z, y) \ \& \ R(y, x)\} \\ &= \bar{S}\bar{R}(z, x). \end{aligned}$$

Since we only get a lax-functor we do not obtain a morphism from $\mathcal{Q}$-SET to $D\mathcal{Q}$-SET since $\bar{\downarrow}$ does not preserve maps. If we restrict $\widehat{()}$ to $KAR(MAT(D\mathcal{Q}))$ then we have $\widehat{()}\bar{\downarrow} = \mathbf{1}$ and there is a strong lax-transformation $\bar{\downarrow}\widehat{()} \xrightarrow{\varepsilon} \mathbf{1}$. For morphisms $(X, \rho, \delta) \xrightarrow{R} (Y, \rho, \delta) \xrightarrow{S} (Z, \rho, \delta)$ define $\varepsilon_R(z, x) = \bar{\downarrow}\hat{R}(z, x)$. Since $\widehat{()}$ is a functor we get $\bar{\downarrow}\hat{S}\varepsilon_R = \varepsilon_{SR}$ and clearly we have $\varepsilon_S R \leq \varepsilon_S \bar{\downarrow}\hat{R}$ so ε is a strong lax transformation.

2.3.2 Singletons and Strictness

An integral part of Higg's proof of the equivalence between the category of $\mathcal{H}$-valued sets and the category of sheaves on $\mathcal{H}$ was the notion of a singleton $\mathcal{H}$-valued set. In this section we explore singletons for the category of $\mathcal{Q}$-valued sets and its relationship to the strictness condition of Gylys[9].

Definition 2.3.4 For $\mathcal{Q}$ a quantaloid, a *singleton* is a $\mathcal{Q}$-valued set of the form $(\{*\}, \rho, \delta)$.

Theorem 2.3.5 The full subcategory of $\mathcal{Q}$-SET generated by the singleton objects is equivalent to the category $MAP(KAR(\mathcal{Q}))$.

Proof: Denote the full subcategory of $\mathcal{Q}$-SET generated by the singleton objects by $Sing(\mathcal{Q})$. Examine $(\{*\}, \rho, \delta) \xrightarrow{\alpha} (\{*\}, \rho, \gamma)$. $\delta(*, *)$ and $\gamma(*, *)$ each pick out (symmetric) idempotent arrows of $\mathcal{Q}$ (call them q_1 and q_2 respectively). $\alpha(*, *)$ is then a pair of $\mathcal{Q}$ arrows $(p, p^{\#})$ with $p : dom(q_2) \rightarrow dom(q_1)$ and $p^{\#} : dom(q_1) \rightarrow dom(q_2)$, satisfying, $q_1 p = p = p q_2$; and $q_1 \leq p^{\#} p$ and $p p^{\#} \leq q_2$. That is $(p, p^{\#})$ is an arrow in $MAP(KAR(\mathcal{Q}))$. Given an arrow in $MAP(KAR(\mathcal{Q}))$. This then is clearly a full and faithfull functor from $Sing(\mathcal{Q})$ to $MAP(KAR(\mathcal{Q}))$ which is a bijection on the objects. ∎

We can thus represent a singleton by $[q]$, where q is a (symmetric) idempotent in $\mathcal{Q}$. That is an object in $KAR(\mathcal{Q})$. The essence of a singleton is that it represents the unique amalgamation for a matching family.

Example 2.3.6 For (X, ρ, δ), a $\mathcal{Q}$-valued set, and x in X such that $\delta(x, x)$ is a (symmetric) idempotent in $\mathcal{Q}$ then $[\delta(x, x)]$ is a singleton. We will call such an element of X a *(symmetric) idempotent element* of X and denote the singleton $[\delta(x, x)]$ by $[x]$.

Definition 2.3.7 For (X, ρ, δ), a $\mathcal{Q}$-valued set, a *singleton morphism* for (X, ρ, δ) is a morphism $\alpha : [q] \rightarrow (X, \rho, \delta)$, from a singleton $[q]$ into (X, ρ, δ). We will denote the arrows $\alpha(x, *), \alpha^{\#}(*, x)$ by $\alpha(x)$ and $\alpha^{\#}(x)$ respectively.

Our interest is drawn towards those $\mathcal{Q}$-valued sets that can be described by their singleton morphisms, which represent a matching family for a cover. The singleton morphisms of interest are the monomorphic morphisms $\gamma : [q] \hookrightarrow (X, \rho, \delta)$.

Example 2.3.8 If for a $\mathcal{Q}$-valued set (X, ρ, δ), $x \in X$ satifies $\delta(x, y)$ & $\delta(y, y) = \delta(x, y)$ for all $y \in X$, then there is a monomorphic singleton morphism $\alpha_x : [x] \hookrightarrow (X, \rho, \delta)$ given by $\alpha(y, *) = \delta(y, x)$.

Definition 2.3.9 A $\mathcal{Q}$-valued set, (X, ρ, δ), is *atomic* if

$$\bigvee_{\gamma} \gamma\gamma^{\#} = \delta,$$

where we take the supremum over the monomorphic singleton morphisms $\gamma : [q] \hookrightarrow (X, \rho, \delta)$.

Lemma 2.3.10 For a singleton morphisms $\alpha : [q] \rightarrow (X, \rho, \delta)$ we have

$$\bigvee_{\gamma} \alpha^{\#}\gamma\gamma^{\#} = \alpha^{\#}$$
$$\bigvee_{\gamma} \gamma\gamma^{\#}\alpha = \alpha,$$

where the supremum is taken over all the monomorphic singleton morphisms $\gamma : [q'] \hookrightarrow (X, \rho, \delta)$

Proof:

$$\alpha^{\#} = \alpha^{\#}\alpha\alpha^{\#} \leq \bigvee_{\gamma} \alpha^{\#}\gamma\gamma^{\#} \leq \alpha^{\#}\delta = \alpha^{\#}.$$

and similarly for the other equation. ∎

Lemma 2.3.11 For (X, ρ, δ) a $\mathcal{Q}$-valued set, there is an *atomic* $\mathcal{Q}$-valued set $(\bar{X}, \bar{\rho}, \bar{\delta})$ where,

1. $\bar{X}$ is the set $\{\alpha : [q] \hookrightarrow (X, \rho, \delta)\}$

2. $\bar{\rho}(\alpha) = dom(q)$ where $\alpha : [q] \hookrightarrow (X, \rho, \delta)$

3. $\bar{\delta}(\alpha, \beta) = \alpha^{\#}\beta.$

Proof:

$$\begin{aligned}\bar{\delta}\bar{\delta}(\alpha,\beta) &= \bigvee_{\gamma}\{\bar{\delta}(\alpha,\gamma) \;\&\; \bar{\delta}(\gamma,\alpha)\}\\ &= \bigvee_{\gamma}\{\alpha^{\#}\gamma\gamma^{\#}\beta\}\\ &= \alpha^{\#}\beta.\end{aligned}$$

If Q is involutive then

$$\begin{aligned}\bar{\delta}(\alpha,\beta) &= \alpha^{\circ}\beta\\ &= (\beta^{\circ}\alpha)^{\circ}\\ &= (\bar{\delta}(\beta,\alpha))^{\circ}\\ &= \bar{\delta}^{\circ}(\alpha,\beta).\end{aligned}$$

Thus it is a Q-valued set. To show that it is atomic observe that $\bar{\delta}(\alpha,\alpha) \;\&\; \bar{\delta}(\alpha,\beta) = \bar{\delta}(\alpha,\beta)$ and $\alpha^{\circ}\alpha(*,*)$ is an idempotent. Thus by examples (2.3.6) and (2.3.8) above we have $\bar{\delta} = \bigvee_{\gamma} \gamma^{\#}\gamma$. ∎

Lemma 2.3.12 For $R : (X,\rho,\delta) \to (Y,\rho,\delta)$ a morphism in $KAR(MAT(Q))$, there is a morphism $\bar{R} : (\bar{X},\bar{\rho},\bar{\delta}) \to (\bar{Y},\bar{\rho},\bar{\delta})$ in $KAR(MAT(Q))$ defined by $\bar{R}(\alpha,\beta) = \alpha^{\#}R\beta$ and $\bar{R}^{\#}(\beta,\alpha) = \beta^{\#}R^{\#}\alpha$.

Proof:

$$\begin{aligned}\bar{\delta}\bar{R}(\alpha,\beta) &= \bigvee_{\gamma}\{\bar{\delta}(\alpha,\gamma) \;\&\; \bar{R}(\gamma,\beta)\}\\ &= \bigvee_{\gamma}\{\alpha^{\#}\gamma\gamma^{\#}R\beta\}\\ &= \alpha^{\#}R\beta.\end{aligned}$$

$$\begin{aligned}\bar{R}\bar{\delta}(\alpha,\beta) &= \bigvee_{\gamma}\{\bar{R}(\alpha,\gamma)\ \&\ \bar{\delta}(\gamma,\beta)\} \\ &= \bigvee_{\gamma}\{\alpha^{\#}R\gamma\gamma^{\#}\beta\} \\ &= \alpha^{\#}R\beta.\end{aligned}$$

Similarly we can show that the equalities hold for $\bar{R}^{\#}$. ∎

The following shows that for a map R, we can guarantee that $\overline{R}$ is a map if the domain of R is atomic.

$$\begin{aligned}\bar{\delta}(\alpha,\beta) &= \alpha^{*}\beta \\ &= \alpha^{\#}\delta\beta \\ &\leq \alpha^{\#}R^{\#}R\beta \\ &= \alpha^{\#}R^{\#}\delta R\beta \\ &= \bigvee_{\gamma}\{\alpha^{\#}R^{\#}\gamma\gamma^{\#}R\beta\} \qquad \text{since } dom(R) \text{ is atomic} \\ &= \bar{R}^{\#}\bar{R}(\alpha,\beta).\end{aligned}$$

For $\mathcal{Q}$ a quantaloid denote the subcategory of $KAR(MAT(\mathcal{Q}))$ generated by the atomic $\mathcal{Q}$-valued sets by *atomic*($\mathcal{Q}$). We can show that $\overline{()} : KAR(MAT(\mathcal{Q})) \rightarrow$ *atomic*($\mathcal{Q}$) is a lax-functor. All we need to show is that $\overline{()}$ acts on a composite in the appropriate way.

$$\begin{aligned}\overline{R}\,\overline{S}(\alpha,\beta) &= \bigvee_{\gamma}\{\overline{R}(\alpha,\gamma)\ \&\ \overline{S}(\gamma,\beta)\} \\ &= \bigvee_{\gamma}\{\alpha^{*}R\gamma\gamma^{*}S\beta\} \\ &\leq \alpha^{\#}R\delta S\beta \\ &= \alpha^{\#}RS\beta \\ &= \overline{RS}(\alpha,\beta).\end{aligned}$$

with equality if and only if the middle $\mathcal{Q}$-valued set is atomic. Clearly $MAP(atomic(\mathcal{Q}))$ is equivalent to the full subcategory of $MAP(KAR(MAT(\mathcal{Q})))$ generated by the atomic $\mathcal{Q}$-valued sets.

Lemma 2.3.13 If (X, ρ, δ) is an atomic $\mathcal{Q}$-valued set then (X, ρ, δ) is isomorphic to $(\bar{X}, \bar{\rho}, \bar{\delta})$.

Proof: We define the morphism $(X, \rho, \delta) \xrightarrow{\varepsilon} (\bar{X}, \bar{\rho}, \bar{\delta})$ as follows

$$\begin{aligned} \varepsilon(\alpha, x) &= \alpha^{\#}(x) \\ \varepsilon^{\#}(x, \alpha) &= \alpha(x). \end{aligned}$$

To show that ε is an isomorphism we first show that ε is a morphism then that it is an epimorphism and a monomorphism.

$$\begin{aligned} \varepsilon\delta(\alpha, x) &= \bigvee_{\bar{x}} \{\varepsilon(\alpha, \bar{x}) \ \& \ \delta(\bar{x}, x)\} \\ &= \bigvee_{\bar{x}} \{\alpha^{\#}(\bar{x}) \ \& \ \delta(\bar{x}, x)\} \\ &= \alpha^{\#}(x) \\ &= \varepsilon(\alpha, x). \end{aligned}$$

$$\begin{aligned} \bar{\delta}\varepsilon(\alpha, x) &= \bigvee_{\gamma} \{\bar{\delta}(\alpha, \gamma) \ \& \ \varepsilon(\gamma, x)\} \\ &= \bigvee_{\gamma} \{\alpha^{\#}\gamma\gamma^{\#}(*, x)\} \\ &= \alpha^{\#}(x) \\ &= \varepsilon(\alpha, x). \end{aligned}$$

Similarly for $\varepsilon^{\#}$, so ε is a morphism in $KAR(MAT(\mathcal{Q}))$. Now we show that ε is an isomorphism. We will show that it is both an epimorphism and a monomorphism.

$$\begin{aligned} \varepsilon\varepsilon^{\#}(\alpha, \beta) &= \bigvee_{x} \{\varepsilon(\alpha, x) \ \& \ \varepsilon^{\#}(x, \beta)\} \\ &= \bigvee_{x} \{\alpha^{\#}(*, x) \ \& \ \beta(x, *)\} \\ &= \alpha^{\#}\beta(*, *) \\ &= \bar{\delta}(\alpha, \beta), \end{aligned}$$

which tells us that ε is an epimorphism. Now to show that we have a monomorphism

$$\begin{aligned}
\varepsilon^{\#}\varepsilon(x,y) &= \bigvee_{\gamma}\{\varepsilon^{\#}(x,\gamma) \mathbin{\&} \varepsilon(\gamma,y)\} \\
&= \bigvee_{\gamma}\{\gamma(x,*) \mathbin{\&} \gamma^{\#}(*,y)\} \\
&= \bigvee_{\gamma}\{\gamma\gamma^{\#}(x,y)\} \\
&= \delta(x,y).
\end{aligned}$$

It now follows that the two $\mathcal{Q}$-valued sets are isomorphic. ∎

This immediately gives us the following corollary.

Corollary 2.3.14 For $\mathcal{Q}$ a quantaloid $\bar{()}$: *atomic*($\mathcal{Q}$) $\rightarrow$ *atomic*($\mathcal{Q}$) is isomorphic to the identity functor.

When an involutive quantaloid is pseudo-rightsided then every $\mathcal{Q}$-valued set is atomic. Automatically we have $\alpha\alpha^{\circ} \leq \delta$. Pseudo-rightsidedness tells us that $\delta(x,x)$ is a symmetric idempotent and so there is a singleton morphism $\alpha_x : [x] \hookrightarrow (X,\rho,\delta)$ which takes the values $\alpha(y,*) = \delta(y,x)$. Equality now follows.

We have seen that $\overline{()}$ is a lax-functor from $KAR(MAT(\mathcal{Q}))$ into *atomic*($\mathcal{Q}$) and that the inclusion of *atomic*($\mathcal{Q}$) into $KAR(MAT(\mathcal{Q}))$ composed with $\overline{()}$ is isomorphic to the identity functor. There also happens to be a strong lax-transformation $\eta : \mathbf{1} \rightarrow \overline{()}$. For $R : (X,\rho,\delta) \rightarrow (Y,\rho,\delta)$ define $\eta_R : (X,\rho,\delta) \rightarrow (\bar{Y},\bar{\rho},\bar{\delta})$ by $\eta_R(\alpha,x) = \alpha^{\#}R(*,x)$. Now assume we have arrows $(X,\rho,\delta) \xrightarrow{R} (Y,\rho,\delta) \xrightarrow{S} (Z,\rho,\delta)$ then

$$\begin{aligned}
\overline{S}\eta_R(\beta,x) &= \bigvee_{\alpha}\{\overline{S}(\beta,\alpha) \mathbin{\&} \eta_R(\alpha,x)\} \\
&= \bigvee_{\alpha}\{\beta^{\#}S\alpha\alpha^{\#}R(*,x)\} \\
&\leq \beta^{\#}S\delta R(*,x) \\
&= \beta^{\#}SR(*,x) \\
&= \eta_{SR}(\beta,x)
\end{aligned}$$

and

$$\begin{aligned}\eta_S R(\beta, x) &= \bigvee_y \{\eta_S(\beta, y) \ \&\ R(y, x)\} \\ &= \bigvee_y \{\beta^{\#} S(*, y) \ \&\ R(y, x)\} \\ &= \beta^{\#} SR(*, x) \\ &= \eta_{SR}(\beta, x)\end{aligned}$$

So η is a strong lax-transformation. Observe that if (X, ρ, δ) is atomic then η is a transformation and recall that $\overline{()}$ is then a functor. Later we will use these to show that atomic $\mathcal{Q}$-valued sets and the strictness property of Gylys[9] are essentially the same.

Definition 2.3.15 For (X, ρ, δ), a $\mathcal{Q}$-valued set, and x a (symmetric) idempotent element of X, if

$$\begin{aligned}\delta(x, x) \ \&\ \delta(x, \bar{x}) &= \delta(x, \bar{x}) \\ \delta(\bar{x}, x) \ \&\ \delta(x, x) &= \delta(\bar{x}, x)\end{aligned}$$

for every $\bar{x}$ in X, then we say that x is *strict.* A $\mathcal{Q}$-valued set (X, ρ, δ) is said to be *strict* if every x in X, is strict. Note that every arrow $\delta(\bar{x}, x)$ is then an arrow in $KAR(\mathcal{Q})$ ($\delta(\bar{x}, x) : [x] \rightarrow [\bar{x}]$). Denote the subcategory of $\mathcal{Q}$-*Set* generated by the strict objects by $Strict(\mathcal{Q})$.

Example 2.3.16 If x is a strict element of (X, ρ, δ) then there is a singleton morphism $\alpha_x : [x] \rightarrow (X, \rho, \delta)$, given by $\alpha_x(\bar{x}, *)$ equal to $\delta(\bar{x}, x)$ and $\alpha_x^{\#}(*, \bar{x})$ equal to $\delta(x, \bar{x})$. Note that α_x is a monomorphism.

Example 2.3.17 Every singleton $\mathcal{Q}$-valued set is strict since $\delta(*, *)$ is an idempotent.

Example 2.3.18 If the $\mathcal{Q}$-valued set (X, ρ, δ) is strict then for every x in X there is an associated singleton morphism α_x, as in example 2. In other words there is a singleton morphism that equals $\delta(x, \bar{x})$, for every x and $\bar{x}$ in X.

Lemma 2.3.19 For q a (symmetric) idempotent in $\mathcal{Q}$ and $\alpha : [q] \to (X, \rho, \delta)$ a singleton morphism, if x in X is strict then

$$\begin{aligned} \delta(x,x) \,\&\, \alpha(x) &= \alpha(x) \\ \alpha^{\#}(x) \,\&\, \delta(x,x) &= \alpha^{\#}(x). \end{aligned}$$

.

Proof:

$$\begin{aligned} \alpha(x) &= \bigvee_{\bar{x}} \{\delta(x,\bar{x}) \,\&\, \alpha(\bar{x})\} \\ &= \bigvee_{\bar{x}} \{\delta(x,x) \,\&\, \delta(x,\bar{x}) \,\&\, \alpha(\bar{x})\} \\ &= \delta(x,x) \,\&\, \bigvee_{\bar{x}} \{\delta(x,\bar{x}) \,\&\, \alpha(\bar{x})\} \\ &= \delta(x,x) \,\&\, \alpha(x). \end{aligned}$$

The argument for $\alpha^{\#}(x)$ is similar. ∎

This tells us that the morphisms $\alpha(x)$ and $\alpha^{\#}(x)$ are morphisms in $KAR(\mathcal{Q})$. So we will want to describe a *matching family* as a family of morphisms in $KAR(\mathcal{Q})$ satisfying some conditions. This we will do in the next section.

Lemma 2.3.20 If an involutive quantaloid $\mathcal{Q}$ is pseudo-rightsided then each $\mathcal{Q}$-valued set is strict.

Proof: This follows immediately from the fact that pseudo-rightsidedness of $\mathcal{Q}$ gives $\delta(x,y) = \delta(x,y) \& \delta(y,x) \& \delta(x,y)$. ∎

Example 2.3.21 Every complete Heyting algebra $\mathcal{H}$ satisfies Freyd's modular law and thus every $\mathcal{H}$-valued set is strict.

Example 2.3.22 A bounded complete distributive category of relations $\mathcal{Q}$ satisfies Freyd's modular law and so every $\mathcal{Q}$-valued set is strict.

Lemma 2.3.23 If (X,ρ,δ) is a strict $\mathcal{Q}$-valued set then $\bigvee_{\gamma}\{\gamma\gamma^{\#}\} = \delta$

Proof: Clearly we have

$$\bigvee_{\gamma}\{\gamma\gamma^{\#}\} \leq \delta.$$

Since (X,ρ,δ) is a strict $\mathcal{Q}$-valued set we have monomorphic singleton morphisms α_x, with $\alpha_x\alpha_x^{\#}(x,y) = \delta(x,x) \,\&\, \delta(x,y)$, so $\bigvee_{x}\alpha_x\alpha^{\#} = \delta$. ∎

Putting all the pieces together immediately gives us the following.

Theorem 2.3.24 For a quantaloid $\mathcal{Q}$, *Strict*$(\mathcal{Q})$ is equivalent to *atomic*$(\mathcal{Q})$

Proof: The functors for the equivalence are given by

$$\mathit{Strict}(\mathcal{Q}) \underset{\iota}{\overset{\overline{()}}{\rightleftarrows}} \mathit{atomic}(\mathcal{Q})$$

First we have to show that for (X,ρ,δ) an atomic $\mathcal{Q}$-valued set, $\overline{\delta}$ is strict. Observe that

$$\delta(\alpha,\alpha) \,\&\, \delta(\alpha,\beta) = \alpha^{\#}\alpha\alpha^{\#}\beta = \alpha^{\#}\beta = \delta(\alpha,\beta)$$
$$\delta(\alpha,\beta) \,\&\, \delta(\beta,\beta) = \alpha^{\#}\beta\beta^{\#}\beta = \alpha^{\#}\beta = \delta(\alpha,\beta)$$

Thus $\overline{\delta}$ is strict. We already have that $\overline{()}$ is isomorphic to the identity on *atomic*$(\mathcal{Q})$ and since strict implies atomic we have that for a $\mathcal{Q}$-valued set $\overline{()} : \mathit{Strict}(\mathcal{Q}) \to \mathit{Strict}(\mathcal{Q})$ is isomorphic to the identity on *Strict*$(\mathcal{Q})$. ∎

2.4 Sheaves for an Involutive Quantaloid

Much of the study of *sheaves for a quantaloid* $\mathcal{Q}$, has depended on the definition of $\mathcal{Q}$-valued sets. Few of these definitions have been given in terms of set valued functors. In this section we give a definition of sheaf for an involutive quantaloid as a set valued functor which has a unique *amalgamation* for every *matching family.* We begin by first describing the category of *presheaves* for an involutive quantaloid $\mathcal{Q}$.

2.4.1 Presheaves for a Quantaloid

We saw in the previous section that the subcategory of $\mathcal{Q}$-valued sets generated by the singleton objects is equivalent to $MAP(KAR(\mathcal{Q}))$. We also saw that an atomic $\mathcal{Q}$-valued set was equivalent to the $\mathcal{Q}$-valued set constructed out of the monomorphic singleton morphisms and that if $\mathcal{Q}$ is pseudo-rightsided every $\mathcal{Q}$-valued set was strict. Recall that for $\mathcal{H}$ a Heyting algebra considered as a one object **SUP**-enriched category, $MAP(KAR(\mathcal{H}))$ is equivalent to $\mathcal{H}$ when we interpret it as a multiple object category. Note that every arrow in $MAP(KAR(\mathcal{H}))$ is a monomorphism. This is the key that we will use to describe the category of presheaves for $\mathcal{Q}$.

Definition 2.4.1 For $\mathcal{Q}$ an involutive quantaloid define $\overline{\mathcal{Q}}$ to be the category $MONO(MAP(^{*}KAR(\mathcal{Q})))$. That is the category with objects the symmetric idempotents in $\mathcal{Q}$ and arrows the monomorphic maps (Recall that when $\mathcal{Q}$ is involutive we ask that a map p satisfy $p^{\#} = p^{*}$).

Example 2.4.2 Recall that the category of relations, **REL**, together with the functor that sends a relation to the opposite relation is an involutive **SUP**-category. $\overline{\mathbf{REL}}$ is then the category with objects symmetric transitive interpolative relations δ_X, and arrows maps $f : \delta_X \to \delta_Y$, such that $\delta_Y f = f = f\delta_X$ and $f^{\circ}f = \delta_X$. By eliminating those elements that are not related to themselves $(\delta_X(x,x) = 0)$, we can associate δ_X with a set. The morphism f then is a one to one function from δ_X to δ_Y. We then have that $\overline{REL}$ is equivalent to the category with objects sets and morphisms one to one functions.

Definition 2.4.3 For $\mathcal{Q}$ an involutive quantaloid, the category of *pre-sheaves* for $\mathcal{Q}$ is the functor category $SET^{\overline{\mathcal{Q}}^{op}}$.

2.4.2 Sheaves for a Quantaloid

We now give a definition of a sheaf in terms of the amalgamation of matching families. This definition is applicable to all involutive quantaloids but we will achieve an equivalence with the category of $\mathcal{Q}$-valued sets only for pseudo-rightsided quantaloids.

Definition 2.4.4 For $q : A \longrightarrow A$, a symmetric idempotent in $\mathcal{Q}$, a *cover* of q is a family of arrows in ${}^*KAR(\mathcal{Q})$

$$\langle p_i : q \longrightarrow q_i \rangle,$$

such that

$$\bigvee_i \{p_i^* p_i\} = q.$$

Definition 2.4.5 For a cover $\langle p_i : q \longrightarrow q_i \rangle$ and F a presheaf, a *matching family* is a family $\langle x_i \in F(q_i) \rangle$ such that

1. $p_i p_j^* \ \leq \ \bigvee \{p_1 p_2^* : x_{i_{|p_1}} = x_{j_{|p_2}}\}$ for all i, j
2. If $x_{i_{|p_1}} = x_{j_{|p_2}}$ then $p_1 p_2^* p_i \ \leq \ p_j$ for all i, j.

Where $p_1 : r \longrightarrow q_i$ and $p_2 : r \longrightarrow q_j$ are arrows in $\overline{\mathcal{Q}}$.

If $\mathcal{H}$ is a complete Heyting algebra then it is easy to see that a cover $\langle h_i \rangle$ and a matching family $\langle x_i \in F(h_i) \rangle$ in the traditional sense is a matching family and cover in this sense. On the other hand if we have a cover and a matching family in this new sense then we need not have a matching family in the classic sense since $x_{i|_{h_i \wedge h_j}}$ need not equal $x_{j|_{h_i \wedge h_j}}$. We do have $\bigvee\{k \mid x_{i_{|k}} = x_{j_{|k}}\} = h_i \wedge h_j$. This does represent a cover and matching family. The cover being the k such that $x_{i_{|k}} = x_{j_{|k}}$ and the matching family $< x_{i_{|k}} >$. So we have many covers and matching families patched together.

Definition 2.4.6 For a matching family $\langle x_i \in F(q_i)\rangle$ of the cover $\langle p_i : q \to q_i\rangle$ an *amalgamation* is a $y \in F(q)$ such that for every $i \in I$;

$$p_i \;=\; \bigvee\{p_1 p_2^* : x_{i_{|p_1}} \;=\; y_{|p_2}\}.$$

Definition 2.4.7 For $\mathcal{Q}$ a quantaloid a presheaf F is a *sheaf* if every matching family has a unique amalgamation. Denote the full subcategory of $SET^{\overline{\mathcal{Q}}^{op}}$ determined by the sheaves by $SHV(\mathcal{Q})$.

For $\mathcal{H}$ a Heyting algebra and F a sheaf in this sense, if we have $h_i = \bigvee\{k \mid x_{i_{|k}} = y_{|_k}\}$, then the k's are a cover for h_i and the $x_{i_{|k}}$ are a matching family for the cover. Thus there is a unique amalgamation of this. This immediately implies that $x_i = y_{|_{h_i}}$ since both are an amalgamation for this family.

2.4.3 The Associated Sheaf Functor

For F an object in $SET^{\overline{\mathcal{Q}}^{op}}$, define $\Psi(F)$ to be the triple (X_F, ρ_F, Ψ_F) where

$$X_F \;=\; \textstyle\sum_q F(q)$$
$$\rho_F(x) \;=\; dom(q) \text{ if } x \in F(q)$$

And for $x \in F(q_1)$ and $y \in F(q_2)$

$$\Psi_F(x,y) \;= \{p_1 \;\&\; p_2^* \mid p_1 : r \to q_1; p_2 : r \to q_2;\ \text{and } x_{|p_1} = y_{|p_2}\}^{\downarrow}.$$

For a natural transformation $F \xrightarrow{\tau} G$ define $\Psi_\tau : \Psi_F \to \Psi_G$ by

$$\Psi_\tau(x,y) \;= \{p_1 \& p_2^* : x_{|p_1} \;=\; \tau_{y_{|p_2}}\}^{\downarrow}.$$

It follows immediately that $\Psi_\tau(x,y) = \Psi_G(x, \tau_y)$. Where for $y \in F(q)$, τ_y is short for $\tau_q(y)$. Also define Ψ_τ° by

$$\Psi_\tau^\circ(y,x) \;=\; \{p_2 \& p_1^* : x_{|p_1} \;=\; \tau_{y_{|p_2}}\}^{\downarrow}.$$

As before we immediately have $\Psi_\tau^\circ = \Psi_G(\tau_y, x)$.

Theorem 2.4.8 For a pseudo-rightsided quantaloid $\mathcal{Q}$, Ψ is a functor from $SET^{\overline{\mathcal{Q}}^{op}}$ to $D\mathcal{Q}$-SET.

Proof: Clearly $\Psi_F : (X_F, \rho_F) \to (X_F, \rho_F)$ is an endo matrix. We first show that Ψ_F is a symmetric idempotent.

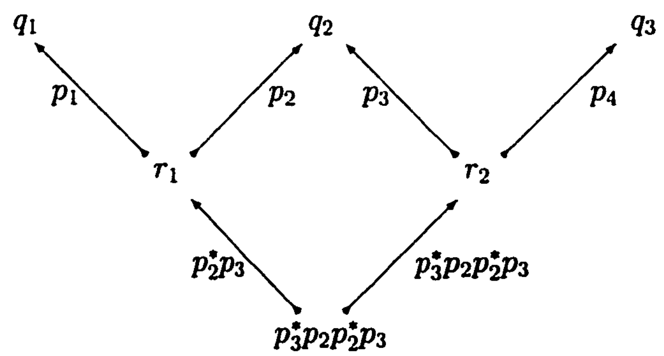
$$\begin{aligned}
\Psi_F^{\circ}(y,x) &= \Psi_F(x,y)^* \\
&= \{\{p_1 \,\&\, p_2^* \mid x_{|p_1} = y_{|p_2}\}^{\downarrow}\}^* \\
&= \{(p_1 \,\&\, p_2^*)^* \mid x_{|p_1} = y_{|p_2}\}^{\downarrow} \\
&= \{p_2 \,\&\, p_1^* \mid x_{|p_1} = y_{|p_2}\}^{\downarrow} \\
&= \Psi(y,x).
\end{aligned}$$

Thus we have that Ψ_F is symmetric.

$$\begin{aligned}
\Psi_F\Psi_F(x,y) &= \bigcup_z\{\Psi_F(x,z) \,\&\, \Psi(z,y)\} \\
&= \bigcup_z\{q_1q_2 \mid (q_1 \in F(x,z) \text{ and } q_2 \in F(z,y)\}^{\downarrow} \\
&= \bigcup_z\{p_1p_2^*p_3p_4^* \mid x_{|p_1} = z_{|p_2} \text{ and } z_{|p_3} = y_{|p_4}\}^{\downarrow}.
\end{aligned}$$

Now assume that $x_{|p_1} = y_{|p_2}$ (that is $p_1p_2^* \in \Psi_F(x,y)$). Then clearly $p_1p_2^* = p_1p_1^*p_1p_2^* \in \Psi_F\Psi_F(x,y)$. Which shows that $\Psi_F \leq \Psi_F\Psi_F$. To obtain idempotency we need the following lemma.

Lemma 2.4.9 if $p_1 : r_1 \hookrightarrow q_1$, $p_2 : r_1 \hookrightarrow q_2$, $p_3 : r_2 \hookrightarrow q_2$, $p_4 : r_2 \hookrightarrow q_3$, then we have the following diagram in $\overline{Q}$.



Proof:

To show that $p_3^* p_2 p_2^* p_3$ is an idempotent observe that we have

$$\begin{aligned} p_3^* p_2 (p_3^* p_2) p_3^* p_2 (p_3^* p_2) &= p_3^* p_2 p_2^* p_3 p_3^* p_2 p_2^* p_3 \\ &\leq p_3^* p_2 p_2^* p_3. \end{aligned}$$

Since Q is pseudo-rightsided the above becomes an equality. Now we clearly have that $p_3^* p_2 p_2^* p_3$ is a monomorphism from $p_3^* p_2 p_2^* p_3$ to r_2 To show that $p_2^* p_3$ is a morphism from $p_3^* p_2 p_2^* p_3$ to r_1 the only difficult part is to show that $p_2^* p_3 p_3^* p_2 p_2^* p_3$ is equal to $p_2^* p_3$. Again we must rely on the pseudo-right sidedness of Q since we do have that $p_2^* p_3 p_3^* p_2 p_2^* p_3 \leq p_2^* p_3$ by adjointness. $\diamond$

This lemma tells us that if : $x_{|_{p_1}} = z_{|_{p_2}}$ and $z_{|_{p_3}} = y_{|_{p_4}}$ then $x_{|_{p_1 p_2^* p_3}} = y_{|_{p_4 p_3^* p_2 p_2^* p_3}}$. Thus if $p_1 p_2^* p_3 p_4^* \in \Psi_F \Psi_F(x, y)$ then $(p_1 p_2^* p_3)(p_3^* p_2 p_2^* p_3 p_4^*)$ is an element in $\Psi_F(x, y)$. As we saw pseudo-rightsidedness implies that $p_2^* p_3 p_3^* p_2 p_2^* p_3 = p_2^* p_3$. So we have $p_1 p_2^* p_3 p_4^*$ is an element in $\Psi_F(x, y)$. This shows that $\Psi_F \Psi_F \leq \Psi_F$. Thus we have Ψ_F is an idempotent and so a Q-valued set.

The next lemma tells us directly what the value of $\Psi_F(x, x)$ is. This will help when we examine how Ψ acts on morphisms.

Lemma 2.4.10 If $x \in F(q)$ then $\Psi_F(x,x) = \downarrow q$.

Proof: For every $q_i \overset{p}{\hookrightarrow} q$ we have $pp^* \leq q$ and q is a mono subobject of itself so the result follows. $\diamondsuit$

The next lemma explicitly tell us that for a natural transformation $\tau : F \longrightarrow G$, Ψ_τ is a morphism in $KAR(MAT(DQ))$.

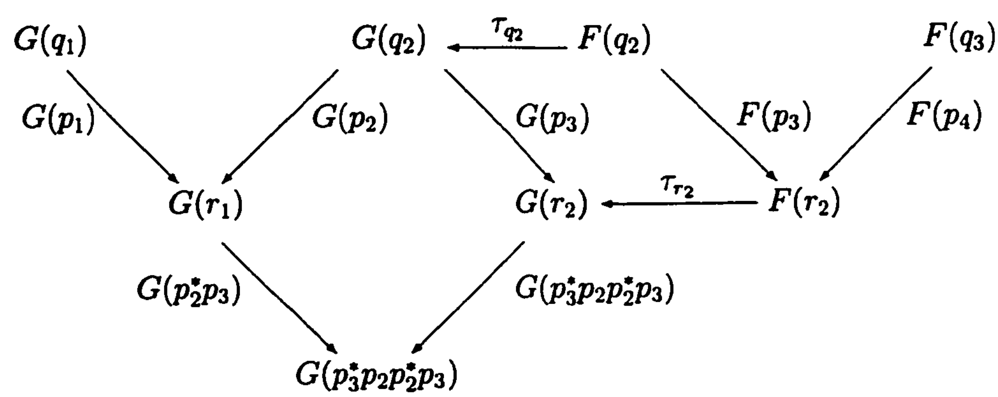
$q_2^\downarrow$

Lemma 2.4.11 $\bigcup_z \{\Psi_G(x,\tau_z) \& \Psi_F(z,y)\}^\downarrow = \Psi_G(x,\tau_y)$ and $\bigcup_y \{\Psi_F(x,y) \& \Psi_G(\tau_y,z)\}^\downarrow = \Psi_G(\tau_x,z)$.

Proof: First we clearly have

$$\bigcup_z \{\Psi_G(x,\tau_z) \& \Psi_F(z,y)\}^\downarrow \supseteq \Psi_G(x,\tau_y) \& \Psi_F(y,y).$$

Now assume that for $x \in G(q_1)$, $z \in F(q_2)$ and $y \in F(q_3)$ we have that $x_{|_{p_1}} = \tau_{z_{|_{p_2}}}$ and $z_{|_{p_3}} = y_{|_{p_4}}$. Pictorially we have

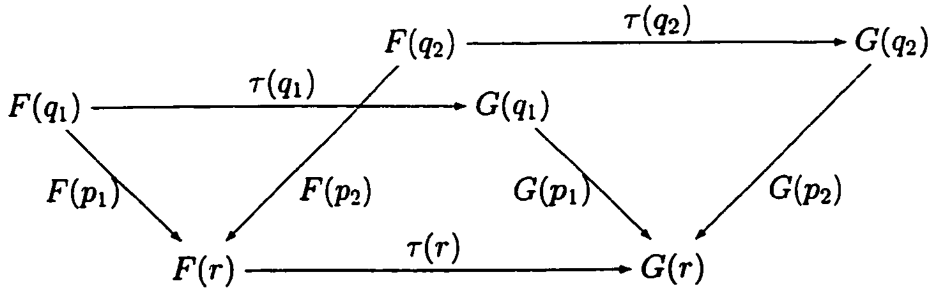


Now if we trace x and y through the diagram we see that $x_{|_{p_1p_2^*p_3}} = \tau_{y_{|_{p_4p_3^*p_2p_2^*p_3}}}$. So we have that $p_1p_2^*p_3p_4^*$ is an element of $\Psi_G(x,\tau_y)$. Thus $\bigcup_z \{\Psi_G(x,\tau_z) \& \Psi_F(z,y)\}^\downarrow \subseteq \Psi_G(x,\tau_y)$, hence we have the desired equality. The argument for the other equality is similar. This is how the composite of morphisms is defined in DQ-SET, thus we have that Ψ_τ is a morphism in $KAR(MAT(DQ))$. $\diamondsuit$

Now we need to show that Ψ_τ is a map.

Lemma 2.4.12 $\Psi_G(\tau_x, \tau_y) \supseteq \Psi_F(x, y)$.

Proof: Assume that for $x \in F(q_1)$ and $y \in F(q_2)$, $x_{|p_1} = y_{|p_2}$. Pictorially



By naturality of τ, $\tau_{x_{|p_1}} = \tau_{y_{|p_2}}$ and so we have $\Psi_G(\tau_x, \tau_y) \supseteq \Psi_F(x, y)$. ◊

We can now show that Ψ_τ is a map.

$$\begin{aligned}
\Psi_\tau^\circ \Psi_\tau(x, y) &= \bigcup_z \{\Psi_\tau^\circ(x, z) \& \Psi_\tau(z, y)\}^\downarrow \\
&= \bigcup_z \{\Psi_G(\tau_x, z) \& \Psi_G(z, \tau_y)\}^\downarrow \\
&= \Psi_G(\tau_x, \tau_y) \\
&\supseteq \Psi_F(x, y).
\end{aligned}$$

$$\begin{aligned}
\Psi_\tau \Psi_\tau^\circ(x, y) &= \bigcup_z \{\Psi_\tau(x, z) \& \Psi_\tau^\circ(z, y)\}^\downarrow \\
&= \bigcup_z \{\Psi_G(x, \tau_z) \& \Psi_G(\tau_z, y)\}^\downarrow \\
&\subseteq \bigcup_w \{\Psi_G(x, w) \& \Psi_G(w, y)\}^\downarrow \\
&= \Psi_G(x, y).
\end{aligned}$$

Thus Ψ_τ is a DQ-Set morphism. Now let $G \xrightarrow{\sigma} H$ be another natural transformation and examine the composite transformation.

$$\begin{aligned}
\Psi_\sigma\Psi_\tau(x,y) &= \bigcup_z\{\Psi_\sigma(x,z)\&\Psi_\tau(z,y)\}^\downarrow \\
&= \bigcup_z\{\Psi_H(x,\sigma_z)\&\Psi_G(z,\tau_y)\}^\downarrow \\
&= \Psi_H(x,\sigma\tau_y) \qquad \text{by 2.4.11} \\
&= \Psi_{\sigma\tau}(x,y).
\end{aligned}$$

$$\begin{aligned}
\Psi_\tau^\circ\Psi_\sigma^\circ(y,x) &= \bigcup_z\{\Psi_\tau^\circ(y,z)\&\Psi_\sigma^\circ(z,x)\}^\downarrow \\
&= \bigcup_z\{\Psi_G(\tau_y,z)\&\Psi_H(\sigma_z,x)\}^\downarrow \\
&= \Psi_H(\sigma\tau_y,x) \qquad \text{by 2.4.11} \\
&= \Psi_{\sigma\tau}(y,x).
\end{aligned}$$

Finally for the identity transformation $\tau_F : F \to F$ we have

$$\Psi_{\tau_F}(x,y) = \Psi_F(x,\tau_F(y)) = \Psi_F(x,y)$$

and so Ψ is a functor. ∎

Composing Ψ with $\widehat{()}$: *DQ-SET* $\to$ *Q-SET* gives us a functor from $Set^{\overline{\mathcal{Q}}^{op}}$ to *Q-SET*. We will refer to the composite $\widehat{()}\Psi$ simply as $\widehat{\Psi}$.

We now construct the right adjoint to $\widehat{\Psi}$. We will presently see that composing with $\widehat{\Psi}$ is the associated sheaf functor. We define Φ : *Q-Set* $\to Set^{\overline{\mathcal{Q}}^{op}}$ as follows;

• Objects • Given a $\mathcal{Q}$-Set (X, ρ, δ),

for q a (symmetric) idempotent,

$$\Phi_\delta(q) \;=\; \{\alpha : [q] \hookrightarrow (X, \rho, \delta)\}$$

for $q_1 \overset{p}{\hookrightarrow} q_2$ a morphism

$$\Phi_\delta(p)(\alpha) \;=\; [q_1] \overset{\alpha_p}{\hookrightarrow} [q_2] \overset{\alpha}{\hookrightarrow} (X, \rho, \delta).$$

• Arrows • Given a morphism $R : (X, \rho, \delta) \longrightarrow (Y, \rho, \delta)$

$$\Phi_R(\alpha) \;=\; [q] \overset{\alpha}{\hookrightarrow} (X, \rho, \delta) \overset{R}{\to} (Y, \rho, \delta).$$

Recall that for a symmetric idempotent q, $[q]$ is the associated $\mathcal{Q}$-Set. Clearly Φ is functorial.

Theorem 2.4.13 $\widehat{\Psi} \dashv \Phi$.

Proof: Define the unit $\eta : 1 \longrightarrow \Phi\widehat{\Psi}$ as follows; for $F : \overline{\mathcal{Q}}^{op} \longrightarrow Set$ and q a (symmetric) idempotent,

$$\eta_{F_q}(x) \;=\; \alpha_x : [q] \hookrightarrow \widehat{\Psi}_F,$$

where $\alpha_x(y, *) \;=\; \widehat{\Psi}_F(y, x)$.

To show that η is natural we show that for every natural transformation $\tau : F \longrightarrow G$, the following diagram commutes

$$\begin{array}{ccc} F_q & \xrightarrow{\;\eta_{F_q}\;} & \Phi\widehat{\Psi}_{F_q} \\ {\scriptstyle \tau_q}\downarrow & & \downarrow{\scriptstyle \Phi\widehat{\Psi}_{\tau_q}} \\ G_q & \xrightarrow{\;\eta_{G_q}\;} & \Phi\widehat{\Psi}_{G_q} \end{array}$$

For $x \in F_q$ the top of the square takes x to $\widehat{\Psi}_\tau \circ \alpha_x$, while the bottom takes x to α_{τ_x}

$$
\begin{aligned}
\widehat{\Psi}_\tau \circ \alpha_x(y,*) &= \bigvee_z \{\widehat{\Psi}_\tau(y,z) \& \alpha_x(z,*)\} \\
&= \bigvee_z \{\widehat{\Psi}_G(y,\tau_z) \& \widehat{\Psi}_F(z,x)\} \\
&= \widehat{\Psi}_G(y,\tau_x) \\
&= \alpha_{\tau_x}.
\end{aligned}
$$

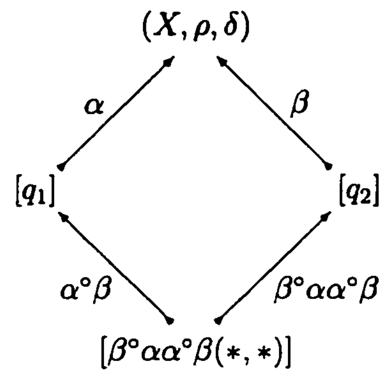
Thus η is natural. We now define the counit $\varepsilon : \widehat{\Psi}\Phi \longrightarrow 1$ as follows, for (X,ρ,δ) a $\mathcal{Q}$-Set let $\varepsilon_\delta(x,\alpha) = \alpha(x,*)$.

Lemma 2.4.14 $\widehat{\Psi}\Phi_\delta(\alpha,\beta) = \alpha^\circ\beta$.

Proof: Assume $\alpha\alpha_{p_1} = \beta\alpha_{p_2}$. Then we have

$$
\begin{aligned}
\alpha_{p_1}\alpha^\circ_{p_2} &= \delta_{q_1}\alpha_{p_1}\alpha^\circ_{p_2} \\
&= \alpha^\circ\alpha\alpha_{p_1}\alpha^\circ_{p_2} \\
&= \alpha^\circ\beta\alpha_{p_2}\alpha^\circ_{p_2} \\
&\leq \alpha^\circ\beta\delta_{q_2} \\
&= \alpha^\circ\beta.
\end{aligned}
$$

This tells us that $\widehat{\Psi}\Phi_\delta(\alpha,\beta) \leq \alpha^\circ\beta$. Now examine the following diagram



Using the adjunction inequalities the composite $\beta\beta^\circ\alpha\alpha^\circ\beta$ is less than or equal to $\alpha\alpha^\circ\beta$. We now have

$$\begin{aligned}
\alpha\alpha^\circ\beta &= \alpha\alpha^\circ\beta\beta^\circ\alpha\alpha^\circ\alpha\alpha^\circ\beta && \text{by pseudo-rightsidedness}\\
&= \alpha\alpha^\circ\beta\beta^\circ\alpha\alpha^\circ\beta && \text{since } \alpha \text{ is mono}\\
&\le \delta\beta\beta^\circ\alpha\alpha^\circ\beta && \text{adjointness}\\
&= \beta\beta^\circ\alpha\alpha^\circ\beta.
\end{aligned}$$

It follows that $\beta\beta^\circ\alpha\alpha^\circ\beta$ and $\alpha\alpha^\circ\beta$ are equal. Thus $\alpha^\circ\beta\beta^\circ\alpha\alpha^\circ\beta$ $(p_1p_2^*)$, is equal to $\alpha^\circ\beta$ by pseudo-rightsidedness. This is less than $\widehat{\Psi}\Phi_\delta(\alpha,\beta)$, hence we have equality. ♢

It is now clear that for a morphism $R : (X,\rho,\delta) \to (Y,\rho,\delta)$, $\widehat{\Psi}\Phi_R(\alpha,\beta) = \alpha^\circ R\beta$. Now from work done in the previous chapter, ε is a natural isomorphism. To show that this forms an adjunction we show that the triangle equalities hold. First examine $\Phi_\delta \xrightarrow{\eta\Phi} \Phi\widehat{\Psi}\Phi_\delta \xrightarrow{\Phi\varepsilon} \Phi_\delta$. $\eta_{\Phi_\delta}(\beta : [q] \to (X,\rho,\delta))$ is the arrow $R_\beta(\gamma,*) = \widehat{\Psi}\Phi_\delta(\gamma,\beta) = \gamma^\circ\beta$. So we have $\Phi_\varepsilon\eta_{\Phi_\delta}(\beta)$ is equal to $\varepsilon_\delta R_\beta$.

$$\begin{aligned}
\varepsilon_\delta R_\beta(x,*) &= \bigvee_\gamma\{\varepsilon_\delta(x,\gamma) \ \&\ R_\beta(\gamma,*)\}\\
&= \bigvee_\gamma\{\gamma(x,*) \ \&\ \gamma^\circ\beta(*,*)\}\\
&= \bigvee_\gamma\{\gamma\gamma^\circ\beta(x,*)\}\\
&= \bigvee_{\alpha'}\{\delta_X\beta(x,*)\} \quad \text{atomic}\\
&= \beta(x,*).
\end{aligned}$$

Thus we have that the composite $\Phi_\delta \xrightarrow{\eta\Phi} \Phi\widehat{\Psi}\Phi_\delta \xrightarrow{\Phi\varepsilon} \Phi_\delta$ is the identity transformation on Φ. Now examine the other triangle $\widehat{\Psi}_F \xrightarrow{\widehat{\Psi}\eta_F} \widehat{\Psi}\Phi\widehat{\Psi}_F \xrightarrow{\varepsilon_{\widehat{\Psi}_F}} \widehat{\Psi}_F$. Note that $\varepsilon_{\widehat{\Psi}_F}(x,\alpha) = \alpha(x,*)$, where $\alpha : [q] \to \widehat{\Psi}_F$, and

$$
\begin{aligned}
\widehat{\Psi}_{\eta_F}(\beta, y) &= \bigvee\{p_1 p_2^* : \alpha_{p_1}\beta = \alpha_{p_2}\alpha_y\} \\
&= \widehat{\Psi}\Phi\widehat{\Psi}_F(\beta, \alpha_y) \\
&= \beta^\circ \alpha_y(*, *).
\end{aligned}
$$

So the composite becomes

$$
\begin{aligned}
\varepsilon_{\widehat{\Psi}_F}\widehat{\Psi}_{\eta_F}(x, y) &= \bigvee_\alpha\{\varepsilon_{\widehat{\Psi}_F}(x, \alpha) \ \& \ \widehat{\Psi}_{\eta_F}(\alpha, y)\} \\
&= \bigvee_\alpha\{\alpha(x, *) \ \& \ \alpha^\circ\alpha_y(*, *)\} \\
&= \bigvee_\alpha\{\alpha(x, *) \ \& \ \bigvee_z\{\alpha^\circ(*, z) \ \& \ \alpha_y(z, *)\}\} \\
&= \bigvee_{\alpha, z}\{\alpha(x, *) \ \& \ \alpha^\circ(*, z) \ \& \ \alpha_y(z, *)\} \\
&= \bigvee_z\{\widehat{\Psi}_F(x, z) \ \& \ \widehat{\Psi}(z, y)\} \qquad \text{atomic} \\
&= \widehat{\Psi}_F(x, z).
\end{aligned}
$$

So we have $\widehat{\Psi} \dashv \Phi$. ∎

The results of the next subsection show that the composite $\Phi\widehat{\Psi}$ is the associated sheaf functor.

2.4.4 $\mathcal{Q}$-SET is Equivalent to $SHV(\mathcal{Q})$

Recall that ε_δ is an isomorphism if (X, ρ, δ) is a strict $\mathcal{Q}$-*Set* and further that pseudo-rightsidedness implies that every $\mathcal{Q}$-valued set is strict.

Lemma 2.4.15 For $\mathcal{Q}$ a pseudo-rightsided quantaloid, F is a sheaf if and only if η_F is an isomorphism.

Proof: Assume that F is a sheaf. Then for a $\mathcal{Q}$-SET morphism $\alpha : [q] \hookrightarrow \widehat{\Psi}_F$ the following family of arrows,

$$\langle \alpha(x) : q \to \widehat{\Psi}_F(x,x)\rangle_{x\in X}$$

is a cover of q and $\langle x \in F(\widehat{\Psi}_F(x,x))\rangle$ is a matching family since we have

$$\alpha(x) \mathbin{\&} \alpha^{\circ}(x) \;\leq\; \widehat{\Psi}_F(x,y) \;=\; \bigvee\{p_1p_2^* : x_{|p_1} = y_{|p_2}\}.$$

If $x_{p_1} = y_{|p_2}$ then we have

$$\begin{aligned} p_1p_2^* \mathbin{\&} \alpha(y) &\leq \bigvee\{p_1p_2^* : x_{|p_1} = y_{|p_2}\} \mathbin{\&} \alpha(y) \\ &= \widehat{\Psi}_F(x,y) \mathbin{\&} \alpha(y) \\ &\leq \alpha(x). \end{aligned}$$

So there is a unique amalgamation x_0 such that

$$\begin{aligned} \alpha(x) &= \bigvee\{p_1p_2^* : x_{|p_1} = x_{0|p_2}\} \\ &= \alpha_{x_0}(x). \end{aligned}$$

Thus η_F is an isomorphism. Now assume that η_F is an isomorphism and let $\langle p_i : q \to q_i\rangle_{i\in I}$ be a cover of q and $\langle x_i \in F(q_i)\rangle$ a matching family.

Define $\alpha : [q] \to \widehat{\Psi}_F$ by

$$\alpha(x) = \begin{cases} p_i & \text{if } x = x_i \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha^{\circ}(x) = \begin{cases} p_i^* & \text{if } x = x_i \\ \bot & \text{otherwise.} \end{cases}$$

We need to show that α is a morphism.

$$\begin{aligned} \alpha\delta_q(x,*) &= \alpha(x) \mathbin{\&} q \\ &= \alpha(x). \end{aligned}$$

$$
\begin{aligned}
\widehat{\Psi}_F\alpha(x,*) &= \bigvee_y\{\widehat{\Psi}(x,y)\ \&\ \alpha(y)\} \\
&= \bigvee_y\{p_1p_2^* : x|_{p_1} = y|_{p_2}\}\ \&\ \alpha(y) \\
&= \bigvee_y\{p_1p_2^*\alpha(y) : x|_{p_1} = y|_{p_2}\} \\
&\le \alpha(x).
\end{aligned}
$$

But we know that for the symmetric idempotent q_x, we have that $\alpha(x) = q_xq_x\alpha(x)$. So it follows that $\widehat{\Psi}_F\alpha(x,*)$ is equal to $\alpha(x)$.

$$
\begin{aligned}
\alpha^\circ\alpha(*,*) &= \bigvee_x\{\alpha^\circ(x)\ \&\ \alpha(x)\} \\
&= \bigvee_x(p_i^*p_i) \ = \ q.
\end{aligned}
$$

If $\alpha(x) = \bot$ then we clearly have $\alpha(x)\ \&\ \alpha^\circ(y) \le \widehat{\Psi}_F(x,y)$. So assume x and y are such that $x = x_i$ and $y = x_j$ for some i, j.

$$
\begin{aligned}
\alpha\alpha^\circ(x,y) &= \alpha(x)\ \&\ \alpha^\circ(y) \\
&= p_ip_j^* \\
&\le \bigvee\{p_1p_2^* : x_{|p_1} = y_{|p_2}\} \\
&= \widehat{\Psi}_F(x,y).
\end{aligned}
$$

Thus there is a unique $y \in F(q)$ such that

$$
p_i \ = \ \alpha(x_i) \ = \ \alpha_y(x_i) \ = \ \bigvee\{p_1p_2^* : x_{i_{|p_1}} = y_{|p_2}\}.
$$

∎

Theorem 2.4.16 For $\mathcal{Q}$ a pseudo-right sided quantaloid, $SHV(\mathcal{Q})$ is equivalent to $\mathcal{Q}$-SET.

Proof: We know that for any F, $\widehat{\Psi}_F$ is strict. Also since $\mathcal{Q}$ is pseudo-rightsided every $\mathcal{Q}$-valued set is strict thus the counit of the adjunction ε is a natural isomorphism. So we want to show that for (X,ρ,δ) a $\mathcal{Q}$-SET, Φ_δ is a sheaf. We will show that $\Phi_\delta \xrightarrow{\eta_{\Phi_\delta}} \Phi\widehat{\Psi}\Phi_\delta$ is an isomorphism from which the result follows. Let $A : [q] \hookrightarrow \widehat{\Psi}\Phi_\delta$. We wish to find an $\alpha : [q] \hookrightarrow (X,\rho,\delta)$ such that $A(\beta,*)$ is equal to $\widehat{\Psi}\Phi_\delta(\beta,\alpha) = \beta^\circ\alpha$. To this end examine

$$\begin{aligned}
A(\beta,*) &= \widehat{\Psi}\Phi_\delta A(\beta,*) \\
&= \bigvee_\gamma \{\widehat{\Psi}\Phi_\delta(\beta,\gamma) \mathbin{\&} A(\gamma,*)\} \\
&= \bigvee_\gamma \{\beta^\circ\gamma(*,*) \mathbin{\&} A(\gamma,*)\} \\
&= \bigvee_\gamma \{\bigvee_x \{\beta^\circ(*,x) \mathbin{\&} \gamma(x,*)\} \mathbin{\&} A(\gamma,*)\} \\
&= \bigvee_x \{\beta^\circ(*,x) \mathbin{\&} \bigvee_\gamma \{\gamma(x,*) \mathbin{\&} A(\gamma,*)\}\}.
\end{aligned}$$

Notice that we have β° composed with 'something'. We define $\alpha : [q] \hookrightarrow (X,\rho,\delta)$ as

$$\begin{aligned}
\alpha(x,*) &= \vee_\gamma \{\gamma(x,*) \mathbin{\&} A(\gamma,*)\} \\
\alpha^\circ(*,x) &= \vee_\gamma \{A^\circ(*,\gamma) \mathbin{\&} \gamma^\circ(*,x)\}.
\end{aligned}$$

To show that α is a morphism we have

$$\begin{aligned}
\delta\alpha(x,*) &= \bigvee_y \{\delta(x,y) \mathbin{\&} \alpha(y,*)\} \\
&= \bigvee_y \{\delta(x,y) \mathbin{\&} \bigvee_\gamma \{\gamma(y,*) \mathbin{\&} A(\gamma,*)\}\} \\
&= \bigvee_\gamma \{\bigvee_y \{\delta(x,y) \mathbin{\&} \gamma(y,*)\} \mathbin{\&} A(\gamma,*)\} \\
&= \bigvee_\gamma \{\gamma(x,*) \mathbin{\&} A(\gamma,*)\} \\
&= \alpha(x,*),
\end{aligned}$$

$$\begin{aligned}
\alpha\delta_q(x,*) &= \alpha(x,*) \,\&\, q \\
&= \bigvee_\gamma\{\gamma(x,*) \,\&\, A(\gamma,*)\} \,\&\, q \\
&= \bigvee_\gamma\{\gamma(x,*) \,\&\, A(\gamma,*) \,\&\, q\} \\
&= \bigvee_\gamma\{\gamma(x,*) \,\&\, A(\gamma,*)\} \\
&= \alpha(x,*).
\end{aligned}$$

The results for α° are similar. To finish off we show that α is a monomorphic map.

$$\begin{aligned}
\alpha\alpha^\circ(x,y) &= \alpha(x,*) \,\&\, \alpha^\circ(*,y) \\
&= \bigvee_\xi\{\xi(x,*) \,\&\, A(\xi,*)\} \,\&\, \bigvee_\gamma\{A^\circ(*,\gamma) \,\&\, \gamma(*,y)\} \\
&= \bigvee_{\xi\gamma}\{\xi(x,*) \,\&\, A(\xi,*) \,\&\, A^\circ(*,\gamma) \,\&\, \gamma^\circ(*,y)\} \\
&= \bigvee_{\xi\gamma}\{\xi(x,*) \,\&\, AA^\circ(\xi,\gamma) \,\&\, \gamma^\circ(*,y)\} \\
&\leq \bigvee_{\xi\gamma}\{\xi(x,*) \,\&\, \widehat{\Psi}\Phi(\xi,\gamma) \,\&\, \gamma^\circ(*,y)\} \\
&= \bigvee_{\xi\gamma}\{\xi(x,*) \,\&\, \xi^\circ\gamma(*,*) \,\&\, \gamma^\circ(*,y)\} \\
&= \bigvee_{\xi\gamma}\{\xi\xi^\circ\gamma\gamma^\circ(x,y)\} \\
&\leq \delta\delta(x,y) \qquad \text{by adjointness} \\
&= \delta(x,y).
\end{aligned}$$

$$\begin{aligned}
\alpha^\circ\alpha(*,*) &= \bigvee_x\{\alpha^\circ(*,x) \,\&\, \alpha(x,*)\} \\
&= \bigvee_x\{\bigvee_\xi\{A^\circ(*,\xi) \,\&\, \xi^\circ(*,x)\} \,\&\, \bigvee_\gamma\{\gamma(x,*) \,\&\, A(\gamma,*)\}\} \\
&= \bigvee_{\xi\gamma}\{A^\circ(*,\xi) \,\&\, \bigvee_x\{\xi^\circ(*,x) \,\&\, \gamma(x,*)\} \,\&\, A(\gamma,*)\}
\end{aligned}$$

$$\begin{aligned}
&= \bigvee_{\xi\gamma}\{A^\circ(*,\xi) \mathbin{\&} \xi^\circ\gamma(*,*) \mathbin{\&} A(\gamma,*)\} \\
&= \bigvee_{\xi\gamma}\{A^\circ(*,\xi) \mathbin{\&} \widehat{\Psi}\Phi(\xi,\gamma) \mathbin{\&} A(\gamma,*)\} \\
&= A^\circ A(*,*) \\
&= q.
\end{aligned}$$

Thus α is a monomorphic singleton arrow with the property that

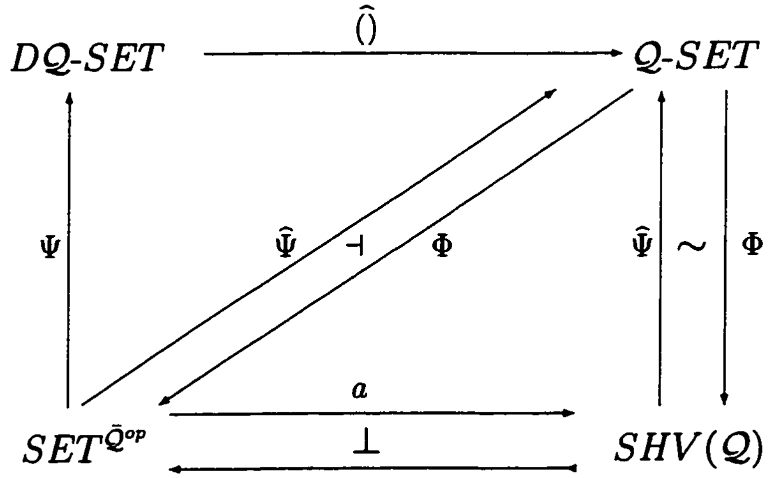
$$A(\beta,*) \;=\; \widehat{\Psi}\Phi(\beta,\alpha).$$

If $\bar{\alpha}$ is another such morphism then we have $\alpha^\circ\bar{\alpha}$ equal to $\alpha^\circ\alpha$, and $\bar{\alpha}^\circ\alpha$ equals $\bar{\alpha}^\circ\bar{\alpha}$ which gives us $\bar{\alpha} \geq \alpha\alpha^\circ\bar{\alpha} = \alpha$, and $\alpha \geq \bar{\alpha}$ from which the equality follows and so η is an isomorphism. Thus when restricted to the categories $\mathcal{Q}$-SET and $SHV(\mathcal{Q})$, the functors $\widehat{\Psi}$ and Φ give an equivalence of categories. ∎

Corollary 2.4.17 For $\mathcal{Q}$ pseudo-rightsided $SHV(\mathcal{Q})$ is a reflective subcategory of $SET^{\overline{\mathcal{Q}}^{op}}$.

Proof: With the counit of the previous adjunction a natural isomorphism, the result follows. ∎

For $\mathcal{H}$ a Heyting algebra we have that the traditional notion of the category of sheaves is equivalent to this definition since both are equivalent to the category of $\mathcal{H}$-valued sets. The following diagram captures the relationship between $\mathcal{Q}$-SET and Sheaves on $\mathcal{Q}$ for a pseudo-rightsided quantaloid $\mathcal{Q}$.


DQ-SET
Q-SET
SET^{Q̄^op}
SHV(Q)
()̂
Ψ
Ψ̂
⊣
Φ
Ψ̂
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Φ
a
⊥

Chapter 3

Relational Presheaves

3.1 Relational Presheaves and $\mathcal{Q}$-Taxons

3.1.1 $\mathcal{Q}$-Categories

We begin this section with a short review of the work that Rosenthal[22] has done on $\mathcal{Q}$-categories and $\mathcal{Q}$-functors for $\mathcal{Q}$ a supremum-enriched category. In this case $MAT(\mathcal{Q})$ is a supremum-enriched category with the identity arrows given by the Kronecker delta Δ.

Definition 3.1.1 For $\mathcal{Q}$ a **SUP**-category, a matrix $\delta \in MAT(\mathcal{Q})((X,\rho),(X,\rho))$ is a $\mathcal{Q}$-category if

$$\delta\delta \leq \delta \text{ and } \Delta \leq \delta$$

If $\mathcal{Q}$ is involutive then we ask that $\delta = \delta^{\circ}$.

Definition 3.1.2 For $\mathcal{Q}$ a **SUP**-category and (X,ρ,δ) and (Y,ρ,δ), $\mathcal{Q}$-categories a $\mathcal{Q}$-functor $f : (X,\rho,\delta) \to (Y,\rho,\delta)$ is a function $f : X \to Y$ such that

$$\rho_X(x) = \rho_Y(f(x)) \text{ and } \delta_X(x_1,x_2) \leq \delta_Y(f(x_1),f(x_2)).$$

Denote the category of $\mathcal{Q}$-categories and $\mathcal{Q}$-functors by $\mathcal{Q}$-CAT.

Definition 3.1.3 For a **SUP**-category $\mathcal{Q}$ a *relational presheaf* is an infimum preserving lax-functor $F : \mathcal{Q}^{co} \to \mathbf{REL}$. For an arrow $q : A \to B$ in $\mathcal{Q}$ denote the relation F picks out by F_q. If $\mathcal{Q}$ is involutive then we require that $F_{q^\circ} = F_q^\circ$. A morphism of relational presheaves F and G is a colax-transformation $\tau : F \to G$ (that is $\tau_{f,B} \circ F_g \leq G_g \circ \tau_{f,A}$) such that for every $A \in |\mathcal{Q}|$, τ_A is a function.

Denote the category of relational presheaves and their morphisms by $\mathcal{R}(\mathcal{Q})$. Rosenthal refers to a infimum preserving relational presheaf as a *continuous* relational presheaf. For an arrow $q : A \to B$ in $\mathcal{Q}$, F a relational presheaf, and $y \in F(A), x \in F(B)$, we will denote $x\ F_q\ y$ by $F_q(x, y) = 1$. Observe that the colax-transformations are not required to preserve the infima.

Definition 3.1.4 For $\mathcal{C}$ a category and $F, G : \mathcal{C} \to \mathbf{REL}$ lax-functors, a *strong colax-transformation* $\tau : F \to G$ is a modular lax-transformation that satisfies $\tau\tau_F \leq \tau$ and $\tau_G\tau = \tau$.

Theorem 3.1.5 For $\mathcal{C}$ an involutive category and $F, G : \mathcal{C} \to \mathbf{REL}$ involutive lax-functors, a *strong colax-transformation* $\tau : F \to G$ satisfies $\tau_F \leq \tau^\circ\tau$ and $\tau\tau^\circ \leq \tau_G$ (a map) if and only if for every $A \in |\mathcal{C}|$, τ_{1_A} is a function

Proof: Assume $\tau_F \leq \tau^\circ\tau$ and $\tau\tau^\circ \leq \tau_G$. So $(\tau^\circ\tau)_{1_A}(a, a) = 1$ for every $a \in F(A)$.

$$(\tau^\circ\tau)_{1_A}(a, a) = 1 \quad \Leftrightarrow \quad (\exists fg = 1_A)(\exists b)\{\tau_{f^\circ}(b, a) = 1 \text{ and } \tau_g(b, a) = 1\}$$

$$\begin{aligned} \tau_g(b, a) = 1 \quad &\Leftrightarrow \quad \tau_{G_g}\tau_{1_A}(b, a) = 1 \qquad\qquad \text{since } \tau_G\tau = \tau \\ &\Leftrightarrow \quad (\exists a')\{\tau_{G_g}(b, a') = 1 \text{ and } \tau_{1_A}(a', a) = 1\}. \end{aligned}$$

Thus for every $a \in F(A)$ there is an $a' \in G(A)$ with $\tau_{1_A}(a', a) = 1$. Now assume that there exist $a', a'' \in G(A)$ with $\tau_{1_A}(a', a) = 1$ and $\tau_{1_A}(a'', a) = 1$. So we have $\tau\tau^\circ(a', a'') = 1$ and since $\tau\tau_{1_A}^\circ \leq \tau_{G_{1_A}}$ we imediately have $a' = a''$. Thus τ_{1_A} is a function.

Now assume that for every $A \in |\mathcal{C}|$, τ_{1_A} is a function. If for $f : A \to B$ and $a \in F(A), b \in F(B)$ we have $\tau_{F_f}(b,a) = 1$. Then $\tau_{F_{1_A}}(a,a) = 1$ and $\tau_{F_{1_B}}(b,b) = 1$. We know that τ_{1_B} is a function so there exists $b' \in G(B)$ such that $\tau_{1_B}(b',b) = 1$. Thus $\tau_f(b',a) = 1$ (since $\tau\tau_F \leq \tau$). So we now have $(\tau^\circ\tau)_f(b,a) = 1$ ($\tau^\circ_{1_B}(b,b') = 1$). Therefore $\tau_F \leq \tau^\circ\tau$. Finally assume that $(\tau\tau^\circ)_f(b,a) = 1$.

$$(\tau\tau^\circ)_f(b,a) = 1 \;\Leftrightarrow\; (\exists gh = f)(\exists c)\{\tau_g(b,c) = 1 \text{ and } \tau_{h^\bullet}(a,c) = 1\}$$

$$\begin{aligned}\tau_f(b,c) = 1 \;&\Leftrightarrow\; (\tau_{G_f}\tau_{1_{A'}})(b,c) = 1\\ &\Leftrightarrow\; (\exists c')\{\tau_{G_f}(b,c') = 1 \text{ and } \tau_{1_{A'}}(c'c) = 1\}.\end{aligned}$$

Similarly we have $\tau_{h^\bullet}(a,c) = 1$ if and only if there exists c'' such that $\tau_{G_{h^\bullet}}(a,c'') = 1$ and $\tau_{1_{A'}}(c'',c) = 1$. Since $\tau_{1_{A'}}$ is a function we have that $c'' = c'$. So we have that $\tau_{G_g}\tau_{G_h}(b,a) = 1$ which implies that $\tau_{G_f}(b,a) = 1$. And therefore $\tau\tau^\circ \leq \tau_G$. ∎

Lemma 3.1.6 There is a bijective correspondence between relational presheaves and $\mathcal{Q}$-categories.

Proof: For $F : \mathcal{Q}^{co} \to \mathbf{REL}$ define a $\mathcal{Q}$-category (X_F, ρ_F, δ_F) by

$$X_F = \coprod_A F(A)$$

$$\rho_F(x) = A \quad \text{if and only if} \quad x \in F(A)$$

$$\delta_F(x,y) = \bigvee\{q \mid F_q(x,y) = 1\}.$$

Since F is an infimum preserving lax-functor we have

$$\begin{aligned}\delta_F\delta_F(x,y) &= \bigvee_z\{\delta_F(x,z) \;\&\; \delta_F(z,y)\}\\ &= \bigvee\{p_1p_2 \mid F_{p_1}F_{p_2}(x,y) = 1\}\\ &\leq \bigvee\{q \mid F_q(x,y) = 1\}\\ &= \delta_F(x,y).\end{aligned}$$

Now for (X, ρ, δ) a $\mathcal{Q}$-category create an infimum preserving lax-functor F as follows

For an object A, set $F(A)$ equal to $\{x \mid \rho(x) = A\}$

For a morphism $q : A \to B$ and $x \in F(B), y \in F(A)$

set $F_q(x, y) = 1$ if and only if $q \leq \delta(x, y)$.

F so defined is an infimum preserving lax-semifunctor because:

$$\begin{aligned} F_q F_p(x, y) = 1 &\Leftrightarrow \exists_z\, F_q(x, z) = 1 \text{ and } F_p(z, y) = 1 \\ &\Leftrightarrow \exists_z\, q \leq \delta(x, z) \text{ and } p \leq \delta(z, y) \\ &\Rightarrow qp \leq \delta(x, y) \\ &\Leftrightarrow F_{qp}(x, y) = 1. \end{aligned}$$

To show that F preserves infima we have

$$\begin{aligned} F_{\vee q_i}(x, y) = 1 &\Leftrightarrow \vee q_i \leq \delta(x, y) \\ &\Leftrightarrow \text{for every } i,\ q_i \leq \delta(x, y) \\ &\Leftrightarrow \text{for every } i,\ F_{q_i}(x, y) = 1. \end{aligned}$$

If $\mathcal{Q}$ is involutive then

$$\begin{aligned} F_q^\circ(x, y) = 1 &\Leftrightarrow F_q(y, x) = 1 \\ &\Leftrightarrow q \leq \delta(y, x) \\ &\Leftrightarrow q^* \leq \delta(x, y) \\ &\Leftrightarrow F_{q^*}(x, y) = 1. \end{aligned}$$

Now it is easy to show we have a bijection of sets. ∎

Theorem 3.1.7 For a **SUP**-category $\mathcal{Q}$ the categories $\mathcal{R}(\mathcal{Q})$ and $\mathcal{Q}$-CAT are isomorphic.

Proof: For $F, G : \mathcal{Q}^{co} \to$ **REL** relational presheaves and $f : \delta_F \to \delta_G$ a $\mathcal{Q}$-functor define $\tau_f : F \to G$ by $\tau_{f,A}(x) = f(x)$. To show that τ_f is a morphism of relational presheaves we must show that for $g : A \to B$

$$\tau_{f,B} \circ F_g \leq G_g \circ \tau_{f,A}.$$

Since $\tau_{f,B}$ is a function a typical element of the relation $\tau_{f,B}F_g$ has the form $\tau_{f,B}F_g(z,x) = 1$ where $z = f(y)$, $\tau(f(y),y) = 1$ and $F_g(y,x) = 1$ for some y. Since f is a $\mathcal{Q}$-functor it follows that $G_g(f(y),f(x)) = 1$. We know that $\tau_{f,A}(f(x),x) = 1$ thus $G_g\tau_{fA}(f(y),x) = 1$, thus τ is a colax transformation.

Now assume (X,ρ,δ) and (Y,ρ,δ) are $\mathcal{Q}$-categories and that $\tau : F_X \longrightarrow F_Y$ is a colax-transformation. Define a $\mathcal{Q}$-functor f_τ by

$$\text{For } x \in F(A),\ f_\tau(x) = \tau_A(x).$$

Clearly f_τ is a function so assume $q \leq \delta_F(y,x)$. This is the case if and only if $F_q(y,x) = 1$. By definition of τ, $\tau_B(y) = f_\tau(y)$ and since τ is a colax transformation $\tau_B F_q(\tau_B(y),x) = 1$ implies that $G_q\tau_A(\tau_B(y),x) = 1$. But τ_A is a function so we must have $G_q(\tau_B(y),\tau_A(x)) = 1$ which means that $q \leq \delta_G(f_\tau(y),f_\tau(x))$. So f_τ is a $\mathcal{Q}$-functor.

That the constructions above define two functors is easy to see and tracing back and forth easily gives us that the two categories are isomorphic. ∎

3.1.2 $\mathcal{Q}$-Taxons

The previous theorem required the existence of identities only when defining the morphisms of relational presheaves. The colax-transformations were defined using the objects notion of transformation. We also had the added requirement that for each object A and a colax-transformation τ the morphism τ_A was a function. Observe though that it does not follow that in the arrows-based definition of τ all of the morphisms are functions. But we do know that the associated colax-transformation is a map. On the other hand the fact that $\Delta \leq \delta$ and that F was a lax-functor was not needed in the proof. In this section we explore the category of $\mathcal{Q}$-taxons and its relationship to the category of sheaves. In the future, for $\mathcal{Q}$ a quantaloid, or $\mathcal{O}$

an **ORD**-semicategory a relational presheaf will refer to an infimum preserving lax-semifunctor $F : \mathcal{Q}^{co} \to \mathbf{REL}$ or an order preserving lax-semifunctor $F : \mathcal{O}^{co} \to \mathbf{REL}$ (the context will be clear). If $\mathcal{Q}$ is involutive then we assume that the relational presheaves preserve the involution. All the results that follow, easily extend to the involutive case.

For $\mathcal{Q}$ an involutive quantaloid a $\mathcal{Q}$-semifunctor $f : (X, \rho, \delta) \to (Y, \rho, \delta)$ gives rise to a pair of matrices $R_f : (X, \rho, \delta) \to (Y, \rho, \delta)$ and $R_f^\circ : (Y, \rho, \delta) \to (X, \rho, \delta)$ defined by

$$
\begin{aligned}
R_f(y,x) &= \bigvee_{x'}\{\delta_Y(y, f(x')) \,\&\, \delta_X(x', x)\} \\
R_f^\circ(x,y) &= \bigvee_{x'}\{\delta(x,x') \,\&\, \delta_Y(f(x), y)\}.
\end{aligned}
$$

These matrices satisfy

$$
\begin{aligned}
R_f\delta_X &= R_f, \qquad \delta_Y R_f = R_f \\
\delta_X R_f^\circ &= R_f^\circ, \qquad R_f^\circ\delta_Y = R_f^\circ \\
\delta_X &\le R_f^\circ R_f \quad \text{and} \quad R_f R_f^\circ \le \delta_Y.
\end{aligned}
$$

That R_f is a morphism in ${}^*KAR(MAT(\mathcal{Q}))$ is automatic. To show that R_f is a map we have,

$$
\begin{aligned}
R_f^\circ R_f(x_1,x_2) &= \bigvee_{y}\{R_f^\circ(x_1,y) \,\&\, R_f(y,x_2)\} \\
&= \bigvee_{y,x_1,x_2}\{\delta_X(x_1,x') \,\&\, \delta_Y(f(x'),y) \,\&\, \delta_Y(y,f(x'')) \,\&\, \delta_X(x'',x_2)\} \\
&\ge \bigvee_{x',x''}\{\delta_X(x_1,x') \,\&\, \delta_X(x',x'') \,\&\, \delta_X(x'',x_2)\} \\
&= \delta_X(x_1,x_2),
\end{aligned}
$$

and

$$
R_f R_f^\circ(y_1,y_2) = \bigvee_{x}\{R_f(y_1,x) \,\&\, R_f(x,y_2)\}
$$

$$
\begin{aligned}
&= \bigvee_{x,x',x''} \{\delta_Y(y_1, f(x')) \ \&\ \delta_X(x', x) \ \&\ \delta_X(x, x'') \ \&\ \delta_Y(f(x''), y_2)\} \\
&\leq \bigvee_{x,x',x''} \{\delta_Y(y_1, f(x')) \ \&\ \delta_Y(f(x'), f(x)) \\
&\qquad\qquad \&\ \delta_Y(f(x), f(x'')) \ \&\ \delta_Y(f(x''), y_2)\} \\
&\leq \delta(y_1, y_2).
\end{aligned}
$$

Definition 3.1.8 For $\mathcal{Q}$ a quantaloid the category of *$\mathcal{Q}$-semicategories* consists of

- Objects • $\mathcal{Q}$-semicategories.

- Arrows • $\mathcal{Q}$-semifunctors, $f : (X, \rho, \delta) \to (Y, \rho, \delta)$
  a function $f : X \to Y$ that satisfies

$$\rho(x) = \rho(f(x))$$
$$\delta_X(x_1, x_2) \leq \delta_Y(f(x_1), f(x_2)).$$

Denote the category of $\mathcal{Q}$-semicategories by $\mathcal{Q}$-$SCAT$. Recall, from section 1.5.1, a $\mathcal{Q}$-semicategory (X, ρ, δ) is a $\mathcal{Q}$-taxon if and only if δ is an idempotent.

Definition 3.1.9 For $\mathcal{Q}$ a quantaloid the category of *$\mathcal{Q}$-taxons* consists of

- Objects • $\mathcal{Q}$-taxons.

- Arrows • $\mathcal{Q}$-semifunctors, $f : (X, \rho, \delta) \to (Y, \rho, \delta)$

Denote the category of $\mathcal{Q}$-taxons and their morphisms by $\mathcal{Q}$-TAX. Recall that for $\mathcal{Q}$ a pseudo-rightsided quantaloid there is an equivalence between $\mathcal{Q}$-SET and $SHV(\mathcal{Q})$ given by the functors $\widehat{\Psi}$ and Φ.

Lemma 3.1.10 For $\mathcal{Q}$ a pseudo-rightsided quantaloid and $R : (X, \rho, \delta) \to (Y, \rho, \delta)$ a $\mathcal{Q}$-SET morphism $\widehat{\Psi}\Phi(R)$ is a $\mathcal{Q}$-semifunctor.

Recall that if $\alpha : [q] \hookrightarrow (X, \rho, \delta)$ is a singleton then the transformation $\tau_R : \Phi_X \to \Phi_Y$ is given by composition with R. That is $\tau_q(\alpha) = R\alpha$. It was also shown that

$\widehat{\Psi}_{\phi_X}(\alpha,\beta) \leq \widehat{\Psi}_{\phi_X}(R\alpha, R\beta)$. Since the resulting morphism is a morphism of $\mathcal{Q}$-valued sets we have that $\widehat{\Psi}\Phi(R)$ is a $\mathcal{Q}$-semifunctor. ∎

Lemma 3.1.11 For $\mathcal{Q}$ an involutive quantaloid, there is a functor $\iota : \mathcal{Q}\text{-}TAX \to \mathcal{Q}\text{-}SET$.

Proof: The functor is given by sending a $\mathcal{Q}$-semifunctor f to the induced matrix R_f. Clearly R_{1_X} is δ_X and thus is the identity morphism. We need to show that the composition is preserved. For $f : (X,\rho,\delta) \to (Y,\rho,\delta)$ and $g : (Y,\rho,\delta) \to (Z,\rho,\delta)$, composible $\mathcal{Q}$-semifunctors we have

$$\begin{aligned}
R_{gf}(z,x) &= R_{gf}\delta_X(z,x) \\
&= \bigvee_{x'}\{R_{gf}(z,x') \mathbin{\&} \delta_X(x',x)\} \\
&= \bigvee x', x''\{\delta_Z(z, gf(x'')) \mathbin{\&} \delta_X(x'',x') \mathbin{\&} \delta(x',x)\} \\
&\leq \bigvee_{x',y'}\{\delta_Z(z,g(y')) \mathbin{\&} \delta_Y(y', f(x') \mathbin{\&} \delta(x',x)\} \\
&= \bigvee_{x',y',y''}\{\delta_Z(z,g(y')) \mathbin{\&} \delta_Y(y',y'') \mathbin{\&} \delta_Y(y'', f(x')) \mathbin{\&} \delta(x',x)\} \\
&= \bigvee_{y''}\{R_g(z,y'') \mathbin{\&} \mathcal{R}_f(y'',x)\} \\
&= R_gR_f(z,x).
\end{aligned}$$

Thus we have that $R_{gf} \leq R_gR_f$. Since $\mathcal{Q}$ is involutive we know that $\mathcal{Q}\text{-}SET$ is a category and since R_{fg} and R_gR_f are $\mathcal{Q}\text{-}SET$ morphisms they then must be equal. ∎

Corollary 3.1.12 For $\mathcal{Q}$ a pseudo-rightsided quantaloid the functor $\widehat{\Psi}\Phi : \mathcal{Q}\text{-}SET \to \mathcal{Q}\text{-}SET$ factors through $\mathcal{Q}\text{-}TAX$.

Proof: Every $\mathcal{Q}$-valued set is a $\mathcal{Q}$-taxon. This together with the previous lemmas give the desired result. ∎

Theorem 3.1.13 For $\mathcal{Q}$ a pseudo-rightsided quantaloid the category $\mathcal{Q}$-SET is equivalent to the category $\mathcal{Q}$-TAX.

Proof: We know that $\widehat{\Psi}\Phi$ is naturally isomorphic to $\mathbf{1}_{\mathcal{Q}-SET}$. We need to show that $\widehat{\Psi}\Phi$ is naturally isomorphic to $\mathbf{1}_{\mathcal{Q}-TAX}$. For (X,ρ,δ) a $\mathcal{Q}$-taxon define the $\mathcal{Q}$-semifunctor $\varepsilon_X : (X,\rho,\delta) \longrightarrow \widehat{\Psi}\Phi_\delta$ by $\varepsilon_X(x) = \alpha_x$. Recall that α_x is the singleton morphism that satisfies $\alpha_x(x',*) = \delta_X(x',x)$. Now to show that ε is a natural transformation, let $f : (X,\rho,\delta) \longrightarrow (Y,\rho,\delta)$ be a $\mathcal{Q}$-semifunctor.

$$\begin{aligned}
\widehat{\Psi}\Phi_f\varepsilon(\alpha,x) &= \bigvee_\gamma\{\widehat{\Psi}\Phi_f(\alpha,\gamma) \,\&\, \varepsilon(\gamma,x)\} \\
&= \bigvee_\gamma\{\widehat{\Psi}\Phi_f(\alpha,\gamma) \,\&\, \widehat{\Psi}\Phi(\gamma,\alpha_x)\} \\
&= \bigvee_\gamma \alpha^\circ R_f\gamma\gamma^\circ\alpha_x \\
&= \alpha^\circ R_f\alpha_X(*,*) \\
&= \alpha^\circ R_f\delta(*,x) \\
&= \alpha^\circ\delta\mathcal{R}_f(*,x) \\
&= \bigvee_y\{\alpha^\circ\alpha_y R_f(y,x)\} \\
&= \bigvee_y\{\widehat{\Psi}\Phi(\alpha,\alpha_y) \,\&\, R_f(y,x)\} \\
&= \bigvee_y\{\varepsilon(\alpha,y) \,\&\, R_f(y,x)\} \\
&= \varepsilon R_f(\alpha,x).
\end{aligned}$$

This tells us that ε is a natural transformation. Now we will show that it is a natural isomorphism.

$$\begin{aligned}
\varepsilon_\delta^\circ\varepsilon_\delta(x,y) &= \bigvee_\gamma\{\varepsilon_\delta^\circ(x,\gamma) \,\&\, \varepsilon_\delta(\gamma,y)\} \\
&= \bigvee_\gamma\{\widehat{\Psi}\Phi(\alpha_x,\gamma) \,\&\, \widehat{\Psi}\Phi(\gamma,\alpha_y)\}
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{\gamma} \alpha_x^\circ \gamma\gamma^\circ \alpha_x \\
&= \alpha_x^\circ \alpha_y(*,*) && \text{strictness} \\
&= \bigvee_{z} \{\alpha_x^\circ(*,z) \mathbin{\&} \alpha_y(z,*)\} \\
&= \bigvee_{z} \{\delta(x,z) \mathbin{\&} \delta(z,y)\} \\
&= \delta(x,y).
\end{aligned}
$$

So this tells us that $\varepsilon_\delta^\circ \varepsilon_\delta = \delta$

$$
\begin{aligned}
\varepsilon_\delta \varepsilon_\delta^\circ(\beta_1,\beta_2) &= \bigvee_{x} \{\varepsilon_\delta(\beta_1,x) \mathbin{\&} \varepsilon_\delta^\circ(x\beta_2)\} \\
&= \bigvee_{x} \{\widehat{\Psi}\Phi(\beta,\alpha_x) \mathbin{\&} \widehat{\Psi}\Phi(\alpha_x,\beta_2)\} \\
&= \bigvee_{x} \beta_1^\circ \alpha_x \alpha_x^\circ \beta_2 \\
&= \beta_1^\circ \beta_2 && \text{atomic} \\
&= \widehat{\Psi}\Phi(\beta_1,\beta_2).
\end{aligned}
$$

We used the fact that since (X,ρ,δ) is strict we have $\bigvee_x \alpha_x^\circ \alpha_x = \delta_X$. Thus we have $\varepsilon_\delta \varepsilon_\delta^\circ = \widehat{\Psi}\Phi$. ∎

Corollary 3.1.14 For $\mathcal{Q}$ a pseudo-rightsided quantaloid the category of sheaves on $\mathcal{Q}$ is equivalent to the category $\mathcal{Q}$-TAX.

3.1.3 ORD-Relational Presheaves

We now explore the relationship between relational presheaves and modules for **ORD**-semicategories and **SUP**-semicategories.

Lemma 3.1.15 For $\mathcal{O}$ an **ORD**-semicategory a lax-semifunctor $F : \mathcal{O}^{co} \to$ **REL** preserves the order if and only if for every pair $a \in F(A)$ and $b \in F(B)$ the set $\{q \mid F_q(a,b) = 1\}$ is a downset.

Proof: Assume that F preserves the order. So if $p \leq q \in \{q \mid F_q(a,b) = 1\}$, we have $F_q \leq F_p$ thus $p \in \{q \mid F_q(a,b) = 1\}$. If for every pair a, b the set $\{q \mid F_q(a,b) = 1\}$ is a downset then for $p \leq q$ and $F_q(a,b) = 1$ we have $F_p(x,y) = 1$. ∎

Recall that for $\mathcal{O}$ an **ORD**-taxon the composite of transformations $F \xrightarrow{\tau} G \xrightarrow{\sigma} H$: $\mathbf{ORD}^{co} \to \mathbf{REL}$ is given by

$$(\sigma\tau)_f = \sigma_h\tau_g \qquad \text{for some } f = hg.$$

Since $\mathcal{O}$ is an **ORD**-taxon this is the same as

$$(\sigma\tau)_f = \bigcup_{f=gh} \sigma_g\tau_h,$$

because the composite does not depend on the choice of $gh = f$. Also recall that for $F \xrightarrow{\tau} G : \mathcal{O}^{co} \to \mathbf{REL}$, a modular lax transformation between two lax-semifunctors, that τ is a transformation if and only if we have $\tau\tau_F = \tau$ and $\tau_G\tau = \tau$.

Using these ideas we define laxity and the composition of transformations if $\mathcal{O}$ is an **ORD**-semicategory. We begin by first defining how the composite of transformations will work

Lemma 3.1.16 If $\mathcal{O}$ is an **ORD**-taxon, and $\tau : F \to G$ and $\sigma : G \to H$ are transformations then

$$\bigcup_{f \leq gh} \sigma_g\tau_h = \bigcup_{f=gh} \sigma_g\tau_h.$$

Proof: Clearly

$$\bigcup_{f \leq gh} \sigma_g\tau_h \geq \bigcup_{f=gh} \sigma_g\tau_h.$$

Since $(\sigma\tau)$ is order preserving, if $f \leq gh$ and $(\sigma\tau)_{gh}(b,a) = 1$ then we must have $(\sigma\tau)_f(b,a) = 1$. And since $\mathcal{Q}$ is a **SUP**-taxon it must be the case that for some $f = gh$, $\sigma_g\tau_h(b,a) = 1$. ∎

Definition 3.1.17 For $\mathcal{O}$ an **ORD**-semicategory, $F, G, H : \mathcal{O}^{co} \to$ **REL** a *pre-transformation* $\tau : F \to G$ is an $|\mathcal{O}| \times |\mathcal{O}|$ family of order preserving arrows $\langle \tau_{AB} : \mathcal{O}^{co}(A, B) \to \mathbf{REL}(FA, GB)\rangle$. For another pre-transformation $\sigma : G \to H$ the composite pre-transformation $\sigma\tau$ is given by $(\sigma\tau)_f(a, b) = 1$ if and only if there exists $f \leq gh$ such that $\sigma_g\tau_h(a, b) = 1$. That is

$$(\sigma\tau)_f = \bigcup_{f \leq gh} \sigma_g\tau_h.$$

Lemma 3.1.18 A family of arrows $\langle \tau_{AB} : \mathcal{O}^{co}(A, B) \to \mathbf{REL}(FA, GB)\rangle$ is a pre-transformation if and only if for every pair $a \in F(A)$ and $b \in F(B)$ the set of arrows $\{q \mid F_q(a, b) = 1\}$ is a downset.

Proof: The proof is the same as for lemma(3.1.13) ∎

We want to show that the composite of pre-transformations is a pre-transformation. To that end let $a \in F(A)$ and $b \in G(B)$. We have

$$(\sigma\tau)_f(b, a) = 1 \text{ if and only if } \exists_{f \leq gh}\ \sigma_g\tau_h(b, a) = 1$$

Thus if $k \leq f$ then clearly $(\sigma\tau)_k(b, a) = 1$. By the lemma we have that the composite is a pre-transformation.

For $F : \mathcal{O}^{co} \to$ **REL** a lax-semifunctor there is an associated pre-transformation $\tau_F : F \to F$ given by $\tau_{AB} = F_{AB}$. For two pre-transformations $\sigma, \tau : F \to G$ we say that $\sigma \leq \tau$ if for every arrow f in $\mathcal{O}$ we have $\sigma_f \leq \tau_f$.

Definition 3.1.19 For $\mathcal{O}$ an **ORD**-semicategory a pre-transformation $\tau : F \to G$ between relational presheaves is a *transformation* if $\tau\tau_F = \tau$ and $\tau_G\tau = \tau$. τ is a *strong lax-transformation* if $\tau\tau_F \leq \tau$ and $\tau_G\tau = \tau$. τ is a *modular lax-transformation* if $\tau\tau_F \leq \tau$ and $\tau_G\tau \leq \tau$

Denote the semicategory of modular lax-transformations by $MLAX_{ord}(\mathcal{Q}^{co}, REL)$, of strong lax-transformations by $SLAX_{ord}(\mathcal{Q}^{co}, REL)$. The following is an immediate consequence of the definitions

Theorem 3.1.20 For $\mathcal{O}$ an **ORD**-taxon the two definitions of transformations and their composition coincide.

3.1.4 SUP-Relational Presheaves

Lemma 3.1.21 For $\mathcal{Q}$ a **SUP**-semicategory an order preserving lax-semifunctor $F : \mathcal{Q}^{co} \to \mathbf{REL}$ preserves infima if and only if for every pair $a \in F(A)$ and $b \in F(B)$ the set $\{q \mid F_q(b,a) = 1\}$ is a principal downset.

Proof: If F preserves infima then we must have $F_{\vee\{q \mid F_q(x,y)=1\}}(x,y) = 1$ and so $\{q \mid F_q(x,y) = 1\}$ is a principal down set. On the other hand We automatically have $F_{\vee q_i} \le \wedge_i F_{q_i}$. Assume $\wedge_i F_{q_i}(x,y) = 1$. Then for every i, $F_{q_i}(x,y) = 1$ thus since $\{q \mid F_q(x,y) = 1\}$ is a principal down set we have $F_{\vee q_i}(x,y) = 1$. ∎

Recall that for two complete lattices $\mathcal{Q}_1$ and $\mathcal{Q}_2$ the tensor product is given by the set $\{W \in D(\mathcal{Q}_1 \times \mathcal{Q}_2) \mid S \times T \subseteq W \text{ implies } (\bigvee S, \bigvee T) \in W\}$. In **SUP** the coproduct is product so an element of $\coprod_X \mathcal{Q}(A,X) \otimes \mathcal{Q}(X,B)$ is a family $\langle W_X \rangle_{X \in |\mathcal{Q}|}$. For $\mathcal{Q}$ a **SUP**-taxon the composite of the transformations $F \xrightarrow{\tau} G \xrightarrow{\sigma} H : \mathcal{Q}^{co} \to \mathbf{REL}$ is then given by

$$\begin{aligned} (\sigma\tau)_f &= \sigma \circ \tau(W_X) && \text{for some } f = m(W_X) \\ &= \bigcap_{\substack{X \\ (g,h) \in W_X}} \sigma_h \tau_g && \text{for some } f = m(W_X) \end{aligned}$$

Also recall that for the coequalizer m,

$$m(W_X) = \bigvee_{\substack{X \\ (g,h) \in W_X}} hg$$

As we did with **ORD**-semicategories we can define transformations and laxity for $\mathcal{Q}$ a **SUP**-semicategory.

Lemma 3.1.22 For $\mathcal{Q}$ a **SUP**-taxon, $\tau : F \to G$ and $\sigma : G \to H$ two transformations of relational presheaves

$$\bigcup_{f \leq \vee h_i g_i} (\bigcap \sigma_{h_i} \tau_{g_i}) = \bigcup_{f = \vee h_i g_i} (\bigcap \sigma_{h_i} \tau_{g_i})$$

Proof: Clearly we have

$$\bigcup_{f \leq \vee h_i g_i} (\bigcap \sigma_{h_i} \tau_{g_i}) \geq \bigcup_{f = \vee h_i g_i} (\bigcap \sigma_{h_i} \tau_{g_i})$$

Since $\mathcal{Q}$ is a **SUP**-taxon if we have $\bigcap \sigma_{h_i} \tau_{g_i}(b,a) = 1$ then $(\sigma\tau)_{\vee h_i g_i}(b,a) = 1$. This immediately implies that $(\sigma\tau)_f(b,a) = 1$ since the transformations preserve order. Thus there is a family $\langle W_X \rangle$ with $m(W_X) = f$ and $\bigcap^{X}_{(h,g) \in W_X} \sigma_h \tau_g(b,a) = 1$ ∎

Definition 3.1.23 For $\mathcal{Q}$ a **SUP**-semicategory, $F, G, H : \mathcal{Q}^{co} \to$ **REL** a *pre-transformation* $\tau : F \to G$ is a $|\mathcal{Q}| \times |\mathcal{Q}|$ indexed family of infimum preserving arrows $\langle \tau_{AB} : \mathcal{Q}^{co}(A,B) \to \mathbf{REL}(FA, GB) \rangle$. For another pre-transformation $\sigma : G \to H$ the composite pre-transformation $\sigma\tau$ is given by $(\sigma\tau)_f(b,a) = 1$ if and only if there exists a family of arrows (g_i, h_i), with $f \leq \vee h_i g_i$ and $\cap \sigma_{h_i} \tau_{g_i}(b,a) = 1$. That is

$$(\sigma\tau)_f = \bigcup_{f \leq \vee h_i g_i} (\bigcap \sigma_{h_i} \tau_{g_i}).$$

Lemma 3.1.24 A family of arrows $\langle \tau_{AB} : \mathcal{Q}^{co} \to \mathbf{REL}(FA, GB) \rangle$ is called a pre-transformation if and only if for every pair of elements $a \in F(A)$ and $b \in F(B)$ the set $\{q \mid \tau_q(b,a) = 1\}$ is a principal downset.

Proof: The proof is the same as for lemma (3.1.19) ∎

For the composite above we have

$$(\sigma\tau)_f(b,a) = 1 \quad \text{if and only if} \quad \exists_{f \leq \vee h_i g_i} (\bigcap \sigma_{h_i} \tau_{g_i}(b,a) = 1).$$

So if we have a family of arrows f_j such that $(\sigma\tau)_{f_j}(b,a) = 1$ then for each j we have a family of arrows (g_{ij}, h_{ij}) with $f_j \leq \vee_i h_{ij} g_{ij}$ and $\cap \sigma_{h_{ij}} \tau_{g_{ij}}(b,a) = 1$. It now follows by the definition of composition that $(\sigma\tau)_{\bigvee_j f_j}(b,a) = 1$. This tells us that the composite is a pre-transformation.

For a relational presheaf $F : \mathcal{Q}^{co} \rightarrow \mathbf{REL}$ there is an associated pre-transformation τ_F, where $\tau_{AB} = F_{AB}$. For $\sigma, \tau : F \rightarrow G$, two pre-transformations, say that $\sigma \leq \tau$ if for every arrow f in $\mathcal{Q}$ we have $\sigma_f \leq \tau_f$.

Definition 3.1.25 For $\mathcal{Q}$ a **SUP**-semicategory a pre-transformation $\tau : F \rightarrow G$ between relational presheaves is a *transformation* if $\tau\tau_F = \tau$ and $\tau_G\tau = \tau$. τ is a *strong lax-transformation* if $\tau\tau_F = \tau$ and $\tau_G\tau \leq \tau$. τ is a *modular lax-transformation* if $\tau\tau_F \leq \tau$ and $\tau_G\tau \leq \tau$.

Denote the semicategories of modular lax-transformations and strong lax-transformations by $MLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL})$ and $SLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL})$ respectively. The following theorem is an immediate consequence of the definitions.

Theorem 3.1.26 For $\mathcal{Q}$ a **SUP**-taxon the two definitions of transformations coincide.

We can now explore the relationship between relational presheaves and sheaves. In particular for $\mathcal{Q}$ a pseudo-rightsided quantaloid we will answer the question 'when is a relational presheaf a sheaf'?

Theorem 3.1.27 For $\mathcal{O}$ an **ORD**-semicategory, $MLAX_{ord}(\mathcal{O}^{co}, \mathbf{REL})$ is equivalent to $MLAX_{inf}(D\mathcal{O}^{co}, \mathbf{REL})$.

Proof: This is a straight forward consequence of the fact that for an order preserving semifunctor $F : \mathcal{O}^{co} \rightarrow \mathbf{REL}$ each set $\{f \mid F_f(x, y) = 1\}$ is a down set and the supremum in $D\mathcal{O}$ is the union of sets. ∎

For a semicategory $\mathcal{C}$ we can define lax semifunctors from $\mathcal{C}$ into **REL** and transformations between them in the obvious way. Denote the resulting semicategory by $MLAX(\mathcal{C}, \mathbf{REL})$. When we interpret $\mathcal{C}$ as an **ORD**-semicategory with the trivial order on each hom set we obtain the following corollary.

Corollary 3.1.28 $MLAX(\mathcal{C}, \mathbf{REL})$ is equivalent to $MLAX_{inf}(\mathcal{P}\mathcal{C}^{co}, \mathbf{REL})$.

Proof: A semifunctor $F : \mathcal{C} \to \mathbf{REL}$ trivially preserves the order so clearly $MLAX(\mathcal{C}, \mathbf{REL})$ is equivalent to the semicategory $MLAX_{ord}(\mathcal{C}, \mathbf{REL})$. ∎

For $\mathcal{Q}$ a quantaloid the following theorem gives us the relationship between modular lax-transformations and the quantaloid $MOD(MAT(\mathcal{Q}))$. From this result we can explore the relationship with sheaves.

Theorem 3.1.29 For $\mathcal{Q}$ a quantaloid the semicategory $MLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL})$ is equivalent to the semicategory $MOD(MAT(\mathcal{Q}))$.

Proof: First we define the appropriate semifunctors. $\Psi : MLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL}) \to MOD(MAT(\mathcal{Q}))$ is given by sending a relational presheaf F to the module (X_F, ρ_F, δ_F) where

$$X_F = \coprod_{A \in |\mathcal{Q}|} F(A)$$

$$\rho(x) = A \qquad \text{if and only if} \qquad x \in F(A)$$

$$\delta(x, y) = \bigvee\{q \mid F_q(x, y) = 1\}.$$

For G another relational presheaf a modular lax-transformation $\tau : F \to G$ is sent to the morphism R_τ where

$$R_\tau(x, y) = \bigvee\{q \mid \tau_q(x, y) = 1\}$$

We need to show that δ_F is a module and that R_τ is in fact a morphism.

$$\begin{aligned}
\delta_F\delta_F(x, y) &= \bigvee_z\{\delta_F(x, z) \,\&\, \delta_F(z, y)\} \\
&= \bigvee_z\{\vee\{q_1 \mid F_{q_1}(x, z) = 1\} \,\&\, \vee\{q_2 \mid F_{q_2}(z, y) = 1\}\} \\
&= \bigvee_z\{q_1 q_2 \mid F_{q_1}(x, z) = 1 \text{ and } F_{q_2}(z, y) = 1\} \\
&\leq \bigvee\{q \mid F_q(x, y) = 1\} \\
&= \delta_F(x, y).
\end{aligned}$$

Showing that R_τ is a morphism follows the exact same lines. If $\sigma : G \to H$ is a second modular lax-transformation then we have

$$\begin{aligned}
R_\sigma R_\tau(x,y) &= \bigvee_z \{R_\sigma(x,z) \,\&\, R_\tau(z,y)\} \\
&= \bigvee_z \{\vee\{q_1 \mid \sigma_{q_1}(x,z) = 1\} \,\&\, \vee\{q_2 \mid \tau_{q_2}(z,y) = 1\}\} \\
&= \bigvee_z \{q_1 q_2 \mid \sigma_{q_1}(x,z) = 1 \text{ and } \tau_{q_2}(z,y) = 1\} \\
&= \bigvee \{q_1 q_2 \mid \sigma_{q_1}\tau_{q_2}(x,y) = 1\} \\
&= \bigvee \{q \mid (\sigma\tau)_q(x,y) = 1\} \\
&= R_{\sigma\tau}(x,y).
\end{aligned}$$

Thus Ψ is a semifunctor. For $\Phi : MOD(MAT(\mathcal{Q})) \to MLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL})$ we send a module (X, ρ, δ) to the relational presheaf F_X where

$$F_X(A) = \{x \mid \rho(x) = A\}$$

and for $q : A \to B$ and $x \in F_X(B)$, $y \in F_X(A)$ we have $F_{X_q}(x,y) = 1$
if and only if
$$q \;\leq\; \delta(x,y).$$

For a morphism of modules $R : (X, \rho, \delta) \to (Y, \rho, \delta)$ we define a modular lax-transformation $\tau_R : F_X \to F_Y$ by

$$\tau_{R_q}(y,x) = 1 \text{ if and only if } q \leq R(x,y).$$

To show that F_X is a lax semifunctor we first need to show that F_X preserves infimum. This follows automatically since each set $\{q \mid F_{X_q}(x,y) = 1\}$ is a principal down set. For laxity we have for composable arrows pq

$$\begin{aligned}
F_{X_p}F_{X_q}(x,y) = 1 &\Leftrightarrow \exists_z \text{ such that } F_{X_p}(x,z) = 1 \text{ and } F_{X_q}(z,y) = 1 \\
&\Leftrightarrow p \leq \delta(x,z) \text{ and } q \leq \delta(z,y) \\
&\Rightarrow pq \leq \delta(x,y) \\
&\Leftrightarrow F_{X_{pq}}(x,y) = 1.
\end{aligned}$$

And so F is a lax semifunctor. In a similar manner we can show that for the morphism R, τ_R is a modular lax-transformation. For a second morphism $S : (Y, \rho, \delta) \to (Z, \rho, \delta)$ the composite transformations have

$$
\begin{aligned}
(\tau_S \tau_R)_q(z, x) = 1 \quad & \Leftrightarrow \quad q \leq \bigvee \{p_1 p_2 \mid \tau_{S_{p_1}} \tau_{R_{p_2}}(z, x) = 1\} \\
& \Leftrightarrow \quad q \leq \bigvee_y \{p_1 p_2 \mid \tau_{s_{p_1}}(z, y) = 1 \text{ and } \tau_{R_{p_2}}(y, x) = 1\} \\
& \Leftrightarrow \quad q \leq \bigvee_y \{p_1 p_2 \mid p_1 \leq S(z, y) \text{ and } p_2 \leq R(y, x)\} \\
& \Leftrightarrow \quad q \leq RS(z, y) \\
& \Leftrightarrow \quad \tau_{RS_q}(z, y) = 1.
\end{aligned}
$$

Thus Φ is a semifunctor. Clearly we have $\Phi\Psi = 1$ and $\Psi\Phi = 1$. ∎

In a straightforward way we can extend the result to involutive quantaloids and with a minor modification we can show that $SLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL})$ and $RMOD(MAT(\mathcal{Q}))$ are equivalent. We now have the following series of corollaries.

Corollary 3.1.30 For $\mathcal{Q}$ a quantaloid, $\mathcal{Q}$-SET is equivalent to $MAP(KAR(MLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL})))$

Proof:

$$
\begin{aligned}
\mathcal{Q}\text{-}SET &= MAP(KAR(MAT(\mathcal{Q}))) \\
&\cong MAP(KAR(MOD(MAT(\mathcal{Q})))) \\
&\cong MAP(KAR(MLAX_{sup}(\mathcal{Q}^{co}, \mathbf{REL}))).
\end{aligned}
$$

∎

This tells us that an idempotent modular lax-transformation is directly related to a $\mathcal{Q}$-valued set.

Corollary 3.1.31 For $\mathcal{Q}$ a **SUP**-taxon we have $MAP(TAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL}))$ is equivalent to $\mathcal{Q}$-SET.

Proof: $KAR(MLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL}))$ is equivalent to $TAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL})$. ∎

Corollary 3.1.32 If $\mathcal{Q}$ is a pseudo-rightsided **SUP**-taxon then $SHV(\mathcal{Q})$ is equivalent to ${}^{*}MAP(TAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL}))$.

This tells us that a sheaf for a pseudo-rightsided sup-taxon $\mathcal{Q}$, is an **INF**-semifunctor $F : \mathcal{Q}^{co} \to \mathbf{REL}$. Alternatively a sheaf is a **SUP**-semifunctor $F : \mathcal{Q} \to \mathbf{REL}^{co}$. For $\mathcal{H}$ a complete Heyting algebra we know that the category of $\mathcal{H}$-valued sets of Higgs is equivalent to $\mathcal{Q}$-SET when we think of $\mathcal{H}$ as a one object **SUP** enriched semicategory $\mathcal{Q}$. Higgs showed that the category of $\mathcal{H}$-valued sets is equivalent to the category of sheaves on $\mathcal{H}$. Recall that every complete Heyting algebra satisfies Freyd's law of modularity, and thus is pseudo rightsided. This tells us that the category of sheaves on $\mathcal{H}$, $\mathcal{H}$-*Set* is equivalent to the category of sheaves $\mathcal{H}$-SET in this new sense. Thus a sheaf on a Heyting algebra is an **INF** preserving semifunctor $F : \mathcal{H}^{co} \to \mathbf{REL}$.

The previous theorem now gives the following result, which is similar to theorem 3.1.5.

Corollary 3.1.33 If $\mathcal{Q}$ is a pseudo-rightsided **SUP**-taxon then $\mathcal{Q}$-TAX is equivalent to ${}^{*}MAP(TAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL}))$

3.2 A Grothendieck Construction

3.2.1 The Category of Elements

We now examine the Grothendieck construction on lax-semifunctors and modular lax-transformations from a semicategory $\mathcal{C}$ into **REL**. We also wish to know what properties the resulting categories have.

Definition 3.2.1 For $\mathcal{C}$ a semicategory and $F : \mathcal{C} \to$ **REL** a lax-semifunctor the Grothendieck construction of the *semicategory of elements*, $\mathcal{C}_F$, for F consists of

- Objects • Pairs (a, A) such that $a \in F(A)$.
  (usually we will denote the object (a, A) just by a).

- Arrows • $\mathcal{C}_F(a, b) = \{f : A \to B \mid F_f(b, a) = 1 \text{ for } b \in F(B), a \in F(A)\}$.

That this is a semicategory follows since laxity of F guarantees that the composite arrows exist while we inherit associativity from $\mathcal{C}$. There is the usual forgetful semi-functor U_F from $\mathcal{C}_F$ into $\mathcal{C}$. It sends the objects (a, A) to the object A and an arrow $(a, A) \xrightarrow{f} (b, B)$ to f. Note that U_F is faithful.

Definition 3.2.2 For $F \xrightarrow{\tau} G$ a lax-transformation construct a profunctor γ_τ : $\mathcal{C}_F^{op} \times \mathcal{C}_G \longrightarrow SET$ as follows,

$$\gamma_\tau(a, b) = \{f \mid \tau_f(b, a) = 1\}.$$

Note that γ_τ is a subfunctor of $\mathcal{C}(U_F-, U_G-)$. Since it is a subfunctor it is compatible with $\mathcal{C}_F$ and $\mathcal{C}_G$ in the sense that if $(a, A) \xrightarrow{f} (b, B)$ and $(c, C) \xrightarrow{g} (d, D)$ and $h \in \gamma(b, c)$ then $hf \in \gamma(a, c)$ and $gh \in \gamma(b, d)$.

Using the properties above we define a semicategory with objects semi-functors and morphisms profunctors.

Definition 3.2.3 The category $PROF/\mathcal{C}$ is defined as follows

- **Objects** • Pairs $< \mathcal{D}, L >$ such that $\mathcal{D}$ is a semicategory and $L : \mathcal{D} \to \mathcal{C}$ is a faithful semi-functor.

- **Arrows** • $< \mathcal{D}_1, L_1 > \xrightarrow{P} < \mathcal{D}_2, L_2 >$ is a subprofunctor of $\mathcal{C}(L_1-, L_2-)$

To say that P is a profunctor is to say that it is compatible with L_1 and L_2 in the sense that if $(a, A) \xrightarrow{f} (b, B)$ and $(c, C) \xrightarrow{g} (d, D)$ and $h \in P(b, c)$ then $L_2(g)h \in P(a, c)$ and $hL_1(f) \in P(b, d)$.

If we have two arrows $< D_1, L_1 > \xrightarrow{P} < D_2, L_2 > \xrightarrow{Q} < D_3, L_3 >$ then define the composite QP by

$$QP(C, A) = \{fg \mid (\exists_B)(\ g \in Q(B, A) \text{ and } f \in P(C, B))\}.$$

Notice that this is the usual composition of profunctors in the category setting. The composition is

$$QP(C, A) = \coprod_B \{P(B, C) \times Q(A, B)\} / \sim$$

where $\sim$ is generated by the pairs $(Q(h, A)(g), f) \sim (g, P(B, h)(f))$, for all $g \in Q(A', A)$, $f \in P(B', B)$ and $h : B' \to A'$. One of the pairs in the relation is $(1, gf) \sim (g, f)$, so every pair (f, g) can be associated with the composite of its elements. If $gf = g'f'$ then we have

$$\begin{aligned} (f, g) &\sim (1, gf) \\ &= (1, g'f') \\ &\sim (f', g') \end{aligned}$$

On the other hand if $(f, g) \sim (f', g')$ then there is a zig zag of composites that gives us $gf = g'f'$. So two pairs (f, g) and (f', g') are related if and only if $gf = g'f'$.

Theorem 3.2.4 $PROF/\mathcal{C}$ is equivalent to $MLAX(\mathcal{C}, \mathbf{REL})$.

Proof: The construction of the category of elements and the morphisms between them is clearly a semi-functor $EL : MLAX(\mathcal{C}, \mathbf{REL}) \longrightarrow PROF/\mathcal{C}$. Given a faithful semi-functor $\mathcal{D} \xrightarrow{L} \mathcal{C}$ define a lax semi-functor $F_L : \mathcal{C} \longrightarrow REL$ as follows,

On objects $A \in |\mathcal{C}|$ $\qquad F_L(A) = \{D \in |\mathcal{D}| \mid L(D) = A\}$.

On Arrows $f : A \to B$ $\quad F_{L_f}(D_1, D_2) = 1$ if and only if there exists $f' : D_2 \to D_1$ such that $L(f') = f$.

Also if $< \mathcal{D}_1, L_1 > \xrightarrow{P} < \mathcal{D}_2, L_2 >$ is a morphism in $PROF/\mathcal{C}$ then define a lax transformation $\tau_P : F_{L_1} \longrightarrow F_{L_2}$ by,

$$\tau_{P_f}(D_1, D_2) = 1 \quad \text{if and only if} \quad f \in P(D_2, D_1).$$

This gives us a semi-functor $\Gamma : PROF/\mathcal{C} \longrightarrow MLAX(\mathcal{C}, \mathbf{REL})$. The composite $\Gamma.EL$ is the identity semi-functor on $MLAX(\mathcal{C}, \mathbf{REL})$. Define $\varepsilon : \Gamma.EL \longrightarrow \mathbf{1}$ to be the identity transformation. For $(\mathcal{D}, L) \xrightarrow{P} (\mathcal{E}, M)$ define $\eta_P : (\mathcal{D}, L) \longrightarrow EL.\Gamma(\mathcal{E}, M)$ by

$$\eta_P(D, < E, C >) = P(D, E).$$

For $(\mathcal{D}, L) \xrightarrow{P} (\mathcal{E}, M) \xrightarrow{R} (\mathcal{F}, N)$ we have

$$\begin{aligned} \eta_R P(D, < F, C >) &= \{fg \mid (\exists_E)\ (g \in \eta_R(E, < F, C >) \text{ and } f \in P(D, E))\} \\ &= \{fg \mid (\exists_E)\ (g \in R(E, F) \text{ and } f \in P(D, E))\} \\ &= RP(D, F) \\ &= \eta_{RP}(D, F). \end{aligned}$$

$$\begin{aligned} EL\Gamma_R \eta_P(D, < F, C >) &= \{fg \mid (\exists_{<E, C'>})\ (f \in EL\Gamma_R(< E, C' >, < F, C >) \\ &\qquad \text{and} \quad g \in \eta_P(D, < E, C' >))\} \\ &= \{fg \mid (\exists_E)\ (f \in R(E, F) \text{ and } g \in P(D, E))\} \\ &= RP(D, F) \\ &= \eta_{RP}(D, F). \end{aligned}$$

Thus η is a natural transformation. Since the ε is the identity transformation we have

$$\varepsilon_{\Gamma P}\Gamma(\eta_R) = \Gamma_P\Gamma_R = \Gamma_{PR}$$

$$EL(\varepsilon_\tau)\eta_{EL\sigma} = EL_\tau EL_\sigma = EL_{\tau\sigma}.$$

These tell us that $\Gamma \dashv EL$. For $(\mathcal{D}, L) \xrightarrow{P} (\mathcal{E}, M)$ define $\eta'_P : EL\Gamma_{(\mathcal{D},L)} \to (\mathcal{E}, M)$ by $\eta'_P(< D, C >, E) = P(D, E)$. Exactly as with η we see that η' is a natural transformation. For $(\mathcal{D}, L) \xrightarrow{P} (\mathcal{E}, M) \xrightarrow{R} (\mathcal{F}, N)$ we have

$$\eta_R\eta'_P(< D, C >, < F, C' >) = EL\Gamma_{RP}(< D, C >, < F, C' >)$$

and

$$\eta'_R\eta_P(D, F) = RP(D, F).$$

So we have that η' is the inverse of η. Thus $PROF/\mathcal{C}$ is equivalent to $MLAX(\mathcal{C}, \mathbf{REL})$ ∎

3.2.2 Order Preservation and the Semicategory of Elements

Definition 3.2.5 For $\mathcal{O}_1$ and $\mathcal{O}_2$ partial orders a functor $f : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ satisfies *Frobenius reciprocity* if for every $a \in \mathcal{O}_1$ we have $\downarrow f(a) = \{f(x) \mid x \leq a\}$.

For $\mathcal{O}$ an order enriched semicategory, we construct the semicategory of elements for an **ORD** lax-semifunctor F as above. Note that in this case the forgetful semifunctor U_F has the added property that it preserves the order and for a, b objects in $\mathcal{O}_F$, $U_{F_{a,b}}$ satisfies the Frobenius reciprocity condition. In addition to this for a modular lax-transformation $\tau : F \longrightarrow G$ we have that the profunctor Γ_τ takes its values in the category **ORD** instead of in **SET**. That is we have $\Gamma_\tau : \mathcal{O}_F^{op} \times \mathcal{O}_G \longrightarrow \mathbf{ORD}$. This is still a sub semifunctor of $\mathcal{O}(U_F-, U_G-)$.

Definition 3.2.6 For $\mathcal{O}$ an **ORD**-semicategory the category $PROF/\mathcal{O}$ consists of

- Objects • Pairs $\langle \mathcal{O}_1, L_1 \rangle$ such that $\mathcal{O}_1$ is an **ORD**-semicategory and $L_1 : \mathcal{O}_1 \to \mathcal{O}$ is a faithful **ORD**-semifunctor satisfying Frobenius reciprocity.

- Arrows • $\langle \mathcal{O}_1, L_1 \rangle \xrightarrow{P} \langle \mathcal{O}_2, L_2 \rangle$ is a **ORD**-subprofunctor of $\mathcal{O}(L_1-, L_2-)$

The composition of the morphisms

$$\langle \mathcal{O}_1, L_1 \rangle \xrightarrow{P} \langle \mathcal{O}_2, L_2 \rangle \xrightarrow{Q} \langle \mathcal{O}_3, L_3 \rangle$$

is defined on objects A and C by

$$QP(C, A) = \bigcup_B \{fg \mid g \in Q(B, A) \text{ and } f \in P(C, B)\}^{\downarrow}.$$

Theorem 3.2.7 The category $PROF/\mathcal{O}$ is equivalent to $MLAX_{ord}(\mathcal{O}^{co}, \mathbf{REL})$.

Proof: The functors are essentially the same as for Theorem (3.2.4). Frobenius ensures that we have the necessary downset property of modular lax-transformations. We just need to show that the new composite is preserved. For P, Q composable morphisms in $PROF/\mathcal{O}$ we have

$$\begin{aligned}
\tau_{PQ_f}(D_3, D_1) = 1 \;&\Leftrightarrow\; f \in PQ(D_1, D_3)\\
&\Leftrightarrow\; (\exists_{B, f \le gh})\ (g \in P(B, D_3) \text{ and } h \in Q(D_1, B))\\
&\Leftrightarrow\; (\exists_{B, f \le gh})\ (\tau_{P_g}(D_3, B) = 1 \text{ and } \tau_{Q_h}(B, D_1) = 1)\\
&\Leftrightarrow\; (\tau_P \tau_Q)_f(D_3, D_1) = 1.
\end{aligned}$$

Now for composable modular lax-transformations τ and σ

$$\begin{aligned}
EL_{\sigma\tau}(D_1, D_3) &= \{f \mid \sigma\tau_f(D_3, D_1) = 1\}\\
&= \{gh \mid \sigma_g \tau_h(D_3, D_1) = 1\}^{\downarrow}\\
&= \bigcup_B \{gh \mid \sigma_g(D_3, B) = 1 \text{ and } \tau_h(B, D_1) = 1\}^{\downarrow}\\
&= \bigcup_B \{gh \mid g \in EL_\sigma(B, D_3) \text{ and } h \in EL_\tau(D_1, B)\}^{\downarrow}\\
&= EL_\sigma EL_\tau(D_1, D_3).
\end{aligned}$$

∎

3.2.3 Supremum Preservation and the Semicategory of Elements

For $\mathcal{Q}$ a supremum-enriched semicategory and $F : \mathcal{Q}^{co} \to REL$ a supremum preserving lax-semifunctor, we construct the semicategory of elements for F in a similar manner. The forgetful functor $U_F : \mathcal{Q}_F \to \mathcal{Q}$, has the added property that it is supremum preserving. For $\tau : F \to G$ a lax transformation the associated profunctor Γ_τ takes its values in the monoidal category **SUP**. That is $\Gamma_\tau : \mathcal{Q}_f^{op} \times \mathcal{Q}_G \longrightarrow \mathbf{SUP}$.

Definition 3.2.8 For $\mathcal{Q}$ a **SUP**-semicategory the category $PROF/\mathcal{Q}$ consists of

- • Objects • Pairs $\langle \mathcal{Q}_1, L_1 \rangle$ such that $\mathcal{Q}_1$ is a **SUP**-semicategory and $L_1 : \mathcal{Q}_1 \to \mathcal{Q}$ is a faithful **SUP**-semifunctor satisfying Frobenius reciprocity.

- • Arrows • $\langle \mathcal{Q}_1, L_1 \rangle \xrightarrow{P} \langle \mathcal{Q}_2, L_2 \rangle$ is a **SUP**-subprofunctor of $\mathcal{Q}(L_1-, L_2-)$

The composition of the morphisms

$$\langle \mathcal{Q}_1, L_1 \rangle \xrightarrow{P} \langle \mathcal{Q}_2, L_2 \rangle \xrightarrow{Q} \langle \mathcal{Q}_3, L_3 \rangle$$

is defined on objects A and C by

$$QP(C, A) = (\bigvee_B \{fg \mid g \in Q(B, A) \text{ and } f \in P(C, B)\})^{\downarrow}.$$

Theorem 3.2.9 $PROF/\mathcal{Q}$ is equivalent to $MLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL})$

Proof: Again Frobenius ensures that the necessary downset property on the modular lax-transformations holds. Since we are working in **SUP** the downsets are principal.

All we need to show is that the composition as now defined is preserved. For P, Q composable morphisms in $PROF/\mathcal{Q}$ we have

$$\begin{aligned}
\tau_{PQ_f}(D_3, D_1) = 1 &\Leftrightarrow f \in PQ(D_1, D_3) \\
&\Leftrightarrow f \leq \bigvee_B \{gh \mid g \in P(B, D_3) \text{ and } h \in Q(D_1, B)\} \\
&\Leftrightarrow f \leq \bigvee_B \{gh \mid \tau_{P_g}(D_3, B) = 1 \text{ and } \tau_{Q_h}(B, D_1) = 1\} \\
&\Leftrightarrow (\tau_P \tau_Q)_f(D_3, D_1) = 1.
\end{aligned}$$

Now for composable modular lax-transformations τ and σ

$$\begin{aligned}
EL_{\sigma\tau}(D_1, D_3) &= \{f \mid (\sigma\tau)_f(D_3, D_1) = 1\} \\
&= (\bigvee \{gh \mid \sigma_g \tau_h(D_3, D_1) = 1\})^{\downarrow} \\
&= (\bigvee_B \{gh \mid \sigma_g(D_3, B) = 1 \text{ and } \tau_h(B, D_1) = 1\})^{\downarrow} \\
&= (\bigvee_B \{gh \mid g \in EL_\sigma(B, D_3) \text{ and } h \in EL_\tau(D_1, B)\})^{\downarrow} \\
&= EL_\sigma EL_\tau(D_1, D_3). \blacksquare
\end{aligned}$$

3.2.4 $\mathcal{Q}$-TAX and the Semicategory of Elements

In this section we will explore the relationship between $\mathcal{Q}$-taxons and the grothendieck construction. Recall that the semicategory $MLAX_{inf}(\mathcal{Q}^{co}, \mathbf{REL})$ is equivalent to $MOD(MAT(\mathcal{Q}))$. Using this as a guide we convert the elements construction to $\mathcal{Q}$-semicategories. Given a quantaloid $\mathcal{Q}$ and (X, ρ, δ) a $\mathcal{Q}$-semicategory, the category of elements EL_X has

- Objects • $|EL_X| = X$.

- Arrows • $q : x_1 \to x_2$ if and only if
  $q \leq \delta(x_2, x_1)$.

For $\mathcal{Q}_1$ a quantaloid and $L : \mathcal{Q}_1 \to \mathcal{Q}$ a faithful **SUP**-semifunctor that satisfies Frobenius reciprocity, define a $\mathcal{Q}$-semicategory (X_L, ρ_L, δ_L) by

$$X_L = |\mathcal{Q}_1|$$
$$\rho_L = L : |\mathcal{Q}_1| \to |\mathcal{Q}_2|$$
$$\delta_L(x_2, x_1) = L(\top_{x_1 x_2})$$

where $\top_{x_1x_2} : x_1 \to x_2$ is the top arrow in $\mathcal{Q}_1(x_1, x_2)$.

Observe that if $\mathcal{Q}$ is pseudo-rightsided then (X_L, ρ_L, δ_L) is a $\mathcal{Q}$-taxon. Now we want to see what a $\mathcal{Q}$-semifunctor gives us. If $f : (X, \rho, \delta) \to (Y, \rho, \delta)$ is a $\mathcal{Q}$-semifunctor then define a **SUP**-functor $F_f : EL_X \to EL_Y$ by

$$F_f : X \to Y \quad = \quad f : X \to Y$$

For $q : x_1 \to x_2$ in EL_X let

$$F_f(q) \quad = \quad q : f(x_1) \to f(x_2).$$

The action on arrows is well defined since $\delta_X(x_1, x_2) \leq \delta_Y(f(x_1), f(x_2))$. Observe that F_f composed with the forgetful $U_Y : EL_Y \to \mathcal{Q}$ equals the forgetful $U_X : EL_X \to \mathcal{Q}$. That is $U_Y F_f = U_X$. Now for $\langle q_1, L_1\rangle$ and $\langle q_2, L_2\rangle$ two objects in $PROF/\mathcal{Q}$ and a **SUP**-semifunctor $F : \mathcal{Q}_1 \to \mathcal{Q}_2$ that satisfies $L_2F = L_1$ we have a $\mathcal{Q}$-semifunctor $f_F : (X_{L_1}, \rho_{L_1}, \delta_{L_1}) \to (X_{L_2}, \rho_{L_2}, \delta_{L_2})$. For an object A in $\mathcal{Q}_1$, $f_F(A) = F(A)$ and $\delta_{L_1}(x_1, x_2) \leq \delta_{L_2}(f_F(x_1), f_F(x_2))$ since $L_2F = L_1$.

Definition 3.2.10 For $\mathcal{Q}$ a quantaloid, the category **SUP**$/\mathcal{Q}$ consists of

- Objects • $|\mathbf{SUP}/\mathcal{Q}| = |PROF/\mathcal{Q}|$.

- Arrows • $\langle \mathcal{Q}_1, L_1\rangle \xrightarrow{F} \langle \mathcal{Q}_2, L_2\rangle$
  is a **SUP**-semifunctor $F : \mathcal{Q}_1 \to \mathcal{Q}_2$, such that
  $L_2F = L_1$.

It is easy to see that the constructions given above give two functors $EL : \mathcal{Q}\text{-}SCAT \to$ **SUP**$/\mathcal{Q}$ and $\bar{()} : \mathbf{SUP}/\mathcal{Q} \to \mathcal{Q}\text{-}SCAT$. The composite $\bar{()}EL : \mathcal{Q}\text{-}SCAT \to \mathcal{Q}\text{-}SCAT$ is the identity functor $\mathbf{1}_{\mathcal{Q}-SCAT}$.

Theorem 3.2.11 For $\mathcal{Q}$ a quantaloid $\mathcal{Q}$-$SCAT$ is equivalent to $\mathbf{SUP}/\mathcal{Q}$.

Proof: We need to show that there is a natural isomorphism between $\mathbf{1}_{\mathbf{SUP}/\mathcal{Q}}$ and $EL(\bar{})$. For an object $\langle q_1, L_1\rangle$, $EL_{\bar{L}_1}$ is the quantaloid with objects $|\mathcal{Q}_1|$. There is a morphism $q : A \to B$ if and only if $q \leq \delta_{L_1}(B, A)$ if and only if $q \leq L_1(T_{AB})$. Since L_1 is faithful and Frobenius there is a unique arrow q' in $\mathcal{Q}_1(A, B)$ such that $L_1(q') = q$. Define η_{L_1} to be the **SUP**-semifunctor that is the identity on the objects and which maps an arrow q' to the arrow $L_1(q') = q$. The inverse maps the arrow q to the unique arrow q'. ∎

Corollary 3.2.12 For $\mathcal{Q}$ a pseudo-rightsided quantaloid $\mathbf{SUP}/\mathcal{Q}$ is equivalent to $SHV(\mathcal{Q})$.

Proof: Since $\mathcal{Q}$ is pseudo-rightsided, every $\mathcal{Q}$-semicategory is a $\mathcal{Q}$-taxon. Thus we have

$$\mathbf{SUP}/\mathcal{Q} \cong \mathcal{Q}\text{-}SCAT = \mathcal{Q}\text{-}TAX \cong SHV(\mathcal{Q})$$

∎

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