CONTRIBUTIONS TO NONLINEAR ANALYSIS

By

Xian-Zhi YUAN

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
AT
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
FEBRUARY 1994

ISBN 0-315-93804-8

Name Xian-zhi Yuan

Dissertation Abstracts International is arranged by broad, general subject categories. Please select the one subject which most nearly describes the content of your dissertation. Enter the corresponding four-digit code in the spaces provided.

MATHEMATICS — 0405 — U·M·I

SUBJECT TERM — SUBJECT CODE

Subject Categories

THE HUMANITIES AND SOCIAL SCIENCES

COMMUNICATIONS AND THE ARTS
Architecture 0729
Art History 0377
Cinema 0900
Dance 0378
Fine Arts 0357
Information Science 0723
Journalism 0391
Library Science 0399
Mass Communications 0708
Music 0413
Speech Communication 0459
Theater 0465

EDUCATION
General 0515
Administration 0514
Adult and Continuing 0516
Agricultural 0517
Art 0273
Bilingual and Multicultural 0282
Business 0688
Community College 0275
Curriculum and Instruction 0727
Early Childhood 0518
Elementary 0524
Finance 0277
Guidance and Counseling 0519
Health 0680
Higher 0745
History of 0520
Home Economics 0278
Industrial 0521
Language and Literature 0279
Mathematics 0280
Music 0522
Philosophy of 0998
Physical 0523
Psychology 0525
Reading 0535
Religious 0527
Sciences 0714
Secondary 0533
Social Sciences 0534
Sociology of 0340
Special 0529
Teacher Training 0530
Technology 0710
Tests and Measurements 0288
Vocational 0747

LANGUAGE, LITERATURE AND LINGUISTICS
Language
- General 0679
- Ancient 0289
- Linguistics 0290
- Modern 0291

Literature
- General 0401
- Classical 0294
- Comparative 0295
- Medieval 0297
- Modern 0298
- African 0316
- American 0591
- Asian 0305
- Canadian (English) 0352
- Canadian (French) 0355
- English 0593
- Germanic 0311
- Latin American 0312
- Middle Eastern 0315
- Romance 0313
- Slavic and East European 0314

PHILOSOPHY, RELIGION AND THEOLOGY
Philosophy 0422

Religion
- General 0318
- Biblical Studies 0321
- Clergy 0319
- History of 0320
- Philosophy of 0322

Theology 0469

SOCIAL SCIENCES
American Studies 0323

Anthropology
- Archaeology 0324
- Cultural 0326
- Physical 0327

Business Administration
- General 0310
- Accounting 0272
- Banking 0770
- Management 0454
- Marketing 0338

Canadian Studies 0385

Economics
- General 0501
- Agricultural 0503
- Commerce-Business 0505
- Finance 0508
- History 0509
- Labor 0510
- Theory 0511

Folklore 0358
Geography 0366
Gerontology 0351

History
- General 0578
- Ancient 0579
- Medieval 0581
- Modern 0582
- Black 0328
- African 0331
- Asia, Australia and Oceania 0332
- Canadian 0334
- European 0335
- Latin American 0336
- Middle Eastern 0333
- United States 0337

History of Science 0585
Law 0398

Political Science
- General 0615
- International Law and Relations 0616
- Public Administration 0617

Recreation 0814
Social Work 0452

Sociology
- General 0626
- Criminology and Penology 0627
- Demography 0938
- Ethnic and Racial Studies 0631
- Individual and Family Studies 0628
- Industrial and Labor Relations 0629
- Public and Social Welfare 0630
- Social Structure and Development 0700
- Theory and Methods 0344

Transportation 0709
Urban and Regional Planning 0999
Women's Studies 0453

THE SCIENCES AND ENGINEERING

BIOLOGICAL SCIENCES
Agriculture
- General 0473
- Agronomy 0285
- Animal Culture and Nutrition 0475
- Animal Pathology 0476
- Food Science and Technology 0359
- Forestry and Wildlife 0478
- Plant Culture 0479
- Plant Pathology 0480
- Plant Physiology 0817
- Range Management 0777
- Wood Technology 0746

Biology
- General 0306
- Anatomy 0287
- Biostatistics 0308
- Botany 0309
- Cell 0379
- Ecology 0329
- Entomology 0353
- Genetics 0369
- Limnology 0793
- Microbiology 0410
- Molecular 0307
- Neuroscience 0317
- Oceanography 0416
- Physiology 0433
- Radiation 0821
- Veterinary Science 0778
- Zoology 0472

Biophysics
- General 0786
- Medical 0760

EARTH SCIENCES
Biogeochemistry 0425
Geochemistry 0996
Geodesy 0370
Geology 0372
Geophysics 0373
Hydrology 0388
Mineralogy 0411
Paleobotany 0345
Paleoecology 0426
Paleontology 0418
Paleozoology 0985
Palynology 0427
Physical Geography 0368
Physical Oceanography 0415

HEALTH AND ENVIRONMENTAL SCIENCES
Environmental Sciences 0768

Health Sciences
- General 0566
- Audiology 0300
- Chemotherapy 0992
- Dentistry 0567
- Education 0350
- Hospital Management 0769
- Human Development 0758
- Immunology 0982
- Medicine and Surgery 0564
- Mental Health 0347
- Nursing 0569
- Nutrition 0570
- Obstetrics and Gynecology 0380
- Occupational Health and Therapy 0354
- Ophthalmology 0381
- Pathology 0571
- Pharmacology 0419
- Pharmacy 0572
- Physical Therapy 0382
- Public Health 0573
- Radiology 0574
- Recreation 0575
- Speech Pathology 0460
- Toxicology 0383

Home Economics 0386

PHYSICAL SCIENCES
Pure Sciences

Chemistry
- General 0485
- Agricultural 0749
- Analytical 0486
- Biochemistry 0487
- Inorganic 0488
- Nuclear 0738
- Organic 0490
- Pharmaceutical 0491
- Physical 0494
- Polymer 0495
- Radiation 0754

Mathematics 0405

Physics
- General 0605
- Acoustics 0986
- Astronomy and Astrophysics 0606
- Atmospheric Science 0608
- Atomic 0748
- Electronics and Electricity 0607
- Elementary Particles and High Energy 0798
- Fluid and Plasma 0759
- Molecular 0609
- Nuclear 0610
- Optics 0752
- Radiation 0756
- Solid State 0611

Statistics 0463

Applied Sciences

Applied Mechanics 0346
Computer Science 0984

Engineering
- General 0537
- Aerospace 0538
- Agricultural 0539
- Automotive 0540
- Biomedical 0541
- Chemical 0542
- Civil 0543
- Electronics and Electrical 0544
- Heat and Thermodynamics 0348
- Hydraulic 0545
- Industrial 0546
- Marine 0547
- Materials Science 0794
- Mechanical 0548
- Metallurgy 0743
- Mining 0551
- Nuclear 0552
- Packaging 0549
- Petroleum 0765
- Sanitary and Municipal 0554
- System Science 0790

Geotechnology 0428
Operations Research 0796
Plastics Technology 0795
Textile Technology 0994

PSYCHOLOGY
General 0621
Behavioral 0384
Clinical 0622
Developmental 0620
Experimental 0623
Industrial 0624
Personality 0625
Physiological 0989
Psychobiology 0349
Psychometrics 0632
Social 0451

To My Parents and My Wife

Contents

Abstract

In this thesis, we present interconnections among the Knaster-Kuratowski-Mazurkiewicz theorem (in short, KKM theorem), Ky Fan minimax inequalities, fixed point theorems, coincidence theorems, equilibria of generalized games and variational inequalities.

In Chapter 2, we obtain generalizations of the KKM theorem in topological spaces from which a characterization of a generalized HKKM mapping is proved. As applications, generalizations of Ky Fan minimax inequalities, coincidence and fixed point theorems for multivalued mappings are derived in H-spaces, topological vector spaces or in locally convex topological vector spaces.

In Chapter 3, using results from Chapter 2 and combining "approximate method" we show existence theorems for equilibria of generalized games in H-spaces, topological vector spaces, locally convex spaces, Frechet spaces or in finite dimensional spaces under various continuous and non-compact hypotheses. In particular, the question raised by Yannelis and Prabhakar in 1983 is answered under weaker hypotheses.

In Chapter 4, by applying the existence theorems from Chapter 3, we achieve several existence theorems for non-compact variational inequalities and non-compact generalized quasi-variational inequalities in locally convex spaces and in reflexive Banach spaces. These results in turn imply some new existence theorems for generalized complementarity problems and fixed point theorems for multivalued pseudo and nonexpansive mappings in Hilbert spaces.

Furthermore, the stability of Ky Fan points, of coincidence points and of solutions of generalized quasi-variational inequalities are also established.

Acknowledgements

I would like to express my sincere appreciation to my supervisor, Professor Kok-Keong Tan, for his excellent guidance and generous advice to my research, constant encouragement, very careful reading and critical discussions and great help at all states in the preparation of this thesis.

I would like to thank the members of my thesis examining committee, Professor A. C. Thompson and Professor S. Swaminathan for their careful reading and useful comments. Their insistence on the mathematical accuracy and clarity of expression have made a considerable improvement in the style, content and language of this thesis.

I would like to thank my external examiner, Professor E. Tarafdar for his valuable suggestions and comments.

This research work has received financial support from the Izaak Walton Killam Memorial Scholarships Trust through the Faculty of Graduate Studies at Dalhousie University, which is gratefully acknowledged. Thanks also go to Graduate Coordinators, Professor P. Stewart, Professor K. Johnson and Professor W. R. S. Sutherland, for helping me to follow all the required procedures, and to all faculty and friends of mine for their help, academic or otherwise. Specially, my thanks go to Professor and Chair, R. P. Gupta for his encouragement during my study at Dalhousie University.

I would like to thank Professor Jian Yu for his helpful discussions and comments, Professor Xie-Ping Ding and Professor Shih-Sen Chang and my colleagues and friends in China, Canada and elsewhere for their help and encouragements. Moreover I like to thank Professor Sehie Park for sending me his preprints and reprints.

I would like to express my greatest respect and appreciation to my parents who have brought up their children through a great hardship, and to my brothers and sisters who have been taking care of my parents while I am away from home. Their love and support always accompany me throughout my study in Canada.

Finally, my deepest gratitude goes to my wife Mei for her love, support, understanding, and especially her patience and encouragement. Many hours that I have spent on the preparation of dissertation could otherwise have been devoted to her.

Table of Symbols and Abbreviations

$\mathbf{R}$ .... the real line.

$\mathbf{N}$ .... the set of all natural numbers.

N .... the set $\{0, 1, 2, \cdots, n\}$.

2^X .... the family of all non-empty subsets of X.

$\mathcal{F}(X)$ .... the family of all non-empty finite subsets of X.

Δ_N .... the standard n-dimensional simplex in $\mathbf{R}^n$.

$\emptyset$ .... the empty set.

$X_{-i} = \Pi_{j\in I, j\neq i} X_j$ ....

x_{-i} .... the element of X_{-i}.

coX .... the convex hull of X.

$cl_X A$ (or $\overline{A}$) .... the closure of A in X.

Hco$D$ .... the H-convex hull of D in the H-space $(X, \{\Gamma_A\})$.

$\delta_E(X)$ .... the algebraic boundary of X in the topological space E.

$\partial_E(X)$ .... the topological boundary of X in the topological space E

$I_X(y)$ .... the inward set of X at y.

$O_X(y)$ .... the outward set of X at y.

π_i .... the ith projection.

Aff(A) .... the affine span of A.

ri(A) .... the relative interior of A in Aff(A).

E^* .... the dual space of E.

Graph$F$ .... the graph of the mapping F.

C .... a lattice with a least element 0.

$K(X)$ .... the family of all non-empty compact subsets of X.

$L(X)$ .... the family of all bounded real-valued functions on $X \times X$.

$bc(X)$ .... the family of all non-empty bounded and closed subsets of X.

h .... the Hausdorff metric on $bc(X)$ induced by the metric d or norm $\|\cdot\|$).

h^* .... the Hausdorff metric induced by the dual norm $\|\cdot\|^*$.

KKM theorem .... Knaster-Kuratowski-Mazurkiewicz theorem.

usco .... upper semicontinuous with compact values.

$GHKKM$ .... generalized HKKM mapping.

γ-DQCX .... γ-diagonally quasi-convex.

γ-DQCV .... γ-diagonally quasi-concave.

γ-DCX .... γ-diagonally convex.

γ-DCV .... γ-diagonally concave.

Chapter 1

Introduction

Let n be a positive integer, $N = \{0, 1, \cdots, n\}$ and Δ_N denote the unit n-simplex in (n + 1)-dimensional Euclidean space $\mathbf{R}^{n+1}$. For each $S \subset \{0, 1, \cdots, n\}$, we denote by Δ_S the face of Δ_N spanned by the unit vectors e_i for $i \in S$. A closed covering $C = \{C_0, C_1, \cdots, C_n\}$ of Δ_N is called a KKM covering if $\Delta_S \subset \cup_{i \in S} C_i$ for all $\emptyset \neq S \subset N$. In 1929, Knaster, Kuratowski and Mazurkiewicz [191] proved that if $\{C_0, C_1, \cdots, C_n\}$ is a KKM closed covering of Δ_N, then $\cap_{i=0}^{n} C_i$ is non-empty. In 1961, Ky Fan generalized the classical KKM theorem to infinite dimensional Hausdorff topological vector spaces and established an elementary but very basic "*geometric lemma*" for mulitvalued mappings. In 1968, Browder gave a fixed point version of Fan's geometric lemma and this result is now known as the Fan-Browder fixed point theorem. Since then there have been numerous generalizations of the Fan-Browder fixed point theorem and their applications to coincidence and fixed point theory, minimax inequalities, variational inequalities, nonlinear analysis, convex analysis, game theory, mathematical economics and so on.

By applying his geometric lemma in 1972, Ky Fan obtained a minimax inequality which plays a fundamental role in nonlinear analysis and mathematical economics and has been applied to potential theory, partial differential equations, monotone operators, variational inequalities, optimization, game theory, linear and nonlinear programming, operator theory, topological group and linear algebra. In particular, by using Ky Fan's minimax inequality, a more general form of the Fan-Glicksberg fixed point theorem is derived for multivalued mappings which are inward (or outward) as defined by Fan in 1969 (which are more general than Halpern's definitions for inward (or outward) mappings in 1965).

Recently, Horvath obtained some generalizations of Fan's geometric lemma and his minimax inequality in 1983 and 1987 by replacing the convexity assumption with topological properties: pseudo-convexity and contractibilty. By extending Horvath's concepts, Bardaro and Ceppitelli [46] in 1988 obtained generalizations of Ky Fan minimax inequalities to topological spaces which have the so called *H-Structure* (such spaces are called H-spaces).

Following this line, a number of generalizations of Ky Fan's minimax inequalities are given by Horvath [154], Baradaro and Ceppitelli [47], Ding and Tan [85], Ding, Kim and Tan [86]-[87], Chang and Ma [51], Park [243], Tarafdar [303], Tan, Yu and Yuan [289] in topological spaces which need not have a linear structure but with an H-structure.

The importance of fixed point theory in mathematics is well known. An example to illustrate and emphasize the close relationship between nonlinear analysis (in particular, fixed point theory) and economic science (in particular, mathematical economics) is as follows:

It was Leon Walras who, at the end of the last century, despite great opposition, dared to suggest using mathematics in economics. He described certain economic agents as automata seeking to optimize evaluation functions (utility, profit, etc) and posed the problem of economic equilibria. However, this area did not blossom until the birth of nonlinear analysis in 1912, with Brouwer's fixed point theorem [39], the usefulness of which was recognized by John von Neumann [234] when he developed the foundations of game theory in 1928. In the wake of von Neumann came the works of John Nash, Kakutani, Aumann, Shapley and many others who provided the tools used by Arrow, Debreu, Gale, Nikaido and many others to complete Walras' construction, culminating in the 1950's in the proof of the existence of economic equilibria. Debreu's classic book [73] "*Theory of Value*" is a very good survey-type exposition of economic equilibria at that time.

The natural extension of fixed point theory is the study of coincidence points. Let X

and Y be Hausdorff topological spaces and $S, T : X \to 2^Y$ be mappings. The *coincidence problem* for (S, T) is to find $(x_0, y_0) \in X \times Y$ such that $y_0 \in S(x_0) \cap T(x_0)$. Geometrical problems of this type in an appropriate context turn out to be intimately related to some basic problems arising in convex analysis. This important fact was discovered by J. von Neumann [234] in 1928, who established a coincidence theorem in $\mathbf{R}^n$ and made a direct use of it in the proof of his well-known minimax principle. Since then, geometrical problems of a similar kind (as well as their analytic counterparts) have attracted many people as well as finding new applications in various fields. In particular, since Eilenberg and Montgomery [94] studied coincidence theory in topological settings in 1946, this topic has been comprehensively developed by contributions due to Kakutani, Nash, Ky Fan, Kneser, Gale, Debreu, Nikaido, Sion, Gorniewicz, Granas, Liu, Chang, Song, Ben-El-Medchaiek, Deguire, Kryszewski, Ko, Shih, Tan, Powers and others. This topic has many applications in mathematics and other subjects, for example, see Aubin [7], Aubin and Cellina [9] and Zeidler [336]. In 1988, Ichiishi [156] successfully used Fan's coincidence theorem to give another proof of Scarf's existence theorem [259] for the non-emptiness of the core of balanced n-person game without side payments.

In this thesis, we first show in Chapter 2 that the classical KKM theorem holds in topological spaces. Then, a characterization of a generalized HKKM mapping (which is a generalization of the KKM mapping) is given in topological spaces which in turn gives several Ky Fan's minimax inequalities in H-spaces or in Hausdorff topological vector spaces. Moreover, several fixed point theorems and coincidence theorems for non-self multivalued mappings are derived under weaker continuity and boundary conditions. As applications, several matching theorems for closed coverings of convex sets are also derived. Furthermore , the concepts of the KF point and KF essential point are first introduced and the stability of KF points and coincidence points are established. These results improve or unify many corresponding results in the literature. For instance, our Ky Fan type minimax inequalities show the "*lower semicontinuity*" condition which is assumed by many authors (e.g., see Fan [106] etc.) is not needed for the existence of solutions for Ky Fan's minimax inequalities. Furthermore, our generalizations of the Fan-Glicksberg fixed point theorem show that the condition "*the domain is paracomapct*" appearing in many literatures (e.g., See Fan [106], Ko and Tan [192] and the references therein) is superfluous. These results will be needed in our further developments in Chapter 3.

A game is a situation in which each of several players has partial control over some outcome but generally conflicting preferences over the outcome: each player has a fixed range of strategies among which he selects one so as to bring about the best outcome according to his own preferences. An n-person game is a game in which the strategies of n players can not be made independently: each player must select a strategy in a subset determined by the strategies chosen by the other players. Formally, the situation can be described as follows. Let $N = \{1, 2, \cdots, n\}$ denote the set of players and for each $i \in N$, let X_i denote the set of strategies of the ith player. Each element of $X = \Pi_{i=1}^n X_i$ determines an outcome. The payoff to the ith player is a real-valued function f_i defined on X. Given $x_{-i} \in X_{-i} (= \Pi_{j\in N, j\neq i} X_j$, the strategies of all the others), the choice of the ith player is restricted to a non-empty subset $A_i(x^i)$ of X_i; the ith player chooses x_i in $A_i(x^i)$ so as to maximize $f_i([x_i, x^i])$. An equilibrium point in such an n-person game is a strategy vector $x \in X$ such that for all $i \in N$, $x_i \in A_i(x^i)$ and $f_i(x) = \max_{y_i \in A_i(x^i)} f_i([y_i, x^i])$.

The existence theorems for equilibria of an n-person game with compact strategy sets in $\mathbf{R}^n$ was proved in a seminal paper of Debreu [72] in 1952. The theorem of Debreu extended the earlier work of Nash [228] which also covers the existence theorem of equilibria of the general economic model presented by von Neumann [235] in 1937 (see also von Neumann and Morgenstern [236]) in game theory. Since then there have been many generalizations of Debreu's theorem by Arrow and Debreu [5] in 1954, Mas-Colell [215], Gale and Mas-Colell [126], Borglin and Keiding [37] in 1976 and others. Following Debreu [72] and Shafer and Sonnenschein [37], *a generalized game (or an abstract economy)* is a family $\Gamma = (X_i; A_i; U_i)_{i\in I}$ where I is an any (countable or uncountable) set of players (or agents) such that for each $i \in I$, X_i is the strategy set or choice set, $A_i : X = \Pi_{i\in I} X_i \rightarrow 2^{X_i}$ is the constraint correspondence and $U_i : X \rightarrow \mathbf{R}$ is the payoff or utility function. X_i will be a subset of a topological space or a topological vector space for each $i \in I$. We denote the product $\Pi_{j\in I, j\neq i} X_j$ by X_{-i} and a generic element of X_{-i} by x_{-i}.

Note that a generalized game instead of being given by $(X_i; A_i; U_i)_{i\in I}$ may be given by $\Gamma = (X_i; A_i; P_i)_{i\in I}$ where for each $i \in I$, $P_i : X \to 2^{X_i}$ is the preference correspondence. The relationship between the utility function U_i and the preference correspondence P_i may be exhibited by defining for each $x \in X$, $P_i(x) = (y_i \in X_i : U_i([y_i, x_{-i}]) > U_i(x)\}$, where for each $i \in I$, x_{-i} is the projection of x onto X_{-i} and $[y_i, x_{-i}]$ is the point y in X whose ith coordinate is y_i and $y_{-i} = x_{-i}$. In the case of a generalized game being given by $\Gamma = (X_i; A_i; U_i)_{i\in I}$, a point $\hat{x} \in X$ is called an equilibrium point or a generalized Nash equilibrium point of Γ if $U_i(\hat{x}) = U_i([\hat{x}_i, \hat{x}_{-i}]) = \max_{z_i \in A_i(\hat{x})} U_i([z_i, \hat{x}_{-i}])$ for each $i \in I$ where $\hat{x}$ and $\hat{x}_{-i}$ are respectively projections of $\hat{x}$ onto X_i and X_{-i}. In this case the equilibrium point is a natural extension of the equilibrium point introduced by Nash [227] in 1950. Now let $\Gamma = (X_i; A_i; U_i)_{i\in I}$ be a generalized game and for each $i \in I$, let P_i be obtained as above. Then it can be easily checked that a point $\hat{x} \in X$ is an equilibrium point of Γ if and only for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. This model has been generalized into a more general setting by Tan and Yuan [294] which in turn includes the model of a generalized game introduced by Ding, Kim and Tan [86].

Following the work of Sonnenschein [283] in 1971, Gale and Mas-Colell [124] in 1975 and Borglin and Keiding [37] in 1976 on non-ordered preference relations, many theorems on the existence of maximal elements of preference relations which may not be transitive or complete, have been proved by Aliprantis and Brown [2], Bergstrom [30], Kim [181], Mehta and Tarafdar [221], Shafer and Sonnenschein [263], Sonnenschein [283], Tan and Yuan [294], Tarafdar [304], Toussaint [315], Tulcea [317], Yannelis [325] and Yannelis and Prabhakar [326] and others. These papers generalize Debreu's theorem by considering preference correspondences that are not necessarily transitive or total, by allowing externalities in consumption and by assuming that the commodity space is not necessarily finite-dimensional. In these papers, the domain (and /or codomain) of the preference and constraint correspondences are assumed to be compact or paracompact, and the preference correspondences (respectively, payoff functions) are assumed to have open lower sections or open graphs (respectively, to be continuous).

However, most of these existence theorems for maximal elements and equilibrium points deal with preference correspondences which have open lower sections or are majorized by correspondences with open lower sections. Note that every correspondence with open lower sections must be lower semicontinuous but the converse is not true in general. Moreover, in most cases, preference and constraint correspondences may be upper semicontinuous (or majorized by upper semicontinuous correspondences) instead of being lower semicontinuous (or being majorized by lower semicontinuous), or the preference and constraint mappings are condensing. Furthermore, in the study of equilibrium theory in most economic models, the feasible sets or the budget constraints are generally not (weakly) compact in an infinite dimensional commodities and are not convex in the case of the indivisibility of commodities and the underlying spaces do not have a linear structure. Thus, relaxation of convexity of choice sets and generalizations of spaces enable us to deal with the existence of maximal elements and equilibrium points even though commodities are indivisible.

Therefore it is necessary and important to study the existence of equilibria for generalized games in which the preference and constraint correspondences need not have open lower sections nor open upper sections and also the underlying spaces need not have any linear structures and so on.

The objective of Chapter 3 is to systematically study the existence of maximal elements and equilibria for generalized games under various hypotheses, such as the preference and constrained correspondences are lower semicontinuous, upper semicontinuous or condensing, and the strategy sets may not be compact and the underlying spaces may not have a linear structure. Moreover, we also study some properties of lower semicontinuous multivalued mappings in finite dimensional spaces which in turn give several fixed point theorems and existence theorems for equilibria of generalized games. In particular, the question raised by Yannelis and Prabhakar [326] is answered in the affirmative with weaker assumptions.

The essential idea behind these existence theorems for equilibria of generalized games is to reduce them to qualitative games which in turn are reduced to the existence problem of maximal elements for preference correspondences. Since existence of maximal elements of correspondences have equivalent formulations in fixed point theorems which can be derived from Ky Fan's minimax inequalities, the results in Chapter 2 are applicable.

Even though the topic of variational inequalities has a very long history, it has only been studied systematically since 1960s (e.g., see Fichera [110] and Stampacchia [284] and others). The variational inequality theory is related to the simple fact that the minimum of the differentiable convex functional on a convex set D in a Hilbert space can be characterized by an inequality of the type $\langle I'(u), v-u\rangle \leq 0$ for all $v \in D$, where $I'(u)$ is the derivative of the functional $I(u)$. However, it is remarkable that the variational inequality theory has many diversified applications. During the last three decades which have elapsed since its discovery, the important developments in variational theory are formulations that variational inequalities can be used to study problems of fluid flow through porous media (e.g., see Baiocchi and Capelo [14]), contact problems in elasticity (e.g., see Kikuchi and Oden [178]) transportation problems (see Bertsekas and Gafni [32] and Harker [144]) and economic equilibria (see Dafermos [71]). An additional main area of applications for variational inequalities arises in control problems with a quadratic objective functional, where the control equations are partial differential equations. A detailed discussion of this can be found in Lions [209]. The connection between control problems and quasi-variational inequalities is presented in Aubin [7] and Zeidler [336]. There also exist intimate interconnections between variational inequalities, stochastic differential equations, and stochastic optimization. One can find these in Friedman [118]-[119], Bensoussan and Lions [27] and Bensoussan [26].

In recent years, various extensions and generalizations of variational inequalities have been considered and studied. It is clear that in a variational inequalities formulation, the convex set involved does not depend on solutions. If the convex set does depend on solutions, then variational inequalities are called quasi-variational inequalities. These useful and important generalizations are mainly due to Bensoussan and Lions [28]. Applications of quasi-variational inequalities can be found in Aubin [7], Aubin and Cellina [9] and Zeidler [336].

In 1982, for the study of operation research, mathematical programming and optimization theory, Chan and Pang [48] first introduced the so-called generalized quasi-variational inequalities in finite dimensional Euclidean spaces. Chan and Pang's generalized quasi-variational inequalities can be illustrated as follows

Let $\mathbf{N}$ and $\mathbf{R}$ denote the set of all natural numbers and the set of all real numbers respectively Let X be a non-empty subset of $\mathbf{R}^n$, where $n \in \mathbf{N}$ Let $A : X \to 2^X$ and $B : X \to 2^{\mathbf{R}^n}$ The generalized quasi-variational problem associated with A and B (briefly, denoted by GQVI (X, A, B) here) is to find $(\hat{x}, \hat{u}) \in X \times \mathbf{R}^n$ such that $\hat{x} \in A(\hat{x})$, $\hat{u} \in B(\hat{x})$ and $\sup_{y \in A(x)} \langle \hat{u}, \hat{x} - y \rangle \leq 0$

The existence theorem of Chan and Pang [48] is stated as follows.

Theorem A. Let X be a non-empty compact convex subset of $\mathbf{R}^n$ and $A : X \to 2^X$ and B $X \to 2^{\mathbf{R}^n} \setminus \{\emptyset\}$ are such that $A(x)$ is compact convex and $B(x)$ is contractible and compact for each $x \in X$ Moreover assume that A is continuous and B is upper semicontinuous Then GQVI(X, A, B) has at least one solution

In 1985, Shih and Tan [267] were the first to study the $GQVI(X; A; B)$ in infinite dimensional locally convex Hausdorff topological vector spaces as follows.

Theorem B. Let E be a locally convex Hausdorff topological vector space, E^* be the dual space of E and X be a non-empty compact convex subset of E Let $A : X \to 2^X$ be continuous such that for each $x \in X$, $A(x)$ is a non-empty closed convex subset of X, and $B \cdot X \to 2^{E^*}$ be upper semicontinuous from the relative topology of X to the strong topology of E^* such that for each $x \in X$, $B(x)$ is a non-empty strongly compact subset of E^* Then there exists a point $\hat{y} \in X$ such that

$$\begin{cases} \hat{y} \in A(\hat{y}) \text{ and} \\ \sup_{x \in A(\hat{y})} \inf_{w \in B(\hat{y})} Re\langle w, \hat{y} - x \rangle \leq 0 \end{cases}$$

Since then, there have been a number of generalizations of the existence theorems about GQVI(X, A, B), e g , see Cubiotti [68], Ding and Tan [81], Harker and Pang [145], Kim [180], Shih and Tan [274] and Tian and Zhou [311] and references therein. These results have wide applications to problems in game theory and economics, mathematical programming (e.g., see Aubin [7], Aubin and Ekeland [10], Chan and Pang [48], Harker and Pang [145] and reference therein). Most existence theorems mentioned above, however, are obtained on compact sets in finite dimensional spaces or infinite dimensional locally convex Hausdorff topological vector spaces, and both A and B are either continuous or upper (lower) semicontinuous.

On the other hand, in economic and game applications, it is known that the choice space (or the space of feasible allocations) generally is not compact in any topology of the choice space (even though it is closed and bounded), a key situation in infinite dimensional topological vector spaces. Moreover, we note that there is essentially no existence theorems of solutions of generalized quasi-variational inequalities on non-compact sets in infinite dimensional spaces. This motivates our work in Chapter 4 to give a series of existence theorems on generalized quasi-variational inequalities by relaxing the compactness conditions and continuity. By the existence theorems of generalized quasi-variational inequalities, the stability of solutions for two types of generalized quasi-variational inequalities are also established.

Equally important is the area of mathematical programming known as the complementarity theory, which was introduced and studied by Lemke [205] in 1965. Cottle and Dantzing [63] defined the complementarity problem and called it the fundamental problem. For recent results and applications, see Harker and Pang [145], Noor and Rassias [233] and references therein. However, it was Karamardian [171], who proved that if the set involved in a variational inequality and complementarity problem is a convex cone, then both problems are equivalent. After that, many generalizations have been given by Shih and Tan [266], Ding [79], Isac [160]-[161], Chang and Huang [50] and references therein. For more details on the discussion between the variational inequalities and complementarity problems, we refer to Cottle, Giannessi and Lions' book [64] and references therein.

In Chapter 4, as applications of our generalized quasi-variational inequalities, an existence theorem of generalized complementarity problem is given and by using the concept of a semi-monotone operator introduced by Bae, Kim and Tan [13], some fixed point theorems for set-valued pseudo-contractive mappings and set-valued nonexpansive mappings are obtained. The stability of solutions of generalized quasi-variational inequalities is also investigated

In recently years, a number of literatures have exposited the interconnections among minimax inequalities, equilibria of generalized games and variational inequalities. For instance, Tulcea [317] give a number of minimax inequalities which are derived by the applications of existence theorems for equilibria of generalized games. Dafermos [71] formulated the problems of finding equilibria of generalized games (in particular equilibria of pure exchange equilibria) to the problems of finding solutions of variational inequalities.

In this thesis, we present interconnections among the Knaster-Kuratowski-Mazurkiewicz theorem (in short, KKM theorem), Ky Fan minimax inequalities, fixed point theorem, coincidence theorems, equilibria of generalized games and variational inequalities in the following way:

We reduce the existence problems of variational inequalities to the existence problem for equilibria of generalized games; that means, the solutions of variational inequalities are nothing else, but are exactly the equilibria of their equivalent model of generalized games. This simple fact enable us to consider the existence of solutions for non-compact variational inequalities and generalized quasi-variational inequalities in infinite dimensional Hausdorff topological vector spaces. As we mention above, the existence problems of equilibria for generalized games can be reduced to the existence problems for equilibria of qualitative games, the latter existence problems are equivalent to finding maximal elements of their preference mappings. Note that maximal elements are equivalent forms of their fixed point theorems which can be derived by Ky Fan type minimax inequalities. Therefore we give the interconnections among minimax inequalities, fixed point theorems of multivalued mappings, generalized games in mathematical economics and variational inequalities and generalized quasi-variational inequalities.

We remark that the development of variational inequalities can be viewed as the simultaneous pursuit of two different lines of research: On the one side, it reveals the fundamental facts on the qualitative behaviour of solutions (such as its existence, uniqueness and regularity) to important classes of problems. On the other side, it also enables us to develop highly efficient and powerful new numerical methods to solve, for example, free and moving boundary value problems and the general equilibrium problems. A comprehensive investigation of numerical methods for variational inequalities is contained in Glowinski, Lions and Tremolieres's book [128]. For more details, see Cottle, Giannessi and Lions [64], Crank [66], Harker and Pang [145], Aslam Noor [231]-[232], A. Noor, I. Noor and Rassias [233], Rodrigues [255] and Shi [265] etc. Among the most effective numerical techniques are projection methods and its variant forms, linear approximation method, relaxation method, auxiliary principle and penalty function techniques. In addition to these methods, the finite element technique is also being applied for the approximate solution of variational inequalities have been obtained by many research workers including Falk [96], Mosco and Strang [226] and Noor, Noor and Rassias [233] and references therein.

Further, even though we have some results on abstract general algorithms for solutions of variational inequalities, they are not included here. The author wish to continue these topics soon. Moreover, we do not cover the topics on random analysis and its applications to fixed point theory and existence for equilibria of random generalized games which we refer to Tan and Yuan [293]-[300], Yuan [332]-[335] and the references therein.

Chapter 2

KKM Theorem and Some Related Results

2.1 Introduction

The classical theorem of Knaster-Kuratowski-Mazurkiewicz (often called the KKM theorem, KKM Lemma or KKM Principle in [191]) has numerous applications in various fields of pure and applied mathematics. These studies and applications are called the KKM Theory today.

In 1961, Ky Fan proved the generalization of the classical KKM theorem in infinite dimensional Hausdorff topological vector spaces and established an elementary but very basic "*geometric lemma*" for mulitvalued mappings. In 1968, Browder gave a fixed point form of Fan's geometric lemma and it is now called Fan-Browder fixed point theorem. Since then there have been numerous generalizations of Fan-Browder fixed point theorem and their applications in coincidence and fixed point theory, minimax inequalities, variational inequalities, nonlinear analysis, convex analysis, game theory, mathematical economics and so on.

By applying his geometric lemma in 1972, Ky Fan obtained a minimax inequality which plays a fundamental role in nonlinear analysis and mathematical economics and has been applied to potential theory, partial differential equations, monotone operators, variational inequalities, optimization, game theory, linear and nonlinear programming, operator theory, topological group and linear algebra. In particular, by using Ky Fan's minimax inequality, a more general form of the Fan-Glicksberg fixed point theorem is derived for multivalued mappings which are inward (or outward) as defined by Fan in 1969 which are more general than Halpern's definitions for inward (or outward) mappings in 1965.

Recently, Horvath obtained some generalizations of Fan's geometric lemma and his minimax inequality in 1983 and 1987 by replacing convexity assumption with topological properties: pseudo-convexity and contractibilty. By extending Horvath's concepts, Bardaro and Ceppitelli [46] in 1988 obtained generalizations of Ky Fan minimax inequalities to topological spaces which have so called *H-Structure* (also called H-spaces).

Following this line, a number of generalizations of Ky Fan's minimax inequalities are given by Horvath [154], Baradaro and Ceppitelli [47], Ding and Tan [85], Ding, Kim and Tan [86]-[87], Chang and Ma [51], Park [243], Tarafdar [303], Tan, Yu and Yuan [289] in topological spaces which need not have a linear structure but with an H-structure.

On the other hand, for the need of applications, various generalizations of the classical KKM principle and Sperner's lemma [285] have been given by Fan [101], [102], [104] and [107], Ding and Tan [85], Gale [123], Idzik and Tan [158], Shapley [264], Shih and Tan [269], [270], [271], Ichiishi [156], Ichiishi and Idzik [157]. Recently, Horvath [154] obtained some intersection theorems for closed coverings of a topological space with a contractible structure.

In this chapter, based on the classical KKM principle and its dual form given by Shih and Tan in 1987, we first study the closed (respectively, open) covering properties and intersection properties of topological spaces in section 2. These results generalize the corresponding results of Alexandroff-Pasynkoff [1], Berge [29], Klee [187], Fan [98], Horvath [154] and Spener [285]. As applications, we give a characterization of generalized HKKM mapping which is a generalization of the classical KKM theorem in topological spaces.

In section 3, by applying our generalized HKKM theorem, Ky Fan type minimax inequalities with weaker continuity conditions are given in topological spaces. Our results show that the traditional condition "*lower semicontinuity*" posited by many authors (e.g., see Fan [106] and references therein) is not essential for the existence of solutions for the Ky Fan type minimax inequalities. By employing a new coercive concept called "*escaping sequences*" which is first introduced by Border in 1985, several non-compact minimax inequalities are derived. As consequences, several equivalent fixed point theorems and maximal element theorems are given in H-spaces and topological spaces. In particular, the well-known Fan-Browder fixed point theorem has been improved.

In section 4, we study the stability of solutions of Ky Fan minimax inequality in both compact and non-compact settings.

It is well-known that Fan's best approximation theorem [103] and its generalizations have many applications in fixed point theory and approximation theory (for instance, see Lin and Yen [208], Reich [250] and Sehgal, Singh and Smithson [260] and references therein.

In section 5, we first generalize Fan's best approximation theorem to a Hausdorff topological vector space for multivalued mappings. Then as applications, several coincidence and fixed point theorems are given for non-self multivalued mappings under weaker boundary conditions. These results improve and generalize corresponding results of Komiya [196], Ha [138], [139] etc.

In section 6, we investigate the stability of coincidence points. Our results improve and cover corresponding results given by Fort [116] and Jiang [163] in several ways.

In section 7, we obtain some fixed point theorems and coincidence theorems in topological vector spaces with sufficient continuous linear functionals and in locally convex topological spaces for inward (respectively, outward) upper hemicontinuous multivalued non-self mappings These results unify most results of fixed point theorems and coincidence theorems in the literature. For instance, see Park [240], Ko and Tan [192] etc.). As applications, several matching theorems for closed coverings of convex sets are given which include the well-known Shapley generalization [264] of the classical KKM theorem.

2.2 Knaster-Kuratowski-Mazurkiewicz Theorem

In this section, based on the classical KKM theorem [191] and its "dual" form given by Shih and Tan [271], we first discuss some properties of contractible subsets in topological spaces by employing Horvath's approach [153]. As applications, a characterization of generalized HKKM mappings is given. These results improve Fan's famous geometric result given in his celebrated paper [98], and corresponding results due to Fan [106]-[107], Klee [187], Alexandroff-Pasynkoff [1], Berge [29], Horvath [153], and Chang and Zhang [54], Chang and Yan [53] and Chang and Ma [51].

First we introduce and recall some notations and definitions. Throughout this thesis all spaces are assumed to be *Hausdorff* if this is not specified. Let X and Y be non-empty sets. We shall denote by 2^Y the family of all non-empty subsets of Y, $\mathcal{F}(X)$ the family of all non-empty finite subsets of X. Let X be a topological space. For each non-empty subset A of X, we denote the closure of A in X by $cl_X A$ (in short, clA) or $\bar{A}$ if there is no confusion. A subset A of X is said to be compactly closed (respectively, open) if $A \cap C$ is closed (respectively, open) in each non-empty compact subset C of X.

Let $\mathbf{N}$ and $\mathbf{R}$ denote the set of all natural numbers and the set of all real numbers, respectively. For each $n \in \mathbf{N}$, let $N = \{0, 1, \cdots, n\}$ and $\Delta_N = co\{e_0, \cdots, e_n\}$ be the standard simplex of dimension n, where $\{e_0, \cdots e_n\}$ is the canonical basis of $\mathbf{R}^{n+1}$ and for $J \in \mathcal{F}(N)$, $\Delta_J = co\{e_j : j \in J\}$. A topological space X is said to be *contractible* if the identity mapping I_X of X is homotopic to a constant function.

The classical KKM theorem [191] is stated as follows:

Theorem 2.2.A. Let $C_0, \cdots, C_n$ be closed subsets of the standard n-dimensional simplex Δ_N and let $\{e_0, \cdots, e_n\}$ be the set of its vertices. If for each $J \in \mathcal{F}(N)$, $\Delta_J \subset \cup_{j \in J} C_j$. Then $\cap_{i=0}^n C_i \neq \emptyset$.

In 1987, Shih and Tan [271] (see also Kim [179] or Lassonde [200]) provided the following "dual form" of KKM theorem in the sense that the word "closed" is replaced by the word "open":

Theorem 2.2.A′. Let $C_0, \cdots, C_n$ be open subsets of the standard n-dimensional simplex Δ_N and let $\{e_0, \ldots, e_n\}$ be the set of its vertices If for each $J \in \mathcal{F}(N)$, $\Delta_J \subset \cup_{j\in J} C_j$ Then $\cap_{i=0}^n C_i \neq \emptyset$

The following notions which were introduced by Bardaro and Ceppitelli in [46] were motivated by earlier work of Horvath [153]. A pair $(X, \{\Gamma_A\})$ (also called an H-structure) is said to be an H-space (also called c-space according to Horvath [154]) if X is a topological space and $\{\Gamma_A\}_{A\in\mathcal{F}(X)}$ a given family of non-empty contractible subsets Γ_A of X such that $\Gamma_A \subset F_B$ whenever $A \subset B$ Let $(X, \{\Gamma_A\})$ be an H-space A non-empty subset D of X is said to be (i) H-convex (also called an F-set by Horvath [154]) if $\Gamma_A \subset D$ for each $A \in \mathcal{F}(D)$, (ii) weakly H-convex if $\Gamma_A \cap D$ is contractible for each $A \in \mathcal{F}(D)$ (or equivalently, $(D, \{\Gamma_A \cap D\})$ is an H-space) and (iii) H-compact in X if for each $A \in \mathcal{F}(X)$, there exists a compact, weakly H-convex subset D_A of X such that $D \cup A \subset D_A$ It is clear that the product space of a family of H-spaces is also an H-space.

The following example (e g , see Horvath [154, p 345]) shows that an H-space may be not a convex subset in a topological vector space

Let X be a convex set in a topological vector space E and Y any topological space. Suppose that $f : X \to Y$ is a continuous bijection For given $A \in \mathcal{F}(Y)$, let $D_A := co\{x \in X \; f(x) \in A\}$ Then D_A is convex, so that D_A is contractible Since D_A is also compact, so that $f \; D_A \to f(D_A)$ is an homeomorphism Let $\Gamma_A = f(D_A)$. Then Γ_A is contractible and $\Gamma_A \subset \Gamma_{A'}$ whenever $A \subset A' \in \mathcal{F}(Y)$. Therefore $(Y, \{\Gamma_A\})$ is an H-space Note that the space Y itself may be a torus, the Mobius band or the Klein bottle This example shows that an H-space does not have to be contractible

The following notion is due to Tarafdar [303]: Let D be a non-empty subset of an H-space $(X, \{\Gamma_A\})$ The H-convex hull of D, denoted by Hco(D), is defined by Hco$(D) = \cap\{B \subset X \, . \, B$ is H-convex and $D \subset B\}$ Clearly, Hco(D) is the smallest H-convex subset of X containing D and the intersection of any family of H-convex set is also H-convex

Let X be a topological space, an n-dimensional singular face structure on X (e g., see Horvath [154]) is a mapping $F : \mathcal{F}(N) \to 2^X$ such that (a) for each $J \in \mathcal{F}(N)$, $F(J)$ is not-empty and contractible and (b) for any $J, J' \in \mathcal{F}(N)$, $J \subset J'$ implies $F(J) \subset F(J')$

The following result is contained in the proof of Theorem 1.1 of Horvath [153] (see also Theorem 1 of Horvath [154]) and its proof is omitted.

Lemma 2.2.B. let X be a topological space. For each non-empty subset J of $\{0, 1, \cdots, n\}$, let F_J be a non-empty contractible subset of X with $F_J \subset F_{J'}$ whenever $\emptyset \neq J \subset J' \subset \{0, 1, \cdots, n\}$. Then there exists a continuous function $f : \Delta_N \to X$ such that $f(\Delta_J) \subset F_J$ for each non-empty subset J of $\{0, 1, \cdots, n\}$.

Proposition 2.2.1. Let X be a topological space. Let $F : \mathcal{F}(N) \to 2^X$ be a singular face structure on X and $\{M_i : i = 0, \cdots, n\}$ be a family of closed (respectively, open) subsets of X such that for any $J \in \mathcal{F}(N)$, $F(J) \subset \cup_{i \in J} M_i$.
Then $\cap_{i=0}^n M_i \neq \emptyset$.

Proof. By Lemma 2.2.B, there is a continuous function $f : \Delta_N \to X$ such that for each $J \in \mathcal{F}(N)$, $f(\Delta_J) \subset F(J)$. Let $C_i = f^{-1}(M_i)$ for each $i = 0, \cdots, n$. Then $\{C_i\}_{i=0}^n$ is a family of closed (respectively, open) subsets of Δ such that for any $J \in \mathcal{F}(N)$, $\Delta_J \subset \cup_{i \in J} C_i$. By Theorem 2.2.A (respectively, Theorem 2.2.A′), $\cap_{i=0}^n C_i \neq \emptyset$. Take any $x_0 \in \cap_{i=0}^n C_i$, then $f(x_0) \in \cap_{i=0}^n M_i \neq \emptyset$. □

As an application of Proposition 2.2.1, we have:

Theorem 2.2.2. Let X be a contractible topological space, $\{M_i : i = 0, \cdots, n\}$ be a closed (respectively, open) covering of X and $\{F_i : i = 0, \cdots, n\}$ a family of contractible subsets of X such that (i) for any $i \in \{0, \cdots, n\}$, $F_i \cap M_i = \emptyset$ and (ii) for any $J \in \mathcal{F}(N)$ with $J \neq N$, $\cap_{i \in J} F_i$ is non-empty and contractible. Then $\cap_{i=0}^n M_i \neq \emptyset$

Proof. Define $F : \mathcal{F}(N) \to 2^X$ by $F(N) = X$ and $F(J) = \cap_{i \notin J} F_i$ if $J \in \mathcal{F}(N)$ with $J \neq N$. Then F is a singular face structure on X. Note that for any $J \in \mathcal{F}(N)$, $F(J) \subset \cup_{i \in J} M_i$ since $F(J) \subset \cup_{i \in J} M_i \cup \cup_{i \notin J} M_i$ and $F(J) \cap M_i = \emptyset$ whenever $i \notin J$. Then Proposition 2.2.1 implies that $\cap_{i=0}^n M_i \neq \emptyset$. □

Theorem 2.2.2 is clearly a generalization of the KKM lemma. For $X = \Delta_N$ and $F_i = co(\{e_j : j \neq i\}$ for $i \in N$, Theorem 2.2.2 reduces to a theorem of Alexandroff-Pasynkoff in [1]. For $X = \Delta_N$ and $F_i = X \setminus M_i$ which is a closed convex subset for each $i \in N$, Theorem 2.2.2 is a generalization of a corresponding result due to Klee [187] and is known as Berge's intersection theorem [29].

Theorem 2.2.3. Let X be a contractible topological space and Y be a topological space, $\{M_i : i = 0, \cdots, n\}$ be an open (respectively, closed) covering of Y and $\{F_i : i = 0, \cdots, n\}$ be a family of contractible subsets of X. Let $S : X \to Y$ be continuous such that

(a) for each $i \in \{0, 1, \cdots, n\}$, $F_i \subset S^{-1}(M_i)$ and

(b) for each $J \in \mathcal{F}(N)$ with $J \neq N$, $\cap_{i \in J} F_i$ is non-empty and contractible.

Then $\cap_{i=0}^{n} M_i \neq \emptyset$.

Proof. Suppose the contrary, so that $\cup_{i=0}^{n} M_i^C = Y$, where $M_i^C = Y \setminus M_i$ denotes the complement of M_i in Y for each $i = 0, 1, \cdots, n$. Thus $\{M_i^C : i = 0, \cdots, n\}$ is an open (respectively, closed) covering of Y. So that $\{S^{-1}(M_i^C) : i = 0, \cdots, n\}$ is an open (respectively, closed) covering of X and by the condition (a), for each $i = 0, 1, \cdots, n$, $F_i \cap S^{-1}(M_i^C) = \emptyset$. Therefore F_i and $S^{-1}(M_i^C)$ for $i = 0, \cdots, n$ satisfy all hypotheses of Theorem 2.2.2. By Theorem 2.2.2, $\cap_{i=0}^{n} S^{-1}(M_i^C) \neq \emptyset$ which contradicts that $\{M_i : i = 0, \cdots, n\}$ is a covering of Y. Thus $\cap_{i=0}^{n} M_i \neq \emptyset$. □

We remark that Horvath [154, p.343] proved Theorem 2.2.3 under the additional assumptions that $X = Y$ and X is a normal space.

Since Chang and Yang in [53] gave a generalization of the KKM theorem in which the domain need not be a subset of its range, there are several generalizations in this direction. For example, Chang and Ma [51] extended this definition into H-spaces and later Zhou [337] gave a more generalized definition and obtained a characterization of the generalized HKKM mapping which is also a generalization of the corresponding result given by Chang and Zhang [54].

Definition. Let X be a non-empty set and Y a topological space. A mapping $G : X \to 2^Y \cup \{\emptyset\}$ is said to be transfer closed valued (e. g., see Zhou and Tian [339]) if for each $x \in X$ and $y \notin G(x)$, there exist $x' \in X$ and an open neighborhood $N(y)$ of y in Y such that $y' \notin G(x')$ for each $y' \in N(y)$. It is obvious that if a mapping $G : X \to 2^Y$ is transfer closed valued, then for each $x \in X$ and $y \in Y$ with $y \notin G(x)$, there exists some $x' \in X$ such that $y \notin clG(x')$.

The following lemma was first proved by Zhou and Tian [339] for the case when the domain X is a topological space and in the present form by Zhou [337]; for completeness, we include its simple proof.

Lemma 2.2.4. Let X be a non-empty set, Y a topological space and $G : X \to 2^Y$ Then $\cap_{x\in X} G(x) = \cap_{x\in X} clG(x)$ if and only if the mapping G is transfer closed valued

Proof. Sufficiency. It is clear that $\cap_{x\in X} G(x) \subset \cap_{x\in X} cl_Y G(x)$. It is sufficient to show that $\cap_{x\in X} clG(x) \subset \cap_{x\in X} G(x)$. Suppose that $y \notin \cap_{x\in X} G(x)$ Then there exists some $x \in X$ such that $y \notin G(x)$. Since G is transfer closed valued on X, there exists some $x' \in X$ such that $y \notin cl_Y G(x')$, so that $y \notin \cap_{x\in X} clG(x)$. Therefore $\cap_{x\in X} clG(x) = \cap_{x\in X} G(x)$.

Necessity. Suppose $(x, y) \in X \times Y$ such that $y \notin G(x)$; then $y \notin \cap_{z\in X} G(z) = \cap_{z\in X} clG(z)$ so that there exists $x' \in X$ such that $y \notin clG(x')$. But then there exists an open neighborhood $N(y)$ of y in Y such that $N(y) \cap G(x') = \emptyset$ so that $y' \notin G(x')$ for all $y' \in N(y)$ Thus G is transfer closed valued. □.

Let D be a non-empty subset of an H-space $(X, \{\Gamma_A\})$. A map $F : D \to 2^X$ is called HKKM if $\Gamma_A \subset \cup_{x\in A} F(x)$ for each $A \in \mathcal{F}(X)$. When X is a non-empty convex subset of a topological vector space and $\Gamma_A = coA$, the convex hull of A for each A$\in \mathcal{F}(X)$, then $(X, \{\Gamma_A\})$ becomes an H-space. In this case, the notion of an HKKM mapping $F : D \to 2^X$ coincides with the notion of a KKM mapping F, i.e., $coA \subset \cup_{x\in A} F(x)$ for each $A \in \mathcal{F}(D)$.

Definition. Let X be a non-empty set and Y a topological space A mapping $G : X \to 2^Y$ is said to be a *generalized HKKM* mapping (in short, GHKKM) if for each finite subset $A = \{x_1, \cdots, x_n\}$ of X, there exist a corresponding finite subset $B = \{y_1, y_2 \cdots, y_n\}$ (y_i's need not be distinct here) in Y and a family $\{\Gamma_C\}_{C\in\mathcal{F}(B)}$ of non-empty contractible subsets of Y such that $\Gamma_C \subset \Gamma_{C'}$ whenever $C \subset C' \in \mathcal{F}(Y)$ such that

$$\Gamma_{\{y_{i_j},\ j\in J\}} \subset \cup_{j=1}^{s} G(x_{i_j})$$

for $\emptyset \neq J \subset \{0, 1, \cdots, n\}$.

It is clear that each $HKKM$ mapping is $GHKK$, the following example show that the converse does not hold:

Example. Let $E = (-\infty, +\infty)$ and $X = [-2, +2]$. Define $G : X \to 2^E$ by

$$G(x) = [-(1 + \frac{x^2}{5}), \frac{x^2}{5}],$$

for each $x \in X$. Since $\cup_{x \in X} H(x) = [-9/5, 9/5]$ and $x \notin G(x)$ for each $x \in [-2, -9/5) \cup (9/5, 1]$. This shows that G is not a KKM mapping. Next we prove that G is a generalized KKM mapping. In fact, for any finite subset $\{x_1, \cdots, x_n\} \subset X$, take $\{y_1, \cdots, y_n\} \subset [-1, 1]$, then for any finite subset $\{y_{i_1}, \cdots, y_{i_k}\} \subset \{y_1, \cdots, y_n\}$, we have

$$co\{y_{i_1}, \cdots, y_{i_k}\} \subset [-1, 1] = \cap_{x \in X} G(x) \subset \cup_{j=1}^{k} G(x_{i_j})$$

Thus G is GHKKM. For more details, we refer to [53].

Theorem 2.2.5. Let X be a non-empty set, and let both Y and Z be topological spaces. Let $S : Y \to Z$ be continuous and $G : X \to 2^Z$ be such that:

(1) the composition mapping $S^{-1} \circ G : X \to 2^Y$ defined by $(S^{-1} \circ G)(x) = \cup_{z \in G(x)} \{y \in Y : z = S(y)\}$ for each $x \in X$, is a generalized HKKM mapping;

(2) for each $x \in X$, $G(x)$ is closed (respectively, open) in Y.

Then the family $\{G(x) : x \in X\}$ has the finite intersection property, i.e., for each $A \in \mathcal{F}(X)$, $\cap_{x \in A} G(x) \neq \emptyset$.

Proof. For any finite subset $\{x_0, x_1, \cdots, x_n\}$ of X, since $S^{-1} \circ G : X \to 2^Y$ is a generalized HKKM mapping, there exist a finite subset $B = \{y_0, y_1, \cdots, y_n\}$ of Y and a family $\{\Gamma_C\}_{C \in \mathcal{F}(B)}$ of non-empty contractible subsets of Y such that $\Gamma_C \subset \Gamma_{C'}$ whenever $C \subset C'$ such that

$$\Gamma_{\{y_{i_0}, y_{i_1}, \cdots, y_{i_s}\}} \subset \cup_{j=0}^{s} (S^{-1} \circ G)(x_{i_j})$$

for each finite subset $\{y_{i_0}, y_{i_1}, \cdots, y_{i_s}\}$ of $\{y_0, y_1, y_2, \cdots, y_n\}$, where $(0 \leq s \leq n)$. Let $M_i = S^{-1}(G(y_i))$ for each $i = 0, 1, \cdots, n$; and define a mapping $F : \mathcal{F}(N) \to Y$ by $F(J) = \Gamma_{\{y_k, k \in J\}}$ for each $J \in \mathcal{F}(N)$. Since S is continuous, M_i is closed (respectively, open) in Y for $i = 0, 1, \cdots, n$ by the assumption (2). Moreover the mapping F is a singular face structure on Y. Therefore all hypotheses of Proposition 2.2.1 are satisfied. By Proposition 2.2.1, $\cap_{i=0}^{n} M_i \neq \emptyset$. Take any $y_0 \in \cap_{i=0}^{n} M_i$, then $S(y_0) \in \cap_{i=0}^{n} G(x_i) \neq \emptyset$. □

As an application of Theorem 2.2.5, we have the following result due to Zhou [337]:

Theorem 2.2.6. Let X be a non-empty set and Y a topological space. Let $G : X \to 2^Y$ be such that

(a) G is transfer closed valued on X;

(b) there exists a non-empty finite subset X_0 of X such that the set $Y_0 = \cap_{x \in X_0} clG(x)$ is non-empty and compact in Y.

Then the intersection $\cap_{x \in X} G(x)$ is non-empty and compact if and only if the mapping clG is a generalized HKKM mapping.

Proof. Necessity: Suppose $\cap_{x \in X} G(x)$ is non-empty and compact. Take any $y_0 \in \cap_{x \in X} G(x)$. Note that the singleton set $\{y_0\}$ is contractible. For each $A = \{x_1, \cdots, x_n\} \in \mathcal{F}(X)$, take $B = \{y_1, \cdots, y_n\}$ with $y_i = y_0$ for all $i = 1, 2, \cdots, n$ and let $\Gamma_{B'} = \{y_0\}$ for all $B' \in \mathcal{F}(B)$. Since $y_0 \in clG(x)$ for all $x \in X$, it is clear that the mapping clG is generalized HKKM.

Sufficiency: Since the mapping clG is a generalized HKKM by Theorem 2.2.5 with $Y = Z$ and S being the identity map on Y, the family $\{clG(x) : x \in X\}$ has the finite intersection property. Now define a mapping $G'(x) = cl_Y G(x) \cap Y_0$ for each $x \in X$. Then the family of non-empty compact subsets $\{G'(x) : x \in X\}$ has the finite intersection property, so that $\cap_{x \in X} cl_Y G(x) = \cap_{x \in X} G'(x) \neq \emptyset$. Since G is transfer closed, $\cap_{x \in X} G(x) = \cap_{x \in X} clG(x)$ by Lemma 2.2.4. Therefore $\cap_{x \in X} G(x) \neq \emptyset$ □

An an immediate consequence of of Theorem 2.2.6, we have the following:

Theorem 2.2.7. Let X be a non-empty set and Y a compact topological space. Let $G : X \to 2^Y$ be transfer closed valued on X. Then the intersect $\cap_{x \in X} G(x)$ is non-empty if and only if the mapping clG is a generalized HKKM mapping.

Theorem 2.2.7 is a generalization of the corresponding results given by Chang and

Yang [53] and Chang and Ma [51].

2.3 Ky Fan Minimax Inequalities in H-Spaces

The minimax inequality of Fan ([105]) is fundamental in proving many existence theorems in nonlinear analysis. There have been numerous generalizations of Fan's minimax inequality by weakening the compactness assumption or the convexity assumption. In [46], using Horvath's approach [153], Bardaro and Ceppitelli obtained some minimax inequalities in topological spaces which have "H-space" structure. Following this line, there are many generalizations given by Horvath [153], Tarafdar [303], Ding and Tan [81], Ding, Kim and Tan [87], Chang and Ma [51], Park [243], Tan, Yu and Yuan [289]. These results generalize most of the corresponding results given by Fan [98] and [106], Degundji and Granas [90], Lassonde [199], Simons [276], Zhou and Chen [338] to topological spaces which have the so-called H-structure. However, all results mentioned above require lower semicontinuity to guarantee the existence of solutions. Our results shows that the *lower semicontinuity* is not essential for the existence of solutions for Ky Fan's minimax inequalities.

In this section, by weakening the compactness and continuity assumption on H-spaces, we obtain some new minimax inequalities. Then several non-compact minimax inequalities are obtained by using the concept "*escaping of sequence*" introduced by Border [34] which is different from other non-compact minimax inequalities given by Allen [4], Aubin [7], Aubin and Ekeland [10], Lassonde [199], Fan [106], Ding and Tan [81], Chang and Zhang [54], Yen [327] and Tian and Zhou [311]. Finally, several fixed point theorems and existence theorems for maximal elements are given in H-spaces (respectively, in topological vector spaces) which are equivalent to the minimax inequalities in H-spaces (respectively, in topological vector spaces). These results will be needed in our further developments.

Theorem 2.3.1. Let X be a non-empty set and Y a compact topological space and $\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that:

(a) the mapping $x \mapsto \{y \in Y : \phi(x,y) \leq 0\}$ is transfer closed valued;

(b) the mapping $x \to cl_Y\{y \in Y : \phi(x,y) \leq 0\}$ is generalized HKKM on X.

Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Proof. Define a mapping $G : X \to 2^Y$ by $G(x) = \{y \in X : \phi(x,y) \leq 0\}$ for each

$x \in X$. Then we have: (1) the mapping G is transfer closed valued and (2) the mapping clG is generalized HKKM. By Theorem 2.2.7, $\cap_{x \in X} G(x) \neq \emptyset$. Take any $y^* \in \cap_{x \in X} G(x)$, then $\sup_{x \in X} \phi(x, y^*) \leq 0$ for all $x \in X$. □

Remark: It is clear that the condition (a) of Theorem 2.3.1 is equivalent to the following condition which first appeared in Tan, Yu and Yuan [289]:

Fact $(a)'$: *for each* $y \in Y$ *with* $\{x \in X : \phi(x, y) > 0\} \neq \emptyset$, *there exists* $x' \in X$ *such that* $y \in int_Y\{y' \in Y : \phi(x', y') > 0\}$.

Theorem 2.3.2. Let X be a non-empty subset of a compact H-space $(Y, \{\Gamma_A\})$ and $\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that

(a) the mapping $x \mapsto \{y \in Y : \phi(x, y) \leq 0\}$ is transfer closed valued on X;

(b) the map $x \mapsto cl_Y\{y \in Y : \phi(x, y) \leq 0\}$ is HKKM on X.

Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Proof. Since $(Y, \{\Gamma_A\})$ is an H-space, each HKKM mapping is automatically a generalized HKKM mapping. Therefore all hypotheses of Theorem 2.3.1 are satisfied. By Theorem 2.3.1, there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$. □.

Corollary 2.3.3. Let X be a non-empty subset of a compact H-space $(Y, \{\Gamma_A\})$ and $\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that:

(a) for each $x \in X$, $y \mapsto \phi(x, y)$ is lower semicontinuous on Y;

(b) the map $x \mapsto cl_Y\{y \in Y : \phi(x, y) \leq 0\}$ is HKKM on X.

Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Proof. Suppose $y \in Y$ is such that $\{x \in X : \phi(x, y) > 0\} \neq \emptyset$. Fix any $x' \in X$ with $\phi(x', y) > 0$. By (a), there exists an open neighborhood $N(y)$ of y such that $\phi(x', y') > 0$ for each $y' \in N(y)$. Hence $y \in \text{int}_Y\{y' \in X : \phi(x', y') > 0\}$. Now the conclusion follows from Theorem 2.3.2 and the Fact $(a)'$ preceding it. □

Corollary 2.3.4. Let X be a non-empty subset of a non-empty compact convex set Y in a topological vector space and $\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that

(a) the mapping $x \mapsto \{y \in Y : \phi(x, y) \leq 0\}$ is transfer closed valued on X;

(b) the map $x \mapsto cl_Y\{y \in Y : \phi(x, y) \leq 0\}$ is KKM on X.

Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Proof. For each $A \in \mathcal{F}(Y)$, let $\Gamma_A = co(A)$. Then $(Y, \{\Gamma_A\})$ is an H-space and the map $x \mapsto cl_Y\{y \in Y : \phi(x, y) \leq 0\}$ is HKKM on X. Thus the conclusion follows from Theorem 2.3.2. □

Corollary 2.3.5. Let X be a non-empty subset of a non-empty compact convex set Y in a topological vector space and $\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that

(a) the mapping $x \mapsto \{y \in Y : \phi(x, y) \leq 0\}$ is transfer closed valued on X;

(b) for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, $min_{x \in A}\phi(x, y) \leq 0$.

Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Proof. By Corollary 2.3.4, we only need to prove that the map $x \to cl_Y\{y \in Y : \phi(x, y) \leq 0\}$ is KKM on X. Suppose not, then there exist $A \in \mathcal{F}(X)$ and $y \in co(A)$ such that $y \notin \cup_{x \in A} cl_X\{y \in X : \phi(x, y) \leq 0\}$. It follows that $\phi(x, y) > 0$ for each $x \in A$, so that $min_{x \in A}\phi(x, y) > 0$ which is a contradiction. □.

As seen from the proof of Corollary 2.3.3, the condition "for each $x \in X$, $y \to \phi(x, y)$ is lower semicontinuous" implies the condition" for each $y \in X$ with $\{x \in X : \phi(x, y) > 0\} \neq \emptyset$, there exists $x' \in X$ such that $y \in \text{int}_X\{y' \in X : \phi(x', y') > 0\}$". Thus Corollary 2.3.4 and hence Theorems 2.3.1 and Theorem 2.3.2 generalize Theorem 1 of Yen [327] (see also Theorem 2.2 of Simons [275]) and Theorem 2.11 of Zhou and Chen [338]. The following is an example for which Theorem 2.3.2 is applicable while Theorem 1 of Yen [327] and Theorem 2.11 of Zhou and Chen [338] are not:

Example. Let $Y = [0, 1]$ and X be the set of all rational numbers in $[0, 1]$. Define $\phi : X \times Y \to \mathbf{R}$ by

$$\phi(x, y) = \begin{cases} x - y, & \text{if } y \text{ is rational,} \\ 2, & \text{if } y \text{ is irrational} \end{cases}$$

for each $(x, y) \in X \times Y$. Suppose $(x, y) \in X \times Y$ and $\phi(x, y) > 0$. If y is irrational, then clearly $y < 1$. If y is rational, then since $\phi(x, y) = x - y > 0$, we also have $y < x \leq 1$. In either case, take $x' = 1$ and note that $\{y' \in Y : \phi(x', y') > 0\} = [0, 1)$, so that $y \in [0, 1) = \text{int}_Y\{y' \in Y : \phi(x', y') > 0\}$. Thus the condition $(a)'$ and hence the condition (a) of Theorem 2.3.2 is satisfied. Moreover, for each $x \in X$, $cl_X\{y \in Y : \phi(x,y) \leq 0\} = cl_X\{y \in Y : y$ is rational and $y \geq x\} = [x,1]$. It follows that the map $x \to cl_Y\{y \in Y : \phi(x,y) \leq 0\}$ is KKM on X. Thus the condition (b) of Theorem 2.3.2 is also satisfied. Therefore Theorem 2.3.2 is applicable. However, for each $x \in X$, the map $y \to \phi(x,y)$ is not lower semicontinuous and hence T' :orem 1 of Yen [327] and Theorem 2.11 of Zhou and Chen [338] are not applicable.

The Example above shows that for each $x \in X$, the lower semicontinuity of the mapping $y \mapsto \phi(x,y)$ is not essential for the existence of solutions for minimax inequalities.

In order to obtain our main resul s on minimax inequalities, we need the concept of an escaping sequence introduced in Border [34, p.34]: Let X be a topological space such that $X = \cup_{n=1}^{\infty} X_n$ where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact sets. A sequence $(y_n)_{n=1}^{\infty}$ in X is said to be escaping from X (relative to $\{X_n\}_{n=1}^{\infty}$) if for each $n = 1, 2, \cdots$, there exists a positive integer M such that $y_k \notin X_n$ for all $k \geq M$.

Theorem 2.3.6. Let X be a non-empty set and Y a topological space such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$ where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets and of compact spaces respectively. Let $\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that

(a) for each $n \in \mathbf{N}$, the mapping $x \to \{y \in Y_n : \phi(x,y) \leq 0\}$ is transfer closed valued on X_n;

(b) for each $n \in \mathbf{N}$, the map $x \mapsto cl_{Y_n}\{y \in Y_n : \phi(x,y) \leq 0\}$ is generalized HKKM on X_n;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{R}$ and $x_{n_0} \in X_{n_0}$ such that $\phi(x_{n_0}, y_{n_0}) > 0$.

Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Proof. For each $n \in \mathbf{N}$, by Theorem 2.3.1, there exists $y_n \in Y_n$ such that $\phi(x, y_n) \leq 0$ for all $x \in X_n$.

Suppose the sequence $(y_n)_{n=1}^{\infty}$ were escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$. By (c), there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $\phi(x_{n_0}, y_{n_0}) > 0$ which is a contradiction.

Therefore the sequence $(y_n)_{n=1}^{\infty}$ is not escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, so that some subsequence of $(y_n)_{n=1}^{\infty}$ must lie entirely in some Y_{n_1}. Since Y_{n_1} is compact, there exist a subnet $\{z_\alpha\}_{\alpha\in\Gamma}$ of $(y_n)_{n=1}^{\infty}$ in Y_{n_1} and a point $y^* \in Y_{n_1}$ such that $z_\alpha \to y^*$. Denote $z_\alpha = y_{n(\alpha)}$ for each $\alpha \in \Gamma$. If $x \in X$ is given, there exists $n_2 \geq n_1$ such that $x \in X_{n_2}$. If $\phi(x, y^*) > 0$, then $\{x' \in X_{n_2} : \phi(x', y^*) > 0\} \neq \emptyset$, by (a) and Fact $(a)'$ proceeding Theorem 2.3.2, there exists $x' \in X_{n_2}$ such that $y^* \in \mathrm{int}_{Y_{n_2}}\{y' \in Y_{n_2} : \phi(x', y') > 0\}$. Since $z_\alpha \to y^*$, there exists $\alpha_0 \in \Gamma$ such that $n(\alpha_0) \geq n_2$ and $z_{\alpha_0} \in \mathrm{int}_{Y_{n_2}}\{y' \in Y_{n_2} : \phi(x', y) > 0\}$, hence $\phi(x', z_{\alpha_0}) > 0$. But $x' \in X_{n_2} \subset X_{n(\alpha_0)}$ so that $\phi(x', z_{\alpha_0}) = \phi(x', y_{n(\alpha_0)}) \leq 0$ which is a contradiction. Therefore $\phi(x, y^*) \leq 0$ for all $x \in X$. □

By Theorem 2.3.6, we have the following.

Theorem 2.3.7. Let X be a non-empty subset of a topological space Y such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$ where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets with $X_i \subset Y_i$ and Y_i is a compact H-space for each $i = 1, 2, \cdots$. Let $\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that

(a) for each $n \in \mathbf{N}$, the mapping $x \mapsto \{y \in Y_n : \phi(x, y) \leq 0\}$ is transfer closed valued on X_n,

(b) for each $n \in \mathbf{N}$, the map $x \mapsto cl_{Y_n}\{y \in Y_n : \phi(x, y) \leq 0\}$ is HKKM on X_n,

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $\phi(x_{n_0}, y_{n_0}) > 0$.

Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Proof. Since each HKKM mapping is also generalized HKKM, all hypotheses (a), (b) and (c) in Theorem 2.3.6 are satisfied. By Theorem 2.3.6, the conclusion follows. □

Similar to Corollaries 2.3.3, 2.3.4 and 2.3.5, we have the following Corollaries 3.8, 3.9 and 3.10.

Corollary 2.3.8. Let X be a non-empty subset of a topological space Y such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$ where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets for which $X_i \subset Y_i$ and Y_i is a compact H-space for each $i = 1, 2, \cdots$. Let

$\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that

(a) for each $n \in \mathbf{N}$ and for each $x \in X_n$, $y \mapsto \phi(x, y)$ is lower semicontinuous on X_n;

(b) for each $n \in \mathbf{N}$, the map $x \mapsto cl_{Y_n}\{y \in Y_n : \phi(x, y) \leq 0\}$ is HKKM on X_n;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exists $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $\phi(x_{n_0}, y_{n_0}) > 0$. Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Corollary 2.3.9. Let E be a topological vector space. Let X be a non-empty subset of a non-empty set Y such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$ where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets for which $X_i \subset Y_i$ and Y_i is compact convex in E for each $i = 1, 2, \cdots$. Let $\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that

(a) for each $n \in \mathbf{N}$, the mapping $x \mapsto \{y \in Y_n : \phi(x, y) \leq 0\}$ is transfer closed valued;

(b) for each $n \in \mathbf{N}$, the map $x \mapsto cl_{Y_n}\{y \in Y_n : \phi(x, y) \leq 0\}$ is KKM on X_n;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $\phi(x_{n_0}, y_{n_0}) > 0$. Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Corollary 2.3.10. Let E be a topological vector space. Let X be a non-empty subset of a non-empty set Y such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$ where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequence of non-empty sets for which $X_i \subset Y_i$ and Y_i is compact convex in E for each $i = 1, 2, \cdots$. Let $\phi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ be such that

(a) for each $n \in \mathbf{N}$, the mapping $x \mapsto \{y \in Y_n : \phi(x, y) \leq 0\}$ is transfer closed valued on X_n;

(b) for each $n \in \mathbf{N}$, $A \in \mathcal{F}(X_n)$ and $y \in co(A)$, $min_{x \in A}\phi(x, y) \leq 0$;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $\phi(x_{n_0}, y_{n_0}) > 0$. Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Corollary 2.3.10 generalizes Theorem 3.1 of Tan and Yu ([288]).

Now we give equivalent formulations to our minimax inequalities. We first show that Theorem 2.3.6 implies the following:

Theorem 2.3.11. Let X be a non-empty set and Y a topological space such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$, where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets and of compact spaces respectively. Let B be a non-empty subset of $X \times Y$ such that

(a) for each $n \in \mathbf{N}$, the mapping $x \mapsto \{y \in Y_n : (x, y) \notin B\}$ is transfer closed valued on X_n;

(b) for each $n \in \mathbf{N}$, the map $x \mapsto cl_{Y_n}\{y \in Y_n : (x, y) \notin B\}$ is generalized HKKM on X_n;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $(x_{n_0}, y_{n_0}) \in B$. Then there exists $y^* \in Y$ such that $\{x \in X : (x, y^*) \in B\} = \emptyset$.

Proof. Let $\phi : X \times Y \to \mathbf{R}$ be defined by

$$\phi(x, y) = \begin{cases} 1, & \text{if } (x, y) \in B, \\ 0, & \text{if } (x, y) \notin B. \end{cases}$$

Then the hypotheses of Theorem 2.3.6 are all satisfied. Hence by Theorem 2.3.6, there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$, i.e., $(x, y^*) \notin B$ for all $x \in X$ so that $\{x \in X : (x, y^*) \in B\} = \emptyset$. □

It is clear that Theorem 2.3.11 implies the following:

Theorem 2.3.12. Let X be a non-empty set and Y be a topological space such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$, where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets and of compact spaces respectively. Let C be a non-empty subset of $X \times Y$ such that

(a) for each $n \in \mathbf{N}$, the mapping $x \mapsto \{y \in Y_n : (x, y) \in C\}$ is transfer closed valued on X_n;

(b) for each $n \in \mathbf{N}$, the map $x \mapsto cl_{Y_n}\{y \in Y_n : (x, y) \in C\}$ is generalized HKKM on X_n;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $(x_{n_0}, y_{n_0}) \notin C$. Then there exists $y^* \in Y$ such that $X \times \{y^*\} \subset C$.

Definition. Let X and Y be two topological spaces and a mapping $F : X \to 2^Y \cup \{\emptyset\}$.

(i): F is transfer open inverse valued on X if for each $y \in Y$ and $x \in X$ with $x \in F^{-1}(y) = \{x \in X : y \in F(x)\}$, there exist some $y' \in Y$ and a non-empty open neighborhood $N(x)$ of x in X such that $N(x) \subset F^{-1}(y')$. It is clear that $F : X \to 2^Y \cup \{\emptyset\}$ is transfer open inverse valued on X if and only if the mapping $G : Y \to 2^X \cup \{\emptyset\}$ defined by $G(y) = X \setminus F^{-1}(y)$ for each $y \in Y$ is transfer closed valued.

(ii) a point $x \in X$ is said to be a maximal element of the mapping F provided $F(x) = \emptyset$.

The example after Theorem 2.3.18 below shows that a transfer open inverse valued mapping may be not open inverse valued.

Now we shall show that Theorem 2.3.12 implies the following maximal element theorem:

Theorem 2.3.13. Let X be a non-empty set and Y be a topological space such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$, where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets and of compact spaces respectively, Suppose the map $F : Y \to 2^X \cup \{\emptyset\}$ is such that

(a) for each $n \in \mathbf{N}$, the mapping $F : Y_n \to 2^{X_n} \cup \{\emptyset\}$ is transfer open inverse valued on X_n;

(b) for each $n \in \mathbf{N}$, the map $x \mapsto cl_{Y_n}\{y \in Y_n : x \notin F(y)\}$ is generalized HKKM on X_n;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in N$ and $x_{n_0} \in X_{n_0}$ such that $x_{n_0} \in F(y_{n_0})$. Then there exists $y^* \in Y$ such that $F(y^*) = \emptyset$.

Proof. Let $C = \{(x, y) \in X \times Y : x \notin F(y)\}$, then all the conditions of Theorem

2.3.12 are satisfied. Hence by Theorem 2.3.12, there exists $y^* \in Y$ such that $X \times \{y^*\} \subset C$, i.e., $x \notin F(y^*)$ for all $x \in X$ so that $F(y^*) = \emptyset$. □

We shall now prove that Theorem 2.3.13 implies Theorem 2.3 6 so that Theorems 2.3.6, 2.3.11, 2.3.12 and Theorem 2.3.13 are all equivalent.

The proof of "Theorem 2.3.13 ⇒ Theorem 2.3.6": Define $F : Y \to 2^X \cup \{\emptyset\}$ by $F(y) = \{x \in X : \phi(x, y) > 0\}$ for each $y \in Y$. Then the conditions of Theorem 2.3.13 are satisfied. Hence by Theorem 2.3.13, there exists $y^* \in Y$ such that $F(y^*) = \emptyset$, i.e., $\phi(x, y^*) \le 0$ for all $x \in X$. □

As an immediate consequence of Theorem 2.3.13, we have:

Corollary 2.3.14. Let X be a non-empty subset of a convex subset Y of a topological vector space E such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$, where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets for which $X_i \subset Y_i$ and Y_i is compact convex for each $i = 1, 2, \cdots$. Suppose the map $F : Y \to 2^X \cup \{\emptyset\}$ is such that

(a) for each $n \in \mathbf{N}$, $F : Y_n \to 2^{X_n} \cup \{\emptyset\}$ is transfer open inverse valued,

(b) for each $y \in Y$, $y \notin coF(y)$;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $x_{n_0} \in F(y_{n_0})$.

Then there exists $y^* \in Y$ such that $F(y^*) = \emptyset$.

Proof. Suppose that there exist $n \in \mathbf{N}$, $A \in \mathcal{F}(X_n)$ and $y \in co(A)$ such that $y \notin \cup_{x \in A} cl_{Y_n}\{y' \in Y_n : x \notin F(y')\}$. Then $x \in F(y)$ for all $x \in A$ so that $y \in co(A) \subset coF(y)$ which contradicts (b). Therefore all conditions of Theorem 2.3 13 are satisfied and hence there exists $y^* \in Y$ such that $F(y^*) = \emptyset$. □

Corollary 2.3.14 generalizes Theorem 3.2 of Tan and Yu [288] and Theorem 7.10 of Border [34, p.35].

In what follows, we shall give two fixed point theorems and an existence theorem for maximal elements.

Theorem 2.3.15. Let X be a non-empty subset of an H-space $(Y, \{\Gamma_A\})$ such that

$X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$, where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets for which $X_i \subset Y_i$ and Y_i is a non-empty compact and weakly H-convex subset of Y for each $i = 1, 2, \cdots$. Suppose the map $F : Y \to 2^X$ is such that

(a) for each $n \in \mathbf{N}$, the mapping $F : Y_n \to 2^{X_n} \cup \{\emptyset\}$ is transfer open inverse valued;

(b) for each $y \in Y$, $F(y)$ is H-convex;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $x_{n_0} \in F(y_{n_0})$. Then there exists $y^* \in Y$ such that $y^* \in F(y^*)$.

Proof. For each $n \in \mathbf{N}$, since $(Y, \{\Gamma_A\})$ is an H-space and Y_n is weakly H-convex, so that $(Y_n, \{\Gamma_A \cap Y_n\})$ is an H-space. If the condition (b) of Theorem 2.3.13 holds, then there is $\hat{y} \in Y$ such that $F(\hat{y}) = \emptyset$ which is a contradiction. Therefore the condition (b) of Theorem 2.3.13 does not hold, i.e., there exist $n \in \mathbf{N}$, $A \in \mathcal{F}(Y_n)$ and $y^* \in \Gamma_A$ such that $y^* \notin cl_{Y_n}\{y \in Y_n : x \notin F(y)\}$ for all $x \in A$, hence $x \in F(y^*)$ for all $x \in A$. By (b), $F(y^*)$ is H-convex so that $y^* \in \Gamma_A \subset F(y^*)$. □

Theorem 2.3.16. Let X be a non-empty subset of an H-space $(Y, \{\Gamma_A\})$ such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$, where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets for which $X_i \subset Y_i$ and Y_i is a non-empty compact and weakly H-convex subset of Y for each $i = 1, 2, \cdots$. Suppose the map $F : Y \to 2^X$ is such that

(a) for each $n \in \mathbf{N}$, the mapping $F : Y_n \to 2^{X_n} \cup \{\emptyset\}$ is transfer open inverse valued;

(b) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in N$ and $x_{n_0} \in X_{n_0}$ such that $x_{n_0} \in F(y_{n_0})$. Then there exists $y^* \in Y$ such that $y^* \in \mathrm{Hco}F(y^*)$.

Proof. As in the preceding proof, there exist $n \in \mathbf{N}$, $A \in \mathcal{F}(X_n)$ and $y^* \in \Gamma_A$ such that $y^* \notin cl_{Y_n}\{y \in Y_n : x \notin F(y)\}$ for all $x \in A$, hence $x \in F(y^*)$ for all $x \in A$ and $y^* \in \Gamma_A \subset \mathrm{Hco}F(y^*)$. □

Theorem 2.3.17. Let X be a non-empty subset of an H-space space $(Y, \{\Gamma_A\})$ such that $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$, where $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are increasing sequences of non-empty sets for which $X_i \subset Y_i$ and Y_i is a non-empty compact and weakly H-convex subset of Y for each $i = 1, 2, \cdots$. Suppose the map $F : Y \to 2^X$ is such that

(a) for each $n \in \mathbf{N}$, the mapping $F : Y_n \to 2^{X_n} \cup \{\emptyset\}$ is transfer open inverse valued;

(b) for each $y \in Y$, $F(y)$ is H-convex and $y \notin F(y)$;

(c) for each sequence $(y_n)_{n=1}^{\infty}$ in Y with $y_n \in Y_n$ for each $n \in \mathbf{N}$ which is escaping from Y relative to $\{Y_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in X_{n_0}$ such that $x_{n_0} \in F(y_{n_0})$.

Then there exists $y^* \in Y$ such that $F(y^*) = \emptyset$.

Proof. Suppose $F(y) \neq \emptyset$ for all $y \in Y$, by Theorem 2.3.15, there exists $\hat{y} \in Y$ such that $\hat{y} \in F(\hat{y})$ which is a contradiction of condition (b). □

Finally, as an immediate consequence of Theorem 2.3.15, we have the following generalization of the Fan-Browder fixed point theorem (e.g, see Fan [98] or Browder [42]):

Theorem 2.3.18. Let X be a non-empty compact convex subset of a topological vector space E and $F : X \to 2^X$ is such that:

(a) for each $x \in X$, $F(x)$ is convex; and

(b) F is transfer open inverse valued.

Then F has a fixed point.

The following example shows that Theorem 2.3.18 is really a generalization of the Fan-Browder fixed point theorem.

Example. Let $X = [0, 1]$ and define a mapping $F : X \to 2^{[0,1]}$ by

$$F(x) = \begin{cases} [x, 1], & \text{if } x \text{ is rational} \\ [0, 1], & \text{if } x \text{ is irrational} \end{cases}$$

Then it is clear that F is not open inverse valued but F is transfer open inverse valued.

2.4 Stability of Ky Fan Points

Let (X, d) be a compact metric space with the fixed point property for continuous mappings. In [115], Fort introduced the concept of essential fixed points of a continuous mapping f on X. He proved that (1) every continuous mapping on X can be arbitrarily approximated by a continuous mapping on X whose fixed points are all essential; and (2) if each fixed point of a continuous mapping f on X is essential, then the fixed point set $S(f) = \{x \in X : f(x) = x\}$ of f is stable: for each $\epsilon > 0$, there is $\delta > 0$ such that for each continuous mapping g on X, if $\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\} < \delta$, then $h(S(f), S(g)) < \epsilon$ where h is the Hausdorff metric defined on all non-empty bounded closed subsets of X induced by the metric d; i.e., the fixed point set $S(g)$ of g is "close" to the fixed point set $S(f)$ of f whenever g is "close" to f.

In this section, the concepts of the *KF point* and *essential KF point* are first introduced. We then study the stability of KF points (which are the solutions of Ky Fan minimax inequalities) in both compact and non-compact settings.

We shall recall some definitions. If X is a topological space, we shall denote by $K(X)$ and $\mathcal{P}_0(X)(= 2^X)$ the space of all non-empty compact subsets of X and the space of all non-empty subsets of X respectively, both endowed with the Vietoris topology (see, Klein and Thompson [189]). If Z is another topological space, then a mapping $T : X \to 2^X$ is said to be (i) upper (respectively, lower) semicontinuous at $x \in X$, if for each open set G in Z with $G \supset T(x)$ (respectively, $G \cap T(x) \neq \emptyset$), there exists an open neighborhood $O(x)$ of x in X such that $G \supset T(x')$ (respectively, $G \cap T(x') \neq \emptyset$) for each $x' \in O(x)$; (ii) T is said to be almost lower semicontinuous at $x \in X$, if there exists $z \in T(x)$ such that for each open neighborhood $N(z)$ of z in Z, there exists an open neighborhood $O(x)$ of x in X with the property that $N(z) \cap T(x') \neq \emptyset$ for each $x' \in O(x)$ and (iii) T is an usco if T is upper semicontinuos with non-empty compact values.

A space X is said to be Cech-complete if it can be embedded as a G_δ subset of some compact Hausdorff space (e.g., see Engelking [95]). It is known that (i): A Cech-complete space is a Baire space; (ii): Locally compact spaces are also Cech-complete, because a non-compact locally compact space has a compactification with one-point remainder; (iii):

The space of all irrational numbers with the topology of a subspace of the real line is an example of a Cech-complete space that is not locally compact and moreover (iv): Each completely metrizable space is Cech-complete. Thus, the Cech-complete spaces contain within them the two important types of Baire spaces.

A space X is said to belong to the class $\mathcal{L}$ (see Kenderov [174]) if for each Cech-complete space Z, every usco mapping $S : Z \to K(X)$ is almost lower semicontinuous on some dense G_δ subset of Z.

Note that there are a number of spaces under which each usco multifunction from X to Z is almost lower semicontinuous at the points of some dense G_δ subset of X. For example,

(a) X is a Baire space and Z is metrizable (e.g., see Fort [115, Theorem 2]);

(b) X is Cech-complete and Z is a Banach space with weak topology (which is non-metrizable if it is infinite dimensional) by Theorem 2 of Christensen [60];

(c) X is a Baire space and Z is the dual space of a Banach space with the weak*-topology provided Z has the randon - Nikodym property (e.g., see Christensen and Kenderov [61]).

It follows from (a) and (b) above that the class $\mathcal{L}$ contains all metrizable spaces and all Banach spaces equipped with the weak topology. The class $\mathcal{L}$ has nice stability properties: it is closed under taking subspaces, countable products, countable sums of closed sets, and perfect images.

As a special case of Corollary 2.3.5, we have the following:

Theorem 2.4.A. Let X be a non-empty compact convex subset of a topological vector space and $f : X \times X \to \mathbf{R}$ be such that

(i) $f(x,x) \leq 0$ for all $x \in X$;

(ii) for each fixed $x \in X$, $y \mapsto f(x,y)$ is lower semicontinuous;

(iii) for each fixed $y \in X$, $x \mapsto f(x,y)$ is quasi-concave (i.e., for each $\lambda \in R$, the set $\{x \in X : f(x,y) > \lambda\}$ is convex).

Then there exists $\hat{y} \in X$ such that $f(x,\hat{y}) \leq 0$ for all $x \in X$.

It is clear that Theorem 2.4.A above is equivalent to the celebrated Ky Fan minimax inequality [105] stated as follows:

Theorem 2.4.B. Let X be a non-empty compact convex subset of a topological vector space and $f : X \times X \to \mathbf{R}$ be such that

(a) for each fixed $x \in X$, $y \mapsto f(x,y)$ is lower semicontinuous;

(b) for each fixed $y \in X$, $x \mapsto f(x,y)$ is quasi-concave.

Then $\inf_{y\in X}\sup_{x\in X} f(x,y) \leq \sup_{x\in X} f(x,x)$.

We shall call such a point $\hat{y}$ in Theorem 2.4.A as Ky Fan point (in short, KF point) of f in X and denote by $S(f)$ the set of all KF points of f in X. Thus $S(f)$ is non-empty by Theorem 2.4.A. Also, $S(f) = \cap_{x\in X}\{y \in X : f(x,y) \leq 0\}$ by the condition (ii) of Theorem 2.4.A is closed in X and is thus also compact. Therefore, for each function $f : X \times X \to \mathbf{R}$ satisfying the conditions (i), (ii) and (iii) of Theorem 2.4.A, one can associate a non-empty compact subset $S(f)$, the set of all solutions $y \in X$ of the inequality $\sup_{x\in X} f(x,y) \leq 0$.

In this section, we shall first discuss the stability of $S(f)$ with f varying where f is a bounded real-valued function on $X \times X$ satisfying the conditions (i), (ii) and (iii) in Theorem 2.4.A and X is a non-empty compact convex subset of a topological space. Next, if X is a Cech-complete space which belongs to the class $\mathcal{L}$ (see the definition below), we shall study the stability of the set $S(A,f) = \{y \in A : \sup_{x\in A} f(x,y) \leq 0\}$ with both f and A varying, where $f : X \times X \to \mathbf{R}$ is bounded and lower semicontinuous and A is a non-empty compact subset of X. When X is a closed convex subset of a Frechet space, as an application, the stability of the set $S(A,f)$ is investigated, where $f : X \times X \to \mathbf{R}$ satisfies the conditions (i) and (iii) of Theorem 2.4.A and the subset A is, in addition, convex.

2.4.1 *Stability in Compact Setting*

Throughout this section, X denotes a non-empty compact convex subsets of a topological vector space. Let $L(X)$ be the family of all bounded real-valued functions on $X \times X$.

For $f, g \in L(X)$, define

$$\rho(f,g) = \sup_{x,y \in X} |f(x,y) - g(x,y)|.$$

Clearly, $(L(X), \rho)$ is a complete metric space. Let

$$M = \{f \in L(X) : f \text{ satisfies the conditions (i), (ii) and (iii) of Theorem 2.4.A } \}.$$

It is easy to show that M is closed in $L(X)$. Thus we have:

Lemma 2.4.1. (M, ρ) is a complete metric space.

Now for each $f \in M$, the set $S(f) = \{y \in X\colon \sup_{x \in X} f(x,y) \leq 0\}$ is non-empty and compact by Theorem 2.4.A. Furthermore, we have:

Lemma 2.4.2. $F : M \to K(X)$ is upper semicontinuous.

Proof. Let $\{(f_\alpha, y_\alpha)\}_{\alpha \in \Gamma}$ be a net in Graph$S$ with $(f_\alpha, y_\alpha) \to (f_0, y_0) \in M \times X$, then $f_\alpha \to f_0$, $y_\alpha \to y_0$ and $f_\alpha(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and for all $x \in X$. Fix $x \in X$. Since $y \mapsto f_0(x,y)$ is lower semicontinuous at y_0, for any $\epsilon > 0$, there exists an open neighborhood $O(y_0)$ of y_0 in X such that for each $y' \in O(y_0)$, $f_0(x,y_0) < f_0(x,y') + \epsilon/4$. As $f_\alpha \to f_0$, there exists $\alpha_0 \in \Gamma$ such that for any $\alpha \geq \alpha_0$, $\rho(f_0, f_\alpha) < \epsilon/4$ so that for each $y' \in O(y_0)$, $f_\alpha(x,y_0) < f_0(x,y_0) + \epsilon/4 < f_0(x,y') + \epsilon/2 < f_\alpha(x,y') + 3\epsilon/4$. Since $y_\alpha \to y_0$, there exists $\alpha_1 \geq \alpha_0$ such that $y_{\alpha_1} \in O(y_0)$; it follows that $f_0(x,y_0) = f_0(x,y_0) - f_{\alpha_1}(x,y_0) + f_{\alpha_1}(x,y_0) - f_{\alpha_1}(x,y_{\alpha_1}) + f_{\alpha_1}(x,y_{\alpha_1}) < \rho(f_0, f_{\alpha_1}) + 3\epsilon/4 < \epsilon$. Since $\epsilon > 0$ is arbitrary, $f_0(x,y_0) \leq 0$ for all $x \in X$. This implies that $(f_0, y_0) \in$ Graph$S$ and hence Graph$S$ is closed in $M \times X$. Therefore S is upper semicontinuous since X is compact. □

Definition. For each $f \in M$, (i) a point $y \in S(f)$ is KF-essential relative to M if for each open neighborhood $N(y)$ of y in X, there exists an open neighborhood $O(f)$ of f in M such that $S(f') \cap N(y) \neq \emptyset$ for each $f' \in O(f)$; (ii) f is weakly essential relative to M if there exists $y \in S(f)$ which is KF-essential relative to M and (iii) f is essential relative to M if every $y \in S(f)$ is KF-essential relative to M.

The following result is due to Fort [115]:

Lemma 2.4.3. If X is metrizable, Z is a Baire space and $S : Z \to K(X)$ is an usco mapping, then the set of points where S is lower semicontinuous is a dense G_δ set in Z.

Theorem 2.4.4. (i) S is almost lower semicontinuous at $f \in M$ if and only if f is weakly essential relative to M.

(ii) S is lower semicontinuous at $f \in M$ if and only if f is essential relative to M.

(iii) S is continuous at $f \in M$ if and only if f is essential relative to M.

Proof. (i) S is almost lower semicontinuous at $f \in M$ if and only if there exists $y \in S(f)$ such that y is KF-essential relative to M if and only if f is weakly essential relative to M.

(ii) S is lower semicontinuous at $f \in M$ if and only if each $y \in S(f)$ is KF-essential relative to M if and only if f is essential relative to M.

(iii) This follows from (ii) and Lemma 2.4.2. □

If X is metrizable by a metric d, then the Vietoris topology on $K(X)$ coincides with the topology generated by the Hausdorff metric h induced by d (e.g., see Corollary 4.2.3 of [189]). Then S is continuous at $f \in M$ if and only if for each $\epsilon > 0$, there is $\delta > 0$ such that for each $g \in M$, $h(S(f), S(g)) < \epsilon$ whenever $\rho(f, g) < \delta$; i.e., $S(f)$ is stable: $S(g)$ is "close" to $S(f)$ whenever g is "close" to f. Theorem 2.4.4 (iii) shows that $S(f)$ is stable if and only if f is essential relative to M.

We shall give a sufficient condition that $f \in M$ is essential relative to M:

Theorem 2.4.5. If $f \in M$ is such that $S(f)$ is a singleton set, then f is essential relative to M.

Proof. Suppose $S(f) = \{x\}$. Let G be any open set in X such that $S(f) \cap G \neq \emptyset$, then $x \in G$ so that $S(f) \subset G$. Since S is upper semicontinuous at f by Lemma 2.4.2, there is an open neighborhood $O(f)$ of f in M such that $S(f') \subset G$ for each $f' \in O(f)$; in particular, $G \cap S(f') \neq \emptyset$ for each $f' \in O(f)$. Thus S is lower semicontinuous at f. By Theorem 2.4.4 (ii), f is essential relative to M. □

Theorem 2.4.6. (i) Suppose that X belongs to class $\mathcal{L}$. Then there exists a dense G_δ subset Q of M such that f is weakly essential relative to M for each $f \in Q$.

(ii) Suppose that X is metrizable. Then there exists a dense G_δ subset Q of M such that f is essential relative to M for each $f \in Q$.

Proof. (i) Since M is a complete metric space, M is Cech-complete. By Lemma 2.4.2, the mapping $S : M \to K(X)$ is upper semicontinuous. Since X belongs to class $\mathcal{L}$, S is almost lower semicontinuous on some dense G_δ subset Q of M. By Theorem 2.4.4 (i), f is weakly essential relative to M for each $f \in Q$.

(ii) By Lemma 2.4.2 and Lemma 2.4.3, S is lower semicontinuous on some dense G_δ subset Q of M. By Theorem 2.4.4 (ii), f is essential relative to M for each $f \in Q$. □

We remark that if we define $M = \{f \in L(X) : f$ satisfies conditions (a) and (b) of Theorem 2.4.B $\}$ and $S(f) = \{y \in X : \sup_{x\in X} f(x,y) \leq \sup_{x\in X} f(x,x)\}$ for each $f \in M$, then all the results in this section remain valid.

2.4.2 *Stability in Non-Compact Setting*

In section 2.4.1, we have studied the stability of the solution set $S(f)$ of f in X with f varying but X fixed. In this section we shall study the stability of the solution set $S(f) \cap A$ of f in A with both f and A varying.

Throughout this section, X denotes a topological space and $L(X)$ denotes the space of all bounded real-valued lower semicontinuous functions on $X \times X$. For each $f_1, f_2 \in L(X)$, let $\rho(f_1, f_2) = \sup_{(x,y)\in X\times X} |f_1(x,y) - f_2(x,y)|$, then clearly ρ is a metric on $L(X)$.

Let $Y = K(X) \times L(X)$. Now for each $u = (A, f) \in Y$.

Definition. A point y in A is called a Ky Fan's point (in short, KF point) of f in A if $\sup_{x\in A} f(x,y) \leq 0$.

A point y in A satisfying $(*)$ is called a Ky Fan's point (in short, KF point) of f in A.

Before we study the stability of the set $S(u)$ of KF points of f in A for $u = (A, f) \in Y$, we shall give several lemmas which will be used to prove our main results later. The proof of the following result is routine and is hence omitted.

Lemma 2.4.7. $(L(X), \rho)$ is a complete metric space.

Lemma 2.4.8. Suppose X is a non-empty subset of a topological vector space. If $\{A_\alpha\}_{\alpha\in\Gamma}$ is a net of compact and convex sets in $K(X)$ which converges to $A \in K(X)$ in the Vietoris topology, then A is also convex.

Proof. Suppose that A were not convex. Then there exist $x_1, x_2 \in A$ and $\lambda_1 \in (0,1)$ such that $\lambda_1 x_1 + (1-\lambda_1)x_2 \notin A$. Since A is compact, there exist an open set G in X containing A and an open neighborhood $O(\lambda_1 x_1 + (1-\lambda_1)x_2)$ of $\lambda_1 x_1 + (1-\lambda_1)x_2$ in X such that $O(\lambda_1 x_1 + (1-\lambda_1)x_2) \cap G = \emptyset$. Note that there exist an open neighborhood $O(x_1)$ of x_1 in X and an open neighborhood $O(x_2)$ of x_2 in X such that $\lambda_1 O(x_1) + (1-\lambda_1)O(x_2) \subset O(\lambda_1 x_1 + (1-\lambda_1)x_2)$. Since x_1, $x_2 \in A$ and $A_\alpha \to A$, there exists $\alpha_0 \in \Gamma$ such that for each $\alpha \geq \alpha_0$, $O(x_1) \cap A_\alpha \neq \emptyset$ and $O(x_2) \cap A_\alpha \neq \emptyset$. Since $G \supset A$, there exists $\alpha_1 \in \Gamma$ such that for each $\alpha \geq \alpha_1$, $G \supset A_\alpha$. Now let $\alpha_2 \in \Gamma$ be such that $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. Then for each $\alpha \geq \alpha_2$, $O(x_1) \cap A_\alpha \neq \emptyset$, $O(x_2) \cap A_\alpha \neq \emptyset$ and $A_\alpha \subset G$. Choose any $z_1 \in O(x_1) \cap A_{\alpha_2}$ and $z_2 \in O(x_2) \cap A_{\alpha_2}$. Since A_{α_2} is convex, $\lambda z_1 + (1-\lambda)z_2 \in A_{\alpha_2} \subset G$. But $\lambda z_1 + (1-\lambda)z_2 \in \lambda O(x_1) + (1-\lambda)O(x_2) \subset O(\lambda x_1 + (1-\lambda)x_2)$ which contradicts $O(\lambda_1 x_1 + (1-\lambda)x_2) \cap G = \emptyset$. Hence A must be convex. □

The following result is Lemma 3.3 of Beer [17]; as it was stated without a proof, we shall include its simple proof for completeness:

Lemma 2.4.9. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a net in $K(X)$ which converges to $A \in K(X)$ in the Vietoris topology. Then every net $\{x_\alpha\}_{\alpha\in\Gamma}$ with $x_\alpha \in A_\alpha$ for each $\alpha \in \Gamma$ has a cluster point in A.

Proof. Suppose that the net $\{x_\alpha\}_{\alpha\in\Gamma}$ has no cluster point in A. Then for each $x \in A$, there exist an open neighborhood $O(x)$ of x in X and an $\alpha(x) \in \Gamma$ such that $x_\alpha \notin O(x)$ for all $\alpha \geq \alpha(x)$. Since $A \subset \cup_{x\in A} O(x)$ and A is compact, there exist $x_1, x_2, \cdots, x_n \in A$ such that $A \subset \cup_{i=1}^n O(x_i)$. Now let α' be such that $\alpha' \geq \alpha(x_i)$ for $i = 1, 2, \cdots, n$. Then for each $\alpha \geq \alpha'$, $x_\alpha \notin O(x_i)$ for $i = 1, 2, \cdots, n$. Since $\cup_{i=1}^n O(x_i)$ is an open set which contains A and $A_\alpha \to A$ in the Vietoris topology, there exists $\alpha'' \in \Gamma$ such that for any $\alpha \geq \alpha''$, $x_\alpha \in A_\alpha \subset \cup_{i=1}^n O(x_i)$. Now let $\alpha''' \in \Gamma$ be such that $\alpha''' \geq \alpha'$ and $\alpha''' \geq \alpha''$; then $x_{\alpha'''} \notin O(x_i)$ for $i = 1, 2, \cdots, n$ which contradicts the fact that

$x_{\alpha'''} \in A_\alpha \subset \cup_{i=1}^n O(x_i)$. Hence $\{x_\alpha\}_{\alpha\in\Gamma}$ has a cluster point in A. □

Now define the subspace M of Y by $M = \{(A,f) \in Y$: there exists $y \in A$ such that $\sup_{x\in A} f(x,y) \leq 0\}$. Then we define a mapping $S : M \to \mathcal{P}_0(X)$ by $S(u) = \{y \in A : \sup_{x\in A} f(x,y) \leq 0\}$ for each $u = (A,f) \in M$.

Lemma 2.4.10. M is closed in Y.

Proof. Suppose that $\{(A_\alpha, f_\alpha)\}_{\alpha\in\Gamma}$ is a net in M such that $(A_\alpha, f_\alpha) \to (A,f) \in Y$. For each $\alpha \in \Gamma$, let $y_\alpha \in A_\alpha$ be such that $\sup_{x\in A_\alpha} f_\alpha(x,y_\alpha) \leq 0$. Since $A_\alpha \to A$ in the Vietoris topology, the net $\{y_\alpha\}_{\alpha\in\Gamma}$ has a cluster point $y_0 \in A$ by Lemma 2.4.9. Now we shall show that $\sup_{x\in A} f(x,y_0) \leq 0$. Suppose that this were not true, then there exist $\epsilon_0 > 0$ and $x_0 \in A$ such that $f(x_0,y_0) > \varepsilon_0$. Since f is lower semicontinuous at (x_0,y_0), there exist an open neighborhood $O(x_0)$ of x_0 in X and an open neighborhood $O(y_0)$ of point y_0 in X such that $f(x,y) > \epsilon_0$ for any $(x,y) \in O(x_0) \times O(y_0)$. Since $f_\alpha \to f$, there exists $\alpha_0 \in \Gamma$ such that for any $\alpha \geq \alpha_0$, $|f_\alpha(x,y) - f(x,y)| < \epsilon_0/2$ for all $(x,y) \in X \times X$, so that $f_\alpha(x,y) > f(x,y) - \epsilon_0/2$ for all $(x,y) \in X \times X$. Therefore $f_\alpha(x,y) > f(x,y) - \epsilon_0/2 > \epsilon_0 - \epsilon_0/2 = \epsilon_0/2$ for each $(x,y) \in O(x_0) \times O(y_0)$. As $A_\alpha \to A$, there exists $\alpha_1 \geq \alpha_0$ such that $O(x_0) \cap A_\alpha \neq \emptyset$ for all $\alpha \geq \alpha_1$. Note that $y_0 \in A$ is a cluster point of $\{y_\alpha\}_{\alpha\in\Gamma}$, there exists $\alpha_2 \geq \alpha_1$ such that $y_{\alpha_2} \in O(y_0)$. Choose any $x_{\alpha_2} \in O(x_0) \cap A_{\alpha_2}$, we have $f_{\alpha_2}(x_{\alpha_2}, y_{\alpha_2}) > \varepsilon_0/2$ which contradicts the choice of $y_{\alpha_2} \in A_{\alpha_2}$ that $\sup_{x\in A_{\alpha_2}} f_\alpha(x,y_\alpha) \leq 0$. Therefore we must have that $\sup_{x\in A} f(x,y_0) \leq 0$. Hence $(A,f) \in M$ and M is closed in Y. □

Lemma 2.4.11. If X is Cech-complete, then M is Cech-complete.

Proof. The space $L(X)$ is Cech-complete since $L(X)$ is a complete metric space by Lemma 2.4.7. Since X is Cech-complete, $K(X)$ is also Cech-complete by Lemma 2.2 of Beer [17]. Therefore the product space $K(X) \times L(X)$ is Cech-complete by Theorem 3.9.8 of Engelking [95]. By Lemma 2.4.10 and Theorem 3.9.6 of Engelking [95], M is also Cech-complete. □

Lemma 2.4.12. $S(u) \in K(X)$ for each $u \in M$.

Proof. For each $u = (A,f) \in M$, since $S(u) \subset A$, it is sufficient to prove that $S(u)$

is closed in A. Let $(y_\alpha)_{\alpha\in\Gamma}$ be a net in $S(u)$ which converges to a point $y_0 \in A$. By the definition of S, we have $\sup_{x\in A} f(x,y_\alpha) \leq 0$ for each $\alpha \in \Gamma$. By the lower semicontinuity of $y \mapsto \sup_{x\in A} f(x,y)$, we have $\sup_{x\in A} f(x,y_0) \leq 0$. Hence $y_0 \in S(u)$ so that $S(u)$ is a closed subset of A. □.

Lemma 2.4.13. The correspondence $S : M \to K(X)$ is upper semicontinuous.

Proof. Suppose that S were not upper semicontinuous at some point $u = (A, f) \in M$, then there exist an open subset G of X with $G \supset S(u)$ and a net $\{u_\alpha\}_{\alpha\in\Gamma}$ in M with $u_\alpha \to u \in M$ such that for each $\alpha \in \Gamma$, there exists $y_\alpha \in S(u_\alpha)$ with $y_\alpha \notin G$. Denote $u_\alpha = (A_\alpha, f_\alpha)$ and $u = (A, f)$, then $f_\alpha \to f$ and $A_\alpha \to A$. Since $y_\alpha \in A_\alpha$ for each $\alpha \in \Gamma$, by Lemma 2.4.9, the net $\{y_\alpha\}_{\alpha\in\Gamma}$ has a cluster point $y_0 \in A$. Since $y_\alpha \notin G$ for each $\alpha \in \Gamma$, we have $y_0 \notin G$. Therefore $\sup_{x\in A} f(x, y_0) > 0$, so that there exist $\epsilon_0 > 0$ and $x_0 \in A$ such that $f(x_0, y_0) > \epsilon_0$. Since $(x,y) \mapsto f(x,y)$ is lower semicontinuous at (x_0, y_0), there exist an open neighborhood $N(x_0)$ of x_0 in X and an open neighborhood $N(y_0)$ of y_0 in X such that for each $(x,y) \in N(x_0) \times N(y_0)$, $f(x,y) > \epsilon_0$. Since $f_\alpha \to f$, there exists $\alpha_1 \in \Gamma$ such that for each $\alpha \geq \alpha_1$, $|f_\alpha(x,y) - f(x,y)| < \epsilon_0/2$ for all $(x,y) \in X \times X$. Therefore $f_\alpha(x,y) > f(x,y) - \epsilon_0/2$ for all $(x,y) \in X \times X$. Since $N(x_0) \cap A \neq \emptyset$ and $A_\alpha \to A$, there exists $\alpha_2 \geq \alpha_1$ such that for each $\alpha \geq \alpha_2$, $N(x_0) \cap A_\alpha \neq \emptyset$. Note that because y_0 is a cluster point of the net $\{y_\alpha\}_{\alpha\in\Gamma}$ there exists $\alpha_3 \geq \alpha_2$ with $y_{\alpha_3} \in N(y_0)$. Now choose any $x_{\alpha_3} \in N(x_0) \cap A_{\alpha_3}$, we have $f_{\alpha_3}(x_{\alpha_3}, y_{\alpha_3}) > f(x_{\alpha_3}, y_{\alpha_3}) - \epsilon_0/2$. Therefore $f_{\alpha_3}(x_{\alpha_3}, y_{\alpha_3}) > f(x_{\alpha_3}, y_{\alpha_3}) - \epsilon_0/2 > \epsilon_0 - \epsilon_0/2 = \epsilon_0/2 > 0$ which contradicts the fact that $y_{\alpha_3} \in S(u_{\alpha_3})$. Therefore s must be upper semicontinuous. □

Now let M_1 be a non-empty closed subset of M.

Definition. For each $u \in M_1$, (i) a point $y \in S(u)$ is KF-essential relative to M_1 if for each open neighborhood $N(y)$ of y in X, there exists an open neighborhood $O(u)$ of u in M_1 such that $S(u') \cap N(y) \neq \emptyset$ for each $u' \in O(u)$; (ii) u is weakly essential relative to M_1 if there exists $y \in S(u)$ which is KF-essential relative to M_1 and (iii) u is essential relative to M_1 if every $y \in S(u)$ is KF-essential relative to M_1.

Theorem 2.4.14. (i) S is almost lower semicontinuous at $u \in M_1$ if and only if u is weakly essential relative to M_1.

(ii) S is lower semicontinuous at $u \in M_1$ if and only if u is essential relative to M_1.

(iii) S is continuous at $u \in M_1$ if and only if u is essential relative to M_1.

Proof. (i) S is almost lower semicontinuous at $u \in M_1$ if and only if there exists $y \in S(u)$ such that y is KF-essential relative to M_1 if and only if u is weakly essential relative to M_1.

(ii) S is lower semicontinuous at $u \in M_1$ if and only if each $y \in S(u)$ is KF-essential relative to M_1 if and only if u is essential relative to M_1.

(iii) This follows from (ii) and Lemma 2.4.13. □

A proof analogous to that of Theorem 2.4.5 and therefore omitted gives us the following result:

Theorem 2.4.15. If $u \in M_1$ is such that $S(u)$ is a singleton set, then u is essential relative to M_1.

Theorem 2.4.16. (i) Let X be Cech-complete and belong to the class $\mathcal{L}$. Then there exists a dense G_δ subset Q of M_1 such that u is weakly essential relative to M_1 for each $u \in Q$.

(ii) Let X be completely metrizable. Then there exists a dense G_δ subset Q of M_1 such that u is essential relative to M_1 for each $u \in Q$.

Proof. Note that S is an usco by Lemma 2.4.12 and Lemma 2.4.13.

(i) Since X is Cech-complete, Lemma 2.4.11 implies that M is also also Cech-complete. Since M_1 is closed in M, M_1 is also Cech-complete by Theorem 3.9.6 of Engelking [95]. Since X is of class $\mathcal{L}$, there is a dense G_δ subset Q of M_1 such that S is almost lower semicontinuous at each $u \in Q$. By Theorem 2.4.14 (i), u is weakly essential relative to M_1 for each $u \in Q$.

(ii) By Lemma 2.4.3, there exists a dense G_δ subset Q of M_1 such that S is lower semicontinuous at each $u \in Q$. By Theorem 2.4.14 (ii), u is essential relative to M_1 for each $u \in Q$. □

If X is a complete metric space with metric d, then $K(X)$ is a complete metric space when equipped with the Hausdorff metric h induced by d. By Corollary 4.2.3 in [189, p.41], the Vietoris topology on $K(X)$ coincides with the topology induced by the Hausdorff metric h. By Lemma 2.4.7, it follows that $Y = K(X) \times L(X)$ and hence M and M_1 are also complete metric spaces when equipped with the metric D defined by

$$D(u, u') = \rho(f, f') + h(A, A')$$

for $u = (A, f)$ and $u' = (A', f')$. We note then, the mapping $S : M_1 \to K(X)$ is continuous at $u = (A, f) \in M_1$ if and only if for each $\epsilon > 0$, there is $\delta > 0$ such that $h(S(u), S(u')) < \epsilon$ whenever $u' \in M_1$ and $D(u, u') < \delta$; i.e., the solution set $S(u)$ of u is stable: $S(u')$ is close to $S(u)$ whenever u' is close to u for all $u' \in M_1$. Theorem 2.4.14 (iii) implies that if $u \in M_1$, then u is essential relative to M_1 if and only if the solution set $S(u)$ is stable.

Now let X be a non-empty closed and convex subset of a Frechet space E equipped with a translation invariant metric d. Denote

$CK(X) = \{A \in K(X) : A \text{ is convex }\}$,

$CL(X) = \{f \in L(X) : f \text{ satisfies (i) and (iii) of Theorem 2.4.A}\}$,

$M' = CK(X) \times CL(X)$.

The following is an application of the results obtained in this section:

Theorem 2.4.17. (i) M' is a non-empty closed subset of M.

(ii) There exists a dense G_δ subset Q of M' such that u is essential relative to M' for each $u \in Q$.

Proof. (i) Clearly M' is non-empty. If $u = (A, f) \in M'$, then by Ky Fan's minimax inequality Theorem 2.4.A, there exists $\hat{y} \in A$ such that $\sup_{x \in A} f(x, \hat{y}) \leq 0$; thus $u \in M$ so that $M' \subset M$. Now if $\{(A_n, f_n)\}_{n=1}^{\infty}$ is a sequence in M' such that $(A_n, f_n) \to (A, f) \in M$, then $f_n \to f$ and $A_n \to A$. Since for each $y \in X$, $x \mapsto f_n(x, y)$ is quasi-concave, it is also easy to see that $x \mapsto f(x, y)$ is also quasi-concave. By Lemma 2.4.8, A is also convex. Thus $(A, f) \in M'$ so that M' is closed in M.

Now (ii) follows from (i) and Theorem 2.4.16 (ii). □

Finally, we remark that if we define $M = \{(A, f) \in Y$: there exists $y \in A$ such that $\sup_{x \in A} f(x, y) \leq \sup_{x \in X} f(x, x)\}$, $S(u) = \{y \in A : \sup_{x \in A} f(x, y) \leq \sup_{x \in X} f(x, x)\}$ for each $u = (A, f) \in M$ and $CL(X) = \{f \in L(X) : f$ satisfies (a) and (b) of Theorem 2.4.B $\}$, then all the results in this section remain valid.

2.5 Coincidence Points for Non-self Mappings in Topological Vector Spaces

The natural extension of fixed point theory is the study of coincidence points. Let X and Y be topological space and $S, T : X \to 2^Y$. The *coincidence problem* for (S, T) is to find $(x_0, y_0) \in X \times Y$ such that $y_0 \in S(x_0) \cap T(x_0)$. Geometrical problems of this type in an approximate context turn out to be intimately related to some basic problems arising in convex analysis. This important fact was discovered by J. von Neumann in 1937, who established a coincidence theorem in $\mathbf{R}^n$ which was then applied to prove his well-known minimax principle. Since then, geometrical problems of a similar kind (as well as their analytic counterparts) have attracted broad attention. Also, new applications in various mathematical areas have been found. In particular, since Eilenberg and Montgomery [94] studied coincidence theory in topological settings in 1946, this topic has been comprehensively developed by the contributions of Kakutani [170], Nash [227], Fan[97], Kneser [190], Gale [122], Debreu [72], Nikaido [229], Sion [279], Gorniewicz, Granas [131] and Kryszewski [132], Granas and Liu [134], Chang and Song [52], Ben-El-Medchaiek and Deguire [24], Ko and Tan [192], Powers [247] and other contributors. This topic has many applications in mathematics and other subjects, for example, see Aubin [7], Aubin and Cellina [9] and Zeidler [336].

In this section, we first consider the relations between Halpern's inward (respectively, outward) mappings in [141] and Fan's inward (respectively, outward) definitions [103]. Several facts involved in the study of fixed point theorems for non-self mappings are also exhibited. Next a general multivalued version of Fan's best approximation theorem [103] is given in topological vector space. As applications, a number of approximation theorems, fixed point theorems and coincidence theorems are given in topological vector spaces. These results improve or unify most of the well known results in Browder [44], Fan [98], [103], [105], [106], Komiya [196], Park [242], Reich [248], Halpern and Bergman [143] and Ha [139].

2.5.1 *Some Facts.*

In this section, some facts concerning boundary conditions which are involved in the study of non-self mappings are exhibited.

We introduce some notation. Let X and Y be topological spaces and $A : X \to 2^Y$. Then A is continuous if A is both upper semicontinuous and lower semicontionous. If Y is a non-empty subset of a topological vector space, then A is convex valued (respectively, closed convex valued) if $A(x)$ is convex (respectively, closed and convex) for each $x \in X$.

The following definitions are due to Halpern (e.g., see [141]). Let X be a non-empty subset of a vector space E and $y \in E$. Then the inward set $I_X(y)$ and the outward set $O_X(y)$ of X at y are defined by

$I_X(y) = \{x \in E : \text{ there exist } u \in X \text{ and } r > 0 \text{ such that } x = y + r(u - y)\}$,

and

$O_X(y) = \{x \in E : \text{ there exist } u \in X \text{ and } r > 0 \text{ such that } x = y - r(u - y)\}$.

If E is a topological vector space, then the closure of $I_X(y)$ and $O_X(y)$ in E, denoted by $\overline{I_X}(y)$ and $\overline{O_X}(y)$ respectively, are called the weakly inward set and weakly outward set of X at y respectively.

Let X be a non-empty subset of a topological vector space E. Then a mapping $T : X \to 2^E$ is called (i) inward (respectively, outward) if for each $x \in X$, $T(x) \cap I_X(x) \neq \emptyset$ (respectively, $T(x) \cap O_X(x) \neq \emptyset$) and (ii) weakly inward (respectively, weakly outward) if for each $x \in X$, $T(x) \cap \overline{I_X}(x) \neq \emptyset$ (respectively, $T(x) \cap \overline{O_X}(x) \neq \emptyset$).

We note that if T is a (weakly) inward mapping, then the mapping $G : X \to 2^E$ defined by $G(x) = 2x - T(x)$ for each $x \in X$ is (weakly) outward and vice versa. Also, x is a fixed point of T if and only if it is a fixed point of G. Hence fixed point results for (weakly) inward mappings are equivalent to such results for (weakly) outward mappings. Thus we shall mainly give details of proofs for (weakly) inward mappings.

Let X be a non-empty convex set in a (real or complex) vector space E. Following Fan [103], the algebraic boundary $\delta_E(X)$ of X in E is the set of all $x \in X$ for which there exists $y \in E$ such that $x + ry \notin X$ for all $r > 0$. If X is a subset of a topological vector space, the topological boundary $\partial_E(X)$ is the complement of $\text{int}_E X$ in E. It is easy to see that $\delta_E(X) \subset \partial_E(X)$ and in general $\delta_E(X) \neq \partial_E(X)$ as there exists a convex subset X of a topological vector space E such that $\partial_E(X) = X$ while $\delta_E(X) \neq X$, e.g., see [313, Example 4 in Chapter 3].

Let E and W be two topological vector spaces and X a non-empty subset of E. Let $G, F : X \to 2^W$. A point $x \in X$ is said to be a coincidence point of G and F if $G(x) \cap F(x) \neq \emptyset$.

Let E be a topological vector space and E^* be its continuous dual. E is said to have sufficiently many continuous linear functionals if for each $x \in E$ with $x \neq 0$, there exists $\phi \in E^*$ such that $Re\phi(x) \neq 0$, i.e., E^* separates points in E. By the Hahn-Banach theorem, if E is a locally convex topological vector space, then E has sufficiently many continuous linear functionals. There are topological vector spaces with sufficiently many continuous linear functionals which are not locally convex, e.g., the Hardy space H^p, $0 < p < 1$.

Since most fixed point theorems of inward (outward) mappings depend on the boundary conditions of the domains, for example see Halpern and Bergman [143], Browder [41] and Fan [103] and Park [240], it is our purpose in this section to discuss the relations between various boundary conditions appearing in the literature. Following the idea of Fan [103], we first have the following:

Propositions 2.5.1. Let X be a non-empty convex subset of a vector space E and $F : X \to 2^E$. Then the following two conditions (a) and (b) are equivalent.

(a) For each $x \in \delta_E(X)$, there exist $y \in X$, $u \in F(x)$ and $r > 0$ such that $u - x = r(y - x)$.

(b) For each $x \in X$, there exist $u \in F(x)$ and $r \in (0,1)$ such that $rx + (1-r)u \in X$.

Proof. (a) $\Longrightarrow$ (b). Fix an arbitrary $x \in X$. If there exists $u \in F(x)$ such that $u \in X$, then because X is convex, $rx + (1-r)u \in X$ holds for every $r \in (0,1)$. Now assume $F(x) \subset E \setminus X$. (1) If $x \in \delta_E(X)$, then by (a), there exist $y \in X$ and $u \in F(x)$ and $r > 0$ such that $u - x = r(y - x)$. It follows that $ry + (1-r)x = u \notin X$. Since y, $x \in X$ and X is convex, we must have $r > 1$. Let $\lambda = \frac{r-1}{r}$, then $\lambda \in (0,1)$ and $\lambda x + (1-\lambda)u = y \in X$. (2) If $x \notin \delta_E(X)$, then by the definition of $\delta_E(X)$, for the point $u-x$, where $u \in F(x)$, there exist $r>0$ such that $y=x+r(u-x) \in X$. Since $x, y \in X$ and $u=\frac{1}{r}y+\frac{r-1}{r}x \notin X$, we must have $r<1$. Let $\lambda=1-r$, then we have $\lambda x+(1-\lambda)u=y \in X$. Thus the condition (b) is verified.

(b) $\Longrightarrow$ (a). Suppose $x \in \delta_E(X)$. By (b), there exist $u \in F(x)$ and $r \in (0,1)$ such that $y=rx+(1-r)u$. If we take $\lambda=\frac{1}{1-r}$, then $\lambda>0$ and $u-x=\lambda(y-x)$. □

Proposition 2.5.1 improves the result given by Fan [103] to multivalued mappings. Fan also gave the following result in [103]:

Proposition 2.5.2. Let X be a non-empty convex subset of a vector space E. Suppose $F, G: X \to 2^E$. Then the following two conditions are equivalent:

(a) For each point $x \in \delta_E(X)$, there exist three points $y \in X$, $u \in F(x)$, $v \in G(x)$ and a real number $r>0$ such that $y-x=r(u-v)$.

(b) For each $x \in X$, there exist three points $y \in X$, $u \in F(x)$, $v \in G(x)$ and a real number $r>0$ such that $y-x=r(u-v)$.

By Proposition 2.5.1 and Proposition 2.5.2, we have the following:

Proposition 2.5.3. Let X be a non-empty convex subset of a vector space E. Suppose $F, G: X \to 2^E$. Then the following are equivalent:

(a) For each $x \in \delta_E(X)$, there exist $y \in X$, $u \in F(x)$, $v \in G(x)$ and $r>0$ such that $y-x=r(u-v)$.

(b) For each $x \in X$, there exist $y \in X$, $u \in F(x)$, $v \in G(x)$ and $r>0$ such that $y-x=r(u-v)$.

(c) For each $x \in X$, there exist $u \in F(x)$, $v \in G(x)$ and $r \in (0,1)$ such that $x+(1-r)(u-v) \in X$.

Proof. By Proposition 2.5.2, the condition (a) is equivalent to the condition (b).

Now define $W: X \to 2^E$ by $W(x)=\{x\}+F(x)-G(x)$ for each $x \in X$. Then it is clear that the condition (a) is equivalent to the following:

$(a)'$ For each $x \in \delta_E(X)$, there exist $y \in X$, $u \in W(x)$ and $r>0$ such that $y-x=r(u-x)$.

Now by Proposition 2.5.1, the condition $(a)'$ is equivalent to the following condition:

$(c)'$: For each $y \in X$, there exist $u \in W(y)$ and $r \in (0, 1)$ such that $ry+(1-r)u \in X$.

By the definition of W, it is also obvious that the condition $(c)'$ is equivalent to the condition (c). Therefore conditions (a), (b) and (c) are equivalent. □

We also have the following:

Proposition 2.5.4. Let X be a non-empty subset of a vector space E and $F, G : X \to 2^E$. Then the following two conditions are equivalent:

(i) For each $x \in \delta_E(X)$ and $\phi \in E^*$ such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in X$, there exist $u \in F(x)$ and $v \in G(x)$ with $Re\phi(u) \geq Re\phi(v)$.

(ii) For each $x \in X$ and $\phi \in E^*$ such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in X$, there exist $u \in F(x)$ and $v \in G(x)$ with $Re\phi(u) \geq Re\phi(v)$.

Proof. We only need to show that (i) $\Longrightarrow$ (ii). Suppose $x \in X$. Let $\phi \in E^*$ be such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in X$. If $x \notin \delta_E(X)$, then by the definition of $\delta_E(X)$, for each $y \in E$, there exist $r > 0$ such that $x + ry \in X$; it follows that $Re\phi(x) \leq Re\phi(x + ry)$, so that $Re\phi(y) \geq 0$ for all $y \in E$, and therefore ϕ is necessarily the zero linear functional. Since $Re\phi \equiv 0$, $Re\phi(u) \geq Re\phi(v)$ is satisfied for any $u \in F(x)$ and $v \in G(x)$. Next, if $x \in \delta_E(X)$, then for any $\phi \in E^*$ satisfying $Re\phi(x) \leq Re\phi(y)$ for all $y \in X$, by (i), there exist $u \in F(x)$ and $v \in G(x)$ such that $Re\phi(u) \geq Re\phi(v)$. Thus the condition (ii) is verified. □

Let X be a non-empty closed convex subset of a topological vector space E. We remark that (1) Corollary 6.3.1 of Rockafellar [253, p.47] implies that Fan's definition of *algebraic boundary* $\delta_E(X)$ of X in E is equivalent to the definition given by Browder [41, p.285]: $\delta_E(X) = \{x \in X :$ there exists a finite dimensional flat M such that $x \in \partial_E(X \cap M)$ and (2) if $\text{int}_E(X) \neq \emptyset$, then $\partial_E(X) = \delta_E(X)$ (see, e.g., [313, Theorem 2.27(a)]. Therefore our Proposition 2.1.3 really generalizes Fan's result [106].

Since Halpern gave the definitions of inward (outward) mappings in his Ph.D. thesis [141] (e.g., see Halpern and Bergman [143]), many fixed point theorems are obtained. In [105], Fan also gave another definition for the inward (outward) mappings as follows:

Definition. Let X be a non-empty convex subset of a topological vector space

E and $F : X \to 2^E$. The mapping F is an inward mapping (respectively, outward mapping) if for each $x \in X$ and any continuous linear functional ϕ on E such that $Re\phi(x) \leq \inf_{y\in X} Re\phi(y)$, there exists a point $u \in F(x)$ such that $Re\phi(u) \geq Re\phi(x)$ (respectively, $Re\phi(u) \leq Re\phi(x)$).

It is easy to see that the condition (a) (respectively, (b)) of Proposition 2.5.2 implies the condition (i) (respectively, (ii)) of Proposition 2.5.4. Therefore Fan's definition of inward mappings (respectively, outward mappings) in a topological vector space includes the definitions of inward mappings (respectively, outward mappings) for a single (or multivalued) mapping given by Halpern (also see Fan [103], Halpern and Bergman [143]).

Let X be a non-empty convex subset of a topological vector space E and $W, T, F : X \to 2^E$. Consider the following conditions:

(a) For each $x \in X$, there is $y \in W(x)$ and $r \in [0, 1)$ such that $rx + (1 - r)y \in X$.

(b) For each $x \in X$, there is $y \in X$, $u \in T(x)$, $v \in F(x)$ and $r \in (0, 1]$ such that $y - x = r(u - v)$.

Then it is not difficult to see that T and F satisfy the condition (b) if and only if the mapping $W = I + T - F$ satisfies the condition (a); and moreover, $x \in W(x)$ if and only if $T(x) \cap F(x) \neq \emptyset$.

2.5.2 *Best Approximation Theorems.*

In this section, we will give a general extension of Fan's best approximation theorem [103] in topological vector spaces. As applications, several coincidence theorems are derived which in turn imply some fixed point theorems.

We shall need the following result which is contained in the proof of Theorem 2 and the remark immediately following its proof in Ha [139]:

Theorem 2.5.A. Let X be a non-empty compact subset of a topological vector space E which has sufficiently many continuous linear functionals and $F : X \to 2^E$ be upper semicontinuous with compact and convex values. If F has no fixed point, then there exist $\delta > 0$ and a continuous seminorm P on E such that $\inf_{x\in X} \inf_{u\in F(x)} P(x-u) \geq \delta$.

We shall need also the following Lemma 2 of Ha [138]:

Theorem 2.5.B. Let Z be an n-simplex and let K be a non-empty compact convex subset of a topological vector space. If $A : Z \to 2^K$ is upper semicontinuous with closed and convex values and $p : K \to Z$ is continuous, then there exists $x_0 \in Z$ such that $x_0 \in p(A(x_0))$.

We shall now prove the following coincidence theorem:

Theorem 2.5.5. Let X be a contractible space and Y be a compact convex subset of a topological vector space E. Let $A : X \to 2^Y$ be upper semicontinuous with closed and convex values. Suppose that $B : Y \to 2^X$ is such that

(a) $B^{-1}(x)$ is open for each $x \in X$; and

(b) for each open set O in Y, the set $\cap_{y\in O} B(y)$ is empty or contractible.

Then there exist $w_0 \in X$ and $z_0 \in Y$ such that $w_0 \in B(z_0)$ and $z_0 \in A(w_0)$.

Proof. We first show that there exist an n-simplex Δ_N and two functions $f : \Delta_N \to X$ and $\psi : Y \to \Delta_N$ such that $f(\psi(y)) \in B(y)$ for all $y \in Y$.

Since Y is compact, by (a), there exists a finite subset $\{x_0, \cdots, x_n\}$ of X such that $Y = \cup_{i=0}^n B^{-1}(x_i)$. Now for each non-empty subset J of $N := \{0, \cdots, n\}$, we define

$$F_J = \begin{cases} \cap\{B(y) : y \in \cap_{j\in J} B^{-1}(x_j)\}, & \text{if } \cap_{j\in J} B^{-1}(x_j) \neq \emptyset, \\ X, & \text{otherwise.} \end{cases}$$

Note that if $y \in \cap_{j\in J} B^{-1}(x_j)$, then $\{x_j : j \in J\} \subset B(y)$. Therefore by (b), if $\cap_{j\in J} B^{-1}(x_j) \neq \emptyset$, then $F_J = \cap\{B(y) : y \in \cap_{j\in J} B^{-1}(x_j)\}$ is non-empty and contractible. It is clear that $F_J \subset F_{J'}$ whenever $\emptyset \neq J \subset J' \subset \{0, \cdots, n\}$. Thus F satisfies all hypotheses of Lemma 2.2.B. By Lemma 2.2.B, there is a continuous function $f : \Delta_N \to X$ such that $f(\Delta_J) \subset F_J$ for all $J \in \mathcal{F}(N)$. Let $\{\psi_i : i \in N\}$ be a continuous partition of unity subordinated to the covering $\{B^{-1}(x_i) : i \in N\}$, i.e., for each $i \in N$, $\psi_i : Y \to [0,1]$ is continuous, $\{y \in Y : \psi_i(y) \neq 0\} \subset B^{-1}(x_i)$ such that $\Sigma_{i=0}^n \psi_i(y) = 1$ for all $y \in Y$ Define $\psi : Y \to \Delta_N$ by $\psi(y) = (\psi_0(y), \psi_1(y), \cdots, \psi_n(y))$ for each $y \in Y$.

Then $\psi(y) \in \Delta_{J(y)}$ for all $y \in Y$, where $J(y) = \{i \in \{0, \cdots, n\} : \psi_i(y) \neq 0\}$. Therefore $f(\psi(y)) \in f(\Delta_{J(y)}) \subset F_{J(y)} \subset B(y)$.

Since A is upper semicontinuous with closed and convex values and f is continuous, the composition $A \circ f : \Delta_N \to 2^Y$ is also upper semicontinuous with closed and convex values, and $\psi : Y \to \Delta_N$ is continuous. By Theorem 2.5.B, there exists $x_0 \in \Delta_N$ such that $x_0 \in \psi \circ (A \circ f(x_0))$. Let $w_0 := f(x_0)$, then $w_0 = f(x_0) \in f \circ (\psi \circ (A \circ f(x_0)) = f \circ (\psi \circ (A(w_0))$ so that there exists $z_0 \in A(w_0)$ is such that $w_0 = f \circ \psi(z_0) \in B(z_0)$. □

As a special case of Theorem 2.5.5, we have the following result which is Theorem 1 of Komiya [196]:

Corollary 2.5.6. Let X be a non-empty convex subset of a topological vector space E and Y be a non-empty compact convex subset of a topological vector space W. Suppose $A : X \to 2^Y$ is upper semicontinuous with closed and convex values and $B : Y \to 2^X$ has convex values such that $B^{-1}(x)$ is open in Y for each $x \in X$. Then there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in B(y_0)$ and $y_0 \in A(x_0)$.

We now prove a multivalued generalization of Fan's best approximation theorem [103, Theorem 2] in topological vector spaces.

Theorem 2.5.7. Let E and W be two topological vector spaces. Let X be a non-empty compact convex subset of E and $G, F : X \to 2^W$. Suppose further there exists continuous $H : X \times W \to \mathbf{R}$ such that:

(i) for each fixed $x \in X$, the set $\{y \in W : H(x, y) < r\}$ is convex for each $r \in \mathbf{R}$;

(ii) G is continuous with compact values and the set $G^{-1}(y)$ is convex for each $y \in G(X)$, where $G(X)$ is a convex set;

(iii) F is continuous with compact values.

Then there exists a point $x_0 \in X$ such that

$$\inf_{v \in G(X)} [\inf_{u \in F(x_0)} H(x_0, v - u)] = \inf_{v \in G(x_0)} [\inf_{u \in F(x_0)} H(x_0, v - u)].$$

Proof. Suppose that the conclusion were false. Define two mappings $A : G(X) \to 2^X$ and $B : X \to 2^{G(X)}$ by $A(y) = G^{-1}(y)$ for each $y \in G(X)$ and $B(x) = \{y \in G(X) :$

$\inf_{u\in F(x)} H(x,y-u) < \inf_{v\in G(x)} \inf_{u\in F(x)} H(x,v-u)\}$ for each $x \in X$. Then by assumption, $B(x) \neq \emptyset$ for each $x \in X$. Since G is upper semicontinuous with closed values, the graph of G is closed in $E \times W$, Graph$A$ is also closed in $W \times E$. Since X is compact, the mapping $A : G(X) \to 2^X$ is upper semicontinuous and has closed convex values. Since F is continuous with compact values and $H : X \times W \to \mathbf{R}$ is continuous, Theorem 1 of Aubin [7, p.67] and Theorem 2 of Aubin and Ekeland [10, p.69] imply that $(x,y) \mapsto \inf_{u\in F(x)} H(x,y-u) - \inf_{v\in G(x)} \inf_{u\in F(x)} H(x,v-u)$ is continuous. It follows that for each $y \in G(X)$, the set $B^{-1}(y) = \{x \in X : \inf_{u\in F(x)} H(x,y-u) < \inf_{v\in G(x)} \inf_{u\in F(x)} H(x,v-u)\}$ is open in X. Since $G(X)$ is convex, by (i) $B(x)$ is convex for each $x \in X$. Therefore A and B satisfy all hypotheses of Corollary 2.5.6. By Corollary 2.5.6, there exist $x_0 \in X$ and $y_0 \in G(X)$ such that $x_0 \in A(y_0)$ and $y_0 \in B(x_0)$, i.e. $y_0 \in G(x_0)$ and

$$\inf_{u\in F(x_0)} H(x_0, y_0 - u) < \inf_{v\in G(x_0)} [\inf_{u\in F(x_0)} H(x_0, v-u)]$$

which is impossible. Therefore the conclusion must hold. □

If the mapping G in Theorem 2.5.7 is single-valued, then we have:

Theorem 2.5.8. Let E and W be two topological vector spaces. Let X be a non-empty compact convex subset of E, $F : X \to 2^W$ and $G : X \to W$. Suppose further there exists a continuous function $H : X \times W \to \mathbf{R}$ such that:

(i) for each fixed $x \in X$, $y \to H(x,y)$ is convex;

(ii) G is continuous and the set $G^{-1}(y)$ is convex for each $y \in G(X)$, where $G(X)$ is convex;

(iii) F is continuous with compact and convex values.

Then there exists a point $x_0 \in X$ such that

$$\inf_{w\in \overline{I_{G(X)}(G(x_0))}} \inf_{u\in F(x_0)} H(x_0, w-u) = \inf_{u\in F(x_0)} H(x_0, G(x_0) - u).$$

In particular, if $\inf_{u\in F(x_0)} H(x_0, G(x_0) - u) > 0$ and $H(x, ry) \leq rH(x,y)$ for each $r \in (0,1)$ and each $(x,y) \in X \times W$, then $G(x_0) \in \delta_E(G(X))$.

Proof. By Theorem 2.5.7, there exists $x_0 \in X$ such that

$$\inf_{v\in G(X)} \inf_{u\in F(x_0)} H(x_0, v-u) = \inf_{u\in F(x_0)} H(x_0, G(x_0) - u).$$

we shall prove that

$$\inf_{w\in \overline{I_{G(X)}}(G(x_0))} \inf_{u\in F(x_0)} H(x_0, w-u) \geq \inf_{u\in F(x_0)} H(x_0, G(x_0) - u).$$

Fix an arbitrary $w \in I_{G(X)}(G(x_0)) \setminus G(X)$. As $G(X)$ is convex, there exist $z \in G(X)$ and $r > 1$ such that $w = G(x_0) + r(z - G(x_0))$. Suppose that

$$\inf_{u\in F(x_0)} H(x_0, w-u) < \inf_{u\in F(x_0)} H(x_0, G(x_0) - u).$$

Since $F(x_0)$ is compact and H is continuous, there exist $z_1, z_2 \in F(x_0)$ such that $H(x_0, w-z_1) = \inf_{u\in F(x_0)} H(x_0, w-u) < \inf_{u\in F(x_0)} H(x_0, G(x_0)-u) = H(x_0, G(x_0) - z_2)$. Let $\overline{z} = (1-\frac{1}{r})z_2 + \frac{1}{r}z_1$, then $\overline{z} \in F(x_0)$ since $F(x_0)$ is also convex. Therefore we have

$$\begin{aligned}
H(x_0, G(x_0) - z_2) &= \inf_{u\in F(x_0)} H(x_0, G(x_0) - u) = \inf_{v\in G(X)} \inf_{u\in F(x_0)} H(x_0, v-u) \\
&\leq H(x_0, z - \overline{z}) = H(x_0, \frac{1}{r}w + (\frac{r-1}{r})G(x_0) - \frac{1}{r}z_1 - (\frac{r-1}{r})z_2) \\
&\leq \frac{1}{r}H(x_0, w - z_1) + \frac{r-1}{r}H(x_0, G(x_0) - z_2) \\
&< H(x_0, G(x_0) - z_2)
\end{aligned}$$

which is a contradiction. Thus we must have

$$\inf_{w\in I_{G(X)}(G(x_0))} \inf_{u\in F(x_0)} H(x_0, w-u) \geq \inf_{u\in F(x_0)} H(x_0, G(x_0) - u).$$

By the continuity of $w \to \inf_{u\in F(x_0)} H(x_0, w-u)$, we have

$$\inf_{w\in \overline{I_{G(X)}}(G(x_0))} \inf_{u\in F(x_0)} H(x_0, w-u) \geq \inf_{u\in F(x_0)} H(x_0, G(x_0) - u).$$

Hence

$$\inf_{w\in \overline{I_{G(X)}}(G(x_0))} \inf_{u\in F(x_0)} H(x_0, w-u) = \inf_{u\in F(x_0)} H(x_0, G(x_0) - u).$$

If $\inf_{u\in F(x_0)} H(x_0, G(x_0)-u) > 0$ and $H(x, ry) \le rH(x,y)$ for each $r \in (0,1)$ and for each, $(x,y) \in X \times W$, we shall show that $G(x_0) \in \delta_E(G(X))$. Note that $F(x_0)$ is compact, thus there exists a point $u_0 \in F(x_0)$ such that $H(x_0, G(x_0)-u_0) = \inf_{u\in F(x_0)} H(x_0, G(x_0)-u)$. We first show that $u_0 \notin G(X)$. Suppose that $u_0 \in G(X)$. Then for any $r \in (0,1)$, $rG(x_0)+(1-r)u_0 \in G(X)$. It follows that

$$\begin{aligned}
H(x_0, G(x_0)-u_0) &= \inf_{u\in F(x_0)} H(x_0, G(x_0)-u) = \inf_{v\in G(X)} \inf_{u\in F(x_0)} H(x_0, v-u) \\
&\le \inf_{u\in F(x_0)} H(x_0, rG(x_0)+(1-r)u_0-u) \\
&\le H(x_0, rG(x_0)+(1-r)u_0-u_0) \le H(x_0, r(G(x_0)-u_0)) \\
&\le rH(x_0, G(x_0)-u_0) < H(x_0, G(x_0)-u_0)
\end{aligned}$$

which is a contradiction. Therefore $u_0 \notin G(X)$.

Now suppose that $G(x_0) \notin \delta_E(G(X))$, then by the definition of $\delta_E(G(X))$, for points $u_0 \in F(x_0)$ and $G(x_0) \in G(X)$, there exists $r > 0$ such that $x_1 = G(x_0) + r(u_0 - G(x_0)) \in G(X)$. Since $u_0 = \frac{1}{r}x_1 + \frac{r-1}{r}G(x_0) \notin G(X)$ and both x_1, $G(x_0) \in G(X)$ and $G(X)$ is convex, we must have $r < 1$. Therefore

$$\begin{aligned}
H(x_0, G(x_0)-u_0) &= \inf_{u\in F(x_0)} H(x_0, G(x_0)-u) = \inf_{v\in G(X)} \inf_{u\in F(x_0)} H(x_0, v-u) \\
&\le \inf_{u\in F(x_0)} H(x_0, (1-r)G(x_0)+ru_0-u) \\
&\le H(x_0, (1-r)G(x_0)+ru_0-u_0) \\
&\le H(x_0, (1-r)G(x_0)-(1-r)u_0) \\
&\le (1-r)H(x_0, G(x_0)-u_0) \\
&< H(x_0, G(x_0)-u_0)
\end{aligned}$$

which is a contradiction. Hence $G(x_0) \in \delta(G(X))$. □

Theorem 2.5.8′. Let E and W be two topological vector spaces. Let X be a non-empty compact convex subset of E, $F : X \to 2^W$ and $G : X \to W$. Suppose further there exists a continuous function $H : X \times W \to \mathbf{R}$ such that:

(i) for each fixed $x \in X$, $y \to H(x,y)$ is convex;

(ii) G is continuous and the set $G^{-1}(y)$ is convex for each $y \in G(X)$, where $G(X)$ is convex;

(iii) F is continuous with compact convex values.

Then there exists a point $x_0 \in X$ such that

$$\inf_{w \in \overline{O_{G(X)}(G(x_0))}} \inf_{u \in F(x_0)} H(x_0, w-u) = \inf_{u \in F(x_0)} H(x_0, G(x_0)-u).$$

In particular, if $\inf_{u \in F(x_0)} H(x_0, G(x_0)-u) > 0$, and $H(x, ry) \leq rH(x,y)$ for each $r \in (0,1)$ and each $(x,y) \in X \times W$, then $G(x_0) \in \delta_E(G(X))$

Proof. If we define the mappings $F_1 : X \to 2^W$ by $F_1(x) = 2G(x) - F(x)$ for each $x \in X$ and $H_1 : X \times W \to \mathbf{R}$ by $H_1(x,y) = H(x,-y)$ for each $(x,y) \in X \times W$ Then the mappings G, F_1 and H_1 satisfy all hypotheses of Theorem 2.5 8 and by the same argument used in Theorem 2.5.8, the conclusion follows. □

By Theorem 2.5.8 and Theorem 2.5.8′, we have the following:

Corollary 2.5.9. Let E be a topological vector space which has sufficiently many continuous linear functionals. Let X be a non-empty compact convex subset of E and $F : X \to 2^E$ be continuous with compact and convex values. If F satisfies the following condition (i) or (i)′, then F has a fixed point in X.

(i): For each $x \in \delta_E(X) \setminus F(x)$, there exist a real number $r \in (0,1)$ and $u \in F(x)$ such that $rx + (1-r)u \in \overline{I_X}(x)$ (respectively, $rx + (1-r)u \in \overline{O_X}(x)$).

(i)′: For each $x \in \delta_E(X) \setminus F(x)$ and $u \in F(x)$, there exists a number r (real or complex, depending on whether the vector space E is real or complex) with $|r| < 1$ such that $rx + (1-r)u \in \overline{I_X}(x)$ (respectively $rx + (1-r)u \in \overline{O_X}(x)$).

Proof. Suppose that F has no fixed point, then by Theorem 2 5 A, there exist $\delta > 0$ and a continuous seminorm P on E such that $\inf_{u \in F(x)} P(x-u) \geq \delta$ for all $x \in X$

Define continuous mappings $H : X \times E \to \mathbf{R}$ and $G : X \to E$ by $H(x,y) = P(y)$ for each $(x,y) \in X \times E$ and $G(x) = x$ for each $x \in X$ By Theorem 2.5 8 (respectively, Theorem 2.5.8′), there exists $x_0 \in \delta_E(X)$ such that

$$\inf_{v \in \overline{I_X}(x_0)} \inf_{u \in F(x_0)} P(v-u) = \inf_{u \in F(x_0)} P(x_0 - u) \geq \delta > 0$$

(respectively,

$$\inf_{v \in \overline{O_X}(x_0)} \inf_{u \in F(x_0)} P(v-u) = \inf_{u \in F(x_0)} P(x_0 - u) \geq \delta > 0).$$

If F satisfies the condition (i), then $x_0 \in \delta_E(X)\setminus F(x_0)$ By (i), there exist $r \in (0,1)$ and $u_0 \in F(x_0)$ such that $rx_0 + (1-r)u_0 \in \overline{I_X}(x_0)$ (respectively, $rx_0 + (1-r)u_0 \in \overline{O_X}(x_0)$) so that

$$\begin{aligned}
0 &< \inf_{u\in F(x_0)} P(x_0 - u)\\
&= \inf_{v\in \overline{I_X}(x_0)} \inf_{u\in F(x_0)} P(v-u)(\text{ respectively, } = \inf_{v\in \overline{O_X}(x_0)} \inf_{u\in F(x_0)} P(v-u))\\
&\le \inf_{u\in F(x_0)} P(rx_0 + (1-r)u_0 - u))\\
&\le \inf_{u\in F(x_0)} P(rx_0 + (1-r)u_0 - (ru + (1-r)u_0))(\text{ since } F(x_0) \text{ is convex})\\
&= r \inf_{u\in F(x_0)} P(x_0 - u)(\text{ since } P \text{ is a seminorm})\\
&< \inf_{u\in F(x_0)} p(x_0 - u)
\end{aligned}$$

which is a contradiction Therefore F must have a fixed point in X

Now suppose that F satisfies the condition (i)′, then $x_0 \in \delta_E(X)\setminus F(x_0)$. Since $F(x_0)$ is compact, there exists a point $u_1 \in F(x_0)$ such that $\inf_{u\in F(x_0)} P(x_0 - u) = P(x - u_1)$ By (i)′, there exists a number r with $|r| < 1$ such that $rx_0 + (1-r)u_1 \in \overline{I_X}(x_0)$ (respectively, $\overline{O_X}(x_0)$) Thus

$$\begin{aligned}
0 &< \inf_{u\in F(x} F(x_0 - u)\\
&= \inf_{v\in \overline{I_X}(x_0)} \inf_{u\in F(x_0)} P(v-u)(\text{ respectively, } = \inf_{v\in \overline{O_X}(x_0)} \inf_{u\in F(x_0)} p(v-u))\\
&\le \inf_{u\in F(x_0)} F(rx_0 + (1-r)u_1 - u)\\
&\le \inf_{u\in F(x_0)} P(rx_0 + (1-r)u_1 - u_1)\\
&\le |r| P(x_0 - u_1)\\
&< P(x_0 - u_1) = \inf_{u\in F(x_0)} P(x_0 - u)
\end{aligned}$$

which is a contradiction Therefore F must have a fixed poi t. □

We note that the condition of (i)′ is different from (i) since the number r in the condition (ii)′ may be real or complex

Since for each non-empty subset X in a topological vector space E, its algebraic boundary $\delta_E(X)$ in E is usually smaller than the topological boundary $\partial_E(X)$, by Proposition 2 5.1, Corollary 2 5 9 generalizes Theorem 3 of Fan [103] which in turn improves

Theorem 1 of Browder [44], Lemma 1.6 of Reich [248] and Theorem 4.1 of Halpern and Bergman [143] in the following ways: (1) the underlying space is a topological vector space instead of a locally convex space and (2) the mapping F is set-valued instead of single-valued and (3) the boundary condition of Corollary 2.5.9 is weaker than theirs. Moreover, Corollary 2.5.9 also generalizes Theorem 4 of Park [242] which in turn generalizes many fixed point theorems for single-valued or set-valued inward (outward) mappings in the literature.

2.5.3 *Coincidence Theorems*

In this section, as applications of Theorem 2.5.7, several coincidence theorems for set-valued inward and outward mappings are derived which in turn imply fixed point theorems of inward and outward set-valued mappings in topological vector spaces. Finally, a coincidence theorem in locally convex spaces is also given.

As an application of Theorem 2.5.7, we first have the following coincidence theorem in topological vector spaces:

Theorem 2.5.10. Let E and W be two topological vector spaces. Let X be a non-empty compact convex subset of E and $G, F : X \to 2^W$. Suppose further there exists a continuous function $H : X \times W \to \mathbf{R}$ such that:

(i) for each fixed $x \in X$, the set $\{y \in W : H(x,y) < r\}$ is convex for each $r \in \mathbf{R}$;

(ii) G is continuous with non-empty closed values and the set $G^{-1}(y)$ is convex for each $y \in G(X)$, where $G(X)$ is convex;

(iii) F is continuous with compact values.

(iv) for each $x \in X$, if $G(x) \cap F(x) = \emptyset$, there exists a point $y \in G(X)$ such that $\inf_{u \in F(x)} H(x, y-u) < \inf_{v \in G(x)}[\inf_{u \in F(x)} H(x, v-u)]$.

Then there exists a point $x_0 \in X$ such that $G(x_0) \cap F(x_0) \neq \emptyset$.

Proof. Suppose the conclusion is not true, i.e., for each $x \in X$, $G(x) \cap F(x) = \emptyset$. Note that G, F and H satisfy all hypotheses of Theorem 2.5.7. By Theorem 2.5.7, there exists $x_0 \in X$ such that

$$\inf_{v\in G(X)}[\inf_{u\in F(x_0)} H(x_0, v-u)] = \inf_{v\in G(x_0)}[\inf_{u\in F(x_0)} H(x_0, v-u)].$$

But by (iv), there exists $y \in G(X)$ such that

$$\begin{aligned}\inf_{u\in F(x_0)} H(x_0, y-u) &< \inf_{v\in G(x_0)}[\inf_{u\in F(x_0)}(x_0, v-u)] = \inf_{v\in G(X)}[\inf_{u\in F(x_0)} H(x_0, v-u)] \\ &\leq \inf_{u\in F(x_0)} H(x_0, y-u),\end{aligned}$$

which is a contradiction. □.

Theorem 2.5.10 generalizes Theorem 4 of Sessa and Mehta [261] and Proposition 2.2 of Browder [44, p.4750] to topological vector spaces and set-valued mappings. We also note that Theorem 2.5.10 generalizes Theorem 2 of Fan [103, p.235] to topological vector spaces from normed linear spaces.

We now give some coincidence theorems for inward (respectively, outward) and weakly inward (respectively, weakly outward) mappings in topological vector spaces.

Theorem 2.5.11. Let E and W be two topological vector spaces. Let X be a non-empty compact convex subset of E, $F : X \to 2^W$ and $G : X \to W$. Suppose further there exists another continuous function $H : X \times W \to \mathbf{R}$ such that:

(i) for each fixed $x \in X$, $y \to H(x,y)$ is convex;

(ii) G is continuous and the set $G^{-1}(y)$ is convex for each $y \in G(X)$, where $G(X)$ is convex;

(iii) F is continuous with compact and convex values;

(iv) for each $x \in X$ with $G(x) \notin F(x)$, there exists a point $y \in \overline{I_{G(X)}}(G(x))$ such that $\inf_{u\in F(x)} H(x, y-u) < \inf_{u\in F(x)} H(x, G(x)-u)$.

Then there exists a point $x_0 \in X$ such that $G(x_0) \in F(x_0)$.

Proof. Let $x \in X$ be such that $G(x) \notin F(x)$. By (iv) and continuity of $y \to \inf_{u\in F(x)} H(x, y-u)$, there exists y in $I_{G(X)}(G(x))$ such that

$$\inf_{u\in F(x)} H(x, y-u) < \inf_{u\in F(x)} H(x, G(x)-u).$$

If y lies in $G(X)$, then the hypothesis (iv) of Theorem 2.5.10 is valid. If $y \notin G(X)$, since $y \in I_{G(X)}(G(x))$ and $G(X)$ is convex, there exist $u_0 \in G(X)$ and $r \in (0,1)$, $u_0 = (1-r)G(x) + ry$. Now for any $u \in F(x)$, by (i) we have

$$H(x, u_0 - u) = H(x, (1-r)G(x) + ry - u) \leq (1-r)H(x, G(x) - u) + rH(x, y - u).$$

Since $F(x)$ is compact and convex, $H(x, \cdot)$ is continuous, there exist $u_1 \in F(x)$ and $u_2 \in F(x)$ such that $\inf_{u \in F(x)} H(x, G(x) - u) = H(x, G(x) - u_1)$ and $\inf_{u \in F(x)} H(x, y - u) = H(x, y - u_2)$. Since $F(x)$ is convex,

$$\begin{aligned}
\inf_{u \in F(x)} H(x, u_0 - u) &\leq H(x, u_0 - ((1-r)u_1 + ru_2)) \\
& H(x, (1-r)(G(x) - u_1) + r(y - u_2)) \\
&\leq (1-r)H(x, G(x) - u_1) + rH(x, y - u_2) \\
&= \inf_{u \in F(x)} (1-r)H(x, G(x) - u) + \inf_{u \in F(x)} rH(x, y - u) \\
&< \inf_{u \in F(x)} H(x, G(x) - u),
\end{aligned}$$

Hence all the hypotheses of Theorem 2.5.10 hold so that there exists a point $x_0 \in X$ such that $G(x_0) \in F(x_0)$. □

We shall show that in Theorem 2.5.11, $\overline{I_{G(X)}}(G(x))$ can be replaced by $\overline{O_{G(X)}}(G(x)$:

Theorem 2.5.11′. Let E and W be two topological vector spaces. Let X be a non-empty compact convex subset of E, $F : X \to 2^W$ and $G : X \to W$. Suppose further there exists another continuous function $H : X \times W \to \mathbf{R}$ such that:

(i) for each fixed $x \in X$, $y \to H(x,y)$ is a convex function on W;

(ii) G is continuous and the set $G^{-1}(y)$ is convex for each $y \in G(X)$, where $G(X)$ is convex;

(iii) F is continuous with compact and convex values.

(iv) for each $x \in X$ with $G(x) \notin F(x)$, there exists a point $y \in \overline{O_{G(X)}}(G(x))$ such that $\inf_{u \in F(x)} H(x, y - u) < \inf_{u \in F(x)} H(x, G(x) - u)$.

Then there exists a point $x_0 \in X$ such that $G(x_0) \in F(x_0)$.

Proof. Define $F_1 : X \to 2^W$ and $H_1 : X \times W \to \mathbf{R}$ by $F_1(x) = 2G(x) - F(x)$ and $H_1(x,y) = H(x, -y)$ for each $(x,y) \in X \times W$. Then G, F_1 and H_1 satisfy all conditions (i), (ii) and (iii) of Theorem 2.5.11. Let $x \in X$ be such that $G(x) \notin F_1(x)$, then $G(x) \notin F(x)$ so that by (iv), there exists $y \in \overline{O_{G(X)}}(G(x))$ such that $\inf_{u \in F(x)} H(x, y - u) < \inf_{u \in f(x)} H(x, G(x) - u)$. Let $z = 2G(x) - y$, then $z \in \overline{I_{G(X)}}(G(x))$. But then

$$\begin{aligned} \inf_{u \in F_1(x)} H_1(x, z-u) &= \inf_{u \in F(x)} H(x, y-u) \\ &< \inf_{u \in F(x)} H(x, G(x)-u) \\ &= \inf_{u \in F_1(x)} H_1(x, G(x)-u). \end{aligned}$$

This shows that G, F_1 and H_1 also satisfy the condition (iv) of Theorem 2.5.11. Therefore by Theorem 2.5.11, there exists a point $x_0 \in X$ such that $G(x_0) \in F_1(x_0)$ which implies that $G(x_0) \in F(x_0)$. $\square$

By letting $E = W$ and $G = I_X$, the identity map on X in Theorem 2.5.11 and Theorem 2.5.11′ respectively, we have the following result which generalize Theorem 1 and Theorem 2 of Browder [44] in the following ways: (a) the underlying spaces are topological vector spaces instead of locally convex topological vector spaces and (2) the mapping F is set-valued instead of being single-valued.

Corollary 2.5.12. Let E be a topological vector space. Let X be a non-empty compact convex subset of E and $F : X \to 2^E$. Suppose further there exists a continuous function $H : X \times E \to \mathbf{R}$ such that:

(i) for each fixed $x \in X$, $y \to H(x, y)$ is convex;

(ii) F is continuous with compact and convex values.

(iii) for each $x \in X$ with $x \notin F(x)$, there exists a point $y \in \overline{I_X(x)}$ (respectively, $y \in \overline{O_X}(x)$) such that $\inf_{u \in F(x)} H(x, y - u) < \inf_{u \in F(x)} H(x, x - u)$.

Then there exists a point $x_0 \in X$ such that $x_0 \in F(x_0)$.

By Corollary 2.5.12, we have the following:

Corollary 2.5.13. Let E be a topological vector space which has sufficiently many continuous linear functionals, X be a non-empty compact convex subset of E and $F : X \to 2^E$ be continuous with compact and convex values. Suppose further that for each $x \in X$, $F(x) \cap \overline{I_X}(x) \neq \emptyset$ (respectively, $F(x) \cap \overline{O_X}(x) \neq \emptyset$). Then F has a fixed point.

Proof. Suppose that F has no fixed point, then by Theorem 2.5.A, there exist $\delta > 0$ and a continuous seminorm P on E such that $\inf_{u \in F(x)} P(x-u) \geq \delta$ for all $x \in X$. Now define $H : X \times E \to \mathbf{R}$ by $H(x,y) = P(y)$ for each $(x,y) \in X \times E$. Since $F(x) \cap \overline{I_X}(x) \neq \emptyset$ (respectively, $F(x) \cap \overline{O_X}(x) \neq \emptyset$), F and G satisfy the condition (iii) of Corollary 2.5.12. Clearly, F and H also satisfy the conditions (i) and (ii) of Corollary 2.5.12. Hence by Corollary 2.5.12, there exists $x \in X$ such that $x \in F(x)$ which is a contradiction. Thus F must have a fixed point in X. □

As another application of Theorem 2.5.7, we present another coincidence theorem in locally convex spaces.

Theorem 2.5.14. Let E be a topological vector space and W be a locally convex topological vector space. Let X be a non-empty compact convex subset of E and $G, F : X \to 2^W$ be such that

(i) G is continuous with closed convex values and the set $G^{-1}(y)$ is convex for each $y \in G(X)$, where $G(X)$ is convex;

(ii) F is continuous with compact convex values.

Then we have that:

(1) Either there exists a point $x_0 \in X$ such that $G(x_0) \cap F(x_0) \neq \emptyset$, or there exist a point $x_0 \in X$ and a continuous seminorm P on W such that for all $y \in G(X)$,

$$\inf_{u \in F(x_0)} P(y-u) \geq \inf_{v \in G(x_0)} [\inf_{u \in F(x_0)} P(v-u)] > 0.$$

(2) If $F(x) \cap G(X) \neq \emptyset$ for all $x \in X$, then there exists a point $x_0 \in X$ such that $G(x_0) \cap F(x_0) \neq \emptyset$.

Proof. Case (1). Suppose for each $x \in X$, $G(x) \cap F(x) = \emptyset$. Let $x \in X$ be arbitrarily fixed. Since $G(x)$ is closed and convex and $F(x)$ is compact and convex, by separate theorem, we have $\delta_x > 0$ and a continuous seminorm P_x on W such that $\inf_{v \in G(x)} \inf_{u \in F(x)} P_x(v-u) \geq \delta_x$ for all $x \in X$. By the continuity of G and F, the map $y \mapsto \inf_{v \in G(y)} \inf_{u \in F(y)} P_x(v-u)$ is continuous at x so that there exists an open neighborhood $N(x)$ of x in X such that for each $z \in N(x)$, we have $\inf_{v \in G(z)} \inf_{u \in F(z)} P_x(v-u) \geq \delta_x$. Since the family $\{N(x) : x \in X\}$ is an open covering of the compact set X, there exists a finite subset $\{x_1, \cdots, x_n\}$ of X such that $\{N_{x_i} : 1 \leq i \leq m\}$ covers X. Let $P = \max\{P_{x_i} : 1 \leq i \leq m\}$ and $\delta = \min\{\frac{\delta_{x_i}}{2} : 1 \leq i \leq m\} > 0$. Then P is a continuous seminorm on W and $\inf_{v \in G(x)} \inf_{u \in F(x)} p(v-u) \geq \delta$ for all $x \in X$. Now we define $H : X \times W \to \mathbf{R}$ by $H(x,y) = P(y)$ for each $(x,y) \in X \times W$, then G, F and H satisfy all hypotheses of Theorem 2.5.7. By Theorem 2.5.7, there exists $x_0 \in X$ such that

$$\inf_{v \in G(X)} [\inf_{u \in F(x_0)} P(v-u)] = \inf_{v \in G(x_0)} [\inf_{u \in F(x_0)} P(v-u)] \geq \delta > 0,$$

which implies that the conclusion of (1) holds.

Case (2). Now assume that $F(x) \cap G(X) \neq \emptyset$ for all $x \in X$. If $G(x) \cap F(x) = \emptyset$ for all $x \in X$, then by (1), there exist $x_0 \in X$ and a continuous seminorm P on W such that for all $y \in G(x)$, $\inf_{u \in F(x)} P(y-u) \geq \inf_{v \in G(x)} \inf_{u \in F(x_0)} P(v-u) > 0$. Take any $y_0 \in F(x_0) \cap G(X)$, we have

$$0 = [\inf_{u \in F(x_0)} P(y_0 - u)] \geq \inf_{v \in G(x_0)} [\inf_{u \in F(x_0)} P(v-u)] > 0$$

which is a contradiction. Therefore the conclusion (2) must hold. □

Theorem 2.5.14 improves Theorems 2 and 3 of Ha [139, p.15] to multivalued mappings which in turn improves Theorem 2 of Fan [103].

By the remark of Ha [139, p.14] and the proof of Theorem 2.5.14, it is easy to see that Theorem 2.5.14 is still true if we assume that the space W is a topological vector space (not necessary locally convex space) which has sufficiently many continuous linear functionals.

2.6 Stability of Coincident Points for Multivalued Mappings

In [163], Jiang introduced the concept of essential fixed points for multivalued mappings and proved a corresponding approximation theorem. The concept of essentiality for fixed points is a stability property. In [116] and [163], the stability of fixed points with respect to perturbations of mappings were studied.

In this section, the concept of essential coincident points of multivalued mappings is given first. We then study the stability of coincident points and fixed points of multivalued mappings with perturbations of mappings and of constrained sets. Some new approximation theorems are also established.

Let (X, d) be a metric space and $K(X)$ be the space of all non-empty compact subsets of X equipped with the Hausdorff metric h which is induced by the metric d. For any $\epsilon > 0$, $x_0 \in X$ and $A \in K(X)$, let $U(\epsilon, A) = \{x \in X : d(u, x) < \epsilon$ for some $u \in A\}$ and $O(x_0, \epsilon) = \{x \in X : d(x_0, y) < \epsilon\}$. Let Y be a topological space. Recall that a subset $Q \subset Y$ is called a residual set if it is a countable intersection of open dense subsets of Y The following Lemma 2.6.1 is due to Fort [116, Theorem 2].

Lemma 2.6.1. Let X be a metric space, Y a topological space and $F : Y \to K(X)$ an usco mapping. Then the set of points where F is lower semicontinuous is a residual set in Y.

Lemma 2.6.2. Let X be a metric space, Y be a complete metric space and $F : Y \to K(X)$ be an usco mapping. Then the set of points where F is lower semicontinuous is a dense residual set in Y.

Proof. Since Y is complete, a residual set in Y is dense; the result now follows from Lemma 2.6.1. □

Throughout the remainder of this section, X denotes a complete metric space and $C = \{f : X \to K(X) : f$ is upper semicontinuous on X and $f(X) = \cup_{x \in X} f(x)$ is bounded $\}$. For each f, $f' \in C$, let $\rho(f, f') = \sup_{x \in X} h(f(x), f'(x))$ Clearly, ρ is a metric on C.

Lemma 2.6.3. (C, ρ) is a complete space

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be each Cauchy sequence in C, then for any $\epsilon > 0$, there is a positive integer $N(\epsilon)$ such that

$$\rho(f_n, f_m) = \sup_{x \in X} h(f_n(x), f_m(x)) < \epsilon \tag{2.1}$$

for any $n, m \geq N(\epsilon)$ It follows that for each $x \in X$, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $K(X)$ Since $K(X)$ is complete, by Theorem 4 3 9 in [189], there is $f \; X \to K(X)$ such that $\lim_{m\to\infty} f_m(x) = f(x)$

For each $x \in X$ and each $\epsilon > 0$ and each $n \geq N(\epsilon)$, (2 1) implies that $f_n(x) \subset U(2\epsilon, f(x))$ and $f(x) \subset U(2\epsilon, f_n(x))$ Fix $n \geq N(\epsilon)$, since $f_n \in C$, there is $\delta > 0$ such that $f_n(x') \subset U(\epsilon, f_n(x))$ whenever $d(x, x') < \delta$ Thus,

$$f(x') \subset U(2\epsilon, f_n(x')) \subset U(3\epsilon, f_n(x)) \subset U(5\epsilon, f(x))$$

whenever $d(x, x') < \delta$ Therefore f is upper semicontinuous on X It is easy to show that $f(X) = \cup_{x \in X} f(x)$ is bounded so that $f \in C$ and that $f_n \to f$ Hence C is complete. □

Set $Y = C \times C \times K(X)$ For each $y = (f, g, A) \in Y$, $y' = (f', g', A') \in Y$, let

$$D(y, y') = \rho(f, f') + \rho(g, g') + h(A, A')$$

Clearly D is a metric on Y By Lemma 2 6 3, C is a complete metric space By Theorem 4 3 9 in [189], $K(X)$ is a complete metric space Hence Y is also a complete metric space

Define $M = \{y = (f, g, A) \in Y \;$ there is $x \in A$ such that $f(x) \cap g(x) \neq \emptyset\}$. Then we have

Lemma 2.6.4. (M, D) is a complete metric space

Proof. Since $M \subset Y$ and Y is complete, it is sufficient to prove that M is closed in Y Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in M and $y_n \to y \in Y$ Set $y_n = (f_n, g_n, A_n)$, $n = 1, 2, \cdots$ and $y = (f, g, A)$, then $f_n \to f$, $g_n \to g$ and $A_n \to A$ For each $n = 1, 2, \cdots$, since $y_n \in M$, there is $x_n \in A_n$ such that $f_n(x) \cap g_n(x) \neq \emptyset$. Since A_n

and A are compact and $A_n \to A$, by A.5.1 (ii) of Mas-Colell [216, p.10], $\cup_{n=1}^{\infty} A_n \cup A$ is compact. Since $x_n \in A_n \subset \cup_{n=1}^{\infty} A_n \cup A$, we may assume without loss of generality that $x_n \to x \in \cup_{n=1}^{\infty} A_n \cup A$. If $x \notin A$, since A is compact, there is $a > 0$ such that $U(a, A) \cap O(x, a) = \emptyset$. Since $A_n \to A$ and $x_n \to x$, there is N_1 such that $A_n \subset U(a, A)$ and $x_n \in O(x, a)$ for all $n \geq N_1$, which contradicts the assumption that $x_n \in A_n$. Hence we must have $x \in A$.

If $f(x) \cap g(x) = \emptyset$, since $f(x)$ and $g(x)$ are compact, there is $b > 0$ such that $U(b, f(x)) \cap U(b, g(x)) = \emptyset$. Since $f_n \to f$ and $g_n \to g$, there is N_2 such that $f_n(u) \subset U(b/2, f(u))$ and $g_n(u) \subset U(b/2, g(u))$ for all $n \geq N_2$ and for all $u \in X$. Since f and g are upper semicontinuous at x and $x_n \to x$, there is $N_3 \geq N_2$ such that $f(x_n) \subset U(b/2, f(x))$ and $g(x_n) \subset U(b/2, g(x))$ for all $n \geq N_3$. Thus for all $n \geq N_3$, $f_n(x_n) \subset U(b/2, f(x_n)) \subset U(b, f(x))$ and $g_n(x_n) \subset U(b/2, g(x_n)) \subset U(b, g(x))$ which contradicts the assumption that $f_n(x_n) \cap g_n(x_n) \neq \emptyset$. Hence we must also have $f(x) \cap g(x) \neq \emptyset$. Therefore $y = (f, g, A) \in M$ so that M is closed in Y. □

For each $y = (f, g, A) \in M$, let $S(y) = \{x \in A : f(x) \cap g(x) \neq \emptyset\}$; note that $S(y) \neq \emptyset$.

Lemma 2.6.5. $S(y) \in K(X)$ for each $y \in M$.

Proof. Let $y = (f, g, A) \in M$ be given. Since $S(y) \subset A$ and A is compact, it is sufficient to prove that $S(y)$ is closed in A. Indeed, let $\{x_n\}_{n=1}^{\infty}$ be any sequence in $S(y)$ such that $x_n \to x \in A$; then $f(x_n) \cap g(x_n) \neq \emptyset$ for each $n = 1, 2, \cdots$. If $x \notin S(y)$, then $f(x) \cap g(x) = \emptyset$. Since $f(x)$ and $g(x)$ are compact, there is $\epsilon_0 > 0$ such that $U(\epsilon_0, f(x)) \cap U(\epsilon_0, g(x)) = \emptyset$. Since f, g are upper semicontinuous at x and $x_n \to x$, there exists N such that $f(x_n) \subset U(\epsilon_0, f(x))$ and $g(x_n) \subset U(\epsilon_0, g(x))$ for all $n \geq N$. It follows that $f(x_n) \cap g(x_n) = \emptyset$ for all $n \geq N$, which is a contradiction. Therefore $x \in S(y)$ and hence $S(y)$ is closed in A. □.

By Lemma 2.6.5 the mapping $y \mapsto S(y)$ defines a map $S : M \to K(X)$.

Lemma 2.6.6. S is upper semicontinuous on M.

Proof. Suppose S is not upper semicontinuous at $y \in M$, then there exist $\epsilon_0 > 0$

and a sequence $\{y_n\}_{n=1}^{\infty}$ in M with $y_n \to y$ such that for each $n = 1, 2, \cdots$, there exists $x_n \in S(y_n)$ with $x_n \notin U(\epsilon_0, S(y))$. Let $y_n = (f_n, g_n, A_n)$ and $y = (f, g, A)$, then $f_n \to f$, $g_n \to g$ and $A_n \to A$. Since $x_n \in A_n \subset \cup_{n=1}^{\infty} A_n \cup A$ and $\cup_{n=1}^{\infty} A_n \cup A$ is compact, we may assume without loss of generality that $x_n \to x \in \cup_{n=1}^{\infty} A_n \cup A$. Note that we must have $x \notin U(\epsilon_0, S(y))$. Now the same argument as in the proof of Lemma 2.6.4 shows that $x \in A$ and $f(x) \cap g(x) \neq \emptyset$ so that $x \in S(y)$; this contradicts that $x \notin U(\epsilon_0, S(y))$. Therefore S must be upper semicontinuous. □

Let M_1 be a non-empty closed subset of M. Since M is complete, M_1 is also complete.

Definition. If $y_1 \in M_1$, then a point x in $S(y)$ is called an essential coincident point of y_1 with respect to M_1 provided that for any $\epsilon > 0$, there is $\delta > 0$ such that for any $y' \in M_1$ with $D(y_1, y') < \delta$, there exists $x' \in S(y')$ with $d(x, x') < \epsilon$. y is called essential with respect to M_1 if every $x \in S(y)$ is an essential coincident point of y with respect to M_1.

Theorem 2.6.7. S is lower semicontinuous at $y \in M_1$ if and only if y is essential with respect to M_1.

Proof. Suppose S is lower semicontinuous at $y \in M_1$. Then for any $\epsilon > 0$. there is $\delta > 0$ such that for any $y' \in M_1$ with $d(y, y') < \delta$, we have $S(y) \subset U(\epsilon, S(y'))$ so that for any $x \in S(y)$, there is $x' \in S(y')$ with $d(x, x') < \epsilon$. Thus every $x \in S(y)$ is an essential coincident point of y with respect to M_1 and hence y is essential with respect to M_1.

Conversely, suppose that y is essential with respect to M_1. If S were not lower semicontinuous at $y \in M_1$, then there exist $\epsilon_0 > 0$ and a sequence $\{y_n\}_{n=1}^{\infty}$ in M with $y_n \to y$ such that for each $n = 1, 2, \cdots$, there is $x_n \in S(y)$ with $x_n \notin U(\epsilon_0, S(y_n))$. Since $S(y)$ is compact, we may assume that $x_n \to x \in S(y)$. Since x is an essential coincident point of y with respect to M_1, $y_n \to y$ and $x_n \to x$, there is N such that $d(x_n, x) < \epsilon_0/2$ and $x \in U(\epsilon_0/2, S(y_n))$ for all $n \geq N$. Hence $x_n \in O(x, \epsilon_0/2) \subset U(\epsilon_0, S(y_n))$ for all $n \geq N$ which contradicts the assumption that $x_n \notin U(\epsilon_0, S(y_n))$ for all $n = 1, 2, \cdots$. Hence S must be lower semicontinuous at y. □

We shall prove the following approximation theorem:

Theorem 2.6.8. The set of essential points with respect to M_1 is a dense residual set in M_1. In particular, every point in M_1 can be arbitrarily approximated by an essential points in M_1.

Proof. By Lemma 2.6.5 and Lemma 2.6.6, $S : M \to K(X)$ is an usco mapping. Since M_1 is complete, by Lemma 2.6.2, the set of points where S is lower semicontinuous is a dense residual set in M_1. By Theorem 2.6.7, the set of essential points in M_1 is a dense residual set in M_1. □

By Lemma 2.6.6, Theorems 2.6.7 and 2.6.8, we have the following:

Theorem 2.6.9. S is continuous at $y \in M_1$ if and only if y is essential with respect to M_1. Moreover, the set of points at which S is continuous is a dense residual set in M_1.

We remark that S is continuous at $y \in M_1$, if and only if for each $\epsilon > 0$, there is $\delta > 0$ such that $h(S(y), S(y')) < \epsilon$ for each $y' \in M$ with $D(y, y') < \delta$. Theorem 2.6.9 implies that if $y = (f, g, A) \in M_1$, then y is essential with respect to M_1 if and only if its set $S(y)$ of coincident points is stable: $S(y')$ is close to $S(y)$ whenever y' is close to y.

We shall now give a sufficient condition that $y \in M_1$ is essential with respect to M_1:

Theorem 2.6.10. If $y \in M_1$ is such that $S(y)$ is a singleton set, then y is essential with respect to M_1.

Proof. Suppose $S(y) = \{x\}$. By Lemma 2.6.6, S is upper semicontinuous at y. Thus for any $\epsilon > 0$, there is $\delta > 0$ such that for each $y' \in M_1$, $D(y, y') < \delta$ implies $S(y') \subset U(\epsilon, S(y)) = O(x, \epsilon)$ so that $S(y) = \{x\} \subset U(\epsilon, S(y'))$. This shows that S is also lower semicontinuous at y. By Theorem 2.6.7, y is essential with respect to M_1. □

We note that if the given metric d on X is bounded, then the identity mapping $I : X \to X$ belongs to C.

Remark 2.6.11. If (X, d) is compact and if we take $M_1 = (\{I\} \times C \times \{X\}) \cap M = \{(I, f, X) \in Y$: there exists $x \in X$ such that $x \in f(x)\}$, then Theorem 2.6.9 reduces to Theorem 1 in [4] which is a multivalued generalization of Theorem 1 in [116]. In this case, if (I, f, X) is essential with respect to M_1, then every fixed point x of f in X is an essential fixed point of f (see [163]); i.e., for each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $g : X \to K(X)$ is upper semicontinuous which has a fixed point in X and $\rho(f, g) < \delta$, there is a fixed point x' of g in X with $d(x, x') < \epsilon$. Hence our Theorem 3 generalizes Theorem 1 in [116] and Theorem 1 in [163] in several respects.

Recall that a function $g : X \to X$ is a contraction if there is a constant $k \in (0,1)$ such that $d(g(x), g(y)) \leq kd(x, y)$ for all $x, y \in X$. If g is a contraction on X, then the classical Banach contraction mapping principle implies that g has a unique fixed point. This fact together with Theorem 2.6.10, yield:

Corollary 2.6.12. Suppose (X, d) is compact and $M_1 = \{(I, f, X) \in Y :$ there exists $x \in X$ such that $x \in f(x)\}$. If $g : X \to X$ is a contraction, then (I, g, X) is essential with respect to M_1.

In what follows, let X be a non-empty closed convex subset of a Banach space E and let A be a non-empty compact subset of X. Let $M_2 = \{(I, f, A) \in Y : f(x)$ is a compact convex subset of A for each $x \in X\}$.

Theorem 2.6.13. The set of points $y \in M_2$ which is essential with respect to M_2 is a dense residual set in M_2.

Proof. For each $(I, f, A) \in M_2$, $f : X \to K(X)$ is upper semicontinuous and $f(x)$ is a compact convex subset of A for each $x \in X$; it follows from the Schauder-Tychonoff fixed point theorem (e.g, see Smart [280, Theorem 9.2.3]) that there exists $x \in A$ such that $x \in f(x)$. This shows that $(I, f, A) \in M$. Thus by Theorem 2.6.8, we only need to show that M_2 is closed in M. Indeed, let $\{(I, f_n, A)\}_{n=1}^{\infty}$ be a sequence in M_2 such that $(I, f_n, A) \to (I, f, A) \in M$, then $\rho(f_n, f) = \sup_{x \in X} h(f_n(x), f(x)) \to 0$ as $n \to \infty$. Since for each $x \in X$, $f_n(x) \subset A$ for each $n = 1, 2, \cdots$, it follows that $f(x) = \cap_{n=1}^{\infty}(\overline{\cup_{k \geq n} f_k(x)}) \subset A$ (e.g., see [189, Theorem 4.3.5, p.43]). To complete the proof it remains to show that $f(x)$ is convex for all $x \in X$. If this were false, there exist $x \in X$, $u_1, u_2 \in f(x)$ and $\lambda \in (0, 1)$ such that $\lambda u_1 + (1 - \lambda) u_2 \notin f(x)$.

Since $f(x)$ is compact, there is $\epsilon_0 > 0$ with $O(\lambda u_1 + (1-\lambda)u_2, \epsilon_0) \cap U(\epsilon_0, f(x)) = \emptyset$. Since $\rho(f_n, f) \to 0$ as $n \to \infty$, there is N with $f(x) \subset U(\epsilon_0, f_N(x))$ and $f_N(x) \subset U(\epsilon_0, f(x))$. It follows that there exist $z_1, z_2 \in f_N(x)$ with $d(u_i, z_i) < \epsilon_0$ for $i = 1, 2$. Thus $d(\lambda u_1 + (1-\lambda)u_2, \lambda z_1 + (1-\lambda)z_2) \leq \lambda d(u_1, z_1) + (1-\lambda)d(u_2, z_2) < \epsilon_0$ so that $\lambda z_1 + (1-\lambda)z_2 \in O(\lambda u_1 + (1-\lambda)u_2, \epsilon_0)$. On the other hand, as $f_N(x)$ is convex, $\lambda z_1 + (1-\lambda)z_2 \in f_N(x) \subset U(\epsilon_0, f(x))$ which contradicts $O(\lambda u_1 + (1-\lambda)u_2), \epsilon_0) \cap U(\epsilon_0, f(x)) \neq \emptyset$. Hence $f(x)$ is convex for each $x \in X$. Therefore M_2 is closed in M. $\square$.

2.7 Matching Theorems and Applications

By applying Theorem 1 of Park and Bae [244], (which is a generalization of Fan's existence theorem for maximizable quasi-concave functions on convex spaces), we first prove some coincidence theorems for upper hemicontinuous non-self mappings in topological vector spaces with sufficiently many continuous linear functionals or in locally convex topological vector spaces. These results improve and unify many results in the literature (e.g, see Fan [106], Park [240], Ko and Tan [192] and references wherein). Next, as applications of coincidence theorems, several matching theorems for closed coverings of convex sets are derived in locally convex topological vector spaces or topological vector spaces with sufficiently many continuous linear functionals which in turn imply Shapley's theorem [264].

2.7.1 *Generalizations of the Fan-Glicksberg Fixed Point Theorem*

The basic idea in this section is to apply the existence theorem for maximizable quasi-concave functions on topological vector space with sufficiently many continuous linear functionals. Several fixed point theorems for non-self mappings are given under weaker continuity and boundary hypotheses. For example, our fixed point theorems show that the hypotheses "*the domain is paracomapct*" is superfluous for the existence of non-self upper hemicontinuous multivalued mappings, in fact, this *superfluous condition* is posed in much of the literature (e.g, see Fan [106], Lassonde [199], [106], Ko and Tan [192], Browder [45]). In particular, the well-known Fan-Glicksberg fixed point theorem has been generalized into the non-compact setting in which the underlying space is a topological vector space with sufficiently many continuous linear functionals under weaker continuity and boundary hypotheses.

We recall that a convex space (e.g., see Lassonde [199, p.153]) is a non-empty convex set in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A non-empty subset L of a convex space X is called c-compact if for each $S \in \mathcal{F}(X)$, there exists a non-empty compact set L_S with $L \cup S \subset L_S$. It is clear that a convex subset (respectively, compact and convex subset) of a topological vector space is a convex space (respectively, c-compact subset). For more details about convex spaces, we refer to Lassonde [199, p.153] and Dugundji [89, p.416].

By the Ky Fan minimax inequality, Park and Bae [244] gave a generalization of the existence theorem for maximizable quasi-concave functions on convex spaces which in turn answered the question raised by Bellenger [19]. In this section, we first recall the following result which is essentially a consequence of the existence theorem for maximizable quasi-concave functions on convex spaces (e.g, see Theorem 1 of Park and Bae [244]):

Theorem 2.7.A. Let X be a convex space and suppose that

(a) for each $x \in X$, $T(x)$ is a non-empty convex set of upper semicontinuous quasi-concave real functions on X;

(b) for each upper semicontinuous and quasi-concave real function f on X, the set $T^{-1}(f)$ is compactly open in X;

(c) there exists a c-compact subset L of X and a non-empty compact subset K of X such that for each $z \in X \setminus K$ and $f \in T(z)$, $f(z) < \sup\{f(x) : x \in co(L \cup \{z\})\}$ Then there exists $\hat{z} \in K$ and $\hat{f} \in T(\hat{z})$ such that $\hat{f}(\hat{z}) = \sup\{\hat{f}(x) : x \in X\}$.

We note that Theorem 2.7.A above generalizes Theorem 8 of Fan [106] and Theorem 1 of Bellenger [19] which in turn improves Theorem 0.1 of Simons [276] for the existence theorem for an upper semicontinuous quasi-concave real function which attains a global maximum on a given compact subset of a convex space X. Recently, Ding [80] generalized Theorem 2.7.A to H-spaces by following Park and Bae's idea in [244].

Let X be a convex subset of a topological vector space E, We now state an equivalent form of Theorem 2.7.A as follows:

Theorem 2.7.1. Let X be a convex space and Φ a non-empty convex set of lower semicontinuous convex real functions on X. Let S be a subset of $X \times \Phi$ such that

(a) for each $\phi \in \Phi$, the section $\{x \in X : (x, \phi) \in \Phi\}$ is compactly open in X and

(b) for each $x \in X$, the section $\{\phi \in \Phi : (x, \phi) \in S\}$ is non-empty and convex.

Then either

(I) there exists $(y, \phi) \in S$ such that $\phi(y) = \inf_{x \in X} \phi(x)$; or

(II) for each compact convex set L in X and each non-empty compact set K in X, there exists $(y, \phi) \in S$ such that $y \in X \setminus K$ and $\phi(y) = \inf_{x \in co(L \cup \{y\})} \phi(x)$.

Before we give generalizations of the Fan-Glicksberg fixed point theorem in topological vector spaces with sufficiently many continuous linear functionals or in locally convex topological vector spaces under weaker continuity assumptions, we first recall some facts about various continuity for multivalued mappings.

If E is a topological vector space, E^* is the dual space of all continuous linear functionals on E, the pairing between E^* and E is denoted by $\langle w, x \rangle$ for each $w \in E^*$ and $x \in E$. Suppose X is a non-empty subset of E. A mapping $f : X \to 2^E$ is said to be upper hemicontinuous [7] if for each $\phi \in E^*$ and for each $\lambda \in \mathbf{R}$, the set $\{x \in X : \sup_{u \in f(x)} Re\langle \phi, u \rangle < \lambda\}$ is open in X. We note that each upper semicontinuous mapping is upper hemicontinuous and the sum of two upper hemicontinuous mappings is again upper hemi-continuous. According to Fan [103], a mapping $f : X \to 2^E$ is upper demicontinuous on X if for each $x \in X$ and any open half-space H containing $f(x)$, there exists an open neighborhood N_x of x in X such that $f(u) \subset H$ for all $u \in N_x$. Recall that an open half-space H in E is a set of the form $H := \{v \in E : Re\phi(v) < t\}$ for some non-zero $\phi \in E^*$ and some real number t. It is obvious that every upper semicontinuous mapping is upper demi-continuous, each upper demicontinuous mapping is upper hemicontinuous and the following examples from Shih and Tan [271] show that the converses do not hold in general.

Example 2.7.B. Let $E = \mathbf{R}^2$ and $X = \{t \in \mathbf{R} : t \in [0, 1]\}$. Define $f, g : X \to 2^{\mathbf{R}^2}$ by

$$f(t) = \{(u, v)\} : (u-1)(v-1) \geq 1 \text{ and } u > 1\},$$

$$g(t) = \{(-z, z) \in \mathbf{R}^2 : 0 \leq z \leq t\sqrt{2}\}$$

for each $t \in X$. Then it is not hard to verify that f and g are both upper semi-continuous but $f + g$ is not upper demi-continuous.

Incidently, Example 2.7.B also shows that an upper hemi-continuous mapping need not to be upper demi-continuous, since $f+g$ is necessarily upper hem-continuous.

We also observe that an upper demi-continuous may need not be upper semi-continuous as the following example shows:

Example 2.7.C. Define $f : \mathbf{R^2} \to \mathbf{R^2}$ by

$$f(x,y) = \{(u,v) \in \mathbf{R}^2 : u \geq x \text{ and } v \geq y\}$$

for each $(x,y) \in \mathbf{R^2}$. Then f is upper demi-continuous, but not uper semi-continuous.

We also note that if a mapping is a compact-valued mapping, then the concepts of upper hemi-continuity and upper demicontinuity coincide. The following fact due to Shih and Tan [271, Proposition 2] shows that under certain conditions, the concepts of upper semi-continuity, upper demi-continuity and upper hem-continuity are the same.

Theorem 2.7.D. Let X be a topological space, Z a non-empty compact subset of a real locally convex topological space E, and let $F : X \to 2^Z$ be such that each $F(x)$ is convex. Then the following statements are equivalent:

(1) F is upper semi-continuous.

(2) F is upper demi-continuous.

(3) F is upper hemi-continuous.

By Theorem 2.7.1, we have the following:

Theorem 1.7.2. Let E be a topological vector space which has sufficiently many continuous linear functionals, let X be a non-empty convex subset of E, X_0 a non-empty compact convex subset of X and K a non-empty compact subset of X. Let $F, G : X \to 2^E$ be upper hemicontinuous and such that

(a) for each $x \in X$, $F(x)$ and $G(x)$ are closed convex at least, one of which is compact;

(b) for each $x \in K \cap \delta_E(X)$ and $\phi \in E^*$ with $Re\phi(x) \leq Re\phi(y)$ for all $y \in X$, there exist $u \in F(x)$ and $v \in G(x)$ such that $Re\phi(u) \geq Re\phi(v)$;

and either

(c_1) suppose $\overline{co}K$ is compact (which is automatically satisfied if E is a complete locally convex topological vector space) for each $x \in X \setminus K$ and $\phi \in E^*$ such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in co(K \cup \{x\})$ (in particular, $Re\phi(x) \leq Re\phi(y)$ for all $y \in co(\overline{co}K \cup \{x\})$), there exist $u \in F(x)$ and $v \in G(x)$ with $Re\phi(u) \geq Re\phi(v)$;

or,

(c_2) for each $x \in X \setminus K$ and $\phi \in E^*$ such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in co(X_0 \cup \{x\})$, there exist $u \in F(x)$ and $v \in G(x)$ with $Re\phi(u) \geq Re\phi(v)$.

Then there exists a point $\hat{x} \in X$ such that for each $\phi \in E^*$ and each $t \in \mathbf{R}$, the following does not hold:

$$Re\phi(u) < t < Re\phi(v) \text{ for all } u \in F(\hat{x}) \text{ and } v \in G(\hat{x}).$$

Moreover,

(I) If $F(\hat{x})$ and $G(\hat{x})$ are both compact, then $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$.

(II) If E is a locally convex topological vector space, then $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$.

Proof. Note that by Proposition 2.5.4, it is clear that the condition (b) of Theorem 2.7.2 is equivalent to the following condition:

$(b)'$: for any $x \in K$ and $\phi \in E^*$ such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in X$, there exist $u \in F(x)$ and $v \in G(x)$ with $Re\phi(u) \geq Re\phi(v)$.

Now we follow the idea of Fan [106] and Ko and Tan [192] to prove our assertion. In order to apply Theorem 2.7.1, we take $\Phi = E^*$ and define the subset S of $X \times E^*$ as follows: $(x, \phi) \in S$ if and only if there exists a real number $t \in \mathbf{R}$ such that $Re\phi(u) < t < Re\phi(v)$ for all $u \in F(x)$ and all $v \in G(x)$.

(1): We first show that for each $\phi \in E^*$, the section $S(\phi) = \{x \in X : (x, \phi) \in S\}$ is open in X.

Indeed, let $x \in S(\phi)$, then $(x, \phi) \in S$ and hence there exists a real number t such that $Re\phi(u) < t < Re\phi(v)$ for all $u \in F(x)$ and all $v \in G(x)$.

First we assume that $F(x)$ is compact. Then there exists an $u_0 \in F(x)$ such that $Re\phi(u_0) = \sup_{u \in F(x)} Re\langle \phi, u \rangle$. Let ϵ_1, $\epsilon_2 > 0$ be such that $Re\phi(u_0) + \epsilon_1 + \epsilon_2 < t$. By upper hemicontinuity of F, there exists an open neighborhood N_1 of x in X such that for each $y \in N_1$,

$$\sup_{v\in F(y)} Re\langle\phi, v\rangle < \sup_{u\in F(x)} Re\langle\phi, u\rangle + \epsilon_1$$

so that $Re\phi(u) < Re\phi(u_0) + \epsilon_1$ for all $u \in F(y)$. Also, since $\sup_{v\in G(x)} Re\langle -\phi, v\rangle = -\inf_{v\in G(x)} Re\langle\phi, v\rangle \le -t < +\infty$, by upper hemicontinuity of G, there exists an open neighborhood N_2 of x in X such that for each $y \in N_2$,

$$\sup_{v\in G(y)} Re\langle -\phi, v\rangle < \sup_{v\in G(x)} Re\langle -\phi, v\rangle + \epsilon_2,$$

so that $Re\phi(v) > t - \epsilon_2$ for all $v \in G(y)$. Let $N = N_1 \cap N_2$. Then N is an open neighborhood of x in X such that for each $y \in N$, for each $u \in F(y)$ and for each $v \in G(y)$,

$$Re\phi(u) < Re\phi(u_0) + \epsilon_1 < t - \epsilon_2 < Re\phi(v).$$

Therefore $N \subset S(\phi)$. Similarly if $G(x)$ is compact, we see that (by replacing ϕ by $-\phi$ and by interchanging F and G in the above argument) there exists an open neighborhood N' of x in X such that $N' \subset S(\phi)$. Therefore $S(\phi)$ is open in X. Thus the condition (b) of Theorem 2.7.1 is satisfied.

(2): For each $x \in X$, it is clear that the set $S(x) = \{\phi \in \Phi : (x, \phi) \in S\}$ is convex.

(3): Next we show that for each $z \in X \setminus K$ and $(z, \phi) \in S$, $Re\phi(z) > \inf\{Re\phi(x) : x \in co(\overline{co}K \cup \{z\})\}$ (respectively, $Re\phi(z) > \inf\{Re\phi(x) : x \in co(X_0 \cup \{z\})\}$).

Note that if there exist $x \in X \setminus K$ and $\phi \in E^*$ such that $Re\phi(x) \le Re\phi(y)$ for all $y \in co(K \cup \{x\})$, then it is clear that $Re\phi(x) \le Re\phi(y)$ for all $y \in co(\overline{co}K \cup \{x\})$ by the linearity of real part of the linear continuous functional ϕ. Suppose the contrary that for some $z \in X \setminus K$ and $(z, \phi) \in S$ such that $Re\phi(z) = \inf\{Re\phi(x) : x \in co(\overline{co}K \cup \{z\})\}$ (respectively, $Re\phi(z) = \inf\{Re\phi(x) : x \in co(X_0 \cup \{z\})\}$). Then by (c_1) (respectively, (c_2)), there exist $u \in F(z)$ and $v \in G(z)$ such that $Re\phi(u) \ge Re\phi(v)$. This contradicts the assumption that $(z, \phi) \in S$. Thus the alternative (II) of Theorem 2.7.1 is false.

(4): Similarly, by $(b)'$ (which is equivalent to (b)), for each $z \in K$ and each $(z, \phi) \in S$, $Re\phi(z) > \inf_{x\in X} Re\phi(x)$. It follows from (3) that for each $z \in X$ and $(z, \phi) \in S$, $Re\phi(z) > \inf_{x\in X} Re\phi(x)$. Hence the alternative (I) of Theorem 2.7.1 is also not satisfied.

By Theorem 2.7.1, there must exist $\hat{x} \in X$ such that $\{\phi \in E^* : (\hat{x}, \phi) \in S\} = \emptyset$; i.e., for each $t \in \mathbf{R}$ and each $\phi \in E^*$, the following does not hold:

$(*)$ $$Re\phi(u) < t < Re\phi(v) \text{ for all } u \in F(\hat{x}) \text{ and } v \in G(\hat{x}).$$

Case (i). Suppose conclusion (I) were false, then $F(\hat{x}) \cap G(\hat{x}) = \emptyset$, so that $0 \notin F(\hat{x}) - G(\hat{x})$. Since both $F(\hat{x})$ and $G(\hat{x})$ are compact and convex, the set $D := F(\hat{x}) - G(\hat{x})$ is compact convex. Then for each $a \in D$, $a \neq 0$, as E^* separates points of E, there exists $\phi_a \in E^*$ such that $Re\phi_a(a) < 0$. Let O_a and U_a be disjoint open convex sets containing $Re\phi_a(a)$ and 0 respectively. Then $Re\phi_a^{-1}(O_a)$ and $Re\phi_a^{-1}(U_a)$ are disjoint open convex sets in E containing a and 0 respectively. Since D is compact, there exist $a_1, \cdots, a_n \in D$ such that $D \subset \cup_{i=1}^n Re\phi_{a_i}^{-1}(O_{a_i})$. Let $U = \cap_{i=1}^n Re\phi_{a_i}^{-1}(U_{a_i})$, then U is an open convex set containing 0 such that $U \cap D = \emptyset$. By Theorem 3.4 of Rudin [256, p.58], there exists $\phi \in E^*$ and $r \in \mathbf{R}$ such that $Re\phi(a) \leq r < 0$ for all $a \in D$, i.e., $Re\phi(u) \leq r + Re\phi(v)$ for all $u \in F(\hat{x})$ and $v \in G(\hat{x})$. Let $t := r/2 + \inf_{w \in G(\hat{x})} Re\phi(w)$. Since $r < 0$, it follows that

$$\begin{aligned} Re\phi(u) &\leq r + \inf_{w \in G(\hat{x})} Re\phi(w) \\ &< r/2 + \inf_{w \in G(\hat{x})} Re\phi(w) = t < Re\phi(v) \end{aligned}$$

which contradicts $(*)$.

Case (ii). If E is a locally convex topological vector space, since $F(\hat{x})$ and $G(\hat{x})$ can not be strictly separated by a closed hyperplane in E and at least one of $F(\hat{x})$ and $G(\hat{x})$ is compact, we must have $F(\hat{x}) \cap F(\hat{x}) \neq \emptyset$ by Theorem 3.4 of Rudin [256]. □

From the proof (due to part (3)) of Theorem 2.7.2, the assumption "$\overline{co}K$ is compact" in the assumption (c_1) is not needed if the underlying space E is a locally convex topological vector space, i.e., we have the following:

Theorem 2.7.2$'$. Let E be a locally convex topological vector space, let X be a non-empty convex subset of E, X_0 a non-empty compact convex subset of X and K a non-empty compact subset of X. Let $F, G : X \to 2^E$ be upper hemicontinuous and such that

(a) for each $x \in X$, $F(x)$ and $G(x)$ are closed convex at least one of which is compact;

(b) for each $x \in K \cap \delta_E(X)$ and $\phi \in E^*$ with $Re\phi(x) \leq Re\phi(y)$ for all $y \in X$, there exist $u \in F(x)$ and $v \in G(x)$ such that $Re\phi(u) \geq Re\phi(v)$;

and either

(c_1) for each $x \in X \setminus K$ and $\phi \in E^*$ such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in co(K \cup \{x\})$ (in particular, Re $\phi(x) \leq Re\phi(y)$ for all $y \in co(\overline{co}K \cup \{x\})$), there exist $u \in F(x)$ and $v \in G(x)$ with $Re\phi(u) \geq Re\phi(v)$;

or,

(c_2) for each $x \in X \setminus K$ and $\phi \in E^*$ such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in co(X_0 \cup \{x\})$, there exist $u \in F(x)$ and $v \in G(x)$ with $Re\phi(u) \geq Re\phi(v)$.

Then $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$.

Proof. We first note that the assumptions of X, X_0, K, F and G remain unchanged in the completion $\overline{E}$ of E. Without loss of generality, we may assume that E is a complete locally convex topological vector space. Since $\overline{E}^* = E^*$, it follows that the conclusions of (1) and (2) still hold. By the completeness of E, the set $\overline{co}K$ is non-empty compact and convex since K is non-empty compact. Now, following the proof of Theorem 2.7.2, there must exist $\hat{x} \in X$ such that $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$.

Theorem 2.7.3. Let X be a non-empty convex subset of a locally convex topological vector space E. Let X_0 be a non-empty compact convex subset of X and K be a non-empty compact subset of X. Let $F : X \to 2^E$ be an upper hemicontinuous mapping with closed and convex values such that:

(a) for each $x \in K \cap \delta_E(X)$, $F(x) \cap X \neq \emptyset$;

and either

(b) for each $x \in X \setminus K$, $F(x) \cap co(\overline{co}K \cup \{x\}) \neq \emptyset$

or,

(b)$'$ for each $x \in X \setminus K$, $F(x) \cap co(X_0 \cup \{x\}) \neq \emptyset$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$.

Proof. Let $G = I_X$ be the identity mapping on X. Since $F(x) \cap X \neq \emptyset$ for all $x \in K \cap \delta(X)$, the condition (b) of Theorem 2.7.2$'$ is satisfied. Also condition (b) (respectively, $(b)'$) implies that condition (c_1) (respectively, (c_2)) of Theorem 2.7.2$'$ is satisfied. Hence the conclusion follows from Theorem 2.7.2′. □

By Theorem 2.7.2, we have:

Theorem 2.7.3′. Let E be a topological vector space which has sufficiently many continuous linear functionals, let X be a non-empty convex subset of E and X_0 be a non-empty compact convex subset of X and K a non-empty compact subset of X. Let $F : X \to 2^E$ be upper hemicontinuous with compact convex values such that

(a) for each $x \in K \cap \delta_E(X)$, $F(x) \cap X \neq \emptyset$;

(b) for each $x \in X \setminus K$, $F(x) \cap co(X_0 \cup \{x\}) \neq \emptyset$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$.

We note that Theorem 2.7.3 (respectively, Theorem 2.7.3′) improves the well-known Fan-Glicksberg fixed point theorem in the following ways: (a) the domain X need not be compact; (b) the mapping F is upper hemi-continuous instead of upper semicontinuous; (c) the mapping F need not have compact values (respectively, the space E need not be locally convex) and (d) the mapping F need not be a self-map.

By Theorem 2.7.3, we have the following:

Corollary 2.7.4. Let X be a non-empty compact convex subset of a locally convex topological vector space E. Let $F : X \to 2^E$ be upper hemi-continuous with closed and convex values such that for each $x \in \delta_E(X)$, $F(x) \cap X \neq \emptyset$.
Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$.

Proof. The conclusion follows from Theorem 2.7.3 by taking $X = K$. □

Corresponding to Theorem 2.7.3′, we also have:

Corollary 2.7.4′. Let E be a topological vector space which has sufficiently many continuous linear functionals and X be a non-empty compact convex subset of E. Let $F : X \to 2^E$ be upper hemicontinuous with compact and convex values such that for each $x \in \delta_E(X)$, $F(x) \cap X \neq \emptyset$.
Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$.

Theorem 2.7.5. Let X be a non-empty convex set in a locally convex topological vector space E. Let X_0 be a non-empty compact convex subset of X and K be a non-empty compact subset of X. Let $F, G : X \to 2^E$ be upper hemicontinuous and such that

(a) for each $x \in X$, $F(x)$ and $G(x)$ are closed convex at least one of which is compact;

(b) for each $x \in K \cap \delta_E(X)$, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \geq 0} \lambda(X - x)} \neq \emptyset$ (respectively, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \leq 0} \lambda(X - x)} \neq \emptyset$);

and either

(c) for each $x \in X \setminus K$, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \geq 0} \lambda(\overline{co}K - x)} \neq \emptyset$ (respectively, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \leq 0} \lambda(\overline{co}K - x)} \neq \emptyset$).

or,

(c)$'$ for each $x \in X \setminus K$, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \geq 0} \lambda(X_0 - x)} \neq \emptyset$ (respectively, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \leq 0} \lambda(X_0 - x)} \neq \emptyset$).

Then there exists $\hat{x} \in X$ such that $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$.

Proof. Let $x \in K \cap \delta(X)$ and $\phi \in E^*$ be such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in X$. Since $(F(x) - G(x)) \cap \overline{\cup_{\lambda \geq 0} \lambda(X - x)} \neq \emptyset$, let $u \in F(x)$, $v \in G(x)$, $\{\lambda_\alpha\}_{\alpha \in \Gamma}$ be a net in $[0, \infty)$ and $\{x_\alpha\}_{\alpha \in \Gamma}$ be a net in X such that $\lambda_\alpha(x_\alpha - x) \to u - v$. It follows that

$$\lambda_\alpha Re\phi(x_\alpha - x) = Re\phi(\lambda_\alpha(x_\alpha - x)) \to Re\phi(u - v) = Re\phi(u) - Re\phi(v).$$

Since $\lambda_\alpha Re\phi(x_\alpha - x) \geq 0$ for each $\alpha \in \Gamma$, $Re\phi(u) \geq Re\phi(v)$. Thus the condition (b) of Theorem 2.7.2 is satisfied.

Next let $x \in X \setminus K$ and $\phi \in E^*$ be such that $Re\phi(x) \leq Re\phi(y)$ for all $y \in co(\overline{co}K \cup \{x\})$ (respectively, $y \in co(X_0 \cup \{x\})$). By (c), since $(F(x) - G(x)) \cap \overline{\cup_{\lambda \geq 0} \lambda(K - x)} \neq \emptyset$ (respectively, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \geq 0} \lambda(X_0 - x)} \neq \emptyset$), let $u \in F(x)$, $v \in G(x)$, $\{\lambda_\alpha\}_{\alpha \in \Gamma}$ be a net in $[0, \infty)$ and $\{x_\alpha\}_{\alpha \in \Gamma}$ be a net in $\overline{co}K$ (respectively, in $co(X_0 \cup \{x\})$) such that $\lambda_\alpha(x_\alpha - x) \to u - v$. It follows from $Re\phi(x) \leq Re\phi(x_\alpha)$ for all $\alpha \in \Gamma$ that $Re\phi(u) \geq Re\phi(v)$. Thus the condition (c_1) (respectively, (c_2)) in Theorem 2.7.2$'$ is satisfied. Therefore by Theorem 2.7.2$'$, there exists $\hat{x} \in X$ such that $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$.

By interchanging the roles of F and G and by replacing the union "$\cup_{\lambda \geq 0}$" in both conditions (b) and (c) by "$\cup_{\lambda \leq 0}$", the proof is complete. □

Corresponding to Theorem 2.7.5, we also have:

Theorem 2.7.5′. Let X be a non-empty convex set in a topological vector space E which has sufficient many continuous linear functionals. Let X_0 be a non-empty compact convex subset of X and K a non-empty compact subset of X. Let $F, G : X \to 2^E$ be upper hemicontinuous and such that

(a) for each $x \in X$, $F(x)$ and $G(x)$ are both compact and convex;

(b) for each $x \in K \cap \delta_E(X)$, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \geq 0}\lambda(X - x)} \neq \emptyset$ (respectively, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \leq 0}\lambda(X - x)} \neq \emptyset$);

(c) for each $x \in X \setminus K$, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \geq 0}\lambda(X_0 - x)} \neq \emptyset$ (respectively, $(F(x) - G(x)) \cap \overline{\cup_{\lambda \leq 0}\lambda(X_0 - x)} \neq \emptyset$).

Then there exists a point $\hat{x} \in X$ such that $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$.

As an immediately corollary to Theorem 2.7.5, we have the following:

Corollary 2.7.6. Let X be a non-empty convex set in a locally convex topological vector space E, X_0 a non-empty compact convex subset of X and K a non-empty compact subset of X. Let $F : X \to 2^E$ be upper hemi-continuous and such that

(a) for each $x \in X$, $F(x)$ is closed and convex;

(b) for each $x \in K \cap \delta_E(X)$, $F(x) \cap [x + \overline{\cup_{\lambda \geq 0}\lambda(X - x)}] \neq \emptyset$ (respectively, $F(x) \cap [x + \overline{\cup_{\lambda \leq 0}\lambda(X - x)}] \neq \emptyset$);

and either

(c) for each $x \in X \setminus K$, $F(x) \cap [x + \overline{\cup_{\lambda \geq 0}\lambda(\overline{co}K - x)}] \neq \emptyset$ (respectively, $F(x) \cap [x + \overline{\cup_{\lambda \leq 0}\lambda(\overline{co}K - x)}] \neq \emptyset$).

or,

$(c)'$ for each $x \in X \setminus K$, $F(x) \cap [x + \overline{\cup_{\lambda \geq 0}\lambda(X_0 - x)}] \neq \emptyset$ (respectively, $F(x) \cap [x + \overline{\cup_{\lambda \leq 0}\lambda(X_0 - x)}] \neq \emptyset$).

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$.

Proof. Let $G(x) = \{x\}$ for each $x \in X$. Then the conclusion follows from Theorem 2.7.5. □

Corollary 2.7.6′. Let X be a non-empty convex set in a topological vector space E which has sufficiently many continuous linear functionals, X_0 be a non-empty compact convex subset of X. Let $F : X \to 2^E$ be upper hemi-continuous such that

(a) for each $x \in X$, $F(x)$ is compact and convex;

(b) for each $x \in K \cap \delta_E(X)$, $F(x) \cap [x + \overline{\cup_{\lambda \geq 0} \lambda(X - x)}] \neq \emptyset$ (respectively, $F(x) \cap [x + \overline{\cup_{\lambda \leq 0} \lambda(X - x)}] \neq \emptyset$);

(c) for each $x \in X \setminus K$, $F(x) \cap [x + \overline{\cup_{\lambda \geq 0} \lambda(X_0 - x)}] \neq \emptyset$ (respectively, $F(x) \cap [x + \overline{\cup_{\lambda \leq 0} \lambda(X_0 - x)}] \neq \emptyset$).

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$.

We note that Theorem 2.7.2 and Theorem 2.7.5 generalize corresponding results of Ko and Tan [192] and Ko and Tan [194].

2.7.2 *Matching Theorems for Closed Coverings of Convex sets*

In this section, as an application of Theorem 2.7.5, we shall consider matching theorems for closed coverings of a convex set:

Theorem 2.7.7. Let X be non-empty convex set of a real locally convex topological vector space E. Let X_0 be a non-empty compact convex subset of E and K a non-empty compact subset of X. Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be two locally finite families of closed subsets of X and such that $\cup_{i\in I}A_I = \cup_{j\in J}B_j = X$. Let $\{C_i : i \in I\}$ and $\{D_j : j \in J\}$ be two families of non-empty subsets of E such that any finite union of the C_i's is contained in a compact convex subset of E. Let $S : X \to 2^E$ be upper hemicontinuous such that each $S(x)$ is a non-empty compact convex set. Suppose that for each $x \in (K \cap \delta(X)) \cup (X \setminus K)$, there exist $i \in I$ and $j \in J$ such that

(i) $x \in A_i \cap B_i$;

(ii) for each $x \in X$, setting $M(x) = \overline{co}(C_i + S(x)) - \overline{co}(D_j)$,

$$M(x)\cap\begin{cases} \overline{\cup_{\lambda\geq 0}\lambda(X - x)} \neq \emptyset \quad (\text{resp., } M(x)\cap\overline{\cup_{\lambda\leq 0}\lambda(X - x)} \neq \emptyset), & \text{if } x \in K \cap \delta(X); \\ \overline{\cup_{\lambda\geq 0}\lambda(X_0 - x)} \neq \emptyset \quad (\text{resp.,} M(x)\cap \overline{\cup_{\lambda\leq 0}\lambda(X_0 - x)} \neq \emptyset), & \text{if } x \in X \setminus K. \end{cases}$$

Then there exist two non-empty finite subsets I_0 of I and J_0 of J and a point $\hat{x} \in X$ such that

(a) $\hat{x} \in (\cap_{i\in I_0}A_i) \cap (\cap_{j\in J_0}B_j)$;

(b) $\overline{co}(\cup\{C_i : i \in I_0\}) + S(\hat{x})]$ meets the set $\overline{co}(\cup\{D_j : j \in J_0\})$.

Proof. For each $x \in X$, let $I(x) = \{i \in I : x \in A_i\}$ and $J(x) = \{j \in J : x \in B_j\}$. Then $I(x)$ and $J(x)$ are non-empty and finite since $\cup_{i\in I}A_i = \cup_{j\in J}B_j = X$ and $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ are locally finite. Define $F, G, H : X \to 2^E$ by $F(x) = \overline{co}(\cup\{C_i + S(x) : i \in I(x)\})$; $G(x) = \overline{co}(\cup\{D_j : j \in J(x)\})$ and $H(x) = \overline{co}(\cup\{C_i : i \in I(x)\})$. By hypothesis, for each $x \in X$, $H(x)$ and $S(x)$ are compact convex so that $F(x) = H(x) + S(x)$ is also compact convex. Since $\{A_i : i \in I\}$ is a locally finite family of closed subsets of X, for each $x \in X$, the set $U(x) = X \setminus \cup_{i\notin I(x)}A_i$ is an open neighborhood of x in X. Note that whenever $y \in U(x)$, $y \notin A_i$ for each $i \notin I(x)$ so that $I(y) \subset I(x)$ and therefore $H(y) \subset H(x)$. This shows that H is upper semicontinuous and hence $F = H + S$ is also upper hemi-continuous. Similarly we can show that G is upper semicontinuous (and hence upper hemi-continuous) on X. Thus the condition (a) of Theorem 2.7.5 is satisfied. By (i) and (ii), the conditions (b) and (c) of Theorem 2.7.5 are also satisfied. By Theorem 2.7.5, there exists $\hat{x} \in X$ such that $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$. Let $I_0 = I(\hat{x})$ and $J_0 = J(\hat{x})$, then I_0 and J_0 are non-empty and finite and the conclusions of the Theorem hold. □

We note the proof above was motivated by Ko and Tan [194] which is a modification of Theorem 11 of Fan [106] and of Theorem 1 of Shih and Tan [270]. Also, Theorem 2 7.7 shows that Theorem 4 of Ko and Tan [194] is still true without assuming that X is paracompact. The following is an easy consequence of Theorem 2.7.7.

By the same proof as in Theorem 2.7.7, but by applying Theorem 2.7.5′ instead of Theorem 2.7.5, and by assuming that the family $\{D_j\}_{j\in J}$ also has the same property as that of the family $\{C_i\}_{i\in I}$ (i.e., " any finite union of the D_j's is also contained in a compact convex subset of E"), Theorem 2.7.7 holds if the hypothesis underlying space E is weakened to a topological vector space with sufficiently many continuous linear functionals:

Theorem 2.7.8. Let X be a convex subset of a real locally convex topological vector space E, let X_0 be a non-empty compact convex subset of X and K be a non-empty compact subset of X. Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be two locally finite families of closed subsets of X such that $\cup_{i\in I} A_i = \cup_{j\in J} B_j = X$. Let $\{C_i : i \in I\}$ and $\{D_j : j \subset J\}$ be two families of non-empty subsets of E such that any finite union of C_i's is contained in a compact convex subset of E. Suppose that for each point $x \in (K \cap \delta(X)) \cup (X \setminus K)$, there exist $i \in I$ and $j \in J$ such that

(i) $x \in A_i \cap B_j$;

(ii) setting $M := \overline{co}(C_i) - \overline{co}(D_j)$, then

$$M \cap \begin{cases} x + \overline{\cup_{\lambda\geq 0}\lambda(X - x)} \neq \emptyset & (\text{resp.}, M \cap x + \overline{\cup_{\lambda\leq 0}\lambda(X - x)} \neq \emptyset), \quad \text{if } x \in K \cap \partial_E X; \\ x + \overline{\cup_{\lambda\geq 0}\lambda(X_0 - x)} \neq \emptyset & (\text{resp.}, M \cap (x + \overline{\cup_{\lambda\leq 0}\lambda(X_0 - x)}) \neq \emptyset), \quad \text{if } x \in X \setminus K. \end{cases}$$

Then there exist non-empty finite subsets I_0 of I and J_0 of J such that

$$(\cap_{i\in I_0}A_i)\cap(\cap_{j\in J_0}B_j)\cap(\overline{co}(\cup_{i\in I_0}C_i)-\overline{co}(\cup_{j\in J_0}D_j))\neq\emptyset.$$

Proof. Let $S: X\to 2^E$ be defined by $S(x)=\{-x\}$ for each $x\in X$. Then all hypotheses of Theorem 2.7.7 are satisfied so that there exist non-empty finite subsets I_0 of I and J_0 of J and a point $\hat{x}\in X$ such that

(a) $\hat{x}\in(\cap_{i\in I_0}A_i)\cap(\cap_{j\in J_0}B_j)$;

(b) $(\overline{co}(\cup_{i\in I_0}C_i)-\hat{x})\cap(\overline{co}(\cup_{j\in J_0}D_j)\neq\emptyset$. It follows that

$$\hat{x}\in(\cap_{i\in I_0}A_i)\cap(\cap_{j\in J_0}B_j)\cap(\overline{co}(\cup_{i\in I_0}C_i)-\overline{co}(\cup_{j\in J_0}D_j)). \qquad \square$$

If the space E in Theorem 2.7.8 is a topological vector space which has sufficiently many continuous linear functionals, by the same proof in Theorem 2.7.8 and by applying Theorem 2.7.7$'$, we have:

Theorem 2.7.8$'$. Let X be a convex subset of a topological vector space E which has sufficiently many linear functionals. Let X_0 be a non-empty compact convex subset of X and K be a non-empty compact subset of X. Let $\{A_i : i\in I\}$ and $\{B_j : j\in J\}$ be two locally finite families of closed subsets of X such that $\cup_{i\in I}A_i=\cup_{j\in J}B_j=X$. Let $\{C_i : i\in I\}$ and $\{D_j : j\subset J\}$ be two families of non-empty subsets of E such that any finite union of C_i's and D_j's is contained in a compact convex subset of E. Suppose that for each point $x\in(K\cap\delta(X))\cup(X\setminus K)$, there exist $i\in I$ and $j\in J$ such that

(i) $x\in A_i\cap B_j$;

(ii) setting $M:=\overline{co}(C_i)-\overline{co}(D_j)$, then

$$M\cap\begin{cases} x+\overline{\cup_{\lambda\geq 0}\lambda(X-x)}\neq\emptyset & (\text{resp.}, M\cap x+\overline{\cup_{\lambda\leq 0}\lambda(X-x)}\neq\emptyset), \quad \text{if } x\in K\cap\partial_E X;\\ x+\overline{\cup_{\lambda\geq 0}\lambda(X_0-x)}\neq\emptyset & (\text{resp.}, M\cap(x+\overline{\cup_{\lambda\leq 0}\lambda(X_0-x)})\neq\emptyset), \quad \text{if } x\in X\setminus K.\end{cases}$$

Then there exist non-empty finite subsets I_0 of I and J_0 of J such that

$$(\cap_{i\in I_0}A_i)\cap(\cap_{j\in J_0}B_j)\cap(\overline{co}(\cup_{i\in I_0}C_i)-\overline{co}(\cup_{j\in J_0}D_j))\neq\emptyset.$$

We note that Theorems 2.7.7 and 2.7.8 show that Theorems 4 and 5 of Ko and Tan [194] hold without assuming that the set X is paracompact.

As an application of Theorem 2.7.8, we have the following result which is Theorem 13 of Fan in [106] and is also a generalization of Shapley's theorem [264].

Corollary 2.7.9. Let Δ be an n-dimensional simplex in a Euclidean space. Let $\mathcal{F}$ denote the family of all faces of Δ (of all dimensional $0, 1, \cdots, n$). For each $\tau \in \mathcal{F}$, let $p(\tau)$ and $q(\tau)$ be two given points in Δ, and let $A(\tau)$, $B(\tau)$ be two closed subsets of Δ such that

(a) $\cup_{\tau \in \mathcal{F}} A(\tau) = \cup_{\tau \in \mathcal{F}} B(\tau) = \Delta$;

(b) for each $\tau \in \mathcal{F}$ of dimensional $< n$ and for any point $x \in \tau$, there is a $\rho \in \mathcal{F}$ such that $x \in B(\rho)$ and $q(\rho) \in \tau$.

Then there exist two non-empty subfamily $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{F}$ such that

(c) $[\cap_{\tau \in \mathcal{G}} A(\tau)] \cap [\cap_{\rho \in \mathcal{H}} B(\rho)] \neq \emptyset$,

(d) $co(\{p(\tau) : \tau \in \mathcal{G}\}) \cap co(\{q(\rho) : \rho \in \mathcal{H}\}) \neq \emptyset$.

For a generalization of the above result, we refer to Theorem 6 of Ko and Tan ([194]).

Chapter 3

Generalized Games

3.1 Introduction

The existence of an equilibrium in an abstract economy with compact strategy sets in $\mathbf{R}^n$ was proved in a seminal paper of Debreu [72]. The theorem of Debreu extended the earlier work of Nash in game theory. Since then there have been many generalizations of Debreu's theorem by Arrow and Debreu [5], Borglin and Keiding [37] and others. These papers generalize Debreu's theorem by considering preference correspondences that are not necessarily transitive or total, by allowing externalities in consumption and by assuming that the commodity space is not necessarily finite-dimensional. In these papers, the domain (and /or codomain) of the preference and constraint correspondences is assumed to be compact or paracompact.

Following the work of Sonnenschein [283], Gale and Mas-Colell [124] and Borglin and Keiding [37] on non-ordered preference relations, many theorems on the existence of maximal elements of preference relations which may not be transitive or complete, have been proved by Aliprantis and Brown [2], Bergstrom [30], Kim [181], Mehta and Tarafdar [221], Shafer and Sonnenschein [263], Sonnenschein [283] , Tan and Yuan [294], Tarafdar [303], Toussaint [315], Tulcea [317], Yannelis [325] and Yannelis and Prabhakar [326] and others. However, most of these existence theorems for maximal elements and equilibrium points deal with preference correspondences which have open lower sections or are majorized by correspondences with open lower sections. Note that every correspondence with open lower sections must be lower semicontinuous but the converse is not true in general. Moreover, in most cases, preference and constraint correspondences may be upper semicontinuous (or majorized by upper semicontinuous correspondences) instead of being lower semicontinuous (or being majorized by lower semicontinuous), or the preference and constraint mappings are condensing. Furthermore, in the study of equilibrium theory in most economic models, the feasible sets or the budget constraints are generally not (weakly) compact in infinite dimensional commodity spaces and are not convex in the case of the indivisibility of commodities and the underlying spaces do not have a linear structure. Thus, relaxation of the convexity of choice sets and generalizations of spaces enable us to deal with the existence of maximal elements and equilibrium points even though commodities are indivisible.

Therefore it is necessary and important to study the existence of equilibria for generalized games in which the preference and constraint correspondences need not have open lower sections nor open upper sections and also the underlying spaces need not have any linear structure and so on.

The objective of this chapter is to systematically study the existence of maximal elements and equilibria for generalized games under various hypotheses. In particular, the question raised by Yannelis and Prabhakar [326] is answered in the affirmative with weaker assumptions. The essential idea behind these existence theorems for equilibria of generalized games is to reduce them first to qualitative games and then to the existence problem of maximal elements for preference correspondences. Since existence of maximal elements of correspondences have equivalent formulations in fixed point theorems which can be derived from Ky Fan's minimax inequalities, so that the results in Chapter 2 are applicable. More precisely, in Chapter 3, we have:

In section 2, we first give the existence theorems of maximal elements and equilibria for non-compact generalized games in topological spaces which have the so-called H-structure.

In section 3, a number of approximative equilibria of generalized games in which preference correspondences are KF-majorized and underlying spaces are topological vector spaces are given.

In section 4, as applications of approximative equilibria for generalized games, several existence theorems for equilibria of generalized games in which preference correspondences are lower semicontinuous and the domain spaces are non-compact in locally convex topological vector spaces are given. By developing the so-called "*approximative method*" which was first motivated by Tulcea [316], we establish existence theorems for equilibria of generalized games in which the constraint correspondences are upper semicontinuous. In particular, the results in this section answer the question raised by Yannelis and Prabhakar [326] in the affirmative with weaker assumptions.

In section 5, the concept of $\mathcal{U}$-majorized mapping is first introduced. Then the existence theorems for equilibria of generalized games in which the constraint correspondences are $\mathcal{U}$-majorized are given.

In section 6, several existence theorems for equilibria of generalized games in which the constraint correspondences are Φ-condensing are given.

In section 7, by Michael's selection theorem, we consider the existence theorems for equilibria of generalized games in which the underlying spaces are Frechet spaces.

Finally in section 8, we first discuss some properties of multivalued mappings in finite dimensional spaces. As applications, fixed point theorems and the existence theorems of equilibria for generalized games are given in finite dimensional spaces.

Moreover, we remark that the existence theorems for equilibria of non-compact generalized games in this Chapter will be applied to give the existence theorems of non-compact quasi-variational and generalized quasi-variational inequalities in Chapter 4.

3.2 Equilibria for Ky Fan-Majorized Mappings in H-Spaces

In this section, we first introduce the notions of a Ky Fan mapping and a Ky Fan-majorized mapping (in short, KF mapping and KF-majorized mapping) in H-spaces. Then a selection theorem is derived which is applied to give an existence theorem for maximal elements for KF-majorized mappings in H-spaces (which need not have a linear structure). As an application of a maximal element existence theorem, we prove the existence for equilibria of one-person games and qualitative games. The existence theorem for qualitative games is then applied to give existence theorems of equilibria for N-person games.

We give some notion. If A is a non-empty subset of a topological vector space E and $S,T : A \to 2^E \cup \{\emptyset\}$ are correspondences, then coT, $T \cap S : A \to 2^E \cup \{\emptyset\}$ are correspondences defined by $(coT)(x) = coT(x)$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively. If X and Y are topological spaces and $T : X \to 2^Y \cup \{\emptyset\}$ is a correspondence, then (1) T is said to be upper semicontinuous at $x \in X$ if for any open subset U of Y containing $T(x)$, the set $\{z \in X : T(z) \subset U\}$ is an open neighborhood of x in X, (2) T is upper semicontinuous (on X) if T is upper semicontinuous at x for each $x \in X$; (3) the correspondence $\overline{T} : X \to 2^Y$ is defined by $\overline{T}(x) = \{y \in Y : (x,y) \in cl_{X \times Y}\text{Graph}(T)\}$ (which is also called the adherence mapping of T) and (4) the correspondence $clT : X \to 2^Y$ is defined by $clT(x) = cl_Y(T(x))$ for each $x \in X$. It is east to see that $clT(x) \subset \overline{T}(x)$ for each $x \in X$.

We remark here that throughout Chapter 2, an upper (or lower) semicontinuous correspondence is not require to be non-empty valued.

Let X and Y be two topological spaces and $F : X \to 2^Y \cup \{\emptyset\}$. Then F is said to be *compact* if for each $x \in X$, there exists a neighborhood V_x of x in X such that $F(V_x) = \cup_{z \in V_x} F(z)$ is relatively compact in Y. If X is a subset of a topological vector space E, X is said to have property K if for each compact subset B of X, the convex hull of B is relatively compact in X.

Let X be a topological space and Y a subset of an H-space E, let $\theta : X \to E$ be a map and $\psi : X \to 2^Y \cup \{\emptyset\}$ be a correspondence. Then (1) ψ is said to be of class

KF_θ (respectively, $KF_{\theta,C}$) or ψ is a KF_θ (respectively, $KF_{\theta,C}$) correspondence if for each $x \in X$, Hco$\psi(x) \subset Y$ and $\theta(x) \notin$ Hco$\psi(x)$ and for each $y \in Y$, $\psi^{-1}(y) = \{x \in X : y \in \psi(x)\}$ is open (respectively, compactly open) in X, (2) a correspondence $\psi_x : X \to 2^Y$ is said to be a KF_θ-majorant (respectively, $KF_{\theta,C}$-majorant) of ψ at $x \in X$ if there exists an open neighborhood N_x of x in X such that (a) for each $z \in N_x$, $\psi(z) \subset \psi_x(z)$ and $\theta(z) \notin$ Hco$\psi_x(z)$, (b) for each $z \in X$, Hco$\psi_x(z) \subset Y$ (this condition is redundant if Y is an H-convex subset of E) and (c) for each $y \in Y$, $\psi_x^{-1}(y)$ is open (respectively, compactly open) in X; (3) ψ is KF_θ-majorized (respectively, $KF_{\theta,C}$-majorized) if for each $x \in X$ with $\psi(x) \neq \emptyset$, there exists a KF_θ-majoriant (respectively, $KF_{\theta,C}$-majorant) of ψ at x. If the underlying space E is a topological vector space, it is clear that that the notions of correspondence ψ being of class KF_θ or KF_θ-majorized and correspondence ψ_x being a KF_θ-majorant of ψ at x generalize the corresponding notions of L_θ^*-correspondence or L_θ^*-majorized and correspondence ψ_x being an L_θ^*-majorant of ψ at x respectively introduced by Ding and Tan [84] which in turn generalize corresponding notions given by Borglin and Keiding [37], Yannelis and Prabhakar [326] and Tulcea [316]. For other kinds of mapping, we refer to Tan and Yuan [294] (which is a generalization of Ding and Tan [83]), Deguire, Tan and Yuan [76], Ben-El-Mechaiekh and Deguire [23] and Deguire and Lassonde [75]. In this section, we shall deal with either (I) $X = Y$ and $\theta = I_X$, the identity map on X or (II) $X = \Pi_{i\in I} X_i$ and $\theta = \pi_j : X \to X_j$ is the projection of X onto X_j. In these cases, we shall write KF (respectively, KF_C) in place of KF_θ (respectively, $KF_{\theta,C}$).

The following example shows that an KF-majorized mapping that is not of class KF.

Example. Let $X = [0,1]$ and $\phi : X \to 2^X \cup \{\emptyset\}$ be defined by

$$\phi(x) = \begin{cases} \{y \in X : y \in [0, x^3]\}, & \text{if } x \in (0,1), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then ϕ is not of class KF since $\phi^{-1}(y)$ is not open in X for each $y \in (0,1)$. For each $x \in (0,1)$, let $N_x = X$ which is an open neighborhood of x in X and define

$\phi_x : X \to 2^X \cup \{\emptyset\}$ by

$$\phi_x(z) = \begin{cases} \{y \in X : y \in [0,x)\}, & \text{if } z \in (0,1), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then it is clear that ϕ_x is an KF-majorant of ϕ at x and ϕ is KF-majorized correspondence (see also Kim [181, p.799]).

Now we have.

Lemma 3.2.1. Let X be a topological space and Y be an H-space. Let $\psi : X \to 2^Y \cup \{\emptyset\}$ be a correspondence with compactly open lower sections (i.e., $\psi^{-1}(y)$ is compactly open in X for each $y \in Y$). Define the correspondence $\phi : X \to 2^Y \cup \{\emptyset\}$ by $\phi(x) = \text{Hco}\psi(x)$ for each $x \in X$. Then ϕ also has compactly open lower sections.

Proof. For each $y_0 \in Y$ and each non-empty compact subset C of X, we need to prove that $\phi^{-1}(y_0) \cap C$ is open in C.

Let $x_0 \in \phi^{-1}(y_0) \cap C$. Since $y_0 \in \phi(x_0) = \text{Hco}\psi(x_0)$, there exists a finite subset $Y_0 = \{y_1, \cdots, y_n\}$ of Y such that $y_0 \in \text{Hco}Y_0$ by Lemma 1 of Tarafdar [303], where $y_i \in \psi(x_0)$ for each $i = 1, \cdots, n$. For each $i = 1, \cdots, n$, the set $\psi^{-1}(y_i)$ is compactly open in X and $x_0 \in \psi^{-1}(y_i)$. Let $U = \cap_{i=1}^{n} \psi^{-1}(y_i)$. Then $x_0 \in U \cap C$ and $U \cap C$ is open in C. Now for any $x \in U \cap C$, then $y_i \in \psi(x)$ for all $i = 1, \cdots, n$. Hence $Y_0 \subset \psi(x)$ which implies that $y_0 \in \text{Hco}Y_0 \subset \text{Hco}\psi(x) = \phi(x)$. Therefore $x \in \phi^{-1}(y_0) \cap C$ for all $x \in U \cap C$. Consequently, $\phi^{-1}(y_0) \cap C$ is open in C. $\square$.

By Lemma 3.2.1, we have the following selection theorem:

Lemma 3.2.2. Let X be a regular topological space and Y a non-empty subset of an H-space E. Let $\theta : X \to E$ and $P : X \to 2^Y \cup \{\emptyset\}$ be KF_θ-majorized (respectively, $KF_{\theta,C}$-majorized). If each open subset of X containing the set $B = \{x \in X : P(x) \neq \emptyset\}$ is paracompact, then there exists a correspondence $\phi : X \to 2^Y \cup \{\emptyset\}$ of class KF_θ (respectively, $KF_{\theta,C}$) such that $P(x) \subset \phi(x)$ and $\phi(x)$ is H-convex for each $x \in X$.

Proof. Since P is KF_θ-majorized (respectively, $KF_{\theta,C}$-majorized), for each $x \in B$, let N_x be an open neighborhood of x in X and $\phi_x : X \to 2^Y \cup \{\emptyset\}$ be such that (1) for each $z \in N_x$, $P(z) \subset \phi_x(z)$ and $\theta(z) \notin \text{Hco}\phi_x(z)$, (2) for each $z \in X$, $\text{Hco}\phi_x(z) \subset Y$ and (3) for each $y \in Y$, $\phi_x^{-1}(y)$ is compactly open in X

Since X is regular, for each $x \in B$ there exists an open neighborhood G_x of x in X such that $cl_X G_x \subset N_x$. Let $G = \cup_{x \in B} G_x$ then G is an open subset of X which contains $B = \{x \in X : P(x) \neq \emptyset\}$ so that G is paracompact by assumption. By Theorem VIII 1.4 of Dugundji [89] the open covering $\{G_x\}$ of G has an open precise neighborhood-finite refinement $\{G'_x\}$ Fix an arbitrary $x \in B$ and define $\phi'_x : G \to 2^Y \cup \{\emptyset\}$ by

$$\phi'_x(z) = \begin{cases} \text{Hco}\phi_x(z), & \text{if } z \in G \cap cl_X G'_x, \\ Y, & \text{if } z \in G \setminus cl_X G'_x, \end{cases}$$

then we have: For each $y \in Y$,

$$\begin{aligned} (\phi'_x)^{-1}(y) &= \{z \in G : y \in \phi'_x(z)\} \\ &= \{z \in G \cap cl_X G'_x : y \in \phi'_x(z)\} \cup \{z \in G \setminus cl_X G'_x : y \in \phi'_x(z)\} \\ &= \{z \in G \cap cl_X G'_x : y \in \text{Hco}\phi_x(z)\} \cup (G \setminus cl_X G'_x) \\ &= [(G \cap cl_X G'_x) \cap ((\text{Hco}\phi_x)^{-1}(y)] \cup (G \setminus cl_X G'_x) \\ &= (G \cap (\text{Hco}\phi_x)^{-1}(y)) \cup (G \setminus cl_X G'_x). \end{aligned}$$

It follows that for each non-empty compact subset C of X, $(\phi'_x)^{-1}(y) \cap C = (G \cap ((\text{Hco}\phi_x)^{-1}(y)) \cap C) \cup ((G \setminus cl_X G'_x) \cap C)$ is open in C by (3) and Lemma 3 2 1 above

Now define $\phi : X \to 2^Y \cup \{\emptyset\}$ by

$$\phi(z) = \begin{cases} \cap_{x \in B} \phi'_x(z), & \text{if } z \in G, \\ \emptyset, & \text{if } z \in X \setminus G. \end{cases}$$

Let $z \in X$ be given, clearly (2) implies that $\text{Hco}\phi(z) \subset Y$. If $z \in X \setminus G$, then $\phi(z) = \emptyset$ so that $\theta(z) \notin \text{Hco}\phi(z)$. If $z \in G$, then $z \in G \cap cl_X G'_x$ for some $x \in B$ so that $\phi'_x(z) = \text{Hco}\phi_x(z)$ and hence $\phi(z) \subset \text{Hco}\phi_x(z)$. As $\theta(z) \notin \text{Hco}\phi'_x(z)$ by (1) we must have $\theta(z) \notin \text{Hco}\phi(z)$. Therefore $\theta(z) \notin \text{Hco}\phi(z)$ for all $z \in X$ Now we show that for each $y \in Y$, $\phi^{-1}(y)$ is compactly open in X. Indeed, let $y \in Y$, C be a non-empty compact subset of X and $u \in \phi^{-1}(y) \cap C = \{z \in X : y \in \phi(z)\} \cap C = \{z \in G : y \in \phi(z)\} \cap C$ be arbitrarily fixed. Since $\{G'_x\}$ is a neighborhood-finite refinement, there exists an open neighborhood M_u of u in G such that $\{x \in B : M_u \cap G''_x \neq \emptyset\} = \{x_1, \cdots, x_n\}$. Note that for each $x \in B$ with $x \notin \{x_1, \cdots, x_n\}$, $\emptyset = M_u \cap G'_x = M_u \cap cl_X G''_x$ so that $\phi'(z) = Y$ for all $z \in M_u$. Thus we have $\phi(z) = \cap_{x \in B} \phi'_x(z) = \cap_{i=1}^n \phi'_{x_i}(z)$ for all $z \in M_u$. It follows that

$$\begin{aligned} \phi^{-1}(y) &= \{z \in X : y \in \phi(z)\} = \{z \in G : y \in \cap_{x \in B} \phi'_x(z)\} \\ &\supset \{z \in M_u : y \in \cap_{x \in B} \phi'_x(z)\} = \{z \in M_u : y \in \cap_{i=1}^n \phi'_{x_i}(z)\} \\ &= M_u \cap \{z \in G : y \in \cap_{i=1}^n \phi'_{x_i}(z)\} = M_u \cap [\cap_{i=1}^n (\phi'_{x_i})^{-1}(y)]. \end{aligned}$$

But $M'_u = M_u \cap [\cap_{i=1}^n (\phi'_{x_i})^{-1}(y)] \cap C$ is an open neighborhood of u in C such that $M'_u \subset \phi^{-1}(y) \cap C$ since $(\phi'_{x_i})^{-1}(y)$ is compactly open in X. This shows that for each $y \in Y$, $\phi^{-1}(y)$ is compactly open in X. Therefore ϕ is of class KF_θ.

Now we shall show that $P(z) \subset \phi(z)$ for each $z \in X$. Indeed, let $z \in X$ with $P(z) \neq \emptyset$. Note that $z \in G$. For each $x \in B$, if $z \in G \setminus cl_X G'_x$, then $\phi'_x(z) = Y \supset P(z)$ and if $z \in G \cap cl_X G'_x$, we have $z \in cl_X G'_x \subset cl_X G_x \subset N_x$ so that by (1), $P(z) \subset \phi_x(z) \subset \phi'_x(z)$. It follows that $P(z) \subset \phi'_x(z)$ for each $x \in B$ so that $P(z) \subset \cap_{x \in B} \phi'_x(z) = \phi(z)$. Finally we replace ϕ by Hcoϕ and, by Lemma 3.2.1, the result follows. □

We note that Lemma 3.2.2 generalize Lemma 2 of Ding and Tan [84] which in turn improves Lemma 1 of Ding, Kim and Tan [86].

Now by Lemma 3.2.2, we have the following existence theorem for KF_θ correspondences in topological spaces which generalizes Theorem 4.1 of Tan and Yu [288] and Corollary 1 of Borglin and Keiding [37]:

Theorem 3.2.3. Let X be an H-space such that $X = \cup_{n=1}^\infty C_n$, where $\{C_n\}_{n=1}^\infty$ is an increasing sequence of non-empty compact and weakly H-convex subsets. Suppose $P : X \to 2^X \cup \{\emptyset\}$ is KF_C-majorized. If for each sequence $(y_n)_{n=1}^\infty$ in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$ which is escaping from X relative to $\{C_n\}_{n=1}^\infty$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in C_{n_0}$ such that $x_{n_0} \in P(y_{n_0})$, then there exists $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$.

Proof. Suppose the conclusion were not true, then $X = \{x \in X : P(x) \neq \emptyset\}$. First we note that since X is regular and σ-compact, X is paracompact by Corollary 33.15 of

Cullen [70, p.341]. Hence by Lemma 3.2.2, there exists a correspondence $\psi : X \to 2^X$ of class KF_C such that $P(x) \subset \psi(x)$ and $\psi(x)$ is H-convex for all $x \in X$. Note that the conditions (i) and (ii) of Theorem 2.3.17 are satisfied by ψ. By assumption, for each sequence $(y_n)_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in C_{n_0}$ such that $x_{n_0} \in P(y_{n_0}) \subset \psi(y_{n_0})$. This shows that condition (iii) of Theorem 2.3.17 is also satisfied by ψ. Hence by Theorem 2.3.17, there exists $\hat{y} \in X$ such that $\psi(\hat{y}) = \emptyset$. It follows that $P(\hat{y}) = \emptyset$ which is a contradiction. Hence the conclusion must hold. □

A one-person game is a quadruple $(X; A, B; P)$ where X is a topological space, $A, B : X \to 2^X \cup \{\emptyset\}$ are constraint correspondences and $P : X \to 2^X \cup \{\emptyset\}$ is a preference correspondence. An equilibrium point for $(X; A, B; P)$ is a point $x^* \in X$ such that $x^* \in \overline{B}(x^*)$ and $A(x^*) \cap P(x^*) = \emptyset$.

Let I be a (finite or infinite) set of players (agents). A generalized game (an abstract economy) is a family $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ of quadruples $(X_i; A_i, A_i; P_i)$ where for each $i \in I$, X_i is a topological space, $A_i, B_i : X := \Pi_{j \in I} X_j \to 2^{X_i} \cup \{\emptyset\}$ are constraint correspondences and $P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is a preference correspondence. An equilibrium point for G is a point $x^* \in X$ such that for each $i \in I$, $x_i^* = \pi_i(x^*) \in \overline{B_i}(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ where $\pi_i : X \to X_i$ is the projection. We remark that when $\overline{B_i}(x^*) = cl_{X_i} B_i(x^*)$ (which is the case when B_i has a closed graph in $X \times X_i$; in particular, when $cl_{X_i} B_i$ is upper semicontinuous with closed values) for each $i \in I$, our definition of equilibrium point for a generalized game (an abstract economy coincides with that of Ding and Tan [84]. Also, according to Gale and Mas-Colell [125], a qualitative game is a family $\Gamma = (X_i; P_i)_{i \in I}$ of ordered pairs (X_i, P_i) where for each $i \in I$, X_i is a topological space, $P_i : X = \Pi_{j \in I} X_j \to 2^{X_i} \cup \{\emptyset\}$ is an irreflexive preference correspondence, i.e., $x_i \notin P_i(x)$ for all $x \in X$. A point $x^* \in X$ is said to be an equilibrium point of the qualitative game Γ if $P_i(x^*) = \emptyset$ for all $i \in I$.

As an application of Theorem 3.2.3, we have the following existence theorem of equilibria for one person-games:

Theorem 3.2.4. Let $(X; A, B; P)$ be a one-person game such that X is an H-space and $X = \cup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of compact and weakly H-convex subsets and the following conditions are satisfied:

(1) for each $x \in X$, $A(x)$ is non-empty and $\text{Hco}A_i(x) \subset B_i(x)$;

(2) for each $y \in X$, $A^{-1}(y)$ is compactly open in X;

(3) $A \cap P$ is KF_C-majorized;

(4) for each sequence $(y_n)_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $x_n \in (A(y_n) \cap P(y_n)) \cap C_n$.

Then (X, A, B, P) has an equilibrium point, i.e., there exists $\hat{x} \in X$ such that $\hat{x} \in \overline{B_i}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Proof. Let $F = \{x \in X : x \in \overline{B}(x)\}$, then F is closed in X. Define $\Psi : X \to 2^X \cup \{\emptyset\}$ by

$$\Psi(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \in F, \\ A(x), & \text{if } x \notin F. \end{cases}$$

Let $x \in X$ be such that $\Psi(x) \neq \emptyset$. If $x \notin F$, then $X \setminus F$ is an open neighborhood of x such that for each $z \in X \setminus F$, $z \notin \overline{B}(z)$. Now define $\Phi_x : X \to 2^X \cup \{\emptyset\}$ by $\Phi_x(z) = A(z)$ for each $z \in X$ and $N_x = X \setminus F$, then N_x is an open neighborhood of x in X such that

(i) $\Psi(z) \subset \Phi_x(z)$ and $z \notin \text{Hco}\Phi_x(z)$ for each $z \in N_x$, and

(ii) $\Phi_x^{-1}(y) = A^{-1}(y)$ is compactly open in X.

Therefore Φ_x is an L_C-majorant of Ψ at x. If $x \in F$, then $\Psi(x) = A(x) \cap P(x) \neq \emptyset$. Since $A \cap P$ is KF_C-majorized, there exist an open neighborhood N_x of x in X and a correspondence $\Phi_x : X \to 2^X$ such that $\Psi(z) = A(z) \cap P(z) \subset \Phi_x(z)$ and $z \notin \text{Hco}\Phi_x(z)$ for each $z \in N_x$, and $\Phi_x^{-1}(y)$ is compactly open in X for each $y \in X$. Define the map $\Phi'_x : X \to 2^X \cup \{\emptyset\}$ by

$$\Phi'_x(z) = \begin{cases} A(z) \cap \Phi_x(z), & \text{if } z \in F, \\ A(z), & \text{if } z \notin F. \end{cases}$$

Note that as $(A \cap P)(z) \subset \Phi_x(z)$ for each $z \in N_x$, we have $\Psi(z) \subset \Phi'_x(z)$. It is easy to see that $z \notin \mathrm{Hco}\Phi'_x(z)$ for all $z \in X$.

Moreover, for any $y \in X$, the set $(\Phi'_x)^{-1}(y) = [\Phi_x^{-1}(y) \cup (X \setminus F)] \cap A^{-1}(y)$ is compactly open in X. Therefore Φ'_x is a KF_C-majorant of Ψ at x. Hence, Ψ is an KF_C-majorized correspondence.

By (4), for each sequence $(y_n)_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ there exist $n \in \mathbf{N}$ and $x_n \in C_n$ with $x_n \in \mathrm{Hco}(A(y_n) \cap P(y_n)) \cap C_n \subset \Psi(y_n) \cap C_n$. Hence by Theorem 3.2.3, there exists $\hat{x} \in X$ such that $\Psi(\hat{x}) = \emptyset$; since $A(\hat{x}) \neq \emptyset$ by (1), we must have $\hat{x} \in \overline{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. □

As another application of Theorem 3.2.3, we have the following existence theorem for equilibria for qualitative games.

Theorem 3.2.5. Let $(X_i, P_i)_{i=1}^{N}$ be a qualitative game such that for each $i = 1, \cdots, N$, $X_i = \cup_{j=1}^{\infty} C_{i,j}$ is an H-space (so that $X = \Pi_{i \in I} X_i$ is also an H-spaces), where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of non-empty compact and weakly H-convex subsets. Suppose the following conditions are satisfied:

(1) for each $i = 1, \cdots, N$, $P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is KF_C-majorized;

(2) for each $i = 1, \cdots, N$, the set $F_i = \{x \in X : P_i(x) \neq \emptyset\}$ is open in X;

(3) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ where $C_n = \Pi_{i=1}^{N} C_{i,n}$ for each $n = 1, 2, \cdots$, there exist $n_0 \in N$ and $x_{n_0} \in C_{n_0}$ such that $\pi_i(x_{n_0}) \in P_i(y_{n_0})$ for each $i \in I(y_{n_0})$, where $I(x) = \{j \in \{1, \cdots, N\} : P_j(x) \neq \emptyset\}$.

Then $(X_i, P_i)_{i=1}^{N}$ has an equilibrium point $\hat{x} \in X$; i.e. $P_i(\hat{x}) = \emptyset$ for all $i = 1, \cdots, N$.

Proof. Suppose the conclusion were false; then for each $x \in X$, $I(x) \neq \emptyset$. For each $i = 1, \cdots N$, define $P'_i : X \to 2^X \cup \{\emptyset\}$ by $P'_i(x) = \pi_i^{-1}(P_i(x))$ for each $x \in X$. Define $P : X \to 2^X \cup \{\emptyset\}$ by $P(x) = \cap_{i \in I(x)} P'_i(x)$ for each $x \in X$. Suppose $x \in X$ and fix one $i \in I(x)$. By (1), there exist a correspondence $\psi_i : X \to 2^{X_i} \cup \{\emptyset\}$ and an open neighborhood N_x of x in X such that

(a) for each $z \in N_x$, $P_i(z) \subset \psi_i(z)$ and $\pi_i(z) \notin \mathrm{Hco}\psi_i(z)$,

(b) for each $y \in X_i$, $\psi_i^{-1}(y)$ is compactly open in X.

By (2), we may assume that $N_x \subset F_i$ so that $P_i(z) \neq \emptyset$ for all $z \in N_x$ and hence $i \in I(z)$ for all $z \in N_x$. Define $\psi_x : X \to 2^X$ by $\psi_x(z) = \pi_i^{-1}(\psi_i(z))$ for each $z \in X$. Now if $z \in N_x$, then by (a),

$$P(z) = \cap_{j \in I(z)} P'_j(z) \subset P'_i(z) = \pi_i^{-1}(P_i(z)) = \psi_x(z)$$

and $z \notin \mathrm{Hco}\psi_x(z)$. Moreover, if $y \in X$, then

$$\begin{aligned} \psi_x^{-1}(y) \quad &= \{z \in X : y \in \psi_x(z)\} \\ &= \{z \in X : \pi_i(y) \in \psi_i(z)\} \\ &= \psi_i^{-1}(\pi_i(y)) \end{aligned}$$

is compactly open in X by (b). This shows that ψ_x is a KF_{I_X}-majorant of P at x. Hence P is KF_{I_X}-majorized.

Finally by (3), for each sequence $(y_n)_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in C_{n_0}$ such that $\pi_i(x_{n_0}) \in P_i(y_{n_0})$ for each $i \in I(y_{n_0})$, it follows that

$$x_{n_0} \in \cap_{i \in I(y_{n_0})} \pi_i^{-1}((P_i(y_{n_0})) = \cap_{i \in I(y_{n_0})} P'_i(y_{n_0}) = P(y_{n_0}).$$

Note that $X = \cup_{n=1}^{\infty} C_n$ and C_n is also a compact H-space for $n = 1, 2, \cdots$. Hence by Theorem 3.2.3, there exists $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$ which contradicts our assumption. Therefore the conclusion must hold. □

Theorem 3.2.6. Let $(X_i, A_i, B_i; P_i)_{i=1}^{N}$ be an N-person game such that for each $i = 1, \cdots, N$, $X_i = \cup_{j=1}^{\infty} C_{i,j}$ is an H-space (so that $X = \Pi_{i=1}^{N} X_i$ is also an H-space), where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of non-empty compact and weakly H-convex subsets of X_i. Suppose the following conditions are satisfied:

(1) for each $i = 1, \cdots, N$ and for each $x \in X$, $A_i(x)$ is non-empty and $\mathrm{Hco}A_i(x) \subset B_i(x)$;

(2) for each $i = 1, \cdots, N$ and for each $y \in X_i$, $A_i^{-1}(y)$ is compactly open in X;

(3) for each $i = 1, \cdots N$, the correspondence $A_i \cap P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is KF_C-majorized;

(4) for each $i = 1, \cdots, N$, the set $F_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X;

(5) for each sequence $(y_n)_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ where $C_n = \Pi_{i=1}^{N} C_{i,n}$ for each $n = 1, 2, \cdots$, there exist $n_0 \in N$ and $x_{n_0} \in C_n$ such that for each $i = 1, \cdots, N$, $\pi_i(x_{n_0}) \in A_i(y_{n_0}) \cap P_i(y_{n_0})$ if $A_i(y_{n_0}) \cap P_i(y_{n_0}) \neq \emptyset$ and $\pi_i(y_{n_0}) \in cl_{X_i} B_i(y_{n_0})$ and $\pi_i(x_{n_0}) \in A_i(y_{n_0})$ if $\pi_i(y_{n_0}) \notin \overline{B_i}(y_{n_0})$.

Then $(X_i; A_i, B_i; P_i)_{i=1}^{N}$ has an equilibrium point $\hat{x}$ in X; i.e. for each $= 1, \cdots, N$, $\pi_i(\hat{x}) \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. For each $i = 1, \cdots, N$, let $G_i = \{x \in X : \pi_i(x) \in \overline{B_i}(x)\}$, then G_i is closed in X; define define $Q_i : X \to 2^{X_i}$ by

$$Q_i(x) = \begin{cases} (A_i \cap P_i)(x), & \text{if } x \in G_i; \\ A_i(x), & \text{if } x \notin G_i. \end{cases}$$

We shall show that $(X_i, Q_i)_{i=1}^{N}$ is a qualitative game satisfying the hypotheses of Theorem 3.2.5. Fix any $i \in \{1, \cdots, N\}$. The set

$$\begin{aligned} \{x \in X : Q_i(x) \neq \emptyset\} &= \{x \in G_i : (A_i \cap P_i)(x) \neq \emptyset\} \cup \{x \in X \setminus G_i : A_i(x) \neq \emptyset\} \\ &= (G_i \cap F_i) \cup (X \setminus G_i) \\ &= F_i \cup (X \setminus G_i) \end{aligned}$$

is compactly open in X by (4). Thus the condition (2) of Theorem 3.2.5 is satisfied. Now let $x \in X$ be such that $Q_i(x) \neq \emptyset$.

Case 1. Suppose $x \notin G_i$.

Let $\psi_x = A_i$ and $N_x = X \setminus G_i$, then N_x is an open neighborhood of x in X such that

(i) for each $z \in N_x$, $Q_i(z) = A_i(z) = \psi_x(z)$ and $\pi_i(z) \notin \text{Hco}A_i(z) = \text{Hco}\psi_x(z)$ by (1);

(ii) for each $y \in X_i$, $\psi_x^{-1}(y) = A_i^{-1}(y)$ is compactly open in X by (2).

Thus ψ_x is an KF_C-majorant of Q_i at x.

Case 2: Suppose $x \in G_i$.

Since $Q_i(x) = (A_i \cap P_i)(x) \neq \emptyset$, by (3), there exist $\psi_x : X \to 2^{X_i} \cup \{\emptyset\}$ and an open neighborhood N_x of x in X such that

(i) for each $z \in N_x$, $(A_i \cap P_i)(z) \subset \psi_x(z)$ and $\pi_i(z) \notin \mathrm{Hco}\psi_x(z)$,

(ii) for each $y \in X_i$, $\psi_x^{-1}(y)$ is compactly open in X.

Define $\psi'_x : X \to 2^{X_i} \cup \{\emptyset\}$ by

$$\psi'_x(z) = \begin{cases} A_i(z) \cap \psi_x(z), & \text{if } z \in G_i, \\ A_i(z), & \text{if } z \notin G_i. \end{cases}$$

Then for each $z \in N_x$, we have $Q_i(z) \subset \psi'_x(z)$ and $\pi_i(z) \notin \mathrm{Hco}\psi'_x(z)$ by (i) and (1). Moreover for each $y \in X_i$, the set $(\psi'_x)^{-1}(y) = [\psi_x^{-1}(y) \cup (X \setminus G_i)] \cap A_i^{-1}(y)$ is compactly open in X by (ii) and (2). Thus ψ'_x is a KF_C-majorant of Q_i at x.

This shows that Q_i is KF_C-majorized so that the condition (1) of Theorem 3.2.5 is also satisfied

Finally, let $(y_n)_{n=1}^{\infty}$ be a sequence in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$. By (5), there exist $n_0 \in \mathbf{N}$ and $x_{n_0} \in C_{n_0}$ such that $\pi_i(x_{n_0}) \in Q_i(y_{n_0})$ for each $i = 1, \cdots, N$ with $Q_i(y_{n_0}) \neq \emptyset$. Hence the condition (3) of Theorem 3.2.5 is also satisfied.

By Theorem 3.2.5, $(X_i, Q_i)_{i=1}^{N}$ has an equilibrium point $\hat{x} \in X$, i.e. $Q_i(\hat{x}) = \emptyset$ for all $i = 1, \cdots, N$. Since $A_i(\hat{x}) \neq \emptyset$ for each $i = 1, \cdots, N$, we must have $\pi_i(\hat{x}) \in cl_{X_i} B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for all $i = 1, \cdots, N$. $\square$

3.3 Approximate Equilibria in Topological Vector Spaces

In this section, by developing an "approximation" method which idea was first motivated by Tulcea [316], we obtain an equilibrium existence theorem for a generalized game (abstract economy) in which the constraint correspondences are not assumed to have open graphs nor open lower sections (which are generally assumed in the literatures, e.g., see Ding, Kim and Tan [86], Ding and Tan [84] and Yannelis and Prabhakar [326] and the references wherein). Our result generalizes the corresponding results of Shafer and Sonnenschein [263], Borglin and Keiding [37], Yannelis and Prabhakar [326], Tulcea [316] and Chang [55] in several ways.

We shall need Theorem 1 and Theorem 3 of Ding and Tan [84] which are stated below as Lemma 3.3.1 and Lemma 3.3.2 respectively.

Lemma 3.3.1. Let X be a non-empty paracompact convex subset of a topological vector space and $P : X \to 2^X \cup \{\emptyset\}$ be KF_C-majorized. Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x \in coP(y)$. Then there exists an $\hat{x} \in K$ such that $P(\hat{x}) = \emptyset$.

For other results related to the existence of maximal elements, we refer to Tan and Yuan [294], Kim [181], Lassonde and Deguire [75], Ben-El-Mechaiekh and Deguire [23], Degiure, Tan and Yuan [76] and Tarafdar [303].

Lemma 3.3.2. Let $\Gamma = (X_i, P_i)_{i\in I}$ be a qualitative game such that $X = \Pi_{i\in I}X_i$ is paracompact. Suppose the following conditions are satisfied:

(i) for each $i \in I$, X_i is a non-empty convex subset of a topological vector space;

(ii) for each $i \in I$, $P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is KF_C-majorized;

(iii) $\cup_{i\in I}\{x \in X : P_i(x) \neq \emptyset\} = \cup_{i\in I} int_X\{x \in X : P_i(x) \neq \emptyset\}$;

(iv) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x_i \in coP_i(y)$ for all $i \in I$.

Then Γ has an equilibrium point in K.

The following is an existence of "approximate" equilibrium points for a one-person game:

Theorem 3.3.3. Let X be a non-empty paracompact convex subset of a topological vector space E and $A, B, P : X \to 2^X \cup \{\emptyset\}$ be such that

(i) A is lower semicontinuous on X and for each $x \in X$, $A(x)$ is non-empty and $coA(x) \subset B(x)$;

(ii) $A \cap P$ is KF_C-majorized;

(iii) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, $co(X_0 \cup \{y\}) \cap co(A(y) \cap P(y)) \neq \emptyset$.

Then for each open convex neighborhood V of 0 in E, the one-person game $(X; A, \overline{B_V}; P)$ has an equilibrium point in K, i.e., there exists a point $x_V \in K$ such that $x_V \in \overline{B_V}(x_V)$ and $A(x_V) \cap P(x_V) = \emptyset$ where $B_V(x) = (B(x) + V) \cap X$ for each $x \in X$.

Proof. Let V be an open convex neighborhood of 0 in E and define $A_V, B_V : X \to 2^X$ by $A_V(x) = (A(x) + V) \cap X$ and $B_V(x) = (B(x) + V) \cap X$ for each $x \in X$. Since A is lower semicontinuous, A_V has an open graph in $X \times X$ by Lemma 4.1 of Chang [55] or Tulcea [316]. By (i), $A_V(x) \subset B_V(x)$ for each $x \in X$. Let $F_V = \{x \in X : x \notin \overline{B_V}(x)\}$, then F_V is open in X. Define $\Psi_V : X \to 2^X\{\emptyset\}$ by

$$\Psi_V = \begin{cases} A(x) \cap P(x), & \text{if } x \notin F_V, \\ A_V(x), & \text{if } x \in F_V. \end{cases}$$

Suppose $x \in X$ is such that $\Psi_V(x) \neq \emptyset$.

Case 1. Suppose $x \in F_V$. Let $\Phi_x = A_V$ and $N_x = F_V$, then N_x is an open neighborhood of x in X such that (a) for each $z \in N_x$, $\Psi_V(z) = A_V(z) = \Phi_x(z)$, $z \notin \overline{B_V}(z)$ so that $z \notin A_V(z) = \Phi_x(z)$; (b) for each $y \in X$, $\Phi_x^{-1}(y) = A_V^{-1}(y)$ is open in X since A_V has an open graph in $X \times X$. Thus Φ_x is a KF_C-majorant of Ψ_V at x.

Case 2. Suppose $x \notin F_V$. Then $\Psi_V(x) = A(x) \cap P(x) \neq \emptyset$. Since $A \cap P$ is KF_C-majorized by (ii), there exist an open neighborhood N_x of x in X and a correspondence $\Phi'_x : X \to 2^X$ such that (a) $\Psi_V(z) = A(z) \cap P(z) \subset \Phi'_x(z)$ and $z \notin co\Phi'_x(z)$ for each $z \in N_x$ and (b) for each $y \in X$, $(\Phi'_x)^{-1}(y)$ is compactly open in X. Define $\Phi_x : X \to 2^X$

by

$$\Phi_x = \begin{cases} A_V(z) \cap \Phi'_x(z), & \text{if } z \notin F_V, \\ A_V(z), & \text{if } z \in F_V. \end{cases}$$

Note that (i) for each $z \in N_x$, clearly $\Psi_x^{-1}(y) \subset \Phi_x(z)$ and it is easy to see that $z \notin co\Phi_x(z)$ and (ii) for each $y \in x$, $\Phi_x^{-1}(y) = [F_V \cup (\Phi'_x)^{-1}y)] \cap A_V^{-1}(y)$ is compactly open in X. Hence Φ_x is an KF_C-majorant of Ψ_V at x.

Therefore Ψ_V is a KF_C-majorized correspondence. Moreover by (iii), for each $y \in X \setminus K$, there exists $x \in co(X_0 \cup \{y\}) \cap co(A(y) \cap P(y))$ so that $x \in co(A(y) \cap P(y)) \subset co\Psi_V(y)$. Thus by Lemma 3.3.1, there exists $\hat{x} \in K$ such that $\Psi_V(\hat{x}) = \emptyset$. Since $A(\hat{x}) \neq \emptyset$ by (i), we must have $\hat{x} \in \overline{B_V}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. □

A proof similar to that of Theorem 3.3.3 and theerfore omitted gives the following result:

Theorem 3.3.3'. Let $(X; A, B; P)$ be a one-person game such that $X = \cup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets in a topological vector space E and the following conditions are satisfied:

(1) A is lower semicontinuous on X and for each $x \in X$, $A(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;

(2) $A \cap P$ is KF_C-majorized;

(3) for each sequence $(y_n)_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$ which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in N$ and $x_n \in C_n$ such that $x_n \in co(A(y_n) \cap P(y_n)) \cap C_n$.

Then for each open convex neighborhood V of zero in E, the one-person game $(X; A, \overline{B_V}; P)$ has an equilibrium point in X, i.e., there exists a point $x_V \in X$ such that $x_V \in \overline{B_V}(x_V)$ and $A(x_V) \cap P(x_V) = \emptyset$ where $B_V(x) = (B(x) + V) \cap X$ for each $x \in X$.

The following is an existence theorem for "approximate" equilibrium points for a generalized game:

Theorem 3.3.4. Let I be any (countable or uncountable) set. For each $i \in I$, let X_i be a non-empty convex subset of a topological vector space E_i and $A_i, B_i, P_i : X =$

$\Pi_{j\in I}X_j \to 2^{X_i} \cup \{\emptyset\}$ be such that

(a) A_i is lower semicontinuous and for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;

(b) $A_i \cap P_i$ is KF_C-majorized;

(c) the set $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X.

Suppose that X is paracompact and that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap P_i(y))$ for all $i \in I$. Then given any $V = \Pi_{i\in I}V_i$ where for each $i \in I$, V_i is an open convex neighborhood of zero in E_i, the generalized game $\Gamma_V = (X_i; A_i, B_{V_i}; P_i)_{i\in I}$ has an equilibrium point in K; i.e., there exists a point $x_V = (x_{V_i})_{i\in I} \in K$ such that $x_{V_i} \in \overline{B_{V_i}}(x_V)$ and $A_i(x_V) \cap P_i(x_V) = \emptyset$ for each $i \in I$, where $B_{V_i} = (B_i(x) + V_i) \cap X_i$ for each $x \in X$ and each $i \in I$.

Proof. Let $V = \Pi_{i\in I}V_i$ be given where, for each $i \in I$, V_i is an open convex neighborhood of zero in E_i. Fix any $i \in I$ and define $A_{V_i}, B_{V_i} : X \to 2^{X_i}$ by $A_{V_i}(x) = (coA_i(x) + V_i) \cap X_i$ and $B_{V_i}(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$. By (a), A_i is lower semicontinuous so that coA_i is also lower semicontinuous by Proposition 2.6 of Michael [222, p.366], it follows from Lemma 4.1 of Chang [55] or from Yannelis [325] that A_{V_i} has an open graph in $X \times X$. Now let $F_{V_i} = \{x \in X : x_i \notin \overline{B_{V_i}}(x)\}$, then F_{V_i} is open in X. Define the map $Q_{V_i} : X \to 2^{X_i}$ by

$$Q_{V_i}(x) = \begin{cases} (A_i \cap P_i)(x), & \text{if } x \notin F_{V_i}, \\ A_i(x), & \text{if } x \in F_{V_i}. \end{cases}$$

We shall prove that the qualitative game $\mathcal{T} = (X_i, Q_{V_i})_{i\in I}$ satisfies all conditions of Lemma 3.3.2. First we note that for each $i \in I$, the set

$$\begin{aligned}\{x \in X : Q_{V_i}(x) \neq \emptyset\} &= F_{V_i} \cup \{x \in X \setminus F_{V_i} : A_i(x) \cap P_i(x) \neq \emptyset\} \\ &= F_{V_i} \cup ((X \setminus F_{V_i}) \cap E^i) = F_{V_i} \cup E^i\end{aligned}$$

is open in X by (c). Let $x \in X$ be such that $Q_{V_i}(x) \neq \emptyset$. We consider the following two cases:

Case 1: $x \in F_{V_i}$.

Let $\Psi_x = A_{V_i}$ and $N_x = F_{V_i}$, then N_x is an open neighborhood of x in X such that (i) $Q_{V_i}(z) \subset \Psi_x(z)$ and by (b), $z_i \notin co\Psi_x(z)$ for each $z \in N_x$; (ii) $co\Psi_x(z) \subset X_i$ for each $z \in X$ by (b) and (iii) $\Psi_x^{-1}(y) = A_{V_i}^{-1}(y)$ is open in X for all $y \in X_i$ since A_{V_i} has an open graph. Therefore, Ψ_x is a KF_C-majorant of Q_{V_i} at x.

Case 2: $x \notin F_{V_i}$.

Since $Q_{V_i}(x) = (A_i \cap P_i)(x) \neq \emptyset$ and $A_i \cap P_i$ is KF_C-majorized, there exist an open neighborhood N_x of x in X and a correspondence $\phi_x : X \to 2^{X_i} \cup \{\emptyset\}$ such that (i) $(A_i \cap P_i)(z) \subset \phi_x(z)$ and $z_i \notin co\phi_x(z)$ for each $z \in N_x$ and (iii) $\phi_x^{-1}(y)$ is compactly open in X for each $y \in X_i$ Define $\Psi_x : X \to 2^{X_i} \cup \{\emptyset\}$ by

$$\Psi_x(z) = \begin{cases} A_{V_i}(z) \cap \phi_x(z), & \text{if } z \notin F_{V_i}, \\ A_{V_i}(z), & \text{if } z \in F_{V_i}. \end{cases}$$

Note that as $(A_i \cap P_i)(z) \subset \phi_x(z)$ and $A_i(z) \subset A_{V_i}(z)$ for each $z \in N_x$, we have $Q_{V_i}(z) \subset \Psi_x(z)$ and $co\Psi_x(z) \subset X_i$. It is easy to see that $z_i \notin co\Psi_x(z)$ for all $z \in X$. Moreover, for any $y \in X_i$, the set

$$\begin{aligned} \Psi_x^{-1}(y) &= \{z \in X : y \in \Psi_x(z)\} \\ &= \{z \in X \setminus F_{V_i} : y \in \Psi_x(z)\} \cup \{z \in F_{V_i} : y \in \Psi_x(z)\} \\ &= \{z \in X \setminus F_{V_i} : y \in A_{V_i}(z) \cap \phi_x(z)\} \cup \{z \in F_{V_i} : y \in A_{V_i}(z)\} \\ &= [(X \setminus F_{V_i}) \cap A_{V_i}^{-1}(y) \cap \phi_x^{-1}(y)] \cup [F_{V_i} \cap A_{V_i}^{-1}(y)] \\ &= [\phi_x^{-1}(y) \cup F_{V_i}] \cap A_{V_i}^{-1}(y) \end{aligned}$$

is compactly open in X. Therefore, Ψ_x is a KF_C-majorant of Q_{V_i} at x.

Hence Q_{V_i} is a KF_C-majorized correspondence. Now by assumption, there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap P_i(y))$ for all $i \in I$. Note that if $y \notin F_{V_i}$, then $x_i \in co(A_i(y) \cap P_i(y)) \subset coQ_{V_i}(y)$ and if $y \in F_{V_i}$, then $x_i \in co(A_i(y) \cap P_i(y)) \subset co(A_i(x)) = coQ_{V_i}(y)$. Thus for each $i \in I$, $x_i \in coQ_{V_i}(y)$. Moreover the set $\{x \in X : Q_{V_i}(x) \neq \emptyset\} = F_{V_i} \cup \{x \in X \setminus F_{V_i} : (A_i \cap P_i)(x) \neq \emptyset\} = F_{V_i} \cup E^i$ is open in X by condition (c). Therefore all hypotheses of Lemma 3.3.2 are satisfied, so that by Lemma 3.3.2, there exists a point $x_V = (x_{V_i})_{i\in I} \in K$ such that $Q_{V_i}(x_V) = \emptyset$ for all $i \in I$. Since for each $i \in I$, $A_i(x)$ is non-empty, we must have $x_{V_i} \in \overline{B_{V_i}}(x_V)$ and $A_i(x_V) \cap P_i(x_V) = \emptyset$. □

A proof similar to that of Theorem 3.3.4 gives the following result and is thus omitted:

Theorem 3.3.4′. Let $(X_i, A_i, B_i, P_i)_{i=1}^N$ be an N-person generalized game such that for each $i = 1, \cdots, N$, $X = \cup_{j=1}^\infty C_{i,j}$ where $\{C_{i,j}\}_{j=1}^\infty$ is an increasing sequence of non-empty compact convex subsets of a topological vector space E_i. Suppose the following conditions are satisfied:

(1) for each $i = 1, 2, \cdots, N$, A_i is lower semicontinuous and for each $x \in X = \Pi_{j\in I} X_j$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;

(2) for each $i = 1, 2, \cdots, N$, $A_i \cap P_i$ is is KF_C-majorized;

(3) for each $i = 1, 2, \cdots, N$, the set $F_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X;

(4) for each sequence $(y_n)_{n=1}^\infty$ in X which is escaping from X relative to $\{C_n\}_{n=1}^\infty$ where $C_n = \Pi_{i\in I} C_{i,n}$ for each $n = 1, 2, \cdots$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $\pi_i(x_n) \in A_i(y_n) \cap P_i(y_n)$ for all $i = 1, 2, \cdots, N$.

Then given any $V = \Pi_{i\in I} V_i$ where for each $i = 1, 2, \cdots, N$, V_i is an open convex neighborhood of zero in E_i, the generalized game $\Gamma_V = (X_i; A_i, B_{V_i}; P_i)_{i\in I}$ has an equilibrium point in X; i.e., there exists a point $x_V = (x_{V_i})_{i\in I} \in X$ such that $x_{V_i} \in \overline{B_{V_i}}(x_V)$ and $A_i(x_V) \cap P_i(x_V) = \emptyset$ for each $i = 1, 2, \cdots, N$, where $B_{V_i} = (B_i(x) + V_i) \cap X_i$ for each $x \in X$ and each $i = 1, 2, \cdots, N$.

3.4 Equilibria in Locally Convex Topological Vector Spaces

In this section, by constructing an approximate generalized game which is associated with a given generalized game (this procedure is called "*approximation method*" in our thesis), existence theorems for equilibria of generalized games are obtained In these theorems, the strategy spaces may be non-compact sets in infinite-dimensional locally convex topological vector spaces, the number of agents may be uncountably infinite and the preference correspondences may be non-total or non-transitive and may not have open lower (or upper) sections. Our results generalize many existence theorems for equilibria of generalized games by relaxing the compactness of strategy spaces and by weakening the continuity of constraint and preference correspondences. In particular, the question raised by Yannelis and Prabhakar [326] in 1983 is answered in the affirmative with weaker assumptions.

We shall need the following fact:

Lemma 3.4.1. Let X be a topological space, Y a non-empty subset of a topological vector space E, $\mathcal{B}$ a base for the zero neighborhoods in E and $B : X \to 2^Y$. For each $V \in \mathcal{B}$, Let $B_V : X \to 2^Y$ be defined by $B_V(x) = (B(x) + V) \cap Y$ for each $x \in X$. If $\hat{x} \in X$ and $\hat{y} \in Y$ are such that $\hat{y} \in \cap_{V \in \mathcal{B}} \overline{B_V}(\hat{x})$, then $\hat{y} \in \overline{B}(\hat{x})$.

Proof. Suppose $\hat{y} \notin \overline{B}(\hat{x})$, then $(\hat{x}, \hat{y}) \notin cl_{X\times Y}\text{Graph}(B)$. Let U be an open neighborhood of $\hat{x}$ in X and $V \in \mathcal{B}$ be such that

$$(*) \qquad (U \times (\hat{y} + V)) \cap Graph(B) = \emptyset.$$

Choose $W \in \mathcal{B}$ such that $W - W \subset V$. Since $\hat{y} \in \overline{B_W}(\hat{x})$, by assumption, $(\hat{x}, \hat{y}) \in cl_{X\times Y}Graph(B_W)$ so that $U \times (\hat{y} + W)) \cap Graph(B_W) \neq \emptyset$. Take any $x \in U$ and $w_1 \in W$ with $(x, \hat{y} + w_1) \in Graph(B_W)$ so that $\hat{y} + w_1 \in B_W(x) = (B(x) + W) \cap Y$. Let $z \in B(x)$ and $w_2 \in W$ be such that $y + w_1 = z + w_2 \in Y$, it follows that $z = \hat{y} + w_1 - w_2 \in \hat{y} + W - W \subset \hat{y} + V$ so that $(\hat{y} + V) \cap B(x) \neq \emptyset$ where $x \in U$. This contradicts $(*)$. Thus we must have $\hat{y} \in \overline{B}(\hat{x})$. □

Using Theorem 3.3.4 and Lemma 3.4.1, we shall present one of our main results in this section as follows:

Theorem 3.4.2. Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i\in I}$ be a generalized game such that $X = \Pi_{i\in I} X_i$ is paracompact Suppose the following conditions are satisfied:

(a) for each $i \in I$, X_i is a non-empty convex subset of a locally convex topological vector space E_i,

(b) for each $i \in I$, $A_i : X \to 2^{X_i}$ is lower semicontinuous such that for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;

(c) for each $i \in I$, $A_i \cap P_i$ is KF_C-majorized;

(d) for each $i \in I$, the set $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X;

(e) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap P_i(y))$ for all $i \in I$

Then $\mathcal{G}$ has an equilibrium point in K, i.e. there exists a point $\hat{x} = (\hat{x}_i)_{i\in I} \in K$ such that for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$

Proof. For each $i \in I$, let $\mathcal{B}_i$ be the collection of all open convex neighborhoods of zero in E_i and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i$. Given any $V \in \mathcal{B}$, let $V = \Pi_{j\in I} V_j$, where $V_j \in \mathcal{B}_j$ for each $j \in I$. By Theorem 3.3.4, there exists $\hat{x}_V \in K$ such that $\hat{x}_{V_i} \in \overline{B_{V_i}}(\hat{x}_V)$ and $A_i(\hat{x}_V) \cap P_i(\hat{x}_V) = \emptyset$ for each $i \in I$, where $B_{V_i}(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$. It follows that the set $Q_V := \{x \in K : x_i \in \overline{B_{V_i}}(x)$ and $A_i(x) \cap P_i(x) = \emptyset\}$ is a non-empty closed subset of K by (d).

Now we want to prove $\{Q_V\}_{V\in\mathcal{B}}$ has the finite intersection property. Let $\{V_1, \cdots, V_n\}$ be any finite subset of $\mathcal{B}$. For each $i = 1, \cdots, n$, let $V_i = \Pi_{j\in I} V_{ij}$ where $V_{ij} \in \mathcal{B}_j$ for each $j \in I$; let $V = \Pi_{j\in I}(\cap_{i=1}^n V_{ij})$, then $Q_V \neq \emptyset$. Clearly $Q_V \subset \cap_{i=1}^n Q_{V_i}$ so that $\cap_{i=1}^n Q_{V_i} \neq \emptyset$ Therefore the family $\{Q_V : V \in \mathcal{B}\}$ has the finite intersection property. Since K is compact, $\cap_{V\in\mathcal{B}} Q_V \neq \emptyset$. Now take any $\hat{x} \in \cap_{V\in\mathcal{B}} Q_V$, then for each $i \in I$, $\hat{x}_i \in \overline{B_{V_i}}(\hat{x})$ for each $V_i \in \mathcal{B}_i$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. By Lemma 3.4.1, we also have for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$. □

By Theorem 3.3.4′ and Lemma 3.4.1, we also have the following result which corresponds to Theorem 3.4.2:

Theorem 3.4.2′. Let $(X_i, A_i, B_i, P_i)_{i=1}^N$ be an N-person generalized game such that for each $i = 1, 2, \cdots, N$, $X_i = \cup_{j=1}^{\infty} C_{i,j}$ where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E_i. Suppose the following conditions are satisfied:

(1) for each $i = 1, 2, \cdots, N$, and for each $x \in X = \Pi_{j \in I} X_j$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;

(2) for each $i = 1, 2, \cdots, N$, A_i is lower semicontinuous;

(3) for each $i = 1, 2, \cdots, N$, the correspondence $A_i \cap P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is KF_C-majorized;

(4) for each $i = 1, 2, \cdots, N$, the set $F_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X;

(5) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ where $C_n = \Pi_{i \in I} C_{i,n}$ for each $n = 1, 2, \cdots$, there exist $n \in N$ and $x_n \in C_n$ such that $\pi_i(x_n) \in A_i(y_n) \cap P_i(y_n)$ for all $i = 1, 2, \cdots, N$.

Then $(X_i, A_i, B_i, P_i)_{i=1}^{N}$ has an equilibrium point $\hat{x} \in X$, i.e., for each $i = 1, 2, \cdots, N$, $\pi_i(\hat{x}) \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Corollary 3.4.3. Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be a generalized game such that $X = \Pi_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:

(a) for each $i \in I$, X_i is a non-empty convex subset of a locally convex topological vector space E_i;

(b) for each $i \in I$ and for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;

(c) for each $i \in I$, A_i and P_i have open lower sections;

(d) for each $i \in I$, $A_i \cap P_i$ is KF_C-majorized;

(e) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap P_i(y))$ for all $i \in I$.

Then $\mathcal{G}$ has an equilibrium point in K, i.e., there exists point $\hat{x} \in K$ such that for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. By (c), the map $A_i : X \to 2^{X_i}$ is lower semicontinuous and the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X. Therefore all the hypotheses in Theorem 3.4.2 are satisfied, so that the conclusion follows. □

Corollary 3.4.3′. Let $(X_i, A_i, B_i, P_i)_{i=1}^N$ be an N-person generalized game such that for each $i = 1, 2, \cdots, N$, $X_i = \cup_{j=1}^{\infty} C_{i,j}$ where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E_i. Suppose the following conditions are satisfied:

(1) for each $i = 1, 2, \cdots, N$, and for each $x \in X = \Pi_{j \in I} X_j$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$,

(2) for each $i = 1, 2, \cdots, N$, both A_i and P_i have open lower sections;

(3) for each $i = 1, 2, \cdots, N$, $A_i \cap P_i$ is KF_C-majorized;

(4) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ where $C_n = \Pi_{i \in I} C_{i,n}$ for each $n = 1, 2, \cdots$, there exist $n \in N$ and $x_n \in C_n$ such that $\pi_i(x_n) \in A_i(y_n) \cap P_i(y_n)$ for all $i = 1, 2, \cdots, N$.

Then $(X_i, A_i, B_i, P_i)_{i=1}^N$ has an equilibrium point $\hat{x} \in X$, i.e., for each $i = 1, 2, \cdots, N$, $\pi_i(\hat{x}) \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Corollary 3.4.3 (respectively, Corollary 3.4.3′) improves Theorem 6.1 of Yannelis and Prabhakar [326] in the following ways: (i) the index I need not be countable, (ii) for each $i \in I$, the set X_i need not be metrizable and (iii) for each $i \in I$, $A_i \cap P_i$ need not be of class KF

Corollary 3.4.4. Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be a generalized game such that $X = \Pi_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:

(a) for each $i \in I$ X_i is a non-empty convex subset of a locally convex topological vector space,

(b) for each $i \in I$ and for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;

(c) for each $i \in I$, A_i has an open graph in $X \times X_i$ (respectively, is lower semicontinuous) and P_i is lower semicontinuous (respectively, has an open graph in $X \times X_i$);

(d) for each $i \in I$, $A_i \cap P_i$ is KF_C-majorized;

(e) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap P_i(y))$ for all $i \in I$.

Then $\mathcal{G}$ has an equilibrium point in K, i.e., there exists point $\hat{x} \in X$ such that for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. For each $i \in I$, since A_i has an open graph in $X \times X_i$ (respectively, is lower semicontinuous) and P_i is lower semicontinuous (respectively, has an open graph in $X \times X_i$), the map $A_i \cap P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is also lower semicontinuous by Lemma 4.2 of Yannelis [325], so that the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X. Therefore all conditions of Theorem 3.3.3 are satisfied and the conclusion follows. □

Corollary 3.4.4′. Let $(X_i; A_i, B_i; P_i)_{i=1}^N$ be an N-person game such that for each $i = 1, 2, \cdots, N$, $X_i = \cup_{j=1}^{\infty} C_{i,j}$ where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E_i. Suppose the following conditions are satisfied:

(1) for each $i = 1, 2, \cdots, N$ and for each $x \in X = \Pi_{j\in I} X_j$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;

(2) for each $i = 1, 2, \cdots, N$, A_i has an open graph in $X \times X_i$ (respectively, is lower semicontinuou) and P_i is lower semicontinuous (respectively, has an open graph in $X \times X_i$);

(3) for each $i = 1, 2, \cdots, N$, the correspondence $A_i \cap P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is KF_C-majorized;

(4) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ where $C_n = \Pi_{i\in I} C_{i,n}$ for each $n = 1, 2, \cdots$, there exist $n \in N$ and $x_n \in C_n$ such that $\pi_i(x_n) \in A_i(y_n) \cap P_i(y_n)$ for all $i = 1, 2, \cdots, N$.

Then $(X_i, A_i, B_i, P_i)_{i=1}^N$ has an equilibrium point $\hat{x} \in X$, i.e., for each $i = 1, 2, \cdots, N$, $\pi_i(\hat{x}) \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Corollary 3.4.4 (respectively, 3.4.4′) generalizes Corollary 3 of Borglin and Keiding [37], Theorem 4.1 of Chang [55], Theorem of Shafer and Sonnenschein [263] and Theorem 5 of Tulcea [316].

Remark. In above, we have proved the existence theorem for equilibria of generalized games with non-compact and infinite dimensional strategy spaces, an infinite numbers of agents, and non-total and non-transitive constraint and preference correspondences which need not have open graphs or open lower or upper sections. Since it is well known that if a correspondence has an open graph, then it has open upper and lower sections (see Bergstrom et al [31, p.266] and thus the correspondences having open graphs are lower semicontinuous. However, a continuous correspondence need not have open lower or upper sections in general (see Yannleis and Prabhakar [326, p.237]). Also, in infinite settings, the set of feasible allocations generally is not compact in the commodity spaces. Our result generalizes many results in literatures by relaxing the compactness of strategy spaces and the openness of graphs or lower (upper) sections of constraint correspondences.

In 1983, Yannelis and Prabhakar [326] gave the following existence theorem for equilibria of generalized games:

Theorem 3.4.A. Let $\Gamma = (X_i; A_i; P_i)_{i\in I}$ be a generalized game satisfying for each $i \in I$ (where I is countable):

(i) X_i is a non-empty compact, convex and metrizable subset of a locally convex topological vector space;

(ii): the mapping $A_i : X(= \Pi_{i\in I}X_i) \to 2^{X_i}$ satisfies that $clA_i(x) = \overline{A_i}(x)$ for each $x \in X$ (so that the mapping clA_i is upper semicontinuous);

(iii): A_i and P_i have open lower sections; and

(iv): $x_i \notin cop_i(x)$ for all $x \in X$.

Then Γ has an equilibrium, i.e., there exists $\hat{x} \in X$ is such that $\hat{x}_i \in clA_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

In addition, Yannelis and Prabhakar asked that *if Theorem 3.4.A can be extended to non-metrizable subsets without introducing additional assumptions ?*

Theorem 3.4.2 improves Theorem 3.4.A of Yannelis and Prabhakar in the following ways: (i) the index set I need not be countable, (ii) for each $i \in I$, the set X_i need not be metrizable and need not compact (iii) for each $i \in I$, both A_i and P_i need not have open lower sections. Therefore the question raised by Yannelis and Prabhakar [326, p.242] is answered in the affirmative with weaker assumptions.

Let X and Y be topological spaces. A correspondence $T : X \to 2^Y \cup \{\emptyset\}$ is said to be quasi-regular if

(i) it has open lower sections (i.e., for each $y \in Y$, $T^{-1}(y)$ is open in X);

(ii) $T(x)$ is non-empty and convex for each $x \in X$;

(iii) $\overline{T}(x) = cl_Y T(x)$ for all $x \in X$.

The correspondence T is said to be regular if it is quasi-regular and has an open graph.

Let X be a non-empty set, Y a non-empty subset of a topological vector space E and $F : X \to 2^Y$. A family $(f_j)_{j \in J}$ of correspondences between X and Y, indexed by a non-empty filtering set J (we denote by $\leq$ the order relation in J), is an upper approximating family for the mapping F (e.g, see [317, p.269]) if

(A_I): $F(x) \subset f_j(x)$ for all $x \in X$ and all $j \in J$;

(A_{II}): for each $j \in J$ there is $j^* \in J$ such that for each $h \in J$ with $h \geq j^*$, $f_h(x) \subset f_j(x)$ for each $x \in X$;

(A_{III}): for each $x \in X$ and $V \in \mathcal{B}$, where $\mathcal{B}$ is a base for the zero neighborhoods in E, there is $j_{x,V} \in J$ such that $f_h(x) \subset F(x) + V$ if $h \in J$ and $j_{x,V} \leq h$.

From (A_I)-(A_{III}), it is easy to deduce that:

(A_{IV}): for each $x \in X$ and $k \in J$, $F(x) \subset \cap_{j \in J} f_j(x) = \cap_{k \leq j, k \in J} f_j(x) \subset cl F(x) \subset \overline{F}(x)$.

If X is a subset of a topological vector space E, X is said to have the property (K) if for every compact subset B of X, the convex hull of B is relatively compact in X.

By Theorem 3 and its Remark in Tulcea [317, p.280 and p.281-282], we have the following:

Lemma 3.4.5. Let $(X_i)_{i \in I}$ be a family of paracompact spaces and $(Y_i)_{i \in I}$ be a family of non-empty closed convex subsets, each in a locally convex topological vector space and each having property (K). For each $i \in I$, let $F_i : X_i \to 2^{Y_i}$ be such that F_i is compact and upper semicontinuous with compact convex values. Then there is a common filtering set J (independent of $i \in I$) such that for each $i \in I$, there is a family $(f_{ij})_{j \in J}$ of correspondences between X_i and Y_i with the following properties:

(a) for each $j \in J$, f_{ij} is regular;

(b) $(f_{ij})_{j\in J}$ and $(\overline{f_{ij}})_{j\in J}$ are upper approximating families for F_i and

(c) for each $j \in J$, $\overline{f_{ij}}$ is continuous if Y_i is compact.

By the above approximation theorem for upper semicontinuous correspondences (Tulcea [317, Theorem 3, p.280]), we can also prove the following existence theorem for equilibria of generalized games in which the constraint correspondences are upper semicontinuous.

Theorem 3.4.6. Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i\in I}$ be a generalized game such that $X = \Pi_{i\in I} X_i$ is paracompact. Suppose the following conditions are satisfied:

(a) for each $i \in I$, X_i is a non-empty closed convex subset of a locally convex topological vector space E_i and X_i has the property (K);

(b) for each $i \in I$, B_i is compact and upper semicontinuous with non-empty compact convex values and $A_i(x) \subset B_i(x)$ for each $x \in X$;

(c) for each $i \in I$, P_i is lower semicontinuous and KF_C-majorized;

(d) for each $i \in I$, $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X;

(e) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap P_i(y))$ for all $i \in I$.

Then there exists $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in \overline{B_i}(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$.

Proof. By Lemma 3.4.5, there is a common filtering set J such that for every $i \in I$, there exists a family $(B_{ij})_{j\in J}$ of regular correspondences between X and X_i such that both $(B_{ij})_{j\in J}$ and $(\overline{B_{ij}})_{j\in J}$ are upper approximating of families for B_i.

Let $j \in J$ be arbitrarily fixed. The game $\mathcal{G}_j = (X_i; B_{ij}, \overline{B_{ij}}; P_i)_{i\in I}$ satisfies all hypotheses of Theorem 3.4.2 Hence $\mathcal{G}_j$ has an equilibrium $\bar{x}^j \in K$ such that $B_{ij}(\bar{x}^j) \cap P_i(\bar{x}^j) = \emptyset$, and $\pi_i(\bar{x}^j) \in \overline{B_{ij}}(\bar{x}^j)$ for all $i \in I$.

Since $(\bar{x}^j)_{j\in J}$ is a net in the compact set K, without loss of generality we may assume that $(\bar{x}^j)_{j\in J}$ converges to $x^* \in K$. Then for each $i \in I$, $\pi_i(x^*) = \lim_{j\in J} \pi_i(\bar{x}^j)$. Note that for every $j \in J$ and $x \in X$, $A_i(x) \subset B_i(x) \subset B_{ij}(x)$, we have $A_i(\bar{x}^j) \cap P_i(\bar{x}^j) = \emptyset$ for all $i \in I$. By condition (d), for every $i \in I$, $A_i(x^*) \cap P_i(x^*) = \emptyset$. As $\overline{B_{ij}}$ has closed graph, $(x^*, x_i^*) \in Graph\overline{B_{i,j}}$ for every $i \in I$. For each $i \in I$, since $(\overline{B_{i,j}})_{j\in J}$ is also an upper approximation family for B_i, $\cap_{j\in J}\overline{B_{i,j}}(x) \subset \overline{B_i}(x)$ for each $x \in X$ so that $(x^*, x_i^*) \in Graph\overline{B_i}$. Therefore for each $i \in I$, $A_i(x^*) \cap P_i(x^*) = \emptyset$ and $\pi_i(x^*) \in \overline{B_i}(x^*)$. □

Corresponding to Theorem 3.4.6, we have:

Theorem 3.4.6′. Let $(X_i; A_i; P_i)_{i=1}^N$ be an N-person generalized game such that for each $i = 1, 2, \cdots, N$, $X_i = \cup_{j=1}^{\infty} C_{i,j}$ is closed and has property (K), where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E_i. Suppose the following conditions are satisfied:

(1) for each $i = 1, 2, \cdots, N$, $A_i : X \to 2^{X_i}$ is compact and upper semicontinuous with non-empty compact and convex values;

(2) for each $i = 1, 2, \cdots, N$, the correspondence $P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is lower semicontinuous and KF_C-majorized;

(3) for each $i = 1, 2, \cdots, N$, the set $F_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X;

(4) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ where $C_n = \Pi_{i\in I} C_{i,n}$ for each $n = 1, 2, \cdots$, there exist $n \in N$ and $x_n \in C_n$ such that $\pi_i(x_n) \in A_i(y_n) \cap P_i(y_n)$ for all $i \in I$.

Then $(X_i; A_i; P_i)_{i=1}^N$ has an equilibrium point $\hat{x} \in X$, i.e., for each $i = 1, 2, \cdots, N$, $\pi_i(\hat{x}) \in A_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

If X_i is non-empty compact convex in Theorem 3.4.6, we have

Corollary 3.4.7. Let $\mathcal{G} = (X_i; A_i; P_i)_{i\in I}$ be a generalized game and let $X = \Pi_{i\in I} X_i$. Suppose the following conditions are satisfied for each $i \in I$:

(a) X_i is a non-empty compact convex subset of the locally convex topological vector space E_i;

(b) $A_i : X \to 2^{X_i}$ is upper semicontinuous with non-empty compact and convex values for each $x \in X$;

(c) $P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is lower semicontinuous and KF_C-majorized;

(d) for each $i \in I$, $E^i = \{x \in X; (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X;

Then there exists an $\hat{x} \in X$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Corollary 3.4.7 also generalizes Theorem 5 of Tulcea [317] and Theorem of Shafer-Sonnenschein [263, p 374].

By Corollary 3.4.7, we obtain the well-known fixed point theorem of the Fan-Glicksberg (see Fan [97] or Glicksberg [127]) for upper semicontinuous correspondence in locally convex topological vector space.

Corollary 3.4.8. Let X be a compact and convex subset of a locally convex topological vector space E and let $A : X \to 2^X$ be upper semicontinuous with non-empty closed and convex values. Then A has a fixed point.

Proof. Let $I = \{1\}$ and define $P : X \to 2^X \cup \{\emptyset\}$ by $P_i(x) = \emptyset$ for each $x \in X$ in Corollary 3.4.7, then conclusion is true. □

The following example shows that the condition (d) "for each $i \in I$, $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X" of Theorem 3.4.6 is essential.

Example. Let $I = \{1\}$ and $X = [0,1]$. Define $A, P : X \to 2^X \cup \{\emptyset\}$ by

$$A(x) = \begin{cases} [1/2, 1], & \text{if } x \in [0, 1/2), \\ [0,1], & \text{if } x = 1/2, \\ [0, 1/2], & \text{if } x \in (1/2, 1], \end{cases}$$

and

$$P(x) = \begin{cases} \emptyset, & \text{if } x = 0, \\ [0, x), & \text{if } x \in (0, 1]. \end{cases}$$

Then A is upper semicontinuous with non-empty closed convex values and the fixed point set of A is the singleton set $\{1/2\}$. The correspondence P has convex values with open lower sections since for each $y \in [0,1]$, $P^{-1}(y) = (y, 1]$ which is open in X. Therefore A, P and X satisfy all conditions of Corollary 3.4.7 except that $E = \{x \in [0,1] : A(x) \cap P(x) \neq \emptyset\} = [1/2, 1]$ is closed but not open in $[0,1]$. But $A(1/2) \cap P(1/2) \neq \emptyset$, i.e., the generalized game $\Gamma = ([0,1]; A; P)$ has no equilibrium point.

3.5 Equilibria for $\mathcal{U}$-Majorized Mappings

The objective of this section is to give some existence theorems for maximal elements and equilibria in qualitative games without the compactness (or paracompactness) assumption on the domain of the preferences which are majorized by upper semicontinuous correspondences instead of being majorized by correspondences which have lower open sections. Our intention is to merely illustrate a certain technique that we think will be of use in various problems of mathematical economics. Many other results of the type proved here may be proved under more general conditions.

Let X be a topological space, Y a non-empty subset of a vector space E, let $\theta : X \rightarrow E$ be a map and $\phi : X \rightarrow 2^Y \cup \{\emptyset\}$ a correspondence. Then (1) ϕ is said to be of class $\mathcal{U}_\theta$ if (a) for each $x \in X$, $\theta(x) \notin \phi(x)$ and (b) ϕ is upper semicontinuous with closed and convex values in Y; (2) ϕ_x is a $\mathcal{U}_\theta$-majorant of ϕ at x if there is an open neighborhood $N(x)$ of x in X and $\phi_x : N(x) \rightarrow 2^Y$ such that (a) for each $z \in N(x)$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \phi_x(z)$ and (b) ϕ_x is upper semicontinuous with closed and convex values; (3) ϕ is said to be $\mathcal{U}_\theta$-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists a $\mathcal{U}_\theta$-majorant ϕ_x of ϕ at x. We remark that when $X = Y$ and $\theta = I_X$, the identity map on X, our notions of a $\mathcal{U}_\theta$-majorant of ϕ at x and a $\mathcal{U}_\theta$-majorized correspondence are generalizations of upper semicontinuous correspondences which are irreflexive (i.e., $x \notin \phi(x)$ for all $x \in X$) and have closed convex values. Here we shall deal mainly with either the case (I) $X = Y$ and is a non-empty convex subset of the topological vector space E and $\theta = I_X$, the identity map on X, or the case (II) $X = \Pi_{i \in I} X_i$ and $\theta = \pi_j : X \rightarrow X_j$ is the projection of X onto X_j and $Y = X_j$ is a non-empty convex subset of a topological vector space. In both cases (I) and (II), we shall write $\mathcal{U}$ in place of $\mathcal{U}_\theta$.

We shall need the following:

Theorem 3.5.A. Let X be a topological space and Y a normal space. If $F, G : X \rightarrow 2^Y \cup \{\emptyset\}$ have closed values and are upper semicontinuous at $x \in X$, then $F \cap G$ is also upper semicontinuous at x.

Proof. If $F(x) \cap G(x) \neq \emptyset$, the conclusion follows from Hildenbrand [147, Proposition

B.III.2, p.23-23] (also see Klein and Thompson [189, Theorem 7.3.10, p.86]). If $F(x) \cap G(x) = \emptyset$, since Y is normal, it is easy to see that there exists an open neighborhood N of x in X such that $F(z) \cap G(z) = \emptyset$ for all $z \in N$; thus $F \cap G$ is also upper semicontinuous at x. □

We remark here that in Theorem 3.5.A above, we do not require $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$.

We shall also need the following result which generalizes and extends Lemma 6.1 of Yannelis and Prabhakar [326]:

Lemma 3.5.1. Let X and Y be two topological spaces and A be a closed (respectively, open) subset of X. Suppose $F_1 : X \to 2^Y \cup \{\emptyset\}$, $F_2 : A \to 2^Y \cup \{\emptyset\}$ are lower semicontinuous (respectively, upper semicontinuous) such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then the map $F : X \to 2^Y \cup \{\emptyset\}$ defined by

$$F(x) = \begin{cases} F_1(x), & \text{if } x \notin A\ ; \\ F_2(x), & \text{if } x \in A \end{cases}$$

is also lower semicontinuous (respectively, upper semicontinuous).

Proof. Let U be any closed (respectively, open) subset of Y. Clearly

$$\begin{aligned} \{x \in X : F(x) \subset U\} &= \{x \in A : F_2(x) \subset U\} \cup \{x \in X \setminus A : F_1 \subset U\} \\ &\subset \{x \in A : F_2(x) \subset U\} \cup \{x \in X : F_1(x) \subset U\}. \end{aligned}$$

Conversely, if $x \in A$ and $F_2(x) \subset U$, then $F(x) = F_2(x) \subset U$ and if $x \in A$ and $F_1(x) \subset U$, then $F(x) = F_2(x) \subset F_1(x) \subset U$. If $x \in X \setminus A$, then $F(x) = F_1(x) \subset U$. This shows that we have $\{x \in X : F(x) \subset U\} \supset \{x \in A : F_2(x) \subset U\} \cup \{x \in X : F_1(x) \subset U\}$. Hence $\{x \in X : F(x) \subset U\} = \{x \in A : F_2(x) \subset U\} \cup \{x \in X : F_1(x) \subset U\}$. Since A and U are closed (respectively, open) and F_1 and F_2 are lower semicontinuous (respectively, upper semicontinuous), the set $\{x \in X : F(x) \subset U\}$ is also closed (respectively, open). Therefore, F is lower semicontinuous (respectively, upper semicontinuous). □

We also need the following:

Lemma 3.5.2. Let X be a paracompact space and Y be a non-empty normal subset of a topological vector space E. Let $\theta : X \to E$ and $P : X \to 2^Y \cup \{\emptyset\}$ be $\mathcal{U}$-majorized. Then there exists a correspondence $\Psi : X \to 2^Y \cup \{\emptyset\}$ of class $\mathcal{U}$ such that $P(x) \subset \Psi(x)$ for each $x \in X$.

Proof. Since P is $\mathcal{U}$-majorized, for each $x \in X$, let $N(x)$ be an open neighborhood of x in X and $\psi_x : N(x) \to 2^Y \cup \{\emptyset\}$ be such that (1) for each $z \in N(x)$, $P(z) \subset \psi_x(z)$ and $\theta(z) \notin \psi_x(z)$ and (2) ψ_x is upper semicontinuous with closed and convex values. Since X is paracompact and $X = \cup_{x\in X} N(x)$, by Theorem VIII.1.4 of Dugundji [89, p.162], the open covering $\{N(x)\}$ of X has an open precise neighborhood-finite refinement $\{N'(x)\}$. For each $x \in X$, define $\psi'_x : X \to 2^Y \cup \{\emptyset\}$ by

$$\psi'_x(z) = \begin{cases} \psi_x(z), & \text{if } z \in N'(x); \\ Y, & \text{if } z \notin N'(x), \end{cases}$$

then ψ'_x is also upper semicontinuous on X by Theorem 3.5.A above and is such that $P(z) \subset \psi'_x(z)$ for each $z \in X$.

Now define $\Psi : X \to 2^Y \cup \{\emptyset\}$ by $\Psi(z) = \cap_{x\in X} \psi'_x(z)$ for each $z \in X$. Clearly, Ψ has closed and convex values and $P(z) \subset \Psi(z)$ for each $z \in X$. Let $z \in X$ be given, then $z \in N'(x)$ for some $x \in X$ so that $\psi'_x(z) = \psi_x(z)$ and hence $\Psi(z) \subset \psi_x(z)$; as $\theta(z) \notin \psi_x(z)$, we must also have that $\theta(z) \notin \Psi(z)$. Thus $\theta(z) \notin \Psi(z)$ for all $z \in X$.

Now we shall show that Ψ is upper semicontinuous. For any given $u \in X$, there exists an open neighborhood M_u of u in X such that the set $\{x \in X : M_u \cap N(x) \neq \emptyset\}$ is finite, say $= \{x(u,1), \cdots, x(u,n(u))\}$. Thus we have that

$$\Psi(w) = \cap_{x\in X} \psi'_x(w) = \cap_{i=1}^{n(u)} \psi'_{x(u,i)}(w) \qquad \text{for all } w \in M_u.$$

For $i = 1, \cdots, n(u)$, since each $\psi'_{x(u,i)}$ is upper semicontionuous on X and hence on M_u with closed values and Y is normal, by Theorem 3.5.A above again, $\Psi : M_u \to 2^Y$ is also upper semicontinuous at u. Since M_u is open, $\Psi : X \to 2^Y$ is also upper semicontinuius at u. Hence Ψ is of class $\mathcal{U}$. □

We now prove the following theorem concerning the existence of a maximal element for $\mathcal{U}$-majorized correspondences:

Theorem 3.5.3. Let X be a non-empty convex subset of a locally convex topological vector space and D a non-empty compact subset of X. Let $P : X \to 2^D \cup \{\emptyset\}$ be $\mathcal{U}$-majorized (i.e., $\mathcal{U}_{I_X}$-majorized). Then there exists a point $x \in coD$ such that $P(x) = \emptyset$.

Proof. Suppose the contrary, i.e., for all $x \in coD$, $P(x) \neq \emptyset$. Then for each $x \in coD$, $P(x) \neq \emptyset$ and coD is also paracompact by Lemma 1 of Ding, Kim and Tan [86, p.206] (see also Lassonde [201, p.49]). Now applying Lemma 3.5.2, there exists a correspondence $\Psi : coD \to 2^D$ of class $\mathcal{U}$ such that for each $x \in coD$, $P(x) \subset \Psi(x)$. Since Ψ is upper semicontinuous with non-empty closed and convex values, by a fixed point Theorem of Himmelberg [151, Theorem 2, p.206], there exists $x \in coD$ such that $x \in \Psi(x)$. This contradicts that Ψ is of class $\mathcal{U}$. Hence the conclusion must holds. □

In what follows, we shall give some applications of Theorem 3.5.2 and Theorem 3.5.3. First we have the following:

Theorem 3.5.4. Let X be a non-empty convex subset of a locally convex topological vector space and D be a non-empty compact subset of X. Let $P : X \to 2^D$ be $\mathcal{U}$-majorized and $A : X \to 2^D$ be upper semicontinuous with closed and convex values. Then there exist a point $\hat{x} \in coD$ such that either $\hat{x} \in A(\hat{x})$ and $P(\hat{x}) = \emptyset$ or $\hat{x} \notin A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Proof. Let $F = \{x \in X : x \in A(x)\}$. We first note that F is closed in X since A is upper semicontinuous with closed values. Define $\phi : X \to 2^D$ by

$$\phi(x) = \begin{cases} P(x), & \text{if } x \in F, \\ A(x) \cap P(x), & \text{if } x \notin F. \end{cases}$$

If $x \notin F$ and $A(x) \cap P(x) \neq \emptyset$, then $X \setminus F$ is an open neighborhood of x in X and since P is $\mathcal{U}$-majorized, there exist an open neighborhood $N(x)$ of x in X and a mapping $\psi_x : N(x) \to 2^D$ such that (1) for each $z \in N(x)$, $P(z) \subset \psi_x(z)$ and $z \notin \psi_x(z)$ and (2) ψ_x is upper semicontinuous with closed and convex values. Without loss of generality, we may assume that $N(x) \subset X \setminus F$. We now define the mapping $\Psi_x : X \to 2^D$ by $\Psi_x(z) = A(z) \cap \psi_x(z)$ for each $z \in N(x)$. Then again by Lemma 3.5.1 (note that D is compact so that D is normal), we have (1) Ψ_x is upper semicontinuous with closed and convex values and (2) for each $z \in N(x)$, $z \notin \Psi_x(z)$. Thus Ψ_x is a $\mathcal{U}$-majorant of ϕ at x.

Now suppose that $x \in F$ and $P(x) \neq \emptyset$; then by assumption there exist an open neighborhood $N(x)$ of x in X and $\psi_x : N(x) \to 2^D$ such that (a) $P(z) \subset \psi_x(z)$ and $z \notin \psi_x(z)$ for each $z \in N(x)$ and (b) ψ_x is upper semicontinuous with closed and convex values. Define $\psi'_x : N(x) \to 2^D$ by

$$\psi'_x(z) = \begin{cases} \psi_x(z), & \text{if } z \in N(x) \cap F, \\ A(x) \cap \psi_x(z), & \text{if } z \notin N(x) \setminus F, \end{cases}$$

then (i) for each $z \in N(x)$, it is easy to see that $\phi(z) \subset \psi'_x(z)$ and $z \notin \psi'_x(z)$, (ii) the mapping $A \cap \psi_x : N(x) \setminus F \to 2^D$ defined by $(A \cap \psi_x)(z) = A(z) \cap \psi_x(z)$ for each $z \in N(x) \setminus F$ is upper semicontinuous with closed and convex values by Lemma 3.5.1. It follows that the mapping ψ'_x is also upper semicontinuous with closed and convex values by Lemma 3.5.1 since $N(x) \setminus F$ is open in $N(x)$. This shows that ψ'_x is a $\mathcal{U}$-majorant of ϕ at x.

Therefore ϕ is $\mathcal{U}$-majorized. By Theorem 3.5.3, there exists a point $\hat{x} \in coD \subset X$ such that $\phi(\hat{x}) = \emptyset$. By the definition of ϕ, either $P(\hat{x}) = \emptyset$ and $\hat{x} \in A(\hat{x})$ or $A(\hat{x}) \cap P(\hat{x}) = \emptyset$ and $\hat{x} \notin A(\hat{x})$. $\square$

The following is an equilibrium existence theorem for a qualitative game:

Theorem 3.5.5. Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game such that for each $i \in I$,

(a) X_i is a non-empty convex subset of a locally convex topological vector space E_i and D_i is a non-empty compact subset of X_i;

(b) the set $E^i = \{x \in X : P_i(x) \neq \emptyset\}$ is open in X;

(c) $P_i : E^i \to 2^{D_i} \cup \{\emptyset\}$ is $\mathcal{U}$-majorized;

(d) there exists a non-empty compact and convex subset F_i of D_i such that $F_i \cap P_i(x) \neq \emptyset$ for each $x \in E^i$.

Then there exists a point $x \in X$ such that $P_i(x_i) = \emptyset$ for all $i \in I$.

Proof. Since D_i is a non-empty compact subset of X_i for each $i \in I$, the set $D = \Pi_{i \in I} D_i$ is also a non-empty compact subset of X. Now for each $x \in X$, let

$I(x) = \{i \in I : P_i(x) \neq \emptyset\}$ Define a correspondence $P : X \to 2^D \cup \{\emptyset\}$ by

$$P(x) = \begin{cases} \cap_{i \in I(x)} P_i'(x), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset, \end{cases}$$

where $P_i'(x) = \Pi_{j \neq i, j \in I} F_j \times P_i(x)$ for each $x \in X$

Then by condition (d) and definition of P, for each $x \in X$ with $I(x) \neq \emptyset$, $P(x) \neq \emptyset$. Let $x \in X$ be such that $P(x) \neq \emptyset$ Fix an $i \in I(x)$. By assumption (c), there exist an open neighborhood $N(x)$ of x in E^i and $\phi_i : N(x) \to 2^{D_i}$ such that (i) for each $z \in N(x)$, $P_i(z) \subset \phi_i(z)$ and $\pi_i(z) \notin \phi_i(z)$ and (ii) ϕ_i is upper semicontinuous with closed and convex values. Note that by (b), $N(x)$ is also an open neighborhood of x in X and for each $z \in N(x)$, $P_i(z) \neq \emptyset$ so that $i \in I(z)$ for each $z \in N(x)$. Now we define $\Phi_x : N(x) \to 2^D$ by $\Phi_x(z) = \Pi_{j \neq i, j \in I} F_j \otimes \phi_i(z)$ for each $z \in N(x)$. We observe that (1) for each $z \in N(x)$, $P(z) \subset P_i'(z) \subset \Phi_x(z)$ and $z \notin \Phi_x(z)$; (2) Φ_x has closed and convex values and (3) since $\Pi_{j \neq i, j \in I} F_j$ and $\phi_x(z)$ are compact for each $z \in N(x)$, it is easy to see that Φ_x is also upper semicontinuous. Therefore, Φ_x is a $\mathcal{U}$-majorant of P at x. Thus P is $\mathcal{U}$-majorized. Now by Theorem 3.5.3, there exists a point $x \in coD \subset X$ such that $P(x) = \emptyset$ which implies that $P_i(x) = \emptyset$ for all $i \in I$. □

Theorem 3.5.6. Let $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ be a generalized game where I is any (countable or uncountable) set of players such that for each $i \in I$:

(i) X_i is a non-empty compact and convex subset of a locally convex topological vector space E_i;

(ii) for each $x \in X$, $A_i(x)$ is non-empty, $A_i(x) \subset B_i(x)$ and $B_i(x)$ is convex;

(iii) the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is paracompact (which is satisfied if X_i is metrizable) and open in X;

(iv) the mapping $A_i \cap P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is $\mathcal{U}$-majorized on E^i.

Then Γ has an equilibrium point, i.e., there exists a point $x \in X$ such that $\pi_i(x) \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in I$.

Proof. Let $I_0 = \{i \in I : E^i \neq \emptyset\}$. Suppose $I_0 = \emptyset$. Then for each $i \in I$ and for each $x \in X$, $A_i(x) \cap P_i(x) = \emptyset$. Define $B : X \to 2^X \cup \{\emptyset\}$ by $B(x) = \Pi_{i \in I} \overline{B_i}(x)$ for each $x \in X$. Since each $\overline{B_i}$ has a closed graph, it is easy to see that B also has a closed graph. Since X is compact, B is upper semicontinuous. By the Fan [97] and Glicksberg [127] fixed point theorem, there exists $\hat{x} \in X$ such that $\hat{x} \in B(\hat{x})$, i.e., $\pi_i(\hat{x}) \in \overline{B_i}(\hat{x})$ for all $i \in I$. Thus we may assume without loss of generality that $I_0 \neq \emptyset$.

Case 1: For each $i \in I_0$, by (iv) and Lemma 3.5.2 (note that the set X_i, being compact Haudorff, is normal), there exists a mapping $\psi_i : E^i \to 2^{X_i} \cup \{\emptyset\}$ which is upper semicontinuous with closed and convex values such that $A_i(x) \cap P_i(x) \subset \psi_i(x)$ and $\pi_i(x) \notin \psi_i(x)$ for each $x \in E^i$. Since $\overline{B_i} : X \to 2^{X_i} \cup \{\emptyset\}$ is upper semicontinuous with closed and convex values, the correspondence $\psi_i \cap \overline{B_i} : E^i \to 2^{X_i} \setminus \{\emptyset\}$ is also upper semicontinuous with non-empty closed and convex values by (ii) and Proposition B.III.2 of Hildenbrand [147, p.23-24]. Define a correspondence $\phi_i : X \to 2^{X_i} \setminus \{\emptyset\}$ by

$$\phi_i(x) = \begin{cases} \overline{B_i}(x), & \text{if } x \notin E^i, \\ (\psi_i \cap \overline{B_i})(x), & \text{if } x \in E^i. \end{cases}$$

Then Lemma 3.5.1 implies that ϕ_i is upper semicontinuous with non-empty closed and convex values.

Case 2: For each $i \in I \setminus I_0$, we define a correspondence $\phi_i : X \to 2^{X_i} \cup \{\emptyset\}$ by $\phi_i(x) = \overline{B_i}(x)$ for each $x \in X$. Then ϕ_i is upper semicontinuous with non-empty compact and convex values.

Now define the correspondence $\Psi : X \to 2^X$ by $\Psi(x) = \Pi_{i\in I}\phi_i(x)$ for each $x \in X$. Then Ψ is also upper semicontinuous with non-empty compact and convex values. By the Fan [97] and Glicksberg [127] fixed point theorem again, there exists a point $x \in X$ such that $x \in \Psi(x)$. If there exists $i \in I_0$ such that $x \in E^i$, then $\pi_i(x) \in \phi_i(x) = \overline{B_i}(x) \cap \psi_i(x) \subset \psi_i(x)$ which is a contradiction; it follows that $x \notin E^i$ for all $i \in I_0$. Hence we must have $\pi_i(x) \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in I$. $\square$

It seems natural to replace the condition (iii) of Theorem 3.5.6 by the condition " *the set* $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ *is closed in* X"; however the following simple example shows that this can not be done:

Example. Let $X = [0, 1]$ and define $A, B, P : X \to 2^X \cup \{\emptyset\}$ by

$$B(x) = A(x) = \begin{cases} [1/2, 1], & \text{if } x \in [0, 1/2), \\ [0, 1], & \text{if } x = 1/2, \\ [0, 1/2], & \text{if } x \in (1/2, 1]. \end{cases}$$

and

$$P(x) = \begin{cases} \{x/4\}, & \text{if } x \in [\frac{1}{2}, 1], \\ \emptyset, & \text{if } x \in [0, 1/2). \end{cases}$$

It is easy to see that A and P are both upper semicontinuous with closed and convex values and $x \notin P(x)$ for each $x \in X$; thus $A \cap P$ is $\mathcal{U}$-majorized. Note that the subset $E = \{x \in X : A(x) \cap P(x) \neq \emptyset\} = [1/2, 1]$ is closed in $[0, 1]$ and A, B and P satisfy the hypotheses (i), (ii), (iv) but not (iii) of Theorem 3.5.6. However, at the unique fixed point $1/2$ of the correspondence A, we have $A(\frac{1}{2}) \cap P(\frac{1}{2}) = [0, 1] \cap \{1/8\} \neq \emptyset$. Thus the generalized game $([0, 1]; A, B; P)$ has no equilibrium point.

3.6 Equilibria for Ψ-Condensing Mappings

In this section, we shall prove some existence theorems for maximal elements for Ψ-condensing correspondences which are either KF_C-majorized or $\mathcal{U}$-majorized and whose domain are non-compact sets in locally convex topological vector spaces. As an application, we obtain an existence theorem for equilibrium points for a one-person game from which an existence theorem for N-person games is derived. Finally, we give an existence theorem equilibria of generalized games with a countable or uncountable set of players such that the intersection of constraint and preference correspondences are $\mathcal{U}$-majorized and constraint correspondences are Ψ-condensing.

The object of this part is to present a method for proving the existence of maximal elements and equilibria of generalized games which enables one to remove altogether the compactness (or paracompactness) assumption on the domain (and /or codomain) of the preference and constraint correspondences. This is done by strengthening the assumptions on the preference or constraint correspondences. The basic idea underlying the method may be explained as follows.

Let X be a non-empty subset of a locally convex topological vector space E. We introduce a function $\Psi : 2^X \to \mathbf{R}$ which assigns to each relatively compact subset D of X the value zero, and call it a measure of non-compactness. Intuitively, $\Psi(D)$ measures how far a set is from being relatively compact. The larger the value $\Psi(D)$ the "more non-compact" a set D is in X. Now a multivalued map $T : X \to 2^E \cup \{\emptyset\}$ is said to be Ψ-condensing if for each subset D that is not relatively compact, we have $\Psi(T(D)) < \Psi(D)$.

It should be noted that if $T : X \to 2^E \cup \{\emptyset\}$ is a compact mapping (i.e., if $T(X)$ is contained in a compact subset K of E), then the mapping T is automatically Ψ-condensing for any measure of non-compactness.

Let X be a non-empty set; a mapping $P : X \to 2^X \cup \{\emptyset\}$ is acyclic (e.g., see Bergstrom [30, p.403]) if for each $n \in \mathbf{N}$ with $x_{i+1} \in P(x_i)$ for $i = 1, 2, \cdots n-1$ implies that $x_1 \notin P(x_n)$.

Let C denote a lattice with a least element 0. We now also recall some definitions introduced by Fitzpatick and Petryshyn [112].

Let X be a locally convex topological vector space. Then a mapping $\Psi : 2^X \to C$ is called a measure of non-compactness provided that the following conditions hold for any $A, B \in 2^X$:

(1) $\Psi(A) = 0$ if and only if A is precompact (i.e., it is relative compact);

(2) $\Psi(\overline{co}A) = \Psi(A)$, where $\overline{co}A$ denotes the closed convex hull of A;

(3) $\Psi(A \cup B) = max\{\Psi(A), \Psi(B)\}$.

It follows from (3) that if $A \subset B$, then $\Psi(A) \leq \Psi(B)$. The above notion is a generalization of the set-measure of non-compactness of Kuratowski [198] and the ball-measure of non-compactness of Sadovskii [257] defined either in terms of a family of seminorms when X is a locally convex topological vector space by Gohberg et al [129] or in terms of a single norm when X is a Banach space.

Let $\Psi : 2^X \to C$ be a measure of non-compactness of X and $D \subset X$. A mapping $T : D \to 2^X$ is called Ψ-condensing provided that if $\Omega \subset D$ and $\Psi(T(\Omega)) \geq \Psi(\Omega)$, then Ω is relatively compact.

Note that if $T : D \to 2^X$ is a compact mapping (i.e. $T(D)$ is precompact), then T is Ψ-condensing for any measure of non-compactness Ψ. Various Ψ-condensing mappings which are not compact have been considered by Borisovich et al [35], Gohberg et al [129] and Furi and Vignoli [121]. Moreover, when the measure of non-compactness Ψ is either the set-measure of non-compactness or ball-measure of non-compactness, Ψ-condensing mappings are called condensing mappings, e.g., see Nussbaum [237].

Throughout the rest of this section, E denotes a locally convex topological vector space, D denotes a non-empty closed convex subset of E, C denotes a lattice with a least element 0 and $\Psi : 2^E \to C$ denotes a measure of non-compactness.

Lemma 3.6.1. If $T : D \to 2^D \cup \{\emptyset\}$ is Ψ-condensing, then there exists a non-empty compact convex subset K of D such that $T(x) \subset K$ for each $x \in K$.

Proof. Let x_0 be an element of D and consider the family $\mathcal{F}$ of all closed convex subsets C of D such that $x_0 \in C$ and $T(x) \subset C$ for each $x \in C$. Clearly $\mathcal{F}$ is non-empty. Let $C_0 = \cap_{C \in \mathcal{F}} C$. Then C_0 is a non-empty closed and convex subset of D and $x_0 \in C_0$.

If $x \in C_0$, then $T(x) \subset C$ for all $C \in \mathcal{F}$ so that $T(x) \subset C_0$.

Now we shall prove that C_0 is also compact. Let $C_1 = \overline{co}(\{x_0\} \cup T(C_0))$. Then $C_1 \subset C_0$, which implies that $T(C_1) \subset T(C_0) \subset C_1$. Thus $C_1 \in \mathcal{F}$ and hence $C_0 \subset C_1$. Therefore $C_0 = C_1$. Hence,

$$\begin{aligned}\Psi(C_0) &= \Psi(C_1) = \Psi(\overline{co}(\{x_0\} \cup T(C_0))) = \Psi(\{x_0\} \cup T(C_0)) \\ &= max\{\Psi(\{x_0\}), \Psi(T(C_0))\} = \Psi(T(C_0))\end{aligned}$$

so $\Psi(C_0) \leq \psi(T(C_0))$ which implies that C_0 is compact. □

Theorem 3.6.2. Suppose that $T : D \to 2^D$ satisfies the following conditions:

(1) T is Ψ-condensing and $T(x)$ is non-empty convex for each $x \in D$;

(2) for each $y \in D$, the set $T^{-1}(y) = \{x \in D : y \in T(x)\}$ is compactly open in D.

Then T has a fixed point in D.

Proof. Since T is Ψ-condensing, by Lemma 3.6.1, there exists a non-empty compact convex subset K of D such that $T : K \to 2^K$. For each $y \in K$, $T^{-1}(y)$ is also open in K by (2). Now by the Fan-Browder fixed point theorem (Theorem 2.3.18), there exists $x \in K$ such that $x \in T(x)$. □

Theorem 3.6.3. Suppose that $T : D \to 2^D$ is upper semicontinuous and Ψ-condensing such that $T(x)$ closed and convex for each $x \in D$. Then T has a fixed point.

Proof. Since T is a Ψ-condensing, by Lemma 3.6.1, there exists a non-empty compact convex subset K of D such that $T : K \to 2^K$ is also upper semicontinuous with non-empty closed and convex values. Now by the Fan-Glicksberg fixed point theorem in Fan [97] or Glicksberg [127], there exists $x \in K$ such that $x \in T(x)$. □

Theorem 3.6.2 and Theorem 3.6.3 generalize mant well-known fixed point theorems in locally convex topological vector spaces, e.g., see Dugundji and Granas [91] and Reich [248], Smart [280], Istratescu [162] and Zeidler [336].

We now prove the following theorem on the existence of a maximal element which generalizes the corresponding result of Toussaint [315, Theorem 2.4] and of Yannelis and Prabhakar [326, Corollary 5.1].

Theorem 3.6.4. Let $T : D \to 2^D \cup \{\emptyset\}$ be Ψ-condensing such that either (i) T is $\mathcal{U}$-majorized or (ii) T is KF_C-majorized. Then there exists a point $x^* \in D$ such that $T(x^*) = \emptyset$.

Proof. Suppose $T(x) \neq \emptyset$ for all $x \in D$. By Lemma 3.6.1, there exists a non-empty compact convex subset K of D such that $T : K \to 2^K$. if assumption (i) holds, then by Lemma 3.5.2, there exists an upper semicontinuous mapping $S : K \to 2^K$ such that for each $x \in K$, $S(x)$ is non-empty closed and convex, $T(x) \subset S(x)$ and $x \notin S(x)$. But then by the classical Fan-Glicksberg fixed point theorem ([97] or [127]), there exists a point $x \in K$ such that $x \in S(x)$ which is a contradiction. Now if assumption (ii) holds, since T is KF_C-majorized, by Lemma 3.2.2 (see also Lemma 2 of Ding and Tan [84]), there exists a map $S : K \to 2^K$ such that (a) $T(x) \subset S(x)$ for each $x \in K$, (b) $S^{-1}(y)$ is open in K for each $y \in K$ and (c) $x \notin coS(x)$ for each $x \in K$. By Lemma 3.2.1 (see also Lemma 5.1 of Yannelis and Prabhakar [326]), $(coS)^{-1}(y)$ is also open in K for each $y \in K$. Then by the Fan-Browder fixed point theorem (e.g., see Browder [42]), there exists a point $x \in K$ such that $x \in coS(x)$ which is a contradiction. Therefore the conclusion must hold. □

We now also have the following extension of a theorem of Bergstrom [30] to a locally convex topological vector space and a non-compact setting.

Theorem 3.6.5. Suppose $P : D \to 2^D \cup \{\emptyset\}$ satisfies the following conditions:

(i) P is Ψ-condensing;

(ii) $P^{-1}(y)$ is compactly open for each $y \in D$;

(iii) P is acyclic.

Then there exists $x^* \in D$ such that $P(x^*) = \emptyset$.

Proof. By Lemma 3.6.1, there exists a non-empty compact convex subset K of D such that $P : K \to 2^K \cup \{\emptyset\}$. Clearly, the restriction $P|_K$ of P to K is also acyclic and has open inverse images Therefore the conclusion follows from Bergstrom's theorem [30, p 403]. □.

As an application of Theorem 3.6.4, we shall first prove the following existence theorem for equilibrium points for a one-person game.

Theorem 3.6.6. Let $A, B, P : D \to 2^D \cup \{\emptyset\}$ be such that

(i) $A \cap P$ is KF_C-majorized,

(ii) for each $x \in D$, $A(x)$ is non-empty and $coA(x) \subset B(x)$ and for each $y \in D$, $A^{-1}(y)$ is compactly open in D;

(iii) B is Ψ-condensing.

Then there exists $x \in D$ such that $x \in \overline{B}(x)$ and $A(x) \cap P(x) = \emptyset$; that is, the one person game $(D; A, B; P)$ has an equilibrium point.

Proof. Let $F = \{x \in D : x \in \overline{B}(x)\}$, then F is closed in D. Define $\Lambda : D \to 2^D \cup \{\emptyset\}$ by

$$\Lambda(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \in F, \\ A(x), & \text{if } x \notin F. \end{cases}$$

Suppose $\Lambda(x) \neq \emptyset$. If $x \notin F$, then $D \setminus F$ is an open neighborhood of x in D such that for each $z \in D \setminus F$, $z \notin \overline{B}(z)$. Now define $\Phi_x : D \to 2^D \cup \{\emptyset\}$ by $\Phi_x(z) = A(z)$ for each $z \in D$ and $N_x = D \setminus F$, then N_x is an open neighborhood of x in D such that

(i) $\Lambda(z) \subset \Phi_x(z)$ and $z \notin co\Phi_x(z)$ for each $z \in N_x$, and

(ii) $\Phi_x^{-1}(y) = A^{-1}(y)$ is compactly open in D.

Therefore Φ_x is an KF_C-majorant of Λ at x. On the other hand, if $x \in F$, then $\Lambda(x) = A(x) \cap P(x) \neq \emptyset$. Since $A \cap P$ is KF_C-majorized, there exist an open neighborhood N_x of x in D and a correspondence $\Phi_x : D \to 2^D \cup \{\emptyset\}$ such that $\Lambda(z) = A(z) \cap P(z) \subset \Phi_x(z)$ and $z \notin co\Phi_x(z)$ for each $z \in N_x$, and $\Phi_x^{-1}(y)$ is compactly open in D for each $y \in D$. Define the map $\Phi_x' : D \to 2^D \cup \{\emptyset\}$ by

$$\Phi_x'(z) = \begin{cases} A(z) \cap \Phi_x(z), & \text{if } z \in F, \\ A(z), & \text{if } z \notin F. \end{cases}$$

Note that $\Lambda(z) \subset \Phi_x'(z)$ for each $z \in N_x$. It is easy to see that $z \notin co\Phi_x'(z)$ for all $z \in D$. Moreover, for any $y \in D$, the set $(\Phi_x')^{-1}(y) = [\Phi_x^{-1}(y) \cup (D \setminus F)] \cap A^{-1}(y)$ is compactly open in D. It follows that Φ_x' is a KF_C-majorant of Λ at the point x. Hence Λ is a KF_C-majorized correspondence.

Since $\Lambda(x) \subset A(x) \subset B(x)$ for each $x \in D$, and B is Ψ-condensing by condition (ii), Λ is also Ψ-condensing .

Now By Lemma 3.6.1, there exists a non-empty compact convex subset K of X such that $\Lambda : K \to 2^K \cup \{\emptyset\}$. Clearly, the restriction $\Lambda|_K$ of Λ to K satisfies all hypotheses of Theorem 3.6.4. By Theorem 3.6.4, there exists $x \in K \subset D$ such that $\Lambda(x) = \emptyset$. Since $A(x) \neq \emptyset$, we must have $x \in \overline{B}(x)$ and $A(x) \cap P(x) = \emptyset$. □

Let I be a finite set and X_i be a topological space, and $X = \Pi_{i \in I} X_i$. For a given correspondence $A_i : X \to 2^{X_i}$, define a mapping $A'_i : X \to 2^X$ by $A'_i(x) = \{y \in X : y_i \in A_i(x)\} = \pi_i^{-1}(A_i(x))$ for each $x \in X$, where $\pi_i : X \to X_i$ is the projection. Then it is easy to see that A'_i is of class KF_C if and only if for each $x \in X$, $x_i \notin A_i(x)$ and $A_i^{-1}(y)$ is compactly open for each $y \in X$.

From Theorem 3.6.6, we shall now derive another existence theorem for equilibrium points for N-person games, where $N > 1$.

Theorem 3.6.7. Let $(X_i; A_i, B_i; P_i)_{i=1}^N$ be an N-person game. Suppose for each $i = 1, 2, \cdots, N$.

(i) X_i is a non-empty closed convex subset of a locally convex topological vector space E_i;

(ii) for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;

(iii) for each $y \in X_i$, $A_i^{-1}(y)$ is compactly open in X;

(iv) $A'_i \cap P'_i : X \to 2^X \cup \{\emptyset\}$ is of class KF_C, where $A'_i(x) = \pi_i^{-1}(A_i(x))$ and $P'_i(x) = \pi_i^{-1}(P_i(x))$; and

(v) the mapping $B : X \to 2^X$ defined by $B(x) = \Pi_{i=1}^N \overline{B_i}(x)$ for each $x \in X$ is Ψ-condensing (where $E = \Pi_{i=1}^N E_i$).

Then there exists $x \in X$ such that for each $i = 1, 2, \cdots, N$, $x \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$; that is, the N-person game $(X_i; A_i, B_i; P_i)_{i=1}^N$ has an equilibrium point.

Proof. For each $x \in X$, let $I(x) = \{i \in I : A_i(x) \cap P_i(x) \neq \emptyset\}$. Define the correspondences $A, B, P : X \to 2^X \cup \{\emptyset\}$ by $A(x) = \Pi_{i=1}^N A_i(x)$, $B(x) = \Pi_{i=1}^N \overline{B_i}(x)$ and

$$P(x) = \begin{cases} \cap_{i \in I(x)} P_i'(x), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset \end{cases}$$

for each $x \in X$. Since $B : X \to 2^X \cup \{\emptyset\}$ is Ψ-condensing, by Lemma 3.6.1, there exists a non-empty compact and convex susbet K of X such that $B(x) \subset K$ for each $x \in K$. By (ii), it follows that the A is also a self-mapping on K.

The condition (iii) implies that $A^{-1}(y) \cap K = \cap_{i=1}^N (A_i^{-1}(y_i) \cap K)$ is open in K for each $y \in K$. In order to apply Theorem 3.6.6, it remains to prove that the correspondence $A \cap P : K \to 2^K \cup \{\emptyset\}$ is KF_C-majorized.

Suppose $x \in K$ and $(A \cap P)(x) \neq \emptyset$; then $I(x) \neq \emptyset$. Choose any $y \in A(x) \cap P(x)$, then $x \in \cap_{i \in I} A_i^{-1}(y_i) \cap \cap_{i \in I(x)} P_i^{-1}(y_i) \cap K \subset \cap_{i \in I(x)}((A_i' \cap P_i')^{-1}(y) \cap K) := N_x$ which is an open neighborhood of x in K by (iv). Note that if $z \in N_x$, then $I(x) \subset I(z)$: since for each $i \in I(x)$, $y \in (A_i' \cap P_i')(z)$, we have $y_i \in A_i(z) \cap P_i(z)$ so that $i \in I(z)$. Now fix any $i_0 \in I(x)$. Define $P_K, \psi_x : K \to 2^K \cup \{\emptyset\}$ by $P_K(z) = P(z) \cap K$ and $\psi_x(z) = A_{i_0}'(z) \cap P_{i_0}'(z) \cap K$ for each $z \in K$, then by (iv) again, ψ_x is of class KF_C (in fact, KF) and for each $z \in N_z$, since $A(z) \subset B(z) \subset K$,

$$\begin{aligned} A(z) \cap P(z) \quad &= A(z) \cap P_K(z) = \Pi_{i \in I} A_i(z) \cap \cap_{i \in I(x)} P_i'(z) \cap K \\ &\subset \pi_{i_0}^{-1}(A_{i_0}(z)) \cap \pi_{i_0}^{-1}(P_i(z)) \cap K = A_{i_0}'(z) \cap P_{i_0}'(z) \cap K \\ &= \psi_x(z) \end{aligned}$$

Thus ψ_x is a KF_C-majorized of $A \cap P$ at x. Thus the mappings $A, P_K, B : K \to 2^K \cup \{\emptyset\}$ satisfy all the hypotheses of Theorem 3.6.6. By Theorem 3.6.6, there exists $x_0 \in K$ such that $x_0 \in \overline{B}(x_0)$ and $A(x_0) \cap P_K(x_0) = \emptyset$. Since $x_0 \in K$ and $A(x_0) \subset K$, it follows that $A(x_0) \cap P(x_0) = A(x_0) \cap P(x_0) \cap K = A(x_0) \cap P_K(x_0) = \emptyset$. Note that $B = \Pi_{i \in I} \overline{B_i}$ has a closed graph, we have $\overline{B}(x) = B(x)$ for each $x \in K$ and also $I(x_0) = \emptyset$. Hence the conclusion follows. □

The following is an existence theorem for equilibrium points of generalized games in which the intersection of constraint and preference correspondences are $\mathcal{U}$-majorized and constraint correspondences are Ψ-condensing. We emphasize that the set of players need not be finite.

Theorem 3.6.8. Let $\mathcal{G} = (X_i; A_i, B_i, P_i)_{i\in I}$ be a generalized game where I is any (countable or uncountable) set of players such that

(a) for each $i \in I$, X_i is a non-empty closed convex subset of a locally convex topological vector space E_i;

(b) for each $i \in I$ and for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subset \overline{B_i}(x)$;

(c) for each $i \in I$, the set $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is paracompact and open in X;

(d) for each $i \in I$, $A_i \cap P_i$ is $\mathcal{U}$-majorized ;

(e) the correspondence $B : X \to 2^X$ defined by $B(x) = \Pi_{i\in I}\overline{B_i}(x)$ for each $x \in X$ is Ψ-condensing (where $E = \Pi_{i\in I}E_i$).

Then $\mathcal{G}$ has an equilibrium point in X, i.e., there exists a point $\hat{x} = (\hat{x}_i)_{i\in I} \in X$ such that for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. Without loss of generality, we may assume that $\overline{B_i}(x)$ is convex for each $x \in X$ (otherwise replace $\overline{B_i}$ by $\overline{coA_i}$). Since the correspondence $B : X \to 2^X \cup \{\emptyset\}$ is Ψ-condensing, by Lemma 3.6.1, there exists a non-empty compact and convex subset K of X such that $B : K \to 2^K$.

Let $E^i_K = K \cap E^i$ for each $i \in I$. Let $I_0 = \{i \in I : E^i_K \neq \emptyset\}$. If $I_0 = \emptyset$, then $E^i_K = \emptyset$ for all $i \in I$ so that $(A_i \cap P_i)(x) = \emptyset$ for all $x \in K$. On the other hand, since B has a closed graph, B is upper semicontinuous on K. Also, B has closed and convex values. Thus by the Fan-Glicksberg fixed point theorem, there exists $x \in K$ such that $x \in B(x)$. It follows that $x_i \in \overline{B_i}(x)$ for all $i \in I$ and hence x is an equilibrium point of $\mathcal{G}$. Therefore we may assume that $I_0 \neq \emptyset$.

For each $i \in I$, let $K_i = \pi_i(K)$; note that each K_i is compact and convex and that $\overline{B_i}(x) \subset K_i$ for each $x \in K$.

Case 1: Suppose $i \in I_0$. Note that E^i_K is paracompact (e.g., see Theorem VIII.2.4 of Dugundji [89, p.165]) and open in K. By (d) and Lemma 3.5.2, there exists an upper semicontinuous mapping $\psi_i : E^i_K \to 2^{K_i}$ such that for each $x \in E^i_K$, (i) $\psi_i(x)$ is closed and convex, (ii) $\pi_i(x) \notin \psi_i(x)$ and (iii) $A_i(x) \cap P_i(x) \subset \psi_i(x)$. Since $\overline{B_i} : K \to 2^{K_i}$ is also upper semicontinuous with closed and convex values, the mapping $\psi_i \cap \overline{B_i} : E^i_K \to 2^{K_i}$ is also upper semicontinuous with closed and convex values by Theorem 7.3.10 of Klein and Thompson [189]. Define a correspondence $\phi_i : K \to 2^{K_i}$ by

$$\phi_i(x) = \begin{cases} \overline{B_i}(x), & \text{if } x \notin E^i_K, \\ (\psi_i \cap \overline{B_i})(x), & \text{if } x \in E^i_K. \end{cases}$$

Then Lemma 3.5.1 implies that ϕ_i is upper semicontinuous with non-empty closed and convex values.

Case 2: Suppose $i \in I \setminus I_0$. Define a correspondence $\phi : K \to 2^{K_i}$ by $\phi_i = \overline{B_i}(x)$ for each $x \in K$. Then ϕ is upper semicontinuous with compact and convex values.

Finally we define a correspondence $\Phi : K \to 2^{\prod_{i \in I} K_i}$ by $\Phi(x) = \prod_{i \in I} \phi_i(x)$ for each $x \in K$. Then Φ is also upper semicontinuous and has non-empty compact and convex values. Since $\Phi(x) \subset B(x) \subset K$ for each $x \in K$, $\Phi : K \to 2^K$ is in fact a self-map on K. Now the Fan-Glicksberg fixed point theorem again implies that there exists a point $x \in K$ such that $x \in \Phi(x)$. It follows that $\pi_i(x) \in \overline{B_i}(x)$ for all $i \in I$. If there exists $i \in I_0$ such that $x \in E^i_K$, then $\pi_i(x) \in \phi_i(x) = \overline{B_i}(x) \cap \psi_i(x) \subset \psi_i(x)$ which contradicts (ii). Therefore $x \notin E^i_K$ for all $i \in I_0$. Hence we also have $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in I$. $\square$

3.7 Equilibria in Frechet Spaces

Until now, we have given a number of existence theorems for equilibria of N-person games, qualitative games and generalized games in H-spaces, topological vcetor space and locally convex topological spaces. In this section, the underlying spaces of generalized games are Frechet spaces and we shall obtain existence theorems for generalized games by Michael's selection theorems in [222].

Theorem 3.7.1. Let $\mathcal{G} = (X_i; A_i; P_i)_{i\in I}$ be a generalized game and $X = \Pi_{i\in I}X_i$ be paracompact, where I is any (countable or uncountable) set. Suppose that for each $i \in I$, the following conditions are satisfied:

(i) X_i is a non-empty closed and convex subset of a Frechet space E_i;

(ii) A_i is lower semicontinuous with non-empty closed convex values;

(iii) the mapping $A : X \to 2^X$ defined by $A(x) = \Pi_{i\in I}A_i(x)$ is Ψ-condensing for each $x \in X = \Pi_{i\in I}X_i$, where C is a lattice with a least element 0 and $\Psi : 2^{\Pi_{j\in I}E_j} \to C$ is a measure of non-compactness;

(iv) for each $x \in X$, $\pi_i(x) \notin A_i(x) \cap P_i(x)$;

(v) the set $U_i := \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is closed in X.

(vi) the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is closed and convex.

Then there exists $x^* \in X$ such that for each $i \in I$, $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Proof. Fix an $i \in I$. Define $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \begin{cases} A_i(x) \cap P_i(x), & \text{if } x \in U_i; \\ A_i(x), & \text{if } x \notin U_i. \end{cases}$$

By Lemma 2.5.1, F_i is lower semicontinuous with non-empty closed and convex values. Then by Michael's selection theorem [222, Theorem 3.2″] and Remark of Aubin [7, p.551]), there exists a continuous map $f_i : X \to X_i$ such that $f_i(x) \in F_i(x)$ for each $x \in X$.

Now define $f : X \to X$ by $f(x) = \{f_i(x)\}_{i\in I}$ for each $x \in X$. Then f is continuous and $f(x) \in F(x) = \Pi_{i\in I}F_i(x) \subset \Pi_{i\in I}A_i(x)$. Since A is Ψ-condensing, it follows that

f is also Ψ-condensing. Since $X = \Pi_{i\in I}X_i$ is a non-empty closed and convex subset of the locally convex topological vector space $\Pi_{i\in I}E_i$, f satisfies all hypotheses of Theorem 2.6.3. By Theorem 2.6.3, there exists $x^* \in X$ such that $f(x^*) = x^*$. Note that for each $i \in I$, if $x_i^* \in U_i$, then $\pi_i(x^*) = f_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ which contradicts (iv). Hence for each $i \in I$, we must have $\pi_i(x^*) \notin U_i$ and thus $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$. □

We also have:

Theorem 3.7.2. Let $\mathcal{G} = (X_i; A_i; P_i)_{i\in I}$ be a generalized game, where I is any set. Suppose for each $i \in I$, the following conditions are satisfied:

(i) X_i is a non-empty closed and convex subset of a Frechet space E_i;

(ii) A_i is upper semicontinuous with non-empty closed convex values;

(iii) the mapping $A : X = \Pi_{i\in I}X_i \to 2^X$ defined by $A(x) = \Pi_{i\in I}A_i(x)$ for each $x \in X$ is Ψ-condensing, where C is a lattice with a least element 0 and $\Psi : 2^{\Pi_{j\in I}E_j} \to C$ is a measure of non-compactness;

(iv) the set $U_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is paracompact and open in X;

(v) the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is closed and convex.

Then there exists $x^* \in X$ such that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Proof. Fix an $i \in I$. By (v), Theorem 3.2″ of Michael [222] and Remark of Aubin [7, p.551], let $f_i : U_i \to X_i$ be a continuous function such that $f_i(x) \in A_i(x) \cap P_i(x)$ for each $x \in U_i$. Define $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \begin{cases} \{f_i(x)\}, & \text{if } x \in U_i; \\ A_i(x), & \text{if } x \notin U_i, \end{cases}$$

then by (ii), (iv) and Lemma 2.5.1, F_i is upper semicontinuous with closed convex values.

Now we define $F : X \to 2^X$ by $F(x) = \Pi_{i\in I}F_i(x)$ for each $x \in X$. Then F is upper semicontinuous with closed convex values and $F(x) \subset A(x)$ for each $x \in X$. Since A is Ψ-condensing, F is also Ψ-condensing. Therefore by Theorem 2.6.3, there exists $x^* \in X$

such that $x^* \in F(x^*)$. It follows that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$. □

Note that if the set I is countable, the set $U_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is a subset of a metrizable set $X = \Pi_{i \in I} X_i$ so that U_i is automatically paracompact for each $i \in I$.

By Theorem 3.1‴ of Michael [222, p.368] instead of his Theorem 3.2″ [222, p.367], the same argument used in proving Theorem 3.7.1 and Theorem 3.7.2 can likewise be used to prove the following:

Theorem 3.7.1′. Let $\mathcal{G} = (X_i; A_i; P_i)_{i \in I}$ be a generalized game and $X = \Pi_{i \in I} X_i$ be paracompact, where I is any (countable or uncountable) set. Suppose that for each $i \in I$, the following conditions are satisfied:

(i) X_i is a non-empty closed and convex subset of a finite dimensional space E_i;

(ii) A_i is lower semicontinuous with non-empty convex values (but not necessarily closed);

(iii) the mapping $A : X \rightarrow 2^X$ defined by $A(x) = \Pi_{i \in I} A_i(x)$ is Ψ-condensing for each $x \in X = \Pi_{i \in I} X_i$, where C is a lattice with a least element 0 and $\Psi : 2^{\Pi_{j \in I} E_j} \rightarrow C$ is a measure of non-compactness;

(iv) for each $x \in X$, $\pi_i(x) \notin A_i(x) \cap P_i(x)$;

(v) the set $U_i := \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is closed in X.

(vi) the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is convex (but not necessarily closed).

Then there exists $x^* \in X$ such that for each $i \in I$, $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Theorem 3.7.2′. Let $\mathcal{G} = (X_i, A_i, P_i)_{i \in I}$ be a generalized game where I is any set. Suppose for each $i \in I$, the following conditions are satisfied:

(i) X_i is a non-empty closed and convex subset of a finite dimensional space E_i;

(ii) A_i is upper semicontinuous with non-empty closed convex values;

(iii) the mapping $A : X = \Pi_{i \in I} X_i \rightarrow 2^X$ defined by $A(x) = \Pi_{i \in I} A_i(x)$ for each

$x \in X$ is Ψ-condensing, where C is a lattice with a least element 0 and $\Psi : 2^{\Pi_{j\in I}E_j} \to C$ is a measure of non-compactness;

(iv) the set $U_i := \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is paracompact and open in X;

(v) the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is convex (but not necessarily closed).

Then there exists $x^* \in X$ such that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

As an application of Theorem 3.7.2′, we have the following:

Corollary 3.7.3. Let I be any set. For each $i \in I$, let X_i be a non-empty compact convex subset of a finite dimensional space E_i and $P_i : X = \Pi_{j\in I}X_j \to 2^{X_i} \cup \{\emptyset\}$ be lower semicontinuous on the set $U_i = \{x \in X : P_i(x) \neq \emptyset\}$ such that for each $x \in U_i$, $P_i(x)$ is convex. If for each $i \in I$, U_i is paracompact and is either open or closed in X, then there exists $x^* \in X$ such that for each $i \in I$, either $\pi_i(x^*) \in P_i(x^*)$ or $P_i(x^*) = \emptyset$.

Proof. For each $i \in I$, let $A_i : X \to 2^{X_i}$ be defined by $A_i(x) = X_i$ for each $x \in X$. Then A_i is continuous with closed convex values and A_i is also Ψ-condensing since $X = \Pi_{i\in I}X_i$ is compact. Therefore by Theorem 3.7.2′, there exists $x^* \in X$ such that for each $i \in I$, either $\pi_i(x^*) \in P_i(x^*)$ or $P_i(x^*) = \emptyset$. □

Corollary 3.7.3 generalizes Theorem 1 of Barbolla [16] which in turn improves the fixed point theorem of Gale and Mas-Colell [124] and Florenzano [114] in the following ways: (1) the index set I need not be finite and (2) for each $i \in I$, U_i is either open or closed instead of U_i being open for all $i \in I$ or U_i being closed for all $i \in I$. We remark that our argument in proving Theorem 3.7.2 is different from that of Barbolla [16]. Finally we note that the results in this section improve or generalize the results of Mehta [219] in the case that the set I of agents (or players) is any (countable or uncountable) set instead of a countable set.

3.8 Equilibria in Finite Dimensional Euclidean Spaces

The purpose of this section is two fold: (1) we first obtain some sufficient conditions for the intersection of two lower semicontinuous mappings to be again lower semicontinuous and (2) by applying Michael's selection theorem [222], a fixed point theorem is derived and is applied together with earlier results on intersection of lower semicontinuous maps to obtain existence theorems for equilibrium points of a generalized game and of a qualitative game.

We introduce some notation. Let E be a vector space and $A \subset E$. We shall denote by aff(A) the affine span of A. A is said to be finite dimensional if A is contained in a finite dimensional subspace of E. If E is a topological vector space and $A \subset E$, ri(A) denotes the relative interior of A in aff(A).

First we observe that the proof of Proposition 1.1 of Marano [214, p.286] actually produced the following slightly strengthened version (where the original assumption that ϕ is lower semicontinuous is replaced by the weaker assumption that ϕ is lower semicontinuous at s_0):

Lemma 3.8.1. Let S be a topological space and $\phi : S \to 2^{\mathbf{R^n}}$ be a map with non-empty convex values. If there exists $s_0 \in S$ such that ϕ is lower semicontinuous at s_0 and $0_{\mathbf{R}^n} \in$ int$\phi(s_0)$ where $0_{\mathbf{R}^n}$ is the zero vector in $\mathbf{R}^n$, then there exists a neighborhood U_0 of s_0 in S such that $0_{\mathbf{R}^n} \in$ int$\phi(s)$ for all $s \in U_0$.

Since the translation of a map (respectively, convex set, open set) preserves its lower semicontinuity at a point (respectively, convexity, openness), Lemma 3.8.1 is equivalent to the following

Lemma 3.8.1′. Let S be a topological space and $\phi : S \to 2^{\mathbf{R^n}}$ be a map with convex values. Suppose ϕ is lower semicontinuous at some $s_0 \in S$ and $y_0 \in$ int$\phi(x_0)$, then there exists a neighborhood U_0 of s_0 in S such that $y_0 \in$ int$\phi(s)$ for all $s \in U_0$.

By slightly modifying the proof of Lemma 1 of Monteiro [223], we formulate the following:

Lemma 3.8.2. Let S be a topological space and $\phi : S \to 2^{\mathbf{R^n}} \cup \{\emptyset\}$ be a map with convex values. Suppose ϕ is lower semicontinuous at some $x_0 \in S$ and $B(y_0, t) \subset \phi(x_0)$ where $B(y_0, t)$ is the open ball of positive radius t centered at y_0. Then for any number r with $0 < r < t$, there exists a neighborhood U_0 of x_0 such that $B(y_0, r) \subset \phi(x)$ for all $x \in U_0$. In particular, if ϕ is open-valed, then ϕ has an open graph.

Proof. By restricting ϕ to the set $\{x \in X : \phi(x) \cap \text{int}\phi(x_o) \neq \emptyset\}$ which by the lower semicontinuity of ϕ at x_0, is an open neighborhood of x_0 in S. We may assume without loss of generality that ϕ is non-empty valued. Choose any r' with $r < r' < t$, then we have $\overline{B(y_0, r')} \subset B(y_0, t) \subset \phi(x_0)$ where $\overline{B(y_0, r')}$ is the closure of $B(y_0, r')$ in $\mathbf{R^n}$. Note that $\overline{B(y_0, r')}$ is compact. Choose any ϵ with $0 < \epsilon < r' - r$. Let $\{B(y_i, \epsilon/2) : i = 1, \cdots, m\}$ be a finite cover of $\overline{B(y_0, r')}$ with $y_i \in B(y_0, r')$ for all $i = 1, \cdots, m$. By Lemma 3.8.1$'$, there exists an open neighborhood U_1 of x_0 in S such that $y_0 \in \text{int}\phi(x)$ for all $x \in U_1$; in particular, $\text{int}\phi(x) \neq \emptyset$ for all $x \in U_1$. By lower semicontinuity of ϕ at x_0 again, there exists an open neighborhood U_0 of x_0 in S such that for each $x \in U_0$, $\phi(x) \cap B(y_i, \epsilon/2) \neq \emptyset$ for all $i = 1, \cdots, m$. By replacing U_0 by $U_1 \cap U_0$, we may also assume that $U_0 \subset U_1$. Now given any non-empty subsets A and B of $\mathbf{R^n}$, define $e(A, B) = \sup\{d(a, B); a \in A\}$, where d is the Euclidean metric on $\mathbf{R}^n$ and $d(a, B) = \inf\{d(a, b) : b \in B\}$. It is easy to see that for each $x \in U_0$,

$$e(B(y_0, r'), \phi(x)) \leq e(\cup_{i=1}^{m} B(y_i, \epsilon/2), \phi(x)) \leq \epsilon.$$

By the Lemma of [223], for each $x \in U_0$,

$$d(y_0,\ R^n \setminus B(y_0, r')) \leq d(y_0,\ R^n \setminus \phi(x)) + e(B(y_0, r'), \phi(x)),$$

so that

$$d(y_0,\ R^n \setminus \phi(x)) \geq d(y_0,\ R^n \setminus B(y_0, r')) - e(B(y_0, r'), \phi(x)) \geq r' - \epsilon > r.$$

It follows that $B(y_0, r) \subset \phi(x)$ for all $x \in U_0$. □

Remark 1.1 of Marano [214, p.287] shows that Lemma 3.8.2 does not hold if ϕ takes its values in an infinite dimensional Hilbert space.

Now let X and Y be topological spaces and $\psi,\ \phi : X \to 2^Y$ be lower semicontinuous. We observe that if Graph ϕ is open in $X \times Y$, then it is shown in Yannelis [325, Lemma 4.2] that $\phi \cap \psi$ is also lower semicontinuous. Clearly, if Graph ϕ is open in $X \times Y$, then ϕ is open-valued. The following example shows that (a) Graph ϕ need not be open in $X \times Y$ even though ϕ is open-valued and (b) even when $\phi(x) \cap \psi(x) \neq \emptyset$ for all $x \in X$, if the condition "Graph ϕ is open in $X \times Y$" is weakened and replaced by the condition "ϕ is open-valued", then $\phi \cap \psi$ need not be lower semicontinuous:

Example 3.8.3. Let $F_1, F_2 : [0,1] \to 2^{[0,1]}$ be defined by

$$F_1(x) = \begin{cases} [0,1], & \text{if } x = 0; \\ (0,1] \setminus \{1/n : n = 2, 3, \cdots,\}, & \text{if } x \neq 0 \end{cases}$$

and

$$F_2(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{1/n : n = 1, 2, \cdots\}, & \text{if } x \neq 0\ . \end{cases}$$

Then it is easy to see that (i) F_1 and F_2 are lower semicontinuous, (ii) Graph F_1 is not open in $[0,1] \times [0,1]$ and (iii) F_1 is open-valued such that for each $x \in [0,1]$,

$$F_1(x) \cap F_2(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{1\}, & \text{if } x \neq 0. \end{cases}$$

It follows that $F_1 \cap F_2$ is not lower semicontinuous.

Theorem 3.8.4. Let S be a topological space and $F_1,\ F_2 : S \to 2^{\mathbf{R}^n}$ be lower semicontinuous at $x_0 \in S$ such that F_1 is open and convex-valued. Then $F_1 \cap F_2$ is also lower semicontinuous at x_0.

Proof. Let U be an open subset of $\mathbf{R}^n$ such that $F_1(x_0) \cap F_2(x_0) \cap U \neq \emptyset$. Choose any $y_0 \in F_1(x_0) \cap F_2(x_0) \cap U$. Since $F_1(x_0) \cap U$ is open, y_0 is an interior point of $F_1(x_0) \cap U$. By Lemma 3.8.2, there is an open neighborhood V_1 of x_0 in S and $r > 0$ such that $B(y_0, r) \subset F_1(x)$ for all $x \in V_1$, where $B(y_0, r)$ is an open ball in $\mathbf{R}^n$ of radius r and centered at y_0. Since U is open in $\mathbf{R}^n$, we may assume without loss of generality that $B(y_0, r) \subset U$. Since F_2 is lower semicontinuous at x_0, there is an open neighborhood V_2 of x_0 in S such that $F_2(x) \cap B(y_0, r) \neq \emptyset$ for all $x \in V_2$. Let $V = V_1 \cap V_2$, then

V is an open neighborhood of x_0 in S such that for each $x \in V$, $F_1(x) \cap F_2(x) \cap U \supset F_2(x) \cap B(y_0, r) \neq \emptyset$. Hence $F_1 \cap F_2$ is lower semicontinuous. □

We note that Theorem 3.8.4 is different from Theorem E of Lechicki and Spakowski [203, p.121] in the following ways: (a) int$(F_1 \cap F_2)(x_0)$ is not required to be non-empty, but (b) the mapping F_1 is required to have open values.

The following example from Lechicki and Spakowski [203, Example 3] shows that even if F_2 has closed (or open) convex values, the conclusion of Theorem 3.8.4 fails to hold if $\mathbf{R^n}$ is replaced by an infinite dimensional Banach space:

Example 3.8.5. Let $Y = l^\infty$, the Banach space of all bounded sequences $x = (x_n)_{n=1}^\infty$ of real numbers with $\|x\|_\infty = \sup_{n\in N} |x_n| < \infty$ and $S = [0, 1]$. Define $G_1, G_2 : S \to 2^Y$ by

$$G_1(t) = \{x \in Y : x_1 \geq t, x_k \leq k - t \text{ for } k \geq 2\}$$

and

$$G_2(t) = \{x \in Y : x_1 \leq 1 - t, x_k \leq k(1 - x_1 - t) \text{ and } x_k \leq k + x_1/k - t/k \text{ for } k \geq 2\}.$$

Then G_1 and G_2 are both lower semicontinuous at 0 with closed convex values and $\text{int}_Y(G_1(0) \cap G_2(0)) \neq \emptyset$, but $G_1 \cap G_2$ is not lower semicontinuous at 0 (see Example 3 in [14, p.122]). Now we define $F_1, F_2 : S \to 2^Y$ by $F_1(t) = \text{int}_Y G_1(t)$ and $F_2(t) = \text{int}_Y G_2(t)$ (or $G_2(t)$) for each $t \in S$. Since

$$\overline{F_1(0) \cap F_2(0)} = \overline{(\text{int}_Y G(0)) \cap (\text{int}_Y G_2(0))} = \overline{(\text{int}_Y G_1(0)) \cap G_2(0)} = G_1(0) \cap G_2(0),$$

$F_1 \cap F_2$ is also not lower semicontinuous at $t = 0$ by Proposition 7.3.3 of Klein and Thompson [189] (or Proposition 2.3 of Michael [222, p.366]).

Theorem 3.8.6. Let S be a topological space and F_1, $F_2 : S \to 2^{\mathbf{R^n}}$ be lower semicontinuous at $x_0 \in S$ such that for each $x \in S$, $F_1(x)$ and $F_2(x)$ are both convex and $(\text{int}_{\mathbf{R^n}} F_1(x)) \cap F_2(x) \neq \emptyset$ or $F_1(x) \cap (\text{int}_{\mathbf{R^n}} F_2(x)) \neq \emptyset$ whenever $F_1(x) \cap F_2(x) \neq \emptyset$. Then $F_1 \cap F_2$ is also lower semicontinuous at $x_0 \in S$.

Proof. Without loss of generality, we may assume that $(\text{int}_{\mathbf{R}^n} F_1(x_0)) \cap F_2(x_0) \neq \emptyset$. Since $F_1(x_0)$ and $F_2(x_0)$ are convex, by Theorem of Dolecki [88, p.253],

$$\overline{(\text{int}_{\mathbf{R}^n} F_1(x_0)) \cap F_2(x_0)} = \overline{F_1(x_0) \cap F_2(x_0)} = \overline{F_1(x_0)} \cap \overline{F_2(x_0)}.$$

In order to prove $F_1 \cap F_2$ is lower semicontinuous at x_0, it suffices to prove that $(\text{int}_{\mathbf{R}^n} F_1) \cap F_2$ is lower semicontinuous at x_0. Now let U be any non-empty open subset of R^n such that $(\text{int}_{\mathbf{R}^n} F_1(x_0)) \cap F_2(x_0) \cap U \neq \emptyset$. Choose any $y_0 \in (\text{int}_{\mathbf{R}^n} F_1(x_0)) \cap F_2(x_0) \cap U$. Since $\text{int}_{\mathbf{R}^n} F_1(x_0) \cap U$ is open, y_0 is an interior point of $\text{int}_{\mathbf{R}^n} F_1(x_0) \cap U$. Since F_1 is lower semicontinuous at x_0 and $\text{int}_{\mathbf{R}^n} F(x_0) \neq \emptyset$, we have $\overline{F_1(x_0)} = \overline{\text{int}_{\mathbf{R}^n} F_1(x_0)}$. Therefore $\text{int}_{\mathbf{R}^n} F_1$ is also lower semicontinuous at x_0. By Lemma 3.8.2, there is an open neighborhood V_1 of x_0 in S and $r > 0$ such that $B(y_0, r) \subset \text{int}_{\mathbf{R}^n} F_1(x)$ for all $x \in V_1$, where $B(y_0, r)$ is an open ball in $\mathbf{R}^n$ of radius r and centered at y_0. Since U is open in $\mathbf{R}^n$, we may assume without loss of generality that $B(y_0, r) \subset U$. Since F_2 is lower semicontinuous at x_0, there is an open neighborhood V_2 of x_0 in S such that $F_2(x) \cap B(y_0, r) \neq \emptyset$ for all $x \in V_2$. Let $V = V_1 \cap V_2$, then V is an open neighborhood of x_0 in S such that for each $x \in V$, $(\text{int}_{\mathbf{R}^n} F_1(x)) \cap F_2(x) \cap U \supset F_2(x) \cap B(y_0, r) \neq \emptyset$. Hence $(\text{int}_{\mathbf{R}^n} F_1) \cap F_2$ is lower semicontinuous at x_0, so that $F_1 \cap F_2$ is also lower semicontinuous at x_0. □

Theorem 3.8.6 generalizes a result of Obukhovskii [238] (see also Borisovich et al [35, Corollary 1.3.10, p.725].

Theorem 3.8.7. Let S be a topological space, X be a non-empty subset of a finite dimensional topological vector space and $F_1, F_2 : S \to 2^{\text{aff}(X)}$ be lower semicontinuous at $x_0 \in S$. If for each $x \in S$, $F_1(x)$ is convex and open in aff(X), then $F_1 \cap F_2$ is also lower semicontinuous at x_0.

Proof. Choose any $a_0 \in$ aff(X) and let Y=aff(X) $-a_0$, then Y is a finite dimensional topological vector space and is therefore topologically isomorphic to some $\mathbf{R}^n$. Define $\tilde{F}_1$, $\tilde{F}_2 : S \to 2^Y$ by $\tilde{F}_1(x) = F_1(x) - a_0$, $\tilde{F}_2(x) = F_2(x) - a_0$ for all $x \in S$. Then $\tilde{F}_1$ and $\tilde{F}_2$ are lower semicontinuous at x_0 such that for each $x \in S$, $\tilde{F}_1(x)$ is convex and is open in Y. By Theorem 3.8.4, $\tilde{F}_1 \cap \tilde{F}_2$ is lower semicontinuous at x_0. It follows that $F_1 \cap F_2$ is also lower semicontinuous at x_0. □

By using the same proof as in Theorem 3 8.7, we have the following.

Theorem 3.8.8. Let S be a topological space, X be a non-empty subset of a finite dimensional topological vector space and $F_1, F_2 : S \to 2^{\mathrm{aff}(X)}$ be lower semicontinuous at $x_0 \in S$. If for each $x \in S$, $F_1(x)$ and $F_2(x)$ are convex and $\mathrm{ri}(F_1(x)) \cap F_2(x) \neq \emptyset$ or $F_1(x) \cap \mathrm{ri}(F_2(x)) \neq \emptyset$ whenever $F_1(x) \cap F_2(x) \neq \emptyset$, then $F_1 \cap F_2$ is also lower semicontinuous at x_0.

Next we shall prove a fixed point theorem as follows:

Theorem 3.8.9. Let I be a non-empty countable set. For each $i \in I$, let C_i be a non-empty finite dimensional compact convex subset of a topological vector space E_i and $F_i : C := \Pi_{j \in I} C_j \to 2^{E_i}$ be lower semicontinuous such that

(a) $F_i(x)$ is convex for each $x \in C$;

(b) $F_i(x) \cap \mathrm{ri}(C_i) \neq \emptyset$ for each $x \in C$;

(c) $F_i(C) \subset \mathrm{aff}(C_i)$.

Then the map $F := \Pi_{i \in I} F_i$ has a fixed point in C.

Proof. Let $i \in I$ be arbitrarily fixed. Define $F_i' : C \to 2^{E_i}$ by $F_i'(x) = F_i(x) \cap \mathrm{ri}(C_i)$ for each $x \in C$. Let A be an open subset of E_i such that $\mathrm{ri}(C_i) = \mathrm{aff}(C_i) \cap A$. If B is any open subset of E_i, then by (c),

$$\begin{aligned}\{x \in C : F_i'(x) \cap B \neq \emptyset\} &= \{x \in C : F_i(x) \cap \mathrm{ri}(C_i) \cap B \neq \emptyset\} \\ &= \{x \in C : F_i(x) \cap A \cap B \neq \emptyset\}\end{aligned}$$

is open in C since F_i is lower semicontinuous and $A \cap B$ is open in E_i. This shows that F_i' is lower semicontinuous on C Note that C is metrizable, being a countable product of finite dimensional sets; thus C is perfectly normal. Also for each $x \in C$, $F_i'(x)$ is a convex subset of C_i which is contained in some Euclidean space. Hence by Theorem 3.1‴ of Michael [222], there exists a continuous map $f_i : C \to C_i$ such that $f_i(x) \in F_i'(x)$ for all $x \in C$.

Now define $f : C \to C$ by $f := \Pi_{i\in I} f_i$, then f is continuous. Since C is a compact convex subset of some locally convex space (in fact, a countable product of Euclidean spaces), by the Schauder-Tychonoff fixed point theorem, there exists $x^* \in C$ such that $f(x^*) = x^*$. It follows that $x^* = \Pi_{i\in I} f_i(x^*) \in \Pi_{i\in I} F_i'(x^*) \subset \Pi_{i\in I} F_i(x^*) = F(x^*)$; that is, $F = \Pi_{i\in I} F_i$ has a fixed point in C. □

Corollary 3.8.10. Let X be a non-empty subset of a topological vector space E, C a non-empty finite dimensional compact convex subset of X and let $F : X \to 2^E$ be lower semicontinuous such that $F(C) \subset \text{aff}(C)$ and for each $x \in C$, $F(x)$ is convex and $F(x) \cap \text{ri}(C) \neq \emptyset$. Then F has a fixed point in X.

Corollary 3.8.10 improves Theorem 1 of Cubiotti [68] where F is assumed to have closed values.

We remark that in Theorem 3.8.9: (i) If $I = \{1, 2, \cdots, n\}$ is finite, then $\text{ri}(\Pi_{i=1}^n X_i) = \Pi_{i=1}^n \text{ri}(X_i)$; in this case, Theorem 3.8.9 can also be obtained from Corollary 3.8.10. (ii) If $I = \{1, 2, \cdots, \}$ is infinite and the set $\{j \in I : X_j$ is not a singleton set $\}$ is infinite, then $\text{ri}(\Pi_{j=1}^\infty X_j) = \emptyset$; in this case, Theorem 3.8.9 can not be deduced from Corollary 3.8.10. Thus Theorem 3.8.9 is a true generalization of Theorem 1 of Cubiotti [68].

We shall also need the following affine version of Corollary 6.3.2 of Rockafeller [253, p.46]:

Lemma 3.8.11. Let C be a non-empty convex subset of a finite dimensional topological vector space E. Then every open set in aff(C) which meets clC also meets ri(C).

Proof. Choose any $a_0 \in \text{aff}(C)$ and let $Y = \text{aff}(C) - a_0$, then Y is a finite dimensional topological vector space and is therefore topologically isomorphic to some $\mathbf{R}^n$. Let U be an open set in aff(C) such that $U \cap cl_E(C) \neq \emptyset$. Then $U - a_0$ is open in Y and $\emptyset \neq (U - a_0) \cap (cl(C) - a_0) = (U - a_0) \cap cl(C - a_0)$. By Corollary 6.3.2 in [253, p.46], $(U - a_0) \cap \text{ri}(C - a_0) \neq \emptyset$. As $\text{ri}(C - a_0) = \text{ri}(C) - a_0$, it follows that $U \cap \text{ri}(C) \neq \emptyset$. □

We shall now prove the following existence theorem for an equilibrium point of a generalized game.

Theorem 3.8.12. Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i\in I}$ be a generalized game where I is countable Suppose that for each $i \in I$, the following properties hold·

(1) X_i is a non-empty subset of a topological vector space E_i.

(2) $A_i, B_i : X = \Pi_{i\in I}X_i \to 2^{X_i}$ are lower semicontinuous such that $coA_i(x) \subset B_i(x)$ for each $x \in X$.

(3) $P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is lower semicontinuous on D_i, where $D_i = \{x \in X : P_i(x) \cap A_i(x) \neq \emptyset\}$ is closed in X

(4) There exists a non-empty finite dimensional compact convex subset C_i of X_i such that (a) for each $x \in C := \Pi_{j\in I}C_j$, $P_i(x)$ is open in aff(X_i), (b) for each $x \in C$, $coA_i(x) \cap \text{ri}(C_i) \neq \emptyset$, (c) for each $x \in C \cap D_i$, $coP_i(x) \cap coA_i(x) \cap C_i \neq \emptyset$ and (d) $coA_i(x) \subset$ aff(C_i).

(5) For each $x \in C$, $\pi_i(x) \notin coP_i(x)$.

Then $\mathcal{G}$ has an equilibrium point x^* in C.

Proof. Fix an arbitrary $i \in I$. Define $F_i : C \to 2^{E_i}$ by

$$F_i(x) = \begin{cases} coP_i(x) \cap coA_i(x), & \text{if } x \in C \cap D_i, \\ coA_i(x), & \text{if } x \notin C \cap D_i \ . \end{cases}$$

Since A_i and P_i are lower semicontinuous, coA_i and coP_i are lower semicontinuous Since for each $x \in C$, $coP_i(x)$ is convex and openin aff(X_i) by (4)(a), the map $x \mapsto coA_i(x) \cap coP_i(x)$ is also lower semicontinuous on C by Theorem 3 8.7 Since $C \cap D_i$ is closed in C, F_i is lower semicontinuous by Lemma 3.5.1. Moreover, we have

(i) $F_i(x)$ is convex for each $x \in C$;

(ii) $F_i(C) \subset coA_i(C) \subset$ aff(C_i) by condition (4)(d).

Now we shall show that $F_i(x) \cap \text{ri}(C_i) \neq \emptyset$ for each $x \in C$. Indeed, if $x \in C \setminus D_i$, then $F_i(x) \cap \text{ri}(C_i) = coA_i(x) \cap \text{ri}(C_i) \neq \emptyset$ by (4)(b). Suppose $x \in C \cap D_i$, then by (4)(c), $coP_i(x) \cap coA_i(x) \cap C_i \neq \emptyset$; as $coP_i(x) \cap$ aff(C_i) is open in aff(C_i) by (4)(a), $coP_i(x) \cap \text{ri}(coA_i(x) \cap C_i) \neq \emptyset$ by Lemma 3.8.11. Also, it follows from $coA_i(x) \cap C_i \neq \emptyset$ that $\text{ri}(coA_i(x) \cap C_i) = \text{ri}(coA_i(x)) \cap \text{ri}(C_i)$ by Theorem 6.5 in [20, p.47]. Thus

$$\emptyset \neq coP_i(x) \cap \text{ri}(coA_i(x) \cap C_i)$$

$$= \quad coP_i(x) \cap \mathrm{ri}(coA_i(x)) \cap \mathrm{ri}(C_i)$$
$$\subset \quad coP_i(x) \cap coA_i(x) \cap \mathrm{ri}(C_i) = F_i(x) \cap \mathrm{ri}(C_i).$$

This shows that all hypotheses of Theorem 3.8.9 are satisfied so that there exists $x^* \in C$ such that $x^* \in \Pi_{i\in I} F_i(x^*)$. By (5), we must have $x^* \notin D_i$ and $\pi_i(x^*) \in coA_i(x^*)$ for all $i \in I$ It follows that for each $i \in I$, $\pi_i(x^*) \in B_i(x^*)$ by (2) and $P_i(x^*) \cap A_i(x^*) = \emptyset$. Hence x^* is equilibrium point of $\mathcal{G}$. □

As a special case of Theorem 3.8.12, we have the following result.

Theorem 3.8.13. Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i\in I}$ be a generalized game where I is countable Suppose that for each $i \in I$, the following properties hold:

(1) X_i is a non-empty compact convex subset of a finite dimensional topological vector space.

(2) $A_i, B_i : X = \Pi_{i\in I} X_i \to 2^{X_i}$ are lower semicontinuous such that $coA_i(x) \subset B_i(x)$ for each $x \in X$.

(3) $P_i : X \to 2^{X_i} \cup \{\emptyset\}$ is lower semicontinuous on D_i, where $D_i = \{x \in X : P_i(x) \cap A_i(x) \neq \emptyset\}$ is closed in X.

(4) For each $x \in X$, (a) $P_i(x)$ is open in aff(X_i), (b) $coA_i(x) \cap \mathrm{ri}(X_i) \neq \emptyset$ and (c) $\pi_i(x) \notin coP_i(x)$.

Then $\mathcal{G}$ has an equilibrium point x^* in X.

The following example shows that the assumption "$D_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is closed in X" in condition (3) of Theorem 3.8.12 and Theorem 3.8.13 is necessary.

Example 3.8.14. Define $A : [0,1] \to 2^{[0,1]}$ by $A(x) := [0, 1-x]$ for each $x \in [0,1]$. Clearly, A is continuous with closed convex values. Now define $A_1 : [0,1] \to 2^{[0,1]}$ by

$$A_1(x) = \begin{cases} [0, 1-x], & \text{if } x \in (0,1], \\ \{1\}, & \text{if } x = 0, \end{cases}$$

is also lower semicontinuous by Lemma 3.5.1. We also define $A_2 : [0,1] \to 2^{[0,1]}$ by

$$A_2(x) = \begin{cases} co([0, 1-x] \cup \{1/2\}), & \text{if } x \in (0,1], \\ co(\{1\} \cup \{1/2\}), & \text{if } x = 0, \end{cases}$$

then A_2 is also lower semicontinuous and in fact

$$A_2(x) = \begin{cases} [0, 1-x], & \text{if } x \in (0, 1/2], \\ [0, 1/2], & \text{if } x \in (1/2, 1), \\ [1/2, 1], & \text{if } x = 0, \\ [0, 1/2], & \text{if } x = 1. \end{cases}$$

We now define $P : [0,1] \to 2^{[0,1]}$ by $P(x) = [0, x)$ for each $x \in [0,1]$ then for any $y \in [0,1]$, we have $P^{-1}(y) = (y, 1]$ Since $\{x \in [0,1], P(x) \cap A_2(x) \neq \emptyset\} = (0,1]$ and the fixed point set of A_2 is $(0, 1/2]$, but for each $x \in (0, 1/2]$, $A_2(x) \cap P(x) \neq \emptyset$

Finally we shall derive the following existence theorem for an equilibrium point of a qualitative game.

Theorem 3.8.15. Let $\mathcal{G} = (X_i; P_i)_{i \in I}$ be a qualitative game where I is countable. Suppose that for each $i \in I$, the following properties hold:

(1) X_i is a non-empty subset of a topological vector space E_i

(2) $P_i : X = \Pi_{j \in I} X_j \to 2^{X_i} \cup \{\emptyset\}$ is lower semicontinuous on D_i, where $D_i = \{x \in X : P_i(x) \neq \emptyset\}$ is closed in X.

(3) There exists a non-empty finite dimensional compact convex subset C_i of X_i such that (a) for each $x \in C := \Pi_{j \in I} C_j$, $P_i(x) \subset \text{aff}(C_i)$; (b) for each $x \in C \cap D_i$, $coP_i(x) \cap \text{ri}(C_i) \neq \emptyset$; (c) for each $x \in C$, $\pi_i(x) \notin coP_i(x)$

Then $\mathcal{G}$ has an equilibrium point x^* in C

Proof. Fix an arbitrary $i \in I$. Define $F_i : C \to 2^{E_i}$ by

$$F_i(x) = \begin{cases} coP_i(x), & \text{if } x \in C \cap D_i; \\ \text{aff}(C_i), & \text{if} \in X \setminus (C \cap D_i) \,. \end{cases}$$

Then by (2), (3)(a) and Lemma 3.5.1, F_i is lower semicotiuous on C. Clearly $F_i(C) \subset \text{aff}(C_i)$ by (3)(a) and for each $x \in X$, $F_i(x)$ is convex. We shall now show that for each $x \in C$, $F_i(x) \cap \text{ri}(C_i) \neq \emptyset$. Suppose $x \in C \cap D_i$, then by (3)(b), $F_i(x) \cap \text{ri}(C_i) = coP_i(x) \cap \text{ri}(C_i) \neq \emptyset$. If $x \in X \setminus (C \cap D_i)$, then $F_i(x) \cap \text{ri}(C_i) = \text{aff}(C_i) \cap \text{ri}(C_i) = ri(C_i) \neq \emptyset$ by Theorem 6 2 of Rockafellar [253, p.45].

Therefore all hypotheses of Theorem 3.8.9 are satisfied so that there exists $x^* \in C$ such that $x^* \in \Pi_{i \in I} F_i(x^*)$. By (3)(c), we must have $P_i(x^*) = \emptyset$ for all $i \in I$. Thus x^* is an equilibrium point of $\mathcal{G}$. □

By Theorem 3.8.15, we have the following

Corollary 3.8.16. Let $\mathcal{G} = (X_i; P_i)_{i \in I}$ be a qualitative game where I is countable. Suppose that for each $i \in I$, the following properties hold:

(1) X_i is a non-empty finite dimensional compact convex subset of a topological vector space E_i.

(2) $P_i : X = \Pi_{i \in I} X_i \to 2^{X_i} \cup \{\emptyset\}$ is lower semicontinuous on D_i, where $D_i = \{x \in X : P_i(x) \neq \emptyset\}$ is closed in X.

(3) For each $x \in X$, $coP_i(x) \cap \mathrm{ri}(X_i) \neq \emptyset$.

(4) For each $x \in X$, $\pi_i(x) \notin coP_i(x)$.

Then $\mathcal{G}$ has an equilibrium point $x^* \in X \ D_i$.

Chapter 4

Variational Inequalities

4.1 Introduction

Since Chan and Pang [48] and Shih and Tan [267] gave existence theorems for GQVI in finite dimensional spaces and infinite dimensional locally topological vector spaces respectively, there have been a number of generalizations of the existence theorems for GQVI(X; A; B), e.g., see Cubiotti [68], Ding and Tan [81], Harker and Pang [145], Kim [180], Shih and Tan [267]-[274] and Tian and Zhou [311] and references therein. These results enable people to give wide applications to the problems in game theory and economics, mathematical programming (e.g., see Aubin [7], Aubin and Ekeland [10], Chan and Pang [48], Harker and Pang [145] and reference therein). Most of the existence theorems mentioned above, however, are obtained upon compact sets, in finite dimensional spaces or infinite dimensional locally convex topological vector spaces, and also both A and B are either continuous or upper (lower) semicontinuous.

On the other hand, in economic and game theoretic applications, it is known that the choice space (or say, the space of feasible allocations) generally is not compact in any topology (even though it is closed and bounded), a key situation in infinite dimensional topological vector spaces. On the other hand, we note that there are no essential existence theorems for non-compact generalized quasi-variational inequalities associated with discontinuous functions in infinite dimensional space based on our knowledge right now.

This motivates our work in Chapter 4 of this thesis to give a series of existence theorems about generalized quasi-variational inequalities by relaxing the compactness conditions and continuity. As applications, we obtain some existence theorems for equilibria of constrain games in locally convex topological vector spaces. From the existence theorems for generalized quasi-variational inequalities, the stability of solutions for two types of generalized quasi-variational inequalities are also established.

Furthermore, based on a new concept called "*semi-monotone*" which was first introduced by Bae, Kim and Tan [13], we also discuss and give some interesting variational inequalities in Banach spaces. As applications, an existence theorem for generalized complementarity problems is given and some fixed point theorems for nonexpansive operators are given.

The basic idea in Chapter 4 is as follows: we reduce the existence problems for variational inequalities and generalized quasi-variational inequalities to the existence problem fo equilibria of generalized games; that means, the solutions of variational inequalities are nothing else, but are exactly equilibria of their equivalent model of generalized games. This simple fact enables us to consider the existence of solutions for non-compact variational inequalities and generalized quasi-variational inequalities in infinite dimensional topological vector spaces by the existence theorems for equilibria of non-compact generalized games established in Chapter 4.

More precisely, the contents of Chapter 4 are as follows:

In section 2, as applications of equilibria for non-compact generalized games in Chapter 3, the existence theorems for non-compact variational inequalities are given under various non-compact conditions in locally convex topological vector spaces. As applications, existence theorems for equilibria of constraint games are established. These results improve and unify many corresponding results in the literature.

In section 3, as applications of variational inequalities in section 2, two types of non-compact generalized quasi-variational inequalities are considered under various hypotheses in locally convex topological vector spaces.

In section 4, the stability of solutions for two types of generalized quasi-variational inequalities is established.

In section 5, based on the generalization of monotone operator called a "*semi-monotone operator*" which was first introduced by Bae, Kim and Tan [13], we also discuss and give some variational inequalities for monotone and semi-monotone operators in Banach spaces. As applications, an existence theorem for generalized complementarity problem in the Banach space and some fixed point theorems for multivalued nonexpansive mappings in the Hilbert space are given.

4.2 Variational Inequalities in Locally Convex Topological Vector Spaces

In this section, as applications of equilibria for generalized games, existence theorems for solutions of non-compact variational inequalities are given under different conditions in locally convex topological vector spaces As applications, the existence theorems of constraint games are given These results improve and unify many corresponding results in the literature (e g , see Aubin [7] and Aubin and Ekeland [10] and references therein).

As consequents of Theorem 3 4.2′ (respectively, Theorem 3 4 2) and Theorem 3.4.6′ (respectively, Theorem 3 4 6), we have the following existence theorems for non-compact quasi-variational inequalities in locally convex topological vector spaces

Theorem 4.2.1. For each $i = 1, 2, \cdots, N$, let E_i be a locally convex topological vector space and $X = \cup_{j=1}^{\infty} C_{i,j}$ where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of E_i Let $X = \Pi_{i=1}^{N} X_i$. Suppose the following conditions are satisfied

(i) for each $i = 1, 2, \cdots, N$, $A_i : X \to 2^{X_i}$ is lower semicontinuous with closed graph and convex values,

(ii) for each $i = 1, 2, \cdots, N$, $\psi_i : X \times X_i \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y_i \in X_i$, $x \mapsto \psi_i(x, y_i)$ is lower semicontinuous on X;

(iii) for each $i = 1, 2, \cdots, N$ and for each fixed $x \in X$, $x_i \notin co\{y_i \in X_i : \psi_i(x, y_i) > 0\}$),

(iv) for each $i = 1, 2, \cdots, N$, the set $\{x \in X . \sup_{y_i \in A_i(x)} \psi_i(x, y_i) > 0\}$ is open in X,

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, where $C_n = \Pi_{i \in I} C_{i,n}$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $\pi_i(x_n) \in A_i(y_n) \cap \{z \in X_i : \psi_i(y_n, z) > 0\}$ for each $i = 1, 2, \cdots, N$.

Then there exists $x^* \in X$ such that for each $i = 1, 2, \cdots, N$,

$$x_i^* \in A_i(x^*) \text{ and } \sup_{y_i \in A_i(x^*)} \psi_i(x^*, y_i) \leq 0.$$

Proof. For each $i = 1, 2, \cdots, N$, define $P_i : X \to 2^{X_i} \cup \{\emptyset\}$ by $P_i(x) = \{y_i \in X_i :$

$\psi_i(x, y_i) > 0\}$ for each $x \in X$. First (ii) implies that for each $i = 1, 2, \cdots, N$, P_i has open lower sections in X so that by (iii), P_i is of class KF; it follows that $A_i \cap P_i$ is KF-majorized. The condition (iv) implies that for each $i = 1, \cdots, N$, the set $\{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X. Therefore $\mathcal{G} = (X_i; A_i; P_i)_{i=1}^N$ satisfies all the hypotheses of Theorem 3.4.2′ with $A_i = B_i$ for each $i = 1, 2, \cdots, N$. By Theorem 3.4.2′, there exists an $x^* \in X$ such that for each $i = 1, 2, \cdots, N$, $x_i^* \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$. Since $\{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\} = \{x \in X : \sup_{y_i \in A_i(x)} \psi_i(x, y_i) > 0\}$, it follows that for each $i = 1, 2, \cdots, N$, $x_i^* \in A_i(x^*)$ and $\sup_{y_i \in A_i(x^*)} \psi_i(x^*, y_i) \leq 0$. $\square$.

A proof similar to that of Theorem 4.2.1, with Theorem 3.4.2 being applied instead of Theorem 3.4.2′, gives the following result and is thus omitted:

Theorem 4.2.1′. Let I be any index set. For each $i \in I$, let X_i be a non-empty convex subset of a locally convex topological vector space E_i. Let $X = \Pi_{i \in I} X_i$ be paracompact. Suppose that:

(i) for each $i \in I$, $A_i : X \to 2^{X_i}$ is a lower semicontinuous correspondence with closed graph and convex values;

(ii) for each $i \in I$, $\psi_i : X \times X_i \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y_i \in X_i$, $x \mapsto \psi_i(x, y_i)$ is lower semicontinuous on X;

(iii) for each $i \in I$ and for each $x \in X$, $x_i \notin co(\{y_i \in X_i : \psi_i(x, y_i) > 0\})$;

(iv) for each $i \in I$ and for each $y_i \in X_i$, the set $\{x \in X : \sup_{y_i \in A_i(x)} \psi_i(x, y_i) > 0\}$ is open in X;

(v) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists $x = (x_i)_{i \in I} \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap \{z_i \in X_i : \psi_i(y, z_i) > 0\})$ for all $i \in I$.

Then there exists $x^* \in K$ such that for each $i \in I$,

$$x_i^* \in A_i(x^*) \text{ and } \sup_{y_i \in A_i(x^*)} \psi_i(x^*, y_i) \leq 0.$$

Now by Theorem 3.4.6′ instead of Theorem 3.4.2′, we also have the following results for upper semicontinuous correspondences:

Theorem 4.2.2. For each $i = 1, 2, \cdots, N$, let E_i be a locally convex topological vector space and let $X_i = \cup_{j=1}^{\infty} C_{i,j}$ be closed in E_i and have property (K), where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of E_i. Let $X = \Pi_{i=1}^{N} X_i$. Suppose the following conditions are satisfied:

(i) for each $i = 1, 2, \cdots, N$, $A_i : X \to 2^{X_i}$ is compact and upper semicontinuous with compact and convex values;

(ii) for each $i = 1, 2, \cdots, N$, $\psi_i : X \times X_i \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y_i \in X_i$, $x \mapsto \psi_i(x, y_i)$ is lower semicontinuous on X;

(iii) for each $i = 1, 2, \cdots, N$ and for each $x \in X$, $x_i \notin co(\{y_i \in X_i : \psi_i(x, y_i) > 0\})$;

(iv) for each $i = 1, 2, \cdots, N$, the set $\{x \in X : \sup_{y_i \in A_i(x)} \psi_i(x, y_i) > 0\}$ is open in X;

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, where $C_n = \Pi_{i \in I} C_{i,n}$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $\pi_i(x_n) \in A_i(y_n) \cap \{z \in X_i : \psi_i(y, z) > 0\}$ for all $i = 1, 2, \cdots, N$.

Then there exists $x^* \in X$ such that for each $i = 1, 2, \cdots, N$,

$$x_i^* \in A_i(x^*) \text{ and } \sup_{y_i \in A_i(x^*)} \psi_i(x^*, y_i) \leq 0.$$

Proof. For each $i = 1, 2, \cdots, N$, define $P_i : X \to 2^{X_i} \cup \{\emptyset\}$ by $P_i(x) = \{y_i \in X_i : \psi_i(x, y_i) > 0\}$ for each $x \in X$, then we shall show that $\mathcal{G} = (X_i, A_i, P_i)_{i=1}^{N}$ satisfies all the hypotheses of Theorem 3.4.6′.

First, (ii) implies that for each $i = 1, 2, \cdots, N$, P_i has open lower sections in X so that by (iii), P_i is of class KF and that P_i is lower semicontinuous. The condition (iv) implies that for each $i = 1, 2, \cdots, N$, the set $\{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X. Therefore all hypotheses of Theorem 3.4.6′ are satisfied with $A_i = B_i$ for each $i = 1, 2, \cdots, N$. By Theorem 3.4.6′, there exists an $x^* \in K$ such that for each $i = 1, 2, \cdots, N$, $A_i(x^*) \cap P_i(x^*) = \emptyset$ and $x_i^* \in A_i(x^*)$. Since $\{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\} = \{x \in X : \sup_{y_i \in A_i(x)} \psi_i(x, y_i) > 0\}$, it follows that for each $i = 1, 2, \cdots, N$, $x_i^* \in A_i(x^*)$ and $\sup_{y_i \in A_i(x^*)} \psi_i(x^*, y_i) \leq 0$. $\square$.

Similarly, by applying Theorem 3.4.6 instead of Theorem 3.4.6′, we have:

Theorem 4.2.2′. Let I be any index set. For each $i \in I$, let X_i be a non-empty closed convex subset of a locally convex topological vector space E_i and X_i have property (K). Let $X = \Pi_{i\in I}X_i$ be paracompact. Suppose that:

(i) for each $i \in I$, $A_i : X \to 2^{X_i}$ is compact and upper semicontinuous with compact convex values;

(ii) for each $i \in I$, $\psi_i : X \times X_i \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y_i \in X_i$, $x \mapsto \psi_i(x, y_i)$ is lower semicontinuous on X;

(iii) for each $i \in I$ and for each $x \in X$, $x_i \notin co(\{y_i \in X_i : \psi_i(x, y_i) > 0\})$,

(iv) for each $i \in I$, the set $\{x \in X : \sup_{y_i \in A_i(x)} \psi_i(x, y_i) > 0\}$ is open in X for each fixed $y_i \in X_i$;

(v) there exist a non-empty compact and convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists $x = (x_i)_{i\in I} \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap \{z_i \in X_i : \psi_i(y, z_i) > 0\})$ for all $i \in I$.

Then there exists $x^* \in K$ such that for each $i \in I$,

$$x_i^* \in A_i(x^*) \text{ and } \sup_{y_i \in A_i(x^*)} \psi_i(x^*, y_i) \leq 0.$$

By letting $I = \{1\}$ in Theorem 4.2.1 and Theorem 4.2.2, we have the following existence results for quasi-variational inequalities in non-compact settings:

Corollary 4.2.3. Let $X = \cup_{i=1}^{\infty} C_i$ where $\{C_i\}_{i=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E Suppose the following conditions are satisfied:

(i) $A : X \to 2^X$ is lower semicontinuous with closed graph and convex values;

(ii) $\psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto \psi(x, y)$ is lower semicontinuous on X;

(iii) for each $x \in X$, $x \notin co(\{y \in X : \psi(x, y) > 0\})$,

(iv) the set $\{x \in X : \sup_{y \in A(x)} \psi(x, y) > 0\}$ is open in X,

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in N$ and $x_n \in C_n$ such that $x_n \in (A(y_n) \cap \{z \in X : \psi(y_n, z) > 0\})$.

Then there exists $x^* \in X$ such that $x^* \in A(x^*)$ and $\sup_{y \in A(x^*)} \psi(x^*, y) \leq 0$.

Corollary 4.2.4. Let E be a locally convex topological vector space, let $X = \cup_{i=1}^{\infty} C_i$

be closed in E and have property (K), where $\{C_i\}_{i=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of E. Suppose the following conditions are satisfied:

(i) $A : X \to 2^X$ is compact and upper semicontinuous and with compact convex values;

(ii) $\psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto \psi(x,y)$ is lower semicontinuous;

(iii) for each $x \in X$, $x \notin co(\{y \in X : \psi(x,y) > 0\})$;

(iv) the set $\{x \in X : \sup_{y \in A(x)} \psi(x,y) > 0\}$ is open in X;

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $x_n \in (A(y_n) \cap \{z \in X_i : \psi(y,z) > 0\})$.

Then there exists $x^* \in X$ such that $x^* \in A(x^*)$ and $\sup_{y \in A(x^*)} \psi(x^*, y) \leq 0$.

By Corollary 4.2.4, we have the following slight generalization of Theorem 3 of Tian and Zhou [311]:

Corollary 4.2.5. Let X be a non-empty convex subset of a locally convex topological vector space E. Suppose that

(i) $F : X \to 2^X$ has closed graph and convex values;

(ii) $\psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto \psi(x,y)$ is lower semicontinuous on X for each fixed $y \in X$;

(iii) for each $x \in X$, $x \notin co(\{y \in X : \psi(x,y) > 0\}$;

(iv) there exist a non-empty compact convex subset $Z \subset X$ and a non-empty subset $C \subset Z$ such that

(iv_a) $F(C) \subset Z$;

(iv_b) $F(z) \cap Z \neq \emptyset$ for each $z \in Z$;

(iv_c) for each $x \in Z \setminus C$ there exists $y \in F(x) \cap Z$ with $\psi(x,y) > 0$;

(iv_d) the set $\{z \in Z : \sup_{y \in F(x) \cap Z} \psi(x,y) \leq 0\}$ is closed in Z.

Then there exists $x^* \in F(x^*)$ such that $\sup_{y \in F(x^*)} \psi(x^*, y) \leq 0$.

Proof. Define $F_1 : Z \to 2^Z$ by $F_1(x) = F(x) \cap Z$ for each $x \in Z$. Then $F_1(x)$ is non-empty closed and convex for each $x \in Z$ by (i) and (iv_b). By (i), F has a closed graph so that F_1 is also closed. It follows that F_1 is upper semicontinuous. Now the conditions (ii) and (iii) imply that F_1 and ψ satisfy all hypotheses of Corollary 4.2.4 with $F_1 = A$ and $X = Z$. By Corollary 4.2.4, there exists $x^* \in Z$ such that $x^* \in F_1(x^*)$ and $\sup_{y \in F_1(x^*) \cap Z} \psi(x^*, y) \leq 0$. By (iv_c), x^* must be in C. Therefore $F_1(x^*) = F(x^*) \cap Z = F(x^*)$ by $(v)_a$. Hence, $\sup_{y \in F(x^*)} \psi(x^*, y) \leq 0$. □

Corollary 4.2.5 also generalizes Theorem 3.1 of Zhou and Chen [338], Theorem 15.2.1 of Aubin [7] and Theorem 4 of Fan [106].

We conclude this section with an application of Corollary 4.2.3 to give an existence theorem for equilibria of constrained games.

Let $I = \{1, 2, \cdots, N\}$. Each agent i chooses a strategy x_i in a subset X_i of a locally convex topological vector space E_i. Denote by X the (Cartesian) product $\Pi_{j=1}^N X_j$ and X_{-i} the product $\Pi_{j \in I, j \neq i} X_j$. Denote by x and x_{-i} an element of X and X_{-i} respectively. Each agent i has a payoff (utility) function $u_i : X \to \mathbf{R} \cup \{-\infty, +\infty\}$. Given x_{-i} (the strategies of others), the choice of the ith agent is restricted to a non-empty compact and convex set $A_i(x_{-i}) \subset X_i$, the *feasible strategy set* ; the ith agent chooses $x_i \in A_i(x_{-i})$ so as to minimize $u_i(x_{-i}, x_i)$ over $A_i(x_{-i})$, where (x_{-i}, x_i) is the point $y = (y_j)_{j \in I}$ such that $y_i = x_{-i}$ and $y_i = x_i$. The family $\mathcal{G} = (X_i; A_i; u_i)_{i=1}^N$ is then called a constrained N-person game and an equilibrium for $\mathcal{G}$ is an $x^* \in X$ such that $x_i^* \in A_i(x_{-i}^*)$ and $u_i(x^*) \leq u_i(x_{-i}^*, x_i)$ for all $x_i \in A_i(x_{-i}^*)$ (e.g., $u_i(x^*) = \inf_{x_i \in A_i(x_{-i}^*)} u_i(x_{-i}^*, x_i)$) for each $i = 1, 2, \cdots, N$.

Note that if $A_i(x_{-i}) = X_i$ for each $i = 1, 2, \cdots, N$, the constrained N-person game reduces to the conventional game $\mathcal{G} = (X_i; u_i)_{i \in I}$ and its equilibrium is called a Nash-equilibrium.

Theorem 4.2.6. Let $\mathcal{G} = (X_i; A_i; U_i)_{i=1}^N$ be a constrained game and $X = \Pi_{i=1}^N X_i = \cup_{j=1}^\infty C_j$ where $\{C_j\}_{j=1}^\infty$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E. Suppose the following conditions are satisfied:

(i) the correspondence $A : X \to 2^X$ defined by $A(x) = \Pi_{i=1}^N A_i(x_{-i})$ for each $x = (x_{-i}, x_i) \in X$ is lower semicontinuous with closed graph and convex values;

(ii) the function $\psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ defined by

$$\psi(x,y) = \Sigma_{i=1}^{N}[u_i(x_{-i}, x_i) - u_i(x_{-i}, y_i)]$$

for each $(x,y) \in X \times X$ is such that for each $y \in X$, $x \mapsto \psi(x,y)$ is lower semicontinuous on X, where $x = (x_{-i}, x_i)$ and $y = (y_{-i}, y_i)$;

(iii) for each $x \in X$, $x \notin co(\{y \in X : \psi(x,y) > 0\})$;

(iv) the set $\{x \in X : \sup_{y \in A(x)} \psi(x,y) > 0\}$ is open in X;

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $x_n \in A_i(y_n) \cap \{z \in X_i : \psi(y,z) > 0\}$).

Then there exists $x^* \in X$ such that for each $i = 1, 2, \cdots, N$,

$$x_i^* \in A_i(x_{-i}^*) \text{ and } u_i(x^*) \leq \inf_{x_i \in A(x_{-i}^*)} u_i(x_{-i}^*, z_i).$$

Proof. By (i)-(v), A and ψ satisfy all hypotheses of Corollary 4.2.3. By Corollary 4.2.3, there exists $x^* \in X$ such that $x^* \in A(x^*)$ and $\sup_{y \in A(x*)} \psi(x^*, y) \leq 0$. For each $i \in I$, and $y_i \in A_i(x_{-i}^*)$, let $y = (x_{-i}^*, y_i)$. Then $y \in A(x^*)$ so that $(u_i(x^*) - u_i(x_{-i}^*, y_i)) = \Sigma_{i=1}^{N}[u_i(x^*) - u_i(x_{-i}^*, y_i)] = \psi(x^*, y) \leq \sup_{y \in A(x*)} \psi(x^*, y) \leq 0$. Therefore $(u_i(x^*) - u_i(x_{-i}^*, y_i)) \leq 0$ for all $y_i \in A_i(x_{-i}^*)$. Hence x^* is an equilibrium point of the constrained game $\mathcal{G} = (X_i; A_i; u_i)_{i=1}^{N}$. □

Theorem 4.2.6 generalizes the corresponding results of Aubin [7, p.282-283] and Aubin and Ekeland [10, p.350-351] in the following ways: (i) the feasible correspondence A_i is lower (or upper) semicontinuous instead of continuous and (ii) the strategy set X_i need not be compact.

4.3 Generalized Quasi-Variational Inequalities

In this section, we shall first give some (non-compact) existence theorems of two types of generalized quasi-variational inequalities (in short, GQVI(X; A; B)) where, the domain X is a non-empty convex but not necessarily compact subset of a locally convex topological vector space E and the mapping A need not be continuous and the mapping B need not have any continuity property. These results generalize and improve many known results in the literatures, e.g., Aubin [7], Aubin and Ekeland [10], Chan and Pang [48], Harker and Pang [145], Kim [180], Shih and Tan [267], [274], Tarafdar [301] and Tian and Zhou [311] and the references therein.

Now we introduce some notation and definitions. If E is a topological vector space, we shall denote by E^* the dual space of E. The dual pairing between E^* and E is denoted by $\langle w, x\rangle$ for $w \in E^*$ and $x \in E$ and $Re\langle w, x\rangle$ denotes the real part of $\langle w, x\rangle$. Let X be a non-empty convex subset of a locally convex topological vector E. Then $T : X \to 2^{E^*}$ is monotone (see Browder [43, p.79]) if for each $x, y \in X$ and for each $u \in T(x)$, $w \in T(y)$, $Re\langle w - u, y - x\rangle \geq 0$. Suppose $F : X \to 2^X$, $T : X \to 2^{E^*}$ and $f : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$. By applying Theorems 4.2.1 and 4.2.3, we shall prove the existence of a solution $\hat{x} \in X$ to the following generalized quasi-variational inequalities:

$$\text{(I)} \qquad \begin{cases} \hat{x} \in F(\hat{x}), \\ \sup_{u \in T(\hat{x})} Re\langle u, \hat{x} - y\rangle + f(\hat{x}, y) \leq 0 \text{ for all } y \in F(\hat{x}) \end{cases}$$

or the existence of solutions $\hat{x} \in X$ and $\hat{u} \in E^*$ to the following generalized quasi-variational inequalities:

$$\text{(II)} \qquad \begin{cases} \hat{x} \in F(\hat{x}) \text{ and } \hat{u} \in T(\hat{x}), \\ Re\langle \hat{u}, \hat{x} - y\rangle + f(\hat{x}, y) \leq 0 \text{ for all } y \in F(\hat{x}). \end{cases}$$

Now we recall some definitions (e.g., see Zhou and Chen [338]). Let X be a non-empty convex subset of a topological vector space. A function $\psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is said to be

(1) γ-diagonally quasi-convex (respectively, γ-diagonally quasi-concave) in y, in short

γ-DQCX (respectively, γ-DQCV) in y, if for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, $\gamma \leq \max_{x \in A} \psi(x, y)$ (respectively, $\gamma \geq \inf_{x \in A} \psi(x, y)$);

(2) γ-diagonally convex (respectively, γ-diagonally concave) in y, in short γ-DCX (respectively, γ-DCV) in y, if for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ with $y = \Sigma_{i=1}^m \lambda_i y_i$ (where $\lambda_i \geq 0$ for each $i = 1, 2, \cdots, n$ and $\Sigma_{i=1}^m \lambda_i = 1$), we have $\gamma \leq \Sigma_{i=1}^m \lambda_i \psi(y_i, y)$ (respectively, $\gamma \geq \Sigma_{i=1}^m \lambda_i \psi(y_i, y)$);

Let X and Y be two non-empty convex subsets of E, we also recall that a function $\psi : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ is quasi-convex (respectively, quasi-concave) in y, if for each fixed $x \in X$, for each $A \in \mathcal{F}(Y)$ and each $y \in co(A)$, $\psi(x, y) \leq \max_{z \in A} \psi(x, z)$ (respectively, $\psi(x, y) \geq \inf_{z \in A} \psi(x, z)$).

It is easy to see that

(i) if $\psi(x, y)$ is γ-DCX (respectively, γ-DCV) in y, then $\psi(x, y)$ is γ-DQCX (respectively, γ-DCV) in y;

(ii) if $\psi_i(x, y)$ for each $i = 1, 2, \cdots, N$ is γ-DCX (respectively, γ-DCV) in y, then $\psi(x, y) = \Sigma_{i=1}^m a_i(x) \psi_i(x, y)$ is still γ-DCX (respectively, γ-DCV) in y, where $a_i : X \to \mathbf{R}$ with $a_i(x) \geq 0$ and $\Sigma_{i=1}^m a_i(x) = 1$ for each $x \in X$; and

(iii) the function $\psi(x, y) : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is 0-DQCV in y if and only if $x \notin co(\{y \in X : \psi(x, y) > 0\})$ for each $x \in X$.

Let K be a non-empty convex subset of a topological vector space E and $A : K \to E^*$ be not monotone. We define a function $f : K \times K \to \mathbf{R}$ by $f(x, y) = \langle A(y), x - y \rangle$ for each $(x, y) \in K \times K$. It is clear (e.g., see Zhou and Chen [338, p.216-217]) that for each fixed $x \in K$, the function $f(x, \cdot)$ is both DQCX and DQCV in y, but not quasi-convex or quasi-concave in y.

First we have the following existence for the problem (I) where $T : X \to 2^{E^*}$ is monotone.

Theorem 4.3.1. Let $X = \cup_{i=1}^\infty C_i$ where $\{C_i\}_{i=1}^\infty$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E. Suppose the following conditions are satisfied:

(i) $F : X \to 2^X$ is lower semicontinuous with closed graph and convex values;

(ii) $T : X \to 2^{E^*}$ is monotone such that for each one-dimensional flat $L \subset E$, $T|_{L\cap X}$ is lower semicontinuous when E^* is equipped with the weak*-topology $\sigma(E^*, E)$,

(iii) $f : X \times X \to \mathbf{R} \cup \{\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous on X, and for each $x \in X$, $y \mapsto f(x, y)$ is concave and $f(x, x) = 0$ for each $x \in X$;

(iv) the set $\{x \in X : \sup_{y\in F(x)}[\sup_{u\in T(y)} Re\langle u, x - y\rangle + f(x, y)] > 0\}$ is open in X,

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in N$ and $x_n \in C_n$ such that $x_n \in F(y_n) \cap \{z \in X : \sup_{u\in T(z)} Re\langle u, y_n - z\rangle + f(y_n, z) > 0\}$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{u\in T(\hat{x})} Re[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0 \text{ for all } y \in F(\hat{x})$$

Proof. Define a function $\psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ by

$$\psi(x, y) = \sup_{u\in T(y)} Re\langle u, x - y\rangle + f(x, y)$$

for each $(x, y) \in X \times X$. Then for each $y \in X$, $x \mapsto \psi(x, y)$ is lower semicontinuous on X. Since T is monotone and by (iii), it is easy to see that for each fixed $x \in X$, $y \mapsto \psi(x, y)$ is 0-DCV by Proposition 3.2 of Zhou and Chen [338]. Thus all the hypotheses of Corollary 4.2.3 (hence also of Theorem 4.2.1) are satisfied so that there exists $\hat{x} \in K$ such that $\hat{x} \in F(\hat{x})$ and $\sup_{u\in T(y)}[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0$ for all $y \in F(\hat{x})$.

Since for each one-dimensional flat $L \subset E$, $T|_{L\cap X}$ is lower semicontinuous when E^* is equipped with the weak*-topology $\sigma(E^*, E)$, by the same argument as in the step 2 of Shih and Tan [267, p.338], we can show that

$$\sup_{y\in T(\hat{x})}[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0 \text{ for all } y \in F(\hat{x}). \qquad \square$$

Now by Corollary 4.2.4 (hence also Theorem 4.2.2) and the same argument as in Theorem 4.3.1, we have the following:

Theorem 4.3.2. Let $X = \cup_{i=1}^{\infty} C_i$ be closed and have property (K), where $\{C_i\}_{i=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E. Suppose the following conditions are satisfied:

(i) $F : X \to 2^X$ is compact upper semicontinuous with compact and convex values;

(ii) $T : X \to 2^{E^*}$ is monotone such that for each one-dimensional flat $L \subset E$, $T|_{L\cap X}$ is lower semicontinuous when E^* is equipped with the weak*-topology $\sigma(E^*, E)$;

(iii) $f : X \times X \to \mathbf{R} \cup \{\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous on X and for each $x \in X$, $y \mapsto f(x, y)$ is concave and $f(x, x) = 0$ for each $x \in X$;

(iv) the set $\{x \in X : \sup_{y\in F(x)}[\sup_{u\in T(y)} Re\langle u, x - y\rangle + f(x, y)] > 0\}$ is open in X;

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $x_n \in F(y_n) \cap \{z \in X : \sup_{u\in T(z)} Re\langle u, y_n - z\rangle + f(y_n, z) > 0\}$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{u\in T(\hat{x})}[Re\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0 \text{ for all } y \in F(\hat{x}).$$

As an immediate consequence of Theorem 4.3.2, we have the following:

Corollary 4.3.3. Let X be a non-empty compact convex subset of a locally convex topological vector space E and let $F : X \to 2^X$ be upper semicontinuous with closed and convex values. If $T : X \to 2^{E^*}$ is monotone such that for each one-dimensional flat $L \subset E$, $T|_{L\cap X}$ is lower semicontinuous when E^* is equipped with the weak*-topology $\sigma(E^*, E)$. Suppose $f : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous on X and for each $x \in X$, $y \mapsto f(x, y)$ is concave and $f(x, x) = 0$ for each $x \in X$. Suppose further that the set $\{x \in X : \sup_{y\in F(x)}[\sup_{u\in T(y)} Re\langle u, x - y\rangle + f(x, y)] > 0\}$ is open in X. Then there exists an $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and $\sup_{u\in T(\hat{x})}[Re\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0$ for all $y \in F(\hat{x})$.

Theorem 4.3.2 generalizes Theorem 1 of Shih and Tan [267] to a non-compact setting. We note that our proofs of Theorem 4.3.1 and Theorem 4.3.2 depend on the existence theorems for equilibria of generalized games instead of the partition of unity arguments used by Aubin [7], Aubin and Ekeland [10], Shih and Tan [267] and Zhou and Chen [338]. When $T = 0$ in Corollary 4.2.2, Corollary 4.2.2 generalizes Joly-Mosco Theorem (see Theorem 15.2.2 of Aubin [7]), and also gives the well-known Fan-Glicksberg fixed point theorem (see [97] and [127]).

Recall that for a topological vector space E, the strong topology on its dual space E^* is the topology on E^* generated by the family $\{U(B;\omega) : B$ is a non-empty bounded subset of E and $\omega > 0\}$ as a base for the neighborhood system at zero, where $U(B;\omega) := \{f \in E^* : \sup_{x\in B} |\langle f, x\rangle| < \omega\}$.

We now observe that in Theorem 4.3.1 and Theorem 4.3.2, the interaction between the correspondences T and F (namely, the condition (v)) can be achieved by imposing additional continuity conditions on T and F.

Theorem 4.3.4. Let $X = \cup_{i=1}^{\infty} C_i$ be bounded, where $\{C_i\}_{i=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E. Suppose that $F : X \to 2^X$ is continuous with closed and convex values and $T : X \to 2^{E^*}$ is monotone such that T is lower semicontinuous when E^* is equipped with the strong topology. Suppose that

(i) $f : X \times X \to \mathbf{R} \cup \{\infty, +\infty\}$ is lower semicontinuous such that for each $x \in X$, $y \mapsto f(x,y)$ is concave and $f(x,x) = 0$ for each $x \in X$;

(ii) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in N$ and $x_n \in C_n$ such that $x_n \in F(y_n) \cap \{z \in X : \sup_{u\in T(z)} Re\langle u, y_n - z\rangle + f(y_n, z) > 0\}$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{u\in T(\hat{x})}[Re\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0 \text{ for all } y \in F(\hat{x}).$$

Proof. By Theorem 4.3.1, we need only show that the set

$$\Sigma := \{x \in X : \sup_{y\in F(x)} [\sup_{u\in T(y)} Re\langle u, x - y\rangle + f(x,y)] > 0\}$$

is open in X.

Since X is bounded and $f(\cdot,\cdot)$ is lower semicontinuous, the function $(u, x, y) \mapsto Re\langle u, x-y\rangle + f(x,y)$ is lower semicontinuous from $E^* \times X \times X$ to $\mathbf{R}$. Therefore $(x,y) \mapsto \sup_{u\in T(y)}[Re\langle u, x-y\rangle + f(x,y)]$ is also lower semicontinuous by the lower semicontinuity of $T(\cdot)$ and Proposition III-19 of Aubin and Ekeland [10]. Since F is lower semicontinuous,

$x \mapsto \sup_{y\in F(x)} \sup_{u\in T(y)}[Re\langle u, x-y\rangle + f(x,y)]$ is lower semicontinuous by Proposition III-19 of Aubin [7] again. Thus the set $\Sigma := \{x \in X : \sup_{y\in F(x)} \sup_{u\in T(y)}[Re\langle u, x-y\rangle + f(x,y)] > 0\}$ is open in X. □

Theorem 4.3.4 also generalizes Theorem 2 of Shih and Tan [267] to a non-compact setting.

Now we shall consider the existence of solutions of the problems (I) and (II) where the correspondence $T : X \to 2^{E^*}$ need not be monotone.

We state Kneser's minimax theorem [190] (see also Aubin [7, p.40-41] as follows:

Theorem 4.3.A (Kneser [190]). Let X be a non-empty convex set in a vector space and let Y be a non-empty compact convex subset of a topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, $f(x,y)$ is lower semicontinuous and convex on Y, and for each fixed $y \in Y$, $f(x,y)$ is concave on X. Then

$$\min_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\min_{y\in Y} f(x,y).$$

We first have the following:

Theorem 4.3.5. Let $X = \cup_{i=1}^{\infty} C_i$ where $\{C_i\}_{i=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E. Suppose that

(i) $F : X \to 2^X$ is lower semicontinuous with closed graph and compact and convex values;

(ii) $T : X \to 2^{E^*}$ has compact convex values such that for each fixed $y \in X$, $x \mapsto \inf_{u\in T(x)} Re\langle u, x-y\rangle$ is lower semicontinuous;

(iii) $f : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto f(x,y)$ is lower semicontinuous on X and for each $x \in X$, $y \mapsto f(x,y)$ is 0-diagonally concave;

(iv) the set $\{x \in X : \sup_{y\in F(x)}[\inf_{u\in T(x)} Re\langle u, x-y\rangle + f(x,y)] > 0\}$ is open in X;

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $x_n \in F(y_n) \cap \{z \in X : \inf_{u\in T(y_n)} Re\langle u, y_n -$

$z\rangle + f(y_n, z) > 0\}$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{y \in F(\hat{x})} \inf_{u \in T(\hat{x})} [\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0.$$

If in addition, for each $x \in X$, $y \mapsto f(x, y)$ is concave, then there exists $\hat{u} \in T(\hat{x})$ such that $\sup_{y \in F(\hat{x})} [\langle \hat{u}, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0$.

Proof. Define the functional $\psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ by

$$\psi(x, y) = \inf_{u \in T(x)} [Re\langle u, x - y\rangle + f(x, y)]$$

for each $(x, y) \in X \times X$. Then we have:

(1) for each fixed $y \in X$, $x \mapsto \psi(x, y)$ is lower semicontinuous on X and $x \notin co(\{y \in X : \psi(x, y) > 0\})$ for each $x \in X$ by (iv);

(2) the set $\{x \in X : \sup_{y \in F(x)} \psi(x, y) > 0\}$ is open in X;

(3) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in N$ and $x_n \in C_n$ such that $x_n \in co(\{F(y_n) \cap \{z \in X : \psi(y_n, z) > 0\})$.

Therefore F and ψ satisfy all the conditions of Corollary 4.2.3. By Corollary 4.2.3, there exists an $\hat{x} \in K$ such that $\hat{x} \in F(\hat{x})$ and $\psi(\hat{x}, y) \leq 0$ for all $y \in F(\hat{x})$.

If in addition, for each $x \in X$, $y \mapsto f(x, y)$ is concave, define the function $f_1 : F(\hat{x}) \times T(\hat{x}) \to \mathbf{R} \cup \{-\infty, +\infty\}$ by

$$f_1(x, y) = Re\langle x, \hat{x} - y\rangle + f(\hat{x}, y)$$

for each $(x, y) \in F(\hat{x}) \times T(\hat{x})$. Then for each $y \in X$, $x \mapsto f_1(x, y)$ is lower semicontinuous and for each $x \in X$, $y \mapsto f_1(x, y)$ is concave. By Kneser minimax Theorem 4.3.A,

$$\inf_{u \in T(\hat{x})} \sup_{y \in F(\hat{x})} [Re\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] = \sup_{y \in F(\hat{x})} \inf_{u \in T(\hat{x})} [Re\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0,$$

so that there exists $\hat{u} \in T(\hat{x})$ such that $\sup_{y \in F(\hat{x})} [Re\langle \hat{u}, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0$. $\square$

By the same arguments used in Theorem 4.3.5 and by applying Corollary 4.2.4 instead of Corollary 4.2.3, we have the following result whose proof is omitted:

Theorem 4.3.6. Let $X = \cup_{i=1}^{\infty} C_i$ be closed and have property (K), where $\{C_i\}_{i=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E. Suppose that

(i) $F : X \to 2^X$ is compact and upper semicontinuous with compact and convex values,

(ii) $T : X \to 2^{E^*}$ has compact and convex values such that for each $y \in X$, $x \mapsto \inf_{u \in T(x)} Re\langle u, x - y\rangle$ is lower semicontinuous;

(iii) f $X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous on X for each $y \in X$ and for each $x \in X$, $y \mapsto f(x, y)$ is 0-diagonally concave,

(iv) the set $\{x \in X : \sup_{y \in F(x)}[\inf_{u \in T(x)} Re\langle u, x - y\rangle + f(x, y)] > 0\}$ is open in X;

(vi) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in N$ and $x_n \in C_n$ such that $x_n \in F(y_n) \cap \{z \in X : \inf_{u \in T(y_n)} Re\langle u, y_n - z\rangle + f(y_n, z) > 0\}$.

Then there exists $\hat{x} \in X$ such that

$$\hat{x} \in F(\hat{x}) \text{ and } \sup_{y \in F(\hat{x})} \inf_{u \in T(\hat{x})}[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0.$$

If in addition, for each $x \in X$, $y \mapsto f(x, y)$ is concave, then there exists $\hat{u} \in T(\hat{x})$ such that $\sup_{y \in F(\hat{x})} Re[\langle \hat{u}, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0$.

If X is a bounded subset of a locally convex topological vector space E and $T : X \to 2^{E^*}$ has compact and convex values and is upper semicontinuous when E^* is equipped with the strong topology, then the function $g : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ defined by $g(x, y) = \inf_{u \in T(x)}\langle u, x - y\rangle$ has the property that for each $y \in X$, $x \mapsto g(x, y)$ is lower semicontinuous on X by Lemma 2 of Kim and Tan [184]. Thus, Theorem 4.3.6 generalizes Theorem 3 of Shih and Tan [267] and we have the following:

Corollary 4.3.7. Let $X = \cup_{i=1}^{\infty} C_i$ be bounded, where $\{C_i\}_{i=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E. Suppose that

(i) $F : X \to 2^X$ is lower semicontinuous with closed graph and compact and convex values;

(ii) $T : X \to 2^{E^*}$ has (strongly) compact convex values and is upper semicontinuous when E^* is equipped with the strong topology;

(iii) $f : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto f(x,y)$ is lower semicontinuous and for each $x \in X$, $y \mapsto f(x,y)$ is 0-diagonally concave;

(iv) the set $\{x \in X : \sup_{y \in F(x)}[\inf_{u \in T(x)} Re\langle u, x-y\rangle + f(x,y)] > 0\}$ is open in X;

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $x_n \in F(y_n) \cap \{z \in X : \inf_{u \in T(y_n)} Re\langle u, y_n - z\rangle + f(y_n, z) > 0\}$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{y \in F(\hat{x})} \inf_{u \in T(\hat{x})}[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \le 0.$$

If in addition, for each $x \in X$, $y \mapsto f(x,y)$ is concave, then there exists $\hat{u} \in T(\hat{x})$ such that $\sup_{y \in F(\hat{x})}[\langle \hat{u}, \hat{x} - y\rangle + f(\hat{x}, y)] \le 0$.

Proof. Define $\psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ by

$$\psi(x,y) = \inf_{u \in T(x)}[Re\langle u, x-y\rangle + f(x,y)],$$

for each $(x,y) \in X \times X$. Since X is bounded, by Lemma 2 of Kim and Tan [184], for each $y \in X$, $x \mapsto \inf_{u \in T(x)}\langle u, x-y\rangle$ is lower semicontinuous. Therefore ψ and F satisfy all the hypotheses of Theorem 4.3.5. Thus the conclusion follows from Theorem 4.3.5. □

In the above proof, if we apply Theorem 4.3.6 instead of Theorem 4.3.5, we have:

Corollary 4.3.8. Let $X = \cup_{i=1}^{\infty} C_i$ be bounded, closed and have property (K), where $\{C_i\}_{i=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E_i. Suppose that

(i) $F : X \to 2^X$ is compact and upper semicontinuous with compact and convex values;

(ii) $T : X \to 2^{E^*}$ has (strongly) compact convex values and is upper semicontinuous when E^* is equipped with the strong topology;

(iii) $f : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that for each $y \in X$, $x \mapsto f(x,y)$ is lower semicontinuous and for each $x \in X$, $y \mapsto f(x,y)$ is 0-diagonally concave;

(iv) the set $\{x \in X : \sup_{y \in F(x)}[\inf_{u \in T(x)} Re\langle u, x-y\rangle + f(x,y)] > 0\}$ is open in X;

(v) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in \mathbf{N}$ and $x_n \in C_n$ such that $x_n \in F(y_n) \cap \{z \in X : \inf_{u \in T(y_n)} Re\langle u, y_n - z\rangle + f(y_n, z) > 0\}$

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{y \in F(\hat{x})} \inf_{u \in T(\hat{x})}[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0.$$

If in addition, for each $x \in X$, $y \mapsto f(x,y)$ is concave, then there exists $\hat{u} \in T(\hat{x})$ such that $\sup_{y \in F(\hat{x})}[Re\langle \hat{u}, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0$.

Proof. Define $\psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ by

$$\psi(x,y) = \inf_{u \in T(x)}[Re\langle u, x-y\rangle + f(x,y)]$$

for each $(x,y) \in X \times X$. Since X is bounded, by Lemma 2 of Kim and Tan [184], for each $y \in X$, $x \mapsto \inf_{u \in T(x)}\langle u, x-y\rangle$ is lower semicontinuous. Therefore ψ and F satisfy all hypotheses of Theorem 4.3.6. Thus the conclusion follows from Theorem 4.3.6. □

Now if we impose a continuity condition on the correspondence F, then we have the following:

Theorem 4.3.9. Let $X = \cup_{i=1}^{\infty} C_i$ be bounded, where $\{C_i\}_{i=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of a locally convex topological vector space E. Suppose that

(i) $F : X \to 2^X$ is continuous with non-empty compact and convex values;

(ii) $T : X \to 2^{E^*}$ has (strongly) compact convex values and is upper semicontinuous when E^* is equipped with the strong topology;

(iii) $f : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is lower semicontinuous and for each $x \in X$, $y \mapsto f(x,y)$ is 0-diagonally concave;

(iv) for each sequence $(y_n)_{n=1}^{\infty}$ in X which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n \in N$ and $x_n \in C_n$ such that $x_n \in F(y_n) \cap \{z \in X : \inf_{u \in T(y_n)} Re\langle u, y_n - z\rangle + f(y_n, z) > 0\}$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{y \in F(\hat{x})}[\inf_{u \in T(\hat{x})} Re\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0.$$

If in addition, for each $x \in X$, $y \mapsto f(x,y)$ is concave, then there exists $\hat{u} \in T(\hat{x})$ such that $\sup_{y\in F(\hat{x})}[Re\langle \hat{u}, \hat{x}-y\rangle + f(\hat{x},y)] \leq 0$.

Proof. Define the function $\psi_1 : X \times X \times E^* \to \mathbf{R} \cup \{-\infty, +\infty\}$ by $\psi_1(x,y,u) = Re\langle u, x-y\rangle$ for each $(x,y,u) \in X \times X \times E^*$. Since X is a bounded subset of the locally convex topological vector space E and E^* is equipped with the strong topology, ψ_1 is continuous. Since $T : X \to 2^{E^*}$ is upper semicontinuous with (strong) compact and convex values, by Theorem 1 of Aubin [7, p.67], the function $\psi_2 : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ defined by $\psi_2(x,y) = \inf_{u\in T(x)}\langle u, x-y\rangle$ is also lower semicontinuous on $X \times X$ so that $(x,y) \mapsto \inf_{u\in T(x)} Re\langle u, x-y\rangle + f(x,y)$ is lower semicontinuous by (iii). As $F : X \to 2^X$ is lower semicontinuous, by Theorem 2 of Aubin [7, p.69], the function $x \mapsto \sup_{y\in F(x)} Re \inf_{u\in T(x)}[\langle u, x-y\rangle + f(x,y)]$ is lower semicontinuous from X to $\mathbf{R} \cup \{-\infty, +\infty\}$. It follows that the set $\Sigma = \{x \in X : \sup_{y\in F(x)} \inf_{u\in T(x)} Re[\langle u, x-y\rangle + f(x,y)] > 0\}$ is open in X. Thus F, T and f satisfy all hypotheses of Corollary 4.3.7. Thus the conclusion follows from Corollary 4.3.7. □

Remark: In Theorems 4.3.1, 4.3.2, 4.3.4, 4.3.5, 4.3.6 and 4.3.9, we assume that the correspondence $T : X \to 2^{E^*}$ satisfies some kind of continuity. In fact, under appropriate conditions, the existence theorems for solutions of the problems (I) and (II) still hold without assuming any continuity of T, for more details, see Ricceri [251] and the references therein. We also note that Corollary 4.3.8 (and hence also Theorem 4.3.6) generalizes the Theorem of Kim [180] which in turn improves Theorem 4 of Shih and Tan [267]. For the applications of quasi-variational inequalities and generalized quasi-variational inequalities to game theory and economics theory, we refer to Aubin [7], Aubin and Ekeland [10], Border [34] and references therein.

4.4 Stability of Quasi-Variational Inequalities

In this section, we shall study the stability of the set $S_A(D,S,T) = \{y \in D\colon y \in S(y)$ and $\sup_{x\in S(y)} \sup_{w\in T(y)} Re\langle w, y-x\rangle \le 0\}$ (respectively, the set $S_B(D,S,T) = \{y \in D.$ $y \in S(y)$ and $\sup_{x\in S(y)} \inf_{w\in T(y)} Re\langle w, y-x\rangle \le 0\}$), where D is a non-empty compact convex subset of a complete convex subset X of the normed space E, $S : X \to 2^X$ is a continuous set-valued mapping with non-empty compact convex values and T : $X \to 2^{E^*}$ is monotone and lower semicontinuous with non-empty bounded closed values (respectively, $T : X \to 2^{E^*}$ is upper semicontinuous with non-empty compact values).

Throughout this section, (i) E denotes a normed space with norm $\|\cdot\|$; (ii) E^* denotes the dual space of E with the norm $\|\cdot\|^*$, (iii) X denotes a non-empty complete convex subset of E, (iv) $K(X)$ denotes the family of all non-empty compact subsets of X and (v) h (respectively, h^*) denotes the Hausdorff metric defined on the family $bc(E)$ (respectively, $bc(E^*)$) of all bounded closed subsets of E (respectively, E^*) which is induced by the norm $\|\cdot\|$ (respectively, $\|\cdot\|^*$) Note that $K(X)$ is a complete metric space when equipped with the Hausdorff metric h (e.g , see [6]). If $A \subset E$ is non-empty, $x \in E$ and $\delta > 0$, let $U(\delta, x) = \{y \in E : \|x-y\| < \delta\}$ and $U(\delta, A) = \{y \in E : \|y-a\| < \delta$ for some $a \in A\}$.

We shall study the stability of the solution set $S_A(D,S,T)$ of $CQVI(A)$ and of the solution set $S_B(D,S,T)$ of $GQVI(B)$ with D, S and T varying.

Let $C(X) = \{S : X \to K(X) : S$ is continuous on $X\}$, $L(X) = \{T : X \to 2^{E^*} . T$ is lower semicontinuous with bounded closed values $\}$, and $U(X) = \{T : X \to K(E^*) : T$ is upper semicontinuous $\}$. For each $S_1, S_2 \in C(X)$, define $d_1(S_1,S_2) = \sup_{x\in X} h(S_1(x), S_2(x))$ For each $T_1, T_2 \in L(X)$ (respectively, T_1, $T_2 \in U(X)$), define $d_2(T_1,T_2) = \sup_{x\in X} h^*(T_1(x), T_2(x))$. It can be routinely checked that $(C(X), d_1)$, $(L(X), d_2)$ and $(U(X), d_2)$ are complete metric spaces. For each $u_1 = (D_1, S_1, T_1)$, $u_2 = (D_2, S_2, T_2)$ in $Y_L := K(X)\times C(X)\times L(X)$ (respectively, $Y_U = K(X)\times C(X)\times U(X)$), define $\rho(u_1,u_2) = h(D_1,D_2) + d_1(S_1,S_2) + d_2(T_1,T_2)$. Then (Y,ρ) is a complete metric space for $Y = Y_L$ or $Y = Y_U$

As a special case of Theorem 4.3 4, we have the following generalized quasi-variational inequalities (GQVI) which is Theorem 2 of Shih and Tan [267] (see also Theorem 2 of Shih and Tan [274] with $f \equiv 0$):

Theorem 4.4.A. Let E be a locally convex topological vector space, E^* be the dual space of E and X be a non-empty compact convex subset of E. Let $S : X \to 2^X$ be continuous such that for each $x \in X$, $S(x)$ is a closed convex subset of X and let $T : X \to 2^{E^*}$ be monotone such that T is lower semicontinuous from the relative topology of X to the strong topology of E^*. Then there exists a point $\hat{y} \in X$ such that

$$GQVI(A) \qquad \begin{cases} \hat{y} \in S(\hat{y}) \text{ and} \\ \sup_{x \in S(\hat{y})} \sup_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq 0. \end{cases}$$

By Theorem 4.3.9, we also have the following generalized quasi-variational inequalities which is Theorem 4 of Shih and Tan [267]:

Theorem 4.4.B. Let E be a locally convex topological vector space, E^* the dual space of E and let X be a non-empty compact convex subset of E. Let $S : X \to 2^X$ be continuous such that for each $x \in X$, $S(x)$ is a closed convex subset of X, and $T : X \to 2^{E^*}$ be upper semicontinuous from the relative topology of X to the strong topology of E^* such that for each $x \in X$, $T(x)$ is a strongly compact subset of E^*. Then there exists a point $\hat{y} \in X$ such that

$$GQVI(B) \qquad \begin{cases} \hat{y} \in S(\hat{y}) \text{ and} \\ \sup_{x \in S(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq 0. \end{cases}$$

We call such a point $\hat{y}$ in Theorem 4.4.A (respectively, Theorem 4.4.B) a solution of the generalized quasi-variational inequality (A) (respectively, (B)), in short, GQVI(A) (respectively, GQVI(B)) for (S, T) in X and denote by $S_A(X, S, T)$ (respectively, $S_B(X, S, T)$) the set of all solutions of the generalized quasi-variational inequality (A) (respectively, (B)) for (S, T) in X. Thus under the stated conditions above, the associated set $S_A(X, S, T)$ (respectively, $S_B(X, S, T)$) is non-empty.

Lemma 4.4.1. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $K(X)$ which converges to $A \in K(X)$. Then every sequence $(x_n)_{n=1}^{\infty}$ in X with $x_n \in A_n$ for each $n \in \mathbf{N}$, has a subsequence which converges to a point in A.

Proof. Since A_n and A are compact and $A_n \to A$, by A.5.1 (ii) of Mas-Colell [216, p.10], $\cup_{n=1}^{\infty} A_n \cup A$ is compact. Since $x_n \in A_n \subset \cup_{n=1}^{\infty} A_n \cup A$, the sequence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges to $x \in \cup_{n=1}^{\infty} A_n \cup A$. Now by Lemma 1(2) of Yu [329, p.231], $x \in A$. □

Let $M_L = \{(D,S,T) \in Y_L$: there exists $y \in D$ such that $y \in S(y)$ and
$\sup_{x\in S(y)} \sup_{w\in T(y)} Re\langle w, y-x\rangle \leq 0$ $\}$

and

$M_U = \{(D,S,T) \in Y_U$: there exists $y \in D$ such that $y \in S(y)$ and
$\sup_{x\in S(y)} \inf_{w\in T(y)} Re\langle w, y-x\rangle \leq 0\}$.

Define $S_A : M_L \to 2^X$ (respectively, $S_B : M_U \to 2^X$) by

$$S_A(u) = \{y \in D : y \in S(y) \text{ and } \sup_{x\in S(y)} \sup_{w\in T(y)} Re\langle w, y-x\rangle \leq 0\}$$

for each $u = (D,S,T) \in M_L$ (respectively,

$$S_B(u) = \{y \in D : y \in S(y) \text{ and } \sup_{x\in S(y)} \inf_{w\in T(y)} Re\langle w, y-x\rangle \leq 0\}$$

for each $u = (D,S,T) \in M_U$).

Lemma 4.4.2.

(i) The space M_L is closed in Y_L.

(ii) The space M_U is closed in Y_U.

Proof. Let $((D_n, S_n, T_n))_{n=1}^{\infty}$ be a sequence in M_L (respectively, M_U) such that $(D_n, S_n, T_n) \to (D,S,T) \in Y_L$ (respectively, Y_U). For each $n \in \mathbf{N}$, let $y_n \in D_n$ be such that

(1) $y_n \in S_n(y_n)$ and

(2) $\sup_{x\in S_n(y_n)} \sup_{w\in T_n(y_n)} Re\langle w, y_n - x\rangle \leq 0$
(respectively, $\sup_{x\in S_n(y_n)} \inf_{w\in T_n(y_n)} Re\langle w, y_n - x\rangle \leq 0$).

Since $D_n \to D$, by Lemma 4.4.1, without loss of generality, we may assume that $y_n \to y_0$ as $n \to \infty$ and $y_0 \in D$.

Now we shall show that

(i) $y_0 \in S(y_0)$ and

(ii) $\sup_{x\in S(y_0)} \sup_{w\in T(y_0)} Re\langle w, y_0 - x\rangle \leq 0$
(respectively, $\sup_{x\in S(y_0)} \inf_{w\in T(y_0)} Re\langle w, y_0 - x\rangle \leq 0$}

(i) Suppose that $y_0 \notin S(y_0)$, then there exists $a > 0$ such that $U(a, S(y_0)) \cap U(a, y_0) = \emptyset$ Since $S_n \to S$ as $n \to \infty$ there exists $n_1 \in \mathbf{N}$ such that for each $n \in \mathbf{N}$ with $n \geq n_1$, $S_n(u) \subset U(a/2, S(u))$ for all $u \in X$. Since $y_n \to y_0$, by the upper semicontinuity of S, there exists a positive integer $n_2 \geq n_1$ such that $y_n \in U(a, y_0)$ and $S(y_n) \subset U(a/2, S(y_0))$ for all $n \in \mathbf{N}$ with $n \geq n_2$. Now for any integer $n \geq n_2$, $S_n(y_n) \subset U(a, S(y_0))$ and $y_n \in U(a, y_0)$. Note that $y_n \in S_n(y_n)$ which contradicts that $U(a, S(y_0)) \cap U(a, y_0) = \emptyset$ Therefore we must have $y_0 \in S(y_0)$

(ii) Suppose that

$$\sup_{x\in S(y_0)} \sup_{w\in T(y_0)} Re\langle w, y_0 - x\rangle > 0$$

(respectively,

$$\sup_{x\in S(y_0)} \inf_{w\in T(y_0)} Re\langle w, y_0 - x\rangle > 0),$$

then there exist $x_0 \in S(y_0)$ and $\epsilon > 0$ such that $\sup_{w\in T(y_0)} Re\langle w, y_0 - x_0\rangle > \epsilon > 0$ (respectively, $\inf_{w\in T(y_0)} Re\langle w, y_0 - x_0\rangle > \epsilon > 0$) As T is lower semicontinuous (respectively, upper semicontinuous) and the mapping $(w, y, x) \mapsto Re\langle w, y - x\rangle$ is continuous from $E^* \times X \times X$ to $\mathbf{R}$, the mapping $(y, x) \mapsto \sup_{w\in T(y)} Re\langle w, y - x\rangle$ (respectively, $(y, x) \mapsto \inf_{w\in T(y)} Re\langle w, y - x\rangle$) is lower semicontinuous by Proposition 19 (respectively, Proposition 21) of Aubin and Ekeland [10, p 118] (respectively, [10, p 119]) Since $y_n \to y_0$ as $n \to \infty$, there exist $\delta > 0$ and $n_0 \in \mathbf{N}$ such that for each integer $n \geq n_0$, $y_n \in U(\delta, y_0)$ and for each $(y', x') \in U(\delta, y_0) \times U(\delta, x_0)$, $\sup_{w\in T(y')} Re\langle w, y' - x'\rangle > \epsilon > 0$ (respectively, $\inf_{w\in T(y')} Re\langle w, y' - x'\rangle > \epsilon > 0$) In particular, for each integer $n \geq n_0$ and $x' \in U(\delta, x_0)$,

$$\sup_{w\in T(y_n)} Re\langle w, y_n - x'\rangle > \epsilon > 0 \text{ (respectively, } \inf_{w\in T(y_n)} Re\langle w, y_n - x'\rangle > \epsilon > 0). \quad (4.1)$$

Because $S_n \to S$ as $n \to \infty$, there exists an integer $n_1 \geq n_0$ such that $S(u) \subset U(\delta/4, S_n(u))$ for all $u \in X$. Since $S(y_n) \to S(y_0)$ and $x_0 \in S(y_0)$, there exists an integer $n_2 \geq n_1$ such that $S(y_n) \cap U(\delta/4, x_0) \neq \emptyset$ for all integers $n \geq n_2$. Therefore for all $n \in N$ with $n \geq n_2$, $S(y_n) \cap U(\delta/4, x_0) \neq \emptyset$ and $S(u) \subset U(\delta/4, S_n(u))$ for all $u \in X$ which imply that

$$S_n(y_n) \cap U(\delta/2, x_0) \neq \emptyset. \tag{4.2}$$

Now by (4.1) and (4.2), for each integer $n \geq n_2$, we have

$$\sup_{w \in T(y_n)} Re\langle w, y_n - x' \rangle > \epsilon > 0 \ \text{(respectively,} \inf_{w \in T(y_n)} Re\langle w, y_n - x' \rangle > \epsilon > 0) \tag{4.3}$$

for all $x' \in S_n(y_n) \cap U(\delta/2, x_0)$.

Since $T_n \to T$ as $n \to \infty$, there exists an integer $n_3 \geq n_2$ such that for each $n \geq n_3$, $d_2(T_n, T) < \frac{\epsilon}{2p}$, where $p = \|x_0\| + \delta + \sup\{\|y_n\| : n \geq 1\}$. Fix an $n \geq n_3$. Let $w \in T(y_n)$ (respectively, $w \in T_n(y_n)$) be arbitrarily fixed. Since $h^*(T_n(y_n), T(y_n)) < \frac{\epsilon}{2p}$, there exists $w' \in T_n(y_n)$ (respectively, $w' \in T(y_n)$) with $\|w - w'\|^* < \frac{\epsilon}{2p}$. By (??), choose any $x' \in S_n(y_n) \cap U(\delta/2, x_0)$. Then

$$|Re\langle w' - w, y_n - x' \rangle| \leq \|w' - w\|^*(\|y_n\| + \|x' - x_0\| + \|x_0\|) < \frac{\epsilon}{2},$$

so that

$$\sup_{z \in T(y_n)} Re\langle z, y_n - x' \rangle \geq Re\langle w', y_n - x' \rangle \geq Re\langle w, y_n - x' \rangle - \epsilon/2$$

(respectively,

$$Re\langle w, y_n - x' \rangle \geq Re\langle w', y_n - x' \rangle - \epsilon/2 \geq \inf_{w \in T(y_n)} Re\langle w, y_n - x' \rangle - \epsilon/2).$$

Since $w \in T(y_n)$ (respectively, $w \in T_n(y_n)$) is arbitrary, we have by (4.3),

$$\sup_{z \in T_n(y_n)} Re\langle z, y_n - x' \rangle \geq \sup_{w \in T(y_n)} Re\langle w, y_n - x' \rangle - \epsilon/2 > \epsilon - \epsilon/2 = \epsilon/2 > 0$$

(respectively,

$$\inf_{w \in T_n(y_n)} Re\langle w, y_n - x' \rangle \geq \inf_{w \in T(y_n)} Re\langle w, y - x' \rangle > \epsilon - \epsilon/2 = \epsilon/2 > 0).$$

This contradicts (2) as $x' \in S_n(y_n)$. Hence (ii) must hold.

Therefore $y_0 \in S(u)$. Thus $(D,S,T) \in M_L$ (respectively, $(D,S,T) \in M_U$) so that M_L (respectively, M_U) is closed in Y_L (respectively, Y_U). □

For convenience, we recall and state Theorem 2 of Fort [115, p.101] again as Lemma 4.4.3 (see also Lemma 1.7.1.) below:

Lemma 4.4.3. Suppose W is a metric space, Z is a topological space and $S : Z \to K(W)$ is upper semicontinuous. Then S is continuous at points of a residual set in Z.

Lemma 4.4.4.

(i) $S_A(u) \in K(X)$ for each $u \in M_L$.

(ii) $S_B(u) \in K(X)$ for each $u \in M_U$.

Proof. Suppose $u = (D,S,T) \in M$ (respectively, $u = (D,S,T) \in M_U$). Since $S_A(u) \subset D$ (respectively, $S_B(u) \subset D$), it is sufficient to prove that $S_A(u)$ (respectively, $S_B(u)$) is closed in D. Let $(y_n)_{n=1}^{\infty}$ be a sequence in $S_A(u)$ (respectively, $S_B(u)$) which converges to a point $y_0 \in D$. By the definition of S_A (respectively, S_B), we have (i) $y_n \in S(y_n)$ and (ii) $\sup_{x\in S(y_n)} \sup_{w\in T(y_n)} Re\langle w, y_n - x\rangle \le 0$ (respectively, $\sup_{x\in S(y_n)} \inf_{w\in T(y_n)} Re\langle w, y_n - x\rangle \le 0$). Since S is upper semicontinuous on D with compact values and D is compact, $y_0 \in S(y_0)$. Note that T is lower semicontinuous (respectively, upper semicontintinuous) and the mapping $(w,y,x) \mapsto Re\langle w, y-x\rangle$ is continuous from $E^* \times X \times X$ to R, it follows that the mapping $(y,x) \mapsto \sup_{w\in T(y)} Re\langle w, y-x\rangle$ (respectively, $(y,x) \mapsto \inf_{w\in T(y)} Re\langle w, y-x\rangle$) is lower semicontinuous from $X \times X$ to R by Proposition 19 (respectively, Proposition 21) of Aubin and Ekeland [10, p.118] (respectively [10, p.119]). Since S is also lower semicontinuous, the mapping $y \mapsto \sup_{x\in S(y)} \sup_{w\in T(y)} Re\langle w, y-x\rangle$ (respectively, $y \mapsto \sup_{x\in S(y)} \inf_{w\in T(y)} Re\langle w, y-x\rangle$) is also lower semicontinuous by Proposition 19 of Aubin and Ekeland [10, p.118] again. Thus

$$\sup_{x\in S(y_0)} \sup_{w\in T(y_0)} Re\langle w, y_0 - x\rangle \le \liminf_{n\to\infty} \sup_{x\in S(y_n)} \sup_{w\in T(y_n)} Re\langle w, y_n - x\rangle \le 0$$

(respectively,

$$\sup_{x\in S(y_0)} \inf_{w\in T(y_0)} Re\langle w, y_0 - x\rangle \leq \liminf_{n\to\infty} \sup_{x\in S(y_n)} \inf_{w\in T(y_n)} Re\langle w, y_n - x\rangle \leq 0).$$

Hence $y_0 \in S_A(u)$ (respectively, $y_0 \in S_B(u)$). Therefore $S_A(u)$ (respectively, $S_B(u)$) is closed in D. □

Lemma 4.4.5. The correspondences $S_A : M_L \to K(X)$ (respectively, $S_B : M_U \to K(X)$) is upper semicontinuous.

Proof. Suppose that S_A (respectively, S_B) were not upper semicontinuous at some point $u = (D, S, T) \in M_L$ (respectively, M_U), then there exists an open subset G of X with $G \supset S_A(u)$ (respectively, $G \supset S_B(u)$) and a sequence $\{u_n\}_{n\in\mathbf{N}}$ in M_L (respectively, M_U) with $u_n \to u \in M_L$ (respectively, M_U) such that for each $n \in \mathbf{N}$, there exists $y_n \in S_A(u_n)$ (respectively, $y_n \in S_B(u_n)$) with $y_n \notin G$. Let $u_n = (D_n, S_n, T_n)$, then $D_n \to D$; $S_n \to S$ and $T_n \to T$. Since $y_n \in D_n$, for each $n \in \mathbf{N}$, by Lemma 4.4.1, we may assume without loss of generality that $y_n \to y_0 \in D$. Note that $y_n \notin G$ for all $n \in \mathbf{N}$ so that $y_0 \notin G \supset S_A(u)$ (respectively, $y_0 \notin G \supset S_B(u)$). Now the same argument as in the proof of Lemma 4.4.2 shows that (1) $y_0 \in S(y_0)$ and (2) $\sup_{x\in S(y_0)} \sup_{w\in T(y_0)} Re\langle w, y_0 - x\rangle \leq 0$ (respectively, $\sup_{x\in S(y_0)} \inf_{w\in T(y_0)} Re\langle w, y_0 - x\rangle \leq 0$). Therefore $y_0 \in S_A(u)$ (respectively, $y_0 \in S_B(u)$). This contradicts tht fact that $y_0 \notin G \supset S_A(u)$ (respectively, $y_0 \notin G \supset S_B(u)$). Therefore S_A (respectively, S_B) is upper semicontinuous. □

In what follows, let $M = M_L$ or M_U and $S = S_A$ or S_B, respectively. Now let M_1 be an arbitrarily fixed non-empty closed subset of M.

Definition. If $u \in M_1$, then (i) a point $y \in S(u)$ is essential relative to M_1 if for each open neighborhood $N(y)$ of y in X, there exists an open neighborhood $O(u)$ of u in M_1 such that $S(u') \cap N(y) \neq \emptyset$ for each $u' \in O(u)$ and (ii) u is essential relative to M_1 if every $y \in S(u)$ is essential relative to M_1.

Theorem 4.4.6.

(i) S is lower semicontinuous at $u \in M_1$ if and only if u is essential relative to M_1.

(ii) S is continuous at $u \in M_1$ if and only if u is essential relative to M_1.

Proof. (i) S is lower semicontinuous at $u \in M_1$ if and only if each $y \in S(u)$ is essential relative to M_1 if and only if u is essential relative to M_1.

(ii) This follows from (i) and Lemma 4.4.5. □

Theorem 4.4.7. If $u \in M_1$ is such that $S(u)$ is a singleton set, then u is essential relative to M_1.

Proof. Suppose $S(u) = \{x\}$. Let G be any open set in X such that $S(u) \cap G \neq \emptyset$, then $x \in G$ so that $S(u) \subset G$. Since S is upper semicontinuous at u by Lemma 4.4.5, there is an open neighborhood $O(u)$ of u in M such that $S(u') \subset G$ for each $u' \in O(u)$; in particular, $G \cap S(u') \neq \emptyset$ for each $u' \in O(u)$. Thus S is lower semicontinuous at u. By Theorem 4.4.6 (ii), u is essential relative to M. □

Theorem 4.4.8. There exists a dense G_δ subset Q_1 of M_1 such that u is essential relative to M_1.

Proof. Note that S is an usco by Lemma 4.4.4 and Lemma 4.4.5. By Lemma 4.4.3, there exists a dense G_δ subset Q_1 of M_1 such that S is lower semicontinuous at each $u \in Q_1$. By Theorem 4.4.6 (ii), u is essential relative to M_1 for each $u \in Q_1$. □

Remark: Note that the mapping $S : M_1 \to K(X)$ is continuous at $u = (D, S, T) \in M_1$ if and only if for each $\epsilon > 0$, there is $\delta > 0$ such that $h(S(u), S(u')) < \epsilon$ whenever $u' \in M_1$ and $\rho(u, u') < \delta$; i.e., the solution set $S(u)$ of u is stable in M_1: $S(u')$ is close to $S(u)$ whenever $u' \in M_1$ is close to u. Theorem 4.4.6 (ii) implies that if $u \in M_1$, then u is essential relative to M_1 if and only if the solution set $S(u)$ is stable in M_1. Theorem 4.4.8 implies that there exists a dense G_δ subset Q_1 of M_1 such that for each $u = (D, S, T) \in Q_1$, the solution set $S(u)$ of the $GQVI$ is stable in M_1. In particular, most (in the sense of Baire category) u in M_1 have stable solution set $S(u)$.

Now let

$$
\begin{aligned}
CK(X) &= \{A \in K(X) : A \text{ is convex }\}, \\
CC(X) &= \{S \in C(X) : S(x) \in CK(X) \text{ for each } x \in X\},
\end{aligned}
$$

$$
\begin{aligned}
ML(X) &= \{T \in L(X) : T \text{ is monotone }\}, \\
M'_L &= \{X\} \times CC(X) \times ML(X), \\
M'_U &= \{X\} \times CC(X) \times U(X).
\end{aligned}
$$

It is easy to see that $CC(X)$ is closed in $K(X)$ so that M'_U is closed in M_U. Also $ML(X)$ is closed in $L(X)$ so that M'_L is also closed in M_L. The following is an application of the results obtained in this section:

Theorem 4.4.9. Let X be a non-empty compact and convex subset of the normed space E. Then there exists a dense G_δ subset Q' of M'_L (respectively, M'_U) such that u is essential relative to M'_L (respective, M'_U) for each $u \in Q'$. Thus most (in the sense of Baire category) of the solutions of the $GQVI(A)$ (respectively, $GQVI(B)$) for (S,T) in X are stable in M'_L (respectively, M'_U).

Let $M''_L = \{X\} \times \{O_X\} \times ML(X)$ (respectively, $M''_U = \{X\} \times \{O_X\} \times U(X)$), where $O_X(x) = X$ for all $x \in X$. Clearly, M''_L (respectively, M''_U) is a closed subset of M'_L (respectively, M'_U) if X is compact. The following deals with the stability of Hartman-Stampacchia variational inequalities [146]:

Theorem 4.4.10. Let X be a non-empty compact and convex subset of the normed space E. Then there exists a dense G_δ subset Q'' of M''_L (respectively, M''_U) such that u is essential relative to M''_L (respectively, M''_U) for each $u \in Q''$.

4.5 Variational Inequalities on Reflexive Banach Spaces and Applications

It is well-known that variational inequalities have a close connection with fixed point theory, for example, the famous Fan-Browder fixed point theorem [42] can be derived from variational inequalities for monotone operators (e g , see Browder [45]) We note that there is also an interconnection between variational inequalities and monotone operators In fact, most existence theorems for variational inequalities could be obtained from direct applications of the main theorems on maximal monotone operators Monotone operators have been comprehensively studied in the last three decades The theory of monotone operator is related to the simple fact that the *derivative* f' of a convex real function f is a monotone function Moreover, it is a very powerful tool to handle nonlinear differential equations (e g , see Zeidler [336] and references therein)

A generalization of monotone operators which they called a "*semi-monotone operator*" was first introduced by Bae, Kim and Tan [13] In this section, we give some variational inequalities for monotone and semi-monotone operators in Banach spaces As applications, an existence theorem for a generalized complementarity problem in the Banach space and some fixed point theorems for multivalued nonexpansive mappings in the Hilbert space are given

Definition. Let E be a topological vector space and X a non-empty subset of E Then a map $T : X \to 2^{E^*}$ is semi-monotone on X (see Bae, Kim and Tan [13]) if for each $x, y \in X$, $u \in T(x)$ and $w \in T(y)$, $\inf_{u\in T(x)} Re\langle u, y-x\rangle \leq \inf_{w\in T(y)} Re\langle w, y-x\rangle$ It is clear from the definitions that if T is monotone, then T is semi-monotone

If E is a normed space with norm $\|\ \|$, A is a non-empty subset of E and $x \in E$, $d(x, A) = \inf\{\|x-y\| : y \in A\}$ is the distance from x to A Recall that the Hausdorff metric h on the family $bc(E)$ of all non-empty bounded and closed subsets of E induced by the norm is defined by

$$\begin{aligned} h(A_1, A_2) &= \inf\{r > 0 : A_1 \subset B_r(A_2) \text{ and } A_2 \subset B_r(A_1)\} \\ &= \max\{\sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1)\} \end{aligned}$$

where $B_r(A) = \{x \in E : d(x, A) > r\}$ for any $A \in 2^E$ and $r > 0$. A map $S : X \to 2^E$ is said to be pseudo-contractive [13] on X if for each $x, y \in X$ and $w \in S(y)$, there exists $u \in S(x)$ such that $\|x - y\| \leq \|(1 + r)(x - y) - r(u - w)\|$ for all $r > 0$. (This is a set-valued generalization of pseudo-contractive (single-valued) maps as defined by Browder in [41]). A map $S : X \to bc(E)$ is said to be nonexpansive on X if for each $x, y \in X$, $h(S(x), S(y)) \leq \|x - y\|$.

The following example shows that (i) there is a nonexpansive map T such that $I - T$ is not monotone and (ii) a semi-monotone map need not be monotone:

Example. Let $\mathbf{R}$ be the real linear, d be the usual metric on $\mathbf{R}$ and h be the Hausdorff metric on $bc(\mathbf{R})$ induced by d. Define $T : \mathbf{R} \to bc(\mathbf{R})$ by $T(x) = [-|x|, |x|]$ for each $x \in \mathbf{R}$. Then we have:

(1) For each $x, y \in \mathbf{R}$,

$$h(T(x), T(y)) = ||x| - |y|| \leq |x - y| = d(x, y);$$

it follows that T is nonexpansive.

(2) Suppose $y > x > 0$; then

$(I - T)(x) = x - [-|x|, |x|] = [0, 2x]$ and $(I - T)(y) = [0, 2y]$.

Choose $u = 2x$ and $w = x$, then $u \in (I - T)(x)$ and $w \in (I - T)(y)$. But

$$\langle w - u, y - x\rangle = \langle x - 2x, y - x\rangle = -x(y - x) < 0$$

which shows that $I - T$ is not monotone.

Further more, tha map $I - T$ is necessarily semi-monotone by Lemma 4.5.16 below (see also Proposition of Bae et al [13]). Thus, in general, a semi-monotone map need not be monotone.

We need the following result which can be derived from Corollary 1.3.5 and is equivalent to the minimax inequality of Yen [327, Theorem 1] which in turn improves Ky Fan's minimax inequality [105, Theorem 1]; we omit its proof.

Lemma 4.5.1. Let X be a non-empty compact convex subset of a topological vector space E. Suppose $\Phi, \Psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ satisfy the following conditions:

(1) $\Phi(x,y) \leq \Psi(x,y)$ for all $x, y \in X$;

(2) $\Psi(x,x) \leq 0$ for all $x \in X$;

(3) for each $x \in X$, $y \mapsto \Phi(x,y)$ is lower semicontinuous;

(4) for each $y \in X$, $x \mapsto \Psi(x,y)$ is quasi-concave.

Then there exists $\hat{y} \in X$ such that $\Phi(x, \hat{y}) \leq 0$ for all $x \in X$.

We also need the following result which is a special case of Lemma 3 of Ding and Tan [81]:

Lemma 4.5.2. Let $(E, \|\cdot\|)$ be a Banach space, X a non-empty convex subset of E; let $f : X \to R$ be a convex function and $T : X \to 2^{E^*}$ be lower semicontinuous from the line segments in X to the weak*- topology on E^*. If $\hat{y} \in X$, then the inequality

$$\sup_{u \in T(x)} Re\langle u, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X$$

implies the inequality

$$\sup_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

The same proofs of Lemmas 4.5.1 and 6.2 in Shih and Tan [272] can be modified to obtain the following slight improvement of Lemma 2 in [272] and is thus omitted.

Lemma 4.5.3. Let $(E, \|\cdot\|)$ be a Banach space, X a non-empty convex subset of E; let $f : X \to \mathbf{R}$ be a convex and lower semicontinuous function and $T : X \to 2^{E^*}$ be such that each $T(x)$ is a weak*-compact subset of E^* and T is upper semicontinuous from the line segments in X to the weak*-topology on E^*. If $\hat{y} \in X$, then the inequality

$$\sup_{u \in T(x)} Re\langle u, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X$$

implies the inequality

$$\inf_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \leq f(x)-f(\hat{y}) \text{ for all } x\in X.$$

Lemma 4.5.4. Let $(E, \|\cdot\|)$ be a reflexive Banach space, X a non-empty closed convex subset of E; let $f : X \to \mathbf{R}$ be a convex and lower semicontinuous function and $T : X \to 2^{E^*}$ be monotone. Assume that the following condition is satisfied:

(E) For each sequence $(y_n)_{n=1}^{\infty}$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n\to\infty}\{\sup_{u\in T(x_n)} Re\langle u, y_n - x_n\rangle + f(y_n) - f(x_n)\} > 0.$$

Then there exists $\hat{y} \in X$ such that

$$\sup_{u\in T(x)} Re\langle u, \hat{y}-x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

Proof. Define $\Phi, \Psi : X \times X \to R$ by

$$\Phi(x,y) = \sup\nolimits_{u\in T(x)} Re\langle u, y-x\rangle + f(y) - f(x),$$

$$\Psi(x,y) = \inf\nolimits_{w\in T(y)} Re\langle w, y-x\rangle + f(y) - f(x),$$

Then we have:

(a) since T is monotone,

$$\sup_{u\in T(x)} Re\langle u, y-x\rangle \leq \inf_{w\in T(y)} Re\langle w, y-x\rangle \text{ for all } x, y \in X$$

so that $\Phi(x,y) \leq \Psi(x,y)$ for all $x, y \in X$;

(b) clearly $\Psi(x,x) = 0$ for all $x \in X$;

(c) since f is convex and lower semicontinuous with respect to the norm topology on X, f is also lower semicontinuous with respect to the weak topology on X; it follows that for each $x \in X$, the function $y \to \Phi(x,y)$ is weakly lower semicontinuous;

(d) fix $y \in X$; suppose $x_1, x_2 \in X$ and $t \in R$ are such that $\Psi(x_i, y) > t$ for $i = 1, 2$; let $\alpha \in (0,1)$, then for i=1,2,

$$\inf_{w\in T(y)} Re\langle w, y-x_i\rangle + f(y) - f(x_i) > t$$

so that

$$\begin{aligned}
f(\alpha x_1 &+ (1-\alpha)x_2) - f(y) \le \alpha f(x_1) + (1-\alpha)f(x_2) - f(y) \\
&= \alpha(f(x_1) - f(y)) + (1-\alpha)(f(x_2) - f(y)) \\
&< \alpha \cdot (\inf_{w \in T(y)} Re\langle w, y - x_1\rangle - t) + (1-\alpha)\cdot(\inf_{w\in T(y)} Re\langle w, y - x_2\rangle - t) \\
&\le \inf_{w\in T(y)} Re\langle w, \alpha(y - x_1) + (1-\alpha)(y - x_2)\rangle - t \\
&= \inf_{w \in T(y)} Re\langle w, y - (\alpha x_1 + (1-\alpha)x_2)\rangle - t
\end{aligned}$$

it follows that $\Psi(\alpha x_1 + (1-\alpha)x_2, y) > t$ and hence the function $x \mapsto \Psi(x,y)$ is quasi-concave

For each $N \in \mathbf{N}$, let $X_N = \{x \in X : \|x\| \le N\}$. We may assume that $X_n \neq \emptyset$ for all $N \ge N_0$ Note that for each $N \ge N_0$, X_N is weakly compact and convex since E is reflexive, equip X_N with the weak topology, then by Lemma 4.5.1, there exists $\hat{y}_N \in X_N$ such that

(1) $\quad \Phi(x, \hat{y}_N) \le 0$ for all $x \in X_N$.

Suppose $\|\hat{y}_N\| \to \infty$ as $N \to \infty$, then by the assumption (E), there exists a sequence $(x_N)_{N\ge N_0}$ in X with $\|x_N\| \le \|\hat{y}_N\|$ for all $N \ge N_0$ such that

(2) $\quad \limsup_{N\to\infty} \sup_{u\in T(x_N)} Re\langle u, \hat{y}_N - x_N\rangle + f(\hat{y}_N) - f(x_N) > 0.$

But, for each $N \ge N_0$, $\|x_N\| \le \|\hat{y}_N\| \le N$ implies $x_N \in X_N$ so that by (1), $\Phi(x_N, \hat{y}_N) \le 0$ for all $N \ge N_0$; i e ,

$$\sup_{u\in T(x_N)} Re\langle u, \hat{y}_N - x_N\rangle + f(\hat{y}_N) - f(x_N) \le 0 \text{ for all } N \ge N_0$$

which contradicts (2). Therefore we must have $\|\hat{y}_N\| \to \infty$ as $N \to \infty$. It follows that there exists a positive integer $M > N_0$ and a subsequence $(\hat{y}_{N(i)})_{i=1}^{\infty}$ of $(\hat{y}_N)_{N\ge N_0}$ such that $\|\hat{y}_{N(i)}\| \le M$ for all $i = 1, 2, \cdots$. Thus $(\hat{y}_{N(i)})_{i=1}^{\infty}$ is a sequence in the weakly compact set X_M so that by the Eberlein-Smulian Theorem (e.g. see Dunford and Schwart [92, p.430], there exist another subsequence $(\hat{y}_{N(i(j))})_{j=1}^{\infty}$ of $(\hat{y}_{N(i)})_{i=1}^{\infty}$ and $\hat{y} \in X_M$ such that $(\hat{y}_{N(i(j))})_{j=1}^{\infty}$ converges weakly to $\hat{y}$.

Now let $x \in X$ be given. Choose any positive integer $M' \geq M$ with $x \in X_{M'}$. Take any $j_0 \in \mathbf{N}$ with $N(i(j_0)) \geq M'$; then for all $j \geq j_0$, $x \in X_{M'} \subset X_{N(i(j_0))}$ so that by (1), $\Phi(x, \hat{y}_{N(i(j))}) \leq 0$. Since $(\hat{y}_{N(i(j))})_{j=1}^{\infty}$ converges weakly to $\hat{y}$ and $y \mapsto \Phi(x, y)$ is weakly lower semicontinuous by (c), we must have

$$\Phi(x, \hat{y}) \leq \liminf_{j \to \infty} \Phi(x, \hat{y}_{N(i(j))}) \leq 0.$$

Hence

$$\sup_{u \in T(x)} Re\langle u, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X. \qquad \square$$

Theorem 4.5.5. Let $(E, \|\cdot\|)$ be a reflexive Banach space, X a non-empty closed convex subset of E; let $f : X \to \mathbf{R}$ be a convex and lower semicontinuous function and $T : X \to 2^{E^*}$ monotone . Assume that the following condition is satisfied:

(E) For each sequence $(y_n)_{n=1}^{\infty}$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n \to \infty} \{ \sup_{u \in T(x_n)} Re\langle u, y_n - x_n\rangle + f(y_n) - f(x_n)\} > 0.$$

(I) If T is lower semicontinuous from the line segments in X to the weak topology of E^*, then there exists $\hat{y} \in X$ such that

$$\sup_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

(II) If T is upper semicontinuous from the line segments in X to the weak topology of E^* and each $T(x)$ is weakly compact, then there exists $\hat{y} \in X$ such that

$$\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

If, in addition, $T(\hat{y})$ is also convex, then there exists $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w}, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

Proof. By Lemma 4.5.4, there exists $\hat{y} \in X$ such that

(3) $\sup_{u\in T(x)} Re\langle u, \hat{y}-x\rangle \le f(x)-f(\hat{y})$ for all $x\in X$.

(I) If T is lower semicontinuous from the line segments in X to the weak topology of E^*, then by (3) and Lemma 4.5.2,

$$\sup_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \le f(x)-f(\hat{y}) \text{ for all } x\in X.$$

(II) If T is upper semicontinuous from the line segments in X to the weak topology of E^* and each $T(x)$ is weakly compact, by (3) and Lemma 4.5.3, we have

(4) $\inf_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \le f(x)-f(\hat{y})$ for all $x\in X$.

If, in addition, $T(\hat{y})$ is also convex, define $g: X\times T(\hat{y})\to \mathbf{R}$ by

$$g(x,w) = Re\langle w, \hat{y}-x\rangle + f(\hat{y}) - f(x).$$

Note that for each fixed $x\in X$, $w\mapsto g(x,w)$ is weakly lower semicontinuous and affine and for each fixed $w\in T(\hat{y})$, $x\mapsto g(x,w)$ is concave. Thus by Kneser's minimax Theorem 4.3.A, we have

$$\text{(5)} \qquad \begin{aligned}&\min_{w\in T(\hat{y})}\{\sup_{x\in X} Re\langle w, \hat{y}-x\rangle + f(\hat{y}) - f(x)\}.\\ &= \sup_{x\in X}\inf_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle + f(\hat{y}) - f(x)\}\end{aligned}$$

Since $T(\hat{y})$ is weakly compact, there exists $\hat{w}\in T(\hat{y})$ such that by (4) and (5),

$$\begin{aligned}\sup_{x\in X} &\quad \{Re\langle \hat{w}, \hat{y}-x\rangle + f(\hat{y}) - f(x)\}\\ = &\quad \min_{w\in T(\hat{y})}\sup_{x\in X}\{Re\langle w, \hat{y}-x\rangle + f(\hat{y}) - f(x)\} \le 0,\end{aligned}$$

that is, $Re\langle \hat{w}, \hat{y}-x\rangle \le f(x)-f(\hat{y})$ for all $x\in X$. $\square$

Theorem 4.5.5 generalizes Theorem 2 of Yen [327, p.479-480]

Theorem 4.5.6. Let $(E, \|\cdot\|)$ be a reflexive Banach space, X a non-empty closed convex subset of E; let $f: X\to\mathbf{R}$ be a convex and lower semicontinuous function and $T: X\to 2^{E^*}$ monotone. Assume that the following condition is satisfied:

$(E)_\infty$ For each sequence $(y_n)_{n=1}^\infty$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^\infty$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n\to\infty}\{\sup_{u\in T(x_n)} Re\langle u, y_n - x_n\rangle + f(y_n) - f(x_n)\}/\|y_n\| = \infty.$$

(I) If T is lower semicontinuous from line segments in X to the weak topology of E^*, then for each given $w_0 \in E^*$, there exists $\hat{y} \in X$ such that

$$\sup_{w\in T(\hat{y})} Re\langle w - w_0, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

(II) If T is upper semicontinuous from line segments in X to the weak topology of E^* and each $T(x)$ is weakly compact and convex, then for each given $w_0 \in E^*$, there exists $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w} - w_0, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

Proof. Let $w_0 \in E^*$ be given. Define $T^* : X \to 2^{E^*}$ by $T^*(y) = T(y) - w_0$ for all $y \in X$. By $(E)_\infty$, for each sequence $(y_n)_{n=1}^\infty$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^\infty$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\begin{aligned}
\limsup_{n\to\infty} \quad & \{\sup_{u\in T^*(x_n)} Re\langle u, y_n - x_n\rangle + f(y_n) - f(x_n)\}/\|y_n\| \\
&= \limsup_{n\to\infty}\{\sup_{u\in T(x_n)} Re\langle u - w_0, y_n - x_n\rangle + f(y_n) - f(x_n)\}/\|y_n\| \\
&\geq \limsup_{n\to\infty}\{\sup_{u\in T(x_n)} Re\langle u, y_n - x_n\rangle + f(y_n) - f(x_n)\}/\|y_n\| - 2\|w_0\| \\
&= \infty
\end{aligned}$$

since $|Re\langle w_0, y_n - x_n\rangle|/\|y_n\| \leq \|w_0\| + \|w_0\|\|x_n\|/\|y_n\| \leq 2\|w_0\|$.
It follows that

$$\limsup_{n\to\infty}\{\sup_{u\in T^*(x_n)} Re\langle w_0, y_n - x_n\rangle + f(y_n) - f(x_n)\} = \infty > 0$$

and hence the conclusion follows from Theorem 4.5.5. □

We note that the condition (E) in Lemma 4.5.4 and Theorem 4.5.5 and $(E)_\infty$ in Theorem 4.5.6 are automatically satisfied if the set X is bounded. Also Theorem 4.1 of

Chang and Zhang in [54] is a special case of Theorem 1 (II) We remark that Theorems 4 5 5 and 4 5 6 are very closely related to but not comparable with Theorems 1 and 2 in [272] and Theorems 3 and 4 of Shih and Tan [273]

Recall that a subset X of a vector space E is called a cone if X is a non-empty convex set such that $\alpha X \subset X$ for all $\alpha \geq 0$ If X is a cone in a topological vector space E, X^* will denote the dual cone of X in E^*, i e ,

$$X^* = \{y \in E^* \quad Re\langle y, x\rangle \geq 0 \text{ for all } x \in X\}$$

The same proof of Lemma 2 of Shih and Tan [268] can be modified to obtain the following results and is thus omitted

Lemma 4.5.7. Let X be a cone in a topological vector space E, $T : X \to 2^{E^*}$ and $\hat{y} \in X$ Then the following statements are equivalent

(a) $\sup_{w\in T(y)} Re\langle w, \hat{y} - x\rangle \leq 0$ for all $x \in X$,

(b) $Re\langle w, \hat{y}\rangle = 0$ for all $w \in T(\hat{y})$ and $T(\hat{y}) \subset X^*$.

Lemma 4.5.8. Let X be a cone in a topological vector space E, $T : X \to 2^{E^*}$, $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ Then the following statements are equivalent

(a) $Re\langle \hat{w}, \hat{y} - x\rangle \leq 0$ for all $x \in X$,

(b) $Re\langle \hat{w}, \hat{y}\rangle = 0$ and $\hat{w} \in X^*$

When E is real, Lemma 4 5 6 was also obtained by S C Fang (e g see Chan and Pang [48, p 213])

In view of Lemma 4 5 7 and Lemma 4 5 8 by taking $f \equiv 0$ in Theorem 4 5 5, we have the following theorem on the generalized complementarity problem

Theorem 4.5.9. Let $(E, \|\ \|)$ be a reflexive Banach space, X a closed cone in E and let $T: X \to 2^{E^*}$ be monotone Assume that the following condition is satisfied

$(E)_0$ For each sequence $(y_n)_{n=1}^{\infty}$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n\to\infty} \sup_{u\in T(x_n)} Re\langle u, y_n - x_n\rangle > 0.$$

(I) If T is lower semicontinuous from the line segments in X to the weak topology of E^*, then there exists $\hat{y} \in X$ such that $\text{Re}\langle w, \hat{y}\rangle = 0$ for all $w \in T(\hat{y})$ and $T(\hat{y}) \subset X^*$.

(II) If T is upper semicontinuous from the line segments in X to the weak topology of E^* and each $T(x)$ is weakly compact convex, then there exists $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y}\rangle = 0$ and $\hat{w} \in X^*$.

For more discussion about complementarity problems and their applications, we refer to Isac's new book [161].

The following result is Lemma 1 of Bae, Kim and Tan [13]:

Lemma 4.5.10. Let E be a topological vector space and E^* the dual of E equipped with the strong topology. Let A be a non-empty bounded subset of E and C a non-empty(strongly) compact subset of E^*. Define $f : A \to \mathbf{R}$ by

$$f(x) = \min_{u\in C} Re\langle u, x\rangle \text{ for all } x \in A.$$

Then f is weakly continuous on A.

We shall need the following result:

Lemma 4.5.11. Let E be a topological vector space, X be a non-empty convex subset of E, $f : X \to \mathbf{R}$ be a convex function and $T : X \to 2^{E^*}$ be upper semicontinuous from the line segments in X to the weak* topology on E^* such that each $T(x)$ is weak* compact. If $\hat{y} \in X$, then the inequality

$$\inf_{u\in T(x)} Re\langle u, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X$$

implies the inequality

$$\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

Proof Let $x \in X$ be arbitrarily fixed. For each $t \in [0,1]$, let $z_t = tx + (1-t)\hat{y} = \hat{y} - t(\hat{y} - x)$. Since X is convex, $z_t \in X$ for all $t \in [0,1]$. Thus for all $t \in (0,1]$,

$$t \cdot \inf_{u \in T(z_t)} Re\langle u, \hat{y} - x\rangle = \inf_{u \in T(z_t)} Re\langle u, \hat{y} - z_t\rangle$$

$$\leq f(z_t) - f(\hat{y}) \leq t \cdot f(x) + (1-t) \cdot f(\hat{y}) - f(\hat{y}) = t \cdot (f(x) - f(\hat{y}))$$

so that

(6) $\qquad \inf_{u \in T(z_t)} Re\langle u, \hat{y} - x\rangle \leq f(x) - f(\hat{y})$ for all $t \in (0.1]$.

If $\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle > f(x) - f(\hat{y})$, let $G = \{w \in E^* : Re\langle w, \hat{y} - x\rangle + f(\hat{y}) - f(x) > 0\}$, then G is a weak*-open set in E^* such that $T(\hat{y}) \subset G$. As $z_t \to \hat{y}$ as $t \to 0^+$, by upper semicontinuity of T on $\{z_t : t \in [0,1]\}$, there exists $t_0 \in (0,1]$ such that $T(x_t) \subset G$ for all $t \in (0, t_0)$. As $T(z_t)$ is weak* compact, $\inf_{u \in T(z_t)} Re\langle u, \hat{y} - x\rangle + f(\hat{y}) - f(x) > 0$ for all $t \in (0, t_0)$ which contradicts (6). Thus we must have $\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq f(x) - f(\hat{y})$. $\square$

Lemma 4.5.12. Let $(E, \|\cdot\|)$ be a reflexive Banach space, X a non-empty closed convex subset of E and let $f : X \to \mathbf{R}$ be a convex and lower semicontinuous function and $T : X \to 2^{E^*}$ be semi-monotone such that each $T(x)$ is compact in the norm topology on E^*. Assume that the following condition is satisfied:

$(E)^*$ for each sequence $(y_n)_{n=1}^{\infty}$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n \to \infty}\{\inf_{u \in T(x_n)} Re\langle u, y_n - x_n\rangle + f(y_n) - f(x_n)\} > 0.$$

Then there exists $\hat{y} \in X$ such that

$$\inf_{u \in T(x)} Re\langle u, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

Proof Define $\Phi, \Psi : X \times X \to \mathbf{R}$ by

$$\Phi(x,y) = \inf\nolimits_{u \in T(x)} Re\langle u, y - x\rangle + f(y) - f(x),$$

$$\Psi(x,y) = \inf\nolimits_{w \in T(y)} Re\langle w, y - x\rangle + f(y) - f(x).$$

Then we have:

(a) Since T is semi-monotone, $\Phi(x,y) \leq \Psi(x,y)$ for all $x, y \in X$.

(b) Clearly $\Psi(x,x) = 0$ for all $x \in X$.

(c) Since f is convex and lower semicontinuous with respect to the norm topology on X, f is also lower semicontinuous with respect to the weak topology on X. It follows that for each $x \in X$, the function $y \mapsto \Phi(x,y)$ is weakly lower semicontinuous on A for each non-empty (norm-) bounded subset A of X by Lemma 4.5.10.

(d) For each $y \in X$, it is easy to show that the function $x \to \Psi(x,y)$ is quasi-concave.

Then the proof of Lemma 4.5.4 with the necessary modifications (all "$\sup_{u\in T(x)}$" and all "$\sup_{u\in T(x_N)}$" being replaced by "$\inf_{u\in T(x)}$" and "$\inf_{u\in T(x_N)}$" respectively), we see that there exists $\hat{y} \in X$ such that

$$\inf_{u\in T(x)} Re\langle u, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X. \qquad \square$$

Theorem 4.5.13. Let $(E, \|\cdot\|)$ be a reflexive Banach space, X a non-empty closed convex subset of E and $f : X \to \mathbf{R}$ be a convex and lower semicontinuous function and $T : X \to 2^{E^*}$ be semi-monotone and upper semicontinuous from the line segments in X to the weak topology on E^* such that each $T(x)$ is compact in the norm topology on E^*. Assume that the following condition is satisfied:

$(E)^*$ for each sequence $(y_n)_{n=1}^{\infty}$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n\to\infty}\{\inf_{u\in T(x_n)} Re\langle u, y_n - x_n\rangle + f(y_n) - f(x_n)\} > 0.$$

Then there exists $\hat{y} \in X$ such that

$$\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

If, in addition, $T(\hat{y})$ is also convex, then there exists $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w}, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

Proof. By Lemma 4.5.12, there exists $\hat{y} \in X$ such that

$$\inf_{u\in T(x)} Re\langle u, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

By Lemma 4.5.11, we have

$$\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

If, in addition, $T(\hat{y})$ is also convex, by Kneser's minimax Theorem 4.3.A and by using the same argument as in the proof of Theorem 4.5.5, we get that there exists $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w}, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X. \qquad \square$$

In the proof of Theorem 4.5.6, if we replace all "$\sup_{u\in T^\bullet(x_n)}$" and all "$\sup_{u\in T(x_n)}$" by "$\inf_{u\in T^\bullet(x_n)}$" and "$\inf_{u\in T(x_n)}$" respectively, we have the following application of Theorem 4.5.13:

Theorem 4.5.14. Let$(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E, $f : X \to \mathbf{R}$ be a convex and lower semicontinuous function and $T : X \to 2^{E^*}$ be semi-monotone and upper semicontinuous from the line segments in X to the weak topology on E^* such that each $T(x)$ is convex and compact in the norm topology on E^*. Assume that the following condition is satisfied:

$(E)^*_\infty$ For each sequence $(y_n)_{n=1}^\infty$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^\infty$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n\to\infty}\{\inf_{u\in T(x_n)} Re\langle u, y_n - x_n\rangle + f(y_n) - f(x_n)\}/\|y_n\| = \infty.$$

Then for each given $w_0 \in E^*$, there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w} - w_0, \hat{y} - x\rangle \leq f(x) - f(\hat{y}) \text{ for all } x \in X.$$

By Lemma 4.5.8 and by taking $f \equiv 0$ in Theorem 4.5.13, we have the following theorem on the generalized complementarity problem:

Theorem 4.5.15. Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed cone in E and $T : X \to 2^{E^*}$ be semi-monotone and upper semicontinuous from the line segments in X to the weak topology on E^* such that each $T(x)$ is convex and compact in the norm topology on E^*. Assume that the following condition is satisfied:

$(E)_0^*$ For each sequence $(y_n)_{n=1}^{\infty}$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n\to\infty}\{\inf_{u\in T(x_n)} Re\langle u, y_n - x_n\rangle\} > 0.$$

Then there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w}, \hat{y}\rangle = 0 \text{ and } \hat{w} \in X^*.$$

The following is essentially the Proposition of Bae, Kim and Tan [13]; thus we omit its proof.

Lemma 4.5.16. Let X be a non-empty subset of a Hilbert space H.

(a) If $T : X \to bc(H)$ is nonexpansive such that for each $x \in X$, $T(x)$ is weakly compact, then T is pseudo-contractive on X.

(b) If $T : X \to 2^H$ is pseudo-contractive on X, then $I - T$ is semi-monotone on X where $I(x) = x$ for all $x \in X$.

As another application of Theorem 4.5.13, we have the following fixed point theorem:

Theorem 4.5.17. Let X be a non-empty closed convex subset of a Hilbert space H and let $T : X \to 2^H$ be pseudo-contractive and upper semicontinuous from the line segments in X to the weak topology on H such that each $T(x)$ is compact in the norm topology on H. Assume that the following condition is satisfied:

$(E)^+$ For each sequence $(y_n)_{n=1}^{\infty}$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n\to\infty}\{\inf_{u\in T(x_n)} Re\langle x_n - u, y_n - x_n\rangle\} > 0.$$

Then there exists $\hat{y} \in X$ such that

$$\inf_{w \in T(\hat{y})} Re\langle \hat{y} - w, \hat{y} - x\rangle \leq 0 \text{ for all } x \in X.$$

If $T(\hat{y})$ is also convex, then there exists $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{y} - \hat{w}, \hat{y} - x\rangle \leq 0 \text{ for all } x \in \overline{I_X(\hat{y})},$$

and if, in addition, either $\hat{y}$ is an interior point of X in H or $p(\hat{y}) \in \overline{I_X(\hat{y})}$, where $p(\hat{y})$ is the projection of $\hat{y}$ on $T(\hat{y})$, then $\hat{y}$ is a fixed point of T, i.e $\hat{y} \in T(\hat{y})$.

Proof: By Lemma 4.5.16, $T^* = I - T$ is semi-monotone on X. By Theorem 4.5.13 with $h \equiv 0$, there exists $\hat{y} \in X$ such that

$$\inf_{w \in T^*(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq 0 \text{ for all } x \in X;$$

that is,

$$\inf_{w \in T(\hat{y})} Re\langle \hat{y} - w, \hat{y} - x\rangle \leq 0 \text{ for all} x \in X.$$

If $T(\hat{y})$ is also convex, then by Theorem 4.5.13 again, there exists $\hat{w} \in T(\hat{y})$ such that

$$(7) \qquad Re\langle \hat{y} - \hat{w}, \hat{y} - x\rangle \ \leq 0 \text{ for all } x \in X.$$

If $x \in I_X(\hat{y})$, then $x = \hat{y} + r(u - \hat{y})$ for some $u \in X$ and $r > 0$. Thus $\hat{y} - x = r(\hat{y} - u)$ so that by (7),

$$Re\langle \hat{y} - \hat{w}, \hat{y} - x\rangle \ = r \cdot Re\langle \hat{y} - \hat{w}, \hat{y} - u\rangle \leq 0.$$

It follows that

$$(8) \qquad Re\langle \hat{y} - \hat{w}, \hat{y} - x\rangle \ \leq 0 \text{ for all } x \in \overline{I_X(\hat{y})}.$$

Now, if $\hat{y}$ is an interior point of X in H, then (8) implies that $\hat{y} = \hat{w} \in T(\hat{y})$. Next suppose $p(\hat{y}) \in \overline{I_X(\hat{y})}$ Since $p(\hat{y})$ is the projection of $\hat{y}$ on $T(\hat{y})$, we must have, by Theorem I.2.3 of Kinderlehrer and Stampacchia [185, p.9], $p(\hat{y}) \in T(\hat{y})$ and $Re\langle p(\hat{y}) - \hat{y}, w - p(\hat{y})\rangle \ \geq 0$ for all $w \in T(\hat{y})$. Since $\hat{w} \in T(\hat{y})$, by (8) we have

$$\begin{aligned} 0 \ &\leq \ Re\langle p(\hat{y}) - \hat{y}, \hat{w} - p(\hat{y})\rangle \\ &= \ Re\langle p(\hat{y}) - \hat{y}, \hat{w} - \hat{y} + \hat{y} - p(\hat{y})\rangle \end{aligned}$$

$$
\begin{aligned}
&= Re\langle p(\hat{y}) - \hat{y}, \hat{w} - \hat{y}\rangle - \|\hat{y} - p(\hat{y})\|^2 \\
&= Re\langle \hat{y} - \hat{w}, \hat{y} - p(\hat{y})\rangle - \|\hat{y} - p(\hat{y})\|^2 \leq -\|p(\hat{y}) - \hat{y}\|^2
\end{aligned}
$$

so that $\|p(\hat{y}) - \hat{y}\|^2 \leq 0$ and hence $\hat{y} = p(\hat{y}) \in T(\hat{y})$. □

As an immediate consequence of Lemma 4.5.16 and Theorem 4.5.17, we have

Theorem 4.5.18. Let X be a non-empty closed convex subset of a Hilbert space H and $T : X \to 2^H$ be nonexpansive such that each $T(x)$ is compact convex and $p(y) \in \overline{I_X(y)}$ for each $y \in \partial X$ where $p(y)$ is the projection of y on $T(y)$ and ∂X is the boundary of X in H. Assume that the following condition is satisfied:

$(E)^+$ For each sequence $(y_n)_{n=1}^{\infty}$ in X with $\|y_n\| \to \infty$ as $n \to \infty$, there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$ such that

$$\limsup_{n\to\infty}\{\inf_{u\in T(x_n)} Re\langle x_n - u, y_n - x_n\rangle\} > 0.$$

Then T has a fixed point in X.

Except that the set X is required to be closed in H, the above result is a generalization of Theorem 4.5.18 in [287, p.561] to set-valued and non-self maps.

We note that the condition $(E)^*$ in Lemma 4.5.12 and Theorem 4.5.13, $(E)^*_\infty$ in Theorem 4.5.14 and $(E)^+$ in Theorems 4.5.17 and 4.5.18 are automatically satisfied if the set X is bounded.

Finally we remark that given any increasing sequence $(N_n)_{n=1}^{\infty}$ of positive integers, the conclusions of Lemmas 4.5.4 and 4.5.12 and Theorems 4.5.5, 4.5.6, 4.5.9, 4.5.13, 4.5.14, 4.5.15, 4.5.17 to 4.5.18 remain valid if we replace the phrase "$\cdots$ there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq \|y_n\|$ for all $n = 1, 2, \cdots$" in the conditions (E), $(E)_\infty$, $(E)_0$, $(E)^*$, $(E)^*_\infty$, $(E)^*_0$ and $(E)^+$ by the phrase "$\cdots$ there exists a sequence $(x_n)_{n=1}^{\infty}$ in X with $\|x_n\| \leq N_n$ for all $n = 1, 2, \cdots$".

Chapter 5

Concluding Remarks

To summerize: in Chapter 2, we give a number of existence theorems for minimax inequalities, fixed point theorems, coincidence theorems and stability of coincidence points and of KF points; in Chapter 3, we obtain some existence theorems for equilibria for generalized games in H-spaces, topological vector spaces, locally convex topological spaces, Frechet spaces and finite dimensional spaces; and in Chapter 4, we prove some existence theorems for variational inequalities and generalized quasi-variational inequalities in locally convex topological spaces and reflexive Banach spaces, the stability of quasi-variational inequalities and applications to constrained N-person games, complementarity problems and fixed point theorems for set-valued pseudo contractive maps and set-valued nonexpensive maps. Here, no applications to differential equations nor differential inclusions are given. Further, even though we have some results on abstract general algorithms for solutions of variational inequalities, these are not included here. The author wishes to continue these topics in the near future. Moreover, we do not cover the topics on random analysis and its applications to fixed point theory and existence of equilibria for random generalized games for which we refer to Tan and Yuan [293]-[300], Yuan [332]- [335] and the references therein,

Moreover, in this thesis, I do not touch the topic such as topological Iminimax inequalities which has been attentioned by many authors in recent authors, for most recent results on this topic, we refer to Kindler [186], Konig [195], Ricceri [[252] and the reference therein. We also do not study the existence theorems for generalized quasi-variational inequalities wihch are associated discontinuous mappings (e.g., see Cubiotti [69], Ricceri [251]). The author hopes to continue study in these areas soon.

1. P. Alexandrov, *Combinatorial Topology I*, Graylock Press, Baltimore, 1956.
2. C. Aliprantis and D. Brown, *Equilibria in markets with a Riesz space of commodities*, J. Math. Economics 11 (1983), 189-207.
3. C. Aliprantis, D. Brown and O. Burkinshaw, *Existence and Optimality of Competitive Equilibria*, Springer-Verlag, New York, 1989.
4. G. Allen, *Variational inequalities, complementary problems and dual theorems*, J. Math. Anal. Appl. 58 (1977), 1-10.
5. K. J. Arrow and G. Debreu, *Existence of an equilibrium for a competitive economy*, Econometrica 22 (1954), 265-290.
6. K. J. Arrow and M. D. Intriligator, *Handbooks in Economics*, vol. 1-4, North-Holland, Amsterdam, 1983-1989.
7. J. P. Aubin, *Mathematical Methods of Game and Economic Theory (Revised Edition)*, North-Holland, Amsterdam, 1982.
8. ———, *Viability Theory*, Birkhäuser, Boston, 1991.
9. J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, New York, 1984.
10. J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, 1984.
11. J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser Boston.
12. R. J. Aumann and S. Hart, *Handbook of Game Theory with Economic Applications*, vol. 1, North-Holland, Amsterdam, 1992.
13. J. S. Bae, W. K. Kim and K. K. Tan, *Another generalization of Ky Fan's minimax inequality and its applications*, Bull. Inst. Acad. Sinica (to appear).
14. C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities, Applications to Free Boundary Problems*, John Wiley & Sons, New York, 1984.
15. M. Baye, G. Tian and J. Zhou, *The existence of pure Nash equilibria in games without non-quasiconcave payoffs*, Texas A&M University, Mimeo (1991).
16. R. Barbolla, *Equilibrium and quasi-equilibrium theorems for a general model without ordered preferences*, Economic Letters 19 (1985), 205-210.
17. G. Beer, *On a generic optimization theorem of Kenderov*, Nonlinear Analysis 12 (1988), 647-655.
18. ———, *Topology on Subset*, Preprint, 1993.
19. J. C. Bellenger, *Existence of maximizable quasiconcave functions on paracompact convex spaces*, J. Math. Anal. Appl. 123 (1987), 333-338.
20. H. Ben-El-Mechaiekh, *Note on a class of set-valued maps having continuous selections*, Fixed Point Theory and Applications (M. A. Thera and J. B. Baillon, eds.), Longman Scientific Technical, New York, 1991, pp. 33-45.
21. H. Ben-El-Mechaiekh and P. Deguire, *General fixed point theorems for non-convex set-valued maps*, C. R. Acad. Sci. Paris 312 (1991), 433-438.
22. ———, *Approachability and fixed points for non-convex set-valued maps*, J. Math. Anal. Appl. 170 (1992), 477-450.
23. ———, *Maximal elements for families of multifunctions*, preprint (1993).
24. H. Ben-El-Mechaiekh, P. Deguire and A. Granas, *Une alternative non linéaire en analyse convexe et application* I, C. R. Acad. Sci. Paris. I Math. 295 (1982), 257-259; II, C. R. Acad. Sci. Paris. I Math. 259 (1982), 381-384.
25. ———, *Point fixes et coincidences pour les fonctions multivoques (applications de type M* et M)* III, C. R. Acad. Sci. Paris 305 (1987), 381-384.

26. A. Bensoussan, *Stochastic Control by Functional Analysis Method*, North-Holland, Amsterdam, 1982.
27. A. Bensoussan, M. Goursat and J. L. Lions, *Control impulsinnel et inéquations quasi-variationelles stationaries* 276 (1973), C. R. Acad. Sci., 1279-1284.
28. Bensoussan and J. L. Lions, *Impulse Control and Quasivariational Inequalities*, Gauthiers-Villers, Bordas, Paris (English version Trans-Inter-Scientia), 1984.
29. C. Berge, *Espaces Topologiques Fonctions Multivoques*, Dunod, Paris, 1959.
30. T. Bergstrom, *Maximal elements of acyclic preference relations on compact sets*, vol. 10, 1975.
31. T. R. Bergstrom, R. Parks and T. Rader, *Preferences which have open graphs*, J. Math. Economics 3 (1976), 265-268.
32. D. P. Bertsekas and J. N. Tsitsiklis, *Projection methods for variational inequalities with applications to the traffic assignment problem*, Math. Prog. Study 17 (1982), 139-159.
33. T. F. Bewley, *Existence of equilibria in economies with infinitely many commodities*, J. Economics Theory 4 (1972), 514-540.
34. K. C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, Cambridge, London, New York, Sydney, 1985.
35. Y. G. Borisovich, B. D. Gelman, A. D. Myskis and V. V. Obukhovskii, *Topological methods in the theory of fixed points of multivalued maps*, Uspehi Mat. Nauk 35 (1980), 52-126.
36. ———, *Multivalued mappings*, J. Soviet Math. 24 (1984), 719-791.
37. A. Borglin and H. Keiding, *Existence of equilibrium actions of equilibrium, "A note on the 'new' existence theorems"*, J. Math. Economic. 3 (1976), 313-316.
38. H. Brezis, L. Nirenberg and G. Stampacchia, *A remark on Ky Fan's minimax principle*, Boll. Un. Mat. Ital. 6 (1972), 293-300.
39. L. E. J. Brouwer, *Über abbildungen vom mannigfaltigkeiten*, Math. Ann. 71 (1912), 97-115.
40. F. E. Browder, *Nonlinear mappings of nonexpensive and accretive type in Banach space*, Bull. Amer. Math. Soc. 73 (1967), 875-882.
41. ———, *A new generalization of the Schauder fixed point theorem*, Math. Ann. 174 (1967), 285-290.
42. ———, *The fixed point theory of multivalued mappings in topological vector spaces*, Math. Ann. 177 (1968), 283-301.
43. ———, *Nonlinear Operators and Nonlinear Equations of Evolutions in Banach Spaces*, Proc. Sympos. in Pure Mathematics, vol. 18, Part 2, Amer. Math. Soc., Prov. R. I., 1976.
44. ———, *On a sharpened form of the Schauder fixed-point theorem*, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), 4749-4751.
45. ———, *Coincidence theorems, minimax theorems, and variational inequalities*, Contemporary Math., vol. 26, 1984, p. 67-80.
46. C. Bardaro and R. Ceppitelli, *Some further generalizations of the Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities*, J. Math. Anal. Appl. 132 (1988), 484-490.
47. ———, *Fixed point theorems and vector-valued minimax theorems*, J. Math. Anal. Appl. 146 (1990), 363-373.
48. D. Chan and J. S. Pang, *The generalized quasi-variational inequality problem*, Math. Operations Research 7 (1982), 211-222.
49. S. S. Chang, *Variational Inequalities and Complementarity Problem Theory with Applications*, Shanghai Scientific Literature Press (in Chinese), 1991.
50. S. S. Chang and N. J. Huang, *Generalized strongly nonlinear quasi-complementarity problems in Hilbert spaces*, J. Math. Anal. Appl. 158 (1991), 194-202.

51. S. S. Chang and Y. H. Ma, *Generalized KKM theorem on H-space with applications*, J. Math. Anal. Appl. 163 (1992), 406-421.
52. S. S. Chang and J. W. Song, *Coincidence indices for set-valued compact mapping pairs*, J. Math. Anal. Appl. 148 (1990), 469-488.
53. S. S. Chang and G. S. Yang, *Some further generalizations of Ky Fan's minimax inequality and its applications*, Appl. Math. Mech. 11 (1990), 1027-1034.
54. S. S. Chang and Y. Zhang, *Generalized KKM Theorem and Variational Inequalities*, J. Math. Anal. Appl. 159 (1991), 208-223.
55. S. Y. Chang, *On the Nash equilibrium*, Soochow J. Math. 16 (1990), 241-248.
56. ———, *A generalization of KKM principle and its applications*, Soochow J. Math. 15 (1989), 7-17.
57. G. Chichilnsky, *Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium*, Work Paper, Columbia Univ. (1992).
58. ———, *Intersecting families of sets and the topology of cones in economics*, Bull. Amer. Math. Soc. (N.S.) 29 (1993), 189-207.
59. G. Chichilnsky and G. M. Heal, *Necessary and sufficient conditions for a resolution of the social choice paradox*, J. Econ. Theory 31 (1983), 68-87.
60. J. P. Christensen, *Theorems of Namioka and Johnson type for upper semicontinuous and compact valued set-valued mappings*, Proc. Amer. Math. Soc. 86 (1982), 649-655.
61. H. P. R. Christensen and P. S. Kenderov, *Dense strong continuity of mappings and the Random-Nikodym property*, Math. Scand. 54 (1984), 70-78.
62. R. W. Cottle, *Complementarity and variational problems*, Sympos. Math. 19 (1976), 177-208.
63. P. W. Cottle and G. B. Dantzig, *Complementarity pivot theory of mathematical programming*, Linear Algebra Appl. 1 (1968), 103-125.
64. R. W. Cottle, F. Giannessi and J. L. Lions, *Variational Inequalities and Complementarity Problems: Theory and Applications*, J. Wiley and Sons, New York, London, 1980.
65. R. W. Cottle, J. Kyparisis and J. S. Pang (eds), *Variational Inequality Problems*, vol. 48, Math. Prog. Series B, 1990.
66. J. Crank, *Free and Moving Boundary Problems*, Oxford Univ. Press, London, 1984.
67. P. Cubiotti, *Finite-dimensional quasi-variational inequalities associated with discontinuous functions*, J. Optim. Theory Appl. 72 (1992), 577-582.
68. ———, *Some remarks on the fixed points of lower semicontinuous multifunctions*, J. Math. Anal. Appl. 174 (1993), 407-412.
69. ———, *An existence theorem for generalized quasi-variational inequalities*, Set-Valued Anal. 1 (1993), 81-87.
70. H. F. Cullen, *Introduction to General Topology*, D. C. Heath, Boston, 1968.
71. S. Dafermos, *Exchange price equilibria and variational inequalities*, Math. Prog. 46 (1990), 391-402.
72. G. Debreu, *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci. U.S.A. 38 (1952), 386-393.
73. ———, *Theory of Value· An axiomatic Analysis, New Haven and London*, Yale Univ. Press, 1959.
74. P. Deguire, *Thèoremes de coincidences, Théorie de minimax et applications*, Thèse de doctorat, Université de Montréal, 1987.
75. P. Deguire and M. Lassonde, *Sélections et points fixes généralises dans des espaces produits*, preprint (1993).
76. P. Deguire, K.K. Tan and X. Z. Yuan, *Maximal elements for couple-majorized correspondences*, in preparation (1993).
77. K. Deimling, *Multivalued Differential Eqvations*, Walter de Gruyter & Co. Publ, 1992.

78. X. P. Ding, *Continuous selection theorems, coincidence theorems and intersection theorems concerning sets with H-convex sections*, J. Austral. Math. Soc.(Series A) 52 (1992), 11-25.
79. ———, *Generalized strongly nonlinear quasi-variational inequalities*, J. Math. Anal. Appl. 173 (1993), 577-587.
80. ———, *Existence of maximizable quasi-concave functions on H-Spaces*, Acta Math. Sinica (in Chinese) 36 (1993), 273-279.
81. X. P. Ding and K. K. Tan, *Generalized variational inequalities and generalized quasi-variational inequalities*, J. Math. Anal. Appl. 148 (1990), 497-508.
82. ———, *A minimax inequality with applications to existence of equilibrium points and fixed point theorems* 68 (1992), 233-247.
83. ———, *Fixed point theorems and equilibria of non-compact games*, Fixed Point Theory and Applications (K. K. Tan, ed.), Scientific Press, 1992, pp. 80-96.
84. ———, *On equilibria of non-compact generalized games*, J. Math. Anal. Appl. 177 (1993), 226-238.
85. ———, *Covering properties of H-spaces and applications*, Applied. Math. Mech. (to appear).
86. ———, *Equilibria of non-compact generalized games with L^*-majorized preference correspondences*, J. Math. Anal. Appl. 164 (1992), 508-517.
87. ———, *A selection theorem and its applications*, Bull. Austral. Math. Soc. 46 (1992), 205-212.
88. S. Dolecki, *Tangency and differentiation Some applications of convergence theory*, Ann. Mat. Pura. Appl. 130 (1982), 223-255.
89. J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966.
90. J. Dugundji and A. Granas, *KKM maps and variational inequalities*, Ann. Scuola Norm. Sup. Pisa 5 (1978), 679-682.
91. ———, *Fixed Point Theory*, vol. I Polish Scientific Publishers, Warszawa, 1982.
92. N. Dunford and J. T. Schwartz, *Linear Operators, Part I General Theory*, Interscience Publisher Inc., New York, 1957.
93. G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlang, Berlin, 1976.
94. S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformations*, vol. 68, 1946.
95. R. Engelking, *General Topology*, Polish Scientific, Warsaw, 1977.
96. R. S. Falk, *Error estimates for the approximation of a class of variational inequalities*, Math. programm. 28 (1974), 963-972.
97. K. Fan, *Fixed points and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. 38 (1952), 131-136.
98. ———, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. 142 (1961), 305-310.
99. ———, *Sur un théorème minimax*, C. R. Acad. Sci. Paris Ser. I. Math 259 (1964), 3925-3928.
100. ———, *Applications of a theorem concerning sets with convex sections*, Math. Ann. 163 (1966), 189-203.
101. ———, *Simplicial maps from an orientable n-pseudomanifold into S^m with the octahedral triangulation*, J. Comb. Theory 2 (1967), 588-602.
102. ———, *A covering property of simplexes*, Math. Scand. 22 (1968), 17-20.
103. ———, *Extensions of two fixed point theorems of F E Browder*, Math. Z. 112 (1969), 234-240.
104. ———, *A combinatorial property of pseudomanifolds and covering properties of simplexes*, J. Math. Anal. Appl. 31 (1970), 68-80.

105. ______, *A minimax inequality and its applications*, Inequalities III (O. Shisha, ed.), Academic Press, 1972, pp. 103-113.
106. ______, *Some properties of convex sets related to fixed point theorems*, Math. Ann. 266 (1984), 519-537.
107. ______, *A survey of some results closely related to the Knaster - Kuratowski - Mazuriewicz theorem*, Game Theory and Applications, Econom. Theory Econometrics Math. Econom., Academic Press, 1990, p. 358-370.
108. S. C. Fang, *An iterative methods for generalized complementarity problems*, IEEE Trans. Auto. Control 25 (1980), 1225-1227.
109. S. C. Fang and E. L. Peterson, *Generalized variational inequalities*, J. Optim. Theory Appl. 38 (1982), 363-383.
110. G. Fichera, *Problemi elastostatic con vincoli unilaterali problema di singnorinicon ambigue condizioni al controno*, Atti. Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. La. 7 (1963-64), 91-140.
111. P. M. Fitzpatick and W. V. Petryshyn, *Fixed point theorems for multivalued-valued non-compact inward maps*, J. Math. Anal. Appl. 46 (1974), 756-767.
112. ______, *Fixed point theorems for multivalued noncompact acyclic mappings*, Pacific Journal of Mathematics 54 (1974), 17-23.
113. S. D. Flam. *Abstract economy and games*, Soochow J. Math. 5 (1979), 155-162.
114. M. Florenzano, *On the existence of equilibria in economies with an infinitely dimensional commodity spaces*, J. Math. Econ. 12 (1983), 207-219.
115. M. K. Fort, Jr., *Points of continuity of semicontinuous functions*, Publ. Math. Debrecen 2 (1951), 100-102.
116. ______, *Essential and nonessential fixed points*, Amer. J. Math. 72 (1952), 315-322.
117. D. H. Fremlin, *Topological Riesz Spaces and Measure Theory*, Cambridge Univ. Press, London, New York, 1974.
118. A. Friedman, *Stochastic Differential Equations and Applications*, vol. 1-2, Academic Press, New York, 1975-1976.
119. ______, *Optimal stopping problems in stochastic control*, SIAM Rev. 21 (1979), 71-80.
120. ______, *Variational Principle and Free Boundary Value Problems*, Wiiey, New York, 1982.
121. M. Furi and A. Vignoli, *A fixed point theorem in complete metric spaces*, Boll. Un. Mat. Ital. Ser. 4 (1969), 505-506.
122. D. Gale, *The law of supply and demand*, Math. Scand. 3 (1955), 155-169.
123. ______, *Equilibrium in a discrete exchange economy with money*, Int. J. Game Theory 13 (1984), 61-64.
124. D. Gale and A. Mas-Colell, *An equilibrium existence for a general model without ordered preferences*, J. Math. Economics. 2 (1975), 9-15.
125. ______, *On the role of complete, translative preferences in equilibrium theory*, Equilibrium and Disequilibrium in Economics Theory (G. Schwödiauer, ed.), Reidel, Dordrecht, 1978, p. 7-14.
126. ______, *Corrections to an equilibrium existence theorem for a general model without ordered preferences*, J. Math. Econ. 6 (1979), 297-298.
127. I. L. Glicksberg, *A further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium points*, Proc. Amer. Math. Soc. 3 (1952), 170-174.
128. R. Glowinski, J. Lions and R. Trmolieres, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1982.
129. I. T. Gohberg, L. S. Goldenstein and A. S. Markus, *Investigations of some properties of bounded linear operators with their q-norms*, Uch. Zap. Kishinevsk. In-Ta. 29 (1957), 29-36.
130. L. Gorniewicz, *A Lefschetz-type fixed point theorem*, Fund. Math. 88 (1975), 103-115.

131. L. Gorniewicz and A. Granas, *Some general theorems in coincidence theory I*, J. Math. Pure Appl. 60 (1981), 361-373.
132. L. Gorniewicz, A. Granas and W. Kryszewski, *On the homotopy method in the fixed point index theory of multivalued mappings of compact ANR's*, Preprint (1989).
133. A. Granas, *KKM-maps and their applications to nonlinear problems*, The Scottish Book: Mathematic from the Scottish Cafe (R. Daniel Mauldin, ed.), Birkhäuser, Boston, 1982, p. 45-61.
134. A. Granas and F. C. Liu, *Théorèmes du minimax*, C. R. Acad. Sci. Paris 298 (1984), 329-332.
135. ______, *Coincidences or set-valued maps and minimax inequalities*, J. Math. Pure. Appl. 65 (1986), 119-148.
136. ______, *Théorèmes du minimax*, Nonlinear and Convex Analysis, Proceedings in Honor of Ky Fan (B. L. Lin and S. Simons, eds.), Marcel Dekker, 1987, pp. 115-117.
137. J. Gwinner, *On fixed points and variational inequalities A circular tour*, Nonlinear Analysis 5 (1981), 565-583.
138. C. W. Ha, *Minimax and fixed point theorems*, Math. Ann. 248 (1980), (73-77).
139. ______, *Extension of two fixed point theorems of Ky Fan*, Math. Z. 190 (1985), 13-16.
140. ______, *On a minimax inequality of Ky Fan*, Proc. Amer. Math. Soc. 99 (1987), 680-682.
141. B. Halpern, *Fixed-Point Theorems for Outward Maps*, Doctoral Thesis, U.C. L.A. (1965).
142. ______, *Fixed point theorems for set-valued maps in infinite dimensional spaces*, Math. Ann. 189 (1970), 87-98.
143. B. Halpern and G. Bergman, *A fixed point theorem for inward and outward maps*, Trans. Amer. Math. Soc. 130 (1968), 353-358.
144. P. T. Harker, *Predicting Intercity Freight Flow*, VUN Science Press Utrecht, Holland, 1987.
145. P. T. Harker and J. S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms, and applications*, Math. Prog. 48 (1990), 161-220.
146. P. Hartman and G. Stampacchia, *On some nonlinear elliptic differential functional equations*, Acta Math. 115 (1966), 271-310.
147. W. Hildenbrand, *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, New Jersey, 1974.
148. W. Hildebrand and H. Sonnenschein, *Handbook of Mathematical Economics*, vol. 1-4, North-Holland, Amsterdam, 1986-1991.
149. D. Hilbert, *Über das Dirichletsche Prinzip (lecture on the Dirichlet Principle)*, Math. Ann. (1904), 161-186.
150. ______, *Mathematical Problems, Lecture delivered before the First International Congress of Mathematics at Paris*, Bull. Amer. Math. Soc. 8 (1902), 437-479 (Reprinted in Browder, E.F (ed) (1976), Vol. 1, pp.1-34).
151. C. J. Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. 38 (1972), 205-207.
152. C Horvath, *Points fixes et coïncidences pour les applications multivoques sans convexité*, C. R. Acad. Sci. Paris 296 (1983), 403-406.
153. ______, *Some results on multivalued mappings and inequalities without convexity*, Nonlinear and Convex Analysis (B. L. Lin and S. Simons, eds.), Marcel Dekker, 1987, p. 96-106.
154. ______, *Contractibility and generalized convexity*, J. Math. Anal. Appl. 156 (1991), 341-357.

155. T. Ichiishi, *Game Theory for Economic Analysis*, Academic Press, New York, 1983.
156. ———, *Alternative version of the Shapley s theorem on closed coverings of simples*, Proc. Amer. Math. Soc. 104 (1988), 759-763.
157. T. Ichiishi and T. Idzik, *Theorems on closed coverings of a simplex and their applications to cooperative game theory*, J. Math. Anal. Appl. 146 (1990), 259-270.
158. A. Idzik and K. K. Tan, *Covering properties of simplexes*, J. Math. Anal. Appl. 176 (1993), 608-616.
159. I. S. Iohvidov, *On a lemma of Ky Fan generalizing the fixed point principle of A N Tihonov [Russian]*, Dokl. Akad. Nauk SSSR 159 (1964), 501-504 (English Trans., Soviet Math. Dokl., Vol. 5 (1964), p.1523-1526).
160. G. Isac, *A special variational inequality and the implicit complementarity problem*, J. Fac. Sci. Univ. Tokyo 37 (1990), 109-127.
161. ———, *Complementarity Problems, Lecture Notes in Mathematics*, vol. 1528, Springer-Verlag, 1992.
162. V. I. Istratescu, *Fixed Point Theory*, P. Reidel Publishing Company, London, 1981.
163. J. H. Jiang, *Essential fixed points of the multivalued mappings*, Scientia Sinica 11 (1962), 293-298.
164. ———, *Essential fired points of n-person non-cooperative games (II)*, Scientia Sinica 12 (1963), 651-671.
165. ———, *Fixed point theorems for paracompact convex sets, I and II,*, Acta Math. Sinica 4 (1988), 64-71 and p.234-241.
166. ———, *Fixed point theorems for convex sets*, Acta Math. Sinica 4 (1988), 356-363.
167. ———, *Coincidence theorems and minimax theorems*, Acta Math. Sinica 5 (1989), 307-320.
168. I. Joo, *A simple proof for von Neumann's minimax theorems*, Acad. Sci. Math. Szeged 42 (1980), 91-94.
169. A. Kajii, *Note on equilibrium without ordered preferences in topological vector space*, Economics Letter 27 (1988), 1-4.
170. S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. 8 (1941), 457-459.
171. S. Karamardian, *Generalized complementarity problem*, J. Optim. Theory Appl. 8 (1971), 161-168.
172. H. Keiding, *Existence of economic equilibria*, Lecture Notes in Economics and Math. System, vol. 22, 1984, pp. 223-243.
173. J. L. Kelley, *General Topology*, Von-Nostand, 1955.
174. P. S. Kenderov, *Most of the optimization problems have unique solutions*, Proceedings, Oberwolhfach Conference on Parametric Optimization (B. Brosowki and F. Deutsch, eds.), vol. 72, Birkhäuser, Basel., 1984, pp. 203-216.
175. M. A. Khan and R. Vohra, *Equilibrium in abstract economies without preferences and with a measure of agents*, J. Math. Economics 13 (1984), 133-142.
176. M. A. Khan and R. Papageorgiou, *On Cournot-Nash equilibria in generalized quantitative games with an atomless measure space of agents*, Proc. Amer. Math. Soc. 100 (1987), 505-510.
177. M. A. Khan and C. Yannelis (eds), *Equilibrium Theory in Infinite Dimensional Spaces*, Springer-Verlag, New York, 1991.
178. N. Kikuchi and J. T. Oden, *Contact Problems in Elasticity: Avariational Approach*, SIAM, Philadelphia, 1988.
179. W. K. Kim, *Some applications of the Kakutani fixed point theorem*, J. Math. Anal. Appl. 121 (1987), 119-122.
180. ———, *Remarks on a generalized quasi-variational inequalities*, Proc. Amer. Math. Soc. 103 (1988), 667-668.

181. ———, *Existence of maximal element and equilibrium for a nonparacompact N-person game*, Proc. Amer. Math. Soc. 116 (1992), 797-807.
182. T. Kim and M. K. Richter, *Nontransitive-nontotal consumer theory*, J. Economic Theory 28 (1986), 324-363.
183. T. Kim, K. Prikry and N. C. Yannelis, *Equilibrium in abstract economies with a measure space and with an infinite dimensional strategy space*, J. Approx. Theory 56 (1989), 256-266.
184. W. K. Kim and K. K. Tan, *A variational inequality in non-compact sets and its applications*. Bull. Aust. Math. Soc. 46 (1992), 139-148.
185. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
186. J. Kindler, *Intersection theorems and minimax theorems based on connectedness*, J. Math. Anal. Appl. 178 (1993), 529-546.
187. V. Klee, *On certain intersction properties for convex sets*, Canad. J. Math. 3 (1951), 272-275.
188. ———, *Leray-Schauder theory without local convexity*, Math. Ann. 141 (1960), 286-296.
189. E. Klein and A. C. Thompson, *Theory of Correspondences including applications to mathematical economics*, John Wiley & Sons, 1984.
190. H. Kneser, *Sur une théorème fondamental de la théorie des jeux*, C. R. Acad. Sci. Paris 234 (1952), 2418-2420.
191. H. Knaster, C. Kuratowski and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes fur n-dimensionale simplexe*, Fund. Math. 14 (1929), 132-137.
192. H. M. Ko and K. K. Tan, *A coincidence theorem with applications to minimax inequalities and fixed point theorems*, Tamkang J. Math. 17 (1986), 37-45.
193. ———, *A note on a covering theorem of Ky Fan*, Tamkang J. Math. 22 (1991), 329-333.
194. ———, *Coincidence theorems and matching theorems*, Tamkang J. Math. 23 (1992), 297-309.
195. H. König, *A general minimax theorem based on connectedness*, Arch. Math., 59 (1992), 55-64.
196. H. Komiya, *Coincidence theorems and saddle point theorems*, Proc. Amer. Math. Soc. 196 (1986), 599-602.
197. ———, *A minimax theorem without linear structure*, Hiyoshi Review Nat. Sci. Keio. Univ. 8 (1990), 74-78.
198. C. Kuratowski, *Sur les éspaces completès*, Fund. Math. 15 (1920), 301-309.
199. M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. 97 (1983), 151-201.
200. ———, *Sur le principle KKM*, C. R. Acad. Sci. Paris 310 (1990), 573-576.
201. ———, *Fixed point for Kaktutani factorizable multifunctions*, J. Math. Anal. Appl. 152 (1990), 46-60.
202. M. Lassonde and C. Schenkel, *KKM principle, fixed points and Nash equilibria*, J. Math. Anal. Appl. 164 (1992), 542-548.
203. A. Lechicki and A. Spakowski, *A note on intersections of lower semicontinuous multifunctions*, Proc. Amer. Math. Soc. 95 (1985), 119-122.
204. C. M. Lee and K. K. Tan, *On Fan's extensions of Browder's fixed point theorems for multivalued inward mappings*, J. Austral. Math. Soc. 26 (Ser. A) (1978), 169-174.
205. C. E. Lemke, *Bimatrix equilibrium points and mathematical programming*, Management Sci. 11 (1965), 681-689.
206. B. L. Lin and X. C. Quan, *A non-compact topological minimax theorem*, J. Math. Anal. Appl. 161 (1991), 587-590.
207. B. L. Lin and S. Simons, *Nonlinear and Convex Analysis Proceedings in Honor of Ky Fan*, Marcel Dekker, Inc., 1987.

208. T. C. Lin and C. L. Yen, *Applications of proximity maps to fixed point theorems*, J. Approx. Theory 52 (1988), 141-148.

209. J. L. Lions, *Optimal Control of Systems Governed by Partial differential Equations*, Springer-Verlag, 1971.

210. T. W. Ma, *On sets with convex sections*, J. Math. Anal. Appl. 27 (1969), 413-416.

211. Y. H. Ma and F. B. Zhang, *Generalized minimax inequalities and fixed point theorems*, Appl. Math. Mech. 12 (1991), 465-472.

212. E. Marchi and J. E. Martinez-Legaz, *A generalization of Browder's fixed point theorems and its applications*, Preprint, Department of Mathematics, University of Barcelona, Spain (1989).

213. ———, *Some results on approximate continuous selections, fixed points and minimax inequalities*, Fixed Point Theory and Applications (M. A. Thera and J. B. Baillon, eds.), Longman Scientific Technical, New York, 1991, p. 327-342.

214. S. A. Marano, *Controllability of partial differential inclusions depending on a parameter and distributed parameter control processes*, Le Matematiche 95 (1990), 283-300.

215. A. Mas-Colell, *An equilibrium existence theorem without complete or transitive preferences*, J. Math. Economics 1 (1974), 237-246.

216. ———, *The Theory of General Economic Equilibrium*, Cambridge Univ. Press, Cambridge, 1985.

217. L. McLinden, *Acyclic multifunctions without metrizability*, presented in the First International Conference on Fixed Point Theory and Applications, Marseille-Luminy (June, 1989).

218. G. Mehta, *Fixed points, equilibria and maximal elements in linear topological vector spaces*, Commentationes Mathmaticae Universitatis Carolinae 28 (1987), 377-385.

219. ———, *Maximal elements of condensing preference maps*, Appl. Math. Letter 3 (1990), 69-71.

220. G. Mehta and S. Sessa, *Coincidence theorems and maximal elements in topological vector spaces*, Mathematiea Japonica 37 (1993), 839-845.

221. G. Mehta and E. Tarafdar, *Infinite-dimensional Gale-Debreu theorem and a fixed point theorem of Tarafdar*, J. Econ. Theory 41 (1987), 333-339.

222. E. Michael, *Continuous selections I*, Ann. Math. 63 (1956), 361-382.

223. M. D. P. Monteiro Marques, *Rafle par un convexe semi-continu inférieurement d'Intérieur non vide en dimension finie*, Séminaire d' Analyse Convex, Montpellier, Exposé No. 6 (1984).

224. J. J. Moreau, *Intersection of moving convex sets in a normed space*, Math. Scand. 36 (1975), 159-173.

225. U. Mosco, *Implicit problems and quasi-variational inequalities*, Lecture Notes in Mathematics, vol. 543, Springer-Verlag, 1976, p. 83-156.

226. U. Mosco and G. Strang, *One-side approximation and variational inequalities*, Bull. Amer. Math. Soc. 80 (1974), 308-312.

227. J. Nash, *Equilibrium points in N-person games*, Proc. Nat. Acad. Sci. U.S.A. 36 (1950), 48-49.

228. ———, *Non-cooperative games*, Ann. Math. 54 (1951), 286-295.

229. I Nikaido, *Convex Structures and Economic Theory*, Academic Press, New York, 1967.

230. M. A. Noor and K. I. Noor, *Invitation to variational inequalities*, Journal of Natural Geometry (1994) (to appear).

231. M. A. Noor, *Some recent advances in variational inequalities*, New Developments in Approximation-Solvability of Nonlinear Equations - Abstract Differential and Integral Equations: Theory and Applications (Ram U. Verma, ed.), Marcel Dekker, Inc., New York (to appear).

232. ———, *General nonlinear complementarity problems*, Analysis, Geometry and Groups: A Reimann Legacy Volume (Th. M. Rassias and H. M. Srivastava, eds.), Hadronic Press, Inc (to appear).

233. M. A. Noor, K. I. Noor and Th. M. Rassias, *Some aspects of variational inequalities*, J. Comput. Anal. Math. 47 (1993).

234. J. von Neumann, *Zur theorie der gesellschaftsspiele*, Math. Ann 100 (1928), 295-320.

235. ———, *Uber ein okonomsiches Gleichungssystem und eine Verallgemeinerung des Browerschen Fixpunktsatzes*, Ergebnisse eines Mathematischen Kolloquiums 8 (1937), 73-83; English transl. *A model of general economic equilibrium (G Morgenstern, Trans)*, Review Econom. Stud. 13 (1945-1946), 1-9.

236. J. von. Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, N.J., 1944.

237. R. D. Nussbaum,, *The fixed point index for local condensing maps*, Ann. Mat. Pura Appl. 81 (1971), 217-258.

238. V. V. Obukhovskii, *Equations with multivalued operators and Some of Their Applications*, Voronezh, 1974.

239. J. S. Pang, *On the convergence of a basic iterative method for the implicit complementarity problem*, J. Optim. Theory and Appl. 37 (1982), 149-162.

240. S. Park, *Fixed point theorems on compact convex sets in topological vector spaces*, Contemporary Mathematics, vol. 72, 1988, p. 183-191.

241. ———, *Generalization of Ky Fan's matching theorems and their application*, J. Math. Anal. Appl. 141 (1989), 164-176.

242. ———, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, Fixed Point Theory and Applications (K. K. Tan, ed.), World Scientific, Singapore, 1992, pp. 248-278.

243. ———, *On minimax inequalities on spaces having certain contractible subsets*, Bull. Austral. Math. Soc. 47 (1993), 25-40.

244. S. Park and J. S. Bae, *Existence of maximizable quasiconcave functions on convex spaces*, J. Korean Math. Soc. 28 (1991), 285-292.

245. D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Bucuresti, Romania, Sijthoff and Noordhoff International Publisher, 1976.

246. B. Peleg, *Equilibrium points for open acyclic relations*, Canadian J. Math. 19 (1967), 366-369.

247. M. Powers, *Lefschetz fixed point theorems for a new class of multi-valued maps*, Pacific. J. Math. 68 (1970), 619-630.

248. S. Reich, *Fixed points in locally convex spaces*, Math. Z. 125 (1972), 17-31.

249. ———, *On fixed point theorems obtained from existence theorems for differential equations*, J. Math. Anal. Appl. 54 (1976), 26-36.

250. ———, *Approximate selections, best approximations, fixed points, and invariant sets*, J. Math. Anal. Appl. 62 (1978), 104-113.

251. B. Ricceri, *Un théorème d'existence pour les inéquations*, C. R. Acad. Sci. Paris S'er I Math. 301 (1985), 885-888.

252. ———, *Some topological mini-max theorems via an alternative principle for multifunctions*, Arch. Math. 60 (1993), 367-377.

253. P. T. Rockafellar, *Convex Analysis*, Princeton, New Jersey, Princeton University Press, 1970.

254. ———, *Lagrange multipliers and variational inequalities*, Variational Inequalities and Complementarity Problems, Theory and Applications (Cottle et al., eds.), New York, 1980, pp. 303-322.

255. J. F. Rodrigues, *Obstacle Problems in Mathematical Physics*, North-Holland, Amsterdam, 1987.

256. W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.
257. N. B. Sadovskii, *Ultimately compact and condensing mappings*, Uspehi Mat. Nauk 27 (1972), 81-146.
258. R. Saigal, *Extension of the generalized complementarity problem*, Math. Oper. Res. 1 (1976), 260-266.
259. H. Scarf, *The core of an n-person game*, Econometrica 35 (1967), 50-69.
260. V. M. Sehgal, S. P. Singh and R. E. Smithson, *Nearest points and some fixed point theorems for weakly compact sets*, J. Math. Anal.Appl. 128 (1987), 108-111.
261. S. Sessa and G. Mehta, *General coincidence theorems for maps and multivalued maps in topological vector spaces,*, Inter. J. Math. Math. Sci. (1993) (to appear).
262. W. Shafer, *Equilibrium in economies without ordered preferences or free disposal*, J. Math. Economics. 3 (1976), 135-137.
263. W. Shafer and H. Sonnenschein, *Equilibrium in abstract economies without ordered preferences*, J. Math. Econ. 2 (1975), 345-348.
264. L. S. Shapley, *On balanced games without side payments*, Mathematical programming (T. C. Hu and S. M. Robinson, eds.), 1973, pp. 260-290.
265. P. Shi, *An iterative method for obstacle problems via Green's functions*, Nonlinear Analysis 15 (1990), 339-346.
266. M. H. Shih and K. K. Tan, *The Ky Fan minimax principle, sets with convex sections, and variational inequalities, in Differential Geometry, Calculus of Variations and Their Applications*, Lecture Notes in Pure Appl. Math. 100 (1985), Dekker, 471-481.
267. ———, *Generalized quasi-variational inequalities in locally convex topological vector spaces*, J. Math. Anal. Appl. 108 (1985), 333-343.
268. ———, *Minimax inequalities and applications*, Contemp. Math., vol. 54, 1986, pp. 45-63.
269. ———, *Covering theorems of simplexes and systems of linear inequalities*, Linear and Multilinear Algebra 19 (1986), 309-320.
270. ———, *Covering theorems of convex sets related to fixed point theorems*, Nonlinear and Convex Analysis, proceeding in Honor of Ky Fan (B. L. Lin and S. Simons, eds.), Marcel Dekker, Inc., New York, 1987, pp. 235-244.
271. ———, *Shapley selections and covering theorems of simplexes*, Nonlinear and Convex Analysis (B. L. Lin and S. Simons, eds.), Marcel Dekker, Inc., 1987, pp. 245-251.
272. ———, *The Browder-Hartman-Stampacchia Variational inequalities for multi-valued monotone operators*, J. Math. Anal. Appl. 134 (1988), 431-440.
273. ———, *A minimimax inequality and Browder-Hartman-Stampacchia variational inequalities for multi-valued monotone operators*, Proceedings of Fourth FRANCO-SEAMS Joint Conference, Chiang Mai, Thailand, 1988.
274. ———, *Generalized bi-quasi-variational inequalities*, J. Math. Anal. Appl. 143 (1989), 66-85.
275. S. Simons, *Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed point theorems*, Proc. Sympos. Pure Math., vol. 45, Part 2, 1986, pp. 377-392.
276. ———, *An existence theorem for quasiconcave functions wi th applications*, Nonlinear Analysis 10 (1986), 1133-1152.
277. ———, *Cyclical coincidences of multivalued maps*, J. Math. Soc. Japan 38 (1986), 515-525.
278. ———, *An upward-downward minimax theorem*, Arch. Math. 55 (1990), 275-279.
279. M. Sion, *On nonlinear variational inequalities*, Pacific J. Math. 8 (1958), 171-176.
280. D. R. Smart, *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge, 1974.
281. S. Smale, *Exchange processes with price adjustment*, J. Math. Econom. 3 (1976), 211-226.

282. ———, *A convergent process of price adjustment and global Newton methods*, J. Math. Econom. 3 (1976), 107-120.
283. H. Sonnenschein, *Demand theory without transitive preference with applications to the theory of competitive equilibrium*, Preference, Utility and Demand (J. Chipman et al., eds.), Harcourt Brace Jovanovich, New York, 1971.
284. G. Stampacchia, *Formes bilinéaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris 258 (1964), 4413-4416.
285. E. Sperner, *Neuer Beweis für die invarianz der dimensionszahl und des gebietes*, Abh. Math. Sem. Univ. Hamburg 6 (1928), 265-272.
286. W. Takahashi, *Fixed point, minimax and Hahn-Banach theorems*, Proc. Sympos. Pure Math., vol. 45, 1986, Part 2, pp. 419-427.
287. K. K. Tan, *Comparison theorems on minimax inequalities, variational inequalities, and fixed point theorems*, J. London Math. Soc. 28 (1983), 555-562.
288. K. K. Tan and J. Yu, *A new minimax inequality with applications to existence theorems of equilibrium points*, J. Optim. Theory. Appl. (to appear).
289. K. K. Tan, J. Yu and X. Z. Yuan, *Some new minimax inequalities and applications to existence of equilibria in H-spaces*, Nonlinear Analysis (to appear).
290. K. K. Tan and X. Z. Yuan, *A minimax inequality with applications to existence of equilibrium points*, Bull. Austral. Math. Soc. 47 (1993), 483-503.
291. ———, *Equilibria of generalized games with U-majorized preference correspondences*, Economics Letters 41 (1993), 379-383.
292. ———, *Lower semicontinuity of multivalued mappings and equilibrium points*, Proceedings of the First World Congress of Nonlinear Analysis, Walter de Gruyter Publishers, 1993.
293. ———, *Some random fixed point theorems*, Fixed point Theory and Applications (K. K. Tan, ed.), World Scientific, Singapore, 1992, pp. 334-345.
294. ———, *On deterministic and random fixed points*, Proc. Amer. Math. Soc. 119 (1993), 849-856.
295. ———, *Random fixed point theorems and approximation*, J. Approx. Theory (in press).
296. ———, *Random fixed point theorems and approximation in cones*, J. Math. Anal. Appl. (in press).
297. ———, *Equilibria of Random Generalized Games*, Nonlinear World (in press) (1993).
298. ———, *Random fixed point theorems and approximations in a sphere*, Proc. Amer. Math. Soc. (to appear).
299. ———, *General algorithms of quasi-variational inequalities*, in preparation (1993).
300. ———, *Random fixed point theorems and random non-compact generalized quasi-variational inequalities*, preprint (1993).
301. E. Tarafdar, *On nonlinear variational inequalities*, Proc. Amer. Math. Soc. 67 (1977), 95-98.
302. ———, *Variational problems via a fixed point theorem*, Indian J. Math. 28 (1986), 229-240.
303. ———, *A fixed point theorem in H-space and related results*, Bull. Austral. Math. Soc. 42 (1990), 133-140.
304. ———, *A fixed point theorem and its applications to equilibria*, J. Math. Econ. 20 (1991), 211-218.
305. ———, *A fixed point theorem on H-space and equilibrium point of abstract economy*, J. Austral. Math. Soc. (Ser. A) 53 (1992), 252-260.
306. ———, *Pareto solution of cone inequality and Pareto optimality of a mapping*, Proceedings of the First World Congress of Nonlinear Analysts (in press), Walter de Gruyter Publishers, 1993.

307. ———, *Applications of Pareto optimality of a mapping to mathematical economics*, Proceedings of the First World Congress of Nonlinear Analysts (in press), Walter de Gruyter Publishers, 1993.
308. E. Tarafdar and G. Mehta, *A generalized version of the Gale-Nikaio-Debreu theorem*, Comm. Math. Univ. Carolinae 28 (1987), 655-659.
309. F. Terkelson, *Some minimax theorems*, Math. Scand. 31 (1972), 405-413.
310. G. Tian, *Equilibria in abstract economies with a non-compact infinite dimensional strategy space an infinite number of agents and without ordered preferences*, Economic Letters 33 (1990), 203-206.
311. G. Tian and J. Zhou, *Quasi-variational inequalities with non-compact sets*, J. Math. Anal. Appl. 160 (1991), 583-595.
312. ———, *Transfer continuities, generalizations of the Weierstrass and maximum theorems a full characterization*, Texas A&M University worker paper (1990).
313. J. V. Tiel, *Convex Analysis an introductory text*, John Wiley and Sons Ltd., 1984.
314. R. Toscano and A. Maceri, *On the elastic stability of beams under unilateral constraints*, Meccanica, March (1980), 31-38.
315. S. Toussaint, *On the existence of equilibria in economies with infinitely many commodities and without ordered preferences*, J. Econ. Theory 33 (1984), 98-115.
316. ———, *On the equlibriums of generalized games*, The Center for Mathematical Studies in Economics and Management Science, 1986.
317. C. I. Tulcea, *On the approximation of upper-semicontinuous correspondences and the equilibriums of generalized games*, J. Math. Anal. Appl. 136 (1988), 267-289.
318. H. Tuy, *On the general minimax theorems*, Colloq. Math. 33 (1975), 145-158.
319. A. Tychonoff, *Ein Fixpunktsatz*, Math. Ann. 111 (1935), 767-776.
320. M. Walker, *On the existence of maximal elements*, J. Economic Theory 16 (1977), 470-474.
321. A. Wilansky, *Topology for Analysis*, Waltham Massachusetts: Ginn, 1972.
322. W. T. Wu, *A remark on the fundamental theorem in the theory of games*, Sci. Record (New Ser.) 3 (1959), 229-233.
323. W. T. Wu and J. H. Jiang, *Essential equilibrium points of n-person non-cooperative games*, Sci. Sinina 11 (1962), 1307-1322.
324. N. C. Yannelis, *Maximal elements over non-compact subsets of linear topological spaces*, Economics Letters 17 (1985), 133-136.
325. ———, *Existence of maxima elements and equilibria in linear topological spaces*, J. Math. Econ. 12 (1987), 96-111.
326. N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econ. 12 (1983), 233-245.
327. C. L. Yen, *A minimax inequality and its applications to variational inequalities*, Pacific J. Math. 97 (1981), 477-481.
328. J. Yu, *A generalization of Ky Fan's lemma and its applications (in Chinese)*, Acta Math. Appl. Sinica 11 (1988), 423-428.
329. ———, *On Nash equilibrium in N-person games over reflexive Banach spaces*, J. Optim. Theory and Appl. 73 (1992), 211-214.
330. ———, *Essential weak efficient solution in multiobjective optimization problems*, J. Math. Anal. Appl. 166 (1992), 230-235.
331. ———, *Essential equilibria of n-person non-cooperative games*, preprint (1992).
332. X. Z. Yuan, *Variational methods in random analysis*, J. Math. Res. Exp. 10 (1990), 15-19.
333. ———, *Random nonlinear Hammerstein equations*, J. Math. Anal. Appl. 168 (1992), 249-254.
334. ———, *Random nonlinear monotone operators*, Acta. Math. Sinica (to appear).

335. ———, *Remarks of random equilibria*, PanAmerican Math. Journal (in press) 3 (1993).

336. E. Zeidler, *Nonlinear Functional Analysis and its Applications*, vol. 1-5, Springer-Verlag, New York, 1985-1990.

337. J. Zhou, *Necessary and sufficient conditions for KKM property and variational inequalities for preference relations*, preprint (1992).

338. J. X. Zhou and G. Chen, *Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities*, J. Math. Anal. Appl. 132 (1988), 213-225.

339. J. Zhou and G. Tian, *Transfer method for characterizing the existence of maximal elements of binary relations on compact or non-compact sets*, SIAM J. Optim. 2 (1992), 360-375.