Analysis of nonmetric theories of gravity. II. The weak equivalence principle

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A set of equations representing the (generalized) laws of electromagnetism in a gravitational field for a class of nonmetric theories of gravity are written down. From these equations we calculate the center-of-mass acceleration $\langle \vec{A}_{\text{c.m.}} \rangle$ of a composite test body, consisting of electromagnetically interacting charged point particles, in an external, spherically symmetric and static (SSS) gravitational field. Demanding that the weak equivalence principle be satisfied, so that $\langle \vec{A}_{\text{c.m.}} \rangle$ contains no composition-dependent terms, puts severe constraints on the form of the original generalized equations. Indeed, the principle demands that the laws of electromagnetism in an (SSS) gravitational field take on a "metric" form. The observational constraints on the form of the (generalized) laws of electromagnetism from Eötvös experiments are also discussed.

I. INTRODUCTION

This paper is one of a series of papers involved in a systematic analysis of nonmetric theories of gravity. The class of metric theories of gravity (MTG's) is well defined in the literature.[1] A nonmetric theory of gravity is a theory not belonging to the class of MTG's. Although the techniques and ideas used in this analysis are quite general, they are primarily applied to a subclass of the class of all nonmetric theories of gravity, called metric-affine theories of gravity[1] (MATG's). An MATG is essentially characterized by the following:

(a) It is a geometric theory of gravity; that is, spacetime is characterized by a four-dimensional, Hausdorff, differentiable manifold of signature -2.

(b) The spacetime manifold is endowed with a connection Γ and a $\binom{0}{2}$ tensor field g. The gravitational field is represented (completely) by Γ and g.

(c) The unique curves of freely falling test bodies are associated with the natural geometric curves in the spacetime manifold, called paths. That is, the motion of freely falling particles is governed by the (path) equation

$$\frac{d^2x^a}{d\lambda^2}+\Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda}=0\ . \tag{1.1}$$

[Note: An MTG is a special case of an MATG, where the connection Γ is constructed from the $\binom{0}{2}$ tensor g (which is assumed to be the metric) and is given by the Christoffel symbol. We are specifically interested in the case when Γ and g are not related in this way.]

In addition, an MATG must specify how other fields should act in a gravitational field. In particular the laws of electromagnetism in a gravitational field must be given. The laws of electromagnetism to be used in this work are certainly general enough to incorporate the laws of electromagnetism in a gravitational field for MATG's. Also, it is not necessary (in this paper) to assume that g is the metric; g could be a general $\binom{0}{2}$ tensor (occurring in the laws of electromagnetism). Consequently the results obtained are applicable to a larger class of theories than MATG's.

In this paper we shall be concerned with the weak equivalent principle (WEP), and its relation to modern-day Eötvös experiments. We recall that the WEP states that there is a unique trajectory for any test body at a given point in spacetime, and with any initial velocity, which is composition independent. We shall find that there are severe constraints on the form of a theory (of gravity) in order for the theory to satisfy the WEP.

If we consider the acceleration of a test body due to an external gravitational field (only), we see that for MTG's and MATG's (with equations of motion of test particles given by the geodesic and path equations, respectively), the WEP is satisfied trivially. However, the acceleration of a test particle does not depend solely on the gravitational field, but on all the physical forces, and the WEP states that the acceleration is composition independent with respect to all such physical forces. (Indeed, the Eötvös experiments are performed on real test bodies subject to all the laws of physics.)

In the following we shall consider the acceleration of a test body due to the electromagnetic nature of that body. Other forces producing an acceleration of the body, such as nuclear forces, are beyond the scope of this paper. That is, we shall calculate the center-of-mass acceleration of a composite test body, consisting of electromagnetically interacting charged particles, in an external gravitational field. The calculation uses the techniques developed by Lightman and Lee[2] and Haugan and Will.[3]

However, the calculation is far too difficult for a general gravitational field. In order that the equations be manageable, we idealize the gravitational field to be spherically symmetric and static (SSS). This idealization does not really weaken the calculation since: (a) We shall compare the results of the calculation to the observations of the Eötvös experiments, where the gravitational field of the Sun is approximated to be SSS. (b) If we demand that the acceleration be composition independent for all gravitational fields, it must certainly be composition independent for SSS gravitational fields.

The general form of the connection Γ and the $\binom{0}{2}$ tensor g in an SSS spacetime is[4]

$$\begin{aligned}
&\Gamma^{\sigma}{}_{\mu\nu}=(\alpha)_{,\nu}\delta^{\sigma}_{\mu}+(\bar{\alpha})_{,\mu}\delta^{\sigma}_{\nu}+(\beta)_{,\sigma}\delta_{\mu\nu}\ ,\\
&\Gamma^{\sigma}{}_{00}=(\gamma)_{,\sigma}\ ,\\
&\Gamma^{0}{}_{0\nu}=(\delta)_{,\nu},\quad \Gamma^{0}{}_{\mu 0}=(\bar{\delta})_{,\mu}
\end{aligned}\tag{1.2}$$

and

$$\begin{aligned}
&g_{00}=f\ ,\\
&g_{\mu\nu}=-g\delta_{\mu\nu}\ ,
\end{aligned}\tag{1.3}$$

where f, g, α, $\bar{\alpha}$, β, γ, δ, and $\bar{\delta}$, are arbitrary functions of the Newtonian gravitational potential U (only).

In order to proceed with the calculation we need to establish the laws of electromagnetism in a gravitational field. In particular, we want to write these laws in a general (or generalized) form in order to include the possible laws of electromagnetism in a gravitational field for nonmetric theories of gravity. We shall call these the gravitationally generalized laws of electromagnetism (laws of GGEM), and these laws consist of the gravitationally generalized Maxwell equations (GGM equations) and the gravitationally generalized Lorentz equations (GGL equations). These laws of GGEM are required to satisfy the following conditions:

(a) They should reduce to the special-relativistic laws in the appropriate limit.

(b) The laws of electromagnetism in MTG's are a special case.

(c) The laws are certainly general enough to include all possible laws of GGEM for MATG's.

Such a set of equations representing these generalized laws were obtained and investigated by Coley[5] (their generality was shown within). In particular, these laws of GGEM take on a very simple form in an SSS gravitational field in terms of the functions f, g, α, $\bar{\alpha}$, β, γ, δ, $\bar{\delta}$, and four arbitrary functions (of U) $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{Q}$ (which represent the possible "nonmetric" coupling of electromagnetism to gravity). The GGL equations in an SSS field are given below by Eqs. (2.1) and (2.2), and the GGM equations by Eqs. (2.19) and (2.20). We note that the laws of electromagnetism in MTG's are a special case of these equations, with Γ equal to the metric connection, viz.,

$$\begin{aligned}
&\alpha'=\bar{\alpha}'=-\beta'=\frac{1}{2}\frac{g'}{g}\ ,\\
&\delta'=\bar{\delta}'=\frac{g\gamma'}{f}=\frac{1}{2}\frac{f'}{f}\ ,
\end{aligned}\tag{1.4}$$

where $\alpha'=d\alpha/dU$, and the four functions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{Q}$ are given by

$$\begin{aligned}
&\mathscr{A}=\frac{1}{2}\frac{f'}{f}-\frac{1}{2}\frac{g'}{g}=-\mathscr{B}\ ,\\
&\mathscr{P}=\frac{1}{2}\frac{f'}{f},\quad \mathscr{Q}=0\ .
\end{aligned}\tag{1.5}$$

We shall be interested in calculating the center-of-mass acceleration of a test body which is "dropped" in the gravitational field, that is, a body which is at rest at $t=0$ (which is appropriate if we wish to compare the results to the actual Eötvös experiments performed). Therefore, we choose our coordinate system so that $\vec{X}_{\text{c.m.}}=0$ and $\vec{V}_{\text{c.m.}}=0$ at $t=0$, and calculate the instantaneous center-of-mass acceleration. In order for the gravitational field to be written in the form given by (1.2) and (1.3), we must work in the appropriate coordinate system. In particular, the coordinate system must be at rest with respect to the spherically symmetric mass (generating the gravitational field). This condition, in addition to the conditions $\vec{X}_{\text{c.m.}}=\vec{V}_{\text{c.m.}}=0$ at $t=0$, determine the coordinate system in which we work completely.

Next, let us discuss some of the details of the calculation. We assume that the test body is made up of electromagnetically interacting charged particles. The motion of each individual particle is governed by the gravitationally generalized Lorentz (GGL) force law. The electromagnetic field (which is generated by the charged particles) is governed by the gravitationally generalized Maxwell (GGM) equations. In addition, we assume that the masses of the particles do not themselves contribute to the gravitational field, so that the gravitational field is due to the external SSS source (only).

We shall be looking for a perturbative solution. The first small quantity in which we expand is the squared particle velocity $\vec{v}_k{}^2$. We set

$$\vec{v}_k{}^2 \sim v^2 , \quad (1.6)$$

where we regard v^2 as a typical squared particle velocity. From the virial theorem (see Sec. II)

$$\frac{(\text{typical charge of a particle})^2}{(\text{typical mass})(\text{typical spatial separation of neighboring particles})} \approx \frac{e^2}{ms} \approx v^2 . \quad (1.7)$$

In the calculation we shall expand in terms of this small quantity, and we label terms in the expansion by $O(v^n)$. Formally, we write

$$\vec{v}_k{}^2 \approx \frac{e^2}{ms} \approx O(v^2) . \quad (1.8)$$

A second small quantity is the size of the body (which is typically $\sim s$). We expand all the arbitrary functions of the gravitational field [e.g., $f(U)$, $\alpha(U)$, $\mathscr{A}(U)$, etc.] in a Taylor series about the instantaneous center of mass of the test body, viz.,

$$f(U) = f(\vec{x}) = f_0 + f_0'(\vec{g}\cdot\vec{x}) , \quad (1.9)$$

where f_0 and f_0' represent f and df/dU evaluated at the center of mass.

All the effects of the gravitational field in a small region containing the test body can be represented by writing the functions of the gravitational field in the form indicated by (1.9). f_0 and f_0' are then regarded as constants, and all the spatial dependence (of the gravitational field) is contained in the factor $(\vec{g}\cdot\vec{x})$. Moreover, $\vec{g}$ can also be regarded as constant since, by the definition of a test body, we assume that second derivatives of U are negligible.[6] Therefore, we shall also expand in terms of $(\vec{g}\cdot\vec{x})$, which we shall label by $O(g)$ [or, more precisely, $O(gs)$]. However, we only expand to first order in $O(g)$ in this calculation.

We seek a perturbative solution to the problem, independently expanding in terms of $O(v^2)$ and $O(g)$. (Note that we are not expanding in terms of the Newtonian potential U. We want the solution to all orders in U.) The calculation then proceeds as follows. We solve the GGM equations perturbatively [in powers of $O(g)$ and $O(v^2)$] for the electromagnetic potentials, and substitute these solutions into the GGL equations to obtain an expression for the acceleration of the kth particle $(\vec{a}_k)$ entirely in terms of particle coordinates. We then define a center of mass $\vec{X}_{\text{c.m.}}$ for the test body and obtain an expression for $d^2\vec{X}_{\text{c.m.}}/dt^2$, which is related to the center-of-mass acceleration of the test body, in terms of the constituent particle coordinates. Having obtained an expression for the instantaneous center-of-mass acceleration, we use the viral relations to simplify the results (and express any body dependence of this acceleration in terms of the body's electromagnetic structure).

In the next section we shall calculate the center-of-mass acceleration to second order, or post-post Coulombian order [that is, up to and including terms of order $O(v^4)$ and $O(gv^4)$]. Calculating the center-of-mass acceleration to this order "completes" the calculation for all practical purposes for the following reasons. If we were to continue the calculation to post-post-post Coulombian order [that is, $O(v^6)$ and $O(gv^6)$], we would need to take into account terms arising due to electromagnetic radiation, which is far beyond the scope of the calculation. We could continue the calculation to $O(g^2)$, but these terms represent effects such as post-Newtonian corrections to the tidal forces, and would consequently yield no information on the WEP (moreover, as pointed out above, these terms would be very small, and are really outside the approximation scheme completely). In addition, as we shall see in Sec. III, the calculation to second order contains all the information that the WEP could possibly give on the structure of the GGEM equations.

In Sec. III we shall discuss the theoretical and experimental implications of the calculation with respect to the WEP.

Finally, we shall make a few brief comments on the notation to be used. We shall use latin indices (a,b,c) to range from 0 to 3 and greek indices (μ,ν,σ) to range from 1 to 3 (alternatively, we shall use three-vector notation). We shall also use latin subscripts (i,j,k) to range over the number of particles; that is, $i=1$ to n, where $n=$ the number of particles constituting the test body. This double use of latin index notation should not cause any confusion since it will be clear from the context which notation is employed.

II. CALCULATION OF THE CENTER-OF-MASS ACCELERATION OF A TEST PARTICLE, MADE UP OF ELECTROMAGNETICALLY INTERACTING CHARGED POINT PARTICLES, IN AN EXTERNAL, SPHERICALLY SYMMETRIC, STATIC GRAVITATIONAL FIELD, TO POST-POST-COULOMBIAN ORDER

A. Equations of motion for the kth particle

First we establish the equations of motion for the kth particle (of the composite body made up of electromagnetically interacting charged particles), under the action of the electromagnetic and SSS gravitational fields. This is given by the GGL equations[5]

$$\vec{a}_k(\vec{x}_k)=-\vec{\nabla}(\gamma)-[\vec{\nabla}(\alpha+\bar{\alpha}-\delta-\bar{\delta})\cdot\vec{v}_k]\vec{v}_k-\vec{\nabla}(\beta)\vec{v}_k{}^2+\frac{e_k}{m_{0k}}L(U(\vec{x}_k),U_{,\mu},\vec{v}_k)\left[\frac{1}{g}\vec{A}_L(\vec{x}_k)+\frac{1}{f}\vec{A}_L'(\vec{x}_k)\right], \tag{2.1}$$

where all functions (f,g,α, etc.) are evaluated at $\vec{x}_k$, m_{0k} is the rest mass of the kth particle, e_k its charge, $\vec{x}_k$ its three-position, and $\vec{v}_k$ its three-velocity. $\vec{A}_L$ and $\vec{A}_L'$ are defined by

$$\vec{A}_L=-\left[\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\phi-\vec{\nabla}(\vec{v}_k\cdot\vec{A})+(\vec{v}\cdot\vec{\nabla})\vec{A}\right] \tag{2.2a}$$

and

$$\vec{A}_L'=\left[\left[\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\phi\right]\cdot\vec{v}_k\right]\vec{v}_k \tag{2.2b}$$

[where ϕ and $\vec{A}$ are related to the electromagnetic four-potential A_a by $A_a=(-\phi,\vec{A})$], and represent electromagnetic contributions to the acceleration. (They are, of course, generalizations of the "usual" Lorentz force.) We wish to expand the right-hand side of (2.1) in powers of $O(g)$ and $O(v^2)$. We recall that functions depending on the gravitational field can be written according to (1.9). The gradients of these functions can therefore be written as

$$\vec{\nabla}f(\vec{x}_k)=f_0'\vec{g}\ . \tag{2.3}$$

We can use (1.9) and (2.3) to expand (2.1) in powers of $O(g)$. Also, we must expand $L(U,U_{,\mu},v_k{}^{\nu})$ appropriately. The correct expansion is found to be

$$L(\vec{x}_k,\vec{v}_k)=f_0{}^{1/2}\left[1+\mathscr{P}_0(\vec{g}\cdot\vec{x}_k)-\frac{1}{2}\frac{g_0}{f_0}\vec{v}_k{}^2-\frac{g_0\mathscr{Q}_0}{f_0}\int\frac{d}{dt}(\vec{v}_k{}^2)(\vec{g}\cdot\vec{x}_k)dt\right.$$
$$\left.+\left[-\frac{1}{2}\frac{g_0\mathscr{P}_0}{f_0}+\frac{g_0\mathscr{Q}_0}{f_0}+\frac{1}{2}\frac{g_0f_0'}{f_0{}^2}-\frac{1}{2}\frac{g_0'}{f_0}\right](\vec{g}\cdot\vec{x}_k)\vec{v}_k{}^2\right]+O(g^2)+O(v^4)\ , \tag{2.4}$$

where we have only expanded L to $O(gv^2)$ in order for $\vec{a}_k$ to remain $O(gv^4)$. The expansion of L is in terms of two arbitrary functions (of U) $\mathscr{P}$ and $\mathscr{Q}$. These two functions can be thought of as arbitrary multiplying factors for the $(\vec{g}\cdot\vec{x})$ and $(\vec{g}\cdot\vec{x})\vec{v}^2$ terms in L. (The $\vec{v}^2$ term is fixed in order to obtain the correct special-relativistic limit.)

To see that the expansion for L given by (2.4) is appropriate, we consider two examples of possible GGL equations, and hence two examples of possible L 's, and see how the L 's expand in these two cases.

In the first example we consider the GGL equations in an MTG (written in terms of the coordinate time t) given by

$$\frac{d^2x^\mu}{dt^2}+\{{}^{\mu}_{bc}\}\frac{dx^b}{dt}\frac{dx^c}{dt}-\{{}^{0}_{bc}\}\frac{dx^\mu}{dt}\frac{dx^b}{dt}\frac{dx^c}{dt}=\frac{e}{m}\left[\frac{d\tau}{dt}\right]\left[F_n{}^\mu\frac{dx^n}{dt}-F_n{}^0\frac{dx^n}{dt}\frac{dx^\mu}{dt}\right]. \quad (2.5)$$

In this example L is given by

$$L(\vec{x},\vec{v})=\frac{d\tau}{dt}=\left[g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}\right]^{1/2}. \quad (2.6)$$

In an SSS gravitational field this becomes

$$L(\vec{x},\vec{v})=(f-g\vec{v}^2)^{1/2}. \quad (2.7)$$

Expanding f and g according to (1.9), and using the binomial theorem to expand (2.7) in terms of $O(g)$ and $O(v^2)$, we find that L expands according to (2.4) with $\mathscr{P}_0$ and $\mathscr{Q}_0$ given by

$$\mathscr{P}_0=\frac{1}{2}\frac{f_0'}{f_0}, \qquad \mathscr{Q}_0=0. \quad (2.8)$$

In the second example we take the GGL equations in the form

$$\frac{d^2x^a}{d\lambda^2}+\Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda}=\frac{e}{m}F_n{}^a\frac{dx^n}{d\lambda} \quad (2.9)$$

(the validity of this equation is discussed in Ref. 1). For $a=0$ this equation becomes

$$\frac{d^2t}{d\lambda^2}=-\Gamma^0{}_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt}\left[\frac{dt}{d\lambda}\right]^2+\frac{e}{m}F_n{}^0\frac{dx^n}{dt}\frac{dt}{d\lambda}, \quad (2.10)$$

and for $a=\mu$ (written in terms of the coordinate time t)

$$\frac{d^2x^\mu}{dt^2}+\Gamma^\mu{}_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt}-\Gamma^0{}_{bc}\frac{dx^\mu}{dt}\frac{dx^b}{dt}\frac{dx^c}{dt}=\frac{e}{m}\left[\frac{d\lambda}{dt}\right]\left[F_n{}^\mu\frac{dx^n}{dt}-F_n{}^0\frac{dx^\mu}{dt}\frac{dx^n}{dt}\right]. \quad (2.11)$$

In this second example

$$L(\vec{x},\vec{v})=\frac{d\lambda}{dt}. \quad (2.12)$$

We wish to calculate the value of $d\lambda/dt$ along the trajectory defined by Eq. (2.9). Calculating $F_n{}^0v^n$ by evaluating $(d/dt)(\vec{v}\cdot\vec{v})$ and using (2.11), and using the forms of g and Γ in an SSS gravitational field, Eq. (2.10) becomes

$$\frac{d^2t}{d\lambda^2}=-\frac{d}{d\lambda}(\delta+\bar{\delta})\frac{dt}{d\lambda}+\frac{g}{f}\left[1-\frac{gv^2}{f}\right]^{-1}\left[\frac{1}{2}\frac{d}{d\lambda}(v^2)\frac{dt}{d\lambda}+\frac{d}{d\lambda}(\gamma)\frac{dt}{d\lambda}-\frac{d}{d\lambda}(\delta+\bar{\delta}-\alpha-\bar{\alpha}-\beta)\frac{dt}{d\lambda}\vec{v}^2\right]. \quad (2.13)$$

Expanding the right-hand side of this equation in terms of $O(g)$ and $O(v^2)$ to $O(gv^2)$ yields

$$\frac{d^2t}{d\lambda^2}=\frac{d}{d\lambda}\left[\left[\frac{g_0\gamma_0'}{f_0}-\delta_0'-\bar{\delta}_0'\right](\vec{g}\cdot\vec{x})+\frac{1}{2}\frac{g_0}{f_0}\vec{v}^2-\frac{g_0}{f_0}\left[\delta_0'+\bar{\delta}_0'-\alpha_0'-\bar{\alpha}_0'-\beta_0'-\frac{g_0\gamma_0'}{f_0}\right](\vec{g}\cdot\vec{x})\vec{v}^2\right]\frac{dt}{d\lambda}$$
$$+\Phi_0(\vec{g}\cdot\vec{x})\frac{d}{d\lambda}(\vec{v}^2)\frac{dt}{d\lambda}+O(g^2)+O(v^4), \quad (2.14)$$

where

$$\Phi_0=\frac{g_0}{f_0}\left[\frac{1}{2}\frac{g_0'}{g_0}-\frac{1}{2}\frac{f_0'}{f_0}+\delta_0'+\bar{\delta}_0'-\alpha_0'-\bar{\alpha}_0'-\beta_0'-\frac{g_0\gamma_0'}{f_0}\right]. \quad (2.15)$$

Integrating Eq. (2.14) yields

$$\frac{dt}{d\lambda}=\frac{1}{f_0{}^{1/2}}\left\{1+\left[\frac{g_0\gamma_0'}{f_0}-\delta_0'-\bar{\delta}_0'\right](\vec{g}\cdot\vec{x})+\frac{1}{2}\frac{g_0}{f_0}\vec{v}^2\right.$$
$$+\left[\frac{1}{2}\frac{g_0}{f_0}\left[\frac{g_0\gamma_0'}{f_0}-\delta_0'-\bar{\delta}_0'\right]-\frac{g_0}{f_0}\left[\delta_0'+\bar{\delta}_0'-\alpha_0'-\bar{\alpha}_0'-\beta_0'-\frac{g_0\gamma_0'}{f_0}\right]\right](\vec{g}\cdot\vec{x})\vec{v}^2$$
$$\left.+\Phi_0\int\frac{d}{dt}(\vec{v}^2)(\vec{g}\cdot\vec{x})dt\right\}+O(g^2)+O(v^4), \quad (2.16)$$

where the "arbitrary constant of integration" has been chosen as $f_0^{-1/2}$ in order for (2.16) to reduce to its "metric" form in the appropriate limit.

Finally, using $d\lambda/dt=(dt/d\lambda)^{-1}$, we find that in this second example (with $L=d\lambda/dt$), L expands according to (2.4) with $\mathscr{P}_0$ and $\mathscr{Q}_0$ defined by

$$\mathscr{P}_0=\delta_0'+\bar{\delta}_0'-\frac{g_0\gamma_0'}{f_0}\ ,$$
$$\mathscr{Q}_0=\frac{1}{2}\frac{g_0'}{g_0}-\frac{1}{2}\frac{f_0'}{f_0}+\delta_0'+\bar{\delta}_0'-\alpha_0'-\bar{\alpha}_0'-\beta_0'-\frac{g_0\gamma_0'}{f_0}\ . \tag{2.17}$$

Having found the required general expansion of L, we can now write down the equations of motion for the kth particle to $O(gv^4)$. Substituting the value of L given by Eq. (2.4) into Eq. (2.1), we obtain

$$\begin{aligned}\vec{a}_k(\vec{x}_k)=&-\gamma_0'\vec{g}-(\alpha_0'+\bar{\alpha}_0'-\delta_0'-\bar{\delta}_0')(\vec{g}\cdot\vec{v}_k)\vec{v}_k\\&-\beta_0'\vec{v}_k^{\,2}\vec{g}+\frac{e_k}{m_{0k}}\left\{f_0^{1/2}\left[1+\mathscr{P}_0(\vec{g}\cdot\vec{x}_k)-\frac{1}{2}\frac{g_0}{f_0}\vec{v}_k^{\,2}\right.\right.\\&\qquad+\left[-\frac{1}{2}\frac{g_0\mathscr{P}_0}{f_0}+\frac{g_0\mathscr{Q}_0}{f_0}+\frac{1}{2}\frac{g_0f_0'}{f_0^{2}}-\frac{1}{2}\frac{g_0'}{f_0}\right](\vec{g}\cdot\vec{x}_k)\vec{v}_k^{\,2}\\&\qquad\left.\left.-\frac{g_0\mathscr{Q}_0}{f_0}\int\frac{d}{dt}(\vec{v}_k^{\,2})(\vec{g}\cdot\vec{x}_k)dt\right]\right\}\left\{\frac{1}{g_0}\left[1-\frac{g_0'}{g_0}(\vec{g}\cdot\vec{x}_k)\right]\vec{A}_L(\vec{x}_k)\right.\\&\qquad\left.+\frac{1}{f_0}\left[1-\frac{f_0'}{f_0}(\vec{g}\cdot\vec{x}_k)\right]\vec{A}_L'(\vec{x}_k)\right\}\\&+O(g^2)+O(v^6)\ .\end{aligned} \tag{2.18}$$

B. Solution of the GGM equations

The GGM equations in an SSS gravitational field, for a source consisting of electromagnetic point particles, are given by[5]

$$\nabla^2\phi=\frac{g}{f}\frac{\partial^2\phi}{\partial t^2}+\mathscr{A}_0\vec{g}\cdot\left[\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\phi\right]-4\pi\left[\frac{f}{g}\right]^{1/2}\sum_k e_k\delta^3(\vec{x}-\vec{x}_k)\ , \tag{2.19}$$

and

$$\nabla^2\vec{A}=\frac{g}{f}\frac{\partial^2\vec{A}}{\partial t^2}+\frac{f}{g}(\vec{\nabla}\cdot\vec{A})\vec{\nabla}\left[\frac{g}{f}\right]+\mathscr{B}_0(\vec{\nabla}\times\vec{A})\times\vec{g}-4\pi\left[\frac{g}{f}\right]^{1/2}\sum_k e_k\vec{v}_k\delta^3(\vec{x}-\vec{x}_k)\ , \tag{2.20}$$

where $\mathscr{A}$ and $\mathscr{B}$ are arbitrary functions of the gravitational field.

We expand (2.19) and (2.20) in powers of $O(g)$ and $O(v^2)$, and look for a perturbation solution for $\vec{A}$ and ϕ. $\vec{A}$ and ϕ expand according to

$$\phi\sim O(g^0)[O(v^2)+O(v^4)+\cdots]+O(g)[O(v^2)+O(v^4)+\cdots]+\cdots\ ,$$
$$\vec{A}\sim O(g^0)[O(v^3)+O(v^5)+\cdots]+O(g)[O(v^3)+O(v^5)+\cdots]+\cdots\ . \tag{2.21}$$

Formally, we write

$$\phi=\phi_0+\phi_1+\bar{\phi}_1+\phi_2+\cdots\ ,$$
$$\vec{A}=\vec{A}_0+\vec{A}_1+\cdots\ , \tag{2.22}$$

where

$$\phi_0 \sim O(v^2),\ \phi_1 \sim O(gv^2),\ \bar{\phi}_1 \sim O(v^4),\ \phi_2 \sim O(gv^4)\ ,$$
$$\vec{A}_0 \sim O(v^3),\ \vec{A}_1 \sim O(gv^3)+O(v^5)\ . \tag{2.23}$$

Since we only wish to calculate $\vec{a}_k$ [from Eq. (2.18)] to $O(gv^4)$, we need only calculate $\vec{A}_0$, $\vec{A}_1$, ϕ_0, ϕ_1, $\bar{\phi}_1$, and ϕ_2.

First we calculate $\vec{A}_0$. From (2.20) the equation for $\vec{A}_0$ is

$$\nabla^2 \vec{A}_0 = -4\pi \frac{g_0{}^{1/2}}{f_0{}^{1/2}} \sum_i e_i \vec{v}_i \delta^3(\vec{x}-\vec{x}_i)\ , \tag{2.24}$$

which has the solution

$$\vec{A}_0(\vec{x}_k) = \frac{g_0{}^{1/2}}{f_0{}^{1/2}} \sum_i \frac{e_i \vec{v}_i}{|\vec{x}_{ki}|}\ , \tag{2.25}$$

where $\vec{x}_{ki} = \vec{x}_k - \vec{x}_i$, and the summation is assumed to exclude the case $i = k$.

Therefore, from Eq. (2.20) the equation for $\vec{A}_1$ is

$$\nabla^2 \vec{A}_1 = \frac{g_0}{f_0} \frac{\partial^2 \vec{A}_0}{\partial t^2} + \frac{f_0}{g_0} (\vec{\nabla}\cdot\vec{A}_0) \vec{\nabla} \left[\frac{g}{f} \right] + \mathscr{B}_0 (\vec{\nabla}\times\vec{A}_0)\times\vec{g} - 2\pi \frac{g_0{}^{1/2}}{f_0{}^{1/2}} \left[\frac{g_0'}{g_0} - \frac{f_0'}{f_0} \right] \sum_i e_i \vec{v}_i (\vec{g}\cdot\vec{x}) \delta^3(\vec{x}-\vec{x}_i)\ , \tag{2.26}$$

where we retain terms up to $O(gv^3)$ and $O(v^5)$, and $\vec{A}_0$ is given by (2.25). The solution of Eq. (2.26) is given by

$$\vec{A}_1(\vec{x}_k) = \frac{1}{2} \frac{g_0{}^{1/2}}{f_0{}^{1/2}} \left[\frac{g_0'}{g_0} - \frac{f_0'}{f_0} \right] \sum_i \frac{e_i \vec{v}_i (\vec{g}\cdot\vec{x}_i)}{|\vec{x}_{ki}|} + \frac{g_0}{f_0} \frac{\partial^2 \vec{\Psi}}{\partial t^2} + \left[\frac{g_0'}{g_0} - \frac{f_0'}{f_0} \right] (\vec{\nabla}\cdot\vec{\Psi}) \vec{g} + \mathscr{B}_0 (\vec{\nabla}\times\vec{\Psi})\times\vec{g} \tag{2.27}$$

[where we retain $O(gv^3)$ and $O(v^5)$ terms only], in terms of the vector "superpotential" $\vec{\Psi}$ given by

$$\vec{\Psi}(\vec{x}_k) = \frac{1}{2} \frac{g_0{}^{1/2}}{f_0{}^{1/2}} \sum_i e_i \vec{v}_i |\vec{x}_{ki}|\ , \tag{2.28}$$

where, to $O(gv)$ and $O(v^3)$,

$$\vec{A}_0 = \nabla^2 \vec{\Psi}\ . \tag{2.29}$$

From (2.19) the equation for ϕ_0 is

$$\nabla^2 \phi_0 = -4\pi \frac{f_0{}^{1/2}}{g_0{}^{1/2}} \sum_i e_i \delta^3(\vec{x}-\vec{x}_i)\ , \tag{2.30}$$

which has the solution

$$\phi_0(\vec{x}_k) = \frac{f_0{}^{1/2}}{g_0{}^{1/2}} \sum_i \frac{e_i}{|\vec{x}_{ki}|}\ . \tag{2.31}$$

Therefore, the equation for ϕ_1 is

$$\nabla^2 \phi_1 = \frac{g_0}{f_0} \frac{\partial^2 \phi_0}{\partial t^2} + \mathscr{A}_0 (\vec{g}\cdot\vec{\nabla}\phi_0) - 2\pi \frac{f_0{}^{1/2}}{g_0{}^{1/2}} \left[\frac{f_0'}{f_0} - \frac{g_0'}{g_0} \right] (\vec{g}\cdot\vec{x}) \sum_i e_i \delta^3(\vec{x}-\vec{x}_i)\ , \tag{2.32}$$

which has the solution,

$$\phi_1(\vec{x}_k) = \frac{1}{2} \frac{f_0{}^{1/2}}{g_0{}^{1/2}} \left[\frac{f_0'}{f_0} - \frac{g_0'}{g_0} \right] \sum_i \frac{e_i (\vec{g}\cdot\vec{x}_i)}{|\vec{x}_{ki}|} + \frac{g_0}{f_0} \frac{\partial^2 \Upsilon}{\partial t^2} + \mathscr{A}_0 (\vec{g}\cdot\vec{\nabla}\Upsilon) \tag{2.33}$$

[where we keep $O(gv^2)$ terms only], in terms of the "superpotential" Υ defined by

$$\Upsilon(\vec{x}_k)=\frac{1}{2}\frac{f_0^{1/2}}{g_0^{1/2}}\sum_i e_i\,|\,\vec{x}_{ki}\,|\ , \tag{2.34}$$

where

$$\phi_0=\nabla^2\Upsilon\ . \tag{2.35}$$

Next, the equation for $\bar{\phi}_1$ is

$$\nabla^2\bar{\phi}_1=\frac{g_0}{f_0}\frac{\partial^2\phi_0}{\partial t^2}\ , \tag{2.36}$$

which has the solution

$$\bar{\phi}_1(\vec{x}_k)=\frac{g_0}{f_0}\frac{\partial^2\Upsilon}{\partial t^2}\ . \tag{2.37}$$

Finally, the equation for ϕ_2 is

$$\nabla^2\phi_2=\frac{g}{f}\frac{\partial^2\phi_0}{\partial t^2}+\frac{g_0}{f_0}\frac{\partial^2\phi_1}{\partial t^2}+\frac{g_0}{f_0}\frac{\partial^2\bar{\phi}_1}{\partial t^2}+\mathscr{A}_0(\vec{g}\cdot\vec{\nabla}\bar{\phi}_1)+\mathscr{A}_0\left[\vec{g}\cdot\frac{\partial\vec{A}_0}{\partial t}\right]\ . \tag{2.38}$$

The solution of this equation is given by

$$\begin{aligned}\phi_2(\vec{x}_k)=&\frac{g}{f}\frac{\partial^2\Upsilon}{\partial t^2}-2\frac{g_0}{f_0}\left[\frac{g_0'}{g_0}-\frac{f_0'}{f_0}\right]\left[\vec{g}\cdot\vec{\nabla}\left[\frac{\partial^2\Lambda}{\partial t^2}\right]\right]\\&+\frac{g_0}{f_0}\frac{\partial^2\Omega}{\partial t^2}+\frac{g_0^2}{f_0^2}\frac{\partial^4\Lambda}{\partial t^4}+\frac{g_0}{f_0}\mathscr{A}_0\left[\vec{g}\cdot\vec{\nabla}\left[\frac{\partial^2\Lambda}{\partial t^2}\right]\right]+\mathscr{A}_0\left[\vec{g}\cdot\frac{\partial^2\vec{\Psi}}{\partial t^2}\right]\ ,\end{aligned} \tag{2.39}$$

where we retain $O(gv^4)$ terms only. The "superpotentials" Λ and Ω are defined by

$$\Lambda(\vec{x}_k)=\frac{1}{24}\frac{f_0^{1/2}}{g_0^{1/2}}\sum_i e_i\,|\,\vec{x}_{ki}\,|^3\ ,$$

$$\Omega(\vec{x}_k)=\frac{1}{8}\frac{f_0^{1/2}}{g_0^{1/2}}\left[\frac{g_0\gamma_0'}{f_0}+\mathscr{A}_0\right]\sum_i e_i(\vec{g}\cdot\vec{x}_{ki})\,|\,\vec{x}_{ki}\,|+\frac{1}{4}\frac{f_0^{1/2}}{g_0^{1/2}}\left[\frac{f_0'}{f_0}-\frac{g_0'}{g_0}\right]\sum_i e_i(\vec{g}\cdot\vec{x}_i)\,|\,\vec{x}_{ki}\,|\ , \tag{2.40}$$

so that

$$\nabla^2\Lambda=\Upsilon\ ,\qquad \nabla^2\Omega=\phi_1\ . \tag{2.41}$$

Consequently the solutions for ϕ and $\vec{A}$ to the appropriate order are given by Eqs. (2.25), (2.27), (2.31), (2.33), (2.37), and (2.39). Substituting these solutions into Eq. (2.18) [using (2.2)] gives us an expression for the kth particle's acceleration $\vec{a}_k$ up to $O(gv^4)$. In order to write this expression explicitly in terms of particle coordinates we must evaluate the (appropriate) derivatives of the superpotentials in the solution. The time derivatives of the superpotentials will give rise to terms containing the single-particle acceleration; therefore, in order to obtain a consistent approximation for $\vec{a}_k$ to $O(gv^4)$ we must iterate the equations for ϕ and $\vec{A}$ and the expression for $\vec{a}_k$ [given by (2.2) and (2.18)].

C. The center-of-mass acceleration

Having obtained an expression for the acceleration of a single particle, we now define a center of mass $\vec{X}_{\text{c.m.}}$ and calculate the instantaneous center-of-mass acceleration $\vec{A}_{\text{c.m.}}$ of the body. To some extent the definition of the center of mass to be used is unimportant, so long as it remains within the body. Indeed, since ultimately we shall be interested in a time-averaged acceleration, any point (representing $\vec{X}_{\text{c.m.}}$) that remains within the body will yield the same numerical result on average.

The definition that we shall use for the center of mass is given by the following equations[7]:

$$\vec{X}_{\text{c.m.}}=\frac{1}{m}\sum_k(m_k\vec{x}_k)\ , \tag{2.42}$$

$$m_k=m_{0k}+\frac{1}{2}\frac{g_0}{f_0}m_{0k}\vec{v}_k^{\,2}+\frac{1}{2g_0^{1/2}}\sum_j\frac{e_je_k}{|\,\vec{x}_{jk}\,|}\ , \tag{2.43}$$

and

$$m=\sum_k m_k \ . \tag{2.44}$$

Note that we have defined m_k [in Eq. (2.43)] up to $O(v^2)$ only. If we had included terms in m_k of $O(v^4)$ and $O(g)$, these terms would also contribute to $\langle \vec{A}_{\text{c.m.}} \rangle$ to $O(gv^4)$. However, these additional terms in $\langle \vec{A}_{\text{c.m.}} \rangle$ would all cancel when use is made of the virial relations (to be discussed in the next subsection). A detailed calculation can be made to prove this.[1]

From Eq. (2.42)

$$m\frac{d^2\vec{X}_{\text{c.m.}}}{dt^2}=\sum_k \frac{d^2 m_k}{dt^2}\vec{x}_k+2\sum_k \frac{dm_k}{dt}\vec{v}_k+\sum_k m_k \vec{a}_k \ . \tag{2.45}$$

In order to calculate $md^2\vec{X}_{\text{c.m.}}/dt^2$ to $O(gv^4)$ we need to calculate dm_k/dt to $O(gv^3)$ and d^2m_k/dt^2 to $O(gv^4)$, using our calculated expression for $\vec{a}_k$ to $O(gv^2)$. We need to use $\vec{a}_k$ to $O(gv^4)$ to calculate $\sum_k m_k \vec{a}_k$ to $O(gv^4)$. Collecting all terms together we can write down the expression for $md^2\vec{X}_{\text{c.m.}}/dt^2$ to $O(gv^4)$. However, since this expression contains over 100 terms, it will not be written explicitly here but represented symbolically by

$$m\frac{d^2\vec{X}_{\text{c.m.}}}{dt^2}=\left[m\frac{d^2\vec{X}_{\text{c.m.}}}{dt^2}\right]_{O(g)}+\left[m\frac{d^2\vec{X}_{\text{c.m.}}}{dt^2}\right]_{O(gv^2)}+\left[m\frac{d^2\vec{X}_{\text{c.m.}}}{dt^2}\right]_{O(gv^4)}+O(v^6) \ . \tag{2.46}$$

There are no $O(v^2)$ nor $O(v^4)$ terms present since they essentially come from the special-relativistic contributions to $md^2\vec{X}_{\text{c.m.}}/dt^2$. (It is relatively easy to calculate these terms and show that they do indeed vanish.)

We wish to obtain an expression for the time-averaged center-of-mass acceleration. Taking the time average of Eq. (2.46), we then define the center-of-mass acceleration $\langle \vec{A}_{\text{c.m.}} \rangle$ to be

$$\langle \vec{A}_{\text{c.m.}} \rangle=\frac{1}{\langle m \rangle}\left\langle \left[m\frac{d^2\vec{X}_{\text{c.m.}}}{dt^2}\right]_{O(g)}+\left[m\frac{d^2\vec{X}_{\text{c.m.}}}{dt^2}\right]_{O(gv^2)}+\left[m\frac{d^2\vec{X}_{\text{c.m.}}}{dt^2}\right]_{O(gv^4)}\right\rangle+O(v^6) \ . \tag{2.47}$$

D. Virial relations

In this subsection we shall establish a set of virial relations, which are relations between different terms in $\langle \vec{A}_{\text{c.m.}} \rangle$, obtained by setting $\langle (d/dt)(\Phi^{\mu\nu}) \rangle$ equal to zero, for internal, structure-dependent quantities $\Phi^{\mu\nu}$. This set of virial relations can then be used to simplify the expression for $\langle \vec{A}_{\text{c.m.}} \rangle$.

(a) "Second-order" tensor virial relations.

Setting $\langle (d/dt)(\Phi^{\mu\nu}) \rangle=0$, where $\Phi^{\mu\nu}$ is defined by

$$\Phi^{\mu\nu}=\sum_{i,k}\frac{e_i e_k}{|\vec{x}_{ik}|}v_i{}^{\mu}x_k{}^{\nu}{}_i \ , \tag{2.48}$$

yields the virial relation

$$\left\langle \frac{f_0}{g_0{}^{3/2}}\sum_{i,j,k}\frac{e_i e_j e_k{}^2}{m_{0k}}\frac{x_k{}^{\mu}{}_j x_i{}^{\nu}{}_k}{|\vec{x}_{ik}|\,|\vec{x}_{kj}|^3}\right\rangle+\left\langle \sum_{i,k}\frac{e_i e_k}{|\vec{x}_{ik}|}v_i{}^{\mu}v_k{}^{\nu}{}_i\right\rangle-\left\langle \sum_{i,k}\frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{x}_{ik}\cdot\vec{v}_{ik})v_i{}^{\mu}x_k{}^{\nu}{}_i\right\rangle=0 \ , \tag{2.49}$$

where we have used the calculated value of $a_i{}^{\sigma}$ to $O(v^2)$. Other "second-order" tensor virial relations that we shall use are

$$\left\langle \frac{f_0}{g_0{}^{3/2}}\sum_{i,j,k}\frac{e_i e_j e_k{}^2}{m_{0k}}\frac{(\vec{x}_{ik}\cdot\vec{x}_{jk})x_i{}^{\mu}{}_k x_i{}^{\nu}{}_k}{|\vec{x}_{ik}|^3|\vec{x}_{kj}|^3}\right\rangle+\left\langle \sum_{i,k}\frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{v}_i\cdot\vec{v}_{ik})x_i{}^{\mu}{}_k x_i{}^{\nu}{}_k\right\rangle+\left\langle \sum_{i,k}\frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{v}_i\cdot\vec{x}_{ik})v_i{}^{\mu}{}_k x_i{}^{\nu}{}_k\right\rangle$$
$$+\left\langle \sum_{i,k}\frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{v}_i\cdot\vec{x}_{ik})x_i{}^{\mu}{}_k v_i{}^{\nu}{}_k\right\rangle-3\left\langle \sum_{i,k}\frac{e_i e_k}{|\vec{x}_{ik}|^5}(\vec{v}_{ik}\cdot\vec{x}_{ik})(\vec{v}_i\cdot\vec{x}_{ik})x_i{}^{\mu}{}_k x_i{}^{\nu}{}_k\right\rangle=0 \ , \tag{2.50}$$

$$\left\langle \frac{f_0}{g_0{}^{3/2}}\sum_{i,j,k}\frac{e_i e_j e_k{}^2}{m_{0k}}\frac{x_k{}^{\mu}{}_j x_i{}^{\nu}}{|\vec{x}_{kj}|^3|\vec{x}_{ik}|}\right\rangle+\left\langle \sum_{i,k}\frac{e_i e_k}{|\vec{x}_{ik}|}v_k{}^{\mu}v_i{}^{\nu}\right\rangle-\left\langle \sum_{i,k}\frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{x}_{ik}\cdot\vec{v}_{ik})v_k{}^{\mu}x_i{}^{\nu}\right\rangle=0 \ , \tag{2.51}$$

$$\left\langle \sum_{i,k} e_i e_k \left[\int (\vec{v}_k\cdot\vec{x}_{ki}) \frac{x_k{}^\mu}{|\vec{x}_{ki}|^3} dt \right] \frac{dv_k{}^\nu}{dt} \right\rangle + \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_k\cdot\vec{x}_{ki}) x_k{}^\mu v_k{}^\nu \right\rangle = 0 , \tag{2.52}$$

$$\left\langle \frac{f_0}{g_0{}^{3/2}} \sum_{i,j,k} \frac{e_i e_j e_k{}^2}{m_{0k}} \frac{(\vec{x}_{kj}\cdot\vec{x}_{ik}) x_i{}^\mu{}_k x_i{}^\nu}{|\vec{x}_{kj}|^3 |\vec{x}_{ki}|^3} \right\rangle + \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_k\cdot\vec{v}_{ik}) x_i{}^\mu{}_k x_i{}^\nu \right\rangle + \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_k\cdot\vec{x}_{ik}) v_i{}^\mu{}_k x_i{}^\mu \right\rangle$$
$$+ \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_k\cdot\vec{x}_{ik}) x_i{}^\mu{}_k v_i{}^\nu \right\rangle - 3 \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^5} (\vec{v}_{ik}\cdot\vec{x}_{ik})(\vec{v}_k\cdot\vec{x}_{ik}) x_i{}^\mu{}_k x_i{}^\nu \right\rangle = 0 , \tag{2.53}$$

$$\left\langle \frac{f_0}{g_0{}^{3/2}} \sum_{i,j,k} \frac{e_i e_j e_k{}^2}{m_{0k}} \frac{(\vec{x}_{kj}\cdot\vec{x}_{ik}) x_k{}^\mu x_k{}^\nu}{|\vec{x}_{ik}|^3 |\vec{x}_{kj}|^3} \right\rangle + \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_k\cdot\vec{x}_{ik}) x_k{}^\mu v_k{}^\nu \right\rangle + \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_k\cdot\vec{v}_{ik}) x_k{}^\mu x_k{}^\nu \right\rangle$$
$$+ \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_k\cdot\vec{x}_{ik}) v_k{}^\mu x_k{}^\nu \right\rangle - 3 \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^5} (\vec{x}_{ik}\cdot\vec{v}_{ik})(\vec{v}_k\cdot\vec{x}_{ik}) x_k{}^\mu x_k{}^\nu \right\rangle = 0 , \tag{2.54}$$

and

$$\left\langle \frac{2f_0}{g_0{}^{3/2}} \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_i\cdot\vec{x}_{ik}) v_i{}^\mu x_i{}^\nu \right\rangle + \left\langle \sum_k m_{0k} \vec{v}_k{}^2 v_k{}^\mu v_k{}^\nu \right\rangle + \left\langle \frac{f_0}{g_0{}^{3/2}} \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} \vec{v}_i{}^2 x_i{}^\mu{}_k x_i{}^\nu \right\rangle = 0 . \tag{2.55}$$

Due to the symmetry of the last virial relation (in $v_k{}^\mu v_k{}^\nu$) we obtain the virial identity

$$2\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_i\cdot\vec{x}_{ik})(\vec{g}\cdot\vec{v}_i)\vec{x}_i \right\rangle + \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} \vec{v}_i{}^2 (\vec{g}\cdot\vec{x}_{ik})\vec{x}_i \right\rangle$$
$$= 2\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_i\cdot\vec{x}_{ik})(\vec{g}\cdot\vec{x}_i)\vec{v}_i \right\rangle + \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} \vec{v}_i{}^2 (\vec{g}\cdot\vec{x}_i)\vec{x}_{ik} \right\rangle . \tag{2.56}$$

(b) The "first-order" tensor virial relation with "second-order" corrections.
With

$$\Phi^{\mu\nu} = \tfrac{1}{2} \sum_k m_k x_k{}^\mu v_k{}^\nu , \tag{2.57}$$

where m_k is defined by Eq. (2.43), setting $\langle (d/dt)(\Phi^{\mu\nu}) \rangle = 0$ yields

$$\tfrac{1}{2}\left\langle \sum_k \frac{dm_k}{dt} x_k{}^\mu v_k{}^\nu \right\rangle + \tfrac{1}{2}\left\langle \sum_k m_k v_k{}^\mu v_k{}^\nu \right\rangle + \tfrac{1}{2}\left\langle \sum_k m_k x_k{}^\mu a_k{}^\nu \right\rangle = 0 . \tag{2.58}$$

Calculating dm_k/dt, using $a_k{}^\sigma$ to $O(v^4)$, and keeping all terms to $O(v^4)$, we obtain the virial relation

$$\left\langle \sum_k m_{0k} v_k{}^\mu v_k{}^\nu \right\rangle + \frac{f_0}{2g_0{}^{3/2}} \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} x_i{}^\mu{}_k x_i{}^\nu{}_k \right\rangle + \frac{1}{2}\frac{g_0}{f_0} \left\langle \sum_k m_{0k} \vec{v}_k{}^2 v_k{}^\mu v_k{}^\nu \right\rangle + \frac{1}{2g_0{}^{1/2}} \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} v_i{}^\mu v_k{}^\nu \right\rangle$$
$$- \frac{1}{4g_0{}^{1/2}} \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_i\cdot\vec{v}_k) x_i{}^\mu{}_k x_i{}^\nu{}_k \right\rangle - \frac{3}{4g_0{}^{1/2}} \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^5} (\vec{v}_i\cdot\vec{x}_{ik})(\vec{v}_k\cdot\vec{x}_{ik}) x_i{}^\mu{}_k x_i{}^\nu{}_k \right\rangle$$
$$+ \frac{1}{2g_0{}^{1/2}} \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_k\cdot\vec{x}_{ik}) v_i{}^\mu x_i{}^\nu{}_k \right\rangle + \frac{1}{2g_0{}^{1/2}} \left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3} (\vec{v}_k\cdot\vec{x}_{ik}) x_i{}^\mu{}_k v_i{}^\nu \right\rangle = 0 , \tag{2.59}$$

where the "second-order" virial relations in part (a) have been used to simplify this result. The first two terms in this equation are $O(v^2)$, while the rest are $O(v^4)$.

The calculation now proceeds as follows. First we simplify the $O(gv^2)$ terms in $\langle \vec{A}_{\text{c.m.}} \rangle$ using (2.59). This simplification is achieved at the cost of introducing extra $O(gv^4)$ terms into $\langle \vec{A}_{\text{c.m.}} \rangle$ [arising from the $O(v^4)$ correction terms in (2.59)]. Then the $O(gv^4)$ terms are simplified using the "second-order" tensor virial relations. After an exceedingly long calculation we find that $\langle \vec{A}_{\text{c.m.}} \rangle$ is given by

$$\langle \vec{A}_{\text{c.m.}} \rangle = -\gamma_0' \vec{g} + \frac{\eta}{m}\vec{\eta} + \frac{\omega}{m}\vec{\omega} + \frac{\eta}{m}\vec{W}_1 + \frac{\omega}{m}\vec{W}_2 + \frac{\epsilon_1}{m}\vec{W}_3 + \frac{\epsilon_2}{m}\vec{W}_4 + \frac{\xi}{m}\vec{W}_5 , \tag{2.60}$$

where

$$\vec{\eta}=\vec{g}\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|}\right\rangle, \quad \vec{\omega}=\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{g}\cdot\vec{x}_{ik})\vec{x}_{ik}\right\rangle ,$$
$$\vec{W}_1=\tfrac{1}{4}\vec{\eta}_1+\tfrac{1}{4}\vec{\eta}_2, \quad \vec{W}_2=\tfrac{1}{2}\vec{\omega}_1+\tfrac{1}{2}\vec{\omega}_2+\tfrac{1}{2}\vec{\omega}_3-\tfrac{1}{4}\vec{\omega}_4-\tfrac{3}{4}\vec{\omega}_5 ,$$
$$\vec{W}_3=\tfrac{1}{2}\vec{\eta}_2-\tfrac{1}{2}\vec{\omega}_1-\tfrac{1}{2}\vec{\omega}_2+\tfrac{1}{2}\vec{\omega}_4 , \tag{2.61}$$
$$\vec{W}_4=\tfrac{1}{2}\vec{\eta}_1-\tfrac{1}{2}\vec{\omega}_1+\tfrac{1}{2}\vec{\omega}_2-\tfrac{1}{2}\vec{\omega}_4, \quad \vec{W}_5=\vec{\omega}_6 ,$$

with

$$\vec{\eta}_1=\vec{g}\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|}(\vec{v}_i\cdot\vec{v}_k)\right\rangle, \quad \vec{\eta}_2=\vec{g}\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{x}_{ki}\cdot\vec{v}_i)(\vec{x}_{ki}\cdot\vec{v}_k)\right\rangle ,$$
$$\vec{\omega}_1=\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|}(\vec{g}\cdot\vec{v}_i)\vec{v}_k\right\rangle, \quad \vec{\omega}_2=\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{g}\cdot\vec{v}_k)(\vec{v}_i\cdot\vec{x}_{ki})\vec{x}_{ki}\right\rangle ,$$
$$\vec{\omega}_3=\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{g}\cdot\vec{x}_{ki})(\vec{v}_i\cdot\vec{x}_{ki})\vec{v}_k\right\rangle ,$$
$$\vec{\omega}_4=\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^3}(\vec{v}_i\cdot\vec{v}_k)(\vec{g}\cdot\vec{x}_{ki})\vec{x}_{ki}\right\rangle , \tag{2.62}$$
$$\vec{\omega}_5=\left\langle \sum_{i,k} \frac{e_i e_k}{|\vec{x}_{ik}|^5}(\vec{v}_i\cdot\vec{x}_{ki})(\vec{v}_k\cdot\vec{x}_{ki})(\vec{g}\cdot\vec{x}_{ki})\vec{x}_{ki}\right\rangle ,$$
$$\vec{\omega}_6=\frac{{g_0}^{3/2}}{f_0}\left\langle \sum_i m_{0i}\vec{v}_i^{\,2}(\vec{g}\cdot\vec{v}_i)\vec{v}_i\right\rangle ,$$

and the multiplying factors are given by

$$\eta=\frac{1}{{g_0}^{1/2}}\left[\frac{g_0\gamma_0'}{f_0}-\mathscr{A}_0+\beta_0'\right] ,$$
$$\omega=\frac{1}{{g_0}^{1/2}}\left[\frac{g_0\gamma_0'}{f_0}+\mathscr{A}_0+\mathscr{P}_0-\frac{1}{2}\frac{f_0'}{f_0}-\frac{1}{2}\frac{g_0'}{g_0}-\delta_0'-\bar{\delta}_0'+\alpha_0'+\bar{\alpha}_0'\right] , \tag{2.63}$$
$$\epsilon_1=\frac{1}{{g_0}^{1/2}}\left[\mathscr{A}_0+\frac{1}{2}\frac{g_0'}{g_0}-\frac{1}{2}\frac{f_0'}{f_0}\right], \quad \epsilon_2=\frac{1}{{g_0}^{1/2}}\left[\mathscr{B}_0+\frac{1}{2}\frac{f_0'}{f_0}-\frac{1}{2}\frac{g_0'}{g_0}\right] ,$$
$$\xi=\frac{1}{{g_0}^{1/2}}\left[-\mathscr{D}_0-\frac{1}{2}\frac{f_0'}{f_0}+\frac{1}{2}\frac{g_0'}{g_0}+\delta_0'+\bar{\delta}_0'-\alpha_0'-\bar{\alpha}_0'-\beta_0'-\frac{g_0\gamma_0'}{f_0}\right] .$$

Finally, we remark that the $O(gv^2)$ terms $\vec{\eta}$ and $\vec{\omega}$ and the $O(gv^4)$ terms $\vec{W}_1$, $\vec{W}_2$, $\vec{W}_3$, $\vec{W}_4$, and $\vec{W}_5$ are completely independent of one another.

III. SUMMARY AND CONCLUSIONS

A. Theoretical discussion

In this paper we have taken the gravitationally generalized laws of electromagnetism (GGEM), and obtained a (predicted) expression for the quantity $\langle \vec{A}_{\text{c.m.}} \rangle$ [given by Eq. (2.60)], which is related to the acceleration of a "real" test body. In Eq. (2.60) we see that only the first (Newtonian-type) term is composition independent, all the other terms represent accelerations depending on the internal structure of the test body. Moreover, the composition-dependent accelerations $\vec{\eta}$, $\vec{\omega}$, $\vec{W}_1$, $\vec{W}_2$, $\vec{W}_3$, $\vec{W}_4$, and $\vec{W}_5$ are all independent of one another. Therefore, the only way that the WEP can be (theoretically) satisfied, so that $\langle \vec{A}_{\text{c.m.}} \rangle$ does not contain any composition-dependent terms, is for the five multiplying factors η, ω, ϵ_1, ϵ_2, and ξ to be independently zero, viz.,

$$\eta=\omega=\epsilon_1=\epsilon_2=\xi=0\ . \qquad (3.1)$$

Equation (3.1) represents conditions for the WEP to be satisfied. These results can be thought of as physical constraints on the original (mathematical) equations.

Let us analyze the results given by Eq. (3.1). We have not specified the initial location of our test body with respect to the external gravitating source. Since we wish the WEP to hold everywhere, Eq. (3.1) should be satisfied at any point the test body is deposited, and so we can remove the zero subscripts from all quantities appearing in this equation [through Eq. (2.63)].

In order to write Eq. (3.1) in full it is convenient to split the functions α, $\overline{\alpha}$, β, γ, δ, and $\overline{\delta}$ into metric and nonmetric parts (it is always possible to decompose Γ according to $\Gamma^a{}_{bc}=\{{}^a_{bc}\}+A^a{}_{bc}$, for some tensor A). The metric parts of these functions are given by (1.4). We shall use the caret notation to denote the nonmetric part of a particular function (e.g., α' can be written as $\alpha'=\frac{1}{2}g'/g+\hat{\alpha}'$—the prime denotes differentiation with respect to U). Therefore, Eq. (3.1) becomes (in full)

$$\left[\frac{g\hat{\gamma}'}{f}+\hat{\beta}'\right]-\left[\mathscr{A}-\frac{1}{2}\frac{f'}{f}+\frac{1}{2}\frac{g'}{g}\right]=0\ , \qquad (3.2a)$$

$$\left[\mathscr{A}-\frac{1}{2}\frac{f'}{f}+\frac{1}{2}\frac{g'}{g}\right]+\left[\frac{g\hat{\gamma}'}{f}+\mathscr{P}-\frac{1}{2}\frac{f'}{f}\right]-(\hat{\delta}'+\hat{\overline{\delta}}'-\hat{\alpha}'-\hat{\overline{\alpha}}')=0\ , \qquad (3.2b)$$

$$\mathscr{A}-\frac{1}{2}\frac{f'}{f}+\frac{1}{2}\frac{g'}{g}=0\ , \qquad (3.2c)$$

$$\mathscr{B}+\frac{1}{2}\frac{f'}{f}-\frac{1}{2}\frac{g'}{g}=0\ , \qquad (3.2d)$$

and

$$\mathscr{D}+\left[\frac{g\hat{\gamma}'}{f}+\hat{\beta}'\right]-(\hat{\delta}'+\hat{\overline{\delta}}'-\hat{\alpha}'-\hat{\overline{\alpha}}')=0\ . \qquad (3.2e)$$

From the structure of the calculation, the most that can be expected from the WEP is that it should give us information on the four electromagnetic-gravitational coupling functions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{D}$, and on the two quantities $(g\hat{\gamma}'/f+\hat{\beta}')$ and $(\hat{\delta}'+\hat{\overline{\delta}}'-\hat{\alpha}'-\hat{\overline{\alpha}}')$. From (3.2) we see that the six quantities are constrained by five equations, and since the GGM equations are invariant under $(f,g)\rightarrow(\mu f,\mu g)$ ("conformally invariant") these results contain all the information that could possibly be hoped for. Consequently, if we continued the calculation of $\langle\vec{A}_{\text{c.m.}}\rangle$ to higher order, no further information could be obtained on the structure of the laws of GGEM. (Moreover, the higher-order terms [i.e., $O(g^2)$ and $O(v^6)$ terms] are really outside the scope of the approximation scheme.) Therefore, as remarked earlier, we can regard the calculation to be "complete."

We can simplify (3.2) by either exploiting the "conformal invariance" of the GGM equations [and choosing a "gauge" in which $(g\hat{\gamma}'/f+\mathscr{P}-\frac{1}{2}f'/f)$ is zero], or by imposing the constraints that $(g\hat{\gamma}'/f+\hat{\beta}')$ and $(\hat{\delta}'+\hat{\overline{\delta}}'-\hat{\alpha}'-\hat{\overline{\alpha}}')$ are both zero. These two constraints are theoretically justified by (3.2) and by the desire for the equation governing the motion of photons deduced from the optical limit of the GGM equations to be equivalent to the mass$\rightarrow$0, speed$\rightarrow$1 limit of the GGL equations for an uncharged particle. (In addition, these two constraints are also supported, at least to some order of observational accuracy, by experiments that measure the deflection of light.)

Subtracting out the conditions that $(g\hat{\gamma}'/f+\hat{\beta}')$ and $(\hat{\delta}'+\hat{\overline{\delta}}'-\hat{\alpha}'-\hat{\overline{\alpha}}')$ are zero, we obtain the following four (independent) equations for $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{D}$:

$$\frac{g\hat{\gamma}'}{f}+\mathscr{P}-\frac{1}{2}\frac{f'}{f}=0\ , \qquad (3.3a)$$

$$\mathscr{A}-\frac{1}{2}\frac{f'}{f}+\frac{1}{2}\frac{g'}{g}=0\ , \qquad (3.3b)$$

$$\mathscr{B}+\frac{1}{2}\frac{f'}{f}-\frac{1}{2}\frac{g'}{g}=0\ , \qquad (3.3c)$$

and

$$\mathscr{D}=0\ . \qquad (3.3d)$$

We can rewrite the relations (3.2)/(3.3) in the following way:

$$\frac{1}{2}(\alpha'+\overline{\alpha}'-\delta'-\overline{\delta}')=\mathscr{B}=-\mathscr{A}=\frac{1}{2}\left\{\frac{(\mu g)'}{(\mu g)}-\frac{(\mu f)'}{(\mu f)}\right\},$$

$$\mathscr{D}=0,\quad \beta'=-\frac{1}{2}\frac{(\mu g)'}{(\mu g)}\ ,\qquad \gamma'=\frac{1}{2}\frac{(\mu f)'}{(\mu g)},\quad \mathscr{P}=\frac{1}{2}\frac{(\mu^{-1}f)'}{(\mu^{-1}f)}\ , \qquad (3.4)$$

where μ is a scalar field defined by $\mu'/2\mu=g\hat{\gamma}'/f$. From the forms of $\mathscr{A}$ and $\mathscr{B}$ in (3.4) we see that the GGM equations take on a "metric" form with respect to μg_{ab} [this of course follows from Eqs. (3.2c) and (3.2d) and the "conformal invariance" of the GGM equations]. From the forms of $\mathscr{P}$ and $\mathscr{D}$ in (3.4) and Eqs. (2.4), (2.7), and (2.8), we observe that L is of the form $L=(\mu^{-1}f-\mu^{-1}g\vec{v}^2)^{1/2}$. L

occurs in Eq. (2.1) through the terms $(1/f)L$ and $(1/g)L$, which can now be written as $[1/(\mu f)](\mu f-\mu g\vec{v}^2)^{1/2}$ and $[1/(\mu g)](\mu f-\mu g\vec{v}^2)^{1/2}$. From (3.4) we observe that all other terms in (2.1) representing the gravitational field can be written as μ multiplied by the appropriate metric form. Consequently, we have shown that relations (3.4) are precisely those for which the GGL equations take on a "metric" form with respect to μg_{ab}. Therefore, the WEP demands that the laws of GGEM must take on a "metric" form with respect to a tensor conformally related to g_{ab}.

There are two ways to view the results above. First, if we analyze the work outlined in this paper in isolation, we have considered a class of relativistic theories of gravity characterized by a set of laws of GGEM [represented by Eqs. (2.1), (2.18), (2.19), and (2.20)], and shown that the WEP demands that these laws of GGEM be metric with respect to μg_{ab}. Moreover, since μg_{ab} is the only object occurring in the laws under investigation, it takes on the role of "physical metric"; that is, it is μg_{ab} that is interpreted to be of physical importance in the laws under consideration. Thus we could state the results of this paper as follows: *For the class of relativistic theories of gravity under investigation, the WEP demands that the laws of GGEM (in this class of theories) must take on their metric form.*

On the other hand, we could attempt to view the results above in the context of an overall analysis of nonmetric theories of gravity, whose structure would include physical laws other than just the laws of GGEM investigated here. In such an analysis, g_{ab} may take on a role of physical importance (for example, g_{ab} may be of physical significance in the measuring process—see Ref. 4). We have not shown that the laws of GGEM under investigation here must take on their metric form with respect to g_{ab}. To show this we must prove $\hat{\gamma}'=0$, or, alternatively, $\mu=$constant, results which cannot in general be obtained from the WEP due to the "conformal invariance" of the laws of GGEM. Further analysis must therefore be done outside the framework of the WEP. Immediately we see that for a nonzero $\hat{\gamma}'$ the Einstein equivalence principle is broken. Evidence to prove that $\hat{\gamma}'$ is zero would come from an analysis of solar system experiments involving test particle motions and clock measurements (which will verify the result up to the PN order of approximation), and from an analysis involving the consistency of gravitational red-shift experiments. This analysis will be presented in a later paper.

As we mentioned above, in general no further information can be obtained from the above analysis. However, if we investigate some particular cases (that is, theories in which the laws of GGEM are more specific than the general case considered up to now), additional, more specialized information may be obtained. From Eqs. (3.3) we observe that $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{D}$ must take on their metric form with respect to g_{ab}, while, due to the presence of the $\hat{\gamma}'$ term, $\mathscr{P}$ need not. We could interpret Eq. (3.3a) in other ways. First, in the context of MTG's, with GGL equations of the form

$$\frac{d^2x^\mu}{dt^2}+\left\{{}^{\mu}_{bc}\right\}\frac{dx^b}{dt}\frac{dx^c}{dt}-\left\{{}^{0}_{bc}\right\}\frac{dx^\mu}{dt}\frac{dx^b}{dt}\frac{dx^c}{dt}=\frac{e}{m}L(\vec{x},\vec{v})\left[F_n{}^\mu\frac{dx^n}{dt}-F_n{}^0\frac{dx^\mu}{dt}\frac{dx^n}{dt}\right], \quad (3.5)$$

we could calculate the conditions the WEP imposes on L. In this case $\hat{\gamma}'=0$, and so $\mathscr{P}=\frac{1}{2}f'/f$. Therefore, for MTG's equations (3.3) imply that the laws of GGEM take on their metric form. Indeed for any class of theories with $\hat{\gamma}'=0$, the WEP implies the laws of GGEM take on their metric form.

Second, for a theory with given GGL equations (that is, with L, and thus $\mathscr{P}$ and $\mathscr{D}$, specified), we could calculate the conditions the WEP then imposes on the remaining nonmetric coupling functions. That is, we regard (3.3a) as an equation for $\hat{\gamma}'$, in terms of the given function $\mathscr{P}$, viz.,

$$\hat{\gamma}'=\frac{f}{g}\left[\frac{1}{2}\frac{f'}{f}-\mathscr{P}_{\text{given}}\right]. \quad (3.6)$$

In Sec. II A, we considered two examples of possible GGL equations (which, in fact, we regard as the two most "reasonable" possibilities). What information does Eq. (3.6) yield in these two examples?

(a) In the first example $L=d\tau/dt$. In this case $\mathscr{P}$ is given by [from Eq. (2.8)]

$$\mathscr{P}_{(a)}=\frac{1}{2}\frac{f'}{f}, \quad (3.7)$$

so that Eq. (3.6) yields

$$\hat{\gamma}'_{(a)}=0. \quad (3.8)$$

(b) In the second example $L=d\lambda/dt$. In this case $\mathscr{P}$ is given by [from Eq. (2.17)]

$$\mathscr{P}_{(b)}=\hat{\delta}'_{(b)}+\hat{\bar{\delta}}'_{(b)}-\frac{g}{f}\hat{\gamma}'_{(b)}+\frac{1}{f}\frac{f'}{f}, \quad (3.9)$$

and Eq. (3.6) yields

$$\hat{\delta}'_{(b)}+\hat{\bar{\delta}}'_{(b)}=0. \quad (3.10)$$

Equations (3.8) and (3.10) represent "strong" constraints on the possible form of $\hat{\Gamma}$.

We have shown that the WEP (theoretically) implies that the GGEM equations take on a metric form in an SSS gravitational field. However, it is theoretically conceivable (for a particular theory of gravity), that the GGEM equations are metric in the SSS approximation, but nonmetric in other, more general, cases. Therefore, we cannot conclude that the GGEM equations must be metric in general. Nonetheless, the calculation does represent a very severe constraint on the form of the most general GGEM equations.

In our calculation we have considered only "electromagnetic" test bodies. Since we wish to apply our results to real test bodies consisting of actual atoms, subject to nuclear interactions as well as the laws of electromagnetism, we should include in $\langle \vec{A}_{\text{c.m.}} \rangle$ terms which involve nuclear energies. We should then ask whether it is theoretically possible for the nuclear terms to cancel the electromagnetic terms. Lightman and Lee[2] considered this question, and concluded that there is no credible mechanism that could lead to such a cancellation. (If nuclear energies were included in the calculation, the WEP could then be used to constrain the possible form of gravitational-nuclear interactions.)

B. Experimental implications

Experimental support for the WEP comes from the so-called Eötvös experiments, which measure the relative acceleration toward the Sun of two different substances. The two experiments of highest precision were performed in Princeton[8] and Moscow.[9] In the Princeton experiment, gold and aluminum were used as the test substances, with the (null) result

$$\left| \frac{\langle \vec{A}_{\text{c.m.}} \rangle_{\text{Al}} - \langle \vec{A}_{\text{c.m.}} \rangle_{\text{Au}}}{\langle \vec{A}_{\text{c.m.}} \rangle} \right| \simeq \frac{|\langle \vec{A}_{\text{c.m.}} \rangle_{\text{Al}} - \langle \vec{A}_{\text{c.m.}} \rangle_{\text{Au}}|}{|\vec{g}|} < 10^{-11} . \quad (3.11)$$

In the Moscow version of the experiment, platinum and aluminum were used to obtain the result

$$\frac{|\langle \vec{A}_{\text{c.m.}} \rangle_{\text{Al}} - \langle \vec{A}_{\text{c.m.}} \rangle_{\text{Pt}}|}{|\langle \vec{A}_{\text{c.m.}} \rangle|} < 10^{-12} . \quad (3.12)$$

Let us investigate to what order the Eötvös experiments test the GGEM equations.[10] We note that the composition-dependent accelerations $\vec{\eta}$, $\vec{\omega}$, $\vec{\eta}_1$, $\vec{\eta}_2$, $\vec{\omega}_1$ $\vec{\omega}_2$, $\vec{\omega}_3$, $\vec{\omega}_4$, $\vec{\omega}_5$, and $\vec{\omega}_6$ [defined by (2.61) and (2.62)] occurring in $\langle \vec{A}_{\text{c.m.}} \rangle$ [as given by Eq. (2.60)] are, in general, completely independent of one another. But in any given experiment they may be related in some way. In particular, the bodies used in the aforementioned Eötvös experiments were approximately spherical (at least in the time-averaged sense), so that the above composition-dependent accelerations satisfy the following conditions:

$$\begin{aligned} &\tfrac{1}{2}\vec{\eta} = \tfrac{3}{2}\vec{\omega} = \vec{g}E_e , \\ &\tfrac{1}{4}\vec{\eta}_1 = \tfrac{3}{4}\vec{\omega}_1 = \tfrac{3}{4}\vec{\omega}_4 = \vec{g}E_{m(1)} , \\ &\tfrac{1}{4}\vec{\eta}_2 = \tfrac{3}{4}\vec{\omega}_2 = \tfrac{3}{4}\vec{\omega}_3 = \tfrac{3}{4}\vec{\omega}_5 = \vec{g}E_{m(2)} , \\ &\tfrac{1}{2}\vec{\omega}_6 = \vec{g}E_{m(3)} , \end{aligned} \quad (3.13)$$

where E_e, $E_{m(1)}$, $E_{m(2)}$, and $E_{m(3)}$ are defined in the above relations. Using (3.13), we can then write $\langle \vec{A}_{\text{c.m.}} \rangle$ as

$$\langle \vec{A}_{\text{c.m.}} \rangle = -\gamma'\vec{g} + \frac{E_e}{m}[2(\eta + \tfrac{1}{3}\omega)]\vec{g} + \frac{E_{m(1)}}{m}(\eta + \tfrac{1}{3}\omega + \tfrac{2}{3}\epsilon_2)\vec{g} + \frac{E_{m(2)}}{m}(\eta + \tfrac{1}{3}\omega + \tfrac{4}{3}\epsilon_1 + \tfrac{2}{3}\epsilon_2)\vec{g} + \frac{E_{m(3)}}{m}(2\xi)\vec{g} \quad (3.14)$$

From this equation we can now interpret E_e as the electrostatic self-energy of the body, and the E_m's as related to the magnetostatic self-energy of the body.

Since $\langle \vec{A}_{\text{c.m.}} \rangle$ has been calculated to all orders in U, in order to compare (3.14) with experiment all functions of the gravitational field must be expanded in powers of U. Using the numerical values for the differences between the electrostatic and magnetostatic self-energies of aluminum and platinum estimated in Refs. 2 and 3 and a numerical estimate for U (for Earth-bound experiments), we can now use (3.12) and (3.14) to calculate the experimental limits [on the arbitrary functions of the gravitational field in (3.14)] by demanding that each composition-dependent term (to each power of U) must separately satisfy the experimental lower limit obtained from the Moscow version of the experiment.

The actual experimental constraints obtained are rather complicated and will not be reproduced here (although the results are available). The results essentially amount to the following. Equations (3.3a), (3.3b), (3.3c), and (3.3d), for $\mathscr{P}$, $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{D}$, respectively, are experimentally verified to first order in U (very strongly). $\mathscr{P}$ is weakly constrained to second order in U. Eötvös experiments must improve in accuracy by three orders of magnitude to significantly constrain $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{D}$ to second order in U. We conclude that experimental evidence supports the postulate that the GGEM equations are of a metric form.

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[1]A. A. Coley, Ph.D. thesis, University of London, 1980 (unpublished).

[2]A. P. Lightman and D. L. Lee, Phys. Rev. D 8, 364 (1973).

[3]M. P. Haugan and C. M. Will, Phys. Rev. D 15, 2711 (1977).

[4]A. A. Coley, Phys. Rev. D (to be published).

[5]A. A. Coley, Phys. Rev. D 27, 728 (1983).

[6]K. S. Thorne, D. L. Lee, and A. P. Lightman, Phys. Rev. D 7, 3564 (1973).

[7]See Ref. 1 for a discussion on why this definition (of the center of mass of the composite test body) is used.

[8]P. G. Roll, R. Krotkov, and R. H. Dicke, Ann. Phys. (N.Y.) 26, 442 (1964).

[9]V. B. Braginski and V. I. Panov, Zh. Eksp. Theor. Fiz. 61, 873 (1971) [Sov. Phys. JETP 34, 463 (1972)].

[10]To see that a direct comparison can be made between the expression for $\langle \vec{A}_{\text{c.m.}} \rangle$ (obtained in the calculation) and the experimental results see Ref. 2.