Qualitative analysis of soft inflation

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In this paper we perform a qualitative analysis of the dynamical system resulting from the soft inflation scenario. This allows us to examine the various types of asymptotic behavior displayed by the system; we pay particular attention to the inflationary solutions given by Berkin, Maeda, and Yokoyama [Phys. Rev. Lett. **65**, 141 (1990)]. We find that as $\phi\rightarrow\infty$, no unique attracting critical point exists, but rather there exists a higher-dimensional set of critical points. It is shown that the solution given by Berkin, Maeda, and Yokoyama is representative of a class of solutions which asymptotically undergoes power-law inflation. Under the assumption that new inflation occurs, we show that, asymptotically, there exists a global attractor. Under the assumption that chaotic inflation occurs, we show that there exists an attractor for finite values of the field ϕ and that solutions which inflate will experience infinitely many inflationary eras.

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I. INTRODUCTION

Inflationary cosmology was originally investigated in the hope that some outstanding problems in cosmology might be solved. However, to date there is no fully acceptable model for the source of inflation. In a recent paper [1], a *soft inflationary* scenario was proposed in which the matter content is described by two coupled scalar fields, one of which has a decaying potential and the other which serves as the inflaton during expansion [1,2]. Inflation with two scalar fields has been considered previously [3]. However, the effect of the decaying exponential potential in soft inflation is to reduce the rate of inflation in a manner similar to that in *extended inflation* [4]. As the inflaton rolls down a flat plateau, the second scalar field evolves on the exponential potential resulting in power-law inflation [5]. The advantages of soft inflation are (i) when the inflaton is of new inflation type [6] the fine-tuning of initial conditions is lessened and density perturbations are suppressed, and (ii) when the inflaton is of *chaotic inflation* type [7], the restrictions placed upon the coupling parameter are reduced considerably. Thus soft inflation allows the constraints placed on previous models to be loosened.

In this article we shall show that the field equations governing soft inflation can be written as a dynamical system allowing us to analyze in a mathematically rigorous way the evolution of the model and the corresponding asymptotic behaviors.

The action under investigation is

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2}-\frac{1}{2}(\nabla\phi)^2-\frac{1}{2}(\nabla\psi)^2-e^{-\beta\kappa\phi}V(\psi)\right], \tag{1.1}$$

where $\kappa^2=8\pi G$, ϕ and ψ (the inflaton) are scalar fields, $V(\psi)$ is a potential, and β is the coupling constant. Variation of the action in a flat Friedmann-Robertson-Walker (FRW) universe yields the following set of nonlinear second-order ordinary differential equations:

$$\ddot{\phi}+3H\dot{\phi}-\beta\kappa e^{-\beta\kappa\phi}V(\psi)=0, \tag{1.2a}$$

$$\ddot{\psi}+3H\dot{\psi}+e^{-\beta\kappa\phi}\frac{dV(\psi)}{d\psi}=0, \tag{1.2b}$$

where the constraint equation is

$$H^2=\frac{\kappa^2}{3}\left[\frac{1}{2}(\dot{\phi})^2+\frac{1}{2}(\dot{\psi})^2+e^{-\beta\kappa\phi}V(\psi)\right], \tag{1.3}$$

and where an overdot denotes differentiation with respect to time and $H=\dot{a}/a$ is the Hubble parameter where a is the length scale.

Berkin *et al.* [1] have found a unique stable power-law inflationary solution as the field $\phi\rightarrow+\infty$. The solution is given by

$$\begin{aligned}&\kappa\phi=\kappa\phi_c+(2/\beta)\ln(t/\kappa),\\&a=a_0(t/t_0)^{2/\beta^2},\\&f(\psi)=f(\psi_0)-(1-\beta^2/6)\ln(a/a_0),\end{aligned} \tag{1.4}$$

where for a general potential V we have

$$\exp(\beta\kappa\phi_c)\equiv\frac{\beta^4\kappa^4V(\psi)}{12(1-\beta^2/6)},$$

$$f(\psi)\equiv\kappa^2\int d\psi\frac{V}{V'}.$$

We wish to investigate whether this solution is generic. This can be done using qualitative techniques of dynamical systems theory.

First we define new independent and dependent variables:

$$\frac{dt}{d\tau}=e^{\beta\kappa\phi/2}, \tag{1.5a}$$

$$\Phi=\phi'=\dot{\phi}e^{\beta\kappa\phi/2}, \tag{1.5b}$$

$$\Psi=\psi'=\dot{\psi}e^{\beta\kappa\phi/2}, \tag{1.5c}$$

where a prime denotes differentiation with respect to the new time τ. Calculating both Φ' and Ψ', the resulting equations form a four-dimensional autonomous system of ordinary differential equations:

$$\phi'=\Phi, \tag{1.6a}$$

$$\psi'=\Psi, \tag{1.6b}$$

$$\Phi'=\frac{\beta\kappa}{2}\Phi^2-\frac{\sqrt{6}\kappa}{2}\Phi[\Phi^2+\Psi^2+2V(\psi)]^{1/2}+\beta\kappa V(\psi), \tag{1.6c}$$

$$\Psi'=\frac{\beta\kappa}{2}\Psi\Phi-\frac{\sqrt{6}\kappa}{2}\Psi[\Phi^2+\Psi^2+2V(\psi)]^{1/2}-\frac{dV(\psi)}{d\psi}, \tag{1.6d}$$

where the constraint equation is

$$H^2=e^{-\beta\kappa\phi}\frac{\kappa^2}{6}[\Phi^2+\Psi^2+2V(\psi)]. \tag{1.7}$$

We see that the singular points (defined by $\phi'=\psi'=\Phi'=\Psi'=0$) at finite values are given by

$$\Phi=0,\quad \Psi=0,\quad V(\psi)=0,\quad \frac{dV(\psi)}{d\psi}=0. \tag{1.8}$$

We note that Eqs. (3.5)–(3.7) in Berkin and Maeda [2] therefore hold at all finite singular points in the full system. Further analysis depends on the chosen form of the potential $V(\psi)$. In this work we shall consider potentials arising from both new and chaotic inflation.

II. CHAOTIC INFLATION

In chaotic inflation we choose the potential $V(\psi)=(\lambda_n/n)\psi^n$ where n is even and λ_n is constant [2]. In particular, here we consider $n=2$. From above we see that for finite values of ϕ, we have a nonisolated line singularity along the ϕ axis $(\phi_0,0,0,0)$. Linearizing about the singular line we find that all eigenvalues have $\mathrm{Re}(\lambda)=0$ and hence all critical points are "nonlinear." We note that the system (ψ,Φ,Ψ) is independent of ϕ. Thus for each $\phi=\phi_0$ we need only consider a three-dimensional system to determine the qualitative behavior. Progress is made by converting to cylindrical coordinates:

$$\Psi=r\cos\theta, \tag{2.1a}$$

$$\psi=\frac{r}{\sqrt{\lambda_2}}\sin\theta, \tag{2.1b}$$

$$\Phi=z. \tag{2.1c}$$

The inverse transformation is

$$r^2=\Psi^2+\lambda_2\psi^2, \tag{2.2a}$$

$$\theta=\arctan\left[\frac{\sqrt{\lambda_2}\psi}{\Psi}\right], \tag{2.2b}$$

$$z=\Phi. \tag{2.2c}$$

The equations then become (hereafter dropping the subscript on λ for convenience)

$$r'=r\cos^2\theta\frac{\kappa}{2}[\beta z-\sqrt{6}(z^2+r^2)^{1/2}], \tag{2.3a}$$

$$\theta'=\sqrt{\lambda}-\cos\theta\sin\theta\frac{\kappa}{2}[\beta z-\sqrt{6}(z^2+r^2)^{1/2}], \tag{2.3b}$$

$$z'=z\frac{\kappa}{2}[\beta z-\sqrt{6}(z^2+r^2)^{1/2}]+\frac{\beta\kappa}{2}r^2\sin^2\theta. \tag{2.3c}$$

It can be shown that if $\beta<\sqrt{6}$ [1] then

$$\beta z-\sqrt{6}(z^2+r^2)^{1/2}\le 0. \tag{2.4}$$

Hence we can see that $r'\le 0$ everywhere. We define the compact set

$$S=\{(r,\theta,z)\,|\,r\le\epsilon,\,-1\le z\le 1\},$$

where $\epsilon^2+1=6/\beta^2$. On the boundary $r=\epsilon$, for $(-1\le z\le 1)$, $r'\le 0$. On the boundary $z=-1$, for $(0\le r\le\epsilon)$, it is easily seen that $z'\ge 0$. On the boundary $z=1$, for $(0\le r\le\epsilon)$, after some algebra it can be seen that $z'\le 0$. Hence the set S is a positively invariant compact set in $\mathcal{R}^3$; i.e., it is a trapping set. We can choose $V=r$ as a Lyapunov function. Since $r'\le 0$ everywhere, $V'=r'\le 0$ everywhere in S. Then, by the global Lyapunov theorem [8], we have that

$$\forall a\in S,\quad \omega(a)\subseteq W=\{x\in S\,|\,\dot{V}(x)=0\}, \tag{2.5}$$

where

$$W=\left\{(r=0),\ \left[\theta=\frac{\pi}{2}\right],\ \left[\theta=-\frac{\pi}{2}\right],(r=0,z=0)\right\}. \tag{2.6}$$

But the omega-limit set of a, $\omega(a)$, is the union of complete orbits. The only whole orbit in W is the singular point $\{r=0,z=0\}$. Hence the $\omega(a)$ for any point a in the trapping set S is the singular point. Therefore, for each $\phi=\phi_0$ the singular point $(\phi_0,0,0,0)$ is a sink.

Let us now consider the conditions for these singular points to be inflationary. Using the fact that $\dot{a}=aH$ and the appropriate coordinate transformations we calculate

$$\frac{\ddot{a}}{a}=e^{-\beta\kappa\phi}\frac{\kappa^2}{3}\left[\frac{\lambda}{2}\psi^2-\Phi^2-\Psi^2\right]. \tag{2.7}$$

For inflation to occur $\ddot{a}/a$ must be greater than zero. Hence the condition for which inflation takes place is given by

$$\frac{\lambda}{2}\psi^2-\Phi^2-\Psi^2>0. \tag{2.8}$$

This inequality describes the interior of a cone aligned along the ψ axis. So any orbits inside the cone will experience an acceleration in their expansion. We note that the apex of the cone is at the singular point (see Fig. 1).

Let us define the compact set

$$S=\{(r,\theta,z)\,|\,r\le\epsilon,-\epsilon\le z\le\epsilon\}\ ,$$

where

$$\epsilon=\frac{2\sqrt{\lambda}}{\kappa(\beta+\sqrt{12})}>0\ .$$

After some algebra it can be shown that $\theta'>0$ inside S. Along with the fact that $r'\le 0$, this shows that orbits spiral around the singular point infinitely many times in a sufficiently small neighborhood of the singular point. Thus for any point $\{a\}$ in the inflationary regime, we can show that the orbit through $\{a\}$ will eventually leave the inflationary regime.

We can show that if $\beta<\sqrt{2}$ then every orbit (except the orbit $r=0$) that enters the inflationary regime will do so infinitely many times as it spirals its way to the singular point. We choose

$$S=\left\{(r,\theta,z)\,|\,r\le\epsilon,|z|\le\frac{r}{\sqrt{2}}\right\}$$

where $\epsilon>0$. The set S is a compact set that contains part of the inflationary region in such a way that at $\theta=\pi/2$ or $3\pi/2$ the inflationary cone is bounded by S. On the boundary $r=\epsilon$ we have that $r'\le 0$. The differential equation (2.3) defines a vector field $\vec{v}$ on the surfaces $z=\pm r/\sqrt{2}$ with inward normals (with respect to S) $\vec{n}_+$ and $\vec{n}_-$. It can be shown that if $\beta<\sqrt{2}$, that $\vec{n}\cdot\vec{v}\ge 0$, so the angle between $\vec{n}$ and $\vec{v}$ is less than 90° which implies that the vector field $\vec{v}$ is directed into S, that is trajectories are flowing into the set S. Hence for $\beta<\sqrt{2}$ the set S is a trapping set, and thus any orbit that enters S must also enter the inflationary regime infinitely many times as it spirals its way to the singular point.

We are also interested in the behavior of the field ϕ at infinity [9]. By making use of a Poincaré-like transformation (and a new time transformation) given by

$$x=\frac{1}{\phi},\quad u=\frac{\psi}{\phi},\quad v=\frac{\Phi}{\phi},\quad w=\frac{\Psi}{\phi},\quad \left[\frac{d\tau}{d\bar{\tau}}=\frac{1}{\phi}\right],\qquad (2.9)$$

the transformed set of equations become

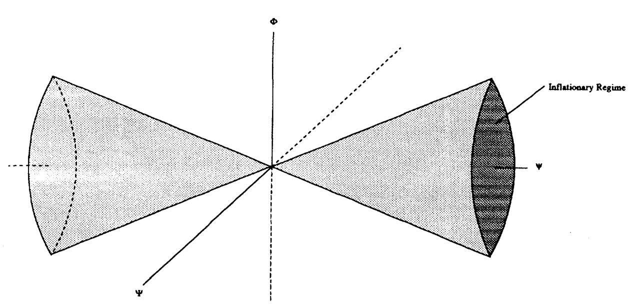

FIG. 1. The cone describes the inflationary regime defined by Eq. (2.8).

$$\dot{x}=-vx^2\ ,\qquad (2.10a)$$

$$\dot{u}=x(w-uv)\ ,\qquad (2.10b)$$

$$\dot{v}=\frac{\beta\kappa}{2}v^2-\frac{\sqrt{6}\kappa}{2}v(v^2+w^2+\lambda u^2)^{1/2}-xv^2+\frac{\beta\kappa}{2}\lambda u^2\ ,\qquad (2.10c)$$

$$\dot{w}=\frac{\beta\kappa}{2}vw-\frac{\sqrt{6}\kappa}{2}w(v^2+w^2+\lambda u^2)^{1/2}-xvw-\lambda ux\ .\qquad (2.10d)$$

We are interested in the singular points on the hypersurface $x=0$ (that is, as $\phi\to+\infty$, $x\to 0^+$). We note that the set $x=0$ is an invariant set. Thus the problem becomes less difficult because $x=0$ divides the phase space into three invariant sets.

In the set $x=0$ we find that the critical values depend on the value of the parameter β and are given by

$$\begin{array}{lllll} u=u_0, & v=v_0=\left[\dfrac{\lambda}{(6-\beta^2}\right]^{1/2}\beta|u_0|, & & w=0, & \beta<\sqrt{6},\\ u=0, & v=v_0, & & w=0, & \beta=\sqrt{6},\\ u=0, & v=v_0=\left[\dfrac{6}{\beta^2-6}\right]^{1/2}|w_2|, & & w=w_0, & \beta>\sqrt{6}. \end{array}\qquad (2.11)$$

Note that in each case the critical points are again nonisolated. Motivated by Berkin *et al.* [1], hereafter we shall consider the case $\beta<\sqrt{6}$.

Linearizing the system about the singular line we find that the eigenvalues are

$$\lambda_1=0,\quad \lambda_2=\lambda_3=-\frac{\kappa}{2}\sqrt{\lambda(6-\beta^2)}|u_0|<0\ .\qquad (2.12)$$

There exists a center manifold which is tangent to the eigenvector associated with the zero eigenvalue, namely $v=w=0$; the u axis is a center manifold for all u_0. The nonlinear system is thus topologically equivalent in a neighborhood of the singular point to the linear system restricted to the center manifold [10]. We also note that $u=u_0$ is a two-dimensional invariant set, and thus we have effectively foliated the phase space and now need only consider the planar system (v,w) (with parameter u_0). For $u_0\neq 0$, the singular point is hyperbolic and so (by the Hartman-Grobman theorem [10]) the nonlinear system is topologically equivalent to the linear system which is an attracting stellar node. For $u_0=0$, the equations can be integrated exactly and the same behavior results. Thus inside the invariant set $x=0$ the singular line is a sink.

We note that $\dot{x}=-vx^2$ and that the singular line has positive v coordinate. Hence in any sufficiently small neighborhood of the singular line $\dot{x}<0$, so in the set $x>0$ the orbits are attracted to the line. However, in the set $x<0$ orbits are repeled away from the line. Thus as $\phi\to+\infty$ the singular line is a sink and as $\phi\to-\infty$ the singular line is a source.

We next consider whether these singular points at positive infinity are inflationary. In the appropriate coordinates we have that

$$\frac{\ddot{a}}{a}=e^{-\beta\kappa\phi}x^{-2}\left[\frac{\lambda}{2}u^2-v^2-w^2\right] . \quad (2.13)$$

We note that in a neighborhood of the singular point the condition

$$\frac{\lambda}{2}u^2-v^2-w^2>0 \quad (2.14)$$

must hold true if inflation is to occur. Equation (2.14) represents a cone along the u axis. If in any $u=u_0$ stable manifold we substitute the coordinate values of the singular point into the condition (2.14) we find that in order to have inflation $\beta<\sqrt{2}$ (which is precisely the same condition given in Berkin *et al.* [1] to guarantee power-law inflation). Thus there is a neighborhood about the singular line such that it is a stable attractor and for $\beta<\sqrt{2}$ it is also inflationary.

III. NEW INFLATION

We consider a potential of the form $V(\psi)=V_0-(\lambda/4)\psi^4$, which gives rise to new inflation [2]. From Eq. (1.7) we see that there are no critical points at a finite distance from the origin, since their existence would imply that $V_0=0$, which would not lead us to new inflation but to chaotic inflation. However, the interesting behavior in these models occurs when the field $\phi\to\infty$ [9]. By making use of the Poincaré-like transformation as before [see Eq. (2.9)], except with a new time transformation

$$\frac{d\tau}{d\overline{\tau}}=\frac{1}{\phi^3} , \quad (3.1)$$

we are able to examine the phase portraits as $\phi\to\infty$. The transformed set of equations become

$$\dot{x}=-vx^4 , \quad (3.2a)$$

$$\dot{u}=x^3(w-uv) , \quad (3.2b)$$

$$\dot{v}=\frac{\beta\kappa}{2}v^2x^2-\frac{\sqrt{6}\kappa}{2}vx\left[v^2x^2+w^2x^2-\frac{\lambda}{2}u^4+2V_0x^4\right]^{1/2} -v^2x^3-\frac{\beta\kappa}{4}\lambda u^4+\beta\kappa V_0x^4 , \quad (3.2c)$$

$$\dot{w}=\frac{\beta\kappa}{2}vwx^2-\frac{\sqrt{6}\kappa}{2}wx\left[v^2x^2+w^2x^2-\frac{\lambda}{2}u^4+2V_0x^4\right]^{1/2} -vwx^3+\lambda u^3x . \quad (3.2d)$$

We look for critical points on the equator of the Poincaré sphere given by $x=0$. The critical points form a two-dimensional submanifold of the phase space given by

$$x=0,\quad u=0,\quad v=v_0,\quad w=w_0 , \quad (3.3)$$

where v_0 and w_0 are constants. In order to analyze the stability of this submanifold, we first notice that Eqs. (1.2) and (1.3) yield

$$\dot{H}=-\frac{\kappa^2}{2}(\dot{\phi}^2+\dot{\psi}^2)\le 0 , \quad (3.4)$$

where the overdot denotes differentiation with respect to the original time coordinate. Thus H can be considered as a Liapunov function [10].

We note that $H(x)$ is a C^1 scalar function, and that

$$\dot{H}=-\frac{\kappa^2}{2}(\dot{\phi}^2+\dot{\psi}^2)=-\frac{\kappa^2}{2}\left[\frac{v^2+w^2}{x^2}\right]e^{-\beta\kappa/x} . \quad (3.5)$$

We let Ω_l designate a bounded component of the region $H(x)<l$; then within Ω_l, $\dot{H}(x)\le 0$. We see that R, the set of all points within Ω_l where $\dot{H}(x)=0$, is given by

$$R=\{(x,u,0,0)\} . \quad (3.6)$$

From Eqs. (3.2), we find that the set M, the largest invariant set in R, is given by

$$M=\{(0,0,0,0)\} , \quad (3.7)$$

that is, M consists of a single point, the origin. Therefore, by application of a theorem by La Salle and Lefschetz [11] (to H), we have that every solution of the above system in Ω_l tends to the origin as $t\to+\infty$; i.e., the origin is asymptotically stable.

For the solutions to be of inflationary type, i.e., $\ddot{a}/a>0$, we require that

$$\frac{\ddot{a}}{a}=-\frac{\kappa^2}{3}[\Phi^2+\Psi^2-V(\psi)]>0 , \quad (3.8)$$

which yields, in terms of the new variables,

$$v^2x^2+w^2x^2+\frac{\lambda}{4}u^4-V_0x^4<0 . \quad (3.9)$$

In addition, the physical region is defined by

$$v^2x^2+w^2x^2-\frac{\lambda}{2}u^4+2V_0x^4\ge 0 . \quad (3.10)$$

We wish to investigate whether there are trajectories in the physical region satisfying the inflationary condition. We note that the (new) variables are not dimensionless and that typical values of the (dimensional) parameters V_0 and λ are of the order of $V_0\sim 10^{75}$ g cm^{-3} and $\lambda\sim\frac{1}{2}$ [12]. This indicates that there are indeed regions within physical phase space close to the singular point in which the trajectories are inflationary.

IV. CONCLUSIONS

We have used the geometrical techniques of dynamical systems theory to investigate the generic behavior of the differential equations resulting from soft inflation. We found that as the field $\phi\to+\infty$ there exists no unique critical point which can act as an attractor, but a one- or two-dimensional submanifold of critical points; consequently the nature of these submanifolds needed further analysis. For the chaotic inflation case, we found that for finite values of ϕ_0 there does not exist an asymptotically stable inflationary solution, but for $\beta<\sqrt{2}$ there exist trajectories that enter the inflationary regime infinitely many times. Also, as the field $\phi\to+\infty$, for $\beta<\sqrt{2}$ there exist regions $U \subset \mathcal{R}^4$ such that for any initial point in U the orbit asymptotically approaches a stable singular point evolving through some inflationary regime as it approaches the singular point; hence the solution given by Berkin *et al.* [1] is representative of a class of solutions in which the model undergoes power-law inflation as $\phi \to +\infty$. For the new inflation case, we found that there exist no critical points for finite values of the field ϕ, but as the field $\phi \to +\infty$, the two-dimensional submanifold of critical points contains a global attractor at the origin and that there are regions within physical phase space (close to the singular point) in which the trajectories are inflationary.

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