The Interaction of Water Waves with Submerged Spheres and Circular Cylinders

by

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Dedicated to

My Father **Lakshmi Nandan Bora**

My Mother **Madhuri Bora**

and

My Wife **Swapnali (Bula)**

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a :	radius of the sphere
A :	wave amplitude
b :	radius of the cylinder
C_{ij} :	restoring coefficient
d :	depth of water
f_x :	total force in x-direction
f_{x1} :	surge radiation force
f_{xd} :	exciting force in x-direction
f_z :	total force in z-direction
f_{z3} :	heave radiation force
f_{zd} :	exciting force in z-direction
g :	acceleration due to gravity
h :	depth of the center of sphere from the free surface
H :	distance between the center of the sphere and the bottom of the surface

$H_m^{(1)}(.)$:	Hankel function of 1st kind
$H_m^{(2)}(.)$:	Hankel function of 2nd kind
i :	$= \sqrt{-1}$.
$I_m(.)$:	modified Bessel function
$J_m(.)$:	Bessel function of 1st kind
k_0 :	finite depth wave number
K :	$= \frac{\sigma^2}{g}$
K_1 :	Bessel function with imaginary argument
P :	pressure
$P_n(.)$:	Legendre polynomial of 1st kind
$P_n^m(.)$:	associated Legendre polynomial of 1st kind
(r, θ, z) :	cylindrical coordinates
(r, θ, ψ) :	spherical coordinates
R :	$= \sqrt{x^2 + y^2}$
t :	time
(x, y, z) :	Cartesian coordinates
X_1 :	surge displacement
X_3 :	heave displacement
X_5 :	pitch displacement
η :	water elevation
η_l :	linear elevation (for cylinder)

η_q : quadratic elevation (for cylinder)

λ_{11} : surge damping coefficient

λ_{33} : heave damping coefficient

μ_{11} : surge added-mass

μ_{33} : heave added-mass

ϕ : complex velocity potential

ϕ_l : complex linear potential (for cylinder)

ϕ_q : complex quadratic potential (for cylinder)

ϕ_D : diffraction velocity potential

ϕ_I : incident velocity potential

ϕ_S : scattered potential (for cylinder)

ϕ_1 : surge velocity potential

ϕ_3 : heave velocity potential

ϕ_5 : pitch velocity potential.

Φ : real velocity potential

Φ_l : real linear potential (for cylinder)

Φ_q : real quadratic potential (for cylinder)

π : circumference/diameter

ρ : density of water

σ : wave frequency

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Abstract

The evaluation of hydrodynamic coefficients and loads on submerged or floating bodies has a lot of significance in designing these structures. Some special type of geometries such as circular cylinders, elliptic cylinders and spherical structures (hemisphere, sphere, spheroid) can be considered to derive analytical solutions to the wave diffraction and radiation problem. The work presented here is mainly the result of water wave interaction with submerged spheres. We also present some analysis and discussion regarding the hydrodynamic interaction with circular cylinders.

In the first part of this study, analytical expressions for various hydrodynamic coefficients and loads due to the effects of diffraction and radiation are derived separately. The case of the combined effect of diffraction and radiation is also been considered in a similar way. The solution to the boundary value problem is obtained by considering two separate problems, namely the diffraction and the radiation problem. The exciting force components are derived by solving the diffraction problem: the added-mass and damping coefficients are evaluated by solving the radiation problem. Theory of multipole expansions has been used to expressed the velocity potentials in terms of an infinite series of associated Legendre polynomials with unknown coefficients. The orthogonality of those polynomials has been exploited to simplify the expressions. The responses due to surge, heave and pitch, induced by wave excitation, are determined from the equation of motion. Since the infinite series appearing in various expressions have excellent truncation properties, these series are evaluated by considering only a finite number of terms. Gaussian quadrature has been used to evaluate the integrals. Numerical estimates for the analytical expressions for the hydrodynamic coefficients and loads are presented for various depth to radius ratios.

In the second part of the study, an analysis is presented for the second-order diffraction problem for a large vertical cylinder. Expressions are derived for first-order and second-order potentials. The second-order loads are divided into three components: waterline force, dynamic force and quadratic force. The first-order potentials contribute to the waterline and dynamic forces whereas the second-order potentials contribute to the quadratic force. Our emphasis is on obtaining the second-order potential: we also discuss the quadratic force. Numerical results for various analytical expressions are presented in tabular and graphical forms, for different wave parameters.

Chapter 1

Introduction

Since the days of Havelock, the study of water waves has been considered a major part of fluid dynamics. The forces exerted by the surface waves on a structure in the water are very important for designing these structures. Accurate prediction of wave loads becomes indispensable in order to design safe structures. Research is aimed at the evaluation of the wave forces generated by waves on structures with different geometries. The different structures that have caught the attention of mathematicians and engineers are circular cylinders, elliptic cylinders, spheres, hemispheres, spheroids and caissons. From the practical point of view, the investigations of wave forces on circular cylinders and spherical structures are more important than those on the other structures.

Due to the inherent nonlinear nature of the ocean waves, no perfect nonlinear mathematical theory is presently available to predict the wave forces on an arbitrary body in the water. Depending on the size and the shape of the structure, there are basically three methods to evaluate wave forces.

They are:

1) Morison equation;

2) Froude-Krylov theory; and

3) Diffraction theory.

These theories along with analytical studies and experiments lead to some reasonably accurate wave force estimates.

The Morison equation is applicable in evaluating wave forces on structures which are small compared to the dominant wavelength. The Morison equation assumes the total force is composed of two forces, namely the inertia and drag forces, linearly added together. The coefficients of these two forces are the inertia (or mass) and drag coefficients (due to the viscosity) which must be determined experimentally.

The Froude-Krylov theory can be applied when the drag force is small in comparison with the inertia force but the size of the structure is still relatively small compared to the dominant wavelength. In this case, the force is computed using the incident wave pressure and the pressure-area method on the surface of the structure. This method has the advantage that for certain symmetric structures the force may be obtained in a closed form and the force coefficients can easily be determined.

When the size of the structure is comparable to the dominant wavelength, or when the structure is large enough to span a significant fraction of a wavelength, the incoming waves undergo significant diffraction or scattering upon arriving at the structure. The presence of the structure alters the conditions of the wave field surrounding it. Hence the diffraction of the waves is a major factor which cannot be neglected. That compels us to take into account the diffraction of the waves, from the surface of the structure, in evaluating the wave forces. This is generally known the diffraction theory. Adopting this method, analytical solution in closed forms are possible for a number of structures.

A rigid floating or submerged structure may undergo six degrees of freedom: three translational and three rotational. Assuming a suitable coordinate system, OXYZ, the translational motions are in x, y and z directions (here longitudinal along x, transverse along y and vertical along z) which are referred to as surge, sway and heave respectively. The rotational motions about the x, y and z directions are referred to as roll, pitch and yaw respectively. Here, the z axis is considered to be vertically downwards from the still water level. Often the structure is restrained to have fewer than six degrees of freedom, for example, the type of mechanical connection used to fasten the structure to the sea-floor. Physically, the vertical and longitudinal motions are of primary importance for a floating or submerged body. All these motions are illustrated in Appendix C.

Structures in water require motion analysis, in addition to estimation of the wave forces. Solution of the equations of motion for various degrees of freedom is required in most cases. Because of the nonlinear damping and exciting forces as well as a nonlinear restoring force, the equations are usually nonlinear. However, to the relief of the researchers, in most instances, these nonlinearities can be eliminated without having any practical effect on the solution, or they can be linearized, so that useful and important results can be obtained through a simplified solution.

The motions of a floating or submerged structure are influenced by the added-mass effect in the water and the damping introduced by the motion of the structure in the water. Correct determination of these quantities is very important in order to analyze the motion. In fact added-mass and damping coefficients must be determined before a motion can be analyzed. For smaller structures these coefficients can be found from various experiments already published. However, for large structures, these quantities usually must be obtained analytically.

Many scientific investigations have been performed since the 1930's in the field of floating and submerged structures. These studies have resulted in understanding the problems related to the wave forces on these structures. As mentioned earlier in this chapter, only a few types of structures have earned the attention of the researchers. Studies have mainly focused on determining the hydrodynamic effects on geometries such as circular cylinders, spheres and spheroids.

1.1 Spherical Structures

Havelock can be considered as the pioneer in the area of hydrodynamic loading on spherical structures. Havelock (1931) calculated the wave resistance of a submerged spheroid by replacing it with a distribution of sources and sinks, or of doublets, using the linearized free surface condition. Much later, Havelock (1955) discussed the fluid motion due to a half-immersed floating sphere undergoing small heaving oscillations. He obtained the velocity potential as a series, with the unknown coefficients given by an infinite set of equations. Newman (1967) derived the second-order steady horizontal force and vertical moment for a freely floating ship in regular waves. He used momentum relations to derive general results for an arbitrary ship and for the far-field velocity potential of the body.

Chey (1970) found that the first-order linearized wave theory was not adequate to produce accurate results. Prior to that, study on second-order theory for two-dimensional bodies had been carried out by Bessho (1957), Tuck (1965) and Salvensen (1966). Chey (1970) developed a new second-order theory for a three-dimensional body which provided a better description of the free-surface conditions. In his study, only total resistance and deep-submergence resistance were measured, with no attempt made to measure the viscous resistance.

Farell and Güven (1973) presented some results originating from towing tank measurements of the viscous resistance of a spheroidal model, by means of the wake-survey technique. Farell (1973) used the theory of infinitesimal waves to calculate the wave resistance, by obtaining the velocity potential of a flow about a submerged prolate spheroid in axial horizontal motion below a free surface which also exactly satisfied the body boundary condition. Davis (1974) investigated the scattering effect of a submerged sphere on a plane short surface wave. The amplitudes of the outgoing cylindrical waves, which were generated, were found to be exponentially small with the factor determined by the highest point of the sphere.

Gray (1978) considered a fully submerged, rigid, stationary sphere, reducing the problem to the solution of an infinite set of linear algebraic equations for the expansion coefficients in spherical harmonics of the velocity potential. This approach was to formulate the problem as an integral equation. The scattering cross section was evaluated numerically and was shown to peak for values of the product of the radius and wave number somewhat less than unity. Srokosz (1979) investigated a submerged sphere, considering it to absorb power from an incident wave through an integrated moving and power take-off system. It was shown that the power absorbed depended on the hydrodynamic properties of the sphere: in particular on the added-mass and the damping coefficients.

Hulme (1982) considered heave and surge motions of a floating hemisphere, to derive added-mass and damping coefficients associated with the periodic motions. He has also briefly discussed the derivation of the long- and short-wave asymptotics of these coefficients. This method can also be used to treat the physically distinct, but mathematically similar, problem of the diffraction of waves by a fixed hemisphere. Considering a submerged vehicle as a neutrally buoyant sphere, Wang (1986) discussed the free motions of a submerged vehicle with a spherical hull form, but with different metacentric heights. The associated radiation and diffraction problems were solved independently, in order to examine the motions and the stability of the submerged hull form. The works of Hulme and Wang were based on the multipole expansions method (Thorne:1953) which proved to be very successful for periodic motions without forward speed. However, this method did not seem to be applicable to the problem of a body with forward speed.

Wu and Eatock Taylor (1987) analyzed the hydrodynamic problem of a submerged spheroid in waves, based on linearized potential theory. The problem of a submerged spheroid in head or following seas was considered and the subsequent formulation was presented. It was suggested that this method could be extended to deal with the problem of oscillating bodies at forward speed. Wu and Eatock Taylor (1989) analyzed the problem of wave radiation and diffraction by submerged spheroids, using linearized three-dimensional potential flow theory. The solution was obtained by expanding the velocity potential into a series of Legendre polynomials in a spheroidal coordinate system. However, this solution also could not be extended for the cases with forward speed. As forward speed significantly affects the body-surface, free-surface and radiation boundary conditions imposed on the velocity potential corresponding to the oscillations of the body, Wu and Eatock Taylor (1988) considered a submerged sphere advancing in regular deep water waves at constant forward speed. Linearized potential theory was adopted and a distribution of sources over the surface of the sphere was expanded into a series of Legendre polynomials. Although linearized potential theory has very little physical significance for a spherical structure, the solution clarified doubts about the influence of forward speed on hydrodynamic forces. Later, Wu and Eatock Taylor (1990) considered a submerged sphere moving in a circular path at constant angular velocity, the analysis being based on the linearized velocity potential theory. The potential was expressed by means of a Green's function and a distribution of sources over the body surface, written in terms of Legendre polynomials.

First- and second-order wave effects on a submerged spheroid were investigated by Lee and Newman (1991). Based on a three-dimensional panel code, they presented numerical results for the linearized force and moment acting on a submerged slender spheroid in regular waves, the subsequent pitch and heave motions and the second-order mean force and moment. Linton (1991) investigated the problems of radiation (both heave and sway) and diffraction of water waves by a submerged sphere in finite depth using the multipole method. The resultant infinite system of linear equations were solved numerically. Although the method adopted is applicable to simple geometries only, it was successful in providing an approximation to problems involving almost spherical bodies.

Wu (1994) considered the hydrodynamic problem of a sphere submerged below a free surface and undergoing large amplitude oscillation. Velocity potential theory was applied and the body surface boundary condition was satisfied on its instantaneous position, with linearized free-surface boundary conditions. Wu *et al.* (1994) presented a solution for the wave induced drift forces acting on a submerged sphere in finite water depth, based on linearized potential theory. The theory of multipole expansions was used in terms of an infinite series of Legendre polynomials with unknown coefficients. The series expression for the second order mean forces (drift forces) was provided by integrating the fluid pressure over the body surface. The horizontal drift force was also expressed by a series solution, obtained by using the far-field method. Detailed description of the works carried out dealing with wave interaction with spherical structures is summarized by Bora *et al.* (1996).

In the first part of our work, we analyze the effects of diffraction and radiation by a submerged sphere in water of finite depth. We restrict ourselves only to first-order. We present an analytical procedure for the boundary value problem: we evaluate the hydrodynamic coefficients and motions for a submerged sphere in finite depth due to surge, heave and pitch motions in the presence of an incident wave. The multipole expansions method of Thorne (1953) was used to express the velocity potentials in terms of an infinite series of Legendre polynomials with unknown coefficients. The orthogonality property of associated Legendre polynomials was utilized in obtaining the expressions for most of the potentials and forces. The exciting forces along x and z directions were evaluated. The analytical expressions for the surge added-mass, heave added-mass and the damping coefficients due to surge and heave motions were derived and computed. We present the total wave loadings due to the combined effects of diffraction and radiation by a submerged sphere. These results (the total loads due to the combined effects of diffraction and radiation) are significantly absent in the published literature so far. The total effect of diffraction and radiation is very important: this gives a better description of the happenings in the vicinity of the submerged body, compared to the effect of either diffraction or radiation alone.

1.2 Circular Cylinders

Circular cylinders have attracted the maximum attention from the research community, because of their extensive use in offshore engineering. An extensive literature is available regarding the estimation of wave forces arising out of interaction of water waves with a circular cylinder. Havelock (1940) worked on linear diffraction theory for deep water waves by examining the diffraction of plane water waves by a stationary obstacle with vertical sides. His main objective was to explore the application possibilities for the problem of a ship advancing through a train of plane waves. Ursell (1949) investigated the two-dimensional motion of a fluid of infinite depth when a circular cylinder was immersed with its axis in the free surface and oscillating about the axis with small amplitudes. It was assumed that the effects of viscosity and surface tension were negligible. The wave amplitude at large distance from the cylinder and the added-mass of the cylinder due to the fluid motion were deduced from the potential and stream functions. Morison *et al.* (1950) obtained an empirical equation for determining the force on a circular cylinder in terms of inertial and drag forces, under the assumption that the incident wave field was not sufficiently affected by the presence of the cylinder. Their result, better known as the Morison equation, has paved the way for the others to proceed with more complicated problems. Morison *et al.* performed their experiment on a single pile without bracing and hence leaving a great scope to deal with the wave force problem in a similar way but with multiple piles. MacCamy and Fuchs (1954) extended Havelock's deep water theory to shallow water waves to evaluate the wave forces exerted on a cylindrical pile immersed in the ocean. However, due to the highly nonlinear property of water waves, this solution has limited applications. They have also presented some simple deductions based on the assumption of very small ratio of cylinder diameter to incident wavelength. The linear diffraction theory by MacCamy and Fuchs was extended to Stokes' fifth order theory by Chakrabarti (1972), without taking care of the nonlinear kinematic free surface boundary condition. The combined nonlinear free surface boundary condition, consisting of the kinematic and dynamic conditions, makes the solution of the problem very complicated. Yamaguchi and Tsuchiya (1974), Raman *et al.* (1975-77) and Rahman (1981) have proposed complete solutions but these studies still exhibit difficulty in handling the boundary conditions. Lighthill (1979) was successful in obtaining an expression for the second-order diffraction force due to regular waves in finite depth. Molin (1979) showed that the previous theories failed because some components of the second-order potential were omitted.

A more appropriate method for handling the non-homogeneous equations evolving from the boundary conditions has been discussed by Garrison (1978) and Shen

(1977). It was suggested that the boundary value problem be broken up into two boundary value problems, each having one non-homogeneous boundary condition; the solutions from these problems are then summed up.

Hunt and Baddour (1981-82) investigated the nonlinear standing and progressive wave forces on a vertical cylinder in deep water. The standing wave problem was solved inside and outside a vertical circular cylinder. The solution for the second-order progressive waves in deep water bounded by a vertical cylinder was obtained as integrals of Bessel functions. Rahman (1984) formulated an exact second-order theory to calculate the wave forces on offshore structures, extending Lighthill's deep water wave theory to shallow water. In some of the solutions mentioned above, there were deficiencies as the formulations involved a free surface integral which oscillated rapidly and converged slowly. Hence it was very difficult to obtain a convergent solution. Eatock Taylor and Hung (1987) overcame this behavior of the integral by adopting an asymptotic form. Their formulation is very similar to the one presented by Molin (1979). Special consideration has been devoted to the far-field behavior of the second order potential. The use of asymptotic forms has led to obtaining a convergent solution to the awkward free surface integral arising from the second order potential. Garrison (1984) made use of Green's theorem and a double-frequency Green's function to formulate the second-order problem in regular waves by expressing the velocity potential as a distribution of wave sources and doublets over the body surface and the free surface. Kim and Yue (1989) solved the second-order diffraction problem for the nonlinear sum-frequency potential for an axisymmetric body in the presence of plane monochromatic waves. Their results showed that the second-harmonic component of the diffraction field was significant at large depths. To generalize the second-order theory to irregular waves, Kim and Yue (1990) considered the general second-order wave-body interactions in the presence of bichromatic incident waves including the radiation problem and the second-order motion. Since the calculation of the complete second-order solution was rather complicated, Newman (1990) initiated some approximation methods. He derived an approximation of the second-order diffraction potential for water waves of small amplitudes at large depths; but the applicability of this result is still doubtful for complex geometries. Chau and Eatock Taylor (1992), in a detailed analysis of the second-order diffraction problem of a uniform vertical cylinder in regular waves, were able to provide results for the free surface as well, in addition to the cylinder surface. The asymptotic analysis developed by Newman for mono-directional waves was extended by Kim (1993) to the case of multi-directional wave. Kareem *et al.* (1994) investigated the diffraction of nonlinear random waves by a fixed, surface-piercing vertical cylinder in deep water. The incident wave field was considered as a stationary random process and the Stokes perturbation expansion method was utilized in the analysis. The second order velocity potential was explicitly obtained by applying a modified form of Weber's Integral Theorem to invert the non-homogeneous second-order free-surface condition.

Rahman and Bhatta (1993) presented an approximate method for estimating the hydrodynamic forces to the second-order on a pair of bottom mounted, surface-piercing circular cylinder in waves of arbitrary uniform depth. The theoretical results were based upon the large spacing approximation and the method involved replacing scattered diverging waves by plane waves. Isaacson *et al.* (1988-92) have investigated a full nonlinear solution for diffraction by adopting a time-stepping procedure. Isaacson and Cheung (1990) applied a perturbation method to the time-stepping procedure. However, this solution gave rise to some difficulties due to the inept handling of the radiation condition at second-order. Isaacson and Cheung's (1991) method for the two-dimensional vertical plane problem has been extended by them later on to three dimensions. Though the special case of regular wave diffraction around a surface-piercing circular cylinder is presented in their work, they have theoretically extended it to any structure of arbitrary shape.

More recently, Newman (1996) has derived the second-order potential by Weber transformation of the corresponding forcing function on the free surface. This forcing function is reduced to a form which involves a simple factor inversely proportional to the radial coordinate and an oscillatory function which decays more rapidly in the far-field.

Generally, analytical solutions do not exist for problems with arbitrary geometry and more complex boundary conditions. However, over the past few years, the application of numerical methods has helped hydrodynamicists to solve some complicated problems. The numerical methods frequently used to solve wave diffraction and radiation problems are mainly : the Finite Element Method (*FEM*), Green's function Method and Boundary Element Method (*BEM*).

Although *FEM* is mainly used in the stress analysis of complex geometries, it has also found its place in the solution of water wave problems. Mei (1978) has given an extensive review of the works done in this area. Bai (1975) considered the diffraction of oblique waves incident upon a long infinite cylinder on the free surface. The numerical method was based on a variational principle equivalent to the linearized boundary value problem. Finite element techniques were used to represent the velocity potential. The diffraction forces and moments were computed for oblique wave incident upon a vertical flat plate, a horizontal flat plate and rectangular cylinders. However, linear theory tends to underestimate the diffraction force and it has been noticed that for steep water the error can be significant. Clark *et al.* (1991) presented a new *FEM* approach for calculating nonlinear wave loads on offshore structures in extreme seas. Stokes' second-order wave theory was used to model the diffraction wave field. The boundary value problem for the second-order velocity potential, including the radiation condition, was solved to obtain wave-loads and free surface elevations. Bai (1977) presented numerical results for the added-mass and damping coefficients of semi-submerged two-dimensional heaving cylinders in water of finite depth. The added-mass and damping coefficients were computed for a circular cylinder oscillating in water of several different depths.

Green's function method, also known as the panel method in fluid dynamics, is the early form of *BEM*. This method is quite powerful in handling linearized problems. Adopting this method, some important works on water wave diffraction theory have been carried out by Garrison (1969), and Faltinsen and Michelsen (1974). The typical procedure for this method begins with a special Green's function that satisfies the governing equation and nearly all the boundary conditions except that on the body surface.

BEM has most actively been applied to those problems where better accuracy is required, *e.g.* problems with the domain extending to infinity. Brebbia (1980) has been instrumental in applying *BEM* to the complex water wave diffraction problems. The mathematical model for *BEM* problems is a Helmholz integral. Lee (1988) adopted the direct *BEM* to calculate the wave-exciting and motion-induced hydrodynamic forces for fixed and floating ocean structures in a fluid of finite or infinite depth. The fluid potential was expressed by means of a Helmholtz integral, which involved the normal fluid velocities at the fluid-structure interface. This method is suitable for solving problems having high ratio of domain volume to boundary surface area. Rahman *et al.* (1992) adopted *BEM* to solve the problem of water wave diffraction by a large fixed rectangular shaped structure floating in the ocean and subjected to regular incident waves. He obtained satisfactory results. Bora *et al.* (1996) have discussed in details the formulation and results of most of the important works carried out regarding wave-loading on circular cylinders.

In this second half of our work, we analyze the effects of diffraction and radiation by a floating circular cylinder. Adopting an analytical procedure, we obtain expressions for all the four forces – linear force, dynamic force, waterline force and quadratic force which comprise the total force, and also for the moments of these forces. We mainly focus on the quadratic force and its impact. This second order theory is mainly based on Lighthill's method (1979). Interestingly, the contribution of this quadratic force to the total force has not been discussed earlier in details.

1.3 Other Geometries

Though our work is related to the evaluation of wave forces due to the presence of a spherical structure or a circular cylinder in water, it is a good idea to discuss in brief, the important works carried out in this area with elliptic cylinder and caisson.

1.3.1 Elliptic Cylinder

There has not been a significant number of works carried out regarding estimation of wave forces on elliptic cylinders. Chen and Mei (1971) investigated the problem of scattering of linear progressive waves by an elliptic cylinder. The associated problem of wave forces and moments on a stationary floating elliptic platform has also been studied by Chen and Mei (1973) for the case with long wavelengths. Williams (1982) presented a linear theory of wave diffraction by a fixed vertical cylinder of elliptic cross-section in water of finite depth. He obtained approximate expressions for the force and moment coefficients under the assumption that the eccentricity of the ellipse was small.

1.3.2 Caissons

Compared to other structures, caissons have received very scant attention from engineers and scientists. Only a handful of attempts have been made to approximate wave forces on caisson (usually rectangular) fixed vertically on the sea-bed. Rahman and Chakravartty (1986), using a series of transformations, predicted wave forces on rectangular caissons. In this method, the rectangular caisson was transformed to represent approximately a circular cylinder because of the difficulty encountered in satisfying the boundary conditions on structures with rectangular or square cross-sections. Rahman (1987), adopting the same method as in his previous work (1986), showed the effect of linear forces for the case of rectangular caissons as well as the square caissons. Due to the lack of practical use for this type of structures with regard to water waves, a complete analysis of wave forces has not been formulated yet.

Chapter 2

Mathematical Formulations for a Submerged Sphere

We assume that the fluid is homogeneous, inviscid and incompressible and the fluid motion is irrotational. The waves are also assumed to be of small amplitude. Here we consider the coefficients related to the motion with three degrees of freedom, namely, two translational motions in the x and z directions, i.e. surge and heave motions, respectively, and the rotational motion about y direction, i.e. pitch motion. We consider a surface wave of amplitude A incident on a sphere of radius a submerged in water of finite depth d. The body is assumed to have motions with three degrees of freedom in the presence of the incident wave with angular frequency σ. The wave is parallel to the x-axis at the time of incidence on the sphere and is propagating along the positive direction.

We consider two sets of coordinate systems. One is a right-handed Cartesian coordinate system (x, y, z), in which the x-y plane coincides with the undisturbed free surface and the z-axis is taken vertically downwards from the still water level (SWL). The other coordinate system is the spherical coordinate system (r, θ, ψ) with the origin at the geometric center of the sphere. Figure 2.1 shows the axes systems along a sphere of radius a in water of depth d with its geometric center located at $(0, 0, h)$ with respect to the Cartesian coordinate system.

The relationship between the coordinate systems is :

$$\begin{aligned} R &= \sqrt{x^2 + y^2} \\ r &= \sqrt{R^2 + (z-h)^2} \\ \tan\theta &= \frac{R}{z-h} \qquad \text{for } 0 \leq \theta \leq \pi \\ \tan\psi &= \frac{y}{x} \qquad \text{for } -\pi \leq \psi \leq \pi \end{aligned}$$

For an incompressible and inviscid fluid, and for small amplitude wave theory with irrotational motion, we can express the fluid motion by introducing a velocity potential $\Phi(r, \theta, \psi, t)$. This Φ can be written as:

$$\Phi(r, \theta, \psi, t) = Re[\phi(r, \theta, \psi)e^{-i\sigma t}] \tag{2.1}$$

where Re stands for the real part.

The motion is assumed harmonic. Also, from Bernoulli's equation, we get pressure, $P(r, \theta, \psi, t)$ as

$$P = -\rho\frac{\partial\Phi}{\partial t} \tag{2.2}$$

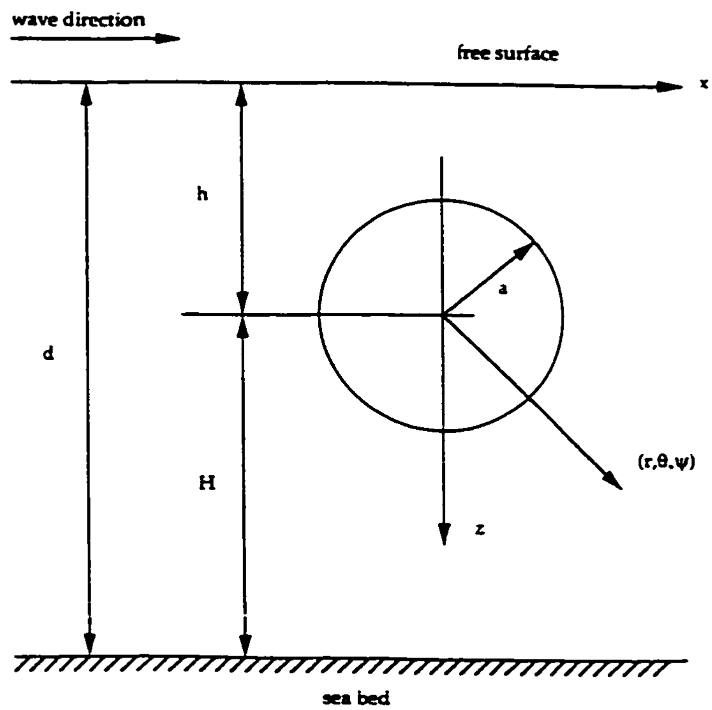

Figure 2.1: Reference Coordinate System

Note that the problem can be considered as a combination of two fundamental problems: the diffraction problem of an incident wave interacting with a fixed body; and the radiation problem of a body forced to oscillate in otherwise still water. Because of the linearity of the situation, the time-independent velocity potential $\phi(r,\theta,\psi)$ can be decomposed into five velocity potentials $\phi_I, \phi_D, \phi_1, \phi_3$ and ϕ_5 where ϕ_I is the incident potential, ϕ_D is the velocity potential due to the diffraction of an incident wave acting on the sphere; and ϕ_1, ϕ_3 and ϕ_5 are velocity potentials due to the radiation of surge, heave and pitch respectively.

Thus ϕ can be written as $\phi = \phi_I + \phi_D + X_1\phi_1 + X_3\phi_3 + X_5\phi_5$ where X_1, X_2 and X_3 are the displacements for surge, heave and pitch motions respectively. Here $\phi_I, \phi_D, \phi_j, j = 1,3,5$ are all functions of r, θ and ψ and $X_j, j = 1,3,5$ is the independent parameter.

To obtain the velocity potential ϕ, the following boundary problem must be solved:

1) Laplace's equation in spherical coordinates:

$$\nabla^2\phi = 0 \tag{2.3}$$

2) free surface condition:

$$\frac{\partial\phi}{\partial z} + K\phi = 0 \quad \text{on } z = 0 \tag{2.4}$$

3) bottom boundary condition:

$$\frac{\partial\phi}{\partial z} = 0, \qquad z = d \tag{2.5}$$

4) radiation condition:

$$\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial}{\partial R} - ik_0\right)\phi = 0 \tag{2.6}$$

where $K = \frac{\sigma^2}{g}$ and k_0 is the finite depth wave number defined by

$$k_0 \sinh k_0 d - K \cosh k_0 d = 0 \tag{2.7}$$

and the incident and diffraction potentials satisfy the body surface condition

$$\frac{\partial\phi_I}{\partial\mathbf{n}} = -\frac{\partial\phi_D}{\partial\mathbf{n}} \quad \text{on } r = a \tag{2.8}$$

where $\mathbf{n}$ denotes the normal vector from the body surface to the fluid.

The radiation potentials satisfy the body surface condition

a) for surge motion:

$$\frac{\partial \phi}{\partial r} = i\sigma \sin\theta \cos\psi \quad \text{on } r = a \tag{2.9}$$

b) for heave motion:

$$\frac{\partial \phi}{\partial r} = i\sigma \cos\theta \quad \text{on } r = a \tag{2.10}$$

c) for pitch motion:

$$\frac{\partial \phi}{\partial r} = 0, \quad \text{on } r = a \tag{2.11}$$

The boundary conditions (2.9)-(2.11) have arisen from the equation

$$\frac{\partial \phi_j}{\partial \mathbf{n}} = (-i\sigma)n_j, j = 1, 3, 5 \tag{2.12}$$

2.1 Incident Potential

Incoming waves of amplitude A and frequency σ, propagating in the positive x-direction, can be described by the following incident velocity potential,

$$\phi_I = \frac{Ag}{\sigma}\frac{\cosh k_0(z-d)}{\cosh k_0 d} e^{ik_0 R\cos\psi} \tag{2.13}$$

Using McLachlan (1941) (Appendix A), this may be expressed as,

$$\phi_I = \frac{Ag}{\sigma}\frac{\cosh k_0(z-d)}{\cosh k_0 d} \sum_{m=0}^{\infty} \epsilon_m i^m J_m(k_0 R) \cos m\psi \tag{2.14}$$

where $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m \geq 1$.

Using Thorne (1953), the incident potential can be expressed in terms of associated Legendre's polynomial as:

$$\begin{aligned}
\phi_I &= \frac{Ag}{2\sigma\cosh kd}\sum_{m=0}^{\infty}\epsilon_m i^m \cos m\psi \times \\
&[e^{k_0(h-d)}\sum_{s=m}^{\infty}(k_0r)^s\frac{P_s^m(\cos\theta)}{(s+m)!} + e^{-k_0(h-d)}\sum_{s=m}^{\infty}(-1)^{m+s}(k_0r)^s\frac{P_s^m(\cos\theta)}{(s+m)!}] \\
&= \frac{Ag}{2\sigma\cosh k_0d}\sum_{m=0}^{\infty}\epsilon_m i^m \cos m\psi \times \\
&\sum_{s=m}^{\infty}\{(-1)^{s+m}e^{k_0(d-h)} + e^{k_0(h-d)}\}\frac{(k_0r)^s}{(s+m)!}P_s^m(\cos\theta) \qquad (2.15)
\end{aligned}$$

or we can write for our convenience,

$$\phi_I(r,\theta,\psi) = \sum_{m=0}^{\infty}\hat{\phi}_I(r,\theta)\cos m\psi \qquad (2.16)$$

where

$$\hat{\phi}_I = \frac{Ag}{2\sigma\cosh k_0d}\epsilon_m i^m\sum_{s=m}^{\infty}\{(-1)^{s+m}e^{k_0(d-h)} + e^{k_0(h-d)}\}\frac{(k_0r)^s}{(s+m)!}P_s^m(\cos\theta)$$

Changing s to s+m, we have

$$\hat{\phi}_I = \frac{Ag}{2\sigma\cosh k_0d}\epsilon_m i^m\sum_{s=0}^{\infty}\{(-1)^s e^{k_0(d-h)} + e^{k_0(h-d)}\}\frac{(k_0r)^{s+m}}{(s+2m)!}P_{s+m}^m(\cos\theta) \qquad (2.17)$$

which can be modified to write as

$$\hat{\phi}_I(r,\theta) = \frac{Ag}{\sigma}\epsilon_m i^m\sum_{s=0}^{\infty}\chi_s\frac{(k_0r)^{s+m}}{(s+2m)!}P_{s+m}^m(\cos\theta) \qquad (2.18)$$

where

$$\begin{aligned}
\chi_s &= \frac{(-1)^s e^{k_0(d-h)} + e^{-k_0(d-h)}}{2\cosh k_0d} \\
&= \begin{cases} \frac{\cosh k_0(d-h)}{\cosh k_0d}, & s = 0,2,4,6,\ldots \\ -\frac{\sinh k_0(d-h)}{\cosh k_0d}, & s = 1,3,5,\ldots \end{cases} \qquad (2.19)
\end{aligned}$$

Hence, the incident potential ϕ_I can be written in the final form as

$$\phi_I(r,\theta,\psi) = \sum_{m=0}^{\infty}\frac{Ag}{\sigma}\epsilon_m i^m\sum_{s=0}^{\infty}\chi_s\frac{(k_0r)^{s+m}}{(s+2m)!}P_{s+m}^m(\cos\theta)\cos m\psi \qquad (2.20)$$

2.2 Diffraction Potential

The diffraction velocity potential ϕ_D satisfies eqns (2.3)-(2.6) and eqn (2.8). We can express this potential by making it ψ-independent as:

$$\phi_D(r,\theta,\psi) = \sum_{m=0}^{\infty} \hat{\phi}_D(r,\theta)\cos m\psi \tag{2.21}$$

where the ψ-independent potential is

$$\hat{\phi}_D(r,\theta) = \sum_{n=m}^{\infty} a^{n+2} A_{mn} G_n^m \tag{2.22}$$

Here, A_{mn} are the unknown complex coefficients and G_n^m are the multipole potentials. Multipole potentials are solutions of Laplace's equation which satisfy the free surface and bottom boundary conditions and behave like outgoing waves from the singular point which, in this case, is the centre of the sphere. We can express G_n^m as,

$$\begin{aligned} G_n^m &= \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{P_n^m(\cos\alpha)}{r_1^{n+1}} + \frac{1}{(n-m)!} \times \\ & \int_0^\infty \frac{(K+k)[e^{-k(d+H)} + (-1)^{n+m} e^{-kh}]}{k\sinh kd - K\cosh kd} k^n \cosh k(z-d) J_m(kR)\,dk \end{aligned} \tag{2.23}$$

The quantities α and r_1 are defined as :

$$\begin{aligned} r_1 &= \sqrt{R^2 + (d+H-z)^2} \\ \tan\alpha &= \frac{R}{d+H-z} \end{aligned}$$

where R, d and H have already been defined.

The line integration in the expression for G_n^m passes under the singular point of the integrand at $k = k_0$. The potentials G_n^m and ϕ_D satisfy Laplace's equation, the free surface condition, the bottom surface condition and the radiation condition.

The second and third terms in eqn (2.23) can be expanded, in the region near the body surface, into a series of associated Legendre's polynomials by

$$\frac{P_n^m(\cos\alpha)}{r_1^{n+1}} = \sum_{s=0}^{\infty} B_{ns}^m \left(\frac{r}{2H}\right)^{s+m} P_{s+m}^m(\cos\theta) \tag{2.24}$$

and

$$\begin{aligned} \frac{1}{(n-m)!} & \int_0^\infty \frac{(K+k)[e^{-k(d+H)} + (-1)^{n+m} e^{-kh}]}{k\sinh kd - K\cosh kd} k^n \cosh k(z-d) J_m(kR)dk \\ &= \sum_{s=0}^{\infty} C_s(n,m) \left(\frac{r}{2H}\right)^{s+m} P_{s+m}^m(\cos\theta) \end{aligned} \tag{2.25}$$

where B_{ns}^m and $C_s(n,m)$ are given by

$$B_{ns}^m = \frac{1}{(2H)^{n+1}} \frac{(s+n+m)!}{(s+2m)!(n-m)!} \tag{2.26}$$

$$\begin{aligned} C_s(n,m) &= \frac{(2H)^{s+m}}{(n-m)!(s+2m)!} \\ & \int_0^\infty \frac{(K+k)[e^{-k(d+H)} + (-1)^{n+m} e^{-kh}]}{k\sinh kd - K\cosh kd} u_s(kH)dk \end{aligned} \tag{2.27}$$

with $u_s(kH)$ as

$$u_s(kH) = \begin{cases} \cosh kH, & s = 0, 2, 4, \dots \\ -\sinh kH, & s = 1, 3, 5, \dots \end{cases} \tag{2.28}$$

Hence the multipole potentials G_n^m can finally be written as

$$G_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{s=0}^{\infty} [B_{ns}^m + C_s(n,m)] \left(\frac{r}{2H}\right)^{s+m} P_{s+m}^m(\cos\theta) \tag{2.29}$$

Using the body boundary condition (2.8), we may write,

$$\sum_{n=m}^{\infty} a^{n+2} A_{mn} \frac{\partial G_n^m}{\partial r}\Big|_{r=a} = -\frac{\partial \hat{\phi}_I}{\partial r}\Big|_{r=a} \tag{2.30}$$

From the expressions for G_n^m and $\hat{\phi}_I$ from eqns (2.29) and (2.18) respectively, we can evaluate

$$\begin{aligned}
\frac{\partial G_n^m}{\partial r}|_{r=a} &= -(n+1)\frac{P_n^m(\cos\theta)}{a^{n+2}} + \sum_{s=0}^{\infty} B_{ns}^m(\frac{1}{2H})^{s+m}(s+m)a^{s+m-1}P_{s+m}^m(\cos\theta) \\
&+ \sum_{s=0}^{\infty} C_s(n,m)(\frac{1}{2H})^{s+m}(s+m)a^{s+m-1}P_{s+m}^m(\cos\theta) \\
\frac{\partial\hat{\phi}_I}{\partial r}|_{r=a} &= \frac{Ag}{\sigma}\epsilon_m i^m \sum_{s=0}^{\infty} \chi_s \frac{k_0^{s+m}}{(s+2m)!}(s+m)a^{s+m-1}P_{s+m}^m(\cos\theta)
\end{aligned}$$

Using these two expressions in eqn (2.30), we get

$$\begin{aligned}
\sum_{n=m}^{\infty} A_{mn}[-(n+1)P_n^m(\cos\theta) + \sum_{s=0}^{\infty}\{B_{ns}^m + C_s(n,m)\}(\frac{a}{2H})^{s+m}(s+m)a^{n+1}P_{s+m}^m(\cos\theta)] \\
= -\frac{Ag}{\sigma}\epsilon_m i^m \sum_{s=0}^{\infty} \chi_s \frac{(k_0a)^{s+m}}{(s+2m)!}(s+m)a^{-1}P_{s+m}^m(\cos\theta)
\end{aligned}$$

Multiplying this by $P_n^m(\cos\theta)\sin\theta$ and integrating with respect to θ in $0 \le \theta \le \pi$ and using the orthogonality property of associated Legendre's polynomials (Appendix B), we arrive at

$$\begin{aligned}
\sum_{n=m}^{\infty} A_{mn}[-(n+1)\delta_{ns} &+ (B_{ns}^m + C_s(n,m))(\frac{a}{2H})^{s+m}(s+m)a^{n+1}] \\
&= -\frac{Ag}{\sigma}\epsilon_m i^m \chi_s \frac{(k_0a)^{s+m}}{(s+2m)!}(s+m)a^{-1}
\end{aligned}$$

which in compact form gives rise to

$$\sum_{n=m}^{\infty} A_{mn}E_{ns}^m = T_s^m \quad \text{for } s = m, m+1, m+2, \ldots \tag{2.31}$$

where

$$T_s^m = -\frac{Agk_0}{\sigma}\epsilon_m i^m (k_0a)^{s-1}\frac{s}{(s+m)!}\ \chi_{s-m} \tag{2.32}$$

$$E_{ns}^m = -(n+1)\delta_{ns} + D_n^m(s-m) \tag{2.33}$$

$$D_n^m(s) = a^{n+1}(s+m)(\frac{a}{2H})^{s+m}[C_s(n,m) + B_{ns}^m] \tag{2.34}$$

The diffraction potential ϕ_D has the final form

$$\phi_D = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} a^{n+2}A_{mn}[\frac{P_n^m(\cos\theta)}{r^{n+1}}+\sum_{s=0}^{\infty}\{B_{ns}^m+C_s(n,m)\}(\frac{r}{2H})^{s+m}P_{s+m}^m(\cos\theta)]\cos m\psi \tag{2.35}$$

Equation (2.31) is a complex matrix equation in the unknowns A_{mn}. Since the infinite series appearing in (2.32) and (2.34) have excellent truncation properties, the infinite matrices can be truncated, after a finite number of terms, and we solve (2.31) numerically. Commercially available complex matrix inversion routines are used to obtain the solution of the modified equation. Once these coefficients are known, the diffraction problem is completely known.

2.2.1 Exciting Forces

The forces associated with the incident and diffraction potentials are the exciting forces which play a very important role in the wave field for a structure in water. The exciting forces $F_j^{(e)}$ can be obtained from:

$$F_j^{(e)} = 2i\rho a^2\sigma A\int_o^{\pi}\int_0^{\pi}\phi_{ID}|_{r=a}n_j\sin\theta d\theta d\psi \tag{2.36}$$

where $j = 0$ corresponds to heave motion and $j = 1$ corresponds to surge motion and we have written $\phi_{ID} = \phi_I + \phi_D$, where

$$n_j = -P_1^j(\cos\theta)\cos j\psi, \quad j = 0, 1 \tag{2.37}$$

From eqns (2.20) and (2.35), we have the following

$$\frac{\partial\phi_I}{\partial r}|_{r=a} = \sum_{m=0}^{\infty}\frac{Ag}{\sigma}\epsilon_m i^m\sum_{s=0}^{\infty}\chi_s(s+m)\frac{(k_0a)^{s+m}}{a(s+2m)!}P_{s+m}^m(\cos\theta)\cos m\psi$$

$$\frac{\partial\phi_D}{\partial r}|_{r=a} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} A_{mn}[-(n+1)P_n^m(\cos\theta) + \sum_{s=0}^{\infty}(B_{ns}^m + C_s)(\frac{a}{2H})^{s+m} \times (s+m)a^{n+1}P_{s+m}^m(\cos\theta)]\cos m\psi$$

Applying the body surface condition $\frac{\partial\phi_D}{\partial r} = -\frac{\partial\phi_I}{\partial r}$ at $r = a$, we have

$$\chi_s = -\frac{\sigma a(s+2m)!}{Ag\epsilon_m i^m (k_0 a)^{s+m}(s+m)}\sum_{n=m}^{\infty} A_{mn}[-(n+1)+(B_{ns}^m + C_s)(s+m)(\frac{a}{2H})^{s+m}a^{n+1}]$$

We can derive the following from the same expressions (2.20) and (2.35)

$$\phi_I|_{r=a} = \sum_{m=0}^{\infty}\frac{Ag}{\sigma}\epsilon_m i^m\sum_{s=0}^{\infty}\chi_s\frac{(k_0 a)^{s+m}}{(s+2m)!}P_{s+m}^m(\cos\theta)\cos m\psi$$

$$\phi_D|_{r=a} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} aA_{mn}[P_n^m(\cos\theta) + \sum_{s=0}^{\infty}(B_{ns}^m + C_s)(\frac{a}{2H})^{s+m} \times a^{n+1}P_{s+m}^m(\cos\theta)]\cos m\psi$$

After some simplifications we get

$$\phi_{ID}|_{r=a} = a\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{2n+1}{n}A_{mn}P_n^m(\cos\theta)\cos m\psi \quad (2.38)$$

Therefore, the exciting forces are given by

$$\begin{aligned} F_j^{(e)} &= -2i\rho\sigma a^2 A\int_0^{\pi}\int_0^{\pi}\phi_{ID}|_{r=a}P_1^j(\cos\theta)\cos j\psi\cos\psi\sin\theta d\theta d\psi \\ &= -\frac{2i\rho\sigma a^2 A\pi}{\epsilon_j}\int_0^{\pi}\sum_{n=j}^{\infty} a\frac{2n+1}{n}A_{jn}P_n^j(\cos\theta)\sin\theta d\theta \quad (2.39)\end{aligned}$$

where $\epsilon_j = 1$ for $j = 0$, $\epsilon_j = 2$ for $j \geq 1$.

Using the orthogonality property of associated Legendre's polynomials, we obtain

$$\begin{aligned} F_j^{(e)} &= -2i\rho\sigma\pi Aa^3\epsilon_j 2\frac{(1+j)!}{(1-j)!}A_{j1} \\ &= -4i\rho\sigma\pi a^3 AA_{j1} \quad (2.40)\end{aligned}$$

since the terms ϵ_j and $\frac{(1+j)!}{(1-j)!}$ cancel out for the respective values of j.

Hence the surge exciting force $F_x^{(e)} = f_{xd}$ is given by

$$f_{xd} = -4i\rho\sigma\pi A a^3 A_{11} \tag{2.41}$$

and the heave exciting force $F_z^{(e)} = f_{zd}$ is given by

$$f_{zd} = -4i\rho\sigma\pi A a^3 A_{01} \tag{2.42}$$

Nondimensionalizing the forces given by eqns (2.41) and (2.42), we can write the non-dimensional forces as:

$$\frac{f_{xd}}{4i\rho\sigma A\pi a^3} = -A_{11} \tag{2.43}$$

and

$$\frac{f_{zd}}{4i\rho\sigma A\pi a^3} = -A_{01} \tag{2.44}$$

2.3 Radiation Problem

Having solved the diffraction problem for the submerged sphere, now we turn our attention to the radiation problem. As mentioned earlier we will consider surge, heave and pitch potentials only. All these potentials mainly satisfy the same set of equations except for the body boundary condition which is different for each motion. Surge and heave potentials are both related with translational motions and have resemblance in their expressions. Hence we proceed to find the expressions for surge and heave potentials at the same time and we consider the respective boundary conditions. One very important point to note is that, due to the body symmetry of a sphere, no moment forces act upon the body.

2.3.1 Surge and Heave Potentials

The radiation velocity potential ϕ_m must satisfy :

$$\nabla^2\phi_m = 0 \quad \text{in the fluid} \tag{2.45}$$

$$\frac{\partial\phi_m}{\partial z} + k\phi_m = 0 \quad \text{on } z = 0 \tag{2.46}$$

$$\frac{\partial\phi_m}{\partial z} = 0 \quad \text{on } z = d \tag{2.47}$$

$$\frac{\partial\phi_m}{\partial r} = (-i\sigma)n_j, \quad j = 1, 3, 5 \text{ on } r = a \tag{2.48}$$

$$\lim_{R\to\infty} R^{\frac{1}{2}}\{\frac{\partial}{\partial R} - ik\}\phi_m = 0 \tag{2.49}$$

The kinematic boundary condition on the body surface for the radiation problem in case of surge and heave motions, can be written as

$$\frac{\partial\phi_m}{\partial r} = i\sigma P_1^m(\cos\theta)\cos m\psi \tag{2.50}$$

where $m = 0$ corresponds to heave motion and $m = 1$ to surge motion. The ψ-dependence of ϕ_m can be removed by assuming

$$\phi_m(r,\theta,\psi) = \hat{\phi}_m(r,\theta)\cos m\psi \tag{2.51}$$

The velocity potential $\hat{\phi}_m(r,\theta)$ will be expanded in multipole potentials which have already been discussed while dealing with the diffraction potential. From

Thorne (1953) section 5, removing the time dependence term, we write

$$\hat{\phi}_m(r,\theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^{n+m-1}}{(n-m)!}\int_0^\infty \frac{K+k}{K-k} k^n e^{-k(z+d)} J_m(kR)dk + i\frac{(-1)^{m+n}}{(n-m)!} 2\pi K^{n+1} e^{-k(z+d)} J_m(KR) \tag{2.52}$$

where $\hat{\phi}_m$ can be finally expressed as

$$\hat{\phi}_m(r,\theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^{n+m-1}}{(n-m)!}\int_0^\infty \frac{K+k}{K-k} k^n (e^{-2kd}(-1)^m \sum_{s=m}^{\infty}(-kr)^s \times \frac{P_s^m(\cos\theta)}{(s+m)!})dk + i\frac{(-1)^{m+n}}{(n-m)!} 2\pi K^{n+1} e^{-2Kd} \sum_{s=m}^{\infty}(-1)^m(-Kr)^s \frac{P_s^m(\cos\theta)}{(s+m)!} \tag{2.53}$$

which can be organized to write as

$$\hat{\phi}_m(r,\theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{s=m}^{\infty} \frac{(-1)^{m+s-1}}{(n-m)!(s+m)!} r^s P_s^m(\cos\theta) PV \int_0^\infty \frac{K+k}{K-k} k^{n+s} e^{-2kd} dk + i\sum_{s=m}^{\infty} \frac{(-1)^{n+s}}{(n-m)!(s+m)!} 2\pi K^{n+s+1} e^{-2Kd} r^s P_s^m(\cos\theta) \tag{2.54}$$

where PV means the principal value of the integral is to be considered. Alternately, we can write $\hat{\phi}_m$ as

$$\hat{\phi}_m(r,\theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{s=m}^{\infty}[A_s + iB_s] r^s P_n^m(\cos\theta) \tag{2.55}$$

where,

$$A_s = \frac{(-1)^{m+s-1}}{(n-m)!(s+m)!} PV \int_0^\infty \frac{K+k}{K-k} k^{n+s} e^{-2kd} dk \tag{2.56}$$

$$B_s = \frac{(-1)^{n+s}}{(n-m)!(s+m)!} 2\pi K^{n+s+1} e^{-2Kd} \tag{2.57}$$

Hence the radiation potential ϕ_m can be written as

$$\phi_m(r,\theta.\psi) = [\frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{s=m}^{\infty}(A_s + iB_s) r^s P_s^m(\cos\theta)] \cos m\psi \tag{2.58}$$

Applying the body boundary condition, we may write

$$i\sigma P_1^m(\cos\theta) = [\frac{-(n+1)P_n^m(\cos\theta)}{a^{n+2}} + \sum_{s=m}^{\infty}(A_s + iB_s)sa^{s-1}P_s^m(\cos\theta)], \quad m = 0, 1. \tag{2.59}$$

After simplifying and using the orthogonality of associated Legendre's polynomials, we obtain

$$\sum_{n=m}^{\infty} \frac{2na^{n-1}}{2n+1}\frac{(n+m)!}{(n-m)!}(A_n + iB_n) = \frac{2(n+1)}{(2n+1)a^{n+2}}\frac{(n+m)!}{(n-m)!} + \frac{2i\sigma}{3}\frac{(1+m)!}{(1-m)!}, \quad m = 0, 1 \tag{2.60}$$

which is an infinite system of linear algebraic equations in an infinite number of unknowns. Solution of these will enable us to find the radiation potentials and subsequently the surge and heave hydrodynamic coefficients. We can also write $A_n + iB_n = D_n$ to have complex coefficients D_n. Then we can rewrite (2.60) as

$$\sum_{n=m}^{\infty} \frac{2na^{n-1}}{2n+1}\frac{(n+m)!}{(n-m)!}D_n = \frac{2(n+1)}{(2n+1)a^{n+2}}\frac{(n+m)!}{(n-m)!} + \frac{2i\sigma}{3}\frac{(1+m)!}{(1-m)!}, \quad m = 0, 1 \tag{2.61}$$

or we can equate real and imaginary parts from eqn (2.60), and we obtain

$$\sum_{n=m}^{\infty} \frac{2na^{n-1}}{2n+1}\frac{(n+m)!}{(n-m)!}A_n = \frac{2(n+1)}{(2n+1)a^{n+2}}\frac{(n+m)!}{(n-m)!} \tag{2.62}$$

$$\sum_{n=m}^{\infty} \frac{2na^{n-1}}{2n+1}\frac{(n+m)!}{(n-m)!}B_n = \frac{2}{3}\sigma\frac{(1+m)!}{(1-m)!} \tag{2.63}$$

2.3.2 Pitch Potential

The pitch potential ϕ_5 due to the pitch motion satisfies eqns (2.45)-(2.49). We note that for pitch motion, eqn (2.48) is $\frac{\partial \phi_m}{\partial r} = 0$ at $r = a$.

As before, we can express ϕ_5 as:

$$\phi_5(r,\theta,\psi) = \hat{\phi}_5(r,\theta)\cos m\psi \tag{2.64}$$

where

$$\hat{\phi}_5(r,\theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{n=m}^{\infty}(\bar{A}_n + i\bar{B}_n)r^n P_n^m(\cos\theta) \tag{2.65}$$

with

$$\bar{A}_n = \frac{(-1)^{n+m-1}}{(n-m)!(n+m)!} PV \int_o^\infty \frac{K+k}{K-k} k^{m+n} e^{-2kd} dk \tag{2.66}$$

and

$$\bar{B}_n = \frac{(-1)^{n+m}}{(n-m)!(n+m)!} 2\pi K^{n+m-1} e^{-2Kd} \tag{2.67}$$

Applying the body surface condition (2.48), we obtain

$$\frac{P_n^m(\cos\theta)}{a^{n+2}} - \sum_{n=m}^{\infty}(\bar{A}_n + i\bar{B}_n)\frac{n}{n+1}a^{n-1}P_n^m(\cos\theta) = 0$$

Using the orthogonality property of associated Legendre's polynomials,

$$\frac{2}{(2n+1)a^{n+2}}\frac{(n+m)!}{(n-m)!} - (\bar{A}_n + i\bar{B}_n)\frac{n}{n+1}a^{n-1}\frac{(n+m)!}{(n-m)!} = 0$$

gives us

$$(\bar{A}_n + i\bar{B}_n) = \frac{2(n+1)}{n(2n+1)a^{2n+1}} \tag{2.68}$$

We can also write in terms of complex coefficient as

$$\bar{D}_n = \frac{2(n+1)}{n(2n+1)a^{2n+1}} \tag{2.69}$$

and equating real and imaginary parts from eqn (2.68),

$$\bar{A}_n = \frac{2(n+1)}{n(2n+1)a^{2n+1}} \tag{2.70}$$

$$\bar{B}_n = 0 \tag{2.71}$$

2.4 Determination of Hydrodynamic Coefficients and Motion

The coefficients related with the radiation play a big role in allowing us to know the impact of motions due to radiation. The evaluation of added-mass and damping coefficients is of utmost importance in analyzing the contribution of radiation to the total boundary value problem.

2.4.1 Surge Hydrodynamic Coefficients

From Sarpkaya and Isaacson (1981), the components of the radiated force can be written as,

$$F_i^{(R)} = -\sum_j (\mu_{ij}\frac{\partial^2 X_j}{\partial t^2} + \lambda_{ij}\frac{\partial X_j}{\partial t}) \tag{2.72}$$

where μ_{ij} and λ_{ij} are respectively called the added-mass and damping coefficients. Those coefficients are taken to be real. These are termed added-mass and damping coefficients respectively, since they assume corresponding roles in the equations of motion.

The equation of motion can be written as

$$(M_{ij} + \mu_{ij})\frac{\partial^2 X_j}{\partial t^2} + \lambda_{ij}\frac{\partial X_j}{\partial t} + C_{ij}X_j = F_i^{(e)} \tag{2.73}$$

where M_{ij} is the mass matrix, C_{ij} the hydrodynamic stiffness matrix and $F_i^{(e)}$ are the exciting forces associated with the diffraction potential.

The exciting force can be considered as the forcing function of the motion. It is emphasized that this equation relates to an unrestricted floating or submerged body. The added-mass μ_{ij} are analogous to those coefficients for a body accelerating in an unbounded fluid: but they are not the same. The damping coefficients λ_{ij} are associated with a net outward flux of energy in the radiated waves and thus represent only damping due to the (radiating) fluid motion. μ_{ij} and λ_{ij} are not dimensionless coefficients but possess appropriate dimensions.

The radiated force F_{r1} due to the surge motion can be written as the real part of $f_{r1}e^{-i\sigma t}$ where f_{r1} is given by

$$\begin{aligned} f_{r1} &= 2i\rho a^2\sigma A\int_0^{\pi}\int_0^{\pi}\hat{X}_1\phi_1(a,\theta,\psi)n_1\sin\theta d\theta d\psi \\ &= -2i\rho a^2\sigma A\int_0^{\pi}\int_0^{\pi}\hat{X}_1\phi_1(a,\theta,\psi)\sin^2\theta\cos\psi d\theta d\psi \end{aligned} \quad (2.74)$$

This radiated force can be conveniently decomposed into components in phase with the velocity and the acceleration,

$$F_{r1} = -(\mu_{11}\frac{\partial^2 X_1}{\partial t^2} + \lambda_{11}\frac{\partial X_1}{\partial t}) \quad (2.75)$$

Also, as $X_1 = Re\{\hat{X}_1 e^{-i\sigma t}\}$, we can write,

$$\begin{aligned} F_{r1} &= Re\{(-\sigma^2)\mu_{11}\hat{X}_1 + \lambda_{11}(-i\sigma)\hat{X}_1\} \\ &= Re\{\sigma^2\mu_{11}\hat{X}_1 + i\sigma\lambda\hat{X}_1\} \end{aligned} \quad (2.76)$$

which implies

$$\sigma^2\mu_{11}\hat{X}_1 + i\sigma\lambda_{11}\hat{X}_1 = -2i\rho a^2\sigma A\int_0^{\pi}\int_0^{\pi}\hat{X}_1\phi_1(a,\theta,\psi)\sin^2\theta\cos\psi d\theta d\psi$$

which will give us

$$\mu_{11} + i\frac{\lambda_{11}}{\sigma} = -\frac{2i\rho a^2 A}{\sigma}\int_0^{\pi}\int_0^{\pi}\phi_1(a,\theta,\psi)\sin^2\theta\cos\psi d\theta d\psi \tag{2.77}$$

Hence, the added-mass and the damping coefficients are respectively given by

$$\mu_{11} = -\frac{2\rho A a^2}{\sigma}\int_0^{\pi}\int_0^{\pi} Re[i\phi_1(a,\theta,\psi)]\sin^2\theta\cos\psi d\theta d\psi \tag{2.78}$$

$$\lambda_{11} = -2\rho A a^2\int_0^{\pi}\int_0^{\pi} Im[i\phi_1(a,\theta,\psi)]\sin^2\theta\cos\psi d\theta d\psi \tag{2.79}$$

The surge potential $\phi_1(r,\theta,\psi)$ can be written from (2.58) as,

$$\phi_1(r,\theta,\psi) = [\frac{P_n^1(\cos\theta)}{r^{n+1}} + \sum_{n=1}^{\infty} D_n r^n P_n^1(\cos\theta)]\cos\psi \tag{2.80}$$

Hence, at $r = a$, we may write

$$\phi_1(a,\theta,\psi) = [\frac{P_n^1(\cos\theta)}{a^{n+1}}\sum_{n=1}^{\infty} D_n a^n P_n^1(\cos\theta)]\cos\psi \tag{2.81}$$

Therefore, using eqn (2.81) in eqns (2.78) and (2.79) and simplifying by use of associated Legendre polynomials, we obtain the added-mass and damping coefficients as

$$\begin{aligned}\mu_{11} &= -\frac{2\rho A a^2}{\sigma}\int_0^{\pi}\int_0^{\pi}[\sum_{n=1}^{\infty} -B_n a^n P_n^1(\cos\theta)]\cos\psi \times P_1^1(\cos\theta)\sin\theta\cos\psi d\theta d\psi \\ &= \frac{4}{3}\frac{\rho a^3\pi A}{\sigma}B_1\end{aligned} \tag{2.82}$$

and

$$\begin{aligned}\lambda_{11} &= -2\rho A a^2\int_0^{\pi}\int_0^{\pi}[\frac{P_n^1(\cos\theta)}{a^{n+1}} + \sum_{n=1}^{\infty} A_n a^n P_n^1(\cos\theta)]\cos\psi \times P_1^1(\cos\theta)\sin\theta d\theta d\psi \\ &= -\frac{4}{3}\rho\pi A[1 + A_1 a^3]\end{aligned} \tag{2.83}$$

Alternately, we can represent μ_{11} and λ_{11} as

$$\frac{\mu_{11}}{\frac{3}{4}\frac{\rho a^3 \pi A}{\sigma}} = B_1$$

and

$$\frac{\lambda_{11}}{\frac{4}{3}\rho\pi A} = -[1 + A_1 a^3]$$

From the equation of motion, we get

$$M_{11}\frac{\partial^2 X_1}{\partial t^2} = -\mu_{11}\frac{\partial^2 X_1}{\partial t^2} - \lambda_{11}\frac{\partial X_1}{\partial t} + F_x^{(e)} \tag{2.84}$$

where M_{11} is the mass of the displaced fluid, μ_{11} is the surge added-mass, λ_{11} is the surge damping coefficient and $F_x^{(e)}$ is the x-component of the exciting force.

In complex form, the equation of motion can be summed up as:

$$(M_{11} + \mu_{11})(-i\sigma)^2 \hat{X}_1 + (-i\sigma)\hat{X}_1\lambda_{11} = f_{xd}$$

which simplifies to

$$(M_{11} + \mu_{11})\sigma^2 \hat{X}_1 + i\sigma\hat{X}_1\lambda_{11} = -f_{xd} \tag{2.85}$$

where

$$f_{xd} = -2i\rho\sigma a^2 A \int_0^{\pi} \int_0^{\pi} \phi_{ID}(a,\theta,\psi)\sin^2\theta\cos\psi d\theta d\psi$$

This implies

$$\hat{X}_1 = -\frac{f_{xd}}{\sigma^2(M_{11} + \mu_{11} + \frac{i\lambda_{11}}{\sigma})} \tag{2.86}$$

2.4.2 Heave Hydrodynamic Coefficients

The radiated force F_{r3} due to the heave motion can be written as the real part of $f_{r3}e^{-i\sigma t}$ where f_{r3} is given by

$$f_{r3} = -2i\rho a^2 A\sigma \int_0^{\pi} \int_0^{\pi} \hat{X}_3\phi_3(a,\theta,\psi)\sin\theta\cos\theta d\theta d\psi \tag{2.87}$$

Considering $X_3 = Re\{\hat{X}_3 e^{-i\sigma t}\}$, we have, proceeding as in the previous subsection,

$$\mu_{33} + i\frac{\lambda_{33}}{\sigma} = -\frac{2i\rho A a^2}{\sigma} \int_0^{\pi} \int_0^{\pi} \phi_3(a,\theta,\psi)\sin\theta\cos\theta d\theta d\psi \tag{2.88}$$

where μ_{33} and λ_{33} are the heave added-mass and the damping coefficient due to heave motion respectively. Hence,

$$\mu_{33} = -\frac{2\rho A a^2}{\sigma} \int_0^{\pi} \int_0^{\pi} Re[i\phi_3(a,\theta,\psi)]\sin\theta d\theta d\psi \tag{2.89}$$

$$\lambda_{33} = -2\rho A a^2 \int_0^{\pi} \int_0^{\pi} Im[i\phi_3(a,\theta,\psi)]\sin\theta\cos\theta d\theta d\psi \tag{2.90}$$

The heave potential $\phi_3(r,\theta,\psi)$ can be written from (2.58) as

$$\phi_3(r,\theta,\psi) = \frac{P_n^0(\cos\theta)}{r^{n+1}} + \sum_{n=0}^{\infty} D_n r^n P_n^0(\cos\theta) \tag{2.91}$$

Hence at $r = a$, we may write

$$\phi_3(a,\theta,\psi) = \frac{P_n^0(\cos\theta)}{a^{n+1}} + \sum_{n=0}^{\infty} D_n a^n P_n^0(\cos\theta) \tag{2.92}$$

Therefore, using eqn (2.92) in eqns (2.89) and (2.90) and simplifying by the associated Legendre polynomials, we obtain the heave coefficients as

$$\begin{aligned} \mu_{33} &= -\frac{2\rho a^2 A}{\sigma} \int_0^{\pi} \int_0^{\pi} \left(\sum_{n=0}^{\infty} B_n a^n P_n^0(\cos\theta)\right) \times P_1^0(\cos\theta)\sin\theta d\theta d\psi \\ &= \frac{4}{3}\frac{\rho a^3 \pi A}{\sigma} B_1 \end{aligned} \tag{2.93}$$

and

$$\begin{aligned}\lambda_{33} &= -2\rho a^2 A \int_0^{\pi} \int_0^{\pi} [\frac{P_n^0(\cos\theta)}{a^{n+1}} + \sum_{n=0}^{\infty} A_n a^n P_n^0(\cos\theta)] P_1^0(\cos\theta) \sin\theta d\theta d\psi \\ &= -\frac{4}{3}\frac{\rho\pi A}{\sigma}(1 + A_1 a^3) \end{aligned} \tag{2.94}$$

Or else we can represent μ_{33} and λ_{33} as:

$$\frac{\mu_{33}}{\frac{4}{3}\rho a^3 A\pi} = B_1$$

and

$$\frac{\lambda_{33}}{\frac{4}{3}\frac{\rho A\pi}{\sigma}} = -[1 + A_1 a^3]$$

The equation of motion in the complex form can be written as:

$$\nu_{33} + (M_{33} + \mu_{33})(-i\sigma)^2 \hat{X}_3 + (i\sigma)\hat{X}_3\lambda_{33} = f_{zd} \tag{2.95}$$

where ν_{33} is the restoring coefficient and and f_{zd} may be expressed as,

$$f_{zd} = -2i\rho\sigma a^2 A \int_0^{\pi} \int_0^{\pi} \phi_{ID}(a, \theta, \psi) \sin\theta \cos\theta d\theta d\psi$$

which implies

$$\hat{X}_3 = \frac{f_{zd}}{\nu_{33} - \sigma^2(M_{33} + \mu_{33} + i\frac{\lambda_{33}}{\sigma})} \tag{2.96}$$

2.5 Evaluation of Forces

This section is concerned with the evaluation of wave forces due to the combined effects of diffraction and radiation. We will derive the forces acting along x and z directions. The component of the horizontal force f_x can be computed from

$$f_x = f_{xd} + f_{x1} \tag{2.97}$$

where f_{xd} is the x-component of the diffraction force and f_{x1} the force due to the surge motion. The mathematical expression for each case is given by

$$f_{xd} = -2i\rho\sigma a^2 A \int_0^\pi \int_0^\pi \phi_{ID}(a,\theta,\psi)\sin^2\theta\cos\psi d\theta d\psi \tag{2.98}$$

and

$$f_{x1} = -2i\rho\sigma a^2 A\hat{X}_1 \int_0^\pi \int_0^\pi \phi_1(a,\theta,\psi)\sin^2\theta\cos\psi d\theta d\psi \tag{2.99}$$

The vertical force component f_z can be written as

$$f_z = f_{zd} + f_{z3} \tag{2.100}$$

where f_{zd} is the z-component of diffraction force and f_{z3} the force due to the heave motion. The mathematical expression for each case is given by

$$f_{zd} = -2i\rho\sigma a^2 A \int_0^\pi \int_0^\pi \phi_{ID}(a,\theta,\psi)\sin\theta\cos\theta d\theta d\psi \tag{2.101}$$

and

$$f_{z3} = -2i\rho\sigma a^2 A\hat{X}_3 \int_0^\pi \int_0^\pi \phi_3(a,\theta,\psi)\sin\theta\cos\theta d\theta d\psi \tag{2.102}$$

Substituting the value of $\phi_1(a,\theta,\psi)$ from eqn (2.81) into eqn (2.99), we can evaluate f_{x1} as

$$\begin{aligned} f_{x1} &= -2i\rho\sigma a^2\pi A\hat{X}_1 \int_0^\pi \int_0^\pi \Big[\frac{P_n^1(\cos\theta)}{a^{n+1}} + \sum_{n=1}^{\infty} D_n a^n P_n^1(\cos\theta)\Big]\cos\psi \\ &\times \; P_1^1(\cos\theta)\cos\psi\sin\theta d\theta d\psi \\ &= -\frac{4}{3}i\rho\pi A\sigma\hat{X}_1(1 + D_1 a^3) \end{aligned} \tag{2.103}$$

Substituting the value of $\phi_3(a, \theta, \psi)$ from eqn (2.92) into eqn (2.102), we can evaluate f_{z3} as,

$$
\begin{aligned}
f_{z3} &= -2i\rho\sigma a^2 A\hat{X}_3 \int_0^{\pi}\int_0^{\pi}[\frac{P_n^0(\cos\theta)}{a^{n+1}} + \sum_{n=0}^{\infty} D_n a^n P_n^0(\cos\theta)]P_1^0(\cos\theta)\sin\theta d\theta d\psi \\
&= -\frac{4}{3}Ai\rho\sigma\pi\hat{X}_3[1 + D_1 a^3] \qquad (2.104)
\end{aligned}
$$

Hence, the total force along x-axis is,

$$
\begin{aligned}
f_x &= f_{xd} + f_{x1} \\
&= -4i\rho\sigma\pi a^3 A_{11} - \frac{4}{3}i\rho\pi\sigma\hat{X}_1(1 + D_1 a^3) \qquad (2.105)
\end{aligned}
$$

and the total force along z-axis is,

$$
\begin{aligned}
f_z &= f_{zd} + f_{z3} \\
&= -4i\rho\sigma\pi a^3 AA_{01} - \frac{4}{3}i\rho\sigma\pi A\hat{X}_3(1 + D_1 a^3) \qquad (2.106)
\end{aligned}
$$

where $\hat{X}_1$ and $\hat{X}_3$ are given by,

$$
\begin{aligned}
\hat{X}_1 &= -\frac{f_{xd}}{\sigma^2(M_{11} + \mu_{11} + i\frac{\lambda_{11}}{\sigma})} \qquad (2.107) \\
\hat{X}_3 &= \frac{f_{zd}}{\nu_{33} - \sigma^2(M_{33} + \mu_{33} + i\frac{\lambda_{33}}{\sigma})} \qquad (2.108)
\end{aligned}
$$

with $\mu_{11}, \lambda_{11}, \mu_{33}$ and λ_{33} as already obtained. Thus, eqns (2.105) and (2.106) respectively give us the total horizontal and vertical forces due to the combined effect of diffraction and radiation. The evaluation of the forces along x and z axis helps us in understanding the combined effect of diffraction and radiation.

The eqns (2.105) and (2.106) respectively representing the forces f_x and f_z can also be written as

$$\frac{f_x}{4i\rho\sigma\pi a^3 A} = -A_{11} - \frac{1}{3}\frac{\hat{X}_1}{a^3}[1 + D_1 a^3] \qquad (2.109)$$

$$\frac{f_z}{4i\rho\sigma\pi a^3 A} = -A_{01} - \frac{1}{3}\frac{\hat{X}_3}{a^3}[1 + D_1 a^3] \qquad (2.110)$$

Chapter 3

Numerical Results and Discussions for Submerged Sphere

In this chapter we present numerical results for the analytical expressions for various hydrodynamic coefficients and loadings (derived in chapter 2) on a submerged sphere. The complex matrix equation (2.31) must be solved in order to determine the unknown coefficients A_{mn} for $m = 0$ and $m = 1$. To compute the horizontal exciting force, f_{xd}, we need to solve eqn (2.43). The vertical exciting force, f_{zd}, is evaluated by solving eqn (2.44). This infinite system of equations represented by eqn (2.31) is made finite and solved it numerically by truncating as

$$\sum_{n=0}^{N_p} A_{mn} E_{ns}^{m} = T_s^m \tag{3.1}$$

where E^m_{ns} and T^m_s are given by

$$\begin{aligned} T^m_s &= -\frac{Agk_0}{\sigma}\epsilon_m i^m (k_0 a)^{s-1}\frac{s}{(s+m)!}X_{s-m} \\ E^m_{ns} &= -(n+1)\delta_{ns} + D^m_n(s-m) \\ D^m_n(s) &= a^{n+1}(s+m)(\frac{a}{2H})^{s+m}[C_s(n,m) + B^m_{ns}] \end{aligned}$$

To compute the radiated forces and the hydrodynamic coefficients due to the motion, we need to find the coefficients $D_n = A_n + iB_n$ from eqn (2.61).The added-mass and the damping coefficients for surge and heave motions are obtained by solving eqns (2.82), (2.83), (2.93) and (2.94). The main task is to find the coefficients A_{mn} and D_n which help us in computing the various loading and hydrodynamic coefficients.

These system of equations are solved by using a complex matrix inversion subroutine from IMSL on HP9000 computer system in the Applied Mathematics Department at the Technical University of Nova Scotia, Halifax, Canada. We select $N_p = 20, N_n = 20$ for our computations. We have observed throughout our numerical calculations that addition of more terms beyond 20 terms does not have any significant effect. Once A_{01}, A_{11} are known, we can compute the exciting forces due to surge and heave motions.

Tables (3.1)-(3.4) give us the exciting force coefficients for both fixed submergence and fixed depth. The results have been compared with the results of Wang (1986) and Wu *et al.* (1994) and they seem to agree with those sets of results.

Table 3.1: Surge exciting forces (h/a=1.25)

	←		d/a		→
Ka	2.5	3.0	5.0	11.0	20.0
.10	3.1539	2.7864	2.1872	1.5893	1.4897
.20	2.1152	2.1152	1.5902	1.3151	1.2621
.30	1.6347	1.3976	1.1861	1.1361	1.1102
.40	1.2862	1.1471	0.9861	0.9858	0.9826
.50	1.1134	0.9876	0.8862	0.8852	0.8834
.60	0.9217	0.8692	0.8682	0.7809	0.8124
.70	0.7692	0.7418	0.7398	0.6947	0.7395
.80	0.6824	0.6675	0.6482	0.6345	0.6315
.90	0.5824	0.5791	0.5789	0.5786	0.5785
1.00	0.5037	0.4981	0.4925	0.4911	0.4901
1.20	0.3476	0.3403	0.3391	0.3379	0.3377

Table 3.2: Heave exciting forces (h/a=1.25)

	←		d/a		→
Ka	2.5	3.0	5.0	11.0	20.0
.1	0.8241	0.9582	1.2041	1.3979	1.4671
.20	0.7965	0.9297	1.1505	1.3192	1.3294
.30	0.7752	0.9042	1.1421	1.2547	1.2609
.40	0.7598	0.8847	1.1167	1.1147	1.1162
.50	0.7421	0.8624	0.9917	0.9867	0.9872
.60	0.7134	0.8261	0.9256	0.9269	0.9283
.70	0.6790	0.7931	0.8291	0.8304	0.8317
.80	0.6224	0.7391	0.7398	0.7404	0.7409
.90	0.5631	0.6112	0.6123	0.6136	0.6149
1.00	0.4832	0.4841	0.4846	0.4850	0.4852
1.10	0.4162	0.4221	0.4247	0.4261	0.4275
1.20	0.3281	0.3289	0.3286	0.3284	0.3283

Table 3.3: Surge exciting forces (d/a=6)

	←	h/a	→
Ka	1.25	1.75	3.00
.1	2.0117	1.8694	1.7021
.2	1.5106	1.2864	0.9462
.3	1.2461	0.9862	0.6741
.4	1.0967	0.7421	0.3909
.5	0.8984	0.6842	0.3646
.6	0.7791	0.5098	0.2517
.7	0.7364	0.4726	0.2021
.8	0.6274	0.3622	0.1271
.9	0.5097	0.2671	0.0983
1.0	0.4892	0.2491	0.0608
1.2	0.3972	0.1977	0.0323
1.4	0.2947	0.1389	0.0086
1.6	0.2566	0.1082	0.0016
1.8	0.2314	0.0627	0.0009

Table 3.4: Heave exciting forces (d/a=6)

	←	h/a	→
Ka	1.25	1.75	3.00
.1	1.2561	1.0692	0.6841
.2	1.2293	0.9542	0.6194
.3	1.2007	0.7781	0.4382
.4	1.1467	0.7392	0.3922
.5	0.9724	0.6107	0.2965
.6	0.8862	0.5566	0.2264
.7	0.6833	0.4192	0.1791
.8	0.6374	0.3643	0.1267
.9	0.5277	0.2818	0.1082
1.0	0.4721	0.2364	0.0927
1.4	0.2021	0.1028	0.0237
1.8	0.1161	0.0711	0.0081
2.0	0.0986	0.0529	0.0072
2.4	0.0583	0.0294	0.0039
2.8	0.0185	0.0129	0.0014
3.0	0.0011	0.0081	0.0005

Exciting force coefficients for the submerged sphere are presented in Figures (3.1)-(3.12) at a fixed submergence ($h/a = 1.25$) for a range of water depths, e.g. $d/a = 2.5, d/a = 3.0, d/a = 5.0, d/a = 11.0$ and $d/a = 20.0$. These results are calculated using eqns (2.43) and (2.44). The results obtained by Wang (1986) for infinite water depth have also been included. There seems to be agreement between both sets of results for deep water within the three figures although different expansions of the velocity potential have been used. In long waves ($Ka < 0.1$), the shallow water heave exciting force at this submergence reduces significantly from that in deep water. The converse is true for surge exciting force where the values in water of depth $2.5a$ are more than double of those in depth $20a$.

Figures (3.1)-(3.5) give the surge exciting forces for various values of d/a at a fixed submergence $h/a = 1.25$. Figure (3.6) presents all the surge exciting forces together for the fixed submergence. Figures (3.7)-(3.11) give the heave exciting forces for the same set of values of d/a at $h/a = 1.25$. Figure (3.12) presents those force coefficients together.

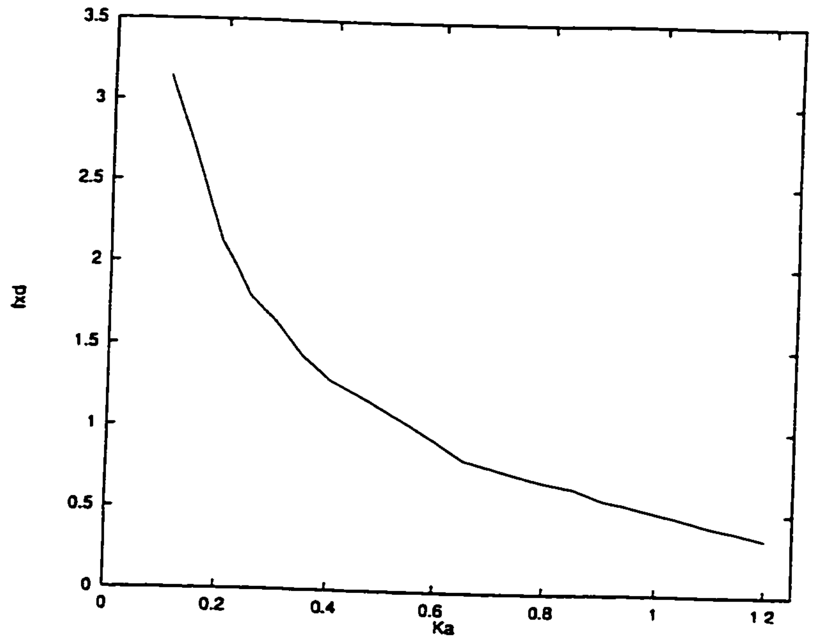

Figure 3.1: Surge exciting force for h/a=1.25 and d/a=2.5

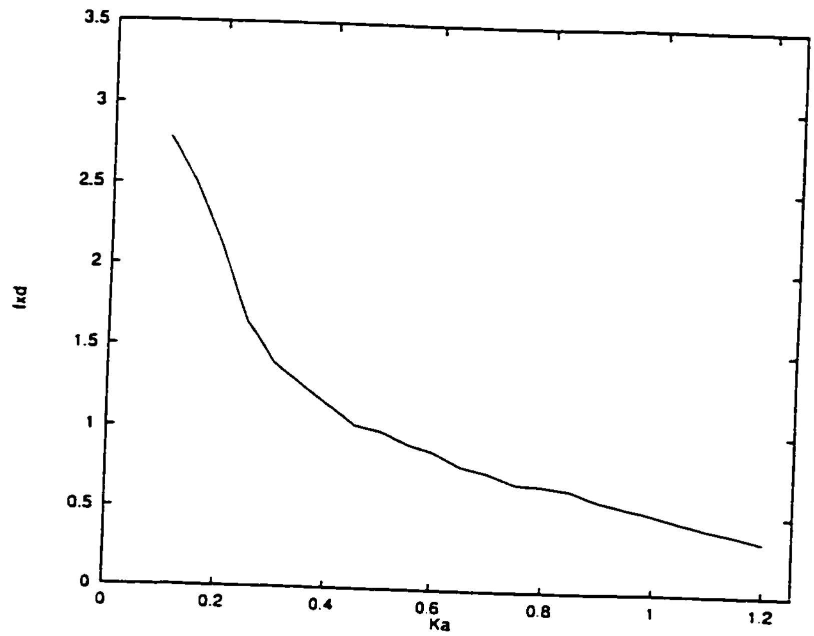

Figure 3.2: Surge exciting force for h/a=1.25 and d/a=3

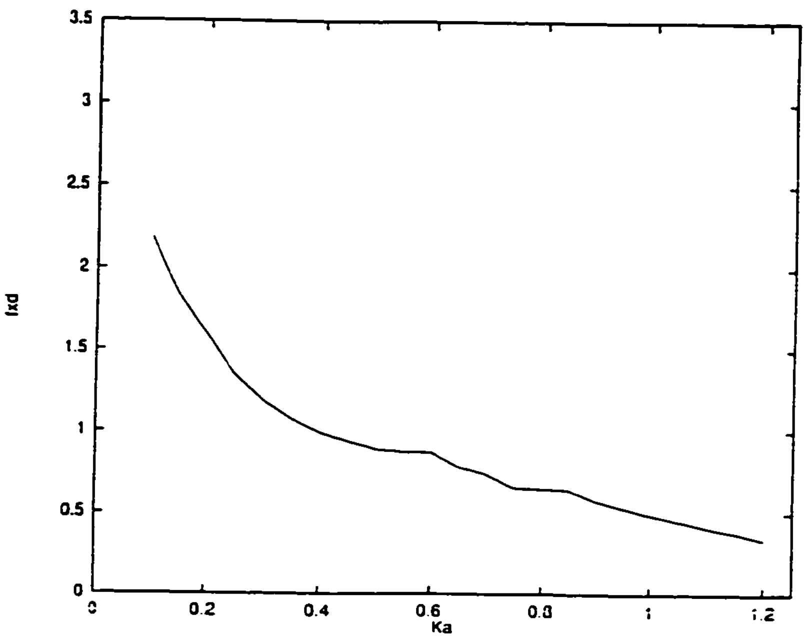

Figure 3.3: Surge exciting force for h/a=1.25 and d/a=5

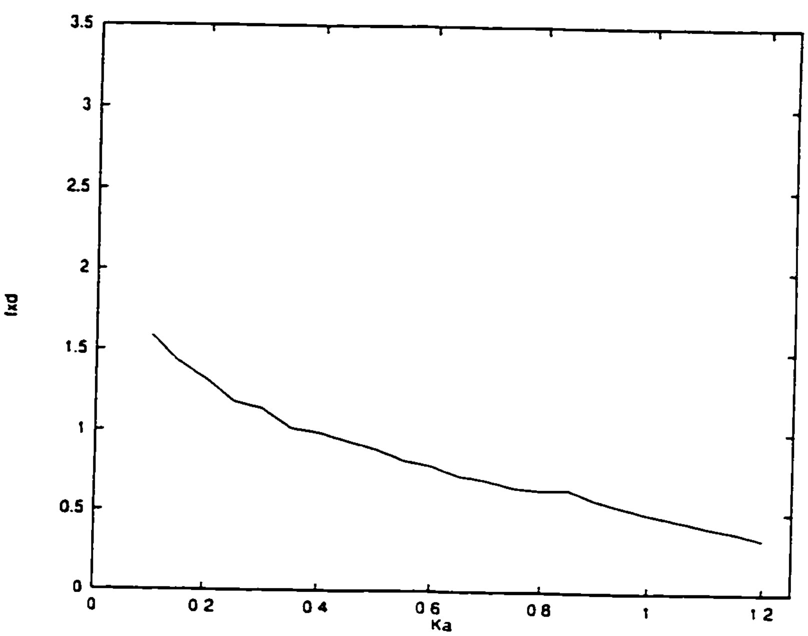

Figure 3.4: Surge exciting force for h/a=1.25 and d/a=11

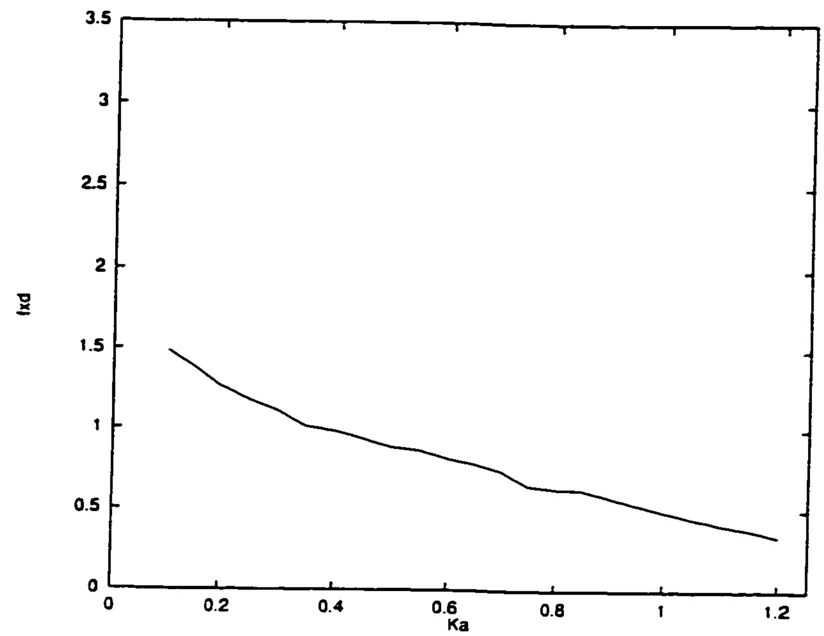

Figure 3.5: Surge exciting force for h/a=1.25 and d/a=20

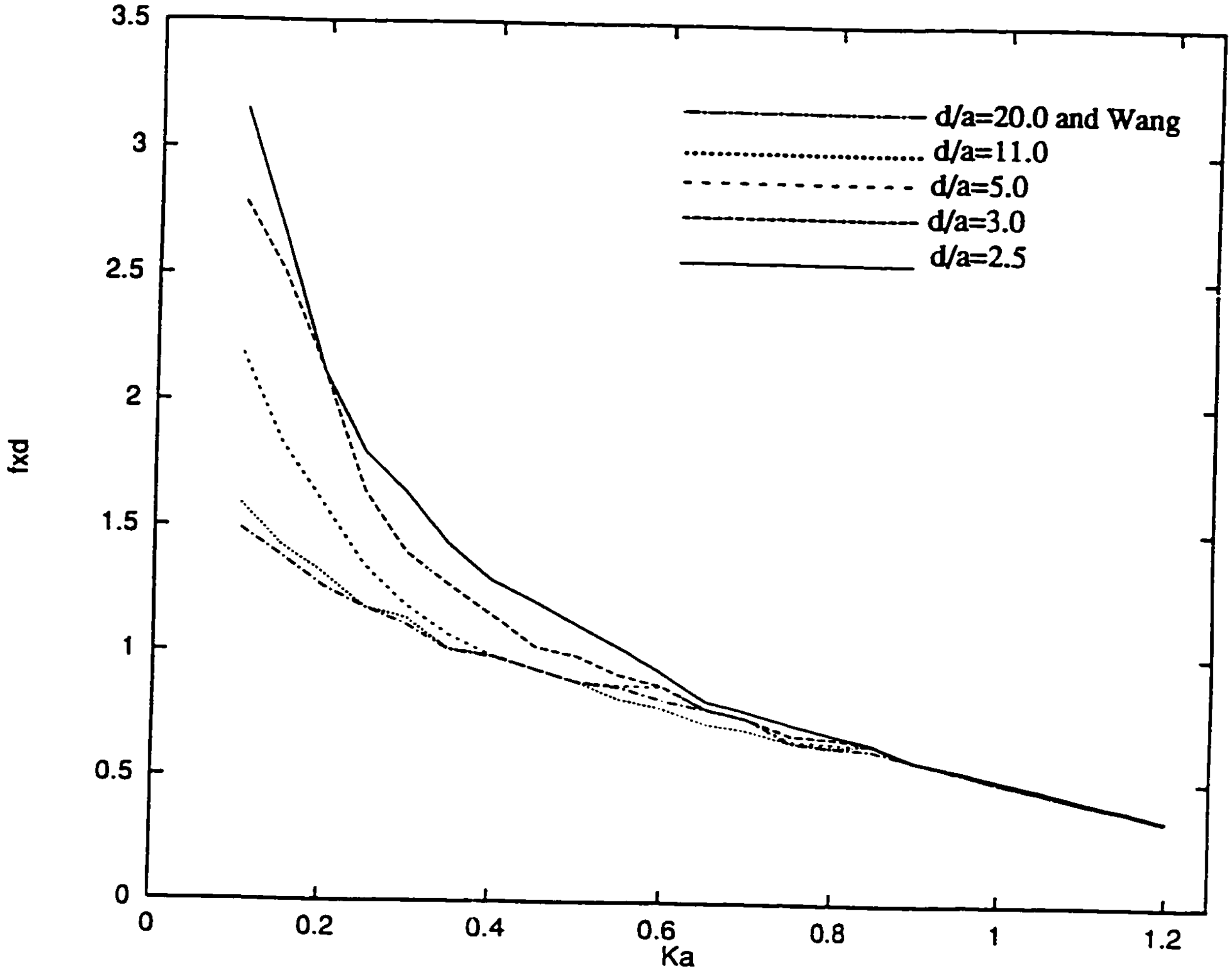

Figure 3.6: Surge exciting forces for h/a=1.25

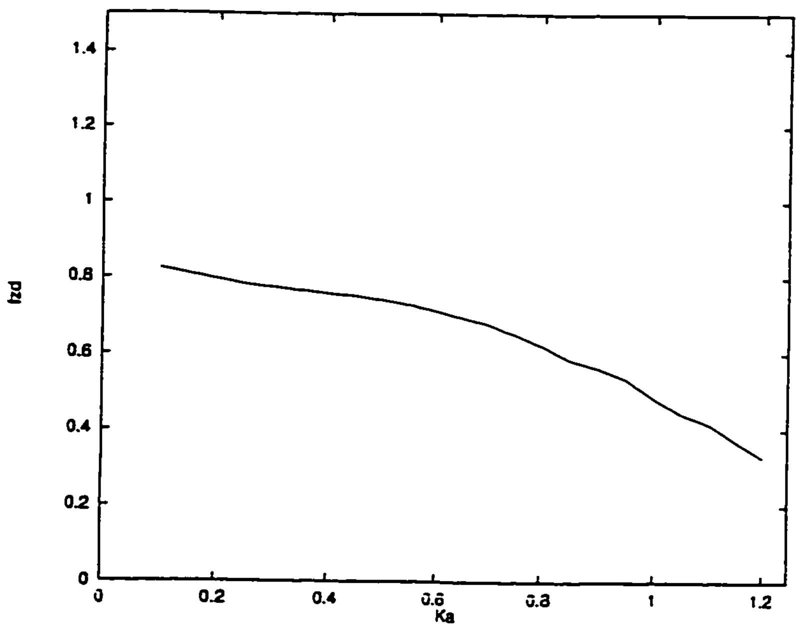

Figure 3.7: Heave exciting force for h/a=1.25 and d/a=2.5

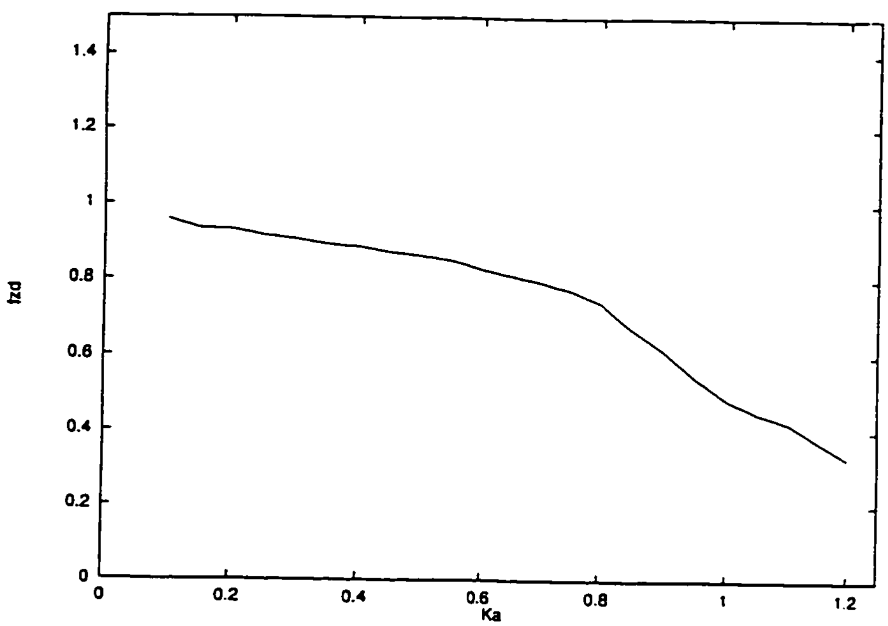

Figure 3.8: Heave exciting force for h/a=1.25 and d/a=3

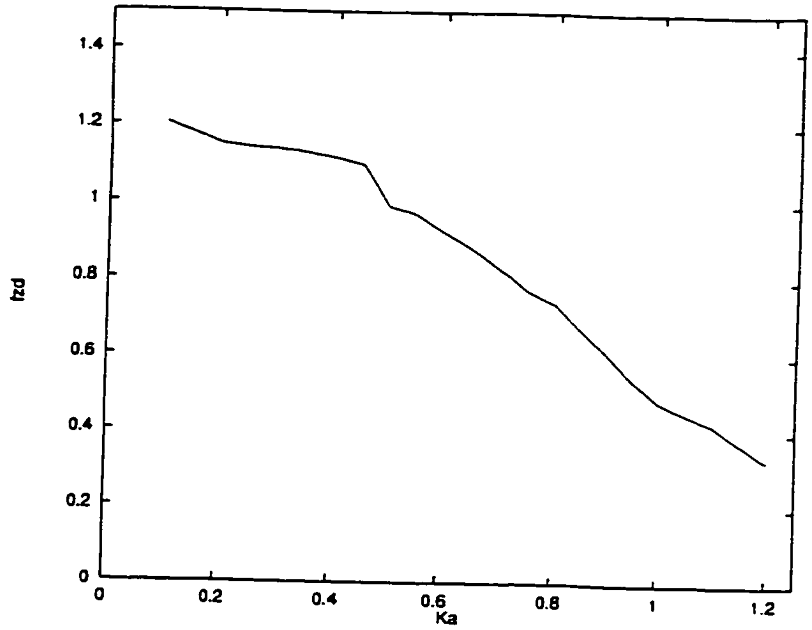

Figure 3.9: Heave exciting force for h/a=1.25 and d/a=5

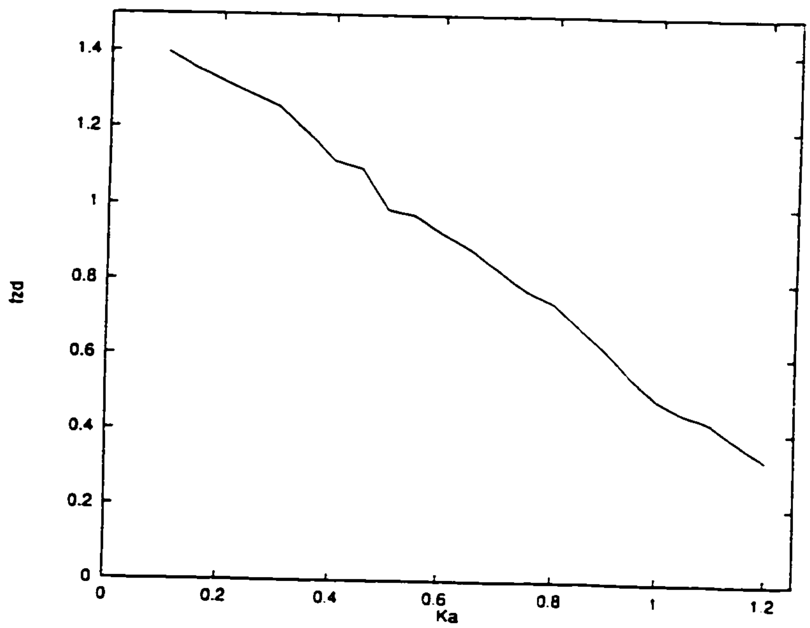

Figure 3.10: Heave exciting force for h/a=1.25 and d/a=11

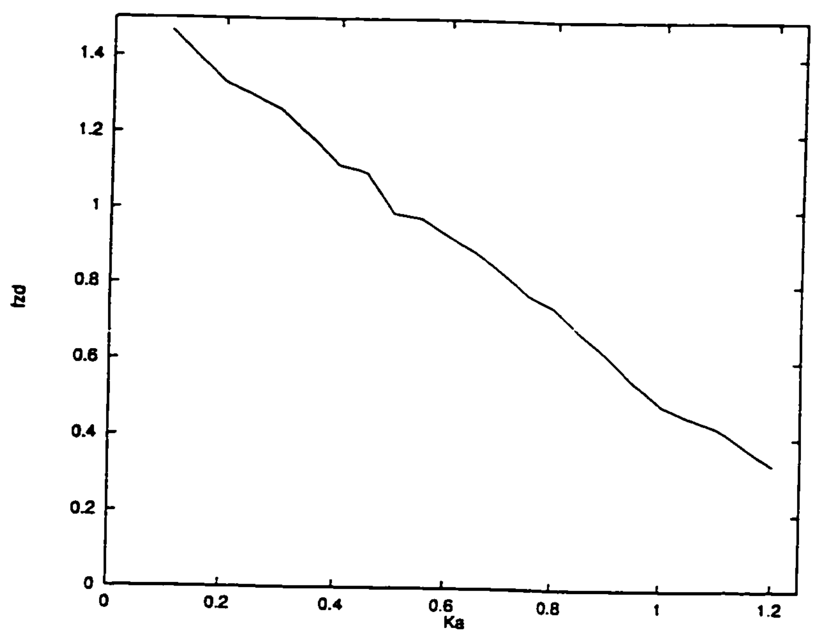

Figure 3.11: Heave exciting force for h/a=1.25 and d/a=20

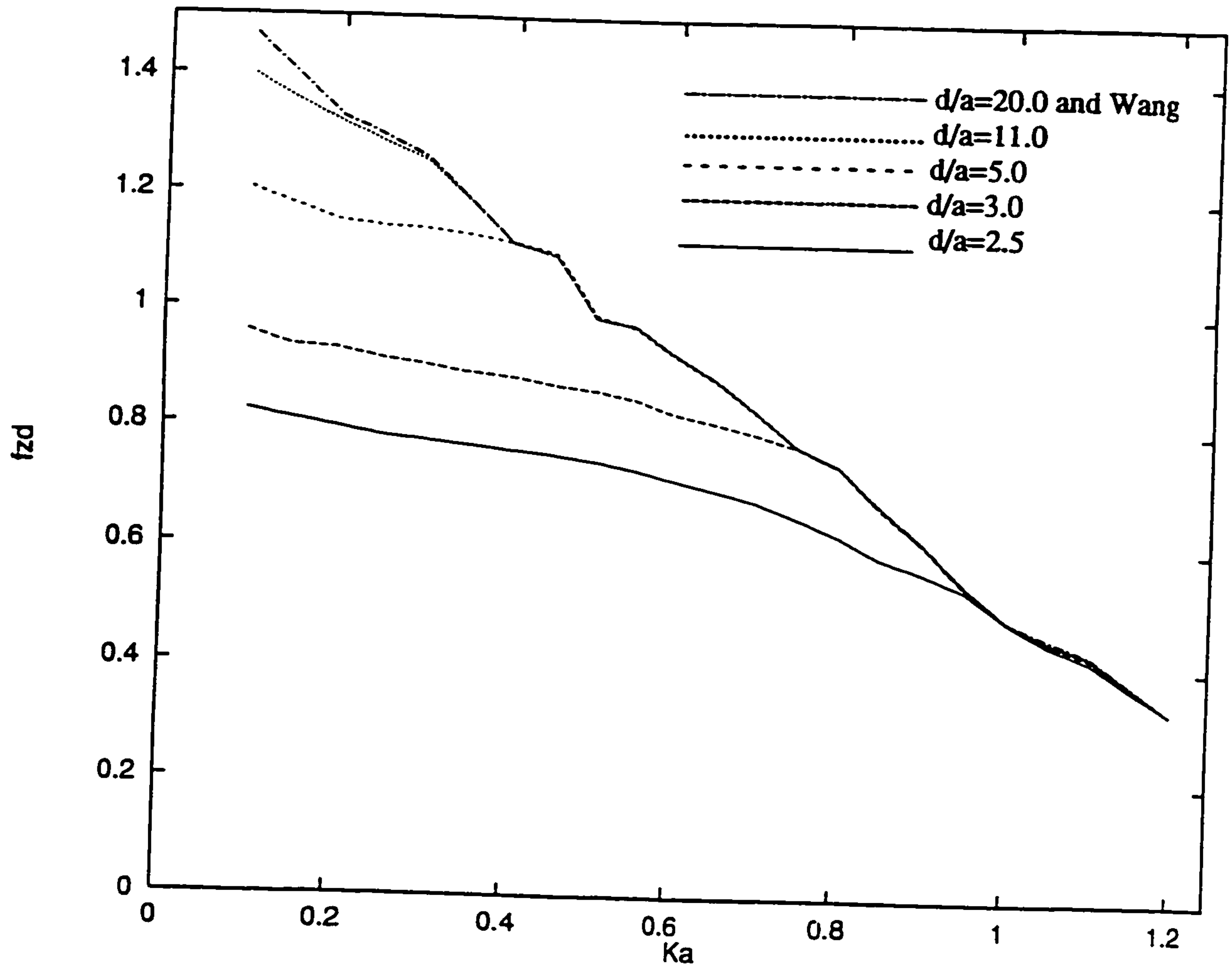

Figure 3.12: Heave exciting forces for h/a=1.25

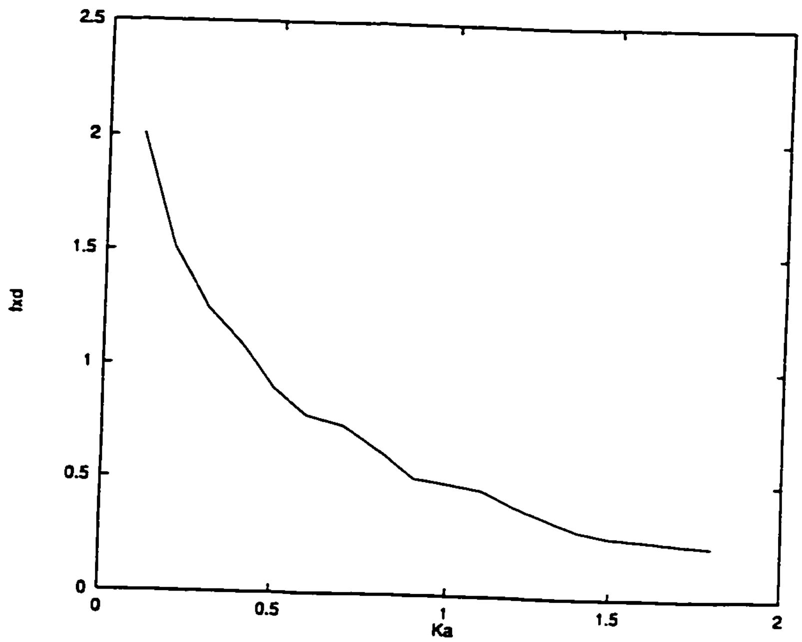

Figure 3.13: Surge exciting force for d/a=6 and h/a=1.25

Figures (3.13)-(3.20) give exciting force coefficients on the sphere in fixed water depth ($d/a = 6.0$) for a range of submergence values. The results indicate that the force coefficients decrease for increased submergence value. Figures (3.13)-(3.16) give the surge exciting forces for different submergence values h/a for a fixed depth d/a whereas Figures (3.16)-(3.20) present the heave exciting forces for the same set of submergence values for the fixed depth d/a.

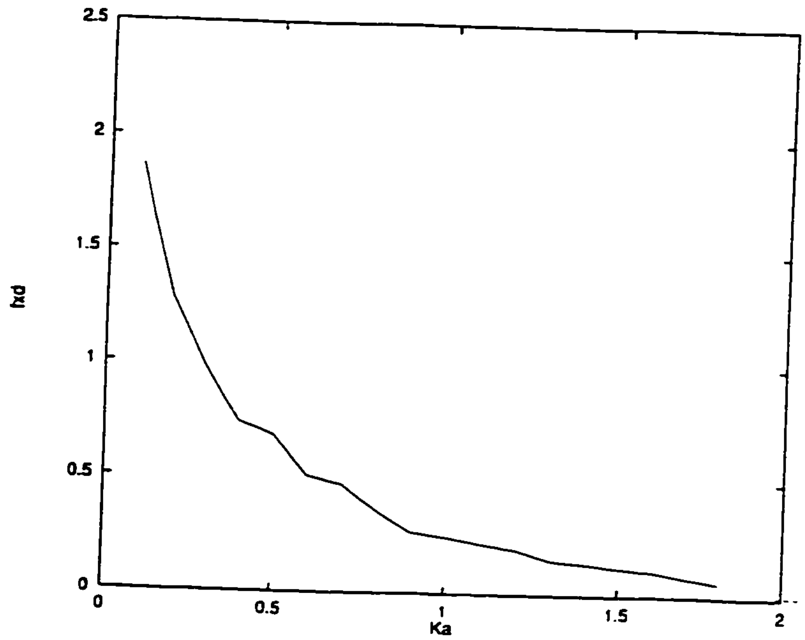

Figure 3.14: Surge exciting force for d/a=6 and h/a=1.75

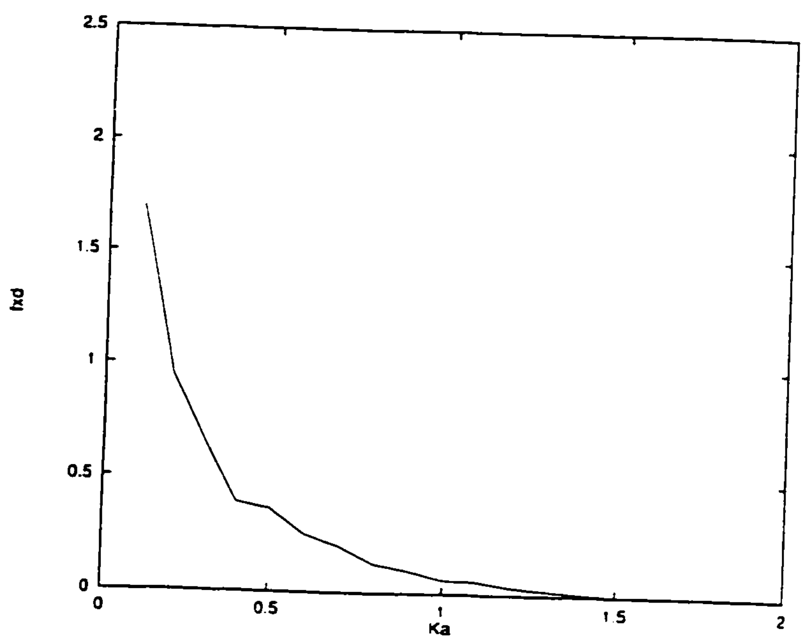

Figure 3.15: Surge exciting force for d/a=6 and h/a=3

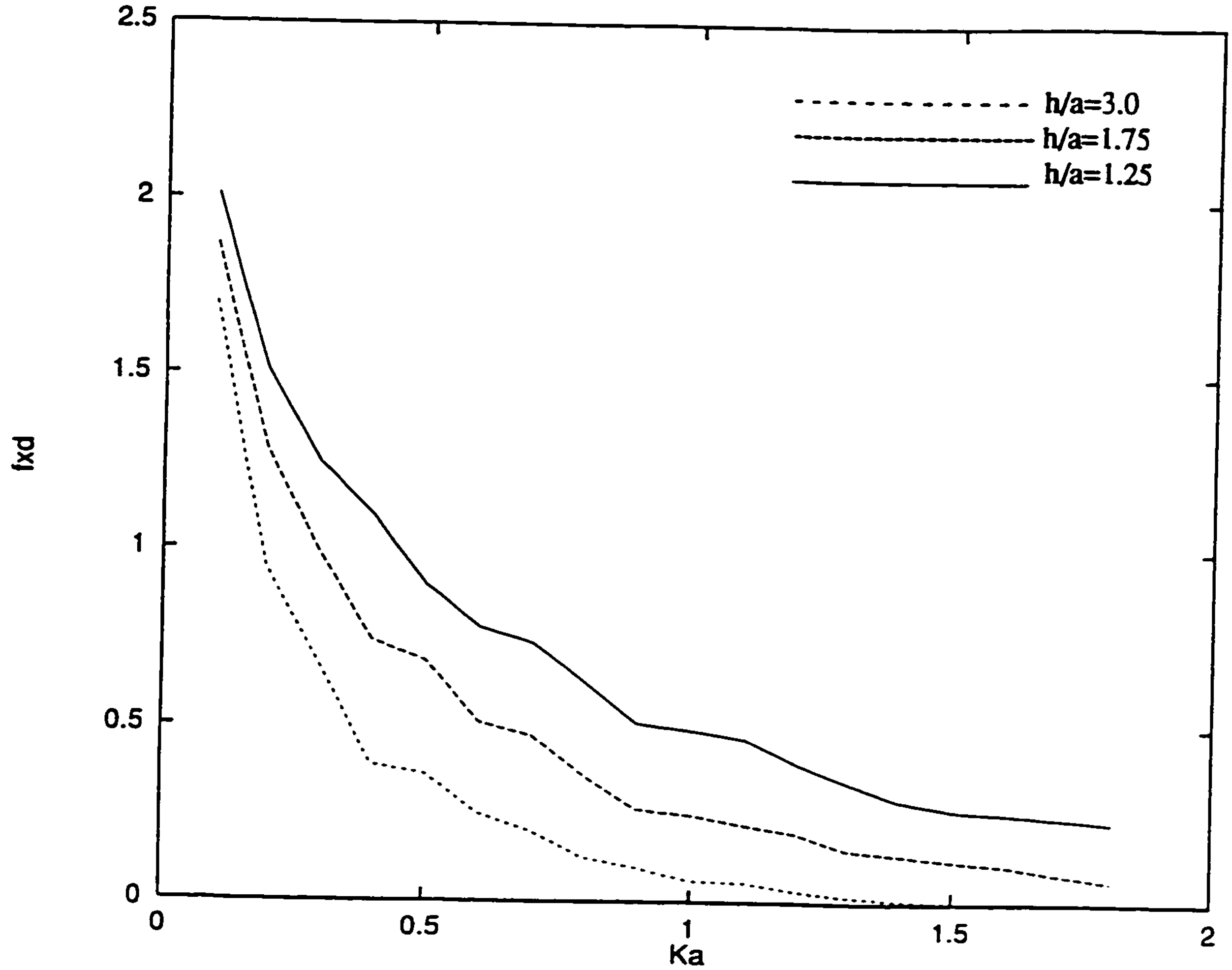

Figure 3.16: Surge exciting forces for d/a=6

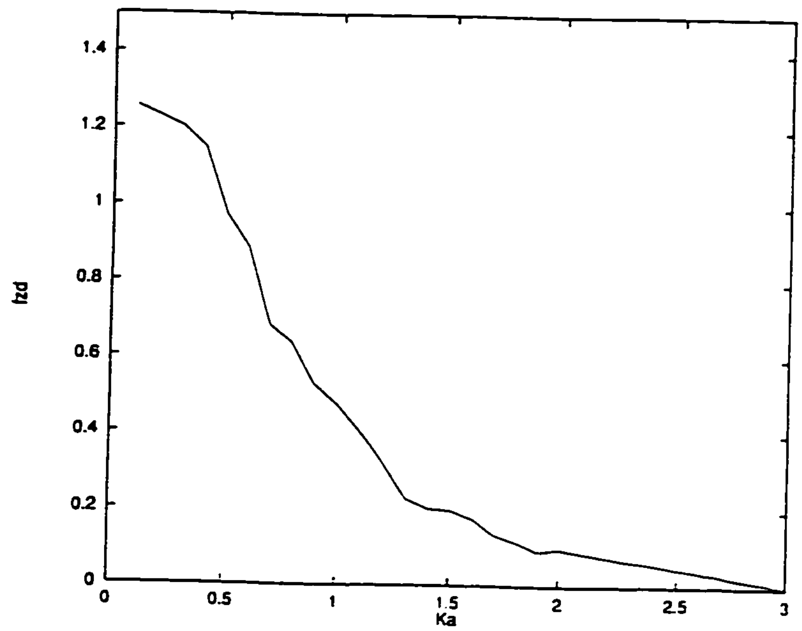

Figure 3.17: Heave exciting force for d/a=6 and h/a=1.25

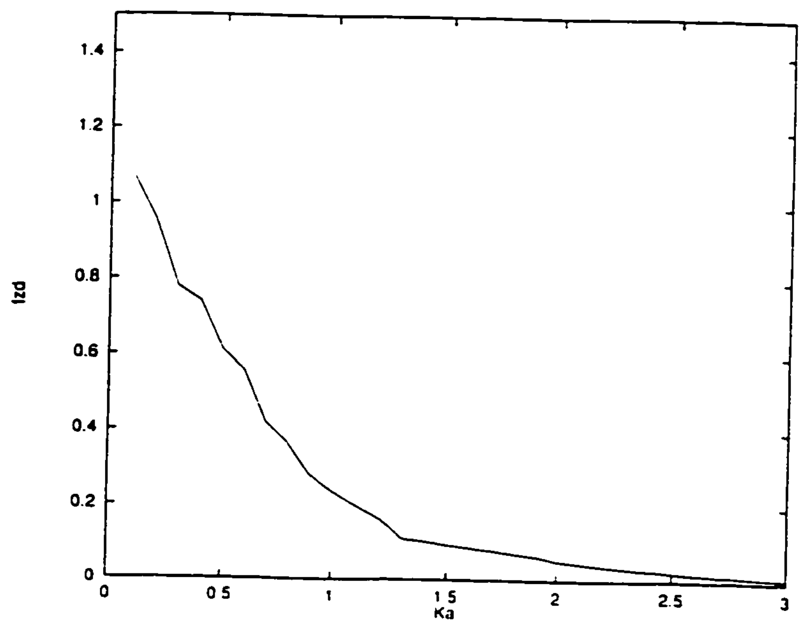

Figure 3.18: Heave exciting force for d/a=6 and h/a=1.75

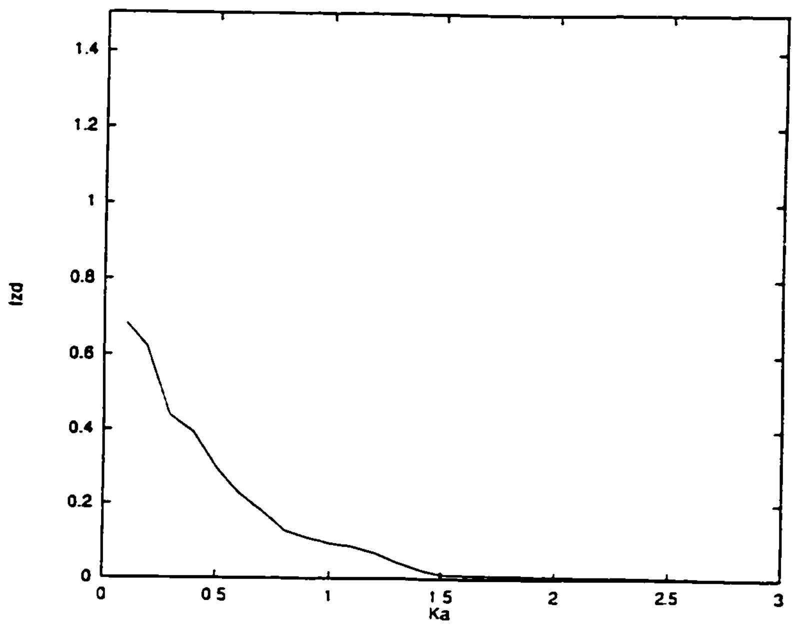

Figure 3.19: Heave exciting force for d/a=6 and h/a=3

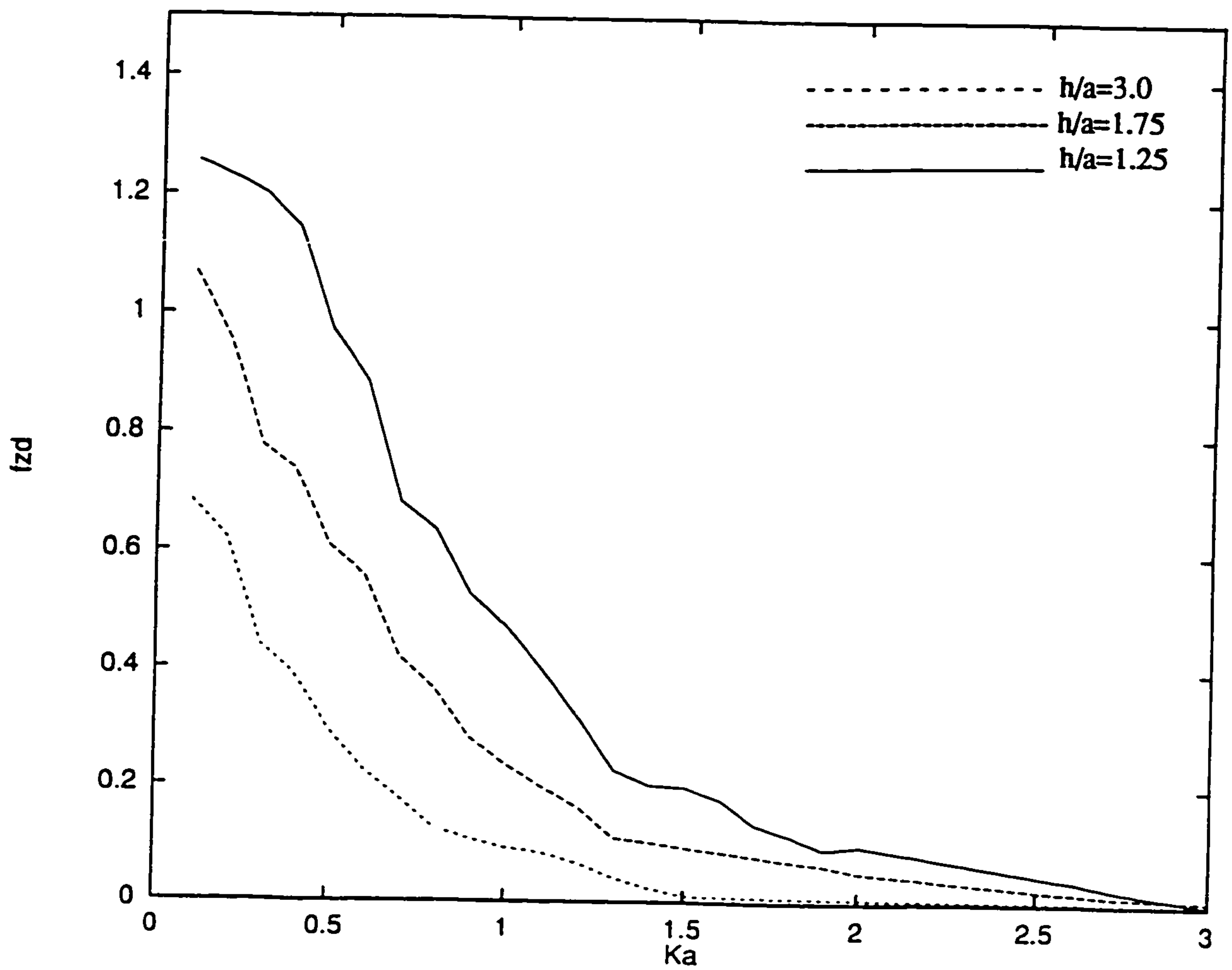

Figure 3.20: Heave exciting forces for d/a=6

Tables (3.5)-(3.8) present the results for the added-mass and damping coefficients for both surge and heave motions for different submergence values. The results show good agreement with those obtained by Wang (1986). From tables (3.5) and (3.7), we see that the added-mass μ_{11} and μ_{33} steadily decrease after reaching the maximum values in the range $0.4 \leq Ka \leq 0.5$. After $Ka = 1.5$ they vary very little. Tables (3.6) and (3.8) show that the damping coefficients start from zero and after certain value of Ka, they decrease uniformly to reach zero again when $Ka = 5.0$. Also, we notice that the damping coefficients are smaller compared to the added-mass for all the submergence values.

Table 3.5: Surge added-mass μ_{11} for different submergence (h/a) values

	←		h/a	→
Ka	1.5	1.75	2.0	3.0
0	0.5287	0.5179	0.5118	0.5034
0.1	0.5403	0.5266	0.5187	0.5066
0.2	0.5545	0.5363	0.5255	0.5082
0.3	0.5656	0.5422	0.5283	0.5069
0.4	0.5693	0.5416	0.5255	0.5030
0.5	0.5646	0.5347	0.5187	0.4986
0.6	0.5527	0.5234	0.5092	0.4949
0.7	0.5359	0.5107	0.4989	0.4920
0.8	0.5160	0.4966	0.4895	0.4905
0.9	0.4962	0.4841	0.4815	0.4893
1.0	0.4776	0.4732	0.4752	0.4896
1.2	0.4475	0.4578	0.4675	0.4903
1.4	0.4286	0.4497	0.4648	0.4915
1.6	0.4189	0.4475	0.4652	0.4925
1.8	0.4158	0.4481	0.4676	0.4930
2.0	0.4171	0.4505	0.4698	0.4938
3.0	0.4381	0.4653	0.4787	0.4950
4.0	0.4523	0.4721	0.4825	0.4955
5.0	0.4582	0.4750	0.4839	0.4966

Table 3.6: Surge damping coefficients λ_{11} for different submergence (h/a) values

	←		h/a	→
Ka	1.5	1.75	2.0	3.0
0	0.	0.	0.	0.
0.1	0.0018	0.0017	0.0016	0.0013
0.2	0.0113	0.0098	0.0088	0.0057
0.3	0.0285	0.0237	0.0200	0.0106
0.4	0.0506	0.0398	0.0317	0.0138
0.5	0.0734	0.0544	0.0412	0.0147
0.6	0.0934	0.0655	0.0472	0.0138
0.7	0.1082	0.0722	0.0496	0.0120
0.8	0.1172	0.0745	0.0489	0.0099
0.9	0.1205	0.0733	0.0460	0.0076
1.0	0.1190	0.0695	0.0418	0.0057
1.2	0.1063	0.0574	0.0317	0.0030
1.4	0.0873	0.0438	0.0223	0.0014
1.6	0.0678	0.0318	0.0148	0.0006
1.8	0.0504	0.0220	0.0094	0.0003
2.0	0.0363	0.0148	0.0058	0.0001
3.0	0.0053	0.0015	0.0004	0.0000
4.0	0.0005	0.0001	0.0000	0.0000
5.0	0.0000	0.0000	0.0000	0.0000

Table 3.7: Heave added-mass μ_{33} for different submergence (h/a) values

	←		h/a	→
Ka	1.5	1.75	2.0	3.0
0.0	0.5586	0.5362	0.5239	0.5070
0.1	0.5834	0.5539	0.5375	0.5131
0.2	0.6139	0.5742	0.5518	0.5166
0.3	0.6365	0.5859	0.5570	0.5133
0.4	0.6421	0.5831	0.5506	0.5055
0.5	0.6272	0.5667	0.5350	0.4969
0.6	0.5955	0.5414	0.5147	0.4895
0.7	0.5541	0.5127	0.4939	0.4845
0.8	0.5095	0.4846	0.4752	0.4890
0.9	0.4680	0.4598	0.4598	0.4794
1.0	0.4316	0.4394	0.4481	0.4793
1.2	0.3788	0.4123	0.4346	0.4805
1.4	0.3497	0.3998	0.4306	0.4827
1.6	0.3381	0.3971	0.4321	0.4847
1.8	0.3374	0.4000	0.4362	0.4863
2.0	0.3428	0.4055	0.4412	0.4874
3.0	0.3852	0.4331	o.4587	0.4901
4.0	0.4091	0.4457	0.4654	0.4910
5.0	0.4203	0.4513	0.4686	0.4918

Table 3.8: Heave damping coefficients λ_{33} for different submergence (h/a) values

	←		h/a	→
Ka	1.5	1.75	2.0	3.0
0	0.	0.	0.	0.
0.1	0.0040	0.0036	0.0033	0.0026
0.2	0.0245	0.0208	0.0182	0.0116
0.3	0.0631	0.0505	0.0416	0.0215
0.4	0.1129	0.0847	0.0658	0.0276
0.5	0.1627	0.1149	0.0848	0.0293
0.6	0.2037	0.1361	0.0958	0.0275
0.7	0.2304	0.1473	0.0991	0.0237
0.8	0.2423	0.1490	0.0964	0.0193
0.9	0.2414	0.1439	0.0896	0.0150
1.0	0.2318	0.1340	0.0805	0.0115
1.2	0.1966	0.1078	0.0604	0.0059
1.4	0.1554	0.0809	0.0421	0.0028
1.6	0.1172	0.0579	0.0279	0.0013
1.8	0.0856	0.0399	0.0177	0.0005
2.0	0.0609	0.0267	0.0109	0.0002
3.0	0.0085	0.0026	0.0007	0.0000
4.0	0.0009	0.0002	0.0003	0.0000
5.0	0.0001	0.0000	0.0000	0.0000

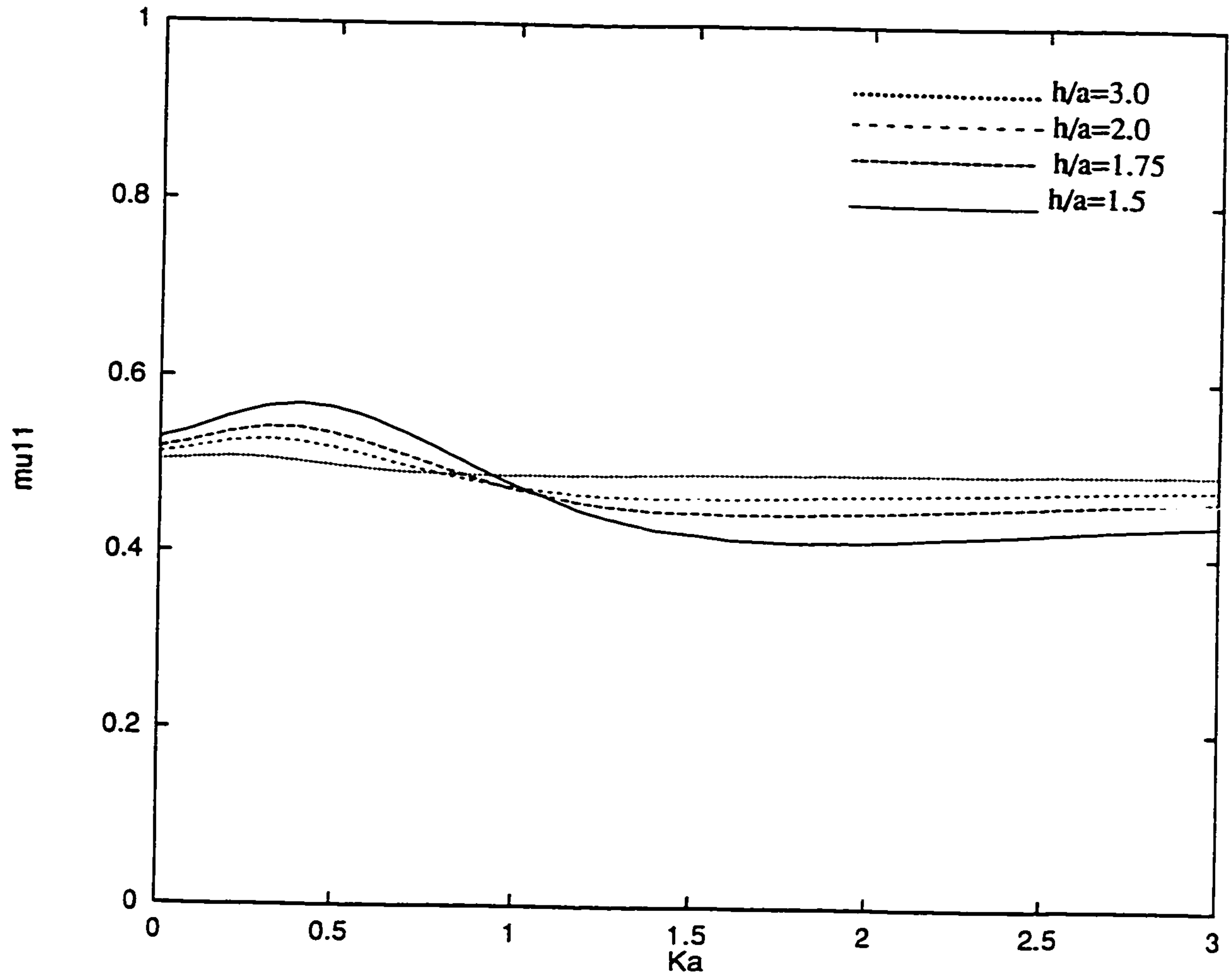

Figure 3.21: Surge added-mass for various values of h/a

Figures (3.21)-(3.24) respectively give the surge added-mass μ_{11}, surge damping coefficients λ_{11}, heave added-mass μ_{33} and heave damping coefficients λ_{33} for various submergence values.

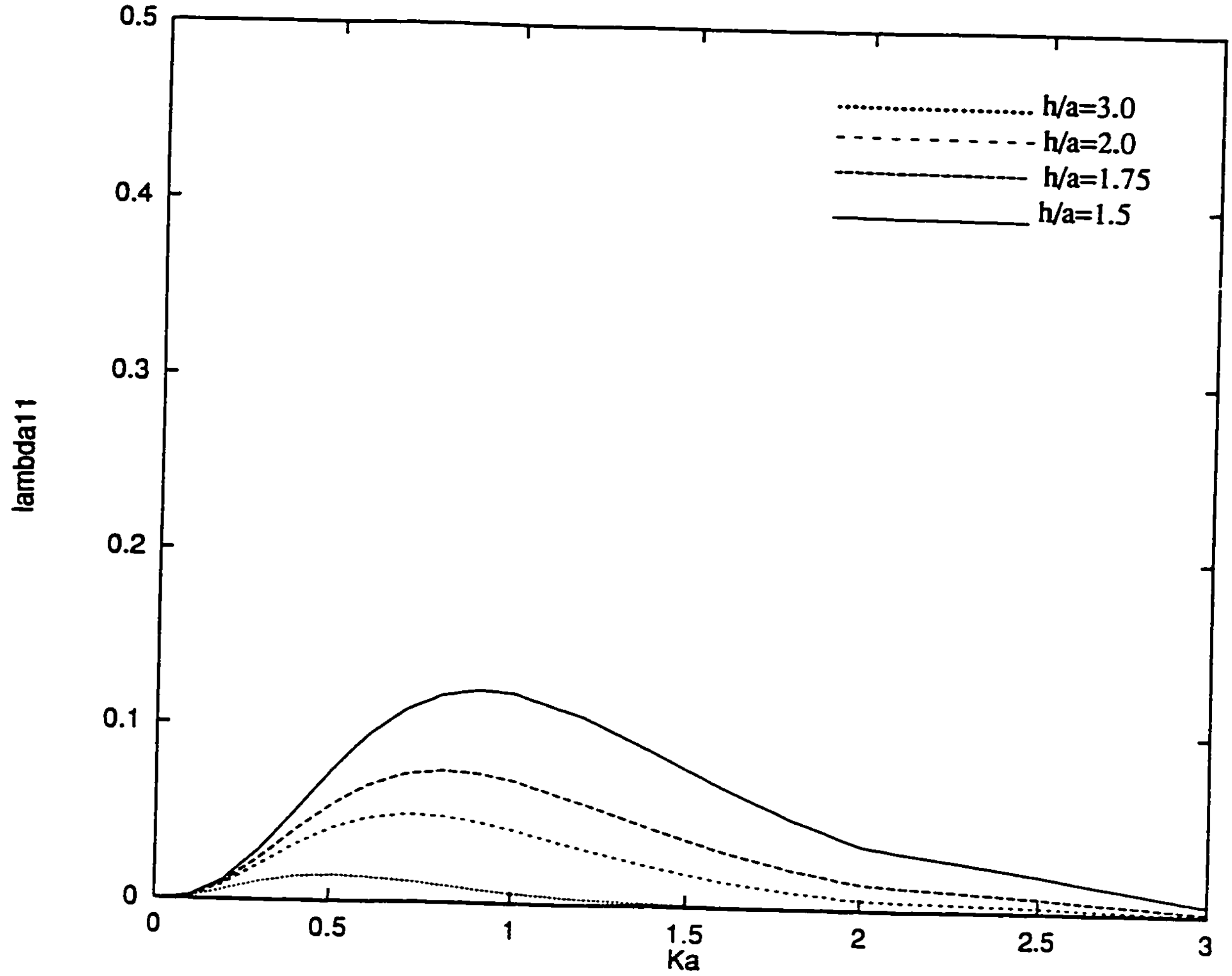

Figure 3.22: Surge damping coefficients for various values of h/a

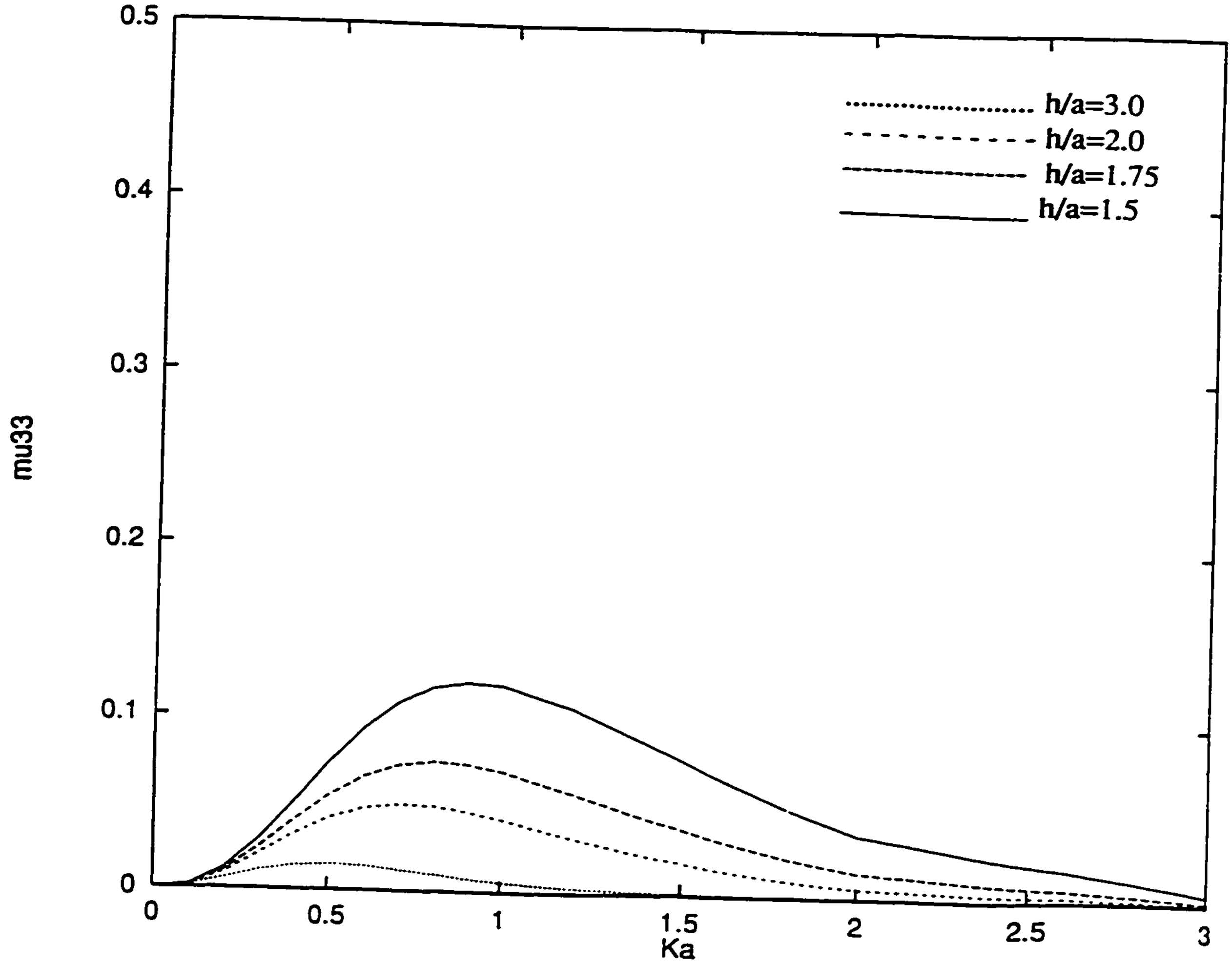

Figure 3.23: Heave added-mass for various values of h/a

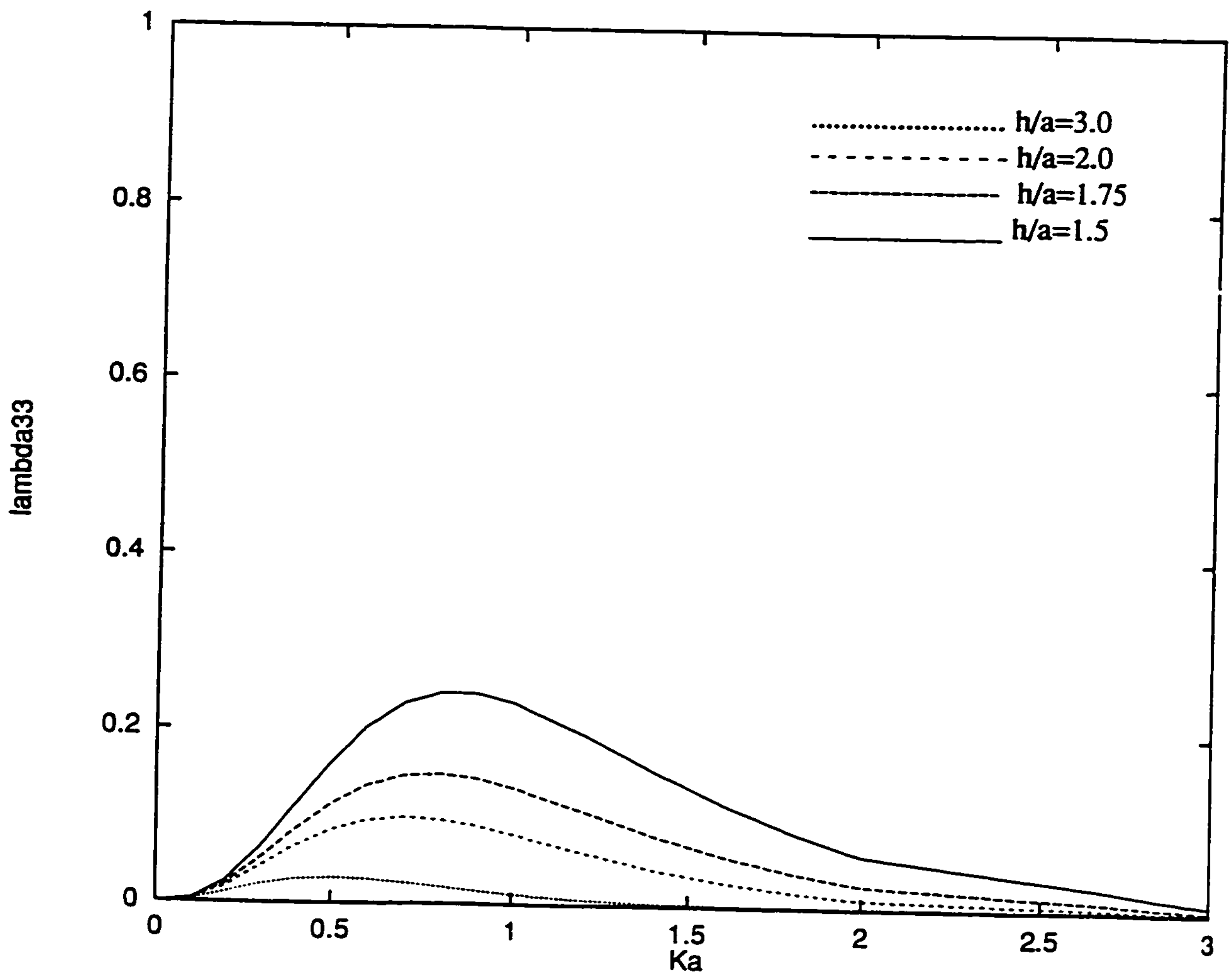

Figure 3.24: Heave damping coefficients for various values of h/a

Chapter 4

Formulation for a Floating Circular Cylinder

We consider a rigid vertical cylinder of radius b acted upon by a train of regular surface waves propagating on the surface of fluid of uniform depth d. A fixed coordinate system Oxyz is employed with x and y axes in the horizontal plane and the z-axis vertically upwards. The corresponding cylindrical coordinate system (r, θ, z) is also assumed. We assume that the fluid is homogeneous, inviscid and incompressible and that the fluid motion is irrotational in the region bounded by the free surface, rigid bottom boundary and the surface of the cylinder. When the amplitude is large, the small amplitude theory does not hold good. In practice, finite amplitude wave theory, namely nonlinear wave theory, is of primary importance. In linear wave theory, the wave amplitudes to the second and higher orders are considered negligible, whereas in finite amplitude wave theory these higher order terms are retained so as to give an accurate representation of the wave motion. A schematic diagram of a vertical cylinder is depicted in Figure 4.1.

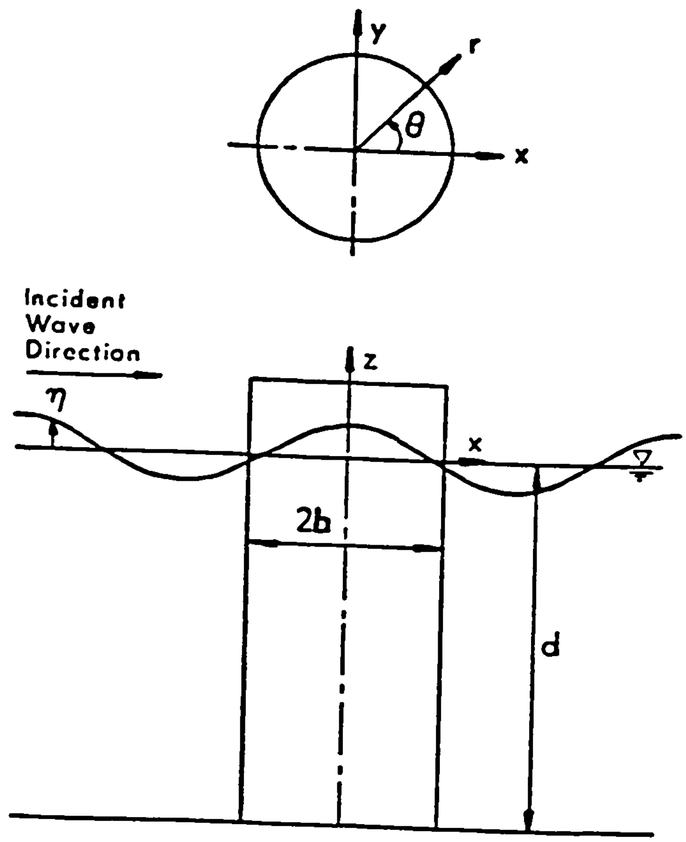

Figure 4.1: A Schematic diagram of a vertical cylinder

Relating the two coordinate systems by

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z,$$

The boundary value problem is given by the following set of equations, with Laplace's equation in cylindrical coordinates expressed as,

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{4.1}$$

in the region $b \le r < \infty,\ -d \le z \le \eta,\ -\pi \le \theta \le \pi$.

The dynamic boundary condition is

$$\frac{\partial\Phi}{\partial t} + g\eta + \frac{1}{2}[(\frac{\partial\Phi}{\partial r})^2 + (\frac{1}{r}\frac{\partial\Phi}{\partial\theta})^2 + (\frac{\partial\Phi}{\partial z})^2] = 0,\ z = \eta; r \ge b, \tag{4.2}$$

the kinematic boundary condition is

$$\frac{\partial\eta}{\partial t} + (\frac{\partial\Phi}{\partial r})(\frac{\partial\eta}{\partial r}) + \frac{1}{r^2}(\frac{\partial\eta}{\partial\theta})(\frac{\partial\Phi}{\partial\theta}) = \frac{\partial\Phi}{\partial z} \text{ on } z = \eta; r \ge b, \tag{4.3}$$

the boundary condition on the body surface is

$$\frac{\partial\Phi}{\partial r} = 0 \text{ on } r = b; -d \le z \le \eta \tag{4.4}$$

and the bottom boundary condition is

$$\frac{\partial\Phi}{\partial z} = 0 \text{ on } z = -d \tag{4.5}$$

Let us assume $\Phi(r,\theta,z,t) = Re[\phi(r,\theta,z)e^{i\sigma t}] = Re[(\phi_I + \phi_S)e^{i\sigma t}]$ where ϕ_S is the scattered potential which satisfies the radiation condition. Thus, the radiation condition can be stated as:

$$\lim_{kr\to\infty} \sqrt{kr}(\frac{\partial}{\partial r} \pm ik)\phi_S = 0 \tag{4.6}$$

4.1 Mathematical Analysis

We know that the irrotational fluid flow pressure distribution can be obtained from Bernoulli's equation, which is

$$\frac{P}{\rho} + gz + \frac{\partial \Phi}{\partial t} + \frac{1}{2}[(\frac{\partial \Phi}{\partial x})^2 + (\frac{\partial \Phi}{\partial y})^2 + (\frac{\partial \Phi}{\partial z})^2] = 0 \tag{4.7}$$

From the above equation we can define the following three pressure distributions:

(i) the hydrostatic pressure distribution, which is given by

$$P_0 - \rho g z \tag{4.8}$$

at height z above the level $z = 0$, where the hydrostatic pressure is P_0,

(ii) the dynamic pressure distribution of Bernoulli, given by

$$\frac{1}{2}\rho(U^2 - q^2) \tag{4.9}$$

at a point where the fluid speed is q and U is the fluctuating fluid velocity that would be found where the body is, if the body were absent, and

(iii) the transient pressure distribution given by

$$-\rho\frac{\partial \Phi}{\partial t} \tag{4.10}$$

Equation (4.7) can be written as

$$\frac{P}{\rho} + gz + \frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 = 0 \tag{4.11}$$

where

$$\left.\begin{array}{l} u = \frac{dx}{dt} = \frac{\partial \Phi}{\partial x} \\ v = \frac{dy}{dt} = \frac{\partial \Phi}{\partial y} \\ w = \frac{dz}{dt} = \frac{\partial \Phi}{\partial z} \end{array}\right\} \tag{4.12}$$

Here, u,v and w respectively represent the velocity components along the x, y and z axes.

The total derivative of eqn (4.11) with respect to time t gives,

$$\frac{d}{dt}(\frac{P}{\rho}) + g\frac{dz}{dt} + \frac{d}{dt}(\frac{\partial \Phi}{\partial t}) + \frac{1}{2}\frac{d}{dt}(\nabla\Phi)^2 = 0 \tag{4.13}$$

which can be written as

$$\frac{d}{dt}(\frac{P}{\rho}) + [\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z}] + \frac{\partial}{\partial t}(\nabla\Phi)^2 + \frac{1}{2}q.\nabla(\nabla\Phi)^2 = 0 \tag{4.14}$$

where $q = (u, v, w)$. Equation (4.14) is evaluated at $z = \eta$, which gives

$$\frac{d}{dt}(\frac{P_a}{\rho}) + (\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z})_{z=\eta} + \frac{\partial}{\partial t}(\nabla\Phi)^2_{z=\eta} + \frac{1}{2}q.\nabla(\nabla\Phi)^2_{z=\eta} = 0 \tag{4.15}$$

where P_a is the atmospheric pressure. The first term in eqn (4.15) vanishes if the atmospheric pressure is constant. Thus, we also get at $z = \eta$:

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} + \frac{\partial}{\partial t}(\nabla\Phi)^2 + \frac{1}{2}q.\nabla(\nabla\Phi)^2 = 0 \tag{4.16}$$

Retaining up to the second-order term in the variable Φ in eqn (4.16), we obtain

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = -\frac{\partial}{\partial t}(\nabla\Phi)^2 \tag{4.17}$$

4.2 Perturbation of Solution

We can choose to write

$$\Phi = \Phi_l + \Phi_q + \ldots \quad (4.18)$$

$$\eta = \eta_l + \eta_q + \ldots \quad (4.19)$$

where Φ_l and η_l satisfy linear diffraction conditions and Φ_q and η_q are the quadratic corrections of the order of the sources of the disturbances. The linear and quadratic terms are of order of $0(\frac{2A}{L})$ and $0\{(\frac{2A}{L})^2\}$ respectively. $\frac{2A}{L}$ is defined to be the wave steepness in which A is the wave amplitude and L is the wavelength. If higher order corrections are ignored, then we can have a Taylor series expansion about $z = 0$:

$$\begin{aligned} \Phi[x, y, \eta(x, y, t), t] &= \Phi(x, y, 0, t) + \eta(\frac{\partial \Phi}{\partial z})_{z=0} + \ldots \\ &= (\Phi_l + \Phi_q + \ldots)_{z=0} + (\eta_l + \eta_q + \ldots) \times \\ &\quad (\frac{\partial \Phi_l}{\partial z} + \frac{\partial \Phi_q}{\partial z} + \ldots)_{z=0} + \ldots \end{aligned} \quad (4.20)$$

Hence the total velocity potential can be written as

$$\Phi = \Phi_l + \Phi_q + \eta_l(\frac{\partial \Phi_l}{\partial z})_{z=0} + \text{higher order terms} \quad (4.21)$$

where Φ_l and $\Phi_q + \eta_l(\frac{\partial \Phi_l}{\partial z})_{z=0}$ are the first and second-order terms respectively. Similar expansion by Taylor's series gives us the following three expressions.

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi_l}{\partial z} + [\frac{\partial \Phi_q}{\partial z} + \eta_l \frac{\partial}{\partial z}(\frac{\partial \Phi_l}{\partial z})_{z=0}] + \ldots \quad (4.22)$$

$$\frac{\partial^2 \Phi}{\partial t^2} = \frac{\partial^2 \Phi_l}{\partial t^2} + [\frac{\partial^2 \Phi_q}{\partial t^2} + \eta_l(\frac{\partial^2 \Phi_l}{\partial z^2})_{z=0}] + \ldots \quad (4.23)$$

$$\nabla \Phi = \nabla \Phi_l + [\nabla \Phi_q + \eta_l \frac{\partial}{\partial z}(\nabla \Phi_q)_{z=0}] + \ldots \quad (4.24)$$

Using all these relations in eqn (4.17), we obtain the following:

linear part:

$$\frac{\partial^2 \Phi_q}{\partial t^2} + g\frac{\partial \Phi_l}{\partial z} = 0 \tag{4.25}$$

quadratic part:

$$\frac{\partial^2 \Phi_l}{\partial t^2} + g\frac{\partial \Phi_q}{\partial z} = -\frac{\partial}{\partial t}(\nabla \Phi_l)^2 - \eta_l\left[\frac{\partial}{\partial z}\left(\frac{\partial^2 \Phi_l}{\partial t^2} + g\frac{\partial \Phi_l}{\partial z}\right)_{z=0}\right] \tag{4.26}$$

with

$$\eta_l = -\frac{1}{g}\left(\frac{\partial \Phi_l}{\partial t}\right) \tag{4.27}$$

4.3 Incident Wave Potential

Due to the linearized boundary condition at $z = 0$, the problem of determining the flow field becomes a purely linear problem. Also the flow potential satisfies the familiar Laplace's equation, which is linear. With such linear equations, sinusoidal waves of amplitude A, propagating freely in the x-direction, have a vertical free surface displacement of

$$\eta = A\sin(\sigma t - kx) \tag{4.28}$$

The incident velocity potential takes the form

$$\Phi_I = \frac{A\sigma}{k}\frac{\cosh k(z+d)}{\sinh kd}\cos(\sigma t - kx) \tag{4.29}$$

in water of depth d. The linearized boundary condition (4.25) leads to the well-known dispersion relationship

$$\sigma^2 = gk \tanh kd \tag{4.30}$$

4.4 Second Order Wave Loading

It has been demonstrated by Lighthill (1979) that a second-order contribution to the irrotational flow loading on a structural component consists of the resultant force F_d of the pressure, given by eqn (4.9), associated with the linear velocity potential Φ_l. This can be written as

$$F_d = \int_S \frac{1}{2}\rho(\nabla\Phi_l)^2 n_x dS \tag{4.31}$$

where n_x is the direction cosine between the outward normal and the direction of the force component F_d being determined.

Another second-order force is associated with the linear velocity potential if the structural component penetrates the surface. The whole additional second-order horizontal x-component of force acting at the waterline w can be written as

$$F_w = \int_w \frac{\rho}{2g}(\frac{\partial\Phi_l}{\partial t})^2 ds \tag{4.32}$$

where the integral is the horizontally resolved force per unit length acting at the waterline.

Following Lighthill's technique, the quadratic potential Φ_q, which satisfies the condition (4.26), is uniquely determined as the potential of the linearized motion generated in the presence of the stationary structure due to a fluctuating pressure distribution

$$\rho[(\nabla\Phi_q)^2 + \frac{1}{2}(\tanh^2 kd - 1)(k\Phi_l)^2 \tag{4.33}$$

applied at the surface. Expression (4.33) reduces to Lighthill's expression (4.31) for the deep water case (because we know $\lim_{kd\to\infty} \tanh kd \simeq 1$).

Thus the quadratic force takes the integral form

$$F_q = -\int W\rho[(\nabla\Phi_l)^2 + \frac{1}{2}(\tanh^2 kd - 1)(k\Phi_l)^2]dS \tag{4.34}$$

where W represents the vertical velocity distribution and dS is an elementary area of the free surface S. This result is obtained by using Green's theorem together with a proper application of the radiation condition to the velocity fields.

Linear diffraction theory yields the potential function Φ_l from which a force F_l may be computed by linear analysis. By adding to F_l, three more terms F_d, F_w and F_q given respectively by eqns (4.31), (4.32) and (4.34), more accurate values of the horizontal x-component of the force can be obtained. The total force $\vec{F}$ exerted on the body and the moment $\vec{M}$ of this force in vector notation are then given by

$$\vec{F}(t) = -\int_S P\vec{n}ds \tag{4.35}$$

$$\vec{M}(t) = -\int_S P(\vec{r}\times\vec{n})ds \tag{4.36}$$

where $\vec{n}$ is the outward normal vector on S and $\vec{r}$ is the vector from the point about which moments are taken.

4.5 Linear Diffraction Theory

Linear diffraction theory, for a vertical cylinder in an incoming wave train, was given by Havelock for deep water waves and can be written for shallow water and intermediate depth as

$$\Phi_l = Re[\frac{A\sigma}{k}\frac{\cosh k(z+d)}{\sinh kd}e^{i\sigma t}\sum_{m=0}^{\infty}\alpha_m A_m(kr)\cos m\theta] \tag{4.37}$$

where Re stands for the real part and

$$\alpha_0 = 1, \quad , \alpha_m = 2(-i)^m, m > 0. \tag{4.38}$$

and

$$A_m(kr) = J_m(kr) - \frac{J_m'(kb)}{H_m^{(2)'}(kb)}H_m^{(2)}(kr) \tag{4.39}$$

Here, $H_m^{(2)}$ is the m-th order Hankel function of second kind given by

$$H_m^{(2)}(kr) = J_m(kr) - iY_m(kr)$$

(r, θ, z) are the cylindrical coordinates and the prime denotes the differentiation with respect to the arguments.

Since, in the calculation of forces on the structures, only the $\cos\theta$ term in the velocity potential will contribute, we are interested in the $\cos\theta$ term in Φ_l and thus the linear velocity potential may be written as

$$\Phi_l = Re[\frac{A\sigma}{k}\frac{\cosh k(z+d)}{\sinh kd}e^{i\sigma t}(-2i)A_1(kr)\cos\theta] \tag{4.40}$$

From the expression of Φ_l from the above equation, we find

$$\begin{aligned}\frac{\partial \Phi_l}{\partial t} &= Re[\frac{A\sigma^2 i}{k} e^{i\sigma t} \frac{\cosh k(z+d)}{\sinh kd} 2(-i) A_1(kr)\cos\theta] \\ &= Re[\frac{A\sigma^2}{k} e^{i\sigma t} \frac{\cosh k(z+d)}{\sinh kd} A_1(kr)\cos\theta]\end{aligned}$$

This expression can be used to express the horizontal force per unit length of the cylinder when acted upon by a fluid with density ρ, as

$$\begin{aligned}F_l(z,t) &= \int_0^{2\pi}\int_{z=-d}^{0} (\rho\frac{\partial \Phi_l}{\partial t})_{r=b} dz(-\cos\theta)(bd\theta) \\ &= Re[\frac{\rho 2Ab^2\sigma^2\pi}{kb} e^{i\sigma t} \frac{\cosh k(z+d)}{\sinh kd} A_1(kb)] \qquad (4.41)\end{aligned}$$

The total horizontal force on the cylinder is given by

$$\begin{aligned}\bar{F}_l(t) &= Re[\int_{-d}^{0} F_l(z,t)dz] \\ &= Re[C_M(\rho g A\pi b^2) e^{i(\sigma t-\beta)} \tanh kd] \\ &= C_M(\rho g A\pi b^2)\tanh kd\cos(\sigma t-\beta) \qquad (4.42)\end{aligned}$$

where the coefficient C_M specifies the cylinder's inertial reaction to a fluid acceleration advanced in phase by phase lead β over its value on the cylinder axis. These calculations are shown in Appendix D.

In evaluating the integrals (4.41) and (4.42) we have used the relation

$$C_M e^{-i\beta} = \frac{2A_1(kb)}{kb} = \frac{-4i}{\pi k^2 b^2 H_1^{(2)\prime}(kb)} \qquad (4.43)$$

where

$$\sigma^2 = gk\tanh kd \qquad (4.44)$$

The Wronskian of J_1 and $H_1^{(2)}$ is given by

$$J_1(kb)H_1^{(2)'}(kb) - H_1^{(2)}(kb)J_1'(kb) = \frac{-2i}{\pi kb} \tag{4.45}$$

The values of C_M and β can be extracted as

$$C_M = \frac{4}{\pi k^2 b^2 \sqrt{J_1'^2(kb) + Y_1'^2(kb)}} \tag{4.46}$$

and

$$\beta = \tan^{-1}[\frac{J_1'(kb)}{Y_1'(kb)}] \tag{4.47}$$

4.6 Calculation of F_q in Deep Water

Assuming

$$\Phi_q = Re[\phi_q e^{2i\sigma t}] \tag{4.48}$$

and

$$\Phi_l = Re[\phi_l e^{i\sigma t}] \tag{4.49}$$

and substituting in eqn (4.26), we obtain

$$(\frac{\partial \phi_q}{\partial z} - K\phi_q)_{z=0} = -\frac{2i\sigma}{g}(\nabla\phi_l)^2 + \frac{i\sigma}{g^2}\phi_l(-\sigma^2\frac{\partial \phi_l}{\partial z} + g\frac{\partial^2 \phi_l}{\partial z^2})_{z=0} \tag{4.50}$$

where $K = \frac{\sigma^2}{g}$.

Let the potential due to a unit translation oscillation of the body be $Re[\tilde{\phi} e^{2i\sigma t}]$.

Then on the body surface S, it follows that

$$(\frac{\partial \tilde{\phi}}{\partial n})_S = n_x, \quad (\frac{\partial \phi_q}{\partial n})_S = 0 \tag{4.51}$$

where n_x is the x-component of a unit inward normal to S, and also

$$(\frac{\partial\bar{\phi}}{\partial z} - K\bar{\phi})_{z=0} = 0 \tag{4.52}$$

where $\bar{\phi}$ also satisfies the radiation condition.

Applying Green's theorem, we get

$$\int_V\int[\phi_q\nabla^2\bar{\phi} - \bar{\phi}\nabla^2\phi_q]dV = \int_{SUz=0}[\phi_q\frac{\partial\bar{\phi}}{\partial n} - \bar{\phi}\frac{\partial\phi_q}{\partial n}]dS \tag{4.53}$$

taken over the boundaries of the fluid (S and $z = 0$). Here n is the normal outward from the fluid.

Since ϕ_q and $\bar{\phi}$ both satisfy Laplace's equation, we have

$$\int_{SUz=0}[\phi_q\frac{\partial\bar{\phi}}{\partial n} - \bar{\phi}\frac{\partial\phi_q}{\partial n}]dS = 0 \tag{4.54}$$

After performing the indicated integration in (4.54), and applying the conditions (4.51) and (4.52), we have

$$\int_{z=0}\bar{\phi}(\frac{\partial\Phi_q}{\partial z} - K\phi_q)dxdy = \int_S \phi_q n_x dS \tag{4.55}$$

The force F_q can be written as,

$$\begin{aligned} F_q &= \int_S(-\rho\frac{\partial\phi_q}{\partial t})n_x ds \\ &= Re[(-\rho 2i\sigma e^{2i\sigma t})\int_S \phi_q n_x dS] \end{aligned} \tag{4.56}$$

With the help of (4.55), (4.51) and (4.50), we can infer

$$\begin{aligned} F_q &= Re[(-\rho 2i\sigma e^{2i\sigma t})\frac{1}{K}\int_{z=0}(\frac{\partial\bar{\phi}}{\partial z})_{z=0}[\frac{\partial\phi_q}{\partial z} - K\phi_q]dxdy] \\ &= Re[(-\frac{4\sigma^2}{Kg}e^{2i\sigma t})\rho\int_{z=0}(\frac{\partial\bar{\phi}}{\partial z})_{z=0} \times \\ &\quad [(\nabla\phi_l)^2 - \frac{1}{2g}\phi_l(-\sigma^2\frac{\partial\phi_l}{\partial z} + g\frac{\partial^2\phi_l}{\partial z^2})]dxdy] \end{aligned} \tag{4.57}$$

which reduces to

$$F_q = Re[-\int_{z=0} \rho W[(\nabla\Phi_l)^2 + \frac{1}{2}(\tanh^2 kd - 1)(k\Phi_l)^2]dxdy] \qquad (4.58)$$

with

$$W = (\frac{\partial\tilde{\phi}}{\partial z})_{z=0} \qquad (4.59)$$

representing the vertical velocity on the free surface associated with the unit translational oscillation of the body.

The solution $\tilde{\phi}$ of Laplace's equation satisfying (4.52) on the free surface and (4.51) on the cylinder $r = b$, together with the radiation condition, can be written as (Lighthill (1979))

$$\tilde{\phi} = \cos\theta[\frac{iK}{\pi}(PV)\int_{-\infty}^{\infty} \frac{e^{imz}}{m(K - im)} \frac{K_1(|m|r)}{|m|K_1'(|m|b)} dm + 2e^{Kz} \frac{H_1^{(2)}(Kr)}{KH_1^{(2)'}(Kb)}] \qquad (4.60)$$

Here (PV) signifies the Cauchy Principal values of the integral, K_1 is the Bessel function of imaginary argument and $H_1^{(2)}$ is the Hankel function of second kind of order one. This expression is valid only for deep water waves. For finite depth water waves, the reader is referred to the work of Rahman (1997).

To evaluate W from eqn (4.59), following Lighthill's analysis, the appropriate value of W for small kb is found to be

$$W \sim K\frac{b^2}{r}\cos\theta \qquad (4.61)$$

where an effective range, $b < r < B$ for which (eqn (4.61) holds, can be found from

$$B = (2e^{-\gamma})K^{-1} = c(k\tanh kd)^{-1}$$

with c=0.28 and $\gamma = .577$ (Euler's constant). Beyond the limiting radius $r = B$, W is oscillatory and hence makes very little contribution to the quadratic force F_q.

4.7 Exact Calculations for Second Order Wave Loads

We recall that the total horizontal force F_l, is given by eqn (4.41) and was evaluated as

$$F_l = C_M \rho g A \pi b^2 \tanh kd \cos(\sigma t - \beta)$$

with C_M and β given by (4.46) and (4.47). Similarly the moment of this linear force, M_l, can be calculated from the following:

$$M_l = \int_0^{2\pi} [\int_{z=-d}^{0} (z+d)(-\rho \frac{\partial \Phi_l}{\partial t}) dz]_{r=b} (-\cos\theta) b d\theta \tag{4.62}$$

After evaluating the integrals, we obtain

$$M_l = C_M \rho g A \pi b^2 (kd \tanh kd + \text{sech } kd - 1) \cos(\sigma t - \beta) \tag{4.63}$$

The dynamic force, F_d, can be evaluated from

$$F_d = \int_0^{2\pi} [\int_{z=-d}^{0} -\frac{1}{2}\rho (\nabla \Phi_l)^2 dz] (-\cos\theta) b d\theta \tag{4.64}$$

Performing the integration, the total dynamic force F_d is found to be:

$$\begin{aligned} F_d &= \frac{2\rho g A^2}{\pi b k^2} \sum_{l=0}^{\infty} [1 - \frac{2kd}{\sinh 2kd}) + \frac{l(l+1)}{k^2 b^2}(1 + \frac{2kd}{\sinh 2kd})] \times \\ & [E_l - (-1)^l \{C_l \cos 2\sigma t - S_l \sin 2\sigma t\}] \end{aligned} \tag{4.65}$$

where

$$E_l = (J_l' Y_{l+1}' - J_{l+1}' Y_l')/T_l \quad (4.66)$$

$$C_l = (Y_l' J_{l+1}' + Y_{l+1}' J_l')/T_l \quad (4.67)$$

$$S_l = (Y_l' Y_{l+1}' - J_l' J_{l+1}')/T_l \quad (4.68)$$

$$T_l = (J_l'^2 + Y_l'^2)(J_{l+1}'^2 + Y_{l+1}'^2) \quad (4.69)$$

and where the Bessel function arguments are kb. The result, eqn (4.65), is the sum of the steady-state and oscillatory parts, with the expression multiplied by E_l giving the steady-state solution and the rest representing the oscillatory solution.

The moment of this force, M_d, can be calculated from the following formula:

$$M_d = \int_{\theta=0}^{2\pi} [\int_{z=-d}^{0} (z+d)[-\frac{\rho}{2}(\nabla\Phi_l)^2]dz(-\cos\theta)bd\theta \quad (4.70)$$

The integration gives us the result

$$M_d = \frac{2\rho g d A^2}{\pi b k^2} \sum_{l=0}^{\infty} [\frac{l(l+1)}{b^2k^2} B + D] \times [E_l - (-l)^l (C_l \cos 2\sigma t - S_l \sin \sigma t) \quad (4.71)$$

where

$$\left. \begin{aligned} B &= 1 - \tfrac{\coth 2kd}{2kd} + \tfrac{1+2k^2d^2}{2kd \sinh 2kd} \\ D &= 1 - \tfrac{\coth 2kd}{2kd} + \tfrac{1-2k^2d^2}{2kd \sinh 2kd} \end{aligned} \right\} \quad (4.72)$$

The waterline force, F_w, can be evaluated from:

$$F_w = \int_0^{2\pi} \frac{\rho}{2g} (\frac{\partial\Phi_l}{\partial t})^2_{z=0,r=b} (-\cos\theta) bd\theta \quad (4.73)$$

The integration gives us the result

$$F_w = \frac{4\rho g A^2}{\pi b k^2} \sum_{l=0}^{\infty} [E_l + (-1)^l \{C_l \cos 2\sigma t - S_l \sin 2\sigma t\}] \quad (4.74)$$

The moment of this force, M_w, is

$$\begin{aligned} M_w &= \int_0^{2\pi} \frac{\rho d}{2g}(\frac{\partial \Phi_l}{\partial t})^2_{z=0,r=b}(-\cos\theta) b d\theta \\ &= dF_w \\ &= \frac{4\rho g A^2 d}{\pi b k^2}\sum_{l=0}^{\infty}[E_l + (-1)^l\{C_l \cos 2\sigma t - S_l \sin 2\sigma t\}] \end{aligned} \tag{4.75}$$

The expression for quadratic force F_q is:

$$\begin{aligned} F_q &= Re[-2i\sigma\rho e^{2i\sigma t}\frac{1}{K}\int\int_{z=0}\frac{\partial\bar{\phi}}{\partial z}\{-\frac{i\sigma}{g}(\nabla\phi_l)^2 + \frac{i\sigma k^2}{2g}(1-\tanh^2 kd)(\phi_l)^2\}dxdy] \\ &= Re[-\frac{\rho}{2}e^{2i\sigma t}\int\int_{z=0}(\frac{\partial\bar{\phi}}{\partial z})[(\nabla\phi_l)^2 + \frac{k^2}{2}(\tanh^2 kd - 1)](\phi_l)^2 r dr d\theta] \end{aligned} \tag{4.76}$$

which can be rewritten as

$$\begin{aligned} F_q &= Re[-\frac{\rho}{2}e^{2i\sigma t}\int_b^{\infty} r dr \int_0^{2\pi}(\frac{\partial\bar{\phi}}{\partial z})_{z=0} \times \\ & \quad \{(\nabla\phi_l)^2 + \frac{k^2}{2}(\tanh^2 kd - 1)(\phi_l)^2\}_{z=0} d\theta] \end{aligned} \tag{4.77}$$

We now aim to find $(\nabla\phi_l)^2$ and $(\phi_l)^2$.

Using the results for the expressions for $(\frac{\partial\phi_l}{\partial r})^2|_{z=0}$, $(\frac{\partial\phi_l}{\partial\theta})^2|_{z=0}$, $(\frac{\partial\phi_l}{\partial z})^2|_{z=0}$ and $(\nabla\phi_l)^2|_{z=0}$ and the dispersion relation (4.30), we can write

$$\begin{aligned} (\frac{\partial\phi_l}{\partial r})^2|_{z=0} &= A^2 gk \coth kd \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha_m\alpha_n[J'_m(kr) - \frac{J'_m(kb)}{H^{(2)'}_m(kb)}H^{(2)'}_m(kr)] \times \\ & \quad [J'_n(kr) - \frac{J'_n(kb)}{H^{(2)'}_n(kb)}H^{(2)'}_n(kr)]\cos m\theta \cos n\theta \end{aligned} \tag{4.78}$$

$$\begin{aligned} (\frac{\partial\phi_l}{\partial\theta})^2|_{z=0} &= \frac{A^2 g}{k}\coth kd \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} mn\alpha_m\alpha_n[J_m(kr) - \frac{J_m(kb)}{H^{(2)'}_m(kb)}H^{(2)'}_m(kr)] \times \\ & \quad [J'_n(kr) - \frac{J'_n(kb)}{H^{(2)'}_n(kb)}H^{(2)'}_n(kr)]\sin m\theta \sin n\theta \end{aligned} \tag{4.79}$$

$$\left(\frac{\partial\phi_l}{\partial z}\right)^2\Big|_{z=0} = A^2gk\tanh kd\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha_m\alpha_n\left[J_m(kr) - \frac{J_m'(kb)}{H_m^{(2)'}(kb)}H_m^{(2)'}(kr)\right]\times$$
$$\left[J_n'(kr) - \frac{J_n(kb)}{H_n^{(2)'}(kb)}H_n^{(2)'}(kr)\right]\cos m\theta\cos n\theta \qquad (4.80)$$

$$(\nabla\phi_l)^2\Big|_{z=0} = \frac{A^2g}{k}\coth kd\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha_m\alpha_n\left[J_m(kr) - \frac{J_m'(kb)}{H_m^{(2)'}(kb)}H_m^{(2)'}(kr)\right]\times$$
$$\left[J_n(kr) - \frac{J_n'(kb)}{H_n^{(2)'}(kb)}H_n^{(2)'}(kr)\right]\cos m\theta\cos n\theta \qquad (4.81)$$

Also, we can evaluate

$$J_m'(kr) - \frac{J_m'(kb)}{H_m^{(2)'}(kb)}H_m^{(2)'}(kr) = -i\frac{\hat{Q}_m'(kr)}{H_m^{(2)'}(kb)}$$

and

$$J_m(kr) - \frac{J_m'(kb)}{H_m^{(2)'}(kb)}H_m^{(2)}(kr) = -i\frac{\hat{Q}_m(kr)}{H_m^{(2)'}(kb)}$$

where $\hat{Q}_m$ will be defined later on. Substituting these two expressions in eqns (4.78)-(4.81) and simplifying, we have

$$\left(\frac{\partial\phi_l}{\partial r}\right)^2\Big|_{z=0} = -A^2gk\coth kd\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha_m\alpha_n\frac{\hat{Q}_m'(kr)}{H_m^{(2)'}(kb)}\frac{\hat{Q}_n'(kr)}{H_n^{(2)'}(kb)}\times$$
$$\cos m\theta\cos n\theta \qquad (4.82)$$

$$\left(\frac{\partial\phi_l}{\partial\theta}\right)^2\Big|_{z=0} = -\frac{A^2g}{k}\coth kd\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}mn\alpha_m\alpha_n\frac{\hat{Q}_m(kr)}{H_m^{(2)'}(kb)}\frac{\hat{Q}_n(kr)}{H_n^{(2)'}(kb)}\times$$
$$\sin m\theta\sin n\theta \qquad (4.83)$$

$$\left(\frac{\partial\phi_l}{\partial z}\right)^2\Big|_{z=0} = -A^2gk\tanh kd\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha_m\alpha_n\frac{\hat{Q}_m(kr)}{H_m^{(2)'}(kb)}\frac{\hat{Q}_n(kr)}{H_n^{(2)'}(kb)}\times$$
$$\cos m\theta\cos n\theta \qquad (4.84)$$

$$(\nabla\phi_l)^2\Big|_{z=0} = -\frac{A^2g}{k}\coth kd\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha_m\alpha_n\frac{\hat{Q}_m(kr)}{H_m^{(2)'}(kb)}\frac{\hat{Q}_n(kr)}{H_n^{(2)'}(kb)}\times$$
$$\cos m\theta\cos n\theta \qquad (4.85)$$

Using those expressions and also the equation for $(\frac{\partial\tilde{\phi}}{\partial z})_{z=0}$ from Appendix D and simplifying, we get

$$\begin{aligned} F_q &= Re[\frac{2\rho g\pi A^2}{k}\coth kd\int_{r=b}^{\infty}e^{2i\sigma t}\sum_{n=0}^{\infty}[2\hat{Q}'_n\hat{Q}'_{n+1}+ \\ &(\frac{2(n+1)n}{k^2r^2}-1+3\tanh^2 kd)\hat{Q}_n\hat{Q}_{n+1}]\{(-1)^n(C_n+iS_n)\}\times \\ &\{\frac{H_1^{(2)}(\kappa r)}{H_1^{(2)\prime}(\kappa b)}\frac{\sinh^2\kappa d}{\kappa d+\sinh\kappa d\cosh\kappa d}- \\ &\sum_{j=1}\frac{K_1(m_jr)}{K'_1(m_jb)}\frac{\sinh^2 m_jd}{m_jd+\sinh m_jd\cos m_jd}\}r\,dr] \end{aligned} \tag{4.86}$$

where

$$\begin{aligned} A_n(kr) &= Q_n(kr)e^{-i\alpha_n} \\ Q_n(kr) &= \frac{\hat{Q}_n(kr)}{\sqrt{J_1'^2(kb)+Y_n'^2(kb)}} \\ \hat{Q}_n(kr) &= J_n(kr)Y'_n(kb)-J'_n(kb)Y_n(kr) \\ \alpha_n &= \tan^{-1}\frac{J'_n(kb)}{Y'_n(kb)} \end{aligned}$$

Here κ and k are related through the dispersion relation

$$\kappa\tanh\kappa d = K = 4k\tanh kd$$

Similarly, the quadratic moment M_q can be found from the expression

$$\begin{aligned} M_q &= Re[-\frac{\rho}{2}e^{2i\sigma t}\int_b^{\infty}r\,dr\int_0^{2\pi}\frac{\partial\xi}{\partial z}|_{z=0}\times \\ &\{(\nabla\phi_l)^2+\frac{k^2}{2}(\tanh^2 kd-1)(\nabla\phi_l)^2\}d\theta] \end{aligned} \tag{4.87}$$

where $Re[\xi e^{2i\sigma t}]$ is the potential due to a unit rotational oscillation of the cylinder about the ocean bottom.

Chapter 5

Numerical Results and Discussions for Circular Cylinder

In this chapter we present some numerical results for most of the coefficients and forces obtained in the preceding chapter. We first compute the coefficients T_l, C_l, S_l and E_l used in describing the forces and moments. The subroutines available for Bessel functions have been used. We also compute F_l, F_d, F_w, M_l, M_d and M_w, for different values of kb.

In order to compute the linear force F_l and the associated moment M_l from eqns (4.42) and (4.63) respectively, our primary task will be to first find the coefficients C_M and β given by eqns (4.46) and (4.47) respectively. We have considered kb for a wide range and have presented the values of C_M, β, F_l and M_l corresponding to the values of kb (table 5.1).

For various values of kb, Table 5.2 gives the various values of coefficients (which are basically functions of Bessel functions of different kinds) T_l, C_l, S_l and E_l. These coefficients help us in computing the dynamic force and its associated moment and

Table 5.1: Linear forces and linear moments along with C_M and β

kb	C_M	β	F_l	M_l
2.50	0.10739666	-0.37286976	0.33536052	0.62186143
3.00	0.06804680	-0.83984989	0.21337043	0.50350577
3.50	0.05639605	-1.57079637	0.17706844	0.51064982
4.00	0.03597513	-1.30116045	0.11299837	0.38635602
4.50	0.02086744	-1.00241828	0.06555333	0.25964065
5.00	0.01168680	-1.12431347	0.03671457	0.16543712
6.00	0.00986637	-0.86911660	0.03099615	0.17363709
7.00	0.01295755	-1.57079637	0.04070742	0.27276373
8.00	0.00700104	-0.64894944	0.02199446	0.17156078
9.00	0.00624376	-0.71118051	0.01961539	0.17457804
10.00	0.00609685	-1.23904908	0.01915388	0.19153917

the waterline force and its associated moment.

Table 5.3 gives the values of the dynamic force F_d and its associated moment M_d given respectively by eqns (4.65) and (4.71). We have also computed the maximum force for different values of kb.

Similarly, Table 5.4 gives the computed values of waterline force F_w and its associated moment M_w given respectively by eqns (4.74) and (4.75).

Table 5.2: Various coefficients involving Bessel functions

kb	T_l	C_l	S_l	E_l
1.750	0.281559828E+01	0.5459365656	-0.2389927913	0.5459365656
2.750	0.100604499E+01	0.9908821164	-0.1101995985	0.3997354872
3.750	0.671746081E+00	1.2162293496	-0.0971800744	-1.2162293496
4.750	0.163258280E+01	0.0946316576	0.7768984523	-0.7347844788
5.750	0.171704718E+01	-0.5959826260	0.4766549057	-0.1983703436
6.750	0.369420184E+01	-0.2964898213	-0.4275374257	0.0411934910
7.750	0.761516203E+00	0.1073017561	-1.1409013986	1.1217893809
8.750	0.380979049E+00	1.5124219554	-0.5808579552	1.2667079058
9.750	0.411455032E+00	1.4984739127	0.4300874227	-0.6745992694
10.750	0.487532476E+00	0.4226616318	1.3683941410	-1.4215235975

Table 5.3: Dynamic forces and moments

kb	F_d	F_{max}	M_d
1.750	0.00103717	0.02471449	0.00169436
2.750	0.00570604	0.01905005	0.01549726
3.750	0.01275707	0.01277736	0.04917896
4.750	0.00496478	0.00512136	0.02487769
5.750	0.00214754	0.00340896	0.01325056
6.750	0.00077651	0.00168654	0.00569179
7.750	0.00002969	0.00281789	0.00025206
8.750	0.00034089	0.00312537	0.00328958
9.750	0.00173512	0.00242213	0.01875659
10.750	0.00182360	0.00183041	0.02182849

Table 5.4: Waterline forces and moments

kb	F_w	M_w
1.750	0.0550718	0.11179247
2.750	0.02727803	0.08701694
3.750	0.00004071	0.00017710
4.750	0.00031327	0.00172611
5.750	0.00252295	0.01682810
6.750	0.00182008	0.01425125
7.750	0.00542879	0.04754534
8.750	0.00556895	0.05652485
9.750	0.00137402	0.01554021
10.750	0.00001362	0.00016987

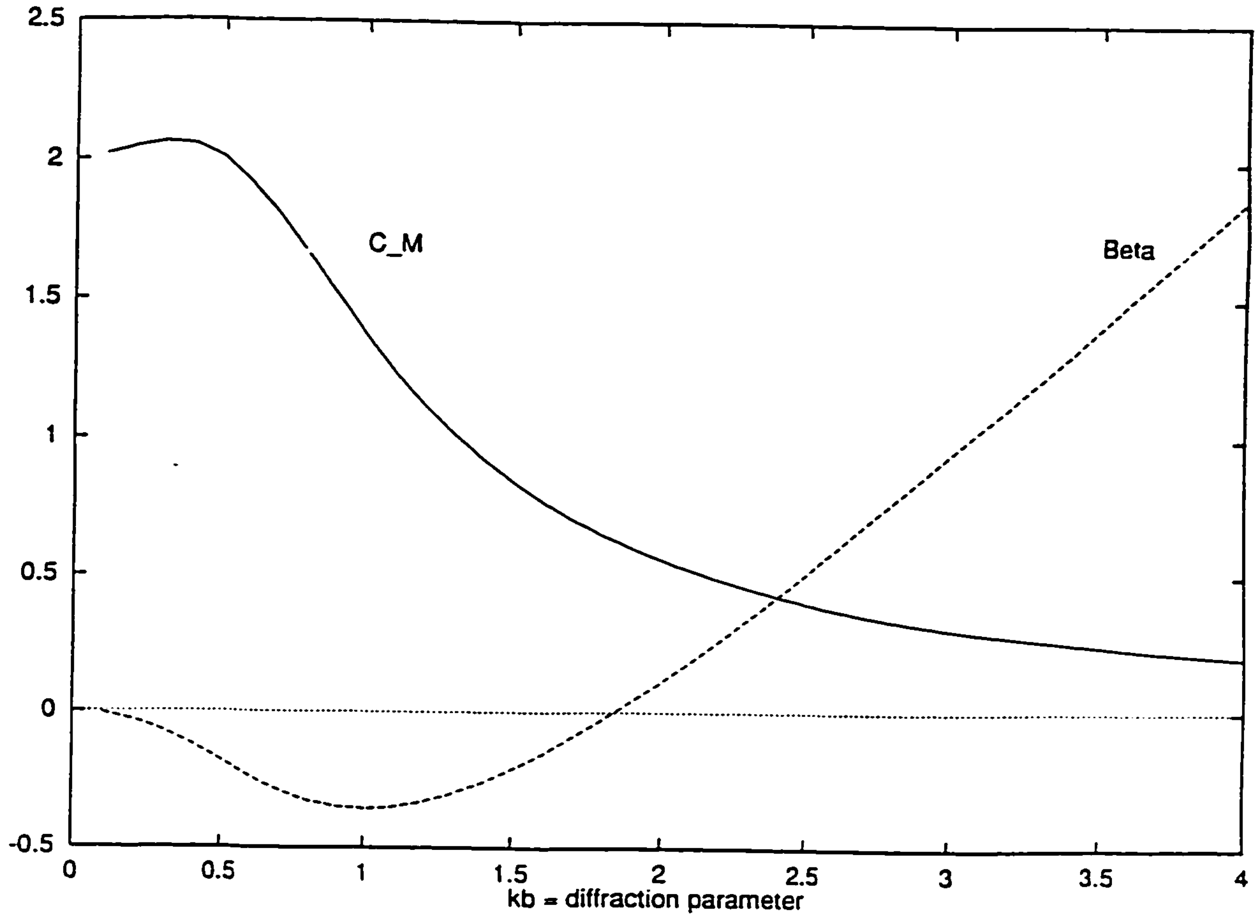

Figure 5.1: Plot of C_M versus β

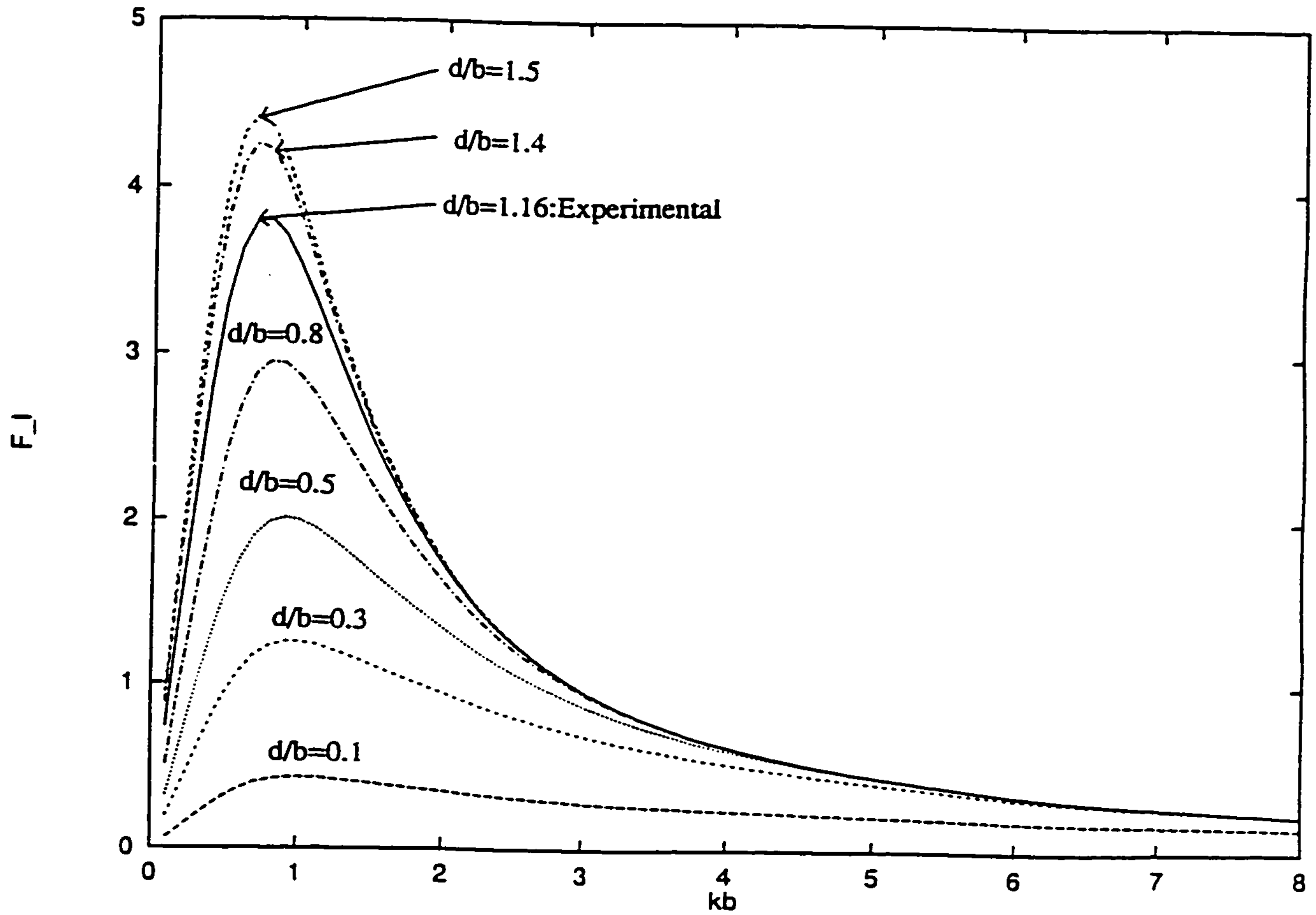

Figure 5.2: Linear forces at various submergence

Figure (5.1) shows the comparison between the coefficients C_M and β. It is interesting to note that β increases rapidly once it becomes positive. C_M attains its maximum value around $kb = .5$ and it decreases steadily thereafter.

The linear forces F_l have been plotted in Figure (5.2) for various submergence values. Experimental values at $d/b = 1.16$ have also been included. The results obtained seem to agree with the available experimental results.

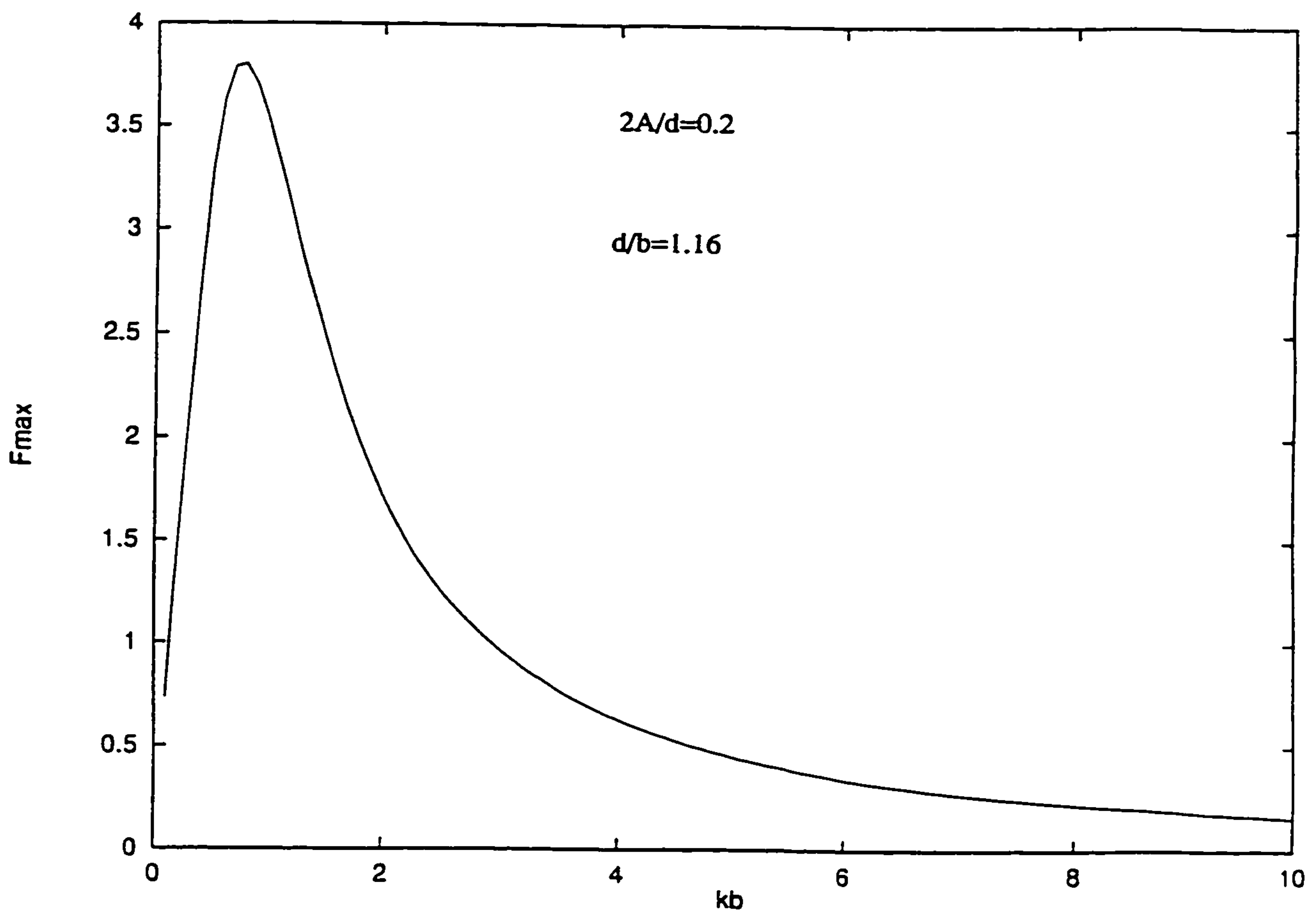

Figure 5.3: Maximum force

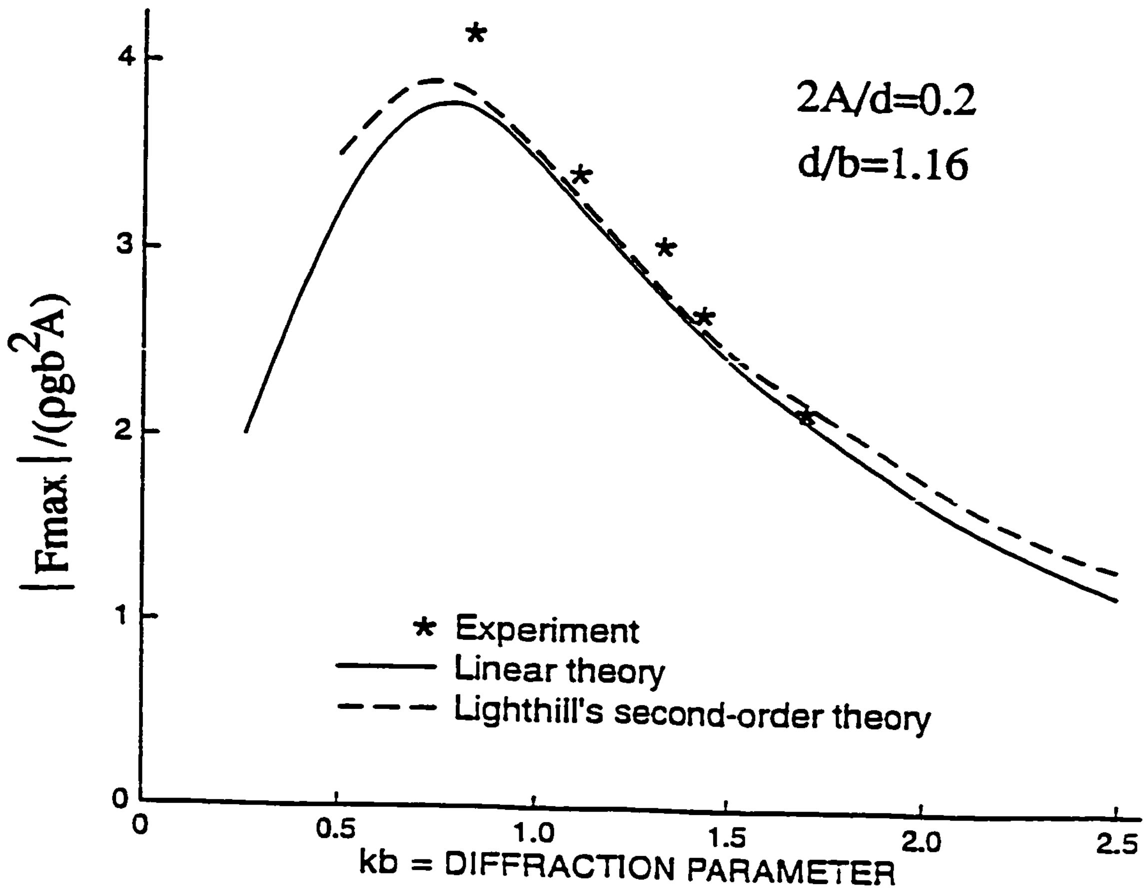

Figure 5.4: Comparison of linear and second-order wave forces with experimental data

Figure (5.3) gives us the maximum force plotted against kb. Here we have considered $2A/d = .2$ and $d/b = 1.16$. It reaches its peak at $kb = 1$ after increasing suddenly. But after that it decreases rapidly till $kb = 8$ after which its decrease is uniform.

Figure (5.4) gives the nondimesional maximum force plotted against kb. This has been taken from the work of Rahman (1997). It compares the linear and second-order wave forces with experimental data.

Chapter 6

Conclusions and Recommendations

A systematic mathematical procedure has been presented in the first portion of this thesis to evaluate the hydrodynamic coefficients and loads on a submerged sphere in finite water depth. The sphere has three degrees of freedom due to surge, heave and pitch motion in water of finite depth in the presence of an incident wave. The whole boundary value problem is divided into two problems, namely, the diffraction problem of an incident wave acting on the submerged sphere and the radiation problem of the sphere forced to move in otherwise still water. The solution of these two problems together provide the analytical solution for the total boundary value problem. The velocity potential has been obtained by using the multipole expansions in terms of an infinite series of Legendre polynomials with unknown coefficients. The exciting force components are obtained by solving the diffraction problem. The added-mass and damping coefficients are obtained by solving the radiation problem. The responses due to surge, heave and pitch induced by wave excitation are determined from the equation of motion of the submerged sphere. Numerical results of the derived analytical expressions for the hydrodynamic coefficients and loads are presented in various figures for different depth to radius ratios and for different submergence. Our main objective was to find simplified analytical solutions to different potentials and forces.

In the second portion of the thesis, a detailed analysis of the second-order diffraction problem for a circular cylinder is presented. The investigation mainly deals with the exact second order calculations using the exact expression for the linear velocity potential of diffraction theory. The second-order forces are decomposed into three components: waterline force, dynamic force and quadratic force. The main focus is on evaluation of the quadratic force. Numerical results of various analytical expressions obtained are presented for different wave parameters.

The combined linear problem due to diffraction and radiation or even the diffraction or radiation problem separately can be extended to two or more submerged spheres. The analysis of interactions among several structures nearer to each other would be important in many practical cases. The same theory of multiple expansions may be applied to expand the total potential. The body surface condition for various velocity potentials might involve the Kronecker delta. If the problem with a group of spheres with the same radius yields good results, then an extension of the analysis can be considered where the spheres can be supposed have different radii. Also it might be advisable to use asymptotic solutions to get a clear idea and to have a simplified solution.

Similarly, the cylinder problem can be extended to two or more cylinders. The work presented here may be extended to study the hydrodynamic coefficients and loads up to second-order due to a pair or more circular cylinders.

Further investigations may include other types of structures too. As we have seen, the solutions for sphere and circular cylinder involve Bessel functions and Legendre polynomials respectively. A solution for the elliptic cylinder will involve Mathieu functions.

The theory and solutions for a sphere or a vertical circular cylinder in water provide an important step in understanding the effects of wave diffraction and radiation on large bodies. The solutions obtained have a wide range of applications, but eventually it becomes necessary to consider the case of bodies of arbitrary geometry in order to deal with the challenges offered by the variety and complexity of design configurations encountered in modern marine and offshore structures. Therefore, for irregularly shaped bodies the results obtained here represent merely the preliminary stage of the analysis. To be more precise, for an irregularly shaped body, the results obtained here may only be used qualitatively.

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Appendix A

Expression for Incident Potential

The following calculations and relations are used to derive expressions (2.14) and (2.15) of the incident potential ϕ_I.

Given the relation

$$e^{iz\cos\theta} = J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos n\theta$$

and substituting $R = r\sin\theta$, we obtain

$$\begin{aligned} e^{ik_0 R\cos\psi} &= e^{ik_0 r\sin\theta\cos\psi} \\ &= J_0(k_0R) + 2\sum_{m=1}^{\infty} i^m J_m(k_0R)\cos m\psi \end{aligned} \quad (A.1)$$

which gives the exponential part of the expression (2.14).

We can write

$$\begin{aligned} \cosh k_0(z-d) &= \cosh k_0\{(z-h)+(h-d)\} \\ &= \frac{1}{2}e^{k_0[(z-h)+(h-d)]} + e^{-k_0[(z-h)+(h-d)]} \end{aligned}$$

which can be used to imply

$$e^{k_0[(z-h)+(h-d)]} = e^{k_0(h-d)}e^{k_0(z-h)}$$

$$e^{-k_0[(z-h)+(h-d)]} = e^{-k_0(h-d)}e^{-k_0(z-h)}$$

Thorne has deduced the following relation:

$$e^{\pm k(z-f)}J_m(kR) = (\pm)^m \sum_{u=m}^{\infty} (\pm kr)^u \frac{P_u^m(\cos\theta)}{(u+m)!} \tag{A.2}$$

which implies that we can write

$$e^{k_0(h-d)}e^{k_0(z-h)}J_m(k_0R) = e^{k_0(h-d)} \sum_{s=m}^{\infty} (+1)^m \frac{P_s^m(\cos\theta)}{(s+m)!}(k_0r)^s$$

$$e^{-k_0(h-d)}e^{-k_0(z-h)}J_m(k_0R) = e^{-k_0(h-d)} \sum_{s=m}^{\infty} (-1)^{m+s} \frac{P_s^m(\cos\theta)}{(s+m)!}(k_0r)^s$$

These lead to expression (2.15).

Appendix B

Legendre Polynomial and Bessel Function

Associated Legendre Polynomials

The polynomial $P_n^m(x)$, known as associated Legendre's polynomial, is a solution of the following differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \{n(n+1) - \frac{m^2}{1-x^2}\}y = 0 \tag{B.1}$$

It has the following orthogonality properties:

$$\int_{-1}^{1} P_n^m(x)P_l^m(x)dx = 0 \tag{B.2}$$

$$\int_{-1}^{1} [P_n^m(x)]^2 dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \tag{B.3}$$

$P_n^m(\cos\theta)$ is related to Legendre polynomial $P_n(\cos\theta)$ by

$$P_n^m(\cos\theta) = (-1)^m \sin^m\theta \frac{d^m P_n(x)}{dx^m}, \quad x = \cos\theta \tag{B.4}$$

where $P_n(x)$ is a solution of

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \tag{B.5}$$

Bessel Function

The Bessel function of first kind, of integer order m, is given by,

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{m+2k}}{2^{m+2k} k!(m+k)!} \tag{B.6}$$

which is a solution of Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)y = 0 \tag{B.7}$$

Relating Bessel Function to Associated Legendre Polynomial

From Thorne (1953),

$$e^{-k(z-d)} J_m(kR) = (-1)^m \sum_{s=m}^{\infty} (-kr)^s \frac{P_s^m(\cos\theta)}{(s+m)!} \tag{B.8}$$

$$e^{-K(z-d)} J_m(KR) = (-1)^m \sum_{s=m}^{\infty} (-Kr)^s \frac{P_s^m(\cos\theta)}{(s+m)!} \tag{B.9}$$

We infer that the expression $\hat{\phi_m}(r, \theta)$ in eqn (2.52) to be expressed in terms of associated Legendre's polynomials:

$$
\begin{aligned}
e^{-k(z+d)} J_m(kR) &= e^{-k(z-d)} e^{-2kd} J_m(kR) \\
&= e^{-2kd} (-1)^m \sum_{s=m}^{\infty} (-kr)^s \frac{P_s^m(\cos\theta)}{(s+m)!} \qquad \text{(B.10)} \\
e^{-K(z+d)} J_m(KR) &= e^{-K(z-d)} e^{-2Kd} J_m(KR) \\
&= e^{-2Kd} (-1)^m \sum_{s=m}^{\infty} (-Kr)^s \frac{P_s^m(\cos\theta)}{(s+m)!} \qquad \text{(B.11)}
\end{aligned}
$$

Appendix C

Motions for Structure

Figure C.1shows the various motions due to radiation for a floating structure. We have already discussed these six motions, related to a floating or submerged body in water. This diagram shows a structure floating in water. The motions would be similar for a submerged body, in which case the structure may be connected to the sea-floor by some mechanical means. The structure is assumed to be rigid and has six independent degrees of motions. Usually out of the three translational motions, the longitudinal and the vertical motions are of most importance. Among the rotational motions, different motion has different significance depending upon the geometry.

We have considered the surge, heave and pitch motions in our study. Usually three degrees of motion are sufficient to describe the radiation problem for a given structure. Usually structures like articulated towers, moored tankers, tension leg platforms, spheres, spheroids, cylinders, etc. undergo these motions. To determine the stress distributions for these kind of structures, it becomes necessary to know

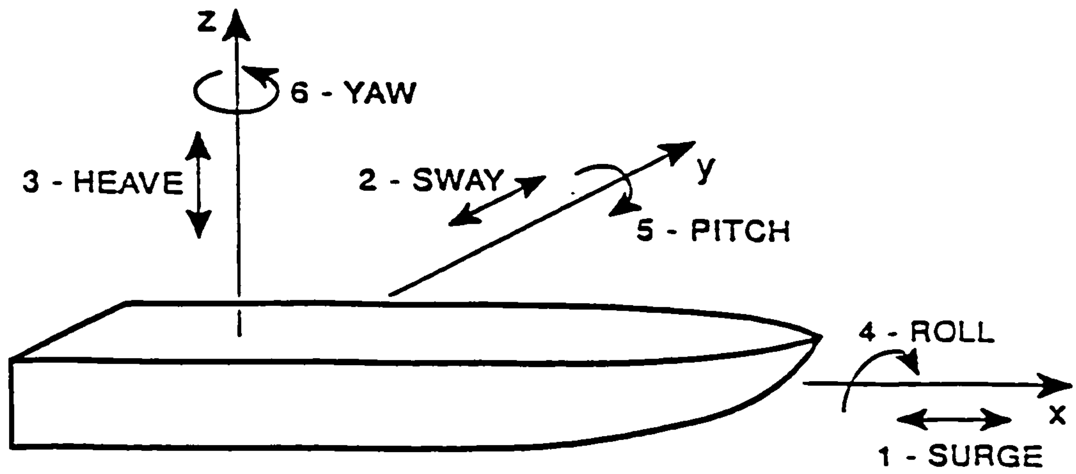

Figure C.1: Motions for a structure in water

their associated motions with associated wave forces.

Appendix D

Expressions for Linear and Quadratic Forces for the Cylinder

Here we show how we calculate the linear force F_l and the quadratic force F_q.

Linear Force

From the complex linear velocity potential ϕ_l in Chapter 4, we can write

$$\nabla\phi_l = \frac{\partial\phi_l}{\partial r}\hat{i} + \frac{1}{r}\frac{\partial\phi_l}{\partial\theta}\hat{j} + \frac{\partial\phi_l}{\partial z}\hat{k} \tag{D.1}$$

which gives us

$$(\nabla\phi_l)^2 = (\frac{\partial\phi_l}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial\phi_l}{\partial\theta})^2 + (\frac{\partial\phi_l}{\partial z})^2 \tag{D.2}$$

We find the coefficient $A_1(kr)$ at $r = b$ in the following way:

$$\begin{aligned} A_1(kr) &= J_1(kr) - \frac{J_1'(kb)}{H_1^{(2)'}(kb)}H_1^{(2)}(kr) \\ &= \frac{J_1(kr)H_1^{(2)'}(kb) - J_1'(kb)H_1^{(2)}(kr)}{H_1^{(2)'}(kb)} \\ &= \frac{J_1(kr)(J_1'(kb) - iY_1'(kb)) - (J_1(kr) - iY_1(kr))J_1'(kb)}{H_1^{(2)'}(kb)} \\ &= \frac{-i(J_1(kr)Y_1'(kb) - J_1'(kb)Y_1(kr))}{H_1^{(2)'}(kb)} \end{aligned}$$

which implies

$$\begin{aligned}
A_1(kr)|_{r=b} &= \frac{-i(J_1(kb)Y_1'(kb) - J_1'(kb)Y_1(kb))}{H_1^{(2)'}(kb)} \\
&= \frac{-iW\{J_1(kb), Y_1(kb)\}}{H_1^{(2)'}(kb)} \\
&= \frac{-2i}{\pi kb H_1^{(2)'}(kb)}
\end{aligned}$$

where W represents the Wronskian of J_1 and Y_1.

Also, because of the expression

$$\begin{aligned}
F_l &= \frac{2bA\rho\sigma^2\pi}{k^2} Re[e^{i\sigma t}\frac{(-2i)}{\pi kb H_1^{(2)'}(kb)}] \\
&= -\frac{4bA\sigma^2\rho\pi}{k^2}\frac{1}{\pi kb} Re[\frac{ie^{i\sigma t}}{H_1^{(2)'}(kb)}]
\end{aligned}$$

using dispersion relation (4.30) and the following relation,

$$\begin{aligned}
\frac{1}{H_1^{(2)'}(kb)} &= \frac{1}{J_1'(kb) - iY_1'(kb)} \\
&= \frac{J_1'(kb) + iY_1'(kb)}{J_1'^2(kb) + Y_1'^2(kb)}
\end{aligned}$$

we get

$$\begin{aligned}
F_l &= -\frac{4bAg\tanh kd\pi}{k^2}\frac{1}{\pi b} Re[\frac{i(\cos\sigma t + i\sin\sigma t)(J_1'(kb) + Y_1'(kb))}{J_1'^2(kb) + Y_1'^2(kb)}] \\
&= \frac{4Ag\pi\tanh kd}{k^2}\frac{1}{\pi b}\frac{\cos\sigma t Y_1'(kb) + \sin\sigma t J_1'(kb)}{J_1'^2(kb) + Y_1'^2(kb)} \\
&= \frac{4bAg\pi\tanh kd}{k^2}\frac{1}{\pi b}\frac{\cos\sigma t\cos\beta + \sin\sigma t\sin\beta}{\sqrt{J_1'^2(kb) + Y_1'^2(kb)}} \\
&= C_M(\rho g A\pi b^2)\tanh kd\cos(\sigma t - \beta)
\end{aligned}$$

where we have chosen

$$C_M = \frac{4}{\pi k^2 b^2\sqrt{J_1'^2(kb) + Y_1'^2(kb)}}$$

$$\beta = \tan^{-1}[\frac{J_1'(kb)}{Y_1'(kb)}]$$

Quadratic Force

Letting $m = n+1$ and using the orthogonality property of cosine and sine functions in eqns (4.82)-(4.85), and also observing that the expression for F_q has terms consisting $\cos m\theta \cos n\theta$ or $\sin m\theta \sin n\theta$ multiplied by $\cos\theta$, only the product of $\cos\theta$ by $\cos\theta$ will be non-zero.

Also since

$$\alpha_n = 2(-i)^n, \; \alpha_{n+1} = 2(-i)^{n+1}$$

we can obtain

$$\alpha_n \alpha_{n+1} = -4(-1)^n i$$

Moreover, we can also write,

$$\frac{1}{H_n^{(2)'}(kb) H_{n+1}^{(2)'}(kb)} = i(C_n + iS_n)$$

All these relations help us to find the solution of F_q in the form of eqn (4.86).

Determination of $\tilde{\phi}$

We assume $Re[\tilde{\phi} e^{2i\sigma t}]$ to be the potential due to a unit translational oscillation of the body. $\tilde{\phi}$ must satisfy

$$\frac{\partial^2\bar{\phi}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{\phi}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\bar{\phi}}{\partial\theta^2}+\frac{\partial^2\bar{\phi}}{\partial z^2} = 0 \tag{D.3}$$

$$\frac{\partial\bar{\phi}}{\partial z}-K\bar{\phi} = 0, \quad z=0 \tag{D.4}$$

$$\frac{\partial\bar{\phi}}{\partial r} = \cos\theta, \quad r=b \tag{D.5}$$

$$\frac{\partial\bar{\phi}}{\partial z} = 0, \quad z=-d \tag{D.6}$$

$$\lim_{\kappa r\to\infty}\sqrt{\kappa r}[\frac{\partial}{\partial r}+i\kappa]\bar{\phi} = 0 \tag{D.7}$$

Assume a solution of the form

$$\bar{\phi}=[P(r)\cosh\kappa(z+d)+U(r)\cos m(z+d)]\cos\theta \tag{D.8}$$

where P(r) and U(r) are arbitrary functions of r only and κ, m are constants. Applying the above conditions gives

$$\bar{\phi}=[pH_1^{(2)}(\kappa r)\cosh\kappa(z+d)+\sum_{j=1}^{\infty}u_jK_1(m_jr)\cos m_j(z+d)]\cos\theta \tag{D.9}$$

where

$$\kappa\tanh\kappa d = K$$

$$m_j\tan m_jd = -K$$

Here p, u_j are constants. It happens that $\cosh\kappa(z+d)$ and $\cos m_j(z+d), j = 1,2,\ldots,\infty$ form an orthogonal set on the interval $(-d,0)$. This makes it easy to determine the constants p, u_j. Therefore, it follows that,

$$\bar{\phi} = [\frac{2H_1^{(2)}(\kappa r)\cosh\kappa(z+d)}{\kappa H_1^{(2)'}(\kappa b)}\frac{\sin\kappa d}{\kappa d+\sinh\kappa d\cosh\kappa d}$$

$$+ \sum_{j=1}^{\infty}\frac{2K_1(m_j r)\cos m_j(z+d)}{m_j K_1'(m_j b)}\frac{\sin m_j d}{m_j d+\sin m_j d\cos m_j d}]\cos\theta \qquad \text{(D.10)}$$

Differentiating above expression with respect to z and the evaluating at z=0, implies

$$\frac{\partial\bar{\phi}}{\partial z} = [\frac{2H_1^{(2)}(\kappa r)}{H_1^{(2)'}(\kappa b)}\frac{\sinh^2\kappa d}{\kappa d+\sinh\kappa d\cosh\kappa d} \qquad \text{(D.11)}$$

$$- \sum_{j=1}^{\infty}\frac{2K_1(m_j r)}{K_1'(m_j b)}\frac{\sin^2 m_j d}{m_j d+\sin m_j d\cos m_j d}]\cos\theta \qquad \text{(D.12)}$$

Appendix E

Some Programs

```
c
c    The variables used in this program are listed below
c
c    Input
c
c kb = kb .GT. 0.0E0
c
c fnu = order of initial y function (fnu .GE. 0.0E0)
c
c N = number of members in the sequence (N .GE.1)
c
c   Output
c
c NZ = number of components of J set to zero
c
c J = a vector whose first N components contain values for
c     the sequence J(K) = J/sub(alpha+K-1)/(X), K=1,...,N.
c
c Y = a vector whose first N components contain values for
c     the sequence Y(I) = Y/sub(fnu+I-1)/(X), I=1,...,N.
c
c Tsub = vector of coefficients of size N.
c
c Csub = vector of coefficients of size N.
c
c Ssub = vector of coefficients of size N.
c
c Esub = vector of coefficients of size N.
c
```

```
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c
c Routines called: besj, besy, recurrence
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc

Integer  I, N, NZ, LL, KK

double precision CM, beta, Fl, Ml, Md, Fd, Fw, Mw, Fmax

double precision BB, GG, tau, kh, kh2

double precision sine, cosine, theta

double precision term1, term2, term3, term4, term5

real JJ1(30),YY1(30),JJ12(30),YY12(30),JJprime(30), YYprime(30)

real JJm1(30), YYm1(30), YYp1(30), JJp1(30)

real  YYpp1(30), JJpp1(30),YYpp12(30),JJpp12(30)

real  YY(30),JJ(30), JJp12(30),YYp12(30)

double precision   Tsub(10),Csub(10), Ssub(10), Esub(10)

real HHb, Hh, hb, Hbhb

real  fnu, order, kb

external besj, besy, recurrence

        data pi /3.1416/


c set the values to be used
kb = 2.5

fnu   = 1.0
```

```
N     = 1


c set up the output files

open(unit=43, file='out.dat', status='unknown')
open(unit=45, file='moment.dat', status='unknown')
open(unit=55, file='bessel.dat', status='unknown')
open(unit=60, file='dynamic.dat', status='unknown')

c calculate other constants

write(43,10)
10 format(2X,'kb',8X,'CM',13X,'beta',14X,'Fl',15X,'Ml')


do while (kb .lt. 70.0)

kh = 1.16*kb

call recurrence(kb, fnu, fnu, N, JJprime, YYprime)

do k = 1,N
JJ1(k) = JJprime(k)
YY1(k) = YYprime(k)
enddo

do k = 1,N
CM = 4.0/(pi*kb**2*sqrt(JJ1(k)**2+YY1(k)**2))
beta = -atan2(JJ1(k),YY1(k))
Fl = abs(CM*pi*dtanh(kh))
Ml = abs(CM*pi*(1.1*kb*dtanh(kh)+1.0/dcosh(kh)-1.0))
enddo

write(43,33)kb, CM, beta, Fl, Ml
33 format(f5.2,4f16.8)

kb = kb + 0.1

enddo
```

```
write(45,11)
11 format(4X,'kb',6X,'fnu',6X,'Fd',7X,'Fmax',8X,'Md')

write(60,12)
12 format(4X,'kb',4X,'fnu',5X,'Fw',5X,'Mw')

write(55,13)
13 format(4X,'kb',4X,'fnu',7X,'T_l',13X,'C_l',15X,'S_l',16X,'E_l')

kb = 1.750

fnu = 0.0

Hh = 0.2

HHb = 0.232

hb = 1.16

Hbhb = 0.26912

do 77 II=1,3

    do 88 LL=1,80

c first term in derivative expression

  call recurrence(kb, fnu, fnu, N, JJ, YY)

  do k = 1,N
JJp1(k) = JJ(k)
YYp1(k) = YY(k)
  enddo

  do k = 1,N
JJp12(k) = JJp1(k)**2
YYp12(k) = YYp1(k)**2
  enddo

c second term in derivative expression
```

```
  order = fnu + 1.0

  call recurrence(kb, order , order, N, JJ, YY)

  do k = 1,N
JJpp1(k) = JJ(k)
YYpp1(k) = YY(k)
JJpp12(k) = JJpp1(k)**2
YYpp12(k) = YYpp1(k)**2
  enddo

c compute coefficients now

  do k = 1,N
Tsub(k) = (JJp12(k)+YYp12(k))*(JJpp12(k)+YYpp12(k))
Csub(k) = (YYp1(k)*JJpp1(k)+YYpp1(k)*JJp1(k))/Tsub(k)
Ssub(k) = (YYp1(k)*YYpp1(k)-JJp1(k)*JJpp1(k))/Tsub(k)
Esub(k) = (JJp1(k)*YYpp1(k)-JJpp1(k)*YYp1(k))/Tsub(k)
write(55,44)kb,fnu,Tsub(k),Csub(k),Ssub(k),Esub(k)
44 format(f7.3,f7.3,E16.9,3f15.10)
  enddo

  do k=1,N
    theta = -datan2(Ssub(k),Csub(k))
  enddo

  kh = 1.16*kb

  kh2 = 2.0*kh

  do k = 1,N
term1 = HHb*(1.0/(kb)**2)*(1.0/pi)
term2 = (1.0-kh2/dsinh(kh2))+fnu*(fnu+1)/kb**2*(1.0+kh2/dsinh(kh2))
Fd = abs(term1*term2*(Esub(k)-(-1)**fnu*(Csub(k)*dcos(theta)-Ssub(k)*dsin(theta
Fmax = 2.0*term1*term2*(-1)**fnu*(sqrt(Csub(k)**2+Ssub(k)**2))
BB = (1.0-1.0/(kh2*dtanh(kh2)))+ (1.0+2.0*(kh)**2)/(kh2*dsinh(kh2))
GG = (1.0-1.0/(kh2*dtanh(kh2)))+(1.0-2.0*(kh)**2)/(kh2*dsinh(kh2))
term3 = Hbhb*(1.0/(pi*kb))*(fnu*(fnu+1)*BB+GG)
Md = abs(term3*(Esub(k)-(-1)**fnu*(Csub(k)*dcos(theta)-Ssub(k)*dsin(theta))))
write(45,22)kb,fnu, Fd, Fmax, Md
22 format(f7.3,f6.2,3f15.8)
          enddo
```

```
  do k = 1,N
term4 = HHb*(2.0/pi)*(1.0/kb**2)
Fw = abs(term4*(Esub(k)+(-1)**fnu*(Csub(k)*dcos(theta)-Ssub(k)*dsin(theta))))
term5 = Hbhb*(2.0/pi)*(1.0/kb)
Mw = abs(term5*(Esub(k)+(-1)**fnu*(Csub(k)*dcos(theta)-Ssub(k)*dsin(theta))))
write(60,66)kb,fnu,Fw,Mw
66 format(f7.3,f7.3,2f15.8)
  enddo

        kb  = kb + 0.2

  if (LL .eq. 80) kb = 1.75

88     continue

    fnu = fnu + 1.0

77 continue

close(43)
close(45)
close(55)
close(60)

stop

end


cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c        c
c This routine calculates the derivatives of the bessel functions        c
c using a recurrence relation. These derivatives are needed to compute   c
c the coefficients under the summation sign.                             c
c        c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c Input :
c
c x = x .GT. 0.0E0 argument of the bessel function
c
```

```
c fnu = order of initial Y function (fnu .GE. 0.0E0)
c
c alpha = order of initial J function (alpha .GE. 0.0E0)
c
c NN = number of members in the sequence (NN .GE. 1)
c
c Ouput:
c
c YYprime = vector containing the derivative for the Bessel function
c           J.
c JJprime = vector containing derivative for the Bessel function Y.
c
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c
c Routines called: besy, besj
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc

subroutine recurrence(x, fnu, alpha, NN, JJprime, YYprime)

integer NN, k

double precision JJprime(30),YYprime(30),JJ0(30),YY0(30)

double precision  JJ2(30),YY2(30),YY(30),JJ(30)

double precision x

real fnu, order1, order2, alpha

external besy, besj


c if alpha and fnu are zero then we need
c to stop the calculation immediately


if ((alpha .eq. 0.0) .and.(fnu .eq. 0.0)) then

order1 = 1.0
```

```
call besy(x,order1,NN,YY)
call besj(x,order1,NN,JJ,NZ)
do k = 1,NN
JJprime(k) = -JJ(k)
YYprime(k) = -YY(k)
enddo

return
stop
Else if ((alpha .lt. 1.0) .and. (fnu .lt. 1.0)) then

order1 = abs(1.0 - fnu)
call besy(x,order1,NN,YY)
call besj(x,order1,NN,JJ,NZ)
do k=1,NN
JJprime(k) = (-1)**order1*JJ(k)
YYprime(k) = (-1)**order1*YY(k)
enddo

return
stop
endif


c if all is fine then, the first calls
c to the routines may be made.


call besy(x, fnu-1.0, NN, YY)

call besj(x, fnu-1.0, NN, JJ, NZ)


do k = 1,NN
JJ0(k) = JJ(k)
YY0(k) = YY(k)
enddo


c the next call is made with the orders
c incremented by one.
```

```
call besy(x, fnu+1.0, NN, YY)

call besj(x, fnu+1.0, NN, JJ, NZ)

do k = 1,NN
JJ2(k) = JJ(k)
YY2(k) = YY(k)
enddo


c the calculation of the derivatives is
c next performed.

do k = 1,NN
JJprime(k) = 1.0/2.0*(JJ0(k)-JJ2(k))
YYprime(k) = 1.0/2.0*(YY0(k)-YY2(k))
enddo

return


end

      SUBROUTINE BESY(X,FNU,N,Y)
C***ROUTINES CALLED  ASYJY,BESY0,BESY1,BESYNU,I1MACH,R1MACH,XERROR,
C                    YAIRY
C
      EXTERNAL YAIRY
      INTEGER I, IFLW, J, N, NB, ND, NN, NUD, NULIM
      INTEGER I1MACH
      REAL       AZN,CN,DNU,ELIM,FLGJY,FN,FNU,RAN,S,S1,S2,TM,TRX,
     1           W,WK,W2N,X,XLIM,XXN,Y
      REAL BESY0, BESY1, R1MACH
      DIMENSION W(2), NULIM(2), Y(*), WK(7)
      SAVE NULIM
      DATA NULIM(1),NULIM(2) / 70 , 100 /
C***FIRST EXECUTABLE STATEMENT  BESY
      NN = -I1MACH(12)
      ELIM = 2.303E0*(FLOAT(NN)*R1MACH(5)-3.0E0)
      XLIM = R1MACH(1)*1.0E+3
      IF (FNU.LT.0.0E0) GO TO 140
      IF (X.LE.0.0E0) GO TO 150
```

```
      IF (X.LT.XLIM) GO TO 170
      IF (N.LT.1) GO TO 160
C
C     ND IS A DUMMY VARIABLE FOR N
C
      ND = N
      NUD = INT(FNU)
      DNU = FNU - FLOAT(NUD)
      NN = MIN0(2,ND)
      FN = FNU + FLOAT(N-1)
      IF (FN.LT.2.0E0) GO TO 100
C
C     OVERFLOW TEST  (LEADING EXPONENTIAL OF ASYMPTOTIC EXPANSION)
C     FOR THE LAST ORDER, FNU+N-1.GE.NULIM
C
      XXN = X/FN
      W2N = 1.0E0-XXN*XXN
      IF(W2N.LE.0.0E0) GO TO 10
      RAN = SQRT(W2N)
      AZN = ALOG((1.0E0+RAN)/XXN) - RAN
      CN = FN*AZN
      IF(CN.GT.ELIM) GO TO 170
   10 CONTINUE
      IF (NUD.LT.NULIM(NN)) GO TO 20
C
C     ASYMPTOTIC EXPANSION FOR ORDERS FNU AND FNU+1.GE.NULIM
C
      FLGJY = -1.0E0
      CALL ASYJY(YAIRY,X,FNU,FLGJY,NN,Y,WK,IFLW)
      IF(IFLW.NE.0) GO TO 170
      IF (NN.EQ.1) RETURN
      TRX = 2.0E0/X
      TM = (FNU+FNU+2.0E0)/X
      GO TO 80
C
   20 CONTINUE
      IF (DNU.NE.0.0E0) GO TO 30
      S1 = BESY0(X)
      IF (NUD.EQ.0 .AND. ND.EQ.1) GO TO 70
      S2 = BESY1(X)
      GO TO 40
   30 CONTINUE
```

```
      NB = 2
      IF (NUD.EQ.0 .AND. ND.EQ.1) NB = 1
      CALL BESYNU(X, DNU, NB, W)
      S1 = W(1)
      IF (NB.EQ.1) GO TO 70
      S2 = W(2)
   40 CONTINUE
      TRX = 2.0E0/X
      TM = (DNU+DNU+2.0E0)/X
C     FORWARD RECUR FROM DNU TO FNU+1 TO GET Y(1) AND Y(2)
      IF (ND.EQ.1) NUD = NUD - 1
      IF (NUD.GT.0) GO TO 50
      IF (ND.GT.1) GO TO 70
      S1 = S2
      GO TO 70
   50 CONTINUE
      DO 60 I=1,NUD
        S = S2
        S2 = TM*S2 - S1
        S1 = S
        TM = TM + TRX
   60 CONTINUE
      IF (ND.EQ.1) S1 = S2
   70 CONTINUE
      Y(1) = S1
      IF (ND.EQ.1) RETURN
      Y(2) = S2
   80 CONTINUE
      IF (ND.EQ.2) RETURN
C     FORWARD RECUR FROM FNU+2 TO FNU+N-1
      DO 90 I=3,ND
        Y(I) = TM*Y(I-1) - Y(I-2)
        TM = TM + TRX
   90 CONTINUE
      RETURN
C
  100 CONTINUE
C     OVERFLOW TEST
      IF (FN.LE.1.0E0) GO TO 110
      IF (-FN*(ALOG(X)-0.693E0).GT.ELIM) GO TO 170
  110 CONTINUE
      IF (DNU.EQ.0.0E0) GO TO 120
```

```
      CALL BESYNU(X, FNU, ND, Y)
      RETURN
  120 CONTINUE
      J = NUD
      IF (J.EQ.1) GO TO 130
      J = J + 1
      Y(J) = BESY0(X)
      IF (ND.EQ.1) RETURN
      J = J + 1
  130 CONTINUE
      Y(J) = BESY1(X)
      IF (ND.EQ.1) RETURN
      TRX = 2.0E0/X
      TM = TRX
      GO TO 80
C
C
C
  140 CONTINUE
      CALL XERROR( 'IN BESY, ORDER, FNU, LESS THAN ZERO', 35, 2, 1)
      RETURN
  150 CONTINUE
      CALL XERROR( 'IN BESY, X LESS THAN OR EQUAL TO ZERO', 37, 2, 1)
      RETURN
  160 CONTINUE
      CALL XERROR( 'IN BESY, N LESS THAN ONE', 24, 2, 1)
      RETURN
  170 CONTINUE
      CALL XERROR( 'IN BESY, OVERFLOW, FNU OR N TOO LARGE OR X TOO SMALL
     1', 52, 6,   1)
      RETURN
      END

      FUNCTION BESY0(X)
C***FIRST EXECUTABLE STATEMENT  BESY0
      IF (NTY0.NE.0) GO TO 10
      NTY0 = INITS (BY0CS, 13, 0.1*R1MACH(3))
      NTM0 = INITS (BM0CS, 21, 0.1*R1MACH(3))
      NTTH0 = INITS (BTH0CS, 24, 0.1*R1MACH(3))
C
      XSML = SQRT (4.0*R1MACH(3))
      XMAX = 1.0/R1MACH(4)
```

```
C
 10   IF (X.LE.0.) CALL XERROR ( 'BESY0   X IS ZERO OR NEGATIVE', 29,
     1  1, 2)
      IF (X.GT.4.0) GO TO 20
C
      Y = 0.
      IF (X.GT.XSML) Y = X*X
      BESY0 = TWODPI*ALOG(0.5*X)*BESJ0(X) + .375 + CSEVL (.125*Y-1.,
     1  BY0CS, NTY0)
      RETURN
C
 20   IF (X.GT.XMAX) CALL XERROR ( 'BESY0   NO PRECISION BECAUSE X IS BI
     1G', 37, 2, 2)
C
      Z = 32.0/X**2 - 1.0
      AMPL = (0.75 + CSEVL (Z, BM0CS, NTM0)) / SQRT(X)
      THETA = X - PI4 + CSEVL (Z, BTH0CS, NTTH0) / X
      BESY0 = AMPL * SIN (THETA)
C
      RETURN
      END

      FUNCTION BESY1(X)
C***ROUTINES CALLED  BESJ1,CSEVL,INITS,R1MACH,XERROR
      DIMENSION BY1CS(14), BM1CS(21), BTH1CS(24)
      DATA NTY1, NTM1, NTTH1, XMIN, XSML, XMAX / 3*0, 3*0./
C***FIRST EXECUTABLE STATEMENT  BESY1
      IF (NTY1.NE.0) GO TO 10
      NTY1 = INITS (BY1CS, 14, 0.1*R1MACH(3))
      NTM1 = INITS (BM1CS, 21, 0.1*R1MACH(3))
      NTTH1 = INITS (BTH1CS, 24, 0.1*R1MACH(3))
C
      XMIN = 1.571*EXP ( AMAX1(ALOG(R1MACH(1)), -ALOG(R1MACH(2)))+.01)
      XSML = SQRT (4.0*R1MACH(3))
      XMAX = 1.0/R1MACH(4)
C
 10   IF (X.LE.0.) CALL XERROR ( 'BESY1   X IS ZERO OR NEGATIVE', 29,
     1  1, 2)
      IF (X.GT.4.0) GO TO 20
C
      IF (X.LT.XMIN) CALL XERROR ( 'BESY1   X SO SMALL Y1 OVERFLOWS',
     1  31, 3, 2)
```

```
      Y = 0.
      IF (X.GT.XSML) Y = X*X
      BESY1 = TWODPI*ALOG(0.5*X)*BESJ1(X) +
     1  (0.5 + CSEVL (.125*Y-1., BY1CS, NTY1))/X
      RETURN
C
 20   IF (X.GT.XMAX) CALL XERROR ( 'BESY1   NO PRECISION BECAUSE X IS BI
     1G', 37, 2, 2)
C
      Z = 32.0/X**2 - 1.0
      AMPL = (0.75 + CSEVL (Z, BM1CS, NTM1)) / SQRT(X)
      THETA = X - 3.0*PI4 + CSEVL (Z, BTH1CS, NTTH1) / X
      BESY1 = AMPL * SIN (THETA)
C
      RETURN
      END

      SUBROUTINE BESYNU(X,FNU,N,Y)
C***ROUTINES CALLED  GAMMA,R1MACH,XERROR
      EXTERNAL GAMMA
C
      INTEGER I, INU, J, K, KK, N, NN
      REAL A, AK, ARG, A1, A2, BK, CB, CBK, CC, CCK, CK, COEF, CPT,
     1 CP1, CP2, CS, CS1, CS2, CX, DNU, DNU2, ETEST, ETX, F, FC, FHS,
     2 FK, FKS, FLRX, FMU, FN, FNU, FX, G, G1, G2, HPI, P, PI, PT, Q,
     3 RB, RBK, RCK, RELB, RPT, RP1, RP2, RS, RS1, RS2, RTHPI, RX, S,
     4 SA, SB, SMU, SS, ST, S1, S2, TB, TM, TOL, T1, T2, X, X1, X2, Y
      REAL GAMMA, R1MACH, SINH, COSH
      DIMENSION A(120), RB(120), CB(120), Y(*), CC(8)
      SAVE X1, X2, PI, RTHPI, HPI, CC
      DATA X1, X2 / 3.0E0, 20.0E0 /
      DATA PI,RTHPI        / 3.14159265358979E+00, 7.97884560802865E-01/
      DATA HPI             / 1.57079632679490E+00/
      DATA CC(1), CC(2), CC(3), CC(4), CC(5), CC(6), CC(7), CC(8)
     1                     / 5.77215664901533E-01,-4.20026350340952E-02,
     2-4.21977345555443E-02, 7.21894324666300E-03,-2.15241674114900E-04,
     3-2.01348547807000E-05, 1.13302723200000E-06, 6.11609500000000E-09/
C***FIRST EXECUTABLE STATEMENT  BESYNU
      AK = R1MACH(3)
      TOL = AMAX1(AK,1.0E-15)
      IF (X.LE.0.0E0) GO TO 270
      IF (FNU.LT.0.0E0) GO TO 280
```

```
      IF (N.LT.1) GO TO 290
      RX = 2.0E0/X
      INU = INT(FNU+0.5E0)
      DNU = FNU - FLOAT(INU)
      IF (ABS(DNU).EQ.0.5E0) GO TO 260
      DNU2 = 0.0E0
      IF (ABS(DNU).LT.TOL) GO TO 10
      DNU2 = DNU*DNU
   10 CONTINUE
      IF (X.GT.X1) GO TO 120
C
C     SERIES FOR X.LE.X1
C
      A1 = 1.0E0 - DNU
      A2 = 1.0E0 + DNU
      T1 = 1.0E0/GAMMA(A1)
      T2 = 1.0E0/GAMMA(A2)
      IF (ABS(DNU).GT.0.1E0) GO TO 40
C     SERIES FOR F0 TO RESOLVE INDETERMINACY FOR SMALL ABS(DNU)
      S = CC(1)
      AK = 1.0E0
      DO 20 K=2,8
        AK = AK*DNU2
        TM = CC(K)*AK
        S = S + TM
        IF (ABS(TM).LT.TOL) GO TO 30
   20 CONTINUE
   30 G1 = -(S+S)
      GO TO 50
   40 CONTINUE
      G1 = (T1-T2)/DNU
   50 CONTINUE
      G2 = T1 + T2
      SMU = 1.0E0
      FC = 1.0E0/PI
      FLRX = ALOG(RX)
      FMU = DNU*FLRX
      TM = 0.0E0
      IF (DNU.EQ.0.0E0) GO TO 60
      TM = SIN(DNU*HPI)/DNU
      TM = (DNU+DNU)*TM*TM
      FC = DNU/SIN(DNU*PI)
```

```
      IF (FMU.NE.0.0E0) SMU = SINH(FMU)/FMU
   60 CONTINUE
      F = FC*(G1*COSH(FMU)+G2*FLRX*SMU)
      FX = EXP(FMU)
      P = FC*T1*FX
      Q = FC*T2/FX
      G = F + TM*Q
      AK = 1.0E0
      CK = 1.0E0
      BK = 1.0E0
      S1 = G
      S2 = P
      IF (INU.GT.0 .OR. N.GT.1) GO TO 90
      IF (X.LT.TOL) GO TO 80
      CX = X*X*0.25E0
   70 CONTINUE
      F = (AK*F+P+Q)/(BK-DNU2)
      P = P/(AK-DNU)
      Q = Q/(AK+DNU)
      G = F + TM*Q
      CK = -CK*CX/AK
      T1 = CK*G
      S1 = S1 + T1
      BK = BK + AK + AK + 1.0E0
      AK = AK + 1.0E0
      S = ABS(T1)/(1.0E0+ABS(S1))
      IF (S.GT.TOL) GO TO 70
   80 CONTINUE
      Y(1) = -S1
      RETURN
   90 CONTINUE
      IF (X.LT.TOL) GO TO 110
      CX = X*X*0.25E0
  100 CONTINUE
      F = (AK*F+P+Q)/(BK-DNU2)
      P = P/(AK-DNU)
      Q = Q/(AK+DNU)
      G = F + TM*Q
      CK = -CK*CX/AK
      T1 = CK*G
      S1 = S1 + T1
      T2 = CK*(P-AK*G)
```

```
      S2 = S2 + T2
      BK = BK + AK + AK + 1.0E0
      AK = AK + 1.0E0
      S = ABS(T1)/(1.0E0+ABS(S1)) + ABS(T2)/(1.0E0+ABS(S2))
      IF (S.GT.TOL) GO TO 100
  110 CONTINUE
      S2 = -S2*RX
      S1 = -S1
      GO TO 160
  120 CONTINUE
      COEF = RTHPI/SQRT(X)
      IF (X.GT.X2) GO TO 210
C
C     MILLER ALGORITHM FOR X1.LT.X.LE.X2
C
      ETEST = COS(PI*DNU)/(PI*X*TOL)
      FKS = 1.0E0
      FHS = 0.25E0
      FK = 0.0E0
      RCK = 2.0E0
      CCK = X + X
      RP1 = 0.0E0
      CP1 = 0.0E0
      RP2 = 1.0E0
      CP2 = 0.0E0
      K = 0
  130 CONTINUE
      K = K + 1
      FK = FK + 1.0E0
      AK = (FHS-DNU2)/(FKS+FK)
      PT = FK + 1.0E0
      RBK = RCK/PT
      CBK = CCK/PT
      RPT = RP2
      CPT = CP2
      RP2 = RBK*RPT - CBK*CPT - AK*RP1
      CP2 = CBK*RPT + RBK*CPT - AK*CP1
      RP1 = RPT
      CP1 = CPT
      RB(K) = RBK
      CB(K) = CBK
      A(K) = AK
```

```
      RCK = RCK + 2.0E0
      FKS = FKS + FK + FK + 1.0E0
      FHS = FHS + FK + FK
      PT = AMAX1(ABS(RP1),ABS(CP1))
      FC = (RP1/PT)**2 + (CP1/PT)**2
      PT = PT*SQRT(FC)*FK
      IF (ETEST.GT.PT) GO TO 130
      KK = K
      RS = 1.0E0
      CS = 0.0E0
      RP1 = 0.0E0
      CP1 = 0.0E0
      RP2 = 1.0E0
      CP2 = 0.0E0
      DO 140 I=1,K
        RPT = RP2
        CPT = CP2
        RP2 = (RB(KK)*RPT-CB(KK)*CPT-RP1)/A(KK)
        CP2 = (CB(KK)*RPT+RB(KK)*CPT-CP1)/A(KK)
        RP1 = RPT
        CP1 = CPT
        RS = RS + RP2
        CS = CS + CP2
        KK = KK - 1
  140 CONTINUE
      PT = AMAX1(ABS(RS),ABS(CS))
      FC = (RS/PT)**2 + (CS/PT)**2
      PT = PT*SQRT(FC)
      RS1 = (RP2*(RS/PT)+CP2*(CS/PT))/PT
      CS1 = (CP2*(RS/PT)-RP2*(CS/PT))/PT
      FC = HPI*(DNU-0.5E0) - X
      P = COS(FC)
      Q = SIN(FC)
      S1 = (CS1*Q-RS1*P)*COEF
      IF (INU.GT.0 .OR. N.GT.1) GO TO 150
      Y(1) = S1
      RETURN
  150 CONTINUE
      PT = AMAX1(ABS(RP2),ABS(CP2))
      FC = (RP2/PT)**2 + (CP2/PT)**2
      PT = PT*SQRT(FC)
      RPT = DNU + 0.5E0 - (RP1*(RP2/PT)+CP1*(CP2/PT))/PT
```

```
      CPT = X - (CP1*(RP2/PT)-RP1*(CP2/PT))/PT
      CS2 = CS1*CPT - RS1*RPT
      RS2 = RPT*CS1 + RS1*CPT
      S2 = (RS2*Q+CS2*P)*COEF/X
C
C     FORWARD RECURSION ON THE THREE TERM RECURSION RELATION
C
  160 CONTINUE
      CK = (DNU+DNU+2.0E0)/X
      IF (N.EQ.1) INU = INU - 1
      IF (INU.GT.0) GO TO 170
      IF (N.GT.1) GO TO 190
      S1 = S2
      GO TO 190
  170 CONTINUE
      DO 180 I=1,INU
        ST = S2
        S2 = CK*S2 - S1
        S1 = ST
        CK = CK + RX
  180 CONTINUE
      IF (N.EQ.1) S1 = S2
  190 CONTINUE
      Y(1) = S1
      IF (N.EQ.1) RETURN
      Y(2) = S2
      IF (N.EQ.2) RETURN
      DO 200 I=3,N
        Y(I) = CK*Y(I-1) - Y(I-2)
        CK = CK + RX
  200 CONTINUE
      RETURN
C
C     ASYMPTOTIC EXPANSION FOR LARGE X, X.GT.X2
C
  210 CONTINUE
      NN = 2
      IF (INU.EQ.0 .AND. N.EQ.1) NN = 1
      DNU2 = DNU + DNU
      FMU = 0.0E0
      IF (ABS(DNU2).LT.TOL) GO TO 220
      FMU = DNU2*DNU2
```

```
  220 CONTINUE
      ARG = X - HPI*(DNU+0.5E0)
      SA = SIN(ARG)
      SB = COS(ARG)
      ETX = 8.0E0*X
      DO 250 K=1,NN
        S1 = S2
        T2 = (FMU-1.0E0)/ETX
        SS = T2
        RELB = TOL*ABS(T2)
        T1 = ETX
        S = 1.0E0
        FN = 1.0E0
        AK = 0.0E0
        DO 230 J=1,13
          T1 = T1 + ETX
          AK = AK + 8.0E0
          FN = FN + AK
          T2 = -T2*(FMU-FN)/T1
          S = S + T2
          T1 = T1 + ETX
          AK = AK + 8.0E0
          FN = FN + AK
          T2 = T2*(FMU-FN)/T1
          SS = SS + T2
          IF (ABS(T2).LE.RELB) GO TO 240
  230   CONTINUE
  240   S2 = COEF*(S*SA+SS*SB)
        FMU = FMU + 8.0E0*DNU + 4.0E0
        TB = SA
        SA = -SB
        SB = TB
  250 CONTINUE
      IF (NN.GT.1) GO TO 160
      S1 = S2
      GO TO 190
C
C     FNU=HALF ODD INTEGER CASE
C
  260 CONTINUE
      COEF = RTHPI/SQRT(X)
      S1 = COEF*SIN(X)
```

```
      S2 = -COEF*COS(X)
      GO TO 160
C
C
  270 CALL XERROR( 'IN BESYNU, X NOT GREATER THAN ZERO', 34, 2, 1)
      RETURN
  280 CALL XERROR( 'IN BESYNU, FNU NOT ZERO OR POSITIVE', 35, 2, 1)
      RETURN
  290 CALL XERROR( 'IN BESYNU, N NOT GREATER THAN 0', 31, 2, 1)
      RETURN
      END

      FUNCTION CSEVL(X,CS,N)
C***FIRST EXECUTABLE STATEMENT  CSEVL
       IF(N.LT.1) CALL XERROR( 'CSEVL   NUMBER OF TERMS LE 0', 28, 2,2)
       IF(N.GT.1000) CALL XERROR ( 'CSEVL   NUMBER OF TERMS GT 1000',
     1   31,3,2)
       IF (X.LT. -1.0 .OR. X.GT. 1.0) CALL XERROR( 'CSEVL   X OUTSIDE (-
     11,+1)', 25, 1, 1)
C
       B1=0.
       B0=0.
       TWOX=2.*X
       DO 10 I=1,N
       B2=B1
       B1=B0
       NI=N+1-I
       B0=TWOX*B1-B2+CS(NI)
 10    CONTINUE
C
       CSEVL = 0.5 * (B0-B2)
C
       RETURN
      END

      FUNCTION GAMMA(X)
C***ROUTINES CALLED  CSEVL,GAMLIM,INITS,R1MACH,R9LGMC,XERROR
      DIMENSION GCS(23)
C SQ2PIL IS ALOG (SQRT (2.*PI) )
      DATA SQ2PIL /0.91893 85332 04672 74E0/
      DATA NGCS, XMIN, XMAX, DXREL /0, 3*0.0 /
C
```

```
C
C***FIRST EXECUTABLE STATEMENT  GAMMA
      IF (NGCS.NE.0) GO TO 10
C
C ----------------------------------------------------------------------
C INITIALIZE.  FIND LEGAL BOUNDS FOR X, AND DETERMINE THE NUMBER OF
C TERMS IN THE SERIES REQUIRED TO ATTAIN AN ACCURACY TEN TIMES BETTER
C THAN MACHINE PRECISION.
C
      NGCS = INITS (GCS, 23, 0.1*R1MACH(3))
C
      CALL GAMLIM (XMIN, XMAX)
      DXREL = SQRT (R1MACH(4))
C
C ----------------------------------------------------------------------
C FINISH INITIALIZATION.  START EVALUATING GAMMA(X).
C
 10   Y = ABS(X)
      IF (Y.GT.10.0) GO TO 50
C
C COMPUTE GAMMA(X) FOR ABS(X) .LE. 10.0.  REDUCE INTERVAL AND
C FIND GAMMA(1+Y) FOR 0. .LE. Y .LT. 1. FIRST OF ALL.
C
      N = X
      IF (X.LT.0.) N = N - 1
      Y = X - FLOAT(N)
      N = N - 1
      GAMMA = 0.9375 + CSEVL(2.*Y-1., GCS, NGCS)
      IF (N.EQ.0) RETURN
C
      IF (N.GT.0) GO TO 30
C
C COMPUTE GAMMA(X) FOR X .LT. 1.
C
      N = -N
      IF (X.EQ.0.) CALL XERROR ( 'GAMMA   X IS 0', 14, 4, 2)
      IF (X.LT.0. .AND. X+FLOAT(N-2).EQ.0.) CALL XERROR ( 'GAMMA   X IS
     1 A NEGATIVE INTEGER', 31, 4, 2)
      IF (X.LT.(-0.5) .AND. ABS((X-AINT(X-0.5))/X).LT.DXREL) CALL
     1  XERROR ( 'GAMMA   ANSWER LT HALF PRECISION BECAUSE X TOO NEAR NE
     2GATIVE INTEGER', 68, 1, 1)
C
```

```
      DO 20 I=1,N
        GAMMA = GAMMA / (X+FLOAT(I-1))
 20   CONTINUE
      RETURN
C
C GAMMA(X) FOR X .GE. 2.
C
 30   DO 40 I=1,N
        GAMMA = (Y+FLOAT(I))*GAMMA
 40   CONTINUE
      RETURN
C
C COMPUTE GAMMA(X) FOR ABS(X) .GT. 10.0.  RECALL Y = ABS(X).
C
 50   IF (X.GT.XMAX) CALL XERROR ( 'GAMMA   X SO BIG GAMMA OVERFLOWS',
     1  32, 3, 2)
C
      GAMMA = 0.
      IF (X.LT.XMIN) CALL XERROR ( 'GAMMA   X SO SMALL GAMMA UNDERFLOWS'
     1  , 35, 2, 1)
      IF (X.LT.XMIN) RETURN
C
      GAMMA = EXP((Y-0.5)*ALOG(Y) - Y + SQ2PIL + R9LGMC(Y) )
      IF (X.GT.0.) RETURN
C
      IF (ABS((X-AINT(X-0.5))/X).LT.DXREL) CALL XERROR ( 'GAMMA   ANSWER
     1 LT HALF PRECISION, X TOO NEAR NEGATIVE INTEGER'  , 61, 1, 1)
C
      SINPIY = SIN (PI*Y)
      IF (SINPIY.EQ.0.) CALL XERROR ( 'GAMMA   X IS A NEGATIVE INTEGER',
     1 31, 4, 2)
C
      GAMMA = -PI / (Y*SINPIY*GAMMA)
C
      RETURN
      END

      INTEGER FUNCTION I1MACH(I)
C***FIRST EXECUTABLE STATEMENT  I1MACH
      IF (I .LT. 1  .OR.  I .GT. 16)
     1   CALL XERROR ( 'I1MACH -- I OUT OF BOUNDS',25,1,2)
C
```

```
      I1MACH=IMACH(I)
      RETURN
C
      END

      FUNCTION INITS(OS,NOS,ETA)
C***ROUTINES CALLED  XERROR
      DIMENSION OS(NOS)
C***FIRST EXECUTABLE STATEMENT  INITS
      IF (NOS.LT.1) CALL XERROR ( 'INITS   NUMBER OF COEFFICIENTS LT 1',
     1 35, 2, 2)
C
      ERR = 0.
      DO 10 II=1,NOS
        I = NOS + 1 - II
        ERR = ERR + ABS(OS(I))
        IF (ERR.GT.ETA) GO TO 20
 10   CONTINUE
C
 20   IF (I.EQ.NOS) CALL XERROR ( 'INITS   ETA MAY BE TOO SMALL', 28,
     1  1, 2)
      INITS = I
C
      RETURN
      END

      REAL FUNCTION R1MACH(I)
C***ROUTINES CALLED  XERROR
C
      INTEGER SMALL(2)
      INTEGER LARGE(2)
      INTEGER RIGHT(2)
      INTEGER DIVER(2)
      INTEGER LOG10(2)
C
      REAL RMACH(5)
C
      EQUIVALENCE (RMACH(1),SMALL(1))
      EQUIVALENCE (RMACH(2),LARGE(1))
      EQUIVALENCE (RMACH(3),RIGHT(1))
      EQUIVALENCE (RMACH(4),DIVER(1))
      EQUIVALENCE (RMACH(5),LOG10(1))
```

```
C***  FIRST EXECUTABLE STATEMENT R9LGMC
      IF (NALGM.NE.0) GO TO 10
      NALGM = INITS (ALGMCS,6,R1MACH(3))
 10   IF (X.LT.10.0) CALL XERROR ('R9LGMC X MUST BE GE 10',23,1,2)
      IF (X.GE.XMAX) GO TO 20
C
      R9LGMC = 1.0/(12.0*X)
      IF (X.LT.XBIG) R9LGMC = CSEVL (2.0*(10./X)**2-1., ALGMCS, NALGM)/X
      RETURN
C
 20   R9LGMC = 0.0
      CALL XERROR ( 'R9LGMC  X SO BIG R9LGMC UNDERFLOWS', 34, 2, 1)
      RETURN
C
      END

      SUBROUTINE XERROR(MESSG,NMESSG,NERR,LEVEL)
C***ROUTINES CALLED  XERRWV
      CHARACTER*(*) MESSG
C***FIRST EXECUTABLE STATEMENT  XERROR
      CALL XERRWV(MESSG,NMESSG,NERR,LEVEL,0,0,0,0,0.,0.)
      RETURN
      END

      SUBROUTINE XERRWV(MESSG,NMESSG,NERR,LEVEL,NI,I1,I2,NR,R1,R2)
C***ROUTINES CALLED  FDUMP,I1MACH,J4SAVE,XERABT,XERCTL,XERPRT,XERSAV,
C                    XGETUA
      CHARACTER*(*) MESSG
      CHARACTER*20 LFIRST
      CHARACTER*37 FORM
      DIMENSION LUN(5)
C     GET FLAGS
C***FIRST EXECUTABLE STATEMENT  XERRWV
      LKNTRL = J4SAVE(2,0,.FALSE.)
      MAXMES = J4SAVE(4,0,.FALSE.)
C     CHECK FOR VALID INPUT
      IF ((NMESSG.GT.0).AND.(NERR.NE.0).AND.
     1    (LEVEL.GE.(-1)).AND.(LEVEL.LE.2)) GO TO 10
         IF (LKNTRL.GT.0) CALL XERPRT('FATAL ERROR IN...',17)
         CALL XERPRT('XERROR -- INVALID INPUT',23)
         IF (LKNTRL.GT.0) CALL FDUMP
         IF (LKNTRL.GT.0) CALL XERPRT('JOB ABORT DUE TO FATAL ERROR.',
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```
     1  29)
        IF (LKNTRL.GT.0) CALL XERSAV(' ',0,0,0,KDUMMY)
        CALL XERABT('XERROR -- INVALID INPUT',23)
        RETURN
   10 CONTINUE
C     RECORD MESSAGE
      JUNK = J4SAVE(1,NERR,.TRUE.)
      CALL XERSAV(MESSG,NMESSG,NERR,LEVEL,KOUNT)
C     LET USER OVERRIDE
      LFIRST = MESSG
      LMESSG = NMESSG
      LERR = NERR
      LLEVEL = LEVEL
      CALL XERCTL(LFIRST,LMESSG,LERR,LLEVEL,LKNTRL)
C     RESET TO ORIGINAL VALUES
      LMESSG = NMESSG
      LERR = NERR
      LLEVEL = LEVEL
      LKNTRL = MAX0(-2,MIN0(2,LKNTRL))
      MKNTRL = IABS(LKNTRL)
C     DECIDE WHETHER TO PRINT MESSAGE
      IF ((LLEVEL.LT.2).AND.(LKNTRL.EQ.0)) GO TO 100
      IF (((LLEVEL.EQ.(-1)).AND.(KOUNT.GT.MIN0(1,MAXMES)))
     1.OR.((LLEVEL.EQ.0)   .AND.(KOUNT.GT.MAXMES))
     2.OR.((LLEVEL.EQ.1)   .AND.(KOUNT.GT.MAXMES).AND.(MKNTRL.EQ.1))
     3.OR.((LLEVEL.EQ.2)   .AND.(KOUNT.GT.MAX0(1,MAXMES)))) GO TO 100
         IF (LKNTRL.LE.0) GO TO 20
            CALL XERPRT(' ',1)
C           INTRODUCTION
            IF (LLEVEL.EQ.(-1)) CALL XERPRT
     1('WARNING MESSAGE...THIS MESSAGE WILL ONLY BE PRINTED ONCE.',57)
            IF (LLEVEL.EQ.0) CALL XERPRT('WARNING IN...',13)
            IF (LLEVEL.EQ.1) CALL XERPRT
     1      ('RECOVERABLE ERROR IN...',23)
            IF (LLEVEL.EQ.2) CALL XERPRT('FATAL ERROR IN...',17)
   20    CONTINUE
C        MESSAGE
         CALL XERPRT(MESSG,LMESSG)
         CALL XGETUA(LUN,NUNIT)
         ISIZEI = LOG10(FLOAT(I1MACH(9))) + 1.0
         ISIZEF = LOG10(FLOAT(I1MACH(10))**I1MACH(11)) + 1.0
         DO 50 KUNIT=1,NUNIT
```

```
            IUNIT = LUN(KUNIT)
            IF (IUNIT.EQ.0) IUNIT = I1MACH(4)
            DO 22 I=1,MIN(NI,2)
               WRITE (FORM,21) I,ISIZEI
   21          FORMAT ('(11X,21HIN ABOVE MESSAGE, I',I1,'=,I',I2,')   ')
               IF (I.EQ.1) WRITE (IUNIT,FORM) I1
               IF (I.EQ.2) WRITE (IUNIT,FORM) I2
   22       CONTINUE
            DO 24 I=1,MIN(NR,2)
               WRITE (FORM,23) I,ISIZEF+10,ISIZEF
   23          FORMAT ('(11X,21HIN ABOVE MESSAGE, R',I1,'=,E',
     1         I2,'.',I2,')')
               IF (I.EQ.1) WRITE (IUNIT,FORM) R1
               IF (I.EQ.2) WRITE (IUNIT,FORM) R2
   24       CONTINUE
            IF (LKNTRL.LE.0) GO TO 40
C              ERROR NUMBER
               WRITE (IUNIT,30) LERR
   30          FORMAT (15H ERROR NUMBER =,I10)
   40       CONTINUE
   50    CONTINUE
C        TRACE-BACK
         IF (LKNTRL.GT.0) CALL FDUMP
  100 CONTINUE
      IFATAL = 0
      IF ((LLEVEL.EQ.2).OR.((LLEVEL.EQ.1).AND.(MKNTRL.EQ.2)))
     1IFATAL = 1
C     QUIT HERE IF MESSAGE IS NOT FATAL
      IF (IFATAL.LE.0) RETURN
      IF ((LKNTRL.LE.0).OR.(KOUNT.GT.MAX0(1,MAXMES))) GO TO 120
C        PRINT REASON FOR ABORT
         IF (LLEVEL.EQ.1) CALL XERPRT
     1   ('JOB ABORT DUE TO UNRECOVERED ERROR.',35)
         IF (LLEVEL.EQ.2) CALL XERPRT
     1   ('JOB ABORT DUE TO FATAL ERROR.',29)
C        PRINT ERROR SUMMARY
         CALL XERSAV(' ',-1,0,0,KDUMMY)
  120 CONTINUE
C     ABORT
      IF ((LLEVEL.EQ.2).AND.(KOUNT.GT.MAX0(1,MAXMES))) LMESSG = 0
      CALL XERABT(MESSG,LMESSG)
      RETURN
```

```
      END

      SUBROUTINE XERSAV(MESSG,NMESSG,NERR,LEVEL,ICOUNT)
C***ROUTINES CALLED  I1MACH,S88FMT,XGETUA
      INTEGER LUN(5)
      CHARACTER*(*) MESSG
      CHARACTER*20 MESTAB(10),MES
      DIMENSION NERTAB(10),LEVTAB(10),KOUNT(10)
      SAVE MESTAB,NERTAB,LEVTAB,KOUNT,KOUNTX
C     NEXT TWO DATA STATEMENTS ARE NECESSARY TO PROVIDE A BLANK
C     ERROR TABLE INITIALLY
      DATA KOUNT(1),KOUNT(2),KOUNT(3),KOUNT(4),KOUNT(5),
     1     KOUNT(6),KOUNT(7),KOUNT(8),KOUNT(9),KOUNT(10)
     2     /0,0,0,0,0,0,0,0,0,0/
      DATA KOUNTX/0/
C***FIRST EXECUTABLE STATEMENT  XERSAV
      IF (NMESSG.GT.0) GO TO 80
C     DUMP THE TABLE
         IF (KOUNT(1).EQ.0) RETURN
C        PRINT TO EACH UNIT
         CALL XGETUA(LUN,NUNIT)
         DO 60 KUNIT=1,NUNIT
            IUNIT = LUN(KUNIT)
            IF (IUNIT.EQ.0) IUNIT = I1MACH(4)
C           PRINT TABLE HEADER
            WRITE (IUNIT,10)
   10       FORMAT (32H0          ERROR MESSAGE SUMMARY/
     1      51H MESSAGE START             NERR     LEVEL     COUNT)
C           PRINT BODY OF TABLE
            DO 20 I=1,10
               IF (KOUNT(I).EQ.0) GO TO 30
               WRITE (IUNIT,15) MESTAB(I),NERTAB(I),LEVTAB(I),KOUNT(I)
   15          FORMAT (1X,A20,3I10)
   20       CONTINUE
   30       CONTINUE
C           PRINT NUMBER OF OTHER ERRORS
            IF (KOUNTX.NE.0) WRITE (IUNIT,40) KOUNTX
   40       FORMAT (41H0OTHER ERRORS NOT INDIVIDUALLY TABULATED=,I10)
            WRITE (IUNIT,50)
   50       FORMAT (1X)
   60    CONTINUE
         IF (NMESSG.LT.0) RETURN
```

```
C       CLEAR THE ERROR TABLES
        DO 70 I=1,10
   70      KOUNT(I) = 0
        KOUNTX = 0
        RETURN
   80 CONTINUE
C     PROCESS A MESSAGE...
C     SEARCH FOR THIS MESSG, OR ELSE AN EMPTY SLOT FOR THIS MESSG,
C     OR ELSE DETERMINE THAT THE ERROR TABLE IS FULL.
      MES = MESSG
      DO 90 I=1,10
         II = I
         IF (KOUNT(I).EQ.0) GO TO 110
         IF (MES.NE.MESTAB(I)) GO TO 90
         IF (NERR.NE.NERTAB(I)) GO TO 90
         IF (LEVEL.NE.LEVTAB(I)) GO TO 90
         GO TO 100
   90 CONTINUE
C     THREE POSSIBLE CASES...
C     TABLE IS FULL
         KOUNTX = KOUNTX+1
         ICOUNT = 1
         RETURN
C     MESSAGE FOUND IN TABLE
  100    KOUNT(II) = KOUNT(II) + 1
         ICOUNT = KOUNT(II)
         RETURN
C     EMPTY SLOT FOUND FOR NEW MESSAGE
  110    MESTAB(II) = MES
         NERTAB(II) = NERR
         LEVTAB(II) = LEVEL
         KOUNT(II)  = 1
         ICOUNT = 1
         RETURN
      END

      SUBROUTINE XGETUA(IUNITA,N)
C***ROUTINES CALLED  J4SAVE
      DIMENSION IUNITA(5)
C***FIRST EXECUTABLE STATEMENT  XGETUA
      N = J4SAVE(5,0,.FALSE.)
      DO 30 I=1,N
```

```
         INDEX = I+4
         IF (I.EQ.1) INDEX = 3
         IUNITA(I) = J4SAVE(INDEX,0,.FALSE.)
   30 CONTINUE
      RETURN
      END

      SUBROUTINE YAIRY(X,RX,C,BI,DBI)
C***REFER TO  BESJ,BESY
C***ROUTINES CALLED  (NONE)
C
      INTEGER I, J, M1, M1D, M2, M2D, M3, M3D, M4D, N1, N1D, N2, N2D,
     1 N3, N3D, N4D
      REAL AA, AX, BB, BI, BJN, BJP, BK1, BK2, BK3, BK4, C, CON1, CON2,
     1 CON3, CV, DAA, DBB, DBI, DBJN, DBJP, DBK1, DBK2, DBK3, DBK4, D1,
     2 D2, EX, E1, E2, FPI12, F1, F2, RTRX, RX, SPI12, S1, S2, T, TC,
     3 TEMP1, TEMP2, TT, X
      DIMENSION BK1(20), BK2(20), BK3(20), BK4(14)
      DIMENSION BJP(19), BJN(19), AA(14), BB(14)
      DIMENSION DBK1(21), DBK2(20), DBK3(20), DBK4(14)
      DIMENSION DBJP(19), DBJN(19), DAA(14), DBB(14)
      SAVE N1, N2, N3, M1, M2, M3, N1D, N2D, N3D, N4D,
     1 M1D, M2D, M3D, M4D, FPI12, SPI12, CON1, CON2, CON3,
     2 BK1, BK2, BK3, BK4, BJP, BJN, AA, BB, DBK1, DBK2, DBK3, DBK4,
     3 DBJP, DBJN, DAA, DBB
C***FIRST EXECUTABLE STATEMENT  YAIRY
      AX = ABS(X)
      RX = SQRT(AX)
      C = CON1*AX*RX
      IF (X.LT.0.0E0) GO TO 120
      IF (C.GT.8.0E0) GO TO 60
      IF (X.GT.2.5E0) GO TO 30
      T = (X+X-2.5E0)*0.4E0
      TT = T + T
      J = N1
      F1 = BK1(J)
      F2 = 0.0E0
      DO 10 I=1,M1
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + BK1(J)
        F2 = TEMP1
```

```
   10 CONTINUE
      BI = T*F1 - F2 + BK1(1)
      J = N1D
      F1 = DBK1(J)
      F2 = 0.0E0
      DO 20 I=1,M1D
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + DBK1(J)
        F2 = TEMP1
   20 CONTINUE
      DBI = T*F1 - F2 + DBK1(1)
      RETURN
   30 CONTINUE
      RTRX = SQRT(RX)
      T = (X+X-CON2)*CON3
      TT = T + T
      J = N1
      F1 = BK2(J)
      F2 = 0.0E0
      DO 40 I=1,M1
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + BK2(J)
        F2 = TEMP1
   40 CONTINUE
      BI = (T*F1-F2+BK2(1))/RTRX
      EX = EXP(C)
      BI = BI*EX
      J = N2D
      F1 = DBK2(J)
      F2 = 0.0E0
      DO 50 I=1,M2D
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + DBK2(J)
        F2 = TEMP1
   50 CONTINUE
      DBI = (T*F1-F2+DBK2(1))*RTRX
      DBI = DBI*EX
      RETURN
C
```

```
 60 CONTINUE
    RTRX = SQRT(RX)
    T = 16.0E0/C - 1.0E0
    TT = T + T
    J = N1
    F1 = BK3(J)
    F2 = 0.0E0
    DO 70 I=1,M1
      J = J - 1
      TEMP1 = F1
      F1 = TT*F1 - F2 + BK3(J)
      F2 = TEMP1
 70 CONTINUE
    S1 = T*F1 - F2 + BK3(1)
    J = N2D
    F1 = DBK3(J)
    F2 = 0.0E0
    DO 80 I=1,M2D
      J = J - 1
      TEMP1 = F1
      F1 = TT*F1 - F2 + DBK3(J)
      F2 = TEMP1
 80 CONTINUE
    D1 = T*F1 - F2 + DBK3(1)
    TC = C + C
    EX = EXP(C)
    IF (TC.GT.35.0E0) GO TO 110
    T = 10.0E0/C - 1.0E0
    TT = T + T
    J = N3
    F1 = BK4(J)
    F2 = 0.0E0
    DO 90 I=1,M3
      J = J - 1
      TEMP1 = F1
      F1 = TT*F1 - F2 + BK4(J)
      F2 = TEMP1
 90 CONTINUE
    S2 = T*F1 - F2 + BK4(1)
    BI = (S1+EXP(-TC)*S2)/RTRX
    BI = BI*EX
    J = N4D
```

```
      F1 = DBK4(J)
      F2 = 0.0E0
      DO 100 I=1,M4D
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + DBK4(J)
        F2 = TEMP1
  100 CONTINUE
      D2 = T*F1 - F2 + DBK4(1)
      DBI = RTRX*(D1+EXP(-TC)*D2)
      DBI = DBI*EX
      RETURN
  110 BI = EX*S1/RTRX
      DBI = EX*RTRX*D1
      RETURN
C
  120 CONTINUE
      IF (C.GT.5.0E0) GO TO 150
      T = 0.4E0*C - 1.0E0
      TT = T + T
      J = N2
      F1 = BJP(J)
      E1 = BJN(J)
      F2 = 0.0E0
      E2 = 0.0E0
      DO 130 I=1,M2
        J = J - 1
        TEMP1 = F1
        TEMP2 = E1
        F1 = TT*F1 - F2 + BJP(J)
        E1 = TT*E1 - E2 + BJN(J)
        F2 = TEMP1
        E2 = TEMP2
  130 CONTINUE
      BI = (T*E1-E2+BJN(1)) - AX*(T*F1-F2+BJP(1))
      J = N3D
      F1 = DBJP(J)
      E1 = DBJN(J)
      F2 = 0.0E0
      E2 = 0.0E0
      DO 140 I=1,M3D
        J = J - 1
```

```
        TEMP1 = F1
        TEMP2 = E1
        F1 = TT*F1 - F2 + DBJP(J)
        E1 = TT*E1 - E2 + DBJN(J)
        F2 = TEMP1
        E2 = TEMP2
  140 CONTINUE
      DBI = X*X*(T*F1-F2+DBJP(1)) + (T*E1-E2+DBJN(1))
      RETURN
C
  150 CONTINUE
      RTRX = SQRT(RX)
      T = 10.0E0/C - 1.0E0
      TT = T + T
      J = N3
      F1 = AA(J)
      E1 = BB(J)
      F2 = 0.0E0
      E2 = 0.0E0
      DO 160 I=1,M3
        J = J - 1
        TEMP1 = F1
        TEMP2 = E1
        F1 = TT*F1 - F2 + AA(J)
        E1 = TT*E1 - E2 + BB(J)
        F2 = TEMP1
        E2 = TEMP2
  160 CONTINUE
      TEMP1 = T*F1 - F2 + AA(1)
      TEMP2 = T*E1 - E2 + BB(1)
      CV = C - FPI12
      BI = (TEMP1*COS(CV)+TEMP2*SIN(CV))/RTRX
      J = N4D
      F1 = DAA(J)
      E1 = DBB(J)
      F2 = 0.0E0
      E2 = 0.0E0
      DO 170 I=1,M4D
        J = J - 1
        TEMP1 = F1
        TEMP2 = E1
        F1 = TT*F1 - F2 + DAA(J)
```

```
          E1 = TT*E1 - E2 + DBB(J)
          F2 = TEMP1
          E2 = TEMP2
  170 CONTINUE
      TEMP1 = T*F1 - F2 + DAA(1)
      TEMP2 = T*E1 - E2 + DBB(1)
      CV = C - SPI12
      DBI = (TEMP1*COS(CV)-TEMP2*SIN(CV))*RTRX
      RETURN
      END

      SUBROUTINE ASYJY(FUNJY,X,FNU,FLGJY,IN,Y,WK,IFLW)
C***REFER TO  BESJ,BESY
C***ROUTINES CALLED  I1MACH,R1MACH
      INTEGER I, IFLW, IN, J, JN,JR,JU,K, KB,KLAST,KMAX,KP1, KS, KSP1,
     * KSTEMP, L, LR, LRP1, ISETA, ISETB
      INTEGER I1MACH
      REAL ABW2, AKM, ALFA, ALFA1, ALFA2, AP, AR, ASUM, AZ,
     * BETA, BETA1, BETA2, BETA3, BR, BSUM, C, CON1, CON2,
     * CON3,CON548,CR,CRZ32, DFI,ELIM, DR,FI, FLGJY, FN, FNU,
     * FN2, GAMA, PHI,  RCZ, RDEN, RELB, RFN2,  RTZ, RZDEN,
     * SA, SB, SUMA, SUMB, S1, TA, TAU, TB, TFN, TOL, TOLS, T2, UPOL,
     *  WK, X, XX, Y, Z, Z32
      REAL R1MACH
      DIMENSION Y(*), WK(*), C(65)
      DIMENSION ALFA(26,4), BETA(26,5)
      DIMENSION ALFA1(26,2), ALFA2(26,2)
      DIMENSION BETA1(26,2), BETA2(26,2), BETA3(26,1)
      DIMENSION GAMA(26), KMAX(5), AR(8), BR(10), UPOL(10)
      DIMENSION CR(10), DR(10)
      EQUIVALENCE (ALFA(1,1),ALFA1(1,1))
      EQUIVALENCE (ALFA(1,3),ALFA2(1,1))
      EQUIVALENCE (BETA(1,1),BETA1(1,1))
      EQUIVALENCE (BETA(1,3),BETA2(1,1))
      EQUIVALENCE (BETA(1,5),BETA3(1,1))
      SAVE TOLS, CON1, CON2, CON3, CON548, AR, BR, C, ALFA1, ALFA2,
     1 BETA1, BETA2, BETA3, GAMA
      DATA TOLS            /-6.90775527898214E+00/
      DATA CON1,CON2,CON3,CON548/
C***FIRST EXECUTABLE STATEMENT  ASYJY
      TA = R1MACH(3)
      TOL = AMAX1(TA,1.0E-15)
```

```
      TB = R1MACH(5)
      JU = I1MACH(12)
      IF(FLGJY.EQ.1.0E0) GO TO 6
      JR = I1MACH(11)
      ELIM = 2.303E0*TB*(FLOAT(-JU)-FLOAT(JR))
      GO TO 7
    6 CONTINUE
      ELIM = 2.303E0*(TB*FLOAT(-JU)-3.0E0)
    7 CONTINUE
      FN = FNU
      IFLW = 0
      DO 170 JN=1,IN
        XX = X/FN
        WK(1) = 1.0E0 - XX*XX
        ABW2 = ABS(WK(1))
        WK(2) = SQRT(ABW2)
        WK(7) = FN**CON2
        IF (ABW2.GT.0.27750E0) GO TO 80
C
C     ASYMPTOTIC EXPANSION
C     CASES NEAR X=FN, ABS(1.-(X/FN)**2).LE.0.2775
C     COEFFICIENTS OF ASYMPTOTIC EXPANSION BY SERIES
C
C     ZETA AND TRUNCATION FOR A(ZETA) AND B(ZETA) SERIES
C
C     KMAX IS TRUNCATION INDEX FOR A(ZETA) AND B(ZETA) SERIES=MAX(2,SA)
C
        SA = 0.0E0
        IF (ABW2.EQ.0.0E0) GO TO 10
        SA = TOLS/ALOG(ABW2)
   10   SB = SA
        DO 20 I=1,5
          AKM = AMAX1(SA,2.0E0)
          KMAX(I) = INT(AKM)
          SA = SA + SB
   20   CONTINUE
        KB = KMAX(5)
        KLAST = KB - 1
        SA = GAMA(KB)
        DO 30 K=1,KLAST
          KB = KB - 1
          SA = SA*WK(1) + GAMA(KB)
```

```
   30   CONTINUE
        Z = WK(1)*SA
        AZ = ABS(Z)
        RTZ = SQRT(AZ)
        WK(3) = CON1*AZ*RTZ
        WK(4) = WK(3)*FN
        WK(5) = RTZ*WK(7)
        WK(6) = -WK(5)*WK(5)
        IF(Z.LE.0.0E0) GO TO 35
        IF(WK(4).GT.ELIM) GO TO 75
        WK(6) = -WK(6)
   35   CONTINUE
        PHI = SQRT(SQRT(SA+SA+SA+SA))
C
C     B(ZETA) FOR S=0
C
        KB = KMAX(5)
        KLAST = KB - 1
        SB = BETA(KB,1)
        DO 40 K=1,KLAST
          KB = KB - 1
          SB = SB*WK(1) + BETA(KB,1)
   40   CONTINUE
        KSP1 = 1
        FN2 = FN*FN
        RFN2 = 1.0E0/FN2
        RDEN = 1.0E0
        ASUM = 1.0E0
        RELB = TOL*ABS(SB)
        BSUM = SB
        DO 60 KS=1,4
          KSP1 = KSP1 + 1
          RDEN = RDEN*RFN2
C
C     A(ZETA) AND B(ZETA) FOR S=1,2,3,4
C
          KSTEMP = 5 - KS
          KB = KMAX(KSTEMP)
          KLAST = KB - 1
          SA = ALFA(KB,KS)
          SB = BETA(KB,KSP1)
          DO 50 K=1,KLAST
```

```
            KB = KB - 1
            SA = SA*WK(1) + ALFA(KB,KS)
            SB = SB*WK(1) + BETA(KB,KSP1)
   50     CONTINUE
          TA = SA*RDEN
          TB = SB*RDEN
          ASUM = ASUM + TA
          BSUM = BSUM + TB
          IF (ABS(TA).LE.TOL .AND. ABS(TB).LE.RELB) GO TO 70
   60   CONTINUE
   70   CONTINUE
        BSUM = BSUM/(FN*WK(7))
        GO TO 160
C
   75   CONTINUE
        IFLW = 1
        RETURN
C
   80   CONTINUE
        UPOL(1) = 1.0E0
        TAU = 1.0E0/WK(2)
        T2 = 1.0E0/WK(1)
        IF (WK(1).GE.0.0E0) GO TO 90
C
C     CASES FOR (X/FN).GT.SQRT(1.2775)
C
        WK(3) = ABS(WK(2)-ATAN(WK(2)))
        WK(4) = WK(3)*FN
        RCZ = -CON1/WK(4)
        Z32 = 1.5E0*WK(3)
        RTZ = Z32**CON2
        WK(5) = RTZ*WK(7)
        WK(6) = -WK(5)*WK(5)
        GO TO 100
   90   CONTINUE
C
C     CASES FOR (X/FN).LT.SQRT(0.7225)
C
        WK(3) = ABS(ALOG((1.0E0+WK(2))/XX)-WK(2))
        WK(4) = WK(3)*FN
        RCZ = CON1/WK(4)
        IF(WK(4).GT.ELIM) GO TO 75
```

```
      Z32 = 1.5E0*WK(3)
      RTZ = Z32**CON2
      WK(7) = FN**CON2
      WK(5) = RTZ*WK(7)
      WK(6) = WK(5)*WK(5)
  100 CONTINUE
      PHI = SQRT((RTZ+RTZ)*TAU)
      TB = 1.0E0
      ASUM = 1.0E0
      TFN = TAU/FN
      RDEN=1.0E0/FN
      RFN2=RDEN*RDEN
      RDEN=1.0E0
      UPOL(2) = (C(1)*T2+C(2))*TFN
      CRZ32 = CON548*RCZ
      BSUM = UPOL(2) + CRZ32
      RELB = TOL*ABS(BSUM)
      AP = TFN
      KS = 0
      KP1 = 2
      RZDEN = RCZ
      L = 2
      ISETA=0
      ISETB=0
      DO 140 LR=2,8,2
C
C     COMPUTE TWO U POLYNOMIALS FOR NEXT A(ZETA) AND B(ZETA)
C
        LRP1 = LR + 1
        DO 120 K=LR,LRP1
          KS = KS + 1
          KP1 = KP1 + 1
          L = L + 1
          S1 = C(L)
          DO 110 J=2,KP1
            L = L + 1
            S1 = S1*T2 + C(L)
  110     CONTINUE
          AP = AP*TFN
          UPOL(KP1) = AP*S1
          CR(KS) = BR(KS)*RZDEN
          RZDEN = RZDEN*RCZ
```

```
          DR(KS) = AR(KS)*RZDEN
  120   CONTINUE
        SUMA = UPOL(LRP1)
        SUMB = UPOL(LR+2) + UPOL(LRP1)*CRZ32
        JU = LRP1
        DO 130 JR=1,LR
          JU = JU - 1
          SUMA = SUMA + CR(JR)*UPOL(JU)
          SUMB = SUMB + DR(JR)*UPOL(JU)
  130   CONTINUE
        RDEN=RDEN*RFN2
        TB = -TB
        IF (WK(1).GT.0.0E0) TB = ABS(TB)
        IF (RDEN.LT.TOL) GO TO 131
        ASUM = ASUM + SUMA*TB
        BSUM = BSUM + SUMB*TB
        GO TO 140
  131   IF(ISETA.EQ.1) GO TO 132
        IF(ABS(SUMA).LT.TOL) ISETA=1
        ASUM=ASUM+SUMA*TB
  132   IF(ISETB.EQ.1) GO TO 133
        IF(ABS(SUMB).LT.RELB) ISETB=1
        BSUM=BSUM+SUMB*TB
  133   IF(ISETA.EQ.1 .AND. ISETB.EQ.1) GO TO 150
  140 CONTINUE
  150 TB = WK(5)
      IF (WK(1).GT.0.0E0) TB = -TB
      BSUM = BSUM/TB
C
  160 CONTINUE
      CALL FUNJY(WK(6), WK(5), WK(4), FI, DFI)
      TA=1.0E0/TOL
      TB=R1MACH(1)*TA*1.0E+3
      IF(ABS(FI).GT.TB) GO TO 165
      FI=FI*TA
      DFI=DFI*TA
      PHI=PHI*TOL
  165 CONTINUE
      Y(JN) = FLGJY*PHI*(FI*ASUM+DFI*BSUM)/WK(7)
      FN = FN - FLGJY
  170 CONTINUE
      RETURN
```

```
      END

      FUNCTION BESJ0(X)  C***ROUTINES CALLED CSEVL,INITS,R1MACH,XERROR
      DIMENSION BJ0CS(13), BM0CS(21), BTH0CS(24)
      DATA PI4 / 0.7853981633 9744831E0 /
      DATA NTJ0, NTM0, NTTH0, XSML, XMAX / 3*0, 2*0./
C***FIRST EXECUTABLE STATEMENT  BESJ0
      IF (NTJ0.NE.0) GO TO 10
      NTJ0 = INITS (BJ0CS, 13, 0.1*R1MACH(3))
      NTM0 = INITS (BM0CS, 21, 0.1*R1MACH(3))
      NTTH0 = INITS (BTH0CS, 24, 0.1*R1MACH(3))
C
      XSML = SQRT (4.0*R1MACH(3))
      XMAX = 1.0/R1MACH(4)
C
 10   Y = ABS(X)
      IF (Y.GT.4.0) GO TO 20
C
      BESJ0 = 1.0
      IF (Y.GT.XSML) BESJ0 = CSEVL (.125*Y*Y-1., BJ0CS, NTJ0)
      RETURN
C
 20   IF (Y.GT.XMAX) CALL XERROR ( 'BESJ0   NO PRECISION BECAUSE ABS(X)
     1 IS BIG', 42, 1, 2)
C
      Z = 32.0/Y**2 - 1.0
      AMPL = (0.75 + CSEVL (Z, BM0CS, NTM0)) / SQRT(Y)
      THETA = Y - PI4 + CSEVL (Z, BTH0CS, NTTH0) / Y
      BESJ0 = AMPL * COS (THETA)
C
      RETURN
      END

      FUNCTION BESJ1(X)
C***ROUTINES CALLED  CSEVL,INITS,R1MACH,XERROR
      DIMENSION BJ1CS(12), BM1CS(21), BTH1CS(24)
      DATA NTJ1, NTM1, NTTH1, XSML, XMIN, XMAX / 3*0, 3*0./
C***FIRST EXECUTABLE STATEMENT  BESJ1
      IF (NTJ1.NE.0) GO TO 10
      NTJ1 = INITS (BJ1CS, 12, 0.1*R1MACH(3))
      NTM1 = INITS (BM1CS, 21, 0.1*R1MACH(3))
      NTTH1 = INITS (BTH1CS, 24, 0.1*R1MACH(3))
```

```
C
      XSML = SQRT (8.0*R1MACH(3))
      XMIN = 2.0*R1MACH(1)
      XMAX = 1.0/R1MACH(4)
C
 10   Y = ABS(X)
      IF (Y.GT.4.0) GO TO 20
C
      BESJ1 = 0.
      IF (Y.EQ.0.0) RETURN
      IF (Y.LT.XMIN) CALL XERROR ( 'BESJ1   ABS(X) SO SMALL J1 UNDERFLOW
     1S', 37, 1, 1)
      IF (Y.GT.XMIN) BESJ1 = 0.5*X
      IF (Y.GT.XSML) BESJ1 = X * (.25 + CSEVL(.125*Y*Y-1., BJ1CS, NTJ1))
      RETURN
C
 20   IF (Y.GT.XMAX) CALL XERROR (  'BESJ1   NO PRECISION BECAUSE ABS(X)
     1 IS BIG', 42, 2, 2)
      Z = 32.0/Y**2 - 1.0
      AMPL = (0.75 + CSEVL (Z, BM1CS, NTM1)) / SQRT(Y)
      THETA = Y - 3.0*PI4 + CSEVL (Z, BTH1CS, NTTH1) / Y
      BESJ1 = SIGN (AMPL, X) * COS (THETA)
C
      RETURN
      END

      SUBROUTINE GAMLIM(XMIN,XMAX)
C***ROUTINES CALLED  R1MACH,XERROR
C***FIRST EXECUTABLE STATEMENT  GAMLIM
      ALNSML = ALOG(R1MACH(1))
      XMIN = -ALNSML
      DO 10 I=1,10
        XOLD = XMIN
        XLN = ALOG(XMIN)
        XMIN = XMIN - XMIN*((XMIN+0.5)*XLN - XMIN - 0.2258 + ALNSML)
     1    / (XMIN*XLN + 0.5)
        IF (ABS(XMIN-XOLD).LT.0.005) GO TO 20
 10   CONTINUE
      CALL XERROR ( 'GAMLIM  UNABLE TO FIND XMIN', 27, 1, 2)
C
 20   XMIN = -XMIN + 0.01
C
```

```
      ALNBIG = ALOG(R1MACH(2))
      XMAX = ALNBIG
      DO 30 I=1,10
        XOLD = XMAX
        XLN = ALOG(XMAX)
        XMAX = XMAX - XMAX*((XMAX-0.5)*XLN - XMAX + 0.9189 - ALNBIG)
     1    / (XMAX*XLN - 0.5)
        IF (ABS(XMAX-XOLD).LT.0.005) GO TO 40
 30   CONTINUE
      CALL XERROR ( 'GAMLIM  UNABLE TO FIND XMAX', 27, 2, 2)
C
 40   XMAX = XMAX - 0.01
      XMIN = AMAX1 (XMIN, -XMAX+1.)
C
      RETURN
      END

      FUNCTION J4SAVE(IWHICH,IVALUE,ISET)
      LOGICAL ISET
      INTEGER IPARAM(9)
      SAVE IPARAM
      DATA IPARAM(1),IPARAM(2),IPARAM(3),IPARAM(4)/0,2,0,10/
      DATA IPARAM(5)/1/
      DATA IPARAM(6),IPARAM(7),IPARAM(8),IPARAM(9)/0,0,0,0/
C***FIRST EXECUTABLE STATEMENT  J4SAVE
      J4SAVE = IPARAM(IWHICH)
      IF (ISET) IPARAM(IWHICH) = IVALUE
      RETURN
      END

      SUBROUTINE XERABT(MESSG,NMESSG)
      CHARACTER*(*) MESSG
C***FIRST EXECUTABLE STATEMENT  XERABT
      STOP
      END

      SUBROUTINE XERCTL(MESSG1,NMESSG,NERR,LEVEL,KONTRL)
      CHARACTER*20 MESSG1
C***FIRST EXECUTABLE STATEMENT  XERCTL
      RETURN
      END
```

```
      SUBROUTINE XERPRT(MESSG,NMESSG)
C***ROUTINES CALLED  I1MACH,S88FMT,XGETUA
      INTEGER LUN(5)
      CHARACTER*(*) MESSG
C     OBTAIN UNIT NUMBERS AND WRITE LINE TO EACH UNIT
C***FIRST EXECUTABLE STATEMENT  XERPRT
      CALL XGETUA(LUN,NUNIT)
      LENMES = LEN(MESSG)
      DO 20 KUNIT=1,NUNIT
         IUNIT = LUN(KUNIT)
         IF (IUNIT.EQ.0) IUNIT = I1MACH(4)
         DO 10 ICHAR=1,LENMES,72
            LAST = MIN0(ICHAR+71 , LENMES)
            WRITE (IUNIT,'(1X,A)') MESSG(ICHAR:LAST)
   10    CONTINUE
   20 CONTINUE
      RETURN
      END

      SUBROUTINE BESJ(X,ALPHA,N,Y,NZ)
C***ROUTINES CALLED  ALNGAM,ASYJY,I1MACH,JAIRY,R1MACH,XERROR
C
      EXTERNAL JAIRY
      INTEGER I,IALP,IDALP,IFLW,IN,INLIM,IS,I1,I2,K,KK,KM,KT,N,NN,
     1        NS,NZ
      INTEGER I1MACH
      REAL       AK,AKM,ALPHA,ANS,AP,ARG,COEF,DALPHA,DFN,DTM,EARG,
      REAL R1MACH, ALNGAM
      DIMENSION Y(10), TEMP(3), FNULIM(2), PP(4), WK(7)
      DATA RTWO,PDF,RTTP,PIDT                    / 1.34839972492648E+00,
C***FIRST EXECUTABLE STATEMENT  BESJ
      NZ = 0
      KT = 1
C     I1MACH(14) REPLACES I1MACH(11) IN A DOUBLE PRECISION CODE
C     I1MACH(15) REPLACES I1MACH(12) IN A DOUBLE PRECISION CODE
      TA = R1MACH(3)
      TOL = AMAX1(TA,1.0E-15)
      I1 = I1MACH(11) + 1
      I2 = I1MACH(12)
      TB = R1MACH(5)
      ELIM1 = 2.303E0*(FLOAT(-I2)*TB-3.0E0)
C     TOLLN = -LN(TOL)
```

```
      TOLLN = 2.303E0*TB*FLOAT(I1)
      TOLLN = AMIN1(TOLLN,34.5388E0)
      IF (N-1) 720, 10, 20
   10 KT = 2
   20 NN = N
      IF (X) 730, 30, 80
   30 IF (ALPHA) 710, 40, 50
   40 Y(1) = 1.0E0
      IF (N.EQ.1) RETURN
      I1 = 2
      GO TO 60
   50 I1 = 1
   60 DO 70 I=I1,N
        Y(I) = 0.0E0
   70 CONTINUE
      RETURN
   80 CONTINUE
      IF (ALPHA.LT.0.0E0) GO TO 710
C
      IALP = INT(ALPHA)
      FNI = FLOAT(IALP+N-1)
      FNF = ALPHA - FLOAT(IALP)
      DFN = FNI + FNF
      FNU = DFN
      XO2 = X*0.5E0
      SXO2 = XO2*XO2
C
C     DECISION TREE FOR REGION WHERE SERIES, ASYMPTOTIC EXPANSION FOR X
C     TO INFINITY AND ASYMPTOTIC EXPANSION FOR NU TO INFINITY ARE
C     APPLIED.
C
      IF (SXO2.LE.(FNU+1.0E0)) GO TO 90
      TA = AMAX1(20.0E0,FNU)
      IF (X.GT.TA) GO TO 120
      IF (X.GT.12.0E0) GO TO 110
      XO2L = ALOG(XO2)
      NS = INT(SXO2-FNU) + 1
      GO TO 100
   90 FN = FNU
      FNP1 = FN + 1.0E0
      XO2L = ALOG(XO2)
      IS = KT
```

```
      IF (X.LE.0.50E0) GO TO 330
      NS = 0
  100 FNI = FNI + FLOAT(NS)
      DFN = FNI + FNF
      FN = DFN
      FNP1 = FN + 1.0E0
      IS = KT
      IF (N-1+NS.GT.0) IS = 3
      GO TO 330
  110 ANS = AMAX1(36.0E0-FNU,0.0E0)
      NS = INT(ANS)
      FNI = FNI + FLOAT(NS)
      DFN = FNI + FNF
      FN = DFN
      IS = KT
      IF (N-1+NS.GT.0) IS = 3
      GO TO 130
  120 CONTINUE
      RTX = SQRT(X)
      TAU = RTWO*RTX
      TA = TAU + FNULIM(KT)
      IF (FNU.LE.TA) GO TO 480
      FN = FNU
      IS = KT
C
C     UNIFORM ASYMPTOTIC EXPANSION FOR NU TO INFINITY
C
  130 CONTINUE
      I1 = IABS(3-IS)
      I1 = MAX0(I1,1)
      FLGJY = 1.0E0
      CALL ASYJY(JAIRY,X,FN,FLGJY,I1,TEMP(IS),WK,IFLW)
      IF(IFLW.NE.0) GO TO 380
      GO TO (320, 450, 620), IS
  310 TEMP(1) = TEMP(3)
      KT = 1
  320 IS = 2
      FNI = FNI - 1.0E0
      DFN = FNI + FNF
      FN = DFN
      IF(I1.EQ.2) GO TO 450
      GO TO 130
```

```
C
C     SERIES FOR (X/2)**2.LE.NU+1
C
  330 CONTINUE
      GLN = ALNGAM(FNP1)
      ARG = FN*XO2L - GLN
      IF (ARG.LT.(-ELIM1)) GO TO 400
      EARG = EXP(ARG)
  340 CONTINUE
      S = 1.0E0
      IF (X.LT.TOL) GO TO 360
      AK = 3.0E0
      T2 = 1.0E0
      T = 1.0E0
      S1 = FN
      DO 350 K=1,17
        S2 = T2 + S1
        T = -T*SXO2/S2
        S = S + T
        IF (ABS(T).LT.TOL) GO TO 360
        T2 = T2 + AK
        AK = AK + 2.0E0
        S1 = S1 + FN
  350 CONTINUE
  360 CONTINUE
      TEMP(IS) = S*EARG
      GO TO (370, 450, 610), IS
  370 EARG = EARG*FN/XO2
      FNI = FNI - 1.0E0
      DFN = FNI + FNF
      FN = DFN
      IS = 2
      GO TO 340
C
C     SET UNDERFLOW VALUE AND UPDATE PARAMETERS
C
  380 Y(NN) = 0.0E0
      NN = NN - 1
      FNI = FNI - 1.0E0
      DFN = FNI + FNF
      FN = DFN
      IF (NN-1) 440, 390, 130
```

```
  390 KT = 2
      IS = 2
      GO TO 130
  400 Y(NN) = 0.0E0
      NN = NN - 1
      FNP1 = FN
      FNI = FNI - 1.0E0
      DFN = FNI + FNF
      FN = DFN
      IF (NN-1) 440, 410, 420
  410 KT = 2
      IS = 2
  420 IF (SX02.LE.FNP1) GO TO 430
      GO TO 130
  430 ARG = ARG - X02L + ALOG(FNP1)
      IF (ARG.LT.(-ELIM1)) GO TO 400
      GO TO 330
  440 NZ = N - NN
      RETURN
C
C     BACKWARD RECURSION SECTION
C
  450 CONTINUE
      NZ = N - NN
      IF (KT.EQ.2) GO TO 470
C     BACKWARD RECUR FROM INDEX ALPHA+NN-1 TO ALPHA
      Y(NN) = TEMP(1)
      Y(NN-1) = TEMP(2)
      IF (NN.EQ.2) RETURN
      TRX = 2.0E0/X
      DTM = FNI
      TM = (DTM+FNF)*TRX
      K = NN + 1
      DO 460 I=3,NN
        K = K - 1
        Y(K-2) = TM*Y(K-1) - Y(K)
        DTM = DTM - 1.0E0
        TM = (DTM+FNF)*TRX
  460 CONTINUE
      RETURN
  470 Y(1) = TEMP(2)
      RETURN
```

```
C
C     ASYMPTOTIC EXPANSION FOR X TO INFINITY WITH FORWARD RECURSION IN
C     OSCILLATORY REGION X.GT.MAX(20, NU), PROVIDED THE LAST MEMBER
C     OF THE SEQUENCE IS ALSO IN THE REGION.
C
  480 CONTINUE
      IN = INT(ALPHA-TAU+2.0E0)
      IF (IN.LE.0) GO TO 490
      IDALP = IALP - IN - 1
      KT = 1
      GO TO 500
  490 CONTINUE
      IDALP = IALP
      IN = 0
  500 IS = KT
      FIDAL = FLOAT(IDALP)
      DALPHA = FIDAL + FNF
      ARG = X - PIDT*DALPHA - PDF
      SA = SIN(ARG)
      SB = COS(ARG)
      COEF = RTTP/RTX
      ETX = 8.0E0*X
  510 CONTINUE
      DTM = FIDAL + FIDAL
      DTM = DTM*DTM
      TM = 0.0E0
      IF (FIDAL.EQ.0.0E0 .AND. ABS(FNF).LT.TOL) GO TO 520
      TM = 4.0E0*FNF*(FIDAL+FIDAL+FNF)
  520 CONTINUE
      TRX = DTM - 1.0E0
      T2 = (TRX+TM)/ETX
      S2 = T2
      RELB = TOL*ABS(T2)
      T1 = ETX
      S1 = 1.0E0
      FN = 1.0E0
      AK = 8.0E0
      DO 530 K=1,13
        T1 = T1 + ETX
        FN = FN + AK
        TRX = DTM - FN
        AP = TRX + TM
```

```
        T2 = -T2*AP/T1
        S1 = S1 + T2
        T1 = T1 + ETX
        AK = AK + 8.0E0
        FN = FN + AK
        TRX = DTM - FN
        AP = TRX + TM
        T2 = T2*AP/T1
        S2 = S2 + T2
        IF (ABS(T2).LE.RELB) GO TO 540
        AK = AK + 8.0E0
  530 CONTINUE
  540 TEMP(IS) = COEF*(S1*SB-S2*SA)
      IF(IS.EQ.2) GO TO 560
  550 FIDAL = FIDAL + 1.0E0
      DALPHA = FIDAL + FNF
      IS = 2
      TB = SA
      SA = -SB
      SB = TB
      GO TO 510
C
C     FORWARD RECURSION SECTION
C
  560 IF (KT.EQ.2) GO TO 470
      S1 = TEMP(1)
      S2 = TEMP(2)
      TX = 2.0E0/X
      TM = DALPHA*TX
      IF (IN.EQ.0) GO TO 580
C
C     FORWARD RECUR TO INDEX ALPHA
C
      DO 570 I=1,IN
        S = S2
        S2 = TM*S2 - S1
        TM = TM + TX
        S1 = S
  570 CONTINUE
      IF (NN.EQ.1) GO TO 600
      S = S2
      S2 = TM*S2 - S1
```

```
      TM = TM + TX
      S1 = S
  580 CONTINUE
C
C     FORWARD RECUR FROM INDEX ALPHA TO ALPHA+N-1
C
      Y(1) = S1
      Y(2) = S2
      IF (NN.EQ.2) RETURN
      DO 590 I=3,NN
        Y(I) = TM*Y(I-1) - Y(I-2)
        TM = TM + TX
  590 CONTINUE
      RETURN
  600 Y(1) = S2
      RETURN
C
C     BACKWARD RECURSION WITH NORMALIZATION BY
C     ASYMPTOTIC EXPANSION FOR NU TO INFINITY OR POWER SERIES.
C
  610 CONTINUE
C     COMPUTATION OF LAST ORDER FOR SERIES NORMALIZATION
      AKM = AMAX1(3.0E0-FN,0.0E0)
      KM = INT(AKM)
      TFN = FN + FLOAT(KM)
      TA = (GLN+TFN-0.9189385332E0-0.0833333333E0/TFN)/(TFN+0.5E0)
      TA = XO2L - TA
      TB = -(1.0E0-1.5E0/TFN)/TFN
      AKM = TOLLN/(-TA+SQRT(TA*TA-TOLLN*TB)) + 1.5E0
      IN = KM + INT(AKM)
      GO TO 660
  620 CONTINUE
C     COMPUTATION OF LAST ORDER FOR ASYMPTOTIC EXPANSION NORMALIZATION
      GLN = WK(3) + WK(2)
      IF (WK(6).GT.30.0E0) GO TO 640
      RDEN = (PP(4)*WK(6)+PP(3))*WK(6) + 1.0E0
      RZDEN = PP(1) + PP(2)*WK(6)
      TA = RZDEN/RDEN
      IF (WK(1).LT.0.10E0) GO TO 630
      TB = GLN/WK(5)
      GO TO 650
  630 TB=(1.259921049E0+(0.1679894730E0+0.0887944358E0*WK(1))*WK(1))
```

```
     1 /WK(7)
      GO TO 650
  640 CONTINUE
      TA = 0.5E0*TOLLN/WK(4)
      TA=((0.0493827160E0*TA-0.1111111111E0)*TA+0.6666666667E0)*TA*WK(6)
      IF (WK(1).LT.0.10E0) GO TO 630
      TB = GLN/WK(5)
  650 IN = INT(TA/TB+1.5E0)
      IF (IN.GT.INLIM) GO TO 310
  660 CONTINUE
      DTM = FNI + FLOAT(IN)
      TRX = 2.0E0/X
      TM = (DTM+FNF)*TRX
      TA = 0.0E0
      TB = TOL
      KK = 1
  670 CONTINUE
C
C     BACKWARD RECUR UNINDEXED
C
      DO 680 I=1,IN
        S = TB
        TB = TM*TB - TA
        TA = S
        DTM = DTM - 1.0E0
        TM = (DTM+FNF)*TRX
  680 CONTINUE
C     NORMALIZATION
      IF (KK.NE.1) GO TO 690
      TA = (TA/TB)*TEMP(3)
      TB = TEMP(3)
      KK = 2
      IN = NS
      IF (NS.NE.0) GO TO 670
  690 Y(NN) = TB
      NZ = N - NN
      IF (NN.EQ.1) RETURN
      K = NN - 1
      Y(K) = TM*TB - TA
      IF (NN.EQ.2) RETURN
      DTM = DTM - 1.0E0
      TM = (DTM+FNF)*TRX
```

```
      KM = K - 1
C
C     BACKWARD RECUR INDEXED
C
      DO 700 I=1,KM
        Y(K-1) = TM*Y(K) - Y(K+1)
        DTM = DTM - 1.0E0
        TM = (DTM+FNF)*TRX
        K = K - 1
  700 CONTINUE
      RETURN
C
C
C
  710 CONTINUE
      CALL XERROR( 'BESJ - ORDER, ALPHA, LESS THAN ZERO.', 36, 2, 1)
      RETURN
  720 CONTINUE
      CALL XERROR( 'BESJ - N LESS THAN ONE.', 23, 2, 1)
      RETURN
  730 CONTINUE
      CALL XERROR( 'BESJ - X LESS THAN ZERO.', 24, 2, 1)
      RETURN
      END

      SUBROUTINE JAIRY(X,RX,C,AI,DAI)
      INTEGER I, J, M1, M1D, M2, M2D, M3, M3D, M4, M4D, N1, N1D, N2,
     1 N2D, N3, N3D, N4, N4D
      REAL A, AI, AJN, AJP, AK1, AK2, AK3, B, C, CCV, CON1, CON2, CON3,
     1 CON4, CON5, CV, DA, DAI, DAJN, DAJP, DAK1, DAK2, DAK3, DB, EC,
     2 E1, E2, FPI12, F1, F2, RTRX, RX, SCV, T, TEMP1, TEMP2, TT, X
      DIMENSION AJP(19), AJN(19), A(15), B(15)
      DIMENSION AK1(14), AK2(23), AK3(14)
      DIMENSION DAJP(19), DAJN(19), DA(15), DB(15)
      DIMENSION DAK1(14), DAK2(24), DAK3(14)
C***FIRST EXECUTABLE STATEMENT  JAIRY
      IF (X.LT.0.0E0) GO TO 90
      IF (C.GT.5.0E0) GO TO 60
      IF (X.GT.1.20E0) GO TO 30
      T = (X+X-1.2E0)*CON4
      TT = T + T
      J = N1
```

```
      F1 = AK1(J)
      F2 = 0.0E0
      DO 10 I=1,M1
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + AK1(J)
        F2 = TEMP1
   10 CONTINUE
      AI = T*F1 - F2 + AK1(1)
C
      J = N1D
      F1 = DAK1(J)
      F2 = 0.0E0
      DO 20 I=1,M1D
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + DAK1(J)
        F2 = TEMP1
   20 CONTINUE
      DAI = -(T*F1-F2+DAK1(1))
      RETURN
C
   30 CONTINUE
      T = (X+X-CON2)*CON3
      TT = T + T
      J = N2
      F1 = AK2(J)
      F2 = 0.0E0
      DO 40 I=1,M2
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + AK2(J)
        F2 = TEMP1
   40 CONTINUE
      RTRX = SQRT(RX)
      EC = EXP(-C)
      AI = EC*(T*F1-F2+AK2(1))/RTRX
      J = N2D
      F1 = DAK2(J)
      F2 = 0.0E0
      DO 50 I=1,M2D
        J = J - 1
```

```
        TEMP1 = F1
        F1 = TT*F1 - F2 + DAK2(J)
        F2 = TEMP1
   50 CONTINUE
      DAI = -EC*(T*F1-F2+DAK2(1))*RTRX
      RETURN
C
   60 CONTINUE
      T = 10.0E0/C - 1.0E0
      TT = T + T
      J = N1
      F1 = AK3(J)
      F2 = 0.0E0
      DO 70 I=1,M1
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + AK3(J)
        F2 = TEMP1
   70 CONTINUE
      RTRX = SQRT(RX)
      EC = EXP(-C)
      AI = EC*(T*F1-F2+AK3(1))/RTRX
      J = N1D
      F1 = DAK3(J)
      F2 = 0.0E0
      DO 80 I=1,M1D
        J = J - 1
        TEMP1 = F1
        F1 = TT*F1 - F2 + DAK3(J)
        F2 = TEMP1
   80 CONTINUE
      DAI = -RTRX*EC*(T*F1-F2+DAK3(1))
      RETURN
C
   90 CONTINUE
      IF (C.GT.5.0E0) GO TO 120
      T = 0.4E0*C - 1.0E0
      TT = T + T
      J = N3
      F1 = AJP(J)
      E1 = AJN(J)
      F2 = 0.0E0
```

```
      E2 = 0.0E0
      DO 100 I=1,M3
        J = J - 1
        TEMP1 = F1
        TEMP2 = E1
        F1 = TT*F1 - F2 + AJP(J)
        E1 = TT*E1 - E2 + AJN(J)
        F2 = TEMP1
        E2 = TEMP2
  100 CONTINUE
      AI = (T*E1-E2+AJN(1)) - X*(T*F1-F2+AJP(1))
      J = N3D
      F1 = DAJP(J)
      E1 = DAJN(J)
      F2 = 0.0E0
      E2 = 0.0E0
      DO 110 I=1,M3D
        J = J - 1
        TEMP1 = F1
        TEMP2 = E1
        F1 = TT*F1 - F2 + DAJP(J)
        E1 = TT*E1 - E2 + DAJN(J)
        F2 = TEMP1
        E2 = TEMP2
  110 CONTINUE
      DAI = X*X*(T*F1-F2+DAJP(1)) + (T*E1-E2+DAJN(1))
      RETURN
C
  120 CONTINUE
      T = 10.0E0/C - 1.0E0
      TT = T + T
      J = N4
      F1 = A(J)
      E1 = B(J)
      F2 = 0.0E0
      E2 = 0.0E0
      DO 130 I=1,M4
        J = J - 1
        TEMP1 = F1
        TEMP2 = E1
        F1 = TT*F1 - F2 + A(J)
        E1 = TT*E1 - E2 + B(J)
```

```
        F2 = TEMP1
        E2 = TEMP2
  130 CONTINUE
      TEMP1 = T*F1 - F2 + A(1)
      TEMP2 = T*E1 - E2 + B(1)
      RTRX = SQRT(RX)
      CV = C - FPI12
      CCV = COS(CV)
      SCV = SIN(CV)
      AI = (TEMP1*CCV-TEMP2*SCV)/RTRX
      J = N4D
      F1 = DA(J)
      E1 = DB(J)
      F2 = 0.0E0
      E2 = 0.0E0
      DO 140 I=1,M4D
        J = J - 1
        TEMP1 = F1
        TEMP2 = E1
        F1 = TT*F1 - F2 + DA(J)
        E1 = TT*E1 - E2 + DB(J)
        F2 = TEMP1
        E2 = TEMP2
  140 CONTINUE
      TEMP1 = T*F1 - F2 + DA(1)
      TEMP2 = T*E1 - E2 + DB(1)
      E1 = CCV*CON5 + 0.5E0*SCV
      E2 = SCV*CON5 - 0.5E0*CCV
      DAI = (TEMP1*E1-TEMP2*E2)*RTRX
      RETURN
      END


cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc


      FUNCTION ALNGAM(X)
C***ROUTINES CALLED  GAMMA,R1MACH,R9LGMC,XERROR
      EXTERNAL GAMMA
      DATA SQ2PIL / 0.9189385332 0467274E0/
      DATA SQPI2L / 0.2257913526 4472743E0/
      DATA PI     / 3.1415926535 8979324E0/
```

```
      DATA XMAX, DXREL / 0., 0. /
C***FIRST EXECUTABLE STATEMENT  ALNGAM
      IF (XMAX.NE.0.) GO TO 10
      XMAX = R1MACH(2)/ALOG(R1MACH(2))
      DXREL = SQRT (R1MACH(4))
C
 10   Y = ABS(X)
      IF (Y.GT.10.0) GO TO 20
C
C ALOG (ABS (GAMMA(X))) FOR  ABS(X) .LE. 10.0
C
      ALNGAM = ALOG (ABS (GAMMA(X)))
      RETURN
C
C ALOG (ABS (GAMMA(X))) FOR ABS(X) .GT. 10.0
C
 20   IF (Y.GT.XMAX) CALL XERROR ( 'ALNGAM  ABS(X) SO BIG ALNGAM OVERFLO
     1WS', 38, 2, 2)
C
      IF (X.GT.0.) ALNGAM = SQ2PIL + (X-0.5)*ALOG(X) - X + R9LGMC(Y)
      IF (X.GT.0.) RETURN
C
      SINPIY = ABS (SIN(PI*Y))
      IF (SINPIY.EQ.0.) CALL XERROR ( 'ALNGAM  X IS A NEGATIVE INTEGER',
     1  31, 3, 2)
C
      IF (ABS((X-AINT(X-0.5))/X).LT.DXREL) CALL XERROR ( 'ALNGAM  ANSWER
     1 LT HALF PRECISION BECAUSE X TOO NEAR NEGATIVE INTEGER', 68, 1, 1)
C
      ALNGAM = SQPI2L + (X-0.5)*ALOG(Y) - X - ALOG(SINPIY) - R9LGMC(Y)
      RETURN
C
      END
```

IMAGE EVALUATION TEST TARGET (QA-3)

150mm

6"

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