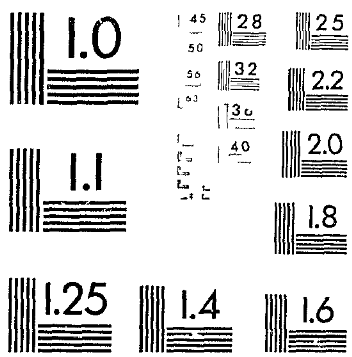

1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
4.0
45
50
56
63

CLASSIFICATION OF CERTAIN NON-COMMUTATIVE THREE-TORI

by

Hassan A. Rouhani

Submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

at

Dalhousie University

Halifax, Nova Scotia

March, 1988

ISBN 0-315-49841-2

DALHOUSIE UNIVERSITY

FACULTY OF GRADUATE STUDIES

The undersigned hereby certify that they have read and recommended to the Faculty of Graduate Studies for acceptance a thesis entitled

CLASSIFICATION OF CERTAIN NON-COMMUTATIVE THREE-TORI

by Hassan A. Rouhani

in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Dated April 6, 1988

External examiner [redacted]

Research Supervisor

Examining Committee

DALHOUSIE UNIVERSITY

Date April 6, 1988

Author Hassan A. Rouhani

Title Classification of Certain Non-Commutative Three-Tori

Department or School Department of Mathematics, Statistics and Computing Science

Degree Ph.D. Convocation May Year 1988

[redacted]

Signature of Author

DEDICATION

I which to dedicate this thesis with all my love and affection to my mother May Hassan and my wife Margaret Rouhani, who have been a great source of inspiration to me.

TABLE OF CONTENTS

ABSTRACT

The main result of this thesis is to classify the isomorphism classes of certain non-commutative 3-tori obtained by taking the C*-algebra crossed product of $C(T^2)$, where T^2 is the 2-torus, by the irrational affine quasi-rotations of T^2. Each such quasi-rotation is represented by a pair (a,A), where $a \in T^2$ and $A \in GL(2, Z)$, and its associated crossed product C*-algebra, denoted by B(a,A), is shown to be determined, up to isomorphism, by an analogue of the rotation angle, namely its primitive eigenvalue $X_A(a)$, by its orientation $\det A = \pm 1$, and by a positive integer m(A), which comes from $K_1(B(a, A))$ and determines the conjugacy class of A in GL(2,Z).

Finally, we briefly consider certain irrational non-affine quasi-rotations by first constructing a minimal quasi-rotation of T^2 which does not have topologically quasi-discrete spectrum. We show that the corresponding crossed product algebras associated with certain quasi-rotations are simple and have a unique tracial state. A conjecture is reformulated concerning the isomorphism classes of crossed products associated with these non-affine quasi-rotations.

LIST OF SYMBOLS

$\mathbf{Z}$	The integers
$\mathbf{R}$	The real numbers
$\mathbf{C}$	The complex numbers
$\mathbf{T}$	The unit circle in $\mathbf{C}$
M_n	The algebra of $n \times n$ complex matrices
$\tilde{A}$	The C*-algebra obtained by adjoining the identity
$\tilde{1}$	The identity adjoined to A
gcd(m,n)	The positive greatest common divisor of the integers m and n (not both zero).

ACKNOWLEDGEMENTS

I would like to thank my supervisor Professor Peter Fillmore for his many helpful comments and suggestions, and for his patience. I wish to express my gratefulness to the Killam Trust for their financial assistance.

My special thanks to my wife, Margie, for her constant encouragement and warm care during my research.

I also wish to give special thanks to Professor Keith Taylor and the Department of Mathematics at the University of Saskatchewan for giving me the opportunity to teach there, while I was doing my research, and for providing a most friendly atmosphere in which to work.

I am very grateful to Mr. Eric Pohorecky for his excellent work in typesetting my thesis with TEX, especially considering the limited time in which he had to accomplish this arduous task.

INTRODUCTION

An Outline of the Historical Background

The range of the trace method is a method that gives us an isomorphism invariant which is considerably more sensitive than homotopy invariant functors like K-theory or Ext-theory for C*-algebras possessing a trace. In fact R. T. Powers suggested to Rieffel the importance of computing the trace on projections in the C*-algebra . The range of the trace on all the projections in the C*-algebra will be an isomorphism invariant when the algebra has a unique normalized trace ([23], pp.417ff), as for example in the case of the irrational rotation C*-algebras .

Given a unital C*-algebra A which possesses a normalized trace (or tracial state) τ, one obtains an induced map

$$\tau_* : K_0(A) \to R$$

such that

$$\tau_*[e] = (\tau \otimes \mathrm{tr}_n)(e)$$

where $e \in \mathrm{Proj}\, M_n(A)$ and $\mathrm{tr}_n : M_n(C) \to C$ is the usual trace on $M_n(C)$ given by

$$\mathrm{tr}_n(x) = \sum_{i=1}^{n} x_{ii}, \qquad x \in M_n(C).$$

The range of the trace, $\tau_*(K_0(A))$, will be unambiguous in two frequent cases: when A has a unique normalized trace, or when the tracial range $\tau_* K_0(A)$ is independent of the normalized trace τ on A. In these cases the range of the trace will be an isomorphism invariant object; that is, if A and B are isomorphic C*-algebras with traces τ_1 and τ_2, respectively, then $(\tau_1)_* K_0(A) = (\tau_2)_* K_0(B)$. Furthermore, if A and B are strongly Morita equivalent (henceforth abbreviated as "SME"), then for any normalized traces τ_1, τ_2, respectively, one has

$$(\tau_1)_* K_0(A) = r\,(\tau_2)_* K_0(B)$$

for some $r > 0$ (Rieffel [23], Proposition 2.5). The notion of strong Morita equivalence was shown by Brown, Green, and Rieffel ([2], Theorem 1.2) to be equivalent to stable isomorphism for separable C^*-algebras . By definition, A and B are stably isomorphic if and only if $A \otimes K \cong B \otimes K$, where K is the C^*-algebra of all compact operators on a separable infinite-dimensional Hilbert space.

The tracial range can help in "classifying" certain families of C^*-algebras for which the K-theories are all the same. By "classify" here one means deriving criteria in terms of the parameters of the family which completely determine when two algebras in the family are "equivalent", i.e. isomorphic, or SME, or perhaps equivalent in some other sense. The first such family of C^*-algebras is that of the irrational rotation C^*-algebras A_θ, for $0 < \theta < 1$, θ irrational, which are crossed products of C(T), the continuous functions on the unit circle T, by the irrational rotation $z \mapsto e^{2\pi i\theta}z$. One can also describe A_θ as the only C^*-algebra (up to isomorphism) generated by two unitaries u, v satisfying the commutation relation $vu = e^{2\pi i\theta}uv$. Since the rotation $z \mapsto e^{2\pi i\theta}z$ is minimal (θ being irrational), the crossed product $A_\theta = C(T) \times_\theta Z$ is a simple C^*-algebra (see Power [20]). Since the rotation map has normalized Lebesgue measure on T as the only invariant probability measure, it follows that A_θ has a unique normalized trace, obtained by extending integration with respect to this measure to the whole crossed product (cf. Lemma 1.3.4 below).

These algebras had a peculiar story. Twenty years ago, in 1967, it was conjectured by Effros and Hahn ([5], p.81) that A_θ is projectionless, i.e. it has no projections except 0 and 1 (θ being irrational). However, as Rieffel pointed out, R. T. Powers soon showed in unpublished work that there is a self-adjoint element in A_θ with disconnected spectrum, so that one may deduce from the spectral theorem that A_θ does in fact contain projections other than 0 and 1 ([23], p.416). Rieffel himself made an elegant and simple construction of a projection in A_θ of trace θ (ibid., p.418ff). But Rieffel was interested in, among other things, determining completely the range of the trace on the projections of A_θ. Using Pimsner and Voiculescu's embedding of A_θ into an AF-algebra B constructed by Elliott for which the range of the trace on $K_0(B)$ was known to be $\tau_* K_0(B) = Z + \theta Z$, Rieffel was able to show using the projection which he constructed of trace θ (which became known in the literature as a "Rieffel projection"), that the range of the trace for A_θ is $\tau_* K_0(A_\theta) = Z + \theta Z$ ([23], Theorem 1.2). This result was also pointed out by Pimsner and Voiculescu as a consequence of Rieffel's construction of a projection of trace θ.

The approach followed by Rieffel, Pimsner, and Voiculescu of embedding A_θ into an AF-algebra (which is difficult) is a method strictly adapted to the algebras A_θ. At about the same time, however, Pimsner and Voiculescu derived the remarkable result that the K-groups of any crossed product by Z, $K_i(A \times_\alpha Z)$, fits into a six term exact loop involving only the K-groups of A ([18], Theorem 2.4). It looks like this:

$$\begin{tikzcd}
 & K_0(A) \arrow[r, "\alpha_* - id_*"] & K_0(A) \arrow[dr, "i_*"] & \\
K_1(A \times_\alpha Z) \arrow[ur, "\delta_1"] & & & K_0(A \times_\alpha Z) \arrow[dl, "\delta_0"] \\
 & K_1(A) \arrow[ul, "i_*"] & K_1(A) \arrow[l, "\alpha_* - id_*"'] &
\end{tikzcd}$$

and this helps us to identify the generators of the group $K_0(A \times_\alpha Z)$, so that we may then be able to compute the trace of each of these generators to determine the range of the trace $\tau_* K_0(A \times_\alpha Z)$ for the crossed product $A \times_\alpha Z$. Since $A_\theta = C(T) \times_{\alpha_\theta} Z$, Pimsner and Voiculescu were able to apply their sequence (henceforth referred to as the "PV-sequence"), and, on page 116 of [18] they computed that Rieffel's projection e, having trace θ, is mapped by δ_0 to a generator of $K_1(C(T)) \cong Z$. Note that here $\alpha_* = id_*$ (= id) on both levels since the rotation $z \mapsto e^{2\pi i\theta}z$ is homotopic to the identity, and so, as also $K_i(C(T)) \cong Z$, it follows that $K_i(A_\theta) \cong Z^2$ for $i = 0,\ 1$. Thus, e and 1 (the identity projection) generate $K_0(A_\theta)$, and hence $\tau_* K_0(A_\theta) = Z + \theta Z$. This method avoids the (difficult) embedding of A_θ into an AF-algebra.

From this computation of the range of the trace for A_θ, one easily sees that the classification of the *-isomorphism classes of these algebras is immediate: $A_\theta \cong A_{\theta'}$ (for two irrational numbers $0 < \theta, \theta' < 1$) if and only if $\theta' = \theta$ or $\theta' = 1 - \theta$. For their classification up to strong Morita equivalence (which is a little more involved), Rieffel proved: $A_\theta \overset{\text{SME}}{\sim} A_{\theta'}$ (θ , θ' irrational) if and only if

$$\theta' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \theta = \frac{a\theta + b}{c\theta + d}$$

for some $\begin{bmatrix} a\, b \\ c\, d \end{bmatrix} \in GL(2, Z)$, where $GL(2, Z)$ is the group of invertible matricies over Z, acting on the set of irrational numbers in the above manner.

Without using these techniques and results it is not clear why, for instance, A_θ cannot also be generated by two other unitaries u_1, v_1 such that $v_1 u_1 = e^{2\pi i\theta'} u_1 v_1$ for some other irrational number θ' independent of θ. We may speculate and ask: Could not these unitaries u_1, v_1 be hidden somewhere in the C^*-closure of the dense *-subalgebra

$$B = \left\{ \sum_{n,m} a_{nm} u^n v^m \mid n, m \in Z, a_{nm} \in C, \text{ and } a_{nm} = 0 \text{ for all but finitely many } (n, m) \right\}$$

where A_θ is generated by u, v such that $vu = e^{2\pi i\theta} uv$? This question seems insurmountable by directly tackling the problem. It is very elusive. Since we are not able to grasp or write down explicitly all the elements of A_θ, particularly those in the C^*-closure of B, we cannot deal directly with the above question; a similar long-standing difficulty is to determine when $\sum_{-\infty}^{\infty} a_n u^n$ determines an element of C(T) where $u(z) = z$, $z \in T$ (convergence of series is in $L^2(T)$). This is where the idea of invariants reveals its usefulness and application; one needs only to know certain "fingerprints" of the algebra(s) to answer some difficult questions.

At this point let us mention a general method for computing the tracial range due to Pimsner [17].

Pimsner showed, using the PV-sequence and the concept of the determinant associated with a trace, that the tracial range for any crossed product $A \times_\alpha Z$ fits into a short exact sequence of the form

$$0 \longrightarrow \tau_* K_0(A) \longrightarrow \tau_* K_0(A \times_\alpha Z) \longrightarrow \underline{\Delta}_\tau^\alpha (K_1(A)^\alpha) \longrightarrow 0$$

where $K_1(A)^\alpha = \{x \in K_1(A) \mid \alpha_* x = x\}$ and $\underline{\Delta}_\tau^\alpha$ is induced from the determinant function Δ_τ associated with the trace τ ([17], Theorem 3). Actually, Pimsner proves this for all crossed products by F_n, the free non-abelian group on $n \geq 1$ generators, but we only mention the case $n = 1$ here.

As an application, Pimsner used this result to compute the range of the trace for crossed products of the circle T by an orientation preserving homeomorphism φ of T. Such a homeomorphism φ has an associated number, θ, called its rotation number, which is usually reduced modulo Z. If α_φ is the associated automorphism of C(T), then he showed that

$$\tau_* K_0\left(C(T) \times_{\alpha_\varphi} Z\right) = Z + \theta Z$$

for any trace τ on $C(T) \times_{\alpha_\varphi} Z$ obtained from a φ-invariant probability measure on T ([17], Proposition 6). This result was proved independently by Putnam, Schmidt, and Skau for certain homeomorphisms of T called "Denjoy" (pronounced "Donj-wa") homeomorphisms ([21], Theorem 5.2).

Exel [7] computed the range of the trace for the special case that $\tau_* K_0(A) \subseteq Z$, so that his result is contained in Pimsner's, though Exel's proof differs somewhat. However, as an application, Exel computes the tracial range for crossed products of the form $C(G) \times_\eta Z$ where G is a compact connected topological group and $\eta = \lambda_g \circ \alpha$, where $\alpha \in \mathrm{Aut}(G)$ and λ_g is left translation by $g \in G$ (such an η is called an affine transformation on G). If τ denotes the trace on the crossed product obtained from Haar measure on G, he obtains the equality

$$\tau_* K_0 (C(G) \times_\eta Z) = \{t \in R \mid e^{2\pi i t} = \chi(g) \text{ for some } \chi \in \mathrm{Hom}(G,T)^\alpha\}$$

where $\mathrm{Hom}(G,T)^\alpha$ is the set of α-invariant characters on G ([7], p.84).

Interest arose in the computation of the tracial range for C*-algebras analogous to the irrational rotation algebras. For example, N. Riedel's minimal rotation algebras on compact abelian metric groups are completely characterized by their sets of eigenvalues, which generalizes the result of Rieffel for the A_θ's (see [22]). The algebras we shall study in this thesis are also determined by certain (non-singular) eigenvalues, together with a certain integer invariant, although they are not rotation algebras nor are they necessarily associated with minimal transformations (see Chapter 4). To each countable subgroup G of the unit circle T, Riedel associates the minimal rotation C*-algebra $A_G = C(\widehat{G}) \times_{\bar{R}_G} Z$, the crossed product of the continuous functions on the dual group $\widehat{G}$ by the automorphism $\bar{R}_G$ induced by the translation (or "rotation") $R_G : \widehat{G} \to \widehat{G}$ defined by $R_G(\sigma) = \rho_G \sigma$ for $\sigma \in \widehat{G}$, where $\rho_G \in \widehat{G}$ is the element induced by the inclusion $G \hookrightarrow T$. Like A_θ, the algebra A_G is simple and has a unique normalized trace ([22], Proposition 2.1). In fact, A_G is generated by unitaries U and $\pi(\lambda)$, where $\pi : G \to L(H)$ is a unitary representation of G, such that $U\pi(\lambda) = \lambda\pi(\lambda)U$ for all $\lambda \in G$. Riedel proved that if G is any countable subgroup of T, then the tracial range is $\tau_* K_0 (A_G) = \sigma^{-1}(G)$, where $\sigma : R \to T$ is the canonical mapping $\sigma(t) = e^{2\pi i t}$ ([22], Corollary 3.6) This result, however, was proved earlier by Elliott [6]. Using the duality theory for abelian groups (namely, $\widehat{\widehat{G}} \cong G$) one can also show that this result follows from the above mentioned application of Exel's result (assuming that $\widehat{G}$ is connected). With this proved, Riedel concludes that the group G (of "eigenvalues") is a complete invariant for the algebra A_G: For countable subgroups G_1 and G_2 of T, $A_{G_1} \cong A_{G_2}$ if and only if $G_1 = G_2$ ([22], Corollary 3.7). In his paper Riedel does not have a result on strong Morita equivalence, but, as we shall describe below, Ji obtains such a result for the algebras A_G when G is a finitely generated torsion-free subgroup of T.

In this thesis we will be mainly interested in crossed products of the 2-torus T^2 by "quasi-rotations" (defined in §4.1 below), computing their tracial range and classifying their *-isomorphism classes. These algebras are non-commutative 3-tori since they are crossed products of $C(T^2)$ by Z, in analogy with A_θ which are non-commutative 2-tori. We shall only mention recent progress regarding tori.

J. Packer proved, using results of Connes, that if $\varphi : T^n \to T^n$ is a minimal homeomorphism which has (topologically) quasi-discrete spectrum, then

$$(\tau_\nu)_* K_0\left(C(T^n) \times_{\alpha_\varphi} Z\right) = \sigma^{-1}(E_\varphi)$$

where ν is the unique φ-invariant probability measure on T^n, E_φ the group of eigenvalues of φ ($E_\varphi \subseteq T$), and $\sigma : R \to T$ is as above ([14], Theorem 3.3). She applies this result to classify the crossed products of $C(T^2)$ by the Anzai transformations defined by

$$\varphi_\theta(x, y) = (e^{2\pi i\theta}x, xy),$$

where $0 < \theta < 1$ is irrational. Letting $H_\theta = C(T^2) \times_{\alpha_\theta} Z$, where α_θ is the automorphism on $C(T^2)$ associated with φ_θ, she proved that for irrational numbers θ, θ' in $(0,1)$

(i) $\quad H_\theta \cong H_{\theta'} \quad \Leftrightarrow \quad \theta' = \theta \quad \text{or} \quad \theta' = 1 - \theta,$

(ii) $\quad H_\theta \overset{SME}{\sim} H_{\theta'} \quad \Leftrightarrow \quad \theta' = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\theta \quad \text{for some} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, Z),$

([14], Theorem 4.1). The Anzai transformations are special cases of our affine quasi-rotations, and we shall prove a generalization of (i) in this thesis (see Theorem 4.3.2 below). We have some partial results for their strong Morita equivalence which we will discuss elsewhere.

In his Ph. D thesis, [11], Ji obtained some results about crossed products of the n-torus T^n by certain homeomorphisms which he called "Furstenberg transformations" (see [8], Theorem 2.1, p.581). These have the form

$$F_{f,\theta}(z_1, \ldots, z_n) = (z_1 f_1(z_2, \ldots, z_n), \ldots, z_{n-1} f_{n-1}(z_n), e^{2\pi i\theta} z_n),$$

where $f_j(z_{j+1}, \ldots, z_n)$ are continuous functions homotopic to $z_{j+1}^{d_j}$ for some non-zero integer d_j. These clearly include the Anzai transformations. Denoting the crossed product associated to $F_{f,\theta}$ by $A_{f,\theta} = C(T^n) \times_{F_{f,\theta}} Z$, Ji proved that

$$\tau_* K_0 (A_{f,\theta}) = Z + \theta Z$$

for any tracial state τ on $A_{f,\theta}$ ([11], Theorem 2.23).

The first classification result which Ji proved is for "descending" Furstenberg transformations on T^n of the form

$$F_{K,\theta}(z_1, \ldots, z_n) = (z_1 z_2^{k_1}, \ldots, z_{n-1} z_n^{k_{n-1}}, e^{2\pi i\theta} z_n),$$

where $K = (k_1, \ldots, k_{n-1})$ is an $(n-1)$-tuple of non-zero integers such that $k_{i+1}|k_i$ (k_{i+1} divides k_i), $i = 1, \ldots, n-2$, and where $0 < \theta < 1$ is irrational. He proved that if $F_{K,\theta}$ and $F_{K',\theta'}$ are two such descending Furstenberg transformations and θ, θ' are irrational, then the following are equivalent:

(i) $\quad A_{F_{K,\theta}} \otimes M_m \cong A_{F_{K',\theta'}} \otimes M_{m'},$

(ii) $\quad m = m' \quad \text{and} \quad \theta' \in \{\theta, 1-\theta\} \quad \text{and} \quad |k_i'| = |k_i| \quad (\forall i)$

([11], p.39).

Ji then generalized this result to descending transformations $F_{K,\Theta}$ on T^{n+m} of the form

$$F_{K,\Theta}(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) = (z_1 z_2^{k_1}, \ldots, z_{n-1} z_n^{k_{n-1}}, e^{2\pi i\theta_1} z_{n+1}, \ldots, e^{2\pi i\theta_m} z_{n+m})$$

where $K = (k_1, \ldots, k_{n-1})$ with $k_{i+1}|k_i$ and $k_i \neq 0$ ($\forall i$), and $\Theta = (\theta_1, \ldots, \theta_m)$, $0 \leq \theta_i < 1$. So $F_{K,\Theta}$ is a "skew product" in the first n variables and a rotation in the last m variables. For these he obtained that

$$\tau_* K_0 \left(C(T^{n+m}) \times_{F_{K,\Theta}} Z \right) = Z + \theta_1 Z + \cdots + \theta_m Z$$

for any tracial state τ on the crossed product ([11], Proposition 5.3). From this and certain other technical lemmas, Ji proved the following:

Suppose that $\Theta = (\theta_1, \ldots, \theta_m)$ and $\Theta' = (\theta'_1, \ldots, \theta'_m)$ are each rationally independent (modulo Z), $n \geq 1$, $m \geq 2$, and assume that $\theta'_1 \in \{\theta_1, 1 - \theta_1\}$. Then the following conditions are equivalent:

(i) $A_{F_{K,\Theta}} \cong A_{F_{K',\Theta'}}$,

(ii) $|k'_i| = |k_i|$ $(\forall i)$ and

$$Z + \theta_1 Z + \cdots + \theta_m Z = Z + \theta'_1 Z + \cdots + \theta'_m Z$$

([11], p.70).

As for the strong Morita equivalence classification, Ji proved that if $F_{k,\theta} : T^2 \to T^2$ is the Furstenberg transformation

$$F_{k,\theta}(x, y) = \left(e^{2\pi i \theta} x, x^k y \right)$$

where θ is irrational, then the following are equivalent:

(i) $A_{F_{k,\theta}} \overset{SME}{\sim} A_{F_{k',\theta'}}$,

(ii) $|k| = |k'|$ and $\theta' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \theta$ for some $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, Z)$

(cf.[11], Theorem 4.11). Although Packer proved this for $k = 1$, as we pointed out above, Ji's proof is different. When θ is rational, Ji showed that $A_{F_{k,\theta}}$ is strongly Morita equivalent to $A_{F_{k,0}}$ ([11], Theorem 4.12).

Finally, Ji proved a strong Morita equivalence for minimal rotations on T^n, which is the special case of Riedel's algebras A_G discussed above for when G is a finitely generated torsion-free subgroup of the circle. These are crossed products of the form

$$C(T^n) \times_{\rho(\theta_1, \ldots, \theta_n)} Z$$

where $\rho(\theta_1, \ldots, \theta_n) : T^n \to T^n$ is the rotation

$$\rho(\theta_1, \ldots, \theta_n)(z_1, \ldots, z_n) = \left(e^{2\pi i \theta_1} z_1, \ldots, e^{2\pi i \theta_n} z_n \right) ,$$

where $z_i \in T$. If $(\theta_1, \ldots, \theta_n)$ is rationally independent (modulo Z), $\rho(\theta_1, \ldots, \theta_n)$ is minimal (one can show this using Proposition 1.1.4 below and induction on n). As we noted above, Riedel proved that for $(\theta_1, \ldots, \theta_n)$ rationally independent, $C(T^n) \times_{\rho(\theta'_1, \ldots, \theta'_n)} Z \cong C(T^n) \times_{\rho(\theta_1, \ldots, \theta_n)} Z$ if and only if $Z + \theta_1 Z + \cdots + \theta_n Z = Z + \theta'_1 Z + \cdots + \theta'_n Z$ ([22], Cor.3.7; this result seems to be contained in [6]). Note that we need to assume that $(\theta_1, \ldots, \theta_n)$ is rationally independent so that $G = \sigma(\theta_1 Z + \cdots + \theta_n Z) \cong Z^n$ which has dual group T^n.

As for the strong Morita equivalence of these algebras, Ji proved that if both $(\theta_1, \ldots, \theta_n)$ and $(\theta'_1, \ldots, \theta'_n)$ are rationally independent n-tuples in $(0,1)$ (modulo Z), then the following are equivalent:

(i) $\qquad C(T^n) \times_{\rho(\theta_1, \ldots, \theta_n)} Z \overset{\text{SME}}{\sim} C(T^n) \times_{\rho(\theta'_1, \ldots, \theta'_n)} Z$,

(ii) $\qquad (\theta_1, \ldots, \theta_n)$ and $(\theta'_1, \ldots, \theta'_n)$ are in the same $GL(n+1, Z)$ orbit

([11], Theorem 4.22). Condition (ii) is defined quite naturally as an extension of the case $n = 1$ we considered above, the case for the irrational rotation algebras ([11], p.45).

In the light of these recent developments let us state our results. In chapter 1 we present the basic concepts and results that we shall need for our work. Then we compute the K-groups of all crossed products $C(T^2) \times_\alpha Z$ in chapter 2. In chapter 3 we prove some lemmas which we will need to be able to classify crossed product algebras associated with irrational affine quasi-rotations on T^2 (§4.3, Theorem 4.3.2). This latter result, which is our main theorem, can be briefly described as follows. Every affine transformatin on the abelian group T^2 has the form

$$\varphi(z) = aA(z) \qquad (\forall z \in T^2)$$

where $a \in T^2$ and $A \in GL(2, Z)$. We shall show that there is an invariant $m(A)$, a positive integer, which, together with the tracial range and the orientation $\det A = \pm 1$, yields a complete invariant for the associated crossed products. Let us write $B(a, A) = C(T^2) \times_{\alpha_\varphi} Z$ where φ is as above. Then for irrational affine quasi-rotations (defined in §4.1) the following conditions are equivalent:

$$(1)\qquad B(a,A) \cong B(a',A'),$$

$$(2)\qquad (i)\qquad \tau_* K_0(B(a,A)) = \tau'_* K_0(B(a',A')),$$

$$(ii)\qquad \det A = \det A',$$

$$(iii)\qquad m(A) = m(A').$$

Condition (i) is equivalent to saying that $\lambda_{a',A'} = \lambda_{a,A}$ or $\bar{\lambda}_{a,A}$, where $\lambda_{a,A}$ is a "primitive" non-singular eigenvalue of φ (§4.1). In condition (ii), $\det A = \pm 1$ depending on whether φ is orientation preserving or reversing. In (iii) the invariant $m(A)$ is a positive integer which comes from the torsion part of the K_1-group of the crossed product $B(a,A)$ which, as it turns out, determines the conjugacy class of A in $GL(2,Z)$ (cf. Chapter 3).

In chapter 5, we construct a minimal quasi-rotation φ on T^2 which does not have (topologically) quasi-discrete spectrum (§5.1). This answers in the negative the following general question raised by Ji in his thesis ([11], p.76): If $\lambda = e^{2\pi i\theta}$, where θ is irrational, and $\varphi(x,y) = (\lambda x, f(x)y)$ where $f: T \to T$ has degree 1, then does φ necessarily have to be topologically conjugate to φ_θ or to φ_θ^{-1}, where $\varphi_\theta(x,y) = (\lambda x, xy)$ is the Anzai transformation? We shall show that the crossed product associated with this quasi-rotation has a unique normalized trace (Proposition 5.1.5; see also Question 5.1.6 and Proposition 5.1.7). However, we do not know how to prove (or disprove) that this latter algebra is not isomorphic to that associated with the Anzai transformation.

In section 5.2 we establish a criterion for when two irrational quasi-rotations of the form

$$\varphi_{\lambda,f}(x,y) = (\lambda x, f(x)y),$$

where $\lambda \in T$ is irrational and $f: T \to T$ is continuous of degree 1, are topologically conjugate (Proposition 5.2.2). This determines when the two C^*-dynamical systems $(C(T^2), \alpha_{\lambda,f}, Z)$ and $(C(T^2), \alpha_{\lambda,g}, Z)$ are equivariantly isomorphic, where $\alpha_{\lambda,f}$ is the automorphism on $C(T^2)$ associated with $\varphi_{\lambda,f}$. This leads us to conjecture that $C(T^2) \times_{\alpha_{\lambda,f}} Z \cong$

$C(T^2) \times_{\alpha_{\lambda,\varepsilon}} Z$ if and only if f and g differ by a "boundary" in some sense (see Conjecture 5.2.4).

CHAPTER 1

Preliminaries

In this chapter we shall present the basic concepts, notations, and facts which we shall need in subsequent chapters.

§1.1. Homeomorphisms of T^2.

Let $f : T^2 \to T$ be a continuous function. We shall prove in Lemma 1.1.2 that f has the form

$$f(x,y) = x^m y^n e^{2\pi i F(x,y)}$$

for some integers m, n and some continuous real-valued function F on T^2. We shall call the 1×2 integral matrix $[m\ n]$ the bidegree of f and denote this by

$$D(f) = [m\ n]\ .$$

If φ is a homeomorphism of the 2-torus T^2 we can associate with it a "degree" matrix $D(\varphi) \in GL(2,Z)$, where $GL(2,Z)$ is the group of invertible 2×2 matrices over Z (so that their determinant $= \pm 1$). To do this we write φ as

$$\varphi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$$

where $\varphi_i : T^2 \to T$ are continuous , $i = 1, 2$. Now define

$$D(\varphi) = \begin{bmatrix} D(\varphi_1) \\ D(\varphi_2) \end{bmatrix} \in M_2(Z)\,,$$

where $M_2(Z)$ is the algebra of 2×2 matrices over Z. It is easy to verify that if φ and ψ are two homeomorphisms of T^2, then

$$D(\varphi \circ \psi) = D(\varphi)D(\psi)\,.$$

If $f : T^2 \to T$ is continuous, then $D(f \circ \varphi) = D(f)D(\varphi)$. So if φ is a homeomorphism , then $D(\varphi)D(\varphi^{-1}) = I_2$ (the identity 2×2 matrix), so that $D(\varphi) \in GL(2, Z)$.

For example, if

$$\varphi(x, y) = (ax^m y^n, bx^p y^q),$$

where $a, b \in T$, then

$$D(\varphi) = \begin{bmatrix} m & n \\ p & q \end{bmatrix}.$$

In fact, such a mapping φ is a homeomorphism if and only if $\det D(\varphi) = \pm 1$.

Now, to be able to define the bidegree of f we shall need the following lemma.

LEMMA 1.1.1. *Let* $f : T^2 \to T$ *be a continuous map such that* $x \mapsto f(x, 1)$ *and* $y \mapsto f(1, y)$ *have degree zero as maps on the circle* T. *Then there exists a continuous function* $H : T^2 \to R$ *such that*

$$f(x, y) = e^{2\pi i H(x,y)}, \qquad \forall x, y \in T.$$

PROOF. Consider the homotopy $F : [0, 1] \times [0, 1] \to T$ given by

$$F(s, t) = f(\sigma(s), \sigma(t))$$

where $\sigma(t) = e^{2\pi i t}$, and consider the following diagram:

$$\begin{array}{ccc} & & R \\ & G \nearrow & \downarrow \sigma \\ [0,1] \times [0,1] & \xrightarrow{F} & T. \end{array}$$

Since the path $s \mapsto f(\sigma(s), 1)$ has degree zero, it has a unique lifting $\gamma : [0, 1] \to R$ such that

$$f(\sigma(s), 1) = \sigma(\gamma(s)), \qquad (\forall s)$$

and

$$\gamma(0) = \gamma(1) = 0 .$$

On applying the Covering Homotopy Theorem ([28], §5.3) to F we obtain a homotopy lifting G (as in above diagram) such that

$$F(s,t) = \sigma(G(s,t)) ,$$

and

$$G(s,0) = \gamma(s) , \qquad \forall s,\, t \in [0,1] .$$

Thus,

$$G(0,0) = G(1,0) = 0 .$$

Since the path $t \mapsto G(0,t)$ is a lifting for $t \mapsto f(1,\sigma(t))$ which starts at zero, i.e. $G(0,0) = 0$, it follows that

$$G(0,1) = 0 ,$$

which is the degree of the loop $t \mapsto f(1,\sigma(t))$.

<u>Assertion</u>:

(i) $\quad G(s,0) = G(s,1)$,

(ii) $\quad G(0,t) = G(1,t) \qquad (\forall s,\, t)$.

The first equation holds because both $s \mapsto G(s,0)$ and $s \mapsto G(s,1)$ are liftings for the loop $s \mapsto f(\sigma(s),1)$ starting at zero. Hence by the uniqueness part of the lifting lemma, (i) holds. Similarly, the second equation holds. These two equations imply that G induces a well-defined continuous map $H : T^2 \to T$ such that

$$H(\sigma(s),\sigma(t)) = G(s,t) , \qquad 0 \le s,t \le 1 .$$

Thus,

$$\begin{aligned} f(\sigma(s),\sigma(t)) &= F(s,t) \\ &= \sigma(G(s,t)) \\ &= \sigma(H(\sigma(s),\sigma(t)), \qquad\qquad (\forall s,\ t) \end{aligned}$$

so that

$$f(x,y) = \sigma(H(x,y)) = e^{2\pi i H(x,y)}\,.$$

□

LEMMA 1.1.2. *Every unitary* $f \in C(T^2)$ *has the form*

$$f(x,y) = x^m y^n e^{2\pi i F(x,y)}$$

for some $F : T^2 \to R$ *continuous and unique integers* m, n.

PROOF. Let m denote the degree of the map $x \mapsto f(x,1)$ and n the degree of $y \mapsto f(1,y)$. Then the map

$$(x,y) \mapsto x^{-m} y^{-n} f(x,y)\,,$$

satisfies the hypotheses of the preceding lemma so that the result clearly follows. The uniqueness of m and n is obvious.

□

Definition. Let X be a compact metric space and $\varphi : X \to X$ a homeomorphism . A complex number λ is an eigenvalue of φ if there exists a continuous function $f : X \to C$, not everywhere zero, such that $f \circ \varphi = \lambda f$. One calls f an eigenfunction of φ of eigenvalue λ.

The homeomorphism φ is said to be minimal if whenever F is a closed subset of X which is φ-invariant, in the sense that $\varphi(F) \subseteq F$, then F is empty or $F = X$. Equivalently, the orbit of every element $x \in X$, namely $\{x, \varphi(x), \varphi\circ\varphi(x), \varphi\circ\varphi\circ\varphi(x), \ldots\}$, is dense in X.

PROPOSITION 1.1.3. *Let $\varphi : X \to X$ be a minimal homeomorphism of a compact metric space X.*

(a) $f\circ\varphi = \lambda f$ *and* $f \neq 0 \quad\Rightarrow\quad |\lambda| = 1$ *and* $|f| =$ *non-zero constant.*

(b) $f\circ\varphi = \lambda f$ *and* $g\circ\varphi = \lambda g$ $(g \neq 0) \quad\Rightarrow\quad f = cg$ *for some complex number* c.

(c) The eigenvalues of φ form a subgroup of T*, and are in fact countable.*

(d) If X *is connected, then φ has no torsion eigenvalues (other than 1).*

Note that using the minimality of φ it is easy to see that if $f \in C(X)$ and $f\circ\varphi = f$, then f is constant.

PROOF.

(a) Taking supremum on both sides of $f\circ\varphi = \lambda f$ we obtain $|\lambda| = 1$. So, by minimality, from $|f|\circ\varphi = |f|$ we obtain that $|f|$ is constant, which is non-zero since $f \neq 0$.

(b) Form fg^{-1} so that $(fg^{-1})\circ\varphi = fg^{-1}$ and hence fg^{-1} is constant.

(c) This is obvious, except for countability (see Walters [25], p.124).

(d) Suppose that $f\circ\varphi = \lambda f$ and $\lambda^k = 1$ for some non-zero integer k. Then $f^k\circ\varphi = \lambda^k f^k = f^k$, where $f^k = f\cdots f$ (k-fold product of f). Hence, by minimality, f^k is a non-zero constant. By the connectedness of X and continuity of f, f is a non-zero constant. Hence from $f\circ\varphi = \lambda f$ we get $\lambda = 1$.

□

<u>Definition</u>. Let $\varphi : X \to X$ be a homeomorphism of a space X. Consider the sets

$$G_0(\varphi) = \{ \lambda \in C \mid \lambda \text{ is an eigenvalue of } \varphi \},$$

$$G_1(\varphi) = \{ f \in C(X) \mid f \circ \varphi = \lambda f \text{ for some } \lambda \in G_0(\varphi), \text{ and } |f| = 1 \},$$

$$G_2(\varphi) = \{ g \in C(X) \mid g \circ \varphi = fg \text{ for some } f \in G_1(\varphi), \text{ and } |g| = 1 \},$$

$$\vdots$$

$$G_j(\varphi) = \{ g \in C(X) \mid g \circ \varphi = fg \text{ for some } f \in G_{j-1}(\varphi), \text{ and } |g| = 1 \},$$

$$\vdots$$

for $j \geq 1$. Their union $G(\varphi) = \bigcup_{j \geq 0} G_j(\varphi)$ is known as the set of quasi-eigenfunctions of φ. The homeomorphism φ is said to have (topologically) quasi-discrete spectrum if the C*-algebra generated by its quasi-eigenfunctions is all of C(X). It is said to have (topologically) discrete spectrum if the C*-algebra generated by $G_1(\varphi)$, its eigenfunctions, is C(X).

Using a Zorn's lemma argument one can show that every homeomorphism φ of X has a minimal subset Y: that is, a closed non-empty subset Y of X such that $\varphi(Y) = Y$ and the restriction of φ to Y is minimal. The following proposition gives us a useful criterion to test a certain class of homeomorphisms for minimality.

<u>PROPOSITION 1.1.4</u>. *Let X be a compact metric space and $S : X \to X$ a minimal homeomorphism . Let $h : X \to T$ be a continuous map on X into the circle. Let H denote the homeomorphism of $X \times T$ given by*

$$H(x, z) = (Sx, h(x)z).$$

Then H is minimal if, and only if, for any non-zero integer n the equation

$$F(Sx) = h(x)^n F(x)$$

has no continuous solution $F : X \to T$.

<u>PROOF</u>. The following proof has been briefly outlined in [15] and we have provided its details here.

Suppose first that there exists a continuous solution F to the above equation for some $n \neq 0$. Then the non-constant continuous function $f(x, z) = F(x)z^{-n}$ satisfies $f \circ H = f$, hence H cannot be minimal.

Conversely, suppose that H is not minimal and let M be a minimal subset of $X \times T$. Let us define a T-action on $X \times T$ by $z(x, w) = (x, zw)$. We can easily see that for any $z \in T$ we have $zM = M$ or $zM \bigcap M$ is empty: Since zM is H-invariant so is the closed subset $zM \bigcap M$ of M, so that by minimality of the set M we have $zM \bigcap M$ is equal to M or is empty. Now we have the union

$$X \times T = \bigcup_{z \in T} zM .$$

Since M is closed, the subgroup

$$G = \{ z \in T \mid zM = M \}$$

of the circle is closed.

<u>Assertion</u>: G is a finite group.

Let us show that there exists a positive integer n such that $z^n = 1$ for all $z \in G$. If this is false, then there exists an increasing sequence $n_1 < n_2 < n_3 < \cdots$ of positive integers such that n_j is the order of an element $z_j \in G$, i.e. z_j is a primitive n_j-th root of unity in T. But then G will contain the dense set

$$\{ z_j^k \mid 1 \leq k \leq n_j \, , \, j = 1, 2, 3, \ldots \}$$

so that $G = T$ and hence $M = X \times T$ meaning that H is minimal, a contradiction. Thus there exists a positive integer n such that $z^n = 1$ for all $z \in G$. But since $G \subseteq T$, G must be finite.

From the above union for $X \times T$ we easily see that for each $x \in X$ there exists $z \in T$ such that $(x,z) \in M$. This allows us to define a map $F : X \to T$ by

$$F(x) = z^n, \quad \text{if } (x,z) \in M.$$

Well-defined: If (x,z), $(x,w) \in M$, then since we have $w\bar{z}(x,z) = (x,w)$ so that $w\bar{z}M = M$, we get $w\bar{z} \in G$. Thus, $z^n = w^n$.

Continuity: Suppose $x_k \to x$ is a convergent sequence in X and suppose (x_k, z_k) and (x,z) belong to M $(\forall k)$. We must show that $z_k^n \to z^n$. Assume that $\{z_k^n\}$ does not converge to z^n. Then there exist integers $\{k_j\}$ and a positive real number r such that

$$|z_{k_j}^n - z^n| \geq r > 0, \qquad \text{for all } j.$$

But $\{z_{k_j}\}$ being a bounded sequence in T implies that it must have a convergent subsequence, say $z_{k_{j_l}} \to w$. So

$$(x_{k_{j_l}}, z_{k_{j_l}}) \to (x,w) \in M \quad \text{as } l \to \infty.$$

But (x,z) also belongs to M, hence $w^n = z^n$. Thus $z_{k_{j_l}}^n \to z^n$, which contradicts the above inequality.

Thus F is continuous and satisfies the equation

$$F(Sx) = h(x)^n F(x),$$

as can easily be checked using the definitions.

□

§1.2. K-groups of $C(T^2)$.

Let us now recall the K-groups of $C(T^2)$ and their canonical generators.

First, recall that $K_0\left(C(T^2)\right) \cong Z^2$ and is generated by [1] and [P], where $1 \in C(T^2)$ is the identity projection and $P \in M_2 \otimes C(T^2)$ is the so-called Bott projection defined by

$$P(x,t) = W(x,t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W(x,t)^*,$$

where

$$W(x,t) = E^{i\pi t/2} \begin{bmatrix} \bar{x} & 0 \\ 0 & 1 \end{bmatrix} E^{-i\pi t/2}, \quad x \in T \text{ and } 0 \le t \le 1,$$

and

$$E^{it} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix},$$

where T^2 is identified with $T \times [0,1]$ modulo the equivalence relation $(x,0) \sim (x,1)$, for all $x \in T$.

To prove this, first let us identify $C(T^2)$ with $C(T, C(T))$ via the map

$$C(T^2) \to C(T, C(T))$$

$$f \mapsto \hat{f}$$

given by

$$\hat{f}(t)(x) = f(x,t),$$

where $t \in T$, which is identified with $[0,1]/(0 \sim 1)$. Secondly, consider the split short exact sequence

$$0 \longrightarrow C_0((0,1), C(T)) \xrightarrow{i} C(T, C(T)) \xrightarrow{ev_0} C(T) \longrightarrow 0,$$

where ev_0 is evaluation at 0 (or 1), and i is the canonical inclusion. Being split means that we have a split short exact sequence of the induced K_0-groups

$$\begin{array}{ccccccc} 0 \longrightarrow & K_0(C_0((0,1), C(T))) & \xrightarrow{i_*} & K_0(C(T, C(T))) & \xrightarrow{(ev_0)_*} & K_0(C(T)) & \longrightarrow 0 \\ & \uparrow s^1 & & & & || & \\ & K_1(C(T)) = Z[f_0] & & & & Z[1] & \end{array}$$

where $s^1 : K_1(A) \to K_0(C_0((0,1), A))$ is the Bott periodicity isomorphism defined in Connes [3] (Appendix 1, Lemma 1), and $f_0(z) = z$ is the positive generator of $K_1(C(T))$.

To show that $[\hat{1}]$ and $[\widehat{P}]$ form a Z-basis for $K_0(C(T, C(T)))$ we must check that $(ev_0)_*[\hat{1}] = [1]$ and $i_* \circ s^1[f_0] = [\widehat{P}]$. The former is obvious. To verify the latter, we use the definition of s^1 (as given in Connes [3]) to obtain

$$s^1[f_0] = [We_0W^*] - [e_0] \quad \in K_0(C_0((0,1), C(T))),$$

where

$$e_0(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$W(t) = E^{i\pi t/2} \begin{bmatrix} \bar{f}_0 & 0 \\ 0 & 1 \end{bmatrix} E^{-i\pi t/2}$$

for all $t \in (0,1)$. Note that the elements $[We_0W^*]$, $[e_0]$ belong to K_0 of the unitization of $C_0((0,1), C(T))$, and their difference is in $K_0\left(C_0((0,1), C(T))\right)$. Applying i_* we obtain

$$\begin{aligned} i_* \circ s^1[f_0] &= i_*[We_0W^*] - i_*[e_0] \\ &= [We_0W^*] - [e_0], \end{aligned}$$

where now

$$e_0(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and the '1' here is that of $C(T)$. Now We_0W^* is just $\widehat{P}$, the associated projection in $M_2 \otimes C(T, C(T))$ to the Bott projection $P \in M_2 \otimes C(T^2)$. Thus $[\widehat{P}] - [\hat{1}]$ and $[\hat{1}]$ form a Z-basis, the canonical basis for $K_0\left(C(T, C(T))\right)$, as desired. Thus we have

$$K_0\left(C(T^2)\right) = Z[1] \oplus Z([P] - [1]).$$

Next, one can show that

$$K_1\left(C(T^2)\right) = Z[u] \oplus Z[v]$$

where $u, v \in C(T^2)$ are the unitaries given by $u(x,y) = x$, $v(x,y) = y$, which can be identified with $u(x,t) = x$, $v(x,t) = e^{2\pi i t}$, for $t \in [0,1]/(0 \sim 1)$. On applying the K_1 functor to the above split short exact sequence we get

$$\begin{array}{ccccccc} 0 \longrightarrow & K_1\left(C_0((0,1), C(T))\right) & \xrightarrow{i_*} & K_1\left(C(T, C(T))\right) & \xrightarrow{(ev_0)_*} & K_1\left(C(T)\right) & \longrightarrow 0 \\ & \uparrow s^0 & & & & || & \\ & K_0\left(C(T)\right) = Z[1] & & & & Z[f_0] & \end{array}$$

where $s^0 : K_0(A) \to K_1\left(C_0((0,1), A)\right)$ is the Bott isomorphism which is also defined in Connes [3].

Now we must show that $[\hat{u}]$ and $[\hat{v}]$ form a Z-basis for $K_1(C(T, C(T)))$. To do this it will suffice to show

(a) $(ev_0)_*[\hat{u}] = [f_0]$,

(b) $[\hat{v}] = i_* \circ s^0[1]$,

since the sequence splits. The first equation is clear because $ev_0(\hat{u})(x) = u(x, 0) = x = f_0(x)$. So it remains to verify (b). Now

$$s^0[1] = [\tilde{1} + (f_0 - 1) \otimes 1] \in K_1(C_0((0,1), C(T))^{\sim}),$$

where

$$\tilde{1} + (f_0 - 1) \otimes 1 \in GL_1(C_0((0,1), C(T))^{\sim}).$$

So

$$\begin{aligned} i_* \circ s^0[1] &= i_*[\tilde{1} + (f_0 - 1) \otimes 1] \\ &= [1 + (f_0 - 1) \otimes 1] \\ &= [f_0 \otimes 1], \end{aligned}$$

where $f_0 \otimes 1 \in C(T, C(T))$ is given by

$$(f_0 \otimes 1)(t)(x) = e^{2\pi i t} = v(x, t) = \hat{v}(t)(x).$$

Hence,

$$i_* \circ s^0[1] = [\hat{v}].$$

We may summarize the above as follows.

<u>THEOREM 1.2.1</u>.

(i) $K_0\left(C(T^2)\right) = Z[1] \oplus Z([P] - [1])$,

(ii) $K_1(C(T^2)) = Z[u] \oplus Z[v]$.

§1.3. Crossed Products and Traces.

Let us recall the definition and construction of crossed products, for in this thesis we shall study certain crossed products of $C(T^2)$ by certain homeomorphisms.

Let (A, α, G) be a C^*-dynamical system. This means that A is a C^*-algebra, G is a locally compact topological group (with left-invariant Haar measure μ), and

$$\alpha : G \to \mathrm{Aut}(A)$$

is a strongly continuous homomorphism of groups, i.e. for each $a \in A$ the map $G \to A : t \mapsto \alpha_t(a)$ is norm continuous .

With this system one can associate a unique C^*-algebra called the C^*-crossed product of the system (A, α, G), denoted by $A \times_\alpha G$, which has the universal property: If $\pi : A \to L(H)$ is a non-degenerate representation of A on a Hilbert space H, and if $u : G \to U(H)$ is a strongly continuous unitary representation of G such that one has the covariance relation

$$\pi(\alpha_t(a)) = u_t \pi(a) u_t^* , \qquad a \in A, t \in G ,$$

then there exists a unique representation $\pi \times u : A \times_\alpha G \to L(H)$ such that

$$(\pi \times u)(f) = \int_G \pi(f(t)) u_t \, d\mu(t) ,$$

for all $f \in L^1(G, A)$. A pair (π, u) satisfying the above covariance relation is called a covariant representation of the system (A, α, G).

The construction of $A \times_\alpha G$ is as follows.

One looks at the vector space $L^1(G, A)$ of all measurable functions $f : G \to A$ such that

$$||f||_1 = \int_G ||f(t)|| \, d\mu(t) < \infty ,$$

and defines the twisted convolution by

$$(f * g)(t) = \int_G f(s)\alpha_s(g(s^{-1}t))\, d\mu(s),$$

and involution $*$ by

$$f^*(t) = \frac{1}{\Delta(t)}\alpha_t(f(t^{-1})^*),$$

for $f, g \in L^1(G, A)$, where $\Delta : G \to R$ is the modular function of G. In our case, $G = Z$ so that $\Delta(t) = 1$ for all $t \in G$. In this way $L^1(G, A)$ with its usual norm becomes a Banach $*$-algebra, and one defines $A \times_\alpha G$ to be its enveloping C*-algebra . Recall that if $(B, || \ ||_B)$ is a Banach $*$-algebra, then its enveloping C*-algebra is the completion of B with respect to the C*-seminorm

$$||x||_* = \sup_\pi ||\pi(x)||,$$

where the supremum is taken over all $*$-representations π of B; since $\pi : B \to L(H)$ is continuous , it follows that $||\pi(x)|| \leq ||x||_B$ $(\forall x \in B)$, hence

$$||x||_* \leq ||x||_B, \qquad \forall x \in B.$$

Equivalently, the enveloping C*-algebra of B can be defined by taking the norm-closure of the image of its universal representation (π_u, H_u), which is the direct sum of all non-degenerate $*$-representations of B. Since $C_c(G, A)$, the space of continuous functions $G \to A$ of compact support, is also a $*$-algebra with the above convolution and involution, the fact that it is dense in $L^1(G, A)$ (in the norm $|| \ ||_1$) means that it has the same enveloping C*-algebra as $L^1(G, A)$ (up to isomorphism). Thus one may take $A \times_\alpha G$ to be the enveloping C*-algebra of $C_c(G, A)$ or of $L^1(G, A)$.

<u>PROPOSITION 1.3.1</u> (PEDERSEN [16], 7.6.4). *If (π, u, H) is a covariant representation of (A, α, G), then there exists a non-degenerate $*$-representation $\pi \times u$ of $A \times_\alpha G$ (acting on H) such that*

$$(\pi \times u)(f) = \int_G \pi(f(t))u_t\, d\mu(t),$$

for every $f \in L^1(G, A)$. *Moreover, the correspondence* $(\pi, u, H) \mapsto (\pi \times u, H)$ *is a bijection onto the set of all non-degenerate* $*$*-representations of* $A \times_\alpha G$.

PROPOSITION 1.3.2 (SPECIALIZED CASE OF [16], 7.6.6). *For each* C^**-dynamical system* (A, α, Z) *where* A *is unital, there exists a covariant representation* (π^0, u^0, H^0) *such that:*

(i) $A \times_\alpha Z = C^*(\pi^0(A), u^0(Z))$; *that is, the* C^**-algebra generated by* $\pi^0(a)$ *and* u^0_n, *for* $a \in A$ *and* $n \in Z$.

(ii) Given any covariant representation (π, u, H) *of* (A, α, Z), *there exists a unique* $*$*-representation*

$$\rho : A \times_\alpha Z \to L(H)$$

such that

$$\pi = \rho \circ \pi^0 \quad \text{and} \quad u = \rho \circ u^0 .$$

In our specialized case $G = Z$, A can be naturally embedded in $l^1(Z, A) \subseteq A \times_\alpha Z$ by mapping $a \in A$ to the element $n \mapsto \delta_{0,n} a$, where $\delta_{m,n}$ is the kronecker δ-function.

Let us now turn to traces on crossed products.

A tracial state (or normalized trace) on a C^*-algebra A is a linear function $\tau : A \to C$ which is positive in the sense that $\tau(a^*a) \geq 0$ ($\forall a \in A$), such that $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$, for all $x, y \in A$. One calls τ a faithful trace if $\tau(a^*a) = 0$ implies $a = 0$, $a \in A$. In fact, one can show that $\{a \in A \mid \tau(a^*a) = 0\}$ is a two-sided ideal in A.

Given a C^*-dynamical system (A, α, Z), a trace τ on A is said to be α-invariant if $\tau(\alpha_n(a)) = \tau(a)$, for all $a \in A$ and $n \in Z$. Such a trace induces a trace $\hat{\tau}$ on the associated crossed product $A \times_\alpha Z$ such that

$$\hat{\tau}(f) = \tau(f(0)),$$

for $f \in l^1(Z, A)$. To show this we first construct what is called a conditional expectation.

PROPOSITION 1.3.3 (ZELLER-MEIER [27]; ITOH [10], THEOREM 4.1).

Given the C^**-dynamical system* (A, α, Z)*, there exists a unique continuous linear map*

$$E : A \times_\alpha Z \to A$$

such that

$$E(f) = f(0), \qquad \forall f \in l^1(Z, A).$$

PROOF. Clearly it suffices to show that the linear map $E : l^1(Z, A) \to A$ given by $E(f) = f(0)$ is continuous with respect to the C^*-norm $\| \, \|_*$ on $l^1(Z, A)$ which yields the enveloping C^*-algebra .

Fix any state φ on A. Apply the GNS construction to obtain a cyclic $*$-representation $\pi : A \to L(H)$ with cyclic unit vector ς such that

$$\varphi(a) = <\pi(a)\varsigma, \varsigma>, \qquad a \in A.$$

(cf. [16], 3.3.3). One then induces a $*$-representation R_φ of $l^1(Z, A)$ on the Hilbert space $l^2(Z, H)$ and cyclic unit vector $\eta = \delta_0\varsigma \in l^2(Z, H)$, where $\delta_0(n) = \delta_{0,n}$, such that

$$\varphi(f(0)) = <R_\varphi(f)\eta, \eta>, \qquad \text{for } f \in l^1(Z, A),$$

(cf. [16], 7.7.1). Thus we have

$$\begin{aligned}
\varphi((Ef)^*(Ef)) = \varphi(f(0)^*f(0)) &\le \varphi((f^*f)(0)) \\
&= <R_\varphi(f^*f)\eta, \eta> \\
&= \|R_\varphi(f)\eta\|^2 \\
&\le \|R_\varphi(f)\|^2 \\
&\le \|f\|_*^2 .
\end{aligned}$$

Since this is true for all states φ of A and $f \in l^1(Z, A)$, it follows that

$$\|(Ef)^*(Ef)\| \le \|f\|_*^2,$$

or

$$||Ef|| \leq ||f||_*, \qquad \text{for } f \in l^1(Z, A)$$

(cf. [24], Theorem 4.3b). Hence E is continuous and therefore extends uniquely to the crossed product $A \times_\alpha Z$.

□

Given a trace τ on the C*-algebra A which is α-invariant, its induced trace $\hat{\tau}$ on $A \times_\alpha Z$ is defined by $\hat{\tau} = \tau \circ E$, with E as in the previous proposition, so that

$$\hat{\tau}(f) = \tau(f(0)), \qquad \text{for } f \in l^1(Z, A).$$

Since τ is α-invariant it follows that $\hat{\tau}$ is tracial, i.e. $\hat{\tau}(f * g) = \hat{\tau}(g * f)$ for all $f, g \in l^1(Z, A)$.

We wish to end this section with a lemma about when there is a one-to-one correspondence between traces on C(X) and traces on $C(X) \times_\alpha Z$. This we shall need in chapter 5.

LEMMA 1.3.4 ([11], PROPOSITION 1.12). *Let φ be a homeomorphism on a compact metric space X, such that $f \circ \varphi = \lambda f$ for some $f \in C(X)$, a unitary, and $\lambda \in T$ irrational. Let τ be any tracial state on $C(X) \times_\alpha Z$, where α is the induced automorphism on C(X) given by $\alpha(g) = g \circ \varphi^{-1}$. Then*

$$\tau(gW^n) = 0$$

for all $g \in C(X)$ and $n \neq 0$, where $W \in l^1(Z, C(X))$ is the unitary $W(k) = \delta_{k,1}1$, $k \in Z$.

PROOF. Note that $\alpha(g) = WgW^*$, $g \in C(X)$. Hence, $W^n f W^{-n} = \alpha^n(f) = \lambda^{-n} f$, so that

$$\begin{aligned} \tau(gW^n) &= \tau(gW^n f W^{-n} W^n f^{-1}) \\ &= \lambda^{-n}\tau(gfW^n f^{-1}) \\ &= \lambda^{-n}\tau(f^{-1}gfW^n) \\ &= \lambda^{-n}\tau(gW^n) \end{aligned}$$

and since λ is irrational, $\lambda^{-n} \neq 1$ for $n \neq 0$, therefore $\tau(gW^n) = 0$.

$\square$

Thus,

$$\tau\left(\sum_{-N}^{N} g_n W^n\right) = \tau(g_0)\,, \qquad \text{for } g_n \in C(X)\,.$$

Hence it is clear that there is a one-to-one correspondence between traces on $C(X) \times_\alpha Z$ and α-invariant traces on $C(X)$, i.e. φ-invariant probability measures on X.

§1.4. The Pimsner Voiculescu Sequence.

The first and most remarkable result in the K-theory of crossed product C*-algebras is due to Pimsner and Voiculescu. It states that the K-groups of a crossed product $A \times_\alpha Z$ fit together into a cyclic six-term exact sequence involving the K-groups of A, which are presumed to be known, and certain maps between them.

THEOREM 1.4.1 ([18], THEOREM 2.4). *Let (A, α, Z) be a C*-dynamical system. Identify α with the automorphism α_1. Then the following sequence is exact:*

$$\begin{array}{ccccc} & K_0(A) & \xrightarrow{\alpha_* - id_*} & K_0(A) & \\ {\scriptstyle \delta_1}\nearrow & & & & \searrow{\scriptstyle i_*} \\ K_1(A \times_\alpha Z) & & & & K_0(A \times_\alpha Z) \\ {\scriptstyle i_*}\nwarrow & & & & \swarrow{\scriptstyle \delta_0} \\ & K_1(A) & \xleftarrow{\alpha_* - id_*} & K_1(A) & \end{array}$$

Here $i : A \to A \times_\alpha Z$ is the natural inclusion map and δ_0, δ_1 are the connecting homomorphisms.

Using his analogue of the Thom isomorphism for crossed products of C*-algebras by actions of R, Connes was able to deduce the above exact sequence as a corollary (Connes

[3], pp.48–49). The Thom isomorphism is the isomorphism $K_i(A) \cong K_{i+1}(A \times_\alpha R)$ which generalizes the Bott periodicity theorem $K_i(A) \cong K_{i+1}(A \otimes C_0(R))$.

The above sequence was later generalized to reduced crossed products of A by F_n, the free non-abelian group on $n \geq 1$ generators ([19], Theorem 3.5).

Example. Let A_θ be the irrational rotation C*-algebra , which can be viewed as the crossed product of C(T) by the irrational rotation of T through the angle $2\pi\theta$. Applying the above theorem one obtains its K-groups,

$$K_0(A_\theta) \cong Z \oplus Z, \qquad K_1(A_\theta) \cong Z \oplus Z.$$

As pointed out in the Appendix of [18], one can show that the Rieffel projection e_θ of trace θ does in fact map to a generator of $K_1(C(T)) \cong Z$ by δ_0. This shows that, more specifically,

$$K_0(A_\theta) = Z[1] + Z[e_\theta]$$

so that the tracial range can be computed to be $\tau_* K_0(A_\theta) = Z + \theta Z$. Note that these computations still work even if $0 < \theta < 1$ was assumed to be rational.

The next result says that the Pimsner-Voiculescu sequence (henceforth abbreviated as "PV-sequence") is functorial with respect to covariant maps between C*-dynamical systems (A, α, Z).

PROPOSITION 1.4.2 ([11], 2.8). *Let (A, α, Z) and (B, β, Z) be C*-dynamical systems and $\rho : A \to B$ be an equivariant homomorphism. Then the following diagram of PV-sequences is commutative:*

$$
\begin{tikzcd}
 & K_0(A) \arrow[rrr, "\alpha_*-\mathrm{id}_*"] \arrow[dr, "\rho_*"] & & & K_0(A) \arrow[dl, "\rho_*"'] \arrow[ddr, "i_*"] & \\
 & & K_0(B) \arrow[r, "\beta_*-\mathrm{id}_*"] & K_0(B) \arrow[dr, "i_*"] & & \\
K_1(A \times_\alpha Z) \arrow[r, "\tilde{\rho}_*"] \arrow[uur, "\delta_1"] & K_1(B \times_\beta Z) \arrow[ur, "\delta_1"] & & & K_0(B \times_\beta Z) \arrow[dl, "\delta_0"] & K_0(A \times_\alpha Z) \arrow[l, "\tilde{\rho}_*"'] \arrow[ddl, "\delta_0"] \\
 & & K_1(B) \arrow[ul, "i_*"] & K_1(B) \arrow[l, "\beta_*-\mathrm{id}_*"'] & & \\
 & K_1(A) \arrow[uul, "i_*"] \arrow[ur, "\rho_*"] & & & K_1(A) \arrow[lll, "\alpha_*-\mathrm{id}_*"'] \arrow[ul, "\rho_*"'] &
\end{tikzcd}
$$

where $\tilde{\rho} : A \times_\alpha Z \to B \times_\beta Z$ is the induced map on the associated crossed products.

In the next chapter we shall use the PV-sequence to compute the K-groups of the crossed products $C(T^2) \times_\alpha Z$, where α is an arbitrary automorphism of $C(T^2)$. This was partly done by Ji [11], was indicated by C.C. Moore to Packer [14] (p.48) by using the Chern character isomorphism $\mathrm{ch} : K_*(C(T^n)) \to H^*(T^n)$, and completely done by Igal Megory-Cohen [13] (Chapter 3), but done by us independently using a different and shorter method.

CHAPTER 2

K-Theory of $C(T^2) \times Z$

Throughout this chapter let us fix a notation: We shall let φ denote a homeomorphism of T^3 and let α denote its associated automorphism on $C(T^2)$ defined by $\alpha(f) = f \circ \varphi^{-1}$, $f \in C(T^2)$.

Every homeomorphism φ of T^2 is homotopic to an automorphism of T^2 as a topological group. In fact, in the notation of §1.1, φ is homotopic to its degree matrix $D(\varphi)$ which acts on T^2 quite naturally:

$$\text{If } D(\varphi) = \begin{bmatrix} m & n \\ p & q \end{bmatrix} \in GL(2, Z), \quad \text{then}$$
$$D(\varphi)(x, y) = (x^m y^n, x^p y^q).$$

The fact that φ is homotopic to $D(\varphi)$ follows from Lemma 1.1.2: If we write $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_i : T^2 \to T$, the lemma says that we can homotope φ_1 to $(x, y) \mapsto x^m y^n$, and φ_2 to $(x, y) \mapsto x^p y^q$, so that φ can be homotoped to $D(\varphi)$. This means that the automorphism α on $C(T^2)$ is homotopic to that associated with the group automorphism $D(\varphi)$. If α_φ denotes the automorphism on $C(T^2)$ associated with φ , then α_φ is homotopic to $\alpha_{D(\varphi)}$ through a path of $*$-homomorphisms, and hence, by the homotopy invariance of K-theory, we have $(\alpha_\varphi)_* = (\alpha_{D(\varphi)})_*$ on $K_*(C(T^2))$. From the PV-sequence we then see that

$$K_i\left(C(T^2) \times_{\alpha_\varphi} Z\right) \cong K_i\left(C(T^2) \times_{\alpha_{D(\varphi)}} Z\right),$$

for $i = 0, 1$. For instance, on the K_0 part one has

$$0 \longrightarrow \frac{K_0\left(C(T^2)\right)}{\operatorname{Im}(\alpha_* - id_*)} \longrightarrow K_0\left(C(T^2) \times_\alpha Z\right) \xrightarrow{\delta_0} \ker(\alpha_* - id_*) \longrightarrow 0$$

which splits because $\mathrm{Im}(\delta_0)$, being a subgroup of $K_1\left(C(T^2)\right) \cong Z^2$, is torsion-free. Since the groups on the left and right depend only upon the homotopy class of α , the middle group $K_0\left(C(T^2) \times_\alpha Z\right)$ only depends upon the homotopy class of α .

This shows that to calculate the K-groups of $C(T^2) \times_\alpha Z$ one can assume that φ is an automorphism of T^2, i.e. $\varphi = D(\varphi)$. This we shall assume for the remainder of this section.

First let us take care of the action of α_* on K_1.

<u>LEMMA 2.1</u>. *The matrix of* $\alpha_* : K_1\left(C(T^2)\right) \to K_1\left(C(T^2)\right)$ *relative to the basis* $\{[u],[v]\}$ *is the transpose of* $D(\varphi^{-1})$*:*

$$\alpha_* = D(\varphi^{-1})^T,$$

where $u(x,y) = x$ *and* $v(x,y) = y$.

<u>PROOF</u>. Write $D(\varphi^{-1}) = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$. Then the action of α is

$$\alpha(u) = u \circ \varphi^{-1} = u^m v^n,$$

$$\alpha(v) = v \circ \varphi^{-1} = u^p v^q,$$

so that

$$\alpha_*[u] = m[u] + n[v],$$

$$\alpha_*[v] = p[u] + q[v],$$

hence

$$\alpha_* = \begin{bmatrix} m & p \\ n & q \end{bmatrix} = D(\varphi^{-1})^T.$$

□

Next we need to know the action of α_* on $K_0\left(C(T^2)\right)$. Before we do this let us recall the fact that the group $GL(2,Z)$ is generated by the two matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

(see Kurosh [12], Appendix B). Thus every $D \in GL(2,Z)$ can be written as

$$D = A^{n_1}B^{m_1}\cdots A^{n_k}B^{m_k}$$

for some integers n_i, m_i, $i = 1, \ldots, k$, $k \geq 1$.

LEMMA 2.2. *On* $K_0\left(C(T^2)\right)$ *with basis* $\{[1], [P] - [1]\}$ *one has*

$$(\alpha_A)_* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad (\alpha_B)_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using this lemma we can prove the following.

LEMMA 2.3. *For any* $D \in GL(2,Z)$ *one has on* $K_0\left(C(T^2)\right)$ *(relative to the basis in previous lemma),*

$$\text{(i)} \qquad (\alpha_D)_* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{if} \quad \det D = -1,$$

$$\text{(ii)} \qquad (\alpha_D)_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{if} \quad \det D = 1,$$

PROOF OF 2.3. We may write D as

$$D = A^{n_1}B^{m_1}\cdots A^{n_k}B^{m_k}.$$

On taking determinants we obtain $\det D = (-1)^{n_1+\cdots+n_k}$. Therefore,

$$\begin{aligned} (\alpha_D)_* &= (\alpha_{A^{n_1}B^{m_1}\ldots A^{n_k}B^{m_k}})_* \\ &= (\alpha_{A^{n_1}})_*(\alpha_{B^{m_1}})_*\cdots(\alpha_{A^{n_k}})_*(\alpha_{B^{m_k}})_* \\ &= (\alpha_A)_*^{n_1}(\alpha_B)_*^{m_1}\cdots(\alpha_A)_*^{n_k}(\alpha_B)_*^{m_k} \\ \text{(by Lemma 2.2)} \qquad &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{n_1+\cdots+n_k}. \end{aligned}$$

If $\det D = -1$, then $n_1 + \cdots + n_k$ is odd and so (i) follows. If $\det D = 1$, then $n_1 + \cdots + n_k$ is even and so (ii) follows.

$\square$

Now let us proceed to prove Lemma 2.2.

PROOF OF 2.2. Let us first prove that $(\alpha_A)_* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Let P denote the Bott projection as defined in §1.2. By lifting to the square $[0,1] \times [0,1]$ we have $\alpha_A(P)(s,t) = P(A^{-1}(s,t)) = P(t,s) = W(t,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W(t,s)^*$. Let $a(s) = e^{2\pi i s}$ and consider the unitary

$$U(s,t) = W(t,s)\begin{bmatrix} 0 & \overline{a(st)} \\ a(st) & 0 \end{bmatrix} W(s,t)^* .$$

Using the definitions one easily verifies that $U(s,0) = U(s,1)$ and $U(0,t) = U(1,t)$, for all $s,t \in [0,1]$. Hence U defines a unitary in $M_2 \otimes C(T^2)$. It remains to check that

$$\alpha_A(P) = U(I_2 - P)U^* ,$$

where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This may be seen as follows. We can rewrite the definition of U as

$$W(s,t)^*U(s,t)^*W(t,s) = \begin{bmatrix} 0 & a(st) \\ \overline{a(st)} & 0 \end{bmatrix} .$$

Since this intertwines $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$W(s,t)^*U(s,t)^*W(t,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} W(s,t)^*U(s,t)^*W(t,s) ,$$

which can be arranged to yield

$$\begin{aligned} \alpha_A(P)(s,t) = W(t,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W(t,s)^* &= U(s,t)W(s,t)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} W(s,t)^*U(s,t)^* \\ &= U(s,t)\left\{ I_2 - W(s,t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W(s,t)^* \right\} U(s,t)^* \\ &= U(s,t)\,(I_2 - P(s,t))\,U(s,t)^* , \end{aligned}$$

and so

$$\alpha_A(P) = U(I_2 - P)U^*$$

which implies that

$$(\alpha_A)_*([P] - [1]) = -([P] - [1]).$$

Therefore,

$$(\alpha_A)_* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now let us consider the proof of $(\alpha_B)_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, where $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, i.e. $B(x,y) = (xy, y)$.

It suffices to show $(\alpha_{B^{-1}})_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In this case we have

$$\begin{aligned} \alpha_{B^{-1}}(P)(x,t) &= P(B(x,t)) = P(xe^{2\pi i t}, t) \\ &= W(xe^{2\pi i t}, t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W(xe^{2\pi i t}, t)^* . \end{aligned}$$

Define the unitary U by

$$U(x,t) = W(xe^{2\pi i t}, t)W(x,t)^* .$$

Since, clearly $U(x,0) = U(x,1)$, U defines a unitary in $M_2 \otimes C(T^2)$. Thus,

$$\begin{aligned} \alpha_{B^{-1}}(P)(x,t) &= U(x,t)W(x,t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W(x,t)^*U(x,t)^* \\ &= U(x,t)P(x,t)U(x,t)^* , \end{aligned}$$

and so

$$\alpha_{B^{-1}}(P) = UPU^* ,$$

and

$$(\alpha_{B^{-1}})_*[P] = [P].$$

Therefore,

$$(\alpha_{B^{-1}})_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

Now we are ready to compute the K-groups. But first let us observe the elementary fact that the kernel of a 2×2 integral matrix $\begin{bmatrix} m & n \\ p & q \end{bmatrix}$, acting on $Z \oplus Z$, is either zero (when $mq - np \neq 0$), or is equal to $Z \oplus Z$ (when $m = n = p = q = 0$), or is isomorphic to Z (when $mq - np = 0$ but not all of its entries are zero). In the latter case the kernel is singly generated.

THEOREM 2.4. ([13], CHAPTER 3). *Let φ be a homeomorphism of T^2 and α its associated automorphism on $C(T^2)$.*

(1) If $\det D(\varphi) = 1$, *then*

$$K_0\left(C(T^2) \times_\alpha Z\right) \cong \begin{cases} Z^4 & \text{if } D(\varphi) = I_2\,, \\ Z^3 & \text{if } \det\left(D(\varphi) - I_2\right) = 0 \quad \text{but } D(\varphi) \neq I_2\,, \\ Z^2 & \text{if } \det\left(D(\varphi) - I_2\right) \neq 0\,. \end{cases}$$

(2) If $\det D(\varphi) = -1$, *then*

$$K_0\left(C(T^2) \times_\alpha Z\right) \cong \begin{cases} Z^2 \oplus Z_2 & \text{if } \det\left(D(\varphi) - I_2\right) = 0\,, \\ Z \oplus Z_2 & \text{if } \det\left(D(\varphi) - I_2\right) \neq 0\,. \end{cases}$$

PROOF. From the PV-sequence we obtain the short exact sequence

$$0 \longrightarrow \frac{Z^2}{\operatorname{Im}(\alpha_* - \mathrm{id}_*)} \longrightarrow K_0\left(C(T^2) \times_\alpha Z\right) \xrightarrow{\delta_0} \ker(\alpha_* - \mathrm{id}_*) \longrightarrow 0\,,$$

where the $\alpha_* - \mathrm{id}_*$ on the left is on $K_0\left(C(T^2)\right) \cong Z^2$ and is identified via Lemma 2.3, and the $\alpha_* - \mathrm{id}_*$ on the right is on $K_1\left(C(T^2)\right) \cong Z^2$ so that $\ker(\alpha_* - \mathrm{id}_*)$ is isomorphic to Z^2, Z, or is zero. Since the latter are free, the sequence splits and the result follows.

□

THEOREM 2.5. ([13], CHAPTER 3). *Let φ be a homeomorphism of* T^2 *and* α *its associated automorphism on* $C(T^2)$. *Write* $D(\varphi^{-1}) = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$ *and denote by* J *the quotient group*

$$J = \frac{Z \oplus Z}{Z(m-1,n) + Z(p,q-1)} = \frac{Z^2}{\mathrm{Im}\,(D(\varphi^{-1})^T - I_2)}.$$

Then

$$K_1\left(C(T^2) \times_\alpha Z\right) \cong \begin{cases} Z^2 \oplus J & \text{if } \det D(\varphi) = 1\,, \\ Z \oplus J & \text{if } \det D(\varphi) = -1\,. \end{cases}$$

PROOF. From the PV-sequence we obtain the short exact sequence

$$0 \longrightarrow \frac{Z^2}{\mathrm{Im}(\alpha_* - \mathrm{id}_*)} \longrightarrow K_1\left(C(T^2) \times_\alpha Z\right) \xrightarrow{\delta_1} \ker(\alpha_* - \mathrm{id}_*) \longrightarrow 0\,,$$

where the $\alpha_* - \mathrm{id}_*$ on the left is on $K_1\left(C(T^2)\right) \cong Z^2$ and is identified via Lemma 2.1 so that the resulting quotient group on the left is just J, and the $\alpha_* - \mathrm{id}_*$ on the right is on $K_0\left(C(T^2)\right) \cong Z^2$ which is identified via Lemma 2.3 so that $\ker(\alpha_* - \mathrm{id}_*)$ is isomorphic to Z^2 if $\det D(\varphi) = 1$, and isomorphic to Z if $\det D(\varphi) = -1$. As the sequence splits the result follows.

□

Remark. The above Theorem 2.4 and 2.5 have been proved by Igal Megory-Cohen in his Ph.D. thesis [13] using a different and somewhat longer method. However, we wish to point out that in our Theorem 2.5 he has $Z^3 \oplus J$ instead of our $Z \oplus J$ (see Proposition 2.3.2 (ii), Chapter 3, of his thesis).

From the PV-sequence and the proof of Theorem 2.4 we immediately obtain the following result about the generators of the K_0-group.

COROLLARY 2.6. *Let φ be a homeomorphism of* T^2 *such that* $\det(D(\varphi) - I_2) = 0$, *and α its associated automorphism.*

(i) If $\det D(\varphi) = 1$ *and* $D(\varphi) \neq I_2$, *then* $K_0\left(C(T^2) \times_\alpha Z\right) \cong Z^3$ *is generated by* [1], [P] − [1], *and* x, *where* x *is such that* $\delta_0(x)$ *is a generator of* $\ker(\alpha_* - \mathrm{id}_*)$ *in* $K_1\left(C(T^2)\right)$.

(ii) If $\det D(\varphi) = -1$*, then* $K_0\left(C(T^2) \times_\alpha Z\right) \cong Z^2 \oplus Z_2$ *is generated by* $[1]$, $[P] - [1]$ *(which has order 2), and* x*, where* x *is as in (i).*

CHAPTER 3

Two Lemmas on Conjugacy

Classes in GL(2, Z)

In this chapter we shall classify the conjugacy classes of certain integral matrices in $GL(2,Z)$, which arise as the degree matrices, $D(\varphi)$, of certain transformations φ of T^2 (as a topological group), the "quasi-rotations", defined in the next chapter. It will turn out that these are the matrices A which have 1 as an eigenvalue, i.e., for which $\det(A - I_2) = 0$. To each such integral matrix A we shall show that its conjugacy class in $GL(2,Z)$ is determined by $\det A = \pm 1$ (its "orientation") and by a certain integer which we shall denote by $m(A)$ (cf. Lemmas 3.1.1 and 3.2.1). We do not know if $m(A)$ has a geometrical interpretation. The proofs of Lemmas 3.1.1 and 3.1.2 are not difficult and could be known, but we are not aware of their presence in the literature.

Recall that two elements A and B of the group $GL(2,Z)$ are conjugate if there exists $S \in GL(2,Z)$ such that $SAS^{-1} = B$, which we shall denote by $A \sim B$.

As a consequence of the results of this chapter and the previous one we shall see that for suitable transformations φ, ψ of T^2, $C(T^2) \times_{\alpha_\varphi} Z \cong C(T^2) \times_{\alpha_\psi} Z$ implies $D(\varphi) \sim D(\psi)$; cf. Proposition 3.2.4 below. If, in addition, such transformations are affine, it will follow from the next chapter that φ and ψ are topologically conjugate (i.e. there is a homeomorphism h of T^2 such that $h \circ \varphi = \psi \circ h$). As the next example illustrates, this is not always true, even for automorphisms of the group T^2.

<u>**Example.**</u> Consider $\varphi : T^2 \to T^2$, $\varphi(x,y) = (xy, x)$, so that

$$D(\varphi) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } D(\varphi^{-1}) = D(\varphi)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

We know that $C(T^2) \times_\alpha Z \cong C(T^2) \times_{\alpha^{-1}} Z$, where α is the automorphism on $C(T^2)$ associated with φ, but yet φ and φ^{-1} are not topologically conjugate since $D(\varphi)$ and $D(\varphi^{-1})$ have different traces: If for some homeomorphism h of T^2 one has $h \circ \varphi \circ h^{-1} = \varphi^{-1}$, then $D(h)D(\varphi)D(h)^{-1} = D(\varphi^{-1})$, where $D(h) \in GL(2,Z)$, meaning that $D(\varphi)$ and $D(\varphi^{-1})$ are conjugate in $GL(2,Z)$, a contradiction.

Now let us proceed with the classification.

§3.1. Conjugacy Classes when $\det(A) = 1$.

LEMMA 3.1.1. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,$) *be such that* $\det(A) = 1$ *and* $\det(A - I_2) = 0$. *Let* $e = \gcd(a-1, b)$, *when* $b \neq 0$, *and define* $m(A)$ *by*

$$m(A) = \begin{cases} \dfrac{e^2}{|b|} & \text{if } b \neq 0, \\ |c| & \text{if } b = 0. \end{cases}$$

Then

$$A \sim \begin{bmatrix} 1 & 0 \\ m(A) & 1 \end{bmatrix}.$$

Hence,

$$A \sim B \quad \Leftrightarrow \quad m(A) = m(B),$$

for all matrices A, B *satisfying the above hypotheses.*

PROOF. From $(a-1)(d-1) - bc = 0$ and $ad - bc = 1$ one obtains $a + d = 2$ and $-(a-1)^2 = bc$. If $b = 0$, the lemma is clear. Suppose that $b \neq 0$. Since $e = \gcd(a-1, b)$, there exist integers s, t such that

$$\left(\frac{a-1}{e}\right) t - \left(\frac{b}{e}\right) s = 1,$$

so that

$$S = \begin{bmatrix} \left(\frac{a-1}{e}\right) & \frac{b}{e} \\ s & t \end{bmatrix} \in GL(2,Z).$$

One then checks that $SA = \begin{bmatrix} 1 & 0 \\ \frac{-e^2}{b} & 1 \end{bmatrix} S$:

$$SA = \begin{bmatrix} (\frac{a-1}{e}) & \frac{b}{e} \\ s & t \end{bmatrix} \begin{bmatrix} a & b \\ c & 2-a \end{bmatrix} = \begin{bmatrix} (\frac{a-1}{e})\,a + \frac{bc}{e} & (\frac{a-1}{e})\,b + \frac{b}{e}(2-a) \\ sa + tc & sb + t(2-a) \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ \frac{-e^2}{b} & 1 \end{bmatrix} \begin{bmatrix} \frac{a-1}{e} & \frac{b}{e} \\ s & t \end{bmatrix} = \begin{bmatrix} \frac{a-1}{e} & \frac{b}{e} \\ \frac{-e^2}{b}\,(\frac{a-1}{e}) + s & t - e \end{bmatrix}.$$

These can be seen to be equal using the relations $-(a-1)^2 = bc$ and $(a-1)t - bs = e$.

Thus,

$$SAS^{-1} = \begin{bmatrix} 1 & 0 \\ \pm m(A) & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ m(A) & 1 \end{bmatrix}.$$

□

COROLLARY 3.1.2. *Let* φ *be a homeomorphism of* T^2 *with* $\det(D(\varphi)) = 1$ *and* $\det(D(\varphi) - I_2) = 0$. *Then* φ *is topologically conjugate to a homeomorphism* ψ *with*

$$D(\psi) = \begin{bmatrix} 1 & 0 \\ m(\varphi) & 1 \end{bmatrix},$$

where $m(\varphi) = m(D(\varphi))$.

PROOF. Since $D(\varphi)$ satisfies the hypotheses of the lemma, we have

$$SD(\varphi)S^{-1} = \begin{bmatrix} 1 & 0 \\ m(\varphi) & 1 \end{bmatrix}$$

for some $S \in GL(2, Z)$. We can choose an automorphism σ of T^2 with $D(\sigma) = S$. For example, if $S = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$, let $\sigma(x, y) = (x^m y^n, x^p y^q)$. Letting $\psi = \sigma \circ \varphi \circ \sigma^{-1}$, a topological conjugate of φ, we obtain

$$D(\psi) = D(\sigma)D(\varphi)D(\sigma)^{-1} = \begin{bmatrix} 1 & 0 \\ m(\varphi) & 1 \end{bmatrix}.$$

$\square$

COROLLARY 3.1.3. *Let* φ *be a homeomorphism of* T^2 *with* $\det(\varphi) = 1$ *and* $\det(D(\varphi) - I_2) = 0$. *Then*

$$K_1\left(C(T^2) \times_{\alpha_\varphi} Z\right) \cong Z^3 \oplus Z_{m(\varphi)}\,.$$

PROOF. Since by the preceding corollary ψ is topologically conjugate to φ, we can use Theorem 2.5 to obtain

$$\begin{aligned} K_1\left(C(T^2) \times_{\alpha_\varphi} Z\right) &\cong K_1\left(C(T^2) \times_{\alpha_\psi} Z\right) \\ &\cong Z^2 \oplus \left(\frac{Z \oplus Z}{Z(0,0) + Z(m(\varphi),0)}\right) \\ &\cong Z^3 \oplus Z_{m(\varphi)}\,. \end{aligned}$$

$\square$

Consequently, if φ and ψ are homeomorphisms of T^2 satisfying the hypotheses of the above corollary, and if $C(T^2) \times_{\alpha_\varphi} Z$ and $C(T^2) \times_{\alpha_\psi} Z$ are isomorphic, strongly Morita equivalent, or more generally, have isomorphic K_1-groups, then $m(\varphi) = m(\psi)$ so that $D(\varphi) \sim D(\psi)$ by Lemma 3.1.1.

Example. Let φ be such that $D(\varphi) = \begin{bmatrix} 101 & -125 \\ 80 & -99 \end{bmatrix}$. Then $e = \gcd(100, 125) = 25$ and so $m(\varphi) = \frac{e^2}{|b|} = \frac{(25)^2}{125} = 5$. Hence,

$$K_1\left(C(T^2) \times_{\alpha_\varphi} Z\right) \cong Z^3 \oplus Z_5\,.$$

§3.2. Conjugacy Classes when $\det(A) = -1$.

LEMMA 3.2.1. *Let* $A \in GL(2,\mathbb{Z})$ *be such that* $\det(A) = -1$ *and* $\det(A - I_2) = 0$, *so that* A *has the form*

$$A = \begin{bmatrix} k & x \\ y & -k \end{bmatrix},$$

where $k^2 + xy = 1$. *Let* $e = \gcd(k-1, x)$, *when* $x \neq 0$, *and consider the integer-valued function*

$$m(A) = \begin{cases} \gcd\left(e, \dfrac{e(k+1)}{x}\right) & \text{if } x \neq 0, \\ \gcd(2, y) & \text{if } x = 0. \end{cases}$$

Then $m(A) \in \{1, 2\}$, *and*

(i) $\quad m(A) = 1 \quad \Leftrightarrow \quad A \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

(ii) $\quad m(A) = 2 \quad \Leftrightarrow \quad A \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Consequently, for such matrices A *and* B *one has*

$$A \sim B \Leftrightarrow m(A) = m(B).$$

(Hence there are only two conjugacy classes in this case.)

PROOF. Since $\frac{k-1}{e}$ and $\frac{x}{e}$ are relatively prime integers and $xy = (1-k)(1+k)$ or $(\frac{x}{e})y = (\frac{1-k}{e})(1+k)$, it follows that $\frac{x}{e}$ divides $k+1$; hence $\frac{e(k+1)}{x}$ is an integer (when $x \neq 0$), so that $m(A)$ makes sense.

To see that $m(A) \in \{1, 2\}$, note that $m(A) \mid e \mid (k-1)$ and $m(A) \mid \frac{k+1}{(x/e)} \mid (k+1)$. Hence $m(A) \mid (k+1) - (k-1)$ or $m(A) \mid 2$, as desired.

Now assume that $m(A) = 1$ and suppose that $k \notin \{\pm 1\}$, so that $x \neq 0$. We shall seek an integral matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} k & x \\ y & -k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and $ad - bc = 1$. This implies that

$$ka + yb = c, \quad xa - kb = d, \quad kc + yd = a, \quad xc - kd = b,$$

and one easily checks that the last two of these equations follow from the first two. Substituting the first two equations into $ad - bc = 1$ we get

$$a(xa - kb) - b(ka + yb) = 1,$$

or

$$xa^2 - 2kab - yb^2 = 1,$$

which may be factored as

$$\left[\frac{x}{e}a - \left(\frac{k-1}{e}\right)b\right]\left[ea + \frac{ey}{k-1}b\right] = 1,$$

where $\frac{ey}{k-1} = -\frac{e(k+1)}{x}$ is an integer (since $k \neq 1$). Therefore, the existence of S is guaranteed provided the equations

$$(*) \qquad \frac{x}{e}a - \left(\frac{k-1}{e}\right)b = 1, \qquad ea + \left(\frac{ey}{k-1}\right)b = 1,$$

have integer solutions a, b.

Multiplying the first of these by e and the second by $\frac{x}{e}$ we obtain

$$xa - (k-1)b = e, \qquad xa - (k+1)b = \frac{x}{e},$$

from which follows $2b = e - \frac{x}{e}$. Similarly, if we multiply equations $(*)$ by $k+1$ and $k-1$, respectively, we obtain

$$2a = \frac{e(k+1)}{x} - \left(\frac{k-1}{e}\right).$$

To show that b exists we must show that e and $\frac{x}{e}$ have the same parity, i.e., either both are even or both are odd. This can be shown as follows.

Assume that $\frac{x}{e}$ is odd and e is even. Then x is even and $k-1$ is even (since $2 \mid e \mid (k-1)$). So $k+1$ is even. But then $2 \mid \frac{k+1}{x/e}$ since $\frac{x}{e}$ is odd; hence, $2 \mid m(A) = 1$, a contradiction. Now assume that $\frac{x}{e}$ is even and e is odd. Then since $\frac{k-1}{e}$ and $\frac{x}{e}$ are relatively prime, it follows that $\frac{k-1}{e}$ is odd, so $k-1$ and hence also $k+1$ is odd. But $2 \mid \frac{x}{e} \mid (k+1)$, so that $k+1$ is even, a contradiction.

To show that a exists one shows that $\frac{k+1}{x/e}$ and $\frac{k-1}{e}$ have the same parity. If $\frac{k-1}{e}$ is even, then $\frac{x}{e}$ is odd. Since $k-1$ is even, $k+1$ is even and so $\frac{k+1}{x/e}$ is even since $\frac{x}{e}$ is odd. Conversely, if $\frac{k+1}{x/e}$ is even, then since $1 = m(A) = \gcd(e, \frac{k+1}{(x/e)})$, e must be odd. Now as $k+1$ is even, so is $k-1$, and so $\frac{k-1}{e}$ is even because e is odd.

Now we assume that $m(A) = 2$ and $k \notin \{\pm 1\}$, so that $x \neq 0$. Then e and $\frac{e(k+1)}{x}$ are even so that the integral matrix

$$S = \begin{bmatrix} \frac{e(k+1)}{2x} & \frac{e}{2} \\ \frac{k-1}{e} & \frac{x}{e} \end{bmatrix}$$

clearly has determinant 1. Using the relation $xy = (1-k)(1+k)$ one can easily check that

$$SA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} S .$$

Now let us take care of the case $k = \pm 1$. Then $xy = 0$. In this case, A can be any of the following four matrices

$$\begin{bmatrix} 1 & x \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ x & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & x \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ x & 1 \end{bmatrix}.$$

For all of these matrices, $m(A) = \gcd(x, 2)$. Further, for fixed x, all these matrices are conjugate. Clearly, the first and fourth matrices, as well as the second and third, are conjugate via $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus it suffices to show that the first two matrices are conjugate.

The relations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & -1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{and } ad - bc = 1,$$

are equivalent to the relations

$$-2b = xd, \quad \text{and} \quad ad - b^2 = 1,$$

since $b = c$. Suppose that x is even, say $x = 2z$ for some integer z. Then $b = -zd$, and upon substituting this into the second equation we get $ad - z^2d^2 = 1$, or $d(a - z^2d) = 1$. Now put $d = 1$, $a = 1 + z^2$, $c = b = -z$.

Now suppose that x is odd. Put $d = -2$ so that $b = x$, and so $-2a - x^2 = 1$, from which it follows that a exists since x^2 is odd.

It is easy to check that

$$A = \begin{bmatrix} 1 & x \\ 0 & -1 \end{bmatrix} \sim \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } x \text{ is odd (i.e. } m(A) = 1), \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{if } x \text{ is even (i.e. } m(A) = 2). \end{cases}$$

This completes the proof of the lemma.

□

The matrices satisfying the hypotheses of Lemma 3.2.1 are the "orientation reversing" (i.e. $\det(A) = -1$) square roots of the identity matrix. Using this lemma we can show that there is a quick way to find the conjugacy class of A when its entries have known parity.

COROLLARY 3.2.2. *Let A be as in the hypotheses of Lemma 3.2.1.*

(1) k *is even* $\Rightarrow$ $m(A) = 1$.

(2) Suppose k *is odd. Then*

(i) x *or* y *is odd* $\Rightarrow$ $m(A) = 1$.

(ii) x *and* y *are even* $\Rightarrow$ $m(A) = 2$.

PROOF. If $m(A) \neq 1$, then $m(A) = 2$ so that $2 \,|\, e \,|\, (k-1)$ and hence k is odd. This proves (1). We now prove (2).

(i) Without loss of generality suppose x is odd. Since $m(A) \,|\, e \,|\, x$, it follows that $m(A) = 1$.

(ii) Suppose that x and y are even. Then k is odd. Since x and $(k-1)$ are even, it follows that e is even. We assert that $\frac{e(k+1)}{x}$ is even, so that $m(A) = 2$. To see this, write

$$y = \left(\frac{1-k}{e}\right)\left(\frac{e(1+k)}{x}\right),$$

where we may assume $x \neq 0$. If $\frac{x}{e}$ is even, then $\frac{(1-k)}{e}$ is odd (being relatively prime to $\frac{x}{e}$), so y is even implies that $\frac{e(1+k)}{x}$ is even. Now if $\frac{x}{e}$ is odd, then $k+1$ being even it follows that $\frac{(k+1)}{(x/e)}$ is even, and hence $m(A) = 2$.

$\square$

<u>COROLLARY 3.2.3</u>. *Let* φ *be a homeomorphism of* T^2 *such that* $\det(D(\varphi)) = -1$ *and* $\det(D(\varphi) - I_2) = 0$. *Then*

$$K_1\left(C(T^2) \times_{\alpha_\varphi} Z\right) \cong Z^2 \oplus Z_{m(\varphi)},$$

where $m(\varphi) = m(D(\varphi))$.

<u>PROOF</u>. Arguing as in the proof of Corollary 3.1.2, φ is topologically conjugate to a homeomorphism ψ of T^2 such that

$$D(\psi) = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } m(\varphi) = 1, \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{if } m(\varphi) = 2. \end{cases}$$

On applying Theorem 2.5 to ψ we obtain

$$\begin{aligned} K_1\left(C(T^2) \times_{\alpha_\varphi} Z\right) &\cong K_1\left(C(T^2) \times_{\alpha_\psi} Z\right) \\ &\cong \begin{cases} Z \oplus \left(\frac{Z \oplus Z}{Z(-1,1) + Z(1,-1)}\right) & \text{if } m(\varphi) = 1, \\ Z \oplus \left(\frac{Z \oplus Z}{Z(0,0) + Z(0,-2)}\right) & \text{if } m(\varphi) = 2, \end{cases} \\ &\cong Z^2 \oplus Z_{m(\varphi)}. \end{aligned}$$

$\square$

Combining the results of this section together with those of the previous section we arrive at the following result.

<u>PROPOSITION 3.2.4</u>. *Let* φ_1 *and* φ_2 *be homeomorphisms of* T^2 *such that* $\det(D(\varphi_i) - I_2) = 0$, $i = 1, 2$. *If* $C(T^2) \times_{\alpha_{\varphi_1}} Z$ *and* $C(T^2) \times_{\alpha_{\varphi_2}} Z$ *have isomorphic* K_i*-groups* $(i = 0, 1)$, *then* $\det(D(\varphi_1)) = \det(D(\varphi_2))$ *and* $m(\varphi_1) = m(\varphi_2)$, *so that* $D(\varphi_1) \sim D(\varphi_2)$.

<u>PROOF</u>. Since they have isomorphic K_0-groups, Theorem 2.4 implies that $\det(D(\varphi_1)) = \det(D(\varphi_2))$. Since they have isomorphic K_1-groups, we may combine Corollaries 3.1.3 and 3.2.3 to get $m(\varphi_1) = m(\varphi_2)$. By Lemmas 3.1.1 and 3.2.1 we have $D(\varphi_1) \sim D(\varphi_2)$.

□

We may now summarize the contents of Lemmas 3.1.1 and 3.2.1 as follows:

<u>COROLLARY 3.2.5</u>. *Let* $A, B \in GL(2, Z)$ *be such that* $\det(A - I_2) = \det(B - I_2) = 0$. *Then* $A \sim B$ *if and only if*

(i) $\det(A) = \det(B)$,

(ii) $m(A) = m(B)$.

<u>Remark</u>. The quantity $\det(D(\varphi) - I_2)$ turns out to be the so-called Lefschetz number of φ, which is defined in algebraic topology as the alternating sum of the traces of the induced maps of φ on the cohomology groups of the underlying space (in our case, T^2). The Lefschetz fixed point theorem states that if φ is a diffeomorphism on a smooth manifold which has no fixed points, then its Lefschetz number (our $\det(D(\varphi) - I_2)$) is zero. For T^2, the alternating sum for the Lefschetz number is

$$\det(D(\varphi) - I_2) = 1 - \operatorname{tr} D(\varphi) + \det(D(\varphi)).$$

See Bott and Tu [1], Theorem 11.25.

CHAPTER 4

Classification of Affine Quasi-Rotation C*-algebras

Our ultimate purpose in this chapter is to classify (up to isomorphism) the crossed products $C(T^2) \times_{\alpha_\varphi} Z$ for certain homeomorphisms φ of T^2, namely, the irrational affine quasi-rotations (defined below). Every such affine quasi-rotation is associated with a pair (a, A) where $a \in T^2$ and $A \in GL(2, Z)$ has eigenvalue 1, i.e., $\det(A - I_2) = 0$ (for a more precise statement see Lemma 4.3.1). The associated crossed product, which we shall denote by $B(a, A)$, is completely determined up to isomorphism, as we shall show, by its "rotation angle" $X_A(a) \in T$, its orientation $\det A$ $(= \pm 1)$, and the positive integer $m(A)$ defined in the preceding chapter (Theorem 4.3.2). That is, the isomorphism class of $B(a, A)$ is determined by the triple $(X_A(a), \det(A), m(A))$. (Replacing $X_A(a)$ by its complex conjugate does not change the isomorphism class.) This is shown to be true also for the rational affine quasi-rotations for which $\det(A) = -1$ in §4.4; the case $\det(A) = 1$ is left open.

We also show, as did Rieffel [23] for the irrational rotation C*-algebras, that $M_n \otimes B(a, A) \cong M_{n'} \otimes B(a', A')$ if, and only if, $n = n'$ and their associated triples are equal (up to complex conjugacy of $X_A(a)$).

But before we embark on this we shall introduce some notations and compute the range of the trace for crossed products of $C(T^2)$ by general quasi-rotations of T^2.

§4.1. Quasi-Rotations of T^2.

Definition. A homeomorphism φ of T^2 is said to be a quasi-rotation if

(1) $D(\varphi) \neq I_2$,

(2) φ has a non-singular eigenvalue $\lambda \neq 1$. That is, there exists an invertible $f \in C(T^2)$ and $\lambda \neq 1$ such that $f \circ \varphi = \lambda f$.

Crossed products of $C(T^n)$ by affine rotations of T^n, i.e. $D(\varphi) = I_n$, have been classified by Riedel [22] (Cor. 3.7) as we noted in the introduction. The affine transformations we are interested in (§4.3) will have eigenvalues $\lambda \neq 1$ which will automatically be non-singular, so that they are quasi-rotations.

LEMMA 4.1.1. *Let φ be a quasi-rotation with non-singular eigenvalue $\lambda \neq 1$ so that $f \circ \varphi = \lambda f$, where $f \in C(T^2)$ is invertible. Then*

(i) $D(f) \neq [0 \quad 0]$,

(ii) $\det(D(\varphi) - I_2) = 0$.

PROOF.

(i) Without loss of generality we may assume f is unitary. This is because taking the supremum of the absolute value on both sides of $f \circ \varphi = \lambda f$ yields $|\lambda| = 1$ and hence $|f| \circ \varphi = |f|$, so that $f/|f|$ is also an eigenfunction with eigenvalue λ. Assume that $D(f) = [0 \quad 0]$ so that by Lemma 1.1.2 one has

$$f(x,y) = e^{2\pi i F(x,y)},$$

for some continuous real-valued function F on T^2. The relation $f \circ \varphi = \lambda f$ then becomes

$$e^{2\pi i (F(\varphi(x,y)) - F(x,y))} = \lambda.$$

Thus

$$F(\varphi(x,y)) - F(x,y) = c, \qquad (\forall x, y \in T),$$

where c is a real constant. By induction this becomes

$$F(\varphi^{(k)}(x,y)) - F(x,y) = kc, \qquad (\forall x, y \in T),$$

for every positive integer k. But since the left hand side is bounded, it follows that $c = 0$ and so $\lambda = e^0 = 1$, a contradiction. Therefore, f has non-zero bidegree .

(ii) Taking degrees on both sides of $f \circ \varphi = \lambda f$ we obtain $D(f)D(\varphi) = D(f)$, or $D(f)\big(D(\varphi) - I_2\big) = 0$, where $D(f) \neq [0 \quad 0]$. Therefore, $\det\big(D(\varphi) - I_2\big) = 0$.

□

Definition. Let φ be a quasi-rotation of T^2 and λ a non-singular eigenvalue of φ . We call λ a primitive eigenvalue if it has an associated unitary eigenfunction $f \in C(T^2)$ such that $D(f)$ has relatively prime entries.

LEMMA 4.1.2. *Every quasi-rotation φ of T^2 has a primitive non-singular eigenvalue ($\neq 1$), which is unique up to complex conjugation.*

PROOF. Suppose that $f \circ \varphi = \lambda f$, $\lambda \neq 1$, and $f \in C(T^2)$ is a unitary with $D(f) = [m \quad n] \neq [0 \quad 0]$ (by Lemma 4.1.1). Let $d = \gcd(m, n)$. Using Lemma 1.1.2 we see that there exists a unitary $g \in C(T^2)$ such that $g^d = f$, where g^d is the d-fold pointwise product of g. Thus $g^d \circ \varphi = \lambda g^d$, or $[(g \circ \varphi)\, \bar{g}]^d = \lambda$. By continuity, $(g \circ \varphi)\bar{g} = \lambda_0$ for some d^{th} root λ_0 of λ. Hence $g \circ \varphi = \lambda_0 g$ and $\lambda_0 \neq 1$ is primitive since the entries of $D(g) = [\frac{m}{d} \quad \frac{n}{d}]$ are relatively prime.

To prove the uniqueness part of the lemma suppose that in addition to $g \circ \varphi = \lambda_0 g$ (λ_0 primitive) we have $h \circ \varphi = \mu h$, where μ is primitive and $D(h)$ has relatively prime entries. Taking degrees on both sides of these two equations we get

$$D(g)\,(D(\varphi) - I_2) = 0\,,$$

$$D(h)\,(D(\varphi) - I_2) = 0\,.$$

Since $D(\varphi) - I_2 \neq 0$, it follows that $D(g)$ and $D(h)$ are rationally dependent, that is, there are non-zero integers a and b such that

$$aD(g) + bD(h) = [0 \quad 0]\,.$$

But since D(g) and D(h) have relatively prime entries it follows that $D(g) = \pm D(h)$, and so $D(gh^{\pm 1}) = [0 \quad 0]$. From the above two eigenvalue equations we have

$$(gh^{\pm 1}) \circ \varphi = (\lambda_0 \mu^{\pm 1})(gh^{\pm 1}) .$$

Since $gh^{\pm 1}$ has zero bidegree , Lemma 4.1.1 (i) implies that $\lambda_0 \mu^{\pm 1} = 1$. Hence $\mu = \lambda_0^{\mp 1}$, as desired.

$\square$

<u>Examples.</u>

(1) Let $\lambda = e^{2\pi i\theta}$, $0 < \theta < 1$, and consider the Anzai transformation $\varphi_\theta(x, y) = (\lambda x, xy)$. Since $D(\varphi_\theta) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq I_2$ and $u \circ \varphi_\theta = \lambda u$ with $u(x, y) = x$ and $\lambda \neq 1$, φ_θ is a quasi-rotation . In fact, it is clear that φ_θ is affine. If θ is irrational, then $x \mapsto \lambda x$ is minimal on T, so one can apply Proposition 1.1.4 to see that φ_θ is minimal on T^2. Hence the associated crossed product $C(T^2) \times_{\alpha_{\varphi_\theta}} Z$ is simple (cf. Power [20]) and has a unique faithful trace (using the fact that φ_θ is uniquely ergodic, i.e. has a unique invariant Borel probability measure, and lemma 1.3.4). The isomorphism classes of these algebras (for θ irrational) were studied by Packer [14], and also by Ji [11] in his more general setting of Furstenberg transformations of n-tori, as we mentioned in the introduction. Here we shall classify these crossed products within the broader family of crossed products of $C(T^2)$ by affine quasi-rotations .

(2) Furstenberg ([8], p.597) proved that a minimal homeomorphism φ of T^2 which is not homotopic to the identity, i.e., such that $D(\varphi) \neq I_2$, has an irrational eigenvalue λ, so that any (non-zero) eigenfunction will be invertible. Hence φ is a quasi-rotation . In the next chapter we shall consider the question of when two crossed products associated with minimal homeomorphisms of the form

$$\varphi(x, y) = (\lambda x, f(x)y) ,$$

where $f : T \to T$ has degree $k \neq 0$, are isomorphic.

(3) There are only two orientation-reversing affine quasi-rotations of T^2 (up to topological conjugacy). The first one is of the form $(x, y) \mapsto (ay, bx)$, with degree matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, having primitive eigenvalue $\lambda = ab$, say $\lambda \neq 1$, and eigenfunction $f(x, y) = xy$. The second one has the form $(x, y) \mapsto (\lambda x, \bar{y})$, with degree matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and has primitive eigenvalue λ, say $\lambda \neq 1$, and eigenfunction $u(x, y) = x$. (cf. Chapter 3).

(4) Using certain techniques of Furstenberg we shall construct a quasi-rotation of the form $\varphi(x, y) - (e^{2\pi i\theta}x, e^{2\pi ir(x)}xy)$, for a suitable choice of an irrational number θ and continuous function $r : T \to R$, which does not have topologically quasi-discrete spectrum (recall the definition in §1.1). This will answer a question which Ji [11] (pp. 75–76) asked about whether in general a transformation of the form $(x, y) \mapsto (e^{2\pi i\theta}x, f(x)y)$, where $f : T \to T$ is continuous with degree ± 1, is conjugate to the Anzai transformation φ_θ or to φ_θ^{-1} (in the notation of example (1) above). Since the latter have quasi-discrete spectrum and our constructed φ does not, clearly then they cannot be topologically conjugate, since it can easily be shown that the property of quasi-discrete spectrum is preserved under topological conjugacy. Thus even though the crossed products associated with φ and $\varphi_\theta^{\pm 1}$ have the same tracial range, are simple, have unique normalized traces, have isomorphic K-groups, and have conjugate bidegree $(D(\varphi_\theta^{\pm 1}) = \begin{bmatrix} 1 & 0 \\ \pm 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = D(\varphi))$, nevertheless φ and $\varphi_\theta^{\pm 1}$ are not conjugate. Which is an indication, although not a proof, that their associated crossed product C*-algebras are not isomorphic.

§4.2. Range of the Trace for Quasi-Rotations.

In this section we wish to compute the range of the trace for crossed products of $C(T^2)$ by general quasi-rotations .

Let us note that almost every crossed product of a commutative unital C*-algebra by Z has a normalized trace. If X is a compact metric space and φ is a homeomorphism of X, then a theorem of Krylov and Bogolioubov (cf. [25], p.132) ensures that there is a Borel probability measure μ on X which is φ-invariant, that is, $\mu(\varphi^{-1}(E)) = \mu(E)$ for every Borel subset E of X. Equivalently, this says that

$$\int_X f \circ \varphi \, d\mu = \int_X f \, d\mu, \qquad \forall f \in C(X).$$

Now since $\tau(f) = \int_X f \, d\mu$ is a normalized trace on C(X), the above says that $\tau(\alpha(f)) = \tau(f)$ for all $f \in C(X)$, where $\alpha(f) = f \circ \varphi^{-1}$ is the automorphism on C(X) associated with φ. Thus τ is an α-invariant trace on C(X), so that by our discussion following Proposition 1.3.3, τ induces a normalized trace $\hat{\tau}$ on $C(X) \times_\alpha Z$. This shows that $C(X) \times_\alpha Z$ always has a normalized trace whenever X is a compact metric space.

<u>THEOREM 4.2.1</u>. *Let φ be a quasi-rotation of T^2 with primitive eigenvalue $\lambda = e^{2\pi i\theta}$. Then for any normalized trace τ on $C(T^2) \times_{\alpha_\varphi} Z$ we have*

$$\tau_* K_0 \left(C(T^2) \times_{\alpha_\varphi} Z \right) = Z + \theta Z.$$

Note that we did not assume that θ is irrational, only that θ is not an integer (since $\lambda \neq 1$).

<u>PROOF</u>. Let $f \in C(T^2)$ be a unitary such that $f \circ \varphi = \lambda f$ and D(f) has relatively prime entries. This f induces a homomorphism $\rho : C(T) \to C(T^2)$ given by $\rho(g) = g \circ f$.

If we let β denote the automorphism on C(T) associated with rotation by λ, namely $\beta(g)(x) = g(\bar{\lambda}x)$, for $g \in C(T)$ and $x \in T$, then ρ is an equivariant homomorphism between the C*-dynamical systems $(C(T), \beta, Z)$ and $(C(T^2), \alpha_\varphi, Z)$. To see this we verify that $\rho \circ \beta = \alpha_\varphi \circ \rho$ as follows:

$$\begin{aligned} \alpha_\varphi(\rho(g))(z) &= \rho(g)(\varphi^{-1}(z)) = g \circ f \circ \varphi^{-1}(z) \\ &= g(\bar{\lambda} f(z)) = \beta(g)(f(z)) \\ &= \rho(\beta(g))(z), \end{aligned}$$

for all $z \in T^2$ and $g \in C(T)$.

Using the the naturality of the PV-sequence, this ρ induces a morphism between their associated PV-sequences (by Proposition 1.4.2) to give us a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_0\left(C(T^2)\right) & \xrightarrow{i_*} & K_0\left(C(T^2) \times_{\alpha_\varphi} Z\right) & \xrightarrow{\delta_0'} & K_1\left(C(T^2)\right) & \longrightarrow & \cdots \\
 & & {\scriptstyle\rho_*}\uparrow & & {\scriptstyle\tilde{\rho}_*}\uparrow & & {\scriptstyle\rho_*}\uparrow & & \\
\cdots & \xrightarrow{0} & K_0(C(T)) & \xrightarrow{i_*} & K_0(C(T) \times_\beta Z) & \xrightarrow{\delta_0} & K_1(C(T)) & \xrightarrow{0} & \cdots
\end{array}$$

where $\tilde{\rho}: C(T) \times_\beta Z \to C(T^2) \times_{\alpha_\varphi} Z$ is the induced homomorphism .

If θ is irrational one can construct the Rieffel projection e in $C(T) \times_\beta Z$ having trace θ (cf. [23], pp. 418 ff). If θ is rational one can still construct the Rieffel projection e and it can be shown that $\tau'(e) = \theta$, for any normalized trace τ' on A_θ (cf. Elliott [6], Lemma 2.3, pp. 170–1). In both cases one has $\delta_0[e] = [f_0]$, which is the generator of $K_1(C(T))$, where $f_0(z) = z$, $z \in T$. Since the diagram commutes, one has

$$\delta_0'[\tilde{\rho}(e)] = \delta_0'\tilde{\rho}_*[(e)] = \rho_*\delta_0[e] = \rho_*[f_0] = [f],$$

and, since D(f) has relatively prime entries, [f] is a generator of

$$\ker((\alpha_\varphi)_* - id_*)$$

in $K_1\left(C(T^2)\right)$. Hence the projecion $\tilde{\rho}(e)$ yields a generator in $K_0\left(C(T^2) \times_{\alpha_\varphi} Z\right)$ which, along with two other generators as in Corollary 2.6 (having traces 0 and 1), gives the range of the trace as

$$\begin{aligned}
\tau_* K_0\left(C(T^2) \times_{\alpha_\varphi} Z\right) &= Z + \tau(\tilde{\rho}(e))Z \\
&= Z + \tau'(e)Z \\
&= Z + \theta Z,
\end{aligned}$$

where $\tau' = \tau \circ \widetilde{\rho}$ is a normalized trace on A_θ.

□

Remark. One could use Pimsner's computation of the tracial range as described in the introduction to prove the above theorem using the concept of determinant. But for our purposes the above short proof suffices, and makes our treatment more self-contained by avoiding the concept of the determinant associated with a trace.

Now let us look at some of the consequences of this theorem and those of the preceding two chapters.

COROLLARY 4.2.2. *Let φ_j be a quasi-rotation of T^2 with primitive eigenvalue $\lambda_j = e^{2\pi i\theta_j}$, $j = 1, 2$. If $C(T^2) \times_{\alpha_{\varphi_1}} Z$ is isomorphic to $C(T^2) \times_{\alpha_{\varphi_2}} Z$, then*

(1) $Z + \theta_1 Z = Z + \theta_2 Z$,

(2) $\det(D(\varphi_1)) = \det(D(\varphi_2))$,

(3) $m(\varphi_1) = m(\varphi_2)$.

Consequently, $D(\varphi_1) \sim D(\varphi_2)$.

PROOF. The above theorem yields (1). Proposition 3.2.4 yields (2) and (3).

□

COROLLARY 4.2.3 (PACKER [14], P.49; JI [11], P.39). *For each irrational number $0 < \theta < 1$ consider the Anzai transformation $\varphi(x, y) = (e^{2\pi i\theta}x, x^k y)$ where k is a non-zero integer, and let $H_{\theta,k}$ denote the associated crossed product C^*-algebra. Then*

$$H_{\theta,k} \cong H_{\theta',k'} \qquad \Leftrightarrow \qquad |k| = |k'| \text{ and } \theta' \in \{\theta, 1-\theta\}.$$

Remark. Note that if $k = 0$, then $H_{\theta,k} \cong A_\theta \otimes C(T)$ and the conclusion of this corollary easily holds.

PROOF. ($\Rightarrow$) If $H_{\theta,k} \cong H_{\theta',k'}$, then their tracial ranges being equal (by the previous corollary) implies that $\theta' \in \{\theta, 1-\theta\}$, since the latter are irrationals in $(0,1)$. As these algebras have isomorphic K_1-groups it follows that $Z^3 \oplus Z_k \cong Z^3 \oplus Z_{k'}$, hence $|k| = |k'|$ (Corollary 3.1.3).

($\Leftarrow$) If $\theta' = \theta$ and $k' = k$, then $H_{\theta',k'} \cong H_{\theta,k}$. If $\theta' = \theta$ and $k' = -k$, then the transformation $(x,y) \mapsto (x,\bar{y})$ conjugates the two corresponding Anzai transformations so that $H_{\theta',k'} \cong H_{\theta,k}$. Now suppose $\theta' = 1-\theta$. If $k' = k$, one could take $(x,y) \mapsto (\bar{x},\bar{y})$, which conjugates the two φ's. If $k' = -k$, one takes $(x,y) \mapsto (\bar{x},y)$ which conjugates the two φ's. Hence in all cases, we have $H_{\theta',k'} \cong H_{\theta,k}$.

□

COROLLARY 4.2.4. *Let φ_j be an irrational quasi-rotation of T^2 with primitive eigenvalue $\lambda_j = e^{2\pi i\theta_j}$, $j = 1,2$. If $C(T^2) \times_{\alpha_{\varphi_1}} Z$ and $C(T^2) \times_{\alpha_{\varphi_2}} Z$ are strongly Morita equivalent, then*

(1) $\theta_2 = \begin{bmatrix} m & n \\ p & q \end{bmatrix} \theta_1$, *for some* $\begin{bmatrix} m & n \\ p & q \end{bmatrix} \in GL(2,Z)$,

(2) $\det(D(\varphi_1)) = \det(D(\varphi_2))$,

(3) $m(\varphi_1) = m(\varphi_2)$.

Consequently, $D(\varphi_1) \sim D(\varphi_2)$.

PROOF. Conclusions (2) and (3) follow from Proposition 3.2.4, since two C*-algebras that are strongly Morita equivalent have isomorphic K-groups.

Theorem 4.2.1 allows us to apply Rieffel's argument ([23], Proposition 2.5, p.425) to prove (1). Let $A = C(T^2) \times_{\alpha_{\varphi_1}} Z$ and $B = C(T^2) \times_{\alpha_{\varphi_2}} Z$, and let X be an A-B-equivalence bimodule. For any normalized trace τ on A there is a corresponding (non-normalized) trace τ_X on B induced by X, and there is an isomorphism Φ_X between $K_0(A)$ and $K_0(B)$ such that we have a commutative triangle

$$
\begin{tikzcd}
K_0(A) \arrow[d, "\Phi_X"'] \arrow[dr, "\tau_*"] & \\
 & R \\
K_0(B) \arrow[ur, "(\tau_X)_*"'] &
\end{tikzcd}
$$

We may normalize τ_X by writing $\tau_X = c\tau'$, where $c > 0$ and τ' is a normalized trace on B. By Theorem 4.2.1, and the above triangle, the ranges of $(\tau_X)_*$ and τ_* being equal implies that $Z + \theta_1 Z = c(Z + \theta_2 Z)$.

Write

$$c = n + m\theta_1 , \qquad 1 = c(p + q\theta_2) ,$$

for some integers n, m, p, q. From these we get

$$c\theta_2 = \left(\frac{1 - np}{q}\right) - \left(\frac{mp}{q}\right)\theta_1$$

so that $(1 - np)/q$ and mp/q are integers. These show that p and q are relatively prime, hence, since $(\frac{m}{q})p \in Z$, it follows that $\frac{m}{q} \in Z$. Let us note that $m/q = \pm 1$. To see this, write $\theta_1 = c(k + l\theta_2)$ for some integers k and l. Substituting for c and $c\theta_2$ into this equation we obtain

$$km - lp\left(\frac{m}{q}\right) = 1 ,$$

so that m/q divides 1; hence $m/q = \pm 1$. Finally, we have

$$\theta_2 = \frac{c\theta_2}{c} = \frac{\left(\frac{1-np}{q}\right) - \left(\frac{mp}{q}\right)\theta_1}{n + m\theta_1} = \begin{bmatrix} -\frac{mp}{q} & \frac{1-np}{q} \\ m & n \end{bmatrix}\theta_1 ,$$

and the matrix has determinant $-m/q = \pm 1$, as desired.

□

Let A_θ be the rotation C^*-algebra associated with θ.

COROLLARY 4.2.5.

(i) for $0 \leq \theta < 1$, $A_\theta \not\cong C(T^2) \times_\alpha Z$ *for any automorphism* α *of* $C(T^2)$.

(ii) For θ *irrational,* A_θ *and* $C(T^2) \times_\alpha Z$ *are not strongly Morita equivalent (for any* α *).*

PROOF. (i) Assume $A_\theta \cong C(T^2) \times_\alpha Z$. Then $K_0\left(C(T^2) \times_\alpha Z\right) \cong K_0(A_\theta) \cong Z^2$, and the proof of Theorem 2.4 shows that $K_0\left(C(T^2) \times_\alpha Z\right)$ is generated by [1] and [P], where P is the Bott proje n. These, however, have traces equal to 1. But then $Z = \tau_* K_0(A_\theta) = Z + \theta Z$, for any tracial state τ on A_θ, and since $0 \leq \theta < 1$, we get $\theta = 0$; thus, $A_\theta \cong C(T^2)$ and from $C(T^2) \cong C(T^2) \times_\alpha Z$ being commutative, implies that $\alpha = \mathrm{id}$. Hence $C(T^2) \cong C(T^2) \times_{\mathrm{id}} Z \cong C(T^2) \otimes C(T) \cong C(T^3)$, a contradiction.

(ii) Clearly, if A_θ and $C(T^2) \times_\alpha Z$ are strongly Morita equivalent, then $K_0\left(C(T^2) \times_\alpha Z\right) \cong K_0(A_\theta) \cong Z^2$, so that, as in (i), $K_0\left(C(T^2) \times_\alpha Z\right)$ is generated by [1] and [P]. Therefore $\tau_* K_0\left(C(T^2) \times_\alpha Z\right) \cong Z$ for any normalized trace τ. Using Rieffel's proposition as in the proof of the preceding corollary we deduce that the ranges of the traces are related by $Z = c(Z + \theta Z)$, for some $c > 0$. But since θ is irrational, this is impossible.

$\square$

Remark. We do not know how to prove (ii) of Corollary 4.2.5 when θ is rational. This corollary says that the non-commutative 2-tori A_θ are not isomorphic, as might be expected, to the non-commutative 3-tori $C(T^2) \times_\alpha Z$. If θ is irrational, they are not even strongly Morita equivalent.

We may now generalize Theorem 4.2.1 as follows.

If A is a unital C^*-algebra, then any tracial state τ on $M_n \otimes A$ has the form $(\frac{1}{n}\mathrm{tr}) \otimes \tau'$ for some tracial state τ' on A (see for instance [11], Lemma 3.3). Also, if all tracial states on A have the same range on $K_0(A)$, then all tracial states on $M_n \otimes A$ have the same range on $K_0(M_n \otimes A)$ (cf. [11], Lemma 3.5). In fact, in this case we have

$$\tau_* K_0(M_n \otimes A) = \frac{1}{n}\tau'_* K_0(A) ,$$

for any tracial states τ, τ' on $M_n \otimes A$ and A, respectively. This proves the following corollary.

<u>COROLLARY 4.2.6</u>. *Let φ be a quasi-rotation of T^2 with primitive eigenvalue $\lambda = e^{2\pi i\theta}$. Then*

$$\tau_* K_0\left(M_n \otimes (C(T^2) \times_{\alpha_\varphi} Z)\right) = \frac{1}{n}(Z + \theta Z) ,$$

for any normalized trace τ on $M_n \otimes (C(T^2) \times_{\alpha_\varphi} Z)$.

<u>COROLLARY 4.2.7</u>. *Let φ_j be a quasi-rotation of T^2 with primitive eigenvalue $\lambda_j = e^{2\pi i\theta_j}$, $j = 1,2$. If*

$$M_n \otimes \left(C(T^2) \times_{\alpha_{\varphi_1}} Z\right) \cong M_m \otimes \left(C(T^2) \times_{\alpha_{\varphi_2}} Z\right)$$

then

(1) $n = m$,

(2) $Z + \theta_1 Z = Z + \theta_2 Z$,

(3) $\det(D(\varphi_1)) = \det(D(\varphi_2))$,

(4) $m(\varphi_1) = m(\varphi_2)$.

<u>PROOF</u>. It will suffice to prove (1) since (2), (3), and (4) follow from Corollaries 4.2.2 and 4.2.6. For brevity, let $B_j = C(T^2) \times_{\alpha_{\varphi_j}} Z$, $j = 1,2$. To prove (1) it suffices to prove that if M_m can be embedded in $M_n \otimes B_2$ (unitally), then $m \mid n$. For then, by symmetry, $n \mid m$, so that $n = m$.

Recall that by Corollary 2.6 and the proof of Theorem 4.2.1, $K_0(B_2)$ is generated by a projection $e \in B_2$ of trace θ_2, and two other classes, [1] and $x = [P] - [1]$, where P is the Bott projection.

Let $\left\{e_{ij}^{(n)}\right\}_{i,j=1,\ldots,n}$ be the standard matrix units for M_n, so that $K_0(M_n \otimes B_2)$ has independent generators $[e_{11}^{(n)} \otimes e]$, $[e_{11}^{(n)} \otimes 1]$, and $e_{11}^{(n)} \otimes x = [e_{11}^{(n)} \otimes P] - [e_{11}^{(n)} \otimes 1]$.

Suppose that $\sigma : M_m \to M_n \otimes B_2$ is a unital embedding so that $\sigma_* : K_0(M_m) \to K_0(M_n \otimes B_2)$, where $K_0(M_m) = Z[e_{11}^{(m)}]$. Then

$$\sigma_*[e_{11}^{(m)}] = a[e_{11}^{(n)} \otimes e] + b[e_{11}^{(n)} \otimes 1] + c(e_{11}^{(n)} \otimes x),$$

for some integers a, b, c. Now since $I_m = \sum_{i=1}^{m} e_{ii}^{(m)}$ is the orthogonal sum of equivalent projections, we get from

$$I_n \otimes 1 = \sigma(I_m) = \sum_{i=1}^{m} \sigma(e_{ii}^{(m)})$$

that

$$[I_n \otimes 1] = m[\sigma(e_{11}^{(m)})] \ \in \ K_0(M_n \otimes B_2) \ .$$

Thus,

$$\begin{aligned} n[e_{11}^{(n)} \otimes 1] &= [I_n \otimes 1] = m\sigma_*[e_{11}^{(m)}] \\ &= ma[e_{11}^{(n)} \otimes e] + mb[e_{11}^{(n)} \otimes 1] + mc(e_{11}^{(n)} \otimes x), \end{aligned}$$

so that $n = mb$, i.e. $m \mid n$.

$\square$

<u>**Remark**</u>. The argument in the above proof can be imitated to show that if $M_n \otimes A_\theta \cong M_m \otimes A_{\theta'}$, then $n = m$ and $\theta' = \theta$ or $1 - \theta$ $(0 \le \theta, \theta' < 1)$, whether θ, θ' are rational or not. Recall that Rieffel [23] showed this for θ, θ' irrational; and in [6] and [26] this is shown for θ, θ' rational in the case $n = m = 1$.

<u>§4.3. Main Theorem.</u>

<u>Classification of Irrational Affine Quasi-Rotation C*-algebras .</u>

We are now ready to classify the isomorphism classes of crossed products of $C(T^2)$ by irrational affine quasi-rotations of T^2. By "irrational" quasi-rotation we mean one which has irrational primitive non-singular eigenvalue $\lambda = e^{2\pi i\theta}$ (i.e., θ is irrational). In the next section we shall give a partial result for rational affine quasi-rotations .

Let G be a group. An affine transformation of G is a mapping $\sigma : G \to G$ of the form

$$\sigma(z) = aA(z) , \qquad\qquad z \in G$$

where a $\in$ G and A $\in$ Aut(G). It is an automorphism of G followed by a translation, which may also be a right translation. For example, the irrational rotation $z \mapsto e^{2\pi i\theta} z$ of the circle is an affine transformation , where the automorphism A is the identity. Their associated crossed product, A_θ, is completely determined by θ (say $0 < \theta < \frac{1}{2}$) up to isomorphism. In our case, we don't have the identity automorphism but any automorphism (of T^2) having eigenvalue 1 as described below.

If A $\in$ GL(2, Z), say $A = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$, then its action on T^2 is defined by $A(x, y) = (x^m y^n, x^p y^q)$. This is how one has an isomorphism of groups $\mathrm{Aut}(T^2) \cong GL(2, Z)$. It is easy to check that

$$A_1(A_2 z) = (A_1 A_2)(z) ,$$

for all $A_1, A_2 \in GL(2, Z)$ and $z \in T^2$.

Now if $X = [m \quad n]$ is a 1×2 integral matrix, it yields a continuous function $X : T^2 \to T$ by $X(x, y) = x^m y^n$. Clearly, we have

$$X(Az) = (XA)(z) , \qquad\qquad z \in T^2 ,$$

for any 1×2 integral matrix X, and A $\in$ GL(2, Z). Also, we have $X(zw) = X(z)X(w)$, for $z, w \in T^2$.

Let us suppose that A $\in$ GL(2, Z) is such that $A \neq I_2$ and $\det(A - I_2) = 0$. Then the proof of Lemma 4.1.2 (uniqueness part) shows that there exists a 1×2 integral matrix $X_A = [m \quad n]$ having relatively prime entries such that

$$X_A(A - I_2) = [0 \quad 0] ,$$

and that X_A is unique up to sign. So $X_A A = X_A$.

Now let us determine which affine transformations of T^2 are quasi-rotations .

LEMMA 4.3.1. *Let $\varphi(z) = aA(z)$ be an affine transformation of T^2. Then φ is a quasi-rotation if, and only if,*

(i) $A \neq I_2$,

(ii) $\det(A - I_2) = 0$,

(iii) $X_A(a) \neq 1$.

PROOF. Suppose these three conditions hold. Then

$$X_A \circ \varphi(z) = X_A(aA(z)) = X_A(a)X_A A(z) = X_A(a)X_A(z),$$

for all $z \in T^2$, so that $X_A \circ \varphi = X_A(a)X_A$ where $X_A(a) \neq 1$ is a non-singular eigenvalue which is primitive since X_A has relatively prime entries. Since $D(\varphi) = A \neq I_2$, it follows that φ is a quasi-rotation .

Conversely, suppose φ is a quasi-rotation . Since $I_2 \neq D(\varphi) = A$ it follows that (i) holds. By Lemma 4.1.1(ii), $\det(A - I_2) = 0$, so condition (ii) holds. It remains to check (iii). By Lemma 4.1.2, φ has a primitive non-singular eigenvalue $\lambda \neq 1$ so that $f \circ \varphi = \lambda f$ where $f \in C(T^2)$ is unitary and $D(f)$ has relatively prime entries. Taking "degrees" on both sides we obtain $D(f)D(\varphi) = D(f)$, i.e. $D(f)(A - I_2) = 0$. By uniqueness of X_A, we get $D(f) = \pm X_A$. Replacing f by $\bar{f}$, if necessary, we may assume without loss of generality that $D(f) = X_A = [m \quad n]$. Let us write, by Lemma 1.1.2, f as

$$f(x, y) = x^m y^n e^{2\pi i F(x,y)},$$

where $F : T^2 \to R$ is continuous . This may be written as

$$f(x, y) = X_A(x, y)e^{2\pi i F(x,y)},$$

or

$$f(z) = X_A(z)e^{2\pi i F(z)}, \qquad z \in T^2.$$

Thus the equation $f \circ \varphi = \lambda f$ becomes

$$X_A(\varphi(z))e^{2\pi i F(\varphi(z))} = \lambda X_A(z)e^{2\pi i F(z)},$$

and since $X_A(\varphi(z)) = X_A(a)X_A(z)$, as we computed above, this becomes

$$e^{2\pi i\{F(\varphi(z))-F(z)\}} = \lambda\overline{X_A(a)}, \qquad z \in T^2.$$

Since the right hand side is independent of z, we may argue as in the proof of Lemma 4.1.1(i) to deduce that $\lambda\overline{X_A(a)} = 1$. Hence $X_A(a) = \lambda \neq 1$.

□

Let us denote the crossed product C*-algebra associated with the affine quasi-rotation corresponding to the pair (a, A), satisyfying conditions (i), (ii), and (iii) of Lemma 4.3.1, by $B(a, A)$. The inverse of such a quasi-rotation can easily be checked to correspond to the pair $(A^{-1}(\bar{a}), A^{-1})$, so that $B(a, A) \cong B(A^{-1}(\bar{a}), A^{-1})$, since the crossed product of a C* -algebra by an automorphism α is isomorphic to the crossed product by α^{-1}.

Now we are ready for our main result.

<u>Main Theorem 4.3.2</u>. Let (a_j, A_j) be a pair corresponding to the irrational affine quasi-rotation φ_j of T^2, $j = 1, 2$. Then $B(a_1, A_1) \cong B(a_2, A_2)$ if and only if

(1) $X_{A_2}(a_2) = X_{A_1}(a_1)^{\pm 1}$,

(2) $\det A_1 = \det A_2$,

(3) $m(A_1) = m(A_2)$.

<u>PROOF</u>. By Lemma 4.3.1, $A_j \neq I_2$ and $\det(A_j - I_2) = 0$, so that $X_j = X_{A_j}$, with relatively prime entries, exists such that $X_jA_j = X_j$, $j = 1, 2$.

The above three conditions are necessary in view of Corollary 4.2.2 and the fact that $X_j(a_j)$ is irrational.

Conversely, let us assume conditions (1), (2), and (3). We shall show that φ_1 and φ_1 are topologically conjugate via an affine transformation of the form

$$\psi(z) = kK(z), \qquad k \in T^2,\ K \in GL(2,Z).$$

Since φ_j is a quasi-rotation , we have $\det(A_j - I_2) = 0$, $j = 1, 2$. So in view of Corollary 3.2.5, conditions (2) and (3) imply that $A_1 \sim A_2$, and hence we can choose $K \in GL(2,Z)$ such that $KA_1K^{-1} = A_2$. The equation $X_2A_2 = X_2$ becomes $(X_2K)A_1 = (X_2K)$. Now since X_2 has relatively prime entries then so does X_2K, since $K \in GL(2,Z)$. By uniqueness of X_1 (up to sign) with $X_1A_1 = X_1$, we have $X_2K = \pm X_1$.

Since we may replace K by $-K$, we can choose the $\pm$ sign in $X_2K = \pm X_1$ according to whether $X_2(a_2) = X_1(a_1)^{\pm 1}$, respectively.

We shall find $k \in T^2$ such that $\psi(z) = kK(z)$ satisfies $\psi \circ \varphi_1 = \varphi_2 \circ \psi$. The left hand side is

$$\psi \circ \varphi_1(z) = kK(a_1A_1(z)) = kK(a_1)KA_1(z),$$

and the right hand side is

$$\varphi_2 \circ \psi(z) = a_2A_2(kK(z)) = a_2A_2(k)A_2K(z).$$

These expressions are equal if, and only if,

$$(*) \qquad kK(a_1) = a_2A_2(k),$$

and it suffices to show that this equation has a solution $k \in T^2$.

To do this, first extend the equation $X_2K = \pm X_1$ to

$$\begin{bmatrix} X_2 \\ R_2 \end{bmatrix} K = \begin{bmatrix} \pm X_1 \\ R_1 \end{bmatrix},$$

for some 1×2 integral matrices R_1 and R_2 such that $\begin{bmatrix} X_j \\ R_j \end{bmatrix}$ has determinant ± 1 (which is possible since X_2 has relatively prime entries). Now apply $\begin{bmatrix} X_2 \\ R_2 \end{bmatrix}$ to both sides of $(*)$ to get

$$\begin{bmatrix} X_2 \\ R_2 \end{bmatrix}(k)\begin{bmatrix} X_2 \\ R_2 \end{bmatrix}K(a_1) = \begin{bmatrix} X_2 \\ R_2 \end{bmatrix}(a_2)\begin{bmatrix} X_2 \\ R_2 \end{bmatrix}A_2(k),$$

or

$$\begin{bmatrix} X_2 \\ R_2 \end{bmatrix}(k)\begin{bmatrix} \pm X_1 \\ R_1 \end{bmatrix}(a_1) = \begin{bmatrix} X_2 \\ R_2 \end{bmatrix}(a_2)\begin{bmatrix} X_2 \\ R_2' \end{bmatrix}(k),$$

where $R_2' = R_2A_2$. Note that $R_2' \neq R_2$; for otherwise $R_2(A_2 - I_2) = 0$ which shows that $\begin{bmatrix} X_2 \\ R_2 \end{bmatrix}(A_2 - I_2) = 0$ and so $A_2 - I_2 = 0$ (since $\begin{bmatrix} X_2 \\ R_2 \end{bmatrix}$ is invertible), a contradiction. Thus the above equation becomes

$$(X_2(k), R_2(k))(X_1(a_1)^{\pm 1}, R_1(a_1)) = (X_2(a_2), R_2(a_2))(X_2(k), R_2'(k)).$$

By condition (1), the first coordinates of both sides are equal for all k. The second coordinates become

$$R_2(k)R_1(a_1) = R_2(a_2)R_2'(k),$$

or

$$R_2(k)\overline{R_2'(k)} = R_2(a_2)\overline{R_1(a_1)},$$

or

$$(R_2 - R_2')(k) = R_2(a_2)\overline{R_1(a_1)} \in T,$$

and this clearly has a solution k since $R_2 - R_2' \neq [0 \quad 0]$. Thus k exists, and hence ψ, and therefore φ_1 and φ_2 are topologically conjugate. So $B(a_1, A_1) \cong B(a_2, A_2)$.

$\square$

Therefore, the irrational affine quasi-rotation algebras $B(a, A)$ are completely determined up to isomorphism by the triple $(X_A(a), \det A, m(A))$, up to conjugacy of $X_A(a)$, where

$\mathrm{X_A(a)}$ is the primitive eigenvalue coming from the tracial range, $\det A = \pm 1$ is known from the K_0-group, and $m(A)$ is known from the K_1-group.

With the help of Corollary 4.2.7 we arrive at

COROLLARY 4.3.3. *For irrational affine quasi-rotations of* T^2, *we have* $M_k \otimes B(a_1, A_1) \cong M_n \otimes B(a_2, A_2)$ *if and only if*

(1) $k = n$,

(2) $X_{A_2}(a_2) = X_{A_1}(a_1)^{\pm 1}$,

(3) $\det A_1 = \det A_2$,

(4) $m(A_1) = m(A_2)$.

§4.4. The Rational Case: A Partial Solution.

In [26], Yin used an elementary argument to show that if $0 \leq \theta, \theta' \leq 1$ are rational numbers and if $A_\theta \cong A_{\theta'}$, then $\theta' = \theta$ or $\theta' = 1 - \theta$. In doing this he used the important fact, due to Elliott [6] (Lemma 2.3), that all normalized traces of A_θ induce the same map on $K_0(A_\theta)$, where θ is rational. We have already used this fact in proving Theorem 4.2.1. Yin's elementary argument goes as follows.

Given any isomorphism $\sigma : A_\theta \to A_{\theta'}$, look at its induced map $\sigma_* : K_0(A_\theta) \to K_0(A_{\theta'})$ and its matrix relative to the generators $\{[1], [e_\theta]\}$, $\{[1], [e_{\theta'}]\}$, respectively, where e_θ is the Rieffel projection in A_θ of trace θ. Then

$$\sigma_*[1] = [1],$$

$$\sigma_*[e_\theta] = m[1] + n[e_{\theta'}],$$

from which we get, by taking traces of the latter equation, $\theta = m + n\theta'$. Since the matrix of σ_*, $\begin{bmatrix} 1 & m \\ 0 & n \end{bmatrix}$, is in $GL(2, Z)$ it follows that $n = \pm 1$, so that $\theta = m \pm \theta'$, hence the result.

We may apply a similar argument to obtain the following result.

PROPOSITION 4.4.1. *Let (a_j, A_j), $j = 1,2$, be associated with rational affine quasi-rotations of T^2 which are orientation-reversing (i.e. $\det A_j = -1$). Then $B(a_1, A_1) \cong B(a_2, A_2)$ if and only if*

(1) $X_{A_2}(a_2) = X_{A_1}(a_1)^{\pm 1}$,

(2) $m(A_1) = m(A_2)$.

PROOF. These two conditions, together with $\det A_j = -1$ for $j = 1, 2$, are sufficient because one can argue as in the proof of Theorem 4.3.2 to show that the affine quasi-rotations associated with (a_1, A_1) and (a_2, A_2) are topologically conjugate; hence $B(a_1, A_1) \cong B(a_2, A_2)$.

Conversely, suppose that $B(a_1, A_1) \cong B(a_2, A_2)$. Since $\det A_j = -1$ we may use Corollary 2.6(ii) so that $K_0(B(a_j, A_j))$ is generated by $[1]$, $[P_j] - [1]$ (which has order 2), and $[e_{\theta_j}]$, where P_j is the Bott projection, $X_{A_j}(a_j) = e^{2\pi i\theta_j}$, and e_{θ_j} is the Rieffel projection (obtained in the proof of Theorem 4.2.1). Let $\sigma : B(a_1, A_1) \to B(a_2, A_2)$ be an isomorphism so that its induced map on K_0 is given by

$$\sigma_*[1] = [1],$$

$$\sigma_*([P_1] - [1]) = [P_2] - [1], \qquad \text{being elements of order two,}$$

$$\sigma_*[e_{\theta_1}] = r[1] + s([P_2] - [1]) + t[e_{\theta_2}],$$

for some integers r, s, t. Taking traces of the last of these equations gives $\theta_1 = r + t\theta_2$. Since the matrix of σ_* is

$$\begin{bmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & t \end{bmatrix},$$

and is invertible, $t = \pm 1$, hence $\theta_1 = r \pm \theta_2$ which yields (1). Corollary 3.2.3 yields (2).

□

From this and Corollary 4.2.7 we may deduce the following consequence.

COROLLARY 4.4.2. *As in the hypotheses of the preceding proposition, one has $M_k \otimes B(a_1, A_1) \cong M_n \otimes B(a_2, A_2)$ if and only if*

(1) $k = n$,

(2) $X_{A_2}(a_2) = X_{A_1}(a_1)^{\pm 1}$,

(3) $m(A_1) = m(A_2)$.

Remark. Although Proposition 4.4.1 holds for the orientation-reversing case ($\det A_j = -1$), we conjecture that it still holds in the orientation-preserving case ($\det A_j = 1$). So let us briefly indicate our difficulty in proving it.

In the orientation-preserving case $K_0(B(a_j, A_j)) \cong Z^3$ is generated by the Z-independent elements

$$[1], \quad [P_j] - [1], \quad \text{and} \quad [e_{\theta_j}].$$

Let $\sigma : B(a_1, A_1) \to B(a_2, A_2)$ be an isomorphism so that

$$\sigma_*[1] = [1],$$

$$\sigma_*([P_1] - [1]) = a[1] + b([P_2] - [1]) + c[e_{\theta_2}],$$

$$\sigma_*[e_{\theta_1}] = r[1] + s([P_2] - [1]) + t[e_{\theta_2}],$$

and the matrix of σ_* is

$$\begin{bmatrix} 1 & a & r \\ 0 & b & s \\ 0 & c & t \end{bmatrix}.$$

Taking traces of the last two of the above equations, and noting that the above matrix is in $GL(3, Z)$, we obtain

$$bt - cs = \pm 1,$$

$$\theta_1 = r + t\theta_2,$$

$$0 = a + c\theta_2.$$

But from these equations alone, without knowing more constraints on the integers a, b, c, r, s, t, we cannot deduce, as we wish, that $t = \pm 1$. For instance, take $\theta_1 = \frac{2}{5}$, $\theta_2 = \frac{1}{5}$, $t = 2$, $s = 1$, $r = 0$, $a = -1$, $b = 3$, $c = 5$. Then the above relations hold and yet $t = 2 \neq \pm 1$ and $\theta_1 \neq \theta_2$ and $\theta_1 \neq 1 - \theta_2$.

CHAPTER 5

Some Auxiliary Results

and a Conjecture

In the first section of this chapter we shall construct an irrational (non-affine) quasi-rotation of T^2 which is not topologically conjugate to the Anzai transformation nor to its inverse, which will settle a question of Ji ([11], p.76) in the negative (Theorem 5.1.1). Then we shall show that the crossed product algebras by such irrational quasi-rotations have a unique normalized trace (Propositions 5.1.5 and 5.1.7).

The next section considers the irrational (non-affine) quasi-rotations of T^2 defined by

$$(x, y) \mapsto (\lambda x, f(x)y),$$

where $\lambda \in T$ is irrational and $f : T \to T$ is continuous of degree 1, and their associated crossed product C*-algebras $B_{\lambda,f}$. We raise the question of how one can classify these algebras as a function of f, with λ held fixed. First we determine when two such quasi-rotations are topologically conjugate (Proposition 5.2.2), and then formulate our conjecture: $B_{\lambda,f} \cong B_{\lambda,g} \Rightarrow$ f and g are "equivalent" in the sense described below (cf. 5.2.4). This question seems to require the creation of a new invariant.

§5.1. A Quasi-Rotation not having Quasi-discrete Spectrum.

As promised in Example (4) of Section 4.1 we shall devote this section to proving the following result.

THEOREM 5.1.1. *There exists a minimal homeomorphism* φ *of* T^2 *of the form*

$$\varphi(x, y) = (\lambda x, e^{2\pi i r(x)} xy),$$

for suitable $\lambda = e^{2\pi i\theta}$, where θ is irrational, and $r : T \to R$ continuous, *such that φ does not have topologically quasi-discrete spectrum. Hence, φ is not topologically conjugate to the Anzai transformation $\varphi_\theta(x, y) = (\lambda x, xy)$ nor to its inverse.*

To prove this theorem we shall need three lemmas. The proof of the following lemma may be found in [9] (p.135).

<u>LEMMA 5.1.2</u>. *Let φ be a minimal homeomorphism of a compact metric space X. Let $f \in C(X)$. Then the following conditions are equivalent:*

(1) $f = g \circ \varphi - g$, *for some* $g \in C(X)$.

(2) $\left\{\sum_{k=0}^{n} f \circ \varphi^{(k)}\right\}_{n \geq 1}$ *is a uniformly bounded sequence of functions on* X.

We shall construct the irrational number θ for Theorem 5.1.1 as follows.

Let $\nu_1 = 1$ and recursively define $\nu_{k+1} = 2^{\nu_k} + \nu_k + 1$. Put

$$\theta = \sum_{k=1}^{\infty} 2^{-\nu_k}.$$

<u>Assertion:</u> θ is irrational.

To prove this assume $\theta = \frac{a}{b}$ where a, b are positive integers. Choose a positive integer r such that $b2^{-r} < 1$. By induction it is easy to see that

$$r + j \leq \nu_{r+j} - \nu_r, \qquad \text{for } j \geq 1.$$

Now

$$2^{\nu_r} b \left(\theta - \sum_{k=1}^{r} 2^{-\nu_k}\right) = 2^{\nu_r} b \sum_{k>r} 2^{-\nu_k},$$

where the left hand side is a positive integer, and the right hand side is

$$\begin{aligned} 2^{\nu_r} b \sum_{k>r} 2^{-\nu_k} &= b \sum_{k>r} 2^{-(\nu_k-\nu_r)} \\ &= b \sum_{j=1}^{\infty} 2^{-(\nu_{r+j}-\nu_r)} \\ &\le b \sum_{j=1}^{\infty} 2^{-(r+j)} \\ &= b2^{-r} \\ &< 1, \end{aligned}$$

so that we have a positive integer less than 1, a contradiction. Thus θ is irrational .

<u>LEMMA 5.1.3</u>. ([8], P.585; [5], P.18).

There exists $\lambda = e^{2\pi i\theta}$, *where* θ *is irrational , and a continuous function* $r : T \to R$ *such that*

$$F(\lambda x) - F(x) = r(x), \qquad (x \in T)$$

has a real $L^2(T)$*-solution* F *which is not equal to a continuous function almost everywhere (with respect to Lebesgue measure). Hence, the equation has no* $C(T)$*-solutions.*

<u>PROOF</u>. (The following construction is due to Furstenberg.) Let θ be the irrational number constructed above, and in the above notation let $n_k = 2^{\nu_k}$ for $k \ge 1$, so that one easily checks the inequality

$$0 < n_k\theta - [n_k\theta] \le 2^{-n_k}, \qquad (k \ge 1),$$

where $[t]$ denotes the greatest integer less than or equal to t. Letting $n_{-k} = -n_k \quad (k \ge 1)$, we set

$$r(t) = \sum_{k \ne 0} \frac{1}{|k|} \left(e^{2\pi i n_k\theta} - 1\right) e^{2\pi i n_k t}, \quad t \in R,$$

where the series converges uniformly since

$$\begin{aligned}\left|e^{2\pi i n_k\theta}-1\right| &= \left|e^{2\pi i(n_k\theta-[n_k\theta])}-1\right| \\ &\le \left|e^{2\pi i 2^{-n_k}}-1\right| \\ &\le 2\pi 2^{-n_k}, \qquad (k\ge 1)\end{aligned}$$

so that r is a continuous function .

Now let

$$F(t)=\sum_{k\neq 0}\frac{1}{|k|}e^{2\pi i n_k t}, \qquad t\in R,$$

so that $F\in L^2(T)$. It is then easy to check that

$$F(t+\theta)-F(t)=r(t), \qquad (\text{a.e. } t\in R).$$

Now if F is equal almost everywhere to a continuous function g , then by Fejér's theorem the arithmetic (Cesàro) means of the partial sums of the Fourier series converge uniformly to g . But it is easy to check that they fail to converge at $t=0$, since $\sum_{k\neq 0}\frac{1}{|k|}=\infty$. Hence the result.

To prove the last part of the lemma, assume that $f\in C(T)$ and $f(\lambda x)-f(x)=r(x)$, for almost every $x\in T$. Then upon subtracting we have $(F-f)(\lambda x)=(F-f)(x)$, (a.e.). However, since $x\mapsto\lambda x$ is ergodic, since θ is irrational , it follows that $F-f$ is constant (a.e.), and so F is equal almost everywhere to a continuous function , a contradiction to what we just proved.

□

LEMMA 5.1.4. *Let $\lambda=e^{2\pi i\theta}$ and r be as in the precedin 'ımma, and write* $h(x)=\lambda x$. *Then for any real number α the sequence of functions*

$$\sum_{k=0}^{n}(r+\alpha)\circ h^{(k)}=(n+1)\alpha+\sum_{k=0}^{n}r\circ h^{(k)}, \qquad n\ge 1,$$

is not uniformly bounded. (Here $h^{(k)} = h \circ h \circ \cdots \circ h, k$ *times.)*

PROOF. Fix $\alpha \in R$. Assume that the sequence of functions in the statement of the lemma is uniformly bounded. Since h is minimal, θ being irrational , Lemma 5.1.2 gives us a continuous function g on T such that

$$g \circ h - g = r + \alpha .$$

Now Lemma 5.1.3 has that $r = F \circ h - F$, where $F \in L^2(T)$ and F is not equal to a continuous function (a.e.). Thus $f \circ h - f = \alpha$, where $f = g - F \in L^2(T)$. By induction we obtain

$$f \circ h^{(n)} - f = n\alpha , \qquad \text{(a.e.)}$$

for all $n \geq 1$. Now $||f \circ h^{(n)}||_2 = ||f||_2$ (L^2-norms) by the Lebesgue invariance of $x \mapsto \lambda^n x$. Hence

$$n|\alpha| \leq ||f \circ h^{(n)}||_2 + ||f||_2 = 2||f||_2 < \infty ,$$

for all $n \geq 1$. Therefore, $\alpha = 0$ and substituting this back into the above we obtain $g \circ h - g = r$, where $g \in C(T)$, which contradicts the second part of Lemma 5.1.3.

□

PROOF OF THEOREM 5.1.1. With $\lambda = e^{2\pi i\theta}$ and r as in Lemma 5.1.3, consider the homeomorphism of T^2 defined by

$$\varphi(x, y) = (\lambda x, e^{2\pi i r(x)} xy) .$$

It is clear that λ is a primitive eigenvalue of φ so that $\{\lambda^k \mid k \in Z\}$ are all the eigenvalues of φ (cf. Lemma 4.1.2). To see that φ is minimal, we apply Proposition 1.1.4. So we must show that for any non-zero integer n the equation

$$F(\lambda x) = \left(e^{2\pi i r(x)} x\right)^n F(x)$$

has no continuous solution $F : T \to T$. Now if d is the degree of F then computing the degrees of both sides of the above equation we get $d = n + d$, which is impossible if $n \neq 0$.

By Proposition 1.1.3 (b) it is easy to see that the only eigenfunctions of φ (of modulus 1) are

$$G_1(\varphi) = \{au^k \mid k \in Z, |a| = 1\},$$

where $u(x, y) = x$ (recall the definition of the sets $G_j(\varphi)$ in Section 1.1).

Since the C*-algebra generated by u is not all of $C(T^2)$, to show that φ does not have topologically quasi-discrete spectrum it will suffice to check that there is no $g \in C(T^2)$ with $|g| = 1$ satisfying

$$g \circ \varphi = au^k g, \qquad |a| = 1,$$

for any non-zero integer k. (If $k = 0$, then g is just an eigenfunction.) This shows that $G_2(\varphi) = G_1(\varphi)$ and so $\bigcup_{j \geq 0} G_j(\varphi)$ is equal to $G_1(\varphi)$, which does not generate $C(T^2)$ as a C*-algebra .

Assume that for some $k \neq 0$ there is a solution g such that $g \circ \varphi = au^k g$, and $g \in C(T^2)$ with $|g| = 1$. By Lemma 1.1.2 we can write

$$g(x, y) = x^m y^n e^{2\pi i R(x,y)},$$

for some $R : T^2 \to R$ continuous and some integers m , n . So the above equation becomes, upon substituting g ,

$$\lambda^m x^{m+n} y^n e^{2\pi i \{R(\varphi(x,y)) + nr(x)\}} = a x^{m+k} y^n e^{2\pi i R(x,y)},$$

so that by looking at the x -degree of both sides we have $n = k$, so the above equation becomes

$$e^{2\pi i \{R(\varphi(x,y)) - R(x,y) + kr(x)\}} = a\lambda^{-m}.$$

Since the right hand side is constant, it follows that

$$R(\varphi(x, y)) - R(x, y) + kr(x) = c,$$

is a real constant (since R , φ , r are continuous). By induction this becomes

$$\frac{R(\varphi^{(p)}(x,y)) - R(x,y)}{-k} = r(x) + r(\lambda x) + \cdots + r(\lambda^{p-1}x) + p\left(\frac{-c}{k}\right),$$

for all $p \geq 1$. But the left hand side is a uniformly bounded sequence of functions , so the right hand side contradicts Lemma 5.1.4 (by our choice of r and λ). This proves that φ does not have topologically quasi-discrete spectrum. Since obviously the Anzai transformation $\varphi_\theta^{\pm 1}$ has topologically quasi-discrete spectrum, it follows that φ is not topologically conjugate to $\varphi_\theta^{\pm 1}$; since the property of having topologically quasi-discrete spectrum is easily seen to be invariant under conjugation by a homeomorphism .

□

In a similar manner one can easily show that for every non-zero integer n the irrational quasi-rotation

$$\varphi_n(x,y) = (\lambda x, e^{2\pi i r(x)} x^n y),$$

does not have topologically quasi-discrete spectrum and hence cannot be topologically conjugate to $(x,y) \mapsto (\lambda x, x^n y)$, nor to its inverse, where λ is as in Theorem 5.1.1.

Theorem 5.1.1 answers a question which Ji posed ([11], p.76) in the negative; he asked if in general a transformation of the form $(x,y) \mapsto (\lambda x, f(x)y)$, where f has degree $n \neq 0$, is topologically conjugate to the Anzai transformation $(x,y) \mapsto (\lambda x, x^n y)$, or to its inverse.

The above suggests that (in the notation of Theorem 5.1.1) the C^*-algebra $C(T^2) \times_{\alpha_\varphi} Z$ is not isomorphic to $C(T^2) \times_{\alpha_{\varphi_\theta}} Z$. However, we do not know how to prove this. For none of the invariants we know so far distinguish these algebras . They are both simple, have unique tracial states (as is shown in Proposition 5.1.5 below), they have isomorphic K-groups, and have the same tracial range.

<u>PROPOSITION 5.1.5</u>. *Let φ be as in Theorem 5.1.1. Then the associated crossed product $C(T^2) \times_{\alpha_\varphi} Z$ has a unique normalized trace.*

PROOF. In view of Lemma 1.3.4 it is sufficient to check that φ is uniquely ergodic, i.e., has a unique invariant probability Borel measure on T^2.

Since φ has the form

$$\varphi(x,y) = (\lambda x, e^{2\pi i r(x)} xy),$$

we may apply a result of Furstenberg (cf. [15], p.17, Theorem 3) so that it suffices to show that φ is ergodic with respect to Lebesgue product measure $m \times m$ on T^2. This means that if E is a Borel subset of T^2 which is φ-invariant, then E has Lebesgue measure 0 or 1. To show, in turn, that φ is ergodic it suffices to show that the equation

$$G(\lambda x) = \left(e^{2\pi i r(x)} x\right)^n G(x), \qquad (\text{a.e. on } T),$$

for any $n \neq 0$, has no measurable solution $G : T \to T$ (cf. [15], ergodicity criteria on pp. 84 f). So let us assume that such a G exists, so that $G \in L^2(T)$. By Lemma 5.1.3 we have $r(x) = F(\lambda x) - F(x)$, where F is measurable. Thus the above equation becomes

$$G(\lambda x) e^{-2\pi i n F(\lambda x)} = x^n G(x) e^{-2\pi i n F(x)},$$

or

$$(*) \qquad f(\lambda x) = x^n f(x), \qquad (\text{a.e.}),$$

where

$$f(x) = G(x) e^{-2\pi i n F(x)}$$

is measurable with $|f| = 1$, (a.e.). So now it remains to check that the equation $(*)$ has no such solution. Assume it has a solution $f \in L^2(T)$ so that it can be represented by its Fourier series (which is L^2-convergent), say

$$f(x) = \sum_{k=-\infty}^{\infty} a_k x^k.$$

Substituting this into $(*)$ we obtain

$$\sum_k a_k \lambda^k x^k = \sum_k a_k x^{k+n},$$

so that $a_k\lambda^k = a_{k-n}$ or $|a_k| = |a_{k-n}|$, $k \in Z$. But since $\sum_k |a_k|^2 < \infty$ and $n \neq 0$, we necessarily have $a_k = 0$, for all k. Thus $f = 0$ (a.e.), a contradiction to $|f| = 1$ (a.e.).

□

<u>**Question 5.1.6**</u>. Let λ' be irrational and ψ the quasi-rotation of T^2 defined by

$$\psi(x, y) = (\lambda' x, f(x)y),$$

where $f : T \to T$ is continuous with non-zero degree. Does ψ necessarily have to be uniquely ergodic?

If the answer is affirmative, then the associated crossed product $C(T^2) \times_{\alpha_\psi} Z$ would have a unique normalized trace.

By Proposition 1.1.4, ψ is minimal so that the associated algebra is always a simple C^* -algebra .

The proof of Proposition 5.1.5 gives only a partial answer to this question . Suppose we write f as

$$f(x) = x^m e^{2\pi i R(x)},$$

where $m \neq 0$ and $R : T \to R$ is continuous . Let us say that R can be "split" (with respect to λ') if it can be written as

$$R(x) = F(\lambda' x) - F(x), \qquad \text{(a.e.)},$$

for some measurable real-valued function F on T .

<u>PROPOSITION 5.1.7</u>. *Suppose that* $\psi(x, y) = (\lambda' x, e^{2\pi i R(x)} x^m y)$, *where* λ' *is irrational*, m *a non-zero integer, and* $R : T \to R$ *is continuous and can be split (with respect to* λ'*). Then the associated crossed product* $C(T^2) \times_{\alpha_\psi} Z$ *has a unique normalized trace.*

<u>PROOF</u>. One follows exactly the same steps in the proof of Proposition 5.1.5.

□

If R cannot be split as above, we do not know how to prove (or disprove) that ψ is uniquely ergodic, so that the associated algebra has unique normalized trace.

§5.2. A Conjecture.

For λ irrational and $f : T \to R$ continuous of degree 1 consider the quasi-rotation

$$\varphi_{\lambda,f}(x,y) = (\lambda x, f(x)y)\,.$$

Let $\alpha_{\lambda,f}$ denote its associated automorphism on $C(T^2)$, and consider the crossed product

$$B_{\lambda,f} = C(T^2) \times_{\alpha_{\lambda,f}} Z\,.$$

We are interested in fixing λ and considering this family of crossed products as a function of $f : B_f = B_{\lambda,f}$.

All the invariants we considered are the same for the B_f's. As we showed in Theorem 4.2.1, the range of the trace is the same: $\tau_* K_0(B_f) = Z + \theta Z$, where $\lambda = e^{2\pi i\theta}$, for all normalized traces τ on B_f. Since θ is irrational , $\varphi_{\lambda,f}$ is minimal (Proposition 1.1.4) so that B_f is a simple C^*-algebra . The K-groups are $K_0(B_f) \cong Z^3 \cong K_1(B_f)$, by Theorems 2.4 and 2.5, since $D(\varphi_{\lambda,f}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. We are not aware of any invariant that helps us distinguish the isomorphism classes of the B_f's. We shall, however, conjecture that $B_f \cong B_g \Rightarrow \varphi_{\lambda,f}$ and $\varphi_{\lambda,g}$ are topologically conjugate . In this section we shall reformulate this latter condition in terms of f and g (and λ), remembering that in the remainder of this section we shall fix λ . To do this we shall need the following lemma.

LEMMA 5.2.1. *Let $\varphi(x,y) = (\lambda x, f(x)y)$, where λ is irrational and $f : T \to T$ is continuous of degree 1. If the equation*

$$b(\varphi(x,y)) = a(x)b(x,y)$$

has continuous solutions $b : T^2 \to T$, $a : T \to T$, where $D(b) = [0 \quad 0]$, then b is a function of x only.

<u>PROOF</u>. The equation implies that a(x) has degree 0 since b has zero bidegree . Thus we may write

$$b(x,y) = e^{2\pi i B(x,y)}, \qquad a(x) = e^{2\pi i A(x)},$$

for some continuous real-valued functions A and B . Upon substituting these into the equation it becomes

$$e^{2\pi i\{B(\varphi(x,y)) - B(x,y) - A(x)\}} = 1,$$

so that, by continuity, the function in the parenthesis $\{\cdots\}$ is an integer; since we may add any integer to A without changing a(x), we may suppose without loss of generality that

$$B(\varphi(x,y)) - B(x,y) = A(x).$$

By induction, this equation implies that

$$B\left(\varphi^{(n)}(x,y)\right) - B(x,y) = \sum_{j=0}^{n-1} A(\lambda^j x),$$

for all $n \geq 1$, which is a uniformly bounded sequence of functions (since B is bounded). By Lemma 5.1.2 we may write A as

$$A(x) = F(\lambda x) - F(x), \qquad (x \in T)$$

for some $F \in C(T)$, real-valued. Hence,

$$B(\varphi(x,y)) - B(x,y) = F(\lambda x) - F(x),$$

or

$$B(\varphi(x,y)) - F(\lambda x) = B(x,y) - F(x).$$

On letting $C(x,y) = B(x,y) - F(x)$, this equation becomes

$$C(\varphi(x,y)) = C(x,y).$$

Since φ is minimal and C is continuous, it follows that C is constant. Hence the function $B(x,y) = F(x) + C$ depends on x only, and hence so does b.

□

__Remark.__ The conclusion of Lemma 5.2.1 is false if we drop the assumption, "$D(b) = [0 \quad 0]$". For if $b(x,y) = y$, then

$$b(\varphi(x,y)) = f(x)y = f(x)b(x,y),$$

and b is a function of y with $D(b) = [0 \quad 1]$.

PROPOSITION 5.2.2. *Let* λ, μ *be irrational in* T *and suppose that* $f, g : T \to T$ *have degree 1. Then* $\varphi_{\lambda,f}$ *and* $\varphi_{\mu,g}$ *are topologically conjugate if, and only if, there exists* $m = \pm 1$ *such that* $\mu = \lambda^m$ *and*

$$g(a_0 x^m) = \lambda^p \frac{b(\lambda x)}{b(x)} f(x)^m,$$

for some $p \in Z$, $a_0 \in T$, *and* $b : T \to T$ *continuous of zero degree.*

PROOF. ($\Rightarrow$) Let $h : T^2 \to T^2$ be any homeomorphism such that $h \circ \varphi_{\lambda,f} = \varphi_{\mu,g} \circ h$. Applying D to both sides we get

$$D(h) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} D(h),$$

so that $D(h)$ has the form

$$D(h) = \begin{bmatrix} m & 0 \\ p & m \end{bmatrix}, \qquad \text{where } m = \pm 1.$$

Thus h has the form

$$h(x,y) = (x^m a(x,y), x^p y^m b(x,y)),$$

for some $a, b : T^2 \to T$ continuous with zero bidegrees. Now

$$\begin{aligned} h \circ \varphi_{\lambda,f}(x,y) &= h(\lambda x, f(x)y) \\ &= (\lambda^m x^m a(\varphi_{\lambda,f}(x,y)),\ \lambda^p x^p f(x)^m y^m b(\varphi_{\lambda,f}(x,y)))\ , \end{aligned}$$

and

$$\varphi_{\mu,g} \circ h(x,y) = (\mu x^m a(x,y),\ g(x^m a(x,y)) x^p y^m b(x,y))\ .$$

Since these expressions are equal we get

$$(1)\ a(\varphi_{\lambda,f}(x,y)) = \mu\lambda^{-m} a(x,y)\,,$$

$$(2)\ \lambda^p f(x)^m b(\varphi_{\lambda,f}(x,y)) = g(x^m a(x,y)) b(x,y)\,.$$

Apply Lemma 5.2.1 to (1) so that $a(x,y)$ is a function of x only. Hence (1) becomes

$$a(\lambda x) = \mu\lambda^{-m} a(x)\,, \qquad x \in T\,.$$

Since $a(x)$ has degree zero we may write it as $a(x) = e^{2\pi i F(x)}$; also write $\mu\lambda^{-m} = e^{2\pi i\delta}$, for some real number δ. Thus the equation involving $a(x)$ becomes

$$e^{2\pi i\{F(\lambda x) - F(x) - \delta\}} = 1\,,$$

so that by continuity of F we have

$$F(\lambda x) - F(x) - \delta = k\,,$$

for some integer k. As before, we deduce that $\delta = -k$ so that $\mu = \lambda^m$. Hence, $a(\lambda x) = a(x)$ for all $x \in T$, and since λ is irrational , $a(x) = a_0$, a constant.

Putting this information into equation (2) it may be rewritten as

$$b(\varphi_{\lambda,f}(x,y)) = \frac{g(a_0 x^m)}{\lambda^p f(x)^m} b(x,y)\,.$$

We again apply Lemma 5.2.1 to this equation, since $D(b) = [0 \quad 0]$, to deduce that b is a function of x only: $b(x,y) = b(x)$. So this equation becomes

$$b(\lambda x) = \frac{g(a_0 x^m)}{\lambda^p f(x)^m} b(x)\,,$$

or

$$g(a_0x^m) = \lambda^p \frac{b(\lambda x)}{b(x)} f(x)^m ,$$

where b has degree zero.

Conversely, one easily checks that the homeomorphism

$$h(x, y) = (a_0x^m, x^p b(x) y^m)$$

satisfies the condition

$$h \circ \varphi_{\lambda,f} = \varphi_{\mu,g} \circ h ,$$

as the above computation clearly shows.

$\square$

The condition in Proposition 5.2.2 (for $m = 1$) defines an equivalence relation $\sim_t$ on the set of continuous functions $f : T \to T$ of degree 1 as follows:

$$g \sim_t f \quad \Leftrightarrow \quad g(a_0x) = \lambda^p \frac{b(\lambda x)}{b(x)} f(x) ,$$

for some $a_0 \in T$, $p \in Z$, and $b : T \to T$ continuous of degree zero. This condition is clearly equivalent to saying that $g(a_0x) = \frac{b(\lambda x)}{b(x)} f(x)$, for some $a_0 \in T$, and $b : T \to T$ continuous . This condition looks remarkably like a condition on f and g : that f and g "differ" by a boundary element in a homological or cohomological sense, the factor $b(\lambda x)/b(x)$ being a "boundary". What the (co)homology group is in this case, we do not know.

<u>COROLLARY 5.2.3</u>. *The C^*-dynamical systems $(C(T^2), \alpha_{\lambda,f}, Z)$ and $(C(T^2), \alpha_{\lambda,g}, Z)$ are equivariantly isomorphic* $\Leftrightarrow$ $g \sim_t f$.

<u>CONJECTURE 5.2.4</u>. $B_f \cong B_g \Rightarrow g \sim_t f$, *for all* $f, g : T \to T$ *of degree* 1.

Note that in this conjecture λ is fixed.

Is there an invariant for B_f that can be expressed in terms of f? Or in terms of its $\sim_t$-equivalence class?

REFERENCES

1. Bott, R., and Tu, L., Differential Forms in Algebraic Topology, Graduate Texts in Mathematics No. 82, Springer-Verlag, 1982.
2. Brown, L., Green, P., and Rieffel, M., Stable isomorphism and strong Morita equivalence of C*-algebras, Pacific J. Math. 71 (1977), 349–363.
3. Connes, A., An analogue of the Thom isomorphism for crossed products of a C*-algebra by an action of R, Advances in Math. 39 (1981), 31–55.
4. Douglas, R., Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1972.
5. Effros, E.G., and Hahn, F., Locally compact transformation groups and C*-algebras, Memoirs of A.M.S., No. 75, 1967.
6. Elliott, G., On the K-theory of the C*-algebra generated by a projective representation of a torsion-free abelian group, Operator Algebras and Group Representations, Vol. 1, Pitman, London (1984), 157–184.
7. Exel, R., Rotation numbers for automorphisms of C*-algebras, Pacific J. Math. 127 (1987), 31–89.
8. Furstenberg, H., Strict ergodicity and transformation of the torus, Amer. J. Math. 83, No. 4 (1961), 573–601.
9. Gottschalk, W. H., and Hedlund, G.A., Topological Dynamics, Amer. Math. Soc. Coll. Publ., Vol. 36 (1955).
10. Itoh, S., Conditional expectations in C*-crossed products, Transactions A.M.S. 267 (1981), 661–667.
11. Ji, R., On the crossed product C*-algebras associated with Furstenberg transformations on tori, Ph.D. thesis, State University of New York at Stony Brook, 1986.
12. Kurosh, A.G., The Theory of Groups, Vol.II ; K.A. Hirsch, transl., 2nd English Ed., Chelsea Publ. Co., New York, 1960.

13. Megory-Cohen, I., Properties of hyperbolic crossed product algebras, Ph.D. thesis, University of California at Berkeley, 1985.

14. Packer, J., K-theoretic invariants for C*-algebras associated to transformations and induced flows, J. Funct. Anal. 67 (1986), 25–59.

15. Parry, W., Topics in Ergodic Theory, Cambridge Univ. Press, 1981.

16. Pedersen, G., C*-algebras and their Automorphism Groups, London Math. Soc. Monographs 14, Academic Press, London-New York, 1979.

17. Pimsner, M., Ranges of traces on K_0 of reduced crossed products by free groups, Springer-Verlag, in Lecture Notes in Mathematics, Vol. 1132, 1985.

18. Pimsner, M., and Voiculescu, D., Exact sequences for K-groups and Ext-groups of certain crossed product C*-algebras, J. Operator Theory 4 (1980), 93–118.

19. Pimsner, M., and Voiculescu, D., K-groups of reduced crossed products by free groups, J. Operator Theory 8 (1982), 131–156.

20. Power, S.C., Simplicity of C*-algebras of minimal dynamical systems, J. London Math. Soc. 18 (1978), 534–8.

21. Putnam, I., Schmidt, K., and Skau, C., C*-algebras associated with Denjoy homeomorphisms of the circle, J. Operator Theory 16 (1986), 99–126.

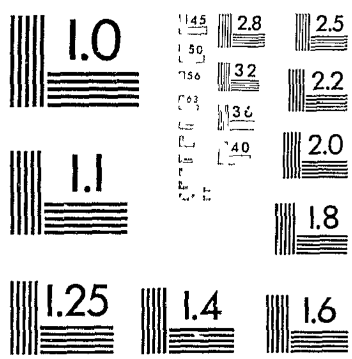
22. Riedel, N., Classification of the C*-algebras associated with minimal rotations, Pacific J. Math. 101 (1982), 153–161.

23. Rieffel, M., C*-algebras associated with irrational rotations, Pacific J. Math. 93 (1981), 415–429.

24. Rudin, W., Functional Analysis, McGraw-Hill, Inc., New Delhi, 1973.

25. Walters, P., Ergodic Theory – Introductory Lectures, Lecture Notes in Mathematics 458, Springer-Verlag, 1975.

26. Yin, H.-S., A simple proof of the classification of rational rotation C*-algebras, Proc. A.M.S. 98 (1986), 469–470.


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
4.0

27. Zeller-Meier, G., Products Croisés d'une C*-algèbre par un groupe d'automorphismes, J. Math. Pures Appl. 47 (1968), 101–239.

28. Greenberg, M.J., Lectures on Algebraic Topology, Mathematics Lecture Note Series, Benjamin, Inc., (1967).