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Homogeneous and isotropic, relativistic two-fluid cosmological models are investigated. In these models two separate fluids act as the source of the gravitational field, as represented by the FRW line element. The general theory of two-fluid FRW models in which neither fluid need be comoving or perfect is developed. However, attention is focused on the physically interesting special class of flat FRW models in which one fluid is a comoving radiative perfect fluid and the second a noncomoving imperfect fluid. The first fluid is taken to model the cosmic microwave background and the second to model the observed material content of the universe. One of the motivations of the present work is to model the observed velocity of our galaxy relative to the cosmic microwave background that was recently discovered by G. F. Smoot, M. V. Gorenstein, and R. A. Muller [Phys. Rev. Lett. **39**, 898 (1977)]. Several models within this special class are found and analyzed. The models obtained are theoretically satisfactory in that they are represented by solutions of Einstein's field equations and the laws of thermodynamics in which all the physical quantities occurring in the solutions are suitably well behaved. In addition, the models are in agreement with current observations. Consequently it is believed that the models obtained are physically acceptable models of the universe.

I. INTRODUCTION

In this article we shall consider cosmological models that have two fluids (possibly imperfect) as the source of the gravitational field. In particular, we shall be dealing with isotropic and homogeneous models in which the metric is the general FRW line element given, in a "spherical polar coordinate system," by

$$ds^2 = -c^2\,dt^2 + R^2(t)\{dr^2/(1-kr^2) + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\}\,, \tag{1.1}$$

where t is the cosmic time, R is the expansion factor, and k the normalized curvature constant (i.e., $k = -1, 0, +1$, depending on whether the model is open, flat, or closed). We shall also wish to study such models in "axial coordinates" in which the flat ($k = 0$) line element takes on the form

$$ds^2 = -c^2\,dt^2 + R^2(t)\{dx^2 + dy^2 + dz^2\}\,. \tag{1.2}$$

Although we shall be dealing with isotropic and homogeneous models, the analysis can, of course, be applied to general two-fluid cosmological models. In addition, we shall, for physical reasons that will be discussed below, focus our attention on models in which one of the two cosmological fluids is a comoving perfect fluid (black-body) radiation field.

The motivation behind this research is twofold. First, it has been established that cosmological models, in particular FRW models, can be interpreted as solutions of Einstein's field equations for a variety of different sources. In the earliest solutions the source was taken to be a comoving perfect fluid. Later, and mainly in the 1960's, authors interpreted the gravitational field to be due to two cosmological fluids, both perfect and comoving (see Sec. II). More recently,[1] FRW models have been investigated in which the source is a noncomoving imperfect fluid either (i) with or without heat conduction or (ii) with or without electromagnetic field. It is thus the aim to complete this mathematical analysis and investigate FRW models in which two-fluid sources are present, neither of which need be comoving or perfect.

It will be noted that models of this type are already implicitly available, for if we take a known two-fluid model, then we can "reinterpret" each of the two fluids separately using the techniques developed in Coley and Tupper.[1] However, in Secs. III and IV a general analysis of two-fluid cosmological fluids will be presented.

It will also be noted that this does not, strictly speaking, complete the general investigation of the interpretation of FRW models, since articles have been written in which n (comoving, perfect) fluids have constituted the source of the gravitational field (see Sec. II). Thus in a full analysis there would be n fluids, in general noncomoving and not necessarily perfect. However, such an analysis will not be undertaken here. First, an investigation involving n fluids (rather than two) would not introduce any new interesting or significant features from a mathematical point of view. Second, there is not such a strong physical motivation for studying n (>2) fluid models.

The second motivation for the present work is strictly physical. The presently accepted view of the evolution of the universe is that, except for very early times (when $T > 10^{10}$–10^{12} K, $T \sim 10^{10}$ K corresponds to $t \sim 10$ sec), the universe is reasonably described by " a FRW model." The conventional wisdom is that the universe evolved initially from a radiationlike state to a matterlike universe ("dust") at later times.

The first FRW models to appear had as sources either comoving radiation perfect fluids *or* comoving matter per-

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fect fluids; each model was supposedly applicable to different eras in the evolution of the universe. Later, attempts were made to take a known radiation model and a known matter model and smoothly (or, at least, continuously) match up the models at $\rho_r = \rho_m$ (where ρ denotes the energy density and the indices refer to the radiation and matter fluids) in order to obtain a qualitative description of the evolution of the universe in terms of a single model (see, for example, Refs. 2 and 3).

The discovery in 1965[4,5] of the 2.7 K isotropic cosmic microwave background, which was presumed to be a remnant of the "primeval fireball," stimulated renewed interest in the subject, and led many authors to investigate FRW models which included both matter and radiation fields (for all times). In these models the source of the gravitational field is assumed to be two comoving perfect fluids; a brief review of this approach will be given in Sec. II.

Recently it was discovered[6] that there is an observed motion of our galaxy relative to the microwave background radiation. This, in turn, stimulates our present interest in models in which there are two cosmological fluids, one representing the background radiation field and the second a matter field constituting the observed galaxies, and in which there is a relative motion between the two fields. We shall take the cosmic microwave background radiation field as comoving and thus seek models in which the matter field is noncomoving. Since the isotropy and homogeneity of both the cosmic microwave background and the observed matter is established to a reasonable experimental accuracy, we shall wish to study models in which isotropy and homogeneity is preserved, that is, FRW models. Thus we shall wish to investigate FRW models that have two fluids present, a comoving radiation field, and a tilting matter field. However, this is possibly only if one of the fluids (here assumed to be the matter field) is assumed to be imperfect.

There is one more aspect to this type of research worth mentioning here. There are two approaches possible. First, the expansion factor $R(t)$ in Eq. (1.1) can be specified and solutions are then sought in which two fluids constitute the source. The problem of finding such a model is essentially an algebraic mathematical problem; the outstanding problems that then need to be resolved require a determination of whether the resulting fluids are physically interesting. The second approach is to specify (physical) equations of state for the fluids present; seeking a model then consists of solving differential equations for the remaining unknown quantities in the model [for example, in the standard two-perfect-fluids case we have to solve an ordinary differential equation for $R(t)$]. Both approaches have been taken in the literature, and both will be discussed in this article.

As mentioned above, in Sec. III the theory of two general fluid sources in FRW models will be investigated. We shall discuss both "radial" and "axial" systems. In Sec. IV we shall restrict attention to the physically important case in which one fluid is comoving, perfect (black-body) radiation fluid and the second fluid a noncomoving imperfect fluid. Several acceptable models will be found, which will be discussed in Sec. V. The notation to be used in this article is similar to that found in Coley and Tupper[1] and McIntosh.[7–9]

II. THE STANDARD THEORY OF TWO-FLUID COSMOLOGIES

Einstein's field equations, with metric (1.1), and for a comoving perfect fluid source are (in cgs units)

$$8\pi G\rho = 3\left(\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2}\right) - \Lambda c^2,$$
$$\frac{8\pi Gp}{c^2} = -\left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2}\right) + \Lambda c^2, \tag{2.1}$$

where Λ is the cosmological constant and a dot denotes differentiation with respect to t. From these equations follow the conservation law

$$\frac{d}{dt}(\rho R^3) + \frac{p}{c^2}\frac{d}{dt}(R^3) = 0. \tag{2.2}$$

To complete the solution an equation of state relating ρ and p is needed. The dimensionless function $\epsilon(t)$ is introduced where

$$\epsilon(t) \equiv p/\rho c^2. \tag{2.3}$$

[If $\Lambda \neq 0$, $\epsilon(t) = (8\pi Gp/c^2 - \Lambda c^2)/(8\pi G\rho + \Lambda c^2)$, and Λ will appear in all subsequent equations. For the rest of this section we shall not include the cosmological constant.]

Suppose that the model contains both matter (with density ρ_m, pressure p_m) and radiation (ρ_r, p_r, and temperature T_r), then we can write

$$\rho = \rho_r + \rho_m, \quad p = p_r + p_m. \tag{2.4}$$

Thus the universe consists of two (comoving) cosmological perfect fluids. The temperature in these models is usually taken to be that of the radiation of the cosmic microwave background (i.e., the temperature is taken to be T_r), since it is argued that the thermal balance is maintained by the radiation. Indeed, Szekeres and Barnes[10] argue that since the entropy of the universe is almost entirely carried by photons (the ratio of number densities of photons to baryons is 10^8 based on a background temperature of 2.7 K) the thermodynamics is almost entirely dictated by the photon field. If thermal equilibrium is assumed during the expansion all components consequently share the common temperature T_r.

If the radiation field is black body we also have that

$$\rho_r = aT_r^4, \quad p_r/c^2 = \tfrac{1}{3}\rho_r, \tag{2.5}$$

where $a = 7.57 \times 10^{-15}$ erg cm^{-3} deg^{-4} is Stefan's constant.

For the two-fluid model we can write the conservation law (2.2) as

$$E_r + E_m = 0, \tag{2.6}$$

where

$$E_r \equiv \frac{1}{R^3}\left\{c^2\frac{d}{dt}(\rho_r R^3) + p_r\frac{d}{dt}(R^3)\right\},$$
$$E_m = \frac{1}{R^3}\left\{c^2\frac{d}{dt}(\rho_m R^3) + p_m\frac{d}{dt}(R^3)\right\}, \tag{2.7}$$

where E_r is the rate of energy transfer per unit volume from matter to radiation and E_m that from radiation to matter. These expressions were first introduced by Davidson[11] and used by McIntosh.[7]

If the two fluids do not interact then the radiation and matter are both independently conserved, i.e.,

$$E_r = E_m = 0\,. \tag{2.8}$$

Since we have five variables, ρ_m, ρ_r, p_m, p_r, and R, two field equations [two of Eqs. (2.1) or (2.2) or one of (2.8)] and essentially two equations of state [(2.5) and one of (2.8)], if we assume that the two fluids are separately conserved we only need one more equation of state to determine the model. This final assumption is usually taken to be that the pressure of the matter field p_m is zero; that is, the pressure from the random motions of galaxies and interstellar matter is negligible. Thus the matter is taken to be "dust" and consequently

$$p_m = 0\,. \tag{2.9}$$

Indeed, the inclusion of radiation in the FRW models is only slightly affected by its interaction with matter and, in fact, such an effect is of about the same magnitude as that of including a nonzero p_m.[12]

As an illustration, with the above assumptions we obtain

$$\rho_r = c_1 R^{-4}, \quad \rho_m = c_2 R^{-3}, \tag{2.10}$$

and thus

$$\epsilon(t) = 1/3(1+\bar{c}R) \quad \text{or} \quad R(t) = c^*(1-3\epsilon)/\epsilon \tag{2.11}$$

(where c_1, c_2, $\bar{c}$, and c^* are constants). There remains one ordinary differential equation to solve for $R(t)$. We note that $\bar{c}R \ll 1$ initially so that $\epsilon = \frac{1}{3}$, and that $\bar{c}R$ is large at later times (regardless of k) so that $\epsilon = 0$; hence the model expands out of a pure radiation state towards a final matter (dust) state. This is, in fact, a general feature of all such two-fluid models, and is regarded as a desirable feature.

Lemaitre[13] was the first to find a model of this type. Lemaitre only considered the case $k = 0$. Other early solutions were found by Alpher and Herman[14] (for the case $k = +1$) and by Chernin[15] (for general values of k, i.e., $k = -1, 0$, or $+1$). Chernin's solutions were shown by McIntosh[8] to be equivalent to results obtained by Tolman.[16] Further models of this type, in which the universe consists of two noninteracting fluids, were found by Cohen,[17] McIntosh,[8] and more recently by Nowotny[18] (all three for general k). Payne[19] used models of this type to investigate the effect of a cosmic microwave background with present temperature greater than 3 K (increasing T_0 decreases the age of the universe). Harrison[12] argues that if $T_r > T_b \sim 5\times 10^9$ K the models above break down due to lepton and hadron pair production, and so developed a model in which a (quantum mechanical type) single-fluid FRW model with equation of state $\epsilon(t) = \text{const}$ is matched continuously at $T_r = T_b$ (at $t = t_b \sim 10$ sec) to a Lemaitre model.

A slightly different approach to the problem is to consider a particular functional form for $R(t)$ (although, traditionally, this approach is not usually taken). There are, of course, certain constraints that should be imposed from the outset. Let us consider a $k = 0$ FRW model here. For small t we wish the model to be approximated by radiation so that $R(t) \sim t^{1/2}$ and the equation of state is $\epsilon(t) \sim \frac{1}{3}$. For large t we wish the universe to be approximated by the Einstein–de Sitter dust universe so that $R(t) \sim t^{2/3}$ and $\epsilon(t) \sim 0$ [moreover, we wish $R(t)$ to be a monotonically increasing function of t]. Formally, we require that $R(t) - t^{1/2}$ and $\epsilon(t) \mapsto \frac{1}{3}$ as $t \to 0$ and $R(t) \mapsto t^{2/3}$ and $\epsilon(t) \mapsto 0$ as $t \to \infty$. We note that the specification of $R(t)$ will give rise to an algebraic relationship between ρ and p [i.e., will give rise to an equation of state $\epsilon(t)$] that may or may not be physical. Other physical constraints to be imposed include $\rho \geqslant 0$, $p \geqslant 0$ (for all t), and possibly we should restrict $\epsilon(t)$ to be a monotonically decreasing function of t. Note that we are allowing the equation of state of the material content of the universe to change with time.

As a simple illustration of the above we consider the expression[20]

$$R(t) = t^{1/2}(1 + lt^{3/5})^{5/18}, \tag{2.12}$$

where l is a positive constant such that $lt^{3/5}$ is dimensionless. With this choice of $R(t)$ we note that all the constraints outlined in the previous paragraph are satisfied, since $\dot{\epsilon} < 0$ for all t where the time varying equation of state $\epsilon(t)$, obtained from Einstein's field equations (2.1), is given by

$$3\epsilon(t) = (1 + \tfrac{8}{15} lt^{3/5})/(1 + \tfrac{8}{3} t^{3/5} + \tfrac{16}{9} l^2 t^{6/5})\,. \tag{2.13}$$

With $R(t)$ specified we can calculate the forms of ρ and p explicitly. If we write $\rho = \rho_r + \rho_m$ and $p = p_r + p_m$ we cannot find ρ_r, ρ_m, p_r, and p_m independently unless we specify equations of state between ρ_r and p_r and between ρ_m and p_m. Suppose we again put $p_r = (c^2/3)\rho_r$, and, for simplicity, $p_m = 0$, then we obtain

$$\rho_r = (3/32\pi G)(t^{-2})(1 + lt^{3/5})^{-2}(1 + \tfrac{8}{15} lt^{3/5}), \tag{2.14}$$

$$\rho_m = (l/5\pi G)(t^{-7/5})(1 + lt^{3/5})^{-2}(1 + \tfrac{5}{6} lt^{3/5})\,. \tag{2.15}$$

We note that in this model $E_m \neq 0$. In fact, we find that

$$E_m = (c^2 l/50\pi G)(t^{-12/5})(1 + lt^{3/5})^{-3}(1 - \tfrac{1}{6} lt^{3/5}), \tag{2.16}$$

so that E_m is positive for small t and will become negative for sufficiently large t.

With $l = 1.06\times 10^{-7}$ we find that the model described above is in very good agreement with actual observations (see Ref. 20, Sec. V, and Table I).

This approach was in fact taken by McIntosh[9] in which he considered a $k = 0$ model of the above kind. In this model a particular functional form was taken for $R(t)$ and McIntosh attempted to show that this produced a model that was physically viable. Unfortunately, as pointed out by Jacobs,[3] in this particular model $\rho_m = \rho_r$ at $t_e \sim 8\times 10^{15}$ sec, whereas conventional wisdom[14] suggests that $t_e \sim 10^{12}$–10^{14} sec. The time t_e is when the universe enters the matter dominated stage and Gamow[21] has suggested that this is when galaxy formation begins.

This defect is not present in McIntosh's later models nor in the other models mentioned here. Nor is this defect necessarily present in models in which $R(t)$ is specified *a priori*, as can be seen from the model represented by Eqs. (2.12)–(2.16) and the models in Ref. 20 (see Secs. IV and V and Ref. 20). However, it is a general feature of two-fluid models that the including of radiation in a matter universe will tend to decrease t_0 and the inclusion of matter in a radiation universe will tend to decrease T_0.

The above comments serve to illustrate that not all models should be restricted by the severe constraints satisfied by

TABLE I. The observed or theoretical values of quantities appearing in this article are given below together with their sources where appropriate. We recall that a zero subscript denotes a quantity's current value.

Quantity	Numerical value	Refs./sources
Hubble constant H_0	55–100 km sec^{-1} Mpc^{-1}	various
Age of universe t_0	2–6 $\times 10^{17}$ sec	various
	2.5 $\times 10^{17}$ sec	Age of uranium and thorium isotopes
	5 $\times 10^{17}$ sec	Age determined for globular clusters
Temperature of cosmic microwave background T_0	2.5–3 K	Refs. 4, 5
ρ_0(total energy)	10^{-30}–5×10^{-29} g cm^{-3}	Ref. 24
$\rho_{r,0}$	10^{-34}–10^{-33} g cm^{-3}	(based on T_0 between 2.5 and 3 K)
$\rho_{m,0}$	5×10^{-31}–10^{-29} g cm^{-3}	Refs. 14, 24, 29
$p_{m,0}$	1–5×10^{-15} dyn cm^{-2}	
Within galaxy ρ_g	1–3×10^{-24} g cm^{-3}	Ref. 30
p_g	5–7×10^{-12} dyn cm^{-2}	
Time t_e at which $\rho_r=\rho_m$	10^{12}–10^{14} sec	Refs. 3, 14
Time at which elements form	10^2–10^3 sec	
Critical values of T	$T > 10^{10}$ K	Radiation plus matter model breaks down due to lepton and hadron production (Ref. 12)
	$T > 10^{12}$ K	p_m no longer negligible (Ref. 25)
	$T > 2\times 10^{12}$ K	Models break down
E_m (early times)	positive	Ref. 22
$E_{m,0}$	$-(10^{-30}$–$10^{-31})$ erg cm^{-3} sec^{-1}	Ref. 24
The velocity of the galaxy relative to the cosmic microwave background	200–600 km sec^{-1}	Ref. 6

the standard noninteracting two-fluid models discussed at the beginning of the section. Indeed, very soon after the noninteracting models were developed models were sought in which there was some energy transfer between the radiation and matter fields (i.e., $E_m \neq 0$) and (correspondingly) models were sought in which $p_m \neq 0$. Generically it is thought that at present there is a conversion or net rate of gain of energy per unit volume from radiation to matter (i.e., $E_{m,0} < 0$) due to the nuclear burning of stars in galaxies, and that $E_m \to 0$ as $t \to \infty$. It is believed that a reliable estimate for $E_{m,0}$ at present is $E_{m,0} \simeq -(10^{-31}$–$10^{-30})$ erg cm^{-3} sec^{-1}. It is also speculated[22] that $E_m > 0$ for small t (in the radiation dominated era) due to pair production and annihilation. It should be stressed that the above are only speculations and other forms for E_m may be acceptable.

Models in which the two fluids interact and consequently the energies of each are not separately conserved were investigated by many authors. McIntosh[7] developed general k models that exhibit the above generic behavior of E_m and include absorption and emission. Models with $k=0$ were investigated in detail with equations of state of the form (i) $\epsilon(t) = \frac{1}{3} e^{-\beta t}$ and (ii) $\epsilon(t) = \frac{1}{3}(1+\mu t)^{-\lambda}$ (where β, μ, λ are positive constants chosen so that E_m is of the "correct" sign in the appropriate time periods). As in the models discussed earlier, these models generally evolve from an $\epsilon = \frac{1}{3}$ radiation dominated universe to an $\epsilon = 0$ dustlike final state.

Other interacting two-fluid models were found by May and McVittie[23] and Sistero.[24] In May and McVittie p_m is defined as an arbitrary function of t but is later restricted by $\epsilon(t)=\frac{1}{3}(1+\mu t)^{-\lambda}$ (and it is shown that McIntosh's solutions are the only ones possible in terms of elementary functions). The behavior of E_m is investigated in all models. Sistero also assumes p_m is an arbitrary function of t through $p_m=f(R)\rho_m$, where f is a non-negative function, and examines for general k the cases $\epsilon(t)=\frac{1}{3}(1+\mu t)^{-1}$ and $f(R)=(\alpha-\beta R+\gamma R^3)^{-1}$ in detail.

Other models in the literature that are variations on the above theme include (a) models with multifluids including those proposed by Vajk[25] [up to four noninteracting (possibly relativistic) fluids for general k], McIntosh[26] [n non-interacting fluids with equations of state $p_i=(\gamma_i-1)\rho_i$ and general k], Szekeres and Barnes[10] (radiation plus multicomponent Synge gas for general k), and Sistero[27] (three interacting fluids including two radiation fields—photons and neutrinos—for general k); (b) models with a nonzero cosmological constant including those proposed by May[22] (generalization of May and McVittie[23]) and McIntosh[26] (in which a nonzero cosmological constant is treated in terms of an additional fluid in an n–fluid model); and (c) other two-fluid models in which neither fluid is a radiation field (McIntosh[26]).

III. GENERAL TWO-FLUID MODELS

Einstein's field equations for two general viscous fluids are

$$\begin{aligned}(c^4/8\pi G)G^{ij} &= (\rho_r+c^{-2}p_r)v^iv^j+p_rg^{ij}-2\eta_r\sigma_r^{ij}+q_r^iv^j+q_r^jv^i\\ &\quad+(\rho_m+c^{-2}p_m)u^iu^j+p_mg^{ij}-2\eta_m\sigma_m^{ij}\\ &\quad+q_m^iu^j+q_m^ju^i,\end{aligned}\tag{3.1}$$

where $\sigma_{r,m}^{ij}$ is the shear tensor, $\eta_{r,m}$ the shear viscosity coefficient, $q_{r,m}^j$ the heat conduction vector, and v^i and u^i are the velocities of the r and m (radiation and matter) fields, respectively. We could investigate models in which these velocities are radially or axially directed. For illustration, in this section we shall consider the case when v^i and u^i both have nonzero components in the radial direction and can be written

$$v^i=(\alpha_r,\beta_rR^{-1},0,0),\quad u^i=(\alpha_m,\beta_mR^{-1},0,0),\tag{3.2}$$

where

$$\alpha_r^2-\beta_r^2=c^2,\quad \alpha_m^2-\beta_m^2=c^2.\tag{3.3}$$

Corresponding to (3.2) we also assume that the $q_{r,m}^j$ are of the form

$$\begin{aligned}q_i^r&=(Q_r/c)(\beta_r,-\alpha_rR,0,0),\\ q_i^m&=(Q_m/c)(\beta_m,-\alpha_mR,0,0),\end{aligned}\tag{3.4}$$

so that $q_i^rv^i=q_i^mu^i=0$ and $Q_{r,m}^2\equiv(q_iq^i)_{r,m}$. In addition, there will be an appropriate set of thermodynamic laws governing the two fields (see Sec. IV).

With the metric taken in the form given by Eq. (1.1), Einstein's field equations become

$$\begin{aligned}3(\dot R^2+kc^2)/c^2R^2 &= \frac{8\pi G}{c^4}\Big\{\rho_r\alpha_r^2+\frac{1}{c^2}p_r\beta_r^2-\frac{4}{3c^2}\beta_r^2\eta_rX_r\\ &\quad-\frac{2}{c}Q_r\alpha_r\beta_r+\rho_m\alpha_m^2+\frac{1}{c^2}p_m\beta_m^2\\ &\quad-\frac{4}{3c^2}\beta_m^2\eta_mX_m-\frac{2}{c}Q_m\alpha_m\beta_m\Big\},\\ -(\dot R^2+2R\ddot R+kc^2)/c^2R^2 &= \frac{8\pi G}{c^4}\Big\{\rho_r\beta_r^2+\frac{1}{c^2}p_r\alpha_r^2-\frac{4}{3c^2}\alpha_r^2\eta_rX_r\\ &\quad-\frac{2}{c}Q_r\alpha_r\beta_r+\rho_m\beta_m^2+\frac{1}{c^2}p_m\alpha_r^2\\ &\quad-\frac{4}{3c^2}\alpha_m^2\eta_mX_m-\frac{2}{c}Q_m\alpha_m\beta_m\Big\},\\ -(\dot R^2+2R\ddot R+kc^2)/c^2R^2 &= \frac{8\pi G}{c^4}\Big\{p_r+\frac{2}{3}\eta_rX_r+p_m+\frac{2}{3}\eta_mX_m\Big\},\\ 0 &= \Big\{\rho_r+\frac{1}{c^2}p_r-\frac{4}{3c^2}\eta_rX_r\Big\}\alpha_r\beta_r-\frac{Q_r}{c}(\alpha_r^2+\beta_r^2)\\ &\quad+\Big\{\rho_m+\frac{1}{c^2}p_m-\frac{4}{3c^2}\eta_mX_m\Big\}\alpha_m\beta_m\\ &\quad-\frac{Q_m}{c}(\alpha_m^2+\beta_m^2),\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}X_r&=(\dot\alpha_r/c+\beta_r'R^{-1}-\beta_rR^{-1}r^{-1})(1-kr^2)^{1/2},\\ X_m&=(\dot\alpha_m/c+\beta_m'R^{-1}-\beta_mR^{-1}r^{-1})(1-kr^2)^{1/2},\end{aligned}\tag{3.6}$$

where a prime denotes differentiation with respect to r. All quantities are assumed to depend on r and t only.

Solutions of Eqs. (3.5) are already known in certain special cases. If we have one comoving perfect fluid (for example, $\rho_r\neq0$, $\alpha_r=c$, $\eta_r=Q_r=0$, $\rho_m=p_m=\eta_m=Q_m=0$) we have the standard one-fluid FRW models. If we have one noncomoving viscous fluid (for example, $\rho_r=p_r=\eta_r=Q_r=0$) we obtain the models of Ref. 1. If we have two comoving perfect fluids ($\alpha_r=\alpha_m=c$, $\eta_r=\eta_m=Q_r=Q_m=0$) we recover the solutions outlined in Sec. II. We can use the above solutions to obtain more general solutions in the following manner: We take a solution in which there are two comoving perfect fluids; each of the "perfect fluids" in this model is "equated" with a noncomoving viscous fluid according to the prescription in Ref. 1; thus we obtain a solution containing two noncomoving viscous fluids. As mentioned in Sec. I there will exist general solutions to Eqs. (3.5). However, in Sec. IV and the remainder of this article we shall seek solutions of Eqs. (3.5) in a particular configuration of physical interest.

IV. RADIATION AND VISCOUS FLUID MODELS

Motivated by the arguments outlined in Sec. I, we shall look for two-fluid models of the following description. We shall assume that the first fluid is a comoving, perfect fluid with radiative equations of state [Eq. (2.5)]. This fluid will model the observed cosmic microwave background. The second fluid will be taken to be a noncomoving, imperfect fluid modeling the observed matter in the universe. We shall focus our attention on $k=0$ FRW models with line element given by Eq. (1.2) and will assume that the matter is moving axially relative to the comoving radiation, thus modeling the observed relative velocity between the center of our galaxy and the cosmic microwave background.

In this physical configuration Einstein's equations are

$$\frac{c^4}{8\pi G}G_{ij} = \frac{\rho_r}{3}(4v_i v_j + c^2 g_{ij}) + \left(\rho_m + \frac{1}{c^2}p_m\right)u_i u_j + p_m g_{ij} - 2\eta_m \sigma^m_{ij} + q^m_i u_j + q^m_j u_i , \tag{4.1}$$

where $v_i = (-c,0,0,0)$. We shall assume u_i has an axial component and is of the form

$$u_i = (-\alpha,0,0,\beta R), \tag{4.2}$$

where $\alpha^2 - \beta^2 = c^2$, and α and β are functions of z and t. We also assume that

$$q^m_i = (Q_m/c)(\beta,0,0,-\alpha R), \tag{4.3}$$

so that $q^m_i u^i = 0$ and $Q^2_m \equiv q^m_i q^i_m$.

Using Eqs. (4.1)–(4.3), Einstein's field equations for $k=0$ become ($\beta \neq 0$)

$$\left(\frac{3c^2}{8\pi G}\right)\frac{\dot R^2}{R^2} = c^2\rho_r + \alpha^2\rho_m + \beta^2\frac{p_m}{c^2} - \frac{4}{3}\beta^2\frac{\eta_m}{c^2}\left(\frac{\dot\alpha}{c} + \frac{\beta'}{R}\right) - \frac{2\alpha\beta Q_m}{c}, \tag{4.4a}$$

$$\left(\frac{-c^2}{8\pi G}\right)\left(\frac{\dot R^2 + 2R\ddot R}{R^2}\right) = c^2\frac{\rho_r}{3} + \beta^2\rho_m + \alpha^2\frac{p_m}{c^2} - \frac{4}{3}\alpha^2\frac{\eta_m}{c^2}\left(\frac{\dot\alpha}{c} + \frac{\beta'}{R}\right) - \frac{2\alpha\beta Q_m}{c}, \tag{4.4b}$$

$$\left(\frac{-c^2}{8\pi G}\right)\left(\frac{\dot R^2 + 2R\ddot R}{R^2}\right) = c^2\frac{\rho_r}{3} + p_m + \frac{2c^2}{3}\frac{\eta_m}{c^2}\left(\frac{\dot\alpha}{c} + \frac{\beta'}{R}\right), \tag{4.4c}$$

$$0 = \rho_m + \frac{1}{c^2}p_m - \frac{4}{3}\frac{\eta_m}{c^2}\left(\frac{\dot\alpha}{c} + \frac{\beta'}{R}\right) - \frac{1}{c}\frac{(\alpha^2+\beta^2)}{\alpha\beta}Q_m , \tag{4.4d}$$

where a prime denotes differentiation with respect to z.

The temperature T_r associated with the radiation field satisfies $\rho_r = aT_r^4$. The physical quantities associated with the imperfect fluid will satisfy the set of thermodynamic laws set out below. Henceforward we shall drop the suffix m (pertaining to the matter field) on all physical quantities in the imperfect fluid (i.e., η, Q, n, S, T, κ) since there should be no confusion, retaining the m suffices on ρ_m and p_m only. In general we shall not take T_r and T equal in the models. This means that the two fluids will not be in thermal equilibrium throughout the history of the universe, which is what we expect for imperfect fluid solutions with nonzero heat conduction vector. However, we shall assume that the following set of thermodynamic laws, based on the assumption that deviations from thermodynamic equilibrium are not too large, are valid.

The thermodynamic laws are[1] the baryon conservation law,

$$(nu^\mu)_{;\mu} = 0; \tag{4.5}$$

Gibb's relation,

$$Td\left(\frac{S}{n}\right) = d\left(\frac{\rho_m}{n}\right) + p_m\, d\left(\frac{1}{n}\right); \tag{4.6}$$

positive entropy production,

$$(Su^\mu + T^{-1}q^\mu)_{;\mu} \geqslant 0; \tag{4.7}$$

and the temperature gradient law,

$$q^\mu = (-\kappa h^{\mu\nu}/c^2)(T_{,\nu} + Ta_\nu/c^2), \quad \kappa \geqslant 0. \tag{4.8}$$

In the above n is the particle density (of the matter field), T the temperature, S the entropy density, $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu/c^2$ the projection tensor, $a_\nu = u_{\nu;\alpha}u^\alpha$ the acceleration vector, and κ the thermal conductivity. We note that Eq. (4.7) is automatically satisfied in the models under consideration if Eq. (4.8) holds.

In addition, we insist that the energy conditions $\rho_r > 0$, $\rho_m > 0$, $\rho_m - p_m/c^2 \geqslant 0$, $p_m \geqslant 0$ are all satisfied and we impose the "boundary conditions" (I) $\alpha \to c$ as $t\to\infty$, (IIa) $\alpha\to\infty$ as $t\to 0$, or (IIb) $\alpha \to Ac$ as $t\to 0$, where $A>1$.

Solving Eqs. (4.4), we obtain

$$\rho_m = \frac{1}{c^2}\left\{\frac{1}{8\pi G}\left[\frac{\dot R^2}{R^2}(3\alpha^2-\beta^2) - \frac{2\beta^2\ddot R}{R}\right] - \frac{\rho_r}{3}(3\alpha^2+\beta^2)\right\}, \tag{4.9a}$$

$$3p_m = \frac{1}{8\pi G}\left\{\frac{\dot R^2}{R^2}(5\beta^2 - 3\alpha^2) - \frac{2\ddot R}{R}(3\alpha^2 - 2\beta^2)\right\} - \frac{\rho_r}{3}(3\alpha^2+\beta^2), \tag{4.9b}$$

$$\frac{\eta}{c^2}\left(\frac{\dot\alpha}{c} + \frac{\beta'}{R}\right) = \frac{-\beta^2}{2c^2}\left\{\frac{1}{8\pi G}\left[\frac{2\dot R^2}{R^2} - \frac{2\ddot R}{R}\right] - \frac{4}{3}\rho_r\right\}, \tag{4.9c}$$

$$Q = \frac{1}{c}\left\{\frac{1}{8\pi G}\left[\frac{2\dot R^2}{R^2} - \frac{2\ddot R}{R}\right] - \frac{4}{3}\rho_r\right\}\alpha\beta. \tag{4.9d}$$

The right-hand sides of Eqs. (4.9a) and (4.9b) are positive, which always ensures that the terms in braces on the right-hand sides of Eqs. (4.9c) and (4.9d) are positive, so that Q is the same sign as β and η is non-negative if and only if

$$\dot\alpha/c + \beta'/R \leqslant 0. \tag{4.10}$$

We note that Eq. (4.8) reduces to the single expression

$$Q = \frac{\kappa}{c^2}\left[\beta \dot{T} + \frac{\alpha T'}{cR} + T\dot{\beta} + \frac{T\alpha'}{cR} + \beta T \frac{\dot{R}}{R}\right]. \qquad (4.11)$$

For $\kappa \geqslant 0$ we must have the expression in square brackets divided by β non-negative (since $Q\beta \geqslant 0$).

One final note concerning notation before the various models are established. The models set out below may also be solutions of Einstein's equations with a comoving perfect radiation fluid and a comoving perfect fluid matter field acting as the source; and in this sense the new models may be thought of as "reinterpretations" of standard-type two-fluid models. That is, the right-hand side of Eq. (4.1) may be formally equivalent to

$$(\rho_r^*/3)(4v_i v_j + c^2 g_{ij}) + (\rho_m^* + c^{-2} p_m^*) u_i u_j + p_m^* g_{ij}, \qquad (4.12)$$

where $u_i = v_i = (-c,0,0,0)$ and $\rho_r^* = \rho_r$. The asterisk notation is being used to denote the pressure and density in the standard-type two-fluid FRW model. Using this notation, the left-hand side of Eq. (4.4a) can be written as $c^2\rho_r + c^2\rho_m^*$, and the left-hand sides of Eqs. (4.4b) and (4.4c) can be written as $(c^2/3)\rho_r + p_m^*$. In addition, Einstein's equations (4.9) can be written in the new notation as

$$\rho_m = (\alpha^2/c^2)\rho_m^* + (\beta^2/c^4) p_m^*, \qquad (4.13a)$$

$$3p_m = \beta^2 \rho_m^* + [(3\alpha^2 - 2\beta^2)/c^2]\, p_m^*, \qquad (4.13a)$$

$$2\eta(\dot{\alpha}/c + \beta'/R) = -\beta^2(\rho_m^* + p_m^*/c^2), \qquad (4.13c)$$

$$cQ = \alpha\beta(\rho_m^* + p_m^*/c^2). \qquad (4.13d)$$

Note that if ρ_m^* and p_m^* are both non-negative, then so are ρ_m and p_m.

Models will exist in which the physical quantities occurring in the models depend upon both z and t. However, such models will not be explicitly sought here. Henceforward, we shall look for models in which the physical quantities are functions of t alone (i.e., α, β, and T are functions of t only). This is in keeping with the types of cosmological models that we seek, and is also a mathematical simplification that enables us to find solutions more easily. With this assumption the equations to be solved simplify as follows.

(a) Einstein's equations: Equations (4.9a), (4.9b), (4.9c), and (4.9d) determine ρ_m, p_m, η, and Q, respectively. Condition (4.10), which ensures η is non-negative, reduces to

$$\dot{\alpha} \leqslant 0. \qquad (4.14)$$

(b) Thermodynamical laws: We can integrate the baryon conservation law (4.5) to obtain

$$n = n_0 R^{-3}\alpha^{-1}, \qquad (4.15)$$

where n_0 is a constant. If α, β, and n are functions of t alone, $T = T(t)$ guarantees that the Gibb's relation has a solution [T is the integrating factor that ensures that the right-hand side of Eq. (4.6) is an exact differential]. With $T = T(t)$, Eq. (4.6) determines S. The temperature gradient law (4.11) then determines κ. The condition for $\kappa \geqslant 0$ [ensuring Eq. (4.7)] reduces to

$$\dot{T}/T + \dot{\beta}/\beta + \dot{R}/R \geqslant 0. \qquad (4.16)$$

(c) Other restrictions: We recall that all energies must be non-negative. We expect T to be a decreasing function of t. The conditions on α are

(I) $\alpha \to c$ as $t \to \infty$,

(IIa) $\alpha \to \infty$ as $t \to 0$, (4.17)

or

(IIb) $\alpha \to \text{const} > c$ as $t \to 0$.

A. Model I

We assume that

$$E_m = 0 \quad \text{and} \quad p_m^* = 0. \qquad (4.18)$$

From Eqs. (2.7) we find that

$$\rho_r = c_1 R^{-4} \quad \text{and} \quad \rho_m^* = c_2 R^{-3}. \qquad (4.19)$$

Einstein's equations now reduce to an ordinary differential equation for $R(t)$, whose solution is given by[8]

$$t + \text{const} = (2/3\lambda^2)(\lambda R - 2\mu)(\lambda R + \mu)^{1/2}, \qquad (4.20)$$

where $\lambda = (8\pi G/3)c^2$, $\mu = (8\pi G/3)c_1$. We note that $\rho_r = aT_r^4$ so that $T_r \sim R^{-1}$. From Eq. (4.14) $\dot{\alpha} < 0$ so we observe that T_r and T cannot be equal, otherwise Eq. (4.16) is violated. Let us choose α, T in the following manner:

$$\rho_m^* = c_3 T^{3/p}, \quad T = (c_2/c_3)^{p/3} R^{-p}, \qquad (4.21)$$

and

$$\alpha = \frac{c(1 + hR^{-q})}{(1 + 2hR^{-q})^{1/2}}, \quad \beta = \frac{chR^{-q}}{(1 + 2hR^{-q})^{1/2}}, \qquad (4.22)$$

where $p(>\frac{3}{4})$, q, and h are positive constants. With this choice of α the conditions (4.17) are satisfied, and Eq. (4.14) is satisfied implying $\eta > 0$, since

$$\dot{\alpha}/c = -qh^2 R^{-2q-1}\dot{R}/(1 + 2hR^{-q})^{3/2}. \qquad (4.23)$$

From Eqs. (4.21)–(4.23), Eq. (4.16) becomes

$$(1 - p - q) + (2 - 2p - q)hR^{-q} > 0, \qquad (4.24)$$

which simply implies that $1 - p - q \geqslant 0$ (providing $q \neq 0$). As an illustration let us choose $p = \frac{6}{7}$, $q = \frac{1}{7}$, whence from Eqs. (4.5), (4.6), (4.9), and (4.11) we obtain

$$\begin{aligned}
\rho_m &= c_2(1 + hR^{-1/7})^2 R^{-3}/(1 + 2hR^{-1/7}), \\
3p_m/c^2 &= c_2 h^2 R^{-23/7}/(1 + 2hR^{-1/7}), \\
\eta &= (7c^2/2)c_2(1 + 2hR^{-1/7})R^{-3}(\mu + \lambda R)^{-1/2}, \\
Q/c &= c_2 h(1 + hR^{-1/7})R^{-22/7}/(1 + 2hR^{-1/7}), \\
n &= (n_0/c)(1 + 2hR^{-1/7})^{1/2}(1 + hR^{-1/7})^{-1}R^{-3}, \\
\kappa/c^2 &= \kappa_0(1 + hR^{-1/2})(1 + 2hR^{-1/7})^{1/2} \\
&\quad \times(\mu + \lambda R)^{-1/2} R^{-12/7},
\end{aligned} \qquad (4.25)$$

where $\kappa_0 \equiv 7c_2^{9/7}c_3^{-2/7}h^{-1}$. We note that ρ_m and p_m are always positive with $(3p_m/c^2\rho_m) \to 1$ as $t \to 0$ and $(3p_m/c^2\rho_m) \to 0$ as $t \to \infty$.

B. Model II

We assume that

$$p_m^* = 0, \quad E_m \neq 0, \quad 3\epsilon(t) = \rho_r/(\rho_r + \rho_m^*). \qquad (4.26)$$

In particular, we shall investigate

$$3\epsilon(t) = (1 + \mu t)^{-\lambda}, \qquad (4.27)$$

where μ and λ are positive constants. Standard two-fluid models of this type were investigated by McIntosh[7] and May and McVittie,[23] and the solutions (4.28) below are due to them. We note that with Eq. (4.27) $\epsilon \to \frac{1}{3}$ as $t \to 0$ and $\epsilon \to 0$ as $t \to \infty$. We also note that $E_m \neq 0$; indeed, $E_m > 0$ for small t

and $E_m < 0$ for large t providing $\lambda < \frac{2}{3}$.

For illustrative purposes we shall investigate the model $\lambda = \frac{1}{2}$, $\mu = 3.5\times 10^{-9}$. The observational predictions of this model were studied by McIntosh.[7] Although the model is in reasonable agreement with actual observations, McIntosh showed that the model $\lambda = \frac{7}{11}$, $\mu = 4.4\times 10^{-11}$ is a better model in that it is in better agreement with observations. We shall consider the former model due to its simplicity, and since it serves to illustrate the general nature of such a class of models.

The model is characterized by[7]

$$\begin{aligned}
&\epsilon(t) = \tfrac{1}{3}(1+\mu t)^{-1/2}, \quad \dot{\epsilon} = -\tfrac{9}{2}\mu\epsilon^3,\\
&R(t) = \frac{c_1(1+5\epsilon)^{5/6}(1-3\epsilon)^{1/2}}{\epsilon^{4/3}},\\
&K \equiv \frac{\dot{R}}{R} = \frac{6\mu\epsilon^2}{(1+5\epsilon)(1-3\epsilon)},\\
&\rho_r = (9/8\pi G)K^2\epsilon, \quad \rho_m^* = (3/8\pi G)K^2(1-3\epsilon),
\end{aligned} \tag{4.28}$$

where c_1 is a positive constant.

We assume that α is of the form

$$\frac{\alpha}{c} = \frac{(1+h\epsilon^q)}{(1+2h\epsilon^q)^{1/2}}, \quad \frac{\beta}{c} = \frac{h\epsilon^q}{(1+2h\epsilon^q)^{1/2}}, \tag{4.29}$$

where h and q are positive constants. Note that as $t\to\infty$, $\alpha/c\to 1$ and as $t\to 0$ $(\epsilon\to\frac{1}{3})$, $\alpha/c\to(1+h\,3^{-q})(1+2h\,3^{-q})^{-1/2} > 1$. Also

$$\alpha\dot{\alpha}/\beta^2 = -\tfrac{9}{2}\mu q(1+h\epsilon^q)\epsilon^2(1+2h\epsilon^q)^{-1}, \tag{4.30}$$

so that $\dot{\alpha} < 0$, which implies that $\eta > 0$ in Eq. (4.9c). We also assume T is of the form

$$T = T_0 R^{-p}\epsilon^s, \tag{4.31}$$

where p and s are positive constants.

The condition for κ to be non-negative is $\dot{T}/T + \alpha\dot{\alpha}/\beta^2 + \dot{R}/R \geqslant 0$, which becomes

$$\begin{aligned}
&\{4(1-p) - 3(s+q)(1+5\epsilon)(1-3\epsilon)\}\\
&\quad + 2h\epsilon^q\{4(1-p) - 3(s+q/2)(1+5\epsilon)(1-3\epsilon)\} \geqslant 0,
\end{aligned} \tag{4.32}$$

which is certainly satisfied if the first term in the braces is positive for all ϵ. Since $(1+5\epsilon)(1-3\epsilon)$ is always positive and has a maximum value of $\frac{16}{15}$, Eq. (4.32) is (strictly) satisfied if

$$1 - p - \tfrac{4}{5}(s+q) \geqslant 0. \tag{4.33}$$

In this model we wish to relate the temperature of the radiation T_r and the temperature of the matter T. The temperatures T_r and T cannot be equal for all t otherwise Eq. (4.16) would be violated for particular eras. Nor would we necessarily expect that $T_r = T$ for all t, since the two fields would then always be in thermal equilibrium. However, as $t\to\infty$, E_m and E_r (and Q) tend to zero so that we might expect that there will be thermal equlibrium as $t\to\infty$. Therefore, in this model we shall add the restriction that

$$\text{as } t\to\infty, \quad T/T_r \to 1. \tag{4.34}$$

From (4.28), as $t\to\infty$ $(\epsilon\to 0)$, $K\to\epsilon^2$, $R\to\epsilon^{-4/3}$, $\rho_r\to\epsilon^5$, $T_r\to\epsilon^{5/4}$, $\rho_m^*\to\epsilon^4$, $T\to\epsilon^{(4p+3s)/3}$, so that Eq. (4.34) implies that

$$\tfrac{5}{4} = \tfrac{4}{3}p + s. \tag{4.35}$$

(Note that $\rho_m^* \sim T^{16/5}$ as $t\to\infty$ as a consequence.) Since $s\geqslant 0$ we have that $p \leqslant \frac{15}{16}$, and Eqs. (4.33) and (4.35) imply that $p - 12q \geqslant 0$.

As an illustration let us choose $p=\frac{3}{4}$, $s=\frac{1}{4}$, $q=\frac{1}{16}$. With these values $T/T_r\to 1$ as $t\to\infty$, and η and κ are always positive, so that the model is physically acceptable. From Eqs. (4.5), (4.9), and (4.11) the full solution becomes

$$\begin{aligned}
\rho_m &= (27\mu^2/2\pi G)(1+h\epsilon^{1/16})^2(1+2h\epsilon^{1/16})^{-1}\\
&\quad\times\epsilon^4(1+5\epsilon)^{-2}(1-3\epsilon)^{-1},\\
3p_m/c^2 &= (27\mu^2h^2/2\pi G)(1+2h\epsilon^{1/16})^{-1}\\
&\quad\times\epsilon^{33/8}(1+5\epsilon)^{-2}(1-3\epsilon)^{-1},\\
T &= T_0 c_1^{-3/4}(1+5\epsilon)^{-5/8}(1-3\epsilon)^{-3/8}\epsilon^{5/4},\\
\eta &= (24c^2\mu/\pi G)(1+2h\epsilon^{1/16})^{1/2}\epsilon^2(1+5\epsilon)^{-2}(1-3\epsilon)^{-1},\\
Q/c &= (27hq^2/2\pi G)(1+h\epsilon^{1/16})(1+2h\epsilon^{1/16})^{-1}\\
&\quad\times\epsilon^{65/16}(1+5\epsilon)^{-2}(1-3\epsilon)^{-1},\\
n &= (n_0/cc_1^3)(1+2h\epsilon^{1/16})^{1/2}(1+h\epsilon^{1/16})^{-1}\\
&\quad\times\epsilon^4(1+5\epsilon)^{-5/2}(1-3\epsilon)^{-3/2},\\
\kappa/c^2 &= \kappa_0(1+2h\epsilon^{1/16})^{1/2}(1+h\epsilon^{1/16})(1+5\epsilon)^{1/8}\\
&\quad\times(1-3\epsilon)^{3/8}\epsilon^{3/4}[\{2-\tfrac{15}{8}(1+5\epsilon)(1-3\epsilon)\}\\
&\quad+h\epsilon^{1/16}\{4-\tfrac{27}{8}(1+5\epsilon)(1-3\epsilon)\}]^{-1},
\end{aligned} \tag{4.36}$$

where $\kappa_0 = 9c_1^{3/4}/128\mu\pi G T_0$. We note that ρ_m and p_m are always positive and that

$$3p_m/c^2\rho_m \to 0 \quad \text{as } t\to\infty \ (\epsilon\to 0)$$

and

$$\frac{3p_m}{c^2\rho_m} \to \frac{h^2 3^{-1/8}}{(1+h\,3^{-1/16})^2} = \frac{0.872h^2}{(1+0.934h)^2},$$

as $t\to 0$ $(\epsilon\to\frac{1}{3})$. Note that this last expression continues increasing as h increases but is always less than 1.

C. Model III

We consider the model with

$$R(t) = t^{1/2}(1+lt^{3/5})^{5/18}, \tag{4.37}$$

as outlined in Sec. II. From Eqs. (2.14) and (2.15) we have that

$$\begin{aligned}
\rho_r &= (3/32\pi G)(t^{-2})(1+lt^{3/5})^{-2}(1+\tfrac{8}{15}lt^{3/5}),\\
\rho_m^* &= (l/5\pi G)(t^{-7/5})(1+lt^{3/5})^{-2}(1+\tfrac{5}{6}lt^{3/5}),\\
p_m^* &= 0.
\end{aligned} \tag{4.38}$$

We note that $\epsilon(t)$ and $E_m(\neq 0)$ are given by Eqs. (2.13) and (2.16), respectively.

We assume that α and β are of the form

$$\frac{\alpha}{c} = \frac{1+ht^{-q}}{(1+2ht^{-q})^{1/2}}, \quad \frac{\beta}{c} = \frac{ht^{-q}}{(1+2ht^{-q})^{1/2}}, \tag{4.39}$$

where h and q are positive constants. With this choice of α we note that as $t\to\infty$, $\alpha/c\to 1$ and as $t\to 0$, $\alpha/c\to\infty$. From Eq. (4.39) we have that

$$\frac{\alpha\dot{\alpha}}{\beta^2} = -qt^{-1}\frac{(1+ht^{-q})}{(1+2ht^{-q})}, \tag{4.40}$$

so that $\dot{\alpha} < 0$ guaranteeing $\eta > 0$ [Eq. (4.9c)]. We also assume T is of the form

$$T = T_0 t^{-b} R^{-p} \alpha^s , \tag{4.41}$$

where b, p, and s are positive constants.

Condition (4.16), ensuring κ is non-negative, becomes

$$\begin{aligned}&(\tfrac{1}{2} - \tfrac{1}{2}p - b - q) + (\tfrac{3}{2} - \tfrac{3}{2}p - 3b - 2q)ht^{-q} \\ &+ (1 - p - q - 2b - sq)h^2 t^{-2q} \\ &+ (\tfrac{2}{3} - \tfrac{2}{3}p - b - q)lt^{3/5} \\ &+ (2 - 2p - 3b - 2q)hlt^{-q}t^{3/5} \\ &+ (\tfrac{4}{3} - \tfrac{4}{3}p - q - 2b - sq)lh^2 t^{-2q} t^{3/5} \geqslant 0 .\end{aligned} \tag{4.42}$$

We note that this inequality is satisfied for all t (and all h and l) if (i) $s \leqslant 1$ and $\frac{1}{2} - p/2 - b - q \geqslant 0$ or (ii) $s > 1$ and $1 - p - 2b - q - sq \geqslant 0$. There will be various solutions depending on the desired behavior of physical quantities (such as T) as $t \to 0$ or $t \to \infty$. Here, we shall make the following assumptions. First, we shall find that as $t \to 0$, $\epsilon(t) \to \frac{1}{3}$, so that we shall assume that $\rho_m^{-1} T^4 \to$const as $t \to 0$, which implies that $\frac{7}{5} + q - 4b - 2p - 2qs = 0$. Second, we shall assume that as $t \to \infty$, $\rho_m^{-1} T^a \to$const, where $a \leqslant 4$, which implies that $b/2 + p/3 \geqslant \frac{1}{4}$. Finally, for simplicity we shall assume that $b = 0$ and $s \leqslant 1$ so that the conditions to be satisfied become

$$0 \leqslant s \leqslant 1 , \tag{4.43a}$$

$$p \geqslant \tfrac{3}{4} , \tag{4.43b}$$

$$\tfrac{7}{5} + q - 2p - 2qs = 0 , \tag{4.43c}$$

$$1 - p - 2q \geqslant 0 \quad (\text{or } 0 \leqslant q \leqslant \tfrac{1}{8}) . \tag{4.43d}$$

Writing $q = \frac{1}{2} - p/2 - \delta/2$ $(\delta \geqslant 0)$, Eq. (4.43c) becomes

$$2s/5 = (\tfrac{19}{25} - p - \delta/5)/(1 - p - \delta) ,$$

so that Eq. (4.43a) implies $\frac{19}{25} \geqslant p$. This suggests two straightforward solutions:

$$\text{(A)} \quad p = \tfrac{3}{4}, \quad q = \tfrac{1}{8}, \quad s = \tfrac{1}{10} \quad (b = 0) , \tag{4.44a}$$

$$\text{(B)} \quad p = \tfrac{19}{25}, \quad q = \tfrac{3}{25}, \quad s = 0 \quad (b = 0) . \tag{4.44b}$$

Note that in solution (A) $\rho_m \sim T^4$ for large t and in solution (B) $\rho_m \sim T^{75/19}$ for large t.

We shall concentrate on model (B) henceforward. In this case $T = T_0 R^{-19/25}$, η and κ are always positive, and the model satisfies the end conditions outlined above. The model is consequently physically acceptable. Using Eqs. (4.5), (4.9), and (4.11) the solution becomes

$$\begin{aligned}\rho_m &= (l/5\pi G)(1 + lt^{3/5})^{-2}(1 + \tfrac{5}{6}lt^{3/5}) \\ &\quad \times (1 + 2ht^{-3/25})^{-1}(1 + ht^{-3/25})^2 t^{-7/5} , \\ 3p_m/c^2 &= (lh^2/5\pi G)(1 + lt^{3/5})^{-2}(1 + \tfrac{5}{6}lt^{3/5}) \\ &\quad \times (1 + 2ht^{-3/25})^{-1} t^{-41/25} , \\ \eta &= (5c^2 l/6\pi G)(1 + lt^{3/5})^{-2}(1 + \tfrac{5}{6}lt^{3/5}) \\ &\quad \times (1 + 2ht^{-3/25})^{1/2} t^{-2/5} , \\ Q/c &= (lh/5\pi G)(1 + lt^{3/5})^{-2}(1 + \tfrac{5}{6}lt^{3/5}) \\ &\quad \times (1 + 2ht^{-3/25})^{-1}(1 + ht^{-3/25}) t^{-38/25} , \\ n &= (n_0/c)(1 + lt^{3/5})^{-5/6}(1 + 2ht^{-3/25})^{1/2} \\ &\quad \times (1 + ht^{-3/25}) t^{-3/2} , \\ \kappa &= (5c^2/T_0 \pi G)(1 + lt^{3/5})^{-71/90}(1 + \tfrac{5}{6}lt^{3/5}) \\ &\quad \times (1 + 2ht^{-3/25})^{1/2}(1 + ht^{-3/25}) t^{-31/50} \\ &\quad \times [1 + (3h/l) t^{-18/25} + 5ht^{-3/25}]^{-1} .\end{aligned} \tag{4.45}$$

We note that ρ_m and p_m are always positive and monotonically decreasing and that $3p_m/c^2\rho_m \to 1$ as $t \to 0$ and $3p_m/c^2\rho_m \to 0$ as $t \to \infty$. The observational predictions of this model will be analyzed in Sec. V.

D. Model IV

In previous articles[1] we have considered imperfect fluid models for which the metric is that of a standard FRW model with a perfect fluid obeying the equation of state $p = \gamma\rho$. As a final example we consider a model based on the Einstein–de Sitter metric, i.e., we take $R = t^{2/3}$. In this case the field equations become

$$\begin{aligned}&\rho_m c^2 - 3p_m = c^2/6\pi G t^2 , \\ &\tfrac{1}{3}\rho_r(3\alpha^2 + \beta^2) + \rho_m c^2 = \alpha^2/6\pi G t^2 .\end{aligned} \tag{4.46}$$

These equations imply that

$$\frac{3p_m}{c^2\rho_m} < \frac{\beta^2}{\alpha^2}, \quad \frac{\rho_r}{\rho_m} < \frac{3\beta^2}{3\alpha^2 + \beta^2} , \tag{4.47}$$

and the second of these inequalities shows that $\rho_r/\rho_m < \frac{3}{4}$ always so that the model can describe only the later part of the matter-dominated era. Accordingly, there is little point in requiring $3p_m/c^2\rho_m \to 1$ as $t \to 0$ and $\alpha \to \infty$ as $t \to 0$, although we could do this and then assume that the model is applicable only for $t > 10^{12}$ sec approximately.

Bearing in mind the inequalities (4.48) we shall assume that

$$\rho_r/\rho_m = 3\beta^2/4\alpha^2 , \tag{4.48}$$

which leads to

$$3p_m/c^2\rho_m = \beta^2 c^2/4\alpha^4 . \tag{4.49}$$

The field equations now yield

$$\begin{aligned}6\pi G\rho_m &= 4\alpha^4(4\alpha^4 - \beta^2c^2)^{-1}t^{-2} , \\ 6\pi G\rho_r &= 3\alpha^2\beta^2(4\alpha^4 - \beta^2c^2)^{-1}t^{-2} , \\ 6\pi G p_m &= \tfrac{1}{3}\beta^2c^4(4\alpha^4 - \beta^2c^2)^{-1}t^{-2} .\end{aligned} \tag{4.50}$$

As suitable functions for α and β we choose

$$\alpha = c[1 + h^2(t + t_0)^{-2b}]^{1/2} , \quad \beta = ch(t + t_0)^{-b} , \tag{4.51}$$

where b, h, and t_0 are positive constants. We also choose T to be of the form

$$T = T_0(t + t_0)^{-m} , \tag{4.52}$$

where m is positive. The complete solution is

$$\begin{aligned}\rho_r &= (1/2\pi G)h^2(t + t_0)^{-2b}[1 + h(1 + t_0)^{-2b}] \\ &\quad \times [4 + 7h^2(t + t_0)^{-2b} + 4h^4(t + t_0)^{-4b}]^{-1}t^{-2} , \\ \rho_m &= (2/3\pi G)[1 + h^2(t + t_0)^{-2b}]^2[4 + 7h^2(t + t_0)^{-2b} \\ &\quad + 4h^4(t + t_0)^{-4b}]^{-1}t^{-2} , \\ 3p_m/c^2 &= (c^2/6\pi G)h^2(t + t_0)^{-2b}[4 + 7h^2(t + t_0)^{-2b} \\ &\quad + 4h^4(t + t_0)^{-4b}]^{-1}t^{-2} ,\end{aligned} \tag{4.53}$$

$$\eta = (c^2/12\pi G b)[4 + 3h^2(t+t_0)^{-2b}]$$
$$\times [1 + h^2(t+t_0)^{-2b}]^{1/2}[4 + 7h^2(t+t_0)^{-2b}$$
$$+ 4h^4(t+t_0)^{-4b}]^{-1}(t+t_0)t^{-2},$$
$$Q/c = (1/6\pi G)\, h\,(t+t_0)^{-b}[4 + 3h^2(t+t_0)^{-2b}]$$
$$\times [1 + h^2(t+t_0)^{-2b}]^{1/2}$$
$$\times [4 + 7h^2(t+t_0)^{-2b} + 4h^4(t+t_0)^{-2b}]^{-1}t^{-2},$$
$$n = (n_0/c)[1 + h^2(t+t_0)^{-2b}]^{1/2}t^{-2},$$
$$\kappa = (1/18\pi G)c^3 T_0^{-1}(1+t_0)^{m+1}[4 + 3h^2(t+t_0)^{-2b}]$$
$$\times [1 + h^2(t+t_0)^{-2b}]^{1/2}[4 + 7h^2(t+t_0)^{-2b}$$
$$+ 4h^4(t+t_0)^{-4b}]^{-1}[2t_0 + (2 - 3m - 3b)\,t\,]^{-1}t^{-1},$$

where we require

$$2 - 3m - 3b \geqslant 0, \tag{4.54}$$

in order to ensure that $\kappa \geqslant 0$ at all times. Provided that this condition holds, all quantities are positive and ρ_r, ρ_m, and p_m are montonically decreasing functions such that $\rho_r/\rho_m \to 0$ and $3p_m/c^2\rho_m \to 0$ as $t \to \infty$. We note that the choice $m = \frac{33}{50}$, $b = \frac{1}{150}$, which satisfies the condition (4.54), results in $\rho_m \sim T^{3.03}$ for large t.

V. DISCUSSION

In the first three models discussed in Sec. IV, $R(t)$ is a monotonically increasing function of t, changing from $R(t) = t^{1/2}$ for early times, so that the universe was initially in a pure radiation state, to $R(t) = t^{2/3}$ for later times, so that the universe evolves towards a final dustlike state. In all the models ρ_r, ρ_m, and p_m are always positive, monotonically decreasing functions of time and all positive energy conditions are satisfied. Einstein's equations and the laws of thermodynamics (4.5) to (4.8) are satisfied. In addition, η and κ are always positive. We conclude that the models are physically acceptable from a theoretical point of view. In order to show that the models are acceptable in the sense that they agree with the actual nature of the universe, we need to investigate the observational predictions of the models.

In actual fact all of the models are in good agreement with observation, as can be seen from Refs. 7, 8, 20, and 23. As an illustration we shall present a detailed investigation of the observational predictions of model III.

We let the subscript zero denote the present time. All numerical values will be calculated to three significant places only. We shall assume that the value of the arbitrary positive constant l is given by $l = 1.06 \times 10^{-7}$ (see Ref. 20). Based upon a Hubble parameter $H_0 = 55$ km sec^{-1} Mpc^{-1} we find that $t_0^* \equiv H_0^{-1} = 5.67 \times 10^{17}$ sec, so that from the definition of H_0 and Eq. (4.37) we find that $t_0 = 3.78 \times 10^{17}$ sec (the age of the universe). Note that $lt_0^{3/5} \equiv 3.73 \times 10^3$.

We shall assume that the present velocity of our galaxy relative to the cosmic microwave background is three hundred kilometers per second,[6] so that from Eqs. (4.39) we find that $ht_0^{-3/25} = 1.00 \times 10^{-3}$, which fixes h as $h = 1.29 \times 10^{-1}$.

From Eq. (4.45) we find that $\rho_{m,0} = 5.57 \times 10^{-30}$ g cm^{-3}. From Eq. (4.38) we find that $\rho_{r,0} = 4.47 \times 10^{-34}$ g cm^{-3}. Consequently, we find that $T_{r,0} = 2.70$ K from the relationship $\rho_r = aT_r^4$. In addition, Eq. (4.45) yields $(1/c^2)p_{m,0} = 1.86 \times 10^{-36}$ g cm^{-3}. Therefore, from Eq. (2.3), $\epsilon_0 = 2.71 \times 10^{-5}$ and, finally, $p_{m,0}/c^2\rho_{m,0} = 3.33 \times 10^{-7}$.

Let t_e be the time when $\rho_r = \rho_m$. From Eqs. (4.38) and (4.45) we then obtain a quadratic equation in $lt_e^{3/5}$. Taking the positive root, and using the established value of l, we find that $t_e = 1.01 \times 10^{11}$ sec. As remarked earlier, E_m is positive for small t, and from Eq. (2.16) we find that $E_{m,0} = -7.16 \times 10^{-32}$ erg cm^{-3} sec^{-1}.

Comparing the above with the values indicated in Table I we see that the predictions of the model are in excellent agreement with actual observations. Indeed, it could be claimed that the model is in better agreement with observations than existing cosmological models since, in addition to comparing very favorably with regard to the standard observations, the model is also able to predict the relative velocity of the galaxy with respect to the cosmic microwave background. Regardless of such merits, it is clear that the model is a *bona fide* cosmological model. The same is true of the other models outlined in Sec. IV. We conclude that the models in Sec. IV are physically acceptable models of the universe.

In general the temperature of the radiation T_r and the temperature of the matter T need to be taken to be equal. In the models established in the previous section T_r and T are certainly not equivalent, although in model II the possibility that T_r and T are related as $t \to \infty$ was investigated. Indeed, it is important that T_r and T are not equivalent in two-fluid cosmologies in which (at least) one fluid is imperfect with a nonzero heat conduction vector. In such models the two fluids will not be in thermal equilibrium throughout the evolution of the universe. (It might be noted that it is presently believed that the current temperature of the "matter" in the universe is about four times higher than that of the cosmic microwave background.)

In Sec. IV we demanded that the models satisfy the set of thermodynamic laws represented by Eqs. (4.5)–(4.8). It should be noted that these laws are based on the assumption that deviations from thermodynamic equilibrium are not too large. In view of the comments made in the previous paragraph it might be argued that the models outlined here (and, in fact, all models of this type) deviate sufficiently from thermal equilibrium that more general laws of thermodynamics ought to be considered (see, for example, Israel and Stewart[28]). Indeed, it has been suggested before that a more general set of laws of thermodynamics is needed in the cosmological arena.[1] However, the issue of determining the "appropriate thermodynamics" of the universe is a very difficult and controversial question that is at present unanswered. We shall assume here that the laws of thermodynamics that have been used are adequate for our purposes. The fact of the matter is that for reasonable values of t the deviations from thermodynamic equilibrium are not sufficient to raise doubts about the validity of the laws of thermodynamics that have been used [so that Eqs. (4.5)–(4.8) do govern the evolution of the universe for most values of t]. Presumably, if the laws of thermodynamics do break down, they will break down for small values of t, where more general laws will consequently be needed. However, the FRW description of the universe breaks down for very small values of t regardless.

We recall that the motivation for the present work was twofold. We wished to complete the work of Ref. 1 regarding the study of FRW cosmological models, in which the FRW models are interpreted as solutions of Einstein's field equations for a variety of different sources. In the present work we study FRW models in the most general case—that in which the source of the gravitational field is due to two (general) imperfect fluids (see Sec. III). The special cases studied previously can be listed as follows: The case in which there are two comoving perfect fluids was reviewed in Sec. II. The case in which there is one noncomoving imperfect fluid was studied in Ref. 1. The case in which there is one comoving perfect fluid gives rise to the so-called standard FRW models.

Although we have alluded to the general case (as set out in Sec. III), we have, in fact, focused our attention on the special case in which one fluid is a comoving (radiative) perfect fluid and the second a noncomoving (matter) imperfect fluid moving with an *axial* velocity relative to the comoving radiation (Sec. IV). This special case is one of particular physical interest. We shall assume that the comoving perfect fluid models the cosmic microwave background and the noncomoving imperfect fluid models the observed matter of the universe. The motivation behind the study of such models is to model the observed velocity of our galaxy relative to the cosmic microwave background.

There are several reasons why we have chosen to attempt to model this effect in the context of two-fluid FRW cosmological models, namely the following: (1) both the observed material content of the universe and the cosmic microwave background are observed to be (approximately) homogeneous and isotropic; (2) it is generally believed that the universe is described with reasonable accuracy by a FRW radiation model for early t and by the Einstein–de Sitter model for later t; and (3) with the discovery of the cosmic microwave background (which was presumed to be a remnant of the radiation era), it became desirable to model the universe as consisting of two fluids, each existing forever, and each "dominating" in the appropriate evolutionary phase of the universe.

We remark that the only way that our objective can be reconciled with the desire to remain within the context of a FRW model is for one of the fluids to be a noncomoving imperfect fluid. The models of Sec. IV are of this form. As mentioned previously, these models are theoretically reasonable and are in excellent agreement with observation. Moreover, through these models, we have achieved our objective of modeling the observed motion of our galaxy relative to the cosmic microwave background. We note that the assumption of a noncomoving imperfect fluid implies that there is a general motion of all matter relative to the cosmic microwave background.

The present work can be generalized somewhat. First, although we have concentrated on FRW models for the reasons given above, we could, of course, repeat the analysis in a more general setting. Indeed, it might be argued that such an analysis would be more appropriate in a nonisotropic and inhomogeneous model. Presumably one would investigate models that approximate FRW models (at least for later times) in order that agreement with present observations is retained. Second, in the actual models that have been described in Sec. IV we have assumed that $k = 0$ and that physical quantities appearing in the models depend on t only. Although both of these assumptions may be relaxed, they have been made here partly for simplicity, but mainly because they give rise to models that exhibit precisely the type of behavior we seek.

We have one final note. This present article represents a natural development of the work by the authors as set out in Ref. 1. In Ref. 1 the imperfect fluid moves relative to a "hypersurface orthogonal preferred observer"; however, such an observer has no physical role within the models. In the present article the imperfect fluid moves relative to the observed cosmic microwave background (and thus a physical interpretation is given to the hypersurface orthogonal preferred observer within the models). So, from a philosophical point of view, the present article presents a more suitable environment for the study of imperfect fluid FRW models.

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