

WELL-COVERED VECTOR SPACES OF GRAPHS*

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Abstract. For any field \mathbf{F} , the set of all functions $f : V(G) \rightarrow \mathbf{F}$ whose sum on each maximal independent set is constant forms a vector space over \mathbf{F} . In this paper, we show that the dimension can vary depending on the characteristic of the field. We also investigate the dimensions of these vector spaces and show that while some families, such as chordal graphs, have unbounded dimension, other families, such as nonempty circulant graphs of prime order, have bounded dimension.

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1. Introduction. A *weighting* of a graph G is a function $f : V(G) \rightarrow \mathbf{F}$ that assigns a value from the field \mathbf{F} to each vertex of G . Following [1], a *well-covered weighting* f of a graph G is a weighting such that $\sum_{x \in M} f(x)$ is constant for every maximal independent set M of G . For a well-covered weighting, we denote the common weight of the maximal independent sets as $f(G)$. In [1], the following is noted.

Observation 1. The well-covered weightings of a graph form a vector space.

This is clear since if f and g are well-covered weightings and k and l are elements of the field \mathbf{F} , then $kf + lg$ is also a well-covered weighting.

We remark that a *well-covered graph* [8] is a graph in which all maximal independent sets have the same cardinality. Thus, well-covered graphs G are precisely those graphs G for which $\mathbf{1}_G : V(G) \rightarrow \mathbf{F} : v \mapsto 1$ is a well-covered weighting over any field \mathbf{F} of characteristic 0. The definition of the well-covered space can be traced to Caro and Yuster [2] in a more general setting. Let $H = (V, E)$ be a hypergraph and \mathbf{F} be a field. A function $f : V \rightarrow \mathbf{F}$ is called *stable* if for each $e \in E$, the sum of the values of f on the members of e is the same. The stable functions form a vector space. One instance that Caro and Yuster consider is the space of well-covered weightings for a graph G . They denote this by $U(MIS : G, \mathbf{F})$ and the dimension by $\text{udim}(MIS : G, \mathbf{F})$ (*MIS* stands for maximal independent sets). In this paper we restrict ourselves to just well-covered weightings so we use $WC(G, \mathbf{F})$ and $\text{wcdim}(G, \mathbf{F})$ (we call the former the *well-covered space* of G and the latter the *well-covered dimension* of G). If the field has characteristic 0, then we eliminate the reference to \mathbf{F} as well.

In general, our graph theoretic notation follows [3]. The complement of graph G is denoted by \overline{G} . The disjoint union of graphs G and H is denoted by $G \cup H$, and the join of G and H (which is $\overline{\overline{G} \cup \overline{H}}$) is denoted by $G + H$. A *maximum independent set* is one of maximum size (which is $\beta(G)$, the *independence number* of G). A *clique* is a complete subgraph (not necessarily maximal). We often obscure the difference between a subset of vertices of a graph and the subgraph they induce. Finally, for a vertex v of G , $N(v) = \{u \in V(G) : uv \text{ is an edge of } G\}$ is the *neighborhood* of v and

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$N[v] = \{v\} \cup N(v)$ is the *closed neighborhood* of G . For matrix theoretic notation, we follow [7]. We denote the all ones vector of length n by $\mathbf{1}_n$, or simply $\mathbf{1}$ if the length is understood, and similarly use $\mathbf{0}_n$ to denote the all zeros vector of length n . (Vectors throughout are written as column vectors.)

If I_1, \dots, I_{t+1} are the maximal independent sets of G , then well-covered weightings are precisely the solutions to the *associated linear system*

$$\begin{aligned} \sum_{v \in I_1} x_v &= \sum_{v \in I_{t+1}} x_v, \\ \sum_{v \in I_2} x_v &= \sum_{v \in I_{t+1}} x_v, \\ &\dots \\ \sum_{v \in I_t} x_v &= \sum_{v \in I_{t+1}} x_v \end{aligned}$$

(we call I_{t+1} the *common* maximal independent set for the linear system). This homogenous linear system can be written in matrix form as

$$A_G \mathbf{x} = \mathbf{0}$$

(we call the $t \times n$ matrix A_G an *associated matrix* for the graph G). Note that $\text{wcdim}(G)$ equals the nullity of A_G (over \mathbf{F}) and hence is equal to the $|V(G)| - \text{rank}(A_G)$ (where, of course, the rank is taken over \mathbf{F}). This formulation clearly shows that $\text{wcdim}(G)$ depends only on the characteristic of \mathbf{F} , rather than the whole field.

As an illustration, consider W_5 , the 5-wheel, which consists of a 5-cycle with a central vertex joined to each vertex on the 5-cycle. It is easy to discover (see Lemma 9) that all the vertices on the 5-cycle must have the same weight, and it is also easy to see that the central vertex must have weight equal to the sum of the weights of any maximal independent set of the 5-cycle, that is, twice the weight assigned to each vertex of the 5-cycle. Thus (writing the well-covered weightings as 6-tuples), we see that $WC(W_5, \mathbf{F})$ is spanned by $(1, 1, 1, 1, 1, 2)$ and hence has well-covered dimension 1. This example also shows that a basis for $WC(G, \mathbf{F})$ cannot always be chosen with values in $\{-1, 0, 1\}$ (when $\text{char}(\mathbf{F}) \neq 2, 3$). As another example, we derive an upper bound on the well-covered dimension involving the chromatic number $\chi(G)$ of a graph G .

THEOREM 2. *Let G be a graph of order n . Then $\text{wcdim}(G) \leq n - \chi(G) + 1$.*

Proof. For a graph G , let $\{I_i | i = 1, 2, \dots, k\}$ be a sequence of nonempty, independent sets such that I_1 is a maximal independent set of G and for $j > 1$, I_j is a maximal independent set in $G - \cup_{i=1}^{j-1} I_i$. We extend each I_i to a maximal independent set I'_i of G . If we choose one vertex $v_i \in I_i$ for each $i = 1, \dots, k$ of G , then using $I_1 = I'_1$ as the common maximal independent set for the linear system, the submatrix of A_G with rows corresponding to I'_2, \dots, I'_k and columns corresponding to v_2, \dots, v_k is lower triangular with ones on the diagonal, as no v_i can lie in I'_j for $j < i$ (and in particular no v_i lies in I'_1 for any $i = 2, \dots, k$). Thus the rank of A_G is at least $k - 1$, so the nullity of A_G (and hence $\text{wcdim}(G)$) is at most $n - k + 1$. Because I_1, \dots, I_k is a covering of G with k independent sets, $\chi(G) \leq k$, so $\text{wcdim}(G) \leq n - k + 1 \leq n - \chi(G) + 1$. \square

The major result on well-covered spaces can be found in Theorem 3.5 of [2]. There, it is shown that if the characteristic of \mathbf{F} is 0, then for a connected graph $G \not\cong C_7$ of girth 7 or greater, $\text{wcdim}(G, \mathbf{F})$ equals the number of leaves. Moreover, the basis vectors can be taken to be the set $\{f_v | v \text{ is a leaf}\}$, where $f_v(v) = f_v(x) = 1$, x is the unique vertex adjacent to v (x is referred to as a *stem*), and $f_v(w) = 0$ otherwise. The

exceptional case is $G \cong C_7$ in which case $\text{wcdim}(G, \mathbf{F}) = 1$ and the basis vector is the all ones vector. All bases can be constructed in polynomial time, and the restriction on the field can be removed if there is at least one leaf. In particular, Caro and Yuster’s result shows that the well-covered dimension of a tree is equal to the number of leaves.

In this paper, after illustrating how the well-covered dimension can depend on the characteristic of the field, we restrict ourselves to the most interesting case, characteristic 0, and consider families of graphs for which the well-covered dimension is unbounded and those for which it is bounded. Extending Caro and Yuster’s result that the well-covered dimension of a tree is equal to the number of leaves, we calculate the dimension of chordal graphs and show how a corresponding basis can be derived from the chordal graph’s simplicial decomposition. Using linear algebraic techniques, we show on the other hand that nonempty circulant graphs of prime order have bounded dimension over any field of characteristic 0.

2. Characteristic does make a difference. In this section we provide, for every prime p , an infinite number of graphs whose dimension is different over fields of characteristic p and 0.

We begin by defining graphs $G_{p,q,n}$. Let $n \equiv 0 \pmod p$ with $n > p \geq 3$ (we will handle the case $p = 2$ at the end). Let $q > p(p - 1)$, $q \not\equiv 0 \pmod p$. We form $G_{p,q,n}$ on vertex sets V_0, \dots, V_{q-1} , where $V_i = \{v_{i,1}, \dots, v_{i,n}\}$. The *nonedges* of $G_{p,q,n}$ are $v_{i,r}v_{i,s}$ and $v_{i,r}v_{j,r}$, with $r, s = 1, 2, \dots, n, r \neq s, i, j \in \{0, 1, \dots, q-1\}, i - j \in \{1, 2, \dots, p-1\}$ (arithmetic mod q). The complement of $G_{3,7,6}$ (which has fewer edges than $G_{3,7,6}$) is shown in Figure 1. Now it is not difficult to verify that the maximal independent sets of $G_{p,q,n}$ are V_0, \dots, V_{q-1} together with the sets

$$\{v_{i,k}, v_{i+1,k}, \dots, v_{i+p-1,k}\}$$

(here and elsewhere, addition is modulo q). Setting the sum of each of the weights on the maximal independent sets equal to the sum of the weights on the vertices of V_{q-1} , we find that the linear system corresponding to the well-covered weightings is $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{pmatrix} I_n & I_n & \cdots & I_n & 0_n & 0_n & \cdots & 0_n & -J_n \\ 0_n & I_n & \cdots & I_n & I_n & 0_n & \cdots & 0_n & -J_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_n & 0_n & \cdots & I_n & I_n & I_n & \cdots & I_n & -J_n \\ 0_n & 0_n & \cdots & 0_n & I_n & I_n & \cdots & I_n & I_n - J_n \\ I_n & 0_n & \cdots & 0_n & 0_n & I_n & \cdots & I_n & I_n - J_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_n & I_n & \cdots & I_n & 0_n & 0_n & \cdots & 0_n & I_n - J_n \\ \mathbf{1}_n^T & \mathbf{0}_n^T & \cdots & \mathbf{0}_n^T & \mathbf{0}_n^T & \mathbf{0}_n^T & \cdots & \mathbf{0}_n^T & -\mathbf{1}_n^T \\ \mathbf{0}_n^T & \mathbf{1}_n^T & \cdots & \mathbf{0}_n^T & \mathbf{0}_n^T & \mathbf{0}_n^T & \cdots & \mathbf{0}_n^T & -\mathbf{1}_n^T \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_n^T & \mathbf{0}_n^T & \cdots & \mathbf{0}_n^T & \mathbf{0}_n^T & \mathbf{0}_n^T & \cdots & \mathbf{1}_n^T & -\mathbf{1}_n^T \end{pmatrix},$$

the columns are indexed by the vertices

$$v_{0,1}, v_{0,2}, v_{0,3}, \dots, v_{i,1}, v_{i,2}, v_{i,3}, \dots, v_{q-1,n},$$

and the rows are indexed by the maximal independent sets $V_0, V_1, V_2, \dots, V_{q-1}$ and

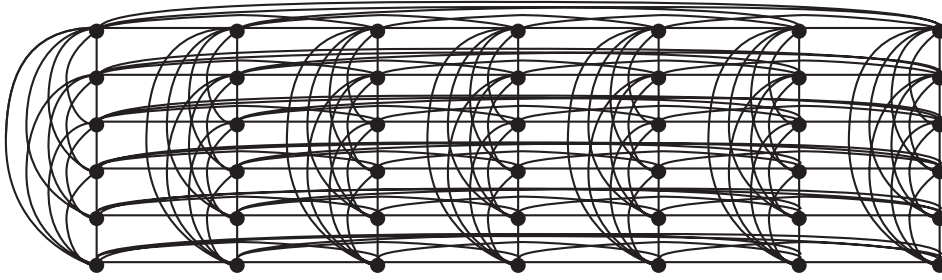


FIG. 1. $\overline{G_{3,7,6}}$.

the sets

$$\{v_{0,1}, v_{0+1,1}, \dots, v_{0+p-1,1}\}, \{v_{0,2}, v_{0+1,2}, \dots, v_{0+p-1,2}\}, \dots, \{v_{1,1}, v_{1+1,1}, \dots, v_{1+p-1,1}\}, \dots, \{v_{q-1,1}, v_{q-1+1,1}, \dots, v_{q-1+p-1,1}\}, \dots \{v_{q-1,1}, v_{q-1+1,1}, \dots, v_{q-1+p-1,1}\}.$$

In the above block form of the matrix, the subscript n denotes the order of the submatrix, with J_n being the $n \times n$ matrix of all ones and 0_n being the $n \times n$ matrix of all zeros. If B denotes the top nq rows of A , then $B = C - D$, where

$$C = \begin{pmatrix} I_n & I_n & \cdots & I_n & 0_n & 0_n & \cdots & 0_n & 0_n \\ 0_n & I_n & \cdots & I_n & I_n & 0_n & \cdots & 0_n & 0_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_n & 0_n & \cdots & I_n & I_n & I_n & \cdots & I_n & 0_n \\ 0_n & 0_n & \cdots & 0_n & I_n & I_n & \cdots & I_n & I_n \\ I_n & 0_n & \cdots & 0_n & 0_n & I_n & \cdots & I_n & I_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_n & I_n & \cdots & I_n & 0_n & 0_n & \cdots & 0_n & I_n \end{pmatrix}$$

is block circulant (with p consecutive identity matrices in each block row) and

$$D = \begin{pmatrix} 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & J_n \\ 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & J_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & J_n \\ 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & J_n \\ 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & J_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & J_n \end{pmatrix}.$$

Suppose first that the characteristic of \mathbf{F} is 0. Then by summing the first n rows and subtracting off rows $nq + 1$ to $nq + p$, we get a row with $n(q - 1)$ zeros followed by $-(n - p)\mathbf{1}_n^T$. Since $n > p$ and the characteristic is 0, we can divide through by $-(n - p)$ to get $\mathbf{1}_n^T$ in the last positions. Adding this row to each of the first nq rows of A , we obtain a matrix whose upper nq rows are the block circulant C . It is clear that C is nonsingular iff the $q \times q$ circulant matrix formed by replacing each I_n and 0_n by 1 and 0, respectively, is nonsingular.

However, it is known (cf. [7, p. 66]) that the determinant of a circulant with first row a_1, \dots, a_m is

$$\prod_{i=1}^m \sum_{i=1}^m a_i x^{i-1},$$

where the product is taken over all m th roots x of unity. In our case, the determinant of C is given by

$$\prod_{i=1}^p x^{i-1}$$

over all x that are q th roots of unity. However, no term in this product is 0, since clearly the term with $x = 1$ is nonzero, and for any other q th root of unity x we have (by multiplying through by $1 - x$) that x is also a p th root of unity, a contradiction since $q \not\equiv 0 \pmod{p}$. Thus we conclude that the matrix A has full row rank over the field of characteristic 0 and hence has nullity 0, i.e., $\text{wcdim}(G_{p,q,n}, \mathbb{Q}) = 0$.

On the other hand, if we weight every vertex with 1, then this yields a weighting over a field of characteristic p since the maximal independent sets have weight p or weight $n \equiv 0 \pmod{p}$. Thus $\text{wcdim}(G_{p,q,n}, \mathbb{Z}_p) > 0$.

Last, we handle $p = 2$. For any $n > 2$, n even, we form the graph $G_{2,n}$ by removing a perfect matching from $K_{n,n}$. We let the partition be $V_0 = \{a_1, \dots, a_n\}$ and $V_1 = \{b_1, \dots, b_n\}$, with a_1b_1, \dots, a_nb_n being the perfect matching that is removed. The maximal independent sets are $\{a_i, b_i\}$ for $i = 1, 2, \dots, n$ and V_1 and V_2 . Setting the sum of each of the weights on the maximal independent sets equal to that of the weights on the vertices of V_2 , we find that the linear system corresponding to the well-covered weightings is

$$Ax = \mathbf{0},$$

where

$$A = \begin{pmatrix} I_n & I_n - J_n \\ \mathbf{1}_n^T & -\mathbf{1}_n^T \end{pmatrix}.$$

Subtracting the top n rows from the bottom yields

$$\begin{pmatrix} I_n & I_n - J_n \\ \mathbf{0}_n^T & (n-2)\mathbf{1}_n^T \end{pmatrix}.$$

Over \mathbb{Q} we can divide out by $n-2$ so that A is row equivalent to

$$\begin{pmatrix} I_n & I_n - J_n \\ \mathbf{0}_n^T & \mathbf{1}_n^T \end{pmatrix} \quad (*),$$

which has rank $n+1$. Hence the nullity is $n-1$, which implies that $\text{wcdim}(G_{2,n}, \mathbb{Q}) = n-1$.

On the other hand, over \mathbb{Z}_2 , since n is even, A is row equivalent to

$$\begin{pmatrix} I_n & I_n - J_n \\ \mathbf{0}_n^T & \mathbf{0}_n^T \end{pmatrix},$$

which has rank n . Hence the nullity is n , which implies that $\text{wcdim}(G_{2,n}, \mathbb{Z}_2) = n$.

For the remainder of the paper, we shall restrict our discussion to fields of characteristic 0, though some of the results will hold over fields of other characteristic as well.

3. Families of graphs with unbounded well-covered dimension. In this section, we shall determine (in polynomial time) the well-covered dimension of cographs and chordal graphs, where the latter extends the result of Caro and Yuster on trees. We begin with the easier case.

3.1. Cographs and anti-well-covered graphs. A *cograph* is a graph that does not contain an induced path on four vertices. It is well known (cf. [5]) that cographs have a recursive definition; the class of cographs is the smallest class of graphs containing K_1 (the complete graph on one vertex) that is closed under disjoint union and join. We shall need to introduce a definition that is of interest in its own right.

DEFINITION 3. A graph for which $f(G) = 0$ for every well-covered weighting f of G is called an *anti-well-covered graph*.

Note that in a well-covered graph G of order n , the all ones vector $\mathbf{1}_n$ is in $WC(G, \mathbf{F})$, and for an anti-well-covered graph, $\mathbf{1}_n$ is in $WC(G, \mathbf{F})^\perp$, the orthogonal complement of the well-covered space of G . The fact that $WC(G, \mathbf{F})^\perp \cap WC(G, \mathbf{F}) = \{\mathbf{0}\}$ ensures that no well-covered graph is anti-well-covered, and this motivates our choice of name for the property.

A graph of dimension 0 is clearly an anti-well-covered graph, but there are others. For example, one can verify that C_6 and Q_3 (the 3-cube) are anti-well-covered. Also, $K_{n,n} - M$, where $n > 2$ and M is a 1-factor, is an anti-well-covered graph with dimension n over any field of characteristic c , where $\gcd(n, c) = 1$ (this follows from the derivation of (*) in the previous section). In order to determine the well-covered dimension of cographs, we will need some simple properties of anti-well-covered graphs.

LEMMA 4. Let G or H be graphs. Then $G \cup H$ is anti-well-covered iff both G and H are anti-well-covered, whereas $G + H$ is anti-well-covered iff either G or H is anti-well-covered.

Proof. The well-covered weightings of the disjoint union of two graphs G and H are precisely those functions on $V(G) \cup V(H)$ whose restrictions to G and H are well-covered weightings, whereas the well-covered weightings of the join of G and H are precisely those functions on $V(G) \cup V(H)$ whose restrictions to G and H are well-covered weightings *with the same sum*. It follows that $G \cup H$ is anti-well-covered iff both G and H are anti-well-covered, whereas $G + H$ is anti-well-covered iff either G or H is anti-well-covered. \square

We now determine how the well-covered dimension behaves under disjoint union and join.

LEMMA 5. Let G and H be graphs. Then

1. $\text{wcdim}(G \cup H) = \text{wcdim}(G) + \text{wcdim}(H)$, and
2. $\text{wcdim}(G + H) = \text{wcdim}(G) + \text{wcdim}(H) - 1$ unless both G and H are anti-well-covered graphs in which case $\text{wcdim}(G + H) = \text{wcdim}(G) + \text{wcdim}(H)$.

Proof. The first result is given in [1]. Let L be the subspace generated by those vectors whose restrictions to G and H are well-covered weightings on the respective graphs. From the proof of Lemma 4, L properly contains the subspace generated by well-covered weightings of $G + H$ iff either G or H is anti-well-covered. (If say G is not anti-well-covered, then we can find well-covered weightings of G with weight equal to any field element, in particular, of unequal weight to some weighting of H .) Thus $\text{wcdim}(G + H) = \text{wcdim}(G) + \text{wcdim}(H)$ if both G and H are anti-well-covered, and $\text{wcdim}(G + H) < \text{wcdim}(G) + \text{wcdim}(H)$ otherwise. In the latter case, note that if we write corresponding linear systems defining the subspaces of G and H as

$$A_G \mathbf{x} = \mathbf{0}$$

and

$$A_H \mathbf{x} = \mathbf{0},$$

then a corresponding linear system for $G + H$ can be given as

$$A_{G+H} \mathbf{x} = \mathbf{0},$$

where

$$A_{G+H} = \begin{pmatrix} A_G & 0 \\ 0 & A_H \\ \mathbf{u} & \mathbf{v} \end{pmatrix}$$

with \mathbf{u} and \mathbf{v} being nonzero vectors of the appropriate dimension. Now

$$\text{rank}(A_{G+H}) \leq \text{rank}(A_G) + \text{rank}(A_H) + 1,$$

so since $\text{wcdim}(K) = \text{nullity}(A_K) = |V(K)| - \text{rank}(A_K)$ for any graph K , we find that

$$\text{wcdim}(G + H) \geq \text{wcdim}(G) + \text{wcdim}(H) - 1.$$

Since $\text{wcdim}(G + H) < \text{wcdim}(G) + \text{wcdim}(H)$, we conclude that $\text{wcdim}(G + H) = \text{wcdim}(G) + \text{wcdim}(H) - 1$. \square

THEOREM 6. *The dimension of a cograph can be determined in polynomial time.*

Proof. A cograph is constructed via the disjoint union and join operation from K_1 . A cograph can be recognized and the order of operations for its construction can be determined in polynomial time [5]. It follows that we can recognize whether a cograph is anti-well-covered in polynomial time as well. The dimension can be determined in polynomial time from Lemma 4. \square

We conclude this section by applying anti-well-covered graphs to determining the dimension of graphs with independence number 2. Graphs with independence number 1 are complete, and it is easy to see that these all have dimension 1, with all vertices having the same weight in any weighting.

THEOREM 7. *Let G be a graph with $\beta(G) = 2$. Then $\text{wcdim}(G)$ is 1 plus the number of bipartite components of order of at least 2 in the complement \overline{G} of G .*

Proof. Let the components of \overline{G} be D_1, \dots, D_t . Noting that K_1 is not anti-well-covered, we observe from Lemma 5 that any D_i of order 1 does not affect the dimension, so we can assume that each D_i has an order of at least 2. Also, every edge of \overline{G} is a maximal independent set of G . Note that under any weighting of G , if xy and yz are edges of \overline{G} , then x and z have equal weight, so that any two vertices connected by a walk of even length have the same weight.

Consider any component D of \overline{G} . If D is not bipartite, it contains an odd cycle. By the argument above (traveling twice around the cycle), any well-covered weighting must be constant on this cycle, and indeed on the component D , and hence the subgraph of G induced by D has dimension 1.

On the other hand, if D is bipartite with bipartition (X, Y) , then we can weight every vertex of X with one weight, weight every vertex of Y with another, and derive a well-covered weighting of the graph. Moreover, every well-covered weighting of G necessarily assigns the same weights to vertices of X and the same weights to the vertices of Y , as vertices of X are at even distances from one another (similarly for the vertices of Y). Thus the subgraph of G induced by D has dimension 2.

Now each D_i induces a well-covered graph (with $\beta = 2$), so in particular, no D_i is anti-well-covered. Since G is the join of the subgraphs induced by D_1, \dots, D_t , we conclude the stated formula for $\text{wcdim}(G)$ from Lemma 5. \square

3.2. Complements of k -trees, chordal graphs, and related vertices.

In this section we show that certain other well known families of graphs also have unbounded well-covered dimension. A k -tree, ($k \geq 2$) is defined recursively: G_0 is a k -clique; for $i > 0$, G_i is formed from G_{i-1} by adding a new vertex that is joined to a $(k-1)$ -clique of G_{i-1} . Every tree is a 2-tree. Here we determine the well-covered dimension of complements of k -trees. The dimension of k -trees themselves will be covered later in this section.

THEOREM 8. *If G is the complement of a k -tree, then G has dimension k .*

Proof. Let G be the complement of a k -tree with G_0 an independent set of size k of G and $G_1, G_2, \dots, G_m \cong \overline{G}$ a sequence of k -trees that build to \overline{G} . Let the vertices of G_0 be v_1, \dots, v_k . Let f be any well-covered weighting of G . By induction on i we show that (i) the maximal independent sets of \overline{G}_i are the independent sets of size k of \overline{G}_i and (ii) $\text{wcdim}(\overline{G}_i) = k$. The latter, for $i = m$, completes the proof.

For $i = 0$, (i) and (ii) are obvious. Suppose now that G_i is formed from G_{i-1} by the addition of vertex v_{k+i} so that, for some independent set X_i of size $k-1$ of \overline{G}_{i-1} , v_{k+i} is joined to all of $G_{i-1} - X_i$ but no vertex of X_i . Now the maximal independent sets of \overline{G}_i are those that do not contain v_{k+i} (which are the maximal independent sets of \overline{G}_{i-1}) and those that contain v_i , of which there is only one, namely $\{v_i\} \cup X_i$. Thus by induction (i) holds. Moreover, an associated linear system for \overline{G}_i can be derived from that of \overline{G}_{i-1} by adding in the equation

$$\sum_{v \in \{v_i\} \cup X_i} x_v = \sum_{v \in G_0} x_v.$$

This introduces a new variable, so it is not hard to see that the associated matrices A_{i-1} and A_i have the same nullity (since A_{i+1} has a rank one larger than that of A_i , but one more column). Part (ii) now follows. \square

We now turn our attention to chordal (or *triangulated*) graphs, that is, graphs without an induced cycle of length of at least 4. Every chordal graph has a *simplicial decomposition*; that is, the graph can be recursively built from a complete graph by adding vertices that are joined to cliques (for more information on chordal graphs, cf. [6, p. 83]). Note that all trees and all k -trees are chordal graphs. We now calculate the dimension of chordal graphs. A new relation on the vertices of a graph plays a key role in calculating the well-covered dimension of chordal graphs. Two vertices x and y of a graph are *related* if there is an independent set I , containing neither x nor y , such that $I \cup \{x\}$ and $I \cup \{y\}$ are both maximal independent sets. Note that x and y must be adjacent or else both could be added to I .

LEMMA 9. *Let f be a well-covered weighting of G . If x and y are related vertices in G , then $f(x) = f(y)$.*

Proof. For an appropriate independent set I , $f(x) + \sum_{z \in I} f(z) = f(y) + \sum_{z \in I} f(z)$, and the result follows. \square

Now we say a vertex x of a graph G is *simplicial* if $N[x]$ is a maximal clique. Let $\mathcal{C}(G) = \{C \mid C \text{ is a maximal clique containing a simplicial vertex of } G\}$. The members of $\mathcal{C}(G)$ are called *simplicial cliques*. Let $\text{sc}(G) = |\mathcal{C}(G)|$. Let C be a simplicial clique of G , and let f_C be the *associated weighting*: $f_C(v) = 1$ if $v \in C$ and $f_C(v) = 0$ otherwise. It was shown in [2] that the number of leaves of a graph is a lower bound to its dimension. We generalize this to simplicial cliques.

LEMMA 10. *Let G be a graph. Then $\{f_C \mid C \in \mathcal{C}\}$ is an independent set of vectors and $\text{wcdim}(G) \geq \text{sc}(G)$.*

Proof. Let $C \in \mathcal{C}$. There is a vertex $v \in C$ that is adjacent only to vertices of C . Therefore, any maximal independent set must contain exactly one vertex of C , and so f_C is a well-covered weighting. Moreover, v is in no other maximal simplicial clique. Therefore, $f_C(v) = 1$, but $f_D(v) = 0$ for all $D \in \mathcal{C}$, $D \neq C$. Consequently $\{f_C | C \in \mathcal{C}\}$ is an independent set of well-covered weightings. The second part of the lemma now follows. \square

Our main result proves that equality indeed holds in Lemma 10 for chordal graphs.

THEOREM 11. *Let G be a chordal graph. Then $\text{wcdim}(G) = \text{sc}(G)$.*

The remainder of the section is devoted to a proof of Theorem 11.

From Lemma 10 we have $\text{wcdim}(G) \geq \text{sc}(G)$. The second part of the proof is now by induction on the size of G . If G is a singleton, then $\text{wcdim}(G) = \text{sc}(G) = 1$. Assume that the result is true for all chordal graphs of sizes 1 through k for some $k \geq 1$. We shall need a few observations about simplicial cliques.

Observation 12. Let w and y be adjacent vertices. If w is a simplicial vertex, then $N[w] \subseteq N[y]$. If both w and y are simplicial vertices, then $N[w] = N[y]$, so that both w and y “generate” the same simplicial clique of G . \square

Let x be a simplicial vertex of G , and put $H = G - \{x\}$. Note that H is also chordal. By induction, $\text{sc}(H) = \text{wcdim}(H)$.

Observation 13. Consider a simplicial clique $C \in \mathcal{C}(H)$. If there is a simplicial vertex $y \in C$ and y is not adjacent to x , then $C \in \mathcal{C}(G)$. Similarly, if $D \in \mathcal{C}(G)$ and there is a simplicial vertex $z \in D$, $z \neq x$, with z not adjacent to x , then $D \in \mathcal{C}(H)$. \square

Observation 14. If $C \in (\mathcal{C}(G) - \mathcal{C}(H))$, then $C = N[x]$.

Proof. By Observation 13, all simplicial vertices of C are adjacent to x , but then, by Observation 12, we have $C = N[x]$. \square

Observation 15. If $C \in (\mathcal{C}(H) - \mathcal{C}(G))$, then either $C = N(x)$, or there is a simplicial vertex $y \in C$, y adjacent to x . Moreover, there is at most one such simplicial clique C .

Proof. Suppose that $C \in (\mathcal{C}(H) - \mathcal{C}(G))$, and let $y \in C$ be a simplicial vertex in H . It follows from Observation 13 that y is adjacent to x (else $C \in \mathcal{C}(G)$) so that in G we have, by Observation 12, $N[x] \subseteq N[y]$. If y is a simplicial vertex of G , then by Observation 12 $N[x] = N[y]$ and thus $C = N[x] - x = N(x)$. If y is not a simplicial vertex of G , then, in H , $C = N[y] = N(x) \cup A$. Suppose that $C, D \in (\mathcal{C}(H) - \mathcal{C}(G))$ with $C \neq D$. There are simplicial vertices $y \in D$, y adjacent to x , and $z \in C$ which is also adjacent to x . But then z and y are adjacent (since both are in the clique $N(x)$), and so by Observation 12, $C = N[z] = N[y] = D$. Thus, there is at most one simplicial clique $C \in (\mathcal{C}(H) - \mathcal{C}(G))$. \square

Observation 16. $\text{sc}(G) - 1 \leq \text{sc}(H) \leq \text{sc}(G)$.

Proof. By Observation 13, every simplicial clique of H that does not contain a simplicial vertex from $N(x)$ is a simplicial clique of G , and by Observation 15 there is at most one simplicial clique of H with a vertex in $N(x)$. Since G has $N[x]$ as a simplicial clique while H clearly does not, we have $\text{sc}(H) \leq \text{sc}(G)$. On the other hand, there is only one simplicial clique of G , namely $N[x]$, that is not a simplicial clique of H , as the only other simplicial vertices of G in $N[x]$ generate the same simplicial clique (by Observation 12). Thus $\text{sc}(G) - 1 \leq \text{sc}(H)$. \square

Now back to the proof of Theorem 11. Let $f(G)$ be a well-covered weighting of G , and let K be the (common) sum of the weights of a maximal independent set. We first show that any well-covered weighting of G can be associated with a well-covered weighting of H . We then use this and the fact that $\text{wcdim}(H) = \text{sc}(H)$ to show

that $wcdim(G) = sc(G)$. From the observations we see that there are three cases to consider.

1. $\mathcal{C}(H) \subset \mathcal{C}(G)$, i.e., no new simplicial clique is created when x is deleted,
2. $\{C\} = \mathcal{C}(H) - \mathcal{C}(G)$ and $C = N(x)$, or
3. $\{C\} = \mathcal{C}(H) - \mathcal{C}(G)$ and $C \neq N(x)$.

Case 1. We have $\mathcal{C}(H) \subset \mathcal{C}(G)$ so that $\{N[x]\} = \mathcal{C}(G) - \mathcal{C}(H)$ and $sc(G) = sc(H) + 1$. Since every simplicial clique of H is a simplicial clique of G , then, from Observation 2, it follows that for all $y \in N(x)$, y is not simplicial in H . We define a weighting w_f on $V(H)$ by

$$w_f(v) = \begin{cases} f(v) & \text{if } v \text{ is not adjacent to } x, \\ f(v) - f(x) & \text{if } v \in N(x). \end{cases}$$

We claim that w_f is in fact a well-covered weighting of H . Let I be a maximal independent set of H . If there exists $s \in I$ such that $s \in N(x)$, then I is a maximal independent set in G , and moreover no other vertex in I is adjacent to x . Therefore,

$$\sum_{v \in I} w_f(v) = \sum_{v \in I-s} w_f(v) + w_f(s) = \sum_{v \in I-s} f(v) + f(s) - f(x) = K - f(x).$$

If I contains no vertex adjacent to x , then $I \cup \{x\}$ is a maximal independent set in G . Therefore

$$f(x) + \sum_{v \in I} w_f(v) = f(x) + \sum_{v \in I} f(v) = K;$$

i.e., $\sum_{v \in I} w_f(v) = K - f(x)$. Thus w_f is a well-covered weighting of H . In H , let h_i , $i = 1, 2, \dots, sc(H)$ be the vector with weight 1 on the coordinates corresponding to the vertices of the i th simplicial clique. By induction, this is a basis for $wcdim(H)$. In G , we extend these vectors to g_i , $i = 1, 2, \dots, sc(H)$, where g_i is the vector with weight 1 on the vertices of the coordinates corresponding to the i th simplicial clique. (That is, each g_i is the same as h_i , but a value for $g_i(x) = 0$ is now defined.) In this case, since every simplicial clique of H is a simplicial clique of G , by Lemma 10, the g_i 's are linearly independent, well-covered weightings of G . Now w_f is a well-covered weighting of H , and so

$$w_f = \sum_{i=1}^{sc(H)} c_i h_i.$$

Now, by the construction of w_f , $f(v) - \sum_{i=1}^{sc(H)} c_i h_i(v) = 0$ for $v \notin N(x)$, and so the well-covered weighting $g = f - \sum_{i=1}^{sc(H)} c_i g_i$ is nonzero only on vertices of $N[x]$. For any $w \in N[x]$, extend w to a maximal independent set $I(w)$ of G . Then $\sum_{u \in I(w)} g(u) = g(w)$, but g is a well-covered weighting so that g is a constant on the simplicial clique $N[x]$ and 0 is everywhere else, i.e., g is a scalar multiple of the associated weighting of the simplicial clique $N[x]$ of G . Thus $f = g + \sum_{i=1}^{sc(H)} c_i g_i$ is a linear combination of the associated weightings for the simplicial cliques of G , and we conclude that $wcdim(G) \leq sc(G)$, and hence (by Lemma 10) $wcdim(G) = sc(G)$ in this case.

Case 2. We have $\{C\} = \mathcal{C}(H) - \mathcal{C}(G)$ and $C = N(x)$. Therefore, there is a $y \in C$ which is simplicial in both H and G . Let I be any maximal independent set of $G - N[x]$. Then both $I \cup \{x\}$ and $I \cup \{y\}$ are maximal independent sets for G ,

i.e., x and y are related and thus have the same weight in any well-covered weighting of G . Note that the restriction f' of f to H is also a well-covered weighting. This follows since any maximal independent set I of H must contain a vertex of $C = N(x)$, and thus I is also a maximal independent set of G . Let h_i , $i = 1, 2, \dots, \text{sc}(H)$, be the vector with weight 1 on the coordinates corresponding to the vertices of the i th simplicial clique of H , and let C correspond to $i = 1$. By induction, this is a basis for $WC(H)$; therefore, $f = \sum_{i=1}^{\text{sc}(H)} d_i h_i$ and $d_1 h_1 = f' - \sum_{i=2}^{\text{sc}(H)} d_i h_i$. In G , we extend these vectors to g_i , $i = 2, \dots, \text{sc}(H)$, with g_i the vector having weight 1 on the coordinates corresponding to the vertices of the i th ($i > 1$) simplicial clique of H (and G). By Lemma 10, each g_i is a well-covered weighting of G and thus so is $g = f - \sum_{i=2}^{\text{sc}(G)} d_i g_i$. Under g , the only vertices with nonzero weights are those of $N[x]$. All of the vertices of C have the same weight under g since g restricted to C is h_1 . But since $f(x) = f(y)$ (y simplicial in C), it follows that g is constant on $N[x]$ and that $\{g_i | i = 2, \dots, \text{sc}(H)\} \cup \{g\}$ spans $WC(G)$. In this case, again we have that $\text{wcdim}(G) = \text{sc}(G)$.

Case 3. We have $\{C\} = \mathcal{C}(H) - \mathcal{C}(G)$ and $C \neq N(x)$. Therefore, there is a simplicial vertex $y \in C$, y adjacent to x , y not simplicial in G , and $C = N[y] - \{x\} = N(x) \cup A$. Also, in H , if $z \in A$ were a simplicial vertex, then, by Observation 1, $N[y] - \{x\} = N[z]$ and $C = N[z]$ would also be a simplicial clique in G . Therefore A contains no simplicial vertices. We define a weight function w_f on $V(H)$ by

$$w_f(v) = \begin{cases} f(v) + f(x) & \text{if } v \in A, \\ f(v) & \text{otherwise.} \end{cases}$$

Let I be a maximal independent set of H . If there exists $s \in I$ such that $s \in N(x)$, then I is a maximal independent set in G . Thus

$$\sum_{v \in I} w_f(v) = \sum_{v \in I} w_f(v) = K.$$

If I contains no vertex adjacent to x , then it must contain exactly one vertex $z \in A$, and $I \cup \{x\}$ must be a maximal independent set in G . Therefore,

$$\begin{aligned} \sum_{v \in I} w_f(v) &= w_f(z) + \sum_{v \in I - \{z\}} w_f(v) \\ &= f(x) + f(z) + \sum_{v \in I - \{z\}} f(v) \\ &= K. \end{aligned}$$

Thus, w_f is a well-covered weighting of H .

In H , let h_i , $i = 1, 2, \dots, \text{sc}(H)$, be the vector with weight 1 on the vertices of the coordinates corresponding to the i th simplicial clique where the simplicial clique containing y has index 1. By induction, this is a basis for $WC(H)$. In G , let g_i , $i = 2, 3, \dots, \text{sc}(H)$, be the vector with weight 1 on the coordinates corresponding to the vertices of the i th simplicial clique. Recall that in this case we have $\text{sc}(G) = \text{sc}(H)$ and the simplicial cliques of H with indices 2 through $\text{sc}(H)$ are also simplicial cliques in G . Thus, $\{g_i | i = 2, 3, \dots, \text{sc}(G)\}$ is a linearly independent set. Now w_f is a well-covered weighting of H , and so

$$w_f = \sum_{i=1}^{\text{sc}(H)} c_i h_i.$$

Therefore,

$$w_f - \sum_{i=2}^{\text{sc}(H)} c_i h_i = c_1 h_1,$$

i.e., all the vertices of $N[y] \cap H$ have weight c_1 in the well-covered weighting $w_f - \sum_{i=2}^{\text{sc}(H)} c_i h_i$ of H . Therefore, in G , the only vertices with nonzero weight in the well-covered weighting $g = f - \sum_{i=2}^{\text{sc}(G)} c_i g_i$ of G are the vertices of $N[y]$ with $g(z) = c_1 - f(x)$ for all $z \in A$ and $g(z) = c_1$ for $z \in N(x)$, and $g(x) = f(x)$.

We now need to show that $c_1 = f(x)$, and for that we need to find an independent set with certain properties. Let I be a minimum-sized independent set of $V(G) - (C \cup \{x\})$ that dominates (i.e., is adjacent to) the maximum number of vertices in C . If I does not dominate all the nonsimplicial vertices of C , then there exists a nonsimplicial $z \in C$ which is not dominated by a vertex of I . However, since z is not simplicial there exists $w \in G - (N[x] \cup N[y])$ with z adjacent to w . Now, since $I \cup \{w\}$ is not independent (I was maximum with this domination property), there exists $i \in I$ such that i is adjacent to w . Let $s \in C \cap N(i)$. The latter is nonempty since otherwise i could be deleted from I , a contradiction. Thus s is adjacent to z since C is a clique, and consequently, $\{z, w, i, s\}$ is a C_4 . Since H is chordal, this cycle must have a chord, specifically $w \sim s$. Since this is true for any i and s , we can replace all the neighbors of w in I by w . This independent set dominates more vertices in C than does I , and this is a contradiction. Therefore, there is an independent set J of $V(G) - (C \cup \{x\})$ which dominates all the nonsimplicial vertices in C and in particular all of A (recall that A has no simplicial vertices). Now, since J dominates all of A , $J \cup \{x\}$ and $J \cup \{y\}$ are maximal independent sets, and so x and y are related and, in particular, $g(x) = g(y) = c_1$. Thus $g(x) = g(w)$ for any $w \in N(x)$. But then for all $z \in A$, $g(z) = c_1 - g(x) = 0$. It follows that the original well-covered weighting f is a linear combination of $\{g_i | i = 2, 3, \dots, \text{sc}(G)\} \cup \{g'\}$, where g' is 1 on the vertices of $N[x]$ and is 0 everywhere else. Thus, in this and all cases, $\text{wcdim}(G) = \text{sc}(G)$, and the theorem is proved. \square

We remark that Theorem 11 holds over *any* field since all of the arguments hold over any characteristic.

4. Families of graphs with bounded well-covered dimension. In this section, we shall determine (in polynomial time) the well-covered dimension of circulant graphs of prime order and partitionable graphs; the techniques here are based in linear algebra. We begin with circulants of prime order.

We shall need some notation for maximal independent sets of a given cardinality. For a graph G , let $\mathcal{I}_t = \{I : I \text{ is a maximal independent set of } G, |I| = t\}$. Here is an upper bound that will be quite useful in this section.

LEMMA 17. *Let G be a graph G of order n , and let $t \leq \beta(G)$. Moreover, if $\text{char}(\mathbf{F}) \neq 0$, suppose that $\text{gcd}(t, \text{char}(\mathbf{F})) = 1$. Let d_t be the dimension of the subspace of \mathbf{F}^n generated by the characteristic vectors of \mathcal{I}_t . Then $\text{wcdim}(G, \mathbf{F}) \leq n - d_t + 1$. Moreover, if $d_t = n$, then the only possible well-covered weightings are constant functions.*

Proof. If \mathbf{w} is a well-covered weighting of G with sum k , then $(\mathbf{w} - \frac{k}{t} \mathbf{j}) \cdot \mathbf{i}_t = k - k = 0$, and so $w \in \text{span}(\langle \mathcal{I}_t \rangle^\perp \cup \{\mathbf{j}\})$. It follows that $\text{wcdim}(G, \mathbf{F})$ is at most the dimension of $\text{span}(\langle \mathcal{I}_t \rangle^\perp \cup \{\mathbf{j}\})$. The dimension of the latter is at most $(n - d_t) + 1$, and so it follows that $\text{wcdim}(G, \mathbf{F}) \leq n - d_t + 1$. If $d_t = n$, then $\text{span}(\langle \mathcal{I}_t \rangle^\perp \cup \{\mathbf{j}\}) = \text{span}(\{\mathbf{j}\})$, so the only possible well-covered weightings are constant functions. \square

THEOREM 18. *Let G be a circulant graph with order p , a prime. If G is not totally disconnected, then $\text{wcdim}(G) = 1$ if G is well-covered and equals 0 otherwise.*

Proof. Let $V(G) = \{0, 1, 2, \dots, p-1\}$. Let S be a maximum independent set that contains 0. Since G is not totally disconnected, then $S \neq \{0, 1, 2, \dots, p-1\}$. Note that $S_i = \{i + j \bmod p : j \in S\}$ is a maximum independent set for $i = 0, 1, \dots, p-1$ and that $S_i \neq S_j$ for $i \neq j$. Let A be the incidence matrix where the rows are indexed by S_i and the columns by $V(G)$. A is clearly a circulant matrix. As in section 2, the determinant of A is given by

$$\prod_{A(0,i)=1} x^{i-1} \quad (**)$$

over all x that are p th roots of unity. Suppose (to reach a contradiction) that for some p th root of unity, q , $\sum_{A(0,i)=1} q^{i-1} = 0$ (we follow the argument given in [4] for vanishing sums of roots of unity). Then the automorphism $\omega \rightarrow \omega^j$ of $\mathbb{Q}[\omega]$ shows that $\sum_{A(0,i)=1} \omega^{i-1} = 0$ for all primitive p th roots of unity. We now sum $(**)$ over all primitive p th roots of unity, noting that, for any primitive p th root of unity and any $1 \leq j \leq p-1$, the sum of the j th power of the primitive p th roots of unity is -1 (since this is equal to the sum of the primitive p th roots of unity). Thus

$$\begin{aligned} 0 &= \sum_{j=1}^{p-1} \sum_{A(0,i)=1} \omega_j^{i-1} \\ &= \sum_{A(0,i)=1} \sum_{j=1}^{p-1} \omega_j^{i-1} \\ &= (p-1) + (|\{i : A(0,i) = 1\}| - 1)(-1). \end{aligned}$$

Therefore, the number of nonzero terms in the first row of A must be p , implying that G is totally disconnected, which is a contradiction. Since $\det(A) \neq 0$, then A is invertible and so the row space of A has dimension p . From Lemma 17, it follows that $\text{wcdim}(G) \leq p - p + 1 = 1$. If G is well-covered, then \mathbf{j} is a well-covered weighting. If G is not well-covered, then the only well-covered weighting is the all-zero weighting. \square

5. Conclusion. The results in the previous sections give rise to a number of questions.

PROBLEM 19. *Is it possible to give a structural characterization of anti-well-covered graphs of positive dimension? Indeed, is there a polynomial algorithm to recognize such anti-well-covered graphs?*

PROBLEM 20. *As indicated in [2], the same questions can be asked of hypergraphs. Can the well-covered dimension of matroids be calculated in polynomial time?*

We can show that the well-covered dimension of a graphic matroid of a graph G is equal to the number of blocks of G .

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