THE EQUIVALENCE PROBLEM FOR ORTHOGONALLY SEPARABLE WEBS ON SPACES OF CONSTANT CURVATURE

by

Caroline Cochran

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "The Equivalence Problem for Orthogonally Separable Webs on Spaces of Constant Curvature" by Caroline Cochran in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Dated: June 9, 2011

External Examiner: ____________________

Research Supervisors: ____________________

____

Examining Committee: ____________________

____

Department Representative: ____________________

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Author: **Caroline Cochran**

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Signature of Author

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ABSTRACT

This thesis is devoted to creating a systematic way of determining all inequivalent orthogonal coordinate systems which separate the Hamilton-Jacobi equation for a given natural Hamiltonian defined on three-dimensional spaces of constant, non-zero curvature. To achieve this, we represent the problem with Killing tensors and employ the recently developed *invariant theory of Killing tensors*.

Killing tensors on the model spaces of spherical and hyperbolic space enjoy a remarkably simple form; even more striking is the fact that their parameter tensors admit the same symmetries as the Riemann curvature tensor, and thus can be considered *algebraic curvature tensors*. Using this property to obtain invariants and covariants of Killing tensors, together with the web symmetries of the associated orthogonal coordinate webs, we establish an equivalence criterion for each space. In the case of three-dimensional spherical space, we demonstrate the surprising result that these webs can be distinguished purely by the symmetries of the web. In the case of three-dimensional hyperbolic space, we use a combination of web symmetries, invariants and covariants to achieve an equivalence criterion. To completely solve the equivalence problem in each case, we develop a method for determining the *moving frame map* for an arbitrary Killing tensor of the space. This is achieved by defining an *algebraic Ricci tensor*.

Solutions to equivalence problems of Killing tensors are particularly useful in the areas of multiseparability and superintegrability. This is evidenced by our analysis of symmetric potentials defined on three-dimensional spherical and hyperbolic space. Using the most general Killing tensor of a symmetry subspace, we derive the most general potential "compatible" with this Killing tensor. As a further example, we introduce the notion of a joint invariant in the vector space of Killing tensors and use them to characterize a well-known superintegrable potential in the plane.

LIST OF ABBREVIATIONS AND SYMBOLS USED

p	A point on a manifold
p_i	Momenta coordinates
q^i	Position coordinates
H	Hamiltonian
$\otimes$	Tensor product
$\odot$	Symmetric tensor product
$\wedge$	Wedge product
$c(t)$	A curve with parameter t
f	Function
f_*	Pushforward map
f^*	Pullback map
X	A vector or vector field
ω	A one-form or one-form field
g	Metric tensor
g_E	Euclidean metric
g_M	Minkowski metric
g_S	Spherical metric
g_H	Hyperbolic metric
$T^{i_1 \ldots i_q}_{j_1 \ldots j_r}$	Tensor field of valence (q, r)
∂_i	Partial differentiation with respect to x^i
$\mathcal{M}$	Differentiable manifold
$\mathcal{M}_i$	Open subset of a manifold $\mathcal{M}$
$T_p\mathcal{M}$	Tangent space at point p on a manifold $\mathcal{M}$
$T_p^*\mathcal{M}$	Cotangent space at point p on a manifold $\mathcal{M}$
e_i	A coordinate-basis vector for $T_p\mathcal{M}$
E_i	A non-coordinate basis vector for $T_p\mathcal{M}$
dx^i	A coordinate-basis form for $T_p^*\mathcal{M}$

θ^i	A non-coordinate basis form for $T_p^*\mathcal{M}$
$T^q_{r,p}(\mathcal{M})$	Tensor field of valence (q,r) defined at a point p on a manifold $\mathcal{M}$
$T\mathcal{M}$	Tangent bundle on the manifold $\mathcal{M}$
$\mathcal{T}^q_{r,p}(M)$	Space of tensors of valence (q,r) at a point p on a manifold $\mathcal{M}$
$\Omega^r_p(\mathcal{M})$	Space of all r-forms at a point p on a manifold $\mathcal{M}$
$\Lambda^q_p(\mathcal{M})$	Space of all q-vectors at a point p on a manifold $\mathcal{M}$
$\mathcal{K}^p(\mathcal{M})$	Space of Killing tensors of valence $(p,0)$ defined on a manifold $\mathcal{M}$
K^{ij}	Killing tensor of type $(2,0)$
$I(\mathcal{M})$	Isometry group of a manifold $\mathcal{M}$
$P(1,n)$	Poincaré group
$O(n)$	Orthogonal group
$SO(n)$	Special orthogonal group
$E(n)$	Euclidean group
$SE(n)$	Special Euclidean group
$\mathfrak{so}(4)$	Lie algebra of $SO(4)$
$O(n,1)$	Lorentz group
$SO(p,q)$	Restricted Lorentz group
$\mathfrak{so}(p,q)$	Lie algebra of $SO(p,q)$
$C^{ijk\ell}$	Coefficient tensor of a Killing tensor
$\mathcal{R}_{ij}$	Algebraic Ricci tensor of the coefficient tensor of a Killing tensor
T^i_{jk}	Torsion tensor
$R^i_{jk\ell}$	Riemann curvature tensor
R_{ij}	Rotational Killing vector
K_σ	Sectional curvature of $\mathcal{M}$ on a section σ
$\delta^{i_1\cdots i_p}_{j_1\cdots j_q}$	Generalized Kronecker delta
$\mathcal{L}$	Lie derivative operator
d	Exterior derivative operator
∇	Covariant derivative operator
$[\ ,\ ]$	Schouten bracket
Γ^i_{jk}	Connection coefficient

$\mathbb{E}^n$	Euclidean space of dimension n
$\mathbb{E}^{n-s,s}$	Pseudo-Euclidean space of dimension n
$\mathbb{M}^n$	Minkowski space of dimension n
$\mathbb{S}^n$	Spherical space of dimension n
$\mathbb{H}^n$	Hyperbolic space of dimension n
D	Dilatation vector
$\mathcal{C}$	Casimir tensor
V	Potential of a Hamiltonian
P	Poisson bivector
X_H	Hamiltonian vector field
Ω	Symplectic form
F	First integral
W	Hamilton's characteristic function
N^i_{jk}	Nijenhuis tensor
H^i_{jk}	Haantjes tensor
CKT	Characteristic Killing tensor
ITKT	Invariant theory of Killing tensors
ODE	Ordinary differential equation
PDE	Partial differential equation

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CHAPTER 1

INTRODUCTION

Equivalence problems of Killing tensors occur naturally in the theory of orthogonal separation of variables of the Hamilton-Jacobi equation. Indeed, the equivalence of coordinate systems up to isometries on the manifold, and the link between orthogonal coordinate systems and Killing tensors, naturally creates an equivalence problem of Killing tensors. This is even more apparent when Hamilton-Jacobi theory is formulated in the modern language of Cartan geometry. The recently developed *invariant theory of Killing tensors* (ITKT), which arises naturally in this new formulation, has been successful in solving equivalence problems of Killing tensors on two and three-dimensional spaces of zero curvature and two-dimensional spaces of non-zero curvature.

In essence, ITKT relies on the intimate relationship between orthogonal coordinate systems and Killing tensors. In particular, on an n-dimensional manifold, an orthogonal coordinate system is characterized by $n-1$ (canonical) *characteristic Killing tensors* (CKTs) having common eigenvectors. However, just as coordinate systems belong to equivalence classes, so too do their corresponding Killing tensors. In particular, each Killing tensor is defined up to an isometry of the given space and thus fall within an equivalence class; the canonical Killing tensor is only a simple representative of its equivalence class.

By representing the problem with Killing tensors, our aforementioned goal can be laid out in the following steps:

(1) Solve the canonical forms problem for CKTs on the given space;

(2) Solve the equivalence problem for CKTs on the given space. This requires solving the following subproblems:

   (i) Develop a classification scheme for CKTs.

(ii) Determine the transformation to canonical form for an arbitrary CKT.

For a natural Hamiltonian defined on a given space, we first need to determine all possible orthogonally separable coordinate systems and the corresponding representative canonical Killing tensor. This constitutes the *canonical forms problem* for a given manifold. Furthermore, if we are given a non-canonical Killing tensor, we need a way of determining which orthogonal coordinate system it represents. Since the Killing tensor is non-canonical, we will also need a way to determine the transformation back to its canonical form so that we can properly define the corresponding coordinate system. These latter two objectives constitute what is called the *equivalence problem for Killing tensors* on the given space.

This thesis is devoted to solving the equivalence problem for two of the most well-known spaces of constant curvature, namely spherical and hyperbolic space. Using the solution to the canonical forms problem for CKTs given by Eisenhart [22] and Olevskii [62], we will develop a classification scheme for CKTs defined on both two and three-dimensional spherical and hyperbolic space using invariants, covariants and symmetries of orthogonal coordinate webs. Furthermore, we will define an important object called the *algebraic Ricci tensor* and use it to determine the transformation to canonical form for an arbitrary CKT.

The content of this thesis is laid out as follows. We first expound the requisite theory from pseudo-Riemannian geometry, defining the principal spaces of this thesis. This is followed by a chapter on Hamiltonian systems, with particular attention paid to Hamilton-Jacobi theory and orthogonal separation of variables. In the third chapter, we first review the theory of Killing tensors, using several illustrative examples. Then, we review the necessary material from invariant theory, including the method of moving frames - both in its classical and modern formulation. In the last section of the chapter we review the fusion of these two topics, namely, the *invariant theory of Killing tensors*. The development and results of the theory are considered in detail, illustrated with numerous examples. In the two chapters that follow, we present a solution to the equivalence problem of Killing tensors defined on three-dimensional spherical space and hyperbolic space, respectively. In the last chapter, we present a series of examples to illustrate the theory, as well as demonstrate its applicability to problems in mathematical physics.

CHAPTER 2

RIEMANNIAN GEOMETRY

Differential geometry is a field of mathematics which studies the properties, theories and applications of differentiable manifolds. A subfield of this body of work, named after Bernhard Riemann, which focusses on differentiable manifolds equipped with a Riemannian metric, is *Riemannian geometry*. Accordingly, the manifolds of this type are called *Riemannian manifolds*.

The problem that we study in this thesis concerns objects defined on spherical and hyperbolic space, both examples of Riemannian manifolds. Therefore, the purpose of this chapter is to formulate a set of definitions and theorems from Riemannian geometry pertinent to this thesis. In doing so, we have tried to follow a natural ordering. We begin with manifolds, define objects on them and then consider mappings. Our consideration of differentiable manifolds permits us to perform calculus on our manifold, and we define the three most important differential operators. By this point, we have covered the foundational information and can define the two principal geometric spaces of the thesis, namely spherical and hyperbolic space. We then state a fundamental theorem of differential geometry, defining several requisite terms, and conclude the chapter with a section on fibre bundle theory.

Having outlined the theory of this chapter, let us begin our exposition of Riemannian geometry. As we stated above, the central structure in this theory is the *Riemannian manifold*. But before we can give its definition, we require the fundamental concepts of *manifold* and *tensor* from differential geometry.

2.1 Manifolds

The fundamental structure in differential geometry is the *manifold*. This is a type of topological space which is locally Euclidean in the sense that on any open subset of the n-dimensional manifold $\mathcal{M}$, we have a homeomorphism which maps $\mathcal{M}$[1] to Euclidean n-space. Having this local property is very convenient, as it enables us to carry over many properties of Euclidean space to a manifold.

We describe a manifold by specifying these homeomorphisms φ_i on open subsets $\mathcal{M}_i$ of $\mathcal{M}$. A subset together with its homeomorphism is called a *chart*, and the collection of these charts on a manifold is called an *atlas*. Let us formulate the definition.

Definition 2.1.1. Given a set $\mathcal{M}$, define a collection of subsets $\mathcal{M}_i$ whose union is all of $\mathcal{M}$. If on each $\mathcal{M}_i$, we have

(i) a homeomorphism $\varphi_i : \mathcal{M}_i \to \mathbb{E}^n$,

(ii) each $\varphi_i(\mathcal{M}_i)$ is an open subset of $\mathbb{E}^n$,

(iii) for any $\mathcal{M}_i \cap \mathcal{M}_j \neq \emptyset$, we have transition functions $\varphi_j \circ \varphi_i^{-1} : \varphi_i(\mathcal{M}_i \cap \mathcal{M}_j) \to \varphi_j(\mathcal{M}_i \cap \mathcal{M}_j)$, and

(iv) each $\varphi_i(\mathcal{M}_i \cap \mathcal{M}_j)$ is an open subset of $\mathbb{E}^n$,

then $\mathcal{M}$ is called an n-dimensional *manifold*.

For $p \in \mathcal{M}_i$, the image $\varphi_i(p) = (x^1, \ldots, x^n) \in \mathbb{E}^n$ is called the *coordinates* of p.

Manifolds can be characterized by the properties of their transition functions. For example, a manifold admitting differentiable[2] transition functions are called *differentiable manifolds*. Hereafter, whenever we speak of a manifold, we mean a differentiable manifold.

2.2 Objects Defined on Manifolds

Let us begin with the simplest objects that can be defined on a manifold, namely curves and functions.

[1] A *homeomorphism* is a map which is one-to-one, onto, continuous and has a continuous inverse.
[2] By differentiable we mean C^∞-differentiable.

2.2.1 Curves and Functions

Definition 2.2.1. Given a manifold $\mathcal{M}$, a differentiable map $c : (a, b) \to \mathcal{M}$ is called a *curve* on $\mathcal{M}$.

The coordinates of a curve $c(t)$ on a chart $(\mathcal{M}_i, \varphi)$ of $\mathcal{M}$ are given by $\varphi(c(t)) = x^i(t)$, where x^i are differentiable functions with respect to t.

Definition 2.2.2. A function f on a manifold $\mathcal{M}$ is a map

$$f : \mathcal{M} \to \mathbb{R}.$$

If $p \in \mathcal{M}$ belongs to a chart $(\mathcal{M}_i, \varphi)$ on an n-dimensional manifold $\mathcal{M}$ with coordinates $\varphi(p) = (u^1, \ldots, u^n)$, and $\varphi^{-1}(u^1, \ldots, u^n) = (x^1, \ldots, x^n)$, then

$$f(p) = f(\varphi^{-1}(u^1, \ldots, u^n)) = f(x^1, \ldots x^n) = y.$$

If f is continuous and differentiable with respect to each x^i, ie.,

$$\left.\frac{\partial f}{\partial x}\right|_p = \left.\frac{\partial(f(\varphi^{-1}))}{\partial u^i}\right|_{\varphi(p)}$$

exists for all $p \in \mathcal{M}_i$, then f is called a *differentiable function*. More generally, a function f is C^r*-differentiable* if the derivatives of f up to and including order r exist and are continuous. If $r = \infty$, the function is said to be *smooth*.

2.2.2 Tangent and Cotangent Vectors

At each point of a curve on $\mathcal{M}$, we have a vector $X|_p$ which is tangent to the curve at the point p.

Definition 2.2.3. Consider a curve $c : (-b, b) \to \mathcal{M}$ on a manifold $\mathcal{M}$, and a function $f : \mathcal{M} \to \mathbb{R}$. A *tangent vector to the curve* $c(t)$ *at a point* $p = c(0)$ is a differential operator

$$X = \frac{d}{dt}$$

which acts on f along $c(t)$ at $t = 0$.

Suppose on a chart of $\mathcal{M}$ a curve $c : (-b, b) \to \mathcal{M}$ has coordinates $x^i(t)$. Then for a differentiable function $f : \mathcal{M} \to \mathbb{R}$, there is a differentiable function

$$g(t) = f(x^i(t))$$

which specifies the values of f along the curve. The action of X on g along $c(t)$ at a point $p = c(0)$ is then given by[3]

$$X(g(0)) = \frac{dg}{dt}\bigg|_{t=0} = \frac{\partial f}{\partial x^i}\frac{dx^i}{dt}\bigg|_{t=0}.$$

Since this holds for any function g, we have

$$X = \frac{d}{dt} = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} = X^i\frac{\partial}{\partial x^i},$$

which denotes the coordinate representation of the tangent vector X to this curve at $p = c(0)$.

If a pair of curves both satisfying $c(0) = p$ have the same tangent vector at p, then they are said to be *equivalent*. Using this equivalence relation, the curves on a manifold can be grouped into equivalence classes.

If we fix a point on a manifold, there may be infinitely many curves on the manifold which pass through this point with the same tangent vector at this point. As such, if we are given a tangent vector to a point on a manifold, we do not identify the vector as the tangent to a single curve, rather, to an equivalence class of curves.

Definition 2.2.4. A *tangent vector at p* on a manifold $\mathcal{M}$ can be identified with the set of all equivalent curves on $\mathcal{M}$ which pass through p.

The collection of all tangent vectors at a point p on a manifold $\mathcal{M}$ forms a vector space, called the *tangent space* $T_p\mathcal{M}$. If x^i are coordinates of p, then a basis for $T_p\mathcal{M}$ is

$$e_1 = \frac{\partial}{\partial x^1}, \ldots, e_n = \frac{\partial}{\partial x^n},$$

[3] Here we assume summation over the index i, in accordance with the *Einstein summation convention*. This stipulates that if an index appears in both an upper and lower position, summation over the index is implied. Unless otherwise stated, we will adopt this convention throughout this thesis.

called a *coordinate basis*, and a tangent vector $X \in T_p\mathcal{M}$ is given by

$$X = X^i \frac{\partial}{\partial x^i}.$$

If $(\tilde{x}^1, \ldots, \tilde{x}^n)$ is another set of coordinates for p, then

$$\tilde{X} = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i} = X^i \frac{\partial}{\partial x^i} = X,$$

and the components of the tangent vector in each coordinate representation are related by

$$\tilde{X}^j = X^i \frac{\partial \tilde{x}^j}{\partial x^i}.$$

Therefore, the vectors

$$E_i = \frac{\partial}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial}{\partial x^j} = \frac{\partial x^j}{\partial \tilde{x}^i} e_j$$

give another basis for $T_p\mathcal{M}$, called a *non-coordinate basis*.

The set of all tangent spaces on a manifold is also a manifold, called a *tangent bundle*. More specifically, for an n-dimensional manifold $\mathcal{M}$,

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}$$

is called the *tangent bundle* of $\mathcal{M}$ with dimension $2n$.

We will now use the notion of a tangent vector to define our next object, called a *cotangent vector* or *one-form*.

Definition 2.2.5. A linear function ω defined at a point $p \in \mathcal{M}$ which maps a tangent vector to a real number is called a *one-form*.

At each point p on a manifold $\mathcal{M}$, we have a vector space of one-forms, called the *cotangent space* $T_p^*\mathcal{M}$. If x^i are coordinates of p, then a coordinate basis for $T_p^*\mathcal{M}$ is

$$dx^1, \ldots, dx^n$$

and a one-form $\omega \in T_p^*\mathcal{M}$ is given by

$$\omega = \omega_i dx^i.$$

If $(\tilde{x}^1, \ldots, \tilde{x}^n)$ is another set of coordinates for p, then

$$\tilde{\omega} = \tilde{\omega}_i d\tilde{x}^i = \omega_i dx^i = \omega,$$

and the components of the one-form in each coordinate representation are related by

$$\tilde{\omega}_j = \omega_i \frac{\partial x^j}{\partial \tilde{x}^i}.$$

Therefore, the one-forms

$$\theta^i = d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j$$

give another basis for $T_p^*\mathcal{M}$, called a *non-coordinate basis*.

The collection of all cotangent spaces on a manifold is also a manifold, called the *cotangent bundle*. More specifically, for an n-dimensional manifold $\mathcal{M}$,

$$T^*\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p^*\mathcal{M}$$

is called the *cotangent bundle* of $\mathcal{M}$ with dimension $2n$.

From Definition 2.2.5, we see that one-forms are linear functions which take as input vectors and map them to a real number:

$$\omega : T_p\mathcal{M} \to \mathbb{R}.$$

Tangent vectors can also be defined in this way; namely, they are linear functions which take as input one-forms and map them to a real number:

$$X : T_p^*\mathcal{M} \to \mathbb{R}.$$

The action of a one-form on a vector, and vice-versa, is given by the expression

$$\omega(X) = X(\omega) = \omega_i X^i, \tag{2.1}$$

called the *contraction* of ω and X. Since X and ω are linear functions of each other, they are said to be *dual*, and thus $T_p(\mathcal{M})$ and $T_p^*(\mathcal{M})$ are called *dual vector spaces*. Defining one-forms and tangent vectors in this way naturally motivates the subject of the next section.

2.2.3 Tensors

We can generalize the notions of vector or one-form on a manifold to a *tensor*. Let us begin by defining a product between a one-form T_1 and a tangent vector T_2 as follows:

$$T_1 \otimes T_2 : T_p^*(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R},$$

where

$$(T_1 \otimes T_2)(X, \omega) = T_1(X)T_2(\omega).$$

The resulting function $T_1 \otimes T_2$ is linear in both of its arguments (ie., *bilinear*). We call $T_1 \otimes T_2$ the *tensor product* of T_1 and T_2, and the resulting function a *tensor*. In general, we have the following.

Definition 2.2.6. A multilinear[4] function

$$T : \times^r T_p^*\mathcal{M} \times^q T_p\mathcal{M} \to \mathbb{R}$$

defined at a point $p \in \mathcal{M}$ which maps q one-forms and r vectors to a real number is called a *tensor of valence* (q, r).

We denote the vector space of all tensors of valence (q, r) defined at a point $p \in \mathcal{M}$ by $\mathcal{T}^q_{r,p}(\mathcal{M})$, called a *tensor space*. If x^i are coordinates on a chart of $\mathcal{M}$, then a coordinate basis for $\mathcal{T}^q_{r,p}(\mathcal{M})$ is

$$\left(\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_q}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_r}\right)_{i_1\ldots i_q, j_1\ldots j_r = 1\ldots n}$$

and a tensor $T \in \mathcal{T}^q_{r,p}(\mathcal{M})$ is given by

$$T = T^{i_1\ldots i_q}_{j_1\ldots j_r} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_q}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_r}.$$

We can specify its action on vectors X and one-forms ω by naturally generalizing (2.1):

$$T(X_1, \ldots, X_r, \omega_1, \ldots, \omega_q) = T^{i_1\ldots i_q}_{j_1\ldots j_r} X_1^{j_1} \ldots X_r^{j_r} \omega_{1,i_1} \ldots \omega_{q,i_q}.$$

If $q \neq 0$ and $r = 0$, then T is called a *contravariant tensor of valence* q. A vector, therefore, can be regarded as a contravariant tensor of valence one. If $q = 0$ and $r \neq 0$, then T is called a *covariant tensor of valence* r. A one-form is therefore a covariant tensor of valence one.

Many of the tensors in this thesis are either *symmetric* or *antisymmetric* tensors. In particular, we say that a contravariant tensor is *symmetric* if it is symmetric in any pair of its indices:

$$T^{i_1\ldots i_\ell \ldots i_k \ldots i_q} = T^{i_1\ldots i_k \ldots i_\ell \ldots i_q}.$$

[4] Multilinearity refers to linearity in each of the function's arguments.

It is *antisymmetric* if it is antisymmetric in any pair of its indices:

$$T^{i_1\dots i_\ell\dots i_k\dots i_q} = -T^{i_1\dots i_k\dots i_\ell\dots i_q}.$$

Analogous definitions of symmetry and antisymmetry can be made for covariant tensors.

Covariant tensors of valence r that are antisymmetric are called *r-forms*. We denote the vector space of all r-forms on a manifold $\mathcal{M}$ at $p \in \mathcal{M}$ by $\Omega^r_p(\mathcal{M})$. Dual to these objects are *q-vectors*, which are antisymmetric contravariant tensors of valence q. When $q = 2$, the object is called a *bivector*. We denote the vector space of all q-vectors on a manifold $\mathcal{M}$ at $p \in \mathcal{M}$ by $\Lambda^q_p(\mathcal{M})$.

Returning now to general tensors, it is useful to know several of the basic operations which can be performed on tensors, namely addition, subtraction, tensor multiplication and contraction. If tensors are of the same valence, we can add (subtract) them together by summing (subtracting) their corresponding components in the natural way. We can multiply tensors of arbitrary valence together using the tensor product $\otimes$ defined earlier. In particular, if $T \in \mathcal{T}^{q_1}_{r_1}(\mathcal{M})$ and $S \in \mathcal{T}^{q_2}_{r_2}(\mathcal{M})$, then their tensor product $T \otimes S \in \mathcal{T}^{q_1+q_2}_{r_1+r_2}(\mathcal{M})$ is specified by:

$$\begin{aligned}&(T\otimes S)(\omega_1,\dots,\omega_{q_1},\eta_1,\dots,\eta_{q_2};X_1,\dots,X_{r_1},Y_1,\dots,Y_{r_2})\\ &\qquad = T(\omega_1,\dots,\omega_{q_1},X_1,\dots,X_{r_1})S(\eta_1,\dots,\eta_{q_2},Y_1,\dots,Y_{r_2}).\end{aligned}$$

If (x^i) are coordinates of $p \in \mathcal{M}$, then for

$$T = T^{i_1\dots i_{q_1}}_{j_1\dots j_{r_1}} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_{q_1}}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_{r_1}}$$

in $\mathcal{T}^{q_1}_{r_1,p}(\mathcal{M})$, and for

$$S = S^{k_1\dots k_{q_2}}_{\ell_1\dots \ell_{r_2}} \frac{\partial}{\partial x^{k_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{k_{q_2}}} \otimes dx^{\ell_1} \otimes \dots \otimes dx^{\ell_{r_2}}$$

in $\mathcal{T}^{q_2}_{r_2,p}(\mathcal{M})$, we have

$$\begin{aligned}T\otimes S &= T^{i_1\dots i_{q_1}}_{j_1\dots j_{r_1}} S^{k_1\dots k_{q_2}}_{\ell_1\dots \ell_{r_2}} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_{q_1}}} \otimes \frac{\partial}{\partial x^{k_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{k_{q_2}}} \otimes \\ &\qquad dx^{j_1} \otimes \dots \otimes dx^{j_{r_1}} \otimes dx^{\ell_1} \otimes \dots \otimes dx^{\ell_{r_2}}.\end{aligned}$$

Let us now consider two special types of tensor products, namely the *symmetric tensor product* and the *wedge product*. Since the tensor product of two symmetric tensors does not necessarily yield a symmetric tensor, we are motivated to define the following special type of tensor product.

Definition 2.2.7. For symmetric tensors $T_1 \in \mathcal{T}^0_r(\mathcal{M})$ and $T_2 \in \mathcal{T}^0_s(\mathcal{M})$, their *symmetric tensor product* yields a symmetric tensor $T_1 \odot T_2$ given by

$$(T_1 \odot T_2)(X_1, \ldots, X_{r+s}) = \frac{1}{(r+s)!} \sum_{\sigma \in S_{r+s}} T_1(X_{\sigma(1)}, \ldots, X_{\sigma(r)}) T_2(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}),$$

where S_n denotes the symmetric group on a set of n elements.

In an analogous way, we can also define the symmetric tensor product on symmetric contravariant tensors.

The symmetric tensor product admits the following properties for symmetric tensors T_1, T_2, T_3 and $a, b \in \mathbb{R}$:

(i) $T_1 \odot T_2 = T_2 \odot T_1, \quad$ (commutivity)

(ii) $T_1 \odot (T_2 \odot T_3) = (T_1 \odot T_2) \odot T_3, \quad$ (associativity)

(iii) $T_1 \odot (aT_2 + bT_3) = aT_1 \odot T_2 + bT_1 \odot T_3, \quad$ (bilinearity)

$(aT_1 + bT_2) \odot T_3 = aT_1 \odot T_3 + bT_2 \odot T_3.$

Similarly, since the tensor product of two antisymmetric tensors does not necessarily yield an antisymmetric tensor, we are motivated to define the following special type of tensor product.

Definition 2.2.8. For an r-form ω and an s-form η, their *wedge product* yields an antisymmetric tensor $\omega \wedge \eta$ given by

$$(\omega \wedge \eta)(X_1, \ldots, X_{r+s}) = \frac{1}{r!s!} \sum_{\sigma \in S_{r+s}} (\text{sgn } \sigma) \omega(X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \eta(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}),$$

where S_n denotes the symmetric group on a set of n elements, and

$$\text{sgn } \sigma = \begin{cases} +1 & \text{when } \sigma \text{ is even} \\ -1 & \text{when } \sigma \text{ is odd.} \end{cases}$$

In an analogous way, we can also define the wedge product on k-vectors.

The wedge product admits the following properties for $\omega \in \Omega^r(\mathcal{M})$, $\eta \in \Omega^s(\mathcal{M})$, $\xi \in \Omega^t(\mathcal{M})$, and $a, b \in \mathbb{R}$:

(i) $\omega \wedge \eta = (-1)^{rs}\eta \wedge \omega$, (anticommutivity)

(ii) $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$, (associativity)

(iii) $\omega \wedge (a\eta + b\xi) = a(\omega \wedge \eta) + b(\omega \wedge \xi)$, (bilinearity)

$(a\eta + b\xi) \wedge \omega = a(\eta \wedge \omega) + b(\xi \wedge \omega)$.

Using the associative property, the wedge product of n one-forms $\{\omega^1 = \omega^1_{i_1} dx^{i_1}, \ldots, \omega^n = \omega^n_{i_n} dx^{i_n}\}$ is the r-form

$$\omega^1 \wedge \cdots \wedge \omega^n = \omega^1_{i_1} \cdots \omega^n_{i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n}. \tag{2.2}$$

Using the anticommutivity property, (2.2) can be rewritten[5] as

$$\begin{aligned} \omega^1 \wedge \cdots \wedge \omega^n &= \tfrac{1}{n!}\delta^{i_1\cdots i_n}_{j_1\cdots j_n}\omega^1_{i_1} \cdots \omega^n_{i_n} dx^{j_1} \wedge \cdots \wedge dx^{j_n} \\ &= \tfrac{1}{n!}\omega^1_{[i_1} \cdots \omega^n_{i_n]} dx^{i_1} \wedge \cdots \wedge dx^{i_n}, \end{aligned} \tag{2.3}$$

where $\delta^{i_1\cdots i_n}_{j_1\cdots j_n} = \delta^{i_1}_{[j_1} \cdots \delta^{i_n}_{j_n]}$ denotes the generalized Kronecker delta. As a consequence of (2.3), we can take the wedge product of a set of one-forms to determine if they form a linearly independent set. In particular, $\{\omega^1, \ldots, \omega^n\}$ are linearly dependent if and only if

$$\omega^1 \wedge \cdots \wedge \omega^n = 0. \tag{2.4}$$

If $p = (x^i)$ is a point on an m-dimensional manifold and $\{dx^1, \ldots, dx^m\}$ is a coordinate basis for $T^*_p(\mathcal{M})$, then the wedge product

$$dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \quad i_1 < \ldots < i_r < m,$$

yields a set of $\binom{m}{r}$ linearly independent r-forms on $\mathcal{M}$, and thus forms a basis for $\Omega^r_p(\mathcal{M})$. It follows that any r-form $\omega \in \Omega^r_p(\mathcal{M})$ can be written as

$$\omega = \frac{1}{r!}\omega_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

[5] Round brackets around the indices of a tensor denote *symmetrization of the indices*. A tensor with symmetrized indices is equivalent to summing over all possible permutations of the indices of this tensor, and dividing by the total number of permutations. Square brackets around the indices of a tensor denote *antisymmetrization of the indices*. A tensor with antisymmetrized indices is equivalent to the difference between all even permutations and all odd permutations of the indices of the tensor, and then dividing by the total number of permutations. Any index placed between vertical bars within round or square brackets indicates that this index is exempt from symmetrization or antisymmetrization [76].

We can use the wedge product to determine if a set of functions are functionally dependent, which will be of use later in this thesis. Recall that a set of functions $\{f_1, \ldots, f_n\}$ are *functionally dependent* if there exists a smooth function $h \neq 0$ such that

$$h(f_1, \ldots, f_n) = 0.$$

A set of functions which are not functionally dependent are called *functionally independent*. If we take the exterior derivative of a set of smooth functions $\{f_1, \ldots, f_n\}$, and then form their wedge product, we obtain

$$\begin{aligned} df_1 \wedge \ldots \wedge df_n &= \left(\frac{\partial f_1}{\partial x^{i_1}} dx^{i_1}\right) \wedge \ldots \wedge \left(\frac{\partial f_n}{\partial x^{i_n}} dx^{i_n}\right) \\ &= \left(\frac{\partial f_1}{\partial x^{i_1}} \cdots \frac{\partial f_n}{\partial x^{i_n}}\right) dx^{i_1} \wedge \ldots \wedge dx^{i_n} \\ &= \frac{1}{n!} \left(\frac{\partial f_1}{\partial x^{[i_1}} \cdots \frac{\partial f_n}{\partial x^{i_n]}}\right) dx^{i_1} \wedge \ldots \wedge dx^{i_n}. \end{aligned}$$

Note that the coefficient of this n-form is the Jacobian determinant, J, of the set of functions. Since $J = 0$ if $\{f_1, \ldots, f_n\}$ are functionally dependent, we find that if

$$df_1 \wedge \ldots \wedge df_n = 0,$$

then $\{f_1, \ldots, f_n\}$ are functionally dependent.

Returning now to general operations on tensors, we note that we can define a further operation on tensors called *contraction* which generalizes (2.1). By summing over a pair of contravariant and covariant indices in a tensor of valence (q, r), we obtain a new tensor of valence $(q-1, r-1)$. In particular, if

$$T^{i_1 \ldots i_k \ldots i_q}_{j_1 \ldots j_\ell \ldots j_r}$$

denote the components of a tensor $T \in T^q_r(\mathcal{M})$ on an n-dimensional manifold $\mathcal{M}$, then

$$T^{i_1 \ldots i_k \ldots i_q}_{j_1 \ldots i_k \ldots j_r}$$

is a contraction on T.

2.2.4 Vector Fields, Integral Curves, Flows; Tensor Fields

Recall that a tensor $T|_p$ of valence (q, r) maps q one-forms and r vectors at p to a real number. In contrast, a *tensor field* is a differentiable map which assigns to each $p \in \mathcal{M}$ a tensor in $\mathcal{T}^q_{r,p}(\mathcal{M})$. On a given manifold $\mathcal{M}$, we let $\mathcal{T}^q_r(\mathcal{M})$ denote the collection of all tensor fields of type (q, r).

There are several subcases of importance to consider: a (0,0)-tensor field is called a scalar function f, a (0,1)-tensor field a *one-form field*, and a (1,0)-tensor field a *vector field*. Let us concentrate on this last concept for a moment. We can envision a vector field as a tangent vector at each point on the manifold. If there exist curves on the manifold in which each tangent vector of the curve coincides with the vectors of this field, we call them integral curves of the vector field. In particular, we say that a curve $c(t)$ is an *integral curve* of a vector field X if

$$\frac{d(c(t))}{dt} = X|_{c(t)}$$

for all $t \in \mathbb{R}$. If $c(t)$ has coordinates $x^i(t)$, then this condition becomes

$$\frac{dx^i}{dt} = X^i(c(t)). \tag{2.5}$$

For a given vector field, the existence of such curves is guaranteed at least in a neighborhood of a point $p = c(0)$ by the *existence and uniqueness theorem of ODEs*, since (2.5) is a system of ODEs in terms of the function $c(t)$. Since this is only a local result, such curves $c(t)$ may only be defined on an interval (t_1, t_2) of $\mathbb{R}$.

A vector field X on a manifold $\mathcal{M}$ gives rise to a *flow* on $\mathcal{M}$. In particular, given a vector field X on a manifold $\mathcal{M}$ and $x \in \mathcal{M}$, there exists a differentiable map $\sigma : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ satisfying

(i) $\dfrac{d\sigma(t, x)}{dt} = X(\sigma(t, x))$,

(ii) $\sigma(0, x) = x$,

(iii) $\sigma(t, \sigma(s, x)) = \sigma(t + s, x), \quad t, s \in \mathbb{R}$,

called the *flow* on $\mathcal{M}$. We call the vector field X the *generator* of the flow. If we recall the image of a vector field described earlier, its integral curves represent the *flow curves* of its flow. If a flow curve passes through $x \in \mathcal{M}$ at time $t = 0$, then $\sigma(t, x)$ denotes the position along a flow curve at time $t > 0$.

The flow on a manifold can be interpreted as a type of *group action* on the manifold. In particular, we define a map $\psi : G \times \mathcal{M} \to \mathcal{M}$ satisfying

(i) $\psi(0, x) = x, \quad \forall x \in \mathcal{M}$,

(ii) $\psi(g, \psi(g, x)) = \psi(gh, x)$,

for $g, h \in G$ as an *action* by a group G on a manifold $\mathcal{M}$. Therefore, we can regard the flow σ on a manifold as an action by the group $G = \mathbb{R}$ on the manifold. It follows that the collection of σ_t forms a transformation group on the manifold in the single parameter t, called the *one-parameter group of transformations*.

There are certain collections of vector fields and one-form fields on a manifold that will be used in subsequent chapters, namely *frames* and *coframes*. If $\mathcal{M}$ is an m-dimensional manifold, a collection of ordered vector fields $\{E_1, \ldots, E_m\}$ on $\mathcal{M}$ which form a basis at each tangent space $T_p\mathcal{M}$ is called a *frame* on $\mathcal{M}$. Analogously, a set of one-form fields $\{\theta^1, \ldots, \theta^m\}$ on $\mathcal{M}$ which form a basis at each cotangent space $T_p^*\mathcal{M}$ is called a *coframe* on $\mathcal{M}$. Property (2.4) can be used to determine if a set of one-forms defines a coframe on a manifold.

Let us return to the more general concept of a tensor field. There are many important examples of tensor fields in differential geometry, one of which is the metric tensor field. Such a tensor field is required to be defined on a manifold if length is to be measured.

2.3 Pseudo-Riemannian Manifolds

At the beginning of this chapter, we noted that Riemannian geometry is the study of manifolds admitting Riemannian metrics. Likewise, pseudo-Riemannian geometry is the study of manifolds with a pseudo-Riemannian or “almost” Riemannian metric. Let us begin with the more general of these two types of manifolds.

Table 2.1: Types of pseudo-Euclidean vectors

Space-like vector	$g(X,X) > 0,$
Time-like vector	$g(X,X) < 0$
Null vector	$g(X,X) = 0$

2.3.1 Pseudo-Riemannian Manifold

Definition 2.3.1. A (0,2)-tensor field $g|_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ defined on a manifold $\mathcal{M}$ satisfying the following two properties

(i) $g(X,Y) = g(Y,X)$, (symmetry)

(ii) $g(X,Y) = 0$ for all $Y \in T_pM$ iff $X = 0$, (non-degeneracy)

at each point $p \in \mathcal{M}$ and $X, Y \in T_p\mathcal{M}$, is a *pseudo-Riemannian metric* on $\mathcal{M}$.

When a smooth manifold $\mathcal{M}$ is equipped with a pseudo-Riemannian metric, we call $\mathcal{M}$ a *pseudo-Riemannian manifold*. Since g is non-degenerate, the eigenvalues of the matrix g^{ij} are nonzero. Thus each eigenvalue has a sign "+" or "-", and we indicate the number of these signs for a given metric by its *signature*. In general, the signature (i,j) indicates that g has i positive and j negative eigenvalues.

An example of a pseudo-Riemannian manifold is n-dimensional pseudo-Euclidean space, $\mathbb{E}^{n-s,s}$. The fixed number $s \in [0,n]$ in the superscript specifies the signature of the metric,[6]

$$ds^2 = -(dx^1)^2 - \ldots - (dx^s)^2 + (dx^{s+1})^2 + \ldots + (dx^n)^2,$$

which we write in pseudo-Cartesian coordinates x^i. Note that if we set $s = 0$, we obtain the usual n-dimensional Euclidean space $\mathbb{E}^{n,0} = \mathbb{E}^n$, where the positive-definiteness of the metric requires vectors to have only positive length. With the more general metric of pseudo-Euclidean space, the lengths of vectors can be positive, negative or null, and we classify them accordingly (see Table 2.1) [50].

Another important case occurs if we set $s = 1$, which gives us n-dimensional Minkowski space $\mathbb{E}^{n-1,1} = \mathbb{M}^n$. When $n = 4$, this space represents the geometric model of Einstein's special theory of general relativity. In this setting, we let the x^1

[6] A metric, g, is also denoted by ds^2.

coordinate represent the dimension of time, while x^2, x^3, x^4 represent the three spatial dimensions.

2.3.2 Riemannian Manifold

The most important subclass of pseudo-Riemannian manifolds are Riemannian manifolds. On these manifolds, the non-degeneracy condition of the metric is replaced by a stricter positive-definitiveness condition.

Definition 2.3.2. A (0,2)-tensor field $g|_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ defined on a differentiable manifold $\mathcal{M}$ satisfying the following two properties

(i) $g(X,Y) = g(Y,X)$, (symmetry)

(ii) $g(X,X) \geq 0$, with equality only when $X = 0$, (positive-definiteness)

at each point $p \in \mathcal{M}$ and $X, Y \in T_p\mathcal{M}$, is a *Riemannian metric* on $\mathcal{M}$.

When a smooth manifold $\mathcal{M}$ is equipped with a Riemannian metric, we call $\mathcal{M}$ a *Riemannian manifold*. There are several well-known examples of Riemannian manifolds, but the simplest example is Euclidean space, $\mathbb{E}^n$, equipped with the Euclidean metric

$$g_E = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2$$

in Cartesian coordinates x^i.

2.4 Maps on Manifolds

At the beginning of this chapter, we defined a function on a manifold $\mathcal{M}$ as a map $f : \mathcal{M} \to \mathbb{R}$. Let us now consider functions between manifolds.

Definition 2.4.1. A function f between manifolds $\mathcal{M}$ and $\mathcal{N}$ is a map

$$f : p \to \tilde{p},$$

where $p \in \mathcal{M}$ and $\tilde{p} \in \mathcal{N}$.

Suppose p belongs to a chart $(\mathcal{M}_i, \varphi)$ on an n-dimensional manifold $\mathcal{M}$ with coordinates $\varphi(p) = (x^1, \ldots, x^m)$. If $\tilde{p}$ belongs to a chart $(\mathcal{N}_i, \psi)$ on an n-dimensional manifold $\mathcal{N}$, then the coordinates of $f(p) = \tilde{p}$ are given by

$$\psi(\tilde{p}) = \psi(f(p)) = \psi(f(\varphi^{-1}(x^1, \ldots, x^m))) = (y^1, \ldots, y^n).$$

Therefore, we can simply write

$$f^i(x^j) = y^i,$$

where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. If the f^i are differentiable with respect to each x^j, then f is a differentiable function. If the f^i is infinitely differentiable, then f is a smooth function.

Consider the following important class of functions between manifolds.

Definition 2.4.2. Suppose $f : \mathcal{M} \to \mathcal{N}$ is a bijective[7] map between manifolds $\mathcal{M}$ and $\mathcal{N}$. If both f and f^{-1} are smooth functions, then f is called a *diffeomorphism*.

If a diffeomorphism can be found between manifolds $\mathcal{M}$ and $\mathcal{N}$, they are said to be *diffeomorphic*. This would imply that their dimensions coincide.

2.4.1 Tensor Transformation Law

If we define a diffeomorphism $f : \mathcal{M} \to \mathcal{M}$ on a manifold, how do the tensors on $\mathcal{M}$ transform under f? Suppose $(x^1, \ldots, x^n)$ are coordinates of $p \in \mathcal{M}$, and $(y^1, \ldots, y^n)$ are coordinates of $f(p)$. Then a tensor T defined at p is transformed under f to $\tilde{T}$ at $f(p)$ according to

$$\tilde{T}^{i_1 \ldots i_q}_{j_1 \ldots j_r} = T^{k_1 \ldots k_s}_{\ell_1 \ldots \ell_t} \frac{\partial y^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial y^{i_q}}{\partial x^{k_s}} \frac{\partial x^{\ell_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{\ell_t}}{\partial y^{j_r}}.$$

We say that an object which satisfies this relation "transforms like a tensor," and we call this rule the *tensor transformation law*. In some texts, this is the way a tensor is defined.

Here, f belongs to the more restrictive class of diffeomorphisms and maps $\mathcal{M}$ to itself. Let us now consider the more general case of a smooth map between different manifolds.

[7] A map which is both one-to-one and onto is called *bijective*.

2.4.2 Pullback and Pushforward Maps

If we define a smooth map $f : \mathcal{M} \to \mathcal{N}$ between manifolds $\mathcal{M}$ and $\mathcal{N}$, we simultaneously create a correspondence between the tensors on the two manifolds. For example, imagine a curve C on a manifold $\mathcal{M}$ and the tangent vectors along C. If f maps this curve to another manifold $\mathcal{N}$, we have a curve $\tilde{C}$ on $\mathcal{N}$ with its respective tangent vectors. Thus f has established a natural correspondence between the tangent vectors to the curves on $\mathcal{M}$ and $\mathcal{N}$. We call such a correspondence the push-forward map between the two tangent spaces, since it maps in the same direction as f. More formally:

Definition 2.4.3 ([51])**.** Consider a smooth map $f : \mathcal{M} \to \mathcal{N}$. The induced map $f_* : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ given by

$$(f_*X)(g) = X(g(f)),$$

for a vector $X \in T_p\mathcal{M}$ and smooth map $g : \mathcal{N} \to \mathbb{R}$, is called the *pushforward map.*

Analogously, there is a correspondence between the one-forms on $\mathcal{M}$ and $\mathcal{N}$ which is dual to the pushforward map. We call such a correspondence the *pullback map* between the two cotangent spaces, since it maps in the opposite direction as f.

Definition 2.4.4 ([51])**.** Consider a smooth map $f : \mathcal{M} \to \mathcal{N}$. The induced map $f^* : T^*_{f(p)}\mathcal{N} \to T_p\mathcal{M}$ given by

$$(f^*\omega)(X) = \omega(f_*X),$$

for one-form $\omega \in T^*_{f(p)}\mathcal{N}$ and vector $X \in T_p\mathcal{M}$, is called the *pullback map.*

By the linearity of f_*, we may extend the pushforward map to act on any contravariant tensor $T \in T^q_{0,p}(\mathcal{M})$:[8]

$$(f_*T)(\omega_1, \ldots, \omega_q) = T(f^*\omega_1, \ldots, f^*\omega_q),$$

for $\omega_i \in T^*_{f(p)}\mathcal{N}$. Such a map locally transforms a contravariant tensor $T^{j_1 \ldots j_q}(x^i) \in \mathcal{T}^q_{0,p}(\mathcal{M})$ into $\tilde{T}^{i_1 \ldots i_q}(y^i) \in \mathcal{T}^q_{0,f(p)}(\mathcal{N})$ in the following way:

$$\tilde{T}^{i_1 \ldots i_q} = T^{j_1 \ldots j_q} \frac{\partial y^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{i_q}}{\partial x^{j_q}}.$$

[8]If f is a diffeomorphism, then the pushforward map can be defined for contravariant tensor *fields.*

In a similar way, the linearity of f^* allows us to extend the pullback map to act on any covariant tensor $T \in T^0_{r,f(p)}(\mathcal{N})$:[9]

$$(f^*T)(X_1, \dots, X_r) = T(f_*X_1, \dots, f_*X_r),$$

for $X_i \in T_p\mathcal{M}$. Such a map locally transforms a covariant tensor $T_{j_1 \dots j_r}(y^i) \in \mathcal{T}^0_{r,f(p)}(\mathcal{N})$ into $\tilde{T}_{i_1 \dots i_r}(x^i) \in \mathcal{T}^0_{r,p}(\mathcal{M})$ in the following way

$$\tilde{T}_{i_1 \dots i_r} = T_{j_1 \dots j_r} \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}}.$$

A useful property of the pullback map is that it preserves the wedge product:

$$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta), \tag{2.6}$$

for k-form ω and r-form η.

We now turn to a specific, but important class of functions defined on a manifold $\mathcal{M}$ with metric g, called *isometries*. These types of maps preserve distance between points on a manifold, a fact that is useful in our solution to the equivalence problem.

2.4.3 Isometries

In the previous section, we saw that a diffeomorphism $f : \mathcal{M} \to \mathcal{M}$ induces the pullback map f^*, which creates a correspondence between covariant tensors at p and $f(p)$. Thus, if our manifold $\mathcal{M}$ has metric g, we have for this covariant tensor

$$f^* : g|_{f(p)} \to \tilde{g}|_p.$$

If $p = (x^i)$ and $f(p) = (y^i)$, then locally we have that $g|_p$ transforms under f according to the tensor transformation law [59]

$$\tilde{g}_{ij}(p) = g_{k\ell}(f(p)) \frac{\partial y^k}{\partial x^i} \frac{\partial y^\ell}{\partial x^j}.$$

In the case when g is unchanged under f, we call f an *isometry* on $\mathcal{M}$. More formally:

[9] Unlike the pushforward map, no extra condition on f is required to define the pullback map for covariant tensor *fields*.

Definition 2.4.5. Suppose $f : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism on a manifold $\mathcal{M}$ with metric g. If g is unchanged under the pullback map, ie.,

$$f^* g = g,$$

then the function f is called an *isometry.*

The set of all isometries on a manifold $\mathcal{M}$ forms a group under composition, called the *isometry group* $I(\mathcal{M})$. For example, the isometry group of Minkowski space is the *Poincaré group*, $P(1, n)$. This group consists of functions $f : \mathbb{M}^n \to \mathbb{M}^n$ which transform a point $x^i \in \mathbb{M}^n$ according to

$$x^i \to \Lambda^i_j i x^j + b^i,$$

where $b \in \mathbb{M}^n$ and $\Lambda \in O(n, 1)$. Here, $O(n, 1)$ refers to the *Lorentz group*, which consists of matrices $\Lambda \in GL(n, \mathbb{R})$ that satisfy $\Lambda g_M \Lambda^t = g_M$.[10]

2.4.4 Submanifolds

In the literature, you will find authors have different ways of defining a *submanifold* $\mathcal{N}$ of a manifold $\mathcal{M}$. Some will use the more general definition of an *immersed submanifold* as their definition of a submanifold, while others will use one of two subclasses of this definition, namely *imbedded submanifolds* or *regular submanifolds.*[11] The choice of the definition will depend on what properties you would like $\mathcal{N}$ to have in common with $\mathcal{M}$ (ie., the same topology, differentiable structure). For our purposes, we will use the following definition to define a submanifold, which is what some authors call a *regular submanifold.* This definition ensures that the subset $\mathcal{N}$ of $\mathcal{M}$ inherits the same topology and differentiable structure as that of $\mathcal{M}$.

Definition 2.4.6. Suppose $\mathcal{M}$ and $\mathcal{N}$ are manifolds with dimensions m and n respectively, where $n \leq m$. Define a smooth and injective map $f : \mathcal{N} \to \mathcal{M}$, such that $f_* : T_p\mathcal{N} \to T_{f(p)}\mathcal{M}$ is also injective. We call the image of f a *submanifold* of $\mathcal{M}$.

[10] Note that g_M refers to the Minkowski metric.
[11] For a discussion of these terms and their differences, please refer to [9].

Simple examples of submanifolds of $\mathbb{E}^3$ would be smooth curves or surfaces. If the submanifold has dimension one less than that of $\mathcal{M}$ (such as a sphere in $\mathbb{E}^3$), then the submanifold is sometimes referred to as a *hypersurface*. The space $\mathcal{M}$ is often called the *ambient space* of the submanifold.

2.5 Calculus on Manifolds

Now that we have defined the type of manifolds we will be working with throughout this thesis, we need to introduce some operations on these manifolds. We begin by defining the *Lie derivative operator*, a tool that will be useful in our study of symmetries of orthogonal coordinate webs in a later chapter. We will see that the expression used for calculating the Lie derivative of a vector field coincides with how the *Lie bracket* operator acts on vector fields. As Schouten showed in 1940 [61], this bracket can be generalized to act on any two contravariant tensor fields by defining the *Schouten bracket*. This particular operator will be of great importance to us, as we will use it to define the principal object of this thesis, namely the *Killing tensor*. Next, we introduce a second type of differential operator, called the *exterior derivative operator*, so that we may define the fundamental *compatibility condition* in a later chapter. Lastly, we introduce the *covariant derivative operator*, so that we can discuss the important concept of curvature of a manifold.

2.5.1 Lie Derivative

An integral part in solving the equivalence problem for Killing tensors rests on the symmetry properties of their associated *orthogonal coordinate webs*. To find such symmetries, we will need to check whether the Killing tensor satisfies a certain condition involving the *Lie derivative*.

Definition 2.5.1. The *Lie derivative operator* is a differential operator $\mathcal{L}_X : \mathcal{T}^q_r(\mathcal{M}) \to \mathcal{T}^q_r(\mathcal{M})$ which measures how a tensor field T changes along the flow of a vector field X in the following way [59]:

$$(\mathcal{L}_X T)^{i_1\ldots i_q}_{j_1\ldots j_r} = X^\lambda \partial_\lambda T^{i_1\ldots i_q}_{j_1\ldots j_r} + \sum_{k=1}^{q} T^{i_1\ldots i_q}_{j_1\ldots\lambda\ldots j_r} \partial_{j_k} X^\lambda - \sum_{k=1}^{r} T^{i_1\ldots\lambda\ldots i_q}_{j_1\ldots j_r} \partial_\lambda X^{i_k}.$$

We call the resulting tensor field the *Lie derivative of T* with respect to X.

The Lie derivative operator satisfies the following properties for $c_1, c_2 \in \mathbb{R}$, differentiable function f, and $T_1, T_2 \in \mathcal{T}_r^q(\mathcal{M})$:

(i) $\mathcal{L}_X(c_1T_1 + c_2T_2) = c_1\mathcal{L}_XT_1 + c_2\mathcal{L}_XT_2$, (linearity)

(ii) $\mathcal{L}_X(T_1 \otimes T_2) = (\mathcal{L}_XT_1) \otimes T_2 + T_1 \otimes (\mathcal{L}_XT_2)$, (Leibniz rule)

(iii) $\mathcal{L}_Xf = X(f)$.

The following example will be useful for later discussions.

Example 2.5.2. The Lie derivative of a $(2,0)$-tensor field T^{ij} is a tensor field of the same type defined by

$$(\mathcal{L}_XT)^{ij} = X^a\partial_aT^{ij} - T^{bj}\partial_bX^i - T^{ib}\partial_bX^j;$$

the Lie derivative of a $(0,2)$-tensor field T^{ij} is a tensor field of the same type defined by

$$(\mathcal{L}_XT)_{ij} = X^a\partial_aT_{ij} + T_{ib}\partial_jX^b + T^{bj}\partial_iX^b.$$

In the special case when T is a vector field, its Lie derivative with respect to the vector field X is also called the *Lie bracket* of X and T.

Definition 2.5.3. The *Lie bracket* is a differential operator $[\ ,\ ] : \mathcal{T}_0^1(\mathcal{M}) \times \mathcal{T}_0^1(\mathcal{M}) \to \mathcal{T}_0^1(\mathcal{M})$ which acts on a pair of vector fields X and Y in the following way [59]:

$$(\mathcal{L}_XY)^i \equiv [X,Y]^i = X^n\partial_nY^i - Y^n\partial_nX^i.$$

It can be shown that the Lie bracket satisfies the following properties for any $X, Y, Z \in \mathcal{T}_0^1(\mathcal{M})$ [59]:

(i) $[X,Y] = -[Y,X]$, (skew-symmetry)

(ii) $[X, aY + bZ] = a[X,Y] + b[X,Z]$, (bilinearity)

$[aX + bY, Z] = a[X,Y] + b[X,Z]$,

(iii) $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$. (Jacobi identity)

Due to these properties, a real vector space V endowed with the Lie bracket is a Lie algebra.[12]

A generalization of the Lie bracket for vector fields is the *Schouten bracket* for contravariant tensor fields.

Definition 2.5.4. The *Schouten bracket* is an operator $[\ ,\ ] : \mathcal{T}_0^p(\mathcal{M}) \times \mathcal{T}_0^q(\mathcal{M}) \to \mathcal{T}_0^{p+q-1}$ which acts on a pair of contravariant tensor fields in the following way [71]:

$$\begin{aligned}
[P,Q]^{i_1\ldots i_{p+q-1}} \quad = \quad & \sum_{k=1}^{p} P^{(i_1\ldots i_{k-1}|\mu|i_k\ldots i_{p-1}}\partial_\mu Q^{i_p\ldots i_{p+q-1})} \\
& + \sum_{k=1}^{p} (-1)^k P^{[i_1\ldots i_{k-1}|\mu|i_k\ldots i_{p-1}}\partial_\mu Q^{i_p\ldots i_{p+q-1}]} \\
& - \sum_{t=1}^{q} Q^{(i_1\ldots i_{t-1}|\mu|i_t\ldots i_{q-1}}\partial_\mu P^{i_q\ldots i_{p+q-1})} \\
& - \sum_{t=1}^{q} (-1)^{pq+p+q+t} Q^{[i_1\ldots i_{t-1}|\mu|i_t\ldots i_{q-1}}\partial_\mu P^{i_q\ldots i_{p+q-1}]}
\end{aligned} \tag{2.7}$$

To see that the Schouten bracket is in fact a generalization of the Lie bracket, let $p = 1$ and q be arbitrary,

$$[P,Q]^{k_1\ldots k_q} = (\mathcal{L}_P Q)^{k_1\ldots k_q}, \tag{2.8}$$

thus when $q = 1$, we get the Lie bracket of vector fields P and Q.

If P and Q are antisymmetric tensor fields, then their Schouten bracket

$$\begin{aligned}
[P,Q]^{i_1\ldots i_{p+q-1}} \quad = \quad & \sum_{k=1}^{p} (-1)^k P^{[i_1\ldots i_{k-1}|\mu|i_k\ldots i_{p-1}}\partial_\mu Q^{i_p\ldots i_{p+q-1}]} \\
& - \sum_{t=1}^{q} (-1)^{pq+p+q+t} Q^{[i_1\ldots i_{t-1}|\mu|i_t\ldots i_{q-1}}\partial_\mu P^{i_q\ldots i_{p+q-1}]}
\end{aligned}$$

is also antisymmetric; and if P and Q are symmetric tensor fields, then their Schouten bracket

$$\begin{aligned}
[P,Q]^{i_1\ldots i_{p+q-1}} \quad = \quad & \sum_{k=1}^{p} P^{(i_1\ldots i_{k-1}|\mu|i_k\ldots i_{p-1}}\partial_\mu Q^{i_p\ldots i_{p+q-1})} \\
& - \sum_{t=1}^{q} Q^{(i_1\ldots i_{t-1}|\mu|i_t\ldots i_{q-1}}\partial_\mu P^{i_q\ldots i_{p+q-1})}.
\end{aligned}$$

[12]Recall that a vector space V is called a *Lie algebra* if it is endowed with a bilinear map $[\ ,\ ] : V \times V \to V$ satisfying

(i) $[u,v] = -[v,u]$,

(ii) $[[u,v],w] + [[w,u],v] + [[v,w],u] = 0$

for all $u, v, w \in V$.

is also symmetric. Thus we have the following result.

Theorem 1 (Nijenhuis)**.**

$$[P, Q] = [P_a, Q_a] + [P_s, Q_s],$$

where the subscripts a and s denote the antisymmetric and symmetric parts of the tensor.

For any *antisymmetric* contravariant tensor fields P, Q, R, the Schouten bracket satisfies

(i) $[P, Q] = -(-1)^{(u+1)(v+1)}[Q, P]$,

(ii) $[P, [Q, R]] - [[P, Q], R] - (-1)^{(u+1)(v+1)}[Q, [P, R]] = 0$.

Thus the set of antisymmetric contravariant tensor fields on a manifold endowed with the Schouten bracket is a graded Lie algebra.[13] Comparatively, for any *symmetric* contravariant tensor fields P, Q, R, the Schouten bracket satisfies

(i) $[P, Q] = -[Q, P]$,

(ii) $[[P, Q], R] + [[R, P], Q] + [[Q, R], P] = 0$,

which naturally generalizes the skew-symmetry and Jacobi identity properties of the Lie bracket of vector fields. As such, the set of symmetric contravariant tensor fields endowed with the Schouten bracket is also a graded Lie algebra. In a later chapter, we will define a principal object of this thesis, called a *Killing tensor*. As we will see, the definition involves taking the Schouten bracket of two symmetric $(2, 0)$-tensor fields.

Example 2.5.5. The Schouten bracket of a pair of symmetric $(2, 0)$-tensor fields yields a $(3, 0)$-tensor field with the following components:

$$[P, Q]^{ijk} = Q^{(|\mu|k}\partial_\mu P^{ij)} - P^{(k|\mu|}\partial_\mu Q^{ij)}.$$

[13] See, for example, [18].

2.5.2 Exterior Derivative

Recall the differential operator from vector calculus which transforms a function (or 0-form) into a 1-form in the following way:

$$\mathrm{d}f = \frac{\partial f}{\partial x^k} dx^k.$$

We can generalize this operator to act on any r-form by defining the *exterior derivative*.

Definition 2.5.6. The *exterior derivative operator* is an operator $\mathrm{d} : \Omega^0_r(\mathcal{M}) \to \Omega^0_{r+1}(\mathcal{M})$ which transforms an r-form ω in the following way [59]:

$$\mathrm{d}\omega = \frac{1}{r!} \partial_k \omega_{i_1 \ldots i_r} dx^k \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_r}.$$

The resulting differential form is called the *exterior derivative* of ω.

It can be shown that the exterior derivative satisfies the following properties

(i) $\mathrm{d}(\omega + \alpha) = \mathrm{d}\omega + \mathrm{d}\alpha$, (linearity)

(ii) $\mathrm{d}(\mathrm{d}\omega) = 0$,

(iii) $\mathrm{d}(\omega \wedge \alpha) = \mathrm{d}\omega \wedge \alpha + (-1)^r \omega \wedge \mathrm{d}\alpha$,

(iv) $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X \omega)$,

(v) $\mathrm{d}(f^* \omega) = f^*(\mathrm{d}\omega)$,

for $\omega \in \mathcal{T}^0_r(\mathcal{M})$, $\alpha \in \mathcal{T}^0_q(\mathcal{M})$, $X \in \mathcal{T}^k_0(\mathcal{M})$ and $f : \mathcal{M} \to \mathcal{N}$ [80].

Later in this thesis we will show that a fundamental step in finding orthogonally separable coordinates for a given Hamiltonian is solving the *compatibility condition*. This condition is a way of determining which Killing tensors from the set of all possible Killing tensors of valence two on the manifold are 'compatible' with the potential of the given Hamiltonian. This important condition, which involves the exterior derivative operator, is based on the second property from this list.

2.5.3 Covariant Derivative

Recall the directional derivative from vector calculus which computes the derivative of a function f in the direction of a vector V:

$$\nabla_V f = \nabla f \cdot V,$$

where ∇f is the gradient of f and "$\cdot$" denotes the dot product. We can generalize the differential operator ∇_V to act on any tensor field T by defining the *covariant derivative*.

Definition 2.5.7. The *covariant derivative operator* is a differential operator $\nabla_X : \mathcal{T}_r^q(\mathcal{M}) \to \mathcal{T}_r^q(\mathcal{M})$ which acts on a tensor field T in the following way:

$$(\nabla_X T)^{i_1 \ldots i_q}_{j_1 \ldots j_r} = X^k \partial_k T^{i_1 \ldots i_q}_{j_1 \ldots j_r} - \sum_{k=1}^{q} X^\ell \Gamma^\lambda_{\ell j_k} T^{i_1 \ldots i_q}_{j_1 \ldots \lambda \ldots j_r} + \sum_{k=1}^{r} X^\ell \Gamma^{i_k}_{\ell \lambda} T^{i_1 \ldots \lambda \ldots i_q}_{j_1 \ldots j_r},$$

where Γ^i_{jk} denote the *connection coefficients* (see below). We call the resulting tensor field the *covariant derivative* of T with respect to X.

As we can see from the definition, determining the covariant derivative of a tensor field requires us to know the continuous functions Γ^i_{jk} in advance. These functions, called *connection coefficients*, connect the bases at different points on the manifold:

$$\nabla_{e_i} e_j = e_k \Gamma^k_{ij}, \quad \nabla_{e_i} dx^j = -dx^k \Gamma^j_{ik}$$

where $e_i = \frac{\partial}{\partial x^i}$. Only on manifolds where this extra information is specified can we compute the covariant derivative. Manifolds which come equipped with this data are called *affine manifolds*, and thus the operator ∇_X is sometimes called an *affine connection*.

The existence and uniqueness of an affine connection on a pseudo-Riemannian manifold with metric g is guaranteed by a fundamental theorem in differential geometry. The theorem asserts that we can always find a unique symmetric affine connection satisfying

$$(\nabla_k g)_{ij} = 0 \tag{2.9}$$

on a pseudo-Riemannian manifold, which we call the *Levi-Civita connection*. Condition (2.9) is called the *metric compatibility condition*, which implies that the metric is constant under covariant differentiation on the manifold.

Like the Lie and exterior operators, the covariant derivative operator satisfies several properties. In particular, we have

(i) $\nabla_X(c_1T_1 + c_2T_2) = c_1\nabla_X T_1 + c_2\nabla_X T_2$, (linearity)

(ii) $\nabla_{(fX+gY)}T_1 = f\nabla_X T_1 + g\nabla_Y T_1$,

(iii) $\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$, (Leibniz rule)

for any $c_1, c_2 \in \mathbb{R}$, $X, Y \in \mathcal{T}_0^1(\mathcal{M})$, $T_1, T_2 \in \mathcal{T}_r^q(\mathcal{M})$, and differentiable functions f, g. The second property is a distinguishing feature between the Lie and covariant derivative, as this rule holds only for constant functions in the case of the Lie derivative.

An important object in this thesis is a *Killing tensor*. One of the ways in which it can be defined involves taking the covariant derivative of a $(0,2)$-tensor field.

Example 2.5.8. Suppose T is a $(0,2)$-tensor field. Then the covariant derivative of T with respect to a vector field X is a $(0,2)$-tensor field with components

$$(\nabla_k T)_{ij} = \partial_k T_{ij} - \Gamma^r_{ki}T_{rj} - \Gamma^r_{kj}T_{ir}.$$

□

2.6 Spaces of Constant Curvature

The spaces that we will be considering are examples of *spaces of constant curvature*. In this section, we clarify the meaning of this term and define the spaces of constant curvature that we will be working with. First, let us consider the following preliminary definitions.

2.6.1 Torsion Tensor

Consider an affine manifold $\mathcal{M}$.

Definition 2.6.1. The tensor field T, which acts on vector fields X and Y in the following way

$$T(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - [X,Y],$$

is called the *torsion tensor*.

If x^i are coordinates on a chart of $\mathcal{M}$, then the torsion tensor $T \in \mathcal{T}_2^1(\mathcal{M})$ in the coordinate bases has components

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}.$$

Since T is antisymmetric in its arguments, this implies

$$T^i_{jk} = -T^i_{kj}.$$

Clearly if ∇ is the symmetric Levi-Civita connection on $\mathcal{M}$, then the torsion tensor will vanish.

On a manifold $\mathcal{M}$ with metric g, suppose we have a frame $\{E_i\}$ with coframe $\{E^i\}$ such that

$$E_i(E^j) = E^j(E_i) = \delta^i_j.$$

Then, in terms of the coordinate frame $\{e_i\} = \{\frac{\partial}{\partial x^i}\}$ and coframe $\{e^i\} = \{dx^i\}$, we have

$$E_j = E_j{}^i e_i, \quad E^i = E^i{}_j e^j.$$

Moreover, let us suppose this frame is orthonormal

$$g(E_i, E_j) = \delta_{ij},$$

and satisfies

$$[E_j, E_k] = C^i_{jk} E_i. \tag{2.10}$$

In this basis, the connection coefficients Γ^k_{ij} are defined by

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k, \quad \nabla_{E_i} E^j = -\Gamma^j_{ik} dx^k,$$

and thus the torsion tensor has components

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} - C^i_{jk}.$$

If we set $\Gamma^k_{ij} E^i = \omega^k{}_j$, and define the torsion two-form $T^i = \frac{1}{2} T^i_{jk} E^j \wedge E^k$, then $\omega^k{}_j$ satisfies

$$dE^i + \omega^i{}_j \wedge E^j = T^i, \tag{2.11}$$

called *Cartan's first structure equation.*

2.6.2 Riemann Curvature Tensor

Given an affine manifold $\mathcal{M}$, we can define an important tensor called the *Riemann curvature tensor* on it.

Definition 2.6.2. The tensor field R, which acts on vector fields X, Y, Z in the following way [59]

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

is called the *Riemann curvature tensor.*

If x^i are coordinates on a chart of $\mathcal{M}$, then the Riemann curvature tensor $R \in \mathcal{T}_3^1(\mathcal{M})$ in the coordinate bases has components [59]

$$R^i_{jk\ell} = \partial_k \Gamma^i_{\ell j} - \partial_\ell \Gamma^i_{kj} + \Gamma^r_{\ell j}\Gamma^i_{kr} - \Gamma^r_{kj}\Gamma^i_{\ell r}.$$

In the frame $\{E_i\}$ with coframe $\{E^i\}$ defined in previous section, R has components

$$R^i_{jk\ell} = E_k(\Gamma^i_{\ell j}) - E_\ell(\Gamma^i_{kj}) + \Gamma^n_{\ell j}\Gamma^i_{kn} - \Gamma^n_{kj}\Gamma^i_{\ell n} - C^n_{k\ell}\Gamma^i_{nj},$$

where the $C^n_{k\ell}$ come from (2.10). If we set $\Gamma^k_{ij} E^i = \omega^k{}_j$, and define the curvature two-form $R^i{}_j = \frac{1}{2} R^i_{jk\ell} E^k \wedge E^\ell$, then $\omega^k{}_j$ satisfies

$$d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = R^i{}_j, \tag{2.12}$$

called *Cartan's second structure equation.*

From the definition, it is easy to see that $R(X,Y,Z) = -R(Y,X,Z)$, which implies

$$R^i_{jk\ell} = -R^i_{j\ell k}$$

for the components of $R \in \mathcal{T}_3^1(\mathcal{M})$. If ∇ is the Levi-Civita connection on $\mathcal{M}$ with metric g, then

$$R_{ijk\ell} = \frac{1}{2}\left(\partial_j \partial_\ell g_{ik} - \partial_i \partial_\ell g_{jk} - \partial_j \partial_k g_{i\ell} + \partial_i \partial_k g_{j\ell}\right) + g_{rs}(\Gamma^r_{ik}\Gamma^s_{j\ell} - \Gamma^r_{i\ell}\Gamma^s_{jk})$$

are the components of $R \in \mathcal{T}_4^0(\mathcal{M})$. Using this expression, we can show R admits the following symmetries and identities:

(i) $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k}$,

(ii) $R_{ijk\ell} = R_{ji\ell k}$,

(iii) $R_{i[jk\ell]} = 0$, (first Bianchi identity)

(iv) $\nabla_{[r} R_{|ij|k\ell]} = 0$. (second Bianchi identity)

Here, we have placed square brackets around some of the indices of R to denote skew-symmetrization of the indices, and vertical bars to indicate which indices are exempt from this skew-symmetrization. Taking into account the first three sets of relations, R has on an n-dimensional manifold, exactly $n^2(n^2-1)/12$ independent components.

If we contract the first and third index of $R \in \mathcal{T}_3^1(\mathcal{M})$, we obtain a symmetric $(0,2)$-tensor field

$$R^i_{ji\ell} = R_{j\ell}$$

called the *Ricci tensor*.

2.6.3 Sectional Curvature

A space is said to have constant curvature if on any two-dimensional subspace of its tangent spaces (called *sections*), the curvature parameter K is constant. We define the curvature parameter as the quotient of two tensors, k and k_1, which, on a pseudo-Riemannian affine manifold $\mathcal{M}$ with metric g and Riemann curvature tensor R, are given by

$$k_1(X,Y) = g(X,X)g(Y,Y) - g(X,Y)^2, \quad k(X,Y) = g(R(X,Y,Y),X).$$

If σ is a two-dimensional subspace of $T_p\mathcal{M}$ spanned by X and Y, we call

$$K_\sigma = \frac{k(X,Y)}{k_1(X,Y)}$$

the *sectional curvature* of $\mathcal{M}$ on section σ. Let us now consider the definition.

Definition 2.6.3. If on a pseudo-Riemannian manifold $\mathcal{M}$ with metric g the curvature parameter K_σ is constant for any σ, then $\mathcal{M}$ is called a *space of constant curvature*.

We now turn to two examples of spaces of constant curvature, namely spherical and hyperbolic space.

2.6.4 Spherical Space

Spherical space, $\mathbb{S}^n$, is the collection of points $(x^1, \ldots, x^{n+1})$ in $\mathbb{E}^{n+1}$ that satisfy the equation:[14]

$$(x^1)^2 + (x^2)^2 + \ldots + (x^{n+1})^2 = 1.$$

The smooth map, $f : \mathbb{S}^n \to \mathbb{E}^{n+1}$, defined by *generalized spherical coordinates*

$$\begin{aligned}
x^1 &= \cos\theta_1, \\
x^2 &= \sin\theta_1 \cos\theta_2, \\
&\vdots \\
x^{n-1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \cos\theta_{n-1}, \\
x^n &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \sin\theta_{n-1},
\end{aligned}$$

realizes $\mathbb{S}^n$ as a submanifold of $\mathbb{E}^{n+1}$. With this parameterization, the Euclidean metric g becomes the spherical metric,

$$g_S = f^* g_E = (dx^1)^2 + \sin^2\theta_1 (dx^2)^2 + \ldots + (\sin^2\theta_1 \cdots \sin^2\theta_{n-1})(dx^n)^2,$$

under the pullback map. If we invert this metric to obtain the contravariant form, and push it forward into $\mathcal{T}^2(\mathbb{E}^{n+1})$ with f_*, we obtain [15]

$$f_* g_S^{-1} = \mathcal{C} = 2 g^{i[j} g^{k]\ell} g_{kp} g_{q\ell} x^p x^q \frac{\partial}{\partial x^i} \odot \frac{\partial}{\partial x^j}, \tag{2.13}$$

called a *Casimir tensor*.[15]

Let us point out some of the properties of $\mathbb{S}^n$. On any section σ of $T_p\mathbb{S}^n$, we have $K_\sigma = 1$, therefore $\mathbb{S}^n$ is a space of constant curvature. The isometry group of $\mathbb{S}^n$ is the *orthogonal group* $O(n)$, which consists of matrices $\Lambda \in GL(n, \mathbb{R})$ that satisfy $\Lambda g_E \Lambda^t = g_E$. When $n = 1, 3$, $\mathbb{S}^n$ can be realized as a Lie group[16] (see, for example, [59]).

[14] N-dimensional spherical space is also referred to as an n-sphere.

[15] A member of a Lie algebra $\mathfrak{g}$ which commutes with any member of $\mathfrak{g}$ is called a *Casimir*.

[16] A set G which is both a group and a differentiable manifold, and whose group operations of multiplication and inversion are differentiable is called a *Lie group*.

2.6.5 Hyperbolic Space

Analogous to spherical space in $\mathbb{E}^n$ is hyperbolic space in $\mathbb{E}^{n-1,1}$. Hyperbolic space, $\mathbb{H}^n$, is the collection of points $(x^0, \ldots, x^n)$ in $\mathbb{E}^{n-1,1}$ that satisfy the equation:

$$(x^0)^2 - (x^1)^2 - \ldots - (x^n)^2 = 1, \quad x^0 > 0.$$

The smooth map, $f : \mathbb{H}^n \to \mathbb{E}^{n-1,1}$, defined by *generalized hyperbolic coordinates*

$$\begin{aligned}
x^0 &= \cosh\theta_1, \\
x^1 &= \sinh\theta_1 \cosh\theta_2, \\
&\vdots \\
x^{n-1} &= \sinh\theta_1 \sinh\theta_2 \cdots \sinh\theta_{n-2} \cosh\theta_{n-1}, \\
x^n &= \sinh\theta_1 \sinh\theta_2 \cdots \sinh\theta_{n-2} \sinh\theta_{n-1},
\end{aligned}$$

realizes $\mathbb{H}^n$ as a submanifold of $\mathbb{E}^{n-1,1}$. With this parameterization, the pseudo-Euclidean metric g becomes the hyperbolic metric,

$$g_H = f^* g_E = (dx^0)^2 + \sinh^2\theta_1 (dx^1)^2 + \ldots + (\sinh^2\theta_1 \cdots \sinh^2\theta_{n-1})(dx^n)^2,$$

under the pullback map. If we invert this metric to obtain the contravariant form, and push it forward into $\mathcal{T}^2(\mathbb{E}^{n-1,1})$ with f_*, we obtain a Casimir tensor

$$f_* g_H^{-1} = \mathcal{C} = 2g^{i[j}g^{k]\ell} g_{pk} g_{q\ell} x^p x^q \frac{\partial}{\partial x^i} \odot \frac{\partial}{\partial x^j}. \tag{2.14}$$

On any section σ of $T_p\mathbb{H}^n$, we have $K_\sigma = -1$, therefore $\mathbb{H}^n$ is a space of constant curvature. Its isometry group is the *Lorentz group* $O(n, 1)$ that we defined in Section 2.4.3.

2.7 The Frobenius Theorem

In a later chapter, we will be focussing on a class of tensors which admit an *integrable distribution* of eigenvectors. Let us now examine the meaning of this term and see how it is connected to the fundamental *Frobenius theorem* of differential geometry.

Recall that the tangent bundle on a manifold is the collection of all the tangent spaces on a manifold. A *distribution* on a manifold, generally speaking, is a kind of subset of the tangent bundle in the following sense.

Definition 2.7.1. Consider a manifold $\mathcal{M}$ of dimension m. Define D_p as a k-dimensional subspace of $T_p\mathcal{M}$ where in a neighborhood U of p, we have k linearly independent vector fields $\{X_i\}$ which span D_q for all $q \in U$. The set

$$D = \{D_p \mid p \in \mathcal{M}\}$$

is called a *distribution* of dimension k on $\mathcal{M}$.

Example 2.7.2. Suppose $\{\omega^1, \ldots, \omega^{m-k}\}$ is a set of linearly independent one-forms on an m-dimensional manifold $\mathcal{M}$. Then the set of vector fields $X \in T_p\mathcal{M}$ satisfying

$$D_p = \{\omega^1(X) = 0, \ldots, \omega^{m-k}(X) = 0\},$$

forms a k-dimensional distribution on $\mathcal{M}$.

Definition 2.7.3. An *integral manifold* of a distribution D on $\mathcal{M}$ is a submanifold $\mathcal{N}$ of $\mathcal{M}$, which for all $p \in \mathcal{N}$, satisfies $D_p = T_p\mathcal{N}$.

Definition 2.7.4. A distribution D is called *integrable* if we have through each $p \in \mathcal{M}$ an integral manifold.

For example, a non-vanishing vector field X on a manifold forms a one-dimensional distribution and the integral curves of X are integral manifolds. The existence of such integral curves, as we mentioned in Section 2.2.4, is guaranteed by the existence and uniqueness theorem of ODEs. It is natural to ask whether this result can be generalized for higher dimensional distributions on a manifold. The answer to this question, as contained in the Frobenius theorem, relies on the following essential property of the distribution.

Definition 2.7.5. A distribution D is called *involutive* if for any vector fields $X, Y \in D$ we have $[X, Y] \in D$ also.

We are now ready to state the theorem.[17]

Theorem 2.7.6 (Frobenius, [2])**.** *Suppose $\mathcal{M}$ is an m-dimensional manifold with a set of linearly independent one-forms $\{\omega^1, \ldots, \omega^{m-k}\}$ on $\mathcal{M}$, and*

$$D = \{X \in T\mathcal{M} : \omega^1(X) = 0, \ldots, \omega^{m-k}(X) = 0\}$$

is a k-dimensional distribution on $\mathcal{M}$. Then the following statements are equivalent.

[17] For a proof of this result, see Chapter 4 of [2].

(i) D is integrable.

(ii) D is involutive.

(iii) For any $p \in \mathcal{M}$, there exists a neighborhood of p and one-forms θ^i_j such that

$$d\omega^i = \theta^i_j \wedge \omega^j.$$

(iv) The following integrability condition *for D holds:*

$$d\omega^i \wedge (\omega^1 \wedge \ldots \wedge \omega^{m-k}) = 0.$$

If we have a k-dimensional involutive distribution of vector fields in a neighborhood U of $p \in \mathcal{M}$, then the integral curves of these vector fields fit together to form a family of k-dimensional submanifolds. This family of submanifolds is called a *foliation* of U, and each submanifold is called a *leaf* of the foliation.

Condition (iii) of Theorem 2.7.6 has the following implication.[18].

Proposition 2.7.7 ([2])**.** *Suppose in an open subset $\mathcal{M}_i \subset \mathbb{E}^m$ we have a set of linearly independent one-forms $\{\omega^1, \ldots, \omega^{m-k}\}$ and one-forms θ^i_j such that*

$$d\omega^i = \theta^i_j \wedge \omega^j.$$

Then, for each $p \in \mathcal{M}_i$ there exists a neighborhood U of p where functions h^i_j and f_j are defined such that

$$\omega^i = h^i_j df^j. \tag{2.15}$$

We call a set of one-forms satisfying (2.15) *surface-forming.*

2.8 Fibre Bundle Theory

The overall framework of ITKT can be described quite naturally using the theory of fibre bundles, as we will see in the next chapter. Therefore, as a final section to this chapter, we include a discussion on some of the key terms of fibre bundle theory. But before we delve into this theory, we need to define the following type of manifold.

[18] For a proof of this result, see, for example, [2]

2.8.1 Product Manifolds

If $\mathcal{M}$ and $\mathcal{N}$ are manifolds of dimensions m and n respectively, then their Cartesian product $\mathcal{M} \times \mathcal{N}$ is an $(m+n)$-dimensional manifold for the following reasons. Since $\mathcal{M}$ and $\mathcal{N}$ are manifolds, for any $(p, q) \in \mathcal{M} \times \mathcal{N}$, p must belong to a chart $(\mathcal{M}_i, \varphi_i)$ on $\mathcal{M}$ and q must belong to a chart $(\mathcal{N}_i, \phi_i)$ on $\mathcal{N}$. Therefore $(\mathcal{M}_i \times \mathcal{N}_j, (\varphi_i, \phi_j))$ are naturally charts on $\mathcal{M} \times \mathcal{N}$, since

$$(\varphi_i, \phi_j) : (p, q) \to \mathbb{R}^{m+n}$$

are injective functions whose image is open in $\mathbb{R}^{m+n}$. Furthermore, on any overlap set $\mathcal{M}_i \times \mathcal{N}_j \cap \mathcal{M}_k \times \mathcal{N}_\ell \neq \emptyset$, $(\varphi_i, \phi_j)(\mathcal{M}_i \times \mathcal{N}_j \cap \mathcal{M}_k \times \mathcal{N}_\ell)$ is open in $\mathbb{R}^{m+n}$, and we have transition functions

$$(\varphi_i \circ \varphi_k^{-1}, \phi_j \circ \phi_\ell^{-1}) : (\varphi_k \circ \phi_\ell)(\mathcal{M}_i \times \mathcal{N}_j \cap \mathcal{M}_k \times \mathcal{N}_\ell) \to (\varphi_i \circ \phi_j)(\mathcal{M}_i \times \mathcal{N}_j \cap \mathcal{M}_k \times \mathcal{N}_\ell)$$

which are differentiable for any i, j, k, ℓ. We call the manifold $\mathcal{M} \times \mathcal{N}$ a *product manifold*.

2.8.2 Fibre Bundles

A fibre bundle is a type of space which is locally a product manifold (ie., the space is *locally trivial*). If the fibre bundle is also a product manifold globally, then it is called a *trivial bundle*, although not all bundles have this structure. Let us consider the definition [59, 70].

Definition 2.8.1. Suppose for manifolds $\mathcal{E}$ and $\mathcal{B}$, we have a surjective map $\pi : \mathcal{E} \to \mathcal{B}$ and a group G of diffeomorphisms $\varphi : F \to F$. If π is *locally trivial*, then $(\mathcal{E}, \mathcal{B}, \pi, F, G)$ is called a *fibre bundle* with fibre F. [19]

The local triviality condition on π means that

(i) there exist open sets $\mathcal{B}_i$ such that $\mathcal{B} = \bigcup_i \mathcal{B}_i$;

(ii) for any $p \in \mathcal{B}$, we have a $\mathcal{B}_i$ such that $\varphi_i : \pi^{-1}(\mathcal{B}_i) \to \mathcal{B}_i \times F$ is a diffeomorphism;

[19] As before, we have assumed our manifolds to be differentiable. Therefore, we are really defining a *differentiable* fibre bundle.

(iii) on any $\mathcal{B}_i \cap \mathcal{B}_j \neq \emptyset$, we have transition functions

$$t_{ij} = \varphi_i^{-1} \circ \varphi_j : F \to F$$

which belong to G.

The components of a fibre bundle have certain names. The space $\mathcal{E}$ is called the *total space* of the bundle, $\mathcal{B}$ is the *base space*, the map π is called the *projection*, and the group G is called the *structure group*. For $p \in \mathcal{M}$, we call the preimage $\pi^{-1}(p) = F_p$ the *fibre* over p. Each neighborhood $\mathcal{M}_i$ is called a *trivializing neighborhood*, and the maps φ_i are called *local trivializations*.

Definition 2.8.2. Given a fibre bundle $(\mathcal{E}, \mathcal{B}, \pi, F, G)$, a smooth map $s : \mathcal{B} \to \mathcal{E}$ such that $\pi(s(p)) = p$ for any $p \in \mathcal{B}$, is called a (global) *cross-section*.

We have already seen the most important example of a fibre bundle in Section 2.2.2, namely the tangent bundle.

Example 2.8.3. On a manifold $\mathcal{M}$, the tangent bundle $T\mathcal{M}$ is an example of a fibre bundle. In particular, we have $(\mathcal{E}, \mathcal{B}, \pi, F, G) = (T\mathcal{M}, \mathcal{M}, \pi : T\mathcal{M} \to \mathcal{M}, T_p\mathcal{M}, GL(n, \mathbb{R}))$, where n is the dimension of $\mathcal{M}$. Since a vector field X on $\mathcal{M}$ is a function $X : \mathcal{M} \to T\mathcal{M}$, it is an example of a cross-section of $T\mathcal{M}$. □

Cross-sections may also be defined *locally* on a fibre bundle. In particular, if U is an open subset of $\mathcal{B}$, then a *local cross-section* is a smooth map $s : U \to \mathcal{B}$ such that $\pi(s(p)) = p$ for any $p \in U$.

2.8.3 Types of Fibre Bundles

There are several important types of fibre bundles to consider.

Definition 2.8.4. Consider a fibre bundle $(\mathcal{E}, \mathcal{B}, \pi, F, G)$ with $\pi : \mathcal{E} \to \mathcal{B}$ and fibre F.

(i) If $\mathcal{E} = \mathcal{B} \times F$, then the bundle is called a *trivial bundle*.

(ii) If F is a vector space, then the bundle is called a *vector bundle*.

(iii) If the fibre F is homeomorphic to the structure group G, then the bundle is called a *principal bundle*.

(iv) Suppose F_p denotes the fibre at a point $p \in \mathcal{B}$. If F_p is the vector space of all frames at p, then the bundle is called a *frame bundle*.

The following example will be useful for later discussions.

Example 2.8.5. Consider a Lie group G with closed Lie subgroup H. For $g \in G$, the set

$$gH = \{gh \mid h \in H\}$$

is called the *left coset* of g. The set of all left cosets is denoted by G/H, and forms a manifold. The group G is an example of a principal bundle, with fibre H. In particular, $(G, G/H, \pi : G \to G/H, H, H)$, where π maps elements of $g \in G$ to a left coset $gH \in G/H$. □

An important result concerning principal bundles which will be used later in this thesis is the following Theorem.

Theorem 2.8.6 (Theorem 9.2, [59])**.** *A principal bundle is trivial if and only if there exists a global cross-section on the bundle.*

CHAPTER 3

HAMILTONIAN SYSTEMS

The field of mechanics is a vast area of science that studies the behavior of bodies acting under forces in a physical system. The field can be subdivided into two main subfields of study: classical mechanics and quantum mechanics. Each of them consist of a set of laws governing the motion of bodies in a system, where the bodies are either macroscopic (classical mechanics) or microscopic (quantum mechanics). For example, classical mechanics is used to describe the motion of planetary bodies, while quantum mechanics is used to describe the motion of atomic particles. Classical mechanics can be mathematically formulated using Lagrangian mechanics, Hamiltonian mechanics, or the more general *Hamiltonian formalism*.

We begin this chapter with a brief introduction to Hamiltonian mechanics, focussing immediately on the class of Hamiltonians to which the theory in this thesis pertains to, and then define the fundamental *equations of motion* for a Hamiltonian. As we will show, this classical theory can be considered a particular case of a more general formulation of Hamiltonian systems, called *Hamiltonian formalism*. In this more general theory, a Hamiltonian is described as a smooth function associated with a Hamiltonian vector field on a Poisson manifold. The system of ODEs describing the flow of this vector field yield the equations of motion for the Hamiltonian. The solution of these equations is central to the theory, thus we devote the final section of this chapter to Hamilton-Jacobi theory, which offers a powerful integration tool for solving the equations of motion. The theory relies on solving a partial differential equation, called the *Hamilton-Jacobi equation*, which in some cases, can be integrated by the method of orthogonal separation of variables. We discuss this important integration method in detail, and provide some significant historical results concerning the separability of the Hamilton-Jacobi equation.

3.1 Hamiltonian Mechanics

The behavior of bodies in a mechanical system can be studied by looking at the energy of the system. The branch of Hamiltonian mechanics concerning natural Hamiltonians is centred around a function H describing the total energy of a physical system:

$$\text{Total energy} = H \quad = \quad \text{kinetic energy} + \text{potential energy}.$$

Since the energy of a system is governed by its state, the Hamiltonian is a function of the two state variables, position and momentum of the bodies:

$$H = H(q^i, p_i).$$

The *position coordinates*, q^i, and *momenta coordinates*, p_i, are called canonical coordinates and together comprise the *phase space* of the system.

Natural Hamiltonians are a particular class of Hamiltonians having the following form.

Definition 3.1.1. Suppose $\mathcal{M}$ is a pseudo-Riemannian manifold equipped with a metric g, with local coordinates q^i. A Hamiltonian $H : T^*\mathcal{M} \to \mathbb{R}$ of the form

$$H(q^i, p_i) = \frac{1}{2} g^{ij} p_i p_j + V(q^i),$$

where $p_i = g_{ij}\dot{q}^j$, is called a *natural Hamiltonian*. When $V = 0$, H is called the *geodesic Hamiltonian*.

Natural Hamiltonians are examples of *conservative* Hamiltonian functions; since they do not depend explicitly on time, the total energy is constant or conserved. The material presented in this chapter pertains to conservative Hamiltonians, and the results in this thesis pertain exclusively to natural Hamiltonians.

If we are interested in determining how the state of a system changes with time, we can use the Hamiltonian to do so. The coordinates (q^i, p_i) describe the state of a system, hence their derivative with respect to time tells us how the state of the system changes with time. *Hamilton's equations* tell us how these latter quantities can be determined from the Hamiltonian.

Definition 3.1.2. The following set of $2n$ first-order ordinary differential equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \tag{3.1}$$

are called *Hamilton's equations of motion*.

3.2 Hamiltonian Formalism

Using the framework of the *Hamiltonian formalism*, we can obtain a more general theory of Hamiltonian systems for which the classical theory of Hamiltonian mechanics is a particular case. In this more general approach, a Hamiltonian system is defined on an n-dimensional manifold with a certain structure, called a *Poisson bivector*. In the classical theory, we consider a special case of this theory; namely, we assume that this structure is non-degenerate and our manifold has $2n$ local coordinates (q^i, p_i).

This more general framework of Hamiltonian systems has the advantage of being coordinate-free, and can be used as a powerful method for solving dynamical systems. To illustrate this latter point, we will consider an example in which a Hamiltonian formalism is used to solve a system of nonlinear ODEs in Yang-Mills theory.

3.2.1 Poisson Manifolds

Let us begin by defining a *Poisson manifold*.

Definition 3.2.1. A bivector $P = P^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ defined on an n-dimensional smooth manifold $\mathcal{M}$ satisfying the following property

$$[P, P] = 0 \tag{3.2}$$

is called a *Poisson bivector*.

Note that P is antisymmetric and $[\ ,\ ]$ is the Schouten bracket (2.5.4). When a smooth manifold M admits a Poisson bivector, we call M a *Poisson manifold*. Equivalently, we may define a Poisson manifold as a smooth manifold admitting the *general Poisson bracket*:

$$\{f, g\} = P^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \tag{3.3}$$

for smooth functions $f, g : \mathcal{M} \to \mathbb{R}$. Using the Schouten bracket, we can rewrite this as

$$\{f, g\} = [[P, f], g],$$

for Poisson bivector P on $\mathcal{M}$.

Since P is antisymmetric, the Poisson bracket is *anticommutative*

$$\{f, g\} = -\{g, f\},$$

and by (3.2), satisfies the *Jacobi identity*:

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0.$$

Using (3.3), it is straightforward to show that the Poisson bracket is also *bilinear*

$$\begin{aligned} \{af + bg, h\} &= a\{f, h\} + b\{g, h\}, \\ \{f, ag + bh\} &= a\{f, g\} + b\{f, h\}, \end{aligned}$$

for $a, b \in \mathbb{R}$, and satisfies the *Leibniz rule*:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Definition 3.2.2. Suppose $\mathcal{M}$ is a Poisson manifold endowed with a Poisson structure P, and $H : \mathcal{M} \to \mathbb{R}$ is a smooth function. A smooth vector field

$$X_H = [P, H]$$

is called a *Hamiltonian vector field*.

A Poisson manifold equipped with a Hamiltonian function, H, and its associated *Hamiltonian vector field*, X_H, defines a *Hamiltonian system* on $\mathcal{M}$. The flow equations associated with X_H are called *Hamilton's equations of motion*.

There is a relationship between the Poisson bracket of functions on a Poisson manifold and the Lie bracket of their associated Hamiltonian vector fields (see, for example, [63]).

Proposition 3.2.3. *For a pair of smooth functions $f : \mathcal{M} \to \mathbb{R}$ and $g : \mathcal{M} \to \mathbb{R}$ on a Poisson manifold $\mathcal{M}$, their associated Hamiltonian vector fields, X_f and X_g, satisfy*

$$[X_f, X_g] = X_{\{f,g\}},$$

where $[\ ,\ ]$ is the Lie bracket.

If for smooth functions $f : \mathcal{M} \to \mathbb{R}$ and $g : \mathcal{M} \to \mathbb{R}$ on $\mathcal{M}$ we have

$$\{f, g\} = 0,$$

they are said to be *in involution*. By Proposition 3.2.3, this is equivalent to

$$[X_f, X_g] = 0$$

for their associated Hamiltonian vector fields. A function f which satisfies

$$\{f, g\} = 0$$

for any smooth function g is called a *Casimir function* on the manifold. By Proposition 3.2.3, this is equivalent to the existence of a Hamiltonian vector field, X_f, which satisfies

$$[X_f, X_g] = 0$$

for any Hamiltonian vector field, X_g, on a Poisson manifold.

3.2.2 Symplectic Manifolds

If we require P to be invertible (ie., $P^{-1} = \Omega$), then a non-degeneracy property must be satisfied. Poisson manifolds admitting this extra property are called *symplectic manifolds*.

Definition 3.2.4. A two-form $\Omega = \Omega_{ij} dx^i \wedge dx^j$ defined on a $2n$-dimensional smooth manifold $\mathcal{M}$ satisfying the following two properties

(i) $\mathrm{d}\Omega = 0$, (closure property)

(ii) $\wedge^n \Omega \neq 0$ for any $x \in M$, (non-degeneracy)

is called a *symplectic form*.

When a smooth manifold $\mathcal{M}$ admits a symplectic form, we call $\mathcal{M}$ a *symplectic manifold*. The $2n$-dimensional phase space of a Hamiltonian is an example of a symplectic manifold. If we define

$$\Omega = dq^i \wedge dp_i \tag{3.4}$$

on the phase space, called the *canonical symplectic 2-form*, then it can be shown that Ω is closed and non-degenerate and hence endows the phase space with a symplectic structure. According to the following theorem, we can always find local coordinates on a chart of a symplectic manifold which transforms its symplectic form into the canonical form (3.4).

Theorem 3.2.5 (Darboux)**.** *Suppose $\mathcal{M}$ is a $2n$-dimensional symplectic manifold endowed with a symplectic form Ω. Then for $p \in \mathcal{M}$, there exist canonical coordinates $(q^1, \dots, q^n; p_1, \dots, p_n)$ such that Ω assumes the canonical form*

$$\Omega = dq^i \wedge dp_i.$$

The inverse of the canonical symplectic 2-form,

$$\Omega^{-1} = P_0 = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}, \tag{3.5}$$

is called the *canonical Poisson bivector*. Thus for two smooth functions, f and g, defined on the phase space of a Hamiltonian, their Poisson bracket is

$$\{f, g\} = P_0^{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_j} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q^i}.$$

A symplectic manifold endowed with the canonical Poisson bivector and a *Hamiltonian vector field* defines a classical Hamiltonian system on the manifold.

Definition 3.2.6. Suppose $\mathcal{M}$ is a symplectic manifold with canonical Poisson bivector (3.5). If $H : \mathcal{M} \to \mathbb{R}$ is a smooth function, then the vector field

$$X_H = [P_0, H] = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}$$

is called a *Hamiltonian vector field.*

According to this definition, we see that solving Hamilton's equations of motion amounts to finding the flow generated by the Hamiltonian vector field. Hereafter, whenever we speak of Hamiltonian systems, we are referring to Hamiltonians in the above classical sense.

3.2.3 First Integrals

For any smooth function f defined on the phase space of a Hamiltonian,

$$X_H(f) = \{f, H\} = \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i} = \mathcal{L}_{X_H} f,$$

hence $\mathcal{L}_{X_H}$ measures how f changes along the flow of the vector field X_H. If f is constant along the flow generated by X_H, we call f a *first integral.* More specifically:

Definition 3.2.7. A function f defined on the phase space of a Hamiltonian H satisfying

$$\{f, H\} = 0$$

is called a *first integral.*[1]

Note that the Hamiltonian itself is always a first integral:

$$\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial p_i} = 0.$$

Therefore H is constant along the flow generated by X_H, or physically speaking, the energy of the system is conserved.

Integration of Hamilton's equations (3.1) is closely related to the first integrals of a system as the following fundamental theorem by Liouville [53] demonstrates.

Theorem 3.2.8 (Liouville)**.** *Suppose for a Hamiltonian on a $2n$-dimensional symplectic manifold we have n functionally independent first integrals $\{F_1 = H, \ldots, F_n\}$ in involution:*

$$\{F_i, H\} = 0, \quad \{F_i, F_j\} = 0, \quad dF_1 \wedge \ldots \wedge dF_n \neq 0.$$

Then Hamilton's equations (3.1) are integrable by quadratures.

Note that a system is "integrable by quadratures" if we can find a solution using algebra and by evaluating integrals. A Hamiltonian satisfying the conditions of Theorem 3.2.8 is said to be *Liouville integrable* or simply *integrable.* If the system admits $m > n$ functionally independent first integrals, n of which are in involution, then the

[1] A first integral is also called a *constant of the motion.*

system is said to be *superintegrable.* If $m = 2n - 1$, then the system is said to be *maximally superintegrable.*

In addition to Liouville's local result, we have the following global information concerning an integrable Hamiltonian due to Arnold [4].

Theorem 3.2.9 (Arnold)**.** *Suppose a Hamiltonian on a $2n$-dimensional symplectic manifold is integrable, with n first integrals $\{F_1 = H, \ldots, F_n\}$.*

(i) If the n-dimensional submanifold

$$\mathcal{M}_{c_1,\ldots,c_n} = \{F_1 = c_1, \ldots, F_{n-1} = c_n, \quad c_i \in \mathbb{R}\},$$

of the phase space is compact and connected, it is diffeomorphic to an n-dimensional torus.

(ii) There exist a set of action-angle coordinates, $(I_1, \ldots, I_n; \varphi_1, \ldots, \varphi_n)$, such that $I_i = I_i(F_i)$ parameterize the tori in the phase space and φ denote the coordinates on a torus.

(iii) In the action-angle coordinates, Hamilton's equations (3.1) are integrable by quadratures. The flow of these vector fields evolves periodically on a torus $\mathcal{M}_{c_1,\ldots,c_n}$.

3.2.4 Example: Yatsun's Integrable Case I

As we discussed at the beginning of this section, Hamiltonian formalism can be used as a powerful tool for solving systems of ODEs in dynamical systems theory. To support this claim, consider the following system of ODEs of second-order from Yang-Mills theory [81, 82]:

$$\begin{aligned} \phi'' + \frac{4}{x^2}\phi(1-\phi)(1-2\phi) - \frac{g^2}{4}\phi^2(1-\phi) &= 0, \\ \phi'' + \frac{3}{x}\phi' - \frac{3}{x^2}\phi(1-\phi)^2 + \lambda\phi^3 &= 0, \end{aligned} \tag{3.6}$$

for constants $g, \lambda \in \mathbb{R}$. If we make the following change of coordinates,

$$t = \ln x, \ q^1 = \phi + 1, \ q^2 = \sqrt{\frac{g^2}{12}}x\phi, \ p_i = \frac{dq^i}{dt},$$

for $i = 1, 2$, the system of equations (3.6) become Hamilton's equations of motion for a natural Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q^1, q^2),$$

with potential

$$\begin{aligned} V_1 &= -2\left((q^1)^4 + \frac{3}{4}(q^1)^2(q^2)^2 + \frac{3\lambda}{2g^2}(q^2)^4\right) + 12\left((q^1)^3 + \frac{1}{2}q^1(q^2)^2\right) \\ &\quad -26\left((q^1)^2 + \frac{1}{4}(q^2)^2\right) + 24q^1. \end{aligned}$$

If $g^2 = 24\lambda$, then H admits the following first integral

$$F = p_1p_2q^2 - p_2^2q^1 - (q^2)^2\left((q^1)^3 + \frac{1}{2}q^1(q^2)^2 - 6((q^1)^2 + \frac{1}{4}(q^2)^2 + 13q^1 - 12)\right),$$

which is functionally independent of the Hamiltonian, and thus the Hamiltonian system is integrable by Theorem 3.2.8.

3.3 Hamilton-Jacobi Theory

If we are interested in determining how the behavior of a body in a physical system evolves with time, we could try solving Hamilton's equations. The solution of which would yield the expressions $q(t)$ and $p(t)$, which describe how the position and momentum of the body changes with time. For many systems, however, these $2n$ nonlinear first-order ordinary differential equations are difficult to solve. The main idea behind Hamilton-Jacobi theory is to resolve this problem by finding a canonical transformation that changes Hamilton's equations into a form which can easily be integrated. As such, Hamilton-Jacobi theory provides a powerful integration method for solving mechanical systems.

At the centre of Hamilton-Jacobi theory is a first-order, partial differential equation given in terms of a generating function, S. The canonical transformation that we are interested in finding is defined by this generating function S, thus we need to solve the partial differential equation in order to find S. As such, we have essentially swapped our problem of solving $2n$ ordinary differential equations for one partial differential equation; and since PDEs are usually difficult to solve, one might think that we have not gained anything from this formulation. But this is not true in some cases.

In certain applications, the PDE is solvable by the method of separation of variables and thus we can obtain the desired solution. The method of Hamilton-Jacobi is useful for these types of situations.

In what follows, we shall describe the integration method of Hamilton-Jacobi for conservative Hamiltonians.[2] After introducing canonical transformations, we will define the central object of this method, the *Hamilton-Jacobi equation*. Finally, we will introduce the method of *orthogonal separation of variables*, the main topic of this thesis.

3.3.1 Canonical Transformations

Canonical transformations play an important role in Hamilton-Jacobi theory. These are transformations on the phase space coordinates

$$\Gamma : (q^i, p_i) \to (Q^i, P_i)$$

which preserve the form of Hamilton's equations. Equivalently, we can define these types of transformations using the symplectic 2-form as follows [17]:

Definition 3.3.1. A transformation $\Gamma : (q^i, p_i) \to (Q^i, P_i)$ such that

$$dq^i \wedge dp_i = dQ^i \wedge dP_i$$

is called a *canonical transformation*.

We can relate the old and new coordinates of a canonical transformation together with one function, F, called a *generating function*. Such functions come in four basic types,[3] but for the purposes of the next section, we will concentrate on generating functions of the following form:

$$F = S(q^i, P_i, t) - Q^i P_i, \tag{3.7}$$

Generating functions for a canonical transformation of this type satisfy the following equations:

$$p_i = \frac{\partial S}{\partial q^i}, \quad Q^i = \frac{\partial S}{\partial P_i}, \quad K = H + \frac{\partial S}{\partial t}.$$

[2]For a discussion of Hamilton-Jacobi theory for time-dependent Hamiltonians, see, for example, Goldstein [27], Chapter 10.

[3]For further discussions on generating functions, see, for example Goldstein [27], Chapter 9.

The first two relations are called *transformation equations* of the canonical transformation, while K represents the new Hamiltonian.

In this thesis, we are concerned with the following special class of canonical transformations.

Definition 3.3.2. Canonical transformations which transform as follows

$$Q^i = Q^i(q), \qquad P_i = \frac{\partial q^j}{\partial Q^i} p_j$$

are called *point transformations.*

As we will see, these types of transformations play a fundamental role in the method of orthogonal separation of variables.

3.3.2 Hamilton-Jacobi Equation

The goal of the Hamilton-Jacobi integration method for conservative Hamiltonians is to find a generating function

$$S = S(q^i, P_i, t),$$

called *Hamilton's principal function*, of a canonical transformation $\Gamma : (q^i, p_i) \to (Q^i, P_i)$ which satisfies

$$H\left(q^i, \frac{\partial S}{\partial q^i}\right) + \frac{\partial S}{\partial t} = 0,$$

called the *Hamilton-Jacobi equation*. Here we have used the fact that S is of the aforementioned type (3.7), and thus satisfies the transformation equations

$$p_i = \frac{\partial S}{\partial q^i}, \quad Q^i = \frac{\partial S}{\partial P_i}.$$

Under such a transformation, the new Hamiltonian becomes

$$K = H + \frac{\partial S}{\partial t} = 0.$$

Canonical transformations by definition preserve the form of Hamilton's equations. Thus in the new canonical coordinates we have

$$\dot{P}_i = -\frac{\partial K}{\partial Q^i} = 0, \qquad \dot{Q}^i = \frac{\partial K}{\partial P_i} = 0,$$

which can easily be integrated. The above relations demonstrate that both Q^i and P_i are constant in time. It is possible to solve a mechanical system using Hamilton-Jacobi theory if we are able to solve the partial differential Hamilton-Jacobi equation for the generating function S. Such a generating function, as we have just shown, leads to the equations of motion that we are trying to find. While solving partial differential equations is typically difficult to do, it is possible in some cases using integration techniques such as the *method of separation of variables*.

3.3.3 Orthogonal Separation of Variables

The method of separation of variables is an integration technique used when trying to solve differential equations. The basic idea is to separate the variables in a differential equation so that we may easily integrate. We can try to apply such a technique when trying to solve the partial differential Hamilton-Jacobi equation for a mechanical system. Whether or not it works will depend on the Hamiltonian itself and on the choice of position coordinates.

The method of additive separation of variables starts by assuming your solution can be written as a sum of functions of the independent variables. Such an expression is called an *ansatz*. For example, consider Hamilton's principal function S for the conservative Hamilton-Jacobi equation. Let us assume that S has the following form:

$$S(q^i, P_i, t) = S_0(P_i, t) + W(q^i, P_i),$$

where W satisfies the non-degeneracy condition

$$\det\left(\frac{\partial^2 W}{\partial q^i \partial P_i}\right) \neq 0. \tag{3.8}$$

Substituting this expression for S into the Hamilton-Jacobi equation leads to

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = -\frac{\partial S_0}{\partial t} = E,$$

from which we see that the variables have been separated. The constant E (the energy of the system) is called a separation constant. We thus have two equations; the first

$$\frac{\partial S_0}{\partial t} = -E$$

can be integrated to yield $S_0(E,t) = -Et$, while the second

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E \tag{3.9}$$

is called the *reduced Hamilton-Jacobi equation*. Consequentially, we have simplified the integration method of Hamilton-Jacobi for conservative Hamiltonians. We now seek a function W, called the *characteristic function*, of a generating function

$$S(q^i, P_i, t) = W(q^i, P_i) - Et$$

which satisfies the reduced Hamilton-Jacobi equation (3.9). If such a function can be found, then Hamilton's equations (3.1) are integrable by quadratures as the following theorem [4] asserts.

Theorem 3.3.3 (Jacobi)**.** *Suppose a solution $W = W(q^i, P_i)$ satisfying (3.8) can be found to the reduced Hamilton-Jacobi equation (3.9). Then the functions Q^i determined by*

$$Q^i = \frac{\partial W}{\partial P_i}$$

define a set of n first integrals in involution, and thus Hamilton's equations (3.1) can be solved by quadratures.

In practice, though, we still need to be able to solve the reduced Hamilton-Jacobi partial differential equation for W. As we did before, this can be accomplished in some cases by the additive method of separation of variables. Now that the independent variable t has been separated for any conservative Hamiltonian, we seek to separate the remaining position coordinates q^i. Unfortunately we cannot come up with a form for W which can be applied in all cases. As mentioned earlier, the form will depend on the Hamiltonian and the choice of position coordinates.

The type of separation which may occur in the reduced Hamilton-Jacobi equation can be summarized as follows.

Definition 3.3.4. Given Hamilton's characteristic function $W = W(q^i, P_i)$, we say that the reduced Hamilton-Jacobi equation is *additively separable* in the coordinate q^i if W can be split in the following way:

$$W(q^1, \ldots, q^n, P_i) = W_1(q^1, P_i) + W'(q^2, \ldots, q^n, P_i).$$

The reduced Hamilton-Jacobi equation is separable in all coordinates q^i, or *completely separable*, if we can write W as

$$W = W_1(q^1, P_i) + W_2(q^2, P_i) + \ldots + W_n(q^n, P_i).$$

In practice, it is difficult to know beforehand what form W will take. The problem, instead, becomes specifying the canonical transformation which enables the reduced Hamilton-Jacobi equation to be solved by the method of separation of variables. The work done in this thesis pertains to finding canonical transformations of a particular type.

Definition 3.3.5. A canonical transformation that is a point transformation and diagonalizes the metric g is an *orthogonal transformation*.

The method of orthogonal separation of variables is based on finding an orthogonal transformation which allows the Hamilton-Jacobi equation to be solved by the method of separation of variables. If such a transformation can be found, the Hamiltonian is said be to *orthogonally separable*. To illustrate the material discussed in the preceding sections, we will show in detail, how to solve the Kepler potential defined on $\mathbb{S}^3$ using Hamilton-Jacobi theory.

Example 3.3.6. Consider the Kepler Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j + \frac{w}{\sqrt{x^2 + y^2 + z^2}} \tag{3.10}$$

on $\mathbb{S}^3 \subset \mathbb{E}^4$, where g denotes the Euclidean metric, and $(x, y, z, w) \in \mathbb{E}^4$ satisfy the spherical constraint $x^2 + y^2 + z^2 + w^2 = 1$. The position and momentum of the bodies is described by the canonical coordinates $q^i = (x, y, z, w)$, $p_i = (p_x, p_y, p_z, p_w)$ which together form the phase space of the system. The evolution of the system is related to the total energy of the system by Hamilton's equations:

$$\dot{x} = p_x, \quad \dot{y} = p_y, \quad \dot{z} = p_z, \quad \dot{w} = p_w,$$

$$\dot{p}_x = \frac{xw}{(x^2 + y^2 + z^2)^{3/2}}, \quad \dot{p}_y = \frac{yw}{(x^2 + y^2 + z^2)^{3/2}},$$

$$\dot{p}_z = \frac{zw}{(x^2 + y^2 + z^2)^{3/2}}, \quad \dot{p}_w = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

Consider the following point transformation,

$$x = \sin t \sin u \cos v, \quad y = \sin t \sin u \sin v, \quad z = \sin t \cos u, \quad w = \cos t.$$

Such a transformation is orthogonal because the metric is diagonal with respect to $\{t, u, v\}$:

$$ds^2 = dt^2 + \sin^2 t du^2 + \sin^2 t \sin^2 u dv^2.$$

Under this canonical transformation, the reduced Hamilton-Jacobi equation assumes the following form

$$\frac{1}{2}\left(\left(\frac{\partial W}{\partial t}\right)^2 + \frac{1}{\sin^2 t}\left(\frac{\partial W}{\partial u}\right)^2 + \frac{1}{\sin^2 t \sin^2 u}\left(\frac{\partial W}{\partial v}\right)^2\right) + \cot t = E.$$

If the Hamilton-Jacobi equation is completely separable in these coordinates, it admits a characteristic function of the form

$$W(t, u, v) = W_1(t) + W_2(u) + W_3(v).$$

Substituting this expression for W into the reduced Hamilton-Jacobi equation and differentiating with respect to t, u, v each respectively yields the following system of equations:

$$\begin{aligned} \left(\frac{dW_3}{dv}\right)^2 &= c_1 \\ \left(\frac{dW_2}{du}\right)^2 &= c_2 - \frac{c_1}{\sin^2 u} \\ \left(\frac{dW_1}{dt}\right)^2 &= 2E - 2\cot t - \frac{c_2}{\sin^2 t}. \end{aligned}$$

After integrating, we find that the characteristic function W has the following form

$$W(t, u, v, c_1, c_2, E) = \pm \int \sqrt{2E - 2\cot t - \frac{c_2}{\sin^2 t}} dt \pm \int \sqrt{c_2 - \frac{c_1}{\sin^2 u}} du \pm \int \sqrt{c_1} dv,$$

and therefore the generating function S has been determined:

$$S = W(t, u, v, c_1, c_2, E) - Et.$$

Using the transformation equations, where P_i represent the integration constants and

$Q^i = \beta^i$, we find

$$\begin{aligned}
\beta^1 = \frac{\partial S}{\partial c_1} &= \pm \int \frac{-du}{2\sin u\sqrt{c_2\sin^2 u - c_1}} \pm \int \frac{dv}{2\sqrt{c_1}}, \\
\beta^2 = \frac{\partial S}{\partial c_2} &= \pm \int \frac{-dt}{2\sin t\sqrt{2E\sin^2 t - 2\cot t\sin^2 t - c_2}} \pm \int \frac{\sin u\ du}{2\sqrt{c_2\sin^2 u - c_1}}, \\
\beta^3 = \frac{\partial S}{\partial E} &= \pm \int \frac{\sin t\ dt}{\sqrt{2E\sin^2 t - 2\cot t\sin^2 t - c_2}} - t.
\end{aligned}$$

□

Example 3.3.6 illustrates how Hamilton-Jacobi theory is an effective analytical technique for finding the equations of motion of a mechanical system. Arguably the key step in the solution to this problem was defining the canonical transformation which enabled the reduced Hamilton-Jacobi equation to be separated. In practice, it is useful to know what transformations (if any) change the Hamilton-Jacobi equation into a separable form.

The separability of the Hamilton-Jacobi equation has been studied extensively over the last two centuries, seeing many important developments. The first key result came from Liouville in 1846 [54] concerning natural Hamiltonians in two-dimensions.

Theorem 2. *If a Hamiltonian defined on a 2-dimensional pseudo-Riemannian manifold has the following form*

$$H = \frac{\frac{1}{2}(\epsilon_1 p_u^2 + \epsilon_2 p_v^2) + C(u) + D(v)}{A(u) + B(v)},$$

then its Hamilton-Jacobi equation is solvable by separation of variables.

Consequentially, Hamiltonians of this form are said to be in *Liouville form.* The converse of this theorem was established by Morera in 1881 [58].

Theorem 3. *Suppose a natural Hamiltonian H is defined on a 2-dimensional pseudo-Riemannian manifold. If the Hamilton-Jacobi equation for H is solvable by separation of variables, then H in the separable coordinates is in Liouville form.*

The next important result came from Stäckel in 1891 [74] when he determined necessary and sufficient conditions for a natural Hamiltonian in n dimensions to be orthogonally separable. This separability was based on the existence of a *Stäckel matrix*, which placed conditions on the form of the metric and potential when expressed in orthogonal separable coordinates. Consequentially, metrics exhibiting this form are said to be of *Stäckel type*. In 1893 [75], Stäckel extended this result by showing that if a system is of Stäckel type (and therefore orthogonally separable), then it admits $n-1$ first integrals $F = K^{ij}p_i p_j + U$ which are functionally independent and in involution. First integrals of this form (4.41) are called *quadratic first integrals*.

A few years later in 1904 [52], Levi-Civita established the following useful separability criterion.

Theorem 4 (Levi-Civita)**.** *The Hamilton-Jacobi equation of a natural Hamiltonian H is separable with respect to a given set of coordinates (q^i, p_i) if and only if H satisfies the following $\frac{1}{2}n(n-1)$ equations:*

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial^2 H}{\partial q^i \partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j}\frac{\partial^2 H}{\partial q^i \partial p_j}\right) + \frac{\partial H}{\partial q^i}\left(\frac{\partial H}{\partial q^j}\frac{\partial^2 H}{\partial p_i \partial p_j} - \frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_i \partial q^j}\right) = 0, \quad \textit{(no sum)}$$

for $i \neq j,\ i, j = 1, \ldots, n$.

Note that the separable coordinates in this case need not be orthogonal. While this criterion is useful for testing whether a set of coordinates are separable for a Hamiltonian, other methods are typically used to determine what sets of coordinates separate the Hamilton-Jacobi equation of a natural Hamiltonian. One of these methods arose from the important results made by Eisenhart in 1934 [22] in which he approached the problem from a geometrical perspective with the introduction of Killing tensors into the theory. This breakthrough led to an integration method (now called *Eisenhart's method*) which allows one to determine the set of all orthogonally separable coordinate systems for the Hamilton-Jacobi equation of the geodesic natural Hamiltonian. This result was extended in 1993 [6] with a theorem by Benenti, which gives a criterion for orthogonal separability of the Hamilton-Jacobi equation for a *general* natural Hamiltonian. Before we discuss Eisenhart and Benenti's results and apply them to the problems studied in this thesis, let us introduce the theory of Killing tensors with the next chapter.

CHAPTER 4

INVARIANT THEORY OF KILLING TENSORS

The determination and use of invariants of Killing tensors to solve equivalence problems in the theory of orthogonal separation of variables first occurred in 1965 [26] when the authors considered orthogonal separation of the Laplace equation on $\mathbb{E}^2$. In 2002 when studying orthogonal separation of the Hamilton-Jacobi equation on $\mathbb{E}^2$ [56], a different set of authors independently formulated and solved an equivalence problem of Killing tensors, reproducing this earlier result. Since 2002, this theory, now formally called the *invariant theory of Killing tensors*, has been developing steadily [56], [57], [40], [19], [72], [1], [39], [37], [15].

The genesis for such a theory arose out of a need to solve two fundamental problems in the theory of orthogonal separation of variables, namely the canonical forms and equivalence problem of characteristic Killing tensors. Following Eisenhart's fundamental result of 1934 [22] and Benenti's theorem of 1993 [6], these two problems were born. When applying Benenti's theorem, one needed to be able to distinguish and classify the compatible characteristic Killing tensors when determining the set of all possible orthogonal separable coordinate systems for a natural Hamiltonian. It was shown in [26] and [56] that such a problem could be solved by viewing the compatible characteristic Killing tensors as a vector space and applying some of the fundamental results of invariant theory.

As such, the invariant theory of Killing tensors can be seen as a merging of two seemingly disparate topics: Killing tensors and invariant theory. We will begin this chapter with an exposition of Killing tensors, focussing on the topics pertinent to the theory of orthogonal separation of variables. This discussion will enable us to define the principal object of this thesis, the characteristic Killing tensor. Following this section, will be a selective coverage of invariant theory. Given the vastness of this theory, we will restrict our coverage to topics used in the invariant theory of Killing tensors. We conclude the chapter with an overview of the invariant theory of Killing tensors, discussing its development and applications.

4.1 Killing Tensors

Killing vectors have long been known for their use in understanding the symmetry of a given metric space. Killing tensors on the other hand (the generalization of the Killing vector), are perhaps a lesser well-known object, but are still of great importance in classical mechanics and relativity theory. As we will see, Killing tensors play an important role in the solution of the equations of motion in a mechanical problem.

4.1.1 Symmetry, Killing Vectors and Killing Tensors

On a manifold with metric g, a Killing vector field indicates a direction in which the metric is unchanged by the Lie derivative. More formally:

Definition 4.1.1. Consider a pseudo-Riemannian manifold $\mathcal{M}$ equipped with a metric g. If a vector field X satisfies the *Killing vector equation*

$$\mathcal{L}_X g = 0, \tag{4.1}$$

then X is called a *Killing vector field*. The most general vector field that satisfies this equation is called the *general Killing vector* of $\mathcal{M}$.

There are other equivalent definitions of a Killing vector field; recall that when $q = 1$, we showed that the Schouten bracket could be rewritten in terms of the Lie derivative (see (2.8)), therefore

$$[X, g]^{ij} = (\mathcal{L}_X g)^{ij} = 0.$$

Equivalently, Killing vectors may be regarded as the generators of infinitesimal isometries on a manifold. In particular, we say a vector field X is a Killing vector field if its flow σ_ϵ is an isometry, i.e., satisfies $\sigma_\epsilon^* g = g$. For a point $p = (x^i)$, this is equivalent to

$$\begin{aligned}
\frac{\partial(x^k + \epsilon X^k)}{\partial x^\mu}\frac{\partial(x^\lambda + \epsilon X^\lambda)}{\partial x^\nu} g_{k\lambda}(x + \epsilon X) &= g_{\mu\nu}(x) \\
(\delta^k_\mu \delta^\lambda_\nu + \epsilon \delta^k_\mu X^\lambda + \epsilon \delta^\lambda_\nu X^k + \epsilon^2 X^k X^\lambda) g_{k\lambda}(x + \epsilon X) &= g_{\mu\nu}(x) \\
\lim_{\epsilon \to 0} \frac{1}{\epsilon}(g_{\mu\nu}(x + \epsilon X) - g_{\mu\nu}(x)) + \partial_\nu X^\lambda g_{\mu\lambda}(x) + \partial_\mu X^k g_{k\nu}(x) &= 0.
\end{aligned}$$

And since $g_{\mu\nu}(x+\epsilon X) = g_{\mu\nu}(x) + \epsilon X^\xi \partial_\xi g_{\mu\nu}(x) + O(\epsilon^2)$, we can simplify further to

$$X^\xi \partial_\xi g_{\mu\nu} + \partial_\nu X^\lambda g_{\mu\lambda} + \partial_\mu X^k g_{k\nu} = 0, \tag{4.2}$$

which is equivalent to equation (4.1) above. Lastly, since our manifold is pseudo-Riemannian, we have a Levi-Civita connection ∇ on $\mathcal{M}$ satisfying

$$(\nabla_k g)_{ij} = \partial_k g_{ij} - \Gamma^r_{ki} g_{rj} - \Gamma^r_{kj} g_{ir} = 0$$

for the metric g. If we rearrange for $\partial_k g_{ij}$ and substitute this expression into the Killing vector equation (4.2), we get

$$\begin{aligned} X^\xi(\Gamma^\ell_{\xi\nu} g_{\ell\mu} + \Gamma^\ell_{\xi\mu} g_{\ell\nu}) + \partial_\mu X^k g_{k\nu} + \partial_\nu X^\lambda g_{\mu\lambda} &= 0 \\ (\partial_\nu X^\ell + X^\xi \Gamma^\ell_{\nu\xi}) g_{\mu\ell} + (\partial_\mu X^\ell + X^\xi \Gamma^\ell_{\mu\xi}) g_{\ell\nu} &= 0 \\ (\nabla_\nu X)^\ell g_{\mu\ell} + (\nabla_\mu X)^\ell g_{\ell\nu} &= 0 \\ (\nabla_\nu X)_\mu + (\nabla_\mu X)_\nu &= 0. \end{aligned}$$

Let us summarize these equivalencies in the following proposition.

Proposition 4.1.2. *Suppose $\mathcal{M}$ is a pseudo-Riemannian manifold equipped with a metric g and Levi-Civita connection ∇. If X is a vector field on $\mathcal{M}$, then the following statements are equivalent:*

(i) *X is a Killing vector field.*

(ii) *The flow σ_t generated by X satisfies $\sigma_t^* g = g$ (ie., is an isometry).*

(iii) $[X, g] = 0$.

(iv) $(\nabla_i X)_j + (\nabla_j X)_i = 0$.

Not every manifold is symmetrical in this way - in fact most are not. Let us look at two examples of manifolds which do admit non-trivial Killing vectors that will be useful for later discussions.

Example 4.1.3. Let us find the Killing vector fields of $\mathbb{E}^4$. Using Example 2.5.2, we have for the Euclidean metric g,

$$\begin{aligned} (\mathcal{L}_X g)_{ij} &= X^a \partial_a g_{ij} + g_{ib} \partial_j X^b + g_{bj} \partial_i X^b \\ &= \partial_j X^i + \partial_i X^j. \end{aligned}$$

Therefore, a Killing vector field X of $\mathbb{E}^n$ satisfies the following over-determined system of PDEs:

$$\partial_j X^i + \partial_i X^j = 0.$$

Solving for the unknown functions $X^i = f^i(x, y, z, w), i = 1 \ldots 4$, we find that any vector of the form

$$\begin{aligned} X &= (c_1 + c_5 y + c_6 z + c_7 w)\frac{\partial}{\partial x} + (c_2 - c_5 x + c_8 z + c_{10} w)\frac{\partial}{\partial y} \\ &\quad + (c_3 - c_6 x - c_8 y + c_9 w)\frac{\partial}{\partial z} + (c_4 - c_7 x - c_9 z - c_{10} y)\frac{\partial}{\partial w} \end{aligned}$$

is a Killing vector field of $\mathbb{E}^4$. □

Example 4.1.4. Let us find the Killing vector fields of $\mathbb{M}^4$. Proceeding in an analogous way to the previous example, but using the Minkowski metric

$$g = -dt^2 + dx^2 + dy^2 + dz^2,$$

we find that any vector of the form

$$\begin{aligned} X &= (c_1 + c_5 x + c_6 y + c_7 z)\frac{\partial}{\partial t} + (c_2 + c_5 t + c_8 y + c_{10} z)\frac{\partial}{\partial x} \\ &\quad + (c_3 + c_6 t - c_8 x + c_9 z)\frac{\partial}{\partial y} + (c_4 + c_7 t - c_9 x - c_{10} y)\frac{\partial}{\partial z} \end{aligned}$$

is a Killing vector field of $\mathbb{M}^4$. □

Since a linear combination of two Killing vectors is still a Killing vector, the set of all Killing vectors on $(\mathcal{M}, g)$ forms a finite-dimensional vector space, with dimension (see, for example, [78])

$$d \leq \frac{n(n+1)}{2}. \tag{4.3}$$

Furthermore, because the Lie bracket of two Killing vectors is again a Killing vector, the set of Killing vectors on $(\mathcal{M}, g)$ has the following additional structure:

Proposition 4.1.5. *The Killing vectors fields on a manifold form a Lie algebra under the Lie bracket.*

Example 4.1.6. Referring to Example 4.1.3, the vector fields

$$X_i = \frac{\partial}{\partial x^i}, \quad R_{ij} = 2\delta_{ij}^{k\ell} g_{\ell m} x^m X_k, \quad i, k, \ell, m = 1, \ldots, 4,$$

form a basis for the vector space of Killing vectors on $\mathbb{E}^4$. The first set of Killing vectors are called *translational vectors*, since they generate symmetry along the x^i-axis of $\mathbb{E}^4$; the second set are called *rotational vectors*, since they generate symmetry about the origin in the $x^i x^j$-plane in $\mathbb{E}^4$. These vectors satisfy the following commutation relations

$$[X_i, X_j] = 0, \quad [X_i, R_{jk}] = 2\delta^{\ell m}_{jk} g_{mi} X_\ell, \quad [R_{ij}, R_{k\ell}] = 4\delta^{mn}_{ij}\delta^{pr}_{k\ell} g_{mp} R_{nr},$$

which specify its Lie algebraic structure, $\mathfrak{se}(4)$. □

The generalization of a Killing vector is a *Killing tensor*.

Definition 4.1.7. Consider a pseudo-Riemannian manifold $\mathcal{M}$ equipped with a metric g. A symmetric tensor field $K \in \mathcal{T}^q_0(M)$ satisfying the *Killing tensor equation*

$$[K, g] = 0 \tag{4.4}$$

is called a (contravariant) *Killing tensor field*. The most general tensor field that satisfies this equation is called the *general Killing tensor field* of $\mathcal{M}$.

Note that $[\ ,\ ]$ is the Schouten bracket and g is the contravariant metric on the manifold $\mathcal{M}$. Using the Levi-Civita connection ∇ on $\mathcal{M}$, we can equivalently define a (covariant) Killing tensor field as a tensor field K satisfying

$$(\nabla_{(j} K)_{i_1 \ldots i_q)} = 0,$$

which generalizes Proposition 4.1.2 (iv).

Example 4.1.8. Let us find the $(2,0)$-Killing tensor fields of $\mathbb{E}^2$. Using Example 2.5.5, we have for the Euclidean metric g,

$$\begin{aligned} [K, g]^{ijk} &= g^{(|\mu|k}\partial_\mu K^{ij)} - K^{(k|\mu|}\partial_\mu g^{ij)} \\ &= \tfrac{1}{6}\left(\partial_k K^{ij} + \partial_j K^{ki} + \partial_i K^{jk} + \partial_k K^{ji} + \partial_j K^{ik} + \partial_i K^{kj}\right) \\ &= \tfrac{1}{3}(\partial_k K^{ij} + \partial_j K^{ki} + \partial_i K^{jk}) \end{aligned}$$

Therefore, a $(2,0)$-Killing tensor field K of $\mathbb{E}^2$ satisfies the following overdetermined system of PDEs:

$$\partial_k K^{ij} + \partial_j K^{ki} + \partial_i K^{jk} = 0.$$

Solving for the unknown functions $K^{ij} = f^{ij}(x, y), i, j = 1, 2$, we find that any tensor of the form

$$K^{ij} = \begin{pmatrix} c_1 + 2c_2y + c_3y^2 & c_4 - c_2x - c_5y - c_3xy \\ c_4 - c_2x - c_5y - c_3xy & c_6 + 2c_5x + c_3x^2 \end{pmatrix}$$

is a $(2, 0)$-Killing tensor field of $\mathbb{E}^2$. □

The set of all Killing tensors defined on a manifold $\mathcal{M}$ with metric g (hereafter denoted $\mathcal{K}^q(M)$) is a real finite-dimensional vector space. This can be seen by noting that the Schouten bracket is a real bilinear operator, thus the Killing tensor equation (4.4) gives rise to a linear system of homogeneous PDEs. Therefore the solutions of this system (i.e., the Killing tensors) form a real, finite-dimensional vector space. A bound on the dimension, d, of $\mathcal{K}^q(M)$ is given by the *Delong-Takeuchi-Thompson formula* [20, 77, 78]:

$$d = \dim(\mathcal{K}^q(M)) \leq \frac{1}{n}\binom{n+q}{q+1}\binom{n+q-1}{q}, \quad q \geq 1. \tag{4.5}$$

It is easy to show that when $q = 1$, this formula simplifies to (4.3) for the vector space of Killing vectors.

Since the Schouten bracket of two Killing tensor fields is again a Killing tensor field by the Jacobi identity, the vector space of Killing tensor fields on a manifold admits the following additional structure:

Proposition 4.1.9. *The set of Killing tensor fields on a manifold endowed with the Schouten bracket is a graded Lie algebra.*

4.1.2 Killing Tensors on Spaces of Constant Curvature

Solving the Killing tensor equation to find the most general Killing tensor on a manifold is generally quite a difficult task; fortunately, on a manifold of constant curvature this is greatly simplified due to the following result [20, 78]:

Proposition 4.1.10. *Any Killing tensor defined on a pseudo-Riemannian manifold of constant curvature can be written as a sum of symmetrized products of Killing vectors defined on the manifold.*

A Killing tensor which can be decomposed into a sum of symmetrized products of Killing vectors is called *reducible*, otherwise it is called *irreducible*. According to Proposition 4.1.10, any Killing tensor on a space of constant curvature is reducible. On spaces of non-constant curvature, examples[1] can be found which prove that not all Killing tensors are reducible. In addition to the reducibility property, the dimension of the vector space $\mathcal{K}^q(M)$ is known.

Proposition 4.1.11 ([20], [77], [78])**.** *For a manifold $\mathcal{M}$ of constant curvature, the dimension d of the vector space $\mathcal{K}^q(M)$ is given by*

$$d = dim(\mathcal{K}^q(M)) = \frac{1}{n}\binom{n+q}{q+1}\binom{n+q-1}{q}, \quad q \geq 1. \tag{4.6}$$

Proposition 4.1.10 suggests a straightforward way of determining the general Killing tensor on a space of constant curvature. Namely, one first computes a basis for the space of Killing vectors, and then takes a linear combination of all symmetrized products of these basis vectors. The next step is to determine any algebraic identities or *syzygies* amongst the parameters in this linear combination. In particular, the existence of algebraic identities amongst the Killing vectors give rise to syzygies amongst the parameters in this linear combination, which effectively lower the dimension of the space.

We will now illustrate the above technique in the next section where we restrict our attentions to the case when the curvature is zero and the metric is pseudo-Euclidean.

4.1.3 Killing Tensors on $\mathbb{E}^{n-s,s}$

According to Proposition 4.1.10, the $(p,0)$-Killing tensors of $\mathbb{E}^{n-s,s}$ can be expressed as a sum of symmetrized products of Killing vectors of $\mathbb{E}^{n-s,s}$. Using pseudo-Cartesian coordinates x^i, a basis for the vector space of Killing vectors of $\mathbb{E}^{n-s,s}$ is given by

$$X_i = \frac{\partial}{\partial x^i}, \quad R_{ij} = 2\delta^{k\ell}_{ij} g_{\ell m} x^m X_k, \quad i,k,\ell,m = 1,\ldots,n, \tag{4.7}$$

where g denotes the pseudo-Euclidean metric. These Killing vectors satisfy the following commutation relations,

$$[X_i, X_j] = 0, \quad [X_i, R_{jk}] = 2\delta^{\ell m}_{jk} g_{mi} X_\ell, \quad [R_{ij}, R_{k\ell}] = 4\delta^{mn}_{ij}\delta^{pr}_{k\ell} g_{mp} R_{nr},$$

[1]See, for example, Section 1.5 of [20].

which specify its Lie algebraic structure, $\mathfrak{se}(n)$, as well as the following algebraic relations

$$R_{(ij)} = 0, \quad X_{[i} \odot R_{jk]} = 0, \quad R_{i[j} \odot R_{k\ell]} = 0. \tag{4.8}$$

Using these vectors, the general Killing tensor of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ is given by [41]

$$K = \sum_{q=0}^{p} \binom{p}{q} C_{\underline{p-q}}^{i_1 \ldots i_{p-q} J_{p-q+1} \ldots J_p} X_{i_1} \odot \cdots \odot X_{i_{p-q}} \odot R_{J_{p-q+1}} \odot \cdots \odot R_{J_p}, \tag{4.9}$$

where the constant coefficients $C^{i_1 \ldots i_{p-q} J_{p-q+1} \ldots J_p}$ are called the *Killing tensor parameters*, and admit the following symmetries:

$$C_{\underline{p-q}}^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p} = C_{\underline{p-q}}^{(i_1 \cdots i_{p-q})(J_{p-q+1} \cdots J_p)}. \tag{4.10}$$

Note that we have adopted the new notation $R_I = R_{ij}$ for the rotation vectors. The components of K are given by [41]

$$\begin{aligned} K_{\underline{p-q}}^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p} \quad = & \sum_{r=0}^{p-q} \binom{p-q}{r} \delta^{(i_{r+1}}_{M_{r+1} \ell_{r+1}} \cdots \delta^{i_{p-q}}_{M_{p-q} \ell_{p-q}} \\ & C_{\underline{r}}^{i_1 \cdots i_r) M_{r+1} \cdots M_{p-q} J_{p-q+1} \cdots J_p} x^{\ell_{r+1}} \cdots x^{\ell_{p-q}}, \end{aligned} \tag{4.11}$$

where $q = 0, \ldots, p$. The following example will be useful for later discussions.

Example 4.1.12. Let us derive the general $(2,0)$-Killing tensor of $\mathbb{E}^{n-s,s}$. Using equation (4.9), the general Killing tensor of $\mathcal{K}^2(\mathbb{E}^{n-s,s})$ has the form

$$K = A^{ij} X_i \odot X_j + B^{ijk} X_i \odot R_{jk} + C^{ijk\ell} R_{ij} \odot R_{k\ell}, \tag{4.12}$$

with components given by

$$K^{ij} = A^{ij} + 2B^{(ij)k} x_k + 4C^{ikj\ell} x_k x_\ell. \tag{4.13}$$

Note that $A^{ij}, B^{ijk}, C^{ijk\ell}$ denote the parameters of the Killing tensor. The algebraic relations given by (4.8) impose the following symmetries on these parameters

$$B^{i(jk)} = B^{[ijk]} = C^{(ij)k\ell} = C^{ij(k\ell)} = C^{i[jk\ell]} = 0,$$

while the symmetrized tensor product imposes

$$A^{[ij]} = 0, \quad B^{ijk} = B^{jki}, \quad C^{ijk\ell} = C^{k\ell ij}.$$

□

Now that we have derived the general $(p,0)$-Killing tensor of $\mathbb{E}^{n-s,s}$, let us see how we can use these tensors to define general $(p,0)$-Killing tensors on a hypersurface of $\mathbb{E}^{n-s,s}$.

4.1.4 Killing Tensors on a Hypersurface of $\mathbb{E}^{n-s,s}$

Suppose $\mathcal{N}$ is a Riemannian hypersurface of $\mathbb{E}^{n-s,s}$ with constant, non-zero curvature, and $f : \mathcal{N} \to \mathbb{E}^{n-s,s}$ is the smooth and injective map realizing $\mathcal{N}$ as a hypersurface in $\mathbb{E}^{n-s,s}$.[2] Then, the general $(p,0)$-Killing tensors on the hypersurface can be determined using the general $(p,0)$-Killing tensor on $\mathbb{E}^{n-s,s}$ according to the following result.

Proposition 4.1.13. *The general $(p,0)$-Killing tensor on $\mathcal{N}$ is given by*

$$K = C_{\underline{0}}^{J_1\cdots J_p} R_{J_1} \odot \cdots \odot R_{J_p}, \tag{4.14}$$

where R_{J_i} are rotation vectors on $\mathbb{E}^{n-s,s}$.

Proof. Since $\mathcal{N}$ is a space of constant, non-zero curvature, it is isomorphic to either spherical or hyperbolic space of the corresponding dimension. Hence, any Killing vector on $\mathcal{N}$ is a rotational vector of the following form:

$$R_{ij} = 2\delta_{ij}^{k\ell} x_\ell X_k.$$

By Proposition 4.1.10, any Killing tensor on $\mathcal{N}$ is expressible as a sum of symmetrized products of Killing vectors on the manifold. Therefore, any Killing tensor on $\mathcal{N}$ has the following form:

$$K = C_{\underline{0}}^{J_1\cdots J_p} R_{J_1} \odot \cdots \odot R_{J_p},$$

where $R_{J_i} = R_{ij}$ are rotation vectors on $\mathbb{E}^{n-s,s}$. $\square$

If we consider equation (4.10) and the symmetries of a rotation vector (4.8), we find that the parameter tensor $C_{\underline{0}}^{J_1\cdots J_p} = C_{\underline{0}}^{j_1 j_2 j_3 j_4 \cdots j_{p-1} j_p}$ admits the following symmetries:

$$\begin{aligned}
C_{\underline{0}}^{J_1\cdots J_p} &= C_{\underline{0}}^{(J_1\cdots J_p)}, \\
C_{\underline{0}}^{(j_1 j_2) j_3 j_4 \cdots j_{p-1} j_p} &= C_{\underline{0}}^{j_1 j_2 (j_3 j_4) \cdots j_{p-1} j_p} = \ldots = C_{\underline{0}}^{j_1 j_2 j_3 j_4 \cdots (j_{p-1} j_p)} = 0, \\
C_{\underline{0}}^{j_1 [j_2 j_3 j_4] j_5 \cdots j_{p-1} j_p} &= C_{\underline{0}}^{j_1 j_2 j_3 [j_4 j_5 j_6] j_7 \cdots j_{p-1} j_p} = \ldots = C_{\underline{0}}^{j_1 j_2 \cdots j_{p-3} [j_{p-2} j_{p-1} j_p]} = 0.
\end{aligned}$$

[2]Hereafter, whenever we refer to a manifold $\mathcal{N}$, we are assuming that is a Riemannian hypersurface of constant, non-zero curvature in an ambient space $\mathbb{E}^{n-s,s}$.

In view of (4.14) and taking into account these symmetries of the coefficient tensor $C_{\underline{0}}^{J_1\cdots J_p} = C_{\underline{0}}^{j_1j_2j_3j_4\cdots j_{p-1}j_p}$, the dimension of the vector space of Killing tensors defined on $\mathcal{N}$ agrees with the dimension determined using formula (4.6).

The theory of orthogonal separation of variables is concerned with a particular class of $(2,0)$-Killing tensors on the manifold. This being the case, let us now focus our attention on Killing tensors of this particular valence and give some useful results pertaining to them.

According to Proposition 4.1.13, the general $(2,0)$-Killing tensor on a hypersurface $\mathcal{N}$ is given by

$$K = C^{IJ} R_I \odot R_J = C^{ijk\ell} R_{ij} \odot R_{k\ell}, \tag{4.15}$$

where the parameters $C^{ijk\ell}$ satisfy

$$C^{(ij)k\ell} = C^{ij(k\ell)} = C^{i[jk\ell]} = 0,\ \ C^{ijk\ell} = C^{k\ell ij}. \tag{4.16}$$

Since the coefficient tensor $C^{ijk\ell}$ admits the same symmetry properties as the Riemann curvature tensor (see Section 2.6.2), it can be called an *algebraic curvature tensor* (see, for example, [8]). By (4.13),

$$K^{ij} = 4C^{ikj\ell} x_k x_\ell \tag{4.17}$$

are its components. For our next result, we need to define the vector

$$D = x^i X_i,$$

called the *dilatation vector*, which satisfies the following commutation relations

$$[X_i, D] = X_i, \quad [D, R_{ij}] = 0,$$

for the Killing vectors X and R in (4.7), and the orthogonality relation

$$g(D, R_{ij}) = 0,$$

where g denotes the metric of the ambient space $\mathbb{E}^{n-s,s}$.

Corollary 4.1.14. *A Killing tensor K of $\mathcal{K}^2(\mathbb{E}^{n-s,s})$ is a Killing tensor of $\mathcal{K}^2(\mathcal{N})$ if and only if it satisfies*

$$[K, D] = 0. \tag{4.18}$$

Proof. Recall that any Killing tensor of $\mathcal{K}^2(\mathbb{E}^{n-s,s})$ must have the following form

$$K = A^{ij} X_i \odot X_j + B^{ijk} X_i \odot R_{jk} + C^{ijk\ell} R_{ij} \odot R_{k\ell}.$$

Since $[R_{ij}, D] = 0$, $[X_i, D] = X_i$ and $B^{ijk} = -B^{ikj}$, we have

$$\begin{aligned} [K, D] &= A^{ij}[D, X_i] \odot X_j + A^{ij} X_i \odot [D, X_j] + B^{ijk}[D, X_i] \odot R_{jk} + \\ &\quad B^{ijk} X_i \odot [D, R_{jk}] + C^{ijk\ell}[D, R_{ij}] \odot R_{k\ell} + C^{ijk\ell} R_{ij} \odot [D, R_{k\ell}] \\ &= -2A^{ij} X_i \odot X_j - B^{ijk} X_i \odot R_{jk} \\ &= -2A^{ij} X_i \odot X_j - (B^{ijk} - B^{ikj}) x_k X_i \odot X_j \\ &= -2A^{ij} X_i \odot X_j - 2B^{ijk} x_k X_i \odot X_j. \end{aligned}$$

Therefore, $[K, D] = 0$ is equivalent to $A^{ij} = B^{(ij)k} = 0$. But since B satisfies the cyclic identity $B^{[ijk]} = 0$, we find that

$$\begin{aligned} B^{[ijk]} &= B^{ijk} + B^{jki} + B^{kij} \\ &= B^{ijk} + B^{ijk} + B^{ijk} \\ 0 &= 3B^{ijk}. \end{aligned}$$

Hence the condition $[K, D] = 0$ is true if and only if

$$K = C^{ijk\ell} R_{ij} \odot R_{k\ell}, \tag{4.19}$$

which is the general Killing tensor of $\mathcal{K}^2(\mathcal{N})$. □

For the purposes of the next section, it is important to note that any Killing tensor of $\mathcal{K}^2(\mathcal{N})$ admits at least one zero eigenvalue.

Proposition 4.1.15. *Any Killing tensor $K = C^{ijk\ell} R_{ij} \odot R_{k\ell}$ satisfies*

$$K^{nr} x_n = 0. \tag{4.20}$$

Proof. Suppose $K = C^{ijk\ell} R_{ij} \odot R_{k\ell}$. Then,

$$\begin{aligned} K^{nr} x_r &= C^{ijk\ell} \delta^n_{ijm} \delta^r_{k\ell s} x^m x^s x_r \\ &= C^{ijk\ell} (\delta^n_i g_{jm} - \delta^n_j g_{im})(\delta^r_k g_{\ell s} - \delta^r_\ell g_{ks}) x^m x^s x_r \\ &= (C^{njr\ell} - C^{nj\ell r} - C^{jnr\ell} + C^{jn\ell r}) g_{jm} g_{\ell s} x^m x^s x_r. \end{aligned}$$

After applying the symmetry properties of C^{ijkl} to the RHS, we find

$$\begin{aligned} K^{nr}x_r &= 4C^{njr\ell}g_{jm}g_{\ell s}x^m x^s x_r \\ &= 4C^{njr\ell}x_j x_\ell x_r. \end{aligned} \tag{4.21}$$

Using the identity $C^{n[jr\ell]} = 0$, the RHS vanishes and hence

$$K^{nr}x_r = 0.$$

□

Corollary 4.1.14 and Proposition 4.1.15 are similar to a proposition given by Delong [20] which states that a function $F = K^{i_1 i_2 \cdots i_r} p_{i_1} p_{i_2} \cdots p_{i_r}$ of an $(r, 0)$-Killing tensor on $\mathbb{E}^{n+1}$ with metric g in involution with the radius function

$$r^2 = g_{ij}x^i x^j,$$

and the dilatation function

$$d = x^i p_i$$

for conjugate momenta p_i, defines a Killing tensor on $\mathbb{S}^n$. If we set $r = 2$ [15], the first condition,

$$\begin{aligned} 0 = \{F, r^2\} &= \{K^{ij}p_i p_j, g_{mn}x^m x^n\} \\ &= K^{ij}g_{mn}\frac{\partial(p_i p_j)}{\partial p_k}\frac{\partial(x^m x^n)}{\partial x^k} \\ &= 4K^{kj}x_k p_j, \end{aligned} \tag{4.22}$$

is equivalent to (4.20), while the second condition,

$$\begin{aligned} 0 = \{F, d\} &= \{K^{ij}p_i p_j, x^n p_n\} \\ &= K^{ij}\frac{\partial(p_i p_j)}{\partial p_k}p_n \delta^n_k - \frac{\partial K^{ij}}{\partial x^k}x^n p_i p_j \delta^n_k \\ &= p_i p_j \left(2K^{ij} - \frac{\partial K^{ij}}{\partial x^k}x^k\right), \end{aligned} \tag{4.23}$$

is equivalent to (4.18):

$$\begin{aligned} 0 = [D, K] &= [D, K^{ij}X_i \odot X_j] \\ &= (\mathcal{L}_D K^{ij})X_i \odot X_j + K^{ij}(\mathcal{L}_D X_i) \odot X_j + K^{ij}X_i \odot (\mathcal{L}_D X_j) \\ &= \left(x^k \frac{\partial K^{ij}}{\partial x^k} - 2K^{ij}\right) X_i \odot X_j. \end{aligned}$$

As a result of these equivalencies, we see that only

$$\{F, d\} = 0$$

must hold for a (2,0)-Killing tensor on $\mathbb{E}^{n+1}$ to define a (2,0)-Killing tensor on $\mathbb{S}^n$, and

$$\{F, r^2\} = 0$$

is a consequence of this condition.

4.1.5 Characteristic Killing Tensors

The theory of orthogonal separation of variables of the Hamilton-Jacobi equation is concerned with a particular subset of $(2,0)$-Killing tensors on the manifold, called *characteristic Killing tensors* (CKTs).

Definition 4.1.16. A Killing tensor, $K \in \mathcal{K}^2(M)$, with

(i) real and distinct eigenvalues, and

(ii) orthogonally integrable eigenvectors[3]

is called a *characteristic Killing tensor*.

Note that if $\{e^i\}$ denote the eigenforms of a Killing tensor, then the corresponding eigenvectors are orthogonally integrable if and only if

$$e^i \wedge de^i = 0 \quad \text{(no sum)}$$

by the Frobenius Theorem 2.7.6. Eigenforms satisfying this condition are said to be *normal*. To determine if a Killing tensor (4.15) on a hypersurface $\mathcal{N} \subset \mathbb{E}^{n+1-s,s}$ is a CKT on $\mathcal{N}$, we have the following proposition.

Proposition 4.1.17. *Suppose $\mathcal{N}$ is an n-dimensional hypersurface in $\mathbb{E}^{n+1-s,s}$. A Killing tensor $K \in \mathcal{K}^2(\mathbb{E}^{n+1-s,s})$ evaluated on $\mathcal{N}$ with orthogonally integrable eigenvectors and either*

[3] A (2,0)-tensor has *orthogonally integrable eigenspaces* if the distributions defined by the orthogonal complements of the eigenspaces are integrable.

(i) one zero eigenvalue and n real and distinct non-zero eigenvalues, or

(ii) two zero eigenvalues and $n-1$ real and distinct non-zero eigenvalues

is a CKT on $\mathcal{N}$.

Proof. Consider a Killing tensor $K \in \mathcal{K}^2(\mathcal{N})$. Then by (4.15), K is of the following form

$$K = C^{ijk\ell} R_{ij} \odot R_{k\ell}, \tag{4.24}$$

and by Proposition 4.1.16 admits at least one zero eigenvalue. Suppose K has either

$$\{0, \lambda_1, \dots, \lambda_n\} \text{ or } \{0, 0, \lambda_1, \dots, \lambda_{n-1}\}$$

for its eigenvalues, where the λ_i are non-zero, real and distinct, and suppose its eigenvectors are orthogonally integrable. Under the pullback map (2.4.4), one zero eigenvalue is eliminated and the remaining eigenvalues are still real and distinct. Since the wedge product is preserved under the pullback map (2.6), normal eigenforms of K pullback to normal eigenforms on $\mathcal{N}$. Hence K defines a CKT on $\mathcal{N}$. □

According to this proposition, a CKT on the ambient space $\mathbb{E}^{n+1-s,s}$ is necessarily pulled back to a CKT on a hypersurface $\mathcal{N}$. However, it also demonstrates that we can have Killing tensors which are not CKTs on the ambient space but *are* CKTs on the hypersurface. Let us illustrate this latter statement with the following example.

Example 4.1.18. The tensor

$$K^{ij} = \begin{pmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is of the form (4.12) and hence is a Killing tensor of $\mathbb{E}^3$. The eigenvalues of K can be obtained from the characteristic equation

$$\det(K^{ij} - \lambda g^{ij}) = 0,$$

where g^{ij} denotes the contravariant form of the Euclidean metric. This yields

$$\lambda_i = 0, 0, x^2 + y^2$$

as the eigenvalues of K. Since the eigenvalues are not distinct, we conclude that K is not a CKT of $\mathbb{E}^3$. Note, however, that K is of the form (4.19) and hence is a Killing tensor of $\mathbb{S}^2$. To see if K is characteristic, we compute the eigenvectors

$$E_{x^2+y^2} = \{(y, -x, 0)\}, \quad E_0 = \{(0, 0, 1), (x, y, z)\},$$

and find $[(y, -x, 0), (y, -x, 0)] = 0$ and $[(0, 0, 1), (x, y, z)] = 0$. Therefore, $E_{x^2+y^2}$ and E_0 define integrable distributions which are the orthogonal complement of the other, and thus orthogonally integrable. Since the eigenvectors are orthogonally integrable and K possesses two zero eigenvalues and one non-zero real eigenvalue, we conclude that K is a CKT on the 2-dimensional hypersurface $\mathbb{S}^2 \subset \mathbb{E}^3$ by Proposition 4.1.17. Indeed, a map $f : \mathbb{S}^2 \to \mathbb{E}^3$ defined by spherical coordinates

$$\begin{aligned} x &= \sin\theta \sin\phi, \\ y &= \sin\theta \cos\phi, \\ z &= \cos\theta, \end{aligned}$$

transforms

$$K_{ij} = \begin{pmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \tilde{K}_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \sin^4\theta \end{pmatrix}$$

under the pullback map f^*. Since K has two real and distinct eigenvalues, and orthogonally integrable eigenvectors, it defines a CKT on $\mathbb{S}^2$ by Definition 4.1.16. $\square$

In the above example, it was easy to find the eigenvalues and eigenvectors of K. In general, though, this can be a very computationally intensive task for a given symmetric matrix having polynomial components. As such, it would be convenient to have a way of determining whether a Killing tensor is characteristic without having to compute the eigenvalues and eigenvectors.

A way of determining the nature of the eigenvalues of a matrix is to compute the discriminant of its characteristic polynomial. The matrices of interest to this thesis are either 3×3 or 4×4, and as such, the characteristic polynomials are of degree three or four. But because the Killing tensors are defined in the ambient space coordinates, at least one eigenvalue is always zero. Hence the characteristic polynomial for dimension $m = 2$ has the form

$$\lambda(a\lambda^2 + b\lambda + c) = 0,$$

while the characteristic polynomial for dimension $m = 3$ has the form

$$\lambda(a\lambda^3 + b\lambda^2 + c\lambda + d) = 0.$$

To determine the nature of the eigenvalues in each case, we compute the discriminant of the polynomial in the brackets. The discriminant of the quadratic polynomial is given by

$$\Delta_1 = b^2 - 4ac,$$

thus a Killing tensor on a 2-dimensional hypersurface has real and distinct eigenvalues if and only if $\Delta_1 > 0$. The discriminant of the cubic polynomial is given by

$$\Delta_2 = b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2,$$

thus a Killing tensor on a 3-dimensional hypersurface has real and distinct eigenvalues if and only if $\Delta_2 > 0$.

In two dimensions, any $(2,0)$-Killing tensor with real and distinct eigenvalues has orthogonally integrable eigenvectors; but in higher dimensions, this is not always the case and this condition must be checked. Therefore, it would be convenient and far more practical to have a criterion for orthogonal integrability without having to compute the eigenvectors.

Proposition 4.1.19 (Tonolo-Schouten-Nijenhuis, [60])**.** *A tensor field T with distinct eigenvalues has orthogonally integrable eigenvectors if and only if*

$$\begin{aligned} N^\ell_{[jk}g_{i]\ell} &= 0, \\ N^\ell_{[jk}T_{i]\ell} &= 0, \\ N^\ell_{[jk}T_{i]m}T^m_\ell &= 0, \end{aligned} \tag{4.25}$$

are satisfied for the Nijenhuis tensor[4] *N*

$$N^i_{jk} = T^i_\ell T^\ell_{[j,k]} + T^\ell_{[j}T^i_{k],\ell}$$

[4]The coordinate-free form of the Nijenhuis tensor is given by

$$N(X,Y) = [AX, AY] + A^2[X,Y] - A[X,AY] - A[AX,Y],$$

where X and Y are vector fields and A is a $(1,1)$-tensor field.

of T.

We call equations (4.25) the *TSN conditions.* For a KT of the form (4.17), the TSN conditions place restrictions on the form of the coefficients $C^{ijk\ell}$. In particular, if we substitute (4.17) into (4.25), we obtain the following constraints on the coefficients $C^{ijk\ell}$ of K:

$$C^{\ell}{}_{(pq[i}C_{jk]r)\ell} = 0, \tag{4.26}$$

$$C_{\ell(pq}{}^{m}C^{\ell}{}_{r[ij}C_{k]st)m} - 2C_{\ell(pq[i}C_{j}{}^{\ell}{}_{|r|}{}^{m}C_{k]st)m} = 0, \tag{4.27}$$

$$\begin{aligned}3C_{\ell(pq}{}^{m}C^{\ell}{}_{r|n|s}C^{n}{}_{t[ij}C_{k]uv)m} + 2C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j|t|}{}^{n\ell}C_{k]uv)m} +& \\ 2C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j}{}^{n}{}_{|t|}{}^{\ell}C_{k]uv)m} = 0,&\end{aligned} \tag{4.28}$$

Using indicial tensor algebra, it is possible to show that (4.26) and (4.27) imply (4.28) [15].[5]

Proposition 4.1.20. *If a Killing tensor* $K^{ij} = 4C^{ikj\ell}x_k x_\ell$ *satisfies conditions (4.26) and (4.27), then* K^{ij} *satisfies (4.28).*

Proof. Suppose a Killing tensor $K^{ij} = 4C^{ikj\ell}x_k x_\ell$ satisfies (4.26) and (4.27). Then, the first condition (4.26) implies

$$C_{\ell(pq}{}^{m}C^{\ell}{}_{r[ij}C_{k]st)m} + 2C_{\ell(pq[i}C_{j}{}^{\ell}{}_{|r|}{}^{m}C_{k]st)m} = 0. \tag{4.29}$$

The system of equations defined by (4.29) and (4.27) implies:

$$C_{\ell(pq}{}^{m}C^{\ell}{}_{r[ij}C_{k]st)m} = 0, \tag{4.30}$$

$$C_{\ell(pq[i}C_{j}{}^{\ell}{}_{|r|}{}^{m}C_{k]st)m} = 0. \tag{4.31}$$

Now let us consider the third term of (4.28). Using the identity $C^{i[jk\ell]} = 0$, we can expand this term as follows:

$$\begin{aligned}C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j}{}^{\ell}{}_{|t|}{}^{n}C_{k]uv)m} &= -C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j|t|}{}^{n\ell}C_{k]uv)m} \\ &\quad +C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j}{}^{n}{}_{|t|}{}^{\ell}C_{k]uv)m}.\end{aligned} \tag{4.32}$$

---

[5] In [40], this result was proven for the case of $\mathbb{E}^3$ by Czapor using Gröbner basis theory and computer algebra. Then in [69], this result was proven for any n-dimensional space of constant, non-zero curvature using representation theory. However, as we demonstrate, this latter result can also be proven more simplistically using indicial tensor algebra.

This condition together with (4.31) implies

$$C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j|t|}{}^{n\ell}C_{k]uv)m} = 0, \tag{4.33}$$

and

$$C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j}{}^{\ell}{}_{|t|}{}^{n}C_{k]uv)m} = C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j}{}^{n}{}_{|t|}{}^{\ell}C_{k]uv)m}. \tag{4.34}$$

The last equation (4.34) together with (4.26) implies

$$C_{\ell(pq}{}^{m}C^{\ell}{}_{r|n|s}C^{n}{}_{t[ij}C_{k]uv)m} - 2C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j}{}^{n}{}_{|t|}{}^{\ell}C_{k]uv)m} = 0; \tag{4.35}$$

furthermore, (4.30) implies

$$C_{\ell(pq}{}^{m}C^{\ell}{}_{r|n|s}C^{n}{}_{t[ij}C_{k]uv)m} + 2C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j}{}^{n}{}_{|t|}{}^{\ell}C_{k]uv)m} = 0. \tag{4.36}$$

The system of equations defined by (4.35) and (4.36) implies

$$C_{\ell(pq}{}^{m}C^{\ell}{}_{r|n|s}C^{n}{}_{t[ij}C_{k]uv)m} = 0, \tag{4.37}$$

$$C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j}{}^{n}{}_{|t|}{}^{\ell}C_{k]uv)m} = 0. \tag{4.38}$$

Finally, conditions (4.33), (4.37) and (4.38) taken together imply the third condition (4.28). □

It is important to note that Proposition 4.1.19 provides a criterion for tensor fields having *distinct* eigenvalues. Since our characteristic Killing tensor fields may have indistinct eigenvalues, Proposition 4.1.19 cannot always be used. Fortunately, we have the following more general result [32].

Proposition 4.1.21 (Haantjes)**.** *Suppose that for a tensor field T, each eigenvalue of multiplicity r has r linearly independent eigenvectors. Then, T has orthogonally integrable eigenspaces if and only if*

$$H^{i}_{jk} = N^{i}_{\ell m}T^{\ell}_{j}T^{m}_{k} - N^{\ell}_{jm}T^{i}_{\ell}T^{m}_{k} - N^{\ell}_{mk}T^{i}_{\ell}T^{m}_{j} + N^{\ell}_{jk}T^{i}_{m}T^{m}_{\ell} = 0 \tag{4.39}$$

is satisfied for the Nijenhuis tensor N of T.

We call H the Haantjes tensor,[6] and equation (4.39) the *Haantjes condition.* For a Killing tensor of the form (4.17), the Haantjes condition places the following restriction on the form of the coefficients $C^{ijk\ell}$ [15]:

$$\begin{aligned} 4C_{\ell(pq}{}^{k}C^{m}{}_{rs|i}C_{j|tu}{}^{n}C^{\ell}{}_{v)mn} + 2C_{\ell(p|m|}{}^{k}C^{n}{}_{qr[i}C_{j]st}{}^{m}C^{\ell}{}_{uv)n} \\ -5C_{\ell(pq}{}^{k}C^{m}{}_{rs[i}C_{j]|m|t}{}^{n}C^{\ell}{}_{uv)n} + C_{\ell(pq}{}^{k}C^{m}{}_{rs[i}C_{j]}{}^{\ell}{}_{t}{}^{n}C_{|n|uv)m} \\ +C_{\ell(pq}{}^{k}C^{m}{}_{rs[i}C_{j]tu}{}^{n}C_{|n|v)m}{}^{\ell} - 3C_{\ell(pq}{}^{k}C^{m}{}_{r|ij|}C^{n}{}_{st|m|}C^{\ell}{}_{uv)n} \\ -2C_{\ell(pq}{}^{k}C^{m}{}_{rs[i}C_{j]t|m|}{}^{n}C^{\ell}{}_{uv)n} = 0. \end{aligned} \tag{4.40}$$

Since the Haantjes proposition applies to the more general case, we will adopt this criterion when determining whether a given Killing tensor of spherical or hyperbolic space is characteristic. In conclusion, the discriminant, together with (4.40), enable us to determine whether a Killing tensor is characteristic without having to find the eigenvalues and eigenvectors.

We can visualize a CKT by its associated *orthogonal coordinate web.* For a CKT defined on an n-dimensional manifold $\mathcal{M}$, the integral curves of the orthogonally integrable eigenvector fields fit together to form a set of n hypersurfaces of dimension $n-1$. This family of hypersurfaces represents a foliation of $\mathcal{M}$, and each hypersurface is a leaf of the foliation. Because of the orthogonality of these eigenvector fields, the hypersurfaces intersect orthogonally and create an *orthogonal coordinate web* for the CKT.

Example 4.1.22. Recall that

$$K^{ij} = \begin{pmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a CKT on $\mathbb{S}^2$. Its eigenvector fields

$$E_{x^2+y^2} = \{(y,-x,0)\}, \quad E_0 = \{(0,0,1),(x,y,z)\}$$

[6]The coordinate-free form of the Haantjes tensor is given by

$$H(X,Y) = N(AX,AY) + A^2N(X,Y) - AN(X,AY) - AN(AX,Y),$$

where X and Y are vector fields, A is a $(1,1)$-tensor field, and N is the Nijenhuis tensor field.

generate the following flows in $\mathbb{E}^3$

$$\begin{aligned}
\sigma_1(t,(c_1,c_2,c_3)) &= (c_2\sin t + c_1\cos t, c_2\cos t - c_1\sin t, c_3),\\
\sigma_2(t,(c_1,c_2,c_3)) &= (c_1,c_2,t+c_3),\\
\sigma_3(t,(c_1,c_2,c_3)) &= (c_1e^t, c_2e^t, c_3e^t),
\end{aligned}$$

for $c_1,c_2,c_3 \in \mathbb{R}$. The flow curves of σ_1 are latitudinal lines on a sphere centred at the origin; the flow curves of σ_2 are vertical lines in $\mathbb{E}^3$ parallel to the z-axis; and the flow curves of σ_3 are lines through the origin in $\mathbb{E}^3$. When we intersect these flow curves with $\mathbb{S}^2$, we obtain an orthogonal web of longitudinal and latitudinal lines on the surface of $\mathbb{S}^2$. □

As the next section demonstrates, CKTs play a fundamental role in the study of orthogonal separation of variables of the Hamilton-Jacobi equation.

4.1.6 Orthogonal Separation of Variables and Killing Tensors

Having defined Killing tensors and the subclass of CKTs, let us now continue with the key developments made in the theory of orthogonal separation of variables of the Hamilton-Jacobi equation begun in Subsection 3.3.3. Recall that in 1893, Stäckel showed that if the Hamilton-Jacobi equation of a natural Hamiltonian is orthogonally separable, then it admits $n-1$ quadratic first integrals

$$F = \frac{1}{2}K^{ij}p_ip_j + U(q^i) \tag{4.41}$$

that are functionally independent and in involution. Building on this work, in 1934 [22] Eisenhart observed that the K^{ij} in (4.41) are Killing tensors on the manifold. Indeed, since F is a first integral,

$$\begin{aligned}
0 = \{F,H\} &= \frac{1}{4}\{K^{ij}p_ip_j, g^{k\ell}p_kp_\ell\} + \frac{1}{2}\{K^{ij}p_ip_j, V\} + \frac{1}{2}\{U, g^{k\ell}p_kp_\ell\}\\
&= \frac{1}{2}p_ip_jp_\ell\left(g^{k\ell}\partial_kK^{ij} - K^{kj}\partial_kg^{i\ell}\right) - p_\ell(g^{k\ell}\partial_kU - K^{k\ell}\partial_kV).
\end{aligned} \tag{4.42}$$

The first term on the RHS of (4.42) implies

$$\partial_kK^{(ij}g^{|k|\ell)} - K^{k(j}\partial_kg^{i\ell)} = 0,$$

which is the Killing tensor equation (4.4). Since Eisenhart was investigating the geodesic case, $U = 0$ and $V = 0$, and thus the second term on the RHS vanishes. Eisenhart also noticed that the $n-1$ K^{ij} are CKTs with the same integrable eigenforms $E_i = h_i du^i$. In this coframe of eigenforms, both the metric

$$g = \epsilon_1 (h_1 du^1)^2 + \cdots + \epsilon_n (h_n du^n)^2 \tag{4.43}$$

and the Killing tensors

$$K_{ij} = \lambda_i g_{ij}, \tag{4.44}$$

are diagonal, where the λ_i are the eigenvalues of K. If we substitute (4.44) into the Killing tensor equation (4.4), we obtain the following n linear first-order partial differential equations

$$\frac{\partial \lambda_i}{\partial u^j} = (\lambda_i - \lambda_j) \frac{\partial \ln g_{ii}}{\partial u^j}, \quad \text{(no sum)} \tag{4.45}$$

called *Eisenhart's equations*, which have

$$\begin{aligned} \frac{\partial^2 \ln h_i^2}{\partial u^i \partial u^j} + \frac{\partial \ln h_i^2}{\partial u^j} \frac{\partial \ln h_j^2}{\partial u^i} &= 0, \quad i \neq j, \\ \frac{\partial^2 \ln h_i^2}{\partial u^j \partial u^k} - \frac{\partial \ln h_i^2}{\partial u^j} \frac{\partial \ln h_i^2}{\partial u^k} + \frac{\partial \ln h_i^2}{\partial u^j} \frac{\partial \ln h_j^2}{\partial u^k} + \frac{\partial \ln h_i^2}{\partial u^k} \frac{\partial \ln h_k^2}{\partial u^j} &= 0, \quad i \neq j \neq k. \end{aligned} \tag{4.46}$$

as integrability conditions. Equations (4.46) yield necessary and sufficient conditions for the metric of an orthogonal coordinate system to be of Stäckel type. Let us summarize these statements in the following theorem.

Theorem 5 (Eisenhart, [22])**.** *Suppose H is the geodesic Hamiltonian. Then the following statements are equivalent.*

(i) There exist coordinates u^i with respect to which the Hamilton-Jacobi equation of H is orthogonally separable.

(ii) H admits $n-1$ functionally independent quadratic first integrals of the form

$$F = \frac{1}{2} K^{ij} p_i p_j,$$

and their Killing tensors K are characteristic with the same eigenforms.

(iii) Equations (4.46) are satisfied for a metric (4.43).

This result enabled Eisenhart to establish a method (now called *Eisenhart's method*) to determine the orthogonal coordinate systems which separate the geodesic Hamiltonian for spaces of constant curvature. Applying this result in [22], Eisenhart derived 11 inequivalent metrics for $\mathbb{E}^3$ and 5 inequivalent metrics for $\mathbb{S}^3$. In 1950 [62], Olevskii applied Eisenhart's method and the geometrical properties of the integral surfaces to derive the orthogonally separable metrics and coordinate systems for the geodesic Laplace-Beltrami equation on $\mathbb{S}^2, \mathbb{S}^3, \mathbb{H}^2$, and $\mathbb{H}^3$. By Robertson's theorem of 1927 [68], determining the metrics which permit orthogonal (product) separation of the Laplace-Beltrami equation is equivalent to finding the metrics which permit (additive) separation of the Hamilton-Jacobi equation when the space is of constant curvature. In 2008 [39], Horwood and McLenaghan applied Eisenhart's method to derive the orthogonally separable coordinate systems for the geodesic Hamilton-Jacobi equation on $\mathbb{M}^3$.[7]

Using Eisenhart's equations (4.45), we can determine a CKT for a given orthogonally separable metric. In particular, if we solve (4.45) for λ_i and substitute these functions along with the metric components into (4.44), we obtain a CKT in the given separable coordinates. Applying this method, Horwood *et al* determined a CKT for each of the orthogonally separable metrics of $\mathbb{E}^3$ in 2005 [40], and for $\mathbb{M}^3$ in 2009 [39]. In both cases, the authors transformed the CKTs into a common coordinate system: Cartesian coordinates for $\mathbb{E}^3$ and pseudo-Cartesian coordinates for $\mathbb{M}^3$.

To illustrate how Eisenhart's equations can be used to determine a CKT for the orthogonally separable coordinate systems on a manifold, let us consider the following example.

Example 4.1.23. In 1950, Olevskii [62] showed there exist two inequivalent systems of orthogonal coordinates which separate the geodesic equation for $\mathbb{S}^2$, namely

[7] The determination of orthogonally separable coordinate systems for the geodesic Laplace-Beltrami equation on $\mathbb{M}^3$ was first considered in [43, 44], although the results are incomplete [37]. The problem was also considered in [33, 34].

spherical and elliptic coordinates. Substituting their metrics

$$\begin{aligned} ds^2 &= d\theta^2 + \sin^2\theta d\phi^2 \\ ds^2 &= \frac{(\rho_2-\rho_1)}{4(\rho_1-a)(\rho_1-b)(\rho_1-c)} d\rho_1^2 + \frac{(\rho_1-\rho_2)}{4(\rho_2-a)(\rho_2-b)(\rho_2-c)} d\rho_2^2 \end{aligned}$$

into Eisenhart's equations and integrating, we obtain the following solution in each case:

$$\begin{array}{lll} \lambda_1 = k_1, & \lambda_2 = k_1 + k_2\sin^2\theta & \text{(spherical)} \\ \lambda_1 = k_1\rho_2 + k_2, & \lambda_2 = k_1\rho_1 + k_2 & \text{(elliptic).} \end{array}$$

To obtain the corresponding Killing tensor, we substitute the λ_i into (4.44) for each metric, which yields

$$\begin{aligned} K^{(S)} &= k_1 g + k_2 \sin^4\theta d\phi^2, \\ K^{(E)} &= k_2 g + k_1\left(\rho_2 g_{11} d\rho_1^2 + \rho_1 g_{22} d\rho_2^2\right). \end{aligned}$$

Since it is more convenient to work in the coordinates of the ambient space, we transform the above expressions into Cartesian coordinates by first raising indices and then using the pushforward map (2.4.3).[8] After renaming the parameters, we find

$$K^{ij}_{(S)} = \begin{pmatrix} c_1y^2 + c_2z^2 & -c_1xy & -c_2xz \\ -c_1xy & c_1x^2 + c_2z^2 & -c_2yz \\ -c_2xz & -c_2yz & c_2(x^2+y^2) \end{pmatrix}, \; c_1, c_2 \in \mathbb{R}$$

is a CKT for the spherical metric, and

$$K^{ij}_{(E)} = \begin{pmatrix} c_1z^2 + c_2y^2 & -c_2xy & -c_1xz \\ -c_2xy & c_2x^2 + c_3z^2 & -c_3yz \\ -c_1xz & -c_3yz & c_1x^2 + c_3y^2 \end{pmatrix},$$

where

$$c_1 \neq c_2 \neq c_3, \; c_1, c_2, c_3 \in \mathbb{R}$$

is a CKT for the elliptic metric. □

[8] The elliptic coordinates can be found in Olevskii [62] on p. 407. Some authors [29] alternatively use the Jacobi elliptic functions to define elliptic coordinates. The form of these coordinates are listed in Appendix B.1.

Recall that Eisenhart's method yields the orthogonally separable metrics for a *geodesic* Hamiltonian (ie., $V = 0$) on a manifold. If, however, $V \neq 0$, then a certain constraint must be satisfied by the potential of the Hamiltonian. This can be derived from the second term on the RHS of equation (4.42). Indeed, the vanishing of $\{F, H\}$ implies

$$g^{k\ell}\partial_k U - K^{k\ell}\partial_k V = 0 \Leftrightarrow \partial_j U = K_j^k \partial_k V,$$

which has the following integrability condition:

$$\partial_{[m}(K_{j]}^k \partial_k V) = 0. \tag{4.47}$$

This brings us to our next theorem.

Theorem 6 (Benenti, [6])**.** *The Hamilton-Jacobi equation of a natural Hamiltonian is orthogonally separable if and only if there exists a characteristic Killing tensor, K, which satisfies*

$$d(K\,dV) = 0. \tag{4.48}$$

Equation (4.48) is called the *compatibility condition*; a tensor which satisfies this condition is said to be *compatible* with the potential V of the Hamiltonian. Note that the component form of (4.48) is (4.47).

Suppose $V = 0$ in Theorem 6. Then by Eisenhart's theorem, orthogonal separability of the Hamilton-Jacobi equation is equivalent to the existence of $n-1$ characteristic Killing tensors with common eigenforms. If we let

$$K = a_1 K_1 + a_2 K_2 + \ldots + a_{n-1} K_{n-1} + a_n g,$$

where $a_i \in \mathbb{R}$, g is the metric, n is the dimension of the manifold, and the K_i are the $n - 1$ characteristic Killing tensors with the same eigenforms, then K represents a characteristic Killing tensor compatible with V. In light of this fact, we see that Eisenhart's theorem can be viewed as a particular case of Benenti's theorem.

Theorem 6 can be used as a starting point for determining the orthogonally separable coordinate systems for the Hamilton-Jacobi equation of a given natural Hamiltonian. In particular, begin by substituting the potential V of the Hamiltonian into (4.48), using the general Killing tensor of the manifold as K. This yields a set of constraints on the parameters of K. After applying these constraints to K, the resulting tensor is the most general Killing tensor compatible with V. Since orthogonal separable coordinate systems are represented by CKT(s), the next step is to determine the most general CKT compatible with V. To achieve this, we apply the results of Section 4.1.5. Once the most general compatible CKT is known, the next step is to determine what orthogonally separable coordinates the CKT(s) within this vector space represent. Naively, we may try to compare the CKTs from this vector space with the CKTs characterizing each orthogonally separable coordinate system for the manifold. This, however, becomes increasingly complicated in higher dimensions as more than one CKT characterizes a coordinate system. Furthermore, the compatible CKTs may not be identical to the CKTs derived using Eisenhart's method - they may differ by a transformation from the isometry group on the manifold.

The complications arising in the application of Theorem 6 was the impetus for the creation and development of the *invariant theory of Killing tensors.* The main idea of the theory, as pointed out in the introduction to this chapter, is to regard these issues as an equivalence problem and apply ideas from invariant theory to resolve them. In the next section we will discuss the requisite material from invariant theory before our exposition on the invariant theory of Killing tensors.

4.2 Invariant Theory

Invariant theory is the study of functions on a space which remain unchanged or *invariant* under a group of transformations acting on the space. The theory has a long history - originating over 150 years ago. Historically, the theory began as a means to study the invariant properties of polynomials under linear transformations. Today, the theory is broader and more complex, drawing on various areas of mathematics and finding many new and important applications in science.

To motivate and illustrate the ideas of this section, let us begin where invariant theory began - with polynomials.

4.2.1 Classical Invariant Theory: Motivational Examples

In the classical setting of invariant theory, we are given a finite group G of linear transformations acting on an n-dimensional vector space of polynomials $\mathcal{P}^n(\mathbb{R}^m)$. In this setting, invariants are functions of the coefficients of the polynomials which remain unchanged under the group G. As an illustration, consider the following simple example taken from Chapter 1 of [64].

Example 4.2.1. Consider the general quadratic polynomial in the variable x,

$$Q(x) = ax^2 + bx + c, \quad x, a, b, c \in \mathbb{R}. \tag{4.49}$$

If we make a change of variables $\tilde{x} = \alpha x + \beta$, then $Q(x)$ transforms into a new quadratic polynomial

$$\begin{aligned} \tilde{Q}(\tilde{x}) &= \tilde{a}\tilde{x}^2 + \tilde{b}\tilde{x} + \tilde{c} \\ &= \tilde{a}(\alpha x + \beta)^2 + \tilde{b}(\alpha x + \beta) + \tilde{c}. \end{aligned} \tag{4.50}$$

Upon comparing (4.49) and (4.50), we see that the coefficients of $Q(x)$ and $\tilde{Q}(\tilde{x})$ are related by the following *transformation equations*

$$\begin{aligned} a &= \tilde{a}\alpha^2, \\ b &= \alpha\tilde{b} + 2\tilde{a}\alpha\beta, \\ c &= \tilde{a}\beta^2 + \tilde{b}\beta + \tilde{c}. \end{aligned} \tag{4.51}$$

Interestingly, we can see from these relations (4.51) that the discriminant of the polynomial $Q(x)$ is related to the discriminant of the polynomial $\tilde{Q}(\tilde{x})$ according to

$$\Delta = b^2 - 4ac = \alpha^2(\tilde{b}^2 - 4\tilde{a}\tilde{c}) = \alpha^2\tilde{\Delta}.$$

Thus under this type of transformation, we see that the discriminant is preserved (up to a multiplicative factor).

The set of all quadratic polynomials in one real variable forms a 3-dimensional vector space, called $\mathcal{P}^2(\mathbb{R})$. Thus, we can view the change of variables in this example as a linear transformation $T : \mathcal{P}^2(\mathbb{R}) \to \mathcal{P}^2(\mathbb{R})$, defined by

$$T(x) = \alpha x + \beta.$$

Table 4.1: Equivalence classes of $\mathcal{P}(\mathbb{R})$

a	Δ	$Q(x)$	Nature of the roots
$\neq 0$	> 0	$x^2 - 1$	distinct real roots
$\neq 0$	0	x^2	equal roots
$\neq 0$	< 0	$x^2 + 1$	complex conjugate roots
0	$\neq 0$	x	single root
0	0	0	constant

The set of all transformations T of this type forms a transformation group, called $SE(1)$. These transformations represent the most general type of transformation which maps a quadratic polynomial in x into a quadratic polynomial in $\tilde{x}$. Geometrically, this type of transformation preserves the nature of the roots of $Q(x)$. For example, a polynomial with distinct roots maps to a new polynomial with distinct roots; a polynomial with equal roots maps to new polynomial with equal roots.

Suppose we define the following equivalence relation on the elements of $\mathcal{P}^2(\mathbb{R})$:
Two polynomials $Q_1(x), Q_2(x) \in \mathcal{P}^2(\mathbb{R})$ are equivalent if one can be mapped to the other under T.
Since T preserves the nature of the roots, the polynomials in each equivalence class must have the same type of roots. Thus, we can characterize each equivalence class by the value of the discriminant Δ and the coefficient a as shown in Table 4.1. The third column in this table contains the "simplest" polynomial in each equivalence class, and we call such a representative a *canonical form*.[9] In light of this table, one could say we have solved the equivalence and canonical forms problem for $\mathcal{P}^2(\mathbb{R})$ under the group of linear transformations. $\square$

In this simple example, we considered the vector space $\mathcal{P}^n(\mathbb{R})$ under the action of the linear transformation $T : \mathcal{P}^n(\mathbb{R}) \to \mathcal{P}^n(\mathbb{R})$ defined by $T(x) = \alpha x + \beta$ when $n = 2$. Since a polynomial is defined by its coefficients, we identified $\mathcal{P}^2(\mathbb{R})$ with the three-dimensional vector space $\mathbb{R}^3$. For arbitrary n, we can identify the vector space $\mathcal{P}^n(\mathbb{R})$ with an $(n+1)$-dimensional real vector space by realizing the coefficients $(a_0, \ldots, a_n)$

[9]The selection of a "simplest" representative depends on one's definition of simplicity, thus canonical forms are not unique.

as coordinates in $\mathbb{R}^{n+1}$. An invariant of this action is any function $\mathcal{I} : \mathcal{P}^n(\mathbb{R}) \to \mathbb{R}$ satisfying

$$\mathcal{I}(a_0, \dots, a_n) = \alpha^k \mathcal{I}(\tilde{a}_0, \dots, \tilde{a}_n)$$

where a_i denote the coefficients of a polynomial $P \in \mathcal{P}^n(\mathbb{R})$. Clearly, $\mathcal{I}_1 = a$, and $\mathcal{I}_2 = \Delta$ satisfy this condition when $n = 2$.

In the less simplistic equivalence problems of classical invariant theory, however, *covariants* are also needed to distinguish between the equivalence classes. As Olver points out [64], covariants are needed when "more subtle algebraic information is required than can be provided by the invariants." To illustrate his point, let us consider the following example.

Example 4.2.2. Consider the vector space of polynomials $\mathcal{P}^2(\mathbb{R}^2)$ under the action of a linear transformation $T : \mathcal{P}^2(\mathbb{R}^2) \to \mathcal{P}^2(\mathbb{R}^2)$ defined by $T(x, y) = (\alpha x + \beta y, \gamma x + \delta y)$. If $P = ax^2 + bxy + cy^2$ represents the general element of this space, then

$$\mathcal{I} = b^2 - 4ac$$

is an invariant of this action. According to this invariant, the polynomials

$$P_1 = x^2 + y^2, \quad P_2 = -x^2 - y^2$$

are equivalent; if we would like a finer distinguishing criteria, it is necessary to use a covariant. To this end, let us define a polynomial $P \in \mathcal{P}^2(\mathbb{R}^2)$ by its coefficients (a, b, c) *and* the coordinates $(x, y) \in \mathbb{R}^2$. Therefore, we are considering the action of T on the product vector space $X = \mathcal{P}^n(\mathbb{R}^2) \times \mathbb{R}^2$. A function $\mathcal{C} = \mathcal{C}(a, b, c, x, y)$ which satisfies

$$\mathcal{C}(a, b, c, x, y) = (\alpha\delta - \beta\gamma)^k \mathcal{C}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{x}, \tilde{y})$$

is called a *covariant* of the action. The polynomial P itself defines a covariant, and can be used to distinguish between P_1 and P_2.[10] □

These two examples embody the fundamental ideas of invariant theory. However, due to their simplicity, they do not illustrate some of the complications which can arise in solving equivalence problems. In particular, the determination of invariants

[10] For a complete solution to this problem, please refer to p. 9 of [64].

and covariants may not be as straightforward as these examples suggest. Moreover, the application of these invariants and covariants to distinguish between equivalence classes can be considerably more complicated. These examples may also prompt us to wonder how many invariants and covariants exist for a given problem. To answer these and other questions, let us return to the more general setting of invariant theory.

4.2.2 Invariants and Orbits

In the modern setting of invariant theory, we consider the following more general definition of an invariant.

Definition 4.2.3 ([63]). Suppose G is a group of transformations acting on a set X. A function $f : X \to \mathbb{R}$ satisfying

$$f(g \cdot x) = f(x),$$

for all $g \in G$ and $x \in X$ is called an *invariant*.

Given an action by a group G on a set X, we group elements of X into equivalence classes or *orbits* [63].

Definition 4.2.4. Suppose G is a group acting on a set X. An *orbit* $\mathcal{O}$ of this action is a set

$$\mathcal{O} = \{x \in X \mid g \cdot x \in X \text{ for } \text{ all } g \in G\},$$

which is both minimal and nonempty.

For an $x \in X$, we define an orbit through x as follows.

Definition 4.2.5. Consider a group G acting on a set X. An *orbit through a point* $x \in X$, denoted by $\mathcal{O}_x$, is a subset of X which are connected to x by the group:

$$\mathcal{O}_x = \{g \cdot x \mid g \in G\}.$$

Elements of X which belong to the same equivalence class or *orbit* are said to be *equivalent*.

The structure of the orbits of a group action is determined by how "well" G acts on X. In particular, if G acts *semi-regularly* on X, then the orbits all have the same dimension. An even simpler orbit structure arises when G acts *regularly* on X, which implies that the orbits all have the same dimension *and* for any $x \in X$ there exists an open neighborhood around x whose intersection with any orbit of X forms a connected subset of the orbit. If the set X is a manifold, and G a Lie group of transformations acting on X, then an orbit of X forms a submanifold of X.

The simplest member of an orbit is called a *canonical form*. For equivalence problems, a canonical form from each orbit is chosen to represent its orbit. For a regular Lie group action, this choice is governed by a *cross-section* [64].

Definition 4.2.6. Suppose G is a Lie group acting regularly on an m-dimensional manifold $\mathcal{M}$, forming s-dimensional orbits. Then an $(m-s)$-dimensional submanifold $\mathcal{N}$ of $\mathcal{M}$ which intersects each orbit $\mathcal{O}$ only once and satisfies

$$T_p\mathcal{N} \cap T_p\mathcal{O} = \emptyset$$

for any $p \in \mathcal{N} \cap \mathcal{O}$ is a *cross-section* of $\mathcal{M}$.

The unique point of intersection of a cross-section with each orbit defines a canonical form for each orbit.

The main task of an equivalence problem is to find a set of functionally independent invariants so that we may distinguish between the orbits. In most cases, this is very difficult to do. Over the years, techniques for computing invariants of a group action have been developed, and can be applied successfully in certain cases. Moreover, recent advances made in computer algebra systems has made many of these once computationally infeasible calculations now possible.

Before attempting to compute the invariants, a natural starting point is to determine the number of functionally independent invariants of a given group action. If the group acts regularly on X, we can calculate this number with the following theorem (Theorem 8.17, Olver [64]).

Theorem 7. *Let G be a Lie group acting regularly on an m-dimensional manifold $\mathcal{M}$ with s-dimensional orbits. Then, in a neighborhood U of each point $p \in U \subset \mathcal{M}$, there exist $m-s$ functionally independent invariants $\mathcal{I}_1(x), \ldots, \mathcal{I}_{m-s}(x)$. Any other invariant $\mathcal{I}$ defined near p can be locally uniquely expressed as an analytic function of* *the fundamental invariants:* $\mathcal{I} = h(\mathcal{I}_1, \dots \mathcal{I}_{m-s})$. *The fundamental invariants serve to distinguish between the orbits near* p. *In particular, two points* $p, \tilde{p} \in U$ *will lie in the same orbit if and only if all the fundamental invariants agree:*

$$\mathcal{I}_1(x) = \mathcal{I}_1(\tilde{x}), \dots, \mathcal{I}_{m-s}(x) = \mathcal{I}_{m-s}(\tilde{x}).$$

Under the conditions stipulated in Theorem 7, the fundamental invariants can be used to determine if $x, \tilde{x} \in X$ are equivalent. Moreover, we can determine the number of fundamental invariants if we know the dimension of the orbits. If the group G acts *freely*[11] on X, the dimension of the orbits agrees with the dimension, d, of the group G. Thus in view of the above theorem, we have

$$\text{number of fundamental invariants} = m - d.$$

Orbits may have different dimensions. In those cases, it may be possible to restrict the group action to an open subset of the vector space where the group *does* act regularly.

Example 4.2.7. Consider the group $G = SE(1)$ acting on the vector space $V = \mathcal{P}^2(\mathbb{R})$ as in Example 4.2.1. According to Table 4.1, we have four different orbits:

$$\begin{aligned}
\mathcal{O}_1 &= \{Q \in \mathcal{P}(\mathbb{R}) \mid b^2 - 4ac > 0\} \\
\mathcal{O}_2 &= \{Q \in \mathcal{P}(\mathbb{R}) \mid b^2 - 4ac = 0\} \\
\mathcal{O}_3 &= \{Q \in \mathcal{P}(\mathbb{R}) \mid b^2 - 4ac < 0\} \\
\mathcal{O}_4 &= \{Q \in \mathcal{P}^2(\mathbb{R}) \mid a = b = c = 0\};
\end{aligned}$$

namely, the 2-dimensional level sets of the discriminant, and the origin, which has dimension 0. If we restrict the space to $\mathcal{P}^2(\mathbb{R})\backslash\{0\}$, then the group action is regular. By Theorem 7, there exists $3 - 2 = 1$ fundamental invariant. We conclude that any invariant of this group action can be generated from the fundamental invariant Δ. $\square$

Once the number of fundamental invariants is known, the next step is to determine a complete set. The *method of infinitesimal generators* and the *method of moving*

[11] Consider a transformation group G acting on X. The *isotropy subgroup* of an $x \in X$ is the set of all $g \in G$ which leaves x unchanged. If all of the isotropy subgroups of X contain only the trivial transformations, G is said to act *freely* on X.

frames are two methods which are usually employed to compute invariants. The first of these methods is useful because, in theory, it yields a complete set of invariants for a group action. These are found by first obtaining a set of *infinitesimal generators* through differentiation of the group action, and then using these generators to define a system of linear PDEs. The solution to this system of PDEs is a complete set of invariants. Moreover, these PDEs can be used to verify whether or not a function is an invariant of a group action, and furthermore, provides a simple way of determining the dimension of the orbits. A disadvantage of this method is having to solve a system of linear PDEs. However, since the method first requires differentiation and then integration, it suggests a completely algebraic approach can be found to determine invariants.

The method of moving frames is an alternative method for obtaining a complete set of invariants of a group action. In the classical version [12], this is done through differentiation of a suitably chosen *moving coframe* of one-form fields on the manifold. In the modern formulation,[12] the set of frames is identified with a group of transformations acting on the manifold, and thus the "moving frame" is more generally an equivariant map from the group to the manifold. A complete set of invariants are obtained through the *method of normalization* [23, 24], which is a completely algebraic method. However, much like the others, this method possesses its own set of computational challenges. Let us consider each of these methods now in turn.

4.2.3 The Method of Infinitesimal Generators

In Chapter 2 we discussed Lie groups and Lie algebras separately, citing no connection. However, associated with each Lie group G is a Lie algebra $\mathfrak{g}$. In particular, since every Lie group G is also a manifold, we can consider its tangent space at the identity of G, namely T_eG. Under the Lie bracket, this vector space has the structure of a Lie algebra. Another equivalent definition of a Lie algebra is as follows. For a Lie group G, the mapping $L_a : G \to G$, where

$$L_h g = hg$$

[12] See [64] and the relevant references therein.

for $h, g \in G$ is called a *left-translation*. This induces the map $L_* : T_gG \to T_{hg}G$ on vectors in the tangent spaces on G. A vector $X \in T_gG$ satisfying

$$L_{h*}X|_g = X|_{hg}$$

is said to be *left-invariant*. The set of left-invariant vectors on a Lie group G is isomorphic to T_eG. Therefore, we may equivalently define the Lie algebra $\mathfrak{g}$ for a Lie group G as the set of all left-invariant vector fields on G.

For an n-dimensional Lie group G, suppose $\{X_1, \ldots, X_n\}$ is a basis for its Lie algebra $\mathfrak{g}$. Then, by the closure of $\mathfrak{g}$ under the Lie bracket, we have

$$[X_j, X_k] = c^i{}_{jk} X_i \tag{4.52}$$

for basis vectors X_i. The coefficients $c^i{}_{jk}$ are called *structure constants*, and equations (4.52) are called the *commutation relations* for this basis of $\mathfrak{g}$. Since $\{X_1, \ldots, X_n\}$ forms a basis for each $T_p\mathcal{M}$, it yields a frame on G.

The link between a Lie algebra and its Lie group is the exponential map, $\exp : \mathfrak{g} \to G$, which maps vectors from T_eG to the Lie group G. In particular, given a vector field $X \in \mathfrak{g}$, the flow generated by X at the identity e of G is given by

$$\sigma_t = \exp(tX)e = \exp(tX),$$

which defines a one-parameter subgroup of G. When $t = 1$,

$$\sigma_1 = \exp(1X) = \exp(X)$$

we have a map from T_eG to G.

The vector fields in the Lie algebra of G can be used to represent the action of G on a manifold. In particular, if G defines a Lie group action on a manifold $\mathcal{M}$, its Lie algebra $\mathfrak{g}$ acts infinitesimally on $\mathcal{M}$ in the following way. For $g \in G$ and $x \in \mathcal{M}$, let $g \cdot x = \Psi(g, x)$ indicate the local action of G on $\mathcal{M}$. If $X \in \mathfrak{g}$, then

$$\psi(X)|_x = \frac{d}{d\epsilon} \left(\Psi(\exp(\epsilon X), x)\right)\Big|_{\epsilon=0} \tag{4.53}$$

forms a Lie algebra of vector fields on $\mathcal{M}$. Since the vectors $\psi(X)$ represent an infinitesimal action by G on $\mathcal{M}$, they are often called *infinitesimal generators*.

Once the infinitesimal generators of a group action are known, they can be used to determine invariants on $\mathcal{M}$.

Proposition 4.2.8 (Theorem 9.28, [64])**.** *Suppose G is a connected Lie group acting on a manifold $\mathcal{M}$. A function $f : \mathcal{M} \to \mathbb{R}$ is an invariant if and only if f satisfies*

$$X(f) = X^i \partial_i f = 0, \tag{4.54}$$

for all the infinitesimal generators $X \in \mathfrak{g}$ on $\mathcal{M}$.

The purpose of the *method of infinitesimal generators* is to determine a set of fundamental invariants for a group action. As part of this process, it is useful to know how many fundamental invariants are possible.

Recall that if G acts regularly on $\mathcal{M}$, then by Theorem 7 the number of fundamental invariants is the difference between the dimension of the manifold and the dimension of the orbits. To determine the dimension of the orbits, we have the following theorem (Proposition 9.26, [64]).

Theorem 8. *Suppose G is a Lie group acting on a manifold $\mathcal{M}$ and let $x \in \mathcal{M}$. Furthermore, suppose $V|_x$ is a vector space spanned by the infinitesimal generators at x, $T\mathcal{O}_x|_x$ is the tangent space to the orbit $\mathcal{O}_x$ of G at x, and $G_x \subset G$ is the isotropy subgroup of x. Then,*

(i) $V|_x = T\mathcal{O}_x|_x$,

(ii) $\dim(G_x) = \dim(G)$ - $\dim\,(\mathcal{O}_x) = d - s$.

Since $V|_x = T\mathcal{O}_x|_x$, their dimensions must also be equal. Therefore, we can determine the dimension of the orbit space by calculating the rank of the coefficient matrix from the system of PDEs (4.54).

Let us consider the following example to illustrate the theory.

Example 4.2.9. Consider the special Lorentz group, $SO(2,1)$, acting on the vector space $\mathcal{K}^2(\mathbb{H}^2)$. Let

$$C^{ij} = \begin{pmatrix} c_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & c_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & c_3 \end{pmatrix}$$

represent the coefficient matrix of the general Killing tensor. Since a Killing tensor $K \in \mathcal{K}^2(\mathbb{H}^2)$ is uniquely determined by its parameters, we use $c_1, c_2, c_3, \gamma_1, \gamma_2, \gamma_3$ as coordinates, therefore $\mathcal{K}^2(\mathbb{H}^2)$ is identified with $\mathbb{R}^6$. To determine a basis of infinitesimal generators for this action, first note that

$$X_1 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, \quad X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

is a basis for the infinitesimal generators of $SO(2,1)$ acting on $\mathbb{R}^3$. The corresponding infinitesimal generators of the parameter space can be found using (4.53). Alternatively, we could Lie differentiate the general Killing tensor K of $\mathcal{K}^2(\mathbb{H}^2)$ with respect to each generator X_i, yielding a new 'deformed' Killing tensor with new parameters in each case. Since these new parameters represent the change in the parameters under X_i, we can use them to define the corresponding infinitesimal generator in the parameter space of K. Here, the corresponding infinitesimal generators in the parameter space are given by

$$\begin{aligned}
\tilde{X}_1 = \pi(\mathcal{L}_{X_1}(K)) &= -2\gamma_1\frac{\partial}{\partial c_2} - 2\gamma_1\frac{\partial}{\partial c_3} - (c_2+c_3)\frac{\partial}{\partial \gamma_1} - \gamma_3\frac{\partial}{\partial \gamma_2} - \gamma_2\frac{\partial}{\partial \gamma_3}, \\
\tilde{X}_2 = \pi(\mathcal{L}_{X_2}(K)) &= 2\gamma_2\frac{\partial}{\partial c_1} + 2\gamma_2\frac{\partial}{\partial c_3} + \gamma_3\frac{\partial}{\partial \gamma_1} + (c_1+c_3)\frac{\partial}{\partial \gamma_2} + \gamma_1\frac{\partial}{\partial \gamma_3}, \\
\tilde{X}_3 = \pi(\mathcal{L}_{X_3}(K)) &= -2\gamma_3\frac{\partial}{\partial c_1} + 2\gamma_3\frac{\partial}{\partial c_2} + \gamma_2\frac{\partial}{\partial \gamma_1} - \gamma_1\frac{\partial}{\partial \gamma_2} + (c_1-c_2)\frac{\partial}{\partial \gamma_3},
\end{aligned}$$

where K represents the general Killing tensor of $\mathcal{K}^2(\mathbb{H}^2)$, and $\pi : \mathcal{K} \to \mathbb{E}^6$ is the projection defined by

$$\pi(K(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)) = \tilde{c}_1\frac{\partial}{\partial c_1} + \tilde{c}_2\frac{\partial}{\partial c_2} + \tilde{c}_3\frac{\partial}{\partial c_3} + \tilde{\gamma}_1\frac{\partial}{\partial \gamma_1} + \tilde{\gamma}_2\frac{\partial}{\partial \gamma_2} + \tilde{\gamma}_3\frac{\partial}{\partial \gamma_3}.$$

Since the $\tilde{X}_i$ satisfy the commutation relations

$$[\tilde{X}_3, \tilde{X}_2] = \tilde{X}_1, \quad [\tilde{X}_1, \tilde{X}_3] = \tilde{X}_2, \quad [\tilde{X}_1, \tilde{X}_2] = \tilde{X}_3,$$

we have verified that the Lie algebra spanned by $\{X_1, X_2, X_3\}$ is isomorphic to the Lie algebra spanned by $\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3\}$. The system of PDEs given by $\tilde{X}_i(F) = 0$ cannot generally be solved by the method of characteristics [37], so instead we employ the more computational method of undetermined coefficients. Assuming solutions which are homogeneous polynomials in the coefficient parameters, we obtain the following three independent functions:

$$\begin{aligned}\mathcal{I}_1 &= c_1 + c_2 - c_3, \\ \mathcal{I}_2 &= c_1^2 + c_2^2 + c_3^2 - 2\gamma_1^2 - 2\gamma_2^2 + 2\gamma_3^2, \\ \mathcal{I}_3 &= c_1c_2c_3 - c_1\gamma_1^2 - c_2\gamma_2^2 - c_3\gamma_3^2 + 2\gamma_1\gamma_2\gamma_3.\end{aligned}$$

The matrix

$$\begin{pmatrix} 0 & -2\gamma_1 & -2\gamma_1 & -c_2 - c_3 & -\gamma_3 & -\gamma_2 \\ 2\gamma_2 & 0 & 2\gamma_2 & \gamma_3 & c_1 + c_3 & \gamma_1 \\ -2\gamma_3 & 2\gamma_3 & 0 & \gamma_2 & -\gamma_1 & c_1 - c_2 \end{pmatrix}$$

has rank 3 almost everywhere, hence the orbits have a maximal dimension of 3. In view of Theorem 7, we expect $6-3=3$ fundamental invariants, therefore $\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ can be used to define the entire space of invariants. □

4.2.4 The Method of Moving Frames

In his famous Erlangen program of 1872 [48], Felix Klein revolutionized geometry with the formulation of a new philosophy of geometry which generalized all previous geometries. In its modern form, a *Klein geometry* consists of a manifold $\mathcal{M}$ together with a Lie group G acting transitively on $\mathcal{M}$. Since the action of G is transitive, there is only one orbit, and thus $\mathcal{M}$ is called a *homogeneous* space. If H is a closed Lie subgroup of G, we can identify $\mathcal{M}$ with the coset space G/H. Thus by Example 2.8.5, G is a principal bundle with $\pi : G \to G/H \simeq \mathcal{M}$ and fibre H.

In a Klein geometry, a submanifold $\mathcal{N}$ of $\mathcal{M}$ is defined by its invariant properties under G. Two submanifolds are considered *equivalent* if there exists a $g \in G$ which maps one into the other. Using this new philosophy of geometry and the tools of Riemannian geometry, Cartan brought geometry to an even more general level. Motivated to solve equivalence problems of submanifolds of a homogeneous space under a transformation group, Cartan developed the theory of moving frames studied by Darboux and Cotton [3]. In modern terms [28], the theory is concerned with the bundle of frames on a manifold. If G is a Lie group acting on a homogeneous space $\mathcal{M}$, then quite often G is isomorphic to the bundle of frames Σ on $\mathcal{M}$. If we consider a submanifold $\mathcal{N}$ of $\mathcal{M}$, defined by the smooth and injective map

$$f : \mathcal{N} \to \mathcal{M},$$

then a cross section $s : \mathcal{N} \to G \simeq \Sigma$ defines a set of frames on $\mathcal{N}$ called a *moving frame*. Such a mapping is also called a *lift*, and allows the bundle diagram

$$\begin{tikzcd} & G \arrow[d, "\pi"] \\ \mathcal{N} \arrow[ur, "s"] \arrow[r, "f"] & \mathcal{M} \simeq G/H \end{tikzcd} \tag{4.55}$$

to commute.

To determine the equivalence of lifts into G, Cartan made use of an important one-form on G, called the *Maurer-Cartan form*. Recall that for a Lie group G, its Lie algebra $\mathfrak{g}$ is the set of all vector fields at the identity e of G, or equivalently, the set of all left-invariant vector fields on G. Therefore, the *dual Lie algebra* $\mathfrak{g}^*$ is the set of all one-forms at the identify e of G, or equivalently the set of all left-invariant one-forms on G. A one-form $\omega \in \mathfrak{g}^*$ is called a *Maurer-Cartan form*. A basis for $\mathfrak{g}^*$ yields a basis at each $T_p^*(\mathcal{M})$, and thus forms a *Maurer-Cartan coframe* on G.

The analogue of the commutation relations (4.52) for $\omega \in \mathfrak{g}^*$ are given by

$$\mathrm{d}\omega^i = -\frac{1}{2}c^i{}_{jk}\omega^j \wedge \omega^k, \tag{4.56}$$

called the *Maurer-Cartan structure equations*. This can be shown using the following identity of the exterior derivative satisfied for vectors X, Y and one-form ω on a manifold:

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]). \tag{4.57}$$

If $X \in \mathfrak{g}$ and $\omega \in \mathfrak{g}^*$, then $X(\omega(Y)) = Y(\omega(X)) = 0$ and thus the identity (4.57) becomes

$$d\omega(X,Y) = -\omega([X,Y]). \tag{4.58}$$

If $\{\omega^i, \dots, \omega^j\}$ is the dual basis for the Lie algebra $\mathfrak{g}^*$, then (4.58) becomes

$$\begin{aligned} \mathrm{d}\omega^i(X_j, X_k) &= -\omega^i(c^\ell{}_{jk}X_\ell) \\ &= -c^i{}_{jk} \end{aligned}$$

which shows (4.56).

The equivalence of two lifts s and $\tilde{s}$ into G can be determined using the Maurer-Cartan form on G.

Theorem 4.2.10 (Lemma 1, [28]). *Consider lifts $s : \mathcal{N} \to G$ and $\tilde{s} : \mathcal{N} \to G$ for a Lie group G and connected manifold $\mathcal{N}$. Then s is equivalent to $\tilde{s}$ under G if and only if the Maurer-Cartan form ω of G satisfies*

$$s^*\omega = \tilde{s}^*\omega. \tag{4.59}$$

Therefore two lifts are equivalent if they pullback the Maurer-Cartan form to the same form on G. A solution to the equivalence problem of submanifolds therefore requires finding a "suitable" lift or moving frame over $\mathcal{M}$ so that when we restrict the Maurer-Cartan forms to this lift we obtain invariants which can be used to distinguish between the orbits. The selection of a "suitable" lift presents a significant challenge to this method. Typically, if the geometry of the problem is well understood, then a lift which naturally adapts to the geometry is chosen. To clarify the general method as well as these finer points, consider the next example taken from pp. 787 - 790 in [28].

Example 4.2.11. Consider submanifolds $\mathcal{N}_1, \mathcal{N}_2 \subset \mathbb{E}^3$. A classic problem in geometry is to determine whether these two submanifolds are equivalent up to the action of the Euclidean group $E(3)$. To determine an equivalence criterion, let us apply the theory of this section. For a submanifold $\mathcal{N} \subset \mathbb{E}^3$, define a "natural" frame on $\mathcal{N}$ by

$$\Sigma_x = (x, E_1, E_2, E_3),$$

where $x \in \mathcal{N}$ is a position vector, $(E_1, E_2) \in T_x(\mathcal{N})$ are orthonormal vectors, and E_3 is a normal vector to $\mathcal{N}$ with unit length. Note that the set of all frames Σ on $\mathcal{N}$ is isomorphic to $E(3)$. In the language of fibre bundles, $E(3)$ is a principal bundle with $\pi : E(3) \to E(3)/O(3) \simeq \mathbb{E}^3$ and fibre $O(3)$. If $f : \mathcal{N} \to \mathbb{E}^3$, then the frame $\Sigma_x : \mathcal{N} \to E(3) \simeq \Sigma$ represents a section of the bundle Σ, and allows the bundle diagram

$$\begin{tikzcd} & E(3) \arrow[d, "\pi"] \\ \mathcal{N} \arrow[ur, "\Sigma_x"] \arrow[r, "f"] & \mathcal{N} \simeq E(3)/O(3) \end{tikzcd} \tag{4.60}$$

to commute. Since (E_1, E_2, E_3) is a frame, we can express the differentials dx and dE_i in terms of this frame as follows

$$dx = \omega^i E_i, \quad dE_i = \omega^i_j E_i, \tag{4.61}$$

where ω^i and ω^i_j are one-forms. Let us simplify these formulas somewhat. First, note that since dx is tangent to $\mathcal{N}$ we must have $\omega^3 = 0$. For the second set of equations, we have

$$dE_i \cdot E_j = \omega^k_i E_k \cdot E_j = \omega^k_i \delta_{kj} = \omega^j_i.$$

Then if we take the exterior derivative of the orthonormality relation $E_i \cdot E_j = \delta_{ij}$, we find $\omega^i_j = -\omega^j_i$, and thus $\omega^i_i = 0$. Therefore the infinitesimal change in the frame is given by

$$\begin{aligned}
dx &= \omega^1 E_1 + \omega^2 E_2 \\
dE_1 &= \omega^2_1 E_2 + \omega^3_1 E_3 \\
dE_2 &= -\omega^2_1 E_1 + \omega^3_2 E_3 \\
dE_3 &= -\omega^3_1 E_1 - \omega^3_2 E_2,
\end{aligned}$$

called the *Gauss-Weingarten equations*. If we take the exterior derivative of equations (4.61), we get

$$0 = d(dx) = d(\omega^i E_i) = d\omega^i E_i - \omega^i \wedge dE_i = d\omega^i E_i - \omega^i \wedge (\omega^j_i E_j) = (d\omega^j - \omega^i \wedge \omega^j_i) E_j,$$

$$0 = d(dE_i) = d(\omega^i_j E_i) = d\omega^j_i E_j - \omega^j_i \wedge dE_j = d\omega^j_i E_j - \omega^j_i \wedge (\omega^k_j E_k) = (d\omega^k_i - \omega^j_i \wedge \omega^k_j) E_k,$$

which implies

$$d\omega^j = \omega^i \wedge \omega^j_i, \quad d\omega^k_i = \omega^j_i \wedge \omega^k_j,$$

giving us the Maurer-Cartan structure equations for this lift. The components of these equations are given by

$$d\omega^1 = \omega^2 \wedge \omega^1_2, \quad d\omega^2 = \omega^1 \wedge \omega^2_1, \quad 0 = \omega^1 \wedge \omega^3_1 + \omega^2 \wedge \omega^3_2, \tag{4.62}$$

$$d\omega^2_1 = \omega^3_1 \wedge \omega^2_3, \quad \omega^3_1 = \omega^2_1 \wedge \omega^3_2, \quad d\omega^3_2 = \omega^1_2 \wedge \omega^3_1, \tag{4.63}$$

where (4.63) are called the *Gauss-Codazzi equations* for $\mathcal{N}$. Since $\{\omega^1, \omega^2\}$ are linearly independent one-forms on $\mathcal{N}$, and $\{\omega^3_1, \omega^3_2\}$ satisfy the last equation in (4.62), then by Cartan's lemma[13] there exist scalars $b^i_j = b^j_i$ such that

$$\omega^3_i = b^i_j \omega^j.$$

[13] Cartan's lemma: Suppose $\omega^1, \ldots, \omega^n$ are linearly independent one-forms on an n-dimensional manifold $\mathcal{M}$. If there exist one-forms $\alpha^1, \ldots, \alpha^n$ on $\mathcal{M}$ satisfying

$$\omega^1 \wedge \alpha^1 + \cdots + \omega^n \wedge \alpha^n = 0,$$

then there exist scalars $b^i_j = b^j_i$ such that $\alpha^i = b^i_j \omega^j$.

Using these expressions, we define the quadratic forms

$$I = dx \cdot dx = (\omega^1)^2 + (\omega^2)^2, \quad II = -dx \cdot dE_3 = b^1_1(\omega^1)^2 + 2b^1_2\omega^1\omega^2 + b^2_2(\omega^2)^2,$$

called the *first* and *second fundamental forms*, which define invariants under the action of $E(3)$. A submanifold $\mathcal{N} \subset \mathbb{E}^3$ is uniquely determined by these two invariants. □

As Example 4.2.11 demonstrates, the classical theory of moving frames has many important applications for geometry.[14] More recently,[15] several important developments have been made in the theory which significantly expands this range of applications. The main idea in these developments, as Olver points out (p. 1, [65]), has been to "decouple the moving frames theory from reliance on any form of frame bundle." In this more recent theory, a moving frame is defined as follows.

Definition 4.2.12. Suppose G is a Lie group acting smoothly on a manifold $\mathcal{M}$. A map $\rho : \mathcal{M} \to G$ which is both smooth and G-equivariant[16] defines a *moving frame* on $\mathcal{M}$.

The existence of a moving frame for a group action depends on how well G acts on $\mathcal{M}$ as the following theorem (Theorem 2, [65]) asserts.

Theorem 4.2.13. *Suppose G is a Lie group acting smoothly on a manifold $\mathcal{M}$. A moving frame exists in a local neighborhood of $p \in \mathcal{M}$ if and only if the action of G is both free and regular on $\mathcal{M}$ in a neighborhood of $p \in \mathcal{M}$.*

If a given group action is not free, there are methods which can be applied to change this. For example, the *method of prolongation* is based on the idea that by expanding the manifold upon which the group acts, the action will eventually become locally free [10]. In particular, if we expand the action to manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_m$, this induces an action by G on the product space $\mathcal{M}_1 \times \ldots \times \mathcal{M}_m$, defined by

$$g \cdot (x^1, \ldots, x^m) = (g \cdot x^1, \ldots, g \cdot x^m),$$

[14]For more applications, please refer to [28].

[15]See, for example, [65] and the relevant references therein.

[16]Consider the action of G on sets X and Y. A map $f : X \to Y$ satisfying

$$f(g \cdot x) = g \cdot f(x)$$

for any $x \in X$ and $g \in G$ is called *G-equivariant.*

where $x_i \in \mathcal{M}_i$ and $g \in G$. Invariants, $\mathcal{J} : \mathcal{M}_1 \times \ldots \times \mathcal{M}_m$, of this group action necessarily satisfy

$$J(g \cdot x^1, \ldots, g \cdot x^m) = \mathcal{J}(x^1, \ldots, x^m),$$

for all $x_i \in \mathcal{M}_i$ and $g \in G$, and are called *joint invariants*.

The method of moving frames can be used to determine invariants (or joint invariants) of a group action. This, however, relies on creating a "suitable" moving frame for the group action. To achieve this, the method of *normalization* can be applied [64]. Let us outline this method with the following algorithm. Suppose G is an r-dimensional Lie group acting (locally) freely and regularly on an m-dimensional manifold $\mathcal{M}$, creating r-dimensional orbits.

(i) *Coordinate cross section.* We can find a local cross-section $K \subset \mathcal{M}$ which intersects $x \in \mathcal{M}$. If $x = (x^1, \ldots, x^r) \in \mathcal{M}$, then let

$$K = \{x^1 = c_1, \ldots, x^r = c_r\}$$

denote the coordinates of the cross-section.

(ii) *Normalization equations.* Suppose $g = (g_1, \ldots, g_r)$ in a neighborhood of $e \in G$, and let $\tilde{x} = g \cdot x = f(g, x)$ denote the group transformation equations. Then

$$\tilde{x}^1 = f_1(g, x) = c_1, \ldots, \tilde{x}^r = f_r(g, x) = c_r$$

are the normalization equations for this cross-section.

(iii) *Moving frame.* Solve the first normalization equation for one of the group parameters, g_i, yielding a solution of the form $g_i = h_i(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_r, x)$. Substitute this solution into the second normalization equation and solve for a new group parameter. Repeat this procedure until a solution $g = f(x)$ is found. If a group parameter cannot be solved for at one of these steps, then replace that component of the cross-section. The final solution defines a moving frame $f : \mathcal{M} \to G$ for this cross-section of the group action.

(iv) *Invariants.* Consider the remaining components $\{f_{r+1}(g, x), \ldots, f_m(g, x)\}$ of the group transformation. Substitution of the moving frame $g = f(x)$ from the previous step into these equations

$$\mathcal{I}_1(x) = f_{r+1}(f(x), x), \ldots, \mathcal{I}_{m-r}(x) = f_m(g(x), x),$$

yields a complete set of fundamental invariants.

To illustrate the normalization method, let us consider the following example.

Example 4.2.14. Consider the action of $SE(1)$ on the vector space $\mathcal{P}^2(\mathbb{R})\backslash\{0\}$ as in Example 4.2.7. Recalling the transformation equations (4.51), let us define a cross-section

$$\{a = 1,\ b = 0\} \tag{4.64}$$

through a point $(\tilde{a}, \tilde{b}, \tilde{c}) \in \mathcal{P}^2(\mathbb{R})$. Solving the normalization equations (4.64) for the group parameters, we find

$$\alpha = \frac{1}{\tilde{a}},\ \beta = -\frac{\tilde{b}}{2\tilde{a}}.$$

Substituting these expressions into c,

$$\mathcal{I} = \tilde{c} - \frac{\tilde{b}^2}{4\tilde{a}}$$

gives us an invariant of the group action. □

The calculations required to determine the invariant in Example 4.2.14 were relatively simple, but this is generally not the case. In fact, a significant drawback of the normalization method is solving the normalization equations for the group parameters. This is a problem when applying the theory to other classical problems. As Olver notes, “unfortunately, most of the standard actions on binary forms are too algebraically complicated for the normalization method to be an effective tool” ([64], p. 165).

Interested in reducing these computational challenges, Kogan [49] developed a technique for determining a moving frame of a group action by first determining the moving frame for a subgroup. As we will discuss in the next section, this method can be used to alleviate some of the computational challenges when applying the method of moving frames to equivalence problems of Killing tensors.

4.3 Invariant Theory of Killing Tensors

The invariant theory of Killing tensors (hereafter ITKT) is a theory in which we apply the ideas of invariant theory to solve equivalence problems of Killing tensors defined on a pseudo-Riemannian manifold $\mathcal{M}$ with metric g. The original motivation for the development of the theory was to solve equivalence problems of CKTs arising in the theory of orthogonal separation of the Hamilton-Jacobi equation. Since then, further applications of the theory have been made. For example, in [1] the authors used joint invariants and the geometric properties of orthogonal coordinate webs to completely characterize a superintegrable potential in $\mathbb{E}^2$. Another application of the theory was made in [13], where the authors in studying the R-separability of the Laplace-Beltrami equations in $\mathbb{E}^3$ formulated and solved an equivalence problem of conformal CKTs under the group of conformal transformations.

These examples represent a natural generalization of ITKT in its two parts, namely invariant theory and the theory of Killing tensors. On the invariant theoretic side, the original study of invariants of Killing tensors has widened to include covariants and joint invariants of Killing tensors. Moreover, the methods for determining these various types of invariants has expanded to include the method of infinitesimal generators, the method of moving frames, and, more recently, a tensorial approach. On the Killing tensor theoretic side, the original study of valence two Killing tensors has expanded to conformal Killing tensors and generalized Killing tensors. The potential applications for these various generalizations suggest an interesting and fruitful future for ITKT.

4.3.1 Hamilton-Jacobi Theory and Invariant Theory of Killing Tensors

ITKT arises quite naturally in the theory of orthogonal separation of variables of the Hamilton-Jacobi equation. Since coordinate systems are considered equivalent up to an isometry on the manifold, one can view these coordinate systems as belonging to equivalence classes, each represented by a canonical coordinate system. And since each coordinate system can be characterized by a CKT, one can view these objects as canonical representatives on an entire orbit of CKTs representing a coordinate system. The CKTs arising from the application of Theorem 6 are not necessarily canonical, as pointed out in Section 4.1.6. Therefore, it is essential in the application of this theorem that we be able to identify which orbit a CKT belongs to and its map back to canonical form so that we may identify and properly define these coordinate systems (see Figure 4.1).

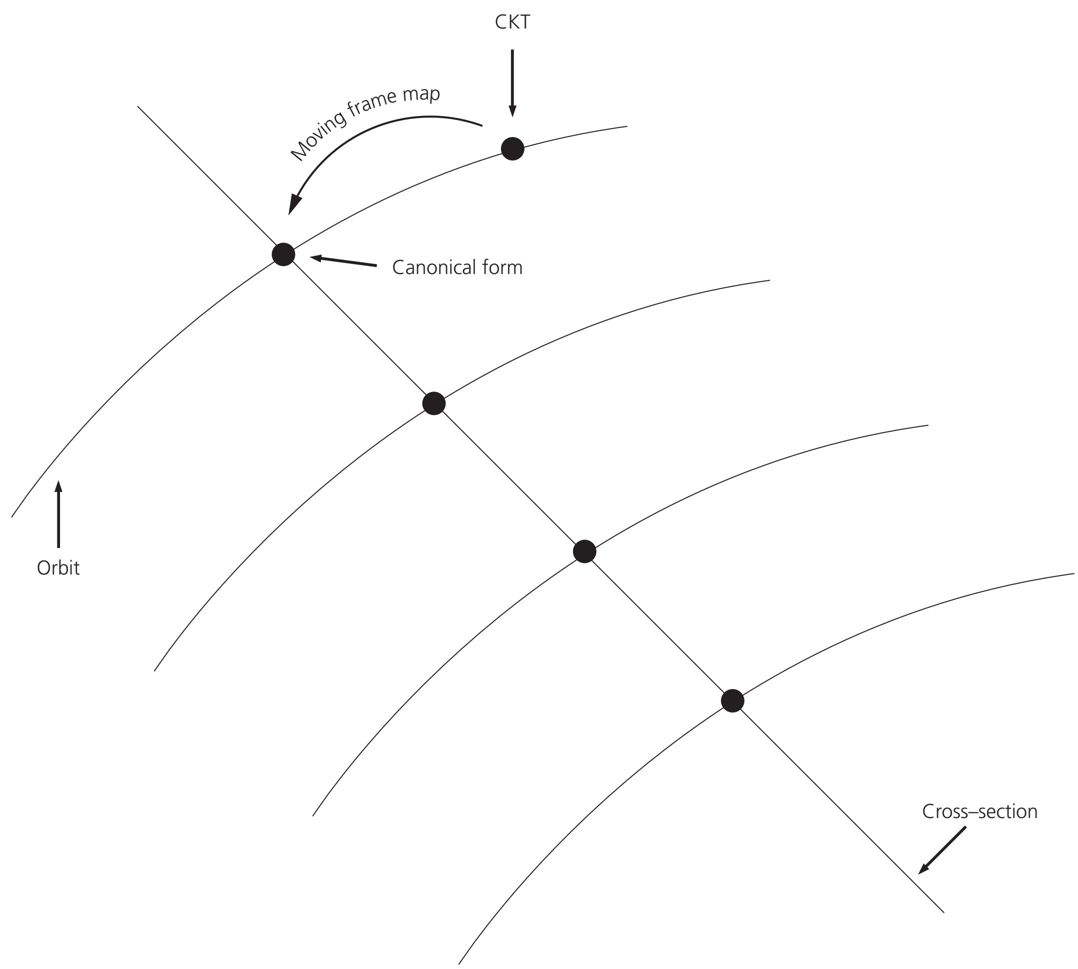
The equivalence problem of Killing tensors on a manifold is naturally in line with the equivalence problems studied by Cartan in Section 4.2.4. This was first observed in [1], and developed further in [41] and [39]. In particular, suppose $\mathcal{M}$ is a space of constant curvature, and $\mathcal{I}(\mathcal{M})$ is the Lie group of isometries of $\mathcal{M}$. Since $\mathcal{M}$ is a space of constant curvature, it is isomorphic to the quotient space G/H, where G is the isometry group of $\mathcal{M}$ and H is a closed subgroup of G. In the language of fibre bundles, G is the principal bundle with $\pi_1 : G \to G/H \simeq \mathcal{M}$ and fibre H. Consider the action of $\mathcal{I}(\mathcal{M})$ on the vector space $\mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$. First note that $\mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$ is a vector bundle with $\pi_2 : \mathcal{K}^p(\mathcal{M}) \times \mathcal{M} \to \mathcal{M} \simeq G/H$. Secondly, this action gives rise to the set of orbits $(\mathcal{K}^p(\mathcal{M}) \times \mathcal{M})/G$. Thus, $\mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$ is a principal bundle with $\pi_3 : \mathcal{K}^p(\mathcal{M}) \times \mathcal{M} \to (\mathcal{K}^p(\mathcal{M}) \times \mathcal{M})/G$ and fibre G. Finally, we define a lift $f : (\mathcal{K}^p(\mathcal{M}) \times \mathcal{M})/G \to G$ so that the bundle diagram

$$\begin{tikzcd} G \arrow[r, "\pi_1"] & G/H \simeq \mathcal{M} \\ (\mathcal{K}^p(\mathcal{M}) \times \mathcal{M})/G \arrow[u, "f"] & \mathcal{K}^p(\mathcal{M}) \times \mathcal{M} \arrow[l, "\pi_3"'] \arrow[u, "\pi_2"'] \end{tikzcd} \tag{4.65}$$

commutes. Note that the lift f defines a cross-section through the orbits of $(\mathcal{K}^p(\mathcal{M}) \times \mathcal{M})/G$, whose intersection with each orbit defines a canonical form along the orbit. The coordinates of these canonical forms are covariants of the group action. If we compose f with π_3, we obtain the moving frame map $f \circ \pi_3 : \mathcal{K}^p(\mathcal{M}) \times \mathcal{M} \to G$, which given a $K \in \mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$, defines a map back to canonical form for K.

Let us now specialize these discussions to the case when $p = 2$ and $K \in \mathcal{K}^2(\mathcal{M})$ is a CKT. Since K is a CKT, then at $p \in \mathcal{M}$, we have a frame of (quasi)-orthonormal eigenvectors $E_{K,p}(\mathcal{M})$. The set of all frames for CKTs on $\mathcal{M}$ forms a frame bundle $E(\mathcal{M})$ with $\pi_4 : E(\mathcal{M}) \to \mathcal{M}$ and fibre $E_{K,p}(\mathcal{M})$. If we define a map $\pi_5 : \mathcal{K}^2(\mathcal{M}) \times$

Figure 4.1: Canonical and equivalence problems for CKTs in ITKT



$\mathcal{M} \to E(\mathcal{M})$, then (4.65) becomes

$$\begin{tikzcd} G \arrow[rr, "\pi_1"] & & G/H \simeq \mathcal{M} & \\ (\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G \arrow[u, "f"] & & \mathcal{K}^2(\mathcal{M}) \times \mathcal{M} \arrow[ll, "\pi_3"] \arrow[u, "\pi_2"'] \arrow[r, "\pi_5"'] & E(\mathcal{M}) \arrow[ul, "\pi_4"'] \end{tikzcd} \tag{4.66}$$

Using the frame of eigenvectors of a CKT, it is possible to solve the equivalence problem of CKTs under the action of the isometry group. In particular, suppose $\{E_i\}$ is the frame of eigenvectors for a CKT $K \in \mathcal{K}^2(\mathcal{M})$ with coframe $\{E^i\}$. In this basis, the metric g and Killing tensor have the form:

$$g = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n), \quad K = \mathrm{diag}(\epsilon_1\lambda_1, \ldots, \epsilon_n\lambda_n),$$

where $\epsilon = \pm 1$ and λ_i are the eigenvalues of K. Suppose further that ∇ is the Levi-Civita connection on $\mathcal{M}$. If we substitute this non-coordinate basis into Cartan's first (2.11) and second (2.12) structure equations, we get

$$\begin{aligned} dE^i + \omega^i{}_j \wedge E^j &= T^i = 0, \\ d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j &= R^i{}_j, \end{aligned} \tag{4.67}$$

where $\Gamma^k_{ij} E^i = \omega^k{}_j$, $T^i = \frac{1}{2} T^i_{jk} E^j \wedge E^k$ and $R^i{}_j = \frac{1}{2} R^i_{jk\ell} E^k \wedge E^\ell$ in this basis, and $\omega^i{}_j = -\omega^j{}_i$. To obtain a set of canonical CKTs for the orbit space $(\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/\mathcal{I}(\mathcal{M})$, we solve the Killing tensor equation

$$K_{(ij;k)} = 0$$

for K subject to the constraint

$$E^i \wedge dE^i = 0, \qquad \text{(no sum)}$$

which ensures that the eigenforms $\{E^i\}$ are normal. To obtain a set of differential invariants to distinguish between the orbits, we solve the structure equations for $\omega^i{}_j$.

While Cartan's approach is interesting from a theoretical perspective, it may be impractical to solve the equivalence problem in this way due to the computational challenges of computing a complete set of differential invariants. Moreover, from an applications perspective, most problems are formulated in the local coordinates of $\mathcal{M}$, rather than in the frame of eigenvectors. In light of these observations, we employ a more algebraic approach to solving equivalence problems of CKTs that is more in line with the solution to the equivalence problems of homogeneous polynomials from Section 4.2.1. In particular, we identify an n-dimensional parameter space of a general Killing tensor on a manifold with $\mathbb{E}^n$, much like we did with the coefficients of polynomials in Section 4.2.1. Considering the action of the group of isometries on this parameter space and determining a set of fundamental invariants, we determine an equivalence criterion for CKTs defined on the manifold.

In what follows, we describe the stages in solving an equivalence problem in ITKT. The first of which is the canonical forms problem.

4.3.2 The Canonical Forms Problem

In 1934, Eisenhart made the important step of introducing Killing tensors into the theory of orthogonal separation of variables when he established Theorem 5. Using Eisenhart's equations

$$\frac{\partial \lambda_i}{\partial u^j} = (\lambda_i - \lambda_j)\frac{\partial \ln g_{ii}}{\partial u^j} \quad \text{(no sum)},$$

it is possible to determine a CKT which characterizes an orthogonally separable coordinate system on a manifold.[17] The determination of a canonical orthogonally separable coordinate system or its associated CKT constitutes the solution to the *canonical forms problem.* Using Eisenhart's method and other techniques, the canonical forms problem has been solved for many of the two and three-dimensional spaces of constant curvature, namely $\mathbb{E}^2$ [58, 26, 66, 56], $\mathbb{S}^2$ [66, 39], $\mathbb{H}^2$ [11, 30, 66, 39], $\mathbb{M}^2$ [66, 57, 14, 43], $\mathbb{E}^3$ [22, 40], $\mathbb{S}^3$ [22, 62, 45], $\mathbb{H}^3$ [31], and $\mathbb{M}^3$ [43, 44, 33, 34, 39]. Moreover, a recursive method for constructing orthogonally separable coordinate systems for $\mathbb{E}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ has been determined and is given in [45], thereby solving the canonical forms problem for the most general cases.

For $\mathbb{E}^2, \mathbb{S}^2, \mathbb{E}^3$, and $\mathbb{S}^3$, the CKT for each orthogonally separable coordinate system has been transformed from separable coordinates to Cartesian coordinates; for $\mathbb{H}^2, \mathbb{M}^2, \mathbb{H}^3$, and $\mathbb{M}^3$, each CKT has been transformed from separable coordinates to

[17] As demonstrated in [37], it is possible to re-derive these equations using the moving frame of eigenvectors of a CKT and Cartan's structure equations (4.67).

pseudo-Cartesian coordinates. Having a common set of coordinates for the CKTs is necessary when discriminating between the orbits with the invariants and covariants. Therefore, Eisenhart's equations and the pullback map (2.4.4) can be used to solve the canonical forms problem of CKTs on a manifold.

The canonical forms generated from Eisenhart's method are not unique, but do define the simplest representatives along the orbits of a group action. The determination of these canonical forms is governed by the choice of a coframe on the manifold, which consists of the eigenforms of the characteristic Killing tensors. The solution of the Killing tensor equation in this coframe of eigenforms yields the canonical form for the orbit. Therefore, Eisenhart's choice of this coframe on the manifold effectively defines a cross-section through the orbits, whose intersection with each orbit yields a canonical form.

An alternative method for determining CKTs, as mentioned above, has been used to derive the canonical CKTs for $\mathbb{S}^3$, $\mathbb{H}^2$ and $\mathbb{H}^3$ [45, 30, 31]. Their approach makes use of the following result originally given in [46], and later simplified in [17] and [7]:

Theorem 4.3.1 (Kalnins & Miller)**.** *The Hamilton-Jacobi equation of the geodesic Hamiltonian on an n-dimensional manifold is orthogonally separable if and only if there exist $n-1$ functionally independent first integrals of the form*

$$F = \frac{1}{2}K^{ij}p_ip_j,$$

which are in involution and such that their Killing tensors commute.

Note that if a pair of Killing tensors, K and $\tilde{K}$, satisfy

$$K^{ij}\tilde{K}_j{}^k = \tilde{K}^{ij}K_j{}^k, \tag{4.68}$$

they are said to *commute*.

Since the first integrals are in involution, we have

$$0 = \{F_1, F_2\} = \{K_1^{ij}p_ip_j, K_2^{k\ell}p_kp_\ell\} = \frac{1}{2}p_ip_jp_\ell(K_2^{k\ell}\partial_k K_1^{ij} - K_1^{kj}\partial_k K_2^{i\ell})$$

which implies

$$\partial_k K_1^{(ij}K_2^{|k|\ell)} - K_1^{k(j}\partial_k K_2^{i\ell)} = 0 \;\Leftrightarrow\; [K_1, K_2] = 0,$$

where [,] denotes the Schouten bracket. Therefore, the vector space

$$c_1 g + c_2 K_1 + \ldots c_n K_{n-1}$$

forms a commutative graded Lie algebra under the Schouten bracket, and is called a *Killing-Stäckel space* [7]. Recall that such a vector space can be readily found for a given orthogonally separable metric by integrating Eisenhart's equations (4.45).

Given a commutative set of Killing tensors, it is possible to simultaneously diagonalize them. In other words, we can find a set of eigenvectors which are common to all of the tensors. If the set of first integrals for these Killing tensors are in involution, then we have the following result [7]:[18]

Theorem 4.3.2. *If a set of $n-1$ functionally independent first integrals of the form*

$$F = \frac{1}{2} K^{ij} p_i p_j$$

are in involution and their Killing tensors have common eigenvectors, then the eigenvectors are orthogonally integrable.

In [45], Kalnins *et al* used complete sets of commuting second-order operators to characterize orthogonally separable systems of coordinates on $\mathbb{S}^3$. These operators were formed by taking products of first-order operators from the Lie algebra of the isometry group of the manifold.[19] In determining which second-order operators characterized a system of coordinates, they first imposed the commutation condition (4.68) on a pair of operators in their most general form and then divided the operators into equivalence classes; operators were deemed equivalent if one could be transformed into the other using the isometry group of the manifold and by taking linear combinations with the Casimir operator. While they similarly grouped these second-order operators into orbits, they did not establish a method for determining which orbit a given operator belonged to. Nor did they develop a way of determining the form of the separable coordinates for an operator which is not in the canonical form of its orbit (ie., determine the moving frame map).

[18]As pointed out by Benenti [7], this result was first stated in the proof of Theorem 6 in [46].

[19]In our language, the first-order operators are Killing vectors on the manifold, and the second-order operators are Killing matrices of order four.

We can, however, make use of these results when solving canonical forms problems in ITKT. For example, the isometry group of the sphere $\mathbb{S}^2$ has a Lie algebra generated by the first-order operators

$$L_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ L_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ L_3 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.$$

These are also a basis of Killing vectors for $\mathbb{S}^2$. A linear combination of products of these operators is used to define a CKT for spherical and elliptic coordinates in [66]:

$$\begin{aligned} \text{spherical:} \quad & L_1^2, \\ \text{elliptic:} \quad & k'^2 L_1^2 - k^2 L_3^2, \quad k^2 + k'^2 = 1. \end{aligned}$$

To define the canonical form K in each case, we let

$$K = c_1\mathcal{C} + c_2 A,$$

where $c_1, c_2 \in \mathbb{R}$, A is the second-order operator, and $\mathcal{C}$ is the Casimir tensor (2.13).[20] Using the second-order operators listed in [45, 30, 31], we can readily solve the canonical forms problems for $\mathbb{S}^3, \mathbb{H}^2$ and $\mathbb{H}^3$.

4.3.3 Equivalence Problem of Killing Tensors

Once the canonical forms have been established for each of the orbits, the next step in solving the equivalence problem of $\mathcal{K}^2(\mathcal{M})$ under the action of $\mathcal{I}(\mathcal{M})$ is to determine a classification scheme for the canonical forms. This is a set of conditions which enables you to determine which orbit a given CKT belongs to. Such a scheme typically uses invariants, covariants or web symmetries, although all three objects may be necessary in building a classification scheme. In what follows we define each of these objects, and describe how they can be used to create a classification scheme for Killing tensors.

Invariants of Killing tensors

As we did for polynomials in Examples 4.2.1 and 4.2.2, invariants and covariants of the group action can be used to distinguish between the orbits. For vector spaces of Killing tensors under the action of the isometry group, we define invariants and covariants as follows.

[20] In Section 2.6.4 we showed that the pushforward of the contravariant spherical metric is the Casimir tensor $\mathcal{C}$.

Definition 4.3.3. Consider the action of the isometry group $\mathcal{I}(\mathcal{M})$ on the vector space of Killing tensors $\mathcal{K}^p(\mathcal{M})$. A function $\mathcal{I} : \mathcal{K}^p(\mathcal{M}) \to \mathbb{R}$ satisfying

$$\mathcal{I}(g \cdot K) = \mathcal{I}(K)$$

for all $g \in \mathcal{I}(\mathcal{M})$ and $K \in \mathcal{K}^p(\mathcal{M})$ is an *invariant* of the action.

Definition 4.3.4. Consider the action of the isometry group $\mathcal{I}(\mathcal{M})$ on the product space $\mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$. A function $\mathcal{C} : \mathcal{K}^p(\mathcal{M}) \times \mathcal{M} \to \mathbb{R}$ satisfying

$$\mathcal{C}(g \cdot K) = \mathcal{C}(K)$$

for all $g \in \mathcal{I}(\mathcal{M})$ and $K \in \mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$ is a *covariant* of the action.

As discussed in Section 4.2.3, the *method of infinitesimal generators* is one possible way of determining invariants and covariants. The next, and perhaps more difficult step, is to create a set of *distinguishing* invariants and covariants from the fundamental invariants and covariants which distinguish between the orbits.

Example 4.3.5. Consider the special rotation group, $SO(2)$, acting on the vector space $K^2(\mathbb{S}^2)$. Let

$$C^{IJ} = \begin{pmatrix} c_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & c_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & c_3 \end{pmatrix} \tag{4.69}$$

represent the coefficient matrix of the general Killing tensor. Using the method of infinitesimal generators and applying the method of undetermined coefficients to solve the resulting PDEs, we obtain the following three fundamental invariants:

$$\begin{aligned} \mathcal{I}_1 &= c_1 + c_2 + c_3, \\ \mathcal{I}_2 &= c_1^2 + c_2^2 + c_3^2 + 2\gamma_1^2 + 2\gamma_2^2 + 2\gamma_3^2, \\ \mathcal{I}_3 &= c_1c_2c_3 - c_1\gamma_3^2 - c_2\gamma_2^2 - c_3\gamma_1^2 + 2\gamma_1\gamma_2\gamma_3. \end{aligned}$$

The auxiliary invariant, $\mathcal{A}$, formed by taking a linear combination of these fundamental invariants

$$\mathcal{A} = \frac{1}{2}(\mathcal{I}_2 - \mathcal{I}_1^2) = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - c_1c_2 - c_1c_3 - c_2c_3,$$

distinguishes between the two orbits (see Table 4.2). $\square$

Table 4.2: Invariant classification of characteristic Killing tensors of $\mathbb{S}^2$

$\mathcal{A}$	Coordinate system
0	spherical
$\neq 0$	elliptic

In addition to $K^2(\mathbb{S}^2)$, the equivalence problem on the vector spaces $K^2(\mathbb{E}^2)$, $K^2(\mathbb{E}^3)$, $K^2(\mathbb{M}^2)$, $K^2(\mathbb{M}^3)$, and $K^2(\mathbb{H}^2)$ has been solved. In these efforts over the last decade, ITKT has experienced many developments, including the addition of new and more effective ways of computing group invariants. In the beginning, this was limited to the method of infinitesimal generators and the method of moving frames, which could be applied successfully in some of the lower-dimensional spaces.[21] In the higher dimensional cases, however, these methods become computationally tedious or even intractable. More recently [41], a tensorial method has been developed to determine invariants and covariants of the group $SE(n-s,s)$ acting on the vector space $\mathcal{K}^p(\mathbb{E}^{n-s,s})$. The essence of the approach is the recognition that invariants can be formed by taking contractions of the parameter matrix of the general Killing tensor (4.9).

Theorem 4.3.6 (Theorem 5, [41])**.** *Consider the action of $SE(n-s,s)$ on the vector space $\mathcal{K}^p(\mathbb{E}^{n-s,s})$. Let (4.9) represent the general element of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$, and define*

$$\begin{aligned} D_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p} &= \textstyle\sum_{r=0}^{p-q}\binom{p-q}{r}|A|^r\delta^{(i_{r+1}}_{M_{r+1}\ell_{r+1}}\cdots\delta^{i_{p-q}}_{M_{p-q}\ell_{p-q}} \\ &\qquad C_{\underline{r}}^{i_1\cdots i_r)M_{r+1}\cdots M_{p-q}J_{p-q+1}\cdots J_p}\hat{\delta}^{\ell_{r+1}}\cdots\hat{\delta}^{\ell_{p-q}}, \end{aligned}$$

where

$$\begin{aligned} \hat{\delta}^i &= \tfrac{(-1)^s}{(n-1)!}\epsilon_{ki_2\cdots i_n}\epsilon^{ij_2\cdots j_n}A^{i_2}_{j_2}\cdots A^{i_n}_{j_n}B^k, \\ A^i_k &= \delta^i_{J_1\ell}\delta^\ell_{Mk}C_{\underline{0}}^{M}{}_{J_2\cdots J_p}C_{\underline{0}}^{J_1\cdots J_p}. \end{aligned}$$

Then, any scalar formed by taking contractions of $D_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p}$, g_{ij}, and $\epsilon_{i_1\cdots i_n}$ is an invariant of this group action.

The proof of this result relies on the construction of a "suitable" moving frame and the method developed by Kogan [49] that we mentioned in Section 4.2.4.[22] In

[21] The method of infinitesimal generators and the method of moving frames were first applied to ITKT to determine invariants in [56] and [19], respectively.

[22] The method developed by Kogan [49] was first applied to ITKT in [73].

particular, after noting that $SE(n-s,s) \simeq SO(n-s,s) \ltimes E(n-s,s)$, Horwood first considered the action of $E(n-s,s)$ on the vector space $\mathcal{K}^p(\mathbb{E}^{n-s,s})$. Using the covariants from this initial step as coordinates, Horwood then considered the action of $SO(n-s,s)$ on the vector space $\mathcal{K}^p(\mathbb{E}^{n-s,s})$. Under this action, the covariants transform like tensors, and thus invariants can be formed through the process of tensor contraction. We can use this theorem to determine invariants of the group $SO(n-s,s)$ acting on the vector spaces $\mathcal{K}^p(\mathbb{S}^{n-s,s})$ and $\mathcal{K}^p(\mathbb{H}^{n-s,s})$.

Example 4.3.7. Consider the special rotation group, $SO(2)$, acting on the vector space $K^2(\mathbb{S}^2)$. A set of fundamental invariants can be easily obtained by taking contractions of the coefficient matrix (4.69):

$$
\begin{aligned}
\mathcal{I}_1 &= C^I_I &&= c_1 + c_2 + c_3, \\
\mathcal{I}_2 &= C^I_J C^J_I &&= c_1^2 + c_2^2 + c_3^2 + 2\gamma_1^2 + 2\gamma_2^2 + 2\gamma_3^2, \\
\mathcal{I}_3 &= C^I_J C^J_K C^K_I &&= c_1^3 + c_2^3 + c_3^3 + 3c_1(\gamma_2^2 + \gamma_3^2) + 3c_2(\gamma_1^2 + \gamma_3^2) + 3c_3(\gamma_1^2 + \gamma_2^2) \\
& && \quad +6\gamma_1\gamma_2\gamma_3.
\end{aligned}
$$

□

In spite of this recent progress, however, solving the equivalence problem for a vector space of Killing tensors is not trivial even once a complete set of fundamental invariants is known. As we mentioned at the beginning of this section, we often require certain combinations of these invariants to distinguish between the orbits, and these particular combinations may not be easy to find. Even worse, there is no guarantee that invariants can distinguish between the orbits. This is because invariants *locally* distinguish between the orbits of a given space. If the orbits have differing dimensions (which is often the case), then other methods must be used which can *globally* distinguish between the orbits.

As we mentioned in Section 4.2.1, covariants may be needed to completely solve an equivalence problem. Covariants are constructed through the method of *prolongation*, which, as we mentioned in Section 4.2.4, enlarges the dimension of the manifold upon which the group acts. By taking the Cartesian product of the parameter space and the underlying manifold $\mathcal{M}$, as we did in Example 4.2.2 and Definition 4.3.4, we have increased the dimension of the space. Using Theorem 4 in [41], we can obtain the covariants of the group $SE(n-s,s)$ acting on the vector space $\mathcal{K}^p(\mathbb{E}^{n-s,s})$.

Theorem 4.3.8. *Consider the action of $SE(n-s,s)$ on the vector space $\mathcal{K}^p(\mathbb{E}^{n-s,s})$. Then, any scalar formed from taking contractions of (4.11), g_{ij}, and $\epsilon_{i_1\cdots i_n}$ is a covariant of the group action.*

The contraction methods presented in Theorems 4.3.6 and 4.3.8 for calculating invariants and covariants in ITKT can be applied in equivalence problems of homogeneous polynomials to efficiently determine invariants and covariants of affine group actions [36]. This development in classical invariant theory greatly reduces the computational challenges previously encountered in the determination of invariants from the method of infinitesimal generators and the method of normalization, and allows the resulting invariants and covariants to conveniently be expressed in a compact indicial form. This demonstrates how a result in ITKT (which is built on ideas from classical invariant theory), has been brought back to further develop classical invariant theory.

We can employ Theorem 4.3.8 to determine the covariants of the group $SO(n-s,s)$ acting on the vector spaces $\mathcal{K}^p(\mathbb{S}^{n-s,s})$ and $\mathcal{K}^p(\mathbb{H}^{n-s,s})$.

Example 4.3.9. Consider the special Lorentz group, $SO(2,1)$, acting on the vector space $K^2(\mathbb{H}^2)$. In Example 4.2.9, we found three fundamental invariants using the method of infinitesimal generators. An easier approach is to apply Theorem 4.3.6 and contract products of the coefficient tensor, which yields the following (slightly different) set of fundamental invariants:

$$
\begin{aligned}
\mathcal{I}_1 &= C^I{}_I &&= -c_1 + c_2 + c_3,\\
\mathcal{I}_2 &= C^I{}_J C^J{}_I &&= c_1^2 + c_2^2 + c_3^2 + 2\gamma_1^2 - 2\gamma_2^2 - 2\gamma_3^2,\\
\mathcal{I}_3 &= C^I{}_J C^J{}_K C^K{}_I &&= -c_1^3 + c_2^3 + c_3^3 + 3c_1(\gamma_2^2 + \gamma_3^2) + 3c_2(\gamma_1^2 - \gamma_3^2) +\\
& && \quad 3c_3(\gamma_1^2 - \gamma_2^2) - 6\gamma_1\gamma_2\gamma_3.
\end{aligned}
$$

Moreover, if we apply Theorem 4.3.8,

$$
\begin{aligned}
\mathcal{C}_1 &= \mathcal{C}_i^i &&= 2(t^2 - x^2 - y^2),\\
\mathcal{C}_2 &= K_i^i &&= 4C^i_{ji\ell}x^j x^\ell,\\
\mathcal{C}_3 &= K_j^i K_i^j &&= 16C^i{}_{kj\ell}C^j{}_{min}x^k x^\ell x^m x^n,
\end{aligned}
$$

where $\mathcal{C}$ denotes the Casimir tensor (2.14), we obtain a set of functionally independent covariants. $\square$

First introduced to ITKT in [72], covariants of Killing tensors are often used when the invariants fail to distinguish between the orbits of a group action. For example, covariants were used to solve the equivalence problem of the vector space $\mathcal{K}^2(\mathbb{E}^3)$ under the action of $SE(3)$ in [37]. As Horwood points out (p. 87, [37]), this was because the invariants derived from Theorem 4.3.6 can only distinguish between the orbits when a certain determinant condition is satisfied for the coefficient tensor. This is only true for five of the eleven webs, and thus the invariants fail to distinguish between all cases. If we prolong the space, and solve the equivalence problem of the vector space $\mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3$ under the action of $SE(3)$, then covariants of this group action derived using Theorem 4.3.8 can distinguish between all eleven webs. This is because the moving frame constructed in the proof of Theorem 4.3.8 is *globally* defined. This follows from the fact that the action of $E(3)$ on $\mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3$ gives rise to the set of orbits $(\mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3)/E(3)$. Thus $\mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3$ is a principal bundle with $\pi : \mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3 \to (\mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3)/E(3)$ and fibre $E(3)$. Recalling Theorem 2.8.6, since this principal bundle is trivial, it admits a *global* cross-section $s : \mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3 \to E(3)$. The intersection of this global cross-section with the orbits define a set of fundamental covariants of this group action. Using these covariants as coordinates, and considering the action of $SO(3)$ on $\mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3$, contractions of the objects in Theorem 4.3.8 in these coordinates yield a set of fundamental covariants of the full $SE(3)$ group action. Moreover, these covariants *globally* distinguish between the orbits.

While covariants have the potential to globally distinguish between the orbits of a group action, one still has to find certain combinations of these covariants to solve the equivalence problem. This can be a very challenging task, particularly for manifolds of dimension three or higher. To overcome this difficulty in ITKT, a method which exploits the geometric symmetry properties of the associated orthogonal coordinate web of a CKT is often used, called the *method of web symmetries*. This approach has the advantage of being more efficient and intuitive than the invariant/covariant methods of this section. Before we provide the details of this method in the next section, let us briefly discuss joint invariants in the context of ITKT.

First introduced in [72], joint invariants of Killing tensors are a natural extension of the theory.

Definition 4.3.10. Consider the action of the isometry group $\mathcal{I}(\mathcal{M})$ on the product space $\mathcal{K}^{p_1}(\mathcal{M})\times\cdots\times\mathcal{K}^{p_q}(\mathcal{M})$. A function $\mathcal{J}:\mathcal{K}^{p_1}(\mathcal{M})\times\cdots\times\mathcal{K}^{p_q}(\mathcal{M})\to\mathbb{R}$ satisfying

$$\mathcal{J}(g\cdot K_1,\ldots g\cdot K_q)=\mathcal{J}(K_1,\ldots,K_q)$$

for all $g\in\mathcal{I}(\mathcal{M})$ and $K_i\in\mathcal{K}^{p_1}(\mathcal{M})\times\cdots\times\mathcal{K}^{p_q}(\mathcal{M})$ is a *joint invariant* of the action.

Joint invariants of Killing tensors is mostly an untapped area of ITKT. To date, only one [1] application of the theory has been made. In this application, the authors studied the action of $SE(2)$ on the vector space $\mathcal{K}^2(\mathbb{E}^2)\times\mathcal{K}^2(\mathbb{E}^2)$ and derived most of the joint invariants of this group action using the method of moving frames. To determine the remaining joint invariants, they adapted the classical Weyl theorem to Killing tensors. Namely, if $\{F_1,F_2\}$ and $\{F_3,F_4\}$ denote the foci of two non-degenerate elliptic-hyperbolic webs, then the distances between these foci define joint invariants. Using the invariant formulae for the coordinates of the foci from [56], the authors obtained expressions for the joint invariants in the coordinates of the parameter space. Then, using the notion of a resultant from the theory of homogeneous polynomials, they interpreted these latter joint invariants as resultants. In this context, the vanishing of a resultant is equivalent to the orthogonal webs having a common focus.

An interesting application of this problem, as mentioned in the introduction to this section, was to use a resultant to completely characterize the well-known superintegrable Kepler potential of $\mathbb{E}^2$:

Theorem 4.3.11 (Theorem 5.1, [1])**.** *Consider a natural Hamiltonian defined in $\mathbb{E}^2$:*

$$H=\frac{1}{2}(p_1^2+p_2^2)+V(q),$$

and suppose $K_1,K_2\in\mathcal{K}^2(\mathbb{E}^2)$ are nondegenerate and compatible with V:

$$d(K_1dV)=0,\quad d(K_2dV)=0.$$

Then V is the Kepler potential if and only if $\{K_1,K_2\}\in\mathcal{K}^2(\mathbb{E}^2)\times\mathcal{K}^2(\mathbb{E}^2)$ admits one vanishing resultant $\mathcal{R}[K_1,K_2]$.

A natural extension of this result would be to study joint invariants on higher dimensional spaces of constant curvature and see if other well-known superintegrable potentials can be characterized in this way.

4.3.4 Web Symmetries

First introduced in [40], the *method of web symmetries* is an effective technique when solving an equivalence problem of Killing tensors on a manifold. The main idea of the method is to use the symmetry properties of an orthogonal coordinate web to characterize its CKT.

Definition 4.3.12. Consider a CKT K defined on a pseudo-Riemannian manifold $\mathcal{M}$ with metric g. The orthogonal coordinate web associated with K admits a *web symmetry* if and only if there exists an isometry $\sigma_t \in I(\mathcal{M})$ such that

$$(\sigma_t)_* K = K.$$

The set of all such isometries is the isotropy subgroup for K. The infinitesimal generator of an isometry σ_t can be used to determine if a CKT admits a web symmetry defined by σ_t.

Proposition 4.3.13 ([13])**.** *If V is the infinitesimal generator of an isometry $\sigma_t \in I(\mathcal{M})$, then K admits a web symmetry defined by σ_t if and only if*

$$\mathcal{L}_V K = 0. \tag{4.70}$$

Proof. If σ_t is the flow generated by a vector field V, then[23]

$$\mathcal{L}_V K = \lim_{t \to 0} \frac{1}{t} \left(K|_{\sigma_t(p)} - (\sigma_t)_* K|_p \right) = 0$$

is equivalent to $(\sigma_t)_* K = K$. □

To determine if a CKT K has any web symmetry, we substitute K into condition (4.70), and use the general Killing vector of the manifold as V. The parameters of V which allow K to satisfy this condition denote the generators of the web symmetries of K.

Example 4.3.14. The CKT

$$K^S = \begin{pmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

[23]Here we make use of the infinitesimal version of the Lie derivative. For more details, see, for example, [59].

characterizes spherical coordinates on $\mathbb{S}^2$. To see if its associated web admits any symmetry, we impose condition (4.70) on K^S, letting

$$V = (ay + bz)\frac{\partial}{\partial x} + (-ax + cz)\frac{\partial}{\partial y} + (-bx - cy)\frac{\partial}{\partial z}.$$

This condition forces $b = c = 0$, proving that the spherical web on $\mathbb{S}^2$ admits the rotational symmetry

$$R_{12} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Inspection of K^S tells us the web also admits the translational symmetry

$$X = \frac{\partial}{\partial z},$$

which comes from the isometry group of the ambient space $\mathbb{E}^3$. Until the orthogonal coordinate web intersects the sphere (ie., before we impose $x^2 + y^2 + z^2 = 1$), it possesses this added symmetry from the ambient space. $\square$

The fact that web symmetry is an invariant property under transformations of the isometry group, and the observation that some of the orthogonal coordinate webs possessed symmetry, motivated the development of the *method of web symmetries*. In this technique, the canonical CKTs belonging to each of the orthogonal separable coordinate systems are first inspected for web symmetry. This preliminary step essentially divides the CKTs into categories labeled by the type of symmetries they admit. In order to distinguish between the CKTs in each category, a "reduced invariant subspace" of $\mathcal{K}^2(\mathcal{M})$ is defined for each category, and a set of "reduced fundamental invariants" are computed. Solving the equivalence problem now requires distinguishing between the CKTs in each symmetry category using their respective reduced invariants, and between the CKTs in the asymmetric category using the regular invariants.

This method proved to be highly successful in the solution to the equivalence problem of Killing tensors on $\mathbb{E}^3$ [40] and $\mathbb{M}^3$ [39]. The classification for $\mathbb{E}^3$ was later refined in [37] by incorporating dilatational symmetry into the technique, a symmetry which is also invariant under transformations of the isometry group (see Table 7.2 in [37]). Tensors T which admit a dilatational symmetry satisfy the condition

$$\mathcal{L}_D T = cT,$$

Table 4.3: Symmetry classification of the webs of $\mathbb{E}^3$ under the action of $SE(3)$

Category	Symmetries	Separable webs	Generators
1	3 translational	Cartesian	X_1, X_2, X_3
2	1 translational and 1 rotational	Circular cylindrical	X_3, R_{12}
3	1 translational	Parabolic cylindrical	X_3
		Elliptic hyperbolic	
4	1 dilatational and 1 rotational	Spherical	R_{12}, D
6	1 rotational	Oblate spheroidal	R_{12}
		Prolate spheroidal	
		Parabolic	
7	1 dilatational	Conical	D
8	none	Paraboloidal	
		Ellipsoidal	

where $c \in \mathbb{R}$ and D denotes the dilatation vector $D = x^i X_i$. We can refine this classification even further by incorporating the number of symmetry generators each web admits. Applying this to $\mathbb{E}^3$, we find that the eleven webs are now almost completely distinguished by symmetry alone (see Table 4.3).

As we mentioned in Section 4.3.3, it is more efficient and less difficult to solve an equivalence problem using a combination of web symmetries and (reduced) invariants and covariants than to try and solve it using invariants and covariants exclusively. This is evidenced by the solution to the equivalence problem of Killing tensors defined on $\mathbb{E}^3$ given in [37] that relied exclusively on invariants and covariants. As Horwood notes at the end of his solution, "...the classification is rather intricate and several of the covariants in the scheme are extremely complicated" (p. 99, [37]). Indeed, finding the combinations of invariants and covariants which distinguish between the orbits typically requires a significant amount of experimentation. Comparatively, the method of web symmetries is more systematic and straightforward, and reduces the amount of invariant/covariant discrimination needed in an equivalence problem.

The relationship between the number of web symmetries and the number of functionally independent covariants for a given CKT is contained in the following proposition.

Proposition 4.3.15. *For a given CKT $K \in \mathcal{K}^2(\mathcal{M})$, the number of functionally independent eigenvalues for K is at most*

$$dim(\mathcal{M}) - p,$$

where p denotes the number of linearly independent web symmetries of K.

Proof. Suppose $K \in \mathcal{K}^2(\mathcal{M})$ is a CKT on an n-dimensional manifold with p web symmetries. Since the existence of a web symmetry is equivalent to an ignorable coordinate,[24] K can be written in terms of $n-p$ coordinates. Therefore, the n eigenvalues of K are functions of only $n-p$ coordinates, and thus at most $n-p$ eigenvalues are functionally independent. □

Since the set of eigenvalues of a Killing tensor are covariant under the group action, this proposition demonstrates that discrimination of the CKTs using the method of web symmetries is equivalent to a discrimination using covariants of the group action.

The *method of web symmetries* will prove to be very useful when solving the classification problem for webs defined on $\mathbb{S}^2, \mathbb{S}^3, \mathbb{H}^2$ and $\mathbb{H}^3$ in the next two chapters. Since we define the Killing tensors in the ambient space coordinates, the webs may admit additional symmetry from the ambient space before applying the constraint of the hypersurface as in Example 4.3.14. This observation will prove to be very useful in developing the classification scheme.

4.3.5 Transformation to Canonical Form

Once a classification scheme has been developed, we can determine which orbit a CKT belongs to. Any CKT K along its orbit is equivalent to the canonical CKT K_0 of the orbit in the sense that a transformation from the isometry group of the manifold exists which can map K to K_0. As part of the solution to equivalence problems of Killing tensors, it is necessary to determine such a transformation for any CKT, referred to as the *transformation to canonical form* or *moving frame map* [65]. Since each CKT corresponds to a system of orthogonal coordinates, this step amounts to

[24] A coordinate q^i which the Hamiltonian no longer depends upon is called an *ignorable* or *cyclic* coordinate [17, 4].

finding the exact form of the coordinates which separate the Hamiltonian, and thus represents a crucial stage in the solution of the problem.

On low-dimensional manifolds such as $\mathbb{E}^2$ and $\mathbb{S}^2$, it is possible to determine the transformation to canonical form using algebraic formulae. In particular, the most general CKT of each orbit is computed using the isometry group of the manifold. Upon comparing this tensor with its canonical form, algebraic expressions can be derived which specify the transformation made by the group. This process is often facilitated by the use of invariants of the group action [56, 40].

On higher-dimensional manifolds, such explicit algebraic formulae are not always possible and methods have been developed to handle such cases. For example, for both $\mathbb{E}^3$ [40] and $\mathbb{M}^3$ [39], the *method of web symmetries* has proven to be useful at this stage of the problem as well. In particular, it was noted that if a symmetric web is transformed by the isometry group, the underlying symmetry is transformed in this way as well. In light of this fact, it is possible to use the form of the symmetry to determine the transformation to canonical form. For the asymmetric webs in each case, the transformation to canonical form is facilitated by the determination of the eigenvectors of the coefficient matrix of the Killing tensor. Thus, in both the symmetric and asymmetric cases the transformation is often deduced from vectors rather than the components of the $(2, 0)$-tensor itself. As such, the calculations are far less complicated.

In this thesis, we are concerned with Killing tensors on three-dimensional spherical and hyperbolic space, each having the form

$$K = C^{ijk\ell} R_{ij} \odot R_{k\ell}.$$

While the method of web symmetries can be applied in this case as well, it is complicated by the fact that $C^{ijk\ell}$ has a 6×6 matrix representation. Therefore, the associated eigenvalue-eigenvector problem may be intractable. As we will see in the next chapter, it is possible to overcome this difficulty in these equivalence problems by defining an "algebraic Ricci tensor" for K. Such a tensor has a 4×4 matrix representation, and thus the associated eigenvalue-eigenvector problem is tractable.

CHAPTER 5

THE EQUIVALENCE PROBLEM FOR ORTHOGONAL WEBS ON $\mathbb{S}^3$

Having defined the requisite theory, we are now prepared to present a solution to the equivalence problem of orthogonal webs on $\mathbb{S}^3$. Before doing so, we will solve the simpler equivalence problem of orthogonal webs defined on $\mathbb{S}^2$. This example will enable us to introduce many of the objects and techniques we will be employing on $\mathbb{S}^3$, which are much easier to visualize in the lower dimension. We will solve the case of $\mathbb{S}^3$ in a similar way. First, we will define a canonical form for each coordinate system, then we will develop a classification scheme for the CKTs of $\mathbb{S}^3$, and finally formulate a method for determining the transformation to canonical form for a given CKT. The chapter will conclude with an algorithm for applying these results.

5.1 Introductory Example: The Equivalence Problem for Orthogonal Webs on $\mathbb{S}^2$

This section presents a solution to the equivalence problem of orthogonal webs defined on $\mathbb{S}^2$.[1] Let us begin by defining an equivalence criterion for Killing tensors $K \in \mathcal{K}^2(\mathbb{S}^n)$.

5.1.1 Equivalence Criterion

The isometry group of $\mathbb{S}^n$ is the special orthogonal group $\mathrm{SO}(n+1)$, which is a Lie subgroup of the orthogonal group $\mathrm{O}(n+1)$. It consists of orthogonal matrices Λ with

[1] We should point out that Horwood [37] has already solved this problem in his solution of $\mathbb{E}^3$, as it naturally falls out as a subcase of the dilatationally symmetric cases. In particular, he develops a classification scheme using invariants, and uses the machinery developed for $\mathbb{E}^3$ to determine the transformation to canonical form. As this example is meant to introduce the techniques used for classifying the CKTs of $\mathbb{S}^3$, we will instead develop a classification scheme using only web symmetries.

positive unit determinant. Its action on a point $x^i \in \mathbb{E}^{n+1}$ is defined by

$$\tilde{x}^i = \Lambda^i{}_j x^j,$$

where $\Lambda \in \mathrm{SO}(n+1)$ and x^i are Cartesian coordinates. For a Killing vector (4.7) or Killing tensor (4.15), this induces the following transformation

$$R_{ij} = \Lambda^k{}_i \Lambda^\ell{}_j \tilde{R}_{k\ell}, \quad K^{ij} = \Lambda^i{}_k \Lambda^j{}_\ell \tilde{K}^{k\ell} \tag{5.1}$$

on their respective components, which in turn, induces the following transformation

$$\tilde{B}^{ij} = \Lambda^i{}_k \Lambda^j{}_\ell B^{k\ell}, \quad \tilde{C}^{ijk\ell} = \Lambda^i{}_p \Lambda^j{}_q \Lambda^k{}_r \Lambda^\ell{}_s C^{pqrs} \tag{5.2}$$

on their respective parameters. If we assume the more compact notation $R_{ij} = R_I$, then the first equation of (5.1) may be rewritten as

$$R_I = \Lambda^J{}_I \tilde{R}_J, \tag{5.3}$$

where $\Lambda^I{}_K = \Lambda^i{}_{[k} \Lambda^j{}_{\ell]}$ represents the second compound of $\Lambda^i{}_j$.[2] Also in this new notation, (5.2) becomes

$$\tilde{B}^I = \Lambda^I{}_J B^J, \quad \tilde{C}^{IJ} = \Lambda^I{}_K \Lambda^J{}_L C^{KL}. \tag{5.4}$$

The action of $\mathrm{SO}(n+1)$ on the vector space $\mathcal{K}^2(\mathbb{S}^n)$ foliates $\mathcal{K}^2(\mathbb{S}^n)$ into the orbit space $\mathcal{K}^2(\mathbb{S}^n)/\mathrm{SO}(n+1)$, and each orbit of $\mathcal{K}^2(\mathbb{S}^n)/\mathrm{SO}(n+1)$ is represented by a canonical form. Killing tensors belonging to the same orbit are connected by a transformation $\Lambda \in \mathrm{SO}(n+1)$ and are called *equivalent*. A solution to the equivalence problem for Killing tensors of $\mathcal{K}^2(\mathbb{S}^n)$ enables one to determine whether two Killing tensors are equivalent, and if so, the transformation $\Lambda \in \mathrm{SO}(n+1)$ which connects them along their orbit.

5.1.2 Classification

The canonical forms of $\mathbb{S}^2$ were derived in Example 4.1.23 using Eisenhart's equations. In Example 4.3.5, we demonstrated how to classify these CKTs using invariants. Alternatively, we can use the intrinsic properties of their associated orthogonal

[2] Please see Appendix A for a discussion of compound matrices.

Table 5.1: Symmetry classification of the webs of $\mathbb{S}^2$ under the action of $SO(2)$

Category	Symmetry	Separable web	Generators
1	rotation	Spherical	R_{12}
2	none	Elliptic	

coordinate web to classify them. Indeed, the fact that the spherical web admits a rotational symmetry while the elliptic web admits none is a distinguishing feature between these two webs. See Table 5.1 for a summary of these results.

5.1.3 Transformation to Canonical Form

Now that we have a classification scheme in place, the next step in the solution to the equivalence problem is to find a method for transforming Killing tensors to their canonical form. Both of the canonical CKTs of the sphere have a coefficient matrix in diagonal form. Thus in order to return a CKT to its canonical form, it is enough to determine a transformation which diagonalizes the coefficient matrix. In particular, if we are seeking a transformation which returns $\tilde{K}$ to its canonical form K, we need to find a $\Lambda \in SO(3)$ which diagonalizes the coefficient matrix $\tilde{C}$. Such a Λ can be derived by first computing the eigenvalues of $\tilde{C}$ and then determining a corresponding orthonormal set of eigenvectors using the Gram-Schmidt algorithm.

5.2 The Equivalence Problem for Orthogonal Webs on $\mathbb{S}^3$

Having solved the equivalence problem of orthogonal webs on $\mathbb{S}^2$ as an introductory example, we now turn to solving the case of $\mathbb{S}^3$.

5.2.1 Canonical Forms

The first step in solving the equivalence problem is to define a canonical form for each orthogonally separable coordinate system. In 1950 [62], Olevskii showed that there are just six metrics which permit orthogonal separation of the Laplace-Beltrami equation on $\mathbb{S}^3$ (please refer to Appendix B.2 for an enumeration of these metrics). By Robertson's theorem of 1927 [68], these metrics equivalently denote the orthogonally separable metrics for the Hamilton-Jacobi equation on $\mathbb{S}^3$. In 1976, Kalnins and his collaborators determined a pair of commuting second-order operators for each of the six coordinate systems [45]. In our setting, these operators represent a commutative pair of Killing tensors in Cartesian coordinates for each metric, and thus can be used to define a canonical form for each coordinate system (see Appendix C.2 for an enumeration of these CKTs).

5.3 Classification

Now that we have defined a canonical form for each coordinate system, the next step is to develop a classification scheme so that we may determine which coordinate system a given CKT represents. Analogous to the case of $\mathbb{S}^2$, the symmetry of the webs can be used to intrinsically characterize the six cases. In this case, however, both the type and the number of symmetries will be needed to distinguish between the webs.

Recall from Section 4.3.4 that the web symmetries of a Killing tensor K are generated by Killing vectors on the manifold. Therefore, to determine the symmetries of a web we impose the following condition

$$\mathcal{L}_V K = 0 \tag{5.5}$$

on the Killing tensor K defining the web, using the general Killing vector V of our manifold. In what follows we demonstrate the surprising result that the six CKTs of $\mathbb{S}^3$ can be classified based purely on the symmetry properties of their associated webs. To this end, we are interested in obtaining the symmetry properties of a web *before* we impose the spherical constraint, which will yield additional web symmetries for the CKTs. Visually, this corresponds to capturing all of the symmetry properties of a web before it is intersected with the hypersurface $\mathbb{S}^3$.

To achieve this, we impose condition (5.5) on each of the six CKTs using the general Killing vector of the ambient space $\mathbb{E}^4$. This will enable us to determine if a web is rotationally and/or translationally symmetric before it intersects the surface of $\mathbb{S}^3$. Upon applying this method, we find that four of the six webs admit at least one rotational web symmetry. We can go even further by noting the number of rotational

Table 5.2: Symmetry classification of the webs of $\mathbb{S}^3$ under the action of SO(4)

Category	Symmetry	Separable webs	Generators
1	2 rotations	Cylindrical	R_{12}, R_{34}
2	1 translation & 1 rotation	Spherical	R_{12}, X_4
3	1 translation	Spheroelliptic	X_4
4	1 rotation	Elliptic-cylindrical I	R_{12}
		Elliptic-cylindrical II	
5	none	Ellipsoidal	

symmetries a CKT admits, which effectively divides the six canonical forms into three categories. Lastly, we find that two of the six webs admit translational symmetry, which provides the final distinguishing feature between each of the six webs. Please refer to Table 5.2 for a summary of these results.

Remark 5.3.1. In an application problem, it is possible that a given CKT K may have the Casimir tensor present. Specifically,

$$K = \alpha\mathcal{C} + K_1,$$

where $\alpha \in \mathbb{R}$, $\mathcal{C}$ is the Casimir tensor and K_1 is a CKT. If K_1 is translationally symmetric, then the addition of the rotationally symmetric Casimir tensor destroys this translational symmetry. Thus in order to determine all of the symmetries of K with or without the presence of the Casimir tensor, it is necessary to check the more general condition

$$\mathcal{L}_V(K + \alpha\mathcal{C}) = 0,$$

for arbitrary α.

It is necessary to prove that the aforementioned symmetry properties of a Killing tensor are invariant under the action of SO(4). To do so, it suffices to solve the equivalence problem of Killing vectors of $\mathcal{K}^1(\mathbb{E}^4)$ under the action of the group SO(4). To begin, we note that the general Killing vector of $\mathbb{E}^4$ is given by

$$K = A^i X_i + B^I R_I,$$

Table 5.3: Invariant classification of Killing vectors on $\mathbb{E}^4$ under the action of SO(4)

Category	Canonical form	$\mathcal{I}_1$
1	R_{12}	$\neq 0$
2	X_1	0

where A^i and B^I denote the Killing vector parameters, and X_i and R_I are the Killing vector fields defined previously. The action of SO(4) on K induces the following transformations

$$\tilde{A}^i = \Lambda^i{}_j A^j, \quad \tilde{B}^I = \Lambda^I{}_K B^K$$

on the Killing vector parameters. Therefore, it follows that

$$\mathcal{I}_1 = B^I B_I, \qquad \mathcal{I}_2 = A^i A_i$$

are invariants in the orbit space $\mathcal{K}^1(\mathbb{S}^3)/SO(4)$. Using either of these two invariants it is possible to distinguish between two different types of symmetry generators. See Table 5.3 for a summary of these results. We can conclude that the translational and rotational web symmetries as defined by the Killing vectors of $\mathcal{K}^1(\mathbb{E}^4)$ are inequivalent under the action of SO(4).

5.4 Transformation to Canonical Form

In addition to a classification scheme, a solution to the equivalence problem also requires establishing a method for determining the *moving frames map* which identifies the group action required to return a given CKT to the canonical form of its orbit. On the two-dimensional manifolds $\mathbb{E}^2, \mathbb{M}^2$ and $\mathbb{S}^2$, algebraic formulas have been derived [56, 57, 37] for determining the moving frame map of a given CKT. On $\mathbb{E}^3$ and $\mathbb{M}^3$, a combination of web symmetry and eigenvalues and eigenvectors of the parameter matrices has be used to determine such a map [40, 39]. In our case, however, the situation is complicated by the fact that our coefficient tensor, $C^{ijk\ell}$, has order six when regarded as a matrix. As such, we will need to devise a different strategy for determining the moving frame map of a CKT on $\mathbb{S}^3$.

It has been noted in (4.16) that the coefficient tensor $C^{ijk\ell}$ has the same symmetries as the curvature tensor, and thus can be called an *algebraic curvature tensor*. In light of this property, let us lower the last three indices of $C^{ijk\ell}$ and contract on the first and third indices

$$Ric = C^{i}{}_{ji\ell} = \mathcal{R}_{j\ell}$$

to obtain an *algebraic Ricci tensor*. The coefficient tensor for each of the six canonical forms listed in the Appendix can be contracted to define a *canonical algebraic Ricci tensor* in each case. The following proposition demonstrates that the Ricci tensor can be used to define the moving frame map for a given CKT.

Proposition 5.4.1. *A Killing tensor (4.15) is in canonical form if and only if its algebraic Ricci tensor is in canonical form.*

Proof. Since the canonical form of an algebraic Ricci tensor is defined by the canonical form of its Killing tensor K, the first direction is trivial. For the other direction, we prove by contradiction. Suppose the Ricci tensor of a Killing tensor K is in canonical form, but K is not. Since $SO(4)$ acts transitively on the orbits of $\mathcal{K}^2(\mathbb{S}^3)/SO(4)$, we can find a group action $\Lambda \in SO(4)$ which sends K to its canonical form $\tilde{K}$. In particular, the components of K transform according to (5.1) which induces the following transformation

$$\tilde{C}^{i}{}_{jk\ell} = \Lambda^{i}{}_{m}\Lambda^{n}{}_{j}\Lambda^{p}{}_{k}\Lambda^{q}{}_{\ell}C^{m}{}_{npq}$$

on the coefficient tensor $C^{i}{}_{jk\ell}$. At the same time, this action on C induces the following transformation

$$\tilde{\mathcal{R}}_{j\ell} = \Lambda^{m}{}_{j}\Lambda^{n}{}_{\ell}\mathcal{R}_{mn}$$

on its Ricci tensor $\mathcal{R}$. Since the Ricci tensor of a canonical Killing tensor is necessarily canonical, we must have $\tilde{\mathcal{R}} = \mathcal{R}$. This implies that $\mathcal{R}$ is invariant under Λ. But since a CKT and its algebraic Ricci tensor have the same symmetries, this implies $K = \tilde{K}$. This is a contradiction. $\square$

According to Proposition 5.4.1, the moving frame map of a CKT can be constructed by determining the moving frame map of the corresponding algebraic Ricci tensor. Note that each canonical algebraic Ricci tensor can be represented by a diagonal matrix of order four. Therefore, the determination of the moving frame map for the algebraic Ricci tensor is an eigenvalue-eigenvector problem for matrices of order four. Before we illustrate this technique with the application in the next section, we summarize our results in the following algorithm.

1. *Compatibility condition.* Begin by substituting the potential into the compatibility condition (4.48) to determine the most general Killing tensor compatible with the potential. Using this Killing tensor, determine the subspace of CKTs.

2. *Classification.* Next, we classify a CKT K by determining whether it admits any symmetry. Namely, impose the constraint

   $$\mathcal{L}_V(K + \alpha\mathcal{C}) = 0,$$

   where $\alpha \in \mathbb{R}$, $\mathcal{C}$ is the Casimir tensor, and V is the general Killing vector of $\mathbb{E}^4$,

   $$V = A^i X_i + B^I R_I.$$

   If K does admit symmetry, determine which type and the number of generators for each type. Consult Table 5.2 to classify the CKT.

3. *Moving frame map.* To determine the moving frame map for K, find the algebraic Ricci tensor $\mathcal{R}$ of the coefficient tensor. Diagonalize $\mathcal{R}$ by solving the corresponding eigenvalue-eigenvector problem. The matrix Λ, which diagonalizes $\mathcal{R}$ defines the moving frame map.

4. *Orthogonally separable coordinates.* Finally, define the orthogonally separable set of coordinates corresponding to K by substituting Λ found in the previous step into the equation

   $$x^i = \Lambda^i{}_j T^j(u^k),$$

   where $x^j = T^j(u^k)$ denote the canonical orthogonally separable coordinates corresponding to K.

CHAPTER 6

THE EQUIVALENCE PROBLEM FOR ORTHOGONAL WEBS ON $\mathbb{H}^3$

In this chapter, we present a solution to the equivalence problem of orthogonal webs defined on $\mathbb{H}^3$. As in the last chapter, we will begin by solving the equivalence problem on the lower dimensional space of $\mathbb{H}^2$. This will give us some insight on how to solve the problem for $\mathbb{H}^3$. Then, proceeding as we did in the last chapter, we will begin with the canonical forms problem for orthogonal webs on $\mathbb{H}^3$. After defining a canonical CKT for each orthogonally separable coordinate system, we will develop a classification scheme for the webs using web symmetries, invariants and covariants. While these methods will be similar to those used on $\mathbb{H}^2$, the scheme will be considerably more complicated. Finally, we will determine a way to obtain the transformation to canonical form for an arbitrary CKT of $\mathbb{H}^3$. This will require the use of both web symmetries and the algebraic Ricci tensor.

6.1 Introductory Example: The Equivalence Problem for Orthogonal Webs on $\mathbb{H}^2$

In this section we present a solution to the equivalence problem of orthogonal webs on $\mathbb{H}^2$.[1] We begin by defining an equivalence criterion for Killing tensors $K \in \mathcal{K}^2(\mathbb{H}^n)$.

6.1.1 Equivalence Criterion

The isometry group of $\mathbb{H}^n$ is the special Lorentz group $\mathrm{SO}(n, 1)$, which is a Lie subgroup of the Lorentz group $\mathrm{O}(n, 1)$. It consists of orthogonal matrices Λ with positive

[1] A solution to this equivalence problem can be found in the solution to the equivalence problem on $\mathbb{M}^3$ [39], as it falls out as a subcase of the dilatationally symmetric cases. The approach that we take will be different, and the resulting number of inequivalent cases will differ. An explanation of this discrepancy will be discussed in this section. Moreover, we will discuss the meaning of the combinations of invariants which distinguish between the webs.

unit determinant. Its action on a point $x^i \in \mathbb{M}^{n+1}$ is defined by

$$\tilde{x}^i = \Lambda^i{}_j x^j,$$

where $\Lambda \in \mathrm{SO}(n,1)$ and x^i are pseudo-Cartesian coordinates. For a Killing vector (4.7) or Killing tensor (4.15), this induces the following transformation

$$R_{ij} = \Lambda^k{}_i \Lambda^\ell{}_j \tilde{R}_{k\ell}, \quad K^{ij} = \Lambda^i{}_k \Lambda^j{}_\ell \tilde{K}^{k\ell} \tag{6.1}$$

on their respective components, which in turn, induces the following transformation

$$\tilde{B}^{ij} = \Lambda^i{}_k \Lambda^j{}_\ell B^{k\ell}, \quad \tilde{C}^{ijk\ell} = \Lambda^i{}_p \Lambda^j{}_q \Lambda^k{}_r \Lambda^\ell{}_s C^{pqrs} \tag{6.2}$$

on their respective parameters. If we assume the more compact notation $R_{ij} = R_I$, then the first equation of (6.1) may be rewritten as

$$R_I = \Lambda^J{}_I \tilde{R}_J, \tag{6.3}$$

where $\Lambda^I{}_K = \Lambda^i{}_{[k} \Lambda^j{}_{\ell]}$ represents the second compound of $\Lambda^i{}_j$.[2] Also in this new notation, (6.2) becomes

$$\tilde{B}^I = \Lambda^I{}_J B^J, \quad \tilde{C}^{IJ} = \Lambda^I{}_K \Lambda^J{}_L C^{KL}. \tag{6.4}$$

The action of $\mathrm{SO}(n,1)$ on the vector space $\mathcal{K}^2(\mathbb{H}^n)$ foliates $\mathcal{K}^2(\mathbb{H}^n)$ into the orbit space $\mathcal{K}^2(\mathbb{H}^n)/\mathrm{SO}(n,1)$, and each orbit of $\mathcal{K}^2(\mathbb{H}^n)/\mathrm{SO}(n,1)$ is represented by a canonical form. Killing tensors belonging to the same orbit are connected by a transformation $\Lambda \in \mathrm{SO}(n,1)$ and are called *equivalent*. A solution to the equivalence problem for Killing tensors of $\mathcal{K}^2(\mathbb{H}^n)$ enables one to determine whether two Killing tensors are equivalent, and if so, the transformation $\Lambda \in \mathrm{SO}(n,1)$ which connects them along their orbit.

6.1.2 Canonical Forms

In 1950 [62], Olevskii proved there are nine inequivalent[3] metrics which permit orthogonal separation of variables of the geodesic Laplace-Beltrami equation on the

[2] Please see Appendix A for a discussion of compound matrices.

[3] Here we are using the definition of equivalence given in Definition 1 of [39].

two-dimensional hyperboloid.[4] By Robertson's theorem of 1927 [68], these equivalently define the orthogonally separable metrics of the geodesic Hamilton-Jacobi equation. After deriving the metrics corresponding to the spherical, equidistant and horicylic coordinate systems, he was left with the following family of metrics,

$$ds^2 = -\frac{1}{4k}(u-v)\left(\frac{du^2}{P(u)} - \frac{dv^2}{P(v)}\right), \quad P(w) = (w-a)(w-b)(w-c). \tag{6.5}$$

By specifying ranges on the separable coordinates u and v so that the resulting metric admitted the correct signature, he determined six additional distinct metrics and their associated separable coordinate systems. In terms of the cubic polynomial $P(w)$, the elliptic-parabolic and hyperbolic-parabolic coordinate systems correspond to the case when $P(w)$ admits a double root; the semihyperbolic coordinate system corresponds to the case when $P(w)$ admits a pair of complex conjugate roots; the semicircular parabolic coordinate system corresponds to the case when $P(w)$ admits a triple root; and finally the elliptic and hyperbolic coordinate systems correspond to the case when $P(w)$ admits real and distinct roots.

In 1996, Grosche *et al* [30] published the first of two papers concerning superintegrability on two and three-dimensional hyperbolic space. As part of their work, they characterized each system of orthogonal coordinates using second-order operators, which in our language, are Killing tensors. Using these operators, we can define a canonical form for each of the orthogonally separable coordinate systems of $\mathbb{H}^2$ (see Appendix C.3). In 2009, Horwood [37] solved the canonical forms and equivalence problem for $\mathbb{M}^3$. In his solution to the equivalence problem, he used the symmetry of the orthogonal coordinate webs in his classification scheme. The 58 webs were first classified as either timelike rotational, spacelike rotational, null rotational, dilatational or asymmetric; to distinguish between the webs in each category, he determined a set of "reduced" invariants on each of the symmetry subspaces of $\mathcal{K}^2(\mathbb{M}^3)$.

The separable webs of $\mathbb{M}^3$ admitting dilatational symmetry define a separable web on $\mathbb{H}^2$ when subject to the constraint $t^2 - x^2 - y^2 = 1$. Thus, as Horwood points out, solving the canonical forms and equivalence problem for $\mathbb{M}^3$ simultaneously solves

[4] These metrics were later independently reproduced in [11], and partially reproduced in [39]. In the latter case, the authors derived only seven inequivalent metrics. This discrepancy can be explained by noting that the authors in both [62] and [11] considered additional ranges on the separable coordinates.

these problems on the subspace $\mathbb{H}^2$. The webs admitting rotational and dilatational symmetry, as given by web I in Sections B.2.3, B.2.4 and B.2.5 of [37], define the symmetric canonical forms of $\mathbb{H}^2$, and the webs given by I, II, IV, V in Section B.2.6 of [37] define the asymmetric canonical forms of $\mathbb{H}^2$. The discrepancy between the number of coordinate systems and canonical forms is a result of defining *global* Killing tensors, namely CKTs which are defined on the entire manifold. As such, the elliptic and hyperbolic coordinate systems are characterized by a single globally defined CKT; the associated orthogonal coordinate web consists of either ellipses and convex hyperbolas or mutually orthogonal confocal hyperbolas depending upon the location on $\mathbb{H}^2$. Much like the elliptic-cylindrical webs of $\mathbb{S}^3$, the type of coordinate system can be determined by the value of the essential parameter.

There is, however, one discrepancy between the work done by Grosche *et al* and Horwood. Namely, in the former work the authors list eight distinct canonical forms, while in the latter the author lists only seven. According to Table II in [39], dilatational web III characterizes a family of coordinates on two-dimensional de Sitter space, dS_2. We claim that this web is also defined on $\mathbb{H}^2$, and thus is the missing asymmetric canonical form. To explain this discrepancy, let us consider the derivation of metric families 10 and 11 in [38].

The metric

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + \frac{u^2(v-w)}{4v^2(v-1)}\mathrm{d}v^2 + \frac{u^2(v-w)}{4w^2(1-w)}\mathrm{d}w^2 \tag{6.6}$$

from metric family 10 in [38] admits the correct signature on the parameter ranges $0 < w < 1 < v$ *and* $w < 0 < 1 < v$.[5] If we would like (6.6) to be of the form

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + u^2(A(v) + B(w))(\mathrm{d}v^2 + \mathrm{d}w^2), \tag{6.7}$$

it is necessary to perform a rescaling in each case.

For the first range, let us rescale $(v, w) \to (\csc^2 v, \mathrm{sech}^2 w)$ so that (6.6) becomes

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + u^2(\csc^2 v - \mathrm{sech}^2 w)(\mathrm{d}v^2 + \mathrm{d}w^2)$$

[5]In [38], only the first parameter range is considered. If the second parameter range is also considered, a second coordinate system can be defined. The CKT for each coordinate system are Webs II and III respectively in Appendix A.6 of [39].

for $0 < \operatorname{sech}^2 w < 1 < \csc^2 v$. For the second range, let us rescale $(v, w) \to (\csc^2 v, -\operatorname{csch}^2 w)$ so that (6.6) becomes

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + u^2(\csc^2 v + \operatorname{csch}^2 w)(\mathrm{d}v^2 + \mathrm{d}w^2)$$

for $-\operatorname{csch}^2 w < 0 < 1 < \csc^2 v$. The former metric is equivalent to metric family (F10.2) of [38], and generates Web II in Appendix A.6 of [39]; the latter metric generates Web III in Appendix A.6 of [39].

Similarly, the metric for case II. of metric family 11 in [38] can be rescaled into two inequivalent metrics: F11.2 for $1 < \csc^2 v < \csc^2 u$, and

$$\mathrm{d}s^2 = w^2(\operatorname{csch}^2 u + \operatorname{sech}^2 v)(-\mathrm{d}u^2 + \mathrm{d}v^2) + \mathrm{d}w^2$$

for $\operatorname{csch}^2 u < 0 < \operatorname{sech}^2 v < 1$. The former metric generates Web III in Appendix A.6 of [39], while the latter metric generates Web II in Appendix A.6 of [39]. If we set $u = r =$ constant in any metric of the form (6.7), we obtain a metric on $\mathbb{H}^2$. Similarly, if we set $w = r =$ constant in any metric of the form

$$\mathrm{d}s^2 = w^2(A(u) + B(v))(-\mathrm{d}u^2 + \mathrm{d}v^2) + \mathrm{d}w^2,$$

we obtain a metric on dS_2. Therefore, if we compute the number of inequivalent metrics of these forms in $\mathbb{M}^3$, we find there are now *eight* distinct separable metrics on both $\mathbb{H}^2$ and dS_2. Using invariants, covariants and web symmetries, it is possible to prove that the eight canonical forms of $\mathbb{H}^2$ are indeed distinct. In what follows, we will prove this claim by establishing a classification scheme for CKTs of $\mathbb{H}^2$. Furthermore, we will provide insight into the set of reduced invariants used by Horwood which enabled him to distinguish between the asymmetric webs.

6.1.3 Classification

As we found for spherical space, the method of web symmetries is an effective first step in distinguishing between the webs of hyperbolic space. To determine the symmetries of the webs, we impose the following condition

$$\mathcal{L}_V K = 0$$

Table 6.1: Symmetry classification of the webs of $\mathbb{H}^2$ under the action of $SO(2,1)$

Category	Symmetries	Separable web(s)	Generators
1	1 spacelike rotation, 1 timelike translation	Spherical	R_{23}, X_1
2	1 timelike rotation, 1 spacelike translation	Equidistant	R_{12}, X_3
3	1 null rotation, 1 null translation	Horicyclic	$X_1 + X_2, R_{23} - R_{13}$
4	none	Elliptic Hyperbolic Semihyperbolic Elliptic-parabolic Hyperbolic-parabolic Semicircular-parabolic	

on each of the canonical CKTs, using the general Killing vector of $\mathbb{M}^3$ as V. This is similar to what we did in spherical space, where we determine if a web is translationally symmetric before it is intersected with the hypersurface. As shown in Table 6.1, we find that three of the webs admit symmetry, while those webs corresponding to metric family 6.5 are asymmetric.

Symmetric webs

The process of classifying the symmetric webs is simplified by the fact that they are distinguishable based on the type of symmetry they admit. Thus, it is necessary to determine the type of web symmetry from its generator in order to classify the web. Unlike those of spherical space, the rotational symmetry generators are not equivalent up to the action of the isometry group of the manifold. Indeed, suppose

$$K = B^{ij} R_{ij},$$

represents the most general rotational generator, where R_{ij} is a Killing vector of $\mathbb{H}^2$ and $B^{ij} = B^I = (b_1, b_2, b_3)$ is the coefficient vector. These generators span the vector space $\mathcal{K}^1(\mathbb{H}^2)$, which, under the action of the special Lorentz group $SO(2,1)$, admits the single invariant

$$\mathcal{I} = b_3^2 - b_2^2 - b_1^2.$$

Table 6.2: Invariant classification of Killing vectors of $\mathbb{M}^3$ under the action of $SO(3,1)$

Category	Type of symmetry	$\mathcal{I}$	Canonical symmetry generator
1	Spacelike rotational	> 0	R_{23}
2	Timelike rotational	< 0	R_{12}
3	Null rotational	0	$R_{23} - R_{13}$
4	Spacelike translational	> 0	X_3
5	Timelike translational	< 0	X_1
6	Null translational	0	$X_1 + X_3$

Using this invariant, we are able to discern three distinct types of rotational symmetry: timelike, spacelike and null.

Like the webs of spherical space, the webs of $\mathbb{H}^2$ defined in pseudo-Cartesian coordinates may admit translational symmetry. The most general translational generator may be expressed as

$$K = A^i X_i,$$

where X_i is a Killing vector of $\mathbb{M}^3$ and $A^i = (a_1, a_2, a_3)$ is the coefficient vector. These generators span a vector subspace of $K^1(\mathbb{M}^3)$, which, under the action of the group $SO(2,1)$, admits the single invariant

$$\mathcal{I} = a_2^2 + a_3^2 - a_1^2.$$

Using this invariant, we are able to discern three distinct types of translational symmetry: timelike, spacelike or null. Using the invariant classification scheme for rotational and translational generators as presented in Table 6.2, we are able to classify the symmetric webs of $\mathbb{H}^2$.

Asymmetric webs

In order to distinguish between the asymmetric webs, we first determine a set of fundamental invariants of the isometry group action on the vector space $\mathcal{K}^2(\mathbb{H}^2)$. Such a set has already been derived in Example 4.3.7, which we now restate for convenience:

$$\mathcal{C}_1 = C^I_I, \quad \mathcal{C}_2 = C^I_J C^J_I, \quad \mathcal{C}_3 = C^I_J C^J_K C^K_I. \tag{6.8}$$

Having determined a complete set of invariants, we must now proceed with the difficult task of finding particular linear combinations of these invariants which distinguish between the asymmetric webs. In the classification scheme of [37], Horwood used these fundamental invariants (6.8) and the following set of covariants of the group action

$$\mathcal{C}_4 = C_{ij}x^i x^j, \quad \mathcal{C}_5 = C_{ik}C^k{}_j x^i x^j, \quad \mathcal{C}_6 = g_{ij}x^i x^j,$$

to define a set of four auxiliary invariants and covariants

$$\begin{aligned}
\mathcal{A}_1 &= \mathcal{C}_1^2 - 3\mathcal{C}_2, \\
\mathcal{A}_2 &= \mathcal{C}_1^2 - 9\mathcal{C}_3, \\
\mathcal{A}_3 &= \mathcal{C}_1^6 - 9\mathcal{C}_1^4\mathcal{C}_2 + 8\mathcal{C}_1^3\mathcal{C}_3 + 21\mathcal{C}_1^2\mathcal{C}_2^2 - 36\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 - 3\mathcal{C}_2^3 + 18\mathcal{C}_3^2, \\
\mathcal{A}_4 &= (\mathcal{C}_1\mathcal{C}_2 - 3\mathcal{C}_3)\mathcal{C}_4 - (\mathcal{C}_1^2 - 3\mathcal{C}_2)\mathcal{C}_5 + (\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2)\mathcal{C}_6,
\end{aligned}$$

to invariantly classify the asymmetric webs of $\mathbb{H}^2$. Finding these exotic combinations of invariants is a difficult task, and until now, arose only tediously through trial and error.

In order to understand how these particular combinations can be determined, let us begin by defining the characteristic polynomial of the coefficient matrix of the general Killing tensor,

$$P(\lambda) = |C^{ij} - \lambda g^{ij}| = 0.$$

In its most general form, P is given by

$$P(\lambda) = \lambda^3 + b\lambda^2 + c\lambda + d, \tag{6.9}$$

a cubic homogeneous polynomial of the vector space $\mathcal{P}(\mathbb{R})$. The coefficients of this polynomial are invariants of the isometry group action on the vector space $\mathcal{K}^2(\mathbb{H}^2)$, and thus can be expressed in terms of the fundamental invariants (6.8),

$$b = -\mathcal{C}_1, \quad c = \frac{1}{2}(\mathcal{C}_1^2 - \mathcal{C}_2), \quad d = \frac{1}{6}(3\mathcal{C}_2\mathcal{C}_1 - 2\mathcal{C}_3 - \mathcal{C}_1^3). \tag{6.10}$$

A cubic polynomial (6.9) under the action of the isometry group $SE(1)$ admits the following set of fundamental invariants

$$\mathcal{I}_1 = b^2 - 3c, \quad \mathcal{I}_2 = 2b^3 - 9bc + 27d.$$

Table 6.3: Invariant classification of cubic polynomials under the action of $SE(1)$

Category	Canonical form	Nature of the roots	$\mathcal{I}_2$	$\mathcal{I}_3$
1	x^3	Triple root	$= 0$	$= 0$
2	$x^3 - x^2$	Double root	$\neq 0$	$= 0$
3	$x^3 + x$	Pair of complex conjugate roots		< 0
4	$x^3 - x$	Distinct real roots		> 0

The nature of the roots of this cubic polynomial can be determined by the value of $\mathcal{I}_3$ and the discriminant

$$\mathcal{I}_3 = \frac{4\mathcal{I}_1^3 - \mathcal{I}_2^2}{27\mathcal{I}_1} = 18bcd - 4b^3d + b^2c^2 - 4c^3 - 27d^2,$$

as shown in Table 6.3.

The invariants $\mathcal{I}_2$ and $\mathcal{I}_3$ are important to our work because they are precisely what is needed to distinguish between the asymmetric webs of $\mathbb{H}^2$. To explain this fact, recall that the asymmetric webs belong to the metric family (6.5), and coordinate systems of this family are distinguished by specifying ranges on the separable coordinates u, v so that the metric admits the correct signature. At the beginning of this section we noted how each coordinate system corresponded to a particular type of cubic polynomial $P(w)$, identified by the nature of its roots. This connection between the coordinate system and the type of polynomial $P(w)$ manifests itself in the characteristic polynomial of the coefficient tensor of the corresponding CKT. For example, the semihyperbolic coordinate system arises by specifying the following values

$$v < a < u, \quad b = \gamma + \delta i, \quad c = \gamma - \delta i$$

on the separable coordinates u, v. Thus we can characterize this coordinate system by the canonical cubic polynomial $P(w) = w^3 + w^2$, where $\mathcal{I}_3 < 0$. If we determine the characteristic polynomial of the canonical CKT corresponding to the semihyperbolic coordinate system, we find that its discriminant is also strictly negative. This is not a coincidence; in fact, in each case we find that the characteristic polynomial of the CKT is equivalent to the polynomial $P(w)$. In light of this observation, we now have a way of determining a set of invariants which distinguish between the separable asymmetric webs of $\mathbb{H}^2$.

Using the invariants $\mathcal{I}_2$ and $\mathcal{I}_3$ we are able to distinguish between almost all of the asymmetric webs (please refer to Table 6.4). The elliptic and hyperbolic coordinate systems, which are characterized by the same CKT, can be distinguished by the value of the essential parameter

$$k = \frac{c_1 + c_3}{c_3 - c_2} \begin{cases} > 0 & \text{elliptic,} \\ < 0 & \text{hyperbolic.} \end{cases}$$

Note that when $c_3 = c_2$, the web becomes symmetric and characterizes equidistant coordinates. Recall that the elliptic-parabolic and hyperbolic-parabolic coordinate systems each correspond to the case when $P(w)$ admits a double root, and the roots satisfy $b = c < a$. If we require that the roots of the characteristic polynomial $P(\lambda)$ of the associated CKT in each case are also ordered in this way, we find that the parameter γ_3 distinguishes between the two webs:

$$\gamma_3 \begin{cases} < 0 & \text{elliptic-parabolic,} \\ > 0 & \text{hyperbolic-parabolic.} \end{cases}$$

In each of the above cases, it is necessary to first transform the CKT to its canonical form before using k or γ_3 to determine which coordinate system the CKT characterizes.

Let us now compare our classification scheme with the work done by Horwood in [37]. Recall that in his classification scheme, he defines a set of auxiliary invariants and covariants $\{\mathcal{A}_1, \ldots, \mathcal{A}_4\}$. If we examine his chart given in Figure 8.3, we suspect that the invariant $\mathcal{A}_3$ is some multiple of the discriminant $\mathcal{I}_3$. Indeed, if we substitute (6.10) into $\mathcal{I}_3$, we find that $\mathcal{A}_3 = -6\mathcal{I}_3$. Thus, web IV corresponds to the elliptic and hyperbolic coordinate systems; web I to semicircular parabolic; web V to semi-hyperbolic; and finally, webs II and III to elliptic-parabolic and hyperbolic-parabolic. In Appendix B.2.6, Horwood lists web III as belonging exclusively to deSitter space, however, as we proved in Section 6.1.2, this web must necessarily belong to $\mathbb{H}^2$ as well in order to characterize hyperbolic-parabolic coordinates. The coefficient matrices of webs II and III admit equivalent characteristic polynomials, as their polynomials $P(w)$ are equivalent; however, the webs themselves are inequivalent, as demonstrated by the covariant $\mathcal{A}_4$. This observation explains why webs II and III cannot be distinguished by the invariants of the parameter space alone, rather a covariant of the

Table 6.4: Invariant classification of the asymmetric webs of $\mathbb{H}^2$ under the action of $SO(2,1)$

Category	Separable web	$\mathcal{I}_2$	$\mathcal{I}_3$	$\mathcal{A}_4$
1	Semicircular parabolic	$=0$	$=0$	
2	Elliptic-parabolic	$=0$	$\neq 0$	<0
3	Hyperbolic-parabolic	$=0$	$\neq 0$	>0
4	Elliptic		>0	
5	Hyperbolic		>0	
6	Semihyperbolic		<0	

prolonged vector space $\mathbb{K}^2(\mathbb{H}^2) \times \mathbb{H}^2$ must be constructed in order to distinguish between the two webs.

Let us conclude this section by outlining how to classify a CKT of $\mathbb{H}^2$.

1. Given a CKT K, first determine whether K admits any symmetry. Namely, impose the constraint
$$\mathcal{L}_V(K + a\mathcal{C}) = 0,$$
where $a \in \mathbb{R}$, $\mathcal{C}$ is the Casimir operator, and V is the general Killing vector of $\mathbb{M}^3$,
$$K = A^i X_i + B^{ij} R_{ij}.$$

2. If K does admit symmetry, determine the type of symmetry using Table 6.2.

3. Consult Table 6.1 to determine which symmetry category K belongs to.

4. If K is an asymmetric web, consult Table 6.4 to classify the web.

6.1.4 Transformation to Canonical Form

Having established a way of classifying the CKTs of $\mathbb{H}^2$, the next step in the solution to the equivalence problem is to outline a method for determining the transformation to canonical form for a given CKT using the isometry group of the manifold. On $\mathbb{S}^2$ and $\mathbb{S}^3$, the CKTs share the property that their coefficient matrices are diagonal, and thus the determination of the transformation to canonical form requires solving an eigenproblem. Unfortunately, some of the CKTs defined on $\mathbb{H}^2$ do not have this property.

These are the horicyclic, semihyperbolic, elliptic-parabolic, hyperbolic-parabolic and semicircular-parabolic CKTs. To determine the transformation to canonical form for these CKTs, we apply the methods derived for the dilatational webs of $\mathbb{M}^3$ in Section of 8.4.7 of [37]. Here, the author uses one of the eigenvectors of the coefficient tensor to find the transformation. Once this eigenvector is transformed to a canonical form by a transformation $\Lambda \in SO(2,1)$, one applies this transformation to the coefficient tensor:

$$\tilde{C}^{ij} = \Lambda^i_k \Lambda^j_\ell C^{k\ell}.$$

Upon comparing $\tilde{C}^{ij}$ with the canonical CKT, one can solve for any remaining unknown group parameters. Please refer to cases I, II, III and V of Section 8.4.7 of [37] for the eigenvector of each CKT, and Appendix C of [37] on how to determine the transformation to canonical form for these vectors. In order to apply these results, we reiterate that webs I, II, III and V in Appendix B.2.6 of [37] correspond to the semicircular-parabolic, elliptic-parabolic, hyperbolic-parabolic and semihyperbolic webs respectively; and web I in Appendix B.2.4 of [37] corresponds to the horicyclic web.

6.2 The Equivalence Problem for Orthogonal Webs on $\mathbb{H}^3$

In what follows we solve the equivalence problem for the orthogonally separable webs defined on $\mathbb{H}^3$. Let us begin with the canonical forms problem.

6.3 Canonical Forms

In 1950 [62], Olevskii proved there are 34 metrics which permit orthogonal separation of variables of the Laplace-Beltrami equation on $\mathbb{H}^3$. Since $\mathbb{H}^3$ is a Riemannian manifold with diagonal Ricci tensor, then by Robertson's theorem [68], this result simultaneously determines the set of metrics which permit orthogonal separation of variables of the geodesic Hamilton-Jacobi equation on $\mathbb{H}^3$. While Olevskii does not derive the CKT corresponding to each system of coordinates, these CKTs can readily be obtained by solving Eisenhart's equations (4.45) for each of the 34 metrics.

In a 1997 paper [31], Grosche *et al* continued their systematic study of superintegrable potentials in hyperbolic space by studying the $\mathbb{H}^3$ case. Included in this paper is a table listing pairs of second-order commuting operators in the pseudo-Cartesian coordinates of the ambient space $\mathbb{M}^4$ which characterize each system of coordinates. In our language, these correspond to a pair of canonical Killing tensors $\{K_1, K_2\}$ for each case. Using these pairs, we can define a set of canonical forms for the orthogonally separable coordinates of $\mathbb{H}^3$ (see Appendix C.4).[6]

6.4 Classification

Using the method of web symmetries and reduced invariants in the symmetric subspaces, we will now formulate a classification scheme for CKTs on $\mathbb{H}^3$. In comparison to the analogous problem for CKTs on $\mathbb{S}^2, \mathbb{S}^3$ and $\mathbb{H}^2$, this classification procedure is considerably more complicated. This is mainly because of the greater number of cases involved, but also because of the additional types of symmetry which can arise, and the close similarities between some of the webs.

Given the complexity of the classification scheme, we have organized it as follows. In Subsection 6.4.1 we begin by formulating a scheme for sorting any CKT of $\mathbb{H}^3$ into a symmetry category. In Subsections 6.4.2 - 6.4.5 we establish a classification procedure for each symmetry category by solving the equivalence problem of CKTs within the symmetry subspace. This step enables us to identify the CKT, thereby completing the classification process.

6.4.1 Web Symmetries

As a first step in developing a classification scheme, let us identify the types of symmetry a CKT can admit. According to Definition 4.3.12, the generators in the Lie algebra of the isometry group of the manifold are used to characterize the web symmetry of a tensor field. However, since we are working in the coordinates of the ambient space $\mathbb{M}^4$, let us instead use the Lie algebra $\mathfrak{se}(3,1)$ of the isometry group $I(\mathbb{M}^4)$. This more general definition of symmetry will allow us to capture the symmetry of

[6]Corrections have been made to some of the pairs of operators listed by Groshe *et al*. See Appendix C.4 for more details.

a CKT *before* it is constrained to $\mathbb{H}^3$, which will prove very useful in developing our classification scheme. A basis for the Lie algebra $\mathfrak{se}(3,1)$ (or equivalently, the vector space $\mathcal{K}^1(\mathbb{M}^4)$) is given by translational vector fields

$$X_i = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, 4,$$

as well as space-like, time-like and null rotational vector fields

$$R_{ij} = 2\delta_{ij}^{k\ell} g_{\ell m} x^m \frac{\partial}{\partial x^k}, \quad i, j, k, \ell = 1, \ldots, 4,$$

where g is the Minkowski metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. To determine all of the web symmetries corresponding to a CKT K, we impose the following condition

$$\mathcal{L}_V K = 0 \tag{6.11}$$

on K, where V is the general Killing tensor of $\mathcal{K}^1(\mathbb{M}^4)$.

Before we formulate a symmetry classification scheme for the CKTs of $\mathbb{H}^3$, it is necessary to demonstrate that the translational and rotational symmetry properties of a Killing tensor are invariant under the action of $SO(3,1)$. This will prove that the symmetries of a web can be used in an equivalence criterion. To accomplish this, we consider the action of $SO(3,1)$ on the vector space $\mathcal{K}^1(\mathbb{M}^4)$. The transitive action of $SO(3,1)$ on $\mathbb{M}^4$ as specified by

$$\tilde{x}^i = \Lambda^i_j x^j,$$

where $\Lambda^i_j \in SO(3,1)$ and x^i denote pseudo-Cartesian coordinates, induces the following transformation

$$\tilde{X}_j = \Lambda^i_j X_i, \quad \tilde{R}_{ij} = \Lambda^k_i \Lambda^\ell_j R_{k\ell}$$

on the Killing vectors of $\mathbb{M}^4$. If

$$K = A^i X_i + B^{ij} R_{ij} \tag{6.12}$$

represents the general Killing vector of $\mathbb{M}^4$, then this in turn, induces the following transformation

$$\tilde{A}^i = \Lambda^i_j A^j, \quad \tilde{B}^{ij} = \Lambda^i_k \Lambda^j_\ell B^{k\ell}$$

on the parameters of the Killing vectors.

Table 6.5: Invariant classification of Killing vectors of $\mathbb{M}^4$ under the action of $SO(3,1)$

Category	Symmetry	Canonical form	$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{C}_1$
1	Time-like rotation	R_{12}	< 0	0	
2	Space-like rotation	R_{23}	> 0	0	
3	Null rotation	$R_{12} - R_{24}$	0	0	$\neq 0$
4	Time-like translation	X_1	0	< 0	
5	Space-like translation	X_4	0	> 0	
6	Null translation	$X_1 + X_3$	0	0	0

Since A^i, B^{ij} and K^i transform like tensors under this group action, we can contract on their products to obtain invariants and covariants in the orbit space $\mathcal{K}^1(\mathbb{M}^4)/SO(3,1)$. This yields the functionally independent set

$$\mathcal{I}_1 = B^i_j B^j_i, \quad \mathcal{I}_2 = A^i A_i, \quad \mathcal{C}_1 = K^i K_i,$$

which we use to invariantly classify the Killing vectors of $\mathbb{M}^4$ (see Table 6.5). This classification demonstrates that a symmetric web will maintain its type of symmetry under the action of $SO(3,1)$.

Now let us return to the central topic of this section, namely the symmetry classification of the separable webs of $\mathbb{H}^3$. Applying condition (6.11) to each canonical CKT of $\mathbb{H}^3$, we can determine the symmetry generators for each web. The results of this calculation effectively sorts each of the CKTs into a symmetry category, as shown in Table 6.6. For a CKT belonging to Categories 1, 4 and 7 - 13, the classification is complete; for the remaining categories, an invariant classification scheme needs to be developed to classify the CKTs within the category. In what follows we present such a scheme for each of these categories.

6.4.2 Category 2

There are four distinct CKTs belonging to this symmetry category, thus it is necessary to solve the equivalence problem on this symmetry subspace. Let us begin by determining the most general CKT of this subspace. Applying the symmetry condition

$$\mathcal{L}_{R_{23}} K = 0,$$

Table 6.6: Symmetry classification of webs of $\mathbb{H}^3$ under the action of $SO(3,1)$

Category	Symmetry	Separable web(s)	Generators
1	Time-like rotation	XIX, XX	R_{12}
2	Space-like rotation	XVII, XVIII, XXI, XXII, XXV, XXVI	R_{23}
3	Null rotation	XXIII, XXIV, XXVII	$R_{12} - R_{24}$
4	Time-like translation	III	X_1
5	Space-like translation	IV, V, VI, VII VIII, IX	X_4
6	Null translation	XV, XVI	$X_1 + X_4$
7	2 null rotations & null translation	II	$R_{12} - R_{24},\ R_{13} - R_{34},$ $X_1 + X_4$
8	Time-like rotation & space-like rotation	I	R_{12}, R_{23}, R_{13}
9	Time-like translation & space-like rotation	X	X_1, R_{23}
10	Null translation & space-like rotation	XIV	$X_1 + X_4, R_{23}$
11	Space-like translation & Space-like rotation	XI	$X_4,\ R_{23}$
12	Space-like translation & null rotation	XIII	$X_4, R_{13} - R_{23}$
13	Time-like rotation & space-like translation	XII	R_{12}, X_4
14	Asymmetric	XXVIII, XXIX, XXXI, XXXII, XXXIII, XXXIV	none

where K is the general Killing tensor of $\mathcal{K}^2(\mathbb{H}^3)$, and requiring that the Haantjes tensor of the resulting tensor field vanish, yields a Killing tensor with the following components

$$\begin{aligned}
K^{11} &= a_4x^2 + a_4y^2 + a_6z^2, \\
K^{22} &= a_1y^2 + a_2z^2 + a_4t^2 - 2\,b_8tz, \\
K^{33} &= a_1x^2 + a_2z^2 + a_4t^2 - 2\,b_8tz, \\
K^{44} &= a_2y^2 + a_2x^2 + a_6t^2, \\
K^{12} &= a_4tx - b_8xz, \\
K^{13} &= a_4ty - b_8yz, \\
K^{14} &= a_6tz + b_8y^2 + b_8x^2, \\
K^{23} &= -a_1xy, \\
K^{24} &= -a_2xz + b_8tx, \\
K^{34} &= -a_2yz + b_8ty.
\end{aligned}$$

Next, we determine the restricted group action on this symmetry subspace. The only subgroup of $SO(3,1)$ which preserves K is $\{R_{14}, R_{23}\}$, hence the restricted group action on $\mathbb{H}^3$ is defined by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \epsilon\cosh\theta & 0 & 0 & \sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\theta & 0 & 0 & \epsilon\cosh\theta \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \tag{6.13}$$

where $\theta \in \mathbb{R}$ and $\epsilon = \pm 1$. To determine a set of fundamental reduced invariants of this symmetry subspace, we employ the method of infinitesimal generators. The vector field R_{14} corresponds to the infinitesimal generator

$$U_{14} = -2b_8\frac{\partial}{\partial a_2} - 2b_8\frac{\partial}{\partial a_4} - (a_4 + a_2)\frac{\partial}{\partial b_8}$$

in the parameter space. Solving the PDE $U_{14}(f) = 0$ for the function $f = f(a_1, a_2, a_4, a_6, b_8)$ yields four fundamental invariants, namely

$$\mathcal{I}_1 = a_1, \quad \mathcal{I}_2 = a_4 - a_2, \quad \mathcal{I}_3 = a_6, \quad \mathcal{I}_4 = b_8^2 - a_2a_4.$$

Upon defining the auxiliary invariants

$$\mathcal{A}_1 = \mathcal{I}_2^2 - 4\mathcal{I}_4, \quad \mathcal{A}_2 = \mathcal{I}_2 - 2\mathcal{I}_3,$$

Table 6.7: Invariant classification of separable webs of $\mathbb{H}^3$ in Category 2

Web	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_6$	Parameter
XVII	> 0			$0 < \delta < 1$
XVIII	> 0			$\delta > 1$
XXI	> 0			$\delta < 0$
XXII	< 0			
XXV	0	$\neq 0$	> 0	
XXVI	0	$\neq 0$	< 0	

we invariantly classify the four CKTs (see Table 6.7). First note that XVIII, XXI and XVII are characterized by the same CKT, therefore the parameter

$$\delta = \frac{a_6 + a_2}{a_4 + a_2}$$

distinguishes between these three cases. Evaluating the first auxiliary invariant for this CKT yields

$$\mathcal{A}_1 = (a_4 + a_2)^2,$$

which must be strictly positive for the Killing tensor to be characteristic. For XXII, we have

$$\mathcal{A}_1 = -4b_8^2,$$

which must be strictly negative for the Killing tensor to be characteristic. For the remaining CKTs, $\mathcal{A}_1 = 0$. To distinguish between XXV and XXVI, we define the covariant

$$\mathcal{C} = (\lambda_1 + \lambda_2 + 2(t^2 - x^2 - y^2 - z^2)\ell)^2 - (\lambda_1 - \lambda_2)^2,$$

where λ_1 and λ_2 are the conjugate pair of eigenvalues of the Killing matrix, and ℓ is the eigenvalue of multiplicity 4 or 5 of the coefficient matrix. For XXV,

$$\mathcal{C} = (a_2 + a_4)^2(t - z)^2 > 0,$$

while for XXVI,

$$\mathcal{C} = -(a_4 - a_6)^2(t - z)^2 < 0$$

which is a distinguishing property.

6.4.3 Category 3

There are three CKTs which belong to this category, thus it is necessary to solve the equivalence problem on this symmetry subspace. Let us begin by determining the most general CKT of this subspace. Applying the symmetry condition

$$\mathcal{L}_{R_{12}-R_{24}}K = 0,$$

where K is the general Killing tensor of $\mathcal{K}^2(\mathbb{H}^3)$, and requiring that the Haantjes tensor of the resulting tensor field vanish, yields a Killing tensor with the following components:

$$\begin{aligned}
K^{11} &= a_4x^2 + a_5y^2 + a_6z^2 - 2\,b_3zy,\\
K^{22} &= a_1y^2 + z^2(a_4 - 2\,a_6) + a_4t^2 + 2\,b_3yz - 2\,b_3ty - 2tz(a_4 - a_6),\\
K^{33} &= a_1x^2 + z^2(2a_1 + a_5) + a_5t^2 - 2\,tz(a_1 + a_5),\\
K^{44} &= y^2(2\,a_1 + a_5) + a_4x^2 - 2\,a_6x^2 + a_6t^2 - 2\,b_3ty,\\
K^{12} &= a_4tx - b_3xy - xz(a_4 - a_6),\\
K^{13} &= a_5ty + b_3x^2 - yz(a_1 + a_5) + b_3z^2 - b_3zt,\\
K^{14} &= a_6zt + y^2(a_1 + a_5) - b_3zy + a_4x^2 - a_6x^2 - b_3yt,\\
K^{23} &= -a_1xy - b_3xz + b_3tx,\\
K^{24} &= -xz(a_4 - 2\,a_6) - b_3xy + tx(a_4 - a_6),\\
K^{34} &= -yz(2a_1 + a_5) + b_3x^2 + ty(a_1 + a_5) + b_3tz - b_3t^2.
\end{aligned}$$

The only subgroup of $SO(3,1)$ which preserves K is $\{R_{12} - R_{24}, R_{14}\}$, hence the restricted group action on $\mathbb{H}^3$ is defined by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \epsilon\cosh\theta & 0 & 0 & \sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\theta & 0 & 0 & \epsilon\cosh\theta \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \tag{6.14}$$

where $\theta \in \mathbb{R}$ and $\epsilon = \pm 1$. To determine a set of fundamental reduced invariants of this symmetry subspace, we employ the method of infinitesimal generators. The vector field R_{14} corresponds to the infinitesimal generator

$$U_{14} = 2(a_6 - a_4)\frac{\partial}{\partial a_4} - 2(a_1 + a_5)\frac{\partial}{\partial a_5} - b_3\frac{\partial}{\partial b_3}$$

Table 6.8: Invariant classification of separable webs of $\mathbb{H}^3$ in Category 3

Web	$\mathcal{A}$	$\mathcal{C}$
XXIII	$\neq 0$	> 0
XXIV	$\neq 0$	< 0
XXVII	0	

in the parameter space. Solving the PDE $U_{14}(f) = 0$ for the function $f = f(a_1, a_4, a_5, a_6, b_3)$ yields four fundamental invariants, namely

$$\mathcal{I}_1 = a_1, \quad \mathcal{I}_2 = a_6, \quad \mathcal{I}_3 = \frac{a_1 + a_5}{a_6 - a_4}, \quad \mathcal{I}_4 = \frac{b_3}{\sqrt{a_6 - a_4}}.$$

Upon defining the auxiliary invariant

$$\mathcal{A} = \mathcal{I}_1 + \mathcal{I}_2.$$

we distinguish between some of the cases (see Table 6.8). In particular, we have

$$\mathcal{A} = \frac{1}{2}(a_2 + a_6)$$

for XXIII, and

$$\mathcal{A} = a_1 + a_6$$

for XXIV. In both cases we must have $\mathcal{A} \neq 0$ for the Killing tensor to be characteristic. To distinguish between these two cases, we define the covariant

$$\mathcal{C} = (\lambda_1 + \lambda_2 + \frac{2}{3}\left(t^2 - x^2 - y^2 - z^2\right)\ell)^2 - (\lambda_1 - \lambda_2)^2,$$

where λ_1 and λ_2 denote the conjugate pair of eigenvalues of the Killing matrix, and ℓ denotes the simple eigenvalue of the algebraic Ricci matrix. For XXIII,

$$\mathcal{C} = (a_2 + a_6)^2(t - z)^2 < 0,$$

while for XXIV,

$$\mathcal{C} = -4(a_1 + a_6)^2(t - z)^2 > 0$$

which is a distinguishing property.

6.4.4 Category 5

There are five distinct CKTs belonging to this category, thus it is necessary to solve the equivalence problem on this symmetry subspace. Let us first determine the most general CKT of this subspace. Applying the symmetry condition

$$\mathcal{L}_{X_4} K = 0,$$

where K is the general Killing tensor of $\mathcal{K}^2(\mathbb{H}^3)$ yields a Killing tensor K with the following components

$$\begin{aligned}
K^{11} &= 2\,b_{13}yx + a_4x^2 + a_5y^2,\\
K^{22} &= a_1y^2 + a_4t^2 - 2\,b_3yt,\\
K^{33} &= a_1x^2 + a_5t^2 + 2\,b_4xt,\\
K^{44} &= 0,\\
K^{12} &= b_{13}yt + a_4xt - b_3yx - b_4y^2,\\
K^{13} &= b_{13}xt + a_5ty + b_3x^2 + b_4yx,\\
K^{14} &= 0,\\
K^{23} &= b_{13}t^2 - a_1xy + b_3xt - b_4yt,\\
K^{24} &= 0,\\
K^{34} &= 0.
\end{aligned}$$

Since the Haanjtes tensor for K vanishes, we find that K represents the most general CKT of this subspace. Upon observing that the non-zero components of K are the elements of the general Killing tensor of $\mathbb{H}^2$, we conclude that the action of $SO(3,1)$ on this symmetry subspace is isomorphic to the action of $SO(2,1)$ on the vector space $\mathcal{K}^2(\mathbb{H}^2)$. Given this fact, it is possible to apply the classification method for the CKTs of $\mathbb{H}^2$ given in Section 6.1.3. In order to apply these results, we first note that $(c_1, c_2, c_3, \gamma_1, \gamma_2, \gamma_3) = (a_1, a_5, a_4, -b_{13}, b_3, -b_4)$ is the correspondence between the parameters of a CKT of $\mathbb{H}^2$ and a CKT of this category. Furthermore, the elliptic/hyperbolic, semihyperbolic, elliptic-parabolic, hyperbolic-parabolic and semi-circular parabolic webs of $\mathbb{H}^2$ correspond to webs IV/V, VI, VII, VIII, IX of this symmetry subspace. Using invariants $\mathcal{I}_2$ and $\mathcal{I}_3$ and covariant $\mathcal{A}_4$ listed in Subsection 6.1.3 and this correspondence, we can readily obtain a classification for the CKTs

Table 6.9: Invariant classification of the separable webs of $\mathbb{H}^3$ in Category 5

Web	$\mathcal{I}_2$	$\mathcal{I}_3$	$\mathcal{A}_4$	Parameter
IX	0	0		
VII	0	$\neq 0$	< 0	
VIII	0	$\neq 0$	> 0	
IV		> 0		$\delta < 0$
V		> 0		$\delta \geq 0$
VI		< 0		

of this category (see Table 6.9). Since IV and V have identical CKTs, we use the parameter

$$\delta = \frac{a_1 + a_5}{a_5 - a_4}$$

to distinguish between these two cases.

6.4.5 Category 6

There are two CKTs belonging to this symmetry category, thus it is necessary to solve the equivalence problem on this symmetry subspace. Let us begin by determining the most general CKT of this subspace. Applying the symmetry condition

$$\mathcal{L}_{X_1+X_4} K = 0,$$

where K is the general Killing tensor of $\mathcal{K}^2(\mathbb{H}^3)$, yields a Killing tensor K with components

$$\begin{aligned}
K^{11} &= 2\,xy(b_5 - b_{11}) + b_8 y^2 - b_{10} x^2, \\
K^{22} &= a_1 y^2 - b_{10} z^2 - b_{10} t^2 + 2\,b_3 yz - 2\,b_3 ty + 2\,b_{10} tz, \\
K^{33} &= a_1 x^2 + b_8 (z^2 + t^2) - 2\,b_1 xz + 2\,b_1 tx - 2\,b_8 tz, \\
K^{44} &= 2\,xy(b_5 - b_{11}) + b_8 y^2 - b_{10} x^2, \\
K^{12} &= ty(b_5 - b_{11}) - b_{10} tx - b_3 xy - b_1 y^2 - yz(b_5 - b_{11}) + b_{10} xz, \\
K^{13} &= tx(b_5 - b_{11}) + b_8 ty + b_3 x^2 + b_1 xy - xz(b_5 - b_{11}) - b_8 yz, \\
K^{14} &= 2\,xy(b_5 - b_{11}) + b_8 y^2 - b_{10} x^2, \\
K^{23} &= (b_5 - b_{11})(t - z)^2 - a_1 xy + b_1 yz - b_3 xz + b_3 tx - b_1 ty, \\
K^{24} &= ty(b_5 - b_{11}) - b_{10} tx - b_3 xy - b_1 y^2 - yz(b_5 - b_{11}) + b_{10} xz, \\
K^{34} &= tx(b_5 - b_{11}) + b_8 ty + b_3 x^2 + b_1 xy - xz(b_5 - b_{11}) - b_8 yz.
\end{aligned} \tag{6.15}$$

Table 6.10: Invariant classification of the separable webs of $\mathbb{H}^3$ in Category 6

Web	$\mathcal{I}$
XV	$\neq 0$
XVI	0

Since K satisfies the Haantjes condition (4.40), K represents the most general CKT of this symmetry subspace. Next, let us determine the restricted group action on this subspace. The only subgroup of $SO(3,1)$ which preserves K is $\{R_{23}, R_{14}\}$, hence the restricted group action on $\mathbb{H}^3$ is defined by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \epsilon\cosh\theta & 0 & 0 & \sinh\theta \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ \sinh\theta & 0 & 0 & \epsilon\cosh\theta \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \tag{6.16}$$

where $\theta, \phi \in \mathbb{R}$ and $\epsilon = \pm 1$. To determine the invariants of this symmetry subspace, we employ the method of infinitesimal generators. The vector fields R_{23} and R_{14} respectively correspond to

$$\begin{aligned} U_{23} &= -b_1\frac{\partial}{\partial b_1} - b_3\frac{\partial}{\partial b_3} - 2b_5\frac{\partial}{\partial b_5} - 2b_8\frac{\partial}{\partial b_8} - 2b_{10}\frac{\partial}{\partial b_{10}} - 2b_{11}\frac{\partial}{\partial b_{11}}, \\ U_{14} &= -b_3\frac{\partial}{\partial b_1} + b_1\frac{\partial}{\partial b_3} + b_8\frac{\partial}{\partial b_5} + 2(b_{11} - b_5)\frac{\partial}{\partial b_8} + 2(b_{11} - b_5)\frac{\partial}{\partial b_{10}} - 2b_{10}\frac{\partial}{\partial b_{11}} \end{aligned}$$

in the parameter space. By observation, it is clear that

$$f(a_1, b_1, b_3, b_5, b_8, b_{10}, b_{11}) = a_1$$

is a solution to the system of PDEs $U_{23}(f) = 0$ and $U_{14}(f) = 0$, and therefore represents an invariant of this subspace. In fact, $\mathcal{I} = a_1$ is the only invariant needed to classify the CKTs of this category (see Table 6.10). In particular, $\mathcal{I} = 0$ for XV, while $\mathcal{I} = a_1 \neq 0$ must hold for the KT of XV to be characteristic.

6.4.6 Category 14

The final and most difficult step in the classification of the webs of $\mathbb{H}^3$ is to solve the equivalence problem for the asymmetric CKTs $K_{XXVIII}, \ldots, K_{XXXIV}$. In solving this problem, we consider the following degenerate forms of these CKTs:

$$
\begin{aligned}
&K^1_{XXIX} = K_{XXIX}|_{a_3=-a_4,\ a_6=a_4}\,, \quad K^2_{XXIX} = K_{XXIX}|_{a_4=-a_1,\ a_5=-a_1}\,,\\
&K^3_{XXIX} = K_{XXIX}|_{a_2=-a_4}\,, \quad K^4_{XXIX} = K_{XXIX}|_{a_6=-a_2,\ a_5=-a_2}\,,\\
&K^5_{XXIX} = K_{XXIX}|_{a_3=-a_5}\,, \quad K^6_{XXIX} = K_{XXIX}|_{a_1=-a_6}\,,\\
&K^7_{XXIX} = K_{XXIX}|_{a_3=a_2,\ a_1=a_2}\,, \quad K^{1,1}_{XXIX} = K^1_{XXIX}|_{a_1=-a_2-2a_4},\\
&K^{3,1}_{XXIX} = K^3_{XXIX}|_{a_1=-a_6}\,, \quad K^{6,1}_{XXIX} = K^6_{XXIX}|_{a_2=-a_4},\\
&K^{6,2}_{XXIX} = K^6_{XXIX}|_{a_2=-(a_3+a_4+a_5)}\,, \quad K^{6,4}_{XXIX} = K^6_{XXIX}|_{a_2=a_3-a_4+a_5}\,,\\
&K^{7,1}_{XXIX} = K^7_{XXIX}|_{a_4=-a_5-2a_2}\,, \quad K^{7,2}_{XXIX} = K^7_{XXIX}|_{a_4=-a_6-2a_2}\,,\\
&K^1_{XXX} = K_{XXX}|_{b_4=0,\ a_1=-a_4}\,, \quad K^2_{XXX} = K_{XXX}|_{b_{12}=0,\ a_3=-a_4}\,,\\
&K^3_{XXX} = K_{XXX}|_{a_2=-a_4}\,, \quad K^4_{XXX} = K_{XXX}|_{a_1=a_3,\ b_4=b_{12}}\,,\\
&K^{2,1}_{XXX} = K^2_{XXX}|_{b_4=0,\ a_1=-a_4}\,, \quad K^{3,1}_{XXX} = K^3_{XXX}|_{a_1=a_3}\,,\\
&K^1_{XXXI} = K_{XXXI}|_{a_3=a_1+a_6-a_4}, \quad K^2_{XXXI} = K_{XXXI}|_{a_2=-a_4},\\
&K^3_{XXXI} = K_{XXXI}|_{a_3=-(a_1+a_4+a_6)}, \quad K^4_{XXXI} = K_{XXXI}|_{a_6=3a_4-a_1+a_3+2a_2},\\
&K^5_{XXXI} = K_{XXXXI}|_{a_6=a_3-a_4-2a_2-a_1}, \quad K^1_{XXXIII} = K_{XXXIII}|_{a_5=a_1+a_6-a_3},\\
&K^2_{XXXIII} = K_{XXXIII}|_{a_2=a_3}, \quad K^3_{XXXIII} = K_{XXXIII}|_{a_1=-(a_3+a_5+a_6)},\\
&K^4_{XXXIII} = K_{XXXIII}|_{a_1=2a_2-a_3+a_5-a_6},\\
&K^5_{XXXIII} = K_{XXXIII}|_{a_1=3a_3+a_5-2a_2-a_6}, \quad K^1_{XXXIV} = K_{XXXIV}|_{a_4=0}.
\end{aligned}
\tag{6.17}
$$

Note that if a CKT is given without a superscript, the complement of its degenerate forms is assumed. For example, if CKT K_{XXXI} is given, it is assumed that $a_3 \neq a_1 + a_6 - a_4,\ a_2 \neq -a_4$. The same reasoning applies to CKTs listed with a superscript but with no comma. For example, if CKT K^3_{XXIX} is given, it is assumed that $a_1 \neq a_6$.

To distinguish between the asymmetric CKTs and their degenerate forms we use invariants and covariants in the orbit space $\mathcal{K}^2(\mathbb{H}^3)/SO(3,1)$. These can readily be obtained by contracting the coefficient tensor $C^{ij}_{k\ell}$ and the general Killing tensor (4.17) respectively:

$$\mathcal{C}_0 = g_{ij}x^ix^j, \quad \mathcal{C}_1 = K^i{}_i, \quad \mathcal{C}_2 = \tfrac{1}{2}\left(K^i{}_jK^j{}_i - (K^i{}_i)^2\right),$$

$$\mathcal{C}_3 = \tfrac{1}{6}\left(2K^i{}_jK^j{}_kK^k{}_i - 3K^i{}_jK^j{}_iK^k{}_k + (K^k{}_k)^3\right),$$

$$\mathcal{I}_1 = \mathcal{R}^i{}_j, \quad \mathcal{I}_2 = \mathcal{R}^i{}_j\mathcal{R}^j{}_i, \quad \mathcal{I}_3 = \mathcal{R}^i{}_j\mathcal{R}^j{}_k\mathcal{R}^k{}_i, \quad \mathcal{I}_4 = \mathcal{R}^i{}_j\mathcal{R}^j{}_k\mathcal{R}^k{}_\ell\mathcal{R}^\ell{}_i, \tag{6.18}$$

$$\mathcal{I}_5 = \mathcal{R}^i{}_j\mathcal{R}^j{}_k\mathcal{R}^k{}_\ell\mathcal{R}^\ell{}_m\mathcal{R}^m{}_i, \quad \mathcal{I}_6 = \mathcal{R}^i{}_j\mathcal{R}^j{}_k\mathcal{R}^k{}_\ell\mathcal{R}^\ell{}_m\mathcal{R}^m{}_n\mathcal{R}^n{}_i.$$

However, it is often necessary to determine certain combinations of these invariants and covariants to distinguish between the webs. To this end, let us consider the characteristic polynomial of the Ricci matrix and coefficient matrix respectively:

$$P_1 = |\mathcal{R} - \lambda g| = -\lambda^4 + \mathcal{A}_1\lambda^3 + \mathcal{A}_2\lambda^2 + \mathcal{A}_3\lambda + \mathcal{A}_4,$$

$$P_2 = |C - \lambda G| = -\lambda^6 + \frac{1}{2}\mathcal{A}_1\lambda^5 + \frac{1}{4}\mathcal{A}_2\lambda^4 + \frac{1}{8}\mathcal{A}_3\lambda^3 + \frac{1}{16}\mathcal{A}_4\lambda^2 + \frac{1}{32}\mathcal{A}_5\lambda + \frac{1}{64}\mathcal{A}_6.$$

The coefficients of these polynomials can be expressed in terms of the contracted invariants (6.18):

$$\mathcal{A}_1 = \frac{1}{2}\left(\mathcal{I}_2 - (\mathcal{I}_1)^2\right), \quad \mathcal{A}_2 = \frac{1}{6}\left(2\mathcal{I}_3 - 3\mathcal{I}_2\mathcal{I}_1 + (\mathcal{I}_1)^3\right),$$

$$\mathcal{A}_3 = \frac{1}{24}\left(6\mathcal{I}_4 - 8\mathcal{I}_3\mathcal{I}_1 - 3(\mathcal{I}_2)^2 + 6\mathcal{I}_2(\mathcal{I}_1)^2 - (\mathcal{I}_1)^4\right),$$

$$\mathcal{A}_4 = \frac{1}{120}(24\mathcal{I}_5 - 30\mathcal{I}_4\mathcal{I}_1 - 20\mathcal{I}_3\mathcal{I}_2 - 10(\mathcal{I}_1)^3\mathcal{I}_2 + 20\mathcal{I}_3(\mathcal{I}_1)^2 + 15(\mathcal{I}_2)^2\mathcal{I}_1 + (\mathcal{I}_1)^5),$$

$$\mathcal{A}_5 = \det(C),$$

and thus are invariants in the orbit space $\mathcal{K}^2(\mathbb{H}^3)/SO(3,1)$. We can depress polynomials P_1 and P_2 by making the respective substitutions $\lambda = u + \frac{1}{4}\mathcal{A}_1$ and $\lambda = u + \frac{1}{12}\mathcal{A}_1$.

The coefficients of the resulting polynomials

$$\begin{aligned}
\tilde{P}_1 &= -u^4 + \tfrac{1}{8}\left(3\mathcal{A}_1^2 + 8\mathcal{A}_2\right)u^2 + \tfrac{1}{8}\left(\mathcal{A}_1^3 + 4\mathcal{A}_1\mathcal{A}_2 + 8\mathcal{A}_3\right)u + \\
&\quad \tfrac{1}{256}(3\mathcal{A}_1^4 + 16\mathcal{A}_1^2\mathcal{A}_2 + 64\mathcal{A}_1\mathcal{A}_3 + 256\mathcal{A}_4) \\
&= -u^4 + \tfrac{1}{8}\Delta_1 u^2 + \tfrac{1}{8}\Delta_2 u + \tfrac{1}{256}\Delta_3 \\
\tilde{P}_2 &= -u^6 + \tfrac{1}{48}(5\mathcal{A}_1^2 + 12\mathcal{A}_2)u^4 + \tfrac{1}{216}(5\mathcal{A}_1^3 + 18\mathcal{A}_1\mathcal{A}_2 + 27\mathcal{A}_3)u^3 + \\
&\quad \tfrac{1}{2304}(5\mathcal{A}_1^4 + 24\mathcal{A}_1^2\mathcal{A}_2 + 72\mathcal{A}_1\mathcal{A}_3 + 144\mathcal{A}_4)u^2 \\
&\quad \tfrac{1}{10368}({\mathcal{A}_1}^5 + 6\,\mathcal{A}_2{\mathcal{A}_1}^3 + 27\,\mathcal{A}_3{\mathcal{A}_1}^2 + 108\,\mathcal{A}_4\mathcal{A}_1 + 324\,\mathcal{A}_5)u + \\
&\quad \tfrac{1}{2985984}(5\,{\mathcal{A}_1}^6 + 36\,\mathcal{A}_2{\mathcal{A}_1}^4 + 216\,\mathcal{A}_3{\mathcal{A}_1}^3 + 1296\,\mathcal{A}_4{\mathcal{A}_1}^2 + 7776\,\mathcal{A}_5\mathcal{A}_1 + \\
&\qquad 46656\,\mathcal{A}_6) \\
&= -u^6 + \tfrac{1}{48}\Delta_4 u^4 + \tfrac{1}{216}\Delta_5 u^3 + \tfrac{1}{2304}\Delta_6 u^2 + \tfrac{1}{10368}\Delta_7 u + \tfrac{1}{2985984}\Delta_8
\end{aligned}$$

give us some of the desired combinations of invariants needed to distinguish between the webs. The remaining combinations are the discriminants of P_1 and P_2, the Jacobian determinant of the covariants $\mathcal{C}_0, \ldots, \mathcal{C}_3$, and combinations of the above coefficient invariants:

$$\Delta_9 = \mathcal{D}(|C - \lambda G|), \quad \Delta_{10} = \mathcal{D}(|\mathcal{R} - \lambda g|), \quad \Delta_{11} = \mathcal{J}(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3),$$

$$\Delta_{12} = \Delta_3 - \frac{1}{3}\Delta_1^2, \quad \Delta_{13} = \Delta_4^2 - 5\Delta_6, \quad \Delta_{14} = 3\Delta_6 + \frac{3}{4}\Delta_4^2, \quad \Delta_{15} = \Delta_8\Delta_5^2 - 25\Delta_7^2\Delta_4.$$

Note that $\mathcal{D}$ represents the discriminant, $(C)_{ij} = C^{IJ}$, $(G)_{ij} = \mathrm{diag}(-1,-1,-1,1,1,1)$, $(\mathcal{R})_{ij} = \mathcal{R}^{ij}$, $(g)_{ij} = g^{ij}$, and $\mathcal{J}$ represents the determinant of the Jacobian matrix:

$$\mathcal{J} = \left|\frac{\partial(y_1, \ldots, y_n)}{\partial(x^1, \ldots, x^n)}\right|.$$

Having defined the invariants and covariants necessary for the classification, we are now ready to state the solution to the equivalence problem.

Proposition 6.4.1. *An invariant classification of the asymmetric webs of* $\mathbb{H}^3$ *is given by Figure 6.1.*

Proof. Consider CKTs $K_{XXVIII}, \ldots, K_{XXXIV}$ and their degenerate cases (6.17). First note that K_{XXVIII} and K_{XXIX}, and K_{XXXII} and K_{XXXIII} are identical, distinguishable only by the value of a discriminating parameter. Therefore, we will only consider K_{XXIX} and K_{XXXIII} in the remainder of the proof, and assume that a discriminating parameter be used to determine the coordinate system for the CKT.

Secondly, note that the covariant Δ_{11} distinguishes between K_{XXXI} and K_{XXXIII}. In particular, we have

$$\Delta_{11} = yz(t+x)^2 a^2 (a-1)(\Gamma_1)^2$$

for K_{XXXI} and

$$\Delta_{11} = -yz(t+x)^2 a^4 (a+1)(\Gamma_2)^2$$

for K_{XXXIII}, where $\Gamma_1 = f(t,x,y,z,a_i)$ and $\Gamma_2 = g(t,x,y,z,a_i)$ are factors of their respective discriminants of $|K - \lambda g|$, and thus are nonzero. If $y, z > 0$, then Δ_{11} is strictly positive for K_{XXXI} and strictly negative for K_{XXXIII}. In what follows, we will assume that Δ_{11} can be used to distinguish between any CKTs of type XXXI and XXXIII.

Let us begin with the invariant Δ_9. K_{XXIX} and K_{XXX} are the only CKTs which do not evaluate to zero for this invariant. To distinguish between these two cases, the sign of the invariant Δ_{10} is used. In particular, it is straightforward to show that $\Delta_{10} > 0$ for K_{XXIX} and $\Delta_{10} < 0$ for K_{XXX}.

The value of Δ_9 is zero for all other CKTs. Within this group, we employ the invariant Δ_{10}. It is straightforward to show that $\Delta_{10} > 0$ for CKTs $\{K^1_{XXIX}, \ldots, K^7_{XXIX}\}$ and $\Delta_{10} < 0$ for CKTs $\{K^1_{XXX}, \ldots, K^4_{XXX}\}$.

The value of Δ_{10} is zero for all other CKTs. Within this group, we employ the invariant Δ_2. It is straightforward to show that the value of Δ_2 is nonzero for CKTs

$$K^{1,1}_{XXIX}, K^{1,2}_{XXIX}, K^{7,1}_{XXIX}, K^{7,2}_{XXIX}, K^{1,1}_{XXX}, K^{2,1}_{XXX}, K_{XXXIV}, K_{XXXI}, K^3_{XXXI},$$
$$K^4_{XXXI}, K^5_{XXXI}, K_{XXXIII}, K^3_{XXXIII}, K^4_{XXXIII}, K^5_{XXXIII}.$$

Since the CKTs within this group are of different types, we employ the invariant Δ_{12} to distinguish between these types. For $\{K_{XXXIV}, K^4_{XXXI}, K^5_{XXXI}, K^4_{XXXIII}, K^5_{XXXIII}\}$, Δ_{12} is zero. To distinguish between these CKTs, we employ the invariant Δ_5, which vanishes for K_{XXXIV} and is nonzero for $\{K^4_{XXXI}, K^5_{XXXI}, K^4_{XXXIII}, K^5_{XXXIII}\}$. For the remaining CKTs of this group $\Delta_{12} \neq 0$. To distinguish between these CKTs, we employ the covariant $\mathcal{C}_1$. The CKTs $\{K^{1,1}_{XXIX}, K^{7,1}_{XXIX}, K^{7,2}_{XXIX}, K^{3}_{XXXI}, K^{3}_{XXXIII}\}$ each satisfy

$$\mathcal{L}_R \mathcal{C}_1 = 0,$$

where R is a time-like rotational Killing vector. Comparatively, the remaining CKTs $\{K^{1,2}_{XXIX}, K^{1,1}_{XXX}, K^{2,1}_{XXX}\}$ each satisfy

$$\mathcal{L}_R \mathcal{C}_1 = 0,$$

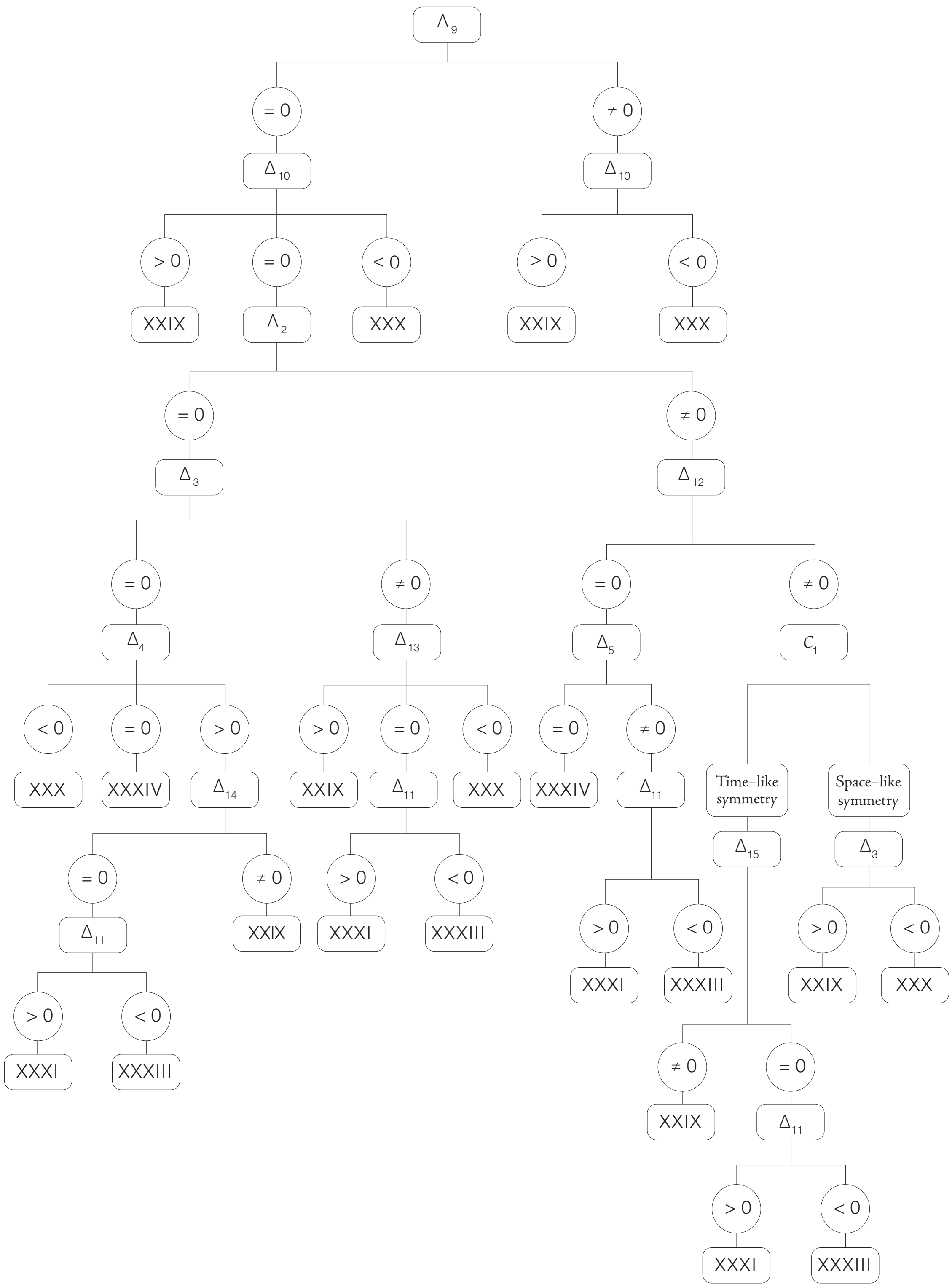
where R is a space-like rotational Killing vector. Since the CKTs within each of these two groups are of different types, we must distinguish between them. Let us consider the first group of CKTs. It is straightforward to show that the invariant Δ_{15} is nonzero for $\{K^{1,1}_{XXIX}, K^{7,1}_{XXIX}, K^{7,2}_{XXIX}\}$ and zero for $\{K^{3}_{XXXI}, K^{3}_{XXXIII}\}$. To distinguish between the CKTs in the second group we employ the invariant Δ_3. It is straightforward to show that Δ_3 is strictly positive for $K^{1,2}_{XXIX}$ and strictly negative for $\{K^{1,1}_{XXX}, K^{2,1}_{XXX}\}$.

The value of Δ_2 is zero for all other CKTs. To distinguish between the CKTs belonging to this group we employ the invariant Δ_3. It is straightforward to show that that value of Δ_3 is nonzero for CKTs $\{K^{3,1}_{XXIX}, K^{6,1}_{XXIX}, K^{4}_{XXX}, K^{1}_{XXXI}, K^{1}_{XXXIII}\}$. Since these CKTs are of different types we consider the value of the invariant Δ_{13}. It is straightforward to show that this invariant is strictly positive for $\{K^{3,1}_{XXIX}, K^{6,1}_{XXIX}\}$, is strictly negative for $\{K^{4}_{XXX}\}$, and vanishes for $\{K^{1}_{XXX1}, K^{1}_{XXXIII}\}$.

The value of Δ_3 is zero for all other CKTs. To distinguish between the CKTs belonging to this group we employ the invariant Δ_4. It is straightforward to show that Δ_4 vanishes for K^{1}_{XXXIV}, is strictly positive for $K^{3,1}_{XXX}$ and strictly negative for $\{K^{6,2}_{XXIX}, K^{6,4}_{XXIX}, K^{2}_{XXXI}, K^{2}_{XXXIII}\}$. Since this group contains different types of CKTs, we employ the invariant Δ_{14}. It is straightforward to show that Δ_{14} is strictly positive for $\{K^{6,2}_{XXIX}, K^{6,4}_{XXIX}\}$, and vanishes for $\{K^{2}_{XXXI}, K^{2}_{XXXIII}\}$.

$\square$

Figure 6.1: Invariant classification of the asymmetric webs of $\mathbb{H}^3$ under the action of $SO(3,1)$



6.5 Transformation to Canonical Form

The next stage in the solution to the equivalence problem for Killing tensors of $\mathbb{H}^3$ is to determine a method for transforming a given CKT into its respective canonical form. Such a determination will enable one to determine the explicit form of the separable coordinates corresponding to a given CKT, and thus is of importance in the application of this theory. In what follows, we discuss a method for transforming one of the symmetry generators from Categories 1 - 13 into its respective canonical form. Next, we determine a method for transforming a CKT from each of these categories into its respective canonical form. We conclude the section with a procedure for transforming the asymmetric CKTs of Category 14 into their respective canonical forms.

6.5.1 Symmetry Generators

Each of the symmetry subspaces of $\mathcal{K}^2(\mathbb{H}^3)$ are defined by at least one symmetry generator, with the respective canonical forms of the generators listed in Table 6.6. For each of these Categories 1 - 13, it is necessary to determine the most general transformation $\Lambda \in SO(3,1)$ which returns one of the symmetry generator(s) of a category to their canonical form. In what follows, we present a method for determining the most general transformation $\Lambda^i{}_j$ which returns one of the symmetry generators of a symmetry category to its canonical form. We will continue with the notation defined in (6.12), where A^i, B^{ij} and $\mathcal{R}^{ij}$ denote the parameters of a non-canonical form, while $\tilde{A}^i$, $\tilde{B}^{ij}$ and $\tilde{\mathcal{R}}^{ij}$ denote the parameters of its canonical form.

1. A canonical timelike rotational symmetry generator of this category has parameters $\tilde{A}^i = 0$ and
$$\tilde{B}^{ij} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
Note that $g\tilde{B}g$ admits a zero eigenvalue with a two-dimensional eigenspace spanned by the spacelike vectors $\tilde{v}_1 = (0,0,1,0)$ and $\tilde{v}_2 = (0,0,0,1)$. $\Lambda^i{}_j$

is given by (D.2), where v^i is one of the spacelike basis vectors of the two-dimensional zero eigenspace of B^{ij} normalized so that $g_{ij}v^iv^j = 1$, and such that $\tilde{B}^{ij} = B^{k\ell}\Lambda^i{}_k\Lambda^j{}_\ell$ forms a consistent system of equations.

2. A canonical spacelike rotational symmetry generator of this category has parameters $\tilde{A}^i = 0$ and
$$\tilde{B}^{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
Note that $g\tilde{B}g$ admits a zero eigenvalue with a two-dimensional eigenspace spanned by a spacelike vector $\tilde{v}_1 = (0,0,0,1)$ and a timelike vector $\tilde{v}_2 = (1,0,0,0)$. $\Lambda^i{}_j$ is given by (D.2), where v^i is the spacelike basis vector of the two-dimensional zero eigenspace of B^{ij} normalized so that $g_{ij}v^iv^j = 1$.

3. A canonical null rotational symmetry generator of this category has parameters $\tilde{A}^i = 0$ and
$$\tilde{B}^{ij} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$
Note that $g\tilde{B}g$ admits a zero eigenvalue with a two-dimensional eigenspace spanned by the spacelike vector $\tilde{v}_1 = (0,0,1,0)$ and the null vector $\tilde{v}_2 = (1,0,0,1)$. $\Lambda^i{}_j$ is given by (D.4), where v^i is the spacelike basis vector of the two-dimensional zero eigenspace of B^{ij} normalized so that $g_{ij}v^iv^j = 1$.

4. A canonical timelike translational symmetry generator of this category has parameters $\tilde{A}^i = (\tilde{a}_1, 0, 0, 0) = ((g_{ij}A^iA^j)^{1/2}, 0, 0, 0)$ and $\tilde{B}^{ij} = 0$. To determine $\Lambda^i{}_j$, we use $v^i = A^i/\tilde{a}_1$ in (D.5), leaving the parameters of Λ arbitrary.

5. A canonical spacelike translational symmetry generator of this category has parameters $\tilde{A}^i = (0,0,0,\tilde{a}_4) = (0,0,0,(g_{ij}A^iA^j)^{1/2})$ and $\tilde{B}^{ij} = 0$. To determine $\Lambda^i{}_j$, we use $v^i = A^i/\tilde{a}_4$ in (D.2), leaving the parameters of Λ arbitrary.

6. A canonical null translational symmetry generator of this category has parameters $\tilde{A}^i = (1,0,0,1)$ and $\tilde{B}^{ij} = 0$. $\Lambda^i{}_j$ is given by (D.10 +) with $v^i = A^i/A^1$ and $e^{\epsilon\theta_1} = A^1$, where A^i are the parameters of a null translational symmetry generator.

7. As in 3.

8. As in 1.

9. As in 4.

10. As in 2.

11. As in 5.

12. A canonical null rotational symmetry generator of this category has parameters $\tilde{A}^i = 0$ and
$$\tilde{B}^{ij} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
Note that $g\tilde{B}g$ admits a zero eigenvalue with a two-dimensional eigenspace spanned by the spacelike vector $\tilde{v}_1 = (0,0,0,1)$ and the null vector $\tilde{v}_2 = (1,-1,0,0)$. $\Lambda^i{}_j$ is given by (D.2), where v^i is the spacelike basis vector of the two-dimensional zero eigenspace of B^{ij} normalized so that $g_{ij}v^iv^j = 1$.

13. As in 5.

6.5.2 Category 1

The restricted group action of this symmetry subspace is given by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \tag{6.19}$$

Since both of the canonical CKTs belonging to this category have diagonal algebraic Ricci matrices, the determination of the transformation to canonical form for CKTs of this subspace is a generalized eigenproblem.

6.5.3 Category 2

The restricted group action of this symmetry subspace is given by (6.13). Since $K_{XVII}, K_{XVIII}, K_{XXI}$ all have diagonal algebraic Ricci matrices, the determination of the transformation to canonical form for CKTs of these types is a generalized eigenproblem. The remaining canonical CKTs of this category do not have a diagonalizable algebraic Ricci matrix.

- For K_{XXV}, the parameter in the group action (6.13) is given by $e^{\epsilon 2\theta} = \frac{a_4+a_2}{2a_6-a_4+a_2}$, where $\epsilon = \pm 1$.

- For K_{XXVI}, the parameter in the group action (6.13) is given by $e^{\epsilon 2\theta} = \frac{-(a_2+a_4)}{2a_6-a_4+a_2}$, where $\epsilon = \pm 1$.

- For K_{XXII} with $b_8 \neq 0$, the parameter in the group action (6.13) is given by $\tanh 2\theta = \frac{\epsilon(a_4-a_6)}{b_8}$, where $\epsilon = \pm 1$.

6.5.4 Category 3

The restricted group action of this symmetry subspace is given by (6.14).

- For K_{XXIII}, the parameter θ in the group action (6.14) is given by $e^{\epsilon 2\theta} = \frac{a_1+a_5}{a_1+a_6}$, where $\epsilon = \pm 1$.

- For K_{XXIV}, the parameter θ in the group action (6.14) is given by $e^{\epsilon 2\theta} = \frac{-(a_1+a_5)}{a_1+a_6}$, where $\epsilon = \pm 1$.

- For K_{XXVII}, the parameters θ and ϵ in the group action (6.14) are arbitrary.

6.5.5 Category 4

The restricted group action of this symmetry subspace is given by $\tilde{x}^i = \Lambda^i{}_j x^j$, where

$$\Lambda^i{}_j = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^a{}_b \end{pmatrix}$$

for $\lambda^a{}_b \in SO(3)$. Since the CKT belonging to this category has a diagonal algebraic Ricci matrix, the determination of the transformation to canonical form is a generalized eigenproblem.

6.5.6 Category 5

In subsection (6.4.4), we noted that this symmetry subspace is isomorphic to the vector space $\mathcal{K}^2(\mathbb{H}^2)$ under the action of $SO(2,1)$. Therefore, the transformation to canonical form for CKTs of this subspace can be determined by applying the transformation theory for dilatational Killing tensors of $\mathbb{M}^3$ found in Section VI.G of [39].

6.5.7 Category 6

The restricted group action of this symmetry subspace is given by (6.16).

- For K_{XV}, the parameters in the group action (6.16) are given by

$$e^{\epsilon 4\theta} = \frac{4(b_5 - b_{11})^2 + (b_8 + b_{10})^2}{a_1^2}$$

  and $\tan 2\phi = \frac{-2(b_5 - b_{11})}{b_8 + b_{10}}$, where $\epsilon = \pm 1$.

- For K_{XVI}, the parameter ϕ in the group action (6.16) is given by $\tan\phi = \frac{b_{11} - b_5}{b_1}$, while θ and ϵ are arbitrary.

6.5.8 Category 7

The restricted group action of this symmetry subspace is given by (6.14). The most general CKT belonging to this symmetry subspace is given by

$$K^{ij} = \begin{pmatrix} b_8y^2 + a_3x^2 & a_3x\,(t-z) & b_8y\,(t-z) & b_8y^2 + a_3x^2 \\ a_3x\,(t-z) & a_3\,(t-z)^2 & 0 & a_3x\,(t-z) \\ b_8y\,(t-z) & 0 & b_8\,(t-z)^2 & b_8y\,(t-z) \\ b_8y^2 + a_3x^2 & a_3x\,(t-z) & -b_8zy + b_8yt & b_8y^2 + a_3x^2 \end{pmatrix},$$

which is unaffected by (6.14).

6.5.9 Category 8

The restricted group action of this symmetry subspace is given by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \epsilon\cosh\theta & 0 & \sinh\theta & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\theta & 0 & \epsilon\cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix},$$

where $\theta \in \mathbb{R}$. Since the CKT belonging to this category has a diagonal algebraic Ricci matrix, the transformation to canonical form is a generalized eigenproblem.

6.5.10 Category 9

The restricted group action of this symmetry subspace is given by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ 0 & -\sin\phi\sin\theta & \cos\theta & -\sin\theta\cos\phi \\ 0 & \cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix},$$

where $\theta, \phi \in \mathbb{R}$. Since the CKT belonging to this category has a diagonal algebraic Ricci matrix, the transformation to canonical form is a generalized eigenproblem.

6.5.11 Category 10

The restricted group action of this symmetry subspace is given by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \epsilon\cosh\theta & 0 & 0 & \sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\theta & 0 & 0 & \epsilon\cosh\theta \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \tag{6.20}$$

where $\theta \in \mathbb{R}$ and $\epsilon = \pm 1$. K_{XIV} is the only CKT belonging to this category.

- For K_{XIV}, the parameters θ and ϵ in the group action (6.20) are arbitrary.

6.5.12 Category 11

The restricted group action of this symmetry subspace is given by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix},$$

where $\theta \in \mathbb{R}$. Since the CKT belonging to this category has a diagonal algebraic Ricci matrix, the transformation to canonical form is a generalized eigenproblem.

6.5.13 Category 12

The restricted group action of this symmetry subspace is given by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \epsilon\cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \epsilon\cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \tag{6.21}$$

where $\theta \in \mathbb{R}$ and $\epsilon = \pm 1$. K_{XIII} is the only CKT belonging to this category.

- For K_{XIII}, the parameters θ and ϵ in the group action (6.21) are arbitrary.

6.5.14 Category 13

The restricted group action of this symmetry subspace is given by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \epsilon\cosh\theta & 0 & 0 & \sinh\theta \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ \sinh\theta & 0 & 0 & \epsilon\cosh\theta \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \tag{6.22}$$

where $\theta, \phi \in \mathbb{R}$ and $\epsilon = \pm 1$.

6.5.15 Category 14

The restricted group action of this symmetry subspace is given by $\tilde{x}^i = \Lambda^i{}_j x^j$, where $\Lambda^i{}_j \in SO(3,1)$. For K_{XXIX}, and some degenerate cases of K_{XXX}, K_{XXXI} and K_{XXXIII}, $g\tilde{\mathcal{R}}g$ is a diagonal matrix, and thus the determination of the transformation is a generalized eigenproblem. For the remaining CKTs, a transformation can be found by first transforming one of the generalized eigenvectors of the algebraic Ricci matrix into its canonical form. After applying this transformation to the algebraic Ricci matrix, the arbitrary parameters of $\Lambda^i{}_j$ can be determined by solving the system of equations given by $\mathcal{R}^{k\ell} = \Lambda^k{}_i \Lambda^\ell{}_j \mathcal{R}^{ij}$.

- For K_{XXX}, if $g\tilde{\mathcal{R}}g$ does not admit four linearly independent eigenvectors, then $g\tilde{\mathcal{R}}g$ admits the spacelike eigenvectors $v_1 = (0,0,1,0)$ and $v_2 = (0,0,0,1)$. $\Lambda^i{}_j$ is given by (D.2), where v^i is one of the spacelike eigenvectors of $g\mathcal{R}g$ normalized so that $g_{ij}v^iv^j = 1$, and such that $\mathcal{R}^{k\ell} = \Lambda^k{}_i \Lambda^\ell{}_j \mathcal{R}^{ij}$ forms a consistent system of equations.

- For K_{XXXI} and K_{XXXIII}, if $g\tilde{\mathcal{R}}g$ does not admit four linearly independent eigenvectors, then $g\tilde{\mathcal{R}}g$ admits the null eigenvector $v = (1,-1,0,0)$. $\Lambda^i{}_j$ is given by (D.8-) with $v^i = A^i/A^1$ and $e^{\epsilon\theta_1} = A^1$, where A^i are the parameters of the null eigenvector of $g\mathcal{R}g$.

- For K_{XXXIV}, the algebraic Ricci matrix $g\tilde{\mathcal{R}}g$ admits the null eigenvector $v_1 = (1,1,0,0)$. $\Lambda^i{}_j$ is given by (D.8+) with $v^i = A^i/A^1$ and $e^{\epsilon\theta_1} = A^1$, where A^i are the parameters of the null eigenvector of $g\mathcal{R}g$.

6.6 Main Algorithm

We now outline an algorithm for determining the set of orthogonally separable coordinates for a given natural Hamiltonian defined on $\mathbb{H}^3$.

1. *Compatibility condition.* Begin by substituting the potential into the compatibility condition (4.48) to determine the most general Killing tensor compatible with the potential. Using this Killing tensor, determine the subspace of CKTs.

2. *Web symmetries.* For a given CKT K, determine if it admits any symmetry. Namely, impose the constraint

$$\mathcal{L}_V(K + \alpha\mathcal{C}) = 0,$$

where $\alpha \in \mathbb{R}$, $\mathcal{C}$ is the Casimir operator, and V is the general Killing vector of $\mathbb{M}^4$. If K does admit symmetry, determine which type and the number of generators for each type. Consult Table 6.6 to determine which category K belongs to.

3. *Classification.*

   - If K belongs to Category 1, 4, or 7 - 13, Table 6.6 can be used to immediately classify K.
   - If K belongs to Category 2, 3, 5 or 6, use Subsection 6.5.1 to determine the transformation $h_1 \in SO(3,1)$ which returns the specified symmetry generator for that category to its canonical form. Next, apply this transformation to K. The resulting tensor, $\tilde{K}$, will now belong to one of the symmetry subspaces 6.4.2 - 6.4.5 and the classification procedure for that category can be applied to identify K.
   - If K belongs to Category 14, apply Proposition 6.4.1 to classify K.

4. *Transformation to canonical form.*

   - If K belongs to Category 1, 4, or 7 - 13, use Subsection 6.5.1 to determine the transformation $h_1 \in SO(3,1)$ which returns the specified symmetry generator for that category to its canonical form. Next, apply this transformation to K to obtain a new tensor, $\tilde{K}$. In Section 6.5, find the category in which $\tilde{K}$ belongs and use the theory of that subsection to determine the restricted group action $h_2 \in SO(3,1)$ which brings $\tilde{K}$ to canonical form. Set $\Lambda^i{}_j = h_2 \circ h_1$.

- If K belongs to Category 2, 3, 5 or 6, consider $\tilde{K}$ from the previous step. In Section 6.5, find the category in which $\tilde{K}$ belongs and use the theory of that subsection to determine the restricted group action $h_2 \in SO(3,1)$ which brings $\tilde{K}$ to canonical form. Set $\Lambda^i{}_j = h_2 \circ h_1$.

- If K belongs to Category 14, consult Subsection 6.5.15 to determine the group action $\Lambda^i{}_j \in SO(3,1)$ which brings K to canonical form.

5. *Orthogonally separable coordinates.* Define the orthogonally separable set of coordinates corresponding to K by substituting $\Lambda^i{}_j$ found in the previous step into the equation

$$x^i = \Lambda^i{}_j T^j(u^k),$$

where $x^j = T^j(u^k)$ denote the canonical orthogonally separable coordinates corresponding to K.

CHAPTER 7

APPLICATIONS

To illustrate the theory of this thesis, as well as demonstrate its applicability to problems in mathematical physics, we consider a rotationally symmetric potential defined on $\mathbb{S}^3$ and a null translationally symmetric potential defined on $\mathbb{H}^3$. In each case, we use the classification scheme to identify compatible CKTs, and the algebraic Ricci tensor to determine the moving frame and define the orthogonally separable coordinates. As a final application of the theory, we determine the most general potential compatible with the general CKT of each symmetry subspace on $\mathbb{S}^3$ and $\mathbb{H}^3$.

7.1 A Rotationally Symmetric Potential on $\mathbb{S}^3$

Consider the following natural Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j + \frac{1}{(x-y)^2}$$

defined on $\mathbb{S}^3 \subset \mathbb{E}^4$, where g^{ij} denotes the Euclidean metric and x, y, z, w are Cartesian coordinates of the ambient space $\mathbb{E}^4$ satisfying the constraint $x^2 + y^2 + z^2 + w^2 = 1$. Using this Hamiltonian, we will now demonstrate how to apply the theory outlined in Section 5.2.

First, we impose the compatibility condition (4.48) to obtain a family of Killing tensors which are compatible with the potential. Of this family, the following restrictions on the parameters yields a subfamily of Killing tensors which satisfies the Haantjes condition (4.40) and generally admits 3 distinct eigenvalues:

$$C_{1212} = C_{3434},\ C_{1313} = C_{2323},\ C_{1414} = C_{2424},$$
$$C_{1323} = C_{1313} - C_{1212},\ C_{1424} = C_{1212} - C_{1414}$$

Therefore we conclude that K must characterize at least one of the six orthogonally separable webs of $\mathbb{S}^3$. After a direct calculation, we find that K admits the following family of rotational Killing vectors

$$V = (c_3 z + c_6 w)\frac{\partial}{\partial x} + (c_6 w - c_3 z)\frac{\partial}{\partial y} + (c_3 y - c_3 x)\frac{\partial}{\partial z} - (c_6 y + c_6 x)\frac{\partial}{\partial w},$$

for arbitrary constants c_3 and c_6. Using the classification scheme outlined in Table 5.2 we conclude that K characterizes a non-canonical cylindrical web.

In order to determine the orthogonally separable coordinates for this Killing tensor, we need to determine the transformation which maps K to its canonical form. As discussed in Section 5.4, such a map can be constructed by diagonalizing the algebraic Ricci tensor of the coefficient tensor. Contracting indices, we obtain the following non-canonical algebraic Ricci tensor components for this family of CKTs:

$$\begin{aligned}
\mathcal{R}_{11} &= C_{1212} + C_{1313} + C_{1414},\\
\mathcal{R}_{22} &= C_{1212} + C_{1313} + C_{1414},\\
\mathcal{R}_{33} &= 2C_{1313} + C_{1212},\\
\mathcal{R}_{44} &= 2C_{1414} + C_{1212},\\
\mathcal{R}_{12} &= C_{1414} - C_{1313},\\
\mathcal{R}_{13} &= 0,\\
\mathcal{R}_{14} &= 0,\\
\mathcal{R}_{23} &= 0,\\
\mathcal{R}_{24} &= 0,\\
\mathcal{R}_{34} &= 0.
\end{aligned}$$

After calculating the eigenvalues and corresponding eigenvectors of $\mathcal{R}_{jk}$ and applying the Gram-Schmidt orthonormalization procedure, we obtain an orthogonal matrix

$$\Lambda^i{}_j = \begin{pmatrix} \frac{-\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{7.1}$$

which brings the algebraic Ricci tensor into its canonical form

$$\tilde{\mathcal{R}}_{jk} = \mathrm{diag}(2C_{1313} + C_{1212},\ 2C_{1313} + C_{1212},\ 2C_{1414} + C_{1212},\ 2C_{1414} + C_{1212}).$$

Therefore, we conclude that

$$\begin{array}{rcl} x & = & -\frac{\sqrt{2}}{2}\cos t\cos u + \frac{\sqrt{2}}{2}\sin t\sin v, \\ y & = & \frac{\sqrt{2}}{2}\cos t\cos u + \frac{\sqrt{2}}{2}\sin t\sin v, \\ z & = & -\cos t\sin u, \\ w & = & \sin t\cos v \end{array},$$

is a system of orthogonally separable coordinates for this Hamiltonian. If we apply the transformation defined by (7.1) to the potential of the Hamiltonian, we get

$$\tilde{V} = \frac{1}{2\tilde{x}^2},$$

which we recognize as a term in one of the multiseparable Smorodinksy-Winternitz potentials on $\mathbb{S}^3$ [29]. We conclude that H defines a rotated multiseparable Hamiltonian on $\mathbb{S}^3$.

7.2 A Null Translationally Symmetric Potential on $\mathbb{H}^3$

Consider the following natural Hamiltonian

$$H = \frac{1}{2}g^{ij}p_ip_j + \frac{1}{(t-x)^2} \tag{7.2}$$

on $\mathbb{H}^3 \subset \mathbb{M}^4$, where g^{ij} denotes the Minkowski metric, and t, x, y, z denote the pseudo-Cartesian coordinates (t, x, y, z) of $\mathbb{M}^4$ satisfying the constraint $t^2 - x^2 - y^2 - z^2 = 1$. To determine the set of all orthogonally separable coordinates for the associated Hamilton-Jacobi equation we follow the algorithm laid out in Section 6.6.

The most general Killing tensor satisfying the compatibility condition (4.48) is given by

$$K^{ij} = a_1K_1^{ij} + a_2K_2^{ij} + a_3K_3^{ij} + a_4K_4^{ij} + a_5K_5^{ij} + a_6K_6^{ij},$$

where

$$K_1^{ij} = \begin{pmatrix} 2\,zy & -2\,zy & zx+zt & yx+yt \\ -2\,zy & 2\,zy & -zx-zt & -yx-yt \\ zx+zt & -zx-zt & 0 & x^2+2\,xt+t^2 \\ yx+yt & -yx-yt & x^2+2\,xt+t^2 & 0 \end{pmatrix},$$

$$K_2^{ij} = \begin{pmatrix} 0 & 0 & -zy & y^2 \\ 0 & 0 & zy & -y^2 \\ -zy & zy & -2\,zx - 2\,zt & yx + yt \\ y^2 & -y^2 & yx + yt & 0 \end{pmatrix}, \; K_3^{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z^2 & -zy \\ 0 & 0 & -zy & y^2 \end{pmatrix},$$

$$K_4^{ij} = \begin{pmatrix} 0 & 0 & -z^2 & zy \\ 0 & 0 & z^2 & -zy \\ -z^2 & z^2 & 0 & -zx - zt \\ zy & -zy & -zx - zt & 2\,yx + 2\,yt \end{pmatrix},$$

$$K_5^{ij} = \begin{pmatrix} -z^2 & z^2 & 0 & -zx - zt \\ z^2 & -z^2 & 0 & zx + zt \\ 0 & 0 & 0 & 0 \\ -zx - zt & zx + zt & 0 & -x^2 - t^2 - 2\,xt \end{pmatrix},$$

$$K_6^{ij} = \begin{pmatrix} y^2 & -y^2 & yx + yt & 0 \\ -y^2 & y^2 & -yx - yt & 0 \\ yx + yt & -yx - yt & x^2 + 2\,xt + t^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since K satisfies the Haantjes condition (4.40) and generally has distinct eigenvalues, it is characteristic and therefore represents at least one of the orthogonally separable coordinate systems of $\mathbb{H}^3$. To determine these coordinate systems, we first check to see if K admits any symmetry by imposing condition (4.70). For arbitrary $a_1, \ldots, a_6$, we find K admits the symmetry generator

$$X = c_1 \frac{\partial}{\partial t} - c_1 \frac{\partial}{\partial x},$$

which, according to Table 6.5, is null translational. Consulting Table 6.6, we find that K belongs to Category 6. Applying step 6 in Subsection 6.5.1, we find a transformation $x^i = \Lambda^i{}_j \tilde{x}^j$ defined by

$$\Lambda^i{}_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{7.3}$$

sends X to $\tilde{X} = (c_1, 0, 0, c_1)$. Upon applying the transformation $x^i = \lambda^i{}_j \tilde{x}^j$ to K, we find the resulting tensor $\tilde{K}$ is the general characteristic Killing tensor (6.15) of Category 6. In light of this fact, we conclude that the orthogonally separable coordinate systems for this Hamiltonian are those characterized by the null-translationally symmetric webs of $\mathbb{H}^3$. These are webs II, XIV, XV and XVI, which characterize the horicyclic, horicyclic-cylindrical, horicyclic-elliptic and horicyclic-parabolic coordinates respectively. To determine the form of the separable coordinates for this potential, we apply the transformation

$$x^i = \Lambda^i{}_j T^j(u^k),$$

where $\Lambda^i{}_j$ is given by (7.3), and $x^j = T^j(u^k)$ are the separable coordinates.

7.3 Symmetry Subspace Potentials

In Section 7.2, we found that the potential of the Hamiltonian is compatible with the general CKT (6.15) of the null-translationally symmetric subspace of $\mathbb{H}^3$. It is then natural to ask, *what is the most general potential compatible with this CKT*? As quoted in [55], we can determine the most general potential compatible with a system of orthogonally separable coordinates u^i by solving the system of equations

$$\frac{\partial^2 V}{\partial u^i \partial u^j} + \frac{\partial V}{\partial u^i}\frac{\partial \ln f_i^2}{\partial u^j} + \frac{\partial V}{\partial u^j}\frac{\partial \ln f_j^2}{\partial u^i} = 0, \quad i \neq j, \tag{7.4}$$

using the metric $ds^2 = f_1^2 du_1^2 + \dots f_n^2 du_n^2$ of the orthogonally separable coordinates. Indeed, a Killing tensor K expressed in orthogonally separable coordinates u^i and substituted it into the compatibility condition (4.48) yields a system of PDEs, which after incorporating Eisenhart's equations (4.45), are equations (7.4). In light of these equations, we can solve for the most general potential compatible with each of the horicyclic, horicyclic-cylindrical, horicyclic-elliptic and horicyclic-parabolic coordinates. After transforming each of these potentials into pseudo-Cartesian coordinates (t, x, y, z), we find that each potential contains the common term

$$V = F(t - z),$$

which implies that this is the most general potential compatible with the null translationally symmetric webs of $\mathbb{H}^3$. Indeed, after applying the transformation $x^i = \lambda^i{}_j \tilde{x}^j$ to the potential in (7.2) we get

$$\tilde{V} = \frac{1}{(\tilde{t} - \tilde{z})^2},$$

which is compatible with the general characteristic Killing tensor $\tilde{K}$ of the null translational symmetric subspace, and clearly belongs to the family of potentials $F(\tilde{t} - \tilde{z})$.

Analogously, it is possible to derive the most general potential compatible with the general CKT of the space-like, time-like and null rotational symmetry subspaces and the space-like and time-like translational symmetry subspaces. Proceeding as we did for the null translational symmetry subspace, we first determine the most general potential for each coordinate system of the subspace. After transforming to pseudo-Cartesian coordinates, we conclude that the common term amongst these potentials represents the most general potential for this subspace. The results of this calculation are presented in Table 7.1.

Similarly, we can repeat the above analysis for the rotational and translational symmetry subspaces on $\mathbb{S}^3$. The most general potentials compatible with these symmetry subspaces are presented in Table 7.2.

Table 7.1: The most general potential compatible with the symmetry subspaces of $\mathbb{H}^3$

Category	Symmetry	Potential	Separable coordinates
1	Time-like rotation	$\frac{F(x/t)}{t^2 - x^2}$	I, XII, XIX, XX
2	Space-like rotation	$\frac{F(y/x)}{x^2 + y^2}$	I, X, XI, XIV, XVII, XVIII, XXI, XXII, XXV, XXVI
3	Null rotation	$\frac{F(x/(t-z))}{(t-z)^2}$	II, XXIII, XXIV, XXVII
4	Time-like translation	$F\left(\frac{x^2+y^2+z^2}{t^2}\right)$	III, X
5	Space-like translation	$F\left(\frac{z^2}{t^2-x^2-y^2}\right)$	IV, V, VI, VII, VIII, IX, XI, XII, XIII
6	Null translation	$F(t-z)$	II, XIV, XV, XVI

Table 7.2: The most general potential compatible with the symmetry subspaces of $\mathbb{S}^3$

Category	Symmetry	Potential	Separable coordinates
1	Rotation	$\frac{F(y/x)}{x^2 + y^2}$	I, II, III, IV
2	Translation	$F\left(\frac{x^2+y^2+z^2}{w^2}\right)$	I, V

CHAPTER 8

CONCLUSION

As we stated in the introduction to this thesis, equivalence problems of Killing tensors occur naturally in theory of orthogonal separation of variables of the Hamilton-Jacobi equation. This, as we showed, becomes readily apparent when the problem is formulated in the modern language of Cartan geometry. The recently developed *invariant theory of Killing tensors*, which naturally follows from this formulation, can be used to solve such equivalence problems on spaces of constant curvature. As we discussed, this has been achieved for the two and three-dimensional spaces of zero curvature, and the two-dimensional spaces of non-zero curvature. The main objective of this thesis was to further develop and apply the theory to solve the equivalence problems of orthogonally separable webs on $\mathbb{S}^3$ and $\mathbb{H}^3$. To achieve these results, we formulated and solved two subproblems in each case, namely the classification problem for CKTs, and the transformation to canonical form.

As a starting point to achieve these objectives, we solved the simpler equivalence problems on the lower dimensional spaces of $\mathbb{S}^2$ and $\mathbb{H}^2$. While solutions to these problems had already been obtained, we presented a new solution based on web symmetry and a better understanding of the meaning of the distinguishing invariants. In particular, by considering the webs as objects in an ambient space *before* their intersection with the hypersurface, we were able to characterize the webs by a greater number of symmetries. In the case of $\mathbb{S}^2$, this led to a classification based on symmetry alone. In the case of $\mathbb{H}^2$, we used a combination of web symmetries, invariants and covariants to achieve a classification scheme. To obtain the necessary combinations of invariants which distinguished between the asymmetric webs, we used the equivalence of the characteristic polynomial of the coefficient tensor of the Killing tensor and the polynomial of the coordinate system to define a set of distinguishing invariants. Moreover, we resolved a discrepancy between the number of inequivalent CKTs for

$\mathbb{H}^2$ in two previous papers [30, 39]. To solve the second subproblem in each case, we noted that the canonical webs of $\mathbb{S}^2$ admit a diagonal coefficient tensor and thus a transformation can be determined by solving an eigenproblem. This is also the case for some of the webs of $\mathbb{H}^2$. For the remaining cases, we cited the work of Horwood *et al* in [39].

We were able to adapt many of these ideas to the higher dimensional problems on $\mathbb{S}^3$ and $\mathbb{H}^3$. In solving the first subproblem for $\mathbb{S}^3$, we used the *method of web symmetries* to classify the webs based on symmetry alone. This, as in the case of $\mathbb{S}^2$, relied on the use of translational symmetry in the ambient space to characterize the webs. The case of $\mathbb{H}^3$ was considerably more complicated. This was due to the greater number of orthogonally separable coordinate systems (34 to be exact), the symmetries of the webs, and the close similarity between some of the webs. To obtain a classification scheme, we first sorted the webs into symmetry categories using the more general definition of web symmetry. Since many categories contained more than one web, we solved the equivalence problem on these symmetry subspaces using reduced invariants and covariants. To obtain a set of invariants and covariants, we used the method of infinitesimal generators and the method of contraction given by Horwood *et al* [39].

To solve the second subproblem for $\mathbb{S}^3$ and $\mathbb{H}^3$, we were disadvantaged by the fact that the coefficient tensor has a 6×6 matrix representation. Therefore, if the canonical CKT admitted a diagonal coefficient matrix, we were unable to reduce the problem to an eigenproblem in general. To overcome this difficulty, we defined an *algebraic Ricci tensor* for a given CKT. Since this tensor has a 4×4 matrix representation, we were able to determine the transformation for all of the webs of $\mathbb{S}^3$ and many of the webs of $\mathbb{H}^3$ by solving an eigenproblem for the algebraic Ricci tensor. To determine the transformation for the remaining webs of $\mathbb{H}^3$, we adapted an idea used by Horwood *et al* in [39] which uses an eigenvector of the algebraic Ricci tensor to help determine a transformation.

The solution of equivalence problems of Killing tensors is useful for determining the set of orthogonally separable coordinate systems of a given natural Hamiltonian. This was demonstrated by the analysis of a rotationally symmetric potential defined on $\mathbb{S}^3$ and a null translationally symmetric potential defined on $\mathbb{H}^3$. Using the solution to the equivalence problem in each case, we were able to derive orthogonally separable coordinate systems for the Hamiltonian. As a further application of the theory, we derived the most general potential compatible with the general CKT of each symmetry subspace of $\mathbb{S}^3$ and $\mathbb{H}^3$.

The results of this thesis suggest several potential directions for future research projects. Two such problems which could build on the results and methods of this thesis are the canonical forms and equivalence problems for Killing tensors defined on $\mathbb{E}^4$ and $\mathbb{M}^4$. Since $\mathbb{S}^3 \subset \mathbb{E}^4$ and $\mathbb{H}^3 \subset \mathbb{M}^4$, the classification of CKTs on these submanifolds can be used to solve the equivalence problem on the dilatational symmetry subspaces of $\mathbb{E}^4$ and $\mathbb{M}^4$, and thus present a partial solution to these problems. Another potential area of growth is the theory of joint invariants of Killing tensors. Building on our result obtained in $\mathbb{E}^2$, it would be interesting to develop this area of ITKT to study superintegrable potentials in other two-dimensional spaces, as well as in higher dimensions.

APPENDIX A

COMPOUND MATRICES

Let us begin by defining some preliminary terms.

For $k, n \in \mathbb{Z}$ satisfying $1 \leq k \leq n$, define $N^{k,n}$ as the collection of all k-tuples of integers that have components from the set $N = \{1, \ldots, n\}$ and arranged in lexicographic order. For example, if $k = 2$ and $n = 3$, then

$$N^{2,3} = \{(1, 2), (1, 3), (2, 3)\}.$$

If A is an $m \times n$ matrix, and $\alpha = (i_1, \ldots, i_r) \in N^{r,m}$ and $\beta = (j_1, \ldots, j_s) \in N^{s,n}$, a *submatrix* $A[\alpha; \beta]$ of A is an $r \times s$ matrix with entries given by

$$a[\alpha; \beta]_{pq} = a_{i_p, j_q}.$$

Example A.0.1. Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

of order 3. If $\alpha = (1, 2, 3) \in N^{3,3}$ and $\beta = (1, 3) \in N^{2,3}$, then

$$A[\alpha; \beta] = A[(1, 2, 3); (1, 3)] = \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

We can now define a *compound matrix*.

Definition A.0.2 ([25])**.** Suppose A is an $m \times n$ matrix and $1 \leq k \leq \min(m, n)$. The kth compound of A is the matrix $A^{(k)}$ of size $\binom{m}{k} \times \binom{n}{k}$ with components equal to

$$\det A[\alpha, \beta], \quad \text{for } \alpha \in N^{k,m}, \beta \in N^{k,n},$$

and organized lexicographically in $A^{(k)}$.

In this thesis we are interested in the 2nd compound of a 4×4 matrix.

Example A.0.3. Suppose A is a 4×4 matrix. The 2nd-compound of A is the 6×6 matrix $A^{(2)}$ whose entries are the determinant of submatrices of order 2 of A. Specifically, taking

$$\alpha, \beta \in N^{2,4} = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\},$$

the components of $A^{(2)}$ are equal to the determinant of the submatrices $A[\alpha; \beta] = A[(i_1, i_2); (j_1, j_2)]$. These components are then arranged in the matrix so that α and β are in lexicographic order, ie., $A^{(2)}_{11} = |A[(1,2);(1,2)]|$, $A^{(2)}_{12} = |A[(1,2);(1,3)]|, \ldots,$ $A^{(2)}_{66} = |A[(3,4);(3,4)]|$. Thus we have

$$A^{(2)} = \begin{pmatrix}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\
\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} \\
\begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{14} \\ a_{42} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} \\
\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\
\begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{41} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{24} \\ a_{41} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix} \\
\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}
\end{pmatrix},$$

where the vertical bars denote the determinant.

Compound matrices $A^{(k)}$ have several attributes in common with their original matrix A. Of importance to this thesis is the following attribute ([25], Theorem 6.16):

Theorem 9. *Suppose A is an orthogonal matrix of order n. For any $1 \leq k \leq n$, the matrix $A^{(k)}$ is also orthogonal.*

APPENDIX B

ORTHOGONALLY SEPARABLE METRICS AND COORDINATE SYSTEMS

The following is a list of the canonical orthogonally separable coordinate systems and metrics for two and three-dimensional spherical and hyperbolic space. In each case, we list the transformation from separable coordinates to the coordinates of the ambient space. For $\mathbb{S}^2$, this is a transformation from (u, v) to $(x, y, z) \in \mathbb{E}^3$; for $\mathbb{S}^3$, this is a transformation from (t, u, v) to $(x, y, z, w) \in \mathbb{E}^4$; for $\mathbb{H}^2$, this is a transformation from (u, v) to $(t, x, y) \in \mathbb{M}^3$; and for $\mathbb{H}^3$, this is a transformation from (u, v, w) to $(t, x, y, z) \in \mathbb{M}^4$.

B.1 The Two-dimensional Sphere

I. Spherical coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}u^2 + \sin^2 u \mathrm{d}v^2 \\ x = \sin u \sin v, \;\; y = \sin u \cos v, \;\; z = \cos u \\ 0 \leq u \leq \pi, \;\; 0 \leq v < 2\pi \end{cases}$$

II. Elliptic coordinates

$$\begin{cases} \mathrm{d}s^2 = (\mathrm{dn}^2(u;k) - \tilde{k}^2\mathrm{sn}^2(v;\tilde{k}))(\mathrm{d}u^2 + \mathrm{d}v^2) \\ x = \mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k}), \;\; y = \mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k}), \;\; z = \mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k}) \\ -K \leq u \leq K, \;\; -2\tilde{K} \leq v \leq 2\tilde{K} \end{cases}$$

B.2 The Three-dimensional Sphere

I. Spherical coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}t^2 + \sin^2 t(\mathrm{d}u^2 + \sin^2 u\mathrm{d}v^2) \\ x = \sin t \sin u \cos v,\ y = \sin t \sin u \sin v, \\ z = \sin t \cos u,\ w = \cos t \\ 0 \le t \le \pi,\ 0 \le u \le \pi,\ 0 \le v < 2\pi \end{cases}$$

II. Cylindrical coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}t^2 + \cos^2 t\mathrm{d}u^2 + \sin^2 t\mathrm{d}v^2 \\ x = \cos t \cos u,\ y = \cos t \sin u, \\ z = \sin t \cos v,\ w = \sin t \sin v \\ 0 \le t \le \pi,\ 0 < u \le 2\pi,\ 0 < v \le 2\pi \end{cases}$$

III. Elliptic-cylindrical coordinates of type 1

$$\begin{cases} \mathrm{d}s^2 = (\mathrm{dn}^2(u;\tilde{k}) - k^2\mathrm{sn}^2(t;k))(\mathrm{d}t^2 + \mathrm{d}u^2) + \mathrm{sn}^2(t;k)\mathrm{dn}^2(u;\tilde{k})\mathrm{d}v^2 \\ x = \mathrm{sn}(t;k)\,\mathrm{dn}(u;\tilde{k})\cos v,\ y = \mathrm{sn}(t;k)\,\mathrm{dn}(u;\tilde{k})\sin v, \\ z = \mathrm{dn}(t;k)\,\mathrm{sn}(u,\tilde{k}),\ w = \mathrm{cn}(t;k)\,\mathrm{cn}(u,\tilde{k}) \\ 0 \le t \le 2K(k),\ -\tilde{K}(k) \le u \le \tilde{K}(k),\ 0 \le v < 2\pi, \\ 0 < k^2 < 1,\ k^2 + \tilde{k}^2 = 1 \end{cases}$$

IV. Elliptic-cylindrical coordinates of type 2

$$\begin{cases} \mathrm{d}s^2 = (\mathrm{dn}^2(u;\tilde{k}) - k^2\mathrm{sn}^2(t;k))(\mathrm{d}t^2 + \mathrm{d}u^2) + \mathrm{cn}^2(t;k)\mathrm{cn}^2(u;\tilde{k})\mathrm{d}v^2 \\ x = \mathrm{cn}(t;k)\mathrm{cn}(u;\tilde{k})\cos v,\ y = \mathrm{cn}(t;k)\mathrm{cn}(u;\tilde{k})\sin v, \\ z = \mathrm{sn}(t;k)\mathrm{dn}(v;\tilde{k}),\ w = \mathrm{dn}(t;k)\mathrm{sn}(u;\tilde{k}) \\ 0 \le t \le 2K(k),\ -\tilde{K}(k) \le u \le \tilde{K}(k),\ 0 \le v < 2\pi, \\ 0 < k^2 < 1,\ k^2 + \tilde{k}^2 = 1 \end{cases}$$

V. Spheroelliptic coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}t^2 + \sin^2 t(\mathrm{dn}^2(u;k) - \tilde{k}^2\mathrm{sn}^2(v;\tilde{k}))(\mathrm{d}u^2 + \mathrm{d}v^2) \\ x = \sin t\ \mathrm{sn}(u;k)\,\mathrm{dn}(v;\tilde{k}),\ y = \sin t\ \mathrm{cn}(u;k)\ \mathrm{cn}(v;\tilde{k}), \\ z = \sin t\ \mathrm{dn}(u;k)\ \mathrm{sn}(v;\tilde{k}),\ w = \cos t \\ 0 \le t \le \pi,\ -K(k) \le u < K(k),\ -2\tilde{K}(k) \le v \le 2\tilde{K}(k), \\ 0 < k^2 < 1,\ k^2 + \tilde{k}^2 = 1 \end{cases}$$

VI. Ellipsoidal

$$\begin{cases} \mathrm{d}s^2 = \dfrac{(t-u)(t-v)}{4(t-a)(t-b)(t-1)}\mathrm{d}t^2 + \dfrac{(u-v)(u-t)}{4(u-a)(u-b)(u-1)}\mathrm{d}u^2 + \\ \qquad \dfrac{(v-t)(v-u)}{4(v-a)(v-b)(v-1)}\mathrm{d}v^2 \\ x^2 = \dfrac{(t-1)(u-1)(v-1)}{(a-1)(b-1)(-1)},\ y^2 = \dfrac{(t-a)(u-a)(v-a)}{-a(1-a)(b-a)}, \\ z^2 = \dfrac{(t-b)(u-b)(v-b)}{-b(1-b)(a-b)},\ w^2 = \dfrac{tuv}{ab} \\ 0 < v < 1 < u < b < t < a \end{cases}$$

B.3 Two-dimensional Hyperbolic Space

I. Spherical coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}u^2 + \sinh^2 u \mathrm{d}v^2 \\ t = \cosh u,\ x = \sinh u \cos v,\ y = \sinh u \cos v \\ u > 0,\ 0 \leq v < 2\pi \end{cases}$$

II. Equidistant coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}u^2 + \cosh^2 u \mathrm{d}v^2 \\ t = \cosh u \cosh v,\ x = \cosh u \sinh v,\ y = \sinh u \\ u, v \in \mathbb{R} \end{cases}$$

III. Horicyclic coordinates

$$\begin{cases} \mathrm{d}s^2 = \dfrac{1}{v^2}(\mathrm{d}u^2 + \mathrm{d}v^2) \\ t = \dfrac{1}{2v}(u^2 + v^2 + 1),\ x = \dfrac{1}{2v}(u^2 + v^2 - 1),\ y = \dfrac{u}{v} \\ u \in \mathbb{R},\ v > 0 \end{cases}$$

IV. Elliptic coordinates

$$\begin{cases} \mathrm{d}s^2 = \dfrac{(u-v)}{4(u-a)(u-b)(u-c)}\mathrm{d}u^2 - \dfrac{(u-v)}{4(v-a)(v-b)(v-c)}\mathrm{d}v^2 \\ t^2 = \dfrac{(u-c)(v-c)}{(a-c)(b-c)},\ x^2 = \dfrac{(u-b)(v-b)}{(a-b)(b-c)},\ y^2 = \dfrac{(u-a)(v-a)}{(a-b)(c-a)} \\ c < b < v < a < u \end{cases}$$

V. Hyperbolic coordinates

$$\begin{cases} \mathrm{d}s^2 = \dfrac{(u-v)}{4(u-a)(u-b)(u-c)}\mathrm{d}u^2 - \dfrac{(u-v)}{4(v-a)(v-b)(v-c)}\mathrm{d}v^2 \\ t^2 = \dfrac{(u-b)(v-b)}{(a-b)(c-b)},\ x^2 = \dfrac{(u-c)(v-c)}{(a-c)(c-b)},\ y^2 = \dfrac{(u-a)(v-a)}{(a-b)(c-a)} \\ v < c < b < a < u \end{cases}$$

VI. Semihyperbolic coordinates

$$\begin{cases} \mathrm{d}s^2 = \dfrac{(u+v)}{4u(u^2+1)}\mathrm{d}u^2 + \dfrac{(u+v)}{4v(v^2+1)}\mathrm{d}v^2 \\ t^2 + x^2 = (1+u^2)(1+v^2),\ t^2 - x^2 = 1 + uv,\ y^2 = uv \\ u, v > 0 \end{cases}$$

VII. Elliptic-parabolic coordinates

$$\begin{cases} \mathrm{d}s^2 = (\sec^2 v - \mathrm{sech}^2 u)(\mathrm{d}u^2 + \mathrm{d}v^2) \\ t = \dfrac{\cosh^2 u + \cos^2 v}{2\cosh u \cos v},\ x = \dfrac{\sinh^2 u - \sin^2 v}{2\cosh u \cos v},\ y = \tan v \tanh u \\ u \in \mathbb{R},\ v \in (\frac{-\pi}{2}, \frac{\pi}{2}) \end{cases}$$

VIII. Hyperbolic-parabolic coordinates

$$\begin{cases} \mathrm{d}s^2 = (\mathrm{csch}^2 u + \csc^2 v)(\mathrm{d}u^2 + \mathrm{d}v^2) \\ t = \dfrac{\cosh^2 u + \cos^2 v}{2\sinh u \sin v},\ x = \dfrac{\sinh^2 u - \sin^2 v}{2\sinh u \sin v},\ y = \cot v \coth u \\ u > 0,\ v \in (0, \pi) \end{cases}$$

IX. Semicircular-parabolic coordinates

$$\begin{cases} \mathrm{d}s^2 = \dfrac{u^2+v^2}{u^2v^2}(\mathrm{d}u^2 + \mathrm{d}v^2) \\ t = \dfrac{(u^2+v^2)^2+4}{8uv},\ x = \dfrac{(u^2+v^2)^2-4}{8uv},\ y = \dfrac{v^2-u^2}{2uv} \\ u, v > 0 \end{cases}$$

B.4 Three-dimensional Hyperbolic Space

I. Cylindrical coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}u^2 + \cosh^2 u\,\mathrm{d}v^2 + \sinh^2 u\,\mathrm{d}w^2 \\ t = \cosh u \cosh v,\ x = \sinh u \cos w,\ y = \sinh u \sin w,\ z = \cosh u \sinh v, \\ u, v \in \mathbb{R},\ 0 \le w < 2\pi \end{cases}$$

II. Horicyclic coordinates

$$\begin{cases} \mathrm{d}s^2 = \dfrac{1}{w^2}(\mathrm{d}u^2+\mathrm{d}v^2+\mathrm{d}w^2) \\ t = \dfrac{1}{2}\left(w + \dfrac{u^2+v^2}{w} + \dfrac{1}{w}\right), \ x = \dfrac{u}{w}, \ y = \dfrac{v}{w}, \ z = \dfrac{1}{2}\left(w + \dfrac{u^2+v^2}{w} - \dfrac{1}{w}\right), \\ u, v \in \mathbb{R}, \ w > 0 \end{cases}$$

III. Spheroelliptic coordinates

$$\begin{cases} \mathrm{d}s^2 = \sinh^2 w(\mathrm{dn}^2(u;k) - \tilde{k}^2\mathrm{sn}^2(v;\tilde{k}))(\mathrm{d}u^2+\mathrm{d}v^2)+\mathrm{d}w^2 \\ t = \cosh w, \ x = \mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k})\sinh w, \\ y = \mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k})\sinh w, \ z = \mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k})\sinh w \\ -k \le u \le k, \ -2\tilde{k} \le v \le 2\tilde{k}, \ w > 0 \end{cases}$$

IV. Equidistant-elliptic coordinates

$$\begin{cases} \mathrm{d}s^2 = \cosh^2 w(-\mathrm{dn}^2(u;k) + \tilde{k}^2\mathrm{sn}^2(v;\tilde{k}))(\mathrm{d}u^2+\mathrm{d}v^2)+\mathrm{d}w^2 \\ t = \mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k})\cosh w, \ x = i\mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k})\cosh w, \\ y = i\mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k})\cosh w, \ z = \sinh w \\ i\tilde{k} < u < i\tilde{k} + 2k, \ 0 \le v < 4\tilde{k}, \ w \in \mathbb{R} \end{cases}$$

V. Equidistant-hyperbolic coordinates

$$\begin{cases} \mathrm{d}s^2 = \cosh^2 w(-\mathrm{dn}^2(u;k) + \tilde{k}^2\mathrm{sn}^2(v;\tilde{k}))(\mathrm{d}u^2+\mathrm{d}v^2)+\mathrm{d}w^2 \\ t = -\mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k})\cosh w, \ x = i\mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k})\cosh w, \\ y = i\mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k})\cosh w, \ z = \sinh w \\ i\tilde{k} < u < i\tilde{k} + 2k, \ 0 \le v < 4\tilde{k}, \ w \in \mathbb{R} \end{cases}$$

VI. Equidistant-semi-hyperbolic coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}u^2 + (v+w)\cosh^2 u\left(\dfrac{\mathrm{d}v^2}{4v(1+v^2)} + \dfrac{\mathrm{d}w^2}{4w(1+w^2)}\right) \\ t = \frac{1}{\sqrt{2}}\cosh u\sqrt{\sqrt{(1+v^2)(1+w^2)} + vw + 1}, \\ x = \frac{1}{\sqrt{2}}\cosh u\sqrt{\sqrt{(1+v^2)(1+w^2)} - vw - 1}, \\ y = \sqrt{vw}\cosh u, \ z = \sinh u \\ u \in \mathbb{R}, \ v, w > 0 \end{cases}$$

VII. Equidistant elliptic-parabolic coordinates

$$\begin{cases} \mathrm{d}s^2 = \cosh^2 w(\tanh^2 u + \tan^2 v)(\mathrm{d}u^2 + \mathrm{d}v^2) + \mathrm{d}w^2 \\ t = \dfrac{1}{2}\cosh w(\cosh u \sec v + \cos v \operatorname{sech} u), \\ x = \frac{1}{2}\cosh w(\cosh u \sec v - 2\operatorname{sech} u \sec v + \cos v \operatorname{sech} u), \\ y = \cosh w \tan v \tanh u, \ \ z = \sinh w \\ u, w \in \mathbb{R}, \ -\frac{\pi}{2} < v < \frac{\pi}{2} \end{cases}$$

VIII. Equidistant hyperbolic-parabolic coordinates

$$\begin{cases} \mathrm{d}s^2 = \cosh^2 w(\operatorname{csch}^2 u + \csc^2 v)(\mathrm{d}u^2 + \mathrm{d}v^2) + \mathrm{d}w^2 \\ t = \dfrac{1}{2}\cosh w(\sinh u \csc v - \sin v \operatorname{csch} u + 2\operatorname{csch} u \csc v, \\ x = \frac{1}{2}\cosh w(\sinh u \csc v - \sin v \operatorname{csch} u), \\ y = \cosh w \cot v \coth u, \ \ z = \sinh w \\ u > 0, \ 0 < v < \pi, \ w \in \mathbb{R} \end{cases}$$

IX. Equidistant semi-circular parabolic coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}u^2 + \dfrac{(v^2 + w^2)}{v^2 w^2}\cosh^2 u(\mathrm{d}v^2 + \mathrm{d}w^2) \\ t = \dfrac{(v^2 + w^2) + 4}{8vw}\cosh u, \ \ x = \dfrac{(v^2 + w^2) - 4}{8vw}\cosh u, \\ y = \dfrac{w^2 - v^2}{2vw}\cosh u, \ \ z = \sinh u \\ u \in \mathbb{R}, \ v, w > 0 \end{cases}$$

X. Spherical coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}u^2 + \sinh^2 u(\mathrm{d}v^2 + \sin^2 v \mathrm{d}w^2) \\ t = \cosh u, \ \ x = \sinh u \sin v \cos w, \\ y = \sinh u \sin v \sin w, \ \ z = \sinh u \cos v \\ u > 0, \ 0 < v < \pi, \ 0 \le w < 2\pi \end{cases}$$

XI. Equidistant-cylindrical coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}u^2 + \cosh^2 u(\mathrm{d}v^2 + \sinh^2 v \mathrm{d}w^2) \\ t = \cosh u \cosh v, \ \ x = \cosh u \sinh v \cos w, \\ y = \cosh u \sinh v \sin w, \ \ z = \sinh u, \\ u, v \in \mathbb{R}, \ 0 \le w < 2\pi \end{cases}$$

XII. Equidistant coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}u^2 + \cosh^2 u(\mathrm{d}v^2 + \cosh^2 v\mathrm{d}w^2) \\ t = \cosh u \cosh v \cosh w, \ \ x = \cosh u \cosh v \sinh w, \\ y = \cosh u \sinh v, \ \ z = \sinh u, \\ u, v, w \in \mathbb{R} \end{cases}$$

XIII. Equidistant-horicyclic coordinates[1]

$$\begin{cases} \mathrm{d}s^2 = \dfrac{\cosh^2 w}{v^2}(\mathrm{d}u^2 + \mathrm{d}v^2) + \mathrm{d}w^2 \\ t = \dfrac{1}{2}\left(v + \dfrac{u^2}{v} + \dfrac{1}{v}\right)\cosh w, \ \ x = \dfrac{1}{2}\left(v + \dfrac{u^2}{v} - \dfrac{1}{v}\right)\cosh w, \\ y = \dfrac{u}{v}\cosh w, \ \ z = -\sinh w \\ u, w \in \mathbb{R}, \ \ v > 0 \end{cases}$$

XIV. Horicyclic-cylindrical coordinates

$$\begin{cases} \mathrm{d}s^2 = \dfrac{1}{u^2}\left(\mathrm{d}u^2 + \mathrm{d}v^2 + v^2\mathrm{d}w^2\right) \\ t = \dfrac{1}{2}\left(u + \dfrac{v^2}{u} + \dfrac{1}{u}\right), \ \ x = \dfrac{v}{u}\cos w, \\ y = \dfrac{v}{u}\sin w, \ \ z = \dfrac{1}{2}\left(u + \dfrac{v^2}{u} - \dfrac{1}{u}\right) \\ u, v > 0, \ \ 0 \le w < 2\pi \end{cases}$$

XV. Horicyclic-elliptic coordinates

$$\begin{cases} \mathrm{d}s^2 = \dfrac{1}{v^2}\left((\cosh^2 u - \cos^2 w)(\mathrm{d}u^2 + \mathrm{d}w^2) + \mathrm{d}v^2\right) \\ t = \dfrac{1}{2}\left(v + \dfrac{1}{v}(\cosh^2 u - \sin^2 w) + \dfrac{1}{v}\right), \ \ x = \dfrac{\cosh u \cos w}{v}, \\ y = \dfrac{\sinh u \sin w}{v}, \ \ z = \dfrac{1}{2}\left(v + \dfrac{1}{v}(\cosh^2 u - \sin^2 w) - \dfrac{1}{v}\right) \\ u, v > 0, \ \ -\pi < w < \pi \end{cases}$$

XVI. Horicyclic-parabolic coordinates[2]

[1] These coordinates are different than those listed for XIII in [31]. They were transformed so that the corresponding CKT admits X_4 as a symmetry, rather than X_2.

[2] These coordinates are different than those listed for XVI in [31]. A correction was made so that the constraint $t^2 - x^2 - y^2 - z^2 = 1$ is satisfied.

$$\begin{cases} \mathrm{d}s^2 = \dfrac{1}{v^2}\left(4(u^2+w^2)(\mathrm{d}u^2+\mathrm{d}w^2)+\mathrm{d}v^2\right) \\ t = \dfrac{1}{2}\left(v + \dfrac{(u^2+w^2)^2}{v} + \dfrac{1}{v}\right),\ x = \dfrac{w^2-u^2}{2v}, \\ y = \dfrac{2uw}{v},\ z = \dfrac{1}{2}\left(v + \dfrac{(u^2+w^2)^2}{v} - \dfrac{1}{v}\right), \\ v, w > 0,\ u \in \mathbb{R} \end{cases}$$

XVII. Elliptic-cylindrical 1 coordinates

$$\begin{cases} \mathrm{d}s^2 = -(\mathrm{dn}^2(u;k) - \tilde{k}^2\mathrm{sn}^2(v;\tilde{k}))(\mathrm{d}u^2+\mathrm{d}v^2) - \mathrm{dn}^2(u;k)\mathrm{sn}^2(v;\tilde{k})\mathrm{d}w^2 \\ t = \mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k}),\ x = i\mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k})\cos w, \\ y = i\mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k})\sin w,\ z = i\mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k}) \\ i\tilde{k} < u < i\tilde{k} + 2k,\ 0 \le v < 4\tilde{k},\ 0 \le w < 2\pi \end{cases}$$

XVIII. Elliptic-cylindrical 2 coordinates[3]

$$\begin{cases} \mathrm{d}s^2 = -(\mathrm{dn}^2(u;k) - \tilde{k}^2\mathrm{sn}^2(v;\tilde{k}))(\mathrm{d}u^2+\mathrm{d}v^2) - \mathrm{cn}^2(u;k)\mathrm{cn}^2(v;\tilde{k})\mathrm{d}w^2 \\ t = \mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k}),\ x = i\mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k})\cos w, \\ y = i\mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k})\sin w,\ z = i\mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k}) \\ i\tilde{k} < u < i\tilde{k} + 2k,\ 0 \le v < 4\tilde{k},\ 0 \le w < 2\pi \end{cases}$$

XIX. Elliptic-cylindrical 3 coordinates

$$\begin{cases} \mathrm{d}s^2 = (\tilde{k}^2\mathrm{sn}^2(v;\tilde{k}) - \mathrm{dn}^2(u;k))(\mathrm{d}u^2+\mathrm{d}v^2) + \mathrm{sn}^2(u;k)\mathrm{dn}^2(v;\tilde{k})\mathrm{d}w^2 \\ t = \mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k})\cosh w,\ x = \mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k})\sinh w, \\ y = i\mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k}),\ z = i\mathrm{cn}(u;k)\mathrm{dn}(v;\tilde{k}) \\ i\tilde{k} < u < i\tilde{k} + 2k,\ 0 \le v < 4\tilde{k},\ w \in \mathbb{R} \end{cases}$$

XX. Hyperbolic-cylindrical 1 coordinates

$$\begin{cases} \mathrm{d}s^2 = (\tilde{k}^2\mathrm{sn}^2(v;\tilde{k}) - \mathrm{dn}^2(u;k))(\mathrm{d}u^2+\mathrm{d}v^2) + \mathrm{cn}^2(u;k)\mathrm{cn}^2(v;\tilde{k})\mathrm{d}w^2 \\ t = -\mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k})\cosh w,\ x = -\mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k})\sinh w, \\ y = i\mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k}),\ z = i\mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k}) \\ i\tilde{k} < u < i\tilde{k} + 2k,\ 0 \le v < 4\tilde{k},\ w \in \mathbb{R} \end{cases}$$

XXI. Hyperbolic-cylindrical 2 coordinates[4]

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[3] These coordinates are different than those listed for XVIII in [31]. A correction was made so that the constraint $t^2 - x^2 - y^2 - z^2 = 1$ is satisfied.

[4] These coordinates are different than those listed for XXI in [31]. A correction was made so that the constraint $t^2 - x^2 - y^2 - z^2 = 1$ is satisfied.

$$\begin{cases} \mathrm{d}s^2 = -(\mathrm{dn}^2(u;k) - \tilde{k}^2\mathrm{sn}^2(v;\tilde{k}))(\mathrm{d}u^2+\mathrm{d}v^2) - \mathrm{dn}^2(u;k)\mathrm{sn}^2(v;\tilde{k})\mathrm{d}w^2 \\ t = -\mathrm{cn}(u;k)\mathrm{cn}(v;\tilde{k}),\ x = i\mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k})\cos w, \\ y = i\mathrm{dn}(u;k)\mathrm{sn}(v;\tilde{k})\sin w,\ z = i\mathrm{sn}(u;k)\mathrm{dn}(v;\tilde{k}) \\ i\tilde{k} < u < i\tilde{k} + 2k,\ 0 \le v < 4\tilde{k},\ 0 \le w < 2\pi \end{cases}$$

XXII. Semi-hyperbolic coordinates

$$\begin{cases} \mathrm{d}s^2 = \frac{1}{4}(u+v)\left(\frac{\mathrm{d}u^2}{u(1+u^2)} + \frac{\mathrm{d}v^2}{v(1+v^2)}\right) + uv\mathrm{d}w^2 \\ t = \frac{1}{\sqrt{2}}\sqrt{\sqrt{(1+u^2)(1+v^2)} + uv + 1},\ x = \sqrt{uv}\cos w, \\ y = \sqrt{uv}\sin w,\ z = \frac{1}{\sqrt{2}}\sqrt{\sqrt{(1+u^2)(1+v^2)} - uv - 1}, \\ u, v > 0,\ 0 \le w < 2\pi \end{cases}$$

XXIII. Elliptic-parabolic 1 coordinates[5]

$$\begin{cases} \mathrm{d}s^2 = \mathrm{sech}^2 u \sec^2 v((\mathrm{sech}^2 u - \sec^2 v)(\mathrm{d}u^2 + \mathrm{d}v^2) + \mathrm{d}w^2) \\ t = \frac{1}{2}\mathrm{sech} u \sec v(\cosh^2 u + \cos^2 v + w^2),\ x = w\mathrm{sech} u \sec v, \\ y = \tanh u \tan v,\ z = \frac{1}{2}\mathrm{sech} u \sec v(\cosh^2 u + \cos^2 v + w^2 - 2) \\ u, w \in \mathbb{R},\ -\frac{\pi}{2} < v < \frac{\pi}{2} \end{cases}$$

XXIV. Hyperbolic-parabolic 1 coordinates[6]

$$\begin{cases} \mathrm{d}s^2 = \mathrm{csch}^2 u \csc^2 v((\cosh^2 u - \cos^2 v)(\mathrm{d}u^2+\mathrm{d}v^2)+\mathrm{d}w^2) \\ t = \frac{1}{2}\mathrm{csch} u \csc v(\sinh^2 u - \sin^2 v + w^2 + 2),\ x = w\mathrm{csch} u \csc v, \\ y = \mathrm{coth} u \cot v,\ z = \frac{1}{2}\mathrm{csch} u \csc v(\sinh^2 u - \sin^2 v + w^2) \\ u > 0,\ 0 < v < \pi,\ w \in \mathbb{R} \end{cases}$$

XXV. Elliptic-parabolic 2 coordinates

$$\begin{cases} \mathrm{d}s^2 = \mathrm{sech}^2 u \sec^2 v((\cosh^2 u - \cos^2 v)(\mathrm{d}u^2 + \mathrm{d}v^2) + \sinh^2 u \sin^2 v\mathrm{d}w^2) \\ t = \frac{1}{2}\mathrm{sech} u \sec v(\cosh^2 u + \cos^2 v),\ x = \tanh u \tanh v \cos w, \\ y = \tanh u \tan v \sin w,\ z = \frac{1}{2}\mathrm{sech} u \sec v(\sinh^2 u - \sin^2 v) \\ u \in \mathbb{R},\ -\frac{\pi}{2} < v < \frac{\pi}{2},\ 0 \le w < 2\pi \end{cases}$$

[5] These coordinates are different than those listed for XXIII in [31]. A correction was made so that the constraint $t^2 - x^2 - y^2 - z^2 = 1$ is satisfied.

[6] These coordinates are different than those listed for XXIX in [31]. A correction was made so that the constraint $t^2 - x^2 - y^2 - z^2 = 1$ is satisfied.

XXVI. Hyperbolic-parabolic 2 coordinates

$$\begin{cases} \mathrm{d}s^2 = \operatorname{csch}^2 u \csc^2 v((\cosh^2 u - \cos^2 v)(\mathrm{d}u^2 + \mathrm{d}v^2) + \cosh^2 u \cos^2 v \mathrm{d}w^2) \\ t = \frac{1}{2}\operatorname{csch} u \csc v(\cosh^2 u + \cos^2 v),\ x = \coth u \cot v \cos w, \\ y = \coth u \cot v \sin w,\ z = \frac{1}{2}\operatorname{csch} u \csc v(\sin^2 v - \sinh^2 u), \\ u > 0,\ 0 < v < \pi,\ 0 \leq w < 2\pi \end{cases}$$

XXVII. Semi-circular parabolic coordinates[7]

$$\begin{cases} \mathrm{d}s^2 = \frac{1}{u^2v^2}\Big((u^2+v^2)(\mathrm{d}u^2+\mathrm{d}v^2) + \mathrm{d}w^2\Big) \\ t = \frac{(u^2+v^2)^2 + 4w^2 + 4}{8uv},\ x = \frac{w}{uv}, \\ y = \frac{v^2-u^2}{2uv},\ z = \frac{(u^2+v^2)^2 + 4w^2 - 4}{8uv}, \\ u, v > 0,\ w \in \mathbb{R} \end{cases}$$

XXVIII. Ellipsoidal coordinates

$$\begin{cases} \mathrm{d}s^2 = \frac{(u-w)(u-v)}{4u(u-1)(u-b)(u-a)}\mathrm{d}u^2 + \frac{(v-w)(v-u)}{4v(v-1)(v-b)(v-a)}\mathrm{d}v^2 \\ \qquad + \frac{(u-w)(v-w)}{4w(w-1)(w-b)(w-a)}\mathrm{d}w^2 \\ t^2 = \frac{uvw}{ab},\ x^2 = \frac{(u-1)(v-1)(w-1)}{(a-1)(b-1)}, \\ y^2 = -\frac{(u-b)(v-b)(w-b)}{(a-b)(b-1)b},\ z^2 = \frac{(u-a)(v-a)(w-a)}{(a-b)(a-1)a} \\ 0 < 1 < w < b < v < a < u \end{cases}$$

XXIX. Hyperboloidal coordinates

$$\begin{cases} \mathrm{d}s^2 = \frac{(u-w)(u-v)}{4u(u-1)(u-b)(u-a)}\mathrm{d}u^2 + \frac{(v-w)(v-u)}{4v(v-1)(v-b)(v-a)}\mathrm{d}v^2 \\ \qquad + \frac{(u-w)(v-w)}{4w(w-1)(w-b)(w-a)}\mathrm{d}w^2 \\ t^2 = -\frac{(u-1)(v-1)(w-1)}{(a-1)(b-1)},\ x^2 = -\frac{uvw}{ab}, \\ y^2 = -\frac{(u-b)(v-b)(w-b)}{(a-b)(b-1)b},\ z^2 = \frac{(u-a)(v-a)(w-a)}{(a-b)(a-1)a}, \\ w < 0 < 1 < b < v < a < u \end{cases}$$

[7] These coordinates are different than those listed for XXVII in [31]. A correction was made so that the constraint $t^2 - x^2 - y^2 - z^2 = 1$ is satisfied. As well, we have transformed the coordinates so that the corresponding CKT admits $R_{12} - R_{24}$ as a symmetry, rather than $R_{13} - R_{34}$.

XXX. Semihyperboloidal coordinates[8]

$$\begin{cases} \mathrm{d}s^2 = \dfrac{(u-w)(u-v)}{4u(u-1)(u-b)(u-a)}\mathrm{d}u^2 + \dfrac{(v-w)(v-u)}{4v(v-1)(v-b)(v-a)}\mathrm{d}v^2 \\ \qquad + \dfrac{(u-w)(v-w)}{4w(w-1)(w-b)(w-a)}\mathrm{d}w^2 \\ (ti+x)^2 = \dfrac{2(u-a)(v-a)(w-a)}{a(a-1)(a-b)}, \\ y^2 = \dfrac{(u-1)(v-1)(w-1)}{(a-1)(b-1)},\ z^2 = -\dfrac{uvw}{ab} \\ w < 0 < v < 1 < u \end{cases}$$

XXXI. Elliptic-paraboloidal coordinates[9]

$$\begin{cases} \mathrm{d}s^2 = \dfrac{(u-v)(u-w)}{4u^2(u-1)(u-a)}\mathrm{d}u^2 + \dfrac{(v-u)(v-w)}{4v^2(v-1)(v-a)}\mathrm{d}v^2 \\ \qquad + \dfrac{(v-w)(u-w)}{4w^2(w-1)(w-a)}\mathrm{d}w^2 \\ t^2 = \dfrac{(a(uv+uw+vw)-uvw)^2}{4a^3uvw}, \\ x^2 = \dfrac{((2a+1)uvw - a(uv+uw+vw))^2}{4a^3uvw},\ y^2 = -\dfrac{(u-1)(v-1)(w-1)}{a-1}, \\ z^2 = \dfrac{(u-a)(v-a)(w-a)}{a^2(a-1)}, \\ 0 < w < 1 < v < a < u \end{cases}$$

XXXII. Hyperbolic-paraboloidal coordinates 1[10]

[8] These coordinates are different than those listed for XXX in [31]. A correction was made so that the constraint $t^2 - x^2 - y^2 - z^2 = 1$ is satisfied. In [31], these coordinates are called "paraboloidal coordinates." However, since the integral surfaces consist of two families of non-ruled semihyperboloids and one family of ruled semihyperboloids [62], we have used a name which is more fitting.

[9] No name is given for these coordinates in [31]. Since the integral surfaces consist of 3 families of elliptic paraboloids, this name is appropriate.

[10] No name is given for these coordinates in [31]. Since the integral surfaces consist of 3 families of hyperbolic paraboloids, this name is appropriate.

$$\begin{cases} \mathrm{d}s^2 = \dfrac{(u-v)(u-w)}{4u^2(u-1)(u-a)}\mathrm{d}u^2 + \dfrac{(v-u)(v-w)}{4v^2(v-1)(v-a)}\mathrm{d}v^2 \\ \qquad + \dfrac{(v-w)(u-w)}{4w^2(w-1)(w-a)}\mathrm{d}w^2 \\ t^2 = -\dfrac{((2a+1)uvw - a(uv+uw+vw))^2}{4a^3uvw}, \\ x^2 = -\dfrac{(a(uv+uw+vw)-uvw)^2}{4a^3uvw}, \\ y^2 = \dfrac{(u-a)(v-a)(w-a)}{a^2(a-1)},\ z^2 = -\dfrac{(u-1)(v-1)(w-1)}{a-1} \\ w < 0 < 1 < v < a < u \end{cases}$$

XXXIII. Hyperbolic-paraboloidal coordinates 2[11]

$$\begin{cases} \mathrm{d}s^2 = \dfrac{(u-v)(u-w)}{4u^2(u+1)(u-a)}\mathrm{d}u^2 + \dfrac{(v-u)(v-w)}{4v^2(v+1)(v-a)}\mathrm{d}v^2 \\ \qquad + \dfrac{(v-w)(u-w)}{4w^2(w+1)(w-a)}\mathrm{d}w^2 \\ t^2 = -\dfrac{((2a-1)uvw + a(uv+uw+vw))^2}{4a^3uvw}, \\ x^2 = -\dfrac{(a(uv+uw+vw)-uvw)^2}{4a^3uvw}, \\ y^2 = \dfrac{(u-a)(v-a)(w-a)}{a^2(a+1)},\ z^2 = -\dfrac{(u+1)(v+1)(w+1)}{a+1} \\ w < -1 < 0 < v < a < u \end{cases}$$

XXXIV. Semicircular-paraboloidal coordinates[12]

$$\begin{cases} \mathrm{d}s^2 = \dfrac{(u-v)(u-w)}{4u^3(u-1)}\mathrm{d}u^2 + \dfrac{(v-u)(v-w)}{4v^3(v-1)}\mathrm{d}v^2 + \dfrac{(v-w)(u-w)}{4w^3(w-1)}\mathrm{d}w^2 \\ (t-x)^2 = -uvw,\ y^2+x^2-t^2 = uv+vw+uw-uvw-(u+v+w), \\ y^2 = -\dfrac{(uv+uw+vw-uvw)^2}{4uvw},\ z^2 = (u-1)(v-1)(w-1) \\ w < 0 < v < 1 < u \end{cases}$$

[11] These coordinates are different than those listed for XXXIII in [31]. A correction was made so that the constraint $t^2 - x^2 - y^2 - z^2 = 1$ is satisfied. Also, no name is given for these coordinates in [31]. Since the integral surfaces consist of 3 families of hyperbolic paraboloids, this name is appropriate.

[12] No name is given for these coordinates in [31]. Since the integral surfaces consist of 3 families of semicircular paraboloids, this name is appropriate.

APPENDIX C

CANONICAL CHARACTERISTIC KILLING TENSORS

The following is a list of the canonical characteristic Killing tensors for two and three-dimensional spherical and hyperbolic space. In each case, we specify how the canonical form is defined. For $\mathbb{S}^2$ and $\mathbb{H}^2$, the canonical forms not admitting translational symmetry are defined by a linear combination of the Killing tensor K_1 from Eisenhart's method and the Casimir tensor $\mathcal{C}$; for the remaining cases, since it is possible to detect the presence of the Casimir tensor, and thus subtract it, the canonical form is defined by a multiple of K_1. For $\mathbb{S}^3$ and $\mathbb{H}^3$, the canonical forms not admitting translational symmetry are defined by a linear combination of the pair of Killing tensors K_1 and K_2 from Eisenhart's method and the Casimir tensor $\mathcal{C}$; for the remaining cases, since it is possible to detect the presence of the Casimir tensor, and thus subtract it, the canonical form is defined by a linear combination of K_1 and K_2. In some cases, the canonical form for two or three different coordinate systems is the same, and thus a discriminating parameter is used to determine the coordinate system.

C.1 The Two-dimensional Sphere

C.1.1 Category 1

The web is defined by $K = c_1K_1$, where K_1 is the canonical CKT for the coordinate system and $c_1 \in \mathbb{R}$.

Spherical web

$$K^{ij} = \begin{pmatrix} c_1y^2 & -c_1xy & 0 \\ -c_1xy & c_1x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

C.1.2 Category 2

The web is defined by $K = \alpha K_1 + \gamma \mathcal{C}$, where K_1 is the canonical CKT for the coordinate system, $\alpha, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Elliptic web

$$K^{ij} = \begin{pmatrix} c_1 z^2 + c_2 y^2 & -c_2 xy & -c_1 xz \\ -c_2 xy & c_2 x^2 + c_3 z^2 & -c_3 yz \\ -c_1 xz & -c_3 yz & c_1 x^2 + c_3 y^2 \end{pmatrix}$$

Essential parameters: $k^2 = \dfrac{c_1 - c_3}{c_2 - c_3}$, $\tilde{k}^2 = 1 - k^2$

C.2 The Three-dimensional Sphere

C.2.1 Category 1

The web is defined by $K = \alpha K_1 + \beta K_2 + \gamma \mathcal{C}$, where K_1, K_2 are canonical CKTs for the coordinate system, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Cylindrical web

$$\begin{aligned}
K^{11} &= c_1 y^2 + c_2 \left(z^2 + w^2\right) \\
K^{22} &= c_1 x^2 + c_2 \left(z^2 + w^2\right) \\
K^{33} &= c_2 \left(x^2 + y^2\right) + c_3 w^2 \\
K^{44} &= c_2 \left(x^2 + y^2\right) + c_3 z^2 \\
K^{12} &= -c_1 xy \\
K^{13} &= -c_2 xz \\
K^{14} &= -c_2 xw \\
K^{23} &= -c_2 yz \\
K^{24} &= -c_2 yw \\
K^{34} &= -c_3 zw
\end{aligned}$$

C.2.2 Category 2

The web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Spherical web

$$K^{ij} = \begin{pmatrix} c_1y^2 + c_4z^2 & -c_1xy & -c_4xz & 0 \\ -c_1xy & c_1x^2 + c_4z^2 & -c_4yz & 0 \\ -c_4xz & -c_4yz & c_4x^2 + c_4y^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

C.2.3 Category 3

The web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Spheroelliptic web

$$K^{ij} = \begin{pmatrix} c_1y^2 + c_2z^2 & -c_1xy & -c_2xz & 0 \\ -c_1xy & c_1x^2 + c_3z^2 & -c_3yz & 0 \\ -c_2xz & -c_3yz & c_2x^2 + c_3y^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Essential parameters: $k^2 = \dfrac{c_2 - c_3}{c_1 - c_3}$, $\tilde{k}^2 = 1 - k^2$

C.2.4 Category 4

The web is defined by $K = \alpha K_1 + \beta K_2 + \gamma \mathcal{C}$, where K_1, K_2 are canonical CKTs for the coordinate system, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Elliptic-cylindrical type 1 web and elliptic-cylindrical type 2 web

$$\begin{aligned}
K^{11} &= c_1y^2 + c_2z^2 + c_3w^2 \\
K^{22} &= c_1x^2 + c_2z^2 + c_3w^2 \\
K^{33} &= c_2\left(x^2 + y^2\right) + c_4w^2 \\
K^{44} &= c_3\left(x^2 + y^2\right) + c_4z^2 \\
K^{12} &= -c_1xy \\
K^{13} &= -c_2xz \\
K^{14} &= -c_3xw \\
K^{23} &= -c_2yz \\
K^{24} &= -c_3yw \\
K^{34} &= -c_4zw
\end{aligned}$$

Discriminating parameter: $\delta = \dfrac{c_4 - c_2}{c_4 - c_3}$

Essential parameters: $k^2,\ \tilde{k}^2 = 1 - k^2$

Type 1: $\delta > 0,\ \ k^2 = \dfrac{c_4 - c_2}{c_4 - c_3}$

Type 2: $\delta < 0,\ \ k^2 = \dfrac{c_4 - c_3}{c_2 - c_3}$

C.2.5 Category 5

The web is defined by $K = \alpha K_1 + \beta K_2 + \gamma \mathcal{C}$, where K_1, K_2 are canonical CKTs for the coordinate system, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Ellipsoidal web

$$\begin{aligned}
K^{11} &= c_1 y^2 + c_2 z^2 + c_3 w^2 \\
K^{22} &= c_1 x^2 + c_4 z^2 + c_5 w^2 \\
K^{33} &= c_2 x^2 + c_4 y^2 + c_6 w^2 \\
K^{44} &= c_3 x^2 + c_5 y^2 + c_6 z^2 \\
K^{12} &= -c_1 xy \\
K^{13} &= -c_2 xz \\
K^{14} &= -c_3 xw \\
K^{23} &= -c_4 yz \\
K^{24} &= -c_5 yw \\
K^{34} &= -c_6 zw
\end{aligned}$$

Parameter relation:

$$(c_3 + c_4)(c_1 c_6 - c_2 c_5) + (c_2 + c_5)(c_3 c_4 - c_1 c_6) + (c_1 + c_6)(c_2 c_5 - c_3 c_4) = 0$$

Essential parameters:

$$a = \frac{c_1(c_2 - c_4) + c_6(c_2 - c_3) - c_2(c_3 + c_4) + 2c_3 c_4}{c_1(c_2 - c_4) + c_4(c_6 - c_2) + c_5(c_4 - c_6)},$$

$$b = \frac{c_2(c_1 - c_4) + c_1(c_5 - c_4) - c_3(c_1 + c_5) + 2c_3 c_4}{c_1(c_2 - c_4) + c_4(c_6 - c_2) + c_5(c_4 - c_6)}$$

C.3 Two-dimensional Hyperbolic Space

C.3.1 Category 1

The web is defined by $K = c_1 K_1$, where K_1 is the canonical CKT for the coordinate system and $c_1 \in \mathbb{R}$.

Spherical web

$$K^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1 y^2 & -c_1 xy \\ 0 & -c_1 xy & c_1 x^2 \end{pmatrix}$$

C.3.2 Category 2

The web is defined by $K = c_1 K_1$, where K_1 is the canonical CKT for the coordinate system and $c_1 \in \mathbb{R}$.

Equidistant web

$$K^{ij} = \begin{pmatrix} c_1 x^2 & c_1 tx & 0 \\ c_1 tx & c_1 t^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

C.3.3 Category 3

The web is defined by $K = \alpha K_1 + \gamma \mathcal{C}$, where K_1 is the canonical CKT for the coordinate system, $\alpha, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Horicyclic web

$$K^{ij} = \begin{pmatrix} c_2 x^2 + c_3 y^2 & c_1 y^2 + c_2 tx & c_3 ty - c_1 xy \\ c_1 y^2 + c_2 tx & c_2 t^2 + (2c_1 - c_3) y^2 & c_1 ty - (2c_1 - c_3) xy \\ c_3 ty - c_1 xy & c_1 ty - (2c_1 - c_3) xy & c_3 t^2 + (2c_1 - c_3) x^2 - 2c_1 tx \end{pmatrix}$$

C.3.4 Category 4

The web is defined by $K = \alpha K_1 + \gamma \mathcal{C}$, where K_1 is the canonical CKT for the coordinate system, $\alpha, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Elliptic web and hyperbolic web

$$K^{ij} = \begin{pmatrix} c_2x^2 + c_1y^2 & c_2tx & c_1ty \\ c_2tx & c_3y^2 + c_2t^2 & -c_3xy \\ c_1ty & -c_3xy & c_3x^2 + c_1t^2 \end{pmatrix}$$

Essential parameters: $k^2,\ \tilde{k}^2 = 1 - k^2$

Elliptic web: $k^2 = \dfrac{c_1 + c_3}{c_2 + c_3}$

Hyperbolic web: $k^2 = \dfrac{c_1 + c_3}{c_1 - c_2}$

Semihyperbolic web

$$K^{ij} = \begin{pmatrix} c_1(x^2 + y^2) & c_1tx - c_2y^2 & c_1ty + c_2xy \\ c_1tx - c_2y^2 & c_1(t^2 - y^2) & c_1xy - c_2ty \\ c_1ty + c_2xy & c_1xy - c_2ty & c_1(t^2 - x^2) + 2\,c_2xt \end{pmatrix}$$

Elliptic-parabolic web

$$K^{ij} = \begin{pmatrix} c_1(x^2 + y^2) & c_1tx + \frac{1}{2}(c_1 + c_2)y^2 & c_1ty - \frac{1}{2}(c_1 + c_2)xy \\ c_1tx + \frac{1}{2}(c_1 + c_2)y^2 & c_1t^2 + c_2y^2 & \frac{1}{2}(c_1 + c_2)ty - c_2xy \\ c_1ty - \frac{1}{2}(c_1 + c_2)xy & \frac{1}{2}(c_1 + c_2)ty - c_2xy & c_1t^2 + c_2x^2 - (c_1 + c_2)tx \end{pmatrix}$$

Hyperbolic-parabolic web

$$K^{ij} = \begin{pmatrix} c_1y^2 - c_2x^2 & \frac{1}{2}(c_1 + c_2)y^2 - c_2tx & c_1ty - \frac{1}{2}(c_1 + c_2)xy \\ \frac{1}{2}(c_1 + c_2)y^2 - c_2tx & c_2(y^2 - t^2) & \frac{1}{2}(c_1 + c_2)ty - c_2xy \\ c_1ty - \frac{1}{2}(c_1 + c_2)xy & \frac{1}{2}(c_1 + c_2)ty - c_2xy & c_1t^2 + c_2x^2 - (c_1 + c_2)tx \end{pmatrix}$$

Semicircular-parabolic web

$$K^{ij} = \begin{pmatrix} c_1(x^2 + y^2) + 2c_2xy & c_1tx + c_2(ty + xy) & c_1ty + c_2(tx - x^2) \\ c_1tx + c_2(ty + xy) & c_1(t^2 - y^2) + 2c_2ty & c_1xy + c_2(t^2 - tx) \\ c_1ty + c_2(tx - x^2) & c_1xy + c_2(t^2 - tx) & c_1(t^2 - x^2) \end{pmatrix}$$

C.4 Three-dimensional Hyperbolic Space

C.4.1 Category 1

Each web is defined by $K = \alpha K_1 + \beta K_2 + \gamma \mathcal{C}$, where K_1, K_2 are canonical CKTs for the coordinate system, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Webs XIX & XX

$$
\begin{aligned}
K^{11} &= a_4x^2 - a_1y^2 - a_3z^2 \\
K^{22} &= a_4t^2 + a_1y^2 + a_3z^2 \\
K^{33} &= -a_1t^2 + a_1x^2 + a_2z^2 \\
K^{44} &= -a_3t^2 + a_3x^2 + a_2y^2 \\
K^{12} &= a_4tx \\
K^{13} &= -a_1ty \\
K^{14} &= -a_3tz \\
K^{23} &= -a_1xy \\
K^{24} &= -a_3xz \\
K^{34} &= -a_2yz
\end{aligned}
$$

Essential parameter: $k = \dfrac{a_3 - a_2}{a_1 - a_2}$

Web XX: Elliptic-cylindrical III coordinates $k > 0$

Web XIX: Hyperbolic-cylindrical I coordinates $k < 0$

C.4.2 Category 2

Each web is defined by $K = \alpha K_1 + \beta K_2 + \gamma \mathcal{C}$, where K_1, K_2 are canonical CKTs for the coordinate system, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Webs XVII, XVIII & XXI

$$
\begin{aligned}
K^{11} &= a_4(x^2 + y^2) + a_6z^2 \\
K^{22} &= a_1y^2 + a_4t^2 + a_2z^2 \\
K^{33} &= a_1x^2 + a_4t^2 + a_2z^2 \\
K^{44} &= a_6t^2 + a_2(x^2 + y^2) \\
K^{12} &= a_4tx \\
K^{13} &= a_4ty \\
K^{14} &= a_6tz \\
K^{23} &= -a_1xy \\
K^{24} &= -a_2xz \\
K^{34} &= -a_2yz
\end{aligned}
$$

Essential parameter: $k = \dfrac{a_2 + a_6}{a_2 + a_4}$

Web XVII: Elliptic-cylindrical I coordinates $0 < k < 1$

Web XVIII: Hyperbolic-cylindrical I coordinates $k > 1$

Web XXI: Hyperbolic-cylindrical II coordinates $k < 0$

Web XXII: Semihyperbolic coordinates

$$
\begin{aligned}
K^{11} &= a_4(x^2+y^2+z^2) \\
K^{22} &= a_1y^2+a_4(t^2-z^2)-2b_8tz \\
K^{33} &= a_1x^2+a_4(t^2-z^2)-2b_8tz \\
K^{44} &= a_4(t^2-x^2-y^2) \\
K^{12} &= a_4tx-b_8xz \\
K^{13} &= a_4ty-b_8yz \\
K^{14} &= b_8(x^2+y^2)+a_4tz \\
K^{23} &= a_1xy \\
K^{24} &= b_8tx+a_4xz \\
K^{34} &= b_8ty+a_4yz
\end{aligned}
$$

Web XXV: Elliptic-parabolic 2 coordinates

$$
\begin{aligned}
K^{11} &= a_4(x^2+y^2+z^2) \\
K^{22} &= a_1y^2+a_2z^2+a_4t^2-(a_4+a_2)tz \\
K^{33} &= a_1x^2+a_2z^2+a_4t^2-(a_4+a_2)tz \\
K^{44} &= a_2(x^2+y^2)+a_4t^2 \\
K^{12} &= a_4tx-\tfrac{1}{2}(a_4+a_2)xz \\
K^{13} &= a_4ty-\tfrac{1}{2}(a_4+a_2)yz \\
K^{14} &= a_4tz+\tfrac{1}{2}(a_4+a_2)(x^2+y^2) \\
K^{23} &= -a_1xy \\
K^{24} &= \tfrac{1}{2}(a_4+a_2)tx-a_2xz \\
K^{34} &= a_4yz-\tfrac{1}{2}(a_4+a_2)ty
\end{aligned}
$$

Web XXVI: Hyperbolic-parabolic 2 coordinates

$$
\begin{aligned}
K^{11} &= a_4(x^2+y^2)+a_6z^2 \\
K^{22} &= a_1y^2+a_4t^2-a_6z^2+(a_4-a_6)tz \\
K^{33} &= a_1x^2+a_4t^2-a_6z^2+(a_4-a_6)tz \\
K^{44} &= a_6(t^2-x^2-y^2) \\
K^{12} &= a_4tx+\tfrac{1}{2}(a_4-a_6)xz \\
K^{13} &= a_4ty+\tfrac{1}{2}(a_4-a_6)yz \\
K^{14} &= a_6tz+\tfrac{1}{2}(a_6-a_4)(x^2+y^2) \\
K^{23} &= -a_1xy \\
K^{24} &= a_6xz++\tfrac{1}{2}(a_6-a_4)tx \\
K^{34} &= \tfrac{1}{2}(a_6-a_4)ty+a_6yz
\end{aligned}
$$

C.4.3 Category 3

Each web is defined by $K = \alpha K_1 + \beta K_2 + \gamma \mathcal{C}$, where K_1, K_2 are canonical CKTs for the coordinate system, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Web XXIII: Elliptic-parabolic I coordinates

$$
\begin{aligned}
K^{11} &= a_4x^2+a_6y^2+a_6z^2 \\
K^{22} &= \tfrac{1}{2}\,(a_2-a_6)\,y^2+(a_4-2\,a_6)\,z^2+a_4t^2+2\,(a_6-a_4)\,zt \\
K^{33} &= \tfrac{1}{2}\,(a_2-a_6)\,x^2+a_2z^2+a_6t^2-(a_2+a_6)\,zt \\
K^{44} &= a_2y^2+(a_4-2\,a_6)\,x^2+a_6t^2 \\
K^{12} &= a_4xt+(a_6-a_4)\,zx \\
K^{13} &= a_6yt-\tfrac{1}{2}\,(a_2+a_6)\,zy \\
K^{14} &= a_6zt+\tfrac{1}{2}\,(a_2+a_6)\,y^2-(a_6-a_4)\,x^2 \\
K^{23} &= -\tfrac{1}{2}\,(a_2-a_6)\,yx \\
K^{24} &= -(a_4-2\,a_6)\,zx-(a_6-a_4)\,xt \\
K^{34} &= -a_2zy+\tfrac{1}{2}\,(a_2+a_6)\,yt
\end{aligned}
$$

Web XXIV: Hyperbolic-parabolic I coordinates

$$
\begin{aligned}
K^{11} &= (a_3 + 2\,a_6)\,x^2 - (2\,a_1 + a_6)\,y^2 + a_6 z^2 \\
K^{22} &= a_1 y^2 + a_3 z^2 + (a_3 + 2\,a_6)\,t^2 - 2\,(a_3 + a_6)\,zt \\
K^{33} &= a_1 x^2 - a_6 z^2 - (2\,a_1 + a_6)\,t^2 + 2\,(a_1 + a_6)\,zt \\
K^{44} &= -a_6 y^2 + a_3 x^2 + a_6 t^2 \\
K^{12} &= (a_3 + 2\,a_6)\,xt - (a_3 + a_6)\,zx \\
K^{13} &= -\,(2\,a_1 + a_6)\,yt + (a_1 + a_6)\,zy \\
K^{14} &= a_6 zt + (-a_1 - a_6)\,y^2 + (a_3 + a_6)\,x^2 \\
K^{23} &= -a_1 yx \\
K^{24} &= -a_3 zx + (a_3 + a_6)\,xt \\
K^{34} &= a_6 zy - (a_1 + a_6)\,yt
\end{aligned}
$$

Web XXVII: Semicircular-parabolic coordinates

$$
\begin{aligned}
K^{11} &= a_4 x^2 + a_6 y^2 + a_6 z^2 + 2\,b_2 zy \\
K^{22} &= -a_6 y^2 + (a_4 - 2\,a_6)\,z^2 + a_4 t^2 - 2\,b_2 zy + 2\,b_2 yt + 2\,(a_6 - a_4)\,zt \\
K^{33} &= -a_6 x^2 - a_6 z^2 + a_6 t^2 \\
K^{44} &= -a_6 y^2 + (a_4 - 2\,a_6)\,x^2 + a_6 t^2 + 2\,b_2 yt \\
K^{12} &= a_4 xt + b_2 yx + (a_6 - a_4)\,zx \\
K^{13} &= a_6 yt - b_2 x^2 - b_2 z^2 + b_2 zt \\
K^{14} &= a_6 zt + b_2 zy - (a_6 - a_4)\,x^2 + b_2 yt \\
K^{23} &= a_6 yx + b_2 zx - b_2 xt \\
K^{24} &= -\,(a_4 - 2\,a_6)\,zx + b_2 yx - (a_6 - a_4)\,xt \\
K^{34} &= a_6 zy - b_2 x^2 - b_2 zt + b_2 t^2
\end{aligned}
$$

C.4.4 Category 4

The web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Web III: Sphero-elliptic coordinates

$$
K^{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1 y^2 + a_3 z^2 & -a_1 xy & -a_3 xz \\ 0 & -a_1 xy & a_1 x^2 + a_2 z^2 & -a_2 yz \\ 0 & -a_3 xz & -a_2 yz & a_3 x^2 + a_2 y^2 \end{pmatrix}
$$

Essential parameter: $\tilde{k}^2 = \dfrac{a_3 - a_1}{a_2 - a_1}$

C.4.5 Category 5

Each web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Webs IV, V

$$K^{ij} = \begin{pmatrix} a_4x^2 + a_5y^2 & a_4tx & a_5ty & 0 \\ a_4tx & a_1y^2 + a_4t^2 & -a_1xy & 0 \\ a_5ty & -a_1xy & a_5t^2 + a_1x^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Essential parameter: $k = \dfrac{a_1 + a_5}{a_5 - a_4}$

Web IV: Equidistant elliptic coordinates $k < 0$

Web V: Equidistant hyperbolic coordinates $k \geq 0$

Web VI: Equidistant semihyperbolic coordinates

$$K^{ij} = \begin{pmatrix} a_4x^2 + a_4y^2 & a_4tx - b_4y^2 & a_4ty + b_4xy & 0 \\ a_4tx - b_4y^2 & a_4t^2 - a_4y^2 & a_4xy - b_4ty & 0 \\ a_4ty + b_4xy & a_4xy - b_4ty & a_4t^2 - a_4x^2 + 2\,b_4tx & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Web VII: Equidistant elliptic-parabolic coordinates

$$K^{ij} = \begin{pmatrix} a_4(x^2 + y^2) & a_4tx - b_4y^2 & a_4ty + b_4xy & 0 \\ a_4tx - b_4y^2 & -(2b_4 + a_4)y^2 + a_4t^2 & (2b_4 + a_4)xy - b_4ty & 0 \\ a_4ty + b_4xy & (2b_4 + a_4)xy - b_4ty & a_4(t^2 - x^2) + 2b_4(tx - x^2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Web VIII: Equidistant hyperbolic-parabolic coordinates

$$K^{ij} = \begin{pmatrix} a_4x^2+(a_4-2b_4)y^2 & a_4tx-b_4y^2 & (a_4-2b_4)ty+b_4xy & 0 \\ a_4tx-b_4y^2 & a_4t^2-a_4y^2 & a_4xy-b_4ty & 0 \\ (a_4-2b_4)ty+b_4xy & a_4xy-b_4ty & (a_4-2b_4)t^2-a_4x^2+2\,b_4tx & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Web IX: Equidistant semicircular-parabolic coordinates

$$K^{ij} = \begin{pmatrix} a_4(x^2+y^2)+2b_{13}\,xy & a_4tx+b_{13}(ty+xy) & a_4ty+b_{13}(tx-x^2) & 0 \\ a_4tx+b_{13}(ty+xy) & a_4(t^2-y^2)+2\,b_{13}ty & a_4xy+b_{13}(t^2-tx) & 0 \\ a_4ty+b_{13}(tx-x^2) & a_4xy+b_{13}(t^2-tx) & a_4(t^2-x^2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

C.4.6 Category 6

Each web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Web XV: Horicyclic elliptic coordinates

$$\begin{aligned} K^{11} &= a_4x^2+a_5y^2 \\ K^{22} &= (a_4-a_5)y^2+a_4\,(t-z)^2 \\ K^{33} &= a_5\,(t-z)^2+(a_4-a_5)x^2 \\ K^{44} &= a_4x^2+a_5y^2 \\ K^{12} &= a_4x\,(t-z) \\ K^{13} &= a_5y\,(t-z) \\ K^{14} &= a_4x^2+a_5y^2 \\ K^{23} &= (a_5-a_4)xy \\ K^{24} &= a_4x\,(t-z) \\ K^{34} &= a_5y\,(t-z) \end{aligned}$$

Web XVI: Horicyclic parabolic coordinates

$$
\begin{aligned}
K^{11} &= a_4\left(x^2+y^2\right)\\
K^{22} &= a_4\left(t-z\right)^2\\
K^{33} &= a_4\left(t-z\right)^2+2\,b_1x\left(t-z\right)\\
K^{44} &= a_4\left(x^2+y^2\right)\\
K^{12} &= a_4x\left(t-z\right)-b_1y^2\\
K^{13} &= a_4y\left(t-z\right)+b_1xy\\
K^{14} &= a_4\left(x^2+y^2\right)\\
K^{23} &= -b_1y\left(t-z\right)\\
K^{24} &= a_4x\left(t-z\right)-b_1y^2\\
K^{34} &= a_4y\left(t-z\right)+b_1xy
\end{aligned}
$$

C.4.7 Category 7

This web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Web II: Horicyclic coordinates

$$
K^{ij} = \begin{pmatrix}
b_8y^2+a_3x^2 & a_3x\left(t-z\right) & b_8y\left(t-z\right) & b_8y^2+a_3x^2\\
a_3x\left(t-z\right) & a_3\left(t-z\right)^2 & 0 & a_3x\left(t-z\right)\\
b_8y\left(t-z\right) & 0 & b_8\left(t-z\right)^2 & b_8y\left(t-z\right)\\
b_8y^2+a_3x^2 & a_3x\left(t-z\right) & -b_8zy+b_8yt & b_8y^2+a_3x^2
\end{pmatrix}
$$

C.4.8 Category 8

This web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Web I: Cylindrical coordinates

$$
K^{ij} = \begin{pmatrix}
a_6z^2 & 0 & 0 & a_6tz\\
0 & a_1y^2 & -a_1xy & 0\\
0 & -a_1xy & a_1x^2 & 0\\
a_6tz & 0 & 0 & a_6t^2
\end{pmatrix}
$$

C.4.9 Category 9

This web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Web X: Spherical coordinates

$$K^{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1y^2 + a_3z^2 & -a_1xy & -a_3xz \\ 0 & -a_1xy & a_1x^2 + a_3z^2 & -a_3yz \\ 0 & -a_3xz & -a_3yz & a_3x^2 + a_3y^2 \end{pmatrix}$$

C.4.10 Category 10

This web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Web XIV: Horicyclic-cylindrical coordinates

$$K^{ij} = \begin{pmatrix} a_1\left(x^2+y^2\right) & a_1\left(xt - zx\right) & a_1\left(yt - zy\right) & a_1\left(x^2+y^2\right) \\ a_1\left(xt - zx\right) & a_1\left(t-z\right)^2 + a_2y^2 & -a_2yx & a_1\left(xt - zx\right) \\ a_1\left(yt - zy\right) & -a_2yx & a_1(t-z)^2 + a_2x^2 & a_1\left(yt - zy\right) \\ a_1\left(x^2+y^2\right) & a_1\left(xt - zx\right) & a_1\left(yt - zy\right) & a_1\left(x^2+y^2\right) \end{pmatrix}$$

C.4.11 Category 11

This web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Web XI:

$$K^{ij} = \begin{pmatrix} a_4\left(x^2+y^2\right) & a_4xt & a_4ty & 0 \\ a_4xt & a_1y^2 + a_4t^2 & -a_1xy & 0 \\ a_4ty & -a_1xy & a_1x^2 + a_4t^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

C.4.12 Category 12

This web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Web XIII:

$$K^{ij} = \begin{pmatrix} a_1(y^2+x^2)+a_2y^2 & a_1xt-a_2y^2 & a_1yt+a_2(yt+xy) & 0 \\ a_1xt-a_2y^2 & a_1(t^2-y^2)+a_2y^2 & a_1xy-a_2(yt+xy) & 0 \\ a_1yt+a_2(yt+xy) & a_1xy-a_2(yt+xy) & a_1(t^2-x^2)+a_2(x+t)^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

C.4.13 Category 13

This web is defined by $K = \alpha K_1 + \beta K_2$, where K_1, K_2 are canonical CKTs for the coordinate system and $\alpha, \beta \in \mathbb{R}$.

Web XII:

$$K^{ij} = \begin{pmatrix} (a_1+a_2)x^2+a_1y^2 & (a_1+a_2)tx & a_1ty & 0 \\ (a_1+a_2)tx & (a_1+a_2)t^2-a_1y^2 & a_1xy & 0 \\ a_1ty & a_1xy & a_1\left(t^2-x^2\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

C.4.14 Category 14

Each web is defined by $K = \alpha K_1 + \beta K_2 + \gamma \mathcal{C}$, where K_1, K_2 are canonical CKTs for the coordinate system, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mathcal{C}$ is the Casimir tensor.

Webs XXVIII and XXIX:

$$
\begin{aligned}
K^{11} &= a_4x^2 + a_5y^2 + a_6z^2 \\
K^{22} &= a_3z^2 + a_1y^2 + a_4t^2 \\
K^{33} &= a_5t^2 + a_1x^2 + a_2z^2 \\
K^{44} &= a_6t^2 + a_3x^2 + a_2y^2 \\
K^{12} &= a_4xt \\
K^{13} &= a_5ty \\
K^{14} &= a_6tz \\
K^{23} &= -a_1xy \\
K^{24} &= -a_3xz \\
K^{34} &= -a_2yz
\end{aligned}
$$

Parameter relation:

$$(a_6 + a_2)(a_5 + a_1)(a_3 + a_4) - (a_4 + a_1)(a_2 + a_5)(a_3 + a_6) = 0$$

Essential parameters: a, b

Distinguishing parameter: $\delta = \dfrac{(a_1 - a_3)(a_2 + a_5)}{(a_2 + a_6)(a_1 - a_2) + (a_2 + a_5)(a_2 - a_3)}$

XXVIII: $\delta > 0$,

$$
a, b = \begin{cases}
\frac{(a_1-a_3)(a_2+a_5)}{(a_2+a_6)(a_1-a_2)+(a_2+a_5)(a_2-a_3)},\ \frac{(a_1-a_3)(a_2+a_6)}{(a_2+a_6)(a_1-a_2)+(a_2+a_5)(a_2-a_3)} & \text{if } a_6 \neq -a_2, \\
 & \quad a_3 \neq a_2 \\
\frac{a_2+a_4}{a_2+a_6},\ \frac{a_2+a_4}{a_2+a_5} & \text{if } a_3 = a_2 \\
\frac{a_5+a_3-a_6+a_4}{a_3+a_5},\ \frac{a_5+a_1-a_6+a_4}{a_1+a_5} & \text{if } a_6 = -a_2
\end{cases}
$$

XXIX: $\delta < 0$,

$$
a, b = \begin{cases}
\frac{(a_5-a_6)(a_1-a_2)}{(a_3-a_2)(a_2+a_5)-(a_1-a_2)(a_2+a_6)},\ \frac{(a_5-a_6)(a_3-a_2)}{(a_3-a_2)(a_2+a_5)-(a_1-a_2)(a_2+a_6)} & \text{if } a_6 \neq -a_2, \\
 & \quad a_3 \neq a_2 \\
\frac{a_2+a_4}{a_2-a_3},\ \frac{a_2+a_4}{a_2-a_1} & \text{if } a_6 = -a_2 \\
\frac{a_1+a_6-a_3-a_4}{a_1+a_6},\ \frac{a_5+a_1-a_3-a_4}{a_1+a_5} & \text{if } a_3 = a_2
\end{cases}
$$

Web XXX:

$$\begin{aligned}
K^{11} &= a_4x^2 - a_1y^2 - a_3z^2 \\
K^{22} &= a_1y^2 + a_3z^2 + a_4t^2 \\
K^{33} &= a_1x^2 + a_2z^2 - a_1t^2 + 2\,b_4tx \\
K^{44} &= a_2y^2 + a_3x^2 - a_3t^2 - 2\,b_{12}tx \\
K^{12} &= a_4tx - b_4y^2 + b_{12}z^2 \\
K^{13} &= -a_1ty + b_4xy \\
K^{14} &= -a_3tz - b_{12}xz \\
K^{23} &= -a_1xy - b_4ty \\
K^{24} &= -a_3xz + b_{12}tz \\
K^{34} &= -a_2yz
\end{aligned}$$

Parameter relation:

$$b_4\,(a_3 - a_2)\,(a_3 + a_4) + b_{12}\,(a_1 + a_4)\,(a_1 - a_2) + b_{12}b_4\,(b_{12} + b_4) = 0$$

Essential parameters: $a = \alpha + \beta i,\ b = \alpha - \beta i$, where

$$\alpha, \beta = \begin{cases} \frac{(a_3+a_4)(b_4(a_3+a_4)+b_{12}(a_2-a_1))}{(b_4+b_{12})((a_3+a_4)^2+b_{12}^2)},\ \frac{-b_{12}(b_4(a_3+a_4)+b_{12}(a_2-a_1))}{(b_4+b_{12})((a_3+a_4)^2+b_{12}^2)} & \text{if } b_{12} \neq 0 \\ \frac{b_4^2+(a_1+a_4)(a_1-a_2)}{b_4^2+(a_1+a_4)^2},\ \frac{-b_4(a_2+a_4)}{b_4^2+(a_1+a_4)^2} & \text{if } b_{12} = 0 \end{cases}$$

Web XXXI

$$\begin{aligned}
K^{11} &= a_4(x^2 + y^2) + a_6z^2 \\
K^{22} &= a_1y^2 + a_3z^2 + a_4t^2 \\
K^{33} &= a_1x^2 + a_2z^2 + a_4t^2 + (a_1 + a_4)tx \\
K^{44} &= a_2y^2 + a_3x^2 + a_6t^2 + (a_3 + a_6)tx \\
K^{12} &= a_4tx - \tfrac{1}{2}(a_1 + a_4)y^2 - \tfrac{1}{2}(a_3 + a_6)z^2 \\
K^{13} &= a_4ty + \tfrac{1}{2}(a_1 + a_4)xy \\
K^{14} &= a_6tz + \tfrac{1}{2}(a_3 + a_6)xz \\
K^{23} &= -a_1xy - \tfrac{1}{2}(a_1 + a_4)ty \\
K^{24} &= -a_3xz - \tfrac{1}{2}(a_3 + a_6)tz \\
K^{34} &= -a_2yz
\end{aligned}$$

Parameter relation:

$$(a_3 + a_6)(a_4 + a_1 - 2a_2 + a_3 - a_6) + 2(a_3 - a_2)(a_6 - a_3 - 2a_4) = 0$$

Essential parameter: $a = \begin{cases} \frac{a_4-a_1+2a_2}{2a_2+a_6-a_3} & \text{if } a_3 \neq 2a_2 + a_6 \\ \frac{a_1+a_4}{a_3+a_6} & \text{if } a_3 = 2a_2 + a_6 \end{cases}$

XXXII and XXXIII

$$
\begin{aligned}
K^{11} &= -a_3x^2 + a_5y^2 + a_6z^2 \\
K^{22} &= a_1y^2 + a_3(z^2 - t^2) \\
K^{33} &= a_1x^2 + a_2z^2 + a_5t^2 + (a_1 + a_5)tx \\
K^{44} &= a_2y^2 + a_3x^2 + a_6t^2 + (a_3 + a_6)tx \\
K^{12} &= -a_3tx - \tfrac{1}{2}(a_1 + a_5)y^2 - \tfrac{1}{2}(a_3 + a_6)z^2 \\
K^{13} &= a_5ty + \tfrac{1}{2}(a_1 + a_5)xy \\
K^{14} &= a_6tz + \tfrac{1}{2}(a_3 + a_6)xz \\
K^{23} &= -a_1xy - \tfrac{1}{2}(a_1 + a_5)ty \\
K^{24} &= -a_3xz - \tfrac{1}{2}(a_3 + a_6)tz \\
K^{34} &= -a_2yz
\end{aligned}
$$

Parameter relation:

$$(a_1 + a_5)(a_6 - a_3 + 2a_2) + (a_5 - a_1 + 2a_3)(a_1 - a_5 - 2a_2) = 0$$

Discriminating parameter: $\delta = \dfrac{a_3 - a_6 - 2a_2}{a_5 - a_1 + 2a_2}$

XXXII: $\delta < 0$

Essential parameter: $a = \begin{cases} -\dfrac{a_3 - a_6 - 2a_2}{a_5 - a_1 + 2a_2} & \text{if } a_5 \neq a_1 - 2a_2 \\ \dfrac{a_3 + a_6}{a_1 + a_5} & \text{if } a_5 = a_1 - 2a_2 \end{cases}$

XXXIII: $\delta > 0$

Essential parameter: $a = \begin{cases} \dfrac{a_3 - a_6 - 2a_2}{a_5 - a_1 + 2a_2} & \text{if } a_5 \neq a_1 - 2a_2 \\ -\dfrac{a_3 + a_6}{a_1 + a_5} & \text{if } a_5 = a_1 - 2a_2 \end{cases}$

Web XXXIV:

$$
\begin{aligned}
K^{11} &= b_6(x^2+y^2)+(b_{13}+b_6)z^2-2b_{13}xy \\
K^{22} &= b_6(t^2-y^2)+(b_{13}-b_6+2a_4)z^2-2b_{13}ty \\
K^{33} &= b_6(t^2-x^2)+(a_4-b_6)z^2 \\
K^{44} &= (b_{13}+b_6)t^2+(b_{13}+2a_4-b_6)x^2+(a_4-b_6)y^2+ \\
&\quad 2a_4y(t-x)-2(a_4+b_{13})tx \\
K^{12} &= (b_{13}+a_4)z^2-b_{13}y(t+x)+b_6tx \\
K^{13} &= b_{13}x\,(x-t)-a_4z^2+b_6ty \\
K^{14} &= b_{13}z\,(t-x)+a_4z\,(y-x)+b_6tz \\
K^{23} &= b_{13}t\,(x-t)-a_4z^2+b_6xy \\
K^{24} &= (-b_{13}+b_6-2a_4)xz+(b_{13}+a_4)tz+a_4yz \\
K^{34} &= (b_6-a_4)yz+a_4z(x-t)
\end{aligned}
$$

APPENDIX D

MOVING FRAME MAPS ON $\mathcal{K}^2(\mathbb{H}^3)$

In what follows, we show how to construct a moving frame map which sends constant vectors $v \in \mathbb{M}^4$ to a canonical form. The exposition follows Appendices C.2-C.4 in [37] for constant vectors in $\mathbb{M}^3$.

D.0.15 Transformation of a Constant Vector to $(0,0,0,1)$

For a constant spacelike vector $v^i = (v_1, v_2, v_3, v_4) \in \mathbb{M}^4$ satisfying $g_{ij}v^iv^j = 1$, let us determine the most general transformation $\Lambda^i{}_j \in SO(3,1)$ which maps $\tilde{v}^i = (0,0,0,1)$ to v. To this end, let us first map v to $v' = (v'_1, 0, v'_3, v'_4)$ using a rotation about the ty-plane. A transformation defined by

$$R_{ty} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_6 & 0 & -\sin\theta_6 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_6 & 0 & \cos\theta_6 \end{pmatrix},$$

such that

$$\sin\theta_6 = \frac{v_2}{\sqrt{v_2^2 + v_4^2}}, \ \cos\theta_6 = \frac{v_4}{\sqrt{v_2^2 + v_4^2}} \tag{D.1}$$

maps v to the vector $v' = (v_1, 0, v_3, \sqrt{v_2^2 + v_4^2})$.[1] Next, let us map v' to the vector $v'' = (v''_1, 0, 0, v''_4)$ using a rotation about the tx-plane. A transformation defined by

$$R_{tx} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_5 & -\sin\theta_5 \\ 0 & 0 & \sin\theta_5 & \cos\theta_5 \end{pmatrix},$$

[1] Here we have assumed $v_2^2 + v_4^2 \neq 0$. If $v_2^2 + v_4^2 = 0$, then $v_2 = 0$ and $v_4 = 0$, and thus θ_6 is arbitrary. See Remark D.0.1 at the end of this section for the transformation in this case.

such that

$$\sin\theta_5 = \frac{v_3}{\sqrt{v_2^2+v_3^2+v_4^2}},\ \cos\theta_5 = \frac{\sqrt{v_2^2+v_4^2}}{\sqrt{v_2^2+v_3^2+v_4^2}}$$

maps v' to the vector $v'' = (v_1, 0, 0, \sqrt{v_2^2+v_3^2+v_4^2})$. Next, let us map v'' to $\tilde{v} = (0,0,0,1)$ using a boost about the xy-plane. A transformation defined by

$$B_{xy} = \begin{pmatrix} \cosh\theta_3 & 0 & 0 & -\sinh\theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta_3 & 0 & 0 & \cosh\theta_3 \end{pmatrix},$$

such that

$$\sinh\theta_3 = v^1,\ \cosh\theta_3 = \sqrt{v_2^2+v_3^2+v_4^2}$$

maps v'' to $\tilde{v} = (0,0,0,1)$. Lastly, a transformation defined by

$$\Lambda_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda \in SO(2,1)$$

maps the vector $\tilde{v}$ to itself. Therefore the most general transformation $\Lambda \in SO(3,1)$ mapping $\tilde{v}$ to v is given by

$$\Lambda = (B_{xy}R_{tx}R_{ty})^{-1}\Lambda_1 = \Lambda_2^{-1}\Lambda_1, \tag{D.2}$$

where

$$\Lambda_2^{-1} = \begin{pmatrix} \sqrt{{v_2}^2+{v_3}^2+{v_4}^2} & 0 & 0 & v_1 \\ \frac{v_1\, v_2}{\sqrt{{v_2}^2+{v_3}^2+{v_4}^2}} & \frac{v_4}{\sqrt{{v_2}^2+{v_4}^2}} & -\frac{v_3\, v_2}{\sqrt{{v_2}^2+{v_3}^2+{v_4}^2}\sqrt{{v_2}^2+{v_4}^2}} & v_2 \\ \frac{v_1\, v_3}{\sqrt{{v_2}^2+{v_3}^2+{v_4}^2}} & 0 & \frac{\sqrt{{v_2}^2+{v_4}^2}}{\sqrt{{v_2}^2+{v_3}^2+{v_4}^2}} & v_3 \\ \frac{v_1\, v_4}{\sqrt{{v_2}^2+{v_3}^2+{v_4}^2}} & -\frac{v_2}{\sqrt{{v_2}^2+{v_4}^2}} & -\frac{v_3\, v_4}{\sqrt{{v_2}^2+{v_3}^2+{v_4}^2}\sqrt{{v_2}^2+{v_4}^2}} & v_4 \end{pmatrix}.$$

Remark D.0.1. In equation (D.1), we assumed $v_2^2+v_4^2 \neq 0$. If $v_2^2+v_4^2 = 0$, then $v = (v_1, 0, v_3, 0)$ and the most general transformation $\Lambda \in SO(3)$ mapping $\tilde{v}$ to v is given by

$$\Lambda = \Lambda_2^{-1}\Lambda_1,$$

where

$$\Lambda_2^{-1} = \begin{pmatrix} v_3 & 0 & 0 & v_1 \\ 0 & \cos\theta_6 & -\sin\theta_6 & 0 \\ v_1 & 0 & 0 & v_3 \\ 0 & -\sin\theta_6 & -\cos\theta_6 & 0 \end{pmatrix}.$$

D.0.16 Transformation of a Constant Vector to $(0,1,0,0)$

For a constant spacelike vector $v^i = (v_1, v_2, v_3, v_4) \in \mathbb{M}^4$ satisfying $g_{ij}v^iv^j = 1$, the most general transformation $\Lambda^i{}_j \in SO(3,1)$ which maps $\tilde{v}^i = (0,1,0,0)$ to v is given by

$$\Lambda = \Lambda_2\Lambda_1, \tag{D.3}$$

where

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and Λ_2 is (D.2).

D.0.17 Transformation of a Constant Vector to $(0,0,1,0)$

For a constant spacelike vector $v^i = (v_1, v_2, v_3, v_4) \in \mathbb{M}^4$ satisfying $g_{ij}v^iv^j = 1$, the most general transformation $\Lambda^i{}_j \in SO(3,1)$ which maps $\tilde{v}^i = (0,0,1,0)$ to v is given by

$$\Lambda = \Lambda_2\Lambda_1, \tag{D.4}$$

where

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and Λ_2 is (D.2).

D.0.18 Transformation of a Constant Vector to $(1, 0, 0, 0)$

For a constant timelike vector $v^i = (v_1, v_2, v_3, v_4) \in \mathbb{M}^4$ satisfying $g_{ij}v^iv^j = -1$, let us determine the most general transformation $\Lambda^i{}_j \in SO(3,1)$ which maps $\tilde{v}^i = (1, 0, 0, 0)$ to v. Following the same technique as described in Subsection D.0.15, we find that such a transformation is given by

$$\Lambda = \Lambda_2^{-1}\Lambda_1, \tag{D.5}$$

where

$$\Lambda_2^{-1} = \begin{pmatrix} v_1 & \frac{v_2}{\sqrt{{v_1}^2 - {v_2}^2}} & \frac{v_1\, v_3}{\sqrt{v_1^2 - v_2^2 - v_4^2}} & \frac{v_1\, v_4}{\sqrt{{v_1}^2 - {v_2}^2 - {v_4}^2}\sqrt{{v_1}^2 - {v_2}^2}} \\ v_2 & \frac{v_1}{\sqrt{{v_1}^2 - {v_2}^2}} & \frac{v_3\, v_2}{\sqrt{{v_1}^2 - {v_2}^2 - {v_4}^2}} & \frac{v_4\, v_2}{\sqrt{{v_1}^2 - {v_2}^2 - {v_4}^2}\sqrt{{v_1}^2 - {v_2}^2}} \\ v_3 & 0 & \sqrt{{v_1}^2 - {v_2}^2 - {v_4}^2} & 0 \\ v_4 & 0 & \frac{v_3\, v_4}{\sqrt{{v_1}^2 - {v_2}^2 - {v_4}^2}} & \frac{\sqrt{{v_1}^2 - {v_2}^2}}{\sqrt{{v_1}^2 - {v_2}^2 - {v_4}^2}} \end{pmatrix}$$

and

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad \lambda \in SO(3).$$

D.0.19 Transformation of a Constant Vector to $\left(\frac{1}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}, 0, 0\right)$

For a constant timelike vector $v^i = (v_1, v_2, v_3, v_4) \in \mathbb{M}^4$ satisfying $g_{ij}v^iv^j = -1$, the most general transformation $\Lambda^i{}_j \in SO(3,1)$ which maps $\tilde{v}^i = \left(\frac{1}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}, 0, 0\right)$ to v is given by

$$\Lambda = \Lambda_2\Lambda_1, \tag{D.6}$$

where

$$\Lambda_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{i\sqrt{2}}{2} & 0 & 0 \\ -\frac{i\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and Λ_2 is (D.5).

D.0.20 Transformation of a Constant Vector to $(1, \pm 1, 0, 0)$

For a constant null vector $v^i = (v_1, v_2, v_3, v_4) \in \mathbb{M}^4$ satisfying $v^1 = 1$ and $g_{ij} v^i v^j = 0$, let us determine the most general transformation $\Lambda^i{}_j \in SO(3,1)$ which maps $\tilde{v}^i = (1, \pm 1, 0, 0)$ to v. To this end, let us first map v to $v' = (1, v_2', v_3', 0)$ using a rotation about the ty-plane. A transformation defined by

$$R_{ty} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_6 & 0 & -\sin\theta_6 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_6 & 0 & \cos\theta_6 \end{pmatrix},$$

such that

$$\sin\theta_6 = -\frac{v_4}{\sqrt{v_2^2 + v_4^2}}, \ \cos\theta_6 = \frac{v_2}{\sqrt{v_2^2 + v_4^2}} \tag{D.7}$$

maps v to the vector $v' = (1, \sqrt{v_2^2 + v_4^2}, v_3, 0)$.[2] Next, let us map v' to the vector $\tilde{v} = (1, \pm 1, 0, 0)$ using a rotation about the tz-plane. A transformation defined by

$$R_{tz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_4 & -\sin\theta_4 & 0 \\ 0 & \sin\theta_4 & \cos\theta_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that

$$\sin\theta_4 = \mp v_3, \ \cos\theta_4 = \pm\sqrt{v_2^2 + v_4^2}$$

maps v' to the vector $\tilde{v} = (1, \pm 1, 0, 0)$. Next, let us determine the most general transformation $\Lambda_1 \in SO(3,1)$ which satisfies $\tilde{v} = \Lambda_1 \tilde{v}$. A transformation satisfying these constraints yields a system of quadratic equations. Solving these equations for the vector $\tilde{v}_+ = (1, 1, 0, 0)$, we find

$$\Lambda_1^+ = \begin{pmatrix} 1 + \frac{1}{2}(k_1^2 + k_2^2) & -\frac{1}{2}(k_1^2 + k_2^2) & \mp k_1\sqrt{1 - k_3^2} + k_2 k_3 & \mp k_2\sqrt{1 - k_3^2} - k_3 k_1 \\ \frac{1}{2}(k_1^2 + k_2^2) & 1 - \frac{1}{2}(k_1^2 + k_2^2) & \mp k_1\sqrt{1 - k_3^2} + k_2 k_3 & \mp k_2\sqrt{1 - k_3^2} - k_1 k_3 \\ -k_1 & k_1 & \pm\sqrt{1 - k_3^2} & k_3 \\ -k_2 & k_2 & -k_3 & \pm\sqrt{1 - k_3^2} \end{pmatrix},$$

[2] Here we assumed $v_2^2 + v_4^2 \neq 0$. If $v_2^2 + v_4^2 = 0$, then $v = (1, 0, \pm 1, 0)$ and θ_6 is arbitrary. See Remark D.0.2 at the end of this section for the transformation in this case.

for $k_1, k_2, k_3 \in \mathbb{R}$ is the most general transformation $\Lambda_1 \in SO(3,1)$ mapping $\tilde{v}_+$ to itself. Solving these equations for the vector $\tilde{v}_- = (1,-1,0,0)$, we find

$$\Lambda_1^- = \begin{pmatrix} 1+\frac{1}{2}(k_1^2+k_2^2) & -\frac{1}{2}(k_1^2+k_2^2) & \mp k_2\sqrt{1-k_3^2}+k_1k_3 & \mp k_1\sqrt{1-k_3^2}-k_2k_3 \\ -\frac{1}{2}(k_1^2+k_2^2) & -1+\frac{1}{2}(k_1^2+k_2^2) & \pm k_2\sqrt{1-k_3^2}-k_1k_3 & \pm k_1\sqrt{1-k_3^2}+k_2k_3 \\ -k_1 & k_1 & -k_3 & \pm\sqrt{1-k_3^2} \\ -k_2 & k_2 & \pm\sqrt{1-k_3^2} & k_3 \end{pmatrix},$$

for $k_1, k_2, k_3 \in \mathbb{R}$ is the most general transformation $\Lambda_1 \in SO(3,1)$ mapping $\tilde{v}_-$ to itself. Therefore, a transformation defined by $\Lambda = (R_{tz}R_{ty})^{-1}\Lambda_1^\pm$ maps $\tilde{v}_\pm$ to v.

It is possible to add one more degree of freedom to this transformation. Note that a transformation defined by

$$B_{yz} = \begin{pmatrix} \cosh\theta_1 & \epsilon\sinh\theta_1 & 0 & 0 \\ \epsilon\sinh\theta_1 & \cosh\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

maps $\tilde{v} = (1,\pm1,0,0)$ to $e^{\pm\epsilon\theta_1}(1,\pm1,0,0)$. Since $\tilde{v}$ and $k\tilde{v}$ are equivalent for any $k>0$, we have that B_{yz} satisfies the equation $\Lambda\tilde{v} = \tilde{v}$. Therefore, the most general transformation $\Lambda \in SO(3,1)$ which maps $\tilde{v}_\pm$ to $e^{\pm\epsilon\theta_1}v$ is given by

$$\Lambda = (R_{tz}R_{ty})^{-1}B_{yz}\Lambda_1^\pm = (\Lambda_2)^{-1}B_{yz}\Lambda_1^\pm, \tag{D.8}$$

where

$$\Lambda_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{v_2}{{v_2}^2+{v_3}^2+{v_4}^2} & -\dfrac{v_3\,v_2}{\sqrt{{v_2}^2+{v_4}^2}\,({v_2}^2+{v_3}^2+{v_4}^2)} & -\dfrac{v_4}{\sqrt{{v_2}^2+{v_4}^2}} \\ 0 & \dfrac{v_3}{{v_2}^2+{v_3}^2+{v_4}^2} & \dfrac{\sqrt{{v_2}^2+{v_4}^2}}{{v_2}^2+{v_3}^2+{v_4}^2} & 0 \\ 0 & \dfrac{v_4}{{v_2}^2+{v_3}^2+{v_4}^2} & -\dfrac{v_3\,v_4}{\sqrt{{v_2}^2+{v_4}^2}\,({v_2}^2+{v_3}^2+{v_4}^2)} & \dfrac{v_2}{\sqrt{{v_2}^2+{v_4}^2}} \end{pmatrix}.$$

Remark D.0.2. In equation (D.7) we assumed $v_2^2+v_4^2 \neq 0$. If $v_2^2+v_4^2 = 0$, then $v_2 = 0$ and $v_4 = 0$ and thus $v = (1,0,\pm1,0)$. The most general transformation $\Lambda \in SO(3,1)$ which maps $\tilde{v}_\pm$ to $e^{\pm\epsilon\theta_1}v$ is given by

$$\Lambda = (R_{tz}R_{ty})^{-1}B_{yz}\Lambda_1^\pm = (\Lambda_2)^{-1}B_{yz}\Lambda_1^\pm,$$

where

$$\Lambda_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta_6 & \sin\theta_6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\sin\theta_6 & \cos\theta_6 \end{pmatrix}.$$

D.0.21 Transformation of a Constant Vector to $(1, 0, \pm 1, 0)$

For a constant null vector $v^i = (v_1, v_2, v_3, v_4) \in \mathbb{M}^4$ satisfying $g_{ij}v^iv^j = 0$, let us determine the most general transformation $\Lambda^i{}_j \in SO(3,1)$ which maps $\tilde{v}^i = (1, 0, \pm 1, 0)$ to v is given by

$$\Lambda = \Lambda_2\Lambda_1, \tag{D.9}$$

where

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and Λ_2 is (D.8) with $v^i = \left(1, \frac{v_2}{v_1}, \frac{v_3}{v_1}, \frac{v_4}{v_1}\right)$ and $e^{\pm\epsilon\theta_1} = v_1$.

D.0.22 Transformation of a Constant Vector to $(1, 0, 0, \pm 1)$

For a constant null vector $v^i = (v_1, v_2, v_3, v_4) \in \mathbb{M}^4$ satisfying $g_{ij}v^iv^j = 0$, let us determine the most general transformation $\Lambda^i{}_j \in SO(3,1)$ which maps $\tilde{v}^i = (1, 0, 0, \pm 1)$ to v is given by

$$\Lambda = \Lambda_2\Lambda_1, \tag{D.10}$$

where

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and Λ_2 is (D.8) with $v^i = \left(1, \frac{v_2}{v_1}, \frac{v_3}{v_1}, \frac{v_4}{v_1}\right)$ and $e^{\pm\epsilon\theta_1} = v_1$.

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