NEUTRAL SIGNATURE CSI SPACES IN FOUR DIMENSIONS

by

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Submitted in partial fulfillment of the requirements

for the degree of Master of Science

at

Dalhousie University

Halifax, Nova Scotia

August 2011

DALHOUSIE UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

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Dated: August 18, 2011

Supervisor: _____

Readers: _____

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DALHOUSIE UNIVERSITY

DATE: August 18, 2011

AUTHOR: Andrew MacDougall

TITLE: NEUTRAL SIGNATURE CSI SPACES IN FOUR DIMENSIONS

DEPARTMENT OR SCHOOL: Department of Mathematics and Statistics

DEGREE: M.Sc.   CONVOCATION: October   YEAR: 2011

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Signature of Author

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Abstract

Scalar curvature invariants are scalars formed by the contraction of the Riemann tensor and its covariant derivatives. The main motivation for studying scalar curvature invariants is that they can be used to classify certain spaces uniquely. For example, the $\mathcal{I}$-non-degenerate spaces in Lorentzian signature. Lorentzian metrics that fail to be $\mathcal{I}$-non-degenerate are Kundt metrics and have a very special structure.

In this thesis we study pseudo-Riemannian spaces with the property that all of their scalar curvature invariants vanish (VSI spaces) or are constant (CSI spaces). These spaces include pseudo-Riemannian Kundt metrics. VSI and CSI spaces are not only of interest from a mathematical standpoint but also have applications to current theoretical physics.

In particular, we focus on studying VSI and CSI spaces in four-dimensional neutral signature. We construct new, very general, classes of CSI and VSI pseudo-Riemannian spaces.

List of Abbreviations and Symbols Used

VSI .... vanishing scalar invariant

CSI .... constant scalar invariant

3D .... three dimensions

4D .... four dimensions

R_{abcd} .... component of the Riemann tensor

R_{ab} .... component of the Ricci tensor

S_{ab} .... component of the traceless Ricci tensor

C_{abcd} .... component of the Weyl tensor

g_{ab} .... component of the metric tensor

D_{abc} .... commutator coefficient

Γ_{abc} .... Christoffel symbols

$\mathcal{I}$ .... set of invariants

$\boldsymbol{\ell}$ .... null vector

$SO(m)$ .... m dimensional special orthogonal group

$SO(k, k+m)$ .... indefinite special orthogonal group with signature $(k, k+m)$

$\boldsymbol{b}$ .... boost weight

Acknowledgements

Chapter 1

Introduction

Scalar curvature invariants are scalars constructed from the Riemann tensor and its covariant derivatives. Scalar curvature invariants can be used to study the in-equivalence of metrics and curvature singularities [1]. In the case of the Lorentzian $\mathcal{I}$-non-degenerate spaces, they can be used to determine the equivalence of metrics [2].

Scalar curvature invariants have been well studied due to their potential use in general relativity. For the most part, scalar curvature invariants have been studied by considering the scalar curvature invariants formed from the Riemann tensor R_{abcd} only (contractions involving products of the undifferentiated Riemann tensor only, the so called *algebraic invariants*) in four-dimensional (4D) Lorentzian spacetimes.

Some spaces can be completely characterized by their scalar curvature invariants. This leads to the natural problem of finding a basis for the scalar curvature invariants formed from the Riemann tensor (up to some order of covariant differentiation). Much work has gone into the problem of constructing a basis in the 4D Lorentzian algebraic case. In this case one can form 14 functionally independent scalar curvature invariants. The smallest set that contains a maximal set of algebraically independent scalars consists of 17 polynomials [3]. As reviewed in the introduction of [4], a number of proposed independent sets were given by Narlikar and Karmarkar [5], Geheniau and Debever [6], Petrov [7], and others. All of these sets were shown to be deficient for various reasons. In [4], a set of algebraic invariants was presented by Carminati and McLenaghan, consisting of 16 curvature invariants, that contains invariants of lowest possible degree and contains a minimal set for any Petrov type and for any specific choice of Ricci tensor type in the perfect fluid and Einstein-Maxwell cases. In general, the expressions relating invariants to the basis members of an independent set can be very complicated, and can be singular in certain algebraic cases.

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Rather than searching for an independent basis, we may search for a *complete* basis $\{I_1, I_2, \ldots, I_n\}$, meaning that any scalar curvature invariant can be expressed as a polynomial of the elements of $\{I_1, I_2, \ldots, I_n\}$ but no member of $\{I_1, I_2, \ldots, I_n\}$ can be expressed in terms of the others. As will be discussed in chapter 2, the smallest known complete set in the 4D algebraic Lorentzian case consists of 38 scalars [8].

In this thesis, we investigate spaces in which all of the polynomial scalar curvature invariants are constant, called constant scalar invariant (CSI) spaces. The CSI spaces that are not locally homogeneous cannot generally be characterized by their scalar curvature invariants, but these spaces have a very specific form enabling us to develop methods for constructing such spaces. Attention is given to both the general case and the case in which all of the scalar curvature invariants vanish, called vanishing scalar invariant (VSI) spaces. A motivation for studying such spaces is that they are of interest in string theory (the twistor approach) and for spaces admitting parallel spinors (see, for example, [9] and [10]).

We begin by reviewing some of the geometric background required to investigate such spaces. In particular, we present some fundamental definitions in the theory of scalar curvature invariants. We review moving frame methods, and then discuss 4D Lorentzian Kundt metrics.

In chapter three the method of algebraic classification of pseudo-Riemannian spaces using boost-weight decomposition is reviewed. This technique is useful for studying degenerate (and, in particular, VSI and CSI) metrics in Lorentzian geometry, and we argue that its generalization is useful in the pseudo-Riemannian case. Based on this decomposition we define some algebraic properties that will allow us to construct VSI metrics from Kundt metrics. Chapter three ends with a review of some classes of VSI spaces.

In chapters four and five we shift focus to spaces in four dimensions with neutral signature. Chapter four is based on joint work with Adam Alcolado. Chapter four discusses two new classes of VSI spaces, whereas chapter five presents two new classes of CSI spaces. The material from chapters four and five is presented in [11].

Chapter 2

Geometric Background

The material in this chapter is a review of some of the mathematical background required for the later work of this thesis. The material is, unless otherwise stated, based on material from [1].

2.1 Decomposition of the Curvature Tensor

The curvature tensor with respect to a basis $\{\mathbf{e}_a\}$ can be uniquely decomposed (into parts which are irreducible representations of the full Lorentz group) as follows

$$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd} \tag{2.1}$$

with

$$E_{abcd} \equiv \frac{1}{2}(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}),$$

$$G_{abcd} \equiv \frac{1}{12}R(g_{ac}g_{bd} - g_{ad}g_{bc}) \equiv \frac{1}{12}Rg_{abcd},$$

$$S_{ab} \equiv R_{ab} - \frac{1}{4}Rg_{ab}$$

$$R \equiv R^a{}_a$$

where R and S_{ab} denote the trace and traceless part of the Ricci tensor.

In equation (2.1) the term C_{abcd} defines *Weyl's conformal tensor*. All terms in the decomposition (2.1) have the same symmetries as the Riemann tensor. There are futher relations:

$$C^a{}_{bad} = 0, \quad E^a{}_{bad} = S_{bd}, \quad G^a{}_{bad} = \frac{1}{4}g_{bd}R.$$

The Weyl tensor is completely traceless. Spaces with vanishing Weyl tensor are said to be *conformally flat*.

2.2 Scalar Invariants

We introduce here the concept of a *scalar polynomial curvature invariant*, a definition that will be used throughout this thesis. The concepts and definitions here are mainly a review of chapter 9 of [1].

A scalar curvature invariant is a scalar constructed from contractions of the Riemann tensor and its covariant derivatives. Given an n-dimensional spacetime, at most n such scalars can be functionally independent. However, the number of algebraically independent scalars (scalars not not satisfying any polynomial relation, called a *syzygy*), is generally much larger. The number of algebraically independent scalars formed by contractions of the metric tensor and its first p covariant derivatives $((p+2)$ derivatives of Riemann tensor), in an n dimensional manifold is given by

$$N(n,p) = \frac{n[n+1][(n+p)!]}{2n!p!} - \frac{(n+p+1)!}{(n-1)!(p+1)!} + n, \tag{2.2}$$

Given a spacetime, constructing a complete (see below) set of algebraically independent scalar curvature invariants does not in general suffice to characterize the spacetime uniquely. However, they are sufficient to (locally) characterize $\mathcal{I}$-non-degenerate spacetimes uniquely, see section 2.4. Furthermore, scalar curvature invariants are often sufficient to prove inequivalence of metrics, and may be used to investigate certain properties of a space such as singularities.

We call a set of scalar curvature invariants $\{I_1, I_2, \ldots, I_n\}$ *complete* if all other scalar curvature invariants can be expressed in terms of the elements of $\{I_1, I_2, \ldots, I_n\}$, but no member of $\{I_1, I_2, \ldots, I_n\}$ can be expressed in terms of the others. The smallest known set of scalar curvature invariants of the Riemann tensor in four dimensions that contains a maximal set of algebraically independent scalars consists of 17 polnomials [3]. The smallest set known to be complete consists of 38 scalars [8].

As an example, we present some well-known invariants of the Weyl tensor. Consider the eigenvalue problem:

$$\frac{1}{2}C_{abcd}X^{cd} = \lambda X^{ab}. \tag{2.3}$$

With each solution (X_{ab}, λ) we associate its complex conjugate solution $(\bar{X}_{ab}, \bar{\lambda})$. Without loss of generality we can rewrite the eigenvalue equation (2.3) as (see page 38 of [1] for a definition of $C^*{}_{abcd}$)

$$\frac{1}{4}C^*{}_{abcd}X^{*cd} = \lambda X^{*ab}.. \tag{2.4}$$

We multiply equation (2.4) by a timelike unit vector u^a and reduce the eigenvalue problem to the form (see page 49 of [1]):

$$Q_{ab}X^b = \lambda X_a. \tag{2.5}$$

where Q_{ab} is given by

$$Q_{ab} = -C^*{}_{abcd}u^b u^d.$$

We can now define the invariants I and J:

$$I \equiv \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2),$$

$$J \equiv \frac{1}{6}(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) = \frac{1}{2}\lambda_1\lambda_2\lambda_3,$$

where the λ's are the solutions to the eigenvalue equation (2.5).

We may use the Ricci tensor to define the well-known Ricci scalar $R = R^a{}_a$, and the three solutions to

$$\lambda^4 - \frac{1}{2}I_6\lambda^2 - \frac{1}{3}I_7\lambda + \frac{1}{8}(I_6^2 - 2I_8) = 0,$$

where the scalar polynomial invariants I_k are given by

$$I_6 = S^a{}_b S^b{}_a,$$

$$I_7 = S^a{}_b S^b{}_c S^c{}_a,$$

$$I_8 = S^a{}_b S^b{}_c S^c{}_d S^d{}_a.$$

Further invariants involve contractions between the Weyl and Ricci curvatures

and are known as mixed invariants. Such invariants can be added to the ones given above to give the 17 invariants previously mentioned, or the complete set of 38.

All invariants displayed above have been constructed without using covariant derivatives of the Riemann tensor. There has been less work done on the construction of complete sets of polynomial invariants of both the Riemann tensor and its covariant derivatives. However, some work has been done by Fulling et al [12].

2.3   Frames

In a four-dimensional space-time a *Lorentz frame* (also called an *orthonormal basis* or *orthonormal tetrad*) $\{\mathbf{E}_a\}$ is a set of three spacelike vectors $\mathbf{E}_\alpha$ and one timelike vector $\mathbf{E}_4 \equiv \mathbf{t}$ such that

$$\{\mathbf{E}_a\} = \{\mathbf{E}_\alpha, t\} = \{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}\}, \quad g_{ab} = x_a x_b + y_a y_b + z_a z_b - t_a t_b$$

$$\Leftrightarrow \mathbf{E}_\alpha \cdot \mathbf{E}_\beta = \delta_{\alpha\beta}, \quad \mathbf{t} \cdot \mathbf{t} = -1, \quad \mathbf{E}_\alpha \cdot \mathbf{t} = 0.$$

We now examine an important class of tetrads, the *complex null tetrads*. A complex null tetrad is a set of two real null vectors $\mathbf{k}, \mathbf{l}$, and two complex conjugate null vectors $\mathbf{m}, \overline{\mathbf{m}}$:

$$\{e_a\} = (\mathbf{m}, \overline{\mathbf{m}}, \mathbf{l}, \mathbf{k}),$$

$$g_{ab} = 2m_{(a}\overline{m}_{b)} - 2k_{(a}l_{b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.6}$$

In other words, the inner product of the frame vectors all vanish except for

$$k^a l_a = -1, \quad m^a \overline{m}_a = 1.$$

Expressing a complex null tetrad $\{\mathbf{e}_a\}$ and its dual $\{\omega^a\}$ in terms a coordinate basis

will give the following form

$$\mathbf{e}_1 = m^i \frac{\partial}{\partial x^i}, \mathbf{e}_2 = \overline{m}^i \frac{\partial}{\partial x^i}, \mathbf{e}_3 = l^i \frac{\partial}{\partial x^i}, \mathbf{e}_4 = k^i \frac{\partial}{\partial x^i},$$

$$\omega^1 = \overline{m}_i dx^i, \omega^2 = m_i dx^i, \omega^3 = -k_i dx^i, \omega^4 = -l_i dx^i.$$

The directional derivatives of a function f with respect to the complex null tetrad are

$$f|_1 = f_{,i} m^i, f|_2 = f_{,i} \overline{m}^i, f|_3 = f_{,i} l^i, f|_4 = f_{,i} k^i.$$

We have the following relationship between an orthonormal and a null tetrad:

$$\sqrt{2}\mathbf{m} = \mathbf{E_1} - i\mathbf{E_2}, \sqrt{2}\overline{\mathbf{m}} = \mathbf{E_1} + i\mathbf{E_2}$$

$$\sqrt{2}\mathbf{l} = \mathbf{E_4} - \mathbf{E_3}, \sqrt{2}\mathbf{k} = \mathbf{E_4} + \mathbf{E_3}.$$

Lorentz transformations induce the following changes of basis:

- null rotations ($\mathbf{l}$ fixed):

$$\mathbf{l}' = \mathbf{l}, \mathbf{m}' = \mathbf{m} + E\mathbf{l}, \mathbf{k}' = \mathbf{k} + E\overline{\mathbf{m}} + \overline{E}\mathbf{m} + E\overline{E}\mathbf{l}, \quad \text{E complex} \qquad (2.7)$$

- null rotations ($\mathbf{k}$ fixed),

$$\mathbf{k}' = \mathbf{k}, \mathbf{m}' = \mathbf{m} + B\mathbf{k}, \mathbf{l}' = \mathbf{l} + B\overline{\mathbf{m}} + \overline{B}\mathbf{m} + B\overline{B}\mathbf{l}, \quad \text{B complex} \qquad (2.8)$$

- spatial rotations in the $\mathbf{m} - \overline{\mathbf{m}}$-plane,

$$\mathbf{m}' = e^{i\Theta}\mathbf{m}, \quad \Theta \text{ real} \qquad (2.9)$$

- boosts in the $\mathbf{k}$-$\mathbf{l}$ plane,

$$\mathbf{k}' = A\mathbf{k}, \quad \mathbf{l}' = A^{-1}\mathbf{l}, \quad A > 0. \qquad (2.10)$$

We see that equations (2.7)-(2.10) contain six real parameters.

Symmetric connection coefficients are uniquely determined by adding the following metric condition:

$$\nabla \mathbf{g} = 0 \Leftrightarrow g_{ab;c} = 0 = g_{ab|c} - 2\Gamma_{(ab)c}, \quad \Gamma_{abc} = g_{ad}\Gamma^d_{bc}.$$

Using the metric conditions, we see that if the space is torsion free then

$$\Gamma_{abc} = \frac{1}{2}(g_{ab|c} + g_{ac|b} - g_{bc|a} + D_{cab} + D_{bac} - D_{abc})$$

with

$$D_{abc} = g_{ad}D^d_{\ bc}$$

where the D_{abc} are the *commutator coefficients* given by

$$[\mathbf{e}_a, \mathbf{e}_b] = D^c_{\ ab}\mathbf{e}_c, \quad D^c_{\ ab} = -D^c_{\ ba},$$

Two special types of frames are:

- *Holonomic frame*: $D_{ijk} = 0 : \Gamma_{i[jk]} = 0$ where $\Gamma^i_{\ jk}$ are the Christoffel symbols.

- *Rigid Frame*: $g_{ab|c} = 0 \quad \Gamma_{(ab)c} = 0$.

These frames are of interest because in a holonomic frame the connection coefficients Γ_{abc} are symmetric in the indices (bc) whereas in a rigid frame they are antisymmetric in the indices (ab).

2.4   4D Lorentzian Kundt Metrics

We now review Kundt spaces in the 4D Lorentzian case, as defined in [13].

Consider a space $\mathcal{M}$ with a Lorentzian metric g. Let $\mathcal{I}_k$ denote the set of all polynomial scalar invariants constructed from the curvature tensor and its first k covariant derivatives.

Definition 2.4.1. $\mathcal{M}$ is called VSI$_k$ if for all invariants $I \in \mathcal{I}_k$, $I = 0$ on $\mathcal{M}$.

Definition 2.4.2. $\mathcal{M}$ is called CSI$_k$ if for all invariants $I \in \mathcal{I}_k$, I is constant on M.

Definition 2.4.3. If a space is CSI_k or VSI_k for all k, it is called CSI or VSI, respectively.

The definition of a Kundt metric (from [14]) is one that will be used throughout this thesis. They play an important role in CSI and VSI spaces.

Definition 2.4.4. A Kundt metric is a metric that possesses a non-zero null vector $\boldsymbol{\ell}$ which is geodesic, expansion-free, twist-free, and shear-free. In other words, there exists a null vector $\boldsymbol{\ell}$ such that

- $\ell^\mu \ell_{\nu;\mu} = 0$.

- $\ell^\mu{}_{;\mu} = \ell^{\mu;\nu} \ell_{(\mu;\nu)} = \ell^{\mu;\nu} \ell_{[\mu;\nu]} = 0$.

Consider the following definitions [2]:

Definition 2.4.5. A (one-parameter) *metric deformation* $\hat{g}_\tau, \tau \in [0, \epsilon)$ of a space $(\mathcal{M}, g)$ is a family of smooth metrics on $\mathcal{M}$ such that

1. $\hat{g}_\tau$ is continuous in τ,

2. $\hat{g}_0 = g$,

3. $\hat{g}_\tau$ for $\tau > 0$ is not diffeomorphic to g.

Definition 2.4.6. Given a space $(\mathcal{M}, g)$ with a set of invariants $\mathcal{I}$, then if there does not exist a metric deformation of g with the same set of invariants as g, then we will call the set of invariants *non-degenerate*. The space metric g will be called *$\mathcal{I}$-non-degenerate*.

Hence a Lorentzian metric that is $\mathcal{I}$-non-degenerate is locally characterized uniquely by its invariants. We now have the following theorem:

Theorem 2.4.7. *Given a space metric, either*

1. *the metric is $\mathcal{I}$-non-degenerate or*

2. *the metric is a Kundt metric.*

Theorem 2.4.7 was proven in [2]. Furthermore, we have the following important theorem on 4D Lorentzian CSI spaces:

Theorem 2.4.8. *A 3D or 4D space is CSI if and only if either*

1. *The space is locally homogeneous.*

   or

2. *The space is Kundt for which there exists a frame such that the positive boost weight components of all curvature tensors vanish and all boost weight zero components are constants.*

Another result that holds for 4D Lorentzian spaces is the following:

Theorem 2.4.9. *For a 4D Lorentzian space,*

$$CSI \Leftrightarrow CSI_3.$$

Proofs of the theorems 2.4.8 and 2.4.9 are given in [13].

Chapter 3

Algebraic Classification of Pseudo-Riemannian Space

3.1 Overview

We begin by generalizing the decomposition of a tensor into boost-weights, a technique useful for studying degenerate metrics in Lorentzian geometries, to the pseudo-Riemannian case. After introducing this decomposition some algebraic properties, namely the $\mathbf{S}_i$ and $\mathbf{N}$ properties, are defined. These properties are used to introduce a method of classifying arbitrary signature Weyl tensors.

These methods introduced allow us to explore VSI spacetimes, spacetimes in which all scalar curvature invariants vanish. In doing so, we will discuss the pseudo-Riemannian Kundt metrics and the role that they play in VSI spacetimes. In particular, we will construct VSI metrics from Kundt metrics by requiring that they possess the $\mathbf{N}$ property, and solving the equations that result for the components.

This chapter reviews material from the papers [14] and [15]

3.2 Boost Weight Formalism

We first review the boost weight classification, originally used in [16],[17], and [18] to study degenerate metrics in the Lorentzian and the pseudo-Riemannian cases respectively. We consider the case of a pseudo-Riemannian space of arbitrary dimension and signature $(k, k+m)$. The symmetry group of frame-rotations is $SO(k, k+m)$, and any element can be written in the form KAN, where K is a compact spin piece, A is an Abelian boost piece, and N is a piece consisting of null-rotations. For $SO(k, k+m)$, $K \in SO(m)$, and there are k-independent boosts (= the real rank of $SO(k, k+m)$). For definitions of $SO(k, k+m)$ and $SO(m)$ see, for example, chapter 1 of [19].

We make use of this decomposition by introducing a null-frame so that we can

write the metric as:

$$ds^2 = 2\left(\boldsymbol{\ell}^1 \boldsymbol{n}^1 + \cdots + \boldsymbol{\ell}^I \boldsymbol{n}^I + \cdots + \boldsymbol{\ell}^k \boldsymbol{n}^k\right) + \delta_{ij} \boldsymbol{m}^i \boldsymbol{m}^j \tag{3.1}$$

where the indices $i = 1, \ldots, m$. Spins act on $\boldsymbol{m}^i$, boosts act on the pair of null-vectors, and null-rotations act on both null-vectors and spatial vectors:

$$\text{Spins:} \quad \tilde{\boldsymbol{\ell}}^I = \boldsymbol{\ell}^I, \ \tilde{\boldsymbol{n}}^I = \boldsymbol{n}^I, \ \tilde{\boldsymbol{m}}^i = M^i{}_j \boldsymbol{m}^j, \ (M^i{}_j) \in SO(m), \tag{3.2}$$

$$\text{Boosts:} \quad \tilde{\boldsymbol{\ell}}^I = e^{\lambda_I} \boldsymbol{\ell}^I, \ \tilde{\boldsymbol{n}}^I = e^{-\lambda_I} \boldsymbol{n}^I, \ \tilde{\boldsymbol{m}}^i = \boldsymbol{m}^j, \tag{3.3}$$

while the null-rotations can be split up at each level. Considering the subset of forms $(\boldsymbol{\ell}^I, \boldsymbol{n}^I, \boldsymbol{\omega}^{\mu_I})$, where $\boldsymbol{\omega}^{\mu_I} = \{\boldsymbol{\ell}^{I+1}, \boldsymbol{n}^{I+1}, \cdots, \boldsymbol{\ell}^k, \boldsymbol{n}^k, \boldsymbol{m}^i\}$, then we can consider the I-th level null-rotations with respect to $\boldsymbol{n}^I$:

$$\text{Null-Rot:} \ \tilde{\boldsymbol{\ell}}^I = \boldsymbol{\ell}^I - z_{\mu_I} \boldsymbol{\omega}^{\mu_I} - \frac{1}{2} z_{\mu_I} z^{\mu_I} \boldsymbol{n}^I, \ \tilde{\boldsymbol{n}}^I = \boldsymbol{n}^I, \ \tilde{\boldsymbol{\omega}}^{\mu_I} = \boldsymbol{\omega}^{\mu_I} + z^{\mu_I} \boldsymbol{n}^I, \tag{3.4}$$

and similarly for $\boldsymbol{\ell}^I$. At the Ith level, there are $2(2k + m - 2I)$ null-rotations, giving $2k(k + m - 1)$ in all.

3.2.1 Boost Weight Decomposition

Consider the k independent boosts:

$$
\begin{aligned}
(\boldsymbol{\ell}^1, \boldsymbol{n}^1) &\mapsto (e^{\lambda_1} \boldsymbol{\ell}^1, e^{-\lambda_1} \boldsymbol{n}^1) \\
(\boldsymbol{\ell}^2, \boldsymbol{n}^2) &\mapsto (e^{\lambda_2} \boldsymbol{\ell}^2, e^{-\lambda_2} \boldsymbol{n}^2) \\
&\vdots \\
(\boldsymbol{\ell}^k, \boldsymbol{n}^k) &\mapsto (e^{\lambda_k} \boldsymbol{\ell}^k, e^{-\lambda_k} \boldsymbol{n}^k).
\end{aligned}
\tag{3.5}
$$

Given some tensor T, we can consider the boost weight of the components of this tensor, $\mathbf{b} \in \mathbb{Z}^k$. If the component $T_{\mu_1 \ldots \mu_n}$ transforms as:

$$T_{\mu_1 \ldots \mu_n} \mapsto e^{-(b_1 \lambda_1 + b_2 \lambda_2 + \ldots + b_k \lambda_k)} T_{\mu_1 \ldots \mu_n},$$

then we say the component $T_{\mu_1...\mu_n}$ is of boost weight $\mathbf{b} \equiv (b_1, b_2, ..., b_k)$. T can now be decomposed into boost weights:

$$T = \sum_{\mathbf{b} \in \mathbb{Z}^k} (T)_{\mathbf{b}},$$

where $(T)_{\mathbf{b}}$ means the projection onto the components of boost weight $\mathbf{b}$. We extend this decomposition to tensor products by noting that they posess the following additive property

$$(T \otimes S)_{\mathbf{b}} = \sum_{\tilde{\mathbf{b}} + \hat{\mathbf{b}} = \mathbf{b}} (T)_{\tilde{\mathbf{b}}} \otimes (S)_{\hat{\mathbf{b}}}. \tag{3.6}$$

3.2.2 The $\mathbf{S}_i$- and N-Properties

Definition 3.2.1. Given a tensor T, with the boost-weight notation above, we will define the following properties:

B1) $(T)_{\mathbf{b}} = 0$, for $\mathbf{b} = (b_1, b_2, b_3, ..., b_k)$, $b_1 > 0$.

B2) $(T)_{\mathbf{b}} = 0$, for $\mathbf{b} = (0, b_2, b_3, ..., b_k)$, $b_2 > 0$.

B3) $(T)_{\mathbf{b}} = 0$, for $\mathbf{b} = (0, 0, b_3, ..., b_k)$, $b_3 > 0$.

$\vdots$

Bk) $(T)_{\mathbf{b}} = 0$, for $\mathbf{b} = (0, 0, ..., 0, b_k)$, $b_k > 0$.

We reference the above conditions in defining algebraic properties that are important in our discussion:

Definition 3.2.2. A tensor T is said to possesses the $\mathbf{S}_1$-property if there exists a null frame such that condition B1) above is satisfied. Similarly, we say that T possesses the $\mathbf{S}_i$-property if there exists a null frame such that conditions B1)-Bi) above are satisfied.

Definition 3.2.3. A tensor T is said to possesses the **N**-property if there exists a null frame such that conditions B1)-Bk) in definition 3.2.1 are satisfied, *and*

$$(T)_{\mathbf{b}} = 0, \text{ for } \mathbf{b} = (0, 0, ..., 0, 0).$$

We can extend these results to tensor products by means of the following proposition:

Proposition 3.2.4. *For the tensor product of tensors T and S,*

1. *Suppose that T and S possess the $\mathbf{S}_i$- and the $\mathbf{S}_j$-property, respectively. We assume without loss of generality that $i \leq j$. Then $T \otimes S$ possesses the $\mathbf{S}_i$-property.*

2. *Let T and S possess the $\mathbf{S}_i$- and $\mathbf{N}$-property, respectively. Then $T \otimes S$ possesses the $\mathbf{S}_i$-property. If $i = k$, then $T \otimes S$ possesses the $\mathbf{N}$-property.*

3. *Let T and S both possess the $\mathbf{N}$-property. Then $T \otimes S$, and any contraction of $T \otimes S$, possesses the $\mathbf{N}$-property.*

We now define some more general, but related, conditions. Let T be a tensor, not necessarily satisfying any of the aforementioned conditions. We note that the boost weights $\mathbf{b} \in \mathbb{Z}^k \subset \mathbb{R}^k$. We consider a transformation $G \in GL(k)$ to a lattice $\Gamma \in \mathbb{R}^k, G : \mathbb{Z}^k \mapsto \Gamma$. If the boost weights of $G\mathbf{b}$ satisfy some conditions in definition 3.2.1, we will say that T posseses the $\mathbf{S}_i^G$-property. In a similar manner we define the $\mathbf{N}^G$-property.

We can also extend the additive rule (3.6) in a natural way. Let T and S be tensors both possessing the $\mathbf{S}_i^G$-property. Then:

$$(T \otimes S)_{G\mathbf{b}} = \sum_{G\hat{\mathbf{b}}+G\tilde{\mathbf{b}}=G\mathbf{b}} (T)_{G\hat{\mathbf{b}}} \otimes (S)_{G\tilde{\mathbf{b}}}.$$

The tensor product will possess the $\mathbf{S}_i^G$-property, with the same $G \in GL(k)$. The $\mathbf{S}_i^G$-property reduces to the $\mathbf{S}_i$ property when G is the identity transformation.

As noted in [14], a tensor, T, satisfying the $\mathbf{S}_i^G$-property or $\mathbf{N}^G$-property is not in general determined by its invariants. It can be that there exists some other tensor, S, possesing the same invariants. Hence, the $\mathbf{S}_i$-property thus implies a kind of degeneracy in the tensor.

3.3 Boost-Weight Classification

Weyl tensors of arbitrary signature can be classified using this boost-weight decomposition ([16],[17]).

For each level, consider the null-rotations that leave the $(2k+m-2I)$ dimensional metric invariant:

$$2\boldsymbol{\ell}^I \boldsymbol{n}^I + \eta_{\mu_I \nu_I} \boldsymbol{\omega}^{\mu_I} \boldsymbol{\omega}^{\nu_I}. \tag{3.7}$$

The metric $\eta_{\mu_I \nu_I}$ will be of signature $(k - I, k - I + m)$.

Now consider a Weyl tensor, C. This can be decomposed into boost weight components, as explained above. In order to find the primary level algebraic type, we consider the components:

$$C = (C)_{(+2,*,*,...,*)} + (C)_{(+1,*,*,...,*)} + (C)_{(0,*,*,...,*)} + (C)_{(-1,*,*,...,*)} + (C)_{(-2,*,*,...,*)},$$

where $(+2, *, *, ..., *)$ means all components of boost-weight $b_1 = +2$.

We now use the standard algebraic classification of Lorentzian tensors at each level. We say that C is of primary (or primary level) algebraic type III if there exists a frame such that $(C)_{(+2,*,*,...,*)} = (C)_{(+1,*,*,...,*)} = (C)_{(0,*,*,...,*)} = 0$.

We will get the second level type by using the prefered form from the primary level. Decompose the highest non-zero primary boost-weight component $(C)_{(b_1,*,*,...,*)}$ as

$$(C)_{(b_1,*,...,*)} = \sum_{b_2=-2}^{+2} (C)_{(b_1,b_2,*,...,*)}.$$

We find the second level type by searching for a frame using the 2nd level null-rotations that preserve primary boost-weights,

The full algebraic type is the sum of the primary, second,..., kth level types. This is written as follows; (I, D, III), means the type at 1st, 2nd, and 3rd level are I, D, and III, respectively.

If a Weyl tensor obeys the $\mathbf{S}_i$,- or $\mathbf{N}$-property, we have the following:

$$\mathbf{S}_i : \quad (\underbrace{II, II, ..., II}_{i}, G, ..., G)$$

$$\mathbf{N} : \quad (II, II, ..., II, III)$$

or simpler.

3.4 The VSI Properties

Definition 3.4.1. A pseudo-Riemannian manifold is a *vanishing scalar invariant* spacetime if all scalar invariants constructed from the Riemann tensor and its covariant derivatives are zero.

Definition 3.4.2. A *curvature operator* T is a tensor considered as a (pointwise) linear operator

$$\mathsf{T} : V \rightarrow V,$$

for some vector space, V, constructed from the Riemann tensor, its covariant derivatives, and the curvature invariants [20].

The following two results will be useful in the work to come on VSI spaces.

Theorem 3.4.3. *A pseudo-Riemannian space is VSI if and only if all the curvature operators are nilpotent.*

Proof. The proof follows from the papers [21],[22]. Let T be a curvature operator. Then:

All invariants $\mathrm{Tr}(\mathsf{T}^k)$ are zero.

$\Updownarrow$

All eigenvalues of T are zero.

$\Updownarrow$

T is nilpotent.

All polynomial curvature invariants are the trace of some curvature operator, and so the theorem follows. □

So if a curvature operator is nilpotent then we know that all its invariants are zero. We therefore have an equivalent and useful definition for a space to be VSI. Furthermore, we can demonstrate a necessary condition for an operator to be nilpotent.

Theorem 3.4.4. *Let T be a tensor of even rank posessing the $\mathbf{N}^G$-property. The operator T (obtained by raising and lowering indices) is nilpotent.*

Proof. This proof follows from an equivalent proof to that in [21],[22]. First observe that since T possesses the $\mathbf{N}^G$-property, so do $T \otimes T$, $T \otimes \cdots \otimes T \otimes T$. Since the metric $\mathbf{g}$ is of boost weight zero, the operators T^k also possess the $\mathbf{N}^G$-property. The coefficients of the eigenvalue equation of T consist of traces of T^k which have zero boost weight. Hence we get that $\mathrm{Tr}(\mathsf{T}^k) = 0$ and so eigenvalues are zero. This implies that T is nilpotent. □

3.5 Pseudo-Riemannian Kundt Metrics

It was shown in [23, 24, 25] that Kundt metrics play an important role for VSI metrics in the Lorentzian case. We will demonstrate that their pseudo-Riemannian generalizations also important for VSI metrics of arbitrary signature. We generalize the concept of a Lorentzian $\mathcal{I}$-non-degenerate space to pseudo-Riemannian space:

Definition 3.5.1. A (one-parameter) *metric deformation* $\hat{g}_\tau, \tau \in [0, \epsilon)$ of a pseudo-Riemannian space $(\mathcal{M}, g)$ is a family of smooth metrics on $\mathcal{M}$ such that

1. $\hat{g}_\tau$ is continuous in τ,

2. $\hat{g}_0 = g$,

3. $\hat{g}_\tau$ for $\tau > 0$ is not diffeomorphic to g.

Definition 3.5.2. Given a pseudo-Riemannian space $(\mathcal{M}, g)$ with a set of invariants $\mathcal{I}$, then if there does not exist a metric deformation of g with the same set of invariants

as g, then we will call the set of invariants *non-degenerate*. The space metric g will be called *$\mathcal{I}$-non-degenerate.*

We recall defintion 2.4.4 from Chapter 2, the definition of a Kundt metric in 4D, and generalize it to arbritary signature and arbritary dimension:

Definition 3.5.3. A metric possessing a non-zero geodesic, expansion-free, twist-free, and shear-free null vector $\boldsymbol{\ell}$ is said to be a pseudo-Riemannian Kundt metric.

This enables us to consider such metrics in null coordinates:

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}v + H(v,u,x^k)\mathrm{d}u + W_i(v,u,x^k)\mathrm{d}x^i\right] + g_{ij}^\perp \mathrm{d}x^i\mathrm{d}x^j. \tag{3.8}$$

The metric (3.8) possesses a null vector field $\boldsymbol{\ell}$ which obeys

$$\ell_{\mu;\nu} = L_{11}\ell_\mu\ell_\nu + L_{1i}\ell_{(\mu}\omega^i{}_{\nu)},$$

and hence,

$$\ell^\mu\ell_{\nu;\mu} = \ell^\mu{}_{;\mu} = \ell^{\nu;\mu}\ell_{(\nu;\mu)} = \ell^{\mu;\nu}\ell_{[\mu;\nu]} = 0; \tag{3.9}$$

i.e., it is geodesic, non-expanding, shear-free and non-twisting.

If we are assuming that the signature of the metric is $(k, k+m)$, then the transversal metric

$$\mathrm{d}s_1^2 = g_{ij}^\perp \mathrm{d}x^i\mathrm{d}x^j,$$

will have signature $(k-1, k-1+m)$.

Using a normalised frame we calculate the boost weight $\mathbf{b} = (1, b_2, ..., b_k)$ and

$(0, b_2, ..., b_k)$ components of the Riemann tensor:

$$R_{\hat{0}\hat{i}\hat{0}\hat{i}} = -\frac{1}{2}W_{\hat{i},vv}, \tag{3.10}$$

$$R_{\hat{0}\hat{i}\hat{0}\hat{i}} = -H_{,vv} + \frac{1}{4}\left(W_{\hat{i},v}\right)\left(W^{\hat{i},v}\right), \tag{3.11}$$

$$R_{\hat{0}\hat{i}\hat{i}\hat{j}} = W_{[\hat{i}}W_{\hat{j}],vv} + W_{[\hat{i};\hat{j}],v}, \tag{3.12}$$

$$R_{\hat{0}\hat{i}\hat{i}\hat{j}} = \frac{1}{2}\left[-W_{\hat{j}}W_{\hat{i},vv} + W_{\hat{i};\hat{j},v} - \frac{1}{2}\left(W_{\hat{i},v}\right)\left(W_{\hat{j},v}\right)\right], \tag{3.13}$$

$$R_{\hat{i}\hat{j}\hat{k}\hat{l}} = \tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}. \tag{3.14}$$

In particular, the Riemann tensor satisfies the $\mathbf{S}_1$-property if $R_{\hat{0}\hat{i}\hat{0}\hat{A}} = -\frac{1}{2}W_{\hat{A},vv} = 0$. Depending on the components of boost weight $\mathbf{b} = (0, b_2, ..., b_k)$, the Riemann tensor may satisfy the other requirements:

$$H_{,vv} - \frac{1}{4}\left(W_{\hat{i},v}\right)\left(W^{\hat{i},v}\right) = \sigma, \tag{3.15}$$

$$W_{[\hat{i};\hat{j}],v} = \mathsf{a}_{\hat{i}\hat{j}}, \tag{3.16}$$

$$W_{(\hat{i};\hat{j}),v} - \frac{1}{2}\left(W_{\hat{i},v}\right)\left(W_{\hat{j},v}\right) = \mathsf{s}_{\hat{i}\hat{j}}, \tag{3.17}$$

and the components $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$.

3.5.1 Walker Metrics

A special class of Kundt metrics are the Walker metrics. Given two mutually orthogonal null vectors, $\boldsymbol{\ell}^1$ and $\boldsymbol{\ell}^2$ that span a 2-dimensional invariant null plane, then they satisfy the recurrence condition

$$[\ell^1_{[a}\ell^2_{b]}]_{;c} = [\ell^1_{[a}\ell^2_{b]}]k_c, \tag{3.18}$$

for some recurrence vector k_c (and the two null vectors are automatically surface forming). This can be interpreted as requiring that the volume form of the invariant 2-plane is recurrent.

Definition 3.5.4. A Walker space is a pseudo-Riemannian space admitting a 2-, or

1-dimensional invariant null plane [26].

Walker metrics can be written in the following canonical form [26]:

$$\mathrm{d}s^2 = \mathrm{d}u^I(2\delta_{IJ}\mathrm{d}v^J + B_{IJ}\mathrm{d}u^J + H_{Ii}\mathrm{d}x^i) + A_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad (3.19)$$

where B_{IJ} is a symmetric matrix which in general depends on all of the coordinates, and H_{Ii} and A_{ij} do not depend on the coordinates v^I. Metrics of this type admit two null-vectors $\boldsymbol{\ell}^1$ and $\boldsymbol{\ell}^2$ which are geodesic, expansion-free, shear-free and vorticity-free, and hence are pseudo-Riemannian Kundt metrics.

3.5.2   The N-Property and VSI Spaces

At this point we are able to state the conditions which are required for these pseudo-Riemannian Kundt metrics to be VSI metrics.

We obtain necessary conditions by considering the Riemann tensor, R as an operator $\mathsf{R} : \wedge^2 T_p M \mapsto \wedge^2 T_p M$, given by: $\mathsf{R} = (R^{\alpha\beta}{}_{\mu\nu})$. For the pseudo-Riemannian Kundt metrics, if $R_{\hat{0}\hat{1}\hat{0}\hat{A}} = -\frac{1}{2}W_{\hat{A},vv} = 0$, the question of when R possesses the **N**-property is related to the values of the components σ, $\mathsf{a}_{\hat{A}\hat{B}}$, $\mathsf{s}_{\hat{A}\hat{B}}$ and $\tilde{R}_{\hat{A}\hat{B}\hat{C}\hat{D}}$.

A sufficient criterion for the Riemann curvature operator to have only zero polynomial invariants is that $\sigma = 0$ and the matrices

$$\left[\mathsf{a}^{\hat{A}}{}_{\hat{B}}\right], \quad \left[\mathsf{s}^{\hat{A}}{}_{\hat{B}}\right], \quad \left[\tilde{R}^{\hat{A}\hat{B}}{}_{\hat{C}\hat{D}}\right],$$

have only zero-eigenvalues and are therefore *nilpotent*.

3.5.3   Nested Kundt Metrics

Consider the transverse metric $\mathrm{d}s_1 = g_{AB}(u, x^C)\mathrm{d}x^A\mathrm{d}x^B$ of signature $(k - 1, k - 1 + m)$. We may consider the case where this metric is also a pseudo-Riemmannian Kundt metric. We can write it in Kundt form, and the transverse space will again be independent of one of the null-coordinates. We may continue this process by considering the case when again the transverse space is pseudo-Riemannian Kundt.

We define null coordinates (v^{a_I}, u^{b_I}), $a_I, b_I = 1, ..., I$ and let x^{μ_I}, $\mu_I = 2I, .., k+m$

be the "transversal" coordinates. As described above, we consider the I-th nested Kundt metric:

$$
\begin{aligned}
\mathrm{d}s_I^2 &= 2\mathrm{d}u^I \left[\mathrm{d}v^I + H^I(v^I, u^a, x^{\rho_I})\mathrm{d}u^I + W^I_{\mu_I}(v^I, u^a, x^{\rho_I})\mathrm{d}x^{\mu_I} \right] \\
&+ g^I_{\mu_I \nu_I}(u^a, x^{\rho_I})\mathrm{d}x^{\mu_I}\mathrm{d}x^{\nu_I},
\end{aligned}
\tag{3.20}
$$

for which the boost weight component $\mathbf{b} = (0, ..0, 1, b_I, ..., b_k)$ is $-\frac{1}{2}W^I_{\hat{\mu}_I, v^I v^I}$. Similarly, we write the boost weight $\mathbf{b} = (0, ..., 0, b_I, ..., b_k)$ components:

$$
H^I_{,v^I v^I} - \frac{1}{4}\left(W^I_{\hat{\mu}_I, v^I}\right)\left(W^{I\ \hat{\mu}_I, v^I}\right) = \sigma^I, \tag{3.21}
$$

$$
W^I_{[\hat{\mu}_I; \hat{\nu}_I], v^I} = \mathsf{a}^I_{\hat{\mu}_I \hat{\nu}_I}, \tag{3.22}
$$

$$
W^I_{(\hat{\mu}_I; \hat{\mu}_I), v^I} - \frac{1}{2}\left(W^I_{\hat{\mu}_I, v^I}\right)\left(W^I_{\hat{\nu}_I, v^I}\right) = \mathsf{s}^I_{\hat{\mu}_I \hat{\nu}_I}, \tag{3.23}
$$

and the components $\tilde{R}^I_{\hat{\mu}_I \hat{\nu}_I \hat{\rho}_I \hat{\lambda}_I}$.

3.6 Pseudo-Riemannian Kundt VSI Metrics

A method of constructing VSI spaces is to construct them from nested Kundt metrics required to possess the **N**-property. We can then iteratively solve the above equations for the componenets. We assume the metric is I-th order nested Kundt and solve for H^I and W^I_i.

It is natural to first consider cases in lower dimensions. To conclude this chapter, we first consider the five dimensional case with $(2, 3)$-signautre, then give a class of VSI metrics of $(k, k + m)$-signature. We then consider Ricci flat metrics. We note that there are also other cases considered in the literature. In the next chapter the 4D neutral signature case is considered in detail

3.6.1 The $(2, 3)$-Signature Case

In this case we may write

$$
\mathrm{d}s^2 = 2\left(\boldsymbol{\ell}^1 \boldsymbol{n}^1 + \boldsymbol{\ell}^2 \boldsymbol{n}^2\right) + (\boldsymbol{m}^1)^2. \tag{3.24}
$$

Considering the pseudo-Riemannian Kundt case, the 1st level pseudo-Riemannian Kundt has a 3-dimensional transverse space, and the 2nd transverse space is 1-dimensional. We may choose $\boldsymbol{\omega}^1 = (\mathrm{d}x)^2$. At the 2nd level, we get the standard Lorentzian Kundt VSI spaces:

$$\mathrm{d}s_1^2 = 2\mathrm{d}v^2 \left(\mathrm{d}v^2 + H^2\mathrm{d}u^2 + W^2\mathrm{d}x\right) + (\mathrm{d}x)^2 \tag{3.25}$$

where

$$H^2 = \frac{(v^2)^2}{8}\left(W^{2,(1)}\right)^2 + v^2 H^{2,(1)}(u^1, u^2, x) + H^{2,(0)}(u^1, u^2, x), \tag{3.26}$$

$$W^2_{\mu_2}\mathrm{d}x^{\mu_2} = v^2 W^{2,(1)}\mathrm{d}x + W^{2,(0)}(u^1, u^2, x)\mathrm{d}x, \tag{3.27}$$

where

$$W^{2,(1)} = -\frac{2\epsilon}{x}. \tag{3.28}$$

When we solve the remaining equations for the 5d Kundt metric we get several cases to consider. We write the metric as:

$$\begin{aligned}
\mathrm{d}s^2 &= 2\mathrm{d}u^1 \left(\mathrm{d}v^1 + H^1\mathrm{d}u^1 + W^1_{\mu_1}\mathrm{d}x^{\mu_1}\right) \\
&\quad + 2\mathrm{d}u^2 \left(\mathrm{d}v^2 + H^2\mathrm{d}u^2 + W^2\mathrm{d}x\right) + (\mathrm{d}x)^2.
\end{aligned} \tag{3.29}$$

The functions H^2 and W^2 are given by eqs. (3.26) and (3.27).

Case 1, $\epsilon = 1$:

$$\begin{aligned}
H^1 &= (v^1)^2 H^{1,(2)} + (v^1)H^{1,(1)}(u^1, u^2, v^2, x) + H^{1,(0)}(u^1, u^2, v^2, x), \\
W^1_{\mu_1}\mathrm{d}x^{\mu_1} &= v^1 \mathbf{W}^{1,(1)} + W^{1,(0)}_{\mu_1}(u^1, u^2, v^2, x)\mathrm{d}x^{\mu_1},
\end{aligned} \tag{3.30}$$

where, by definition $(u^1, v^1, u^2, v^2) = (u, v, U, V)$,

$$\mathbf{W}^{1,(1)} = -\frac{2\mathrm{d}V}{V + Cx + Dx^2} + \frac{2[V - Dx^2]\mathrm{d}x}{[V + Cx + Dx^2]x} \tag{3.31}$$
$$+ \frac{2[DV + x(CD - C_{,U}) + x^2(D^2 - D_{,U})]\mathrm{d}U}{V + Cx + Dx^2},$$

$$H^{1,(2)} = -\frac{x[2(CD - C_{,U}) + x(D^2 - 2D_{,U})]}{2[V + Cx + Dx^2]^2}, \tag{3.32}$$

where $C = C(u, U)$ and $D = D(u, U)$ are arbitrary functions. We also have that

$$\mathbf{W}^{1,(1)} = \mathrm{d}\phi, \quad \phi = 2\ln x - 2\ln[V + Cx + Dx^2] + 2\int D(u, U)\mathrm{d}U.$$

Case 2, $\epsilon = 0$, null:

$$W^1_{\mu_1}\mathrm{d}x^{\mu_1} = v^1 W^{1,(1)}_{u^2}(u^1, u^2, x)\mathrm{d}u^2 + W^{1,(0)}_{\mu_1}(u^1, u^2, v^2, x)\mathrm{d}x^{\mu_1},$$
$$H^1 = v^1 H^{1,(1)}(u^1, u^2, v^2, x) + H^{1,(0)}(u^1, u^2, v^2, x). \tag{3.33}$$

These possess an invariant null-line if $\epsilon = 0$ (null case), and $W^{1,(1)} = 0$, and they possesses a 2-dimensional invariant null plane if $\epsilon = 0$, $W^{1,(0)}_{v^1} = W^{1,(0)}_{v^2} = 0$ and $\partial_{v^2} W^{1,(0)}_x = 0$.

3.6.2 A Class of $(k, k+m)$-Signature VSI Metrics.

We now present a more general class of VSI metrics than the previously considered null-case [15]:

$$\mathrm{d}s^2 = \sum_{I=1}^{k} 2\boldsymbol{\ell}^I \boldsymbol{n}^I + \sum_{i,j=1}^{m} \delta_{ij}\boldsymbol{m}^i \boldsymbol{m}^j, \tag{3.34}$$

where

$$\boldsymbol{\ell}^I \;=\; \mathrm{d}u^I, \tag{3.35}$$

$$\boldsymbol{n}^I \;=\; \mathrm{d}v^I + \left[v^I H^{I,(1)}(u^a, v^{a_I}, x^i) + H^{I,(0)}(u^a, v^{a_I}, x^i) \right] \mathrm{d}u^I$$
$$+ \left[v^I W^{I,(1)}(u^a, v^{a_{I+1}}, x^i)\mathrm{d}u^{I+1} + W^{I,(0)}_{\mu_I}(u^a, v^{a_I}, x^i)\mathrm{d}x^{\mu_I} \right], \tag{3.36}$$

$$\boldsymbol{m}^i \;=\; \mathrm{d}x^i, \tag{3.37}$$

and the indices have the following ranges:

$$a = 1, 2, ..., k; \quad a_I = I + 1, ..., k; \quad i = 1, ..., m,$$
$$\mathrm{d}x^{\mu_I} = \{\mathrm{d}u^{a_I}, \mathrm{d}v^{a_I}, \mathrm{d}x^i\}. \tag{3.38}$$

3.7   A Class of Ricci-Flat Metrics

We consider the case where the only non-zero functions are $H^{I,(0)}$:

$$\boldsymbol{\ell}^I = \mathrm{d}u^I, \quad \boldsymbol{n}^I = \mathrm{d}v^I + H^{I,(0)}(u^a, v^{a_I}, x^i)\mathrm{d}u^I, \quad \boldsymbol{m}^i = \mathrm{d}x^i. \tag{3.39}$$

These are Walker metrics possessing an invariant k-dimensional null plane. The only non-vanishing components of the Ricci tensor are:

$$R_{u^I u^I} = -\Box H^{I,(0)},$$

where

$$\Box = \sum_{J=1}^{k} 2 \left(\frac{\partial}{\partial u^J} - H^{J,(0)} \frac{\partial}{\partial v^J} \right) \frac{\partial}{\partial v^J} + \sum_i \left(\frac{\partial}{\partial x^i} \right)^2.$$

In general the Ricci operator, $\mathsf{R} = (R^\mu{}_\nu)$, is nilpotent with $\mathsf{R}^2 = 0$. The requirement for this metric to be Ricci-flat is the following k equations:

$$\Box H^{I,(0)} = 0. \tag{3.40}$$

Fortunately, the functions $H^{I,(0)}$ have do not depend on v^J, for $J = 1, ..., I$, yielding

the following simplifications:

$$\Box H^{I,(0)} = \left[\sum_{J=I+1}^{k} 2 \left(\frac{\partial}{\partial u^J} - H^{J,(0)} \frac{\partial}{\partial v^J} \right) \frac{\partial}{\partial v^J} + \sum_i \left(\frac{\partial}{\partial x^i} \right)^2 \right] H^{I,(0)} = 0. \quad (3.41)$$

When $I = k$, this is solving the Laplacian over the space $\delta_{ij} dx^i dx^j$. Then $H^{I,(0)}$ can be substituted into the equation for $H^{I-1,(0)}$, which again can in principle be solved. We can repeat this process for $H^{I-2,(0)}$, etc. Thus we have a systematic process for solving these equations, because $H^{I,(0)}$ only involves functions $H^{I+1,(0)}$, $H^{I+2,(0)}$, etc.

3.7.1 Neutral 4D Space

In the neutral 4D case we have, $H^2 = 0$. Thus there is only one differential equation to solve:

$$\Box H^{1,(0)}(u^1, u^2, v^2) = \frac{\partial^2}{\partial u^2 \partial v^2} H^{1,(0)}(u^1, u^2, v^2) = 0. \quad (3.42)$$

We write

$$H^{1,(0)}(u^1, u^2, v^2) = f(u^1, u^2) + g(u^1, v^2), \quad (3.43)$$

for arbitrary functions f and g. This yields standard Ricci-flat neutral metrics:

$$ds^2 = 2du^1 \left[dv^1 + (f(u^1, u^2) + g(u^1, v^2)) du^1 \right] + 2du^2 dv^2. \quad (3.44)$$

This is a VSI space.

3.7.2 The $(2,3)$-Signature Case.

Requiring Ricci-flatness yields two equations:

$$\begin{aligned}
\Box H^{1,(0)}(u^1, u^2, v^2, x) &= \left(\frac{\partial^2}{\partial u^2 \partial v^2} + \frac{\partial^2}{\partial x^2} \right) H^{1,(0)}(u^1, u^2, v^2, x) = 0, \\
\Box H^{2,(0)}(u^1, u^2, x) &= \frac{\partial^2}{\partial x^2} H^{2,(0)}(u^1, u^2, x) = 0,
\end{aligned} \quad (3.45)$$

These equations are Laplacians in flat space (1-, respectively 3-dimensional space) and are easily solved.

Chapter 4

Pseudo-Riemannian Kundt CSI

4.1 4D CSI Spaces

We begin our discussion of Pseudo-Riemannian Kundt CSI spacetimes by recalling the review given in chapter 2 which reviewed the work done on 4D Lorentzian CSI spaces. Recall:

Theorem 4.1.1. *A 4D spacetime is CSI if and only if either:*

1. *the spacetime is locally homogeneous;* or

2. *the spacetime is a* Kundt spacetime *for which there exists a frame such that all curvature tensors have the following properties: (i) all positive boost weight components vanish; (ii) all boost weight zero components are constants.*

In the following definitions let $\mathcal{M}$ be a spacetime equipped with a metric g and let $\mathcal{I}_k$ be the set of all polynomial scalar invariants constructed from the curvature tensor and its covariant derivatives up to order k.

Definition 4.1.2. $\mathcal{M}$ is called CSI_k if for any invariant $I \in \mathcal{I}_k$, I is constant over $\mathcal{M}$

Theorem 4.1.3. *For a 4D Lorentzian spacetime,*

$$CSI \Leftrightarrow CSI_3. \tag{4.1}$$

4.2 Review of n-Dimensional Pseudo-Riemannian Kundt CSI Spacetimes

In chapter 3 it was established that a Kundt CSI spacetime can be written in the form [1]

$$ds^2 = 2du[dv + H(v, u, x^k)du + W_i(v, u, x^k)dx^i] + g_{ij}^{\perp}dx^i dx^j. \tag{4.2}$$

where $dS_H^2 = g_{ij}^{\perp}(x^k)dx^i dx^j$ is the locally homogeneous metric of the 'transverse' space and the metric functions H and W_i, requiring CSI_0, are given by (these are solutions to equations (3.10) and (3.15)

$$W_i(v, u, x^k) = vW_i^{(1)}(u, x^k) + W_i^{(0)}(u, x^k), \tag{4.3}$$

$$H(v, u, x^k) = v^2\tilde{\sigma} + vH^{(1)}(u, x^k) + H^{(0)}(u, x^k), \tag{4.4}$$

$$\tilde{\sigma} \equiv \frac{1}{8}\left(4\sigma + W^{(1)i}W_i^{(1)}\right), \tag{4.5}$$

where σ is a constant.

The additional equations from the CSI_0 requirement are (these are equations (3.16) and (3.17)).

$$W_{[\hat{i};\hat{j}]}^{(1)} = \mathsf{a}_{\hat{i}\hat{j}}, \tag{4.6}$$

$$W_{(\hat{i};\hat{j})}^{(1)} - \frac{1}{2}\left(W_{\hat{i}}^{(1)}\right)\left(W_{\hat{j}}^{(1)}\right) = \mathsf{s}_{\hat{i}\hat{j}}, \tag{4.7}$$

and the components $R_{\hat{i}\hat{j}\hat{m}\hat{n}}^{\perp}$ are all constants. Hatted indices refer to an orthonormal frame in the transverse space. The constant matrices $\mathsf{a}_{\hat{i}\hat{j}}$ and $\mathsf{s}_{\hat{i}\hat{j}}$ are antisymmetric, respectively symmetric, matrices determined by the boost weight 0 components of the Riemann tensor, given by

$$R_{0101} = -\sigma, \quad R_{01ij} = \mathsf{a}_{ij}, \tag{4.8}$$

$$R_{0i1j} = \frac{1}{2}(\mathsf{s}_{ij} + \mathsf{a}_{ij}) \tag{4.9}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}). \tag{4.10}$$

---

[1]This is the same as equation (3.8). Recall that in the VSI case $\tilde{\sigma} = 0$ and $a_{\hat{i}\hat{j}}$ and $s_{\hat{i}\hat{j}}$ are nilpotent.

Equations (4.6) and (4.7) give a set of differential equations for $W_i^{(1)}$, which uniquely determine $W_i^{(1)}$ up to initial conditions. Furthermore, requiring CSI_1 yields the constraints:

$$\boldsymbol{\alpha}_{\hat{i}} = \sigma W_{\hat{i}}^{(1)} - \frac{1}{2}(\mathsf{s}_{\hat{j}\hat{i}} + \mathsf{a}_{\hat{j}\hat{i}})W^{(1)\hat{j}}, \tag{4.11}$$

$$\boldsymbol{\beta}_{\hat{i}\hat{j}\hat{k}} = W^{(1)\hat{n}}R^{\perp}_{\hat{n}\hat{i}\hat{j}\hat{k}} - W_{\hat{i}}^{(1)}\mathsf{a}_{\hat{j}\hat{k}} + (\mathsf{s}_{\hat{i}[\hat{j}} + \mathsf{a}_{\hat{i}[\hat{j}})W_{\hat{k}]}^{(1)}, \tag{4.12}$$

where $\boldsymbol{\alpha}_{\hat{i}}$ and $\boldsymbol{\beta}_{\hat{i}\hat{j}\hat{k}}$ are constants determined by the boost weight 0 components of the covariant derivative of the Riemann tensor (see (60-63) in [13]).

4.2.1 The 4-Dimensional Case

Writing equations (4.6) and (4.7) explicitly in the 4d case gives

$$\frac{1}{2}(W_{\hat{0};\hat{1}}^{(1)} - W_{\hat{1};\hat{0}}^{(1)}) = -\sigma, \tag{4.13}$$

$$W_{\hat{0};\hat{0}}^{(1)} - \frac{1}{2}(W_{\hat{0}}^{(1)})^2 = 0, \tag{4.14}$$

$$W_{\hat{0};\hat{1}}^{(1)} + W_{\hat{1};\hat{0}}^{(1)} - W_{\hat{0}}^{(1)}W_{\hat{1}}^{(1)} = 2\sigma, \tag{4.15}$$

$$W_{\hat{1};\hat{1}}^{(1)} - \frac{1}{2}(W_{\hat{1}}^{(1)})^2 = 0. \tag{4.16}$$

Let us consider some CSI examples in which $\tilde{\sigma}$ is constant and everything else is zero. Setting $\sigma = 0$ we solve the above differential equations.

Equation (4.14) gives the equation

$$\frac{\partial W_0^{(1)}}{\partial x^0} = \frac{1}{2}(W_0^{(1)})^2. \tag{4.17}$$

Solving this using separation of variables gives

$$W_0^{(1)} = -\frac{2\epsilon_0}{x^0 - 2\gamma(u, x^1)}, \qquad (4.18)$$

where $\gamma(u, x^1)$ is arbitrary. Similarly we solve equation (4.16) to obtain

$$W_1^{(1)} = -\frac{2\epsilon_1}{x^1 - 2\delta(u, x^0)}. \qquad (4.19)$$

where $\delta(u, x^0)$ is also an arbitrary function. We note that $\epsilon_0 = \epsilon_1 = 0$ gives the trivial solution $W_0^{(1)} = W_1^{(1)} = 0$. Let us next consider the case $\epsilon_0 = 1, \epsilon_1 = 0$. In this case

$$W_1^{(1)} = 0,$$

$$W_0^{(1)} = -\frac{2}{x^0 - 2\gamma(u, x^1)}.$$

We then see from equation (4.13) that

$$4\frac{\partial \gamma}{\partial x^1} = 0,$$

and so α is some function of u, $K(u)$, giving the solution

$$W_0^{(1)} = -\frac{2}{x^0 - 2K(u)}.$$

Similarly in the case $\epsilon_0 = 0, \epsilon_1 = 1$ we get

$$W_1^{(1)} = -\frac{2}{x^1 - 2K(v)},$$

$$W_0^{(1)} = 0.$$

Finally we consider the case $\epsilon_0 = \epsilon_1 = 1$. Equations (4.13) and (4.14), with $\sigma = 0$, give the following constraints that the arbitrary functions γ, δ must satisfy:

$$-\frac{4\frac{\partial \gamma}{\partial x^1}}{(-x^0 + 2\gamma)^2} + \frac{4\frac{\partial \delta}{\partial x^0}}{(-x^1 + 2\delta)^2} = 0 \qquad (4.20)$$

$$-\frac{4\frac{\partial\gamma}{\partial x^1}}{(-x^0+2\gamma)^2}-\frac{4\frac{\partial\delta}{\partial x^0}}{(-x^1+2\delta)^2}-\frac{4}{(-x^0+2\gamma)(-x^1+2\delta)}=0 \qquad (4.21)$$

We notice from equation (4.20) that

$$\frac{\frac{\partial\delta}{\partial x^0}}{(x^1-2\delta)^2}=\frac{\frac{\partial\gamma}{\partial x^1}}{(x^0-2\gamma)^2}$$

Substituting this into equation (4.21) we can rearrange for $\frac{\partial\delta}{\partial x^0}$ and obtain

$$\frac{\partial\delta}{\partial x^0}=-\frac{1}{2}\frac{x^1-2\delta}{x^0-2\gamma},$$

which seperates and has solution

$$\delta=F(u)(-x^0+2\gamma)+\frac{1}{2}x^1,$$

Where $F(u)$ is arbitrary. We may now substitute this value for δ into equation (4.20) and obtain, after some rearranging

$$\frac{\partial\gamma}{\partial x^1}=-\frac{F(u)(x^0-2\gamma)^2}{\frac{1}{2}x^1-2F(u)(2\gamma-x0)},$$

which has the solution

$$\gamma=-\frac{x^1}{4F(u)}+G(u).$$

Therefore,

$$\delta=-F(u)x^0+2G(u).$$

Finally, substituting these into equations (4.18) and (4.19) we obtain

$$W_0^{(1)}=-\frac{2\epsilon_0}{x^0+\frac{x^1}{2F(u)}-G(u)}.$$

$$W_1^{(1)}=-\frac{2\epsilon_1}{x^1+2F(u)x^0-4G(u)}.$$

4.3   Some Exact CSI Solutions

We now consider 4D neutral signature spaces which are CSI, for which all of the curvature invariants are constant [13]. By solving the appropriate components of the Riemann tensor equal to constants (see eqns. (15) - (22) in [15]), and using similar techniques to those in [13], we can find examples of 4D neutral CSI metrics. In general the Einstein eqns. that need to be solved are quite complicated. Therefore, let us consider two simple examples for illustration. In particular, the neutral signature "Siklos metrics" (in which the only non-zero independent invariant is the Ricci scalar, which is constant) give rise to relatively simple equations that can be solved completely. Lorentzian CSI spacetimes are known to be solutions of supergravity theory when supported by appropriate bosonic fields [27], and it is likely that neutral signature CSI spaces are also of physical interest [9, 10].

4.3.1   CSI Example 1:

Let us write the CSI metric as:

$$\mathrm{d}s^2 = 2\boldsymbol{\ell}^1 \boldsymbol{n}^1 - \exp(2KX)\mathrm{d}T^2 + \mathrm{d}X^2,$$

where

$$\boldsymbol{\ell}^1 = \mathrm{d}u \tag{4.22}$$

$$\boldsymbol{n}^1 = \mathrm{d}v + [v^2\sigma + vH^{(1)}(u,T,X) + H^{(0)}(u,T,X)]\mathrm{d}u \tag{4.23}$$
$$+[Av + W_X^{(0)}(u,T,X)]\mathrm{d}X + [Bv + W_T^{(0)}(u,T,X)]\exp(KX)\mathrm{d}T$$

There are many cases which lead to an Einstein space. In particular, we consider the simple case:

$$A = -2K, \quad B = 0, \quad \sigma = 0.$$

Writing down the Einstein conditions, we obtain $\Lambda = -3K^2$ from the diagonal terms of the Ricci tensor, and differential equations from the off-diagonal terms. All of these have simple curvature structure; in particular, $C_{abcd}C^{abcd} = 0$.

Explicitly, the metric has the form

$$
\begin{aligned}
\mathrm{d}s^2 \;=\;& 2\mathrm{d}u(\mathrm{d}v + [vH^{(1)} + H^{(0)}]\mathrm{d}u \\
& +\; [-2Kv + W_X^{(0)}]\mathrm{d}X \\
& +\; W_T^{(0)}\exp(KX)\mathrm{d}T) - \exp(2KX)\mathrm{d}T^2 + \mathrm{d}X^2 \qquad (4.24)
\end{aligned}
$$

where $H^{(1)}, H^{(0)}, W_X^{(0)}, W_T^{(0)}$ are functions of u, T, X.

By making the transformation

$$
x = \frac{1}{K}\exp(-KX),
$$

the metric is put into the form

$$
\begin{aligned}
\mathrm{d}s^2 \;=\;& 2\mathrm{d}u(\mathrm{d}v + [vH^{(1)} + H^{(0)}]\mathrm{d}u \\
& +\; [-2\frac{v}{x} + \bar{W}_x^{(0)}]\mathrm{d}x \\
& +\; \bar{W}_T^{(0)}\mathrm{d}T) - \frac{1}{K^2 x^2}\mathrm{d}T^2 + \frac{1}{K^2 x^2}\mathrm{d}x^2 \qquad (4.25)
\end{aligned}
$$

so that the transverse metric is explicitly in homogeneous form. Comparing with the general metric (4.2) we see that

$$
W_0^{(1)}(u, x^k) = W_X^{(1)}(u, T, X) = -\frac{2}{x},
$$

and that

$$
W_1^{(1)}(u, x^k) = W_T^{(1)}(u, T, X) = 0.
$$

We now see from (4.13) that $\sigma = 0$ and $\tilde{\sigma} = K^2$. From (4.8) and (4.10) we see that the matrices $\mathsf{a}_{\hat{i}\hat{j}}$ and $\mathsf{s}_{\hat{i}\hat{j}}$ are the zero matrices. Hence, we get that

$$
\boldsymbol{\alpha}_{\hat{0}} = \boldsymbol{\alpha}_{\hat{1}} = 0.
$$

$$
\boldsymbol{\beta}_{\hat{i}\hat{j}\hat{k}} = 0.
$$

The conditions for the metric to be an Einstein space yield the following system

of partial differential equations:

$$4\frac{\partial H^{(1)}}{\partial X}W_X^{(0)}\exp(2KX) + 2K\exp(2KX)\frac{\partial H^{(0)}}{\partial X}$$

$$+ \quad 4K^2\exp(2KX)H^{(0)} + 2\frac{\partial W_T^{(0)}}{\partial X}$$

$$+ \quad 2\frac{\partial W_T^{(0)}}{\partial X}\exp(KX)\frac{\partial W_X^{(0)}}{\partial T}$$

$$+ \quad 2K\exp(2KX)\frac{\partial W_X^{(0)}}{\partial u} - (W_T^{(0)})^2 K^2\exp(2KX)$$

$$- \quad 2v\frac{\partial^2 H^{(1)}}{\partial X^2}\exp(2KX) + 2\frac{\partial W_T^{(0)}}{\partial X}\exp(2KX)W_T^{(0)}K$$

$$- \quad 6K\exp(2KX)v\frac{\partial H^{(1)}}{\partial X} + 2K\exp(2KX)H^{(1)}W_X^{(0)}$$

$$- \quad 2W_T^{(0)}K\exp(KX)\frac{\partial W_X^{(0)}}{\partial T}$$

$$+ \quad 2H^{(1)}\frac{\partial W_X^{(0)}}{\partial X}\exp(2KX) - 4\frac{\partial H^{(1)}}{\partial T}W_T^{(0)}\exp(KX)$$

$$- \quad 2H^{(1)}\frac{\partial W_T^{(0)}}{\partial T}\exp(KX)$$

$$- \quad 2\frac{\partial^2 W_T^{(0)}}{\partial u\partial T}\exp(KX) + 2v\frac{\partial^2 H^{(1)}}{\partial T^2}$$

$$+ \quad 2\frac{\partial^2 H^{(0)}}{\partial T^2} - \left(\frac{\partial W_T^{(0)}}{\partial X}\right)^2 - \left(\frac{\partial W_X^{(0)}}{\partial T}\right)^2$$

$$- \quad \frac{\partial^2 H^{(0)}}{\partial X^2}\exp(2KX) + 2\frac{\partial^2 W_X^{(0)}}{\partial u\partial X}\exp(2KX) = 0 \qquad (4.26)$$

$$2\frac{\partial H^{(1)}}{\partial T} - K\frac{\partial W_T^{(0)}}{\partial X}\exp(KX) + 2W_T^{(0)}K^2\exp(KX) +$$

$$K\frac{\partial W_X^{(0)}}{\partial T} - \frac{\partial^2 W_T^{(0)}}{\partial X^2}\exp(KX) + \frac{\partial^2 W_X^{(0)}}{\partial X\partial T} = 0 \qquad (4.27)$$

$$2\frac{\partial H^{(1)}}{\partial X}\exp(2KX) - \frac{\partial^2 W_T^{(0)}}{\partial X \partial T}\exp(KX)$$
$$+ \frac{\partial W_T^{(0)}}{\partial T}K\exp(KX) + \frac{\partial^2 W_X^{(0)}}{\partial T^2} = 0 \tag{4.28}$$

The v dependence in eqn. (4.26) gives

$$- 2\frac{\partial^2 H^{(1)}}{\partial X^2}\exp(2KX) - 6K\exp(2KX)\frac{\partial H^{(1)}}{\partial X}$$
$$+ 2\frac{\partial^2 H^{(1)}}{\partial T^2} = 0 \tag{4.29}$$

The form of the metric (4.25) is preserved under the following coordinate transformations:

$$(u', v', T', X') = (u, v + h(u, X, T), T, X) \tag{4.30}$$

$$H'^{(1)} = H^{(1)} \tag{4.31}$$
$$H'^{(0)} = H^{(0)} - hH^{(1)} - h_{,u} \tag{4.32}$$
$$W_x'^{(0)} = W_x^{(0)} + 2kh - h_{,X} \tag{4.33}$$
$$W_T'^{(0)} = W_T^0 - \exp(-kx)h_{,T} \tag{4.34}$$

$$(u', v', T', X') = \left(g(u), \frac{v}{g_{,u}(u)}, T, X\right) \tag{4.35}$$

$$H'^{(1)} = (g_{,u}H^{(1)} + g_{,uu})/g_{,u}^2 \tag{4.36}$$

$$H'^{(0)} = H^{(0)}/g_{,u}^2 \tag{4.37}$$

$$W_X'^{(0)} = W_X^{(0)}/g_{,u} \tag{4.38}$$

$$W_T'^{(0)} = W_T^{(0)}/g_{,u} \tag{4.39}$$

Thus, without loss of generality, we can set $W_T^{(0)} = 0$. We note that $H^{(0)}$ does not appear in equations (4.27), (4.28), (4.29). We first use eqns. (4.27) and (4.28) to solve for $W_X^{(0)}$ and $H^{(1)}$, and then use eqn. (4.29) to put constraints on these solutions.

First we can integrate eqn. (4.27) with respect to T to get

$$2H^{(1)} + kW_X^{(0)} + \frac{\partial W_X^{(0)}}{\partial X} + \alpha = 0, \tag{4.40}$$

where $\alpha = \alpha(u, X)$ is an arbitrary function. Now, using using eqns. (4.40) and (4.28) we can solve for $H^{(1)}$ and $W_X^{(0)}$:

$$\begin{aligned}
H^{(1)} &= \frac{1}{2}\bar{\alpha} + \frac{1}{2}\frac{\partial\bar{\alpha}}{\partial X} - \frac{1}{2}\alpha - \frac{1}{2}KC_1T \\
&- \frac{1}{2}KC_2 - \frac{1}{2}K\gamma - \frac{1}{4}KC_3T^2 + \frac{1}{4}C_3K^{-1}\exp(-2KX)
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
W_X^{(0)} &= \frac{1}{2}C_3T^2 + C_1T + \frac{1}{2}C_3K^{-2}\exp(2KX) \\
&- \bar{\alpha} + \rho K^{-1}\exp(-KX) + \gamma
\end{aligned} \tag{4.42}$$

where

$$\bar{\alpha} = \int \frac{\int \exp(KX)\frac{\partial\alpha}{\partial X}dX}{\exp(KX)}dX \tag{4.43}$$

and $\rho = \rho(u), \alpha = \alpha(u, X), \gamma = \gamma(u)$ are arbitrary functions. We can now put constraints on the form of (4.41) using eqn. (4.29). After simpification,

eqn. (4.29) becomes

$$2C_3K(-\exp(-2KX) + 1) = 0 \qquad (4.44)$$

from which we conclude that $C_3 = 0$ (since $K = 0$ is not consistent with eqn. (4.26)).

Finally, we employ the remaining transformational freedom to set $\gamma(u) = 0$. Thus we are left with the following differential equation for $H^{(0)}$

$$
\begin{aligned}
& 2C_2K^2\exp(2KX)\bar{\alpha} + 2C_2K\exp(2KX)\frac{\partial\bar{\alpha}}{\partial X} - 2K\exp(2KX)\frac{\partial\bar{\alpha}}{\partial X}\bar{\alpha} \\
-\ & C_2K\exp(2KX)\alpha + K\exp(2KX)\alpha\bar{\alpha} - C_1^2K^2\exp(2KX)T^2 \\
-\ & C_1^2 + 2C_1K^2\exp(2KX)\bar{\alpha}T + 2C_1K\exp(2KX)\frac{\partial\bar{\alpha}}{\partial X}T - C_1K\exp(2KX)\alpha T \\
-\ & 2C_1C_2K^2\exp(2KX)T - \exp(2KX)\left(\frac{\partial\bar{\alpha}}{\partial X}\right)^2 \\
-\ & 2\exp(KX)\int \exp(KX)\frac{\partial^2\alpha}{\partial u\partial X}\alpha dX - 2K\exp(2KX)\bar{\alpha} - K^2\exp(2KX)\bar{\alpha}^2 \\
-\ & C_2^2K^2\exp(2KX) + \exp(2KX)\frac{\partial\bar{\alpha}}{\partial X}\alpha + 2\frac{\partial^2 H^{(0)}}{\partial T^2} - 2\exp(2KX)\frac{\partial^2 H^{(0)}}{\partial X^2} \\
+\ & 2K\exp(2KX)\frac{\partial H^{(0)}}{\partial X} + 4K^2\exp(2KX)H^{(0)} = 0 \qquad (4.45)
\end{aligned}
$$

A particularly simple subcase is the case where $H^{(1)} = W_X^{(0)} = W_T^{(0)} = 0$, whence we obtain the special solution $H^{(0)} = C\exp(-KX)$, which is the Kaigorodov case.

4.3.2 CSI Example 2:

The CSI metric can be written

$$ds^2 = 2(\boldsymbol{\ell}^1\boldsymbol{n}^1 + \boldsymbol{\ell}^2\boldsymbol{n}^2)$$

where

$$\boldsymbol{\ell}^1 = \mathrm{d}u \tag{4.46}$$

$$\boldsymbol{n}^1 = \mathrm{d}v + [Av^2 + vH^{(1)}(u,U,V) + H^{(0)}(u,U,V)]\mathrm{d}u \tag{4.47}$$

$$+ [vV\beta + W_U^{(0)}(u,U,V)]\mathrm{d}U + [\alpha v/V + W_V^{(0)}(u,U,V)](\mathrm{d}V + BV^2\mathrm{d}U)$$

$$\boldsymbol{\ell}^2 = \mathrm{d}U \tag{4.48}$$

$$\boldsymbol{n}^2 = \mathrm{d}V + BV^2\mathrm{d}U, \tag{4.49}$$

and A, B, α and β are constants.

We look for Einstein spaces in the special case:

$$A = 0, \quad \alpha = -2, \quad \beta = 2B$$

(where B is not specified). The metric then has the explicit form

$$\begin{aligned}
\mathrm{d}s^2 &= 2\mathrm{d}u(\mathrm{d}v + [vH^{(1)} + H^{(0)}]\mathrm{d}u + [2BvV + W_U^{(0)}]\mathrm{d}U \\
&\quad + [-2\frac{v}{V} + W_V^{(0)}](dV + BV^2\mathrm{d}U)) + 2\mathrm{d}U\mathrm{d}V + 2BV^2\mathrm{d}U^2 \tag{4.50}
\end{aligned}$$

Hence $W_U^{(1)} = 0$ and $W_V^{(1)} = \frac{-2}{V}$. We again see from (4.13) that $\sigma = 0$, $\tilde{\sigma} = \frac{1}{2V^2}$ and from (4.8) and (4.10) we see that the matrices $\mathsf{a}_{\hat{i}\hat{j}}$ and $\mathsf{s}_{\hat{i}\hat{j}}$ are the zero matrices. Hence, we get that

$$\boldsymbol{\alpha}_{\hat{0}} = \boldsymbol{\alpha}_{\hat{1}} = 0.$$

$$\boldsymbol{\beta}_{\hat{i}\hat{j}\hat{k}} = 0.$$

The conditions for the metric to be Einstein yields the following system of (independent) differential equations.

$$
\begin{aligned}
&-4BV^4\frac{\partial^2 H^{(0)}}{\partial V^2} + 4v\frac{\partial^2 H^{(1)}}{\partial V \partial U}V^2 + 2BV^4\frac{\partial^2 W_V^{(0)}}{\partial u \partial V}\\
&-\ 4BV^4 v\frac{\partial^2 H^{(1)}}{\partial V^2} - 16BV^3 v\frac{\partial H^{(1)}}{\partial V} + 2H^{(1)}V^4 B\frac{\partial W_V^{(0)}}{\partial V}\\
&+\ 4BV^3 H^{(1)}W_V^{(0)} + 2\frac{\partial W_U^{(0)}}{\partial V}V^4 B\frac{\partial W_V^{(0)}}{\partial V} - 4BV^3\frac{\partial W_V^{(0)}}{\partial V}W_U^{(0)}\\
&-\ 2BV^4\frac{\partial W_V^{(0)}}{\partial V}\frac{\partial W_V^{(0)}}{\partial U} + 4BV^4\frac{\partial H^{(1)}}{\partial V}W_V^{(0)} + 4Vv\frac{\partial H^{(1)}}{\partial U}\\
&+\ 8BV^2 H^{(0)} - 2H^{(1)}V^2\frac{\partial W_U^{(0)}}{\partial V} - 2H^{(1)}V^2\frac{\partial W_V^{(0)}}{\partial U} + 4BV^3\frac{\partial W_V^{(0)}}{\partial u}\\
&-\ 4\frac{\partial H^{(1)}}{\partial V}V^2 W_U^{(0)} - 4\frac{\partial H^{(1)}}{\partial U}W_V^{(0)}V^2 - 4\frac{\partial W_U^{(0)}}{\partial V}VW_U^{(0)}\\
&-\ 2\frac{\partial W_U^{(0)}}{\partial V}V^2\frac{\partial W_V^{(0)}}{\partial U} + B^2 V^6\left(\frac{\partial W_V^{(0)}}{\partial V}\right)^2 + 4W_U^{(0)}\frac{\partial W_V^{(0)}}{\partial U}V\\
&+\ \left(\frac{\partial W_U^{(0)}}{\partial V}\right)^2 V^2 - 4V\frac{\partial H^{(0)}}{\partial U} + \left(\frac{\partial W_V^{(0)}}{\partial U}\right)^2 + 4(W_U^{(0)})^2\\
&-\ 2\frac{\partial^2 W_V^{(0)}}{\partial u \partial U}V^2 - 2\frac{\partial^2 W_U^{(0)}}{\partial u \partial V}V^2 + 4\frac{\partial^2 H^{(0)}}{\partial V \partial U}V^2 = 0
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
&2\frac{\partial H^{(1)}}{\partial U}V - 2\frac{\partial H^{(1)}}{\partial V}BV^3 + 4B^2 V^4\frac{\partial W_V^{(0)}}{\partial V} - 2BVW_U^{(0)}\\
&-\ \frac{\partial W_V^{(0)}}{\partial U}BV^2 - \frac{\partial^2 W_U^{(0)}}{\partial V \partial U}V - 2BV^3\frac{\partial^2 W_V^{(0)}}{\partial V \partial U} + 2\frac{\partial W_U^{(0)}}{\partial U}\\
&+\ \frac{\partial^2 W_V^{(0)}}{\partial U^2}V + BV^3\frac{\partial^2 W_U^{(0)}}{\partial V^2} + B^2 V^5\frac{\partial W_V^{(0)}}{\partial V^2} = 0
\end{aligned}
\tag{4.52}
$$

$$4BV^3\frac{\partial W_V^{(0)}}{\partial V} - 2W_U^{(0)} - 2\frac{\partial W_V^{(0)}}{\partial U}V + 2\frac{\partial H^{(1)}}{\partial V}V^2$$
$$+ \quad \frac{\partial^2 W_U^{(0)}}{\partial V^2}V^2 + BV^4\frac{\partial^2 W_V^{(0)}}{\partial V^2} - \frac{\partial^2 W_V^{(0)}}{\partial V \partial U}V^2 = 0 \qquad (4.53)$$

The v-dependency in eqn. (4.51) gives

$$4\frac{\partial^2 H^{(1)}}{\partial V \partial U}V^2 - 4BV^4\frac{\partial^2 H^{(1)}}{\partial V^2} - 16BV^3\frac{\partial H^{(1)}}{\partial V} + 4V\frac{\partial H^{(1)}}{\partial U} = 0 \qquad (4.54)$$

where $H^{(1)} = H^{(1)}(u, U, V), H^{(0)} = H^{(0)}(u, U, V), W_U^{(0)} = W_U^{(0)}(u, U, V),$
$W_V^{(0)} = W_V^{(0)}(u, U, V).$

Equations (4.52),(4.53),(4.54) do not contain any $H^{(0)}$ terms. The form of the metric (4.50) is invariant under the following two coordinate transformations:

1.

$$(u', v', U', V') = (u, v + h(u, U, V), U, V) \qquad (4.55)$$

$$H'^{(1)} = H'^{(1)} \qquad (4.56)$$
$$H'^{(0)} = H'^{(0)} - hH^{(1)} - h_{,u} \qquad (4.57)$$
$$\qquad (4.58)$$

To see that the other arbitrary functions are preserved under the transformation, we define

$$\overline{W}_V^{(0)} = -\frac{2h}{V} + W_V^{(0)} - h_{,V}, \qquad (4.59)$$

which is just rewriting the form of an arbitrary function (no v's introduced).

Then the other metric components transform as

$$\left(-2\frac{v}{V} + W_V^{(0)}\right)' = -2\frac{v}{V} + \overline{W}_V^{(0)} \tag{4.60}$$

$$\left(W_U^{(0)} + BV^2 W_V^{(0)}\right)' = W_U^{(0)} - h_{,U} + BV^2 h_{,V} + BV^2 \overline{W}_V^{(0)} \tag{4.61}$$

and so the form of the metric is preserved, with all arbitrary functions remaining arbitrary after the transformation.

2.

$$(u', v', U', V') = \left(g(u), \frac{v}{g_{,u}(u)}, U, V\right) \tag{4.62}$$

$$H'^{(1)} = (g_{,uu} + g_{,u}H^{(1)})/g_{,u}^2 \tag{4.63}$$

$$H'^{(0)} = H^{(0)}/g_{,u}^2 \tag{4.64}$$

$$W_U'^{(0)} = W_U^{(0)}/g_{,u} \tag{4.65}$$

$$W_V'^{(0)} = W_V^{(0)}/g_{,u} \tag{4.66}$$

We use this freedom to set $W_V^{(0)} = 0$ and obtain the solutions

$$H^{(1)} = \alpha(u) + \frac{\beta(u)}{V^3} \tag{4.67}$$

$$W_U^{(0)} = V^2\gamma(u, U) + \frac{(-4B\beta(u)U + \delta(u))}{V} + \frac{3}{2}\frac{\beta(u)}{V^2} \tag{4.68}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions. Substituting the above solutions into (4.51)

gives a differential equation for $H^{(0)}$:

$$-4BV^2 \frac{\partial^2 H^{(0)}}{\partial V^2} + 8BV^2 H^{(0)} - 2\left(\alpha + \frac{\beta}{V^3}\right) V^2 \left(-\frac{(-4B\beta U + \delta)}{V^2} + 2V\gamma - \frac{3\beta}{V^3}\right)$$

$$+ \quad \frac{12\beta \left(\frac{-4B\beta U + \delta}{V} + V^2\gamma + \frac{3\beta}{2V^2}\right)}{V^2} + 4\frac{\partial^2 H^{(0)}}{\partial V \partial U} V^2$$

$$- \quad 4\left(-\frac{-4B\beta U + \delta}{V^2} + 2V\gamma - \frac{3\beta}{V^3}\right) V \left(\frac{-4B\beta U + \delta}{V} + V^2\gamma + \frac{3\beta}{2V^2}\right)$$

$$+ \quad V^2 \left(-\frac{-4B\beta U + \delta}{V^2} + 2V\gamma - \frac{3\beta}{V^3}\right)^2$$

$$- \quad 2V^2 \left(-\frac{-4B\frac{d\beta}{dU}U + \frac{d\delta}{du}}{V^2} + 2V\frac{\partial\gamma}{\partial u} - \frac{3\frac{d\beta}{du}}{V^3}\right) = 0 \tag{4.69}$$

Chapter 5

Conclusion

Metrics possessing the CSI property that are not locally homogeneous cannot generally be characterized by their curvature invariants. However, they are degenerate Kundt metrics and hence have a very specific form, allowing us to study necessary conditions for spaces to be CSI (or VSI). Aside from the mathematical motivation for studying geometries with the CSI and VSI properties, presenting spaces with the CSI and VSI structure has applications in theoretical physics, such as in the twistor approach to string theory (see [9]), and in the study of spaces admitting parallel spinors (see [10]).

In this thesis we studied VSI and CSI spaces with a particular focus on the 4D neutral signature case. Two classes of VSI spaces were constructed, and two classes of CSI spaces were constructed. The VSI and CSI solutions are very rich in neutral signature, as is demonstrated by the generality of the spaces constructed.

Before constructing the solutions in neutral signature, the mathematical machinery necessary to study VSI and CSI spaces was reviewed. In particular, after studying some standard material in geometry, we defined the boost-weight classification of tensors in pseudo-Riemannian space. This classification scheme allowed us to discuss the $\mathbf{S}_i$ and $\mathbf{N}$ properties. Defining these properties allowed us to define necessary conditions for spaces to be VSI and CSI.

By imposing the aforementioned necessary conditions we determined the mathematical structure of VSI and CSI spaces. We applied these conditions to the neutral signature case, and imposed some additional constraints (Ricci-flat in the VSI case and Einstein in the CSI case) to generate some additional physically relevant constraints.

The study of VSI and CSI spaces is not yet complete. For example, the pseudo-Riemannian analogue of the $\mathcal{I}$-non-degenerate theorem in pseudo-Riemannian space

has yet to be proven, although it is believed to be true. It may also be useful to construct solutions in other dimensions and signatures using techniques similar to those used in this thesis.

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