LAR'S AND PLAR'S, DURING SARS

by

Adam John Lucas

Submitted in partial fulfillment of the requirements
for the degree of Master of Science

at

Dalhousie University
Halifax, Nova Scotia
April 2022

Dedicated to my friends James G. Lees, a dear brother and excellent blacksmith, and Derek Van Voorst, with whom I delight to disagree.

Table of Contents

Abstract

The Weak Lefschetz Property (WLP) of polynomial quotient rings is studied in commutative algebra for the implications it has for a ring's Hilbert function, which is a chief object of study in commutative algebra. As simplicial complexes are a bridge between spatial and algebraic objects of study, we focus on the WLP and Stanley-Reisner rings, the algebraic allegory to simplical complexes, specifically squarefree Stanley-Reisner rings. We give an introduction to the key ideas here by following the results of a paper which develops several results for the WLP and squarefree Stanley-Reisner rings, while giving a modest generalization of these results and examining other related propositions.

Polarization is an operation on quotient rings of monomial ideals which returns a quotient ring whose ideal is generated entirely by squarefree monomials and has the same Hilbert function and free resolution. We examine the question of whether there is an analogous and natural operation which returns an Artinian ring with a similar or predictable Hilbert function. One property which we hope to preserve between them is called levelness. We provide compelling evidence that such an operation does not exist and we examine the relation between the WLP and levelness.

Acknowledgements

Thank you to my supervisor Dr. Sara Faridi for her invaluable guidance and supervision, to the University of Dalhousie, and in particular the Department of Mathematics and Statistics.

Chapter 1

Introduction

Polynomial quotient rings S/I are *graded* when I is generated by polynomials which are each a linear combination of monomials of the same degree. In a graded S/I, each monomial has a well defined degree, and the number of monomials of degree i in S/I is described by a function called the *Hilbert function* of S/I. The Hilbert function is an invariant which encodes a lot of information about the ring it represents, and so it is naturally the object of much study. We will see that the h- and f-vectors (introduced in Sections 2.2 and 2.3, respectively), which are both related to each other and to the Hilbert function, are also fundamental concepts in the study of graded rings and in particular Stanley-Reisner rings, which are the link between much of algebra, combinatorics, and geometry.

The Weak Lefschetz Property (WLP) is a common property of graded quotient rings, which along with its progenitor, the Hard Lefschetz Theorem (HLT)[30], not to be confused with the sister notion of the Strong Lefschetz Property (SLP), has implications for the h- and f-vectors of this ring [10, 19], and even restricts the individual Betti numbers of an algebra [19]. Every Artinian ideal of $k[x, y]$ has the SLP, and hence the WLP, and this class of ideals is more diverse than it might appear [19]. The WLP also gives a characterization of the number of plane partitions contained within a given box [27]. In general, it can be difficult to affirm or deny the presence of the WLP for a given quotient ring. Here we examine and modestly generalize the results of [34]. In [34], the authors are primarily concerned with polynomial quotient rings $R = S/I$ where I is an ideal generated by quadratic polynomials. This condition turns out not to be necessary. We can also remove the assumption of quadratic generators for most of the results we cover, but there is a "diminishing" usefulness in this because simplicial complexes with low-dimension may nevertheless include many vertices. Within this work-through of [34] we will look at some other results which characterize the WLP for squarefree Stanley-Reisner rings, including how to construct some classes of squarefree Stanley-Reisner rings with the WLP. We also examine some related propositions with the goal of giving the reader an impression of how well understood the WLP is for squarefree Stanley-Reisner rings. This paper also covers some simplicial homology in connection with the WLP, giving a characterization of which squarefree Stanley-Reisner rings have the WLP, but only over a field of characteristic 2. A connection between the WLP and homology or simplicial complexes is interesting since Richard Stanley introduced sufficient conditions for the SLP to be present in the Stanley-Reisner ring of a simplicial complex [39]. The homological result over characteristic 2 that we will examine can be "lifted" to characteristic 0 to achieve a condition which is sufficient to falsify the presence of the WLP for squarefree Stanley-Reisner rings.

As the WLP and SLP are common, it is of interest to study the implications they carry for polynomial quotient rings. Specifically, in commutative algebra there is interest in understanding the basic "shape" and "structure" of a polynomial ring's Hilbert function. As the Hilbert function is a function $f : \mathbb{N} \longrightarrow \mathbb{N}$, we often study questions such as whether a class of polynomial quotient rings produce an inequality $C|f(n)| \leq |f(n-1)|$ for some $n \in [k_1, k_2] \subseteq \mathbb{N}$ and $C \in \mathbb{R}$, or even more complex relations. Each simplicial complex Δ has an associated polynomial quotient ring R called its Stanley-Reisner ring, and when Δ has the characteristic of being *matroid* the Hilbert function satisfies the *log-concave* condition: $f(n-1)f(n+1) \leq f(n)^2$ [26]. Similarly, the Kruscal-Katona Theorem, which gives the relation $h_{j+1} \leq h_j^{(j)}$ (the operation applied to h_j is not simply an exponential) for a Hilbert series (the power series representing a Hilbert function) $h_0 + h_1 t + \cdots + h_n t^n$ and $0 \leq j \leq n$ [20, Theorem 6.4.5]. Studying classes of polynomial quotient rings and their Hilbert functions promises to refine our understanding of the Hilbert function for special, and hopefully common, cases. The Kruskal-Katona Theorem has been used in graph theory and applied to network reliability problems [7, 9].

It is conjectured that if an ideal I of a polynomial ring S is generated by quadratic monomials (or polynomials) and the quotient ring S/I is *Gorenstein*, then S/I has the WLP [33, 32].

Gorenstein is a special case of *levelness* (introduced in Section 4.2), which has to do with the annihilator of the maximal ideal $(x_1, \ldots, x_n)$ of a polynomial quotient ring. Levelness was conceived of as an intermediate step between Gorenstein and another property called Cohen-Macaulay [38]. Levelness is fundamental to the theory of algebras with straightening laws [21]. As we will see in Proposition 4.2.3, knowing if a ring is level simplifies the task of verifying whether that ring has the WLP. We will also see in Proposition 5.2.4 (see [41, 5]) and Corollary 5.2.5 that levelness has interesting connections to *pure* simplicial complexes, whose facets are all of the same degree, and to linear algebra. Levelness also has implications for the structure of Hilbert functions [17].

The presence of the WLP or SLP does place a restriction on the Hilbert function, and in fact a given function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is the Hilbert function of a ring with the WLP or SLP only under certain conditions as well; the quotient rings with the WLP and SLP are somewhat coarsely characterized by their Hilbert functions and vice versa [19]. One property of the Hilbert function which is of interest is the property of *unimodality* (introduced in Section 2.4), where a sequence starts out as an increasing sequence, and then becomes a decreasing one.

The WLP guarantees that the h-vector of a squarefree polynomial ring is unimodal (see Proposition 3.1.9), and even some slightly stronger properties, so it has generated interest among those who study levelness for its "long term" affect on the h-vector [16]. Unimodal sequences are sequences with a single extremal point, similar to the notion of local maxima and minima in calculus. Unimodal sequences are a combinatorial concept, and a wide variety of techniques have been employed to find classes of such sequences and prove that they are unimodal. Such methods include direct combinatorial methods, analytical methods, linear algebraic methods, and homological methods [40]. The variety of techniques for proving unimodality and the relative abundance of such sequences across various fields is a source of interest in the property.

Polarization (introduced in Section 5.1) is an operation which takes a monomial ideal I of a polynomial ring S and returns an ideal J generated by squarefree monomials in a new polynomial ring S'. The benefit to this operation is that it preserves the Hilbert function [20, Corollary 1.6.3]. This property of being squarefree is related to the study of simplicial complexes via squarefree monomial ideals. Every simplicial complex Δ has an associated polynomial quotient ring, the Stanley-Reisner ring, whose squarefree monomials correspond to the faces of Δ [20, Section 1.5.2]. We explore how the notion of polarization induces an operation which returns squarefree monomial quotient rings that correspond to simplicial complexes. This squarefree (Artinian) analogue to polarization does not preserve free resolutions, so we will explore its effect on a more basic property of free resolutions: levelness.

To consolidate our discussion of the WLP and the h- and f-vectors, we will introduce in Chapter 4 the g-vector, and its g-conjecture, which is closely tied to the WLP and a coveted conjecture in combinatorics. In order to tie together the WLP and levelness, we will focus on their relations, mostly how to construct level polynomial quotient rings with the WLP.

Although most questions asked are answered in the negative, we are able to answer them definitively. Since the Hilbert function is not preserved by these attempts, we mainly focus on levelness of the ideal, which is a property subject to a general interest within commutative algebra. We finish by looking at "depolarization", an operation to undo our Artinian analogue to polarization. We generalize one result of [5] and [41] and finish off by showing that the ability to depolarize a squarefree Stanley-Reisner ring does not imply levelness.

Chapter 2

Preliminaries

2.1 Monomial Ideals

A polynomial ring in n variables $S = k[x_1, \ldots, x_n]$ over a field k has what is called a *standard grading*:
For a monomial $m = x_1^{c_1} \ldots x_n^{c_n}$, where $c_1, ..., c_n \in \mathbb{N}$, the sum $c_1 + \cdots + c_n$ is the *degree* of m. Let M be the set of all monomials of S, we have the function:

$$\deg : M \longrightarrow \mathbb{N} \tag{2.1}$$

defined by

$$\deg(x_1^{c_1} \ldots x_n^{c_n}) = c_1 + \cdots + c_n. \tag{2.2}$$

A polynomial $g \in S$ is *homogeneous* of degree i if it may be written as a linear combination of monomials of degree i. For a polynomial

$$f = \sum_{i=1}^{p} m_i \tag{2.3}$$

f may be written as

$$f = \sum_{i=0}^{p} f_i \tag{2.4}$$

where each f_i is a homogeneous polynomial and $f_i \in S_i$, where S_i is the k-vector space spanned by all degree i monomials. We refer to S_i as the *ith graded component* of S. The k-vector space spanned by a list $v_1, \ldots, v_p$ is denoted $\langle v_1, \ldots, v_p \rangle$. So, for example, in the ring $k[x, y]$ we have $S_2 = \langle x^2, y^2, xy \rangle$.

Given a commutative ring R, an *ideal* I of R is a subring of R such that

$$r \in R \quad \text{and} \quad x \in I \Rightarrow rx \in I \tag{2.5}$$

An ideal $I \subseteq S$ allows us to construct a quotient ring $A = S/I$, in the typical way as other quotient rings, modules, or groups. If I is generated by a homogeneous polynomials, then we say I is a *graded* or *homogeneous* ideal. Let I be a graded ideal, then $A = S/I$ inherits a grading from S by the rule $A_i = S_i/I_i$ for $i \in \mathbb{N}$. This works because:

$$I = \oplus_{i=0}^{\infty} I_i, \tag{2.6}$$

where

$$I_i = S_i \cap I. \tag{2.7}$$

Note that:

$$A = S/I = (\oplus_{i=0}^{\infty} S_i)/(\oplus_{i=0}^{\infty} I_i), \tag{2.8}$$

which is equivalent to

$$\oplus_{i=0}^{\infty} S_i/I_i. \tag{2.9}$$

Thus, there is a well defined degree for each $a \in A$.

An ideal I of $S = k[x_1, \ldots, x_n]$ is called a *monomial ideal* if there exists a system of monomials $m_1, m_2, \ldots, m_q$ such that $I = (m_1, \ldots, m_q)$. Because monomials are necessarily homogeneous, a monomial ideal I is graded.

Example 2.1.1. Consider the ring $S = \mathbb{Q}[x, y, z]$ and the monomial ideal $I = (x^2, y^2, z^2, yz)$. The graded components of S are
$S_1 = \langle x, y, z\rangle$, $S_2 = \langle xy, xz, yz, x^2, y^2, z^2\rangle$, $S_3 = \langle xyz, x^2y, x^2z, y^2z, yz^2, x^3, y^3, z^3\rangle$,
$S_4 = \langle x^3y, x^3z, x^2yz, x^2y^2, x^2yz, xy^2z, y^2z^2, y^3z, xy^3, xyz^2, x^2z^2, xz^3, x^4, y^4, z^4\rangle$, and so on. The graded components of $A = S/I$ are $A_1 = \langle x, y, z\rangle$, $A_2 = \langle xy, xz\rangle$, $A_3 = \{0\}$, $A_4 = \{0\}$, and so on. Both S and A are S-modules and $\mathbb{Q}$-vector spaces.

For every graded quotient ring $A = S/I$ there is an important function called the *Hilbert function* of A. David Hilbert studied the graded components of graded quotient rings. Since the graded components of these rings always have a whole number of basis elements, and there are often infinitely many graded components, it is natural to study their dimensions as a function with its range in $\mathbb{N}$ and encode them with a generating function (power series). Hilbert first studied free resolutions in connection to the Hilbert function [22, 37]. The *Hilbert function* of $A = S/I$ is the function

$$H_A : \mathbb{N} \longrightarrow \mathbb{N} \tag{2.10}$$

which follows the definition $i \mapsto \dim_k(A_i)$, where $\dim_k(A_i)$ is the vector space dimension of the k-vector space A_i. Similarly, the *Hilbert Series* is the power series

$$\mathrm{HS}(A,t) = \sum_{i=0}^{\infty} \dim(A_i)t^i = \sum_{i=0}^{\infty} H_A(i)t^i \tag{2.11}$$

This encodes the Hilbert function.

Example 2.1.2. With the same S and I as Example 2.1.1 we have $S_1 = \langle x, y, z\rangle$, $S_2 = \langle xy, xz, yz, x^2, y^2, z^2\rangle$, and $S_3 = \langle xyz, x^2y, x^2z, y^2z, yz^2, x^3, y^3, z^3\rangle$.

The Hilbert series of $S = k[x_1, \dots, x_n]$ by itself is

$$\sum_{i=0}^{\infty} \binom{n+i-1}{i} t^i \tag{2.12}$$

This is the series which represents the n-fold product, or *convolution*,

$$\prod_{j=1}^{n} \left(\sum_{i=0}^{\infty} t^i \right). \tag{2.13}$$

Since, famously:

$$\frac{1}{(1-t)} = \sum_{i=0}^{\infty} t^i \tag{2.14}$$

we see that the closed form of Equation 2.12 is

$$\frac{1}{(1-t)^n}. \tag{2.15}$$

The Hilbert function of S can be derived using the "stars and bars" counting method [28, Page 30]. Stars and bars is a common counting method in discrete mathematics which counts how many ways a collection of k unlabelled stars may be arranged in a line shared by $n-1$ bars with no blank spaces [28, Page 30]. Reading left to right, the n spaces around and between the bars denote each of the n variables and the number of the k stars in the ith space is the number c_i in the monomial $x_1^{c_1} \dots x_n^{c_n}$. Hence, this method also counts the number of monomials of degree k among n variables.

Example 2.1.3. As with Example 2.1.2, when $A = S/I$ we have $A_1 = \langle x, y, z\rangle$, $A_2 = \langle xy, xz\rangle$ and $A_3 = \{0\}$, so the Hilbert series is $1 + 3t + 2t^2$.

We will quickly mention an operation on monomial ideals called the *colon ideal* of ideals J and I. Defined by:

$$(I : J) = \{f \in S | fg \in I, \forall g \in J\} \tag{2.16}$$

This will be used briefly later.

Example 2.1.4. The ideals $(y), (xy, xz, yz) \subseteq S[x, y, z]$ have a colon ideal

$$((xy, xz, yz) : (y)) = (x, z) \tag{2.17}$$

Given two ideals I and J of a ring R we can also define an ideal

$$I + J = \{r + s \mid r \in I, s \in J\} \tag{2.18}$$

An ideal I of a ring R is *prime* if for any $x, y \in R$ then $xy \in I \Rightarrow (x \in I) \vee (y \in I)$.

A ring R is said to have *dimension* p when the maximal length of an ascending chain of prime ideals of R is p. A ring R is said to be *Artinian* when there is no infinite descending chain of ideals.

Example 2.1.5. Consider the ring $\mathbb{R}[x, y]$. The ideal (0) is prime, and so are (x) and (y) , for which $(0) \subseteq (x), (y)$. The ideal (x, y) is also prime and $(x), (y) \subseteq (x, y)$, but there is no prime ideal "above" (x, y), so $\mathbb{R}[x, y]$ has dimension 2 because $(0) \subseteq (x) \subseteq (x, y)$ and $(0) \subseteq (y) \subseteq (x, y)$ (like a computer, we begin counting at zero).

When attempting to find the dimension of a ring or ideal (which itself is just a subring), knowing the minimal prime ideals is useful information. The following proposition helps with this.

Proposition 2.1.6. *If I is an ideal generated by monomials $m_1, \ldots, m_q$ in the polynomial ring $S = k[x_1, \ldots, x_n]$, then the minimal prime ideals containing I are generated by the minimal subsets $K \subseteq \{x_1, \ldots, x_n\}$, where each of the m_i, for $1 \leq i \leq q$, is divided by some variable $x_j \in K$.*

Proof. Let K be a minimal subset of $\{x_1, \ldots, x_n\}$ where each of the m_j, for $1 \leq j \leq q$, is divided by some variable $x_i \in K$, so $I \subseteq (x_i \mid x_i \in K) = F$. Suppose there is a prime ideal P such that $I \subseteq P \subseteq F$. Since $I \subseteq P$ we have $m_i \in P$ for $1 \leq i \leq q$, and since P is prime, a subset of variables which divide each of these monomials must be included in P. Since K was chosen minimally with the same property and $P \subseteq F$, then $P = F$. Thus F is minimal. $\square$

The polynomial ring $S = k[x_1, .., x_n]$ has dimension n because

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \ldots, x_n) \tag{2.19}$$

is a maximal chain of prime ideals in S. Let $I = (m_1, \ldots, m_q) \subseteq S$ be an ideal and let K be a minimal subset of $\{x_1, \ldots, x_n\}$ such that for each m_i, where $1 \leq i \leq q$ we have $x_j \in K$ such that $x_j | m_i$. By the fourth module isomorphism theorem [24, Theorem 1.10 on Page 173], there is a one-to-one correspondence between the ideals of S/I and the ideals of S which contain I. This correspondence respects inclusion, so $\dim(S/I) = n - |K|$, which is the number of prime ideals in the chain

$$(x_i \mid x_i \in K) \subsetneq \cdots \subsetneq (x_1, \ldots, x_n) \tag{2.20}$$

(adding an additional variable with each step in the chain). In practical terms, a ring S/I being Artinian means to us that S/I is a finite dimensional k-vector space.

Artinian ideals are a unique and convenient kind of polynomial ideal to study since its Hilbert series is finite and its Hilbert function, as a sequence, is eventually zero.

2.2 Free Resolutions

For a polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k and a graded ideal I, let $R = S/I$ be a graded quotient ring. The R-module (or S-module) $R(-p)$, for an integer p, denotes a graded free R-module such that $R(-p)_i = R_{i-p}$. We say that $R(-p)$ is *shifted by p degrees*. We can think of it as a free R-module generated by a single element of degree p [37, Proposition 2.3]. In particular, the free module $S(-p)$ may be considered to be generated by a single monomial of degree p. A *homomorphism* between two graded R-modules U and V, denoted $\varphi : U \longrightarrow V$ is a function such that $\forall a, b \in U, \forall r \in R$

1. $\varphi(a + b) = \varphi(a) + \varphi(b)$
2. $\varphi(ra) = r\varphi(a)$

We say that this homomorphism has *degree i* if for every monomial $m \in U$ we have $\deg(\varphi(m)) = i + \deg(m)$.

Example 2.2.1. Let $R(-3)$ and $R(-5)$ be free R-modules as above where $R = S = \mathbb{Q}[x, y]$. Let f_1 be the generator of $R(-5)$ and f_2 be the generator of $R(-3)$. Then the map $\mu : R(-5) \longrightarrow R(-3)$ which sends $f_1 \mapsto x^3 f_2$ has degree 1 because

$\deg(f_1) + 1 = \deg(x^3 f_2)$. A map defined by $f_1 \mapsto x^{100} f_2$ would have degree 98, and likewise $f_1 \mapsto xy f_2$ has degree 0.

Let $i \geq 0$. A *free resolution* of an R-module U is an exact sequence (see [24, Page 175])

$$\ldots \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \tag{2.21}$$

also written

$$\ldots \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} U \to 0 \tag{2.22}$$

where each F_i is a finitely generated free R-module and $U \cong F_0/\operatorname{im}(d_1)$. A *graded* free resolution is a free resolution for which:

1. each F_i is graded;

2. each map d_i has degree zero;

3. for which U is graded; and

4. for which the isomorphism $U \cong F_0/\operatorname{im}(d_1)$ has degree zero.

Graded free resolutions are either infinite or finite in length, and the *minimal graded free resolution* is a graded free resolution as in (2.22) such that $d_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)F_i$ for all $i \geq 0$. It is minimal because $d_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)F_i$ means that the image of F_{i+1} is in the maximal ideal $(x_1, \ldots, x_n)$, which in turn means that the entries in the matrix of d_{i+1} have no non-zero constants. The minimal graded free resolution is unique [37, Theorem 7.5]. The *length* of a free resolution is the largest index $i \geq 0$ such that $F_i \neq 0$. The length of the minimal graded free resolution of U is called the *projective dimension* of U.

The following is *Hilbert's Syzygy Theorem* and it is foundational to our understanding of minimal graded free resolutions.

Theorem 2.2.2 (Theorem 15.2 of [37]). *The minimal graded free resolution of a graded finitely generated S-module, where $S = k[x_1, \ldots, x_n]$, is finite and its length is at most n.*

Example 2.2.3. Let $S = k[x, y]$ and let $I = (x^2y, x^3, y^2)$. Because k is a field, S and the modules $S(-p)$ are vector spaces, then we can describe the maps in the exact sequence below as matrices. For shorthand, we write:

$$S(-p)^q = \bigoplus_{i=1}^{q} S(-p). \tag{2.23}$$

We have the following graded free resolution of S/I:

$$0 \leftarrow S \xleftarrow{(y^2, x^3, x^2y)} S(-2) \oplus S(-3)^2 \xleftarrow{\begin{pmatrix} 0 & -x^2 \\ -y & 0 \\ x & y \end{pmatrix}} S(-4)^2 \leftarrow 0 \tag{2.24}$$

Notice the length of this resolution is 2. The final step in this resolution is

$$0 \leftarrow S \xleftarrow{(y^2, x^3, x^2y)} S(-2) \oplus S(-3)^2 \tag{2.25}$$

for which $S = F_0$. This final step is not necessarily exact. In order to codify the modules involved at each step of this resolution, we construct what's called a *Betti table*:

	0	1	2
total:	1	3	2
0:	1	.	.
1:	.	1	.
2:	.	2	2

This table is read as follows:

> The number $a_{i,j}$ is the number of direct summands of $S(-(a_{i,j} + j))$ in the jth index of the graded free resolution. In the case of $a_{2,1} = 2$, there are 2 summands of $S(-3)$ in the 1st step in the resolution (which starts with index zero). This same column also has $a_{1,1} = 1$ so there is another summand of $S(-2)$, the free module representing the 1st index is hence $S(-2) \oplus S(-3)^2$.

The Betti table encodes the Betti numbers of each module in a graded free resolution. Betti numbers may be familiar from the study of finitely generated Abelian groups and the Fundamental Theorem of Finitely Generated Abelian Groups [13, Page 158].

In the study of graded free resolutions, the following result, originally by David Hilbert, is chief:

Theorem 2.2.4 (Theorem 16.2 of [37])**.** *Let* $\boldsymbol{F}$ *be a graded free resolution of a finitely generated graded* R*-module* U*. Where* $F_i = \oplus_{p\in\mathbb{Z}} R^{c_{i,p}}(-p)$*,*
we have:

$$\mathrm{HS}(U,t) = \mathrm{HS}(R,t)\cdot\left(\sum_{i\geq 0}\sum_{p\in\mathbb{Z}}(-1)^i c_{i,p}t^p\right). \tag{2.26}$$

Specifically, when $R = S$*, where* $S = k[x_1,\ldots,x_n]$*, a common case, we have*

$$\mathrm{HS}(S,t) = \frac{1}{(1-t)^n}, \tag{2.27}$$

and so

$$\mathrm{HS}(U,t) = \frac{\sum_{i\geq 0}\sum_{p\in\mathbb{Z}}(-1)^i c_{i,p}t^p}{(1-t)^n} \tag{2.28}$$

Thus, we have an algebraic connection between graded free resolutions and the Hilbert function of an R-module U. Knowing the Hilbert series of a module R and a graded free resolution of the R-module U, we can find the Hilbert series of U. Knowing the Hilbert series of U and of R places limits on the graded free resolutions of U. In a grander sense however, understanding the Hilbert series and graded R-modules is what motivates our study.

Example 2.2.5. Let $S = k[x,y]$ and $I = (x^2y, x^3, y^3)$ as in Example 5.1.11. We learned in Example 2.2.3 that I has the graded free resolution

$$0 \leftarrow S \xleftarrow{(y^2,x^2y,x^3)} S(-2)\oplus S(-3)^2 \xleftarrow{\begin{pmatrix} 0 & -x^2 \\ -y & 0 \\ x & y \end{pmatrix}} S(-4)^2 \leftarrow 0 \tag{2.29}$$

and so by Theorem 2.2.4 we can calculate the Hilbert series of S/I as

$$\frac{1-(t^2+2t^3)+(2t^4)}{(1-t)^2} = \frac{1-t^2-2t^3+2t^4}{(1-t)^2} \tag{2.30}$$

which simplifies to

$$\mathrm{HS}(S/I,t) = 1+2t+2t^2. \tag{2.31}$$

The *kernel* of a module homomorphism $f : V \longrightarrow W$ is the module

$$\ker(f) = \{v \in V \mid f(v) = 0\}. \tag{2.32}$$

The *cokernel* of $f : V \longrightarrow W$ is the module

$$\operatorname{coker}(f) = W/\operatorname{im}(f). \tag{2.33}$$

The following is called the *Snake Lemma*:

Lemma 2.2.6 (Lemma 37.3 of [37]). *Let R be a commutative ring with identity and let A, A', B, B', C, and C' be R-modules, and let*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd} \tag{2.34}$$

be a commutative diagram of R-module homomorphisms, where the two rows are exact. Then there is an R-module homomorphism δ such that:

$$0 \longrightarrow \ker(\alpha) \xrightarrow{\bar{f}} \ker(\beta) \xrightarrow{\bar{g}} \ker(\gamma) \xrightarrow{\delta} \mathit{coker}(\alpha) \xrightarrow{\bar{f}'} \mathit{coker}(\beta) \xrightarrow{\bar{g}'} \mathit{coker}(\gamma) \longrightarrow 0 \tag{2.35}$$

is an exact sequence.

This lemma allows us to relate the kernels and cokernels of the maps α, β, and γ, to the map δ. In effect, under the right circumstances the map δ can reveal a lot about whether α, β, and γ are injective or surjective, and vice versa. We will use the Snake Lemma in this way later in Chapter 3.

Definition 2.2.7. Let $S = k[x_1, \ldots, x_n]$, and let U be an S-module. Recall that the Hilbert series $\mathrm{HS}(U, t)$ of U, per Theorem 2.2.4, is

$$\mathrm{HS}(U, t) = \frac{\sum_{i \geq 0} \sum_{p \in \mathbb{Z}} (-1)^i c_{i,p} t^p}{(1-t)^n}. \tag{2.36}$$

Let $q \in \mathbb{Z}$ be the largest integer such that

$$\frac{\sum_{i \geq 0} \sum_{p \in \mathbb{Z}} (-1)^i c_{i,p} t^p}{(1-t)^q} = h(t) \tag{2.37}$$

is a polynomial over $k[t]$. The coefficient vector $(h_0, h_1, \ldots, h_r)$, where

$$h(t) = h_0 + h_1 t + \cdots + h_r t^r \tag{2.38}$$

is called the *h-vector* of U. This definition is in [37, Page 63], but in practice we will operate in circumstances that allow us to quickly find the h-vector, we will see an example of this in the next section with Corollary 2.3.10.

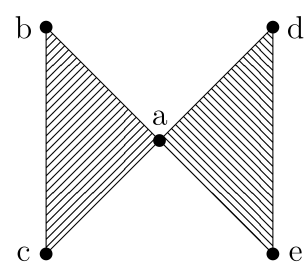
2.3 Simplicial Complexes

Let $V = \{x_1, \ldots, x_n\}$ be considered as a set of vertices (we will consider them as variables soon enough). A *simplicial complex* Δ is a collection of subsets of V such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. Each set $\sigma \in \Delta$ has *dimension* $\dim(\sigma) = |\sigma| - 1$. The sets $\sigma \in \Delta$ are called *faces* of Δ, and the faces of highest dimension, which are not a subset of any other face, are called *facets* of Δ. (Facets are always maximal with respect to inclusion.)

Example 2.3.1. The simplicial complex

$$\Delta = \{\{a,b,c\}, \{a,d,e\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{b,c\}, \{d,e\},$$
$$\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \emptyset\} \tag{2.39}$$

has facets $\{a,b,c\}$ and $\{a,d,e\}$, which for shorthand we sometimes write abc and ade. We can express Δ in the shortened form $\Delta = \langle abc, ade \rangle$, since knowing the facets is enough to reproduce any simplicial complex.



The simplicial complex Δ, rendered spacially

Definition 2.3.2 (f-vector). For a simplicial complex Δ of dimension d, the *f-vector* $\mathbf{f}(\Delta) = (f_0, \ldots, f_d)$ of Δ is the unique vector of integers such that $f_i \in \mathbb{N}$ is the number of i-dimensional faces of Δ.

In the above Example 2.3.1, the f-vector of Δ is $(f_0, f_1.f_2) = (5, 6, 2)$.

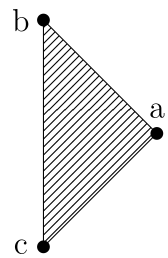
Stars and Links

Let Δ be a simplicial complex and let $\sigma \in \Delta$ be a face. The *star* of σ, denoted $\mathrm{st}(\sigma)$, is a collection of subsets

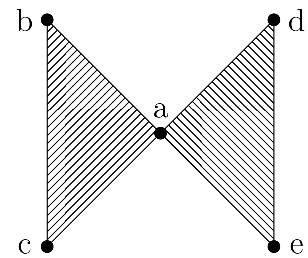
$$\mathrm{st}(\sigma) = \{\tau \in \Delta \mid \sigma \subseteq \tau\}. \tag{2.40}$$

That is, the set of all faces of Δ which σ is a subset of. The *closed star* of σ, denoted $\bar{st}(\sigma)$, is the smallest simplicial complex which contains $\mathrm{st}(\sigma)$.

Example 2.3.3. Let $\Delta = \langle abc, ade \rangle$ as in Example 2.3.1. Observe the following:



The closed star of $\{a, b\}$



The closed star of $\{a\}$

The *minus* operation is similar to regular set subtraction, except instead of removing all elements of $\Delta \cap \sigma$ from Δ, as in regular set subtraction, we define

$$\Delta \backslash \sigma = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset\}. \tag{2.41}$$

In this way, we are deleting every face which contains any vertex of σ.

Later, we will see operations such as $\Delta \backslash \mathrm{st}(\sigma)$. This is simply the regular set-minus operation between one set, the simplicial complex Δ, and the other set, the star $\mathrm{st}(\sigma)$. It is important to note that the result of this is still a simplicial complex:

Lemma 2.3.4. *The collection $\Delta \backslash \mathrm{st}(\sigma)$ is a simplicial complex.*

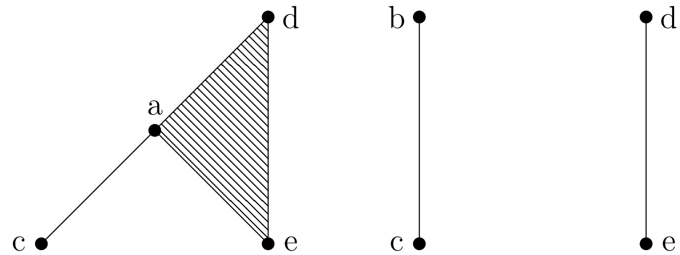
Proof. Let $\sigma \in \Delta \backslash \mathrm{st}(\sigma)$ and let $\tau \subseteq \sigma$. Observe that $\Delta \backslash \mathrm{st}(\sigma) \subseteq \Delta$. If $\tau \notin \Delta \backslash \mathrm{st}(\sigma)$, then $\tau \notin \Delta$, but this is impossible since Δ is a simplicial complex and thus $\sigma \in \Delta$ and $\tau \subseteq \sigma$ implies $\tau \in \Delta$. Thus, $\tau \in \Delta \backslash \mathrm{st}(\sigma)$ and so $\Delta \backslash \mathrm{st}(\sigma)$ is a simplicial complex. $\square$

The *link* of $\sigma \in \Delta$, denoted $\mathrm{lk}(\sigma)$ is

$$\bar{st}(\sigma) \backslash \sigma. \tag{2.42}$$

The link may be thought of, in a colloquial sense, as " the simplicial complex of faces 'in the same facets' as σ but not including σ".

Example 2.3.5. Let $\Delta = \langle abc, ade \rangle$ as in Example 2.3.1. Observe the following:



The simplicial complex $\Delta \backslash \{b\}$ The link of $\{a\}$

Stanley-Reisner Rings

Given a simplicial complex Δ on a vertex set $\{x_1, \ldots, x_n\}$, we can define a quotient ring of the polynomial ring $k[x_1, \ldots, x_n]$ whose squarefree monomials correspond to the faces of Δ. This ring is called the *Stanley-Reisner ring* of Δ. For $\sigma \in \Delta$, we write $\sigma = \{x_{i,1}, \ldots, x_{i,q}\}$ and often write $x_{i,1} \ldots x_{i,q}$ as shorthand. This shorthand illustrates a correspondence between faces and monomials of $k[x_1, \ldots, x_n]$ by the rule

$$M : \{x_{i,1}, \ldots, x_{i,q}\} \mapsto x_{i,1} \ldots x_{i,q} \text{ [the monomial].} \tag{2.43}$$

So the faces of Δ can be represented as monomials.

Consider the ideal

$$I_\Delta = (M(\sigma) \mid \sigma \notin \Delta) \tag{2.44}$$

this is called the *Stanley-Reisner ideal*, it is generated by the minimal nonfaces of Δ respecting inclusion. Let

$$I' = (x_1^2, \ldots, x_n^2) \tag{2.45}$$

be the ideal generated by the squares of each variable in $k[x_1, \ldots, x_n]$. Let

$$I = I_\Delta + I' \tag{2.46}$$

as in (2.44), the quotient ring

$$S/I = S/(I_\Delta + I') \tag{2.47}$$

is the *squarefree Stanley-Reisner ring*, as opposed to S/I_Δ which is the *Stanley-Reisner ring*. This notation is meant to conform with [34], although formulations such as in [41] are in use, and we will use them in Chapter 5. The ring S/I is an

Artinian ring, as it is finitely generated vector space over k, with a basis composed of those monomials which correspond to the faces of Δ, and therefore since each ideal is a subspace and the dimension is finite there cannot be an infinite descending chain of ideals.

Remark 2.3.6. Let Δ be a simplicial complex. Observe that for a subset $\sigma \subseteq \{x_1, \ldots, x_n\}$ we have

$$M(\sigma) \in I \Leftrightarrow \sigma \notin \Delta \tag{2.48}$$

or equivalently:

$$M(\sigma) \notin I \Leftrightarrow \sigma \in \Delta. \tag{2.49}$$

This concept of a Stanley-Reisner ring is the link between polynomial quotient rings and simplicial complexes, and thus a link between algebra and geometry. It will allow us to approach certain algebraic problems spatially, and certain geometric problems algebraically.

Example 2.3.7. Let $S = k[a, b, c, d, e]$ and $\Delta = \langle abc, ade\rangle$ as in Example 2.3.1. In this case $I_\Delta = (bd, ce)$ and $I' = (a^2, b^2, c^2, d^2, e^2)$. So

$$S/I = \langle 1\rangle \oplus \langle a, b, c, d, e\rangle \oplus \langle ab, ac, ae, ad, bc, de\rangle \oplus \langle abc, ade\rangle \tag{2.50}$$

where summands represent the vector spaces spanned by those monomials over k.

Remark 2.3.8. The *h-vector of a simplicial complex* Δ, by which we mean the h-vector of its Stanley-Reisner ring (not the squarefree Stanley-Reisner ring), is related to the concept of the f-vector of Δ by the rule

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} \tag{2.51}$$

as in [20, Page 15].

The following lemma, Lemma 2.3.9, is included in [34], but is also a very well known statement about the f-vector.

Lemma 2.3.9 (Lemma 2.1 of [34]). *If Δ is a simplicial complex on n vertices, where $S = k[x_1, \ldots, x_n]$, and if $A = S/I$ for $I = I' + I_\Delta$ as in* (2.44)*, then the Hilbert series of A is*

$$\mathrm{HS}(A, t) = \sum_{i=0}^{d} f_{i-1} t^i \tag{2.52}$$

where $(f_0, \ldots, f_{d-1})$ *is the* f*-vector of* Δ*.*

Proof. Observe that all nonzero monomials in A must be squarefree, and therefore of the form $M(\sigma)$ for $\sigma \subseteq \{x_1, \ldots, x_n\}$. Thus, the only $\sigma \subseteq \{x_1, \ldots, x_n\}$ for which $M(\sigma)$ is nonzero in A are those with $M(\sigma) \notin I$, or equivalently $\sigma \in \Delta$.

By definition the Hilbert function is

$$H(A, i) = \dim_k(A_i). \tag{2.53}$$

Because A_i is the set of i-degree monomials $M(\sigma)$, where $\sigma \in \Delta$, then the faces of Δ of dimension $i-1$ have i elements and each associated i-degree monomial is in A_i. Thus $\dim_k(A_i) = f_{i-1}$ and

$$H(A, i) = \dim_k(A_i) = f_{i-1}. \tag{2.54}$$

So

$$\mathrm{HS}(A, t) = \sum_{i=0}^{d} f_{i-1} t^i \tag{2.55}$$

as desired.

□

Corollary 2.3.10. *If* Δ *is a simplicial complex on* n *vertices,* $S = k[x_1, \ldots, x_n]$*, and if* $A = S/I$ *for* $I = I' + I_\Delta$*, then*

$$(f_0, \ldots, f_{d-1}) = (h_1, \ldots, h_d). \tag{2.56}$$

Proof. By definition of the Hilbert series and Lemma 2.3.9 we have

$$\mathrm{HS}(A, t) = \dim_k(A_0) + \dim_k(A_1)t + \cdots + \dim_k(A_d)t^d \tag{2.57}$$

where $d = \dim(\Delta) + 1$. Therefore, setting $t = 1$ we get

$$HS(A, 1) = 1 + \dim_k(A_1)(1) + \cdots + \dim_k(A_d)(1)^d \geq 1 \tag{2.58}$$

and so $(1-t)$ does not divide $\mathrm{HS}(A, t)$ in this case. By equation (2.37) we have

$$h(t) = \frac{\sum_{i \geq 0} \sum_{p \in \mathbb{Z}} (-1)^i c_{i,p} t^p}{(1-t)^q} \tag{2.59}$$

where $c_{i,p}$ are the numbers of a minimal graded free resolution of A, as per Theorem 2.2, and q is the largest integer which for which $(1-t)^q$ divides the numerator. By Theorem 2.2.4 we have

$$\mathrm{HS}(A,t) = \frac{\sum_{i\geq 0}\sum_{p\in\mathbb{Z}}(-1)^i c_{i,p}t^p}{(1-t)^n} \tag{2.60}$$

and because $\mathrm{HS}(A,t)$ is a finite power series (polynomial) we must have $q \geq n$. By (2.58) 1 is not a root of $\mathrm{HS}(A,t)$, and so $q = n$. By (2.38)

$$h(t) = h_0 + h_1 t + \cdots + h_d t^d \tag{2.61}$$

then

$$\mathrm{HS}(A,t) = h_0 + h_1 t + \cdots + h_d t^d \tag{2.62}$$

and by Lemma 2.3.9

$$\mathrm{HS}(A,t) = 1 + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d. \tag{2.63}$$

Thus

$$1 + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d = h_0 + h_1 t + \cdots + h_d t^d \tag{2.64}$$

so

$$(1, f_0, \ldots, f_{d-1}) = (h_0, \ldots, h_d) \tag{2.65}$$

as desired. □

Remark 2.3.11. Note that Corollary 2.3.10 only holds in the case of a squarefree Stanley-Reisner ring. Under the normal circumstance of a Stanley-Reisner ring, Corollary 2.3.10 does not hold.

Given a simplicial complex Δ, the *i-skeleton* of Δ is a simplicial complex $\Delta(i)$ whose facets are exactly the i-dimensional faces of Δ. In terms of the squarefree Stanley-Reisner ring, we let $I(i) = I + (S_{i+2})$ (the +2 is due to the difference between the dimension of a face and dimension of a monomial), where (S_{i+2}) is the ideal generated by the basis elements of the k-vector space S_{i+2}, and $S/I(i)$ is the squarefree Stanley-Reisner ring of $\Delta(i)$.

Lemma 2.3.12. *Let Δ be a simplicial complex and let $i \geq 0$. Let $A = S/(I_\Delta + I')$ be the squarefree Stanley-Reisner ring of Δ as in* (2.47). *Let $B = S/(I(i) + I')$ be the squarefree Stanley-Reisner ring of $\Delta(i)$ and let $j \leq i+1$, then $A_j = B_j$ and* $\dim(A_j) = \dim(B_j) = f_{j-1}$, *where $(f_0, \ldots, f_d)$ is the f-vector of Δ.*

Proof. Let $i \geq 0$ and let $j \leq i+1$. By Lemma 2.3.9, $\dim(A_j) = f_{j-1}$ because there is a one to one correspondence between the j-degree monomials of A and the $(j-1)$-dimensional faces of Δ. Because $j \leq i+1$, the simplicial complexes Δ and $\Delta(i)$ have the same $(j-1)$-dimensional faces, thus

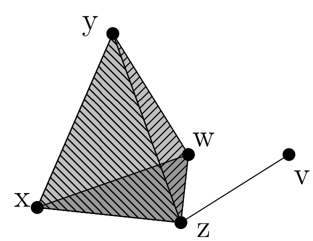
$$\dim(B_j) = f_{j-1} \tag{2.66}$$

and so

$$\dim(B_j) = f_{j-1} = \dim(A_j) \tag{2.67}$$

as desired. If $m \in A_j$ is a monomial, then $m \neq 0$ in A, so $m \notin (I_\Delta + I')$, and since $\deg(m) \leq i+1$ then $m \notin I_\Delta + (S_{i+2}) + I' = I(i) + I'$, thus $m \in B_j$ and $A_j \subseteq B_j$. The other direction is mutatis mutandis, so $A_j = B_j$, as desired. $\square$

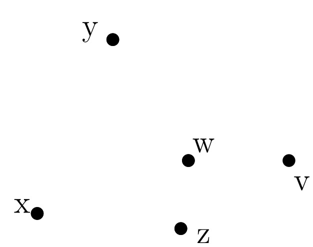
Example 2.3.13. Let $\Delta = \langle xyzw, zv \rangle$ be a simplicial complex:



The simplicial Complex Δ

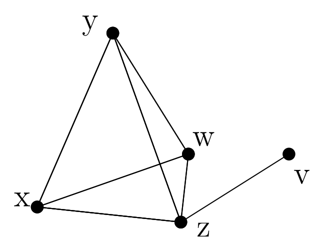
Let $A = S/(I_\Delta + I')$ be its squarefree Stanley-Reisner ring. If $S = k[x, y, z, w, v]$, then $I_\Delta = (xv, yv, wv)$.

The 0-skeleton of Δ is $\langle x, y, z, w, v \rangle$:



The simplicial Complex $\Delta(0)$

The 1-skeleton of Δ is $\langle xy, xz, zw, yz, zv\rangle$:



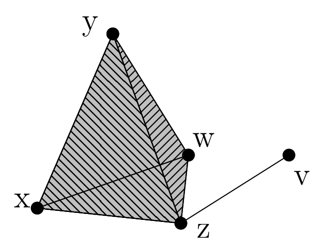
The simplicial Complex Δ

Thus, the Stanley-Reisner ideal of $\Delta(1)$ is

$$I_{\Delta(1)} = (x^2, y^2, z^2, w^2, v^2) + (xv, yv, wv) + (xyz, xyw, xwz, ywz) \tag{2.68}$$

where $A_3 = A_{1+2} = \langle xyz, xyw, xwz, ywz\rangle$.

The 2-skeleton of Δ is $\langle xyw, xyz, zv\rangle$:



The simplicial Complex Δ

where

$$I_{\Delta(2)} = (x^2, y^2, z^2, w^2, v^2) + (xv, yv, wv) + (xyzw). \tag{2.69}$$

Notice that the tetrahedron is not "filled in." This is difficult to render, but $A_4 = A_{2+2} = \langle xyzw\rangle$, so this face is "missing" from our drawing.

We will use this notation later in Section 3.3.

2.4 Unimodality

A sequence $a_0, a_1, \ldots, a_n$ of real numbers is a *unimodal* sequence if for some $0 \le i \le n$ we have

$$a_0 \le a_1 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_n. \tag{2.70}$$

Example 2.4.1. The vector (as a finite sequence):

$$v = (1, 1, 2, e, 3, 3, \pi, 4, 100\pi, 99, 13, 4, -100) \tag{2.71}$$

is unimodal. The vector:

$$(4, 8, 15, 16, 23, 42) \tag{2.72}$$

is also unimodal; this is in spite of the fact that the sequence never decreases.

Unimodality is a common property of many familiar classes of sequences such as the nth row of Pascal's Triangle or the sequence $s(n, 0), \ldots, s(n, n)$ of k-partitions of a set of n objects [40]. What is especially interesting about unimodal sequences is the variety of techniques which have been developed and employed to show that various classes are unimodal. For instance, direct combinatorial arguments work for the rows of Pascal's Triangle, and the sequence of k-partitions of n objects is shown using analytic methods (both of which can be read about in [40]). As we will see in Chapter 3, both geometric and linear-algebraic techniques also exist.

Some functions are also called unimodal. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$, for example, may have only one local maximum and no local minima which are not boundary points in the image over a given domain and be called unimodal. This notion of unimodality is seen in statistics as a way of classifying the distributions of data. Functions can also be bimodal, trimodal, etcetera. Depending on the setting, this definition may vary. For the purposes of the following Proposition 2.4.2, we will consider a unimodal function as one with a single extremal point as described above.

We can use unimodal functions to construct unimodal sequences:

Proposition 2.4.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice differentiable function such that f is unimodal on an interval $[a, b]$ where $a, b \in \mathbb{R}$. Let $a_0 \leq a_1 \leq \cdots \leq a_n$ be a sequence of points in $[a, b]$. The sequence $f(a_0), \ldots, f(a_n)$ is unimodal.*

Proof. Suppose that $f(a_0), \ldots, f(a_n)$ is not unimodal. Then there exists $1 \leq i < j \leq n$ such that

$$f(a_i) > f(a_{i+1}) \geq \cdots \geq f(a_{j-1}) < f(a_j). \tag{2.73}$$

Since f is differentiable it is continuous, and since

$$\frac{f(a_i) - f(a_{j-1})}{a_i - a_j} \tag{2.74}$$

is negative and

$$\frac{f(a_{j-1}) - f(a_j)}{a_{j-1} - a_j} \tag{2.75}$$

is positive then by the Mean Value Theorem f' must be negative for some value on $[a_i, a_{j-1}]$ and positive for some value on $[a_{j-1}, a_j]$. By the Intermediate Value Theorem, f' must be 0 for some value in the interval $[a_i, a_j]$. Therefore, there is a local extremal point of f over the interval $[a, b]$ somewhere on $[a_i, a_j]$. There is only one local extremal point in $[a, b]$ by our hypothesis, however the extremal point on $[a_i, a_j]$ is a local minimum and as a unimodal function f necessarily contains a local maximum on $[a, b]$, therefore there are two extremal points. This contradicts our assumption that f is unimodal on $[a, b]$. By contradiction, $f(a_0), \ldots, f(a_n)$ is unimodal. □

Example 2.4.3. Consider the parabola:

$$f(x) = -x^2 + 2 \tag{2.76}$$

which is a unimodal function on $\mathbb{R}$ with the maximum value 2 at $x = 0$. For any increasing sequence of real numbers

$$a_0 \leq a_1 \leq \cdots \leq a_n \tag{2.77}$$

the sequence

$$f(a_0), f(a_1), \ldots, f(a_n) \tag{2.78}$$

is unimodal. Similarly, if a class $\mathcal{M}$ of sequences can be shown to be a set of sequences

$$f(a_0), f(a_1), \ldots, f(a_n) \tag{2.79}$$

where either

$$a_0 \leq a_1 \leq \cdots \leq a_n \tag{2.80}$$

or

$$a_0 \geq a_1 \geq \cdots \geq a_n \tag{2.81}$$

and each sequence lies in some unimodal interval of f and f is twice differentiable, then $\mathcal{M}$ is a set of unimodal sequences. Thus, unimodal functions and unimodal sequences are closely related.

Proposition 2.4.2 is an example of how things can quickly become nuanced when trying to prove that a sequence or a class of sequences are unimodal. In order to prove Proposition 2.4.2, we had to be careful about which intervals we used, whether those intervals were closed or open (in fact, compact), whether the function was differentiable and continuous, and we also had to invoke the mean and intermediate value theorems. This proof was relatively simple and only used basic calculus, but it illustrates how showing that something is unimodal can quickly get out of hand. Unimodal sequences show up in many settings, but many sequences are only conjectured to be unimodal [6]. Others thought to be unimodal were disproved using fairly exotic counterexamples [40, see the final section]

A sequence of numbers $a_0, \ldots, a_n$ is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq n-1$. A sequence of positive numbers that is log-concave is also unimodal [40]. If two polynomials $g(t)$ and $p(t)$ have log-concave coefficient vectors, then $g(t)p(t)$ is also log-concave [40]. Consequently, two positive log-concave vectors can be used to construct a new positive log-concave, and therefore unimodal, vector from their "product". This method of multiplying the generating functions of positive log-concave sequences is a well-known technique [6]. For readers familiar with graph theory, the coefficient vector of the chromatic polynomial of a graph is log-concave [23].

Example 2.4.4. Although positive log-concave sequences are unimodal, not all positive unimodal sequences are log-concave. For Example:

$$1, 1, 2, 1, 1 \tag{2.82}$$

is not log-concave but unimodal. This sequence is taken from a polynomial in [40].

The quest to understand the Hilbert function and to study unimodality are some of the underlying motivation for our study of the Weak Lefschetz Property and levelness, which are the focus of the remaining chapters. We will see how these concepts relate to the Hilbert function and unimodality as we proceed.

Chapter 3

The Weak Lefschetz Property

A *linear form* of a graded algebra A is a linear combination

$$l = \sum_{i=1}^{n} a_i x_i \tag{3.1}$$

where a_i is in a field k and x_i is a variable of A. The *Lefschetz question* for a graded algebra asks whether there is a linear form l for which multiplication by l has maximal rank between graded components of A. If the Lefschetz question is answered in the affirmative, then A is said to have the *Weak Lefschetz Property* (WLP).

Whether or not the WLP is true of an Artinian graded algebra is something which can always be checked "by hand". However, in practice checking for the WLP can be very difficult as the number of variables or the complexity of the algebra increases. Thus, it is in our interest to find methods of establishing when the WLP holds or does not hold without directly checking. Often, the goal of research around the WLP is to identify classes of graded algebras with it

Along with giving a brief introduction to the WLP, we here introduce and examine some properties of squarefree Stanley-Reisner rings which can be used to rule out the presence of the WLP. This gives us an idea of the WLP for squarefree Stanley-Reisner rings and the qualities of them which impact their answers to the Lefschetz question.

3.1 WLP with the Star and Link

Let k be a field and let $S = k[x_1, \ldots, x_n]$ and $I \subseteq S$ be a homogeneous ideal where $A = S/I$ is an Artinian ring. We say that A has the *Weak Lefschetz Property* (WLP) if there exists a linear form $l = \sum_{i=1}^{n} a_i x_i$, where $a_i \in k$ for $1 \leq i \leq n$, such that the multiplication map

$$\mu : A \longrightarrow A \quad \text{where} \quad \mu(f) = f \cdot l \tag{3.2}$$

has full rank between the graded components of A. Otherwise said, for each $i \geq 0$ the graded component maps

$$\mu_i := \quad \mu|_{A_i} : A_i \longrightarrow A_{i+1} \tag{3.3}$$

are injective when $\dim(A_i) \leq \dim(A_{i+1})$, and surjective when $\dim(A_{i+1}) \leq \dim(A_i)$. The map μ and its graded components are vector space homomorphisms because A is a vector space over the field k. Since vector spaces are modules, μ and its graded components are also module homomorphisms. We will learn some of the history of the WLP in Chapter 4.

A linear form $l = \sum_{i=1}^{n} a_i x_i$ for which each μ_i, where $i \geq 0$, has full rank is called a *Lefschetz element* of A, and there may be several different Lefschetz elements. It is known that for Artinian quotients of monomial ideals with the WLP

$$l = x_1 + \cdots + x_n \tag{3.4}$$

is a Lefschetz element:

Proposition 3.1.1 (Proposition 2.2 of [36]). *Let S/I be a graded Artinian quotient of a polynomial ring of a monomial ideal I. Then S/I has the WLP if and only if $x_1 + \cdots + x_n$ is a Lefschetz element of S/I.*

The setting of this thesis is monomial ideals, so in our context we think of l as $x_1 + \cdots + x_n$ when applicable.

Let Δ be a simplicial complex and let S/I, where

$$I = I_\Delta + (x_1^2, \ldots, x_n^2) \tag{3.5}$$

as in (2.47), be its squarefree Stanley-Reisner ring, and let $m \in S$ be a monomial of degree p. Observe the following exact sequence, remembering the definition of the colon ideal from (2.16), where π is the projection map:

$$0 \to S(-\deg(m))/(I : (m)) \xrightarrow{\cdot m} S/I \xrightarrow{\pi} S/(I + (m)) \to 0. \tag{3.6}$$

Discussion 3.1.2. This sequence is a standard tool for inductive arguments for ideals. In Proposition 3.1.3 we have the sequence

$$0 \to (I : (l))/I \xrightarrow{\iota} S/I \xrightarrow{\mu} S/I \xrightarrow{\pi} S/(I + (l)) \to 0 \tag{3.7}$$

which is very similar to (3.6) except instead of

$$0 \to S(-\deg(m))/(I:(m)) \xrightarrow{\cdot m} \cdots \tag{3.8}$$

it begins with

$$0 \to (I:(l))/I \xrightarrow{\iota} S/I \xrightarrow{\mu} \cdots \tag{3.9}$$

where

$$(I:(l))/I = \ker \mu \tag{3.10}$$

effectively "extending" the sequence by one step. The ideal $(I:(m))$ is the ideal of all polynomials $f \in S$ for which $fm \in I$. Thus, by definition the multiplication map $\cdot m$ in (3.6) is injective. Likewise, the image of the multiplication map is everything in S which is divisible by m but not in I, along with zero. This is exactly the kernel of the projection map $\pi : S/I \longrightarrow S/(I+(m))$.

One reason we look at squarefree Stanley-Reisner rings with the WLP is that they have the following property:

Proposition 3.1.3 (A special case of Proposition 2.1 of [36])**.** *Let I be a squarefree monomial ideal in $S = k[x_1, \ldots, x_n]$, let $l = x_1 + \cdots + x_n$, and let $A = S/(I + (x_1^a, \ldots, x_n^a))$ for some $a \geq 2$. With μ and μ_i as in* (3.2) *and* (3.3)*, if μ_i is surjective then μ_j is surjective for all $j \geq i$.*

Proof. Let π denote a projection map from one quotient module S/K to another S/J given by $x + K \mapsto x + J$ and let ι denote the immersion map defined the same way. Consider the exact sequence:

$$0 \to (I:(l))/I \xrightarrow{\iota} S/I \xrightarrow{\mu} S/I \xrightarrow{\pi} S/(I+(l)) \to 0. \tag{3.11}$$

If μ_i is surjective, then $(S/(I+(l)))_{i+1} = 0$ since it is isomorphic to the cokernel of μ_i. However, if $(S/(I+(l)))_{i+1} = 0$, then we must have $(S/(I+(l)))_{i+2} = 0$ because any nonzero monomial of $(S/(I+(l)))_{i+2}$ of degree $i+2$ is divisible by some $i+1$-degree monomial of S/I, and because all degree $i+1$ monomials in $S/(I+(l))$ are zero then $(S/(I+(l)))_{i+2} = 0$. Observe

$$(S/I)_{i+1} \xrightarrow{\mu_{i+1}} (S/I)_{i+2} \xrightarrow{\pi} (S/(I+(l)))_{i+2} \tag{3.12}$$

becomes

$$(S/I)_{i+1} \xrightarrow{\mu_{i+1}} (S/I)_{i+2} \to 0 \tag{3.13}$$

and so if μ_i is surjective, then μ_{i+1} is surjective. Hence, by induction for any $j \geq i$ we find that μ_j is surjective.

□

A very similar, but weaker, result holds for when μ_i is injective, as we will see in Proposition 3.1.4. Its proof includes a number of concepts which are outside the scope of this thesis, so we will just include the citation. Propositions 3.1.4 and 3.1.6 refer to a property called *level*. We want to mention these propositions here to motivate the WLP by looking at unimodality, but it is too early in this work to define levelness. For now, all that needs to be known is that a squarefree Stanley-Reisner ring can have a property called *level*, and that it is neither too uncommon or too ubiquitous; we will learn more about levelness in Section 4.2.

Proposition 3.1.4 (A special case of Proposition 2.1 of [36])**.** *Let S and I as in Proposition 3.1.3 above. For integers $i, j \geq 0$, if S/I is level (see Section 4.2 in Chapter 4) and μ_i is injective for then μ_j is injective for $j \leq i$.*

Because of Proposition 3.1.3 and Proposition 3.1.4, when a level squarefree Stanley-Reisner ring $S/(I' + I_\Delta)$ has the WLP the f-vector $\mathbf{f}(\Delta) = (f_1, \ldots, f_d)$ (see Definition 2.3.2) of Δ is *unimodal*, meaning that there is an integer $1 \leq j \leq d$ such that

$$f_i \leq f_{i+1} \quad \text{when} \quad i < j \quad \text{and} \quad f_i \geq f_{i+1} \quad \text{when} \quad i \geq j. \tag{3.14}$$

Example 3.1.5. The vector $(1, 4, 4, 7, 3, 2)$ is an example of a unimodal vector since

$$1 \leq 4 \leq 4 \leq 7 \geq 3 \geq 2. \tag{3.15}$$

Proposition 3.1.6. *Let $A = S/(I_\Delta + (x_1^2, \ldots, x_n^2))$ be a level (see Section 4.2) squarefree Stanley-Reisner ring of a d-dimensional simplicial complex Δ and suppose that A has the WLP, then the f-vector and h-vector of A are unimodal.*

Proof. By Corollary 2.3.10, the f-vector of $A = S/(I' + I_\Delta))$ is related to the h-vector by

$$(f_0, \ldots, f_{d-1}) = (h_1, \ldots, h_d) \tag{3.16}$$

so it is sufficient to show that the f-vector is unimodal. Since A has the WLP, every μ_i is either injective or surjective. Let $i \geq 0$ be the smallest $i \in \mathbb{N}$ for which μ_i is surjective. By Proposition 3.1.3, if $j \geq i$, then μ_j is surjective and so $\dim(A_j) \geq \dim(A_{j+1})$, and by Lemma 2.3.9 $\dim(A_j) = f_{j-1}$, so the relation $f_j \geq f_{j+1}$ holds for the f-vector when $j \geq i$. Since i is the smallest $i \in \mathbb{N}$ for which μ_i is surjective and each graded component of μ is either injective or surjective, then μ_{i-1} must be injective if $i > 0$, otherwise $i-1$ does not exist. Hence, by Proposition 3.1.3 μ_j is injective when $0 \leq j \leq i-1$. Likewise, since μ_j is injective for $0 \leq j \leq i-1$, then $\dim(A_{j-1}) \leq \dim(A_j)$ for $j \leq i$ also, hence $f_{j-1} \leq f_j$ for $j \leq i$ as desired.

□

Proposition 3.1.6 means that we can test for failure of the WLP in a level squarefree Stanley-Reisner complex by checking the h-vector or f-vector for unimodality.

If a squarefree Stanley-Reisner ring is not level, then we may still retain the part of the proposition which regards surjectivity. So one may say that no matter what, the WLP ensures that the f and h-vectors are always "half" unimodal, although we will see later on that unimodality is always true in this situation.

Example 3.1.7. Let $\Delta = \langle wx, xy, yz, xz \rangle$ be a simplicial complex. For the squarefree Stanley-Reisner ring of Δ, the multiplication map $\mu_1 : \langle x, y, z, w \rangle \longrightarrow \langle wx, xy, yz, xz \rangle$, has the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{3.17}$$

which is fairly easy to row reduce, ending with the identity matrix. So μ is full rank in each degree, the f-vector is $(1, 4, 4)$, and hence the squarefree Stanley-Reisner ring $S/(I_\Delta + I')$ has the WLP. Notice that $(1, 4, 4)$ is unimodal since

$$1 \leq 4 \geq 4. \tag{3.18}$$

Let $\Delta' = \langle abc, adc, abd, bcd \rangle$ be a simplicial complex. The multiplication map from

$\mu_2 : A_2 \longrightarrow A_3$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.19}$$

and we will see in Example 3.3.10 that this matrix is not full rank over $\mathbb{Z}_2$, so over that field the WLP does not hold.

In the more general setting of a graded Artinian quotient ring, we have a strong connection between unimodality and the WLP:

Definition 3.1.8. Let $\mathbf{p} = (p_1, \ldots, p_d)$ be a sequence of positive integers. We say that $\mathbf{p}$ is an *O-sequence* if there is a graded quotient ring whose Hilbert function is $\mathbf{p}$.

Proposition 3.1.9 (Proposition 3.5 of [35])**.** *Let $\boldsymbol{h} = (1, h_1, \ldots, h_s)$ be a finite sequence of positive integers. Then there is a graded Artinian quotient ring A with the WLP such that the Hilbert series is*

$$\mathrm{HS}(A,t) = 1 + h_1 t + \ldots + h_s \tag{3.20}$$

if and only if $\mathbf{h}$ is a unimodal O-sequence such that the positive part of

$$(h_1 - 1, h_2 - h_1, \ldots, h_s - h_{s-1}) \tag{3.21}$$

is also an O-sequence.

Proposition 3.1.9 differs from Proposition 3.1.6 in that Proposition 3.1.6 uses the WLP to show that the f-vector and h-vector of a squarefree Stanley-Reisner ring are unimodal, whereas Proposition 3.1.9 characterizes which vectors $\mathbf{h}$ are encode the Hilbert function of a graded Artinian ring with the WLP without specifically giving that ring. Proposition 3.1.6 is also concerned with the injectivity and surjectivity of the graded component maps of μ. Since Proposition 3.1.9 implies that a graded Artinian ring with the WLP has a unimodal Hilbert function, it is no surprise that a common direction of research around the WLP is to identify classes of graded Artinian rings with or without the WLP. Proposition 3.1.9 also gives us a connection between the WLP and combinatorics.

We will now explore the WLP for squarefree Stanley-Reisner rings. This section works through the results of Miglore, Nagel, and Schenck in [34], where we generalize and expound their results. Nearly every result in [34] is restricted to the case in which, for $A = S/I$, the ideal I is generated minimally by quadratic monomials. This is due to the authors' interest in finding counterexamples specifically in the class of quadratic monomial ideals. This condition, however, is never necessary for any of their original proofs. The proofs are rewritten here, and in greater detail, to verify this. The authors of [34] were building on work in [31], which is about Artinian quotient rings by ideals whose generators are quadratic and cubic, which seems to be why they did not consider generators of higher degrees. In [34], the authors develop a number of conditions for squarefree Stanley-Reisner rings which are sufficient to conclude that the WLP does not hold.

The following is a generalization of [34, Proposition 2.5]. The function δ is the *connecting map* used in the Snake Lemma 2.2.6. In the original paper, $p = 1$ or 2, here $p \in \mathbb{N}$.

Proposition 3.1.10. *Let $A = S/I$ be the squarefree Stanley-Reisner ring of a simplicial complex Δ, and let $\sigma \in \Delta$ be a face of Δ, let $\Delta' = \Delta \setminus \mathrm{st}(\sigma)$ and $\Delta'' = \mathrm{lk}(\sigma)$. Let $M(\sigma)$ be the squarefree monomial corresponding to σ. Let $A'' = S/(I : (M(\sigma)))$ and $A' = S/(I + (M(\sigma)))$. The rings A'' and A' are the squarefree Stanley-Reisner rings of $\Delta'' = \mathrm{lk}(\sigma)$ and $\Delta' = \Delta \setminus \mathrm{st}(\sigma)$ respectively. Let μ'' and μ' be the maps which denote multiplication by $l = \sum_{i=1}^{n} x_i$ for A''andA$'$ respectively, just like μ for A. For $i \in \mathbb{N}$, $p = \deg(M(\sigma))$, and $p < i$ we get the exact sequence:*

$$0 \to A''_{i-p} \xrightarrow{\cdot M(\sigma)} A_i \to A'_i \to 0 \tag{3.22}$$

then

- *μ_i is injective if and only if both μ''_{i-p} and δ are injective.*
- *μ_i is surjective if and only if both μ'_i and δ are surjective.*

Proof. If $A' = S/(I + (M(\sigma)))$, then $A' = S/((I_\Delta + (M(\sigma))) + I')$, meaning that $A' = S/(I_{\Delta'} + I')$ where σ is a minimal nonface of $\Delta' \subseteq \Delta \setminus \mathrm{st}(\sigma)$, along with all the other minimal nonfaces of Δ. Suppose that m is a squarefree monomial and

$M^{-1}(m) \notin \Delta \setminus \mathrm{st}(\sigma)$. This is equivalent to $M^{-1}(m) \notin \Delta \vee M^{-1}(m) \in \mathrm{st}(\sigma)$, which in turn means $m \in I \vee M(\sigma) \mid m$. Thus, $M^{-1}(m) \notin \Delta \setminus \mathrm{st}(\sigma)$ if and only if $m \in I + M(\sigma)$. Therefore, $S/(I + M(\sigma))$ is the squarefree Stanley Reisner ring of $\Delta \setminus \mathrm{st}(\sigma)$ and $\Delta' = \Delta \setminus \mathrm{st}(\sigma)$. Otherwise stated, $A' = S/(I_{\Delta\setminus\mathrm{st}(\sigma)} + I') = S/(I + (M(\sigma)))$.

If $A'' = S/(I : (M(\sigma)))$ then $(I : (M(\sigma))) = (m \in S \mid mM(\sigma) \in I)$. The ideal $(I : (M(\sigma)))$ includes all of I itself and every variable which divides $M(\sigma)$; this is because $I = I_\Delta + I'$ includes the squares of each variable of S. Because of this, for a variable $x_i \in \{x_1, \dots, x_n\}$ we have $x_i \mid M(\sigma)$ implies that $x_i^2 \mid x_i M(\sigma)$. And, because $x_i^2 \in I$, then $x_i^2 M(\sigma) \in I$, so $x_i \in (I : (M(\sigma)))$. The ideal $(I : (M(\sigma)))$ also includes the variables x_j which are not in $\mathrm{st}(\sigma)$ because $\{x_j\} \cup \sigma \notin \Delta$, so this means $x_j M(\sigma)$ is divisible by some minimal nonface of Δ. Hence, $x_j M(\sigma) \in I \subseteq (I : (M(\sigma)))$ since $I = I_\Delta + I'$ and I_Δ is generated by the minimal nonfaces of Δ. Hence, A'' as a squarefree Stanley-Reisner ring is $A'' = S/(I_{\Delta''} + I')$, where Δ'' is derived from Δ by deleting every face and vertex not connected to σ and by deleting σ itself. Therefore $\Delta'' = \mathrm{lk}(\sigma)$ in accordance with (2.42).

Apply the Snake Lemma (Lemma 2.2.6) to the graded pieces of (3.22) in order to get a diagram like (3.23). Let K'', K', and K be the respective kernels of the graded components of μ'', μ', and μ along the graded submodules of A'', A', and A, and similarly let C'', C', and C be the cokernels.

$$\begin{tikzcd} 0 \arrow[r] & A''_{i-p} \arrow[r] \arrow[d, "\mu''"] & A_i \arrow[r] \arrow[d, "\mu"] & A'_i \arrow[r] \arrow[d, "\mu'"] & 0 \\ 0 \arrow[r] & A''_{i-p+1} \arrow[r] & A_{i+1} \arrow[r] & A'_{i+1} \arrow[r] & 0 \end{tikzcd} \tag{3.23}$$

By the Snake Lemma (Lemma 2.2.6) we get the long exact sequence:

$$0 \to K''_{i-p} \xrightarrow{\cdot M(\sigma)} K_i \xrightarrow{f_1} K'_i \xrightarrow{\delta} C''_{i-p+1} \xrightarrow{f_2} C_{i+1} \xrightarrow{f_3} C'_{i+1} \to 0 \tag{3.24}$$

There are four cases to consider:

1) Suppose that μ''_{i-p} and δ are injective. Since μ''_{i-p} is injective we have $K''_{i-p} = 0$, and because δ is injective, $\ker \delta = 0$, which implies that $\mathrm{im}\ f_1 = \ker \delta = 0$. Because $K''_{i-p} = 0$ we find that f_1 is injective; therefore $K_i \cong \mathrm{im}\ f_1 = \ker \delta = 0$. Thus, $K_i = 0$ and μ_i is injective.

2) Suppose μ_i is injective. By definition $K_i = 0$ and so in the exact sequence $0 \to K''_{i-p} \xrightarrow{\cdot\bar{\sigma}} K_i \xrightarrow{f_1} \dots$ we have $0 \to K''_{i-p} \xrightarrow{\cdot\bar{\sigma}} 0$. Thus $K''_{i-p} = 0$ and so μ''_{i-p} is injective. Since $K_i = 0$ we have $0 \xrightarrow{f_1} K'_i \xrightarrow{\delta} C''_{i-p+1} \xrightarrow{f_2} \dots$, which implies $0 = \ker\delta$, and so δ is injective.

3) Suppose that μ_i is surjective. Then $C_{i+1} = 0$, and because the sequence above is exact, im $\delta = \ker f_2$, but $C_{i+1} = 0$ so $\ker f_2 = C''_{i-p+1}$. Because im $\delta = \ker f_2$ and $C''_{i-p+1} = \ker f_2$, then $\operatorname{im}\delta = C''_{i-p+1}$ and δ is surjective. Since $C_{i+1} = 0 \xrightarrow{f_3} C'_{i+1} \to 0$ is exact, then $C'_{i+1} = 0$ and so μ'_i is surjective.

4) Suppose that μ'_i and δ are surjective. We have $C'_{i+1} = 0$, and because the sequence is exact im $\delta = C''_{i-p+1} = \ker f_2$. Thus, $f_2 \equiv 0$, but im $f_2 = \ker f_3$ and since $C'_{i+1} = 0$ we have $\ker f_3 = C_{i+1}$. Therefore, im $f_2 = 0 = \ker f_3$ and $\ker f_3 = C_{i+1}$, so $C_{i+1} = 0$, which means that μ_i is surjective.

□

Remark 3.1.11. In the original paper, Proposition 3.1.10 was done for $p = 1$ and 2. It should be noted however that for larger p, there is a "diminishing return". For example, to take the link of a 5-dimensional simplex means examining A''_{i-5}, which does not exist for $i < 5$, so in practical terms larger p require larger and more complicated simplicial complexes. It should also be noted that in practice δ can be difficult to work with because of its complicated description.

In the original paper, the first result was not "if and only if," but just "if". The trouble is that this does not tell the whole story, and is not sufficient to prove Corollary 3.1.13. We give the full proof here.

Before seeing Corollary 3.1.13 however, it is worth highlighting a related result to Proposition 3.1.10:

Proposition 3.1.12 (Theorem 1 in [15])**.** *Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k of characteristic zero, let $m \in S$ be a monomial, and let I be a monomial ideal of S. Let $K = I + (m)$ and $J = (I : (m))$. Suppose that for $i \geq 0$ and some $d \in \mathbb{N}$*

$$H_{S/K}(i) < H_{S/K}(i+d) \Rightarrow H_{S/J}(i - \deg(m)) \leq H_{S/J}(i - \deg(m) + d) \tag{3.25}$$

and

$$H_{S/K}(i) > H_{S/K}(i+d) \Rightarrow H_{S/J}(i-\deg(m)) \geq H_{S/J}(i-\deg(m)+d). \tag{3.26}$$

If S/K and S/J have the WLP, then so does S/I.

One may ponder the interaction between Propositions 3.1.12 and 3.1.10.

Proposition 3.1.10 and Corollary 3.1.13 provide a basis for reducing the problem of verifying or dispelling whether the squarefree Stanley-Reisner ring of Δ has the WLP.

The following is a generalization of a result in [34, Corollary 2.6]. In the original paper, the $p \in \mathbb{N}$ is not present but only $p = 1$.

Corollary 3.1.13. *Let $(f_1, \ldots, f_d)$ be the f-vector of a simplicial complex Δ. With A, A', and A'' (along with $\mu' and \mu''$) as in Proposition 3.1.10. The following two statements hold:*

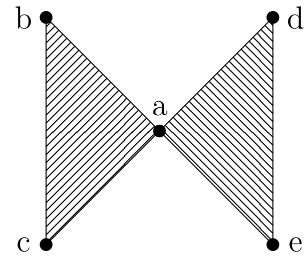
- *If μ'' fails injectivity in degree $i-p$, and $f_i \geq f_{i-1}$, then A fails WLP in degree i due to injectivity.*
- *If μ' fails surjectivity in degree i, and $f_i \leq f_{i-1}$, then A fails WLP in degree i due to surjectivity.*

Proof. There are two cases to consider:

1. Suppose that μ'' fails surjectivity in degree $i-p$ and $f_i \geq f_{i-1}$. By Proposition 3.1.10 μ_i is not injective. Since $f_i \geq f_{i-1}$ the map μ_i is not full rank. Therefore, the WLP fails in degree i.
2. Suppose that μ' fails surjectivity in degree i. By Proposition 3.1.10, μ_i fails surjectivity, and since $f_i \leq f_{i-1}$ this map is not full rank. Therefore, the WLP fails in degree i due to surjectivity.

$\square$

Example 3.1.14. Let $\Delta = \langle abc, ade \rangle$ as in Example 2.3.1:

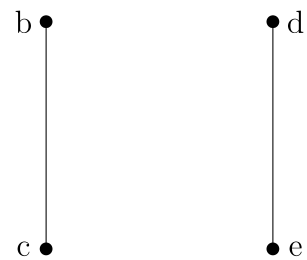


A geometric realization of Δ

In this case,

$$S = k[a,b,c,d,e] \quad \text{and} \quad I = ((bd, be, cd, ce) + (a^2, b^2, c^2, d^2, e^2)) \tag{3.27}$$

thus, $A = S/I$ and we have the link of $\{a\}$:



The link of $\{a\}$

So A'', the squarefree Stanley-Reisner complex of $\mathrm{lk}(\{a\})$, is

$$A'' = k[b,c,d,e]/(b^2, c^2, d^2, e^2, bd, be, cd, ce). \tag{3.28}$$

For Δ, the matrix of $\mu_1 : A_1 \longrightarrow A_2$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{3.29}$$

which is full rank (injective). From Proposition 3.1.10, the connecting map δ and μ''_{1-1} (obviously μ''_0 is injective), the multiplication map of A'', are injective also.

The matrix of the map $\mu_2 : A_2 \longrightarrow A_3$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}. \tag{3.30}$$

This is also full rank (surjective), and thus by Proposition (3.1.10) we know that μ_i' is surjective and δ is surjective. The simplicial complex Δ has the WLP.

3.2 An Archival Interlude

The following is a short account of the necessary graph theoretical ideas to understand this subsection:

A graph G, is a mathematical object which consists of $V = \{v_1, \ldots, v_n\}$ a set of *vertices*, and $E \subseteq \{\{v_i, v_j\} \mid v_i \neq v_j, 1 \leq i, j, \leq n\}$, a set of *edges*. For a given simplicial complex Δ, it is easy to see that $\Delta(1)$ may be considered a graph, so long as we ignore the empty set and singleton sets (consider Example 2.3.13). Two vertices v and w of a graph G are *connected* if there exists a finite sequence of vertices $v, w_1, \ldots, w_p, w$, called a *path*, where any two adjacent vertices in this sequence are in an edge of E. Connected-ness is an equivalence relation on G, and each equivalence class of this relation is called a *component* of G. A component G' of G is called a *tree* if every pair of connected vertices v and w are connected by a unique path. A component G' is *bipartite* if there exists two sets of vertices B_1 and B_2 where:

1. $B_1 \cap B_2 = \emptyset$
2. $B_1 \cup B_2$ is the set of all vertices of the component G';
3. if $v, w \in B_i$, where $i = 1$ or 2, then there is no edge $\{v, w\}$ in G

In December 2021, Hailong Dao and Ritika Nair released a preprint paper, *On the Lefschetz Property for Quotients by Monomial Ideals Containing Squares of Variables* [11]. There are two principal results in this paper, the first is Proposition 3.2.1:

Proposition 3.2.1 (Theorem 3.3 of [11]). *Let Δ be a simplicial complex and let S/I be its Squarefree Stanley-Reisner ring.*

1. *If $f_1 \geq f_0$, then the WLP holds in degree 1 if and only if $\Delta(1)$ has no bipartite components.*
2. *If $f_1 < f_0$, then the WLP holds in degree 1 if and only if each bipartite component of $\Delta(1)$ (if any exist) is a tree and each non-bipartite component satisfies the*

property that the number of edges in the component is equal to the number of vertices in the same component. Further, in this case, the WLP holds in degree 1 *means the WLP holds in all degrees.*

The fact that the WLP holds in degree 1 guarantees the WLP holding in all degrees in Proposition 3.2.1 part 2. This is a consequence of Proposition 3.1.4, which comes from [36].

Proposition 3.2.1 allows us to infer the presence of the WLP in degree 1 based on $\Delta(1)$ as a graph. The technique of examining the graph $\Delta(1)$ is a reduction of the simplicial complex Δ to something less complex, similar to Proposition 3.1.10 and Corollary 3.1.13 which use the closed star and link of Δ. In contrast with Proposition 3.1.10, the graph $\Delta(1)$ is much easier to interact with than the connecting map δ used in the Snake Lemma 2.2.6, although it is restricted to the first degree. Corollary 3.1.13 does not require δ, however it can only establish the failure of the WLP and not confirm it as Proposition 3.2.1. Later, such as in Proposition 3.3.6, we will use the skeleton $\Delta(i)$, but not in such an integral way.

The second result is Proposition 3.2.2:

Proposition 3.2.2 (Theorem 1.2 of [11]). *Let Δ be a simplicial complex and let S/I be its squarefree Stanley-Reisner ring, where Δ is a d-dimensional pseudomanifold. The WLP holds in degree d if and only if*

1. *Δ has a boundary; or*

2. *Δ has no boundary but the dual graph of Δ is not bipartite.*

Exactly what a pseudomanifold or a boundary is falls outside our focus, and we will use neither of these Propositions for the rest of the thesis, so we will not define pseudomanifolds here. In Section 3.3, we will look at some of the homological ideas from [34], which examines the relation between the WLP and squarefree Stanley-Reisner rings from a homological (topological) perspective. Unlike Proposition 3.2.2, which requires a d-dimensional pseudomanifold, Proposition 3.3.6 only works when $S = k[x_1 \dots, x_n]$ and $\mathrm{char}(k) = 2$, but for any simplicial complex Δ. Proposition 3.3.9 is applicable when $\mathrm{char}(k) = 0$, the typical case, but it is a weaker result.

There is not a great body of literature about how simplicial complexes with or without the WLP are characterized, so this paper by Hailong Dao and Ritika Nair is a worthy interlude.

3.3 The Homology Connection

We will now look at some of the connections between the i-th simplicial homology group and the WLP.

Let V and W be finite dimensional vector spaces over a field F with respective bases $v_1, \ldots, v_q$ and $w_1, \ldots, w_p$. *The dual space* of V, denoted $V^\vee$, is the F-vector space of all linear maps $V \longrightarrow F$ with addition and scalar multiplication of maps defined in the natural way. If V is p-dimensional, then $V^\vee$ is also p-dimensional since for $v_1, \ldots, v_q$ a basis of V there are p maps $v_i^\vee$ defined by

$$v_i^\vee(v_j) = 0 \quad \text{if} \quad i = j \quad \text{or} \quad v_i^\vee(v_j) = 1 \text{ else.} \tag{3.31}$$

These maps form a basis of $V^\vee$. For a linear map $f : V \longrightarrow W$ the dual $f^\vee : W^\vee \longrightarrow V^\vee$ is defined by $f^\vee(\varphi) = \varphi \circ f$. The maps $\varphi : W \longrightarrow F$ and $f : V \longrightarrow W$ compose to give $\varphi \circ f : V \longrightarrow F$. The matrix of f, denoted $[f]$, is the transpose of $[f^\vee]$, this is well known in linear algebra [3, Page 101].

Remark 3.3.1. Since the Artinian ring $A = S/I$, where $S = k[x_1, \ldots, x_n]$ and $I = I_\Delta + I'$ as in (2.47), is a vector space, then $A = A_0 \oplus A_1 \oplus \ldots \oplus A_d$ and each A_i is a k-vector space. The multiplication map $\mu_i : A_i \longrightarrow A_{i+1}$, where

$$\mu_i(f) = f \cdot l \tag{3.32}$$

and

$$l = \sum_{i=1}^{n} x_i \tag{3.33}$$

is a linear map, and so it has a dual $\mu_i^\vee$, which we will discuss below.

Let $j \geq 0$ and let $C_j(\Delta)$ denote the free k-module generated by the j-dimensional faces of Δ. Let $\sigma = \{x_{\sigma,1}, \ldots, x_{\sigma,j+1}\}$ be a j-dimensional face of Δ. We know from Lemma 2.3.9 that

$$\dim(C_j(\Delta)) = f_j = \dim(A_{j+1}) = \dim(A_{j+1}^\vee) \tag{3.34}$$

where $A = S/I$ is the squarefree Stanley-Reisner ring of Δ. Notice that the dimension of a face of $\sigma \in \Delta$ differs from the degree of the monomial $M(\sigma)$ by 1, and so the index of $C_j(\Delta)$ and A_{j+1} differ by 1. The map

$$\partial_j : C_j(\Delta) \to C_{j-1}(\Delta) \tag{3.35}$$

defined by

$$\partial_j(\sigma) = \partial_j(\{x_{\sigma,1}, \ldots, x_{\sigma,j+1}\}) = \sum_{i=1}^{j}(-1)^i\{x_{\sigma,1}, \ldots, x_{\sigma,i-1}, x_{\sigma,i+1}, \ldots, x_{\sigma,j+1}\} \tag{3.36}$$

is the j-th *boundary map of* Δ (see Example 3.3.2)[25, Page 341].

The following, Proposition 3.3.4, lays the groundwork for the rest of the section. In Proposition 3.3.4, we will show that

$$(A, \mu) = 0 \to A_1 \xrightarrow{\mu_1} A_2 \xrightarrow{\mu_2} \ldots \to A_{d-1} \xrightarrow{\mu_{d-1}} A_d \tag{3.37}$$

is a *chain complex*, meaning $\ker(\mu_i) \subseteq \operatorname{im}(\mu_{i-1})$ for $i \geq 0$. We also show that the dual map $\mu_i^\vee : A_{i+1}^\vee \to A_i^\vee$ is isomorphic to the simplicial boundary map ∂_i of the simplicial complex Δ.

Let $i \geq 0$ and let $C_i(\Delta) \cong A_{i+1}$ be the free k-module over the i-dimensional faces of Δ. Then

$$0 \leftarrow C_0(\Delta) \xleftarrow{\partial_1} C_1(\Delta) \xleftarrow{\partial_2} C_2(\Delta) \xleftarrow{\partial_3} \ldots \xleftarrow{\partial_{d-1}} C_{d-1}(\Delta) \xleftarrow{\partial_d} C_d(\Delta) \tag{3.38}$$

is the boundary complex of the simplicial complex Δ. The boundary complex is the is the canonical example of a chain complex.

In Proposition 3.3.4, we will compare (3.37) to (3.38) by showing that $\mu_i^\vee \cong \partial_i$ as k-linear maps. Because $\mu_i^\vee \cong \partial_i$, $C_i(\Delta) \cong A_{i+1}$, and $\mu_i^\vee : A_{i+1}^\vee \longrightarrow A_i^\vee$, we can (and will) think of ∂_i as a map from $A_{i+1}^\vee$ to $A_i^\vee$. By Lemma 2.3.9 (see Equation (3.34)), $\partial_i : A_{i+1}^\vee \to A_i^\vee$ is a map from an f_i-dimensional vector space to an f_{i-1}-dimensional one.

Example 3.3.2. Let $\Delta = \langle xyz \rangle$ be a simplicial complex and let $A = k[x, y, z]/I$ be its squarefree Stanley-Reisner ring. Fix an order on the variables $x > y > z$. The map

$$\partial_2 : A_3^\vee \longrightarrow A_2^\vee \tag{3.39}$$

is actually the map

$$\partial_2 : \langle xyz \rangle^\vee \longrightarrow \langle xy, xz, yz \rangle^\vee \tag{3.40}$$

and is defined by

$$\partial_2(xyz) = xy - xz + yz. \tag{3.41}$$

Likewise the map

$$\partial_1 : \langle xy, xz, yz \rangle^\vee \longrightarrow \langle x, y, z \rangle^\vee \tag{3.42}$$

is defined by:

$$\partial_1(xy) = x - y, \tag{3.43}$$

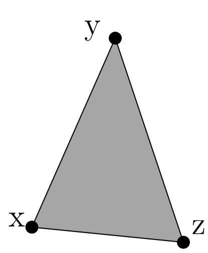
$$\partial_1(xz) = x - z, \tag{3.44}$$

and

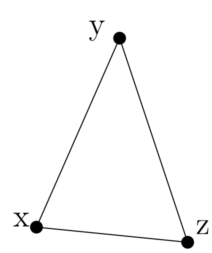
$$\partial_1(yz) = y - z. \tag{3.45}$$

The maps ∂_i may be thought of colloquially as *sending each monomial* $m \in A$ *of degree* j *to the alternating sum of all monomials degree* $j-1$ *which divide* m.

The reason we call this a boundary map is that "each face is sent to the alternating sum of its boundary faces." Normally, the boundary maps ∂_i would be defined in terms of free modules over a field, using the vertices as letters, but since A already falls into this category, it is convenient to cut out this intermediary.



The face xyz



The 1-skeleton of Δ,
showing the boundary of xyz

Remark 3.3.3. From here onwards we will be mostly working over a field k where $\mathrm{char}(k) = 2$, in which case there is no distinction between "+" and "−" in the alternating sum, so we will always write "+" when $\mathrm{char}(k) = 2$.

Proposition 3.3.4 (Lemma 3.1 of [34]). *Let ∂_i be the boundary map of the simplicial complex Δ (except we interpret the faces as monomials, products of vertices, when comparing ∂_i to μ_i). Let $A = k[x_1, \ldots, x_n]/(I_\Delta + I')$, as in* (2.47)*. If $char(k) = 2$, then (A, μ) is a chain complex. Moreover, the dual map $\mu_i^\vee$ has the same matrix as the boundary map ∂_i.*

Proof. Since $l = \sum_{i=1}^n x_i$, then

$$l^2 = \left(\sum_{i=1}^n x_i\right) \cdot \left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n x_i^2 + 2\left(\sum^{0<i<j\leq n} x_i x_j\right). \tag{3.46}$$

Since $x_i^2 \in I$, thus $x_i^2 = 0$ in A, for all $1 \leq i \leq n$, and because $2 \equiv 0$ in $\mathbb{Z}_2$, then

$$l^2 = \sum_{i=1}^n 0 + 0 \cdot \left(\sum^{0<i<j\leq n} x_i x_j\right) = 0. \tag{3.47}$$

Hence, for any homogeneous polynomial f we have $\mu^2(f) = l^2 f = 0$, and so $\operatorname{im}(\mu_i) \subseteq \ker(\mu_{i+1})$ for all $i \geq 0$. Thus (A, μ) is a chain complex as in (3.37).

Observe that $\dim(A_{i+1}) = \dim(C_i(\Delta))$, so the dimension of the respective domains and ranges of $\mu_i^\vee$ and ∂_i are the same.

The matrix of the dual map $\mu_i^\vee : A_{i+1}^\vee \longrightarrow A_i^\vee$ is the transpose of $[\mu_i]$ [13, Theorem 20 of Chapter 11]. If m_j is the basis vector of A_{i+1} for the jth row of the matrix $[\mu_i]$, then this matrix row has a "1" exactly for every basis vector of A_i which divides m_j and has a "0" everywhere else. For the matrix of ∂_i in A, the basis vector m_j of A_{i+1} corresponding to the jth column of the matrix $[\partial_i]$ by definition has a "1" where a basis vector of A_i divides m_j and a "0" everywhere else. Therefore, the dual map $\mu_i^\vee$ has the same matrix as ∂_i.

□

Example 3.3.5. Let $\Delta = \langle abc, ade \rangle$ as in Example 2.3.1. As in Example 3.1.14, the matrix of $\mu_1 : A_1 \longrightarrow A_2$ is:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{3.48}$$

From Proposition 3.3.4, we know that the transpose of this matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \tag{3.49}$$

is the matrix of $\partial_1 : A_2^\vee \longrightarrow A_1^\vee$.

Proposition 3.3.6 (Generalization of [34] (Proposition 3.3))**.** *Let A be an Artinian quotient by $I = I_\Delta + I'$ as in* (2.47)*. Then A fails the WLP over $\mathbb{Z}_2$ in degree $i \geq 0$ if and only if*

1. *surjectivity fails: $H_i(\Delta(i), \mathbb{Z}_2) \neq 0$ and $f_i \leq f_{i-1}$; or*
2. *injectivity fails: coker$(\partial_i) \neq 0$ and $f_{i-1} \leq f_i$.*

Proof. Let $i \geq 0$. By Lemma 2.3.9, A_i is a k-vector space of dimension f_{i-1}. By definition, the WLP holds if in each degree $i \geq 0$ we have μ_i is injective when

$$\dim(A_i) \leq \dim(A_{i+1}) \tag{3.50}$$

or equivalently,

$$f_{i-1} \leq f_i; \tag{3.51}$$

and μ_i is surjective when

$$\dim(A_{i+1}) \leq \dim(A_i) \tag{3.52}$$

or equivalently,

$$f_i \leq f_{i-1}. \tag{3.53}$$

For part 1, let $i \geq 0$ and suppose $f_i \leq f_{i-1}$, then the WLP fails in degree i if and only if μ_i is not surjective. The matrix of ∂_i is the transpose of $[\mu_i]$, so μ_i is not surjective if and only if ∂_i is not injective, in other words $\ker(\partial_i) \neq 0$. Let $j \geq 0$ and let A_j and A'_j be the respective graded components of the squarefree Stanley Reisner rings of Δ and its i-skeleton $\Delta(i)$. Let μ and μ' be the respective multiplication maps for the squarefree Stanley-Reisner rings of Δ and $\Delta(i)$. If $j \leq i$, then by Lemma 2.3.12

we have $A_j = A'_j$ and $\mu_j = \mu'_j$. Let $j \geq 0$, then μ'_j has a dual map $\mu'^{\vee}_j$, which is the boundary map ∂'_j of $\Delta(i)$. Observe that for Δ and $\Delta(i)$ we have $\partial_j = \mu^{\vee}_j = \mu'^{\vee}_j = \partial'_j$ when $j \leq i$. Now, the i-skeleton, $\Delta(i)$, of Δ has no faces of dimension $i+1$, so $\mathrm{im}(\partial'_{i+1}) = \{0\}$. Thus, by the definition of the ith homology group:

$$H_i(\Delta(i), \mathbb{Z}_2) = \ker(\partial'_i)/\mathrm{im}(\partial'_{i+1}) = \ker(\partial'_i)/\{0\} \cong \ker(\partial'_i) \cong \ker(\partial_i). \tag{3.54}$$

Hence, we have $\ker(\mu^{\vee}_i) = \ker(\partial_i) \neq 0$ if and only if $H_i(\Delta(i), \mathbb{Z}_2) \neq 0$. Therefore, because $\ker(\mu^{\vee}_i) \cong H_i(\Delta(i), \mathbb{Z}_2)$ we conclude that μ_i is surjective if and only if $H_i(\Delta(i), \mathbb{Z}_2) \neq 0$, as desired.

For part 2, suppose $f_{i-1} \leq f_i$, then the WLP fails in degree i if and only if $\ker(\mu_i) \neq 0$. By Proposition 3.3.4, the dual of μ_i in degree i is ∂_i. Since the matrix of ∂_i is the transpose of $[\mu_i]$, then $\ker(\mu_i) \neq 0$ if and only if $\ker(\mu^{\vee}_i) = \mathrm{coker}(\partial_i) \neq 0$, as desired.

□

Example 3.3.7. Let $A = \mathbb{Z}_2[a, b, c, x, y, z]$ and let $\Delta = \langle axy, yzc, xzb \rangle$. In degree 2 we have $H_1(\Delta(1), \mathbb{Z}_2) \neq 0$ because of the "hole" xyz, which is in the kernel of ∂_1 but not in the image of ∂_2. By Proposition 3.3.6, surjectivity fails in degree 2.

Example 3.3.8. Let $A = \mathbb{Z}_2[a, b, c, d, x, y, z, w]$ and let $\Delta = \langle axw, bxy, yzc, wdz \rangle$. The Stanley-Reisner ideal is quadratic, and this fulfils Proposition 3.3.6 in the same manner as Example 3.3.7 above.

Example 3.3.9. Let $I = \langle a^2, b^2, c^2, d^2, e^2, f^2, ab, cd, ef \rangle \subseteq k[a, \ldots, f]$ as in [34, Example 2.7]. The associated simplicial complex is

$$\Delta = \langle ade, ace, ceb, bde, adf, acf, cfb, bdf \rangle \cong \mathbb{S}^2. \tag{3.55}$$

This simplicial complex may be visualized as a "hollow diamond" shape in $\mathbb{R}^3$. Therefore $H_2(\Delta(2), \mathbb{Z}_2) \cong H_2(\mathbb{S}^2, \mathbb{Z}_2) \neq 0$. Thus, by Proposition 3.3.6, I fails surjectivity in degree 2, which is consistent with [34, Example 2.7] where they show "by hand" that surjectivity fails in degree 2.

Example 3.3.10. Similarly, let $J = \langle a^2, b^2, c^2, d^2, abcd \rangle$ and $S = \mathbb{Z}_2[a, b, c, d]$. Then S/J is the squarefree Stanley-Reisner ring of the simplicial complex

$$\Delta' = \langle abc, adc, abd, bcd \rangle. \tag{3.56}$$

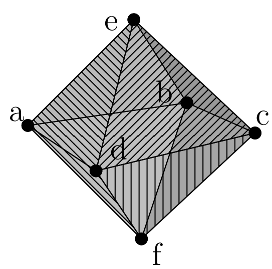
Since $\Delta' \cong \mathbb{S}^2$ we likewise have $H_2(\Delta(2)', \mathbb{Z}_2) \neq 0$. Therefore, Δ' fails the WLP in degree 2 over $\mathbb{Z}_2$.

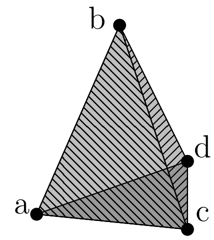
The row reduction of μ in degree 2, over $\mathbb{Z}_2$, is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{3.57}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{3.58}$$

and cannot be continued to achieve full rank.





The simplicial Complex Δ The simplicial Complex Δ'

This all takes place over $\mathbb{Z}_2$. We are not so lucky in characteristic zero, but much like in the last subsection with the star and link, our problem reduces to knowing something about the connecting homomorphism δ.

Proposition 3.3.11 (Generalization of Proposition 3.4 of [34])**.** *For $S = k[x_1, \ldots, x_n]$, where* $\mathrm{char}(k) = 0$*, if $B = S/I(i)$ is the squarefree Stanley-Reisner ring of a simplicial complex $\Delta(i)$, the i-skeleton of a simplicial complex Δ, then for $i \geq 0$ and the long* *exact sequence of abelian groups ($\mathbb{Z}$-modules):*

$$0 \to \ker(\mu_i^\vee) \xrightarrow{\cdot 2} \ker(\mu_i^\vee) \to H_i(\Delta(i), \mathbb{Z}_2) \xrightarrow{\delta} coker(\mu_i^\vee) \xrightarrow{\cdot 2} coker(\mu_i^\vee) \to coker(\partial_i) \to 0 \tag{3.59}$$

the connecting map δ is injective if and only if μ_i is surjective, where μ_i and $\partial_i = \mu_i^\vee$ are as in Proposition 3.3.4. If $f_i \leq f_{i-1}$, then the WLP succeeds or fails in degree i if and only if δ is injective. If $H_i(\Delta(i), \mathbb{Z}_2) = 0$, then δ is injective, and if δ is not injective then $H_i(\Delta(i), \mathbb{Z}_2) \neq 0$.

Proof. Let $i \geq 0$, let id be the identity map between a given module and itself, and let π be the identity/projection map. Take the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \to 0. \tag{3.60}$$

We may construct another exact sequence by the dual module of the tensor of the i-skeleton of Δ: $S/I(i) = B$, whose graded components are free $\mathbb{Z}$-modules (For more on tensors, see [24, Page 207]). We obtain the sequence (which is exact by [24, Proposition 5.4 on Page 209]):

$$0 \to B^\vee \otimes \mathbb{Z} \xrightarrow{id \otimes \cdot 2} B^\vee \otimes \mathbb{Z} \xrightarrow{id \otimes \cdot \pi} B^\vee \otimes \mathbb{Z}_2 \to 0. \tag{3.61}$$

For $j \geq 0$, (3.61) gives us a short exact sequence of the jth graded components

$$0 \to B_j^\vee \to B_j^\vee \to B_j^\vee \otimes \mathbb{Z}_2 \to 0. \tag{3.62}$$

The map $\mu_j^\vee : B_{j+1}^\vee \to B_j^\vee$ induces a map

$$\mu_j^\vee \otimes id : B_{j+1}^\vee \otimes \mathbb{Z}_2 \to B_j^\vee \otimes \mathbb{Z}_2 \tag{3.63}$$

where $id : \mathbb{Z}_2 \to \mathbb{Z}_2$ is the identity map of abelian groups, between graded components of $B^\vee \otimes \mathbb{Z}_2$. The $\mathbb{Z}$-module $B^\vee \otimes \mathbb{Z}_2$ is isomorphic to the $\mathbb{Z}$-module $B'^\vee$, where $B' = S'/I(i)$ and $S' = \mathbb{Z}_2[x_1, \ldots, x_n]$, note that $\operatorname{char}(\mathbb{Z}_2) = 2$. This technique of constructing $B^\vee \otimes \mathbb{Z}_2$ can be thought of as "changing scalars" for $B^\vee$ from a field of characteristic 0 to a field of characteristic 2. Since $B^\vee \otimes \mathbb{Z}_2 \cong B'^\vee$ then we may use Proposition 3.3.4 with B' to conclude that

$$\mu_i^\vee \otimes id : B_{i+1}^\vee \otimes \mathbb{Z}_2 \to B_i^\vee \otimes \mathbb{Z}_2 \tag{3.64}$$

and

$$\partial_i : B'^{\vee}_{i+1} \to B'^{\vee}_i \tag{3.65}$$

form a commutative square

$$\begin{array}{ccc} \mu_i^{\vee} \otimes id : B_{i+1}^{\vee} \otimes \mathbb{Z}_2 & \longrightarrow & B_i^{\vee} \otimes \mathbb{Z}_2 \\ \downarrow & & \downarrow \\ \partial_i : B'^{\vee}_{i+1} & \longrightarrow & B'^{\vee}_i \end{array} \tag{3.66}$$

where $B_{i+1}^{\vee} \otimes \mathbb{Z}_2 \cong B'^{\vee}_{i+1}$ and $B_i^{\vee} \otimes \mathbb{Z}_2 \cong B'^{\vee}_i$. Thus

$$\ker(\mu_i^{\vee} \otimes id) \cong \ker(\partial_i) \tag{3.67}$$

and

$$\operatorname{coker}((\mu_i^{\vee} \otimes id) \cong \operatorname{coker}(\partial_i). \tag{3.68}$$

From (3.61), the sequences

$$0 \to B_{i+1}^{\vee} \to B_{i+1}^{\vee} \to B_{i+1}^{\vee} \otimes \mathbb{Z}_2 \to 0 \tag{3.69}$$

and

$$0 \to B_i^{\vee} \to B_i^{\vee} \to B_i^{\vee} \otimes \mathbb{Z}_2 \to 0 \tag{3.70}$$

are exact sequences. We can use the Snake Lemma (Lemma 2.2.6) with (3.70) and (3.69), via the map $\mu_i^{\vee} : B_{i+1}^{\vee} \to B_i^{\vee}$ as α and β in Lemma 2.2.6 and $\mu_i^{\vee} \otimes id : B_{i+1}^{\vee} \otimes \mathbb{Z}_2 \to B_i^{\vee} \otimes \mathbb{Z}_2$ as γ, forming the diagram:

$$\begin{tikzcd} 0 \arrow[r] & B_{i+1}^{\vee} \arrow[r] \arrow[d, "\mu_i^{\vee}"] & B_{i+1}^{\vee} \arrow[r] \arrow[d, "\mu_i^{\vee}"] & B_{i+1}^{\vee} \otimes \mathbb{Z}_2 \arrow[r] \arrow[d, "\mu_i^{\vee} \otimes id"] & 0 \\ 0 \arrow[r] & B_{i+1}^{\vee} \arrow[r] & B_{i+1}^{\vee} \arrow[r] & B_{i+1}^{\vee} \otimes \mathbb{Z}_2 \arrow[r] & 0 \end{tikzcd} \tag{3.71}$$

The long exact sequence

$$0 \to \ker(\mu_i^{\vee}) \xrightarrow{\cdot 2} \ker(\mu_i^{\vee}) \xrightarrow{f^*} \ker(\mu_i^{\vee} \otimes id) \xrightarrow{\delta^*}$$

$$\operatorname{coker}(\mu_i^{\vee}) \xrightarrow{\cdot 2} \operatorname{coker}(\mu_i^{\vee}) \to \operatorname{coker}((\mu_i^{\vee} \otimes id) \to 0 \tag{3.72}$$

or

$$0 \to \ker(\mu_i^\vee) \xrightarrow{\cdot 2} \ker(\mu_i^\vee) \xrightarrow{f} H_i(\Delta(i), \mathbb{Z}_2) \xrightarrow{\delta} \operatorname{coker}(\mu_i^\vee) \xrightarrow{\cdot 2} \operatorname{coker}(\mu_i^\vee) \to \operatorname{coker}(\partial_i) \to 0 \tag{3.73}$$

is the sequence obtained from this lemma. Notice that the third module of the sequence, $\ker(\mu_i^\vee \otimes id) \cong \ker(\partial_i)$ (from (3.67)) is written as $H_i(\Delta(i), \mathbb{Z}_2)$ because $\ker(\partial_i) \cong H_i(\Delta(i), \mathbb{Z}_2)$, as we showed in (3.54). We also replaced the last module $\operatorname{coker}((\mu_i^\vee \otimes id)$ with $\operatorname{coker}(\partial_i)$, as in (3.68).

Suppose the connecting homomorphism δ is injective, then $\ker(\delta) = 0 = \operatorname{im}(f)$ because the sequence is exact. By (3.73), we get the exact sequence

$$0 \to \ker(\mu_i^\vee) \xrightarrow{\cdot 2} \ker(\mu_i^\vee) \xrightarrow{f} H_i(\Delta(i), \mathbb{Z}_2). \tag{3.74}$$

Since $\operatorname{im}(f) = 0$ and this sequence is exact, then $2 \cdot \ker(\mu_i^\vee) = \ker(f) = \ker(\mu_i^\vee)$. Thus

$$2 \cdot \ker(\mu_i^\vee) = \ker(\mu_i^\vee). \tag{3.75}$$

Because $\ker(\mu_i^\vee)$ is a submodule of a free module over a principle ideal domain, then $\ker(\mu_i^\vee)$ is free [24, Theorem 6.1 on Page 218]. Hence, $2 \cdot \ker(\mu_i^\vee) = \ker(\mu_i^\vee)$ implies $\ker(\mu_i^\vee) = 0$, as this is the only finite-dimensional free $\mathbb{Z}$-module F for which $2 \cdot F = F$ holds. Therefore, because $\ker(\mu_i^\vee) = 0$, then $\mu_i^\vee$ is injective, so μ_i is surjective.

Suppose, conversely, that μ_i is surjective, and therefore $\mu_i^\vee$ is injective, then $\ker(\mu_i^\vee) = 0$ and our exact sequence from (3.73):

$$0 \to \ker(\mu_i^\vee) \xrightarrow{\cdot 2} \ker(\mu_i^\vee) \to H_i(\Delta(i), \mathbb{Z}_2) \xrightarrow{\delta} \operatorname{coker}(\mu_i^\vee) \xrightarrow{\cdot 2} \operatorname{coker}(\mu_i^\vee) \to \operatorname{coker}(\partial_i) \to 0 \tag{3.76}$$

becomes

$$0 \to 0 \to 0 \to H_i(\Delta(i), \mathbb{Z}_2) \xrightarrow{\delta} \operatorname{coker}(\mu_i^\vee) \xrightarrow{\cdot 2} \operatorname{coker}(\mu_i^\vee) \to \operatorname{coker}(\partial_i) \to 0. \tag{3.77}$$

Hence, δ is injective, just as desired. In both cases, since we are working over $\mathbb{Z}$-modules, then we can localize each module to $\mathbb{Q}$ and keep each sequence exact. Hence, over characteristic zero this proposition holds.

Suppose that $f_i \leq f_{i-1}$. Then in order for the WLP to hold μ_i must be surjective, but we already showed this is equivalent to δ being injective. Thus the WLP succeeds or fails in degree $i \leq 0$ when δ is either injective or surjective, respectively. $\square$

Example 3.3.12. Let Δ be the 3-cycle, $\Delta = \langle xy, xz, yz \rangle$, for which

$$I = (x^2, y^2, z^2, xyz) \tag{3.78}$$

is its Stanley-Reisner ideal. Observe the matrix of μ_2:

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}. \tag{3.79}$$

So I_Δ, where $I = I_\Delta + (x_1^2, \ldots, x_n^2)$ as in (2.47), is not generated by quadratics and Δ also fulfils Proposition 3.3.11 because μ_2 is surjective. Alternatively, $H_2(\Delta(2), \mathbb{Z}_2) = 0$, meaning δ is injective (see Equation (3.59)). We can also infer from Proposition 3.3.11 that μ_2 is surjective.

Example 3.3.13. Let $I = (a^2, b^2, c^2, d^2, e^2, de, abc)$, which is the squarefree Stanley-Reisner ring of $\Delta = \langle adc, adb, dbc, ace, abe, bce \rangle$. In degree 2 the map μ has the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.80}$$

which row reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix} \tag{3.81}$$

and so the surjectivity holds in degree 2. The surjectivity of μ_2 is not trivial since

$$H_2(\Delta(2), \mathbb{Z}_2) \cong \mathbb{Z}_2 \neq 0. \tag{3.82}$$

If

$$H_2(\Delta(2), \mathbb{Z}_2) \cong 0 \tag{3.83}$$

then by (3.59) we could conclude that δ is injective. If δ is injective, then Proposition 3.3.11 would imply that μ_2 is surjective also.

And so, by now the reader should have an understanding of the WLP and how it relates to squarefree Stanley-Reisner rings. As we move forward, we will explore the context surrounding the WLP and Artinian analogues of Stanley-Reisner rings. In Chapter 4, we will examine the wider context of the WLP in more depth and then introduce the concept of *levelness*, specifically its connection to the WLP, along with a new notation which generalizes squarefree Stanley-Reisner rings. In Chapter 5, instead of taking a simplicial complex Δ and constructing from it a graded ring, we will examine how one may "naturally" move in the opposite direction: from a graded ring to a ring which may be interpreted as the squarefree Stanley-Reisner ring of some simplicial complex. We will do this by looking at something called *polarization*, which in effect is the process of taking a monomial ideal I of S and constructing a new ideal I' in a new polynomial ring S' which is generated by squarefree monomials.

Chapter 4

Bridging the Gap

In this chapter we will explain the g-conjecture and its connection to the WLP. Then, we will introduce new notation which will be useful to us later, and we will also introduce the concept of levelness and its connection to the WLP as a useful tool when trying to decide whether an Artinian ring has the WLP. In the last section, we will explore some methods of constructing level algebras with the WLP in order to motivate Chapter 5. In Chapter 5, we will introduce and explore some operations aimed at taking a given monomial ideal J and constructing from it a level Artinian quotient ring and a squarefree Stanley-Reisner ring.

4.1 The g-Conjecture

Our ultimate goal in studying the WLP, levelness, Stanley-Reisner rings, and the like, is to understand the structure of the Hilbert function and its associated concepts such as the f-vector and the h-vector. The WLP guarantees structure for the h-vector of a level Artinian graded algebra in that it must necessarily be unimodal because of Proposition 3.1.9. All this is due to the injectivity and surjectivity of the graded components of μ with the WLP, as seen in Section 3.1.6.

There is another vector which is studied called the *g-vector*. For a simplicial complex Δ of dimension d the g-vector is defined by:

$$g_i = h_i - h_{i-1} \quad \text{for} \quad 0 \leq i \leq \lfloor d/2 \rfloor \tag{4.1}$$

where the h_i are coördinates from the h-vector. The g-vector is interesting because of the upcoming Conjecture 4.1.1, which was originally proposed by Peter McMullen in 1971 [29]. It is important to note that the h-vector used to construct the g-vector is the h-vector of the Stanley-Reisner ring (Section 2.3), not the h-vector of the *squarefree* Stanley-Reisner ring (Section 2.3 also). To find the h-vector of a Stanley-Reisner ring, we can use the definition given in Section 2.2, equation (2.37), or the combinatorial version (for a $(d-1)$-dimensional simplicial complex)[20, Page 15]:

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i}. \tag{4.2}$$

Alternatively, there is a formula which gives the value of h_i in terms of the f-vector directly [8, Page 213]

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}. \tag{4.3}$$

We can apply (4.2) and (4.3) to construct the h-vector and g-vector corresponding to any vector $(f_1, \ldots, f_d)$ of non-negative integers so that

$$g_i = h_i - h_{i-1} \quad \text{for} \quad i \leq \lfloor d/2 \rfloor. \tag{4.4}$$

Putting (4.3) and (4.4) together, we get

$$g_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1} - \sum_{j=0}^{i-1} (-1)^{(i-1)-j} \binom{d-j}{(i-1)-j} f_{j-1}. \tag{4.5}$$

Conjecture 4.1.1 (McMullen [29])**.** A vector $(f_1, \ldots, f_d)$ of d non-negative integers is the f-vector of a simplicial complex Δ which is homeomorphic to a sphere, i.e. $\Delta \simeq \mathbb{S}^{d-1}$, if and only if its g-vector, calculated as in (4.4) and (4.5), encodes the Hilbert series of some graded quotient ring S.

There are many different $(d-1)$-spheres, with a lot of variety, and there are infinitely many ways to approximate a sphere with simplicial complexes. A simplicial complex which is homeomorphic to a sphere is called a *triangulation* of that sphere. The WLP comes in from the fact that it is a sufficient condition to say that the g-conjecture holds in cases where Δ has the WLP, or SLP [39]. As the WLP is thought to be common, this is good evidence that the conjecture is true. Richard P. Stanley also proposed the g-conjecture for *Gorenstein* simplicial complexes in the same paper [39], which is a special case of the concept of levelness that we will learn about in the next section. For a class of triangulated spheres called *simplicial polytopes*, the conjecture was proven by Richard Stanley [39], and by corollary the sufficiency of the conjecture followed [4, Theorem 1]; necessity remains to be proven. A summary of the progress toward the g-conjecture can be found in [2]. Adipriasito has published a preprint of a proof for the g-conjecture in full in 2018 [1] and it is still under review.

4.2 The Beginning of Levelness

The notation in this section expands the notation given in Section 2.3. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field of characteristic zero and let Δ be a simplicial complex with vertices $x_1, \ldots, x_n$. Let $\sigma \subseteq \{x_1, \ldots, x_n\}$ and define

$$I_\Delta = (M(\sigma) \mid \sigma \notin \Delta) \tag{4.6}$$

where M is the function with the assignment

$$M : \{x_{i,1}, \ldots, x_{i,k}\} \mapsto x_{i,1} \ldots x_{i,k} \tag{4.7}$$

as in (2.43). This I_Δ is called the *Stanley-Reisner ideal* of Δ. For a list of positive integers $a_1, \ldots, a_n \geq 2$ we denote

$$A(\Delta, a_1, \ldots, a_n) = S/(I_\Delta + (x_1^{a_1}, \ldots, x_n^{a_n})). \tag{4.8}$$

This notation comes from [41], and is well suited to its purpose and thus replicated here. We will use this notation more in Chapter 5.

For an Artinian algebra $A = A(\Delta, a_1, \ldots, a_n)$ the *socle* of A, denoted $\mathrm{Soc}(A)$, is the annihilator of the maximal ideal $(x_1, \ldots, x_n)$ in A. That is,

$$\mathrm{Soc}(A) = \{r \in A \mid r(x_1, \ldots, x_n) = 0\} \tag{4.9}$$

or equivalently

$$\mathrm{Soc}(A) = ((0) : (x_1, \ldots, x_n)). \tag{4.10}$$

Since A is a graded algebra, $\mathrm{Soc}(A)$ is graded with an associated *socle vector*

$$s = (s_0, \ldots, s_e) \quad \text{where} \quad s_i = \dim(\mathrm{Soc}(A)_i). \tag{4.11}$$

When

$$s = (0, 0, \ldots, t) \tag{4.12}$$

we say that A is *level* or *level of type t* [41, Section 2].

Definition 4.2.1. [16, Page 191] An alternate definition of levelness, one where the ring A need not be Artinian, is that A is level when the module in the final step in its minimal free resolution is a sum of free modules of the same degree.

For example, if the final step of A's free resolution is $A(-5)^3$, then it is level of type 3, but $A(-3) \oplus A(-7)^2$ is in the final step would mean A is not level. We will see this in Example 4.2.2.

Since $A(\Delta, a_1, \ldots, a_n)$ is not necessarily a squarefree Stanley-Reisner ring, it does not necessarily have the same h-vector as a squarefree Stanley-Reisner ring. Since $A(\Delta, a_1, \ldots, a_n)$ is Artinian, we have

$$A(\Delta, a_1, \ldots, a_n) = A = A_0 \oplus A_1 \oplus \cdots \oplus A_d. \tag{4.13}$$

So, the Hilbert series of $A(\Delta, a_1, \ldots, a_n)$ is a polynomial in $\mathbb{Z}[t]$ with a finite number of terms and positive coefficients. Thus, 1 is not a root of this polynomial and by (2.37) the coefficients of $\mathrm{HS}(A)$ form the h-vector of A. In the case where $a_1 = a_2 = \cdots = a_n = 2$, where A is a squarefree Stanley-Reisner ring, we know from Lemma 2.3.9 that

$$\mathrm{HS}(A) = \sum_{i=0}^{d} f_{i-1} t^i \tag{4.14}$$

where $(f_0, f_1, \ldots, f_d)$ is the f-vector of Δ. In other words

$$f_{i-1} = \dim(A_i) \quad \text{for} \quad 0 \leq i \leq d \tag{4.15}$$

because the monomials of A are exactly $M(\sigma)$ where $\sigma \in \Delta$. When $a_1 \geq 2, \ldots, a_n \geq 2$ and for some $1 \leq j \leq n$, $a_j > 2$, we have

$$f_{i-1} \leq h_i = \dim(A_i) \quad \text{for} \quad 0 \leq i \leq d \tag{4.16}$$

since A contains every $M(\sigma)$, for $\sigma \in \Delta$ like in the case where $a_1 = a_2 = \ldots = a_n$, but additionally A includes the monomials x_j^2 and possibly other monomials divisible by x_j^2.

If A is level of type t, then t equals h_e, where $(h_0, \ldots, h_e)$ is the h-vector of $A(\Delta, a_1, \ldots, a_n)$. This is because $A(\Delta, a_1, \ldots, a_n)$ is Artinian and h_e represents the number of monomials of the highest degree in $A(\Delta, a_1, \ldots, a_n)$, thus

$$A_e \cdot (x_1, \ldots, x_n) = 0 \tag{4.17}$$

since $A_e \cdot (x_1, \ldots, x_n) \subseteq A_{e+1} = 0$ and $A_e = ((0) : (x_1, \ldots, x_n))$.

Example 4.2.2. Let $S = k[x,y,z]$ and $\Delta = \langle\{x,y\},\{x,z\},\{y,z\}\rangle$. The Stanley-Reisner ideal I_Δ is (xyz), so $A(\Delta,2,2,2) = S/(xyz,x^2,y^2,z^2)$. The betti table of $A(\Delta,2,2,2)$ is

	0	1	2	3
total:	1	4	6	3
0:	1	.	.	.
1:	.	3	.	.
2:	.	1	6	3

Thus, $A(\Delta,2,2,2)$ is level of type 3, and its socle vector is $(0,0,3)$.

Alternatively, we can calculate directly from the annihilator of the ideal (x,y,z). Consider $x+y+z \in (x,y,z)$. If $r \in \mathrm{Ann}((x,y,z))$ we know that $xr = yr = zr = 0$. If $deg(r) < 2$ this is impossible, since every monomial of A of degree ≤ 2 except x^2, y^2, z^2, and xyz is nonzero in A. There is no r among these monomials which satisfies $xr = yr = zr = 0$. For $deg(r) = 2$ there are three monomials in $A(\Delta 2,2,2)$, namely xy, xz, and yz, and each fulfil $r(x,y,z) = 0$ (this is the ideal (x,y,z)). Thus, $s = (0,0,3)$. In an Artinian ring the socle vector is easily calculated by using this process of elimination, though it may take time to check each monomial.

The following proposition is the generalization of Proposition 3.1.3 and 3.1.4.

Proposition 4.2.3 (Proposition 2.1 in [36])**.** *Let* $S = k[x_1,\ldots,x_n]$ *where* $\mathrm{char}(k) = 0$. *Let* $A = S/I$ *be an Artinian graded quotient ring, and let* $l = x_1 + \cdots + x_n$. *If* $i \geq 0$, *then:*

1. *The map* μ_d *is surjective for some* d, *then* μ_i *is surjective for all* $i \geq d$.

2. *If* S/I *is level and the map* μ_d *is injective for some* d, *then* μ_i *is injective for all* $i \leq d$.

3. *If* S/I *is level and* $\dim(A_d) = \dim(A_{d+1})$ *for some* $d \geq 0$, *then* A *has the WLP if and only if* μ_d *is injective.*

Because of Proposition 4.2.3, if $A(\Delta,a_1,\ldots,a_n)$ is level, or more generally $A = S/I$ as in the statement of the proposition is level, then we only need to check the rank of one map in order to determine whether $A(\Delta,a_1,\ldots,a_n)$ has the WLP. Thus, levelness is a desirable property to explore when pursuing the WLP.

4.3 For Want of a Level Algebra

Level algebras originated in a combinatorics paper by Richard Stanley [38] [16, Section 1], and have since found appreciation and publication in both combinatorics and commutative algebra. Like the WLP, levelness is a common property of Artinian algebras that places restrictions on the h-vector of an Artinian ring A. The harshness of these restrictions can range from characterizations of classes of h-vectors which are of level type 2 such as in [42], or complete characterizations for broad classes of Gorenstein (level type 1) algebras [18]. There are also characterizations of the h-vector for specialized classes of rings such as *generically Gorenstein* [20, Theorem 4.4.9], or bounds on the h-vectors of rings of certain *socle degree* [12, Section 4]. The Gorenstein and level cases are an active area of research within the study of Hilbert functions. Methods of constructing classes of level algebras are desirable, and the property is common and well established in the literature, so we will explore this in Chapter 5.

In the next chapter we will explore a process on monomial ideals called polarization with focus on constructing level algebras with an analogous process which yields an Artinian ring. This is ultimately inspired by the previous chapter on squarefree Stanley-Reisner rings. Stanley-Reisner theory is about squarefree monomial ideals, and polarization constructs a squarefree monomial ideal from a monomial ideal which is not squarefree, so this motivates our search for an Artinian analogue to polarization. Here we will view some choice methods for constructing level algebras with the WLP from [17] in preparation for Chapter 5.

For the following propositions, their original proofs were done constructively, meaning that they include a method which perfectly describes a new object, such as an Artinian level algebra with the WLP, in terms of another object which is known to exist.

Proposition 4.3.1 (Proposition 5.16 of [17])**.** *Let* $h = (1, h_1, \ldots, h_{s-2}, h_{s-1}, 1)$ *be a symmetric h-vector of some polynomial quotient ring of a monomial ideal. Let* $(1, h_1 - 1, h_2 - h_1, \ldots, h_{\lfloor s/2 \rfloor} - h_{\lfloor s/2 \rfloor - 1})$ *be also the h-vector of some polynomial quotient ring of a monomial ideal. Then for any integer v where* $0 < v \leq s$*, we have* $(1, h_1, \ldots, h_v)$ *is the h-vector of an Artinian level algebra with the WLP.*

This gives a nice corollary as well, which follows quite readily from Proposition 4.3.1.

Corollary 4.3.2 (Corollary 5.17 of [17]). *Let $h = (1, h_1, \ldots, h_s)$ be the h-vector of a polynomial quotient ring of a monomial ideal for which $(1, h_1 - 1, \ldots, h_s - h_{s-1})$ is again the h-vector of a polynomial quotient ring of a monomial ideal. Then h is the h-vector of an Artinian level algebra with the WLP.*

This next method uses the colon ideal from earlier in (2.16), it also describes the Hilbert function explicitly.

Proposition 4.3.3 (Proposition 5.18 of [17]). *Let A be a level Artinian polynomial quotient ring of a monomial ideal with the WLP. Let*

$$t = \min\{i \mid H(i) \geq H(i+1)\} \quad \textit{and} \quad u = \min\{i \mid H(i) > H(i+1)\}. \tag{4.18}$$

Let d be an integer $0 \leq d \leq u - t$. Then

$$B = A/(0 : l^d) \quad \textit{where} \quad l = x_1 + \cdots + x_n \tag{4.19}$$

is a level algebra with the WLP.

The Hilbert function of B is

$$H_B(i) = H_A(i) \tag{4.20}$$

when $i = 0, \ldots, u - d$ and

$$H_B(i) = H_A(i + d) \tag{4.21}$$

when $i = u - d + 1, \ldots, s - d$.

Example 4.3.4. Let $\Delta = \langle xyzw \rangle$, a tetrahedron, be a simplicial complex and let $A = k[x, y, z, w]/(x^2, y^2, z^2, w^2)$ be its squarefree Stanley-Reisner ideal with Lefschetz element $l = x + y + z + w$. The simplicial complex Δ was chosen carefully to illustrate a nontrivial case of Proposition 4.3.3 such that $t \neq u$. If $t = u$ then

$$B = A/(0 : l^0) = A/(0 : 1) = A \tag{4.22}$$

which is trivial. The proposition is much more interesting when $t \neq u$. This Δ is one of the smallest cases where $t \neq u$, and Δ is easily recognized as a tetrahedron with the WLP. The h-vector of A is $(1, 4, 6, 4, 1)$, so by Proposition 4.3.3 we have $t = 1$ and $u = 2$. Let $d = 1$ so that $0 \leq d \leq 1 = u - t$ as in Proposition 4.3.3, then using a computer program called Macaulay2 we compute:

$$B = A/(0 : l^d) = A/(0 : l) = A/(xz - yz - xw + yw, xy - yz - xw + zw) \tag{4.23}$$

is an Artinian quotient ring with the WLP. We can also compute the Hilbert Series:

$$\mathrm{HS}(B) = 1 + 4t + 4t^2 + t^3 \tag{4.24}$$

so the h-vector is $(1, 4, 4, 1)$. One may check this against the statement of Proposition 4.3.3 and find that

$$H_B(0) = H_A(0) = 1 \tag{4.25}$$

and

$$H_B(1) = H_A(1) = 4 \tag{4.26}$$

for $i \leq u - d = 2 - 1 = 1$. After $i > u - d = 1$, we have

$$H_B(2) = H_A(2 + d) = H_A(2 + 1) = H_A(3) = 4 \tag{4.27}$$

and

$$H_B(3) = H_A(3 + d) = H_A(3 + 1) = H_A(4) = 1 \tag{4.28}$$

as desired.

Chapter 5

Artinian Reductions and levelness

We will now examine the question as to whether or not there is an operation on monomial ideals which produces level algebras. For a polynomial ring $S = k[x_1, \ldots, x_n]$ and a monomial ideal I, we will introduce two operations: Linear Artinian Reduction (LAR) and Polarized Linear Artinian Reduction (PLAR) to construct new monomial quotient rings from S/I. The rings LAR(I) and PLAR(I) would hopefully preserve some properties of the ring S/I as a transformation of S/I into an Artinian ring. We establish that the property of levelness of S/I is not preserved in LAR(I) and PLAR(I), as shown with observations in Examples 5.1.11 through 5.1.16. We also develop some tools for studying PLAR, LAR, and their levelness.

Because, we will see, LAR and PLAR are intuitively defined operations, we will explore whether they can be adapted to preserve levelness, or even whether they can be partially reversed or related along the lines of levelness. This also fails, yet fails conclusively and is therefore of interest.

The operations LAR and PLAR are an attempt to take a monomial ideal I and produce from it a ring PLAR(I) which by construction is the squarefree Stanley-Reisner ring of some simplicial complex Δ, and LAR(I) which is an "Artinian version of I". By itself, LAR(I) is a stepping stone to PLAR(I), but also a method of taking a given monomial ideal I and constructing an Artinian monomial quotient ring LAR(I) which is generated by a finite subset of the generators of S/I.

Studying squarefree Stanley-Reisner rings may be thought of as studying how a geometric object such as a simplicial complex Δ may be related as an algebraic object such as a monomial quotient ring S/I and its ideal I. Conversely, PLAR and LAR are the opposite direction, we seek to start with an ideal I and construct a squarefree Stanley-Reisner ring and an Artinian monomial quotient ring from it.

5.1 P(LAR)

Let $S = k[x_1, \ldots, x_n]$, where k is a field of characteristic zero. A *pure power* monomial is a monomial in the form $x_i^{i_j}$, where $i_j \geq 2$. Let I be a monomial ideal of S which is minimally generated by $n_1, \ldots, n_s, m_1, \ldots, m_q$, where $n_1, \ldots, n_s$ are *pure power* monomials and $m_1, \ldots, m_q$ are not pure powers. For $1 \leq i \leq n$ and a monomial $m \in A$, define

$$\deg_{x_i} m = \max\{r \in \mathbb{N} \mid x_i^r | m\}. \tag{5.1}$$

For each $1 \leq i \leq n$, let

$$a_i = \max\{\deg_{x_i} m_1, \ldots, \deg_{x_i} m_q\} \tag{5.2}$$

then $\mathrm{LAR}(I) = S/(I + I')$ where

$$I' = (x_i^{a_i+1} \mid i \neq n_j \ \forall 1 \leq j \leq s, \quad \text{where} \quad 1 \leq i \leq n). \tag{5.3}$$

The ring $\mathrm{LAR}(I)$ is called the *limited Artinian reduction* of I. LAR is designed to be an operation on an ideal I which makes the resultant quotient module Artinian while preserving the monomial generators of I.

Example 5.1.1. Consider a polynomial ring $S = k[x, y, z, w]$ and an ideal

$$I = (xy, y^2zw, w^4). \tag{5.4}$$

Then by definition

$$\mathrm{LAR}(I) = S/(xy, y^2zw, x^2, y^3, z^2, w^4). \tag{5.5}$$

This definition is similar to the squarefree Stanley-Reisner rings of (2.47), which we have used up until this point. When I is generated by a system of squarefree monomials, $\mathrm{LAR}(I)$ is the same as a squarefree Stanley-Reisner ring. In a sense, LAR is a generalization of squarefree Stanley-Reisner rings.

Let m be a monomial $m = \prod_{i=1}^{n} x_i^{q_i}$. The *polarization* [20, Section 1.6] of m is

$$\mathcal{P}(m) = \prod_{i=1}^{n} x_{i,1}x_{i,2} \ldots x_{i,q_i}. \tag{5.6}$$

For a monomial ideal $I = (u_1, \ldots, u_r)$, where:

$$u_i = \prod_{j=1}^{n} x_j^{q_{i,j}}, \tag{5.7}$$

for $1 \leq i \leq r$ in a ring $S = k[x_1, \ldots, x_n]$, where $q_{i,j} \in \mathbb{N}$ for $1 \leq j \leq n$, let

$$q_j = \max\{q_{1,j}, \ldots, q_{q,j}\}. \tag{5.8}$$

Then the *polarization of the ideal* I is

$$\mathcal{P}(I) = (\mathcal{P}(u_1), \ldots, \mathcal{P}(u_r)) \tag{5.9}$$

as an ideal of $\mathcal{P}(S) = k[x_{1,1}, \ldots, x_{1,q_1}, \ldots, x_{n,1} \ldots, x_{n,q_n}]$.

For an ideal I define

$$PLAR(I) = \mathcal{P}(S)/(\mathcal{P}(I) + (x_{i,j}^2 | x_{i,j} \in \mathcal{P}(S))). \tag{5.10}$$

This is the *Polar Limited Artinian Reduction* of I. Otherwise stated:

$$PLAR(I) = \mathrm{LAR}(\mathcal{P}(I)). \tag{5.11}$$

Example 5.1.2. Let $S = k[x, y, z]$ and $I = (x^2y, zy^3)$. Then:

$$LAR(I) = S/(x^2y, zy^3, x^3, y^4, z^2). \tag{5.12}$$

Likewise $\mathrm{PLAR}(I) = k[x_1, x_2, y_1, y_2, y_3, z_1]/(x_1x_2y_1, z_1y_1y_2y_3, x_1^2, x_2^2, y_1^2, y_2^2, y_3^2, z_1^2)$

We will here invoke the notation used in Theorem 2.2.4. For $S = k[x_1, \ldots, x_n]$ and a monomial ideal I the quotient module S/I may have several graded free resolutions, but the graded free resolution of minimal length is unique [37, Theorem 7.5]. Let F be this minimal graded free resolution. The indices $c_{i,p}$ of

$$F_i = \oplus_{p \in \mathbb{Z}} R^{c_{i,p}}(-p) \tag{5.13}$$

are denoted $\beta_{i,p}(I)$, and called *Betti numbers*, when discussing the minimal graded free resolution F.

Regarding the question of levelness for an ideal I and its polarization $\mathcal{P}(I)$, we have the following theorem:

Theorem 5.1.3 (Special Case of Corollary 1.6.3 of [20])**.** *If I is a monomial ideal, then $\beta_{i,j}(I) = \beta_{i,j}(\mathcal{P}(I))$ for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.*

In this context, we note:

Corollary 5.1.4. *I is level if and only if its polarization $\mathcal{P}(I)$ is level.*

Proof. Let I be an ideal and $\mathcal{P}(I)$ be its polarization. By Definition 4.2.1, if I is level there is exactly one $j > 0$ such that $\beta_{p,j}(I) \neq 0$, where p is the projective dimension of I; in other words, $S(-j)^{\beta_{p,j}(I)}$ is the final module in the minimal graded free resolution of S/I. By Theorem 5.1.3, we have $\beta_{i,j}(I) = \beta_{i,j}(\mathcal{P}(I))$ for all pairs (i, j), so $\mathcal{P}(I)$ is necessarily level also since $S(-j)^{\beta_{p,j}(I)}$ is the final module in the minimal graded resolution of $\mathcal{P}(I)$. Conversely, if $\mathcal{P}(I)$ is level, the same argument applies since $\beta_{i,j}(I) = \beta_{i,j}(\mathcal{P}(I))$ for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.

□

This fact about polarization is what motivates our study of PLAR(I). Since polarization by itself preserves levelness, this thesis seeks an analogous operation which preserves polarization, or something like it, when we take LAR(I).

We move on to the question of levelness between LAR(I) and PLAR(I). The next theorem, Theorem 5.1.5, first appeared in Mats Boij's dissertation in Sweden in 1994, but another proof by Adam Van Tuyl and Fabrizio Zanello in 2009 proves Theorem 5.1.5 incidentally, as they noted [41]. We will see the theorem of Van Tuyl and Zanello as Theorem 5.2.4 later on.

We say that a simplicial complex Δ is *pure* when each facet of Δ has the same dimension

Theorem 5.1.5 (Originally [5], Corollary 4.3 of [41])**.** *Let Δ be a simplicial complex with vertices $x_1, \ldots, x_n$, let $S = k[x_1, \ldots, x_n]$, and let $I_\Delta = (x_{i_1} \ldots x_{i_j} \mid \{x_{i_1} \ldots x_{i_j}\} \notin \Delta)$ be its Stanley-Reisner ideal. Then $S/(I_\Delta + (x_i^2 \mid i \in \{1, \ldots, n\}))$ is level if and only if Δ is pure.*

Example 5.1.6. If S remains the same as the previous example but
$I = (xy, yz, x^2, y^2, z^2)$ as with LAR(I) above, then LAR(I) $= I$, which is not level. We have PLAR(I) in $\mathcal{P}(S) = k[x_1, x_2, y_1, y_2, z_1, z_2]$ and

$$\mathrm{PLAR}(I) = \mathcal{P}(S)/((x_1z_1, y_1z_1, x_1x_2, y_1y_2, z_1z_2, x_1^2, x_2^2, y_1^2, y_2^2, z_1^2, z_2^2)). \tag{5.14}$$

One can perform the computation, or construct the underlying simplicial complex to show that PLAR(I) is level (its simplicial complex is pure even though the one from ob1ained by treating LAR(I) as a squarefree Stanley-Reisner ring is not).

When writing the Stanley-Reisener ideal of a simplicial complex Δ, which is constructed by taking the minimal non-faces of Δ, consider the ideal I' which is generated by the monomials which correspond to the facets of Δ. This is the *facet ideal* of Δ. For example, the simplicial complex $\langle xy, yz, xz\rangle$ has the facet ideal (xy, yz, xz). Thus, every minimally generated squarefree monomial ideal $I = (m_1, \ldots, m_q)$, for $q \geq 1$, has a corresponding *facet complex* $\Delta' = \langle M^{-1}(m_1), \ldots, M^{-1}(m_q)\rangle$ (M^{-1} is the inverse of the function described in (2.43)). If Δ is a simplicial complex and S/I is its squarefree Stanley-Reisner ring, then we can construct a facet complex Δ' with the ideal I_Δ of minimal nonfaces of Δ. For a simplicial complex Δ, a *vertex cover* is a set K of vertices such that for any facet F of Δ it is true that $K \cap F \neq \emptyset$. The following lemma uses a simplicial complex whose facet complex is given:

Lemma 5.1.7 (Proposition 2.4 in [14])**.** *Let Δ be a simplicial complex and let Δ' be the facet complex of the monomial generators of I_Δ. Then, for every minimal vertex cover K of Δ', the set K^c (the complement of K) is a facet of Δ. In other words:*

$$\Delta = \langle K^c \mid \textit{Where } K \textit{ is a minimal vertex cover of } \Delta'\rangle. \tag{5.15}$$

Corollary 5.1.8. *Let Δ and Δ' be as in Lemma 5.1.7. Then Δ is pure if and only if every minimal vertex cover of Δ' has the same cardinality.*

Proof. Let K be a minimal vertex cover of Δ' and suppose Δ has n vertices. By Lemma 5.1.7, K^c is a facet of Δ. If all facets of Δ have the same dimension k, meaning Δ is pure, then each minimal vertex cover K must have $n-(k+1)$ vertices, and so they all have the same number. Conversely, if all vertex covers of Δ' have the same number of vertices, then any minimal vertex cover K has $k \in \mathbb{N}$ vertices, so the facet K^c of Δ has $n-k$ vertices. Therefore, each facet of Δ has the same number of vertices. □

The following gives us a convenient method to construct simple examples where $\mathrm{PLAR}(I)$ is level.

Corollary 5.1.9. *If a monomial ideal I of $S = k[x_1, \ldots, x_n]$ is principal (generated by a single monomial), then* $\mathrm{PLAR}(I)$ *is level.*

Proof. Let I be principal; by definition,

$$\mathrm{PLAR}(I) = \mathcal{P}(S)/(\mathcal{P}(I) + (x_{i,j}^2 | x_{i,j} \in \mathcal{P}(S))) \tag{5.16}$$

and $\mathcal{P}(I)$ is generated by a single squarefree monomial $x_{b_1} \dots x_{b_q}$. Thus, the corresponding facet complex is $\langle\{x_{b_1}, \dots, x_{b_q}\}\rangle$ and its minimal vertex covers are $\{x_{b_1}\}, \dots, \{x_{b_q}\}$. All these vertex covers have the same cardinality, so by Corollary 5.1.8: $PLAR(I)$ is the squarefree Stanley-Reisner ring of a pure simplicial complex. Therefore, by Theorem 5.1.5, PLAR(I) is level. $\square$

Remark 5.1.10. It may be apparent to the reader that in fact we may also say that PLAR(I) is level if it is the squarefree Stanley-Reisner ring of a pure simplicial complex. This is correct, but left to Corollary 5.2.6.

This allows us to easily construct some examples which guarantee PLAR(I) is level. For Examples 5.1.11 , 5.1.12, 5.1.13, and 5.1.14 we will record the minimal free resolutions of S/I, LAR(I), and PLAR(I). We will record the linear map between steps of each sequence as a matrix when there are not "too many" rows and columns (for example, some free modules have a basis exceeding 100 elements).

Example 5.1.11. If $I = (x^2yz)$, with $S = k[x, y, z]$, then we calculate the graded free resolutions:

$$I: \ S \xleftarrow{(x^2yz)} S(-4) \leftarrow 0 \tag{5.17}$$

$$LAR(I): \ S \xleftarrow{(x^2yz, x^3, y^2, z^2)} S(-2)^2 \oplus S(-3) \oplus S(-4) \tag{5.18}$$

$$\xleftarrow{\begin{pmatrix} -z^2 & -x^3 & 0 & -x^2z & 0 & 0 \\ y^2 & 0 & 0 & 0 & -x^3 & -x^2y \\ 0 & y^2 & -yz & 0 & z^2 & 0 \\ 0 & 0 & x & y & 0 & z \end{pmatrix}} S(-3) \oplus S(-5)^5$$

$$\xleftarrow{\begin{pmatrix} 0 & 0 & x^2 \\ -z & 0 & 0 \\ -y & -z & 0 \\ x & 0 & -z \\ 0 & -y & 0 \\ 0 & x & y \end{pmatrix}} S(-6)^3 \leftarrow 0$$

$$PLAR(I): \; \mathcal{P}(S) \xleftarrow{(x_1x_2y_1z_1, x_1^2, x_2^2, y_1^2, z_1^2)} \mathcal{P}(S)(-2)^4 \oplus \mathcal{P}(S)(-4) \tag{5.19}$$

$$\leftarrow \mathcal{P}(S)(-4)^6 \oplus \mathcal{P}(S)(-5)^4$$

$$\leftarrow \mathcal{P}(S)(-6)^{10} \leftarrow \mathcal{P}(S)(-7)^4 \leftarrow 0$$

So in summary:

module:	level?:
I	Y
LAR(I)	Y
PLAR(I)	Y

Example 5.1.12. If $I = (x^2y)$, for $S = k[x,y]$, then we calculate the resolutions:

$$I: \; S \xleftarrow{(x^2y)} S(-3) \leftarrow 0 \tag{5.20}$$

$$LAR(I): \; S \xleftarrow{(x^2y, x^3, y^2)} S(-2) \oplus S(-3)^2 \xleftarrow{\begin{pmatrix} 0 & -x^2 \\ -y & 0 \\ x & y \end{pmatrix}} S(-4)^2 \leftarrow 0 \tag{5.21}$$

$$PLAR(I): \; \mathcal{P}(S) \xleftarrow{\begin{pmatrix} x_1^2 & x_2^2 & y_1^2 & x_1x_2y_1 \end{pmatrix}} \mathcal{P}(S)(-2)^3 \oplus \mathcal{P}(S)(-3) \tag{5.22}$$

$$\xleftarrow{\begin{pmatrix} -x_2^2 & -x_2y_1 & 0 & -y_1^2 & 0 & 0 \\ x_1^2 & 0 & -x_1y_1 & 0 & 0 & -y_1^2 \\ 0 & 0 & 0 & x_1^2 & -x_1x_2 & x_2^2 \\ 0 & x_1 & x_2 & 0 & y & 0 \end{pmatrix}} \mathcal{P}(S)(-4)^6$$

$$\xleftarrow{\begin{pmatrix} y_1 & 0 & 0 \\ -x_2 & -y_1 & 0 \\ x_1 & 0 & -y_1 \\ 0 & x_2 & 0 \\ 0 & x_1 & x_2 \\ 0 & 0 & x_1 \end{pmatrix}} \mathcal{P}(S)(-5)^3 \leftarrow 0$$

In summary:

module	level?
I	Y
LAR(I)	Y
PLAR(I)	Y

What if there is more than one generator?

Example 5.1.13. Let $I = (x^3yz^2, x^2y^2)$ and $S = k[x, y, z]$. This system of generators is reduced. We get:

$$I:\ S \xleftarrow{\begin{pmatrix} x^2y^2 & x^3yz^2 \end{pmatrix}} S(-4) \oplus S(-6) \xleftarrow{\begin{pmatrix} -xz^2 \\ y \end{pmatrix}} S(-7) \leftarrow 0 \tag{5.23}$$

$$LAR(I):\ S \xleftarrow{\left(x^3yz^2, x^2y^2, x^4, y^3, z^3\right)} S(-3)^2 \oplus S(-4)^2 \oplus S(-6) \tag{5.24}$$

$$\xleftarrow{\begin{pmatrix} -x^2 & 0 & -z^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y^3 & 0 & 0 & -x^4 & -x^3y & -x^2y^2 \\ 0 & -y^2 & 0 & -yz^2 & 0 & z^3 & 0 & 0 \\ y & x^2 & 0 & 0 & -xz^2 & 0 & 0 & z^3 \\ 0 & 0 & 0 & x & y & 0 & z & 0 \end{pmatrix}} S(-5) \oplus S(-6)^2 \oplus S(-7)^5$$

$$\xleftarrow{\begin{pmatrix} 0 & 0 & 0 & -z^3 \\ z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^2 \\ -y & -z & 0 & 0 \\ x & 0 & -z & 0 \\ 0 & -y & 0 & 0 \\ 0 & 0 & -x & y \end{pmatrix}} S(-8)^4 \leftarrow 0$$

$$PLAR(I): \; S \tag{5.25}$$

$$\xleftarrow{\begin{pmatrix} x_1x_2x_3y_1z_1z_2 & x_1x_2y_1y_2 & x_1^2 & x_2^2 & x_3^2 & y_1^2 & y_2^2 & z_1^2 & z_2^2 \end{pmatrix}}$$

$$\mathcal{P}(S)(-2)^7 \oplus \mathcal{P}(S)(-4) \oplus \mathcal{P}(S)(-6)$$

$$\leftarrow \mathcal{P}(S)(-4)^{21} \oplus \mathcal{P}(S)(-5)^4 \oplus \mathcal{P}(S)(-6)^3 \oplus \mathcal{P}(S)(-7)^7$$

$$\leftarrow \mathcal{P}(S)(-6)^{41} \oplus \mathcal{P}(S)(-7)^{12} \oplus \mathcal{P}(S)(-8)^{24}$$

$$\leftarrow \mathcal{P}(S)(-7)^4 \oplus \mathcal{P}(S)(-8)^{52} \oplus \mathcal{P}(S)(-9)^{47} \leftarrow$$

$$\mathcal{P}(S)(-9)^{12} \oplus \mathcal{P}(S)(-10)^{70} \oplus \mathcal{P}(S)(-11) \leftarrow \mathcal{P}(S)(-11)^{30} \oplus \mathcal{P}(S)(-12)^6$$

$$\leftarrow \mathcal{P}(S)(-12)^3 \oplus \mathcal{P}(S)(-13)^3 \leftarrow 0.$$

In summary:

module	level?
I	Y
LAR(I)	Y
PLAR(I)	N

Example 5.1.14. Let $I = (a^2, xyz^2s, b^2xw)$, then:

$$S \xleftarrow{(a^2,xyz^2s,b^2xw)} S(-2)\oplus S(-4)\oplus S(-5) \xleftarrow{\begin{pmatrix} -xyz^2 & -xb^2w & 0 \\ 0 & a^2 & -yz^2a \\ a & 0 & b^2w \end{pmatrix}}$$

$$S(-6)^2\oplus S(-8) \xleftarrow{\begin{pmatrix} -b^2w \\ yz^2 \\ a \end{pmatrix}} S(-9) \leftarrow 0$$

$$\mathrm{LAR}(I) : S \xleftarrow{(a^2,xyz^2s,b^2xw)} S(-2)^4\oplus S(-3)^2\oplus S(-4)\oplus S(-4) \leftarrow$$

$$S(-4)^6\oplus S(-5)^{11}\oplus S(-6)^7\oplus S(-7)^2\oplus S(-8)^2 \leftarrow$$

$$S(-6)^7\oplus S(-7)^{23}\oplus S(-8)^{12}\oplus S(-9)^{12} \leftarrow S(-8)^{11}\oplus S(-9)^{17}\oplus S(-10)^{31}$$

$$\leftarrow S(-10)^7\oplus S(-11)^{25}\oplus S(-12)$$

$$\leftarrow S(-12)^6\oplus S(-13) \leftarrow 0$$

$$\mathrm{PLAR}(I) : S \xleftarrow{(x_1^2,\ldots,b_1b_2x_1w_1)} S(-2)^9\oplus S(-4)^2 \leftarrow$$

$$S(-3)\oplus S(-4)^{33}\oplus S(-5)^7\oplus S(-6)^{11}\oplus S(-7) \leftarrow$$

$$S(-5)^{12}\oplus S(-6)^{74}\oplus S(-7)^{42}\oplus S(-8)^{25}\oplus S(-8)^3 \leftarrow$$

$$\oplus S(-7)^{35}\oplus S(-8)^{136}\oplus S(-9)^{89}\oplus S(-10)^{32} \leftarrow$$

$$S(-9)^{84}\oplus S(-10)^{177}\oplus S(-11)^{104} \leftarrow S(-10)^{10}\oplus S(-11)^{122} S(-12)^{170}$$

$$\leftarrow$$

$$S(-12)^{25}\oplus S(-13)^{133} \leftarrow S(-15)^6 \leftarrow 0$$

In summary:

module	level?
I	Y
LAR(I)	N
PLAR(I)	Y

As we can see, LAR(I) being level is not sufficient for PLAR(I) to be level. Whether or not I is level does not translate to PLAR(I) or LAR(I). For the sake of brevity, the following examples only include their summaries.

If $I = (x^2y^2, y^2z^2)$ and $S = k[x, y, z]$ we get:

module	level?
I	Y
LAR(I)	N
PLAR(I)	N

And moreover if $I = (x^2y^2, y^2z)$, which is reduced, then:

module	level?
I	Y
LAR(I)	N
PLAR(I)	N

Thus, as the levelness of any one of I, LAR(I), or PLAR(I) can be independent of the others, there is no hope for a connection, without caveats, between PLAR and LAR.

Example 5.1.15. Let $S = k[x, y]$ and $I = (x^3, xy, y^5)$. The module S/I has a betti table:

−	0	1	2
total:	1	3	2
0:	1	.	.
1:	.	1	.
2:	.	1	1
3:	.	.	.
4:	.	1	1

Thus this module is *not* level since the last module of the free resolution is $S(-4) \oplus S(-6)$

If $S = k[x, y, z]$ and $I = (xy, yz)$ then S/I has a betti table:

–	0	1	2
total:	1	2	1
0:	1	.	.
1:	.	2	1

In this case, not only is S/I level, since the last module is $S(-3)^1$, it is *Gorenstein*.

Example 5.1.16. The relationship between I, $\mathrm{LAR}(I)$, and $\mathrm{PLAR}(I)$ is at best complicated. Observe the case $S = k[x, y, z]$ and $I = (xy, yz)$ where as in the last example S/I is level. Observe $A_I = (x^2, y^2, z^2)$ so $\mathrm{LAR}(I) = S/(xy, yz, x^2, y^2, z^2)$, which has a betti table:

–	0	1	2	3
total:	1	5	6	2
0:	1	.	.	.
1:	.	5	5	1
2:	.	.	1	1

So $\mathrm{LAR}(I)$ is not level. Since $\mathrm{PLAR}(I) = \mathrm{LAR}(I)$ in this case it is also not level.

5.2 Levelness

We now examine how levelness is related in particular to PLAR, LAR, simplicial complexes, and their monomial ideals.

Recall the notation of $A(\Delta, a_1, \ldots, a_n)$ in (4.8), where Δ is a simplicial complex with $n \in \mathbb{N}$ vertices and $a_1 \geq 2, \ldots, a_n \geq 2$:

$$A(\Delta, a_1, \ldots, a_n) = S/(I_\Delta + (x_1^{a_1}, \ldots, x_n^{a_n})). \tag{5.26}$$

Theorem 5.2.1 (Theorem 3.2 of [41])**.** *Let Δ be a simplicial complex, and let there be positive integers $a_1, \ldots, a_n$ such that $A(\Delta, a_1, \ldots, a_n)$ has socle vector $(s_0, \ldots, s_e)$, then for $0 \leq j \leq e$:*

$$s_j = |\{\sigma \in \mathrm{Facets}(\Delta) \mid \sum_{x_i \in \sigma}(a_i - 1) = j\}|. \tag{5.27}$$

Example 5.2.2. As in Example 4.2.2, $\Delta = \langle\{x,y\},\{x,z\},\{y,z\}\rangle$. Without loss of generality, for the case $\sigma = \{x,y\}$, we calculate the sum: $(a_1-1)+(a_2-1) = (2-1)+(2-1) = 2$. Since this is the same for every facet then:

$$s_2 = |\{\{x,y\},\{x,z\},\{y,z\}\}| = 3, \tag{5.28}$$

which is indeed the case.

Example 5.2.3. Consider S as in Example 5.2.2 except $I = (xy, yz, x^2, y^2, z^2)$, as with LAR($I$) above, so LAR($I$) $= I$, which isn't level. We have PLAR(I) in $S' = k[x_1,x_2,y_1,y_2,z_1,z_2]$ and

$$PLAR(I) = S'/((x_1z_1, y_1z_1, x_1x_2, y_1y_2, z_1z_2, x_1^2, x_2^2, y_1^2, y_2^2, z_1^2, z_2^2)). \tag{5.29}$$

One can make the computation, or construct the underlying simplicial complex to show that PLAR(I) is level (its simplicial complex is pure even though the one from LAR(I) is not).

For a simplicial complex which is not pure, Theorem 5.1.5 tells us that $A(\Delta, 2, \ldots, 2)$ is not level, so we may ask if there exists $a_1, \ldots, a_n \geq 2$ such that $A(\Delta, a_1, \ldots, a_n)$ is level. If such a list $a_1, \ldots, a_n$ exists, then we say Δ is a *levelable* simplicial complex, found in [41]. The following theorem tells us when a simplicial complex is levelable.

Theorem 5.2.4 (Theorem 4.1 of [41])**.** *Let Δ be a simplicial complex with facets $F_1, \ldots, F_q$, where $F_i = \{x_{i,1}, \ldots, x_{i,d_i}\}$. Then $A(\Delta, a_1, \ldots, a_n)$, where each $a_i \geq 2$, is level if and only if $(a_1, \ldots, a_n)$ is a solution to the following system of equations:*

$$\begin{gathered}
(x_{1,1} + \cdots + x_{1,d_1}) - (x_{2,1} + \cdots + x_{2,d_2}) = d_1 - d_2 \\
(x_{2,1} + \cdots + x_{2,d_2}) - (x_{3,1} + \cdots + x_{3,d_3}) = d_2 - d_3 \\
\cdots \\
(x_{q-1,1} + \cdots + x_{q-1,d_{q-1}}) - (x_{q,1} + \cdots + x_{q,d_q}) = d_{q-1} - d_q
\end{gathered}$$

Corollary 5.2.5. *Let $S = k[x_1, \ldots, x_n]$ and let Δ be a simplicial complex on the vertices $x_1, \ldots, x_n$ and let $a \geq 2$. Then $A(\Delta, a, \ldots, a)$ is level if and only if Δ is pure.*

Proof. Let $a \geq 2$ be an integer and suppose that Δ is pure of dimension $p-1$ and has facets $F_1, \ldots, F_q$ for $p \geq 2$. Let $F_i = \{x_{i,1}, \ldots, x_{i,d_i}\}$ for $1 \leq i \leq q$ where $d_i - 1$ is the dimension of the facet F_i. Then by Theorem 5.2.4, $A(\Delta, a, \ldots, a)$ is level if

$$(x_{1,1}+\cdots+x_{1,d_1})-(x_{2,1}+\cdots+x_{2,d_2})=d_1-d_2$$
$$(x_{2,1}+\cdots+x_{2,d_2})-(x_{3,1}+\cdots+x_{3,d_3})=d_2-d_3$$
$$\cdots$$
$$(x_{q-1,1}+\cdots+x_{q-1,d_{q-1}})-(x_{q,1}+\cdots+x_{q,d_q})=d_{q-1}-d_q$$

is solved by $(a,\ldots,a)$. Truly, Δ is pure so $d_i=p$ for each facet F_i, $1\leq i\leq q$. Thus our system becomes

$$(x_{1,1}+\cdots+x_{1,p})-(x_{2,1}+\cdots+x_{2,p})=0$$
$$(x_{2,1}+\cdots+x_{2,p})-(x_{3,1}+\cdots+x_{3,p})=0$$
$$\cdots$$
$$(x_{q-1,1}+\cdots+x_{q-1,p})-(x_{q,1}+\cdots+x_{q,p})=0$$

and so $(a,\ldots,a)$ is a solution. Thus $A(\Delta,a,\ldots,a)$ is level.

Suppose conversely that $A(\Delta,a,\ldots,a)$ is level, then $(a,\ldots,a)$ is a solution to the system:

$$(x_{1,1}+\cdots+x_{1,d_1})-(x_{2,1}+\cdots+x_{2,d_2})=d_1-d_2$$
$$(x_{2,1}+\cdots+x_{2,d_2})-(x_{3,1}+\cdots+x_{3,d_3})=d_2-d_3$$
$$\cdots$$
$$(x_{q-1,1}+\cdots+x_{q-1,d_{q-1}})-(x_{q,1}+\cdots+x_{q,d_q})=d_{q-1}-d_q.$$

If we substitute each variable for a, we get

$$(\textstyle\sum_{i=1}^{d_1}a)-(\sum_{i=1}^{d_2}a)=d_1-d_2$$
$$(\textstyle\sum_{i=1}^{d_2}a)-(\sum_{i=1}^{d_3}a)=d_2-d_3$$
$$\cdots$$
$$(\textstyle\sum_{i=1}^{d_{q-1}}a)-(\sum_{i=1}^{d_q}a)=d_{q-1}-d_q$$

which equals

$$(d_1*a)-(d_2*a)=d_1-d_2$$
$$(d_2*a)-(d_3*a)=d_2-d_3$$
$$\cdots$$
$$(d_{q-1}*a)-(d_q*a)=d_{q-1}-d_q$$

and this in turn equals

$$a(d_1 - d_2) = d_1 - d_2$$
$$a(d_2 - d_3) = d_2 - d_3$$
$$\dots$$
$$a(d_{q-1} - d_q) = d_{q-1} - d_q.$$

There are two ways in which this system of equations can hold, either $a = 1$ or $d_i - d_{i+1} = 0$ for $1 \leq i \leq q-1$. By our hypothesis $a \geq 2$, thus $d_i - d_{i+1} = 0$ for $1 \leq i \leq q-1$. Therefore

$$d_i = d_{i+1} \quad \text{for} \quad 1 \leq i \leq q-1 \tag{5.30}$$

which implies that $d_1 = d_2 = \ldots = d_q$, and so Δ is pure, as desired. □

Corollary 5.2.6. *Let I be a monomial ideal in $S = k[x_1, \ldots, x_n]$, then* PLAR(I) *is level if and only if* PLAR(I) *is the squarefree Stanley-Reisner ring of a pure simplicial complex.*

Proof. Let $S = k[x_1, \ldots, x_n]$ and let I be a monomial ideal. Since $\mathcal{P}(I)$ is a squarefree monomial ideal, then it is the Stanley-Reisner ideal of some simplicial complex Δ. Therefore, by Corollary 5.2.5, $A(\Delta, 2, \ldots, 2) = \mathrm{PLAR}(I)$ is level if and only if Δ is pure. □

Example 5.2.7. Let $S = k[a, b, c, d, e]$ and $\Delta = \langle abce, abde, bcd, acd \rangle$, where for example $bcd = \{b, c, d\}$. By Theorem 5.2.4, this simplicial complex is levelable if and only if the system

$$(a + b + c + e) - (a + b + d + e) = c - d = 0 \tag{5.31}$$

$$(a + b + d + e) - (b + c + d) = a + e - c = 1 \tag{5.32}$$

$$(b + c + d) - (a + c + d) = b - a = 0 \tag{5.33}$$

has a solution. By inspection we see that $(3, 3, 4, 4, 2)$ is a solution but $(2, 2, 2, 2, 2)$ is not. The simplicial complex Δ is levelable but PLAR(I_Δ) is not level.

Theorem 5.2.4 and Corollary 5.2.5 are generalizations of Theorem 5.1.5 and provide a proof other than the one in [5].

5.3 Depolarization

Given a squarefree monomial ideal I, of a polynomial ring S, we say I is *depolarizable* if there exists an ideal J, of a polynomial ring S', which is not squarefree, where no monomial generator is pure (e.g. power of a variable), and such that $I \cong P(J)$. This is to ask the question of when it is possible to "undo" polarization: to "depolarize".

Example 5.3.1. Although polarization is a well-defined operation, up to isomorphism, it should not be surprising that depolarization, which is effectively its preimage, is not well defined.

For example, let $S = k[x, y, z, w, l, p]$ and let $I = (wz, zx, xy, yz, xlp)$. There are two distinct depolarizations, the first is

$$S' = k[x, y, z, l, p] \quad \text{and} \quad J = (z^2, zx, xy, yz, xlp) \tag{5.34}$$

and the second one is

$$S' = k[x, y, z, w, p] \quad \text{and} \quad J = (wz, zx, xy, yz, xp^2). \tag{5.35}$$

This example makes use of a principle that can be used to confirm that a monomial ideal I can be depolarized:

Lemma 5.3.2. *A squarefree monomial ideal $I = (m_1, \dots, m_q)$, of $S = k[x_1, \dots, x_n]$, is depolarizable if there exist distinct variables x_i and x_j such that, for any monomial generator m, if $x_i | m$ then $x_j | m$.*

Proof. Let x_i and x_j be as in the statement of the theorem. Define $S' = S/(x_i - x_j)$, so that $x_i = x_j$ in S', and define an embedding $\iota : S \longrightarrow S'$, so that $\iota(x_j) = x_i$. Let $I = (m_1, \dots, m_q)$ be a squarefree monomial ideal of S such that for all $1 \le l \le q$ if $x_i | m_l$, then $x_j | m_l$, where $1 \le i \le n$, and $1 \le j \le n$. Consider $\iota(I) = (\iota(m_1), \dots, \iota(m_q))$. For all $1 \le l \le q$, if $x_i | m_l$ in S, then $x_j | m_l$ in S, so $x_i^2 | \iota(m_l)$ in S'. Because I is generated by squarefree monomials, we have x_i^2 in S' is the maximal power of x_i which divides those monomials $\iota(m_1), \dots, \iota(m_q)$, and no other squares of any variable divide the monomials $\iota(m_1), \dots, \iota(m_q)$. By definition $\mathrm{PLAR}(\iota(I)) = \mathcal{P}(S/(x_i - x_j))/(\mathcal{P}(\iota(I) + \mathcal{P}(\iota(I))')$, where $\mathcal{P}(S') = k[x_{1,1}, \dots, x_{i-1,1}, x_{i,1}, x_{i,2}, x_{i+1,1}, \dots, x_{n,1}] \cong S = k[x_1, \dots, x_n]$ (both polynomial rings have n variables) by the map: $x_{p,1} \mapsto x_p$ if $p \ne i, j$ and $x_{i,1}x_{i,2} \mapsto x_i x_j$. Similarly, $\mathcal{P}(\iota(I)) \cong I$. Thus I is depolarizable.

$\square$

From earlier examples, we know there are simplicial complexes Δ where $\mathrm{PLAR}(I_\Delta)$ is not level. Is it possible to find some $a_1, \ldots, a_n$ either fixed relative to Δ or not, for which $A(\Delta, a_1, \ldots, a_n)$ is level when I_Δ is depolarizable?

Proposition 5.3.3 (Inspired by a simplicial complex found in Theorem 4.6 of [41])**.** *There are simplicial complexes which are not levelable but I_Δ is depolarizable.*

Proof. Let $S = k[a, b, c, d, e, f, g]$ and let $\Delta = \langle aceg, adf, abd, bdf, abe, beg\rangle$. Observe (using Macaulay2):

$$I_\Delta = (abg, abf, cd, cf, bc, dg, de, fg, ef). \tag{5.36}$$

Therefore, a only divides monomial generators that b divides, so I_Δ is depolarizable by Lemma 5.3.2.

By Theorem 5.2.4, we can decide whether I_Δ is depolarizable by looking at the system of equations the theorem prescribes. There are three equations, in particular, of the associated linear system we regard here for our purpose:

$$(a + c + e + g) - (a + d + f) = 1 \tag{5.37}$$

$$(a + d + f) - (b + d + f) = 0 \tag{5.38}$$

$$(b + d + f) - (b + e + g) = 0. \tag{5.39}$$

When solved, it is necessary that $c = 1$. Since we must have $a_i \geq 2$ for all $1 \leq i \leq n$ in Theorem 5.2.4, the fact that $c = 1$ necessarily means that Δ is not levelable.

Therefore, I_Δ is not levelable, but it is depolarizable.

□

The aforementioned question of whether I_Δ being depolarizable implied that Δ is levelable is therefore answered in the negative.

Chapter 6

Conclusion

Although tools for studying the WLP are wanting, there are connections between the common operations of star and link and the presence of the WLP with squarefree Stanley-Reisner rings. We gave modest generalizations of some of these results, a thorough account of their proofs, and some examples. Along with this, we gave a survey of other notable results for the WLP and squarefree Stanley-Reisner rings. The successful reader should understand better the WLP for squarefree Stanley-Reisner rings. Afterward, we connected the WLP to levelness and began to explore it in Artinian monomial quotient rings.

Regarding Artinian reductions and levelness, the natural method of creating an operation which is analogous to polarization fails to produce a minimal graded free resolution which is level, let alone predictable, across I, $\mathrm{LAR}(I)$, and $\mathrm{PLAR}(I)$. Even with the greatest freedom of choice for the $I' = (x_1^{a_1}, \ldots, x_n^{a_n})$, there is still difficulty as there are examples where a ring may be depolarizable and not levelable. So, such a simple Artinian analogue to polarization is unlikely to exist. Since there are simplicial complexes Δ which are not levelable, but which may be depolarized, the hope for an operation on ideals which mirrors the polarization of non-Artinian rings is dim. Future study might look at other operations on ideals aside from polarization and how they may induce an Artinian setting. As the connecting map from the Snake Lemma, is the most restrictive condition for several results in the third section, more information about this map's description, especially when dealing with (squarefree) Stanley-Reisner rings, is desirable. We also seek further results which characterize the WLP for squarefree, or even other classes such as $A(\Delta, a, \ldots, a)$ for $a \geq 2$, both geometrically or algebraically.

Bibliography

[1] Karim Adiprasito. Combinatorial lefschetz theorems beyond positivity. *arXiv preprint arXiv:1812.10454*, 2018.

[2] Karim Adiprasito. Faq on the g-theorem and the hard lefschetz theorem for face rings. *arXiv preprint arXiv:1906.05859*, 2019.

[3] Sheldon Axler. *Linear algebra done right*. Undergraduate Texts in Mathematics. Springer, Cham, third edition, 2015.

[4] Louis J Billera and Carl W Lee. Sufficiency of mcmullen's conditions for f-vectors of simplicial polytopes. *Bulletin (New Series) of the American Mathematical Society*, 2(1):181–185, 1980.

[5] Mats Boij. *Artin level algebras*. ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Takn.dr)–Kungliga Tekniska Hogskolan (Sweden).

[6] Petter Brändén. Unimodality, log-concavity, real-rootedness and beyond. In *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pages 437–483. CRC Press, Boca Raton, FL, 2015.

[7] Jason I. Brown, Charles J. Colbourn, and David G. Wagner. Cohen-Macaulay rings in network reliability. *SIAM J. Discrete Math.*, 9(3):377–392, 1996.

[8] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[9] Charles J. Colbourn. *The combinatorics of network reliability*. International Series of Monographs on Computer Science. The Clarendon Press, Oxford University Press, New York, 1987.

[10] David Cook, II and Uwe Nagel. The weak Lefschetz property for monomial ideals of small type. *J. Algebra*, 462:285–319, 2016.

[11] Hailong Dao and Ritika Nair. On the lefschetz property for quotients by monomial ideals containing squares of variables. *arXiv preprint arXiv:2112.09434, 2021*, 2021.

[12] Alessandro De Stefani. Artinian level algebras of low socle degree. *Comm. Algebra*, 42(2):729–754, 2014.

[13] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[14] Sara Faridi. Simplicial trees are sequentially Cohen-Macaulay. *J. Pure Appl. Algebra*, 190(1-3):121–136, 2004.

[15] Oleksandra Gasanova, Samuel Lundqvist, and Lisa Nicklasson. On decomposing monomial algebras with the Lefschetz properties. *J. Pure Appl. Algebra*, 226(6):Paper No. 106968, 15, 2022.

[16] A. V. Geramita. Level algebras—some remarks. *Rend. Sem. Mat. Univ. Politec. Torino*, 62(3):189–203, 2004.

[17] Anthony V. Geramita, Tadahito Harima, Juan C. Migliore, and Yong Su Shin. The Hilbert function of a level algebra. *Mem. Amer. Math. Soc.*, 186(872):vi+139, 2007.

[18] Anthony V. Geramita, Tadahito Harima, and Yong Su Shin. Extremal point sets and Gorenstein ideals. *Adv. Math.*, 152(1):78–119, 2000.

[19] Tadahito Harima, Juan C. Migliore, Uwe Nagel, and Junzo Watanabe. The weak and strong Lefschetz properties for Artinian K-algebras. *J. Algebra*, 262(1):99–126, 2003.

[20] Jürgen Herzog and Takayuki Hibi. *Monomial ideals*, volume 260 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[21] Takayuki Hibi. Level rings and algebras with straightening laws. *J. Algebra*, 117(2):343–362, 1988.

[22] David Hilbert. *Hilbert's invariant theory papers*. Lie Groups: History, Frontiers and Applications, VIII. Math Sci Press, Brookline, Mass., 1978. Translated from the German by Michael Ackerman, With comments by Robert Hermann.

[23] June Huh. Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs. *J. Amer. Math. Soc.*, 25(3):907–927, 2012.

[24] Thomas W. Hungerford. *Algebra*, volume 73 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1980. Reprint of the 1974 original.

[25] John M. Lee. *Introduction to topological manifolds*, volume 202 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2011.

[26] Matthias Lenz. The f-vector of a realizable matroid complex is strictly log-concave. *arXiv preprint arXiv:1106.2944*, 5, 2011.

[27] Jizhou Li and Fabrizio Zanello. Monomial complete intersections, the weak Lefschetz property and plane partitions. *Discrete Math.*, 310(24):3558–3570, 2010.

[28] Nicholas A. Loehr. *Combinatorics*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, second edition, 2018.

[29] P. McMullen. The numbers of faces of simplicial polytopes. *Israel J. Math.*, 9:559–570, 1971.

[30] William Messing. Short sketch of Deligne's proof of the hard Lefschetz theorem. In *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974)*, pages 563–580, 1975.

[31] Mateusz Michał ek and Rosa M. Miró-Roig. Smooth monomial Togliatti systems of cubics. *J. Combin. Theory Ser. A*, 143:66–87, 2016.

[32] Juan Migliore and Uwe Nagel. Gorenstein algebras presented by quadrics. *Collect. Math.*, 64(2):211–233, 2013.

[33] Juan Migliore and Uwe Nagel. Survey article: a tour of the weak and strong Lefschetz properties. *J. Commut. Algebra*, 5(3):329–358, 2013.

[34] Juan Migliore, Uwe Nagel, and Hal Schenck. The weak Lefschetz property for quotients by quadratic monomials. *Math. Scand.*, 126(1):41–60, 2020.

[35] Juan Migliore and Fabrizio Zanello. The strength of the weak Lefschetz property. *Illinois J. Math.*, 52(4):1417–1433, 2008.

[36] Juan C. Migliore, Rosa M. Miró-Roig, and Uwe Nagel. Monomial ideals, almost complete intersections and the weak Lefschetz property. *Trans. Amer. Math. Soc.*, 363(1):229–257, 2011.

[37] Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.

[38] Richard P. Stanley. Cohen-Macaulay complexes. In *Higher combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976)*, pages 51–62. NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 31, 1977.

[39] Richard P. Stanley. The number of faces of a simplicial convex polytope. *Adv. in Math.*, 35(3):236–238, 1980.

[40] Richard P. Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. 576:500–535, 1989.

[41] Adam Van Tuyl and Fabrizio Zanello. Simplicial complexes and Macaulay's inverse systems. *Math. Z.*, 265(1):151–160, 2010.

[42] Fabrizio Zanello. Level algebras of type 2. *Comm. Algebra*, 34(2):691–714, 2006.