A BRIEF EXPLORATION OF TOTAL COLOURING

by

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Abstract

A total colouring of a graph is an assignment of colours to the edges and vertices such that adjacent objects receive different colours. In this thesis, we prove partial results towards the Total Colouring Conjecture which states that the total chromatic number of a graph is at most degree plus two. In the first part of this thesis there is an almost complete categorization of which total graphs are perfect. Upper bounds on the total chromatic number are found for Cartesian, strong, and tensor graph products. We determine that the total chromatic number of the Cartesian graph product depends strongly on the total chromatic number of the component graphs. Lastly, we explore how vertex multiplication affects the total chromatic number. We establish that for the star graph and cycle graph, no matter how many times a vertex is multiplied, the resulting graph satisfies the Total Colouring Conjecture.

List of Symbols Used

$\alpha(G)$ is the independence number of a graph G

$\alpha'(G)$ is the matching number of a graph G

$\alpha''(G)$ is the total number of a graph G

Δ_G is the maximum degree of a graph G

$\chi(G)$ is the chromatic number of a graph G

$\chi'(G)$ is the edge chromatic number of a graph G

$\chi''(G)$ is the total chromatic number of a graph G

$\omega(G)$ is the clique number of G

$a(\text{mod } b)$ is value of a modulo b

$\deg(v)$ is the degree of a vertex v

$E(G)$ is the edge set of a graph G

$L(G)$ is the line graph of a graph G

$T(G)$ is the total graph of a graph G

$V(G)$ is the vertex set of a graph G

$N(v)$ is the neighbourhood of a vertex v

C_n is the cycle graph on n vertices

F_n is the n-fan graph

$G_{(n,p)}$ is the random graph on n vertices

J_{2n} is the crown graph on $2n$ vertices

K_n is the complete graph on n vertices

$K_{n,m}$ is the complete bipartite graph

P_n is the path graph on n vertices

Q_n is hypercube graph on n vertices

S_n is the star graph on $n+1$ vertices

$S(n,G)$ is the generalized Sierpiński graph of a graph G

W_n is the wheel graph on n vertices

$G \vee H$ is the graph join of two graphs G and H

$G \square H$ is the Cartesian graph product of two graphs G and H

$G \times H$ is the tensor graph product of two graphs G and H

$G \boxtimes H$ is the strong graph product of two graphs G and H

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Chapter 1

Introduction

1.1 Definitions and Notations

A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that assigns each edge two vertices, which are called its *endpoints*. If u and v are vertices, we say that u is *adjacent* to v if they are connected by an edge. We denote this edge uv. If e_1 and e_2 are edges, we say that e_1 is *adjacent* to e_2 if these edges share an endpoint. If u is an endpoint on an edge e, then we say that u is *incident* on e and vice versa.

The *degree of a vertex* v, denoted $\deg(v)$, is the number of edges incident on v. The *maximum degree of a graph* G, denoted Δ_G, is the maximum degree of its vertices. The *neighbourhood* of a vertex v, denoted $N(v)$, is the set of vertices that are adjacent to v. A *loop* is an edge that has the same vertex for both of its endpoints. *Multi-edges* are edges that share the same endpoints.

A *simple graph* is an undirected graph, having no loops or multi-edges. A *finite graph* is a graph on a finite number of vertices. Unless stated otherwise, the graphs to be considered in this thesis are finite and simple. A *planar graph* is a graph that can be drawn such that no two edges intersect anywhere except on a vertex. We now proceed to introduce a variety of different graph colourings.

A *k-vertex-colouring* of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are called *colours*. The set of all the vertices that receive the same colour are called a *colour class*. A k-vertex-colouring is *proper* if adjacent vertices receive different colours. A graph is called *k-vertex-colourable* if it has a proper k-colouring. The *chromatic number*, denoted $\chi(G)$, is the least k such that G is k-vertex-colourable.

A *k-edge-colouring* of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The set of all the edges that receive the same colour are called an *edge colour class*. A *k-edge-colouring* is proper if adjacent edges receive different colours. A graph is called *k-edge-colourable* if it has a proper k-edge-colouring. The *edge chromatic number* (sometimes referred to as the *chromatic index*), denoted $\chi'(G)$, is the least k such that G is k-edge-colourable.

A *k-total-colouring* of a graph G is a labeling $f : V(G) \to S$ and $f : E(G) \to S$, where $|S| = k$. The set of all the vertices and edges that receive the same colour are called a *total colour class.* A *k-total-colouring* is proper if no adjacent or incident elements receive the same colour. A graph is called *k-total-colourable* if it has a proper k-total-colouring. The *total chromatic number*, denoted $\chi''(G)$, is the least k such that G is k-total-colourable.

Unless otherwise stated, all types of colourings are assumed to be proper. We now proceed to introduce independent sets and matchings.

An *independent set* is a set of vertices that includes no two vertices that are adjacent. A *maximum independent set* is an independent set of largest possible size. A *maximal independent set* is an independent set such that every other vertex in the graph is adjacent to a vertex of the set. For a graph G, the *independence number* of G, $\alpha(G)$, is the cardinality of a maximum independent set. With regards to vertex colouring, a colour class is an independent set.

A matching is a set of edges that includes no two edges that are adjacent. A *maximum matching* is a matching of largest possible size. A *maximal matching* is a matching such that every other edge in the graph is adjacent to an edge in the matching. For a graph G, the *matching number* of G, $\alpha'(G)$, is the cardinality of a maximum matching. With regards to edge colouring, an edge colour class is a matching set. A *perfect matching* of a graph is a matching such that every vertex in the graph is incident on an edge in the matching. A *near-perfect matching* is a matching in which a single vertex is left unmatched. We now proceed to introduce a variety of different graph structures.

The *path graph* P_n, is a graph with n vertices that can be enumerated such that two vertices are adjacent if and only if they are consecutive in the enumeration.

The *cycle graph* C_n, is a graph with n vertices that can be enumerated such that two vertices are adjacent if and only if they are consecutive in the enumeration or are the first and last vertex in the enumeration. A *tree graph* is a graph containing no cycles. A *chord* is an edge that is not part of a cycle but connects two vertices of the cycle. A *chordal graph* is one in which all cycles of four or more vertices have a chord.

The *complete graph* K_n, is a graph with n vertices and each pair of vertices is connected by an edge. A *bipartite graph* is a graph that can have its vertex set divided into two disjoint independent sets, U and V, which is called a *bipartition*. The *complete bipartite graph* $K_{n,m}$, is a bipartite graph with disjoint independent sets of size n and m, such that every vertex in U is adjacent to every vertex in V. We now proceed to introduce a variety of different graph products.

The *Cartesian graph product* of two graphs G and H, denoted $G \square H$, is a graph with vertex set $V(G) \times V(H)$, where two vertices, (u, v) and (u', v') in $G \square H$ are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

The *tensor graph product* of two graphs G and H, denoted $G \times H$, has vertex set $V(G) \times V(H)$ and two vertices, (u_1, v_1) and (u_2, v_2) in $G \times H$ are adjacent if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

The *strong graph product* of two graphs G and H, denoted $G \boxtimes H$, has vertex set $V(G) \times V(H)$ and two vertices, (u, v) and (u', v') in $G \boxtimes H$ are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$, or $uu' \in E(G)$ and $vv' \in E(H)$.

1.2 History of Colourings

Graph colouring originated from the Four Colour Theorem, which was conjectured in 1852 by Francis Guthrie. The Four Colour Theorem states that, given any separation of a map into contiguous regions, no more than four colours are required to colour the regions of the map so that no two adjacent regions have the same colour. While the Four Colour Theorem seems simple in principle, it was not until 1976 that the theorem was proved by Kenneth Appel and Wolfgang Haken with a proof that depended heavily on a computer search to rule out a number of possibilities [1, 2].

This idea of colouring maps was then generalized to create the field of graph colouring. For example, a proper vertex colouring is an assignment of colours to the vertices, such that no two adjacent vertices receive the same colour. The chromatic number of a graph G, denoted $\chi(G)$, is the minimum number of colours required for a proper vertex colouring. An edge colouring is an assignment of colours to the edges so that no two adjacent edges receive the same colour. Similarly, the chromatic index of a graph G, denoted $\chi'(G)$, is the minimum number of colours required for a proper edge colouring.

One reason why mathematicians study graph colouring is because it has applications to scheduling problems. Therefore minimizing the number of colours required for a proper vertex colouring is important. Determining the chromatic number of a graph can be difficult. In fact, determining the chromatic number of a graph is an NP-hard problem [11].

An important research area in graph colourings is finding upper and lower bounds for colouring parameters. A clique is a subset of vertices such that every two distinct vertices in the clique are adjacent. Let $\omega(G)$ denote the number of vertices in a maximum clique in G. Then a simple lower bound on the chromatic number is that $\chi(G) \geq \omega(G)$. This lower bound can be far from tight however as Jan Mycielski proved that there exists a graph with clique number 2 and any chromatic number [25]. These graphs are called *Mycielskian graphs*.

Let $\alpha(G)$ denote the number of vertices in a maximum independent set. Then another lower bound on the chromatic number is $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$. This bound results from the fact that a colour class, which is a set of vertices that all receive the same colour, must be an independent set. Similarly, let $\alpha'(G)$ denote the number of vertices in a maximum matching. Then a lower bound on the chromatic index is $\chi(G) \geq \frac{|E(G)|}{\alpha'(G)}$. This bound results from the fact that an edge colour class, which is a set of edges that all receive the same colour, must be a matching.

An example of an upper bound on the chromatic number is Brooks' Theorem. This theorem states that for any connected undirected graph G with maximum degree $\Delta(G)$, the chromatic number of G is at most Δ_G unless G is a complete graph or an odd cycle, in which case the chromatic number is $\Delta_G + 1$. The proof of this theorem relied on a greedy colouring algorithm and spanning trees [7].

We have seen that the chromatic number of a graph is related to the maximum degree, clique number, and independence number of a graph. Now we will review some of the main results from edge colourings. The first, most notable result, is Vizing's Theorem, which was published in 1964 by Vadim Vizing [29]. This theorem states that if G is a simple graph, then $\chi'(G) = \Delta_G$ or $\chi'(G) = \Delta_G + 1$. Graphs having chromatic index Δ_G are called tier 1 graphs whereas graphs having chromatic index $\Delta_G + 1$ are called tier 2 graphs. Vizing's Theorem, while much stronger, has similarities to Brooks' Theorem by relating the chromatic parameters of a graph to the maximum degree of the graph.

An interesting property of edge colouring problems is that they can be translated into vertex colouring problems by using the line graph operation. The line graph, denoted $L(G)$, is a graph where each vertex represents an edge in G, and two vertices are adjacent if the edges they represent are adjacent in G. A proper vertex colouring of $L(G)$ using m colours is equivalent to a proper edge colouring of G using m colours. For example, König used this method to show that all bipartite graphs are tier 1 graphs by showing that the line graph of a bipartite graph is perfect [16] (a graph G is perfect, if G and each of the induced subgraphs of G, has chromatic number $\omega(G)$).

We have seen some connections between the chromatic number and the chromatic index. Total colouring, introduced by Vizing [29] and independently by Bezhad [4] in 1964 is another way of relating the chromatic number and the chromatic index. A total colouring of a graph is an assignment of colours to the edges and vertices such that no two adjacent nor incident elements receive the same colour. Vizing and Bezhad both conjectured [29, 4], what is now known as the Total Colouring Conjecture, that for a simple graph G,

$$\Delta_G + 1 \leq \chi''(G) \leq \Delta_G + 2$$

The vertex with maximum degree needs a colour and a unique colour for every one of its incident edges, thus the lower bound is always satisfied. The Total Colouring Conjecture has immediate resemblance to Vizing's Theorem on edge colourings. Graphs with total chromatic number $\Delta_G + 1$ are called type 1 graphs whereas graphs with total chromatic number $\Delta_G + 2$ are called type 2 graphs. For example, C_3 is a type 1 graph and K_4 is a type 2 graph [5].

The motivation for this thesis is to prove partial results towards the Total Colouring Conjecture. This conjecture has been confirmed for different graph structures. For graphs with $\Delta = 3$, the conjecture was proven by Rosenfeld [26], and for graphs with $\Delta = 4$ or $\Delta = 5$, it was proved by Kostochka [17, 18].

There is a graph that will be useful in determining the total chromatic number. This graph is the total graph, which was defined by Vizing [29]. The total graph, denoted $T(G)$, is analogous to the line graph. Each vertex of $T(G)$ represents an edge of G or a vertex of G and two vertices in $T(G)$ are adjacent if and only if the elements they represent in G are adjacent or incident in G. Similar to edge colourings, a proper vertex colouring of $T(G)$ using m colours is equivalent to a proper total colouring of

G using m colours.

In this thesis, we define a new set related to total colourings, which we call a total set. A total set is a set of vertices and edges such that no two elements are adjacent nor incident to one another. We define the total number of a graph to be the maximum possible size of a total set. Note that this is equal to the independence number of the total graph. We will now summarize the main results found in this thesis. All unreferenced work is the author's.

1.3 Main Results

In Chapter 2, we determine that the crown graph, wheel graph, and n-fan graph, all satisfy the Total Colouring Conjecture. In particular, we prove that under certain conditions they are type 1 graphs. We then prove an upper bound on the total number and show that it can be far from tight. The total number is then used to get a lower bound on the total chromatic number. Lastly, we give sufficient conditions for a graph to have a perfect total graph. In particular, we show that trees have a perfect total graph.

In Chapter 3, we determine that the total chromatic number of the Cartesian graph product depends on the total chromatic number of the component graphs. Interestingly, if both of the components are type 1 or type 2 graphs, then the Cartesian graph product will satisfy the Total Colouring Conjecture. We also give conditions on which Cartesian products are type 1. These results are then applied to show that the hypercube graph and rook graph satisfy the Total Colouring Conjecture.

An upper bound is then found for the strong graph product by applying the results found on Cartesian graph products. We give sufficient conditions for which strong graph products are type 1. This result is applied to show that the kings graph satisfies the Total Colouring Conjecture. Lastly, we briefly explore tensor graph products. We show that the tensor graph product of two paths is a type 1 graph if the paths are sufficiently large. We also determine that a bipartite double of a type 1 or type 2 graph will satisfy the Total Colouring Conjecture.

Lastly, in Chapter 4 we prove that if any vertex of a star graph or cycle graph is multiplied m times that the resulting graph satisfies the Total Colouring Conjecture. We give conditions for which resulting graphs are type 1 and type 2. In most cases the proofs involve the parity of m and the size of the graph. These results are then applied to vertex multiplication in arbitrary graphs. We show that if the vertex being multiplied satisfies certain conditions, then the resulting graph is type 1.

1.4 Research in This Field

In this section, we give an overview of work that has been done on total colourings, or are otherwise relevant to this thesis.

1.4.1 Total Colouring of Chordless Graphs

A *chord* is an edge that is not part of a cycle, but that still connects two vertices of that cycle. A *chordless graph* is a graph that does not contain any chords. In the paper by Machado in 2013, he proved that if G is a chordless graph with $\Delta_G \geq 3$, then G is a type 1 graph [20]. He also determined that under the same conditions that G is a tier 1 graph.

A graph is *2-sparse* if every edge is incident to at least one vertex of degree at most 2. In the proof of Machado's main result, he proved an interesting lemma. He proved that if a graph G is 2-sparse and $\Delta_G \geq 3$, then G is a type 1 and tier 1 graph. This helped confirm the main result because 2-sparse graphs are chordless.

1.4.2 Total Colouring of Planar Graphs

A planar graph is a graph that can be drawn embedded in the plane. In other words, a planar graph is a graph that can be drawn such that no two edges intersect anywhere except at a vertex. For example, the Four Colour Theorem says that the chromatic number of a planar graph is at most 4.

Recall that by the combined results from Rosenfeld and Kostochka, if $\Delta_G \leq 5$, then $\chi''(G) \leq \Delta_G + 2$. Borodin showed that for planar graphs with $\Delta_G \geq 8$ that $\chi''(G) \leq \Delta_G + 2$ [6]. Sanders and Zhao proved the same result for graphs with maximum degree 7 [27]. Thus for total colourings, if G is a planar graph such that $\Delta_G \neq 6$, it is known that $\chi''(G) \leq \Delta_G + 2$. Therefore current research on the total colouring of planar graphs is focused on the case $\Delta_G = 6$ or classifying which planar graphs are type 1.

Recall that a graph is chordal if all cycles of four or more vertices have a chord. Two cycles are *adjacent* if they share at least one common edge. In the paper by X. Wang in 2016, he proved that if G is a planar graph with $\Delta_G = 7$ and G does not have adjacent chordal cycles of length 6, then $\chi''(G) = \Delta_G + 1$ [30]. The proof of this theorem involves a discharging technique. This technique does not seem to help prove that a planar graph with maximum degree 6 will satisfy the total colouring conjecture.

1.4.3 Total Colouring of Toroidal Graphs

While there are many interesting applications from planar graphs, sometimes drawing graphs on a different surface can be beneficial. Consider the three utilities problem, which asks if you can draw a line from three houses to three utilities without the lines ever crossing. This problem is not solvable on a plane, but is solvable on a torus.

A 1-*toroidal graph* is a graph that can be drawn on a torus, such that every edge intersects at most 1 other edge. In a paper by T. Wang, he proved that if G is a 1-toroidal graph where $\Delta_G \geq 11$ and if G has no adjacent triangles, then $\chi''(G) \leq \Delta_G + 2$ [31]. The proof of this result, similar to the proof of X. Wang's result, involves a discharging technique.

1.4.4 Total Colouring of Random Graphs

A graph can be generated by starting with n isolated vertices and adding an edge between them randomly and independently. If the probability p of adding an edge at each step is the same, then the resulting graph is called the *random graph* which is denoted $G_{(n,p)}$. For example, if $p = 1$ then $G_{(n,1)} = K_n$.

Results on vertex colourings of the random graph are generally asymptotic, as are results for total colourings of the random graph. McDiarmid proved that the probability that a graph G formed according to this probabilistic process does not satisfy the Total Colouring Conjecture is small [21]. An interesting corollary followed from this result for $G_{(n,1/2)}$, the random graph where each edge exists with probability 1/2. This corollary is, that the proportion of all graphs on n vertices that does not satisfy the Total Colouring Conjecture is small and decreases as n increases.

1.4.5 Total Colouring of Corona Products

The corona product $G \circ H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1 \leq i \leq |V(G)|$. Mohan, Geetha, and Somasundaram proved that the corona product of two graphs that satisfy the Total Colouring Conjecture will satisfy the Total Colouring Conjecture [23]. In particular, they proved that if one of the components is a cycle graph, complete graph, or bipartite graph, then the corona product will be type 1.

1.4.6 Total Colouring of Thorny Graphs

The thorny graph can be constructed by taking a graph and adding new vertices of degree 1 to each vertex in the original graph. These graphs are interesting, as they can be used to represent the structure of molecules and to design communication networks. Dundar proved that for any graph, its thorny graph satisfies the Total Colouring Conjecture [10]. In particular, he proved that the thorny graph of path graphs, cycle graphs, and complete graphs are type 1. The proof of this result is via an explicit total colouring algorithm.

1.4.7 Equitable Total Colouring of Graphs

Some research in the field of total colouring is on variations of total colouring. The equitable total chromatic number of a graph G is the smallest integer k for which G has a k-total colouring such that the number of vertices and edges coloured with each colour differs by at most one. Equitable total colouring is useful, as it gives an upper bound on the total chromatic number. Chunling proved that $C_n \square C_m$ has an equitable 5-total colouring for all n and m [9].

The graph join of G and H, which we denote $G \vee H$, is a graph with all the edges that connect the vertices of G with the vertices of H. In the paper by Ming, he found the equitable total chromatic number of $P_n \vee S_n$, $P_m \vee F_n$, and $P_m \vee W_n$ [22].

1.4.8 Edge Colouring of Tensor Graph Products

The *tensor graph product* of two graphs G and H, denoted $G \times H$, has vertex set $V(G) \times V(H)$ and two vertices, (u_1, v_1) and (u_2, v_2) in $G \times H$ are adjacent if $u_1u_2 \in$

$E(G)$ and $v_1v_2 \in E(H)$. Jaradat proved the following lemma that we will use in this thesis [14].

Lemma 1.4.1. *If G or H is a tier 1 graph, then $G \times H$ is a tier 1 graph*

The proof of Lemma 1.4.1 relied on showing that the tensor product of G and H will be comprised of bipartite graphs, which are tier 1 graphs. We will use Lemma 1.4.1 to get an upper bound on the total chromatic number of strong graph products, which we define later.

Chapter 2

Total Colouring of Some Classes of Graphs

In this chapter, we determine the total chromatic number of the crown graph and wheel graph. A new subset in a graph is introduced, which is concurrently an independent set and a matching, which we call a total set. Lastly, we give an almost complete categorization of which total graphs are perfect.

2.1 Total Colouring of the Crown Graph

In this section, we show that the crown graph satisfies the Total Colouring Conjecture. The *crown graph* on $2n$ vertices, denoted J_{2n}, is a graph with vertex set $V(J_{2n}) = \{u_i : 0 \leq i < n\} \cup \{v_j : 0 \leq j < n\}$ and edge set $E(J_{2n}) = \{u_i v_j : i \neq j\}$. Equivalently, J_{2n} can be viewed as $K_{n,n}$ with a perfect matching removed. For illustration, the graph J_6 (left) and J_8 (right) are shown in Figure 2.1.

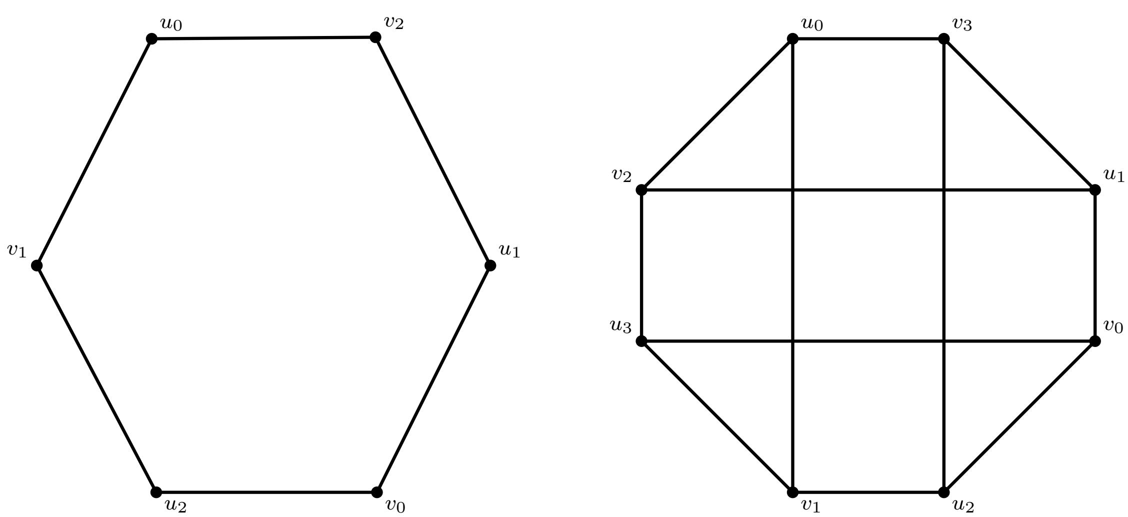

Figure 2.1: Graph J_6 and J_8

The reason why we explore the crown graph is because it will serve as an example of a tensor product, which we explore in Chapter 3. The following lemma will establish the total chromatic number of J_{2n} if n is odd. The case when n is even will follow.

Lemma 2.1.1. *If n is odd, then J_{2n} is a type 1 graph.*

Proof: Suppose n is odd and note that $\Delta_{J_{2n}} = n-1$. Therefore we will prove that J_{2n} is a type 1 graph by giving an explicit total colouring using the colours $\{0, 1, 2, \ldots, n-1\}$. For all i, where $0 \leq i < n$, colour the vertices u_i and v_i with the colour i. This will not result in any conflict because u_i is not adjacent to v_i. Now all that remains is to extend this colouring to the edges.

For each edge u_iv_j, where $i \neq 0$ and $j \neq 0$, assign the colour $(i+j)(\text{mod } n)$ to that edge. Note that this assignment of colours will not conflict with the colours of vertices u_i and v_j. Also note that all the coloured edges incident to the same vertex receive different colours. Now for each i, where $1 \leq i \leq n-1$, since the edge u_iv_i does not exist, the colour $2i(\text{mod } n)$ is not used on any edge incident to u_i (likewise v_i). Thus we will assign the colour $2i(\text{mod } n)$ to the edges u_iv_0 and u_0v_i. For illustration, we apply this total colouring to J_{10} as shown in Figure 2.2.

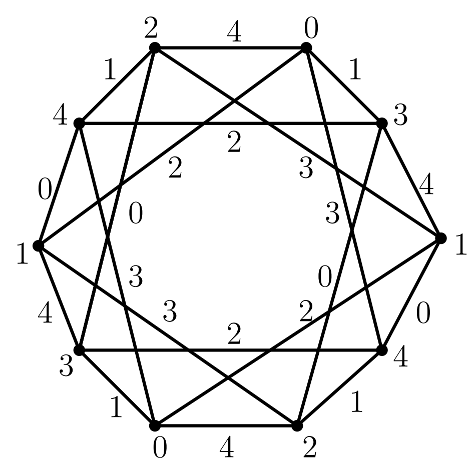

Figure 2.2: Total Colouring of J_{10}

Note that $\{2(\text{mod } n), 4(\text{mod } n), \ldots, (n-1)(\text{mod } n), (n+1)(\text{mod } n), \ldots, (2n-2)(\text{mod } n)\}$ and $\{1, 2, \ldots, n-1\}$ are the same set. This is because $(n+1)(\text{mod } n) \equiv 1$, $(n+3)(\text{mod } n) \equiv 3$, $\ldots$, $(2n-2)(\text{mod } n) \equiv n-2$. Thus all the coloured edges incident to u_0 (and v_0) receive different colours and do not conflict with the colours of vertices u_0 and v_0. We thus have a total colouring J_{2n} using n colours. Therefore if n is odd, then J_{2n} is a type 1 graph. $\square$

Now all that remains is to establish the total chromatic number of J_{2n} when n is even. Note that we cannot use the same colouring argument that was used in the proof of Lemma 2.1.1. This is because if n is even, then n and 2 are not relatively prime, thus the sets $\{2j(\text{mod } n) : j \neq 0\}$ and $\{1, 2, \ldots, n-1\}$ are not the same sets. In the following lemma, we determine an upper bound on the total chromatic number of J_{2n} if n is even.

Lemma 2.1.2. *If n is even, then $\chi''(J_{2n}) \leq \Delta_{J_{2n}} + 2$.*

Proof: A trivial upper bound on the total chromatic number is that $\chi''(G) \leq \chi(G) + \chi'(G)$. This is because we can colour the vertices with $\chi(G)$ colours and the edges with a different $\chi'(G)$ colours. J_{2n} is a bipartite graph, thus $\alpha(J_{2n}) = 2$ and $\alpha'(J_{2n}) = \Delta_{J_{2n}}$. By substituting these values into the upper bound we get that $\chi''(J_{2n}) \leq \chi(J_{2n}) + \chi'(J_{2n}) = \Delta_{J_{2n}} + 2$ as wanted. □

The question remains whether or not J_{2n} is a type 1 graph for even n. We conjecture that the following is true:

Conjecture 2.1.3. *If n is even, then J_{2n} is a type 1 graph.*

In Chapter 3, we will see that J_8 is a type 1 graph. We have also not found a type 2 crown graph, which is why we expect this conjecture to hold.

2.2 Total Colouring of the Wheel Graph

In this section, we show that the wheel graph satisfies the Total Colouring Conjecture. In fact, we show that all wheel graphs except one are type 1 graphs. The *wheel graph* on n vertices, denoted W_n, is obtained by taking the graph join of C_{n-1} and a distinct vertex v. The graph W_4 (right) and W_5 (left) are shown in Figure 2.3.

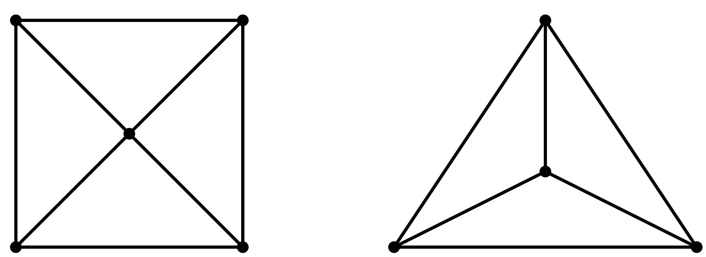

Figure 2.3: Graph W_5 and W_4

The reason why we explore the wheel graph is because it will prove partial results towards which planar graphs are type 1. The following theorem will establish the total chromatic number of W_n for all n.

Theorem 2.2.1. $\chi''(W_n) = \begin{cases} \Delta_{W_n} + 2 & \textit{if } n = 4 \\ \Delta_{W_n} + 1 & \textit{if } n > 4 \end{cases}$

Proof: W_4 is the planar representation of K_4 as can be seen in Figure 2.3. Bezhad proved that K_4 is a type 2 graph [5], therefore W_4 is a type 2 graph.

Suppose that $n > 4$ and note that $\Delta_{W_n} = n-1$. Therefore we will prove that W_n is a type 1 graph by giving an explicit total colouring using the colours $\{0, 1, \ldots, n-1\}$. We label the vertices of C_{n-1} by $\{v_0, v_1, \ldots, v_{n-2}\}$ and label the central vertex v_{n-1}. Colour the vertices v_i, where $0 \leq i < n-1$, with colour i. Then colour the edges $v_i v_{(i+1)(\text{mod } n-1)}$ with the colour $(i+2)(\text{mod } n-1)$. Note that this assignment of colours will not conflict with the colours of vertices v_i and $v_{(i+1)(\text{mod } n-1)}$. Also note that this will result in two of the edges incident to v_i receiving the colours $(i+1)(\text{mod } n-1)$ and $(i+2)(\text{mod } n-1)$.

Therefore the colour $(i+3)(\text{mod } n-1)$ is available to give the edge $v_i v_n$ because $(i+3) \not\equiv (i+2) \not\equiv (i+1)(\text{mod } n-1)$ if $n > 4$. Note that all the coloured edges incident to v_n receive different colours and there are $n-1$ of these colours. Therefore we need a fresh colour to assign the vertex v_n. We will assign v_n the colour $n-1$. Therefore we have a total colouring of W_n using the colours $\{0, 1, \ldots, n-1\}$. $\square$

2.2.1 Total Colouring of the n-Fan Graph

We will now apply the results on wheel graphs to find the total chromatic number of a different graph structure. The *n-fan graph*, denoted F_n, is the graph obtained by identifying a common vertex in n disjoint copies of C_3. For example, the graph F_3 (left) and F_4 (right) are shown in Figure 2.4.

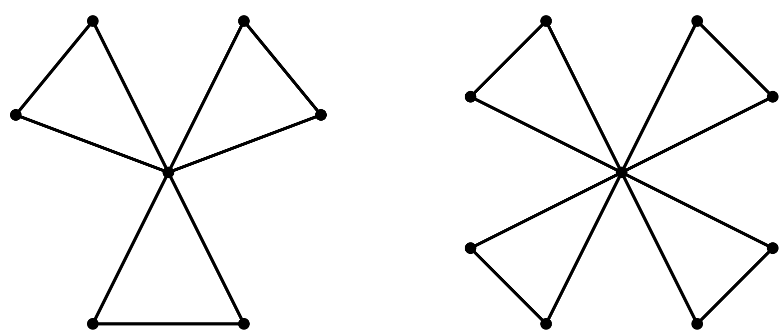

Figure 2.4: Graph F_3 and F_4

The reason why we explore the n-fan, is it will help in the categorization of which chordal graphs are type 1. Also, later in this chapter, the n-fan will serve as an example of a graph that only contains cycles of length 3. The following corollary will establish the total chromatic number of the n-fan for all n.

Corollary 2.2.2. *F_n is a type 1 graph for all n.*

Proof: If $n = 1$, then $F_1 = K_3$. Bezhad proved that K_3 is a type 1 graph [5], thus F_1 is a type 1 graph. Suppose that $n > 1$ and note that the n-fan is a subgraph of W_{2n+1} having the same maximum degree. Hence we can use the total colouring of W_{2n+1} restricted to F_n, which by Theorem 2.2.1, will use $\Delta_{W_{2n+1}} + 1$ colours. Therefore we have a total colouring of F_n using $\Delta_{W_{2n+1}} + 1 = \Delta_{F_n} + 1$ colours as wanted. □

This concludes our exploration of the total chromatic number of certain graphs. The remaining portion of this chapter is on the total set, total number, and total graph. We start by finding bounds on the total number and applying them to the total chromatic number.

2.3 Total Sets and Total Numbers

We want to generalize the notion of an independent set and matching to total colourings. Therefore we define a new set with accompanying terminology that is analogous to these notions.

Definition 2.3.1. *A* ***total set*** *T is a set of vertices and edges such that no two elements in the set are adjacent or incident to one another. A* ***maximal total set*** *is a total set such that every other element is either adjacent or incident to some element in T. A* ***maximum total set*** *is a total set with the largest possible number of elements. This size will be called the* ***total number*** *of G, and will be denoted $\alpha''(G)$.*

A total set of a graph G corresponds to an independent set of $T(G)$ and the total number of G corresponds to the independence number of $T(G)$. Also, finding the total chromatic number of a graph is equivalent to finding the minimum number of disjoint total sets that cover all the vertices and edges. We will see later that total sets are useful in showing that a graph cannot be totally coloured with Δ_G+1 colours.

The general argument involves the total number, the number of disjoint total sets, and the number of elements in the graph. In order to talk about the size of a total set however, we will need some upper and lower bounds for $\alpha''(G)$. The following lemma will give an upper and lower bound for the total number that will allow us to directly apply results found for the independence and matching number.

Lemma 2.3.2. *Let G be a graph. Then* $\max\{\alpha(G), \alpha'(G)\} \leq \alpha''(G) \leq \alpha(G) + \alpha'(G)$

Proof: A total set could consist of entirely edges or entirely of vertices. Thus letting a total set being either a maximum independent set or maximum matching will satisfy the lower bound. Note that at most $\alpha(G)$ vertices can be in a total set, because if more vertices could be in a total set, then this would contradict that $\alpha(G)$ was the maximum size of an independent set. The same argument tells us that at most $\alpha'(G)$ edges can exist in a total set. Therefore we have that a total set cannot have more than $\alpha(G) + \alpha'(G)$ elements. $\square$

It is important to note that the upper bound of Lemma 2.3.2 is sharp. For example, consider the graph K_{2n+1}. Label the vertices of K_{2n+1} by $v_0, v_1, \ldots, v_{2n}$. Then K_{2n+1} has a near-perfect matching by taking the edges $\{v_0v_1, v_2v_3 \ldots, v_{2n-2}v_{2n-1}\}$. Thus $\alpha'(K_{2n+1}) = n$. Every vertex in K_{2n+1} is adjacent to one another, thus $\alpha(K_{2n+1}) = 1$. Therefore the total set $\{v_0v_1, v_2v_3 \ldots, v_{2n-2}v_{2n-1}, v_{2n-1}\}$, has size $n+1 = \alpha(K_{2n+1}) + \alpha'(K_{2n+1})$.

Now that we have a general upper and lower bound for the total number, we want to improve upon it. This is because the upper bound of Lemma 2.3.2 can be far from the total number, as we will see in the following proposition. This is similar to how there are graphs with clique number 2, but arbitrarily high chromatic number.

Proposition 2.3.3. *Let G be a graph. Then $\alpha(G) + \alpha'(G) - \alpha''(G)$ can be arbitrarily large.*

Proof: Let G be a graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ and edge set $E(G) = \{x_iy_i : 1 \leq i \leq k\} \cup \{y_iy_j : 1 \leq i, j \leq k \text{ and } i \neq j\}$ where k is even. Then G has a perfect matching by choosing the edges $\{x_1y_1, x_2y_2, \ldots, x_ky_k\}$. Thus $\alpha'(G) = k$. Now consider the following independent set: $\{x_1, x_2, \ldots, x_k\}$. Thus $\alpha(G) \geq k$.

Suppose that there is an independent set with $k+1$ or more vertices. Then at least one of the vertices of the form y_i must be in the independent set. This is because there are only k vertices of the form x_i. Without loss of generality, assume that y_1 is in the independent set. Then y_i, where $2 \leq i \leq k$, cannot be in the independent set because y_i is adjacent to y_1. Similarly, x_1 cannot be in the independent set. Thus the only elements that could be in the independent set are $\{y_1, x_2, x_3, \ldots, x_k\}$, which is only k vertices, a contradiction that the independent set had $k+1$ or more vertices. Therefore we have that $\alpha(G) = k$.

Now consider the following total set: $\{x_1, x_2, \ldots, x_k, y_1y_2, y_3y_4, \ldots, y_{k-1}y_k\}$. Thus $\alpha''(G) \geq \frac{3k}{2}$. This total set has k vertices and $\frac{k}{2}$ edges. Suppose that a total set with more elements existed. Then it would have to have more than $\frac{k}{2}$ edges, because at most $\alpha(G) = k$ vertices can be in a total set. Suppose that there are $\frac{k}{2} + n$ edges in some total set, where $n > 0$. Then at most $2k - 2(\frac{k}{2} + n) = k - 2n$ vertices could be in the total set. This is because G has a total of $2k$ vertices and the endpoints of each edge in this total set are distinct. Thus a maximum of $(\frac{k}{2} + n) + (k - 2n) = \frac{3k}{2} - n$ elements could exist in a total set with $\frac{k}{2} + n$ edges, which is less than $\frac{3k}{2}$. This is a contradiction that a total set with more elements existed.

Therefore we have that $\alpha''(G) = \frac{3k}{2}$. The upper bound of Lemma 2.3.2 tells us that $\alpha''(G) \leq \alpha(G) + \alpha'(G) = k + k = 2k$, which is a difference of $k/2$ from $\alpha''(G)$. Therefore as k increases, $\alpha(G) + \alpha'(G) - \alpha''(G)$ gets arbitrarily large. □

Since the difference between the total number and the upper bound of Lemma 2.3.2 can be large, there is a need for an improved bound. In the next section, we apply total sets and the total number to get a lower bound on the total chromatic number.

2.3.1 Applying Total Sets to the Total Chromatic Number

Now that we have found an upper and lower bound for the total number, we would like to find a relation between the total number of a graph and its total chromatic number. The following lemma will give that relationship.

Lemma 2.3.4. *Let G be a graph. Then we have that* $\frac{|V(G)|+|E(G)|}{\alpha''(G)} \leq \chi''(G)$

Proof: Note that each total colour class is a total set. Thus each total colour class cannot have more than $\alpha''(G)$ elements. Also note that $\chi''(G)$ denotes the minimum number of disjoint total sets required to totally colour all elements. Let T_c denote the total colour class receiving the colour c and suppose that $\chi''(G) = k$. Then we get the following chain of inequalities:

$$\begin{aligned} |V(G)| + |E(G)| &= \Sigma_{i=1}^{k}|T_i| && \text{since the colour classes cover all elements of G} \\ &\leq k \max\{|T_i|\} \\ &\leq k\alpha''(G) && \text{because } |T_i| \leq \alpha''(G) \end{aligned}$$

Therefore we have that $|V(G)| + |E(G)| \leq \alpha''(G)\chi''(G)$. Rearranging this inequality gives us the desired result. □

2.4 Total Graphs and Perfect Graphs

In this section, we give an almost complete categorization of which graphs have perfect total graphs. This is interesting as it will give a means to tell if a graph is type 1. Recall that for a graph G, each vertex of $T(G)$ represents an edge of G or a vertex of G and two vertices in $T(G)$ are adjacent if and only if the elements they represent in G are adjacent or incident in G. For example, consider the graph C_3. Label the vertices v_1, v_2 and v_3 and label the edges e_1, e_2, and e_3. Then the total graph of C_3 is shown in Figure 2.5.

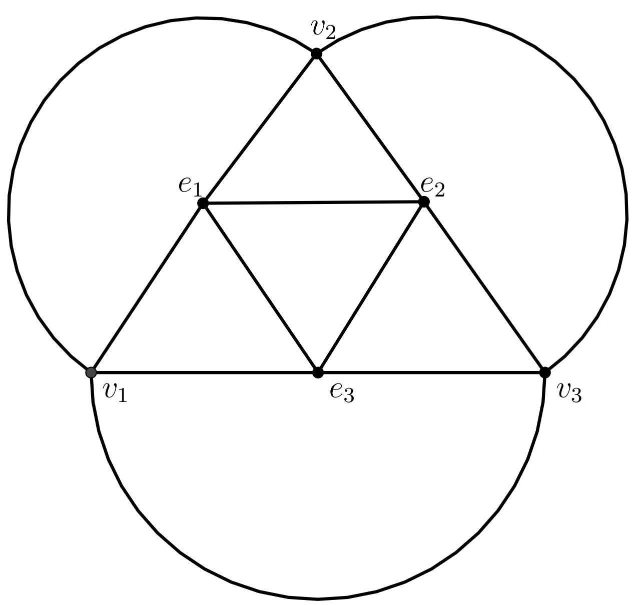

Figure 2.5: Graph $T(C_3)$

Recall from the introduction, that a graph G is *perfect* if G and each of the induced subgraphs of G, have chromatic number equal to the size of its largest clique. For example, it has been shown that bipartite graphs, chordal graphs, and the line graph of bipartite graphs are perfect [15].

This definition of perfect graphs is sometimes not the most convenient however. A graph G is a *Berge graph* if neither G nor its complement has an odd-length induced cycle of length 5 or more. The Strong Perfect Graph Theorem, conjectured by Claude Berge in 1961 [3], tells us that a perfect graph is equivalently a Berge graph. It was not until 2002 that Chudnovsky, Robertson, Seymour, and Thomas proved this conjecture [8]. Showing that a graph or its complement has an induced odd cycle of length 5 or more is a quick way to show that a graph is not perfect.

Similar to edge colourings and the line graph, a proper vertex colouring of $T(G)$ can be translated into a proper total colouring of G. Note that the largest clique in $T(G)$ is of size $\Delta_G + 1$. This is because the vertex v with maximum degree and all of the edges incident on v create a clique of size $\Delta_G + 1$ in $T(G)$. Therefore if the total graph of G is perfect, then $\chi(T(G)) = \omega(T(G)) = \Delta_G + 1$ and G is thus a type 1 graph.

Note however that a graph being type 1 does not imply that it has a perfect total graph. The total graph will have chromatic number equal to the size of its largest clique, but this does not imply that each induced subgraph does. Therefore we want to determine for what graphs G is $T(G)$ a perfect graph. We start this classification by showing that $T(C_n)$ is not perfect when $n > 3$.

Lemma 2.4.1. *If $n > 3$, then $T(C_n)$ is not a perfect graph. If $n = 3$, then $T(C_3)$ is a perfect graph.*

Proof: If $n = 3$, then $T(C_3)$, shown in Figure 2.5, does not contain an induced odd cycle of length 5. Also $T(C_3)$ does not contain an induced complement of an odd cycle of length 5. Therefore $T(C_3)$ is a Berge graph and thus a perfect graph.

Suppose that $n > 3$ and label the vertices of C_n by $v_0, v_1, \ldots, v_{n-1}$ such that v_i is adjacent to $v_{(i+1)(\text{mod } n)}$ where $0 \leq i < n$. We will prove this lemma by case distinction on n being even or odd.

Case 1: Suppose that n is odd. The induced subgraph on the vertices $v_0, v_1, \ldots, v_{n-1}$ is isomorphic to C_n, which is an odd cycle of length 5 or more. Therefore $T(C_n)$ is not a Berge graph, so by the Strong Perfect Graph Theorem, $T(C_n)$ is not a perfect graph.

Case 2: Suppose that n is even. Let e_i be the vertex in $T(C_n)$ representing the edge $v_i v_{(i+1)(\text{mod } n)}$. Then the induced subgraph on the vertices $v_0, v_1, \ldots, v_{n-2}, e_{n-2}, e_{n-1}$ is isomorphic to C_n, which is an odd cycle of length 5 or more. Therefore $T(C_n)$ is not a Berge graph, so by the Strong Perfect Theorem, is not a perfect graph. For example, the total graph $T(C_4)$ is shown in Figure 2.6. It can be seen that the induced subgraph on v_0, v_1, v_2, e_2 and e_3 is isomorphic to C_5. □

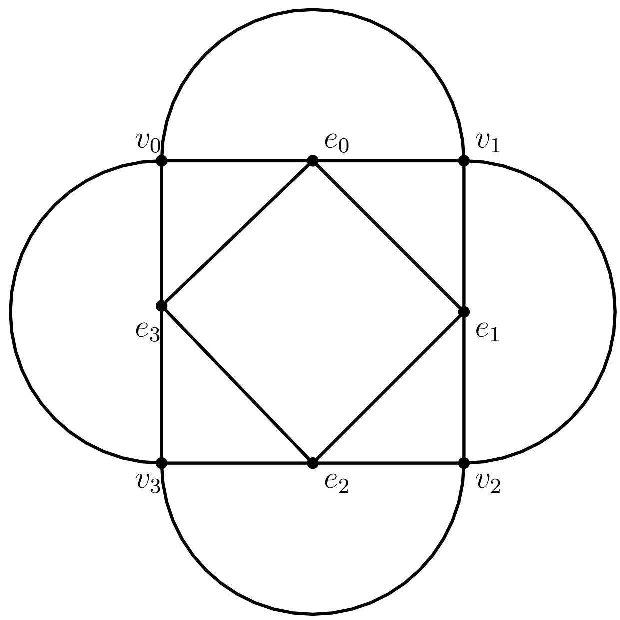

Figure 2.6: Graph $T(C_4)$

Now we have shown that if $n > 3$, then $T(C_n)$ is not a perfect graph. Therefore if G contains a cycle of length 4 or more, then $T(G)$ is not a perfect graph. We will now restrict ourselves to trees. The following lemma establishes that all trees have perfect total graphs.

Lemma 2.4.2. *If G is a tree, then $T(G)$ is perfect.*

Proof: Let G be a tree. If we can prove that any cycle of length 4 or more in $T(G)$ contains a chord, then we have shown that $T(G)$ is chordal, and hence perfect. Let C be a cycle in $T(G)$. We say that C contains an edge e of G, if C contains the vertex that represents e. Since G is a tree and thus contains no cycles, C must contain at least one edge. If C only contains edges, then we have that C is a subset of $L(G)$.

Harary proved that the line graphs of trees are exactly claw-free block graphs, which are chordal [13]. In this case C will have a chord.

Consider then the case that C contains both vertices and edges. Thus, there must be a vertex v and an edge e such that v and e are neighbours in C. Suppose that the other neighbour of v in C is an edge e'. In this case, e,e' and v will all be adjacent in $T(G)$, so C will contain the chord ee'.

Suppose now that the other neighbour of v in C is a vertex u. If u is the other endpoint of e, then C will contain the chord uv. This is the only option for a vertex neighbour of v in C. Consider a different vertex neighbour of v, label this vertex x, as shown in Figure 2.7.

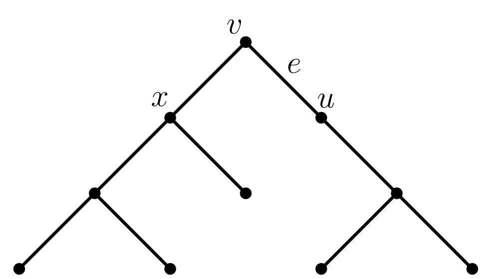

Figure 2.7: Example Tree

Note that there is no path comprised of edges and vertices in $T(G)$ from x to e except through the edge xv. This edge is adjacent to e and v however, so C will contain a chord. Therefore the only option for a vertex neighbour of v in C is the other endpoint of e. In all cases, C contains a chord. Therefore $T(G)$ is chordal and thus perfect. $\square$

We have now determined for all graphs, except those that only contain cycles of length 3, if their total graph is perfect or not. Note that we cannot use the same argument that we used in the proof of Lemma 2.4.2. This is because taking the induced subgraph on the vertices e_1, e_3, v_3, v_2 in Figure 2.5 gives a cycle of length 4 with no chord. We conjecture that the following is true:

Conjecture 2.4.3. *$T(G)$ is perfect if and only if G does not contain a cycle of length 4 or more.*

This concludes our exploration on the total graph and perfect graphs. In the next chapter, we determine the total chromatic number for various graph products.

Chapter 3

Total Chromatic Number of Graph Products

In this chapter, we explore how various graph products affect the total chromatic number. In particular, we start with the Cartesian graph product.

Definition 3.0.4. *The Cartesian graph product of two graphs G and H, denoted $G\Box H$, is a graph with vertex set $V(G)\times V(H)$, where two vertices, (u,v) and (u',v') in $G\Box H$ are adjacent if and only if:*
(1) $u=u'$ and $vv'\in E(H)$, or
(2) $v=v'$ and $uu'\in E(G)$.

Note that $\Delta_{G\Box H}=\Delta_G+\Delta_H$

because at most Δ_H edges can satisfy condition (1) and at most Δ_G edges can satisfy condition (2). For the purpose of this thesis, unless otherwise stated, all graph products are Cartesian products. To explore this definition, and see how it interacts with the total chromatic number, it would be best to see an example of a graph product. The graph product of P_3 with itself is shown in Figure 3.1 below.

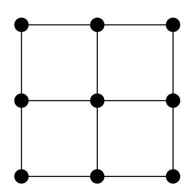

Figure 3.1: Graph $P_3\Box P_3$

3.1 Graph Products with P_2

In this section, we determine how the total chromatic number of a graph is affected after taking its graph product with P_2. First we will need the following total colouring algorithm to prove some of the theorems in this section.

Lemma 3.1.1 (Total Colouring Algorithm for $G\Box P_2$). *Let f be a proper total colouring assignment for G using the colours $\{0,1,\ldots,k-1\}$. The following total* *colouring algorithm produces a proper total colouring of $G\Box P_2$ using $k+1$ colours. Denote the two vertices of P_2 by 0 and 1. First, colour the vertices of the form (u,i), where $0 \le i < 2$, with the colour $(f(u)+i)(\text{mod } k)$. Then, colour the edges of the form $(u_1,i)(u_2,i)$ with the colour $(f(u_1u_2)+i)(\text{mod } k)$. Lastly, colour the edges of the form $(u,x)(u,y)$ with a new colour k.*

Proof: We will show that this is a proper total colouring. We start by showing that there is no colour conflict with the vertices. If (u,i) is adjacent to (v,i), then u is adjacent to v in G. Note that $f(u) \neq f(v)$, because f is a proper total colouring assignment for G. Thus $f((u,i)) = (f(u)+i)(\text{mod } k) \neq (f(v)+i)(\text{mod } k) = f((v,i))$. Next, the vertices $(u,0)$ and $(u,1)$ do not receive the same colour because $f((u,0)) = f(u) \neq (f(u)+1)(\text{mod } k) = f((u,1))$.

Let e be an edge incident to (u,i). If e is an edge to $(u,(i+1)(\text{mod } 2))$, then e is coloured with the colour k. Note that $f((u,i)) = (f(u)+i)(\text{mod } k) \neq k$, so there is no conflict between e and (u,i). If e is an edge from (u,i) to (v,i), then e receives the colour $f(uv)+i$. Note that $f((u,i)) = (f(u)+i) \neq (f(uv)+i)$, because otherwise $f(u) = f(uv)$, but f is a proper total colouring. Thus there are no colour conflicts with the vertices.

All that remains is to check that the edges do not conflict with one another. Let e_1 and e_2 be adjacent edges in $G\Box P_2$. Suppose $e_1 = (u_1,i)(u_2,i)$ and $e_2 = (u_1,i)(u_3,i)$. Then $f(e_1) = (f(u_1u_2)+i)(\text{mod } k) \neq (f(u_1u_3)+i)(\text{mod } k) = f(e_2)$, because otherwise $f(u_1u_2) = f(u_1u_3)$, but f is a proper total colouring assignment for G. Lastly, suppose that $e_1 = (u,i)(v,i)$ and $e_2 = (u,i)(u,i+1(\text{mod } 2))$. Then $f(e_1) = (f(uv)+i)(\text{mod } k) \neq k$, which is the colour assigned to e_2. Thus there are no colour conflicts arising with the edges. Hence we have a proper total colouring of $G\Box P_2$ using the colours $\{0,1,\ldots,k\}$. $\Box$

Now we apply Lemma 3.1.1 to determine the total chromatic number of $G\Box P_2$, if G is a type 1 graph.

Theorem 3.1.2. *If G is a type 1 graph, then $G\Box P_2$ is a type 1 graph.*

Proof: Let G be a type 1 graph and suppose that $\Delta_G = k-1$. By Lemma 3.1.1 we know that $G\Box P_2$ can be totally coloured with $k+1 = \Delta_G + 2 = \Delta_{G\Box P_2} + 1$ colours. Therefore $G\Box P_2$ is a type 1 graph. $\Box$

Lemma 3.1.1 will be generalized in the next section, so it will be good to see an example of the algorithm applied to a graph. C_3 is a type 1 graph [5], thus satisfying the criterion of Theorem 3.1.2. In the following example, we apply Lemma 3.1.1 to $C_3 \square P_2$.

Example 3.1.3. *The first step of Lemma 3.1.1 is to colour the vertices of the first copy of C_3 with its original total colouring assignment using the colours $\{0, 1, 2\}$. Then colour the vertices in the second copy of C_3 with the same total colouring assignment except we add 1 to each vertex modulo 3 as shown in Figure 3.2*

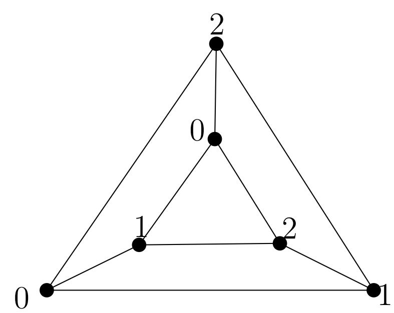

Figure 3.2: Step 1 of the Total Colouring of $C_3 \square P_2$

The next step of Lemma 3.1.1 is to colour the edges of the first copy of C_3 with their original total colouring assignment. Then colour the edges in the second copy of C_3 with the same assignment, except we add 1 to each edge modulo 3, as shown in Figure 3.3

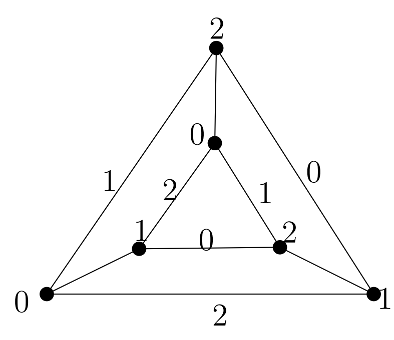

Figure 3.3: Step 2 of the Total Colouring of $C_3 \square P_2$

The last step of Lemma 3.1.1 is to colour the edges connecting the two copies of

C_3. This is where we introduce the new colour. We will use the colour 3 for these edges. This will complete the algorithm and give a proper total colouring of $C_3 \Box P_2$ as shown in Figure 3.4

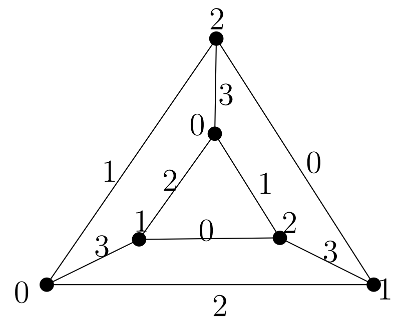

Figure 3.4: Last step of Total Colouring of $C_3 \Box P_2$

Theorem 3.1.4. *Let G be a type 2 graph, then $\chi''(G \Box P_2) \leq \Delta_{G \Box P_2} + 2$.*

Proof: Let G be a type 2 graph and suppose that $\Delta_G = k$. Then it requires $k+2$ colours for a proper total colouring of G. Then by Lemma 3.1.1, $G \Box P_2$ can be totally coloured with $k+3 = \Delta_G + 3 = \Delta_{G \Box P_2} + 2$ colours. Therefore we have that $\chi''(G \Box P_2) \leq \Delta_{G \Box P_2} + 2$. $\square$

Example 3.1.5. *Consider K_4; this graph is a type 2 graph [5]. However, $K_4 \Box P_2$ is a type 1 graph as shown in Figure 3.5.*

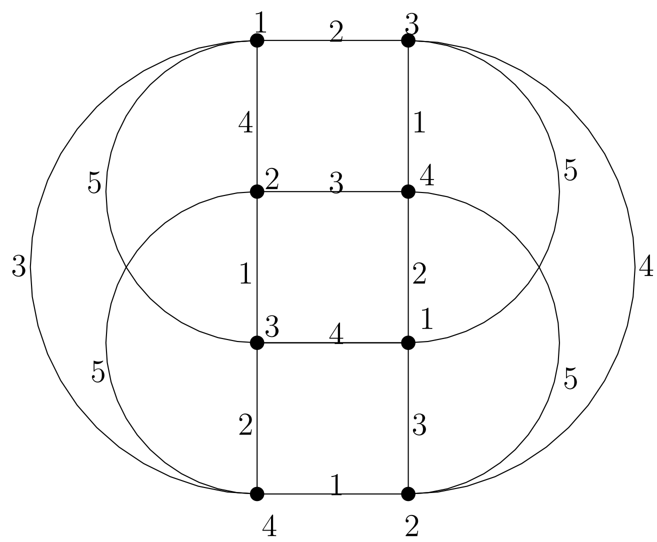

Figure 3.5: Total Colouring of $K_4 \square P_2$

It is unknown whether or not $K_{2n} \square P_2$ is a type 1 graph for all n. However, it is known that $C_{2n} \square P_2$ is a type 1 graph for all n [28]. Thus in particular, $C_4 \square P_2$ is a type 1 graph as shown in Figure 3.6

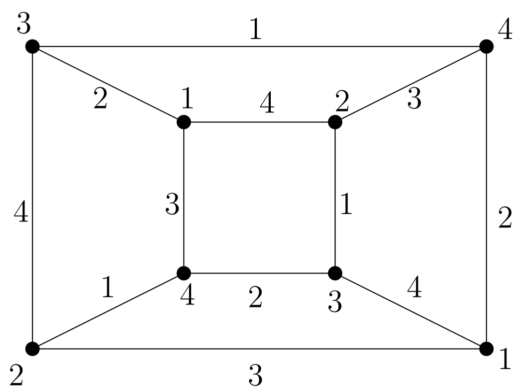

Figure 3.6: Total Colouring of $C_4 \square P_2$

A result that follows from Figure 3.6, is the total chromatic number of the hypercube graph. The *hypercube graph*, which is denoted Q_n, is the graph formed from the vertices and edges of a n-dimensional cube. Alternatively, the hypercube can be viewed inductively as $Q_n = Q_{n-1} \square P_2$. The following corollary will establish the total chromatic number of Q_n.

Corollary 3.1.6. *If $n \geq 3$, then Q_n is a type 1 graph. If $n = 1$ or $n = 2$, then Q_n is a type 2 graph.*

Proof: If $n = 1$, then $Q_1 = P_2$, which is a type 2 graph. If $n = 2$, then $Q_2 = C_4$, which is a type 2 graph. Thus all that remains is if $n \geq 3$. We will prove this by induction on the size of n.

Base Case: If $n = 3$, then Q_3 is the graph from Figure 3.6, which is a type 1 graph.

Induction Step: Let $k \in \mathbb{Z}^+$ and suppose that Q_k is a type 1 graph for $k \geq 3$. We need to show that Q_{k+1} is a type 1 graph. Recall that $Q_{k+1} = Q_k \square P_2$. By the induction hypothesis, Q_k is a type 1 graph. Therefore by Theorem 3.1.2, $Q_k \square P_2 = Q_{k+1}$ is a type 1 graph. Therefore by the induction principle, Q_n is a type 1 graph for all $n \geq 3$. $\square$

We will now show that $C_n \square P_2$ is a type 1 graph if $n \equiv 1 (mod 3)$, but first we need the following lemma.

Lemma 3.1.7. *Let G be a graph with $\chi''(G) = k$. Fix an edge $e \in E(G)$. Let G^* be the graph that results from an edge e in G being subdivided 3 times. Then $\chi''(G^*) \leq k$.*

Proof: Let e be an edge of G with endpoints v_0, v_1. Form G^* by replacing e by a path P of length 5, whose internal vertices are labeled u_0, u_1, u_2. Let P be the induced path on the vertices v_0, u_0, u_1, u_2, v_1. Lastly, let f be a total colouring assignment of G.

Colour the elements of $G^* \setminus \{u_0, u_1, u_2\}$ using the assignment f. All that remains is to colour the elements of P. Without loss of generality, suppose that $f(v_0) = 0, f(v_1) = 1$ and $f(v_0v_1) = 2$. Then we can colour P as shown in Figure 3.7.

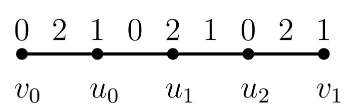

Figure 3.7: Total Colouring of P

Since the colour 2 was assigned the edge v_0v_1 in G, none of the edges in $G^* \setminus \{P\}$, incident to v_0 or v_1, are assigned the colour 2. Also the elements u_0, u_1, u_2, u_0u_1, and u_1u_2 are not adjacent or incident to any element in $G^* \setminus \{P\}$. Therefore this colouring of P does not conflict with the colouring of $G^* \setminus \{P\}$. Thus we have a proper total colouring of G^* with k colours. $\square$

Proposition 3.1.8. *If n is odd and $n \equiv 1(mod\ 3)$, then $C_n \square P_2$ is a type 1 graph.*

Proof: Consider the total colouring of $C_7 \square P_2$ shown in Figure 3.8 using 4 colours. We will extend this to a total colouring of C_{7+6n} where $n > 0$. First, subdivide the edges labelled e_0 and e_1 a total of $6n$ times each and let G^* be this resulting graph. Let u_i and v_i, where $0 \leq i < n$, denote the vertices added by the subdivision of e_0 and e_1 respectively. Note that $G^* \cup \{u_i v_i : 0 \leq i < n\} = C_{7+6n}$. By repeated application of Lemma 3.1.7, we can totally colour G^* with 4 colours. All that remains is to colour the edges $u_i v_i$.

Note that the colours assigned by Lemma 3.1.7 to u_i (v_i) and the edges incident on u_i (v_i) are only the colours 1,2, and 3. Therefore the colour 4 is available to give the edge $u_i v_i$. We now have a total colouring of C_{7+6n} using 4 colours. Therefore if n is odd and $n \equiv 1 (\text{mod } 3)$, then $C_n \square P_2$ is a type 1 graph. $\square$

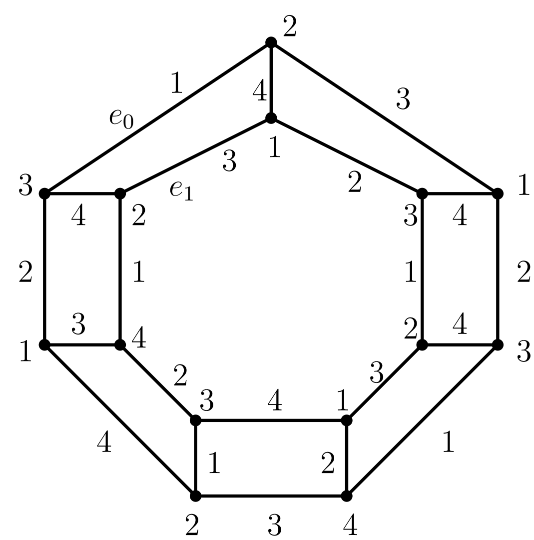

Figure 3.8: Total Colouring of $C_7 \square P_2$

3.2 Graph Products with Tier 1 Graphs

In this section, we determine an upper bound on the total chromatic number of $G \square H$ when G is a type 1 graph. To start, we create a total colouring algorithm that is a generalization of Lemma 3.1.1.

Lemma 3.2.1 (Total Colouring Algorithm for $G \square H$). *Let G and H be graphs. Let f_1 be a total colouring assignment for G and f_2 be a vertex colouring assignment for H both using the colours $\{0, 1, \ldots, k-1\}$. Let f_3 be an edge colouring assignment* *for H using the colours $\{k, k+1, \ldots, k+l-1\}$. The following total colouring algorithm produces a proper total colouring using $k+l$ colours. First, colour the vertex (u, v) in $G\Box H$ with the colour $(f_1(u) + f_2(v))(mod\ k)$. Then, colour the edge $(u_i, v)(u_j, v)$ with the colour $(f_1(u_iu_j) + f_2(v))(mod\ k)$. Lastly, colour the edge $(u, v_1)(u, v_2)$ with the colour $f_3(v_1v_2)$.*

Proof: First we prove that this algorithm gives a proper total colouring of the vertices by supposing otherwise. Recall that two vertices (u_1, v_1) and (u_2, v_2) of $G\Box H$ are connected by an edge if and only if $u_1 = u_2$ and $v_1 \sim v_2$ or $v_1 = v_2$ and $u_1 \sim u_2$. Suppose that there is a colour conflict between a vertex (u_1, v_1) and a vertex (u_2, v_2) where $u_1 = u_2$ and $v_1 \sim v_2$. Then we have that $(f_1(u_1) + f_2(v_1))(\text{mod } k) = (f_1(u_2) + f_2(v_2))(\text{mod } k)$. Thus we have that $f_2(v_1)(\text{mod } k) = f_2(v_2)(\text{mod } k)$ because $u_1 = u_2$. Then $f_2(v_1) = f_2(v_2)$, but this contradicts the fact that f_2 is a proper vertex colouring of H. Therefore this algorithm is a proper vertex colouring of $G\Box H$. The case when $v_1 = v_2$ and $u_1 \sim u_2$ is a similar argument.

Next we show that this algorithm gives a proper total colouring of the edges by supposing otherwise. Suppose that an edge has a conflict with its endpoints. The endpoint (u_i, v) receives the colour $(f_1(u_i)+f_2(v))(\text{mod } k)$. Thus we have $(f_1(u_iu_j)+f_2(v))(\text{mod } k) = (f_1(u_i) + f_2(v))(\text{mod } k)$. Then we have that $f_1(u_i) = f_1(u_iv)$, which contradicts the fact that f_1 is a proper total colouring.

Suppose that an edge has a conflict with another edge. If $(u_i, v)(u_j, v)$ conflicts with $(u_i, v)(u_k, v)$ then $(f_1(u_iu_j) + f_2(v))(\text{mod } k) = (f_1(u_iu_k) + f_2(v))(\text{mod } k)$. Thus we have that $f_1(u_iu_j) = f_1(u_iu_k)$, which contradicts that f_1 is a proper total colouring assignment. Suppose that $(u, v_1)(u, v_2)$ conflicts with $(u, v_2)(u, v_3)$. Then we have $f_3(v_1v_2) = f_3(v_2v_3)$, which contradicts that f_3 is a proper edge colouring of H. Therefore we have a proper total colouring of $G\Box H$ using $k+l$ colours as wanted. $\Box$

In the following theorem, we apply Lemma 3.2.1 to determine the total chromatic number of $G\Box H$ if G is a type 1 graph and H is a tier 1 graph.

Theorem 3.2.2. *Let G be a type 1 graph and H be a tier 1 graph such that $\chi(H) \leq \chi''(G)$. Then $\chi''(G\Box H) = \Delta_{G\Box H} + 1$.*

Proof: Suppose that $\chi''(G) = k = \Delta_G + 1$ and suppose that $\chi'(H) = l = \Delta_H$. Then by Lemma 3.2.1, we know that $G\Box H$ can be totally coloured with $k+l$ colours.

Therefore, since $\Delta_{G\Box H} = \Delta_G + \Delta_H = k + l - 1$, we have totally coloured $G\Box H$ with $\Delta_{G\Box H} + 1$ colours. Therefore $G\Box H$ is a type 1 graph as wanted. $\Box$

The next case to consider is if G is a type 1 graph and H is a tier 2 graph. The following theorem establishes the total chromatic number for this case.

Theorem 3.2.3. *Let G and H be simple graphs. Suppose that $\chi''(G) = \Delta_G + 1 = k$. Also suppose that $\chi(H) \leq k$ and $\chi'(H) = \Delta_H + 1 = l$. Then $\chi''(G\Box H) = \Delta_{G\Box H} + 1$.*

Proof: Let f_1 be a total colouring assignment for G and f_2 be a vertex colouring assignment for H, both using the colours $\{0, 1, \ldots, k-1\}$. Let f_3 be an edge colouring assignment for H, using the colours $\{k, k+1, \ldots, k+l-1\}$. Apply Lemma 3.2.1 to $G\Box H$. This is a total colouring of $G\Box H$ using $k + l = \Delta_{G\Box H} + 2$ colours. We will recolour all instances of the colour 0 from this total colouring, so that we have a total colouring of $G\Box H$ using only $\Delta_{G\Box H} + 1$ colours.

Fix a vertex (u, v). There are at most $\Delta_H = l - 1$ edges of the form $(u, v)(u, v_i)$ incident to (u, v). Edges of this form receive the colour $f_3(vv_i)$. Thus there is at least one colour from the list of colours that f_3 uses, namely $\{k, k+1, \ldots, k+l-1\}$, that is not used on any edge incident to (u, v). Let this colour be denoted c_v. The colour assigned to the edges of the form $(u, v)(u, v_i)$, namely $f_3(vv_i)$, does not depend on u. Therefore c_v does not depend on u; this is why it was defined with the subscript v.

For each vertex (u, v), if (u, v) is assigned the colour 0, replace 0 by the colour c_v. This colour is not used on any vertices. Also this colour is not used on any of the edges incident to (u, v) by definition. Thus this recolouring will not result in any conflicts.

For each edge of the form $(u, v)(u', v)$ that is assigned the colour 0, replace 0 by the colour c_v. As mentioned earlier, c_v does not depend on u; thus the endpoints of $(u, v)(u', v)$ are both missing the colour c_v from their incident edges. Therefore this recolouring will not result in any conflicts.

We now have a proper total colouring of $G\Box H$ using $\Delta_{G\Box H} + 1$ colours as desired. Therefore we have that $\chi''(G\Box H) = \Delta_{G\Box H} + 1$. $\Box$

Interestingly, the algorithms used in Theorem 3.2.2 and Theorem 3.2.3 imply that the graph product of $G\Box H$ does not depend on the edge colourability of G or H. The main factor in determining the total chromatic number of $G\Box H$ is the total colourability of G and H. The difficulty in classifying the total chromatic number of a graph product, as we will see in the next section, is if one or more of the factors are type 2 graphs. If both of the factors are type 1 graphs however, then the graph product is a type 1 graph.

Corollary 3.2.4. *If both G and H are type 1 graphs, then $G\Box H$ is a type 1 graph.*

Proof: First it is important to note that the graph product is symmetric. Suppose that $\chi''(G) = k$ and $\chi(H) = l$. If $k \leq l$, then we can apply Theorem 3.2.2 or Theorem 3.2.3 depending on if H is a tier 1 or tier 2 graph respectively. If $k > l$, then we can apply Theorem 3.2.2 or Theorem 3.2.3 to $H\Box G$ depending on if G is a tier 1 or tier 2 graph respectively. $\Box$

Now we have determined the total chromatic number of a graph product if both of the factors are type 1 graphs.

3.3 Graph Products with Tier 2 Graphs

In this section we determine an upper bound on the total chromatic number of graph products when G is a type 2 graph. Similar to the previous section, we will consider two cases, one when H is a tier 1 graph and one when H is a tier 2 graph. We will start with the former case.

Theorem 3.3.1. *Let G be a type 2 graph and H be a tier 1 graph such that $\chi(H) \leq \chi''(G)$. Then $\chi''(G\Box H) \leq \Delta_{G\Box H} + 2$.*

Proof: Suppose that $\chi''(G) = \Delta_G + 2 = k$ and that $\chi'(H) = \Delta_H = l$. Apply Lemma 3.2.1 to $G\Box H$. This will totally colour $G\Box H$ with $k + l$ colours. Since $\Delta_{G\Box H} = \Delta_G + \Delta_H$, we have that $k + l = \Delta_G + 2 + \Delta_H = \Delta_{G\Box H} + 2$. Therefore we have that $\chi''(G\Box H) \leq \Delta_{G\Box H} + 2$. $\Box$

Theorem 3.3.2. *Let G be a type 2 graph and H a tier 2 graph such that $\chi(H) \leq \chi''(G)$. Then $\chi''(G\Box H) \leq \Delta_{G\Box H} + 2$.*

Proof: Suppose that $\chi''(G) = \Delta_G + 2$ and that $\chi'(H) = \Delta_H + 1 = l$. Apply the algorithm used in Theorem 3.2.3 to $G\Box H$. This will totally colour $G\Box H$ with

$\Delta_G + \Delta_H + 2 = \Delta_{G\Box H} + 2$ colours. Therefore, since $\Delta_{G\Box H} = \Delta_G + \Delta_H$, we have that $\chi''(G\Box H) \leq \Delta_{G\Box H} + 2$. $\square$

The following corollary will combine Theorem 3.3.1 and Theorem 3.3.2 to establish the total chromatic number of $G\Box H$ when both G and H are type 2.

Corollary 3.3.3. *If G and H are both type 2 graphs, then $\chi''(G\Box H) \leq \Delta_{G\Box H} + 2$.*

Proof: Suppose that $\chi''(G) = k$ and $\chi(H) = l$. If $k \leq l$, then we can apply Theorem 3.3.1 or Theorem 3.3.2 depending on if H is a tier 1 or tier 2 graph respectively. If $k > l$, then we can apply Theorem 3.3.1 or Theorem 3.3.2 to $H\Box G$ depending on if G is a tier 1 or tier 2 graph respectively. $\square$

The following theorem summarizes the results found on graph products.

Theorem 3.3.4. *If $\chi''(G) \leq \Delta_G + 2$ and $\chi''(H) \leq \Delta_H + 2$, then $\chi''(G\Box H) \leq \Delta_{G\Box H} + 2$.*

Proof: Follows immediately from Theorem 3.2.2, Theorem 3.2.3, Theorem 3.3.1, and Theorem 3.3.2. $\square$

3.3.1 Total Chromatic Number of the Rook Graph

We will now use the results found for graph products and apply them to the rook graph. The *rook graph*, $K_n\Box K_m$, represents all the legal moves of a rook chess piece. In the following theorem, we show that the total chromatic number of the rook graph satisfies the Total Colouring Conjecture.

Corollary 3.3.5. *$\chi''(K_n\Box K_m) \leq \Delta_{K_n\Box K_m} + 2$. If n is odd and $n \geq m$, or n and m are both odd, then $K_n\Box K_m$ is a type 1 graph.*

Proof: Since $\chi''(K_n) \leq \Delta_{K_n} + 2$ and $\chi''(K_m) \leq \Delta_{K_m} + 2$ [5], by Theorem 3.3.4 we have that $\chi''(K_n\Box K_m) \leq \Delta_{K_n\Box K_m} + 2$. Therefore all that remains is to show $K_n\Box K_m$ is a type 1 graph under certain conditions. Suppose that n is odd. Then K_n is a type 1 graph [5]. We will prove the second part of this theorem by case distinction on m being even or odd.

Case 1: If m is odd then K_m is a type 1 graph [4]. Therefore since K_n and K_m are type 1 graphs, by Corollary 3.2.4, we have that $K_n\Box K_m$ is a type 1 graph.

Case 2: Suppose that m is even and that $n \geq m$. Since $\chi(K_m) = m$, we have that $\chi''(K_n) = n \geq m = \chi(K_m)$. Thus K_n is a type 1 graph and $\chi(K_m) \leq \chi''(K_n)$. Therefore $K_n \Box K_m$ is a type 1 graph by Theorem 3.2.2. $\square$

We have now proved that the rook graph satisfies the Total Colouring Conjecture. However, we conjecture that the following, stronger statement is true.

Conjecture 3.3.6. *$K_n \Box K_m$ is a type 1 graph if $n, m \geq 3$.*

The reason why we suspect that Conjecture 3.3.6 holds is that we have seen earlier an example of the total colouring of $K_4 \Box K_2$, which can be extended to $K_4 \Box K_n$ because $K_4 \Box K_n$ is $n/2$ copies of $K_4 \Box K_2$. Therefore if we can show that $K_{2n} \Box P_2$ is a type 1 graph for all n, we could extend this colouring to $K_{2n} \Box K_m$ to prove the conjecture. The difficulty in proving that $K_{2n} \Box P_2$ is a type 1 graph is knowing the structure of the colouring of K_{2n}.

This concludes the section of Cartesian graph products. We have determined what the total chromatic number of a graph product is when the factors are type 1 and type 2 graphs. In all cases, $G \Box H$ satisfies the Total Colouring Conjecture.

3.4 Total Chromatic Number of Tensor Graph Products

In this section, we determine the total chromatic number of some tensor graph products involving P_n and some tensor graph products involving C_n.

Definition 3.4.1. *The tensor graph product of two graphs G and H, denoted $G \times H$, has vertex set $V(G) \times V(H)$ and two vertices, (u_1, v_1) and (u_2, v_2) in $G \times H$ are adjacent if $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)$.*

Theorem 3.4.2. *$P_n \times P_m$ is a type 1 graph if $n > 2$ or $m > 2$. $P_2 \times P_2$ is a type 2 graph.*

Proof: Label the vertices of P_n by $u_0, u_1, \ldots, u_{n-1}$ and label the vertices of P_m by $v_0, v_1, \ldots, v_{m-1}$ in the natural way. Then (u_i, v_j), where $1 \leq i < n-1$ and $1 \leq i < m-1$, are only adjacent to the vertices $(u_{i-1}, v_{j-1}), (u_{i-1}, v_{j+1})$, (u_{i+1}, v_{j-1}), and (u_{i-1}, v_{j+1}). Thus if $n = 2$, then $P_2 \times P_m$ is comprised of two disjoint P_m graphs which can be seen in Figure 3.9. We use thick edges to distinguish the two paths.

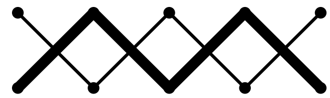

Figure 3.9: Graph $P_2 \times P_5$

If $m > 2$, then P_m is a type 1 graph, thus $P_2 \times P_m$ is a type 1 graph. If $m = 2$, then P_2 is a type 2 graph, thus $P_2 \times P_2$ is a type 2 graph.

Suppose that $n > 2$. Then $P_n \times P_m$ is comprised of two disjoint lattices as is shown in Figure 3.10. We use thick edges to distinguish the two lattices.

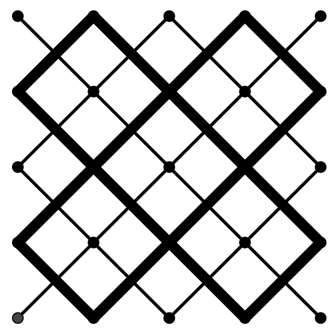

Figure 3.10: Graph $P_5 \times P_5$

These lattices are subgraphs of $P_n \Box P_m$, having the same maximum degree as $P_n \Box P_m$. P_n is a type 1 graph and $\chi(P_m) = 2 < 3 = \chi''(P_n)$. Thus by Theorem 3.2.2, we have that $P_n \Box P_m$ is a type 1 graph. Therefore $P_n \times P_m$ is a type 1 graph when $n > 2$ or $m > 2$. $\square$

3.4.1 Total Chromatic Number of Bipartite Doubles

The *bipartite double* of a graph G is a bipartite graph with bipartition (U, V) and has two vertices u' and u'' for each $u \in V(G)$ such that $u' \in U$ and $u'' \in W$. Two vertices u' and v'' are connected by an edge in the bipartite double if and only if u and v are connected by an edge in G. The bipartite double of a graph G is isomorphic to $G \times P_2$.

Theorem 3.4.3. *Let G be a graph, then $\chi''(G \times P_2) \leq \Delta_{G \times P_2} + 2$*

Proof: $G \times P_2$ is a bipartite graph by construction. As mentioned in the section on crown graphs, all bipartite graphs are type 1 or type 2. Therefore we have that $\chi''(G \times P_2) \leq \Delta_{G \times P_2} + 2$. $\square$

We want to determine for what graphs G is $G \times P_2$ a type 1 graph. To determine this, we will need the following lemma.

Lemma 3.4.4. *If G is a bipartite graph, then $G \times P_2$ consists of two disjoint copies of G.*

Proof: Suppose that G is a bipartite graph and that X and Y are a bipartition of the vertices of G. Label the vertices of P_2 by 0 and 1. Then we will prove that $(X \times \{0\}) \cup (Y \times \{1\})$ and $(X \times \{1\}) \cup (Y \times \{0\})$ are the two disjoint copies of G.

Since 0 is adjacent to 1 in P_2, the edges in $(X \times \{0\}) \cup (Y \times \{1\})$ are the same edge dependencies from X to Y in G. Therefore $(X \times \{0\}) \cup (Y \times \{1\}) \cong G$ and $(X \times \{1\}) \cup (Y \times \{0\}) \cong G$. Thus all that is left to prove is that there are no edges between them.

Vertices in $X \times \{0\}$ are not adjacent to vertices in $X \times \{1\}$ because vertices in X are not adjacent. Also vertices in $X \times \{0\}$ are not adjacent to vertices in $Y \times \{0\}$ because the vertex 0 in P_2 is not adjacent to itself. Therefore no vertex in $X \times \{0\}$ is adjacent to a vertex in $(X \times \{1\}) \cup (Y \times \{0\})$. The same argument shows that no vertex in $Y \times \{1\}$ is adjacent to a vertex in $(X \times \{1\}) \cup (Y \times \{0\})$. Thus $(X \times \{0\}) \cup (Y \times \{1\})$ and $(X \times \{1\}) \cup (Y \times \{0\})$ are disjoint and $G \times P_2$ is two disjoint copies of G. $\square$

Theorem 3.4.5. *Let G be a bipartite graph. If G is a type 1 or type 2 graph, then $G \times P_2$ is a type 1 or type 2 graph respectively.*

Proof: Let G be a bipartite graph. Then $G \times P_2$ is two disjoint copies of G by Lemma 3.4.4. If G is a type 1 graph, we can totally colour each copy of G with $\Delta_G + 1$ colours.

If G is a type 2 graph, we cannot totally colour each copy of G with $\Delta_G + 1$ colours. We can however totally colour each copy of G with $\Delta_G + 2$ colours. Therefore, if G is a type 1 or type 2 graph, then $G \times P_2$ is a type 1 or type 2 graph respectively. $\square$

Theorem 3.4.6. *$C_n \times P_2$ is a type 1 graph if n is divisible by 3. Otherwise $C_n \times P_2$ is a type 2 graph.*

Proof: C_n is type 1 if n is divisible by 3 and is type 2 otherwise [4]. Suppose that n is even and not divisible by 3. Then C_n is a bipartite graph because n is even. Thus by Lemma 3.4.4, $C_n \times P_2$ is two disjoint copies of C_n. Since n is not divisible by 3,

C_n is a type 2 graph. Thus we have that $C_n \times P_2$ is a type 2 graph. If n is even and divisible by 3, then the same argument gives us that $C_n \times P_2$ is a type 1 graph.

Suppose that n is odd. Then note that $C_n \times P_2 = C_{2n}$. If n is divisible by 3, then $2n$ is divisible by 3 and $C_{2n} = C_n \times P_2$ is a type 1 graph. If n is not divisible by 3, then C_{2n} is not divisible by 3, and $C_{2n} = C_n \times P_2$ is a type 2 graph. Therefore we have that $C_n \times P_2$ is a type 1 graph if n is divisible by 3 and $C_n \times P_2$ is a type 2 graph otherwise. $\square$

3.5 Total Chromatic Number of Strong Graph Products

Now that we have determined the total chromatic number for Cartesian graph products, we want to apply these results to determine the total chromatic number for strong graph products.

Definition 3.5.1. *The strong graph product of two graphs G and H, denoted $G \boxtimes H$, has vertex set $V(G) \times V(H)$ and two vertices, (u, v) and (u', v') in $G \boxtimes H$ are adjacent if and only if:*
(1) $u = u'$ and $vv' \in E(H)$, or
(2) $v = v'$ and $uu' \in E(G)$, or
(3) $uu' \in E(G)$ and $vv' \in E(H)$.

Note that $E(G \boxtimes H) = E(G \Box H) \bigcup E(G \times H)$, which implies that $\Delta_{G \boxtimes H} = \Delta_{G \Box H} + \Delta_{G \times H}$. This is because Conditions (1) and (2) of Definition 3.4.1 are the edge conditions for Cartesian products, whereas condition (3) is the edge condition for tensor graph products. We will use the following result from Jaradat for this section.

Theorem 3.5.2 (Jaradat [14]). *Let G and H be two graphs such that at least one of them is tier 1. Then $G \times H$ is a tier 1 graph.*

If one of the terms in a strong graph product is a type 1 graph and one of the terms is a tier 1 graph, then the total chromatic number can be classified as follows:

Theorem 3.5.3. *Let G be a type 1 graph such that $\chi(H) \leq \chi''(G)$. If either G or H is a tier 1 graph, then $G \boxtimes H$ is a type 1 graph.*

Proof: Without loss of generality, assume that G is a tier 1 graph. Then from Theorem 3.5.2, we have that $G \times H$ can be edge coloured with $\Delta_{G\times H}$ colours. From Theorem 3.2.2, we have that $G\Box H$ can be totally coloured with $\Delta_{G\Box H}+1$ colours. Therefore, by combining these colourings, we totally colour $G \boxtimes H$ with $\Delta_{G\times H} + \Delta_{G\Box H} + 1 = \Delta_{G\boxtimes H} + 1$ colours. Therefore, $G \boxtimes H$ is a type 1 graph as wanted. $\Box$

A corollary that follows from this theorem is that the kings graph, $P_n \boxtimes P_m$, is a type 1 graph for sufficiently large n or m. The kings graph represents all legal moves of a king chess piece on an n by m chessboard.

Corollary 3.5.4. *For all positive integers n and m,*

$$\chi''(P_n \boxtimes P_m) = \begin{cases} \Delta_{P_n\boxtimes P_m} + 2, & \text{if } m = 2 \text{ and } n = 2 \\ \Delta_{P_n\boxtimes P_m} + 1, & \text{if } m > 2 \text{ or } n > 2 \end{cases}$$

Proof: If $m = 2$ and $n = 2$ then $P_2 \boxtimes P_2 = K_4$ which is a type 2 graph. Suppose that $n > 2$ or $m > 2$. Then P_n (or P_m) is a type 1 graph and a tier 1 graph. Therefore by Theorem 3.5.3, we have that $P_n \boxtimes P_m$ is a type 1 graph. $\Box$

Example 3.5.5. *Here we show an explicit colouring of $P_3 \boxtimes P_3$. The first step is to colour $P_3\Box P_3$ using the algorithms from the previous section. Then edge colour $P_3 \times P_3$, as shown in Figure 3.11. The last step is to combine these colourings as seen in Figure 3.12.*

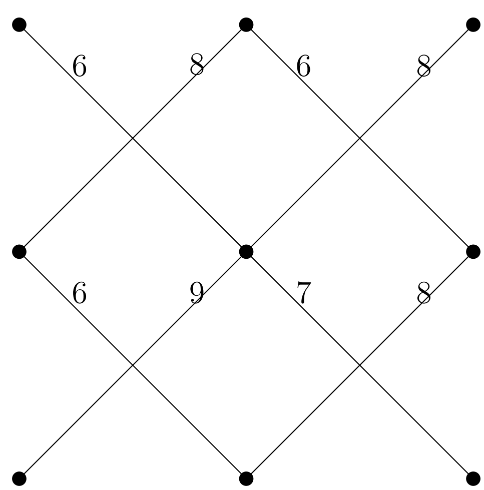

Figure 3.11: Edge Colouring of $P_3 \times P_3$

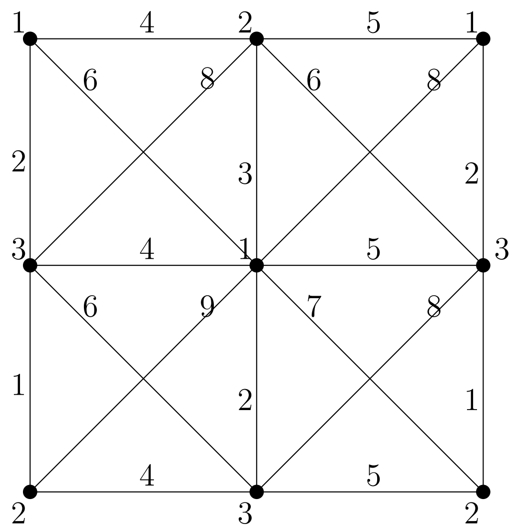

Figure 3.12: Total Colouring of $P_3 \boxtimes P_3$

The following theorem will determine an upper bound on the total chromatic number of $G \boxtimes H$ when neither G nor H are tier 1 graphs.

Theorem 3.5.6. *Let G be type 1 graph such that $\chi(H) \leq \chi''(G)$. If neither G nor H is a tier 1 graph, then $\chi''(G \boxtimes H) \leq \Delta_{G \boxtimes H} + 2$.*

Proof: We apply the same argument used in the proof of Theorem 3.5.3. By Vizing's Theorem, we have that $\chi'(G \times H) \leq \Delta_{G \times H} + 1$. Then by Theorem 3.2.2, we have that $G \Box H$ is a type 1 graph. Therefore by combining these colourings, we can totally colour $G \boxtimes H$ with $\Delta_{G \times H} + 1 + \Delta_{G \Box H} + 1 = \Delta_{G \boxtimes H} + 2$ colours. Therefore we have that $\chi''(G \boxtimes H) \leq \Delta_{G \boxtimes H} + 2$ as wanted. $\square$

Now we have classified an upper bound on the total chromatic number of a strong graph product if one of the factors is a type 1 graph. Next, we will explore the case when one of the factors is a type 2 graph. The following theorem establishes an upper bound on the total chromatic number if one of the factors is a type 2 graph.

Theorem 3.5.7. *Let G be a type 2 graph such that $\chi(H) \leq \chi''(G)$. If either G or H is a tier 1 graph, then $\chi''(G \boxtimes H) \leq \Delta_{G \boxtimes H} + 2$.*

Proof: Without loss of generality, assume that G is a tier 1 graph. Then by Theorem 3.3.1, we have that $\chi''(G \Box H) \leq \Delta(G \Box H) + 2$. Since G is a tier 1 graph, we have that $G \times H$ is a tier 1 graph by Theorem 3.5.2. Therefore by combining these colourings, we can totally colour $G \boxtimes H$ with $\Delta_{G\times H} + \Delta_{G\Box H} + 2 = \Delta_{G\boxtimes H} + 2$ colours, so $\chi''(G \boxtimes H) \leq \Delta_{G\boxtimes H} + 2$. $\square$

Example 3.5.8. *Theorem 3.5.7 does not give an explicit total colouring algorithm. Here we show an explicit total colouring of $K_4 \boxtimes P_2$ First, totally colour $K_4 \Box P_2$ as was previously shown in Figure 3.5. Now we edge colour $K_4 \times P_2$. We know from Theorem 3.5.2 that we can edge colour $K_4 \times P_2$ with 4 colours. An explicit edge colouring of $K_4 \times P_2$ is shown in Figure 3.13. Then we want to combine the edge colouring of $K_4 \times P_2$ onto $K_4 \Box P_2$ as shown in Figure 3.14.*

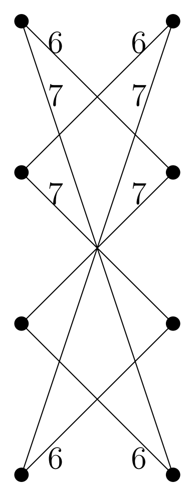

Figure 3.13: Edge Colouring of $K_4 \times P_2$

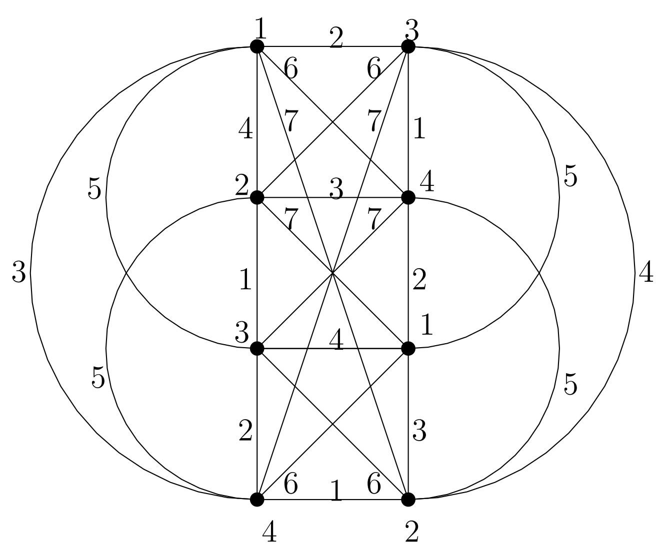

Figure 3.14: Total Colouring of $K_4 \boxtimes P_2$

The last case to consider to is if both graphs are type 2 and tier 2 graphs. In this case we get an upper bound that is outside the Total Colouring Conjecture. The following theorem will determine an upper bound for this case.

Theorem 3.5.9. *Let G be a type 2 graph such that $\chi(H) \leq \chi''(G)$. If neither G nor H is a tier 1 graph, then $\chi''(G \boxtimes H) \leq \Delta_{G\boxtimes H} + 3$.*

Proof: From Theorem 3.3.1, we have that $\chi''(G\square H) \leq \Delta(G\square H) + 2$. Then by Vizing's Theorem, we have that $\chi'(G \times H) = \Delta_{G\times H} + 1$. Therefore by combining these colourings, we can totally colour $G \boxtimes H$ with $\Delta_{G\wedge H} + 1 + \Delta_{G\square H} + 2 = \Delta_{G\boxtimes H} + 3$ colours, so $\chi''(G \boxtimes H) \leq \Delta_{G\boxtimes H} + 3$. $\square$

We are now done our exploration of graph products. In the next chapter, we explore how vertex multiplication affects the total chromatic number of a graph.

Chapter 4

Vertex Multiplication

In this chapter, we are going to investigate what happens to the total chromatic number of a graph when we replace a component of the graph. In particular, we will explore vertex multiplication, and how it affects the total chromatic number.

Definition 4.0.10 (**Vertex Multiplication**). *Given a graph $G = (V, E)$ and a vertex $v \in V$, define the graph $G(v \cdot m)$ as follows. $G(v \cdot m)$ has vertex set $V' = V \setminus \{v\} \cup A_v$, where A_v is a set of m new vertices. The edge set of $G(v \cdot m)$ consists of all edges in E that do not have v as an endpoint, edges between all pairs of vertices in A_v, and edges between any two vertices $u \in N_G(v)$ and $a \in A_v$.*

Vertex multiplication is abstract and seldomly researched. Thus a few examples will be beneficial. Figure 4.1 and Figure 4.2 illustrate the process of vertex multiplication on the house graph.

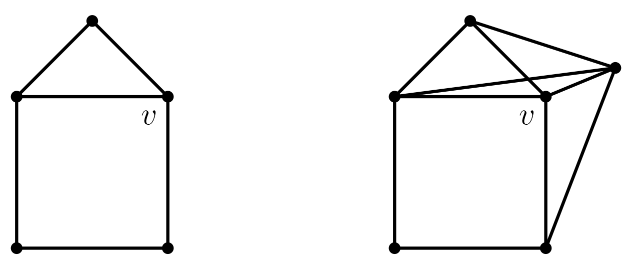

Figure 4.1: Graph G and $G(v \cdot 2)$

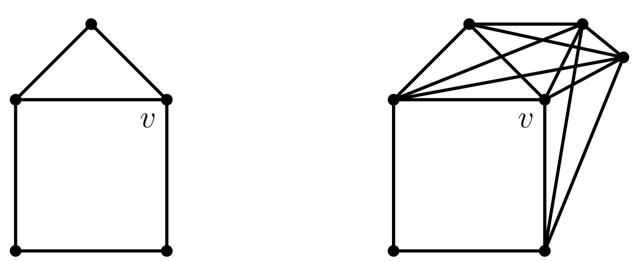

Figure 4.2: Graph G and $G(v \cdot 3)$

To start this chapter, we will establish what happens to the total chromatic number of S_n when a vertex in S_n is multiplied m times. Then we will do the same for C_n. The motivation for these cases is to get an understanding of how vertex multiplication affects general graphs.

4.1 Vertex Multiplication in Star Graphs

In this section, we determine what happens to the total chromatic number of the star graph when any vertex is multiplied m times. The star graph on n vertices, denoted S_n, is graph $K_{1,n}$. For convenience, in S_n we will label the central vertex v and the leaves $\{v_0, v_1, \ldots, v_{n-1}\}$. In A_v, we will label the vertices $\{r_0, \ldots, r_{m-1}\}$. By combining these two labelings, we get a labeling of $S_n(v \cdot m)$. Figure 4.3 illustrates the labeling of S_4 and $S_4(v \cdot 3)$.

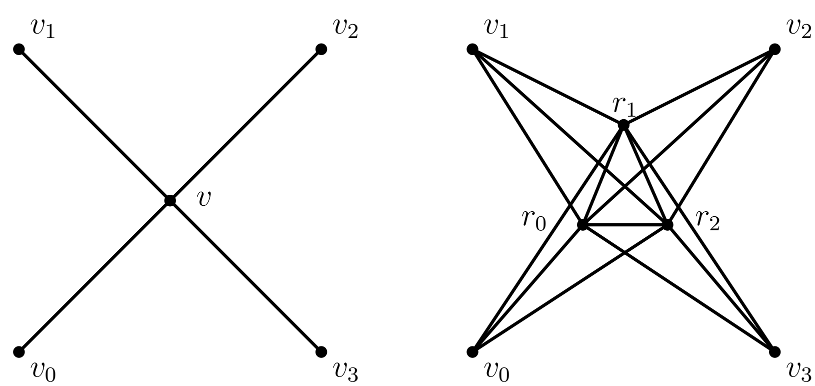

Figure 4.3: Labeling of S_4 and $S_4(v \cdot 3)$

The following lemma will determine what happens to the total chromatic number of $S_2 = P_3$, when v is multiplied m times.

Lemma 4.1.1. *Let v be the central vertex of S_2. Then we have that:*

$$\chi''(S_2(v \cdot m)) = \begin{cases} \Delta_{S_2(v \cdot m)} + 1, & \text{if } m \text{ is odd or } m=2 \\ \Delta_{S_2(v \cdot m)} + 2, & \text{if } m \neq 2 \text{ and } m \text{ is even} \end{cases}$$

Proof: S_2 has 3 vertices, so $S_2(v \cdot m)$ will have $3 + (m-1) = m + 2$ vertices. Since the vertex with maximum degree is being multiplied, we have that $\Delta_{S_2(v \cdot m)} = \Delta_{S_2} + (m-1) = 2 + (m-1) = m + 1$. We will prove this lemma by case distinction on m being even or odd.

Case 1: Suppose that m is odd. Note that $S_2(v \cdot m)$ is a subgraph of K_{m+2} having the same maximum degree. Thus we can use the total colouring of K_{m+2} restricted to $S_2(v \cdot m)$. This is a total colouring using $\Delta_{K_{m+2}} + 1$ colours since $m+2$ is odd [5]. Therefore we have a total colouring of $S_2(v \cdot m)$ using $\Delta_{K_{m+2}} + 1 = \Delta_{S_2(v \cdot m)} + 1$ colours as wanted.

Case 2: Suppose that m is even. For this case, we could use the same argument used in the previous case. This is a total colouring using $\Delta_{K_{m+2}} + 2$ colours since m is even. Thus we can totally colour $S_2(v \cdot m)$ with $\Delta_{K_{m+2}} + 2 = \Delta_{S_2(v \cdot m)} + 2$ colours. Therefore all that remains is to show that $S_2(v \cdot m)$ cannot be totally coloured with $\Delta_{S_2(v \cdot m)} + 1$ colours.

To show that this cannot be done, we will give a proof by contradiction. To find the contradiction, we first need to find $\alpha(S_2(v \cdot m))$ and $\alpha'(S_2(v \cdot m))$. Note that the vertices r_j, where $0 \leq j < m$, are adjacent to all the vertices in $S_2(v \cdot m)$. Whereas the vertices v_0 and v_1, are only adjacent to the vertices r_j. Therefore the largest independent set is $\{v_0, v_1\}$, so $\alpha(S_2(v \cdot m)) = 2$.

Next, $S_2(v \cdot m)$ has a perfect matching with the edges $\{v_0r_0, v_1r_1, r_2r_3, \ldots, r_{m-2}r_{m-1}\}$, so $\alpha'(S_2(v \cdot m)) = \frac{m+2}{2}$. Therefore by Lemma 2.3.2, we know that $\alpha''(S_2(v \cdot m)) \leq \frac{m+2}{2} + 2 = \frac{m+6}{2}$. This graph has a perfect matching however, so if a total set had $\alpha'(S_2(v \cdot m)) = \frac{m+2}{2}$ edges in it, then it could not contain any vertices. Thus a total set with $\frac{m+6}{2}$ elements is not possible. We can get the total set with $\frac{m+4}{2}$ elements however, by taking the elements $\{v_0, v_1, r_0r_1, \ldots, r_{m-2}r_{m-1}\}$.

Note however, that a total colour class of this size can only be picked once because the vertices v_0 and v_1 are now picked. Thus the remaining total sets can have at most $\frac{m+2}{2}$ elements. Suppose that $\chi''(S_2(v \cdot m) = \Delta_{S_2(v \cdot m)} + 1 = m + 2$. Then assuming that there are $(m+1)$ disjoint total sets of size $\frac{m+2}{2}$ and 1 total set of size $\frac{m+4}{2}$, we get that the following number of elements could be maximally coloured:

$$\frac{m+4}{2} + (m+1)\frac{m+2}{2} = \frac{m^2+4m+6}{2}$$

However $|V(S_2(v \cdot m))| = m+2$ and $|E(S_2(v \cdot m))| = \frac{(m+2)(m+1)-2}{2}$. Thus $S_2(v \cdot m)$ has the following number of elements.

$$m + 2 + \frac{(m+2)(m+1)-2}{2} = \frac{m^2+5m+4}{2}$$

If $m > 2$, then $5m + 4 > 4m + 6$, thus $\frac{m^2+5m+4}{2} > \frac{m^2+4m+6}{2}$. Therefore there are more elements in $S_2(v \cdot m)$ than can be maximally coloured by $m+2$ colours. This contradicts our assumption that $S_2(v \cdot m)$ could be totally coloured with $m+2$ colours. Therefore if $m > 2$, $\chi''(S_2(v \cdot m) = \Delta_{S_2(v \cdot m)} + 2$. All that remains is to see if $S_2(v \cdot 2)$ is a type 1 or type 2 graph. It turns out that $S_2(v \cdot 2)$ is a type 1 graph which is illustrated in Figure 4.4. $\square$

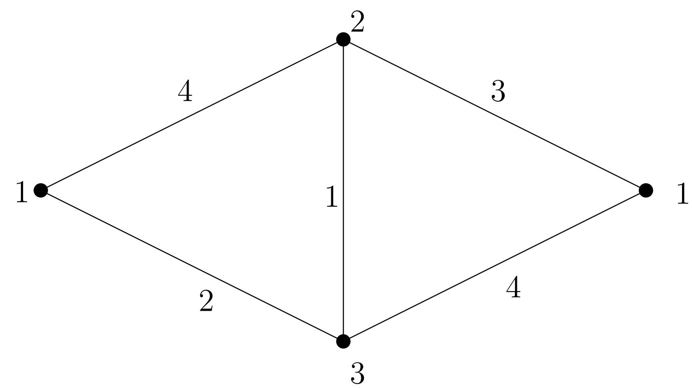

Figure 4.4: Total Colouring of $S_2(v \cdot 2)$

Now we have determined the total chromatic number of $S_2(v \cdot m)$ for all m. Next we will establish an upper bound on the total chromatic number of $S_n(v \cdot m)$ for all n. The following lemma will determine an upper bound on the total chromatic number when n is even. The case when n is odd will follow.

Lemma 4.1.2. *Let v be the central vertex of S_{2n}. Then $\chi''(S_{2n}(v \cdot m)) \leq \Delta_{S_{2n}(v \cdot m)} + 2$. If m is odd or $m \leq 2n$, then $\chi''(S_{2n}(v \cdot m)) = \Delta_{S_{2n}(v \cdot m)} + 1$*

Proof: S_{2n} has $2n+1$ vertices, so $S_{2n}(v \cdot m)$ will have $2n+1+(m-1) = 2n+m$ vertices. Since the vertex with maximum degree is being multiplied, we have that $\Delta_{S_{2n}(v \cdot m)} = \Delta_{S_{2n}} + (m-1) = 2n + (m-1) = 2n+m-1$. We will prove this theorem by case distinction on m being even or odd.

Case 1: Suppose that m is odd. Note that $S_{2n}(v \cdot m)$ is a subgraph of K_{2n+m} having the same maximum degree. Thus we can use the total colouring of K_{2n+m} restricted to $S_{2n}(v \cdot m)$. This is a total colouring using $\Delta_{K_{2n+m}} + 1$ colours since $2n+m$ is odd [5]. Therefore we have a total colouring of $S_{2n}(v \cdot m)$ using $\Delta_{K_{2n+m}} + 1 = \Delta_{S_{2n}(v \cdot m)} + 1$ colours.

Case 2: Suppose that m is even. For this case, we could use the same argument used in the previous case. This is a total colouring using $\Delta_{K_{2n+m}} + 2$ colours since $2n+m$ is even. Thus we can totally colour $S_{2n}(v \cdot m)$ with $\Delta_{K_{2n+m}} + 2 = \Delta_{S_{2n}(v \cdot m)} + 2$ colours. Therefore all that remains is to show that $S_{2n}(v \cdot m)$ is a type 1 graph if $m \leq 2n$.

In a total colouring of K_m, each total colour class leaves one vertex and all its incident edges uncoloured, except the single total colour class is a perfect matching. This is because one colour is assigned to a vertex, then there an odd number of vertices remaining, so a perfect matching with those vertices does not exist.

Take the total colouring of K_m, using the colours $\{2n-1, 2n, \ldots, 2n+m-1\}$, and apply it to the induced subgraph A_v. Without loss of generality, assume that the colour $2n+m-1$ was the colour used on the perfect matching in A_v. Then colour the vertices v_i where $0 \leq i < 2n$ with the colour $2n+m-1$. This will not conflict with the total colouring of A_v since this colour was only used on edges in A_v. Now we need two different colouring extensions: one if $2n = m$ and one if $2n \neq m$.

Subcase 1: Suppose that $2n \neq m$. Then colour the edges $v_{2n-1}r_i$ with the single colour that is not on r_i or any of the edges incident with r_i in the total colouring of A_v. Note that these missing colours will not conflict with the colours of vertices v_{2n-1} and r_i. These missing colours are distinct, so there is also no colour conflict with the edges incident on v_{2n-1}. Now all that remains is to colour the edges $v_i r_j$ where $0 \leq j < m$ and $i \neq 2n-1$.

Colour $v_i r_j$ with the colour $(i+j)(\text{mod } 2n-1)$. The edges incident on v_i will receive the colours $\{i, i+1((\text{mod } 2n-1), \ldots, i+m-1((\text{mod } 2n-1)\}$. If $m = 2n$, then the edge $v_i r_0$ and $v_i r_{m-1}$ would receive the same colour. Since $m \leq 2n$ and $m \neq n$, we have that $m < 2n$, thus this set of colours is distinct. Similarily, the edges incident on r_i will receive distinct colours from $\{0, 1, \ldots, 2n-2\}$. These colours are not used in the total colouring of A_v, so there is no conflict with that prior colouring. Therefore we have a total colouring of $S_{2n}(v \cdot m)$ using $2n+m = \Delta_{S_{2n}(v \cdot m)} + 1$ colours. An example of this colouring applied to $S_6(v \cdot 2)$ is shown in Figure 4.5.

Subcase 2: Suppose that $2n = m$. Then colour the edge $v_i r_i$ with the colour that is not on r_i or any of the edges incident with r_i in the total colouring of A_v. All that remains is to colour the edges $v_i r_{i+j(\text{mod } m)}$ where $1 \leq j < 2n$. We will colour the edges $v_i r_{i+j(\text{mod } m)}$ with the colour $j-1$.

The edges incident on v_i (and r_i) will receive the colours $\{0, 1, \ldots, 2n-2\}$. Since these colours were not used in the total colouring of A_v, they will not conflict with that colouring. Therefore we have a total colouring of $S_{2n}(v \cdot m)$ using $2n+m = \Delta_{S_{2n}} + 1$

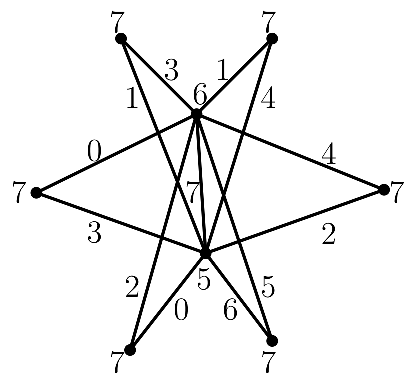

Figure 4.5: Total Colouring of $S_6(v \cdot 2)$

colours as wanted. We have already seen an example of this total colouring in Figure 4.4. Another example of this total colouring is shown in Figure 4.6.

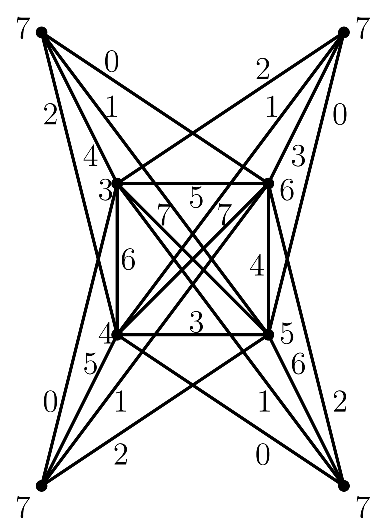

Figure 4.6: Total Colouring of $S_4(v \cdot 4)$

$\square$

Note that if $m > 2n$ and m is even, then we cannot use either of the colouring arguments used in subcase 1 or 2. To prove that $S_{2n}(v \cdot m)$ is type 2 graph when $m > 2n$, we could use an argument on the total number, similar to that used in Lemma 4.1.1. The following lemma will establish an upper bound on the total chromatic number of $S_n(v \cdot m)$ if n is odd.

Lemma 4.1.3. *Let n be odd and let v be the central vertex of S_n. Then $\chi''(S_n(v \cdot m)) \leq \Delta_{S_n(v \cdot m)} + 2$ for all m. If m is even or $m < n$, then $\chi''(S_n(v \cdot m)) = \Delta_{S_n(v \cdot m)} + 1$*

Proof: S_n has $n+1$ vertices, so $S_n(v \cdot m)$ will have $n + 1 + (m - 1) = n + m$ vertices. Since the vertex with maximum degree is being multiplied, we have that $\Delta_{S_n(v \cdot m)} = \Delta_{S_n} + (m - 1) = n + (m - 1) = n + m - 1$. We will prove this lemma by case distinction on m being even or odd.

Case 1: Suppose that m is even. Note that $S_n(v \cdot m)$ is a subgraph of K_{n+m} having the same maximum degree. Thus we can use the total colouring of K_{n+m} restricted to $S_n(v \cdot m)$. This is a total colouring using $\Delta_{K_{n+m}} + 1$ colours since $n + m$ is odd. Therefore we have a total colouring of $S_n(v \cdot m)$ using $\Delta_{K_{n+m}} + 1 = \Delta_{S_n(v \cdot m)} + 1$ colours as wanted.

Case 2: Suppose that m is odd. For this case, we could use the same argument used in the previous case. This is a total colouring using, $\Delta_{K_{n+m}} + 2$ colours since $n + m$ is even. Thus we can totally colour $S_n(v \cdot m)$ with $\Delta_{K_{n+m}} + 2 = \Delta_{S_n(v \cdot m)} + 2$ colours. We will show that $S_n(v \cdot m)$ is a type 1 graph when $m < n$.

Consider the graph $S_{n-1}(v \cdot (m + 1))$. If we remove all the edges $v_i r_0$, where $0 \leq i < n - 1$, from $S_{n-1}(v \cdot (m+1))$, then the resulting graph is $S_n(v \cdot m)$. Therefore $S_n(v \cdot m)$ is a subgraph of $S_{n-1}(v \cdot (m + 1))$ having the same maximum degree. For example, $S_5(v \cdot 3)$ can be seen as a subgraph of $S_4(v \cdot 4)$ in Figure 4.7.

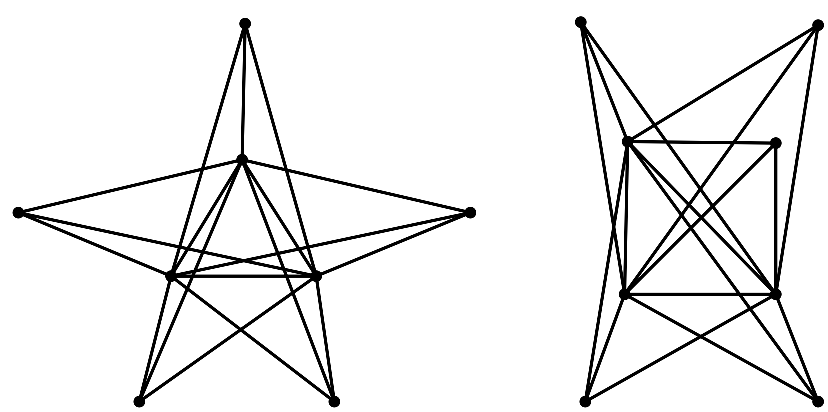

Figure 4.7: Redrawing $S_5(v \cdot 3)$ as a Subgraph of $S_4(v \cdot 4)$

Thus we can use the total colouring of $S_{n-1}(v \cdot (m + 1))$ restricted to $S_n(v \cdot m)$. Since n and m are both odd with $m < n$, we have that $m + 1 \leq n - 1$. Thus by Lemma 4.1.2, $\chi''(S_{n-1}(v \cdot (m+1))) = \Delta_{S_{n-1}(v \cdot (m+1))} + 1$. Therefore we have a proper total colouring of $S_n(v \cdot m)$ using $\Delta_{S_{n-1}(v \cdot (m+1))} + 1 = \Delta_{S_n(v \cdot m)} + 1$ colours. $\square$

Now we have determined $\chi''(S_n(v \cdot m)$ for all n and m when the central vertex is multiplied. The following lemma will establish what happens to the total chromatic number of S_n when one of the leaves is multiplied m times.

Lemma 4.1.4. *Let v_i be a leaf in S_n. Then $S_n(v_i \cdot m)$ is a type 1 graph.*

Proof: We adopt the labeling of the vertices as given at the beginning of this section. Without loss of generality, assume that v_0 is the vertex being multiplied. Note that $\Delta_{S_n(v_0 \cdot m)} = n + m - 1$, so we will prove this lemma by showing that $S_n(v_0 \cdot m)$ can be totally coloured with $n + m$ colours. We will show this by case distinction on m being even or odd.

Case 1: Suppose that m is even. Then $m+1$ is odd, so $\chi''(K_{m+1}) = m+1$. Note that v_0 is only adjacent to v in S_n. Therefore all the vertices in A_v will only be adjacent to v in S_n. Take the total colouring of K_{m+1} using the colours $\{0, 1, \ldots, m\}$ and apply it to the induced subgraph on the vertices $A_v \cup \{v\}$, which is isomorphic to K_{m+1}. All that remains is to colour the vertices v_i and the edges vv_i, where $1 \leq i < n$. Colour the edge vv_i with the colour $m + i$. These $n - 1$ colours are fresh, so they will not conflict with the colouring of $A_v \cup \{v\}$. Suppose that c is the colour assigned to the edge r_0v. Then assign the colour c to v_i. None of the v_i are adjacent to one another, so this is a valid assignment. Thus we have a proper total colouring of $S_n(v_0 \cdot m)$ using $m + n$ colours as wanted. An illustration of this colouring is shown in Figure 4.8.

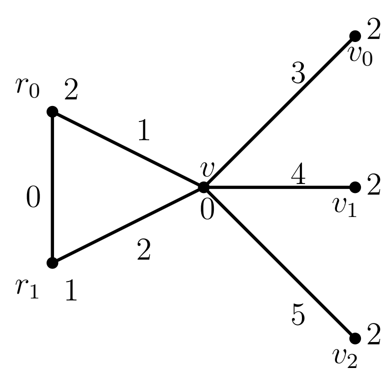

Figure 4.8: Total Colouring of $S_4(v_o \cdot 2)$

Case 2: Suppose that m is odd. Then $m + 1$ is even, so $\chi''(K_{m+1}) = m + 2$. If we repeat the same argument used in case 1, then we will get a proper total colouring of $S_n(v_0 \cdot m)$ using $(m+2)+(n-1) = m+n+1$ colours. In the total colouring of $A_v \cup \{v\}$ however, there is a colour not used on v or any edges incident on v. This is because $A_v \cup \{v\}$ is isomorphic to K_{m+1}. Thus replace the colour $m+2$, that was assigned to the edge vv_1, with the colour not used on v or any edges incident on v. This was the only instance of the colour $m+2$, thus we now have a proper total colouring of $S_n(v_0 \cdot m)$ using $m+n$ colours as wanted. Therefore in both cases we have that $S_n(v_0 \cdot m)$ is a type 1 graph. □

Now we have classified what happens to the total chromatic number of S_n, when any vertex is multiplied m times. In the next section we determine the total chromatic number of $C_n(v \cdot m)$.

4.2 Vertex Multiplication in Cycle Graphs

In this section, we determine what happens to the total chromatic number of C_n when a vertex is multiplied m times. Similar to the previous section, label the vertices in C_n by $\{v_0, v_1, \ldots, v_{n-1}\}$ such that v_i is adjacent to $v_{i \pm 1 (\text{mod } n)}$ where $0 \leq i < n$. Then label the vertices in A_v by $\{r_0, r_1, \ldots r_{m-1}\}$. If a vertex in C_3 is multiplied m times, then $C_3(v \cdot m) = K_{m+2}$, for which we know the total chromatic number. Therefore we will restrict ourselves to cycles of length 4 or more. The following lemma will determine the total chromatic number of $C_n(v \cdot 2)$.

Lemma 4.2.1. *If $n > 3$, then $C_n(v \cdot 2)$ is a type 1 graph.*

Proof: Without loss of generality, assume that v_0 is the vertex that is multiplied. Note that if we subdivide the edge v_1v_2 3 times, then the resulting graph is $C_{n+3}(v_0 \cdot 2)$. Recall from Chapter 3, that if an edge is subdivided 3 times, the total chromatic number does not increase. Thus if we show that $C_4(v \cdot 2), C_5(v \cdot 2)$, and $C_6(v \cdot 2)$ are type 1 graphs, then by applying Lemma 3.1.7, we get that $C_n(v \cdot 2)$ is a type 1 graph for all $n > 3$. The total colourings of $C_4(v \cdot 2), C_5(v \cdot 2)$, and $C_6(v \cdot 2)$ are shown in Figure 4.9, Figure 4.2, and Figure 4.11.

□

The following theorem will determine $\chi''(C_n(v \cdot m))$ for all n and m.

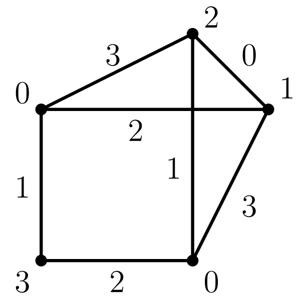

Figure 4.9: Total Colouring of $C_4(v \cdot 2)$

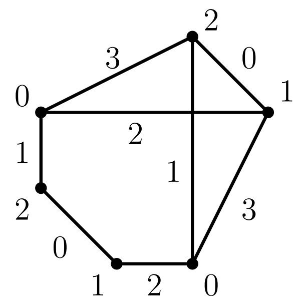

Figure 4.10: Total Colouring of $C_5(v \cdot 2)$

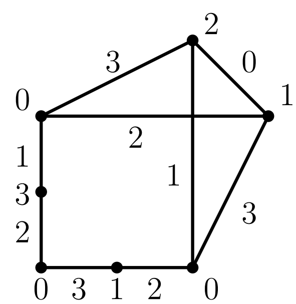

Figure 4.11: Total Colouring of $C_6(v \cdot 2)$

Theorem 4.2.2. *Suppose that $n > 4$. Then $C_n(v \cdot m)$ is a type 1 graph if m is odd and a type 2 graph if m is even.*

Proof: Without loss of generality, assume that v_0 is the vertex being multiplied m times. Consider the induced subgraph on the vertices $\{v_{n-1}, v_1, r_0, r_1, \ldots, r_{m-1}\}$. This induced subgraph is isomorphic to $S_{m+2}(v \cdot 2)$ where v is the central vertex of S_{m+2}.

Case 1: Suppose that m is odd, then $S_{m+2}(v \cdot 2)$ is a type 1 graph by Lemma 4.1.3. Colour this induced subgraph in $C_n(v_0 \cdot m)$ with $m + 2$ colours according to the algorithm used in Lemma 4.1.3. In this colouring, v_{n-1} and v_1 are both missing the same colour. Without loss of generality, suppose that this colour was the colour 0. Assign this colour to the edges $v_{n-1}v_{n-2}$ and v_1v_2. Now all that remains is to colour the elements of the induced path $\{v_2, v_3, \ldots, v_{n-2}\}$. Recall that if we subdivide the edge v_1v_2 in $C_n(v_0 \cdot m)$ 3 times, then the resulting graph is $C_{n+3}(v_0 \cdot m)$.

If we show that in the case $n = 5$ that the current colouring can be extended to the induced path on $\{v_2, v_3\}$, then by applying Lemma 3.1.7, we get that $C_{5+3x}(v_0 \cdot m)$ is a type 1 graph for all x. We can do the same argument if $n = 6$ or $n = 7$ to get that $C_{6+3x}(v_0 \cdot m)$ and $C_{7+3x}(v_0 \cdot m)$ are type 1 graphs for all x. These extended colourings are shown in Figure 4.12, Figure 4.13, and Figure 4.14.

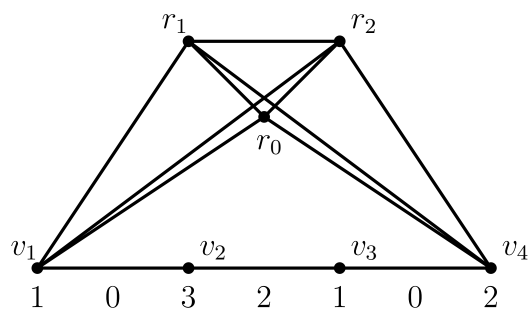

Figure 4.12: Extended Colouring if $n = 5$

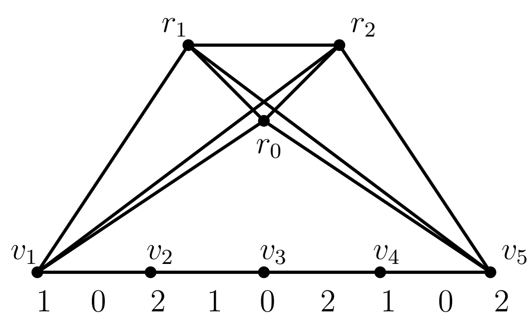

Figure 4.13: Extended Colouring if $n = 6$

Thus in all 3 cases we can extend the total colouring of the induced subgraph $\{v_{n-1}, v_1, r_0, r_1, \ldots, r_{m-1}\}$ to the induced path $\{v_2, v_3, \ldots, v_{n-2}\}$. Therefore by repeated application of Lemma 3.1.7, we have that $C_n(v_0 \cdot m)$ can be totally coloured with $m + 2$ colours. Thus $C_n(v_0 \cdot m)$ is a type 1 graph.

Case 2: If $m > 2$ is even, then $S_{m+2}(v \cdot 2)$ is a type 2 graph by Lemma 4.1.2. Colour the induced subgraph $\{v_{n-1}, v_1, r_0, r_1, \ldots, r_{m-1}\}$, which is isomorphic to $S_{m+2}(v \cdot 2)$, with $m + 3$ colours. All that remains is to colour the elements in the induced

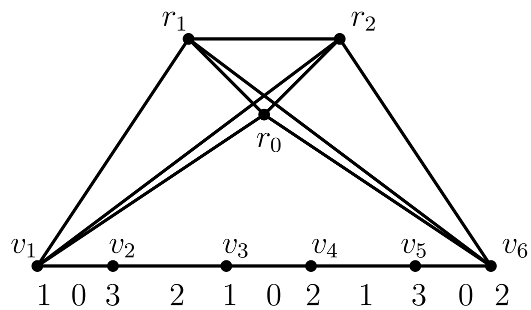

Figure 4.14: Extended Colouring if $n = 7$

path $\{v_2, v_3, \ldots, v_{n-2}\}$. We have already proved in case 1, that the colouring of the induced subgraph $\{v_{n-1}, v_1, r_0, r_1, \ldots, r_{m-1}\}$ can be extended to the induced path $\{v_2, v_3, \ldots, v_{n-2}\}$. Therefore by repeated application of Lemma 3.1.7, we have that $C_n(v_0 \cdot m)$ can be totally coloured with $m+3$ colours. Since the induced subgraph $\{v_{n-1}, v_1, r_0, r_1, \ldots, r_{m-1}\}$ requires $m+3$ colours, there does not exist a total colouring of $C_n(v_0 \cdot m)$ with $m+2$ colours. Therefore $C_n(v_0 \cdot m)$ is a type 2 graph. $\square$

The only remaining case for vertex multiplication in cycles is the case $C_4(v \cdot m)$. We have seen that $C_4(v \cdot 2)$ is a type 1 graph in Figure 4.9. We conjecture that for all other values of m, $C_4(v \cdot m)$ is a type 2 graph based on calculations.

4.3 Vertex Multiplication in Arbitrary Graphs

In the previous two sections, we determined what vertex multiplication does to the total chromatic number of the star graph and cycles. The goal of this section is to generalize these results to determine how vertex multiplication affects graphs in general. For the remainder of this section, all graphs to be considered have maximum degree greater than 2.

Theorem 4.3.1. *Let G be a type 1 graph. If $\deg(v) = 2$ and v is a vertex that is adjacent to a vertex with maximum degree, then $G(v \cdot 2)$ is a type 1 graph.*

Proof: Let f be a minimal total colouring assignment of G using the colours $\{1, 2, \ldots k\}$. Let v_1 and v_2 be the neighbours of the vertex v that is being multiplied and let x and y be the vertices in A_v. First, colour $V(G(v \cdot 2)) \setminus \{v_1, v_2, x, y\}$ with the same colours that f assigned those vertices in G. Next, colour $E(G(v \cdot 2)) \setminus \{v_1x, v_2x, v_1y, v_2y, xy\}$ with the same colours f assigned those edges in G. Then colour the vertices v_1, v_2 and the edge xy with a new colour, $k+1$. Thus all that remains to be coloured in $G(v \cdot 2)$ are the vertices $\{x, y\}$ and the edges $\{v_1x, v_2x, v_1y, v_2y\}$.

Since v is adjacent to a vertex with maximum degree, either $\deg(v_1) = k-1$ or $\deg(v_2) = k-1$. Without loss of generality, assume that v_1 has maximum degree and that the colours $\{1, 2, \ldots, k-2\}$ were the colours used on the edges incident to v_1. We say that a colour is absent at a vertex, if none of its incident edges have received that colour. In particular, $k-1$ and k are both absent at v_1. We will break the proof into what elements are absent at v_2.

Case 1: Suppose both $k-1$ and k are absent at v_2. Then we can colour v_1x and v_2x with colour $k-1$ and k respectively. Then colour v_1y and v_2y with colour k and $k-1$ respectievely. This is valid because k and $k-1$ are absent at v_1 and v_2. All that remains is to colour x and y. The only vertices adjacent to these are v_1 and v_2, which received the colour $k+1$. Thus we can assign x and y the colours 1 and 2 respectively.

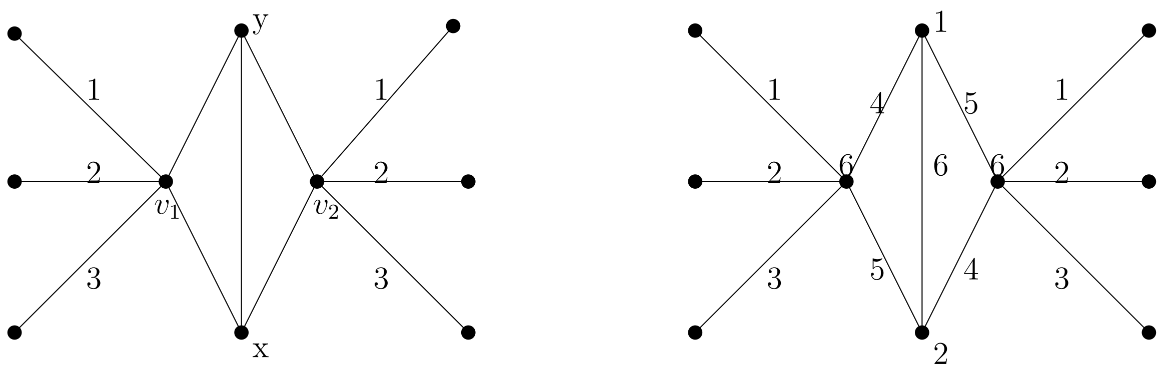

Figure 4.15: Case 1 of Theorem 4.3.1

Case 2: If just $k-1$ is absent at v_2, then colour v_1x and v_2y with colour $k-1$ and colour v_1y and x with colour k. Since k was not absent at v_2, some other colour c must be absent. Thus colour v_2x with colour c and y with colour 1 if $c \neq 1$ and colour 2 otherwise. The same argument is used if just k is absent at v_2.

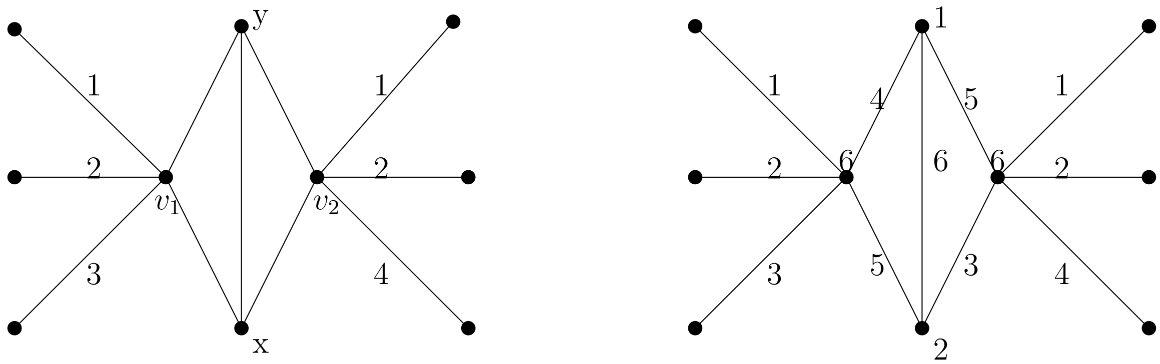

Figure 4.16: Case 2 of Theorem 4.3.1

Case 3: Suppose neither $k-1$ nore k are absent at v_2. Then colour v_1x and y with the colour $k-1$. Then colour v_1y and x with colour k. Lastly colour v_2y and v_2x with the colours c_1 and c_2 respectively, where c_1 and c_2 are the colours absent at v_2.

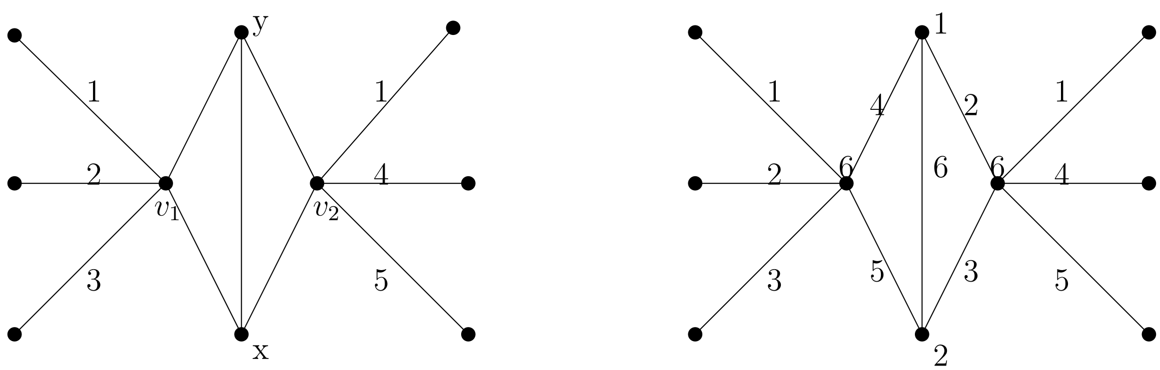

Figure 4.17: Case 3 of Theorem 4.3.1

Therefore in all 3 cases we can extend the total colouring of G to a total colouring of $G(v \cdot 2)$ using $\Delta_{G(v \cdot 2)} + 1$ colours. □

Now we will impose restrictions on the neighbours of a vertex v and $\deg(v)$ to determine the total chromatic number of $G(v \cdot m)$.

Theorem 4.3.2. *Let G be a type 1 graph with $\chi''(G) = \Delta_G + 1 = k$. Suppose that a vertex v with maximum degree or a vertex adjacent to one with maximum degree is being replaced and that none of the neighbours of v are adjacent to one another. Then $G(v \cdot m)$ is a type 1 graph if* $\deg(v) \leq m \leq \deg(v) + 1$.

Proof: Let f be a total colouring of G using the colours $\{m, m+1, \ldots m+k-1\}$ and let $w_0, w_1, \ldots w_{m-1}$ denote the neighbours of v. First, colour the induced subgraph on the vertices $V(G) \setminus \{v\}$ in $G(v \cdot m)$ using the assignment f. Let $r_0, r_1, \ldots r_{m-1}$ be the vertices in A_v. For all of the edges $n_i r_j$, assign the colour $i + j (\text{mod } m)$. This assignment gives all of the edges incident to r_j will receive different colours. Now note that A_v requires m colours if m is odd and $m+1$ colours if m is even.

Case 1: Suppose that m is even. Then using the colours $\{m, m+1, \ldots, m+k-1\}$, colour the induced subgraph A_v. The colours of w_i and r_j may conflict now however. Thus we introduce a fresh colour, $m+k$, to colour the vertices w_i. By assumption, the w_i are not adjacent to one another, so this is a valid assignment. Therefore we have a proper total colouring using $m+k$ colours.

Case 2: Suppose that m is even. Then we need $k+1$ colours to totally colour A_v if $m = k$. Thus totally colour A_v with the colours $\{m, m+1, \ldots, m+k\}$. The colours of w_i and r_j may conflict with one another. Since m is even, there is a total colour class of A_v that is a perfect matching. Without loss of generality, assume that colour $m+k$ was used for that perfect matching. Then assign the vertices w_i with the colour $m+k$. These will not conflict with the r_i since this colour was only used on edges in K_m.

Therefore in both cases, we have that we can totally colour $G(v \cdot m)$ with $k+m$ colours. Since $\Delta_{G(v \cdot m)} = \Delta + m = k + m - 1$, we have that $G(v \cdot m)$ is a type 1 graph as wanted. $\square$

This concludes our exploration of the total chromatic number of vertex multiplication. The complexity in generalizing vertex multiplication to an arbitrary graph, is knowing the structure of the neighbourhood of the vertex being multiplied.

Chapter 5

Conclusions and Future Work

In this thesis, we proved partial results towards the Total Colouring Conjecture. We determined the total chromatic number of the wheel graph and crown graph. We also proved if a graph G contains a cycle of length 4 or more, then $T(G)$ will not be perfect. At the end of Chapter 2, we were left with the following open conjectures.

Conjecture 5.0.3. *J_{2n} is a type 1 graph for all n*

Conjecture 5.0.4. *Let G be a graph, then $T(G)$ is perfect if and only if G does not contain a cycle of length 4 or more.*

In Chapter 3, we gave upper bounds on the total chromatic number for Cartesian, strong, and tensor graph products. We showed that the hypercube, rook, and kings graphs all satisfy the total colouring conjecture. At the end of Chapter 3, we were left with the following open conjectures.

Conjecture 5.0.5. *$K_n \square K_m$ is a type 1 graph if $n > 2$ and $m > 2$.*

Conjecture 5.0.6. *If G and H satisfy the Total Colouring Conjecture then $G \boxtimes H$ will satisfy the Total Colouring Conjecture.*

Lastly, we explored how vertex multiplication affected the total chromatic number. We established that for the star graph and cycle graph, no matter which vertex was multiplied and how many times it was multiplied, the resulting graph satisfies the Total Colouring Conjecture. We then briefly explored how vertex multiplication affects the total chromatic number of an arbitrary graph. At the end of Chapter 4, we were left with the following open conjecture.

Conjecture 5.0.7. *$C_4(v \cdot m)$ is a type 2 graph if $m \neq 2$.*

Researchers interested in total colouring could determine which graphs products result in type 1 or type 2 graphs. If I continued total colouring, I would like to look more into arbitrary vertex multiplication, as it is an interesting way of generating graphs.

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