EINSTEIN-AETHER COSMOLOGICAL SCALAR FIELD MODELS

by

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Dedication

To my parents and my husband, to whom I ever grateful for their continuous love, supports, and prayers to me. I am truly blessed to have to have them in my life.

Abstract

The so-called Einstein-Aether theory is General Relativity coupled (at second derivative order) to a dynamical time-like unit vector field (the "Aether"). It is a Lorentz-violating theory, and has gained much attention in recent years. We study two classes of Einstein-Aether cosmological scalar field models using dynamical systems techniques. In particular, we are interested in exploring the impact of Lorentz violation on the inflationary scenario. We study the local stability of the equilibrium points of the dynamical system corresponding to physically realistic solutions, and find that there are always ranges of values of the parameters of the models for which there exists an inflationary attractor.

In the first application, we investigate the qualitative behaviour of a class of spatially homogeneous Einstein-Aether models with a scalar field. Particularly, we study two models; an isotropic model and an anisotropic model. In both models there always exists a range of the values of the parameters in which there is an attractor which corresponds to an inflationary universe at late times.

In the second application, we study spherically symmetric cosmological models with a scalar field. Particularly, we consider a special case of spatially homogeneous Kantowski-Sachs models using appropriate normalized bounded variables. In this special case, we found that there always exists a range of values in the parameters in which there is one inflationary attractor solution at late times.

List of Abbreviations Used

$\mathbb{R}$	Field of real numbers
GR	General Relativity
EFE	Einstein Field Equations
DE	Differential Equation
FRW	Friedmann Robertson-Walker
$Df(x_0)$	Jacobian Matrix; linearization matrix
ODE	Ordinary Differential Equation
KS	Kantowski-Sacks
CMB	Cosmic Microwave Background
$HG(T)$	Hartman Grobman Theorem

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Chapter 1

Introduction

The aim of studying physical cosmology is to understand the Universe as a whole on large scales. More precisely, cosmology explores the origins of the Universe and its evolution. There are various models of early universe cosmology which incorporate a violation of Lorentz invariance, which include the Einstein-Aether theory [1, 2] and the IR limit of (extended) Horava gravity [3]. When the phenomenology of theories with a preferred frame is studied, it is generally assumed that this frame coincides, at least roughly, with the cosmological rest frame defined by the Hubble expansion of the universe.

Einstein-Aether theory [1, 2] consists of general relativity (GR) coupled, at second derivative order, to a dynamical time-like unit vector field, the "Aether". In Horava gravity, the Aether vector is assumed to be hypersurface-orthogonal; hence every hypersurface-orthogonal Einstein-Aether solution is a Horava solution (most of the solutions studied). In this effective field theory approach, the aether vector field u_a and the metric tensor g_{ab} together determine the local spacetime structure.

The inflationary paradigm [4] provides one of the simplest ways to describe various aspects of the physics of the early universe in standard cosmology. However, despite its successes, some fundamental questions still remain to be answered in this paradigm. We shall discuss the late time dynamics of Einstein-Aether cosmological models. In particular, we explore the impact of Lorentz violation on the inflationary scenario [1, 2, 5, 6].

Cosmological models in Einstein- Aether (Lorentz-violating) theories of gravity are currently of interest. A systematic construction of an Einstein-aether gravity theory with a Lorentz violating dynamical field that preserves locality and covariance in the presence of an additional "Aether" vector field has been presented.

In an isotropic and spatially homogeneous Friedmann universe with expansion scale factor $a(t)$ and co-moving proper time t, the Aether field will be aligned with the cosmic frame and is related to the expansion rate of the universe. The Einstein Field Equations are generalized by the contribution of an additional stress tensor for the Aether field.

In particular, in this thesis we will investigate the qualitative properties of Einstein-Aether (EA) cosmological models with a scalar field at late times and determine whether or not the model has inflationary attractor solutions using dynamical systems theory techniques. Dynamical systems techniques are powerful tools in such an investigation. More precisely, we study the inflationary scenario in the scalar-vector-tensor theory, where the vector is constrained to be a unit and time-like, and investigate whether the inflationary solutions proposed [8] are stable when spatial curvature and anisotropic perturbations are considered.

The governing equations of the most commonly studied cosmological models are a system of ODE. Since our main goal is to give a qualitative description of these models, a dynamical systems approach is undertaken. Usually, a dimensionless time variable, τ, is introduced so that the models are valid for all times since τ takes on all real valued. A normalized set of variables are then chosen for a number of reasons. First, these variables are well-behaved and often have a direct physical interpretation. Second, this normally leads to a bounded dynamical system. Third, one of the equations will be decouple due to a symmetry in the equations. Fourth, it results in a simplified reduced system that is easier to study. The equilibrium points of the reduced system then correspond to dynamically evolving self-similar cosmological models. By using the dimensionless time variable and a normalized set of variables, the governing ODE, define a flow and the evolution of the cosmological models, can then be analysed by studying the orbits of this flow in the state space.

The outline of the thesis is as follows. In chapter (2), we give a brief review of the theory of dynamical systems and some techniques and theorems that we used throughout the thesis to analyze the models under consideration. This thesis contains two parts which are applications of the Einstein-Aether theory.

In part I, we study the "Spatially Homogeneous Einstein-Aether Cosmological Model". We start with a basic introduction to the Einstein-Aether theory. In Chapter (3), we begin by introducing the model and set up the evolution equations, with particular emphasis on Bianchi models of class VI_h with a scalar field. In Chapter (4), we study the isotropic model with two forms for the potential. In Chapter (5), we expand the analysis to study the anisotropic model with three forms for the potential. In Chapter (6), we discuss our final conclusion of this part with emphasis on whether or not the models have an inflationary attractor solutions.

In part II, we study the "Spherically Symmetric Einstein-Aether Cosmological Model with Scalar Field". In Chapter (7), we introduce the model. In Chapter (8), we study the special case of the model called the " Kantowski-Sachs Model". In Chapter (9), we discuss our conclusion for this special case. Finally, in Chapter (10), we state our final conclusion for both parts and some extensions for future work.

Chapter 2

Dynamical Systems Theory

The aim of this chapter is to present a brief review of dynamical systems theory [9, 10, 11, 12, 13].

Consider the dynamical system of differential equations (DE) of the standard form

$$\dot{x} = f(x). \tag{2.0.1}$$

where $\dot{x} = \frac{dx}{dt}$, $x = (x_1, x_2, .., x_n) \in \mathbb{R}^n$ and the map $f : \mathbb{R}^n \to \mathbb{R}^n$. An autonomous system is one in which there is no explicit time dependence. As far as this thesis is concerned, the dynamical systems under consideration are finite dimensional and continuous autonomous systems.

Let us now state briefly some of the key words related to dynamical system which mention in this thesis:

Definition 2.1. Solution of DE: A function $\psi(t) : \mathbb{R} \to \mathbb{R}^n$ is a solution for the DE (2.0.1) if

$$\dot{\psi}(t) = f(\psi(t)) \tag{2.0.2}$$

is satisfied for all $t \in \mathbb{R}$.

Definition 2.2. Equilibrium Point: The point $x_0 \in \mathbb{R}^n$ is an equilibrium point for the DE in (2.0.1) if and only if $f(x_0) = 0$ holds.

Definition 2.3. Linearization Matrix: Linearization matrix is the derivative matrix of $f : \mathbb{R}^n \to \mathbb{R}^n$ (the $n \times n$ matrix) is defined by

$$Df(x_0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{x=x_0}, \quad i, j = 1, ..., n, \tag{2.0.3}$$

where the f_i are the components of the f.

Definition 2.4. Hyperbolic Equilibrium Points: Let x_0 be a equilibrium point for the DE in (2.0.1). Then x_0 is hyperbolic if none of the eigenvalues of the linearization matrix $Df(x_0)$ in (2.0.3), have zero real parts; otherwise it is a non-hyperbolic equilibrium point.

Definition 2.5. The Flow: Let $x(t) = \phi_a(t)$ be a solution of the DE (2.0.1) with an initial condition $x(0) = a$. The flow $\{g^t\}$ is defined in terms of the solution function $\phi_a(t)$ of the DE by

$$g^t{}_a = \phi_a(t). \tag{2.0.4}$$

Definition 2.6. Linear Equivalent: Given two DEs

$$\dot{x} = Ax, \quad \dot{y} = By, \tag{2.0.5}$$

where A and B are matrices. The two DEs in (2.0.5) are linearly equivalent if and only if if there exists an invertible matrix P and positive constant $s > 0$ such that

$$e^{tA} = P^{-1}e^{stB}P \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.0.6}$$

Definition 2.7. The Orbit: The orbit through a, denoted by $\gamma(a)$, is defined by

$$\gamma(a) = \{x \in \mathbb{R}^n | x = g^t{}_a, \text{for all} \quad t \in \mathbb{R}\}. \tag{2.0.7}$$

There are three types of orbits:

- **Point Orbits** correspond to equilibrium points which can be happen if ${g^t}_a = a$ for all $t \in \mathbb{R}$, thus $\gamma(a) = \{a\}$.

- **Periodic Orbits** describe a system that evolves periodically in time which can be happen if there exists a period $T > 0$ such that ${g^T}_a = a$.

- **Non- Periodic Orbits** happen if ${g^t}_a \neq a$ for all $t \neq 0$.

Definition 2.8. Homomorphism Map: A map $H : \mathbb{R}^n \to \mathbb{R}^n$ is a homomorphism on $\mathbb{R}^n$ if and only if it is one to one and onto and it is continuous as well as it's inverse.

Definition 2.9. Topologically Equivalent: Two linear flows e^{tA}, e^{tB} are topologically equivalent if and only if there exists a homomorphism map H and a positive constant s such that

$$H(e^{tA}x) = e^{stB}H(x), \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and for all} \quad t \in \mathbb{R}. \tag{2.0.8}$$

Definition 2.10. Invariant Set: Given a DE as in (2.0.1), a set $S \subseteq \mathbb{R}^n$ is an invariant set for the DE if for any point $a \in S$ the orbit through a lies entirely in S, (i.e., $\gamma(a) \subseteq S$, the orbits stay inside S and never leave).

Definition 2.11. Bifurcation: Given a dynamical system that depends on parameters of the form

$$\dot{x} = f(x, \mu), \tag{2.0.9}$$

where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ represent phase variables and parameters. As the parameters vary, the phase portrait also varies (i.e., A bifurcation is a change of the topological type of the system under parameter variation).

There is one theorem which helps us to study the models in this thesis and provide information regarding the local stability of equilibrium points of the dynamical system. The Hartman Grobman Theorem can be used to analyze the stability of the hyperbolic equilibrium points. On the other hand, we use the multiple scale method to investigate the stability of the non-hyperbolic equilibrium points .

2.1 Hyperbolic Equilibrium Points and the Hartman Grobman Theorem

Given a one dimensional dynamical system $\dot{x} = f(x)$, where $x \in \mathbb{R}$ and x_0 is an equilibrium point. We first define the linear approximation of $f(x)$ such that

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... \tag{2.1.1}$$

which can be written in the generalized form as

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{2.1.2}$$

From the definition of the equilibrium point (i.e., $f(x_0) = 0$) and by ignoring the higher order terms in equation (2.1.2), for values of x in a neighbourhood of x_0 the DE behaves like

$$\dot{x} = f'(x_0)(x - x_0). \tag{2.1.3}$$

In $\mathbb{R}^n$, $f(x)$ can be written as follows

$$f(x) \approx f(x_0) + Df(x_0)(x - x_0) + ..., \tag{2.1.4}$$

where

$$Df(x_0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{x=x_0}, \quad i, j = 1, ..., n. \tag{2.1.5}$$

Then, for x in a neighbourhood of x_0, the DE behaves as

$$\dot{x} = Df(x_0)(x - x_0). \tag{2.1.6}$$

Qualitative analysis of a system starts with finding the location of equilibrium points. Once the equilibrium points of a system of autonomous ordinary differential equations are obtained, it is of interest to consider the dynamics in a local neighbourhood of each of the points. The process of determining the local behaviour is based on the linearization matrix in (2.1.5) at the equilibrium point x_0. Then, it follows that the linear stability of these equilibrium points is classified based on the signs of the eigenvalues of the linearization matrix as follows:

1. If all of the real part of the eigenvalues of the linearization matrix at x_0 are negative ($\lambda_i < 0$), then the equilibrium point of the dynamical system is **asymptotically stable** and the trajectories which start near that point will approach that point.

2. If all of the real part of the eigenvalues of the linearization matrix at x_0 are positive ($\lambda_i > 0$), then the equilibrium point of the dynamical system is a **source** and unstable and the trajectories will go away from that point.

3. If at least one of the real part of the eigenvalues of the linearization matrix at x_0 is positive and the others are negative ($\lambda_i > 0, \lambda_j < 0$), then the equilibrium point of the dynamical system is a **saddle** and it is unstable.

This classification follows from the fact that if the equilibrium point is hyperbolic in nature the flows of the non-linear system and its linear approximation are topologically equivalent in the neighbourhood of the equilibrium points. Using such qualitative characteristics, we can study whether the solution trajectories approach or move away from the equilibrium point over time. This result is stated in the following theorem.

Theorem 2.12. *(**Hartman- Grobman Theorem**): Let a DE $\dot{x} = f(x)$, where the vector field f is of class C^1. If x_0 is a hyperbolic equilibrium point of the DE then there exists a neighbourhood of x_0 on which the flow is topologically equivalent to the flow of the linearization of the DE at x_0.*

Unfortunately, the Hartman Grobmen theorem does not apply for the non-hyperbolic equilibrium points, in which the real part of at least one of the eigenvalues is zero. Therefore, an alternative technique such as multiple scale method can be applied which we use in some cases.

2.2 The Multiple Scale Method

The multiple scale in [14, 15] technique is used to construct uniformly valid approximations to the solutions of perturbation problems in which the solutions depend simultaneously on widely different scales. This is done by introducing fast-scale and slow-scale variables for an independent variable, and subsequently treating these variables, fast and slow, as if they are independent. We will use a classical example to illustrate the idea of the multiple scale method. Consider the ordinary differential equation

$$y'' + \epsilon y' + y = 0 \quad \text{for} \quad t > 0, \quad y(0) = 0, \quad y'(0) = 1. \tag{2.2.1}$$

The exact solution of (2.2.1) is

$$y(t) = \frac{1}{\sqrt{1-\frac{\epsilon^2}{4}}} e^{\frac{-\epsilon t}{2}} \sin\left(t\sqrt{1-\frac{\epsilon^2}{4}}\right). \tag{2.2.2}$$

This solution has an oscillatory component that occurs on a time scale of order $0(1)$, as well as it has a slow variation of order $0\left(\frac{1}{\epsilon}\right)$. To incorporate two time scales into the problem, we introduce the variables $t, \tau = \epsilon^{\alpha} t$. We treat these time scales as independent variables. Based in the solution it is expected to have $\alpha = 1$ in this case. Thus, it follows that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}. \tag{2.2.3}$$

By substituting this into (2.2.1) yields

$$y_{tt} + 2\epsilon y_t y_\tau + \epsilon^2 y_{\tau\tau} + \epsilon(y_t + \epsilon y_{t\tau}) + y = 0 \tag{2.2.4}$$

$$y = 0 \quad y_t + \epsilon y_\tau = 1 \quad \text{for} \quad t = 0 = \tau. \tag{2.2.5}$$

Note that we use the symbol y_t in place of $\frac{\partial y}{\partial t}$ (similarly for y_τ) for simplification. Note that the solution of the ordinary differential equation is not unique and this will enable us to prevent the secular terms from appearing in the expansion at least over the time scales that we are using.

Now we use the power series expansion of the form

$$y \sim y_0(t,\tau) + \epsilon y_1(t,\tau) + ... \tag{2.2.6}$$

By substituting (2.2.6) into (2.2.5) yields

$$y_{0tt} + y_0 + \epsilon(y_{1tt} + y_1 + 2y_{0t\tau} + y_{0t}) + 0(\epsilon^2) + ... \tag{2.2.7}$$

Now collecting terms of ϵ order as follows,

$O(1)$

$$y_{0tt} + y_0 = 0. \tag{2.2.8}$$

The general solution of this equation can be written as

$$y_0(t) = A(\tau)\sin(t) + B(\tau)\cos(t). \tag{2.2.9}$$

$$y_0(t) = A(\tau)\cos(t+\varphi) \quad \text{or as} \quad y_0(t) = A(\tau)e^{ti} + \bar{A}(\tau)e^{-ti}, \tag{2.2.10}$$

where $A(\tau)$ is a complex valued constant and the over bar indicates a complex conjugate.

$O(\epsilon)$

$$y_{1tt} + y_1 = -2y_{0t\tau} - y_{0t}. \tag{2.2.11}$$

From (2.2.9), the differential equation for y_1 is

$$y_{1tt} + y_1 = (2B_\tau + B(\tau))\sin(t) - (2A_\tau + A(\tau))\cos(t). \tag{2.2.12}$$

Then the secular terms are

$$B_\tau = -\frac{B(\tau)}{2}, \tag{2.2.13}$$

$$A_\tau = -\frac{A(\tau)}{2}. \tag{2.2.14}$$

After imposing the initial conditions we get

$$B_\tau = -\frac{B(\tau)}{2} \Rightarrow B(\tau) = \beta_1 e^{-\frac{\tau}{2}} \Rightarrow B(\tau) = 0, \tag{2.2.15}$$

$$A_\tau = -\frac{A(\tau)}{2} \Rightarrow A(\tau) = \beta_2 e^{-\frac{\tau}{2}} \Rightarrow A(\tau) = e^{-\frac{\tau}{2}} \tag{2.2.16}$$

Putting all the result together, we find that the solution of

$$y \sim e^{-\frac{\tau}{2}} \sin(t). \tag{2.2.17}$$

This is first term of approximation that is valid up to the first order of ϵ, which gives us a good approximation for the problem. Moreover, this procedure is easily expandable to more general oscillators, also it can used to investigate if the oscillations decay or grow.

Part I

Spatially Homogeneous Einstein-Aether Cosmological Models: Scalar Fields with Harmonic Potential

Chapter 3

Introduction to the Model

Cosmological models in Lorentz-violating theories of gravity are of recent interest among many scientists. Einstein-Aether theory [1], [3] consists of general relativity coupled, at second derivative order, to a dynamical time-like unit vector field, "Aether". It is one of several proposed models of early universe cosmology which incorporate a violation of Lorentz invariance [2]. The local space-time structure is determined by the Aether vector field, u_a, together with the metric tensor g_{ab} [16].

We shall discuss the late time dynamics of Einstein-Aether cosmological models; in particular, we explore the impact of Lorentz violation on the inflationary scenario [4], which provides one of the simplest ways to describe various aspects of the physics of the early universe in standard cosmology. More precisely, we study the inflationary scenario in the scalar-vector-tensor theory, where the vector is constrained to be unit and time-like, and investigate whether the inflationary solutions proposed [8] are stable when spatial curvature and anisotropic perturbations are considered.

In the phenomenology of the theories, it is assumed that the preferred frame coincides with the cosmological rest frame is defined by the Hubble expansion of the universe.

3.1 Einstein-Aether Cosmology Theory

The action for Einstein-Aether theory is the most general covariant functional of the spacetime metric g_{ab} and Aether field u_a involving no more than two derivatives (not including total derivatives) is [17, 30]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d + \lambda\left(u^c u_c + 1\right) + \mathcal{L}_m\right], \tag{3.1.1}$$

where

$$K^{ab}{}_{cd} \equiv c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c + c_4 u^a u^b g_{cd}. \tag{3.1.2}$$

The action (3.1.1) contains an Einstein-Hilbert term for the metric, a kinetic term for the Aether with four dimensionless coefficients c_i, and λ is a Lagrange multiplier enforcing the time-like constraint on the Aether. The convention used in this thesis for metric signature is $(-+++)$ and the units are chosen so that the speed of light defined by the metric g_{ab} is unity and $\kappa^2 \equiv 8\pi G = 1$. The field equations from varying (3.1.1) with respect to g^{ab}, u^a, and λ are given respectively by [16].

$$G_{ab} = T^{æ}_{ab} \tag{3.1.3}$$

$$\lambda u_b = \nabla_a J^a{}_b + c_4 \dot{u}_a \nabla_b u^a \tag{3.1.4}$$

$$u^a u_a = -1. \tag{3.1.5}$$

Here G_{ab} is the Einstein tensor of the metric g_{ab}. The quantities $J^a{}_b$, $\dot{u}_a$ and the

Aether stress-energy $T^{æ}_{ab}$ are given by

$$J^a{}_m = -K^{ab}{}_{mn}\nabla_b u^n \tag{3.1.6a}$$

$$\dot{u}_a = u^b \nabla_b u_a \tag{3.1.6b}$$

$$\begin{aligned} T^{æ}_{ab} &= 2c_1(\nabla_a u^c \nabla_b u_c - \nabla^c u_a \nabla_c u_b) \\ &- 2[\nabla_c(u_{(a}J^c{}_{b)}) + \nabla_c(u^c J_{(ab)}) - \nabla_c(u_{(a}J_{b)}{}^c)] - 2c_4 \dot{u}_a \dot{u}_b + \\ &+ 2\lambda u_a u_b + g_{ab}\mathcal{L}_u. \end{aligned} \tag{3.1.6c}$$

where

$$\mathcal{L}_u \equiv -K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d, \tag{3.1.7}$$

is the Einstein-Aether Lagrangian.Taking the contraction of (3.1.4) with u^b and with the induced metric $h^{bc} := g^{bc} + u^b u^c$ we obtain the equations

$$\lambda = -u^b \nabla_a J^a_b - c_4 \dot{u}_a \dot{u}^a, \tag{3.1.8a}$$

$$0 = h^{bc}\nabla_a J^a_b + c_4 h^{bc} \dot{u}_a \nabla_b u^a. \tag{3.1.8b}$$

3.2 Models and the Parameters c_i

Regarding this last subcase, there are 4 dimensionless free parameters (constants) that define the Einstein-Aether theory ($c_i; i = 1...4$). Since we have normalized G ($c_1 + 3c_2 + c_3 = 0$), and since there is an invariance in the action in the cosmological application, there are effectively only two independent parameters [1] denoted here by c^2 and d. The arbitrary parameter $d = c_1 - c_4$ does not occur in the tilt-free equations above. The final parameter, $c^2 - 1 = (c_1 + c_3)$ satisfies $0 \leq c^2 \leq 1$, where $c^2 = 1$ is the corresponding GR value. In [17] it was shown that $(c_1 + c_3)$ is positive and bounded above; indeed, it is expected that $(c_1 + c_3)$ is very small (i.e., $\sim 10^{-3}$). Therefore, $c^2 = 1$ is consistent with [1]. In addition, since it is not expected that there would be any significant different qualitative behaviour in the late time dynamics, we could assume self-consistently here that $c^2 = 1$ to simplify the analysis.

3.3 Isotropic Model

In an isotropic and spatially homogeneous Friedmann universe with the expansion scale factor $a(t)$ and the proper time t, "Aether" Aether field will be aligned with the cosmic frame and is related to the expansion rate of the universe. The Friedmann equation is generalized by the contribution of the additional stress tensor for the Aether field. If the universe contains a single self-interacting scalar field ϕ with a self-interacting potential V, then V can now be a function of ϕ and the expansion rate θ. We shall investigate these models in chapter (4).

3.4 Anisotropic Model

In an anisotropic Einstein-Aether model there will be additional terms in the field equations when compared to the isotropic model. For example:

- The effects on the geometry from the anisotropy (and curvature) of the Bianchi spatially homogeneous models will be present in the equations. We shall only be concerned here with a subclass of Bianchi type VI_h models with non-positive spatial curvature (scalar field models with a harmonic potential and a positive spatial curvature are known to be chaotic [13]).

- The effect of the energy momentum tensor of the scalar field, due to the possible dependence on V of the Lorentz violating vector field.

- The Einstein Friedmann equations are generalized by the contribution of an additional stress tensor, $T^{æ}_{ab}$, for the Aether field which depends on the dimensionless parameters of the Aether model (e.g., the "c_i") [18]. In GR, all of the $c_i = 0$. To study the effects of matter, we could perhaps assume the corresponding GR values (or close to them) in the first instance. The parameter c^2, depends on $(c_1 + c_3)$ such that $c^2 \leq 1$, is expected to satisfy $c^2 \sim 1$, consistent with [1, 19]. In addition, since it is not expected that there would be any significant different qualitative behaviour in the late time dynamics, we will assume here that $c^2 = 1$ (the corresponding GR value).

- In anisotropic models, there may be a tilt between the preferred direction of the Aether and that of the anisotropy (in an isotropic and spatially homogeneous Friedmann universe the Aether field is aligned with the cosmic frame). This adds additional terms to the Aether stress tensor $T^{æ}_{ab}$, which can be characterized by a hyperbolic tilt angle, $\alpha(t)$, measuring the boost of the Aether relative to the rest frame of the homogeneous spatial sections [5, 20]. The tilt is expected to decay to the future [21, 22]. Henceforward, we shall assume that the tilt is negligible and so $\alpha = 0$ (and the model does not depend on the parameter d).

3.5 The Model

We shall assume that the spacetime is spatially homogeneous and anisotropic having a normalized hyper-surface orthogonal vector field that is aligned with the Aether, $u_a = (1; 0; 0; 0)$. An interesting one-parameter class of spacetimes is described by the diagonal Bianchi type VI_h metric

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 e^{2mx} dy^2 + c(t)^2 e^{2x} dz^2, \tag{3.5.1}$$

where m is a constant and it is define by $m = h - 1$. If $m = 1$, then we have Bianchi type V; if $m = 0$, then we have a Bianchi type III and when $m = -1$ we have Bianchi type VI_0. With the above assumptions on the metric and Aether vector, the vorticity and the acceleration of the Aether vector are zero and the covariant derivative

$$u_{a;b} = \sigma_{ab} + \frac{1}{3}\theta h_{ab} \tag{3.5.2}$$

is classified by the expansion scalar

$$\theta = \nabla_a u^a = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}, \tag{3.5.3}$$

and $\sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab}$, where we assume $\sigma_1 = \sigma_2$. We shall consider the effective potential of the form

$$V(\theta, \phi, \sigma) = \sum_{r,s} a_{r,s} \theta^r \sigma^s \phi^{2-r-s}, \tag{3.5.4}$$

where $\{a_{r,s}\}$ are constants. For convenience and simplicity we study a potential with arbitrary values for ($a_{00} = \frac{1}{2}n^2, a_{10} = \mu, a_{01} = \nu$). Negative constants $a_{r,s}$ are permitted; however, it might be required that the potential $V(\theta, \phi, \sigma)$ is positive definite. The general Friedmann equation has the form

$$\theta^2 = 3\sigma^2 + 3\rho_\phi + \frac{3(m^2+m+1)c^2}{a^2}. \tag{3.5.5}$$

The Raychaudhuri equation governing the evolution of the expansion is given by

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{1}{2}(\rho_\phi + 3p_\phi). \tag{3.5.6}$$

The shear evolution equation is given by

$$\dot{\sigma} = -\sigma\theta + \frac{1-m}{3\sqrt{3}\sqrt{m^2+m+1}}\left(\theta^2 - 3\sigma^2 - 3\rho_\phi\right). \tag{3.5.7}$$

The Klein-Gordon equation yields

$$\ddot{\phi} = -\theta\dot{\phi} - V_\phi. \tag{3.5.8}$$

The Hubble expansion is $3H = \theta$, and the shear is defined by $\sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab}$. An over dot represents differentiation with respect to coordinate time t. Units have been chosen so that $8\pi G = c^2 = 1$. The Einstein field equation, the conservation equation, together with the Klein-Gordon equation for the scalar field, yield the following autonomous system of ordinary differential equations:

$$\begin{cases} \dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{1}{2}(\rho_\phi + 3p_\phi), \\ \dot{\sigma} = -\sigma\theta + \frac{1-m}{3\sqrt{3}\sqrt{m^2+m+1}}\left(\theta^2 - 3\sigma^2 - 3\rho_\phi\right), \\ \dot{\phi} = \psi, \\ \dot{\psi} = -\theta\psi - V_\phi, \end{cases} \tag{3.5.9}$$

with first integral

$$\theta^2 = 3\sigma^2 + 3\rho_\phi + \frac{3(m^2+m+1)}{a^2}. \tag{3.5.10}$$

In this part of thesis we shall look at the isotropic sub cases of these models: the case when $m = 1, \sigma = 0$ yields a negative spatial curvature model and the limiting case as $(\frac{1}{a^2}) \to 0$ which corresponds to a zero curvature model. We will investigate these isotropic cases in chapter (4). In chapter (5), we shall look at the anisotropic models for $m = 1$, which yields a 4- dimensional system.

3.6 Self Interacting Scalar Field

If the universe contains a self-interaction potential V, which is dependent on a self-interacting scalar field ϕ, together with the expansion rate, then the modified stress tensor for the scalar field [1] is given by

$$\begin{aligned} T^{\phi}_{ab} =& \nabla_a\phi\nabla_b\phi - \left(\frac{1}{2}\nabla_a\phi\nabla^a\phi + V\right) g_{ab} + (\dot{V}_\theta + \theta V_\theta)g_{ab} + \dot{V}_\theta u_a u_b \\ &+ \dot{V}_\sigma \frac{\sigma_{ab}}{6\sigma} + \frac{V_\sigma}{6\sigma}\left[\left(\theta - \frac{\dot{\sigma}}{\sigma}\right)\sigma_{ab} + \dot{\sigma}_{ab} - 6\sigma^2 u_a u_b - \dot{u}^c \sigma_{c(a}u_{b)}\right], \end{aligned} \tag{3.6.1}$$

where V_θ is the derivative of V with respect to θ (similarly for V_σ) and a dot is the covariant derivative along the Aether field, $(\dot{} := u^a\nabla_a)$. For the class of models under consideration the effective energy density and pressure due to a scalar field which interacts with the Aether field velocity vector through its expansion and shear has the form [1, 13]

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V - V_\theta(\theta + \alpha\sqrt{6}\sigma) - V_\sigma\left(\beta\sigma - \frac{1}{\sqrt{6}}\theta\right), \tag{3.6.2}$$

And

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V + V_\theta\left(\theta + \alpha\sqrt{6}\sigma + \frac{\alpha\sqrt{6}\dot{\sigma}}{\theta}\right) + (\beta-1)V_\sigma\frac{\dot{\sigma}}{\theta} + V_\sigma\left(\beta\sigma - \frac{1}{\sqrt{6}}\theta - \frac{1}{\sqrt{6}}\frac{\dot{\theta}}{\theta}\right)$$
$$+\dot{V}_\theta\left(1 + \frac{\alpha\sqrt{6}\sigma}{\theta}\right) + \dot{V}_\sigma\left(\frac{\beta\sigma}{\theta} - \frac{1}{\sqrt{6}}\right), \quad (3.6.3)$$

where we include arbitrary constant α, β to generalize the effective density and the pressure equations. In this thesis, we set $\alpha = 1 = \beta$. Therefore, the effective density and the pressure become

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V - V_\theta(\theta + \sqrt{6}\sigma) - V_\sigma\left(\sigma - \frac{1}{\sqrt{6}}\theta\right), \quad (3.6.4)$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V + V_\theta\left(\theta + \sqrt{6}\sigma + \sqrt{6}\frac{\dot{\sigma}}{\theta}\right) + V_\sigma\left(\sigma - \frac{1}{\sqrt{6}}\theta - \frac{1}{\sqrt{6}}\frac{\dot{\theta}}{\theta}\right)$$
$$+\dot{V}_\theta\left(1 + \sqrt{6}\frac{\sigma}{\theta}\right) + \dot{V}_\sigma\left(\frac{\sigma}{\theta} - \frac{1}{\sqrt{6}}\right). \quad (3.6.5)$$

3.6.1 Harmonic Potential

Many kinds of scalar field potentials have been studied in early universe cosmology, and particularly convex potentials such as the harmonic potential $V(\phi) = \frac{1}{2}n^2\phi^2$. Exponential potentials $V(\phi) = V_0 e^{k\phi}$, the focus of much recent work, are also of physical importance and are of particular interest mathematically due to the existence of symmetry in the evolution equations. This leads to the decoupling of the Raychaudhuri equation and makes a qualitative analysis of such models particularly illuminating [13]. However, a qualitative analysis of scalar field cosmological models with a non-exponential potential is still useful. In an analysis of inflation in scalar field models it is usually the dynamics at intermediate times that are of importance.

For example, for massive scalar field models a slow roll regime is entered, inflation results until oscillations in the scalar field develop, after which the scalar particles decay reheating the plasma. However, a qualitative analysis is still useful; in particular, the question of the degree of generality of solutions that possess an inflationary stage in such models was addressed in [4].

3.6.2 The Potential

The form of the potential includes effects from the interaction of the scalar field ϕ and the velocity vector of the Aether through the expansion and shear $V(\phi, \theta, \sigma)$. Here we are interested in the harmonic potentials of the form

- The potential is a function on ϕ only,

$$V(\phi) = \frac{1}{2}n^2\phi^2, \tag{3.6.6}$$

  where n is a positive constant

- The potential is a function of the scalar field and the expansion rate as follows:

$$V(\theta, \phi) = \frac{1}{2}n^2\phi^2 + \mu\theta\phi, \tag{3.6.7}$$

  where n, μ are positive constants.

- The potential is a function of the scalar field; the expansion rate and the shear,

$$V(\theta, \phi, \sigma) = \frac{1}{2}n^2\phi^2 + \mu\theta\phi + \nu\sigma\phi, \tag{3.6.8}$$

  in which we assume that V is a positively definite potential and μ and n are positive constants but ν can be either positive or negative.

Chapter 4

Isotropic FRW Models

We shall consider the spatially homogeneous and isotropic Einstein-Aether models. Given the metric is Bianchi VI_h in the diagonal form then the Friedmann equation is given as

$$\theta^2 - 3\sigma^2 - 3\rho_\phi = \frac{3}{a^2}(m^2 + m + 1). \tag{4.0.1}$$

The negative curvature Friedmann, Robertson Walker (FRW) model are an invariant set of the dynamical system when $m = 1$ and $\sigma = 0$. In which case the Friedmann equation becomes

$$\theta^2 = 3\rho_\phi - \frac{9}{a^2}. \tag{4.0.2}$$

This corresponds to a $(k = -1)$ FLRW models. The zero curvature $(k = 0)$ FLRW models are found in the limiting case when $\sigma \to 0$ and $\frac{1}{a^2} \to 0$, in which case the Friedmann equation becomes

$$\theta^2 - 3\rho_\phi = 0. \tag{4.0.3}$$

Hence, both $k = 0, -1$ (FRW) models can be describe by the dynamical system

$$\begin{cases} \dot{\theta} = -\frac{1}{3}\theta^2 - \frac{1}{2}(\rho_\phi + 3p_\phi), \\ \dot{\phi} = \psi, \\ \dot{\psi} = -\theta\psi - V_\phi, \end{cases} \tag{4.0.4}$$

with first integral

$$\theta^2 = 3\rho_\phi - \frac{9k}{a^2}. \tag{4.0.5}$$

Note that $\theta = \frac{3\dot{a}}{a}$. The energy density and pressure equations yield

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V - V_\theta\theta, \tag{4.0.6}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V + V_\theta\theta + \dot{V}_\theta. \tag{4.0.7}$$

Therefore,

$$\rho_\phi + 3p_\phi = 2\dot{\phi}^2 - 2V + 2\theta V_\theta + 3\dot{V}_\theta. \tag{4.0.8}$$

We consider a polynomial potential of the form:

$$V(\phi, \theta) = \frac{1}{2}n^2\phi^2 + \mu\theta\phi, \tag{4.0.9}$$

where n, μ are assumed to be positive since we assume that the potential is positive definite. We shall investigate two sub cases under this general case:

1. When the potential is a function of the scalar field ϕ only ($\mu = 0$)

$$V(\phi) = \frac{1}{2}n^2\phi^2.$$

2. When the potential depends on scalar field and the expansion rate with ($\mu > 0$)

$$V(\phi, \theta) = \frac{1}{2}n^2\phi^2 + \mu\phi\theta.$$

We investigate the qualitative behaviour of the spatially isotropic model in the late times using two approaches: an analysis using original variables, an analysis using normalized variables numerically and analytically. Moreover, we investigate the early times behaviour for these model using the original variables and the normalized bounded variables numerically only.

4.1 Case (1a): When $\mu = 0$

4.1.1 Original Variables

We shall look to the isotropic spatially homogeneous model with a potential of the form

$$V(\phi) = \frac{1}{2}n^2\phi^2. \tag{4.1.1}$$

With this particular potential the autonomous system in (4.0.4) reduce to

$$\begin{cases} \dot{a} = \dfrac{1}{3}\left(\sqrt{\dfrac{1}{6}\dot{\phi}^2 + \dfrac{1}{6}n^2\phi^2 - \dfrac{k}{a^2}}\right) a, \\ \dot{\phi} = \psi, \\ \dot{\psi} = -3\left(\sqrt{\dfrac{1}{6}\dot{\phi}^2 + \dfrac{1}{6}n^2\phi^2 - \dfrac{k}{a^2}}\right) \psi - n^2\phi. \end{cases} \tag{4.1.2}$$

Original Variables: Multiple Scales Method

In this subsection, we use the multiple scales method to analyse the behaviour of the model. It is an extremely powerful approach with widespread applications [25]. In this method, we introduce one or more new “slow” time variables for each time scale of our problem. This method gives us a leading order expansion corresponding to the scalar field ϕ in our model. Our objective is to simplify our model with the original variables to a reduced and easier model to study.

Define the variable,

$$r = \frac{1}{a} \Rightarrow \dot{r} = \frac{-\dot{a}}{a^2}. \tag{4.1.3}$$

Then, it follows that

$$\dot{r} = -\frac{1}{3}\left(\sqrt{\frac{1}{6}\dot{\phi}^2 + \frac{1}{6}n^2\phi^2 - kr^2}\right) r. \tag{4.1.4}$$

Combining the second equation of (4.1.2) and (4.1.4) leads to

$$\begin{cases} \ddot{\phi} = -3\left(\sqrt{\frac{1}{6}\dot{\phi}^2 + \frac{1}{6}n^2\phi^2 - kr^2}\right)\dot{\phi} - n^2\phi, \\ \dot{r} = -\frac{1}{3}\left(\sqrt{\frac{1}{6}\dot{\phi}^2 + \frac{1}{6}n^2\phi^2 - kr^2}\right) r. \end{cases} \tag{4.1.5}$$

Next, rescale the variables r and ϕ, where ϵ is small, as follows

$$\phi = \epsilon\Phi, \quad r = \epsilon\mathcal{R}. \tag{4.1.6}$$

Hence, the system (4.1.5) becomes

$$\begin{cases} \ddot{\Phi} = -3\epsilon\left(\sqrt{\frac{1}{6}\dot{\Phi}^2 + \frac{1}{6}n^2\Phi^2 - k\mathcal{R}^2}\right)\dot{\Phi} - n^2\Phi, \\ \dot{\mathcal{R}} = -\frac{1}{3}\epsilon\left(\sqrt{\frac{1}{6}\dot{\Phi}^2 + \frac{1}{6}n^2\Phi^2 - k\mathcal{R}^2}\right)\mathcal{R}. \end{cases} \tag{4.1.7}$$

Multiple Scales Method

We use the multiple scales method with t as the fast time and $\tau = \epsilon t$ as the slow time to determine a leading order approximation to the solution of $\Phi, \mathcal{R}$ of the model in equation (4.1.7). The solution od $\Phi, \mathcal{R}$ are expanded as

$$\Phi \equiv \Phi(t, \tau) \sim \Phi_0(t, \tau) + \epsilon\Phi_1(t, \tau) + ...,$$

$$\mathcal{R} \equiv \mathcal{R}(t, \tau) \sim \mathcal{R}_0(t, \tau) + \epsilon\mathcal{R}_1(t, \tau) + ... \tag{4.1.8}$$

Using the chain rule gives

$$\dot{\mathcal{R}} = \mathcal{R}_{0t} + \epsilon\left(\mathcal{R}_{0\tau} + \mathcal{R}_{1t}\right) + ...,$$
$$\dot{\Phi} = \Phi_{0t} + \epsilon\left(\Phi_{0\tau} + \Phi_{1t}\right) + ..., \tag{4.1.9}$$
$$\ddot{\Phi} = \Phi_{0tt} + \epsilon\left(2\Phi_{0t\tau} + \Phi_{1tt}\right) + ...,$$

Note that, for simplification we used $\left(\frac{d\Phi}{dt} = \Phi_t, \frac{d\Phi}{dtd\tau} = \Phi_{t\tau}\right)$ and so on. Substituting (4.1.8) and (4.1.9) into the model in equation (4.1.7) yields the following

$$\begin{cases} \Phi_{0tt} + \epsilon(2\Phi_{0t\tau} + \Phi_{1tt}) + n^2(\Phi_0 + \epsilon\Phi_1) + 3\epsilon\Bigg(\Phi_{0t} + \epsilon\Bigg(\Phi_{0\tau} \\ \quad + \Phi_{1t}\Bigg)\Bigg)\left(\sqrt{\frac{1}{6}{\Phi_{0t}}^2 + \frac{1}{6}n^2{\Phi_0}^2 - k{\mathcal{R}_0}^2}\right), \\ \mathcal{R}_{0t} + \epsilon(\mathcal{R}_{0\tau} + \mathcal{R}_{1t}) = -\frac{1}{3}\epsilon\mathcal{R}_0\left(\sqrt{\frac{1}{6}{\Phi_{0t}}^2 + \frac{1}{6}n^2{\Phi_0}^2 - k{\mathcal{R}_0}^2}\right). \end{cases} \tag{4.1.10}$$

Equating coefficients of like powers of ϵ to 0, gives the following sequence of partial differential equations:

$$O(1): \quad \Phi_{0tt} + n^2\Phi_0 = 0, \tag{4.1.11a}$$

$$: \quad \mathcal{R}_{0t} = 0 \tag{4.1.11b}$$

$$O(\epsilon): \quad \Phi_{1tt} + n^2\Phi_1 = -3\left(\sqrt{\frac{1}{6}\Phi_{0t}^2 + \frac{1}{6}n^2{\Phi_0}^2 - k{\mathcal{R}_0}^2}\right)\Phi_{0t} - 2\Phi_{0t\tau}, \tag{4.1.11c}$$

$$: \quad \mathcal{R}_{1t} = -\frac{1}{3}\left(\sqrt{\frac{1}{6}{\Phi_{0t}}^2 + \frac{1}{6}n^2\Phi_0^2 - k\mathcal{R}_0^2}\right)\mathcal{R}_0 - \mathcal{R}_{0\tau}. \tag{4.1.11d}$$

Equation (4.2.25a,4.2.25b) have the solutions

$$\begin{cases} \Phi_0 = A(\tau)\cos(nt + \varphi(\tau)), \\ \mathcal{R}_0 = \mathcal{R}(\tau) \quad \text{where} \quad \mathcal{R}(\tau) \quad \text{an arbitrary function of} \quad \tau. \end{cases} \tag{4.1.12}$$

After some algebraic calculations the order of ϵ yields

$$\begin{cases} \Phi_{1tt} + n^2\Phi_1 = \sin(\theta)\left(3A\sqrt{\frac{n^2A^2}{6} - k{\mathcal{R}_0}^2} + 2A_\tau\right) - 2n^2A\varphi_\tau\cos(\theta), \\ \mathcal{R}_{1t} = -\frac{1}{3}\left(\sqrt{\frac{n^2A^2}{6} - k{\mathcal{R}_0}^2}\right)\mathcal{R}_0 - \mathcal{R}_{0\tau}. \end{cases} \tag{4.1.13}$$

Therefore, in order to eliminate these secular terms, we must have the following differential equations

$$\begin{cases} \varphi_\tau = 0, \\ A_\tau = -\frac{3}{2}\left(\sqrt{\frac{n^2A^2}{6} - k\mathcal{R}_0^2}\right)A, \\ \mathcal{R}_{0\tau} = -\frac{1}{3}\left(\sqrt{\frac{n^2A^2}{6} - k\mathcal{R}_0^2}\right)\mathcal{R}_0, \end{cases} \tag{4.1.14}$$

where $\mathcal{R}_0 = \mathcal{R}(\tau)$. The equations in (4.1.14) are separable and it can be solved explicitly when $k = 0$.

$$\begin{cases} A_\tau = -\frac{\sqrt{6}}{4}nA^2, \\ \mathcal{R}_{0\tau} = -\frac{nA}{3\sqrt{6}}\mathcal{R}_0. \end{cases} \tag{4.1.15}$$

Integrating (4.1.15), we obtain

$$\begin{cases} A(\tau) = \frac{4}{\sqrt{6}n\tau + C_1}, \quad \text{where} \quad C_1 \quad \text{is a constant of integration} \\ \mathcal{R}_0(\tau) = \frac{8}{(3\sqrt{6}n\tau + C_1)^{\frac{2}{3}}}. \end{cases} \tag{4.1.16}$$

The multiple scale approximation A and the exact solution ϕ for $\epsilon = 0.1$ are depicted in figure (4.4) which mean that the analogue curve of A gives us how the oscillation of ϕ decays as $t \to \infty$.

4.1.2 Original Variables: Numerical Method-Past Behaviour

In figures 4.1, 4.2, we plot solutions of $\phi(t), a(t)$ for the system (4.1.2) into the past time with $k = 0$.

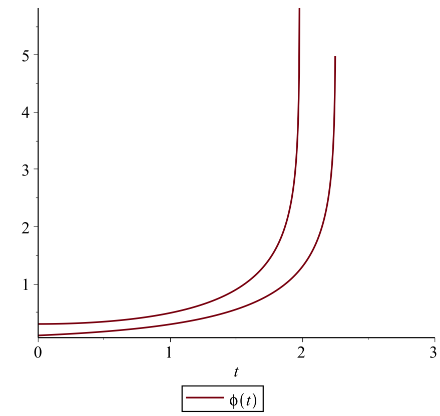

Figure 4.1: Plot of the solution for $\phi(t)$ of the system (4.1.2) with $n = 1$ and $k = 0$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 0.2, (D(\phi))(0) = 0, a(0) = 0.1], [\phi(0) = 0.3, (D(\phi))(0) = 0.1, a(0) = 0.05]]$.

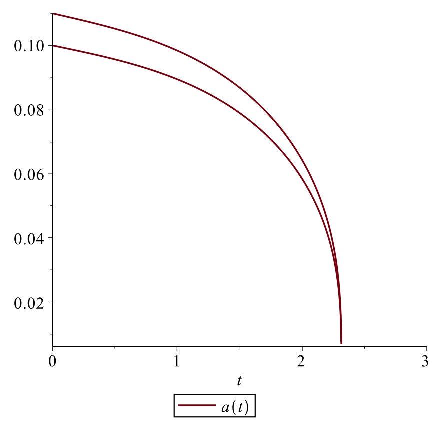

Figure 4.2: Plot of the solution for $a(t)$ of the system (4.1.2) with $n = 1$ and $k = 0$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 0.2, (D(\phi))(0) = 0, a(0) = 0.10], [\phi(0) = 0.3, (D(\phi))(0) = 0.1, a(0) = 0.11]]$.

Discussion: As can be seen in figure 4.1, $\phi(t)$ tends to infinity into the past and in figure 4.2 show that $a(t)$ decreases. In summary, the numerical solutions for the model with the original variables to the past with zero curvature is similar to the numerical solutions for the model with negative curvature, which implies that changing the curvature does not effect the behaviour of the model under consideration at early times.

4.1.3 Original Variables: Numerical Method-Future Behaviour

In figures 4.3, we plot solutions of of $A(t)$ for the system (4.1.14) and in figure 4.4, we plot the solution of of $\Phi(t), A(t)$ for the (4.1.14) and (4.1.7) together with $k = 0$, at the late times.

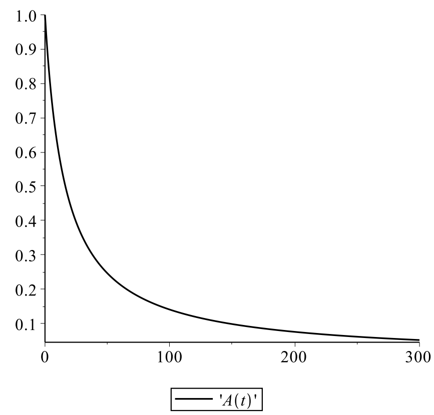

Figure 4.3: Plot of solution of $A(t)$ for the system (4.1.14) with $n = 1$ and $k = 0$ and initial condition: $[A(0) = 1, \mathcal{R}_0(0) = 1]$.

Discussion As can be seen from the numerical solution, in figure 4.4 the analogue of $A(t)$ consists with the solution of $\Phi(t)$ which starts by oscillation than decays to zero.

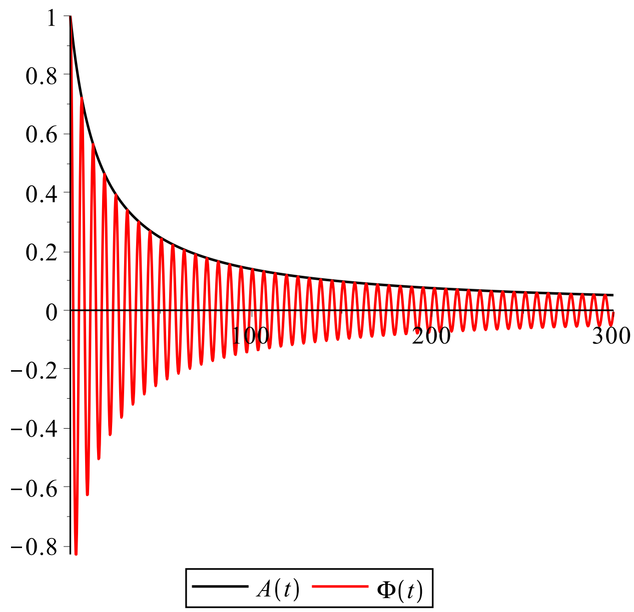

Figure 4.4: Plot of solution of $A(t)$ for the system (4.1.14) with $n = 1$ and $k = 0$ and initial condition: $[A(0) = 1, \mathcal{R}_0(0) = 1]$ together with the plot the solution of Φ for the system (4.1.7) with $n = 1$, $k = 0$ and one initial condition:$[\Phi(0) = 1, D(\Phi(0)) = 0, \mathcal{R}(0) = 1]$, as $t \to \infty$. Black line indicates the oscillation envelope $A(t)$.

4.1.4 Bounded Normalized Variables

Introducing Normalized Variables

We introduce normalized variables to analyze the model. We define a bounded variable D along with the following expressions for Φ and Ψ :

$$D \equiv \frac{\theta}{\sqrt{1+\theta^2}}, \quad \Psi \equiv \sqrt{\frac{3}{2}}\left(\frac{n\phi}{\sqrt{1+\theta^2}}\right), \quad \Phi \equiv \sqrt{\frac{3}{2}}\left(\frac{\dot{\phi}}{\sqrt{1+\theta^2}}\right). \tag{4.1.17}$$

The Friedmann equation becomes

$$D^2 - \Phi^2 - \Psi^2 = -\frac{9k}{a^2(1+\theta^2)} \geq 0, \tag{4.1.18}$$

which is conserved by the 3D system (4.1.20) (illustrated below), when $k = 0$.

Define

$$\Lambda = D^2 - \Phi^2 - \Psi^2,$$

then

$$\begin{aligned}
\Lambda' &= 2DD' - 2\Phi\Phi' - 2\Psi\Psi' \\
&= 2\left[D(1-D^2)\mathcal{X} + D\Phi^2(1+\mathcal{X}) + D\Psi^2\mathcal{X}\right] \\
&= 2\left[-D\mathcal{X}(D^2 - \Phi^2 + \Psi^2) + D(\mathcal{X} + \Phi^2)\right] \\
&= 2\left[-D\mathcal{X}\Lambda - \frac{1}{3}D\left(D^2 - \Phi^2 + \Psi^2\right)\right] \\
&= 2\left[-D\mathcal{X}\Lambda - \frac{1}{3}D\Lambda\right] \\
&= -\frac{2}{3}D\Lambda(3\mathcal{X} + 1) = 0.
\end{aligned}$$

The domain of interest is $0 \leq D \leq 1$, since if

$$\theta \to \infty \quad \Longrightarrow D \to 1,$$

and

$$\theta \to 0 \quad \Longrightarrow D \to 0.$$

Based on the equation (4.1.18) with $k = 0$ and the interest domain of D ($0 \leq D \leq 1$), hence

$$0 \leq \Phi^2 + \Psi^2 \leq D^2 \leq 1. \tag{4.1.19}$$

Then, it follows that the other variables Φ, Ψ are bounded. Now we define a new time variable τ with $\frac{d\tau}{dt} = \sqrt{1+\theta^2}$, and apply it to the bounded variables in (4.1.17), then we obtain the reduced (3D) system:

$$\begin{cases} D' = (1 - D^2)\mathcal{X}, \\ \Psi' = n\sqrt{1 - D^2}\Phi - D\Psi\mathcal{X}, \\ \Phi' = -D\Phi - n\sqrt{1 - D^2}\Psi - \Phi D\mathcal{X}, \end{cases} \tag{4.1.20}$$

where the prime here indicates the differentiation of each variable with respect to the new variable time τ and $\mathcal{X}$ is given by

$$\mathcal{X} = -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2 = \frac{\dot{\theta}}{(\theta^2 + 1)} \tag{4.1.21}$$

Moreover, for $k = -1, \mathcal{X} < 0$, and so D is a monotone decreasing in the interior of the phase space. Note that $\mathcal{X} = 0$ if $\Phi = 0$ and $D = \Psi$.

Bounded Variables:Qualitative Analysis

The equations in system (4.1.20) are relatively simple and it is a well defined system. Also, note that the D' does not decouple and the system is not analytic at $D = 1$ since we have the square root term in the second equation of the 3D system in (4.1.20). Moreover, the origin equilibrium point P_0 is highly degenerate.

Equilibrium Points

In table (4.1), we present the summary of equilibrium points for the system (4.1.20); the values of $\mathcal{X}$ and their stability.

Let us discuss the stability of the equilibrium points of the system (4.1.20):

Pt	(D, Φ, Ψ)	$\mathcal{X}$	Stability	k -values
P_0	$(0,0,0)$	0	Sink	$k=0,-1$
P_1	$(1,1,0)$	-1	Source	$k=0,-1$
P_2	$(1,-1,0)$	-1	Source	$k=0,-1$
P_3	$(1,0,1)$	0	Saddle	$k=0,-1$
P_4	$(1,0,-1)$	0	Saddle	$k=0,-1$
P_5	$(1,0,0)$	$-\frac{1}{3}$	Saddle	$k=-1$

Table 4.1: Equilibrium points of the system (4.1.20); the value of $\mathcal{X}$ and their stability.

Stability of Equilibrium Point (P_0)

Evaluating linearization matrix for the 3D system in equation (4.1.20) at P_0 gives us one zero eigenvalue and other two complex eigenvalues. Since one of the eigenvalues is zero, so we can not tell from the linear stability analysis alone whether or not P_0 is stable. Numerically, we have been able to show that all solutions (orbits) near P_0 go toward it which implies that it is a sink (i.e., see figure 4.9). Below; we analytically show that P_0 is a sink. We have that $D' = (1-D^2)\mathcal{X}$ and we want to show that $D' < 0$ in the interior ($k=-1$) of the phase space. We know that the term $(1-D^2)$ is always positive since the domain of our interest is $0 < D < 1$. However, by using the constraint $\Psi^2 < (D^2 - \Phi^2)$ in $\mathcal{X}$ as follows:

$$\begin{aligned}
\mathcal{X} &= -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2 \\
&\leq -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\left(D^2 - \Phi^2\right) < -\Phi^2 \leq 0.
\end{aligned} \tag{4.1.22}$$

Therefore, $D' < 0$ in the interior of our phase space which implies that the origin P_0 in the 3D system is a sink. Note that $D \to 0$ as $\tau \to \infty$. Similarly, $D \to 1$ as $\tau \to -\infty$. This implies that past behaviour is determined by the $D = 1$ invariant set and the future behaviour of the model is determined by the $k = 0$ invariant set (see figures 4.8, 4.9)

Stability of Equilibrium Points ($P_{1,2,3,4,5}$)

Evaluating the linearization matrix of the system in equation (4.1.20) at $P_{1,2,3,4}$ gives us undefined terms, so we have to examine the stability of these equilibrium points using different techniques. Then, evaluating the linearization matrix at $\Phi = 0$, $\Psi = \pm 1$ or at $\Phi = \pm 1, \Psi = 0$, we found that there is always one positive eigenvalue at $D = 1$. In order to determine the signs of the other eigenvalues we need to study the 2D system of Φ, Ψ. There are two cases:

When $k = 0$ and if $D = 1$ then we have a 1-dimensional set (i.e.,$\Phi^2 + \Psi^2 = 1$).

When $k = -1$ There is a 2-dimensional invariant set when $D = 1$ given by

$$\begin{cases} \Psi' = -\Psi\left(-\dfrac{1}{3} - \dfrac{2}{3}\Phi^2 + \dfrac{1}{3}\Psi^2\right), \\ \Phi' = -\Phi\left(\dfrac{2}{3} - \dfrac{2}{3}\Phi^2 + \dfrac{1}{3}\Psi^2\right), \end{cases} \tag{4.1.23}$$

with the constraint $\Phi^2 + \Psi^2 \leq 1$. There are five equilibrium points for the 2D system in equation (4.1.23).

$$\{(\Psi, \Phi)\} = \{(0,0), (\pm 1, 0), (0, \pm 1)\}. \tag{4.1.24}$$

Note that, $(\pm 1, 0), (0, \pm 1)$ are equilibrium points in the $k = 0$ set.

The linearization matrix for the 2D system (4.1.23) is given by

$$J(\Psi,\Phi)=\begin{bmatrix}\frac{1}{3}+\frac{2}{3}\Phi^2-\Psi^2 & \frac{4}{3}\Phi\Psi\\ -\frac{2}{3}\Phi\Psi & -\frac{2}{3}-\frac{1}{3}\Psi^2+2\Phi^2\end{bmatrix}. \tag{4.1.25}$$

Now we will evaluate all the equilibrium points in the linearization matrix (4.1.25) to see the stability (the behaviour) of them in the unit disk. For $(0,0)$, we get two eigenvalues one positive and the other negative, which implies that the point $(0,0)$ is a saddle. For $P_{1,2}$, we get two positive real eigenvalues, which implies that $P_{1,2}$ are sources in the 2D system as well as in the 3D since we know that the third eigenvalue is positive. Lastly, for $P_{3,4}$, we get two negative real eigenvalues, which implies that $P_{3,4}$ are sinks in the 2D system. However, $P_{3,4,5}$ are saddles in the 3D since we know that the third eigenvalue is positive. The shape of the phase space for the model is

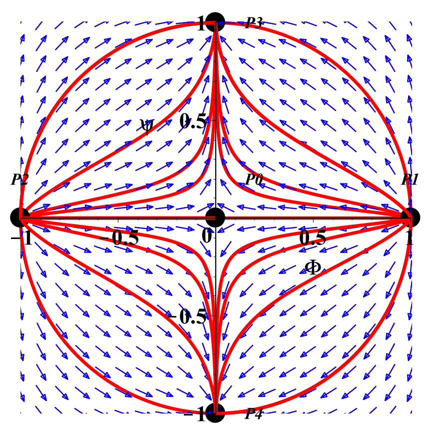

Figure 4.5: Plot of system (4.1.23) as $t \to \infty$

a cone in the 3D space and it is a circle in the 2D. In figure 4.5, we plot the phase portrait for the model in the invariant set $D = 1$. When $k = 0$, the phase space is the boundary (i.e., $\Phi^2 + \Psi^2 = 1$) but if $k = -1$, the phase space is the interior of the circle (i.e., $\Phi^2 + \Psi^2 < 1$). Note that when $k = 0$ we have four equilibrium points but when $k = -1$ we have one equilibrium point in the 2D invariant set and the others are on the boundary. As can be seen from figure 4.5 that $P_{1,2}$ are sources and $P_{3,4}$ are sinks and P_5 is a saddle.

Qualitative Analysis for Bounded Variables (Point P_0)

Our objective now is to reduce the system with the normalized bounded variables in (4.1.20) using the multiple scales method.

Expand $\sqrt{1 - D^2}$ using a Taylor series as follows

$$\sqrt{1 - D^2} = 1 - \frac{1}{2}D^2 + ..., \tag{4.1.26}$$

which is valid for small value of D. Then, neglecting all the higher order terms the system in (4.1.20) becomes

$$\begin{cases} D' = -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2, \\ \Psi' = n\Phi, \\ \Phi' = -D\Phi - n\Psi. \end{cases} \tag{4.1.27}$$

Define the variables as follows:

$$D = \epsilon d, \quad \Phi = \epsilon\phi, \quad \Psi = \epsilon\psi, \tag{4.1.28}$$

where ϵ is small. Plugging (4.1.28) into (4.1.27) yields to

$$\begin{cases} d' = \frac{\epsilon}{3}\left(-d^2 - 2\phi^2 + \psi^2\right), \\ \psi' = n\phi, \\ \phi' = -\epsilon d\phi - n\psi. \end{cases} \tag{4.1.29}$$

The system (4.1.29) can be rewritten as follows:

$$\begin{cases} d' = \dfrac{\epsilon}{3}\left(-d^2 - 2\dfrac{\psi'^2}{n^2} + \psi^2\right), \\ \psi'' = -n^2\psi - \epsilon d\psi'. \end{cases} \tag{4.1.30}$$

Multiple Scales Method

We use the multiple scales method with t as the fast time and $\tau = \epsilon t$ as the slow time to determine a leading order approximation to the solution of ψ, d of the model in equation (4.1.30). The solution of ψ, d are expanded as

$$\begin{aligned} \psi \equiv \psi(t,\tau) &\sim \psi_0(t,\tau) + \epsilon\psi_1(t,\tau) + ..., \\ d \equiv d(t,\tau) &\sim d_0(t,\tau) + \epsilon d_1(t,\tau) + ... \end{aligned} \tag{4.1.31}$$

Using the chain rule gives

$$\begin{aligned} d' &= d_{0t} + \epsilon(d_{0\tau} + d_{1t}) + ..., \\ \psi' &= \psi_{0t} + \epsilon(\psi_{0\tau} + \psi_{1t}) + ..., \\ \psi'' &= \psi_{0tt} + \epsilon(2\psi_{0t\tau} + \psi_{1tt}) + .., \end{aligned} \tag{4.1.32}$$

where ($'$) indicates the derivative of the variables with respect to time t. Substituting (4.1.32) and (4.1.31) into (4.1.30) yields

$$\begin{cases} d_{0t} + \epsilon(d_{0\tau} + d_{1t}) = \dfrac{\epsilon}{3}\left(-{d_0}^2 - \dfrac{2}{n^2}{\psi_{0t}}^2 + {\psi_0}^2\right), \\ \psi_{0tt} + \epsilon(2\psi_{0t\tau} + \psi_{1tt}) = -n^2(\psi_0 + \epsilon\psi_1) - \epsilon d_0\psi_{0t}. \end{cases} \tag{4.1.33}$$

Equating coefficients of like powers of ϵ to 0, gives the following sequence of partial differential equations:

$$O(1): \quad \psi_{0tt} + n^2\psi_0 = 0, \tag{4.1.34a}$$

$$: \quad d_{0t} = 0 \tag{4.1.34b}$$

$$O(\epsilon): \quad \psi_{1tt} + n^2\psi_1 = -2\psi_{0t\tau} - d_0\psi_{0t}, \tag{4.1.34c}$$

$$: \quad d_{1t} = -d_{0\tau} - \frac{1}{3}\left(-d_0^2 - \frac{2}{n^2}\psi_{0t}^2 + \psi_0^2\right). \tag{4.1.34d}$$

Equation (4.2.25a),(4.2.25b) have the following solutions

$$\begin{cases} d_0 = d(\tau) \quad \text{an arbitrary function,} \\ \psi_0 = A(\tau)\cos(\theta) \quad \text{where} \quad \theta = nt + \varphi(\tau). \end{cases} \tag{4.1.35}$$

After some algebraic calculations the order of ϵ yields

$$\begin{cases} d_{1t} = -d_{0\tau} - \frac{1}{3}\left(-d_0^2 + \frac{1}{2}A^2\right), \\ \psi_{1tt} + n^2\psi_1 = n\sin(\theta)\left(2A_\tau + d_0 A\right) - 2An\varphi_\tau\cos(\theta). \end{cases} \tag{4.1.36}$$

Therefore, in order to eliminate these secular terms, we must have the following differential equations

$$\begin{cases} \varphi_\tau = 0, \\ A_\tau = -\frac{d_o}{2}A, \\ d_{0\tau} = -\frac{1}{3}\left(d_0^2 + \frac{1}{2}A^2\right). \end{cases} \tag{4.1.37}$$

As can be seen in (4.1.37) that $d_{0\tau} < 0$. which match the numerical solution in figure 4.7. When we plot the solution of A_τ, we found that $A(\tau)$ decrease. which implies that the approximate solutions of the reduced system (4.1.37) decay to zero. Hence, the approximate solutions of (4.1.30) decay to zero as well. Therefore, P_0 is a sink.

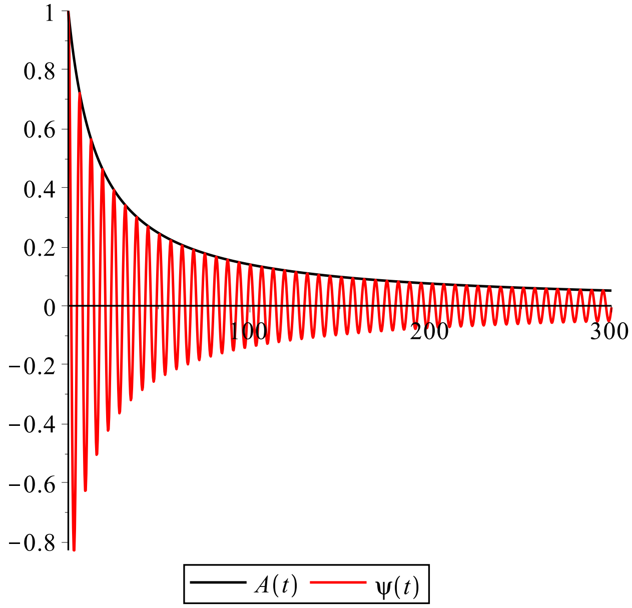

Figure 4.6: Plot of solution of $A(t)$ for the system (4.1.37) with $n = 1$ and $k = 0$ and initial condition: $[A(0) = 1, d_0(0) = 1]$ together with the plot the solution of $\psi(t)$ for the system (4.1.30) with $n = 1$, $k = 0$ and one initial condition:$[\psi(0) = 1, D(\psi(0)) = 0, d(0) = 1]$, as $t \to \infty$. Black line indicates the oscillation envelope $A(t)$.

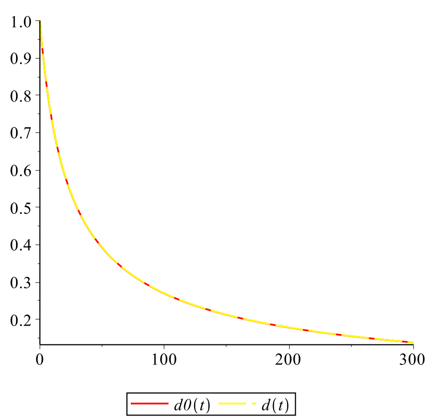

Figure 4.7: Plot of solution of $d_0(t)$ for the system (4.1.37) with $n = 1$ and $k = 0$ and initial condition: $[A(0) = 1, d_0(0) = 1]$ together with the plot the solution of $\psi(t)$ for the system (4.1.30) with $n = 1$, $k = 0$ and one initial condition:$[\psi(0) = 1, D(\psi(0)) = 0, d(0) = 1]$, as $t \to \infty$.

4.1.5 Bounded Variables: Numerical Method-Past Behaviour

In the following figure 4.8, we plot solutions of the system (4.1.20) into the past times with $k = 0$.

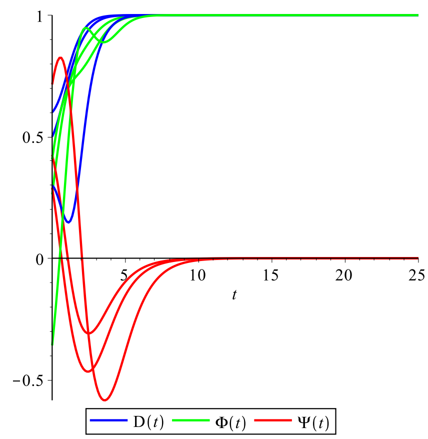

Figure 4.8: Plot for the system (4.1.20) with $n = 1$; $k = 0$ as $t \to -\infty$ with several initial conditions:$[[D(0) = 0.5, \Phi(0) = 0.5\cos(45), \Psi(0) = 0.5\sin(45)], [D(0) = 0.6, \Phi(0) = 0.6\cos(120), \Psi(0) = 0.6\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

Discussion: As can be seen in figures 4.8, $D, \Phi \to 1$ and $\Psi \to 0$ as $t \to -\infty$ which implies that it is going to P_1 in the past time limit. Note that changing the curvature value $k = -1$ does not effect the behaviour of the model into the past, we get similar behaviour as $k = 0$.

4.1.6 Bounded Variables: Numerical Method-Future Behaviour

In figure 4.9 we plot the numerical solutions for the system (4.1.20) with $n = 1$ and with zero curvature into the future times.

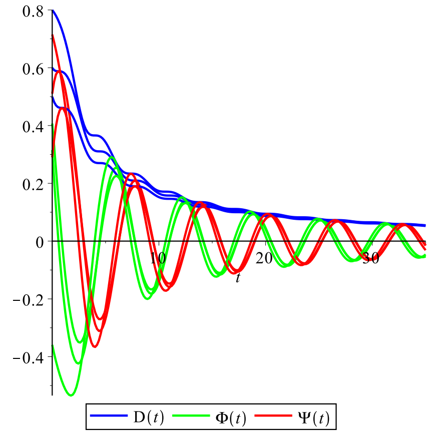

Figure 4.9: Plot for the system (4.1.20) with $n = 1$ and $k = 0$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

Discussion

As can be seen in figure 4.9, Φ, Ψ oscillate and decay to zero, and D decreases and decays to zero in the late time limit, which means that P_0 is sink. Note that when $k = -1$, we get similar behaviour as with $k = 0$. In summary, the numerical solutions for the model with the bounded variables is consistent with the numerical solutions for the model with the original variables in the late time limit (see figures 4.4 and 4.9). In summary, solution approach P_0 at $t \to \infty$.

4.1.7 Inflation

As an indicator of the accelerated expansion of the universe we introduce the deceleration parameter, defined as

$$q \equiv -\left(\frac{3\dot{\theta}}{\theta^2} + 1\right) = -\frac{a\ddot{a}}{\dot{a}^2}. \qquad (4.1.38)$$

With the use of the first equation in (4.0.4) and equation (4.1.17) it follows that the deceleration parameter can be expressed in terms of the normalized bounded variables as

$$q = -\frac{1}{D^2}(\Psi^2 - 2\Phi^2). \tag{4.1.39}$$

The sign of the deceleration parameter indicates the nature of the expansionary evolution. If $q > 0$, the cosmological expansion is decelerating, while negative values of q indicate an accelerating dynamics. The value for q for each of the equilibrium points of the model are

1. P_0: q is undefined in terms of the normalized variables. However, using the original variables we obtain that

$$q = -\frac{1}{\theta^2}\left(-3\dot{\phi}^2 + \frac{3}{2}n^2\phi^2\right), \tag{4.1.40}$$

which is zero at P_0. Hence, P_0 is not inflationary.

2. $P_{1,2}$:

$$q\big|_{P_{1,2}} = 2 > 0, \tag{4.1.41}$$

are always positive which implies that $P_{1,2}$ are not inflationary.

3. $P_{3,4}$:

$$q\big|_{P_{3,4}} = -1 < 0, \tag{4.1.42}$$

are always negative which implies that $P_{3,4}$ are inflationary.

4. P_5:

$$q\big|_{P_5} = 0, \tag{4.1.43}$$

which implies that P_5 is not inflationary.

4.1.8 Slow Roll Inflation

The expansion term tends to slow down the evolution of ϕ, if the following slow roll conditions are valid

$$\dot{\phi}^2 \ll V(\phi), \tag{4.1.44}$$

$$|\ddot{\phi}| \ll |\theta\dot{\phi}|, \tag{4.1.45}$$

Now, we may approximate Friedmann equation using the slow roll conditions as follow

$$\theta^2 = \tfrac{3(n^2\phi^2+\dot{\phi}^2)}{2} \Rightarrow \theta = \sqrt{\tfrac{3}{2}}\sqrt{\dot{\phi}^2 + n^2\phi^2}, \tag{4.1.46}$$

which will be simpler using the slow roll conditions

$$\theta = \sqrt{\frac{3}{2}}n|\phi|. \tag{4.1.47}$$

And the Klein Gordon equation in (4.0.4) becomes

$$\theta\dot{\phi} + n^2\phi = 0. \tag{4.1.48}$$

Plugging (4.1.47) into equation (4.1.48) leads to

$$\dot{\phi} = -\sqrt{\frac{2}{3}}n < 0. \tag{4.1.49}$$

Integrating (4.1.49) yields

$$\phi = -\sqrt{\frac{2}{3}}nt + C_3, \quad \text{where } C_3 \text{ is a constant.} \tag{4.1.50}$$

Hence,

$$\theta = -n^2 t + C_4, \quad \text{where } C_4 \text{ is a constant.} \tag{4.1.51}$$

Now, we can calculate the solution of the dimensionless scale factor for the universe a by integrating the following equation

$$\begin{aligned} \theta &= \frac{3\dot{a}}{a}, && (4.1.52)\\ \int \frac{\dot{a}}{a} &= \frac{1}{3}\left(\int (-n^2 t + C_4)\, dt\right), && (4.1.53)\\ \Rightarrow \ln(a) &= \frac{1}{6} n^2 t^2 + C_4 t + C_5, && (4.1.54)\\ a &= C_6 e^{-\frac{1}{6}n^2 t^2 + C_4 t}, \quad \text{where } C_6 \text{ is a constant.} && (4.1.55) \end{aligned}$$

Thus, the solution for the scale factor of the expanding universe at time zero is $a = C_6$. Moreover, the slow roll inflation will happen for intermediate times (values of t for which the approximations are valid).

4.1.9 Discussion

In the isotropic model (Case (1a): $\mu = 0$) we found there is only one sink which is at the origin P_0. There are inflationary saddles at $P_{3,4}$ but no inflationary sinks nor inflationary sources.

4.2 Case(1b): When $\mu > 0$

4.2.1 Original Variables

We shall look to the isotropic spatially homogeneous model with a potential of the form

$$V(\theta,\phi) = \frac{1}{2}n^2\phi^2 + \mu\theta\phi, \tag{4.2.1}$$

With this particular potential the autonomous system in (4.0.4) reduce to

$$\begin{cases} \dot{a} = \frac{1}{3}\left(\sqrt{\frac{1}{6}\dot{\phi}^2 + \frac{1}{6}n^2\phi^2 - \frac{k}{a^2}}\right)a, \\ \ddot{\phi} = -3(\dot{\phi}+\mu)\left(\sqrt{\frac{1}{6}\dot{\phi}^2 + \frac{1}{6}n^2\phi^2 - \frac{k}{a^2}}\right) - n^2\phi. \end{cases} \tag{4.2.2}$$

Since the evolution equations are invariant under the transformation $\mu \to -\mu$, without lost of generality we can assume that $\mu > 0$. In the following two subsections, we investigate the late time behaviour of the model using two approaches: an analysis of the model with the original variables and analysis using the normalized bounded variables, numerically and analytically.

4.2.2 Original Variables: Numerical Method-Past Behaviour

In figures 4.10, 4.11, we plot the the solutions of $\phi(t), a(t)$ of the system (4.2.2) into the past times with $k = 0$.

Discussion

As can be seen from figure 4.10 that $\phi(t)$ blows up to infinity and in figure (4.11) that $a(t)$ decreases into the past time limit. Also, we have the similar behaviour

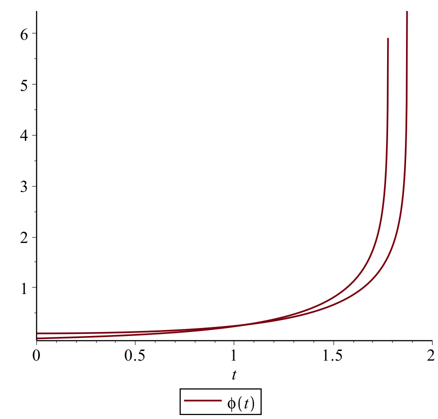

Figure 4.10: Plot of the solution for $\phi(t)$ of the system (4.2.2) with $n = 1$ and $k = 0$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 0.1, (D(\phi))(0) = 0, a(0) = 0.1], [\phi(0) = 0.001, (D(\phi))(0) = 0.1, a(0) = 0.3]]$.

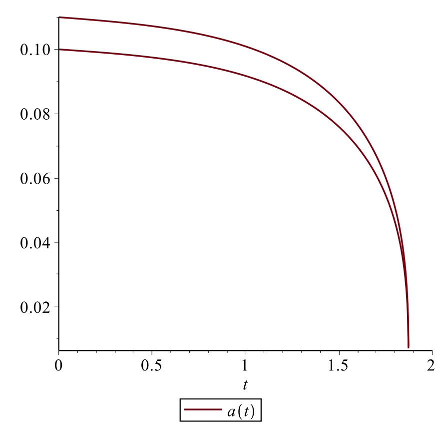

Figure 4.11: Plot for the system (4.2.2) with $n = 1$, $k = -1$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 0.1, (D(\phi))(0) = 0, a(0) = 0.10], [\phi(0) = 0.00001, (D(\phi))(0) = 0, a(0) = 0.11]]$.

with negative curvature ($k = -1$) into the past for any positive values of μ smaller or bigger than the bifurcation value of μ_c which we will be discussed later in this chapter.

4.2.3 Original Variables: Numerical Method-Future Behaviour

In figures from 4.12 to 4.15 we plot the solutions of $\phi(t)$ of the system (4.2.2) at the future time with $k = 0$ with different positive values of μ.

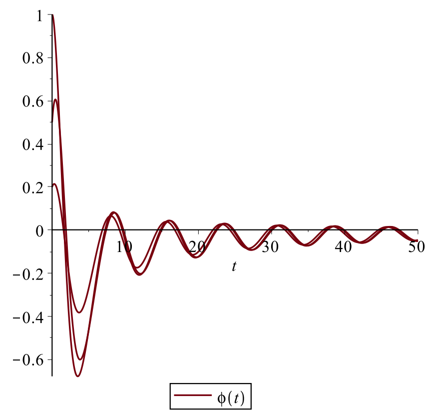

Figure 4.12: Plot for the solution of $\phi(t)$ of the system (4.2.2) with $n = 1$, $\mu = 0.45$, $k = 0$ as $t \to \infty$ with several initial conditions:$[[\phi(0) = 1, (D(\phi))(0) = 0.1, a(0) = 1], [\phi(0) = 0.22, (D(\phi))(0) = 0.1, a(0) = 0.6], [\phi(0) = 0.6, (D(\phi))(0) = 0.5, a(0) = 0.8]]$.

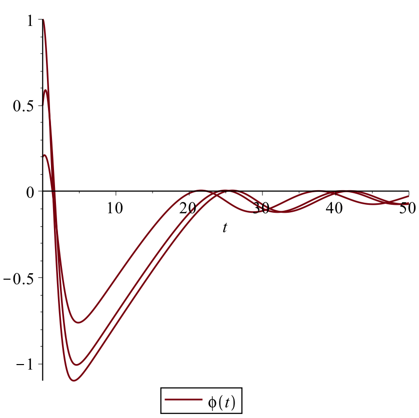

Figure 4.13: Plot for the solution of $\phi(t)$ of the system (4.2.2) with $n = 1$, $\mu = 0.75$, $k = 0$ as $t \to \infty$ with several initial conditions:$[[\phi(0) = 1, (D(\phi))(0) = 0.1, a(0) = 1], [\phi(0) = 0.22, (D(\phi))(0) = 0.1, a(0) = 0.4], [\phi(0) = 0.6, (D(\phi))(0) = 0.5, a(0) = 0.8]$.

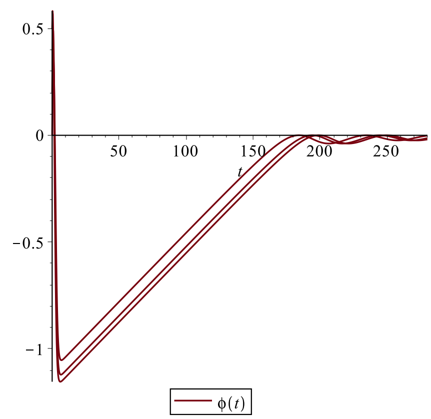

Figure 4.14: Plot for the solution of $\phi(t)$ of the system (4.2.2) with $n = 1$, $\mu = 0.81$, $k = 0$ as $t \to \infty$ with several initial conditions:$[[\phi(0) = 1, (D(\phi))(0) = 0.1, a(0) = 1], [\phi(0) = 0.22, (D(\phi))(0) = 0.1, a(0) = 0.4], [\phi(0) = 0.6, (D(\phi))(0) = 0.5, a(0) = 0.8]]$.

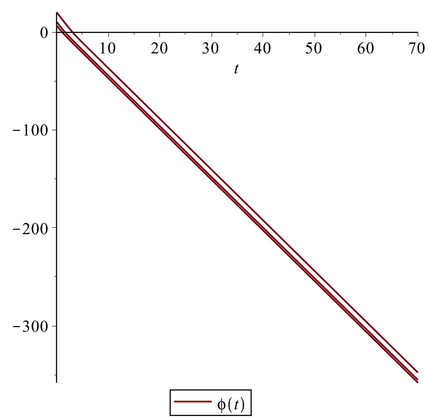

Figure 4.15: Plot for the solution of $\phi(t)$ of the system (4.2.2) with $n = 1$, $\mu = 2$, $k = 0$ as $t \to \infty$ with several initial conditions:$[[\phi(0) = 0.1, (D(\phi))(0) = 0.1, a(0) = 1], [\phi(0) = 0.001, (D(\phi))(0) = 0.1, a(0) = 0.4], [\phi(0) = 0.01, (D(\phi))(0) = 0.5, a(0) = 0.8]]$.

Discussion

As can be seen in figures from 4.12 to 4.15, the numerical solutions suggest that there is bifurcation value, in which the model changes its behaviour as μ passes through a bifurcation value $\mu = \sqrt{\frac{2}{3}}n$. As can be seen from figure 4.12, where we plotted the model with small value of μ, and we have oscillations for $\phi(t)$ then decay to zero. However, when $\mu > \sqrt{\frac{2}{3}}n$ the oscillations disappear and the stability (the behaviour) of the model changes (see figure 4.14). For the value $\mu > \sqrt{\frac{2}{3}}n$, as can be seen in figure 4.15, there is no oscillations and ϕ decreases (note that we get the same behaviour even if we run it for long time). Note that changing the curvature to negative we get similar behaviour of the model in the original variables.

Bifurcation Value $\left(i.e.,\ \mu_c = \sqrt{\frac{2}{3}}n\right)$ Analysis

In this subsubsection we show an analytic proof for the bifurcation value. From the steady state for the model in the equation (4.2.2), set $\dot{\phi} = 0$ leads to

$$\begin{aligned}
n^2\phi + 3\mu\left(\sqrt{\frac{1}{6}n^2\phi^2 - \frac{k}{a^2}}\right) &= 0,\\
n^2\phi &= -3\mu\left(\sqrt{\frac{1}{6}n^2\phi^2 - \frac{k}{a^2}}\right),\\
n^4\phi^2 &= 9\mu^2\left(\frac{1}{6}n^2\phi^2 - \frac{k}{a^2}\right)\\
n^4\phi^2 - \frac{3}{2}\mu^2 n^2\phi^2 &= -9\frac{k}{a^2}\mu^2,\\
n^2\phi^2(n^2 - \frac{3}{2}\mu^2) &= -9\frac{k}{a^2}\mu^2.
\end{aligned} \tag{4.2.3}$$

If $k = 0$ then

$$n^2\Phi^2(n^2 - \frac{3}{2}\mu^2) = 0. \tag{4.2.4}$$

Then,

$$(n^2 - \frac{3}{2}\mu^2) = 0. \tag{4.2.5}$$

Hence, the bifurcation value, in which the stability (the behaviour) of the model changes, is defined by (recall μ is positive):

$$\mu_c = \sqrt{\frac{2}{3}}n. \tag{4.2.6}$$

4.2.4 Normalized Bounded Variable

Introducing Normalized Variables

Introduce the normalized bounded variable D as before, along with the following expressions for Φ, Ψ :

$$D \equiv \frac{\theta}{\sqrt{1+\theta^2}}, \quad \Psi \equiv \sqrt{\frac{3}{2}}\left(\frac{n\phi}{\sqrt{1+\theta^2}}\right), \quad \Phi \equiv \sqrt{\frac{3}{2}}\left(\frac{\dot{\phi}}{\sqrt{1+\theta^2}}\right). \tag{4.2.7}$$

Then the Friedmann equation for this model becomes

$$D^2 - \Phi^2 - \Psi^2 = -\frac{9k}{a^2(1+\theta^2)} \geq 0. \tag{4.2.8}$$

For the zero curvature the Friedmann equation becomes as follow:

$$D^2 = \Phi^2 + \Psi^2, \tag{4.2.9}$$

which is conserved by the 3D system in (4.2.11) (illustration below), when $k = 0$.

Define

$$\Lambda = D^2 - \Phi^2 - \Psi^2,$$

then

$$\begin{aligned}
\Lambda' &= 2DD' - 2\Phi\Phi' - 2\Psi\Psi' \\
&= 2\left[D(1-D^2)\mathcal{X} + D\Phi^2(1+\mathcal{X}) + D\Psi^2\mathcal{X} + \sqrt{\frac{3}{2}}\mu\Phi D\sqrt{1-D^2}\right] \\
&= 2\left[-D\mathcal{X}\Lambda + D\mathcal{X} + D\Phi^2 + \sqrt{\frac{3}{2}}\mu\Phi D\sqrt{1-D^2}\right] \\
&= 2\left[-D\mathcal{X}\Lambda - \frac{D}{3}(D^2 - \Phi^2 - \Psi^2)\right] \\
&= -\frac{2}{3}D\Lambda(3\mathcal{X}+1) = 0.
\end{aligned}$$

The domain of interest is $0 \leq D \leq 1$ since if

$$\theta \to \infty \quad \Longrightarrow D \to 1,$$

and

$$\theta \to 0 \quad \Longrightarrow D \to 0.$$

Based on equation (4.2.9) and the interest domain of D ($0 \leq D \leq 1$), then

$$0 \leq \Phi^2 + \Psi^2 \leq D^2 \leq 1. \tag{4.2.10}$$

Then, it follows that the other variables Φ, Ψ are bounded. Now we define a new time variable τ with $\frac{d\tau}{dt} = \sqrt{1+\theta^2}$, and apply it to (4.2.7), therefore the evolution equation becomes

$$\begin{cases}
D' = (1-D^2)\mathcal{X}, \\
\Psi' = n\Phi\sqrt{1-D^2} - D\Psi\mathcal{X}, \\
\Phi' = -D\Phi - \sqrt{1-D^2}\left(n\Psi + \sqrt{\frac{3}{2}}\mu D\right) - \Phi D\mathcal{X}.
\end{cases} \tag{4.2.11}$$

where the prime here indicates the differentiation of each variable with respect to the new variable time τ and $\mathcal{X}$ is given by the expression as follows:

$$\mathcal{X} = -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2 - \sqrt{\frac{3}{2}}\mu\Phi\left(\sqrt{1-D^2}\right) = \frac{\dot{\theta}}{(\theta^2+1)}. \tag{4.2.12}$$

Bounded Variables:Qualitative Analysis

The equations in 3D system (4.2.11) are relatively simple and it is a well defined system. Note that the D' does not decouple and the system is not analytic at $D = 1$ since we have the square root term in the second equation of the 3D system (4.2.11). Moreover, the origin equilibrium point P_0 is highly degenerate.

Equilibrium Points

In table (4.2), we present the summary of equilibrium points for the system (4.2.11) with the value of $\mathcal{X}$ and their stability.

Pt	(D,Φ,Ψ)	$\mathcal{X}$	Stability		k-value
			$\mu < \sqrt{\frac{3}{2}}n$	$\mu > \sqrt{\frac{3}{2}}n$	
P_0	$(0,0,0)$	0	Sink	Saddle	$k = 0, -1$
P_1	$(1,1,0)$	-1	Source	Source	$k = 0, -1$
P_2	$(1,-1,0)$	-1	Source	Source	$k = 0, -1$
P_3	$(1,0,1)$	0	Saddle	Saddle	$k = 0, -1$
P_4	$(1,0,-1)$	0	Saddle	Sink	$k = 0, -1$
P_5	$(1,0,0)$	$-\frac{1}{3}$	Saddle	Saddle	$k = -1$

Table 4.2: Equilibrium points of the system (4.2.11); the value of $\mathcal{X}$ and their stability.

Let us discuss the stability of the equilibrium points of the system (4.2.11):

Stability of Equilibrium point (P_0) for $k = 0$

Evaluating the linearization matrix of the system (4.2.11) at P_0 gives us the following eigenvalues

$$\lambda_1 = 0, \tag{4.2.13}$$

$$\lambda_{2,3} = \pm\frac{\sqrt{6\mu^2 - 4n^2}}{2}. \tag{4.2.14}$$

Note that, if $\mu > \sqrt{\frac{2}{3}}n$ then $\lambda_2 > 0,\ \lambda_3 < 0$ which implies that P_0 is a saddle. But, if $\mu < \sqrt{\frac{2}{3}}n$ then all the eigenvalues have zero real part which implies that P_0 is non-hyperbolic equilibrium point. Therefore, the "Hartman-Grobman" theorem fails and the behaviour of P_0 for this model is generally not determined by the linearization and thus is more difficult to study. Therefore, P_0 needs to be analyzed more carefully with different techniques (see below).

The numerical solutions of the system (4.2.11) (see figure 4.15) suggest that for any positive value of μ such that $\mu > \sqrt{\frac{2}{3}}n$, P_0 is a saddle. When $\mu < \sqrt{\frac{3}{2}}n$ then P_0 is a sink. In the following pages we will show that analytically.

Qualitative Analysis for P_0 when $\mu < \mu_c$

This section presents analytic proof to show that P_0 is a sink when $\mu < \mu_c$. Recall the model

$$\begin{cases} D' = (1 - D^2)\mathcal{X}, \\ \Psi' = n\Phi\sqrt{1 - D^2} - D\Psi\mathcal{X}, \\ \Phi' = -D\Phi - \sqrt{1 - D^2}\left(n\Psi + \sqrt{\dfrac{3}{2}}\mu D\right) - \Phi D\mathcal{X}. \end{cases} \tag{4.2.15}$$

where $\mathcal{X}$ is

$$\mathcal{X} = -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2 - \sqrt{\frac{3}{2}}\mu\Phi\left(\sqrt{1-D^2}\right). \tag{4.2.16}$$

Using Taylor series, we have

$$\sqrt{1-D^2} = 1 - \frac{1}{2}D^2 + ..., \tag{4.2.17}$$

which is valid for small value of D. Neglect all the higher order terms then the system (4.2.15) becomes

$$\begin{cases} D' = -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2 - \sqrt{\frac{3}{2}}\mu\Phi, \\ \Psi' = n\Phi, \\ \Phi' = -D\Phi - \left(n\Psi + \sqrt{\frac{3}{2}}D\mu\right). \end{cases} \tag{4.2.18}$$

Define the variables as follow:

$$D = \epsilon d, \quad \Phi = \epsilon\phi, \quad \Psi = \epsilon\psi, \tag{4.2.19}$$

where ϵ is small. Now, plugging (4.2.19) into (4.2.18) leads to

$$\begin{cases} d' = \frac{\epsilon}{3}\left(-d^2 - 2\phi^2 + \psi^2\right) - \sqrt{\frac{3}{2}}\mu\phi, \\ \psi' = n\phi, \\ \phi' = -\epsilon d\phi - \left(n\psi + \sqrt{\frac{3}{2}}d\mu\right). \end{cases} \tag{4.2.20}$$

The system (4.2.20) can be rewritten as follow:

$$\begin{cases} d' = \frac{\epsilon}{3}\left(-d^2 - \frac{2}{n^2}\psi'^2 + \psi^2\right) - \sqrt{\frac{3}{2}}\frac{\mu}{n}\psi', \\ \psi'' = -n\left(n\psi + \sqrt{\frac{3}{2}}d\mu\right) - \epsilon d\psi'. \end{cases} \tag{4.2.21}$$

Multiple Scales Method

We use the multiple scales method with t as the fast time and $\tau = \epsilon t$ as the slow time to determine a leading order approximation to the solution of ψ, d of the model in equation (4.2.21). The solutions of ψ, d are expanded as

$$\begin{aligned} \psi \equiv \psi(t,\tau) &\sim \psi_0(t,\tau) + \epsilon\psi_1(t,\tau) + ..., \\ d \equiv d(t,\tau) &\sim d_0(t,\tau) + \epsilon d_1(t,\tau) + ... \end{aligned} \tag{4.2.22}$$

Using the chain rule gives

$$\begin{aligned} d' &= d_{0t} + \epsilon(d_{0\tau} + d_{1t}) + ..., \\ \psi' &= \psi_{0t} + \epsilon(\psi_{0\tau} + \psi_{1t}) + ..., \\ \psi'' &= \psi_{0tt} + \epsilon(2\psi_{0t\tau} + \psi_{1tt}) + .., \end{aligned} \tag{4.2.23}$$

Substituting (4.2.23) and (4.2.22) into (4.2.21) yields

$$\begin{cases} d_{0t} + \epsilon(d_{0\tau} + d_{1t}) = \dfrac{\epsilon}{3}\left(-{d_0}^2 - \dfrac{2}{n^2}{\psi_{0t}}^2 + {\psi_0}^2\right) - \sqrt{\dfrac{3}{2}}\dfrac{\mu}{n}(\psi_{0t} + \epsilon(\psi_{0\tau} + \psi_{1t})), \\ \psi_{0tt} + \epsilon(2\psi_{0t\tau} + \psi_{1tt}) = -n^2(\psi_0 + \epsilon\psi_1) - n\sqrt{\dfrac{3}{2}}\mu(d_0 + \epsilon d_1) - \epsilon d_0\psi_{0t}. \end{cases} \tag{4.2.24}$$

Equating coefficients of like powers of ϵ to 0, gives the following sequence of partial differential equations:

$$O(1): \quad d_{0t} + \sqrt{\frac{3}{2}}\frac{\mu}{n}\psi_{ot} = 0, \tag{4.2.25a}$$

$$: \quad \psi_{0tt} + n^2\psi = -n\sqrt{\frac{3}{2}}\mu d_0, \tag{4.2.25b}$$

$$O(\epsilon): \quad d_{1t} + \sqrt{\frac{3}{2}}\frac{\mu}{n}\psi_{1t} = -d_{0\tau} + \frac{1}{3}\left(-d_0^2 - \frac{2}{n^2}\psi_{0t}^2 + \psi_0^2\right) - \sqrt{\frac{3}{2}}\frac{\mu}{n}\psi_{0\tau}, \tag{4.2.25c}$$

$$: \quad \psi_{1tt} + \left(n^2\psi_1 + \sqrt{\frac{3}{2}}n\mu d_1\right) = -2\psi_{0t\tau} - d_0\psi_{0t}. \tag{4.2.25d}$$

It follows from (4.2.25a), (4.2.25b) that d_0 and ψ_0 have the following solutions

$$\begin{cases} d_0 = -\sqrt{\frac{3}{2}}\frac{\mu}{n}\psi_0 + A(\tau) \quad , \\ \psi_0 = B(\tau)\cos(\sqrt{n^2 - \frac{3}{2}}t + \theta(\tau)) + \frac{\sqrt{6}n\mu A(\tau)}{3\mu^2 - 2n^2}, \end{cases} \tag{4.2.26}$$

where $A(\tau), B(\tau)$ are arbitrary functions of τ. Plugging the solutions from (4.2.26) into (4.2.25c) yields to

$$\begin{aligned} d_{1t} + \sqrt{\tfrac{3}{2}}\tfrac{\mu}{n}\psi_{1t} = &-\tfrac{1}{3}\left(n^2 - \tfrac{3}{2}\mu^2\right)\left(-\tfrac{B^2(\tau)}{n^2} + \tfrac{A(\tau)^2 n^2}{\left(n^2 - \frac{3}{2}\mu^2\right)^2}\right) - A_\tau \\ &-\tfrac{B(\tau)^2}{n^2}\left(n^2 - \tfrac{3}{2}\mu^2\right)\sin^2\left(\sqrt{n^2 - \tfrac{3}{2}\mu^2}t + \theta(\tau)\right). \end{aligned} \tag{4.2.27}$$

Integrating (4.2.27) with respect to t gives

$$\begin{aligned} d_1 = &-\sqrt{\tfrac{3}{2}}\tfrac{\mu}{n}\psi_1 - \tfrac{1}{3}t\left(n^2 - \tfrac{3}{2}\mu^2\right)\left(-\tfrac{B^2(\tau)}{n^2} + \tfrac{A(\tau)^2 n^2}{\left(n^2 - \frac{3}{2}\mu^2\right)^2}\right) - tA_\tau \\ &-\tfrac{B(\tau)^2}{n^2}\sqrt{n^2 - \tfrac{3}{2}\mu^2}\Bigg(-\tfrac{1}{2}\sin\left(\sqrt{n^2 - \tfrac{3}{2}\mu^2}t + \theta(\tau)\right)\cos\left(\sqrt{n^2 - \tfrac{3}{2}\mu^2}t + \theta(\tau)\right) \\ &\tfrac{1}{2}\sqrt{n^2 - \tfrac{3}{2}\mu^2}t + \tfrac{1}{2}\theta\Bigg). \end{aligned} \tag{4.2.28}$$

Substituting (4.2.28) into (4.2.25d) yields

$$\psi_{1tt} + \left(n^2 - \tfrac{3}{2}\mu^2\right)\psi_1 = 2\sqrt{n^2 - \tfrac{3}{2}\mu^2}B(\tau)\theta_\tau \cos\left(\sqrt{n^2 - \tfrac{3}{2}\mu^2}t + \theta(\tau)\right)$$
$$+\sqrt{n^2 - \tfrac{3}{2}\mu^2}\sin\left(\sqrt{n^2 - \tfrac{3}{2}\mu^2}t + \theta(\tau)\right)\left(2B_\tau + \tfrac{n^2 B(\tau)A(\tau)}{\left(n^2 - \frac{3}{2}\mu^2\right)}\right)$$
$$+\tfrac{3}{2}\mu\ nt\left(-\tfrac{1}{3}\left(n^2 - \tfrac{3}{2}\mu^2\right)\left(\tfrac{B(\tau)^2}{2n^2} + \tfrac{A(\tau)^2 n^2}{3\left(n^2 - \frac{3}{2}\mu^2\right)^2}\right) - A_\tau\right). \tag{4.2.29}$$

Therefore, in order to eliminate these secular terms, we must have the following differential equations

$$\begin{cases} \varphi_\tau = 0, \\ A_\tau = -\dfrac{1}{3}\left(n^2 - \dfrac{3}{2}\mu^2\right)\left(\dfrac{B(\tau)^2}{2n^2} + \dfrac{A(\tau)^2 n^2}{3\left(n^2 - \frac{3}{2}\mu^2\right)^2}\right), \\ B_\tau = -\dfrac{n^2 B(\tau)A(\tau)}{2\left(n^2 - \frac{3}{2}\mu^2\right)}. \end{cases} \tag{4.2.30}$$

Now, in figure 4.16, 4.17, we plot the original system in equation 4.2.21 and the reduced system in (4.2.30) with $n = 1, \mu = 0.5$ and compare the behaviour of $d(t), \psi(t)$. with $A(t), B(t)$.

As can be seen from the equations (4.2.30) that $A_\tau < 0$ if $\mu < \sqrt{\frac{3}{2}}n$, which consistent with the numeric plot in figures 4.16. Also, from figure 4.17 show that $B(t)$ decays too, which implies thatB_τ decreases. Therefore, when $\mu < \sqrt{\frac{3}{2}}n$, the solution decay to zero which implies that P_0 is a sink in this case.

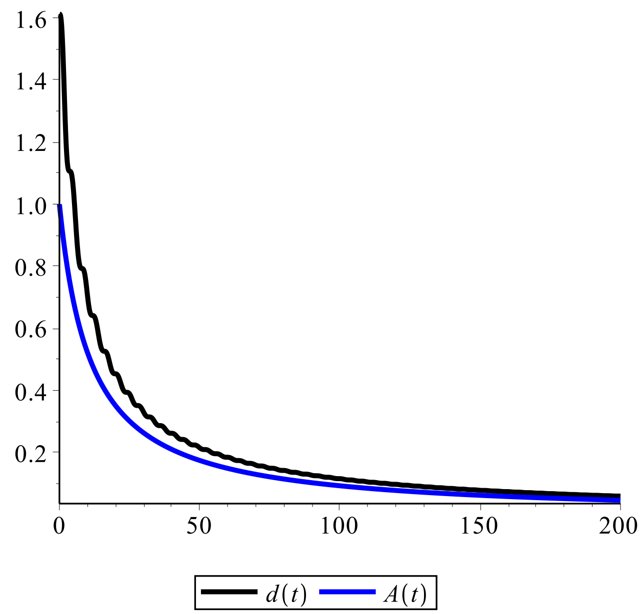

Figure 4.16: Plot of solution of $A(t)$ for the system (4.2.30) with $n = 1$ and $k = 0$ and initial condition: $[A(0) = 1, B(0) = 1]$ together with the plot the solution of $d(t)$ for the system (4.2.21) with $n = 1$, $k = 0$ and one initial condition:$[\psi(0) = 0.99, D(\psi(0)) = 0, d(0) = 1]$, as $t \to \infty$.

Qualitative Analysis P_0: When $\mu > \mu_c$ for the 2D (k=0) invariant set

At equilibrium point P_0, we have $D^2 \cong 0$, and $\Phi^2 + \Psi^2 \cong 0$. For the first equation in the system (4.2.11), we have that $D' = (1 - D^2)\mathcal{X}$. So, we want to analyze the sign of D'. Since the domain of interest for D is $0 \le D \le 1$, then $(1 - D^2) > 0$ (i.e., in our interest domain for $D : 0 \le D < 1$). We have $\mathcal{X}$

$$\mathcal{X} = -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2 - \sqrt{\frac{3}{2}}\mu\Phi\left(\sqrt{1 - D^2}\right), \tag{4.2.31}$$

and since $\Phi^2 + \Psi^2 \le D^2$ then

$$\mathcal{X} \le -\Phi^2 - \mu\Phi\sqrt{\frac{3}{2}}\sqrt{1 - D^2}. \tag{4.2.32}$$

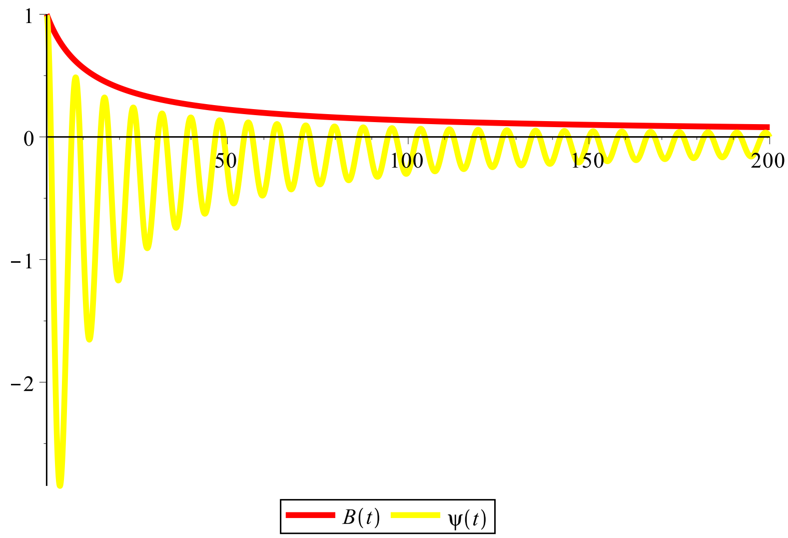

Figure 4.17: Plot of solution of $B(t)$ for the system (4.2.30) with $n = 1$ and $k = 0$ and initial condition: $[A(0) = 1, B(0) = 1]$ together with the plot the solution of $\psi(t)$ for the system (4.2.21) with $n = 1$, $k = 0$ and one initial condition:$[\psi(0) = 0.99, D(\psi(0)) = 0, d(0) = 1]$, as $t \to \infty$.

For $D^2 \simeq 0$ and $\Phi^2 + \Psi^2 \simeq 0$, then

$$\mathcal{X} \cong -\sqrt{\frac{3}{2}}\mu\Phi. \tag{4.2.33}$$

Also, note that $\sqrt{1 - D^2} \cong 1 - \frac{1}{2}D^2$ for $D^2 \cong 0$. So,

$$D' = -\sqrt{\frac{3}{2}}\mu\Phi. \tag{4.2.34}$$

Since we are looking to see what happen to the system when $D^2 \cong 0$, then we can neglect higher order terms of all variables Φ, Ψ, D which yield to

$$\begin{cases} D' = -\mu\sqrt{\frac{3}{2}}\Phi, \\ \Psi' = n\Phi, \\ \Phi' = -\left[n\Psi + \sqrt{\frac{3}{2}}\mu D\right]. \end{cases} \tag{4.2.35}$$

We notice using (4.2.35) that

$$nD' + \sqrt{\frac{3}{2}}\mu\Psi' \cong -n\sqrt{\frac{3}{2}}\mu\Phi + n\mu\sqrt{\frac{3}{2}}\Phi \cong 0. \tag{4.2.36}$$

Thus,

$$\sqrt{\frac{3}{2}}\mu\Psi + nD \cong 0, \tag{4.2.37}$$

which implies

$$D \cong -\frac{\sqrt{\frac{3}{2}}\mu}{n}\Psi. \tag{4.2.38}$$

Hence, the constraint with zero curvature (4.2.9) can be written as follow

$$\Psi^2 + \Phi^2 = \left(\sqrt{\frac{3}{2}}\frac{\mu}{n}\right)^2 \Psi^2 \tag{4.2.39}$$

Then,

$$\Phi^2 = \Psi^2\left[\left(\sqrt{\frac{3}{2}}\frac{\mu}{n}\right)^2 - 1\right], \tag{4.2.40}$$

which is only valid if $\mu > \sqrt{\frac{2}{3}}n$. Substituting the value of D from (4.2.38) into Φ' equation in (4.2.35), we obtain

$$\Phi' \quad = \quad -\sqrt{\frac{2}{3}}\frac{D}{\mu}\left[\frac{3}{2}\mu^2 - n^2\right] < 0 \quad \left(\text{if} \quad \mu > \sqrt{\frac{2}{3}}n\right).$$

Since $\Phi = 0$ is not invariant, $\Phi' < 0$ and therefore Φ decreases. Therefore, $\mathcal{X}$ becomes positive, which implies that $D' > 0$ and D increases. So, we can conclude that when $\mu > \sqrt{\frac{2}{3}}n$, the origin equilibrium point P_0 is unstable and it is a saddle.

Stability of the Equilibrium Points $P_{1,2}$ When $k = 0, -1$

Evaluating the linearization matrix of the system in equation (4.2.11) at $P_{1,2}$ gives us undefined terms, so we have to examine the stability of these equilibrium points using different techniques. Evaluating the linearization matrix at $\Phi = \pm 1, \Psi = 0$, we found that there is always one positive eigenvalue at $D = 1$. In order to determine the signs of the other eigenvalues we need to study the 2D system of Φ, Ψ. We can do that with $k = 0, -1$ as shown below;

When $k = 0$ and if $D^2 = 1$ then we have a 1-dimensional set (i.e.,$\Phi^2 + \Psi^2 = 1$).

When $k = -1$ There is a 2-dimensional invariant set when $D^2 = 1$ given by

$$\begin{cases} \Psi' = -\Psi\left(-\frac{1}{3} - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2\right), \\ \Phi' = -\Phi\left(\frac{2}{3} - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2\right), \end{cases} \tag{4.2.41}$$

with the constraint $\Phi^2 + \Psi^2 \leq 1$. There are five equilibrium points for the 2D system in equation (4.2.41).

$$\{(\Psi, \Phi)\} = \{(0,0), (\pm 1, 0), (0, \pm 1)\}. \tag{4.2.42}$$

Note that, $(\pm 1, 0), (0, \pm 1)$ are in the $k = 0$ set. The linearization matrix for the 2D system (4.2.41) is given by

$$J(\Psi, \Phi) = \begin{bmatrix} \frac{1}{3} + \frac{2}{3}\Phi^2 - \Psi^2 & \frac{4}{3}\Phi\Psi \\ -\frac{2}{3}\Phi\Psi & -\frac{2}{3} - \frac{1}{3}\Psi^2 + 2\Phi^2 \end{bmatrix}. \tag{4.2.43}$$

Now we will evaluate all the equilibrium points in the linearization matrix (4.2.43) to see the stability (the behaviour) of them in the unit disk. For $(0,0)$, we get two eigenvalues one positive and the other negative, which implies that the point $(0,0)$

is a saddle. For $P_{1,2}$, we get two positive real eigenvalues which implies that $P_{1,2}$ are sources in the 2D system as well as in the 3D However, P_5 are saddles in the 2D as well as 3D.

Stability of the Equilibrium Points $P_{3,4}$ When $k = 0, -1$

Evaluating the linearization matrix (4.2.43) at $P_{3,4}$, we get two negative eigenvalues which implies that $P_{3,4}$ are sinks in the 2D invariant set $D^2 = 1$. All we need now is to examine whether the third eigenvalue on the boundary has positive or negative sign. In order to do that we will reduce the system (4.2.11) to easier system to study. Recall the system

$$\begin{cases} D' = (1 - D^2)\mathcal{X}, \\ \Psi' = n\Phi\sqrt{1 - D^2} - D\Psi\mathcal{X}, \\ \Phi' = -D\Phi - \sqrt{1 - D^2}\left(n\Psi + \sqrt{\frac{3}{2}}\mu D\right) - \Phi D\mathcal{X}. \end{cases} \tag{4.2.44}$$

where $\mathcal{X}$ is given by

$$\mathcal{X} = -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2 - \sqrt{\frac{3}{2}}\mu\Phi\left(\sqrt{1 - D^2}\right). \tag{4.2.45}$$

Also, we have the Friedmann equation with zero curvature

$$D^2 - \Phi^2 - \Psi^2 = 0 \tag{4.2.46}$$

Plugging equation (4.2.46) into the expression (4.2.45) leads to

$$\mathcal{X} = -\Phi^2 - \sqrt{\frac{3}{2}}\mu\Phi\sqrt{1 - D^2}, \tag{4.2.47}$$

on the boundary of the phase space and therefore,

$$\begin{cases} D' = -(1-D^2)\left(\Phi^2 + \sqrt{\frac{3}{2}}\mu\Phi\sqrt{1-D^2}\right), \\ \Psi' = n\Phi\sqrt{1-D^2} + D\Psi\left(\Phi^2 + \sqrt{\frac{3}{2}}\mu\Phi\sqrt{1-D^2}\right), \\ \Phi' = -D\Phi\left(1 - \Phi^2 - \sqrt{\frac{3}{2}}\mu\Phi\sqrt{1-D^2}\right) - \sqrt{1-D^2}\left(n\Psi + \sqrt{\frac{3}{2}}\mu D\right). \end{cases} \tag{4.2.48}$$

Since $\Phi \simeq 0$ is an invariant set near $P_{3,4}$, we can neglect the higher order terms of Φ in the above system and leads to

$$\begin{cases} D' = -(1-D^2)\left(\Phi^2 + \sqrt{\frac{3}{2}}\mu\Phi\sqrt{1-D^2}\right), \\ \Phi' = -D\Phi - \sqrt{1-D^2}\left(n\Psi + \sqrt{\frac{3}{2}}\mu D\right). \end{cases} \tag{4.2.49}$$

Since, Φ is an invaraint set then $\Phi' = 0$, which yield

$$\Phi = -T\left(n\Psi + \sqrt{\frac{3}{2}}\mu\right). \tag{4.2.50}$$

Let $T = \sqrt{1-D^2} << 1$, then, $T^2 = 1 - D^2$. And

$$\begin{aligned} 2TT' &= -2DD', \\ TT' &= -D\left(-T^2\left(\Phi^2 + \sqrt{\frac{3}{2}}\mu\Phi T\right)\right), \\ T' &= DT\Phi\left(\Phi + \sqrt{\frac{3}{2}}\mu T\right), \\ T' &= \sqrt{1-T^2}T\Phi\left(\Phi + \sqrt{\frac{3}{2}}\mu T\right). \end{aligned} \tag{4.2.51}$$

Since T is small then expand $\sqrt{1-T^2}$ using a Taylor series as

$$\sqrt{1-T^2} = 1 - \frac{1}{2}T^2 + ... \tag{4.2.52}$$

Now, plugging (4.2.52) into (4.2.51), and using (4.2.50), T' becomes

$$T' = T^3 n\Psi \left(n\Psi + \sqrt{\frac{3}{2}}\mu \right) \left(1 - \frac{T^2}{2} \right). \tag{4.2.53}$$

Examine the sign of the constant of the lowest order term of T

$$T' = T^3 n\Psi \left(n\Psi + \sqrt{\frac{3}{2}}\mu \right). \tag{4.2.54}$$

Therefore, we can now study each of the equilibrium points $P_{3,4}$ below.

1. P_3, since $\Psi = 1$ near P_3, plug $\Psi = 1$ in (4.2.54), thus T' is always positive which means P_3 is always a saddle.

2. P_4: plug $\Psi = -1$ in (4.2.54) then T' have two cases:

   - If $\mu < \sqrt{\frac{2}{3}}n \Longrightarrow T' > 0$ which means P_4 is a saddle.
   - If $\mu > \sqrt{\frac{2}{3}}n \Longrightarrow T' < 0$ which means P_4 is a sink.

Stability of the line of the Equilibrium Points When $\mu = \mu_c$:

When $\mu = \mu_c$, we have a line of sinks of the form

$$\Phi = 0, \Psi = -D, D = D.$$

And the determinant equation of the system when $\mu = \sqrt{\frac{2}{3}}n$ when $\Phi = 0$ is given by

$$\lambda^3 + \frac{5}{3}D\lambda^2 + \frac{2}{3}D^2\lambda = 0. \tag{4.2.55}$$

If $D = 0$ then $\lambda_{1,2,3} = 0$. But if $D \neq 0$ then there are three eigenvalue:

$$\lambda_1 = -D, \quad \lambda_2 = -\frac{2}{3}D, \quad \lambda_3 = 0,$$

which is line of sinks for all $D > 0$.

4.2.5 Bounded Variables: Numerical Method-Past Behaviour

In figure 4.18 we plot solution of the 3D system in (4.2.11) with $n = 1, k = 0$ for different positive values of μ into the past times.

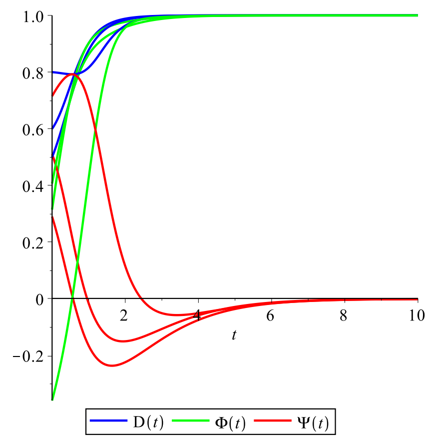

Figure 4.18: Plot for the system (4.2.11) with $n = 1$ and $k = 0$ as $t \to -\infty$ with several initial conditions:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

Discussion: As can be seen in figure 4.18, $D, \Phi \to 1$ and $\Psi \to 0$ as $t \to -\infty$ which implies that it is going to P_1 in to the past time limit. Note that changing the curvature value to negative value does not effect the behaviour of the model into the past (i.e., we get similar behaviour when $k = -1$ in the early times).

4.2.6 Bounded Variables: Numerical Method-Future Behaviour

In figures from 4.19 to 4.22 we plot the numerical solutions for the system in (4.2.11) with $n = 1$ and with zero curvature into the future times for different positive values of $\mu > 0$.

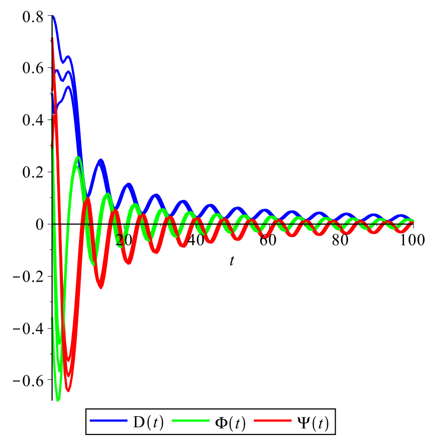

Figure 4.19: Plot for the system (4.2.11) with $n = 1, \mu = 0.45$ and $k = 0$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

Discussion

As can be seen from figure 4.19 to figure 4.20, we get the same numerical results for any values such that $\mu < \mu_c$), the solutions start with oscillations and then decay to zero which implies that P_0 is a sink. When $\mu > \mu_c$, P_4 is a sink (see figure 4.22).

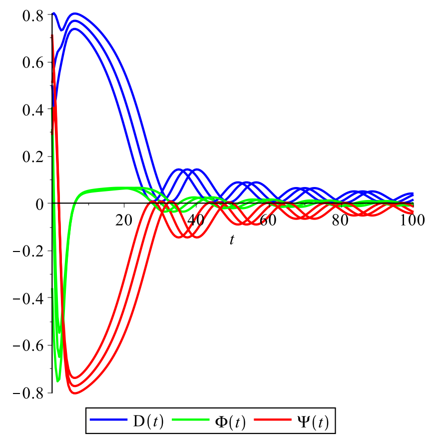

Figure 4.20: Plot for the system (4.2.11) with $n = 1, \mu = 0.75$ and $k = 0$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

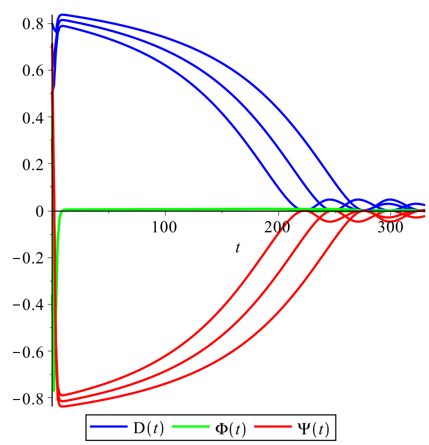

Figure 4.21: Plot for the system (4.2.11) with $n = 1, \mu = 0.814$ and $k = 0$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

Bounded Variable: Numerical Method in (2D Space) When $k = 0$

When $0 \leq \mu < \mu_c$ In figures 4.23, 4.24, we plot the solution of the system in (4.2.11) with the constraint (4.2.9) in the $k = 0$ invariant set 2D space with $n = 1$, $\mu = 0, 0.5$ respectively.

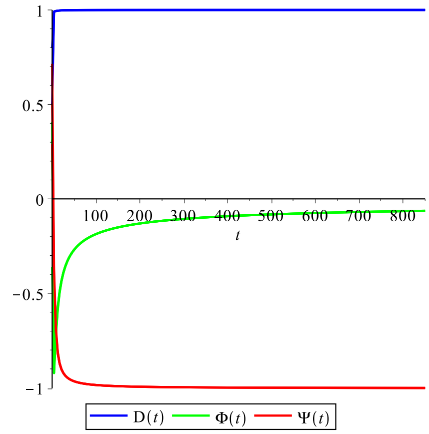

Figure 4.22: Plot for the system (4.2.11) with $n = 1, \mu = 2$ and $k = 0$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

Discussion As you can see from the figures 4.23, 4.24, when $0 \leq \mu < \mu_c$, the trajectories go away from the sources $P_{1,2}$ and approach P_0.

When $\mu = \mu_c$ In figure 4.25, we plot the solution of the system in (4.2.11) with the constraint (4.2.9) in the $k = 0$ invariant set 2D space with $n = 1$, $\mu = 0.814$.

Discussion As you can see from the figures 4.25, when $\mu = \mu_c$the trajectories approach the line of sinks $(\Phi = 0, \Psi = -D, D = D)$.

When $\mu > \mu_c$ In figure 4.26, we plot the solution of the system in (4.2.11) with the constraint (4.2.9) in the $k = 0$ invariant set 2D space with $n = 1$, $\mu = 2$.

Discussion As can be seen from figure (4.26), when $\mu > \mu_c$ the trajectories move away from the line of sink to approach P_4.

4.2.7 Inflation

As an indicator of the accelerated expansion we introduce the deceleration parameter, defined as

$$q \equiv = -\frac{a\ddot{a}}{\dot{a}^2} = -\left(\frac{3\dot{\theta}}{\theta^2} + 1\right). \tag{4.2.56}$$

With the use of equation of $\mathcal{X}$ from (4.2.45) and by using the definition of the normalized variables, it follows that the deceleration parameter can be expressed in terms of the normalized bounded variables as follows;

$$q = -\frac{1}{D^2}\left(-2\Phi^2 + \Psi^2 - 3\sqrt{\frac{3}{2}}\mu\sqrt{1-D^2}\Phi\right). \tag{4.2.57}$$

The sign of the deceleration parameter indicates the nature of the expansionary evolution. If $q > 0$, the cosmological expansion is decelerating, while negative values of q indicate an accelerating dynamics. The value for q for each of the equilibrium points of the model are:

1. P_0: q is undefined in terms of the normalized variables. Using the original variables we obtain that

   $$q = -\frac{9}{\theta^2}\left(-\frac{1}{3}\dot{\phi}^2 - \frac{1}{2}\mu\dot{\phi} + \frac{1}{6}n^2\phi^2\right), \tag{4.2.58}$$

   which is zero at P_0. Hence, P_0 is not inflationary.

2. $P_{1,2}$:

   $$q\big|_{P_{1,2}} = 2 > 0, \tag{4.2.59}$$

   are always positive which implies that $P_{1,2}$ are not inflationary.

3. $P_{3,4}$:

$$q\Big|_{P_{3,4}} = -1 < 0, \tag{4.2.60}$$

are always negative which implies that $P_{3,4}$ are inflationary.

4. P_5:

$$q\Big|_{P_5} = 0, \tag{4.2.61}$$

which implies that P_5 is not inflationary.

4.2.8 Slow Roll Inflation:

The expansion term tends to slow the evolution of ϕ down if the slow roll conditions are valid

$$\dot{\phi}^2 \ll V(\phi), \tag{4.2.62}$$

$$|\ddot{\phi}| \ll |\theta\dot{\phi}|, \tag{4.2.63}$$

With the slow-roll conditions (4.2.62,4.2.63) the Friedmann equation can be simplified to

$$\theta = \frac{3n|\phi|}{\sqrt{6}}. \tag{4.2.64}$$

And the Klein-Gordon equation in (4.2.2) becomes

$$\theta(\dot{\phi} + \mu) + n^2\phi = 0. \tag{4.2.65}$$

Plugging (4.2.64) into equation (4.2.65) we obtain

$$\dot{\phi} = \sqrt{\frac{2}{3}}n - \mu \tag{4.2.66}$$

Note that, when $\mu < \sqrt{\frac{2}{3}}n$ then $\dot{\phi} > 0$ but if $\mu > \sqrt{\frac{2}{3}}n$ then $\dot{\phi} < 0$. Integrating (4.2.66) we obtain

$$\phi = \left(\sqrt{\frac{2}{3}}n - \mu\right)t + C_5, \quad \text{where } C_5 \text{ is a constant.} \tag{4.2.67}$$

Hence,

$$\theta = \frac{3n}{\sqrt{6}}\left(\sqrt{\frac{2}{3}}n - \mu\right)t + C_7. \tag{4.2.68}$$

Now, we can calculate the solution of the dimensionless scale factor for the universe a by integrating the following equation

$$\begin{aligned}
\theta &= \frac{3\dot{a}}{a}, \\
\int \frac{3\dot{a}}{a} &= \int \left(\frac{3n}{\sqrt{6}}\left(\sqrt{\frac{2}{3}}n - \mu\right)t + C_7\right) dt, \\
\Rightarrow \ln(a) &= \frac{3n}{2\sqrt{6}}\left(\sqrt{\frac{2}{3}}n - \mu\right)t^2 + C_7 t + C_8, \quad \text{where} \quad C_7, C_8 \text{ constants} \\
a &= a_0 e^{\frac{n}{2\sqrt{6}}\left(\sqrt{\frac{2}{3}}n - \mu\right)t^2 + C_7 t}.
\end{aligned}$$

Thus, the solution for the scale factor of the expanding universe at $t = 0$ is $a = a_0$. Moreover, the slow roll inflation will happen for intermediate times (values of t for which the approximation are valid).

4.2.9 Discussion

In summary, in the isotropic model with $\mu > 0$, we found there is a bifurcation value (i.e., $\mu_c = \sqrt{\frac{3}{2}}n$). When $\mu < \mu_c$, the equilibrium point P_4 is a saddle and P_0 is a sink, which is consistent with the previous sub case (i.e., when $\mu = 0$, we have the origin

P_0 is a sink). However, when $\mu > \mu_c$ the equilibrium point P_4 is a sink (see figures 4.21). Moreover, if $\mu > \mu_c$, then there is an inflationary attractor at P_4 and there is inflationary saddle at P_3 but there is no inflationary source.

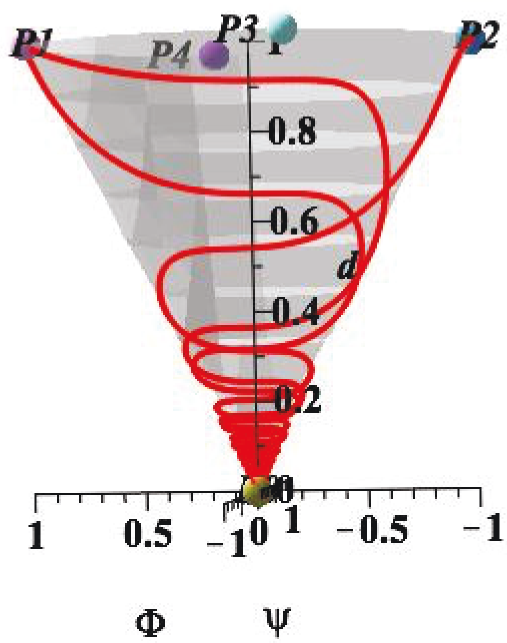

Figure 4.23: Plot for the system (4.2.11) with $n = 1, \mu = 0.5$ and $k = 0$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

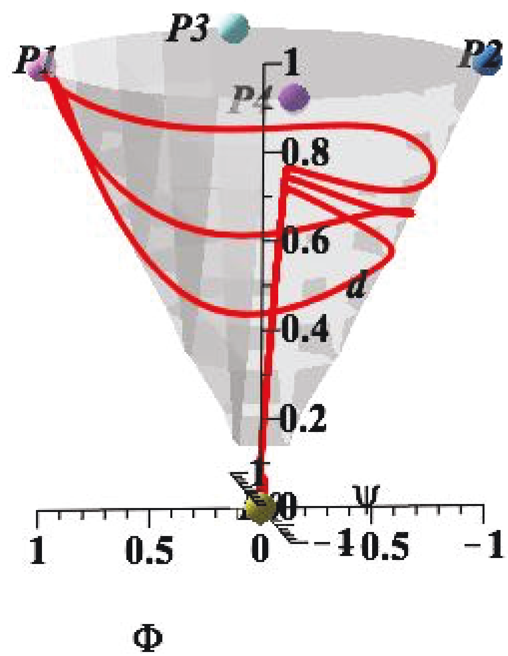

Figure 4.24: Plot for the system (4.2.11) with $n = 1, \mu = 0.5$ and $k = 0$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

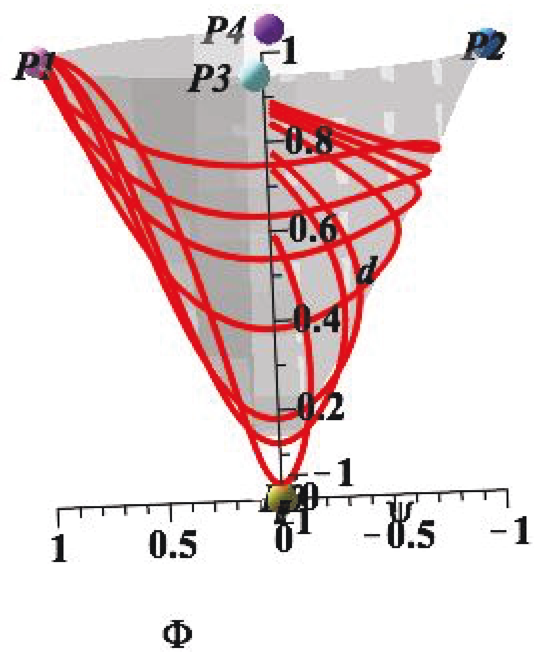

Figure 4.25: Plot for the system (4.2.11) with $n = 1, \mu = 1.5$ and $k = 0$ as $t \to \infty$ with several initial conditions for example:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

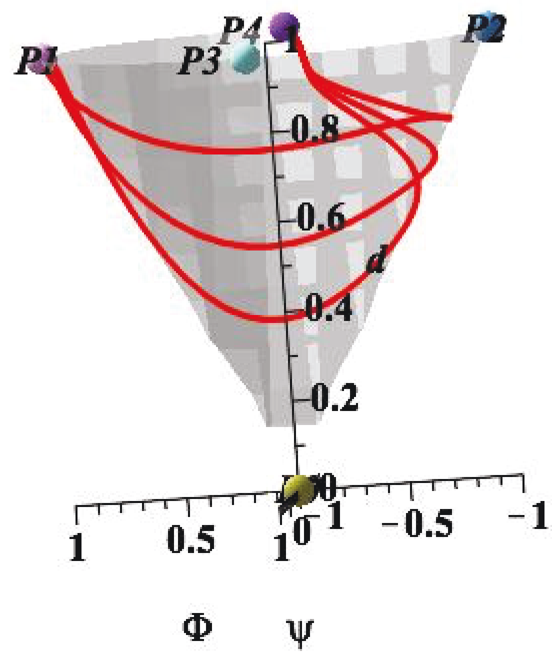

Figure 4.26: Plot for the system (4.2.11) with $n = 1, \mu = 2$ and $k = 0$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.6, \Phi(0) = 0.6\cos(45), \Psi(0) = 0.6\sin(45)], [D(0) = 0.5, \Phi(0) = 0.5\cos(120), \Psi(0) = 0.5\sin(120)], [D(0) = 0.8, \Phi(0) = 0.8\cos(90), \Psi(0) = 0.8\sin(90)]]$.

Chapter 5

Anisotropic Model

In this chapter, we investigate a class of the spatially homogeneous anisotropic Einstein-Aether models. We shall consider a potential of the form:

$$V(\theta, \phi, \sigma) = \sum_{r,s} a_{r,s} \theta^r \sigma^s \phi^{2-r-s}, \tag{5.0.1}$$

where $\{a_{r,s}\}$ are constants. Negative constants $a_{r,s}$ are permitted; however, it maybe required that the potential $V(\theta, \phi, \sigma)$ to be positive-definite. In particular, we shall study the potential of the form

$$V(\theta, \phi, \sigma) = \frac{1}{2} n^2 \phi^2 + \mu\theta\phi + \nu\sigma\phi, \tag{5.0.2}$$

where, without loss of generality, μ is positive (the system has combined $\mu \to -\mu$ and $\phi \to -\phi$ symmetry); in principle ν can be either sign positive or negative.

There are a number of special cases that might be of interest. The Einstein-Aether model with the shear equal to zero (i.e., $\sigma = 0$) that we studied earlier the isotropic model in chapter (4). Two other cases are:

1. The diagonal Bianchi type VI_h (non-zero curvature case), (note that we only set up the equations for future work but we are not study them in this thesis).

2. The Bianchi type V with zero curvature. There are three sub-cases that we study in this chapter:

   (a) When $\nu \neq 0$ and $\mu > 0$.

   (b) When $\nu = 0$ and $\mu > 0$.

   (c) When $\nu \neq 0$ and $\mu = 0$.

5.1 Case(1): Diagonal Bianchi VI_h

The general Friedmann equation (3.5.5) becomes

$$\theta^2 = 3\sigma^2 + 3\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}n^2\phi^2 - \sqrt{6}\phi\sigma\mu + \frac{1}{\sqrt{6}}\nu\theta\phi\right] - \frac{3(m^2+m+1)}{a^2}, \tag{5.1.1}$$

where m is a constant and it is define by $m = h - 1$. If $m = 1$, then we have Bianchi type V; if $m = 0$, then we have a Bianchi type III and we $m = -1$ we have Bianchi type VI_0. The Raychaudhuri equation (3.5.6) can be written as

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{1}{2}(\rho_\phi + 3p_\phi). \tag{5.1.2}$$

The shear evolution equation (3.5.7) becomes

$$\dot{\sigma} = -\sigma\theta + \frac{M}{3\sqrt{3}}\left[\theta^2 - 3\sigma^2 - 3\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}n^2\phi^2 - \sqrt{6}\mu\phi\sigma + \frac{1}{\sqrt{6}}\nu\theta\phi\right)\right]. \tag{5.1.3}$$

The Klein-Gordon equation (3.5.8) becomes

$$\ddot{\phi} + \theta\dot{\phi} + n^2\phi + \mu\theta + \nu\sigma = 0. \tag{5.1.4}$$

5.1.1 The Scalar Field

For the class of model under consideration the effective energy density and pressure has the following form

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}n^2\phi^2 - \sqrt{6}\mu\phi\sigma + \frac{1}{\sqrt{6}}\nu\theta\phi, \tag{5.1.5}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}n^2\phi^2 + \sqrt{6}\mu\phi\left(\sigma + \frac{\dot{\sigma}}{\theta}\right) - \frac{\nu\phi}{\sqrt{6}}\left(\theta + \frac{\dot{\theta}}{\theta}\right)$$
$$+ \mu\dot{\phi}\left(1 + \frac{\sqrt{6}\sigma}{\theta}\right) + \nu\dot{\phi}\left(\frac{\sigma}{\theta} - \frac{1}{\sqrt{6}}\right). \tag{5.1.6}$$

Hence,

$$\rho_\phi + 3p_\phi = \frac{2\sqrt{6}\theta}{2\sqrt{6}\theta - 3\nu\phi}\left[2\dot{\phi}^2 - n^2\phi^2 + 2\sqrt{6}\mu\phi\sigma - \frac{\nu\theta\phi}{\sqrt{6}} + 3\sqrt{6}\mu\phi\frac{\dot{\sigma}}{\theta}\right.$$
$$\left. + \frac{\sqrt{6}\nu\phi\sigma^2}{\theta} + 3\mu\dot{\phi}\left(1 + \frac{\sqrt{6}\sigma}{\theta}\right) + 3\nu\dot{\phi}\left(\frac{\sigma}{\theta} - \frac{1}{\sqrt{6}}\right)\right]. \tag{5.1.7}$$

Therefore, the evolution equation of the expansion rate for this model can be written as follows:

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{\sqrt{6}\theta}{2\sqrt{6}\theta - 3\nu\phi}\left[2\dot{\phi}^2 - n^2\phi^2 + 2\sqrt{6}\mu\phi\sigma - \frac{\nu\theta\phi}{\sqrt{6}} + 3\sqrt{6}\mu\phi\frac{\dot{\sigma}}{\theta}\right.$$
$$\left. + \frac{\sqrt{6}\nu\phi\sigma^2}{\theta} + 3\mu\dot{\phi}\left(1 + \frac{\sqrt{6}\sigma}{\theta}\right) + 3\nu\dot{\phi}\left(\frac{\sigma}{\theta} - \frac{1}{\sqrt{6}}\right)\right]. \tag{5.1.8}$$

Thus, the system as follow:

$$
\begin{cases}
\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{\sqrt{6}\theta}{2\sqrt{6}\theta - 3\nu\phi}\left[2\dot{\phi}^2 - n^2\phi^2 + 2\sqrt{6}\mu\phi\sigma - \frac{\nu\theta\phi}{\sqrt{6}} + 3\sqrt{6}\mu\phi\frac{\dot{\sigma}}{\theta}\right. \\
\quad \left. + \frac{\sqrt{6}\nu\phi\sigma^2}{\theta} + 3\mu\dot{\phi}\left(1 + \frac{\sqrt{6}\sigma}{\theta}\right) + 3\nu\dot{\phi}\left(\frac{\sigma}{\theta} - \frac{1}{\sqrt{6}}\right)\right], \\
\dot{\sigma} = -\sigma\theta + \frac{M}{3\sqrt{3}}\left[\theta^2 - 3\sigma^2 - 3\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}n^2\phi^2 - \sqrt{6}\mu\phi\sigma + \frac{1}{\sqrt{6}}\theta\nu\phi\right)\right], \\
\dot{\phi} = \psi, \\
\dot{\psi} = -\theta(\psi + \mu) - n^2\phi - \nu\sigma, \\
\dot{a} = \frac{1}{3}a\theta,
\end{cases}
\tag{5.1.9}
$$

with first integral

$$
\theta^2 = 3\sigma^2 + 3\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}n^2\phi^2 - \sqrt{6}\mu\phi\sigma + \frac{1}{\sqrt{6}}\nu\theta\phi\right] - \frac{3(m^2+m+1)}{a^2}, \tag{5.1.10}
$$

and M is defined as follows:

$$
M := \frac{1-m}{\sqrt{m^2+m+1}}. \tag{5.1.11}
$$

5.2 Case(2): Bianchi V with $k = 0$

In this model, $M = 0$ and we investigate three sub-cases.

5.3 Case (2a): When $\nu \neq 0$, and $\mu > 0$

5.3.1 Original Variables

The system in (5.1.9) yields

$$\begin{cases} \dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{\sqrt{6}\theta}{2\sqrt{6}\theta - 3\nu\phi}\left[2\dot{\phi}^2 - n^2\phi^2 - \sqrt{6}\mu\phi\sigma - \frac{\nu\theta\phi}{\sqrt{6}}\right. \\ \qquad \left. + \frac{\sqrt{6}\nu\phi\sigma^2}{\theta} + 3\mu\dot{\phi}\left(1 + \frac{\sqrt{6}\sigma}{\theta}\right) + 3\nu\dot{\phi}\left(\frac{\sigma}{\theta} - \frac{1}{\sqrt{6}}\right)\right], \\ \dot{\sigma} = -\sigma\theta, \\ \dot{\phi} = \psi, \\ \dot{\psi} = -\theta(\psi + \mu) - n^2\phi - \nu\sigma, \\ \dot{a} = \frac{1}{3}a\theta. \end{cases} \tag{5.3.1}$$

with first integral

$$\theta^2 = 3\sigma^2 + 3\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}n^2\phi^2 - \sqrt{6}\mu\phi\sigma + \frac{1}{\sqrt{6}}\nu\theta\phi\right]. \tag{5.3.2}$$

We investigate both early and late time behaviour of the model numerically using two approaches: an analysis of the model with the original variables and an analysis of the model using the normalized bounded variables. Moreover, we investigate the late time behaviour of these sub-cases analytically using normalized bounded variables.

5.3.2 Original Variables: Numerical Method-Past Behaviour

In figures from 5.1 to 5.3, we plot the numerical solutions of $(\phi(t), \sigma(t), a(t))$ respectively, for the model in (5.3.1) with $\mu = 2, n = 1$ and $\nu = 0.81$ into the past times.

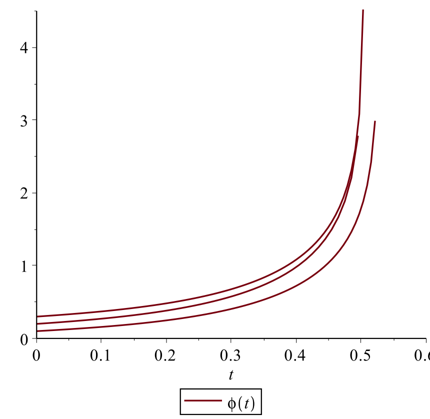

Figure 5.1: Plot for solution of $\phi(t)$ for the model in (5.3.1) with $n = 1, \mu = 2$ and $\nu = 0.81$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 0.3, \sigma(0) = 0.5, D(\phi(0)) = 0.003, a(0) = 0.5, \theta(0) = 0.1], [\phi(0) = 0.1, \sigma(0) = 0.6, D(\phi(0)) = 0.1, a(0) = 0.4, \theta(0) = 0.6], [\phi(0) = 0.2, \sigma(0) = 0.5, D(\phi(0)) = 0.3, a(0) = 0.4, \theta(0) = 0.6]]$.

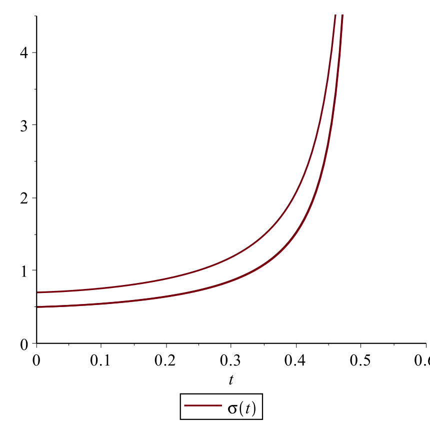

Figure 5.2: Plot for solution of $\sigma(t)$ for the model in (5.3.1) with $n = 1, \mu = 2$ and $\nu = 0.81$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 0.3, \sigma(0) = 0.5, D(\phi(0)) = 0.003, a(0) = 0.5, \theta(0) = 0.1], [\phi(0) = 0.1, \sigma(0) = 0.6, D(\phi(0)) = 0.1, a(0) = 0.4, \theta(0) = 0.6]$.

Discussion

As can be seen from figures 5.1, 5.2, as $t \to -\infty$, $\sigma(t)$ and $\phi(t)$ blow up and from figure 5.3, $a(t)$ decreases. Also, we did numerical solutions for several negative values of ν, for instance; $\nu = -1, -1.8, -2, -2.2, -3$ and we get qualitative similar behaviour as in

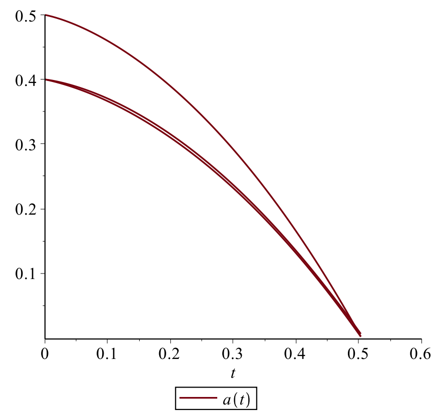

Figure 5.3: Plot for solution of $a(t)$ for the model in (5.3.1) with $n = 1, \mu = 2$ and $\nu = 0.81$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 0.3, \sigma(0) = 0.5, D(\phi(0)) = 0.003, a(0) = 0.5, \theta(0) = 0.1], [\phi(0) = 0.1, \sigma(0) = 0.6, D(\phi(0)) = 0.1, a(0) = 0.4, \theta(0) = 0.6]$.

figures from 5.1 to 5.3. Note that, we did several cases by fixing $\mu = 0.8, 1, 1.8, 2, 2.2,$ and 2.8. So, for each fixed value of μ we took different positive and negative values of ν such as $\nu = 0.5, 0.8, 1, 1.8, 2, 2.2, 2.8, -0.5, -.8, -1, -1.8, -2, -2.2, -2.8$. As a result, we get similar qualitative behaviour as in figures 5.1- 5.3, which is consistent with the result found studying the isotropic model.

5.3.3 Original Variables: Numerical Method-Future Behaviour

We plot the system (5.3.1) for different fixed values of the parameter $\mu > 0$ and vary $\nu \neq 0$ and with zero curvature at the late times. We do this because we want to see the behaviour of the model and compare it with the isotropic model.

Fix $\mu = 0.5$ and $\nu > 0$

In figure 5.4, we plot the system (5.3.1) for fixed value of $\nu = 0.5$ and $\mu = 0.5, n = 1$ into the future.

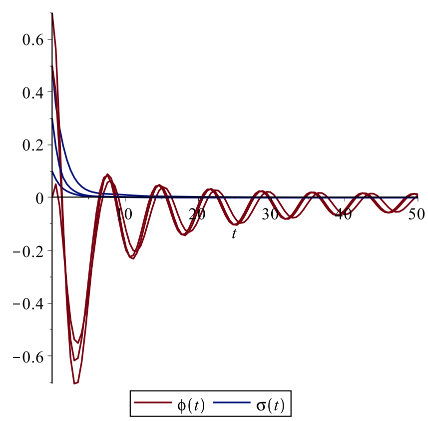

Figure 5.4: Plot for the solution of the system in (5.3.1) with $n = 1, \mu = 0.5$ and $\nu = 0.5$ as $t \to \infty$ with several initial conditions:[[$\phi(0) = 0.0005, \sigma(0) = 0.5, D(\phi(0)) = 0.3, a(0) = 0.3, \theta(0) = 0.1$], [$\phi(0) = 0.05, \sigma(0) = 0.1, D(\phi(0)) = 0.003, a(0) = 0.1, \theta(0) = 0.2$], [$\phi(0) = 0.7, \sigma(0) = 0.3, D(\phi(0)) = 0, a(0) = 0.3, \theta(0) = 0.4$]].

Discussion

As can been seen in figure 5.4, $\phi(t)$ oscillates; $\sigma(t)$ decreases and decays to zero which is consistent with the isotropic model that we studied earlier. Note that we did several cases by fixing μ to be $\mu = 0.5$ and varying $\nu > 0$ such as $\nu = 0.5, 0.8, 1, 1.8, 2, 2.2, 2.8$. As a result, we ended up with similar qualitative behaviour as in figure 5.4. The next step that we are going to take the same fixed value of $\mu = 0.5$ with different negative values of ν.

Fix $\mu = 0.5$ and $\nu < 0$

In figures from 5.5 to 5.7, we plot the system (5.3.1) for fixed value of $\mu = 0.5$ and different negative values of ν into the future.

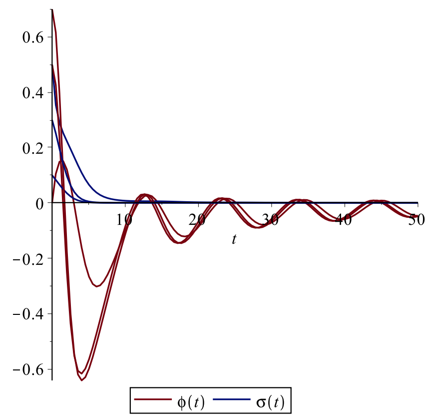

Figure 5.5: Plot for the solution of the system in (5.3.1) with $n = 1, \mu = 0.5$ and $\nu = -0.5$ as $t \to \infty$ with several initial conditions:$[[\phi(0) = 0.0005, \sigma(0) = 0.5, D(\phi(0)) = 0.3, a(0) = 0.3, \theta(0) = 0.1], [\phi(0) = 0.05, \sigma(0) = 0.1, D(\phi(0)) = 0.003, a(0) = 0.1, \theta(0) = 0.2], [\phi(0) = 0.7, \sigma(0) = 0.3, D(\phi(0)) = 0, a(0) = 0.3, \theta(0) = 0.3]]$.

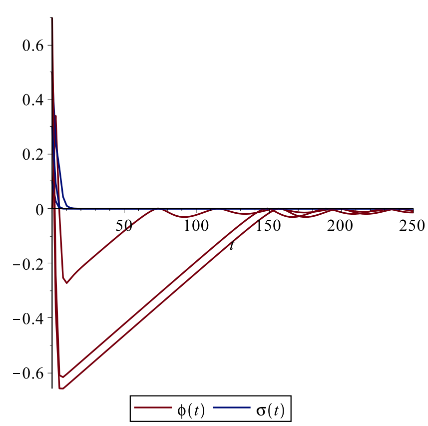

Figure 5.6: Plot for the solution of the system in (5.3.1) with $n = 1, \mu = 0.5$ and $\nu = -1$ as $t \to \infty$ with several initial conditions:$[[\phi(0) = 0.0005, \sigma(0) = 0.5, D(\phi(0)) = 0.3, a(0) = 0.3, \theta(0) = 0.1], [\phi(0) = 0.05, \sigma(0) = 0.1, D(\phi(0)) = 0.003, a(0) = 0.1, \theta(0) = 0.2], [\phi(0) = 0.7, \sigma(0) = 0.3, D(\phi(0)) = 0, a(0) = 0.3, \theta(0) = 0.3]]$.

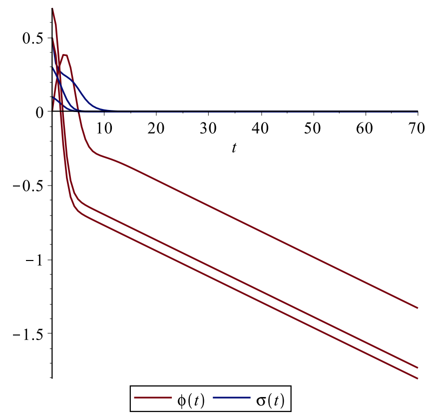

Figure 5.7: Plot for the solution of the system in (5.3.1) with $n = 1, \mu = 0.5$ and $\nu = -1.1$ as $t \to \infty$ with several initial conditions:[[$\phi(0) = 0.0005, \sigma(0) = 0.5, D(\phi(0)) = 0.3, a(0) = 0.3, \theta(0) = 0.1$], [$\phi(0) = 0.05, \sigma(0) = 0.1, D(\phi(0)) = 0.003, a(0) = 0.1, \theta(0) = 0.2$], [$\phi(0) = 0.7, \sigma(0) = 0.3, D(\phi(0)) = 0, a(0) = 0.3, \theta(0) = 0.3$]].

Discussion

As can been seen in figure 5.5, $\phi(t)$ oscillates; $\sigma(t)$ decreases then decays to zero (this behaviour is true for values of ν in the interval $-1 \lesssim \nu < 0$). When $\nu \simeq -1$, the oscillations for ϕ disappear. When $\nu \lesssim -1.1$, as can been seen in figure 5.7, we have that ϕ decreases; σ decreases and decays to zero. This is qualitatively the same behaviour as in the isotropic model when $\sigma = 0$ and $\mu > 0$ but in anisotropic model with $\mu > 0$ and $\nu \neq 0$, we get different bifurcation values. Note that we repeated the numerical with different fixed positive values of μ and vary ν and we get similar qualitative behaviour with different bifurcation values (see table (5.1) for more details).

Bifurcation-Value	ν, μ -Value	The Behaviour
$\nu \cong -1.1$	$\mu = 0.5n$ and $\nu < 0$	For $\nu > -1.1$ ($\phi(t)$ deceases; $\sigma(t)$ decreases to 0). For $\nu < -1.1$ ($\phi(t)$ deceases; $\sigma(t)$ decreases to 0).
$\nu \cong -0.2$	$\mu = 0.81n$ and $\nu < 0$	For $\nu > -0.2$ ($\phi(t)$ deceases; $\sigma(t)$ decreases to 0). For $\nu < -0.2$ ($\phi(t)$ deceases; $\sigma(t)$ decreases to 0).
$\nu \cong 0.4$	$\mu = n$ and $\nu > 0$	For $\nu < 0.4$ ($\phi(t)$ deceases; $\sigma(t)$ decreases to 0). For $\nu > 0.4$ ($\phi(t)$ oscillates; $\sigma(t)$ decreases to 0).
$\nu \cong 1.8$	$\mu = 1.8n$ and $\nu > 0$	For $\nu < 1.8$ ($\phi(t)$ deceases; $\sigma(t)$ decreases to 0). For $\nu > 1.8$ ($\phi(t)$ decreases then increases then oscillates; $\sigma(t)$ decreases to 0).
$\nu \cong 2.1$	$\mu = 2n$ and $\nu > 0$	For $\nu < 2.1$ ($\phi(t)$ deceases; $\sigma(t)$ decreases to 0). For $\nu > 2.1$ ($\phi(t)$ decreases then increases then oscillates; $\sigma(t)$ decreases to 0).

Table 5.1: Summary of different bifurcation values for ν and the behaviour of the model (5.3.1) with fixed values of $n = 1, \mu > 0$ and $\nu \neq 0$

5.3.4 Normalized Bounded Variable

Introducing Normalized Variables

Introduce the normalized bounded variable D as before, along with the following expressions for Φ, Ψ, Σ :

$$D \equiv \frac{\theta}{\sqrt{1+\theta^2}}, \quad \Sigma \equiv \frac{\sigma}{\sqrt{1+\theta^2}}, \tag{5.3.3}$$

$$\Psi \equiv \sqrt{\frac{3}{2}}\left(\frac{n\phi}{\sqrt{1+\theta^2}}\right), \quad \Phi \equiv \sqrt{\frac{3}{2}}\left(\frac{\dot{\phi}}{\sqrt{1+\theta^2}}\right). \tag{5.3.4}$$

Then, the Friedmann equation for this model becomes

$$D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 + \frac{6\mu}{n}\Psi\Sigma - \frac{\nu\Psi D}{n} = 0. \tag{5.3.5}$$

The domain of interest is $0 \leq D \leq 1$, since

$$\theta \to \infty \quad \Longrightarrow D \to 1,$$

and

$$\theta \to 0 \quad \Longrightarrow D \to 0.$$

We have that D is a bounded variable and the other variables Φ, Ψ, Σ are bounded if $6\mu^2 < n^2$ is satisfied. This follows from the Friedmann equation that can be rewritten as follows

$$\left(1+\frac{v^2}{2n^2}\right)D^2 - \Phi^2 - \frac{1}{2}\left(\Psi - \frac{6\mu\Sigma}{n}\right)^2 - \frac{1}{2}\left(\Psi + \frac{vD}{n}\right)^2 - 3\Sigma^2\left(1 - 6\frac{\mu^2}{n^2}\right) = 0. \tag{5.3.6}$$

Note that the system is not well defined for $2D - \frac{\nu\Psi}{n} = 0$, which include the origin $D = 0 = \Psi$ see below (5.3.7, 5.3.8). Defining a new time variable τ with $\frac{d\tau}{dt} = \sqrt{1+\theta^2}$. Therefore, the evolution equations become

$$
\begin{cases}
D' = (1 - D^2)\mathcal{X}, \\
\Psi' = n\sqrt{1-D^2}\Phi - D\Psi\mathcal{X}, \\
\Phi' = -D\Phi(1+\mathcal{X}) - \sqrt{1-D^2}\left[n\Psi + \sqrt{\frac{3}{2}}\left(\mu D + \nu\Sigma\right)\right], \\
\Sigma' = -\Sigma D(1+\mathcal{X}),
\end{cases}
\tag{5.3.7}
$$

where the prime here indicates the differentiation of each variable with respect to the new variable time τ and $\mathcal{X}$ is given by the following expression:

$$
\begin{aligned}
\mathcal{X} := -\frac{1}{3}D^2 - 2\Sigma^2 - \frac{\sqrt{6}D}{\sqrt{6}(2D - \frac{\nu\Psi}{n})}\Bigg[\frac{4}{3}\Phi^2 - \frac{2}{3}\Psi^2 - \frac{2\mu}{n}\Psi\Sigma - \frac{\nu}{3n}D\Psi + \frac{2\nu\Sigma^2\Psi}{nD} \\
+ \sqrt{6}\sqrt{1-D^2}\left[\mu\Phi\left(1 + \frac{\sqrt{6}\Sigma}{D}\right) + \nu\Phi\left(\frac{\Sigma}{D} - \frac{1}{\sqrt{6}}\right)\right]\Bigg] = \frac{\dot{\theta}}{(\theta^2+1)}.
\end{aligned}
\tag{5.3.8}
$$

Also, the Friedmann equation (5.3.5) is conserved by the 4D system in (5.3.7).

Define

$$
\Lambda := D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 + \frac{6\mu}{n}\Psi\Sigma - \frac{\nu\Psi D}{n}. \tag{5.3.9}
$$

then,

$$
\begin{aligned}
\Lambda' &= 2DD' - 2\Phi\Phi' - 6\Sigma\Sigma' - 2\Psi\Psi' + \frac{6\mu}{n}(\Psi'\Sigma + \Psi\Sigma') - \frac{\nu}{n}(\Psi' D + \Psi D') \\
&= 2D\mathcal{X} - 2D^3\mathcal{X} + 2D\Phi^2(1+\mathcal{X}) + 2n\sqrt{1-D^2}\Phi\Psi + \sqrt{6}\mu\sqrt{1-D^2}\Phi(D+\Sigma) \\
&+ 6\Sigma^2 D(1+\mathcal{X}) - n\sqrt{1-D^2}\Phi\Psi + 2\Psi^2 D\mathcal{X} + 6\mu D\Phi\sqrt{1-D^2} - \frac{12\mu}{n}D\Psi\Sigma\mathcal{X} \\
&- \frac{6\mu}{n}\Sigma D\Psi - \nu\sqrt{1-D^2}D\Phi + \frac{2\nu}{n}D^2\Psi\mathcal{X} - \frac{\nu}{n}\Psi\mathcal{X} \\
&= -2D\mathcal{X}\left(D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 + \frac{6\mu}{n}\Psi\Sigma - \frac{\nu\Psi D}{n}\right) + \left(2D - \frac{\nu}{n}\Psi\right)\mathcal{X} \\
&+ 6\Sigma^2 D + 2D\Phi^2 - \frac{6\mu}{n}\Sigma D\Psi + \sqrt{1-D^2}\Phi(\sqrt{6}(\mu D + \nu\Sigma) + 6\mu\Sigma - \nu D). \\
&= -2D\mathcal{X}\Lambda - \frac{2}{3}D^3 + \frac{2\nu}{3n}D^2\Psi + 2D\Sigma^2 + \frac{2}{3}D\Psi^2 + \frac{2}{3}D\Phi^2 - \frac{4\mu}{n}D\Sigma\Psi. \\
&= -2D\mathcal{X}\Lambda - \frac{2}{3}D\left(D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 + \frac{6\mu}{n}\Psi\Sigma - \frac{\nu\Psi D}{n}\right) \\
&= -\frac{2}{3}D\Lambda(3\mathcal{X}+1).
\end{aligned}
$$

Bounded Variables:Qualitative Analysis

The equations in system (5.3.7) are relatively simple. Also, note that the D' does not decouple and the system is not analytic at $D = 1$ since we have the square root term in the second equation of the 4D system (5.3.7).

Equilibrium Points

There are several equilibrium points of the system but we are only interested in the study of the anisotropic generalization of $P_{1,2,3,4}$ and to compare it then with the isotropic model that we studied earlier. In table 5.2, we present the four equilibrium points of the system (5.3.7), $\mu > 0$ and $\nu \neq 0$, and the value of $\mathcal{X}$ for each one.

Pt	(D,Φ,Ψ,Σ)	$\mathcal{X}$	Stability
P_1	$(1,1,0,0)$	-1	Source
P_2	$(1,-1,0,0)$	-1	Source
P_3	$\left(1,0,\frac{-\nu+\sqrt{4\,n^2+\nu^2}}{2n},0\right)$	0	Saddle
P_4	$\left(1,0,-\left(\frac{\nu+\sqrt{4\,n^2+\nu^2}}{2n}\right),0\right)$	0	Sink if $\sqrt{6}\sqrt{4\,n^2+\nu^2}+\sqrt{6}\nu-6\,\mu<0$.

Table 5.2: Equilibrium points of the system (5.3.7) and the value of $\mathcal{X}$ for each one.

Let us discuss the stability of the four equilibrium points ($P_{1,2,3,4}$) of the system (5.3.7).

Stability of Equilibrium Point $P_{1,2}$

Evaluating the linearization matrix of the system (5.3.7) at $P_{1,2}$ leads to undefined terms which means that the linearization fails to determine the stability of $P_{1,2}$. Thus, we need to use different approach to study these equilibrium points $P_{1,2}$. $D=1,\Sigma=0$ is an invariant set then there is 2-dimensional dynamic system given by

$$\begin{cases} \Phi' = \dfrac{2}{3}\Psi\left(\dfrac{2\Phi^2 n - n\Psi^2 - \nu\Psi + n}{2n-\nu\Psi}\right), \\ \Psi' = \dfrac{1}{3}\Phi\left(\dfrac{4\Phi^2 n - 2n\Psi^2 + \nu\Psi - 4n}{2n-\nu\Psi}\right). \end{cases} \tag{5.3.10}$$

Evaluating the linearization matrix of (5.3.10) at $\Phi=\pm1,\Psi=0$ lead two positive eigenvalues. Now we need to find the sign of other two eigenvalues.

By plugging equation (5.3.5) into the expression (5.3.8), and since $\Sigma\to0$ at equilibrium points (so that we neglect any terms with Σ), we obtain a simpler expression for $\mathcal{X}$ as follows:

$$\mathcal{X} := -\frac{1}{3}\Phi^2 - \frac{\nu\Psi D}{3m} - \frac{1}{3}\Psi^2 - \frac{\sqrt{6}D}{\sqrt{6}(2D - \frac{\nu\Psi}{m})}\left[\frac{4}{3}\Phi^2 - \frac{2}{3}\Psi^2 - \frac{\nu}{3m}D\Psi + \Phi\sqrt{6}\sqrt{1-D^2}\left[\mu - \frac{\nu}{\sqrt{6}}\right]\right]. \tag{5.3.11}$$

Define $T = \sqrt{1-D^2} << 1$, then, $T^2 = 1 - D^2$. And

$$\begin{aligned} 2TT' &= -2DD', \\ TT' &= -DT^2\mathcal{X}, \\ T' &= -\sqrt{1-T^2}T\mathcal{X}. \end{aligned} \tag{5.3.12}$$

Since T is small, then we can expand $\sqrt{1-T^2}$ using a Taylor series as

$$\sqrt{1-T^2} = 1 - \frac{1}{2}T^2. \tag{5.3.13}$$

Now, by plugging (5.3.13) into (5.3.12), T' becomes

$$T' = T\mathcal{X}\left(1 - \frac{T^2}{2}\right). \tag{5.3.14}$$

Now, we find the remaining two eigenvalues by looking to the system of D and Σ given by

$$\begin{cases} T' = -T\mathcal{X}, \\ \Sigma' = -\Sigma(1+\mathcal{X}). \end{cases} \tag{5.3.15}$$

Next, substituting the simpler expression for $\mathcal{X}$ from equation (5.3.11) into (5.3.15) leads to

$$\begin{cases} T' = \alpha_1 T + O(T^3), \\ \Sigma' = \alpha_2\Sigma + O(\Sigma^2). \end{cases} \tag{5.3.16}$$

where $\alpha_{1,2}$ are the remaining two eigenvalues given by

$$\begin{cases} \lambda_1 = \alpha_1 = \dfrac{\Phi^2(4n - \nu\Psi)}{3(2n - \nu\Psi)}, \\ \lambda_2 = \alpha_2 = \dfrac{\Phi^2(2n + \nu\Psi)}{3(2n - \nu\Psi)}. \end{cases} \tag{5.3.17}$$

If $\Psi = 0$ then the eigenvalues (5.3.17) become

$$\begin{cases} \lambda_1 = \dfrac{2\Phi^2}{3}, \\ \lambda_2 = \dfrac{\Phi^2}{3}. \end{cases} \tag{5.3.18}$$

When $\Phi^2 = 1, \lambda_{1,2}$ are positive which implies that $P_{1,2}$ are always sources, which is consistent with the analysis and the result found studying the isotropic model.

Stability of Equilibrium Point $P_{3,4}$

Evaluating the linearization matrix of the system (5.3.7) at $P_{3,4}$ leads to undefined terms which means that the linearization fails to determine the stability of $P_{3,4}$. Thus, we need to use different approach to study these equilibrium points $P_{3,4}$. When $D = 1$ is an invariant set then there is a 3 -dimensional invariant set given by

$$\begin{cases} \Phi' = \dfrac{2}{3}\Psi\left(\dfrac{2\Phi^2 n + 6n\Sigma^2 - 3\Sigma\mu\Psi - n\Psi^2 - \nu\Psi + n}{2n - \nu\Psi}\right), \\ \Psi' = \dfrac{1}{3}\Phi\left(\dfrac{4\Phi^2 n + 12n\Sigma^2 - 6\mu\Sigma\Psi - 2n\Psi^2 + \nu\Psi - 4n}{2n - \nu\Psi}\right), \\ \Sigma' = \dfrac{1}{3}\Sigma\left(\dfrac{4\Phi^2 n + 12n\Sigma^2 - 6\mu\Sigma\Psi - 2n\Psi^2 + \nu\Psi - 4n}{2n - \nu\Psi}\right). \end{cases} \tag{5.3.19}$$

Evaluating the linearization matrix of (5.3.19) at $\Phi = 0, \Sigma = 0$ lead to three negative eigenvalues. Now we just need to determine the sign of the fourth eigenvalue. Since $\mathcal{X}$ at $P_{3,4}$ is 0 and $D > 0$ then $\Sigma' < 0$ which implies that $\Sigma \to 0$ as $t \to \infty$, hence we can neglect all terms that has Σ.

Define $T = \sqrt{1-D^2}$, then, $T^2 = 1 - D^2$. And

$$\begin{aligned} 2TT' &= -2DD', \\ TT' &= -DT^2\mathcal{X}, \\ T' &= -\sqrt{1-T^2}T\mathcal{X}. \end{aligned} \tag{5.3.20}$$

Since T is small, then we can expand $\sqrt{1-T^2}$ using a Taylor series as

$$\sqrt{1-T^2} = 1 - \frac{1}{2}T^2. \tag{5.3.21}$$

Now, by plugging (5.3.21) into (5.3.20), T' becomes

$$T' = T\mathcal{X}\left(1 - \frac{T^2}{2}\right). \tag{5.3.22}$$

Since $\Phi = 0$ is an invariant set then $\Phi' = 0$, then it follows that

$$\Phi = -T\left(n\Psi + \sqrt{\frac{3}{2}}\mu\right). \tag{5.3.23}$$

Next, plugging Φ from equation (5.3.23) and the simpler expression for $\mathcal{X}$ from equation (5.3.11) into (5.3.22) leads to

$$T' = \alpha_3 T + \alpha_4(T^3), \tag{5.3.24}$$

where

$$\begin{cases} \alpha_3 = -\dfrac{\Psi\nu(n\Psi^2 + \nu\Psi - n)}{3n(2n - \nu\Psi)}, \\ \alpha_4 = \dfrac{\left(-\sqrt{6}n\nu\,\Psi^2 + 6\,\sqrt{6}n^2\Psi + 3\,\sqrt{6}n\nu - 3\,\mu\,\nu\,\Psi\right)\left(\sqrt{6}n\Psi + 3\,\mu\right)}{18(-\nu\,\Psi + 2\,n)}. \end{cases} \tag{5.3.25}$$

When $\Psi = \pm\left(\frac{-\nu+\sqrt{4\,n^2+\nu^2}}{2n}\right)$ then $\alpha_3 = 0$. Since $\alpha_3 = 0$, there is no linear part. Hence, if $\alpha_4 < 0$ then $T = 0$ is attractor at P_4.

1. For P_3 : When $\Psi = \frac{-\nu+\sqrt{4\,n^2+\nu^2}}{2n}$ then $\alpha_4 > 0$ always which implies that P_3 is a saddle.

2. For P_4: When $\Psi = -\left(\frac{\nu+\sqrt{4\,n^2+\nu^2}}{2n}\right)$ then α_4 is given by

$$\begin{aligned}\alpha_4 &= \frac{1}{36(\nu\,\sqrt{4\,n^2+\nu^2}+4\,n^2+\nu^2)}\Bigg(\Bigg(\sqrt{4\,n^2+\nu^2}(6\,\sqrt{6}n^2+\sqrt{6}\nu^2-3\,\mu\,\nu\,) \\ &+\nu(2\sqrt{6}n^2+\sqrt{6}\nu^2-3\mu\nu)\Bigg)\left(\sqrt{6}\sqrt{4\,n^2+\nu^2}+\sqrt{6}\nu-6\,\mu\right)\Bigg). \quad (5.3.26)\end{aligned}$$

which is negative if $\sqrt{6}\sqrt{4\,n^2+\nu^2}+\sqrt{6}\nu-6\,\mu < 0$. To illustrate this, for example, choose $n = 1, \mu = 0.5$ and $\nu > 0$ then $\alpha_4 > 0$ which mean that P_4 is a saddle in this case (see figure 5.8). However, when $n = 1, \mu = 0.5$ and for any value of $\nu \lesssim -1.1$, then $\alpha_4 < 0$ which means P_4 is sink in this case (see figure 5.12).

Discussion

Among the four equilibrium points that we studied ($P_{1,2,3,4}$), we found that P_4 is a sink when $\alpha_4 < 0(\sqrt{6}\sqrt{4\,n^2+\nu^2}+\sqrt{6}\nu-6\,\mu < 0)$.

Bounded Variables: Numerical Method-Past Behaviour

In figure 5.8, we plot the solution of system (5.3.7) for $\mu = 1, n = 1$ and $\nu = 1$ into the past times.

Discussion As can be seen from figure 5.8, the solution into the past goes to the point P_1, which is consistent with the isotropic model that we studied earlier in chapter (4). Note that, we did this for different positive values of μ and vary $\nu \neq 0$ and we get similar qualitative behaviour.

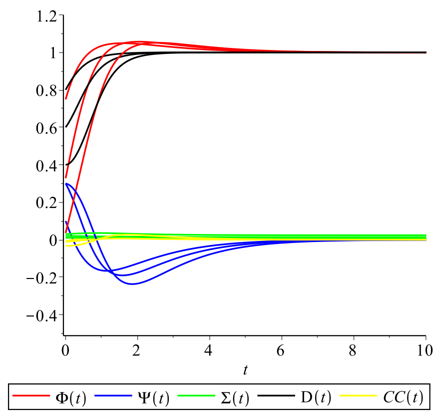

Figure 5.8: Plot for system (5.3.7) with $n = 1, \mu = 1$ and $\nu = 1$ as $t \to -\infty$ with several initial conditions:[[$D(0) = 0.6, \Psi(0) = 0.3, \Sigma(0) = 0.1, \Phi(0) = 0.3281767816$], [$D(0) = 0.4, \Psi(0) = 0.3, \Sigma(0) = 0.03, \Phi(0) = 0.03605551275$], [$D(0) = 0.8, \Psi(0) = 0.1, \Sigma(0) = 0.02, \Phi(0) = 0.7488658091$]]. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t) - \frac{\nu\Psi(t)D(t)}{n}$.

Bounded Variables: Numerical Method Future Behaviour

For completeness, we plot the system (5.3.7) for different positive values of the parameters μ and vary $\nu \neq 0$ at the late times. We do this because we want to see the behaviour of the model and compare it with the standard model, which does not have shear. We fixed μ and vary ν as follows.

Fix $\mu = 0.5$ and $\nu > 0$: In figures from 5.9, we plot the system (5.3.7) for $\mu = 0.5$ and $\nu = 0.5$ and with zero curvature into the future.

Discussion As can been seen in figure 5.9, $\Phi(t)$ and $\Psi(t)$ oscillate; $\Sigma(t)$ decreases and decays to zero, as in the isotropic model that we studied earlier (4). Note that we did several cases by fixing μ to be $\mu = 0.5$ and varying ν such as $\nu =$

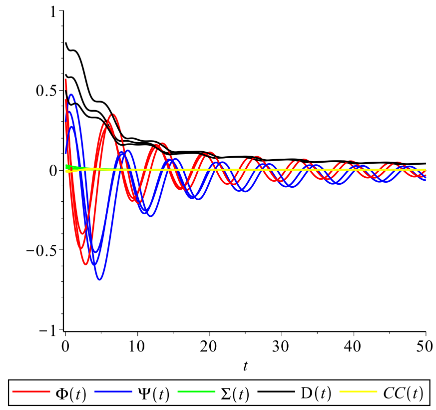

Figure 5.9: Plot for system (5.3.7) with $n = 1, \mu = 0.5$ and $\nu = 0.5$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.6, \Psi(0) = 0.3, \Sigma(0) = 0.1, \Phi(0) = 0.3141655614], [D(0) = 0.5, \Psi(0) = 0.1, \Sigma(0) = 0.03, \Phi(0) = 0.4430575583], [D(0) = 0.8, \Psi(0) = 0.3, \Sigma(0) = 0.02, \Phi(0) = 0.5716642371]]$. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t) - \frac{\nu\Psi(t)D(t)}{n}$.

$0.5, 0.8, 1, 1.8, 2, 2.2, 2.8$. As a result, we get similar qualitatively behaviour as in figure 5.9. The next step that we are going to take the same fixed value of $\mu = 0.5$ with different negative values of ν.

Fix $\mu = 0.5$ and $\nu < 0$: In figures from 5.10 to 5.12, we plot the solution of the system in (5.3.7) with fixed $\mu = 0.5$ and different negative values of ν in to the future

Discussion

As can been seen in figure 5.10, $\Phi(t)$ and $\Psi(t)$ oscillate; $\Sigma(t)$ decreases and decays to zero (this behaviour is true for small values of ν in the interval $-1.1 \lesssim \nu < 0$). When $\nu \simeq -1.1$, the oscillations for $\Phi(t)$ and $\Psi(t)$ disappear and $D(t) \to 1$, $\Psi(t) \to -0.5$ and $\Sigma(t), \Phi(t) \to 0$ which means that P_4 is a sink in this case (see figure 5.12).

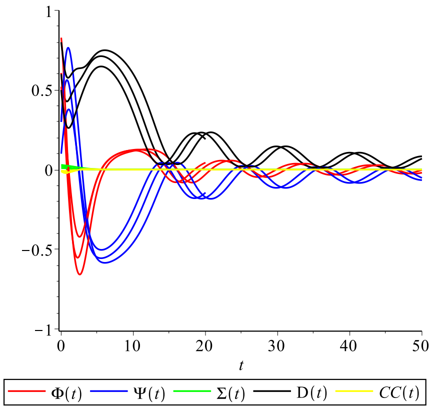

Figure 5.10: Plot for system(5.3.7) with $n = 1, \mu = 0.5$ and $\nu = -0.5$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.5, \Psi(0) = 0.1, \Sigma(0) = 0.03, \Phi(0) = 0.5208646657], [D(0) = 0.6, \Psi(0) = 0.3, \Psi(0) = 0.01, \Phi(0) = 0.6072067193], [D(0) = 0.8, \Psi(0) = 0.3, \Sigma(0) = 0.02, \Phi(0) = 0.8287339742]]$. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t) - \frac{\nu\Psi(t)D(t)}{n}$.

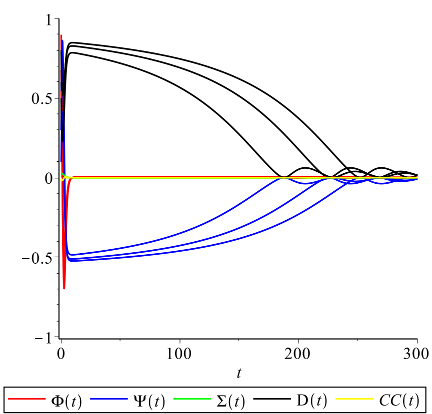

Figure 5.11: Plot for system (5.3.7) with $n = 1, \mu = 0.5$ and $\nu = -1$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.5, \Psi(0) = 0.1, \Sigma(0) = 0.03, \Phi(0) = 0.5443344560], [D(0) = 0.6, \Psi(0) = 0.3, , \Sigma(0) = 0.01, \Phi(0) = 0.6772739475], [D(0) = 0.8, \Psi(0) = 0.3, \Sigma(0) = 0.02, \Phi(0) = 0.8982204629]]$. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t) - \frac{\nu\Psi(t)D(t)}{n}$.

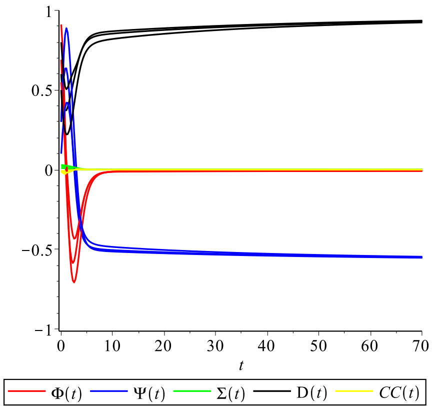

Figure 5.12: Plot for system (5.3.7) with $n = 1, \mu = 0.5$ and $\nu = -1.1$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.5, \Psi(0) = 0.1, \Sigma(0) = 0.03, \Phi(0) = 0.5489080069], [D(0) = 0.6, \Psi(0) = 0.3, , \Sigma(0) = 0.01, \Phi(0) = 0.6904346457], [D(0) = 0.8, \Psi(0) = 0.3, \Sigma(0) = 0.02, \Phi(0) = 0.9114823092]]$. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t) - \frac{\nu\Psi(t)D(t)}{n}$.

5.3.5 Inflation:

We will check if this case has an inflationary attractor using the deceleration parameter in (4.2.56).

$$q = -\frac{1}{D^2}\left[-6\Sigma^2 - \frac{3\sqrt{6}D}{\sqrt{6}(2D - \frac{\nu\Psi}{n})}\left[\frac{4}{3}\Phi^2 - \frac{2}{3}\Psi^2 - \frac{2\mu}{n}\Psi\Sigma - \frac{\nu}{3n}D\Psi + \frac{2\nu\Sigma^2\Psi}{nD} + \sqrt{6}\sqrt{1-D^2}\left[\mu\Phi\left(1 + \frac{\sqrt{6}\Sigma}{D}\right) + \nu\Phi\left(\frac{\Sigma}{D} - \frac{1}{\sqrt{6}}\right)\right]\right]\right]. \tag{5.3.27}$$

The sign of the deceleration parameter indicates the nature of the expansionary evolution. If $q > 0$, the cosmological expansion is decelerating, while negative values of q indicate an accelerating dynamics. The value for q for each of the equilibrium points of the model are:

1. $P_{1,2}$:

$$q\big|_{P_{1,2}} = 2 > 0, \tag{5.3.28}$$

are always positive which implies that $P_{1,2}$ are not inflationary.

2. $P_{3,4}$:

$$q\Big|_{P_{3,4}} = -1 < 0, \tag{5.3.29}$$

which implies that $P_{3,4}$ has an inflationary solutions.

5.3.6 Discussion

In this case when $\Sigma \neq 0,\ \nu \neq 0$, and $\mu > 0$, we looked at only $P_{1,2,3,4}$ and we found there is some different bifurcation values for different values of ν with fixed value of μ (for more details see the table (5.1)). $P_{1,2}$ is always a saddle and never inflationary. P_3 is inflationary saddle and P_4 is inflationary attractor if if $\sqrt{6}\sqrt{4\,n^2+\nu^2}+\sqrt{6}\nu-6\,\mu <$ 0.

5.4 Case(2b):When $\nu = 0, \mu > 0$

For completeness let us consider the sub-case when $\sigma \neq 0,\ \nu = 0$, and $\mu > 0$.

5.4.1 Original Variables

The model is given by

$$\begin{cases} \dot{\phi} = \psi, \\ \dot{\psi} = -n^2\phi - (\psi+\mu)\left(\sqrt{3\sigma^2 + \frac{3}{2}\psi^2 + \frac{3}{2}n^2\phi^2 - 3\sqrt{6}\mu\sigma\phi}\right), \\ \dot{\sigma} = -\sigma\left(\sqrt{3\sigma^2 + \frac{3}{2}\psi^2 + \frac{3}{2}n^2\phi^2 - 3\sqrt{6}\mu\sigma\phi}\right), \\ \dot{a} = \frac{1}{3}a\left(\sqrt{3\sigma^2 + \frac{3}{2}\psi^2 + \frac{3}{2}n^2\phi^2 - 3\sqrt{6}\mu\sigma\phi}\right). \end{cases} \tag{5.4.1}$$

5.4.2 Original Variables: Numerical Method-Past Behaviour

In figures 5.13, 5.14, 5.15, we plot the solution of $(\phi(t), \sigma(t), a(t))$ of system in equations (5.4.1) respectively into the past times with different positive values of the parameter $\mu > 0$.

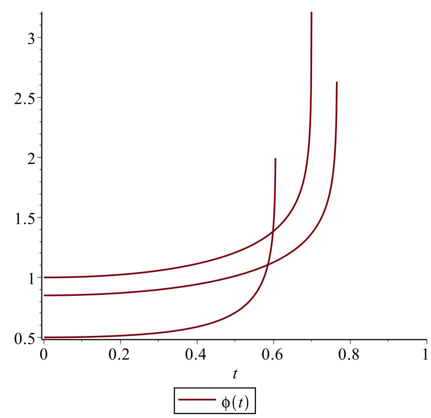

Figure 5.13: Plot for the solution of $\phi(t)$ for system (5.4.1) with $n = 1, \mu = 0.5$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 0.5, a(0) = 1, \theta(0) = 0.1], [\phi(0) = 0.85, D(\phi(0)) = 0.005, \sigma(0) = 0.5, a(0) = 0.7, \theta(0) = 0.2], [\phi(0) = 0.5, D(\phi(0)) = 0.003, \sigma(0) = 0.6, a(0) = 0.75, \theta(0) = 0.3]]$.

Discussion

As can be seen from figures 5.13, 5.14, $\sigma(t)$ and $\phi(t)$ go to infinity but $a(t)$ is decreases. By comparing with the isotropic model when $\sigma = 0$, we have that $\phi(t)$ blows up to infinity which is consistent with the anisotropic model.

5.4.3 Original Variables: Numerical Method-Future Behaviour

In figures from 5.16 to 5.19, we plot the system in (5.4.1) with different positive values of μ to see the behaviour of this model at the late times.

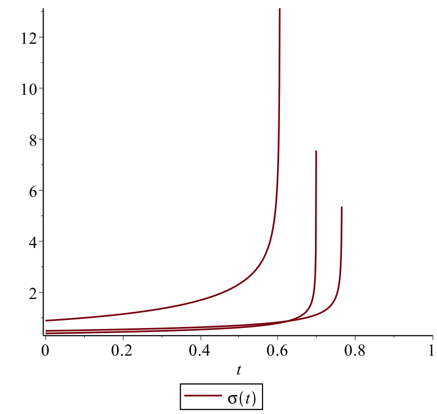

Figure 5.14: Plot of the solution of $\sigma(t)$ of the for system (5.4.1) with $n = 1, \mu = 0.5$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 0.5, a(0) = 1, \theta(0) = 0.1], [\phi(0) = 0.85, D(\phi(0)) = 0.005, \sigma(0) = 0.5, a(0) = 0.7, \theta(0) = 0.2], [\phi(0) = 0.5, D(\phi(0)) = 0.003, \sigma(0) = 0.6, a(0) = 0.75, \theta(0) = 0.3]]$.

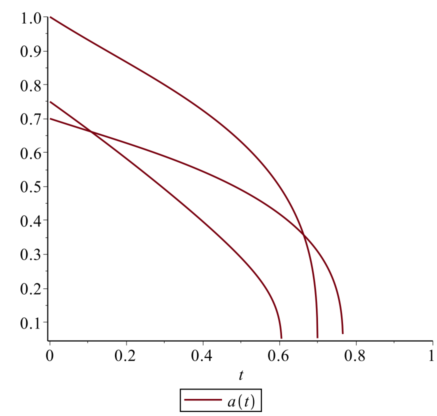

Figure 5.15: Plot of the solution of $a(t)$ of the for system (5.4.1) with $n = 1, \mu = 0.5$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 0.5, a(0) = 1, \theta(0) = 0.1], [\phi(0) = 0.85, D(\phi(0)) = 0.005, \sigma(0) = 0.5, a(0) = 0.7, \theta(0) = 0.2], [\phi(0) = 0.5, D(\phi(0)) = 0.003, \sigma(0) = 0.6, a(0) = 0.75, \theta(0) = 0.3]]$.

Discussion

As can be seen in figures from 5.16 to 5.19, the numerical solutions suggest that there is bifurcation value. As can be seen from figure 5.16, where we plotted the model with

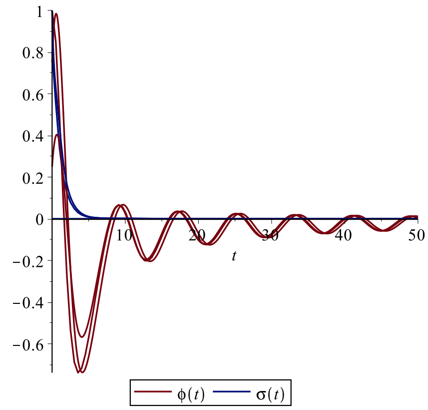

Figure 5.16: Plot for system (5.4.1) with $n = 1, \mu = 0.5$ as $t \to \infty$ with several initial conditions:$[[\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 1, a(0) = 0.8, \theta(0) = 0.3], [\phi(0) = 0.25, D(\phi(0)) = 0.5, \sigma(0) = 0, a(0) = 0.7, \theta(0) = 0.2], [\phi(0) = 0.75, D(\phi(0)) = 1, \sigma(0) = 1, a(0) = 0.2, \theta(0) = 0.1]]$.

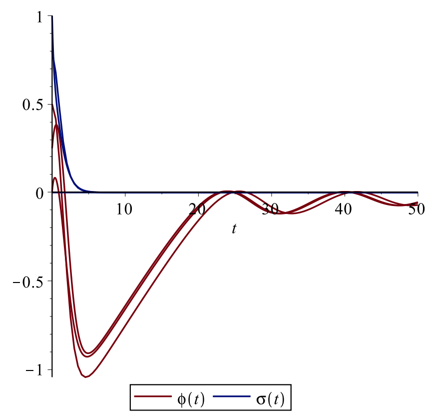

Figure 5.17: Plot for system (5.4.1) with $n = 1, \mu = 0.5$ as $t \to \infty$ with several initial conditions:$[[\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 1, a(0) = 0.8, \theta(0) = 0.2], [\phi(0) = 0.25, D(\phi(0)) = 0.5, \sigma(0) = 0, a(0) = 0.7, \theta(0) = 0.1], [\phi(0) = 0.75, D(\phi(0)) = 1, \sigma(0) = 1, a(0) = 0.2, \theta(0) = 0.3]]$.

small value of μ, and we have oscillations for $\phi(t)$. When $\mu < \sqrt{\frac{2}{3}}n$, in figure 5.17, we still have oscillations but they are not symmetric. However, when μ passes through the bifurcation value the oscillations disappear and the stability (the behaviour) of the model changes (see figure 5.18). For the value $\mu > \sqrt{\frac{2}{3}}n$, as can be seen in figure

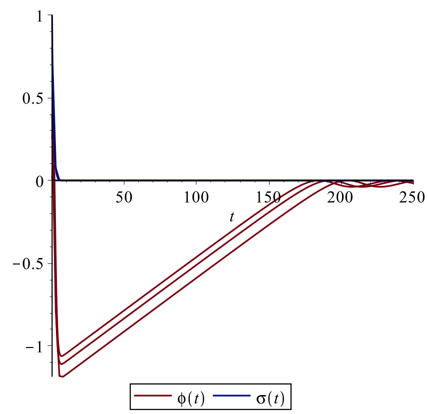

Figure 5.18: Plot for system (5.4.1) with $n = 1, \mu = 0.5$ as $t \to \infty$ with several initial conditions:[[$\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 1, a(0) = 0.8, \theta(0) = 0.2$], [$\phi(0) = 0.25, D(\phi(0)) = 0.5, \sigma(0) = 0, a(0) = 0.2, \theta(0) = 0.3$], [$\phi(0) = 0.75, D(\phi(0)) = 1, \sigma(0) = 1, a(0) = 0.2, \theta(0) = 0.1$]].

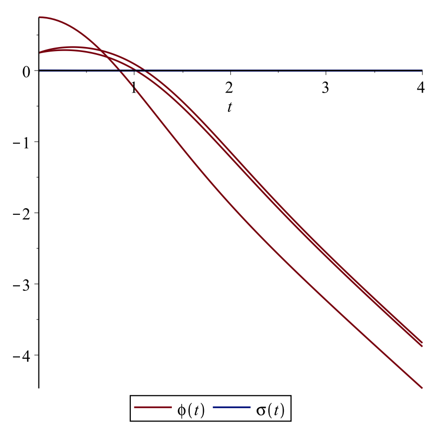

Figure 5.19: Plot for system (5.4.1) with $n = 1, \mu = 0.5$ as $t \to \infty$ with several initial conditions:[[$\phi(0) = 0.5, D(\phi(0)) = 0, \sigma(0) = 0.001, a(0) = 0.8, \theta(0) = 0.2$], [$\phi(0) = 0.25, D(\phi(0)) = 0.5, \sigma(0) = 0, a(0) = 0.7, \theta(0) = 0.1$], [$\phi(0) = 0.75, D(\phi(0)) = 1, \sigma(0) = 0.0001, a(0) = 0.2, \theta(0) = 0.3$]].

5.19, there is no oscillations and $\sigma(t)$ decays, which is consistent with the isotropic model that we studied earlier in chapter (4).

5.4.4 Bifurcation Value Analytic Proof of the System (5.4.1):

In this subsubsection we show an analytic proof for the bifurcation value for this model. Setting $\dot{\phi} = 0$ in the equation (5.4.1) yields to

$$n^2\phi + \mu\left(\sqrt{3\sigma^2 + \frac{3}{2}n^2\phi^2 - 3\sqrt{6}\mu\sigma\phi}\right) = 0. \tag{5.4.2}$$

Since the value of $\sigma = 0$ at the equilibrium points (i.e., $P_{1,2,3,4}$) of the system (5.4.1) then (5.4.2) becomes

$$n^2\phi = -\mu\left(\sqrt{\frac{3}{2}n^2\phi^2}\right), \tag{5.4.3}$$

$$n^4\phi^2 = \frac{3}{2}\mu^2 n^2\phi^2, \tag{5.4.4}$$

$$\Rightarrow \mu^2 = \frac{2}{3}n^2. \tag{5.4.5}$$

Hence, the bifurcation value in which the stability (the behaviour) of the model will change is defined by (recall μ is positive):

$$\mu_c = \sqrt{\frac{2}{3}}n, \tag{5.4.6}$$

which is consistent with the sub-case (1b) that we studied earlier in chapter (4).

5.4.5 Normalized Bounded Variable

Introducing Normalized Variables

Consider the same bounded variables we did earlier. Then, the evolution equations become

$$\begin{cases} D' = (1 - D^2)\mathcal{X}, \\ \Psi' = n\sqrt{1-D^2}\Phi - D\Psi\mathcal{X}, \\ \Phi' = -D\Phi(1+\mathcal{X}) - \sqrt{1-D^2}\left[n\Psi + \sqrt{\frac{3}{2}}\mu D\right], \\ \Sigma' = -\Sigma D(1+\mathcal{X}), \end{cases} \tag{5.4.7}$$

where

$$\mathcal{X} := -\frac{1}{3}D^2 - \frac{2}{3}\Phi^2 + \frac{1}{3}\Psi^2 - 2\Sigma^2 - \sqrt{\frac{3}{2}}\mu\sqrt{1-D^2}\Phi\left(1 + \sqrt{6}\frac{\Sigma}{D}\right) + \frac{e}{6}\Sigma\Psi. \tag{5.4.8}$$

Thus, the Friedmann equation becomes

$$D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 + e\Sigma\Psi = 0, \tag{5.4.9}$$

where e is defined as follow:

$$e \equiv \frac{6\mu}{n} > 0. \tag{5.4.10}$$

The domain of interest is $0 \leq D \leq 1$, since

$$\theta \to \infty \quad \Longrightarrow D \to 1,$$

and

$$\theta \to 0 \quad \Longrightarrow D \to 0.$$

We have that D is a bounded variable and the other variables Φ, Ψ, Σ are bounded if $3\mu^2 < n^2$. This is follows from the Friedmann equation that can be written as follows

$$1 - \Phi^2 - \left(\Psi - \frac{3\mu\Sigma}{n}\right)^2 - 3\Sigma^2\left(1 - 3\frac{\mu^2}{n^2}\right) = 0. \tag{5.4.11}$$

Note that the system (5.4.7) is not well defined at $D = 0$, see (5.4.8). Note that (5.4.9) is conserved using the 4D system (5.4.7) as follows

Define

$$\Lambda = D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 + e\Sigma\Psi,$$

then

$$\begin{aligned}
\Lambda' &= 2DD' - 2\Phi\Phi' - -6\Sigma\Sigma' - 2\Psi\Psi' + e(\Sigma\Psi' + \Sigma'\Psi) \\
&= 2D(1 - D^2)\mathcal{X} + 2D\Phi^2(1 + \mathcal{X}) + 2\sqrt{\frac{3}{2}}\mu D\sqrt{1 - D^2}\Phi + 6\Sigma^2 D(1 + \mathcal{X}) \\
&+ 2D\Psi^2\mathcal{X} + e(n\Sigma\sqrt{1 - D^2}\Phi - \Psi\Sigma D(1 + 2\mathcal{X}) \\
&= -2D\mathcal{X}(D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 + e\Sigma\Psi) + en\Sigma\Phi\sqrt{1 - D^2} \\
&+ 2D\left[\mathcal{X} + \Phi^2 + \sqrt{\frac{3}{2}}\mu\Phi\sqrt{1 - D^2} + 3\Sigma^2 - \frac{e}{2}\Sigma\Psi\right] \\
&= 2\left[-D\mathcal{X}\Lambda - \frac{2D}{3}(D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 + e\Sigma\Psi)\right] \\
&= -\frac{2}{3}D\Lambda(3\mathcal{X} + 1) = 0.
\end{aligned}$$

Note that, since $D \neq 0$ for the above system, the evolutionary system is not analytic at P_0 (i.e., $D = \Phi = \Psi = \Sigma = 0$, is not equilibrium point for the system (5.4.7)).

Bounded Variables:Qualitative Analysis

The equations in system (5.4.7) are relatively simple. Also, note that the D' does not decouple and the system is not analytic at $D = 1$ since we have the square root term in the second equation of the 4D system (5.4.7).

Equilibrium Points

There are several equilibrium points of the system but we are only interested in the study of $P_{1,2,3,4}$ and to compare it then with the isotropic model that we studied earlier. In table 5.3, we present four equilibrium points of the system (5.4.7), $\mu > 0$ and $\nu = 0$, and the value of $\mathcal{X}$ for each one.

.

Pt	(D, Φ, Ψ, Σ)	$\mathcal{X}$	Stability
P_1	$(1, 1, 0, 0)$	-1	Source
P_2	$(1, -1, 0, 0)$	-1	Source
P_3	$(1, 0, 1, 0)$	0	Saddle
P_4	$(1, 0, -1, 0)$	0	Sink when $\mu > \sqrt{\frac{2}{3}}n$

Table 5.3: Equilibrium points of the system (5.4.7) and the value of $\mathcal{X}$.

Now, let us discuss the stability of the equilibrium points ($P_{1,2,3,4}$) of the system (5.4.7).

Stability of the Equilibrium Points $P_{1,2}$

By using eigenvalues in (5.3.18) when $\Phi_0^2 = 1$ then all the eigenvalues are positive which implies that $P_{1,2}$ are always sources.

Stability of the Equilibrium Points $P_{3,4}$

Evaluating the linearization matrix of the system (5.4.7) at $P_{3,4}$ leads to undefined terms which means that the linearization fails to determined the stability of $P_{3,4}$. Thus, we need to use different approach to study these equilibrium points $P_{3,4}$. When $D = 1$ is an invariant set then there is a 3 -dimensional invariant set given by

$$\begin{cases} \Phi' = \dfrac{2}{3}\Psi\left(\dfrac{2\Phi^2 n + 6n\Sigma^2 - 3\Sigma\mu\Psi - n\Psi^2 + n}{2n}\right), \\ \Psi' = \dfrac{1}{3}\Phi\left(\dfrac{4\Phi^2 n + 12n\Sigma^2 - 6\mu\Sigma\Psi - 2n\Psi^2 - 4n}{2n}\right), \\ \Sigma' = \dfrac{1}{3}\Sigma\left(\dfrac{4\Phi^2 n + 12n\Sigma^2 - 6\mu\Sigma\Psi - 2n\Psi^2 - 4n}{2n}\right). \end{cases} \tag{5.4.12}$$

Evaluating the linearization matrix of (5.4.12) at $\Phi = 0, \Sigma = 0$ lead to three negative eigenvalues. Now we just need to determine the sign of the four eigenvalue. By plugging (5.4.9) into (5.4.8), we obtain;

$$\mathcal{X} = -\Phi^2 - 3\Sigma^2 + \frac{3\mu}{n}\Psi\Sigma. - \sqrt{\frac{3}{2}}\mu\Phi\sqrt{1-D^2}\left(1 + \frac{\sqrt{6}\Sigma}{D}\right).$$

Since $\Sigma = 0$ at $P_{3,4}$, (so that we neglect any terms with Σ), we obtain a simpler expression for $\mathcal{X}$ as follows:

$$\mathcal{X} \equiv -\Phi^2 - \sqrt{\frac{3}{2}}\mu\Phi\sqrt{1-D^2}. \tag{5.4.13}$$

Let $T = \sqrt{1-D^2}$ Then, it follows that;

$$T' = T\mathcal{X} \tag{5.4.14}$$

Moreover, since $\Phi' = 0$, then we obtain

$$\Phi = -T\left(n\Psi + \sqrt{\frac{3}{2}}\mu\right). \tag{5.4.15}$$

Next, plugging the value of Φ from (5.4.15) into (5.4.14), then we obtain

$$T' = \sqrt{\frac{3}{2}} n T^3 \left(n\Psi + \sqrt{\frac{3}{2}}\mu \right). \tag{5.4.16}$$

Therefore, now we can study the stability of $P_{3,4}$ as follows

1. For the P_3, it is a always a saddle for any value of μ since T' is small and positive.

2. For the P_4, we have two cases of stability (the behaviour):

   - If $\mu < \sqrt{\frac{2}{3}}n \Longrightarrow T' > 0$, which implies that P_4 is a saddle in this case.
   - If $\mu > \sqrt{\frac{2}{3}}n \Longrightarrow T' < 0$, which implies P_4 is a sink in this case.

   This result is consistent with the isotropic model that we studied earlier.

Bounded Variables: Numerical Method -Past Behaviour

In figure 5.20, we plot the solution of system in (5.4.7) with $\mu = 0.5$ into the past times.

Discussion

As can be seen from figure 5.20, the solution goes toward the equilibrium point P_1 into the past limit in which is consistent with the isotropic model that we studied earlier in chapter (4).

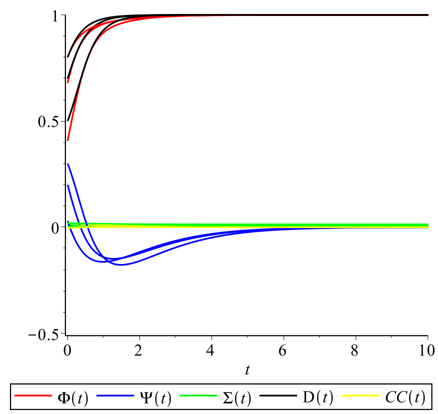

Figure 5.20: Plot for system (5.4.7) with $n = 1, \mu = 0.5$ as $t \to -\infty$ with several initial conditions:$[[D(0) = 0.5, \Psi(0) = 0.3, \Sigma(0) = 0.03, \Phi(0) = 0.4066607923], [D(0) = 0.7, \Psi(0) = 0.2, \Sigma(0) = 0.01, \Phi(0) = 0.6794850992], [D(0) = 0.8, \Psi(0) = 0.03, \Sigma(0) = 0.02\Phi(0) = 0.8009369513]]$. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t)$.

Bounded Variables: Numerical Method-Future Behaviour

In figures from 5.21 to 5.24, we plot the solution of the system (5.4.7) with different positive values of μ to see the behaviour of this model into the future times with the normalized bounded variable and compare it with the numerical solutions that we get from the original variables as well as with the isotropic model

Discussion

As can be seen from figures 5.21 and 5.22 with $\mu < \mu_c$, the solutions for Φ, Ψ, D start with oscillations and decay and Σ decreases and decays to zero. In figure 5.24 when $\mu > \mu_c$, the solution as $t \to -\infty$ go toward P_4 which is consistent with the isotropic model that we studied earlier in chapter (4).

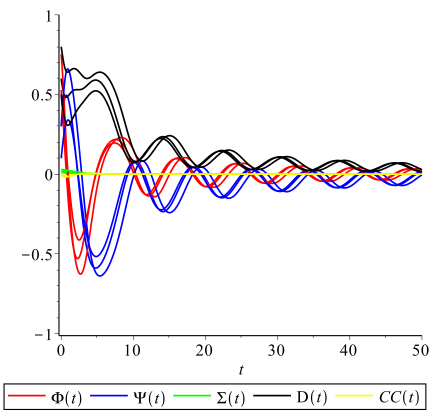

Figure 5.21: Plot for system (5.4.7) with $n = 1, \mu = 0.5$ as $t \to \infty$ with several initial conditions:[[$D(0) = 0.5, \Psi(0) = 0.1, \Sigma(0) = 0.03, \Phi(0) = 0.4962862077$], [$D(0) = 0.6, \Psi(0) = 0.3, \Sigma(0) = 0.01, \Phi(0) = 0.5279204486$], [$D(0) = 0.8, \Psi(0) = 0.3, \Sigma(0) = 0.02\Phi(0) = 0.7528612090$]]. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t)$.

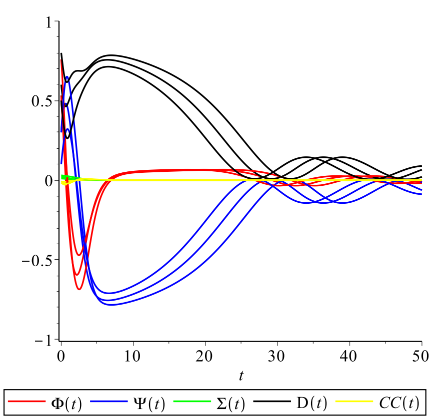

Figure 5.22: Plot for system (5.4.7) with $n = 1, \mu = 0.75$ as $t \to \infty$ with several initial conditions:[[$D(0) = 0.5, \Psi(0) = 0.1, \Sigma(0) = 0.03, \Phi(0) = 0.5007993610$], [$D(0) = 0.6, \Psi(0) = 0.3, \Sigma(0) = 0.01, \Phi(0) = 0.5321653878$], [$D(0) = 0.8, \Psi(0) = 0.3, \Sigma(0) = 0.02\Phi(0) = 0.7588148654$]]. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t)$.

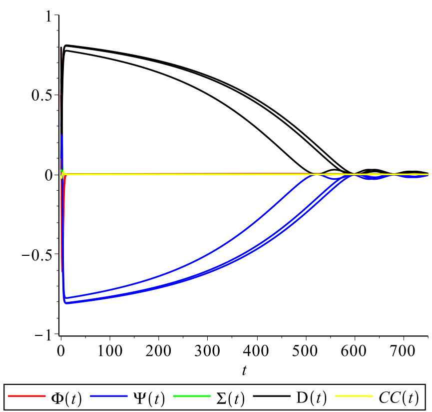

Figure 5.23: Plot for system (5.4.7) with $n = 1, \mu = 0.81$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.5, \Psi(0) = 0.1, \Sigma(0) = 0.03, \Phi(0) = 0.5019482045], [D(0) = 0.6, \Psi(0) = 0.3, \Sigma(0) = 0.01, \Phi(0) = 0.5332466596], [D(0) = 0.8, \Psi(0) = 0.03, \Sigma(0) = 0.1\Phi(0) = 0.7897797161]]$. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t)$.

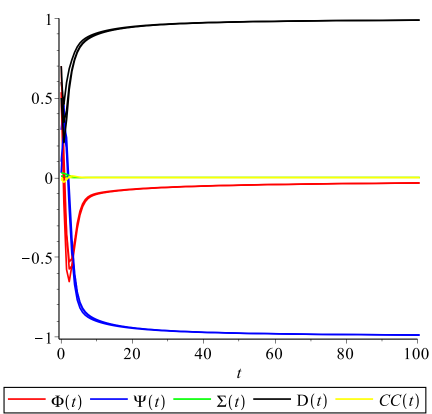

Figure 5.24: Plot for system (5.4.7) with $n = 1, \mu = 0.5$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.5, \Psi(0) = 0.1, \Sigma(0) = 0.03, \Phi(0) = 0.5052722039], [D(0) = 0.7, \Psi(0) = 0.03, \Sigma(0) = 0.01, \Phi(0) = 0.7004284403], [D(0) = 0.6, \Psi(0) = 0.3, \Sigma(0) = 0.01\Phi(0) = 0.5363767333]]$. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 + \frac{6\mu}{n}\Psi(t)\Sigma(t)$.

5.4.6 Inflation:

We will check if this case has an inflation attractor. The deceleration parameter in (5.5.10) when $\nu = 0$ becomes

$$q = -\frac{1}{D^2}\left[-6\Sigma^2 - 2\Phi^2 + \Psi^2 + \frac{3\mu}{n}\Psi\Sigma - \frac{3}{2}\sqrt{6}\sqrt{1-D^2}\mu\Phi\left(1 + \frac{\sqrt{6}\Sigma}{D}\right)\right]. \quad (5.4.17)$$

The sign of the deceleration parameter indicates the nature of the expansionary evolution. If $q > 0$, the cosmological expansion is decelerating, while negative values of q indicate an accelerating dynamics. The value for q for each of the equilibrium points of the model are:

1. $P_{1,2}$:

$$q|_{P_{1,2}} = 2 > 0, \tag{5.4.18}$$

   are always positive which implies that $P_{1,2}$ are not inflationary.

2. $P_{3,4}$:

$$q|_{P_{3,4}} = -1 < 0, \tag{5.4.19}$$

   are always negative which implies that $P_{3,4}$ are inflationary.

5.4.7 Discussion:

In this case when $\Sigma \neq 0,\ \ \mu > 0$ and $\nu = 0$, we found that P_3 is a saddle inflationary solution. Moreover, P_4 is an inflationary attractor when $\mu > \sqrt{\frac{2}{3}}n$ which is consistent with the isotropic model case (1b) when $\Sigma = 0$ and $\mu > 0$.

5.5 Case (2c):When $\mu = 0, \nu \neq 0$

5.5.1 Original Variables

$$\begin{cases} \dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{\sqrt{6}\theta}{2\sqrt{6}\theta - 3\nu\phi}\left[2\dot{\phi}^2 - n^2\phi^2 - \frac{\nu\theta\phi}{\sqrt{6}}\right. \\ \quad \left. + \frac{\sqrt{6}\nu\phi\sigma^2}{\theta} + 3\nu\dot{\phi}\left(\frac{\sigma}{\theta} - \frac{1}{\sqrt{6}}\right)\right], \\ \dot{\sigma} = -\sigma\theta, \\ \dot{\phi} = \psi, \\ \dot{\psi} = -\theta(\psi + \mu) - n^2\phi - \nu\sigma, \\ \dot{a} = \frac{1}{3}a\theta, \end{cases} \tag{5.5.1}$$

with first integral

$$\theta^2 = 3\sigma^2 + 3\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}n^2\phi^2 + \frac{1}{\sqrt{6}}\nu\theta\phi\right]. \tag{5.5.2}$$

5.5.2 Original Variables: Numerical Method-Past Behaviour

In figures 5.25, 5.26, 5.27 , we plot the solution of $\phi(t), \sigma(t)$ and $a(t)$ for the system in (5.5.1) with $\nu = 0.5$ in to the past

Discussion

As can be seen from the figures 5.25,5.26 that $\phi(t)$ and $\sigma(t)$ blow up to infinity but $a(t)$ is decreases which is consistent with the isotropic model. Note this is true for any value of $\nu \neq 0$.

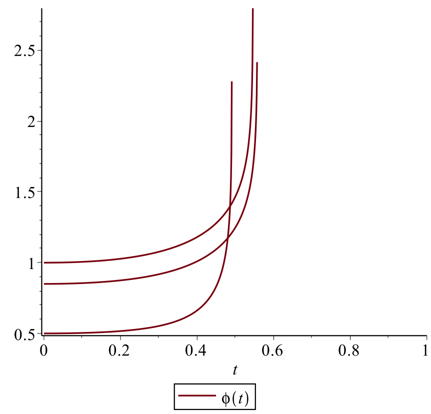

Figure 5.25: Plot for $\phi(t)$ for the system (5.5.1) with $n = 1, \nu = 0.5$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 0.5, a(0) = 1, \theta(0) = 0.1], [\phi(0) = 0.85, D(\phi(0)) = 0.005, \sigma(0) = 0.5, a(0) = 0.7\theta(0) = 0.2], [\phi(0) = 0.5, D(\phi(0)) = 0.003, \sigma(0) = 0.6, a(0) = 0.75, \theta(0) = 0.3]]$.

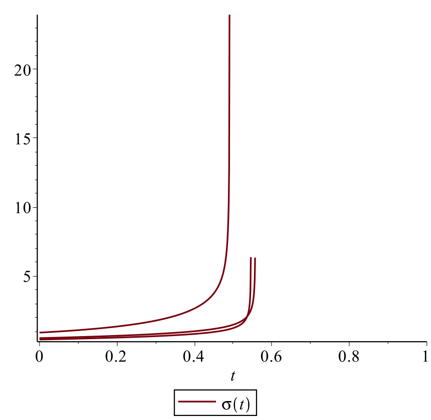

Figure 5.26: Plot for $\sigma(t)$ for the system (5.5.1) with $n = 1, \nu = 0.5$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 0.5, a(0) = 1, \theta(0) = 0.1], [\phi(0) = 0.85, D(\phi(0)) = 0.005, \sigma(0) = 0.5, a(0) = 0.7\theta(0) = 0.2], [\phi(0) = 0.5, D(\phi(0)) = 0.003, \sigma(0) = 0.6, a(0) = 0.75, \theta(0) = 0.3]]$.

5.5.3 Original Variables: Numerical Method-Future Behaviour

In figure 5.28, we plot (5.5.1) with $\nu = 0.5, n = 1$ to see the behaviour of this model into the future

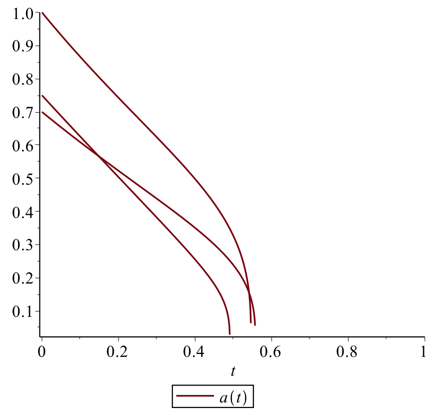

Figure 5.27: Plot for $a(t)$ for the system (5.5.1) with $n = 1, \nu = 0.5$ as $t \to -\infty$ with several initial conditions:$[[\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 0.5, a(0) = 1, \theta(0) = 0.1], [\phi(0) = 0.85, D(\phi(0)) = 0.005, \sigma(0) = 0.5, a(0) = 0.7\theta(0) = 0.2], [\phi(0) = 0.5, D(\phi(0)) = 0.003, \sigma(0) = 0.6, a(0) = 0.75, \theta(0) = 0.3]]$.

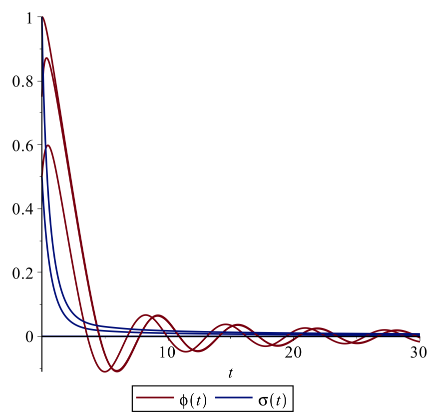

Figure 5.28: Plot for system (5.5.1) with $n = 1, \nu = 0.5$ as $t \to \infty$ with several initial conditions:$[[\phi(0) = 1, D(\phi(0)) = 0, \sigma(0) = 1, a(0) = 0.8, \theta(0) = 0.1], [\phi(0) = 0.59, D(\phi(0)) = 0.5, \sigma(0) = 0, a(0) = 0.7, \theta(0) = 0.2], [\phi(0) = 0.8, D(\phi(0)) = 1, \sigma(0) = 1, a(0) = 0.2, \theta(0) = 0.3]]$.

Discussion

As can be seen from the figure 5.28, $\sigma(t)$ decreases and $\phi(t)$ oscillates and then decays to zero, which is consistent with the isotropic model. Note this is true for any value of $\nu \neq 0$.

5.5.4 Normalized Bounded Variable

Introducing Normalized Variables

Consider a bounded variable D as before, along with the following expressions for Φ, Ψ, and Σ

$$D \equiv \frac{\theta}{\sqrt{1+\theta^2}}, \qquad \Sigma \equiv \frac{\sigma}{\sqrt{1+\theta^2}}, \tag{5.5.3}$$

$$\Psi \equiv \sqrt{\frac{3}{2}}\left(\frac{n\phi}{\sqrt{1+\theta^2}}\right), \quad \Phi \equiv \sqrt{\frac{3}{2}}\left(\frac{\dot{\phi}}{\sqrt{1+\theta^2}}\right). \tag{5.5.4}$$

So, the Friedmann equation for this model becomes

$$\Lambda = D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 - \frac{\nu\Psi D}{n} = 0, \tag{5.5.5}$$

which is conserved by the 4D system in (5.5.6), illustrated below.

Define

$$\Lambda = D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 - \frac{\nu\Psi D}{n},$$

then

$$\begin{aligned}
\Lambda' &= 2DD' - 2\Phi\Phi' - 6\Sigma\Sigma' - 2\Psi\Psi' - \frac{\nu}{n}(D'\Psi + D\Psi') \\
&= 2D(1-D^2)\mathcal{X} + 2\Phi^2 D(1+\mathcal{X}) + 2\sqrt{\frac{3}{2}}\nu\Phi\Sigma\sqrt{1-D^2} + 6D\Sigma^2(1+\mathcal{X}) \\
&+ 2D\Phi^2(1+\mathcal{X}) + 2nD\sqrt{1-D^2}\Psi + \sqrt{6}\mu D\sqrt{1-D^2}\Phi \\
&+ 2D\Psi^2\mathcal{X} - \nu\Phi D\sqrt{1-D^2} + \frac{2\nu}{n}\Phi D^2\mathcal{X} - \frac{\nu}{n}\Psi\mathcal{X} \\
&= -2D\mathcal{X}\left(D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 - \frac{\nu\Psi D}{n}\right) + \mathcal{X}\left(2D - \frac{\nu}{n}\Psi\right) \\
&+ 2D\Phi^2 + 2\sqrt{\frac{3}{2}}\nu\Phi\Sigma\sqrt{1-D^2} + 6\Sigma^2 D - \nu D\Phi\sqrt{1-D^2} \\
&= -2D\mathcal{X}\Lambda - \frac{2}{3}D\left(D^2 - \Phi^2 - 3\Sigma^2 - \Psi^2 - \frac{\nu\Psi D}{n}\right) \\
&= -\frac{2}{3}D\Lambda(3\mathcal{X}+1) = 0.
\end{aligned}$$

Now we define a new time variable τ with $\frac{d\tau}{dt} = \sqrt{1+\theta^2}$. Therefore the evolution equations become

$$\begin{cases}
D' = (1-D^2)\mathcal{X}, \\
\Psi' = n\Phi\sqrt{1-D^2} - D\Psi\mathcal{X}, \\
\Phi' = -D\Phi(1+\mathcal{X}) - \sqrt{1-D^2}\left[n\Psi + \sqrt{\frac{3}{2}}(\nu\Sigma)\right], \\
\Sigma' = -\Sigma D(1+\mathcal{X}),
\end{cases} \tag{5.5.6}$$

where the prime here indicates the differentiation of each variable with respect to the new variable time τ and $\mathcal{X}$ is given by the expression as follow:

$$\begin{aligned}
\mathcal{X} \equiv -\frac{1}{3}D^2 - 2\Sigma^2 - \frac{\sqrt{6}D}{\sqrt{6}(2D - \frac{\nu\Psi}{n})}&\left[\frac{4}{3}\Phi^2 - \frac{2}{3}\Psi^2 - \frac{\nu}{3n}D\Psi + \frac{2\nu\Sigma^2\Psi}{nD}\right. \\
&\left. + \sqrt{6}\sqrt{1-D^2}\left[\nu\Phi\left(\frac{\Sigma}{D} - \frac{1}{\sqrt{6}}\right)\right]\right] = \frac{\dot{\theta}}{(\theta^2+1)}.
\end{aligned} \tag{5.5.7}$$

Note that the system is not well define when $2D - \frac{\Psi\nu}{n} = 0$, which include the origin $D = 0 = \Psi$ see (5.5.6, 5.5.7). But since $D \neq 0$ for the system (5.5.6), the evolutionary system is not analytic at the origin (i.e., $D = \Phi = \Psi = \Sigma = 0$, is not equilibrium point for the system (5.5.6)).

Bounded Variables:Qualitative Analysis

The equations in system (5.4.7) are relatively simple. Also, note that the D' does not decouple and the system is not analytic at $D = 1$ since we have the square root term in the second equation of the 4D system (5.4.7).

Equilibrium Points

There are several equilibrium points of the system but we are only interested in study the $P_{1,2,3,4}$ to compare it with the isotropic model that we studied earlier. In table (5.4), we present the summary of strictly speaking interesting equilibrium points to study, for the system (5.5.6), $\mu = 0$ and $\nu \neq 0$, and the value of $\mathcal{X}$ for each one.

Pt	(D, Φ, Ψ, Σ)	$\mathcal{X}$	Stability
P_1	$(1, 1, 0, 0)$	-1	Source
P_2	$(1, -1, 0, 0)$	-1	Source
P_3	$\left(1, 0, \frac{-\nu+\sqrt{4\,n^2+\nu^2}}{2n}, 0\right)$	0	Saddle
P_4	$\left(1, 0, -\left(\frac{\nu+\sqrt{4\,n^2+\nu^2}}{2n}\right), 0\right)$	0	Saddle

Table 5.4: Equilibrium points of the system (5.5.6) and the value of $\mathcal{X}$.

Now, let us discuss the stability of the equilibrium points ($P_{1,2,3,4}$) of the system (5.5.6).

Stability of the Equilibrium Points $P_{1,2}$

Basically, we use the eigenvalues in (5.3.18) when $\Phi_0^2 = 1$ then we end up with all of the eigenvalues are positive which implies that $P_{1,2}$ are always sources, which is consistent with the isotropic model.

Stability of the Equilibrium Points $P_{3,4}$

Basically, we use same analysis as we did earlier with the case (2a) with $\mu = 0$. When $D = 1$ is an invariant set then there are 3 dynamic system;

$$\begin{cases} \Phi' = \dfrac{2}{3}\Psi\left(\dfrac{2\Phi^2 n + 6n\Sigma^2 - n\Psi^2 - \nu\Psi + n}{2n - \nu\Psi}\right), \\ \Psi' = \dfrac{1}{3}\Phi\left(\dfrac{4\Phi^2 n + 12n\Sigma^2 - 2n\Psi^2 + \nu\Psi - 4n}{2n - \nu\Psi}\right), \\ \Sigma' = \dfrac{1}{3}\Sigma\left(\dfrac{4\Phi^2 n + 12n\Sigma^2 - 2n\Psi^2 + \nu\Psi - 4n}{2n - \nu\Psi}\right). \end{cases} \tag{5.5.8}$$

Evaluating the linearization matrix of (5.5.8) at $\Phi = 0, \Sigma = 0$ lead to three negative eigenvalues. For the fourth eigenvalue it just follows from equation (5.3.26) with $\mu = 0$:

$$\alpha_4 = \frac{1}{36(\nu\sqrt{4n^2 + \nu^2} + 4n^2 + \nu^2)}\Bigg(\Bigg(\sqrt{4n^2 + \nu^2}(6\sqrt{6}n^2 + \sqrt{6}\nu^2) + \nu(2\sqrt{6}n^2 + \sqrt{6}\nu^2\Bigg)\left(\sqrt{6}\sqrt{4n^2 + \nu^2} + \sqrt{6}\nu\right)\Bigg), \tag{5.5.9}$$

Which is always positive therefore $P_{3,4}$ are saddles.

Discussion

Among the four equilibrium points ($P_{1,2,3,4}$) that we studied, we found that $P_{3,4}$ are inflationary saddles.

5.5.5 Bounded Variables: Numerical Method-Past Behaviour

In figure 5.29, we plot the solution of the system in (5.5.6) with $\nu = 0.5$ into the past times.

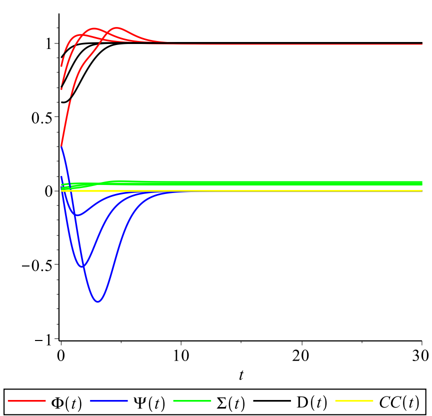

Figure 5.29: Plot for system (5.5.6) with $n = 1, \nu = 2$ as $t \to -\infty$ with several initial conditions:$[[D(0) = 0.9, \Psi(0) = 0.1, \Sigma(0) = 0.04, \Phi(0) = 0.8397618710], [D(0) = 0.7, \Psi(0) = 0.03, \Sigma(0) = 0.02, \Phi(0) = 0.6833008122], [D(0) = 0.6, \Psi(0) = 0.3, \Sigma(0) = 0.01, \Phi(0) = 0.2994995826]]$. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 - \frac{\nu\Psi(t)D(t)}{n}$.

Discussion

As can be seen from figure 5.29, the solution goes toward the equilibrium point P_3 into the past limit in which is consistent with the sub-case (2a) that we studied earlier in chapter (4). Note this is true for any value of $\nu \neq 0$.

Bounded Variables: Numerical Method-Future Behaviour

In figure 5.30, we plot the solution of the system (5.5.6) with $\nu = 0.5$ into the future.

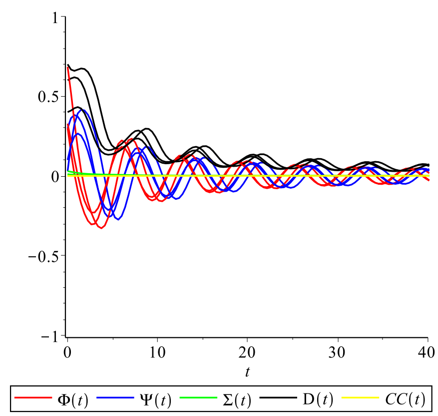

Figure 5.30: Plot for system (5.5.6) with $n = 1, \nu = 1$ as $t \to \infty$ with several initial conditions:$[[D(0) = 0.4, \Psi(0) = 0.1, \Sigma(0) = 0.03, \Phi(0) = 0.3275667871], [D(0) = 0.7, \Psi(0) = 0.03, \Sigma(0) = 0.01, \Phi(0) = 0.6839590631], [D(0) = 0.6, \Psi(0) = 0.3, \Sigma(0) = 0.01, \Phi(0) = 0.2994995826]]$. Note that $CC(t) = D(t)^2 - \Phi(t)^2 - 3\Sigma(t)^2 - \Psi(t)^2 - \frac{\nu\Psi(t)D(t)}{n}$.

Discussion

As can be seen from figure 5.30, the solution oscillate and decay to zero, which is consistent with the sub-case (2a) that we studied earlier in chapter (4). Note this is true for any value of $\nu \neq 0$.

5.5.6 Inflation

In the sub section, we will check if this case has an inflation attractor. The deceleration parameter in The deceleration parameter in (5.5.10) when $\mu = 0$ becomes

$$q = -\frac{1}{D^2}\left[-6\Sigma^2 - \frac{3\sqrt{6}D}{\sqrt{6}(2D - \frac{\nu\Psi}{n})}\left[\frac{4}{3}\Phi^2 - \frac{2}{3}\Psi^2 - \frac{\nu}{3n}D\Psi + \frac{2\nu\Sigma^2\Psi}{nD} + \sqrt{6}\sqrt{1-D^2}\nu\Phi\left(\frac{\Sigma}{D} - \frac{1}{\sqrt{6}}\right)\right]\right]. \tag{5.5.10}$$

The sign of the deceleration parameter indicates the nature of the expansionary evolution. If $q > 0$, the cosmological expansion is decelerating, while negative values of q indicate an accelerating dynamics. The value for q for each of the equilibrium points of the model are:

1. $P_{1,2}$:

$$q\big|_{P_{1,2}} = 2 > 0, \tag{5.5.11}$$

   are always positive which implies that $P_{1,2}$ are not inflationary.

2. $P_{3,4}$:

$$q\big|_{P_{3,4}} = -1 < 0, \tag{5.5.12}$$

   which implies that $P_{3,4}$ have an inflationary solutions.

Discussion:

In this case when $\Sigma \neq 0$, $\nu \neq 0$, and $\mu = 0$, we found that $P_{1,2}$ are always sources and not inflationary. $P_{3,4}$ are inflationary saddles.

Chapter 6

Discussion

In chapters (4)-(5), we have investigated cosmological models in the Einstein-Aether theory and (extended) Horava gravity in which both the Aether vector field and the metric tensor together determine the evolution. We have been especially interested in the possible inflationary behaviour of the models in a class of spatially homogeneous cosmological models. In particular, we have studied scalar field models in which the self-interaction potential, consisting of terms each containing exponentials, depends on the scalar field ϕ and also on the timelike vector field through the expansion rate θ and the shear scalar σ. We derived the evolution equations in two models: the isotropic and an anisotropic models, which consist of the energy momentum conservation law or Klein-Gordon equation, the generalized Friedmann equation, the Raychaudhuri equation, and the evolution equations for the shear. We introduced expansion-normalized variables and obtained the resulting dimensionless evolution equations which reduce to a dynamical system. We studied the behaviour of the models with zero and negative curvature for the isotropic model and only the zero curvature case for the anisotropic model. In particular, we studied the local stability of five equilibrium points ($P_{0,1,2,3,4}$) of the dynamical system corresponding to physically realistic solutions with a restricted set of values of the parameters. We concluded that there are always ranges of values for which there exists an inflationary sink.

We investigated the qualitative late time behaviour of isotropic and anisotropic model using two techniques; the first one with the original variables and the second one with normalized bounded variables. The normalized bounded variables are particularly suitable for numerical and qualitative analysis [13, 21]. In particular, we considered the isotropic model and an anisotropic model (Bianchi type VI_h) with sub-cases for each one. In the isotropic model (Case (1)), when the potential is not a function of the shear, (i.e.,$V(\phi, \theta)$), we studied two sub-cases: case (1a) when $\sigma = 0$, $\mu = 0$ and $\nu = 0$; case (1b) when $\sigma = 0 = \nu$ and $\mu > 0$. In the anisotropic model (Case (2)), when the potential function depends on the shear as well as the scalar field and the expansion rate, (i.e., $V(\phi, \theta, \sigma)$), we studied three sub-cases: case (2a) when $\sigma \neq 0$, $\mu > 0$ and $\nu \neq 0$; case (2b) when $\sigma \neq 0$, $\nu = 0$ and $\mu > 0$ and lastly case (2c) when $\sigma \neq 0$, $\mu = 0$ and $\nu \neq 0$ (where ν could be either positive or negative).

In the isotropic model (Case(1)), we examined the case when the potential does not depend on the shear and we studied two further sub-cases. We paid particular attention to the sinks and whether there are any inflationary attractors. In the sub-case (1a), we found that the equilibrium point P_0 is a sink but not inflationary and $P_{3,4}$ are inflationary saddles. Hence, there is no inflationary attractor for this case. In the sub-case (1b), we found there exists a bifurcation value (i.e., $\mu = \sqrt{\frac{2}{3}}n$) at which the stability of the equilibrium points changes. To illustrate this, when $\mu \leq \sqrt{\frac{2}{3}}n$, P_0 is sink but P_4 is a saddle. However, when $\mu > \sqrt{\frac{2}{3}}n$ P_0 changes from sink to a saddle and P_4 becomes a sink. Also, we conclude, if $\mu > \mu_c$ there is an inflationary attractor at P_4 and there is inflationary saddle at P_3 but there is no inflationary source. In addition, we studied slow roll inflation in these two sub-cases For example; for the sub-case (1b), we found that the slow roll inflation happens when

$$\phi = \left(\sqrt{\frac{2}{3}}n - \mu\right) t + C_6, \tag{6.0.1}$$

where C_3, C_6 are constants. The scale factor of the expnading Universe is

$$a = a_0 e^{\frac{n}{2\sqrt{6}}\left(\sqrt{\frac{2}{3}}n-\mu\right)t^2 + C_7 t},$$

which has the solution $a = a_0$, where a_0 is constance, when $t = 0$.

In the anisotropic model case (2), we considered the Bianchi type I (i.e., $M = 0$) model with zero curvature, in which the potential is a function of the scalar field ϕ, the expansion rate θ and the shear σ. We paid particular attention to the sinks and whether there are any inflationary attractors. We further investigated three sub-cases. In all three sub-cases, we found that P_0 is not an equilibrium point since the system with the shear is not well defined at $D = 0 = \Psi$. In the sub-case (2a), we found there always exists a ranges of values of the parameters μ, ν where the model has an inflationary attractor. For instance, P_4 is a sink when $\sqrt{6}\sqrt{4\,n^2 + \nu^2} + \sqrt{6}\nu - 6\,\mu < 0$. In the sub-case (2b), we found that there is a bifurcation value same as in the case (1b) and P_4 is inflationary attractor when $\mu > \sqrt{\frac{2}{3}}n$. Lastly, in the sub-case (2c) we found that $P_{3,4}$ are an inflationary saddles. In all cases, our major conclusion that we require that $\mu > 0$ to be large enough in order to have an inflationary attractor.

In conclusion, we investigated the stability of the isotropic equilibrium points and their stability as well as the stability of four equilibrium points ($P_{1,2,3,4}$) in the anisotropic model. The result is consistent with the standard inflationary cosmological solutions and previous works in Einstein-Aether cosmological models. A qualitative analysis of these cosmological models are based on the dynamical systems approach. In table (6.1), we summarize all the sinks and the inflationary solutions for each of the two models and their sub-cases.

Cases	Sub Cases	Sinks	Inflation
Case(1) ($\Sigma = 0$) Isotropic Model	Sub-case (1a) $\mu = 0 = \nu$	P_0 in table (4.1)	$P_{3,4}$ in table (4.1)
	Sub-case(1b) $\mu > 0,\ \nu = 0$	In table (4.2): two sinks 1- P_0 if $\mu < \sqrt{\frac{2}{3}}n$ 2- P_4 if $\mu > \sqrt{\frac{2}{3}}n$	$P_{3,4}$ in table (4.2)
Case (2) ($\Sigma \neq 0$) Anisotropic Model	Sub-case (2a)	P_4 in table when (5.2) $\sqrt{6}\sqrt{4\,n^2 + \nu^2} + \sqrt{6}\nu - 6\,\mu < 0$	$P_{3,4}$ in table (5.2)
	Sub-case (2b) $\mu > 0,\ \nu = 0$	Sink P_4 if $\mu > \sqrt{\frac{2}{3}}n$	$P_{3,4}$ in table (5.3)
	Sub-case (2c) $\mu = 0,\ \nu \neq 0$	No sink	$P_{3,4}$ in table (5.4)

Table 6.1: Summary of the sinks points and inflation for all sub-cases studied.

To find where all numerical and qualitative analysis has been done in this chapter for each case see the following table (6.2) In future work we shall investigate the

Cases	Sub Cases	Original Variables	Bounded Variables
$\sigma = 0$	$\mu = 0 = \nu$	In section (4.1.1)	In section (4.1.4)
	$\mu \neq 0,\ \nu = 0$	section (4.2.1)	In section (4.2.4)
$\sigma \neq 0$	$\mu \neq 0,\ \nu \neq 0$	In section (5.3.1)	On section (5.3.4)
	$\mu \neq 0\ \nu = 0$	In section (5.4.1)	In section (5.4.5)
	$\mu = 0,\ \nu \neq 0$	In section (5.5.1)	In section (5.5.4)

Table 6.2: Locations in the manuscript where the analysis with original variable and normalized bounded variables for the isotropic model (and its two sub-cases) and the anisotropic model (and its three sub-cases).

Anisotropic model with a polynomial potential of the following form

$$V(\theta, \phi, \sigma) = \frac{1}{2}n^2\phi^2 + \mu\theta\phi + \nu\sigma\phi + [a_{20}\theta^2 + a_{11}\theta\sigma + a_{02}\sigma^2], \tag{6.0.2}$$

where a_{rs} are positive constants. Without loss of generality, we can assume μ, n are positive constants but ν could be either positive or negative. In particular, it would be of my interest to determine whether the model with the above potential will have the same equilibrium points as in our two models that we studied or not. Also, if it turns out to be the case we would be interest to see the stability of them and compare it with our cases.

Part II

Spherically Symmetric Einstein-Aether Kantowski-Sachs Cosmological Models with Scalar Field

Chapter 7

Introduction to the Model

Since the vacuum in quantum gravity may determine a preferred rest frame at the microscopic level, gravitational Lorentz violation has been studied within the framework of general relativity (GR), where the background tensor field(s) breaking the symmetry must be dynamical [17]. Einstein-Aether theory [1, 2] consists of GR coupled, at second derivative order, to a dynamical timelike unit vector field, the aether. In this effective field theory approach, the Aether vector field and the metric tensor together determine the local spacetime structure.

The aether spontaneously breaks Lorentz invariance by picking out a preferred frame at each point in spacetime while maintaining local rotational symmetry (breaking only the boost sector of the Lorentz symmetry). Since the aether is a unit vector, it is everywhere non-zero in any solution, including flat spacetime.

There has been much interest in the qualitative features of cosmological models in Einstein -Aether theory (and in particular in the presence of curvature and shear), with a polynomial self-interaction scalar field potential, V, and especially a generalization of the harmonic potential [1, 28].

Einstein -Aether theories offer an alternative that permits inflation [5, 20]. Lorentz violation affects the dynamics of the chaotic inflationary model. Generalizations of Einstein -Aether cosmology with a chaotic potential, V, in FLRW models were studied in [1] and more general models (with coupling constants depending on ϕ) in [29]. modifies the picture of the chaotic inflationary scenario.

In this part of the thesis, we study “spherically symmetric Einstein-Aether models”. We shall study scalar field models with an exponential self-interaction potential. Einstein-Aether models with an exponential potential were recently studied [8, 13, 26]. In a companion paper [27], it investigates spherically symmetric Einstein-Aether perfect fluid models.

We shall use the 1+3 frame formalism [30, 31] to write down the evolution equations for non-comoving scalar field spherically symmetric models. We then consider the spatially homogeneous Kantowski- Sachs models using appropriate normalized variables, and obtain the general evolution equations. We then consider a special case and analyse the qualitative behaviour for physically reasonable values of the parameters. We are particularly interested in the future asymptotic behaviour of the models for different values of the parameters and investigate whether the model has an inflationary attractor or not.

7.1 Einstein- Aether Cosmology Theory

The action for Einstein-Aether theory is the most general covariant functional of the spacetime metric g_{ab} and aether field u_a involving no more than two derivatives (not including total derivatives) [17, 30]. The action is [17, 23]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d + \lambda\left(u^c u_c + 1\right)\right], \qquad (7.1.1)$$

where

$$K^{ab}{}_{cd} \equiv c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c + c_4 u^a u^b g_{cd}. \tag{7.1.2}$$

The standard Einstein equations (i.e., $G_{ab} = kT_{ab}$) can incorporate the effects of the Aether by the contribution of an additional stress tensor for the aether field

$$G_{ab} = kT_{ab} + T^{ae}_{ab}, \tag{7.1.3}$$

where

$$\begin{aligned} T^{ae}_{ab} &= 2c_1(\nabla_a u^c \nabla_b u_c - \nabla^c u_a \nabla_c u_b) - 2[\nabla_c(u_{(a} J^c_{b)}) + \nabla_c(u^c J_{(ab)}) - \nabla_c(u_{(a} J^c_{b)})] \\ &\quad -2c_4 \dot{u}_a \dot{u}_b + 2\lambda u_a u_b + g_{ab}\mathcal{L}_u \end{aligned} \tag{7.1.4}$$

and

$$\begin{aligned} K^{ab}_{cd} &\equiv c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c + c_4 u^a u^b g_{cd}, \\ J^a_m &= -K^{ab}_{mn} \nabla_b u^n, \\ \mathcal{L}_u &\equiv -K^{ab}_{cd} \nabla_a u^c \nabla_b u^d. \end{aligned}$$

7.2 Models and the c_i Parameters

We study the model with different dimensionless parameters c_i where $i = 1, ..4$. To simplify the expressions and the equations it is convenient to make a reparameterization of the aether parameters, analogous to the one given in [32]

$$c_\theta = c_2 + \frac{c_1 + c_3}{3}, \quad c_\sigma = c_1 + c_3, \quad c_\omega = c_1 - c_3, \quad c_a = c_4 - c_1. \tag{7.2.1}$$

where the new parameters correspond to terms in the Lagrangian relating to expansion, shear, acceleration and twist of the aether. Since the spherically symmetric models are hypersurface orthogonal the aether field has vanishing twist and is therefore independent of the twist parameter c_ω. Also, for simplicity and convenience we define

$$c^2 = 1 - 2c_\sigma \geq 0.$$

7.3 Spherical Symmetry

All spherically symmetric aether fields are hypersurface orthogonal and, hence, all spherically symmetric solutions of aether theory will also be solutions of the IR limit of Horava gravity. The converse is not true in general, but it does hold in spherical symmetry for solutions with a regular center [18] . The c_i are dimensionless constants in the model. When spherical symmetry is imposed the aether is hypersurface orthogonal, and so it has vanishing twist. Thus it is possible to set c_4 to zero without loss of generality [18]. After the parameter redefinition to eliminate c_4, one is left with a 3- dimensional parameter space. The c_i contribute to the effective Newtonian gravitational constant G; so a renormalization of the parameters in the model can be then used to set $8\pi G = 1$ (i.e., another condition on the c_i can effectively be specified). The remaining parameters in the model can be characterized by two nontrivial constant parameters. In GR $c_i = 0$. We shall study the qualitative properties of models with values for the non-GR parameters which are consistent with current constraints.

7.4 The Model

The evolution equations, follows from the Einstein Field equations, derived from the Einstein aether action [18, 31]. In an Einstein-aether model there will be additional terms in the Friedmann equation:

1. The effects on the geometry from the anisotropy and inhomogeneities (e.g., the curvature) of the spherically symmetric models under consideration.

2. The Einstein field equations are generalised by the contribution of an additional stress tensor, T_{ab}, for the aether field which depends on the dimensionless parameters (c_i) of the aether model. In GR, all of the $c_i = 0$. To study the effects of matter, we could perhaps assume the corresponding GR values (or close to them) in the first instance.

3. The energy momentum tensor of a scalar field, due to the possible dependence of the scalar field potential V on the Lorentz violating vector field; primarily through the expansion, but also through the shear and tilt in spherically symmetric model.

4. When the phenomenology of theories with a preferred frame is studied, it is generally assumed that this frame coincides, at least roughly, with the cosmological rest frame defined by the Hubble expansion of the universe. In particular, in an isotropic and spatially homogeneous Friedmann universe the aether field will be

aligned with the (natural preferred CMB rest frame) cosmic frame and is thus related to the expansion rate of the universe. In principle, the preferred frame determined by the aether can be different from the CMB rest frame in spherically symmetric models. This adds additional terms to the aether stress tensor S_{ab}, which can be characterized by a hyperbolic tilt angle, $\nu(t)$, measuring the boost of the aether relative to the (perfect fluid) CMB rest frame [5, 20]. The tilt is expected to decay to the future in anisotropic but spatially homogeneous models [22].

7.5 Coupling to a Scalar Field

In Einstein-aether theory with inflation caused by a scalar field there exists the possibility that inflation is affected by the aether through a direct coupling of the scalar field with the aether field through quantities like the expansion or shear of the aether. This is in contrast to the situation in a purely metric theory like GR, where such quantities cannot be constructed from the metric in a covariant way [1]. Explicit couplings have been studied for spatially homogeneous and isotropic models in several papers [7], [26], and we extended to the anisotropic case with a phenomenological model in [21]. We will here derive the equations from first principles in the spherically symmetric case.

Assume that a scalar field has a potential, $V(\phi, \theta, \sigma_+)$, that depends on both the expansion and shear of the aether (but not the acceleration $\dot{u}^a$), defined as $\theta = \nabla_a u^a$

and $6\sigma_+^2 = \nabla_a u^b \nabla_b u^a - \frac{1}{3}\theta^2$, where

$$\nabla_a u_b = -u_a \dot{u}_b + \tfrac{1}{3}\theta h_{ab} + \sigma_{ab}, \tag{7.5.1}$$

and $\sigma_+^2 = \frac{1}{6}\sigma_{ab}\sigma^{ab}$, $h_{ab} = g_{ab} + u_a u_b$, $\sigma_{ab} = \sigma_{ba}$, $u^a \sigma_{ab} = u^a \dot{u}_a = u^a h_{ab} = 0$. We here assume that the Aether has vanishing vorticity.

The energy-momentum tensor for the scalar field is of the form [27]

$$\begin{aligned} T^\phi_{ab} =& \nabla_a \phi \nabla_b \phi - \left(\frac{1}{2}\nabla_a \phi \nabla^a \phi + V\right) g_{ab} + (\dot{V}_\theta + \theta V_\theta) g_{ab} + \dot{V}_\theta u_a u_b \\ &+ \dot{V}_{\sigma_+} \frac{\sigma_{ab}}{6\sigma_+} + \frac{V_{\sigma_+}}{6\sigma_+}\left[\left(\theta - \frac{\dot{\sigma}_+}{\sigma_+}\right)\sigma_{ab} + \dot{\sigma}_{ab} - 6\sigma_+^2 u_a u_b - \dot{u}^c \sigma_{c(a} u_{b)}\right], \end{aligned} \tag{7.5.2}$$

where the subscripts in V_θ and V_{σ_+} denotes partial derivatives of $V(\phi, \theta, \sigma_+)$ with respect to θ and σ_+ and a dot the covariant derivative along the aether field, $\dot{} := u^a \nabla_a$. The irreducible decomposition of the tensor (7.5.2) with respect to a comoving aether field is given by

$$T^\phi_{ab} = \rho^\phi u_a u_b + 2q^\phi_{(a} u_{b)} + p^\phi (g_{ab} + u_a u_b) + \pi^\phi_{ab}, \tag{7.5.3}$$

where

$$q^\phi_\alpha = (q^\phi_1, 0, 0), \quad \pi^\phi_{\alpha\beta} = \mathrm{diag}(-2\pi^\phi_+, \pi^\phi_+, \pi^\phi_+), \tag{7.5.4}$$

$$\rho^\phi = \tfrac{1}{2}\left[\mathbf{e}_0(\phi)^2 + \mathbf{e}_1(\phi)^2\right] + V - \theta V_\theta - V_{\sigma_+}\sigma_+, \tag{7.5.5}$$

$$p^\phi = \tfrac{1}{2}\mathbf{e}_0(\phi)^2 - \tfrac{1}{6}\mathbf{e}_1(\phi)^2 - V + \theta V_\theta + \mathbf{e}_0(V_\theta), \tag{7.5.6}$$

$$q^\phi_1 = -\mathbf{e}_0(\phi)\mathbf{e}_1(\phi) - \tfrac{1}{6}V_{\sigma_+}\dot{u}, \tag{7.5.7}$$

$$\pi^\phi_+ = -\tfrac{1}{3}\mathbf{e}_1(\phi)^2 + \tfrac{1}{6}\theta V_{\sigma_+} + \tfrac{1}{6}\mathbf{e}_0(V_{\sigma_+}). \tag{7.5.8}$$

7.6 Exponential Potential

Exponential potentials of the form $V = V_0 e^{-\lambda\phi}$ occur in higher dimensional frameworks, Kaluza-Klein theories, and super gravity [24]. Although in general relativity the exponential potential of the scalar field does not lead to exponential inflation [4], if the potential is not too steep it can lead to a power law inflation. Ultimately, we restrict the steep potentials by using multiple fields in order to have assisted inflation [34, 35, 36, 37, 38]. A late time attractor is a scaling solution for exponential potentials with sufficiently flat potentials [13, 39, 40, 41, 42]. The dynamical system with negative exponential leads to rich physics, such as that which is found in Ekpyrotic behaviour [10]. The main reason of using this kind of exponential potentials is that the dynamical system that results allows us to use dimensionless variables.

Consider the Exponential potential of the form [26]:

$$V(\phi, \theta, \sigma_+) = a_1 e^{-2k\phi} + a_2 \theta e^{-k\phi} + a_3 \sigma_+ e^{-k\phi}, \tag{7.6.1}$$

where constants a_1, a_2 and a_3 are defined such that the potential $V(\theta, \phi, \sigma_+)$ can be assumed to be positive definite. We shall let $a_1 > 0$ but allow a_2, and a_3 to have either positive or negative sign. Thus, the components of the energy momentum become

$$\rho^\phi = \tfrac{1}{2}\left[\mathbf{e}_0(\phi)^2 + \mathbf{e}_1(\phi)^2\right] + a_1 e^{-2k\phi}, \tag{7.6.2}$$

$$p^\phi = \tfrac{1}{2}\mathbf{e}_0(\phi)^2 - \tfrac{1}{6}\mathbf{e}_1(\phi)^2 - a_1 e^{-2k\phi} - k a_2 \mathbf{e}_0(\phi) e^{-k\phi} - a_3 \sigma_+ e^{-k\phi}, \tag{7.6.3}$$

$$q_1^\phi = -\mathbf{e}_0(\phi)\mathbf{e}_1(\phi) - \frac{a_3}{6}\dot{u} e^{-k\phi}, \tag{7.6.4}$$

$$\pi_+^\phi = -\tfrac{1}{3}\mathbf{e}_1(\phi)^2 + \frac{a_3}{6}\big(\theta - k\mathbf{e}_0(\phi)\big) e^{-k\phi}. \tag{7.6.5}$$

7.7 The Evolution Equation

The Einstein Field equations, the Jacobi identities and the contracted Bianchi identities give a system of partial differential equations on the frame and commutator functions [27]:

$$\mathbf{e}_0({e_1}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+){e_1}^1, \tag{7.7.1}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K, \tag{7.7.2}$$

$$\begin{aligned}\mathbf{e}_0(\theta) &= -\tfrac{1}{3}\theta^2 - 6\frac{1-2c_\sigma}{1+3c_\theta}\sigma_+^2 + \frac{1+c_a}{1+3c_\theta}\big(\mathbf{e}_1 + \dot{u} - 2a\big)\dot{u}\\ &+ \frac{1}{1+3c_\theta}\big(-\mathbf{e}_0(\phi)^2 + a_1e^{-2k\phi} + \tfrac{3}{2}a_2ke^{-k\phi}\mathbf{e}_0(\phi) + \tfrac{3}{2}a_3\sigma_+e^{-k\phi}\big),\end{aligned} \tag{7.7.3}$$

$$\begin{aligned}\mathbf{e}_0(\sigma_+) &= -\theta\sigma_+ + \frac{1}{3(1-2c_\sigma)}\big(\mathbf{e}_1(a) - \mathbf{e}_1(\dot{u}) - a\dot{u} - K - (1+2c_a)\dot{u}^2\big)\\ &+ \frac{1}{6(1-2c_\sigma)}\big(-2\mathbf{e}_1(\phi)^2 + a_3\theta e^{-k\phi} - a_3k\mathbf{e}_0(\phi)e^{-k\phi}\big),\end{aligned} \tag{7.7.4}$$

$$\mathbf{e}_0(a) = -\tfrac{1}{3}(\theta - 6\sigma_+)a + \tfrac{1}{3}(\mathbf{e}_1 - \dot{u})(\theta + 3\sigma_+), \tag{7.7.5}$$

Constraints

$$\mathbf{e}_1(\ln N) = \dot{u}, \tag{7.7.6}$$

$$\mathbf{e}_1(\ln K) = 2a, \tag{7.7.7}$$

$$\begin{aligned}2\mathbf{e}_1(a) &= 3a^2 - K - c_a(2\mathbf{e}_1 + \dot{u} - 4a)\dot{u} - \tfrac{1}{3}(1+3c_\theta)\theta^2 + 3(1-2c_\sigma)\sigma_+^2\\ &+ \tfrac{1}{2}\mathbf{e}_0(\phi) + \tfrac{1}{2}\mathbf{e}_1(\phi) + a_1e^{-2k\phi},\end{aligned} \tag{7.7.8}$$

$$\begin{aligned}0 &= -36(1-2c\sigma)a\sigma_+ + 4(1+3c_\theta)\mathbf{e}_1(\theta) + 12(1-2c_\sigma)\mathbf{e}_1(\sigma_+) + 6\mathbf{e}_0(\phi)\mathbf{e}_1(\phi)\\ &+ \big(4a_3\frac{\dot{u}\theta}{\sigma_+} + 72a_3a - 23a_3\dot{u} - 4(3a_2 - a_3)k\mathbf{e}_1(\phi)\big)e^{-k\phi}.\end{aligned} \tag{7.7.9}$$

Klein-Gordon equation

$$\mathbf{e}_0(\mathbf{e}_0(\phi)) = -\theta\mathbf{e}_0(\phi) + \big(\mathbf{e}_1 + \dot{u} - 2a\big)\mathbf{e}_1(\phi) + 2ka_1e^{-2k\phi} + (a_2\theta + a_3\sigma_+)ke^{-k\phi}. \quad (7.7.10)$$

The commutator

$$\mathbf{e}_0(\mathbf{e}_1(\phi)) = \mathbf{e}_1(\mathbf{e}_0(\phi)) - \frac{1}{3}(\theta - 6\sigma_+)\mathbf{e}_1(\phi) + \dot{u}\mathbf{e}_0(\phi). \quad (7.7.11)$$

Chapter 8

The Kantowski-Sachs Models

The Kantowski-Sachs models [33] are spatially homogeneous spherically symmetric models (that have 4 Killing vectors, the fourth being ∂_x). We shall consider the special co-moving aether case. The metric that mentioned earlier in [27] simplifies to

$$ds^2 = -N(t)^2 dt^2 + ({e_1}^1(t))^{-2} dx^2 + ({e_2}^2(t))^{-2}(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{8.0.1}$$

(i.e., N, ${e_1}^1$ and ${e_2}^2$ are now independent of x). The spatial derivative terms $\mathbf{e}_1(\ )$ vanish and as a result $a = 0 = \dot{u}$. Since $\dot{u} = 0$, N is a positive function of t which under a time rescaling can be set to one. We assume here that the aether field is invariant under the same symmetries as the metric and therefore is aligned with the symmetry adapted time coordinate. The evolution equations for the Kantowski-Sachs metric for an Einstein-Aether spherically symmetric cosmology, in the presence of a scalar field, are:

$$\mathbf{e}_0(e_1{}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+)e_1{}^1, \tag{8.0.2a}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K, \tag{8.0.2b}$$

$$\mathbf{e}_0(\theta) = -\tfrac{1}{3}\theta^2 - 6\left(\frac{1-2c_\sigma}{1+3c_\theta}\right)\sigma_+^2 + \frac{1}{1+3c_\theta}\Bigg(-\mathbf{e}_0(\phi)^2 + a_1 e^{-2k\phi}$$
$$+ \tfrac{3}{2}a_2 k e^{-k\phi}\mathbf{e}_0(\phi) + \tfrac{3}{2}a_3\sigma_+ e^{-k\phi}\Bigg), \tag{8.0.2c}$$

$$\mathbf{e}_0(\sigma_+) = -\theta\sigma_+ + \frac{1}{6(1-2c_\sigma)}\big(a_3\theta e^{-k\phi} - a_3 k\mathbf{e}_0(\phi)e^{-k\phi} - 2K\big), \tag{8.0.2d}$$

$$\mathbf{e}_0(\mathbf{e}_0(\phi)) = -\theta\mathbf{e}_0(\phi) + 2ka_1e^{-2k\phi} + (a_2\theta + a_3\sigma_+)ke^{-k\phi}, \tag{8.0.2e}$$

with the following constraint:

$$K + \tfrac{1}{3}(1+3c_\theta)\theta^2 = 3(1-2c_\sigma)\sigma_+^2 + \tfrac{1}{2}\mathbf{e}_0^2(\phi) + a_1e^{-2k\phi}, \tag{8.0.3}$$

where c_σ and c_θ are parameters (see definition of the c_i parameters in section (7.2) from chapter (7)) . Note that all evolution and constraint equations are derived in [27].

8.1 Normalized Variables

We introduce the normalized variables (which are bounded for $1 - 2c_\sigma \geq 0$, note that we do not use the β-normalized variable as in [27, 43] for convenience here):

$$x = \frac{\mathbf{e}_0(\phi)}{\sqrt{2}D}, \quad y = \frac{\sqrt{3}\sigma_+}{D}, \quad z = \frac{\sqrt{K}}{D}, \quad Q = \frac{\theta}{\sqrt{3}D}, \quad W = \frac{e^{-k\phi}}{D}, \tag{8.1.1}$$

where

$$D = \sqrt{K + \frac{\theta^2}{3}}, \tag{8.1.2}$$

and the new time variable $f' \equiv \frac{1}{D}e_0(f)$. To further simplify our model we defined $c^2 \equiv (1-2c_\sigma) = 1-3(c_1+c_3) \geq 0$. Since the evolution equations are invariant under the transformation $y \to -y$ and $c \to -c$, without loss of generality we can assume $c > 0$. Thus, it reduce to full 5 dimensional (5D) system:

$$\begin{aligned} x' = & -\frac{2\sqrt{3}}{3}Qx + \sqrt{2}ka_1W^2 + \frac{\sqrt{6}}{2}kW\left(a_2Q + \frac{a_3}{3}y\right) + \frac{\sqrt{3}}{3}xz^2y \\ & - \frac{\sqrt{3}xQ}{3(1+3c_\theta)}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{\sqrt{3}}{2}a_3yW + \frac{3\sqrt{2}}{2}a_2kWx\right], \end{aligned} \tag{8.1.3a}$$

$$\begin{aligned} y' = & -\sqrt{3}Qy + \frac{\sqrt{3}}{3}yz^2(Q+y) + \frac{\sqrt{3}}{3}yQ^3 \\ & + \frac{\sqrt{3}}{6c^2}\left[-2z^2 + \sqrt{3}a_3QW - \sqrt{2}a_3kWx\right] \\ & - \frac{\sqrt{3}yQ}{3(1+3c_\theta)}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{\sqrt{3}}{2}a_3yW + \frac{3\sqrt{2}}{2}a_2kWx\right], \end{aligned} \tag{8.1.3b}$$

$$\begin{aligned} z' = & \frac{-\sqrt{3}zQ}{3}\left[Qy + \frac{1}{1+3c_\theta}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{\sqrt{3}}{2}a_3yW \right.\right. \\ & \left.\left. + \frac{3\sqrt{2}}{2}a_2kWx\right]\right], \end{aligned} \tag{8.1.3c}$$

$$\begin{aligned} Q' = & \frac{\sqrt{3}z^2}{3}\left[Qy + \frac{1}{1+3c_\theta}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{\sqrt{3}}{2}a_3yW \right.\right. \\ & \left.\left. + \frac{3\sqrt{2}}{2}a_2kWx\right]\right], \end{aligned} \tag{8.1.3d}$$

$$\begin{aligned} W' = & W\left[-\sqrt{2}kx + \frac{\sqrt{3}}{3}(Q+y) - \frac{\sqrt{3}}{3}Q^2y \right. \\ & \left. - \frac{\sqrt{3}Q}{3(1+3c_\theta)}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{3\sqrt{2}}{2}a_2kWx + \frac{\sqrt{3}}{2}a_3yW\right]\right]. \end{aligned} \tag{8.1.3e}$$

The variables (8.1.3) are related through the following constraints

$$-3c_\theta Q^2 + x^2 + c^2y^2 + a_1W^2 = 1, \tag{8.1.4a}$$

$$Q^2 + z^2 = 1, \tag{8.1.4b}$$

which are preserved by the 5D system in equations (8.1.3). From the equations (8.1.4) it follows that Q and z are bounded in the intervals $Q \in [-1,1], z \in [0,1]$ (for expanding universes $Q \geq 0$). However, since c^2 is not necessarily non-negative it follows that x, y and W are unbounded, unless $c^2 > 0$. Thus, x, y, W is bounded when $c^2 > 0$ in the intervals $x \in [-1,1]\,, \quad y \in \left[-\frac{1}{|c|}, \frac{1}{|c|}\right]$ and $W \in \left[0, \frac{1}{\sqrt{a_1}}\right]$ (since $W > 0$). Clearly, the case $c = 0$ is not included here which is the GR case since the equation for y' is not valid in that case. The restriction in the second equation of (8.1.4) allows the elimination of the variable z globally. This leads to the following 4-dimensional dynamical system:

$$x' = -\frac{2\sqrt{3}}{3}Qx + \sqrt{2}ka_1W^2 + \frac{\sqrt{6}}{2}kW\left(a_2Q + \frac{a_3}{3}y\right) + \frac{\sqrt{3}}{3}xy(1-Q^2)$$
$$- \frac{\sqrt{3}xQ}{3(1+3c_\theta)}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{\sqrt{3}}{2}a_3yW + \frac{3\sqrt{2}}{2}a_2kWx\right], \tag{8.1.5a}$$

$$y' = -\frac{2\sqrt{3}}{3}Qy + \frac{\sqrt{3}}{3c^2}(1-Q^2)(c^2y^2-1) + \frac{a_3W}{2c^2}\left(Q - \frac{\sqrt{6}}{3}kx\right)$$
$$- \frac{\sqrt{3}yQ}{3(1+3c_\theta)}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{\sqrt{3}}{2}a_3yW + \frac{3\sqrt{2}}{2}a_2kWx\right], \tag{8.1.5b}$$

$$Q' = \frac{\sqrt{3}(1-Q^2)}{3}\left[Qy + \frac{1}{1+3c_\theta}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{\sqrt{3}}{2}a_3yW\right.\right.$$
$$\left.\left. + \frac{3\sqrt{2}}{2}a_2kWx\right]\right], \tag{8.1.5c}$$

$$W' = W\left[-\sqrt{2}kx + \frac{\sqrt{3}}{3}(Q+y) - \frac{\sqrt{3}}{3}Q^2y\right.$$
$$\left. - \frac{\sqrt{3}Q}{3(1+3c_\theta)}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{3\sqrt{2}}{2}a_2kWx + \frac{\sqrt{3}}{2}a_3yW\right]\right], \tag{8.1.5d}$$

subject to the following constraint

$$-3c_\theta Q^2 + x^2 + c^2y^2 + a_1W^2 = 1. \tag{8.1.6}$$

Using the normalized variables in (8.1.1), then the deceleration parameter equation is given by

$$q \equiv -\frac{1}{(1+3c_\theta)Q^2}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{3\sqrt{2}a_2}{2}kWx + \frac{\sqrt{3}}{2}a_3yW\right]. \tag{8.1.7}$$

We shall study the general case in the future work. For this part of the thesis, we consider the following special case.

8.2 Special Case

Let us assume

$$3c_\theta \equiv c_1 + 3c_2 + c_3 = 0$$

and assume $a_3 = 0$ [7, 21]. Then the system (8.1.5) becomes

$$x' = \sqrt{2}ka_1W^2 + \frac{\sqrt{6}}{2}ka_2WQ + \frac{\sqrt{3}}{3}xy(1-Q^2)$$
$$-\frac{\sqrt{3}xQ}{3}\left[2 - 2c^2y^2 - 2x^2 + a_1W^2 + \frac{3\sqrt{2}}{2}a_2kWx\right], \tag{8.2.1a}$$

$$y' = -\frac{\sqrt{3}yQ}{3}\left[2 - 2c^2y^2 - 2x^2 + a_1W^2 + \frac{3\sqrt{2}}{2}a_2kWx\right]$$
$$+\frac{\sqrt{3}}{3c^2}(1-Q^2)\left(c^2y^2 - 1\right), \tag{8.2.1b}$$

$$Q' = \frac{\sqrt{3}(1-Q^2)}{3}\left[Qy - 2c^2y^2 - 2x^2 + a_1W^2 + \frac{3\sqrt{2}}{2}a_2kWx\right], \tag{8.2.1c}$$

$$W' = W\left[-\sqrt{2}kx + \frac{\sqrt{3}}{3}(Q+y) - \frac{\sqrt{3}}{3}Q^2y\right.$$
$$\left.-\frac{\sqrt{3}Q}{3}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{3\sqrt{2}}{2}a_2kWx\right]\right], \tag{8.2.1d}$$

with the constraint

$$x^2 + c^2y^2 + a_1W^2 = 1, \tag{8.2.2}$$

which is preserved by the 4D system in equations (8.2.1). The declaration equation for the 4D system becomes

$$q \equiv -\frac{1}{Q^2}\left[-2c^2y^2 - 2x^2 + a_1W^2 + \frac{3\sqrt{2}a_2}{2}kWx\right]. \tag{8.2.3}$$

8.2.1 Analysis of the Special Case Using W Substitution:

Solving the constraint (8.2.2) for W leads to

$$W = \frac{\sqrt{1 - c^2y^2 - x^2}}{\sqrt{a_1}}. \tag{8.2.4}$$

Substituting W from (8.2.4) into the system (8.2.1) leads to the following 3D system

$$x' = (1 - c^2y^2 - x^2)(\sqrt{2}k - \sqrt{3}xQ) + \frac{\sqrt{3}}{3}xy(1 - Q^2) + \frac{\sqrt{6}}{2}\frac{a_2kQ\sqrt{1 - x^2 - c^2y^2}(1 - x^2)}{\sqrt{a_1}}, \tag{8.2.5a}$$

$$y' = \frac{\sqrt{3}}{3c^2}(1 - Q^2)(c^2y^2 - 1) - \sqrt{3}yQ(1 - c^2y^2 - x^2) - \frac{\sqrt{6}}{2}\frac{a_2kxQy\sqrt{1 - x^2 - c^2y^2}}{\sqrt{a_1}}, \tag{8.2.5b}$$

$$Q' = \frac{\sqrt{3}(1 - Q^2)}{3}\left[Qy - 3c^2y^2 - 3x^2 + 1 + \frac{3\sqrt{2}}{2}\frac{a_2kx\sqrt{1 - c^2y^2 - x^2}}{\sqrt{a_1}}\right]. \tag{8.2.5c}$$

Equilibrium Points

In table (8.1, 8.2), we present the equilibrium points of the system (8.2.5); their existence conditions and their acceleration parameter (q) value for each one.

Pt	x	y	Q	W	Existence	q
P_1	0	$\frac{1}{c}$	$2c$	0	$c \leq \frac{1}{2}$	$\frac{1}{2c^2}$
P_2	0	$-\frac{1}{c}$	$-2c$	0	$c \leq \frac{1}{2}$	$\frac{1}{2c^2}$
P_3^*	$\sqrt{1-c^2y^{*2}}$	y^*	1	0	$-\frac{1}{c} < y^* < \frac{1}{c};\ c > 0$	2
P_4^*	$-\sqrt{1-c^2y^{*2}}$	y^*	1	0	$-\frac{1}{c} < y^* < \frac{1}{c};\ c > 0$	2
P_5^*	$\sqrt{1-c^2y^{*2}}$	y^*	-1	0	$-\frac{1}{c} < y^* < \frac{1}{c};\ c > 0$	2
P_6^*	$-\sqrt{1-c^2y^{*2}}$	y^*	-1	0	$-\frac{1}{c} < y^* < \frac{1}{c};\ c > 0$	2
P_7	$\frac{\sqrt{3}k(2\sqrt{2}a_1+a_2\sqrt{B})}{3(a_2^2k^2+2a_1)}$	0	1	$\frac{\sqrt{3}(-2a_2k^2+\sqrt{2B})}{3(a_2^2k^2+2a_1)}$	1) $k < \sqrt{\frac{3}{2}}$ 2) $k > \sqrt{\frac{3}{2}}, a_2 < 0$ ${a_2}^2k^2 > \frac{2a_1}{3}(2k^2-3)$	In (8.2.72)
P_8	$\frac{\sqrt{3}k(2\sqrt{2}a_1-a_2\sqrt{B})}{3(a_2^2k^2+2a_1)}$	0	1	$-\frac{\sqrt{3}(2a_2k^2+\sqrt{2B})}{3(a_2^2k^2+2a_1)}$	$a_2 < 0, k > \sqrt{\frac{3}{2}}$ ${a_2}^2k^2 > \frac{2a_1}{3}(2k^2-3)$	In (8.2.73)
P_9	$\frac{-\sqrt{3}k(2\sqrt{2}a_1+a_2\sqrt{B})}{3(a_2^2k^2+2a_1)}$	0	-1	$\frac{\sqrt{3}(2a_2k^2-\sqrt{2B})}{3(a_2^2k^2+2a_1)}$	$k < \sqrt{\frac{3}{2}}, a_2 > 0$ ${a_2}^2k^2 > \frac{2a_1}{3}(2k^2-3)$	In (8.2.74)
P_{10}	$\frac{-\sqrt{3}k(2\sqrt{2}a_1-a_2\sqrt{B})}{3(a_2^2k^2+2a_1)}$	0	-1	$\frac{\sqrt{3}(2a_2k^2+\sqrt{2B})}{3(a_2^2k^2+2a_1)}$	1) $k < \sqrt{\frac{3}{2}}$ 2) $k > \sqrt{\frac{3}{2}}, a_2 > 0$ ${a_2}^2k^2 > \frac{2a_1}{3}(2k^2-3)$	In (8.2.75)

Table 8.1: Equilibrium points ($P_{1,2}, P^*_{3,4,5,6}, P_{7,8,9,10}$) of the system (8.2.5); their existence conditions and their deceleration value (q). We use the notation $B = 3(a_2^2k^2 + 2a_1) - 4a_1k^2$. y^* is an arbitrary parameter and hence the curve $P^*_{3,4,5,6}$ represent lines of equilibrium points ($y^* = 0$ is a special equilibrium point in the curves)

Pt	x	y	Q	W	Existence	q
P_{11}	$\frac{\sqrt{6}}{2k}$	$\frac{\sqrt{-B}}{2kc\sqrt{a_1}}$	1	$-\frac{\sqrt{3}a_2}{2a_1}$	$a_2 < 0, k > \sqrt{\frac{3}{2}}$ $a_2{}^2k^2 < \frac{2a_1}{3}(2k^2-3)$	2
P_{12}	$\frac{\sqrt{6}}{2k}$	$-\frac{\sqrt{-B}}{2kc\sqrt{a_1}}$	1	$-\frac{\sqrt{3}a_2}{2a_1}$	$a_2 < 0, k > \sqrt{\frac{3}{2}}$ $a_2{}^2k^2 < \frac{2a_1}{3}(2k^2-3)$	2
P_{13}	$-\frac{\sqrt{6}}{2k}$	$\frac{\sqrt{-B}}{2kc\sqrt{a_1}}$	-1	$\frac{\sqrt{3}a_2}{2a_1}$	$a_2 > 0, k > \sqrt{\frac{3}{2}}$ $a_2{}^2k^2 < \frac{2a_1}{3}(2k^2-3)$	2
P_{14}	$-\frac{\sqrt{6}}{2k}$	$-\frac{\sqrt{-B}}{2kc\sqrt{a_1}}$	-1	$\frac{\sqrt{3}a_2}{2a_1}$	$a_2 > 0, k > \sqrt{\frac{3}{2}}$ $a_2{}^2k^2 < \frac{2a_1}{3}(2k^2-3)$	2

Table 8.2: Equilibrium points ($P_{11,12,13,14}$) of the system (8.2.5); their existence conditions and their deceleration value (q). We use the notation $B = 3(a_2^2k^2 + 2a_1) - 4a_1k^2$.

Qualitative Analysis

Let us present the analysis for the existence and stability conditions of each equilibrium points in terms of the parameters a_1, k, c and a_2.

Stability of Equilibrium Point P_1

The equilibrium point P_1 exists when $c > 0$. Evaluating the linearization of the system (8.2.5) at P_1 leads to undefined terms. Using the implicit function theorem from vector calculus, we have the constraint surface is

$$f = x^2 + c^2y^2 + a_1W^2 - 1, \tag{8.2.6}$$

Then the

$$\bigtriangledown f\big|_{P_1} = (0, 2, 0). \tag{8.2.7}$$

Therefore, we can not globally use W substitution to analyze the stability of P_1. Thus, we can reduce the 4D system (8.2.1) to a 3D system in a neighbourhood of P_1 by eliminating the variable y. Subsituating y

$$y = \frac{\sqrt{1 - x^2 - a_1 W^2}}{c}, \tag{8.2.8}$$

into the 4D system in (8.2.1) leads to the 3D system

$$x' = a_1 W^2(\sqrt{2}k - \sqrt{3}xQ) + \frac{\sqrt{3}}{3}x\frac{\sqrt{1 - x^2 - a_1 W^2}}{a_1}(1 - Q^2) + \frac{\sqrt{6}}{2}a_2 kWQ(1 - x^2), \tag{8.2.9a}$$

$$Q' = \frac{\sqrt{3}(1 - Q^2)}{3}\left[Q\frac{\sqrt{1 - x^2 - a_1 W^2}}{a_1} - 2 + 3a_1 W^2 + \frac{3\sqrt{2}}{2}a_2 kxW\right], \tag{8.2.9b}$$

$$W' = W\left[-\sqrt{2}kx + \frac{\sqrt{3}}{3}(1 - Q^2)\frac{\sqrt{1 - x^2 - a_1 W^2}}{a_1} - \frac{\sqrt{3}}{3}Q\left[-3 + 3a_1 W^2 + \frac{3\sqrt{2}}{2}a_2 kxW\right]\right] \tag{8.2.9c}$$

Evaluating the linearization of the system (8.2.9) at P_1 leads to three eigenvalues

$$\lambda_1 = \frac{\sqrt{3}(2c^2 + 1)}{3c}, \quad \lambda_2 = -\frac{\sqrt{3}(4c^2 - 1)}{3c}, \quad \lambda_3 = -\frac{\sqrt{3}(4c^2 - 1)}{3c}. \tag{8.2.10}$$

Therefore, P_1 is a source if $c < \frac{1}{2}$; a saddle if $c > \frac{1}{2}$. So, P_1 is never a sink.

Stability of Equilibrium Point P_2

The equilibrium point P_2 exists when $c > 0$. Evaluating the linearization of the system (8.2.5) at P_2 leads to undefined terms. We know that

$$\nabla f\big|_{P_2} = (0, -2, 0). \tag{8.2.11}$$

Therefore, we can not globally use W substitution to analyze the stability of P_2. Thus, we can reduce the 4D system (8.2.1) to a 3D system in a neighbourhood of P_2

by eliminating the variable y. Subsituating y

$$y = -\frac{\sqrt{1-x^2-a_1W^2}}{c}, \tag{8.2.12}$$

into the 4D system in (8.2.1) leads to the 3D system

$$x' = a_1W^2(\sqrt{2}k - \sqrt{3}xQ) - \frac{\sqrt{3}}{3}x\frac{\sqrt{1-x^2-a_1W^2}}{a_1}(1-Q^2) + \frac{\sqrt{6}}{2}a_2kWQ(1-x^2), \tag{8.2.13a}$$

$$Q' = \frac{\sqrt{3}(1-Q^2)}{3}\left[-Q\frac{\sqrt{1-x^2-a_1W^2}}{a_1} - 2 + 3a_1W^2 + \frac{3\sqrt{2}}{2}a_2kxW\right], \tag{8.2.13b}$$

$$W' = W\left[-\sqrt{2}kx - \frac{\sqrt{3}}{3}(1-Q^2)\frac{\sqrt{1-x^2-a_1W^2}}{a_1} - \frac{\sqrt{3}}{3}Q\left[-3 + 3a_1W^2 + \frac{3\sqrt{2}}{2}a_2kxW\right]\right] \tag{8.2.13c}$$

Evaluating the linearization of the system in (8.2.13) at P_2 leads to the three eigenvalues

$$\lambda_1 = -\frac{\sqrt{3}(2c^2+1)}{3c}, \quad \lambda_2 = \frac{\sqrt{3}(4c^2-1)}{3c}, \quad \lambda_3 = \frac{\sqrt{3}(4c^2-1)}{3c}. \tag{8.2.14}$$

Therefore, P_2 is sink if $c < \frac{1}{2}$.

Stability of Equilibrium Point P_3^*

The line of equilibrium point P_3^* exists when $y^* < \frac{1}{c}$ with $c > 0$. We did similar analysis as we did earlier for $P_{1,2}$: since

$$\nabla f\big|_{P_3^*} = \left(\sqrt{1-c^2y^{*2}}, y^*, 0\right). \tag{8.2.15}$$

Therefore, we can not globally use W substitution to analyze the stability of P_3^*. Thus, we can reduce the 4D system (8.2.1) to a 3D system in a neighbourhood of P_3^*

by eliminating the variable y. Evaluating the lineraization of (8.2.9) at P_3^* leads to three eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{2\sqrt{3}(y^* - 2)}{3}, \quad \lambda_3 = -\sqrt{2}k\sqrt{1 - c^2 y^{*2}} + \sqrt{3}. \tag{8.2.16}$$

Since we have one zero real eigenvalue the equilibrium point P_3^* called a normal hyperbolic and the stability of these line of equilibrium points will be determined by studying the signs of the remaining two eigenvalues. There are two lines of sinks:

- If $c < \frac{1}{2}$; $k > \sqrt{\frac{3}{2}}$; $4c^2 + \frac{3}{2k^2} > 1$ then the part of the line that is a sink is $-\frac{1}{c}\sqrt{1 - \frac{3}{2k^2}} < y^* < \frac{1}{c}\sqrt{1 - \frac{3}{2k^2}}$.
- If $c < \frac{1}{2}$; $k > \sqrt{\frac{3}{2}}$; $4c^2 + \frac{3}{2k^2} < 1$ then the part of the line that is a sink is $\frac{1}{c}\sqrt{1 - \frac{3}{2k^2}} < y^* < 2$.

Stability of Equilibrium Point P_4^*

The line of equilibrium point P_4^* exists when $y^* < \frac{1}{c}$ with $c > 0$. We did similar analysis as we did earlier for $P_{1,2}$: since

$$\nabla f\big|_{P_4^*} = \left(-\sqrt{1 - c^2 y^{*2}}, y^*, 0\right). \tag{8.2.17}$$

Therefore, we can not globally use W substitution to analyze the stability of P_4^*. Thus, we can reduce the 4D system (8.2.1) to a 3D system in a neighbourhood of P_4^* by eliminating the variable y. Evaluating the lineraization of (8.2.13) at P_4^* leads to three eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{2\sqrt{3}(y^* - 2)}{3}, \quad \lambda_3 = \left(\sqrt{2}k\sqrt{1 - c^2 y^{*2}} + \sqrt{3}\right). \tag{8.2.18}$$

Since we have one zero real eigenvalue the equilibrium point P_4^* called a normal hyperbolic and the stability of these line of equilibrium points will be determined by studying the signs of the remaining two eigenvalues. It is a saddle when $y^* > 2$ but it is source when $y^* < 2$. P_4^* is non-hyperbolic for $y^* = 2$. So, P_4^* is never a sink.

Stability of Equilibrium Point P_5^*

The line of equilibrium point P_5^* exists when $y^* < \frac{1}{c}$ with $c > 0$. We did similar analysis as we did earlier for $P_{1,2}$: since

$$\nabla f\Big|_{P_5^*} = \left(\sqrt{1 - c^2 y^{*2}}, y^*, 0\right). \tag{8.2.19}$$

Therefore, we can not globally use W substitution to analyze the stability of P_5^*. Thus, we can reduce the 4D system (8.2.1) to a 3D system in a neighbourhood of P_5^* by eliminating the variable y. Evaluating the lineraization of (8.2.9) at P_5^* leads to three eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{2\sqrt{3}(y^* + 2)}{3}, \quad \lambda_3 = -\left(\sqrt{2}k\sqrt{1 - c^2 y^{*2}} + \sqrt{3}\right). \tag{8.2.20}$$

Since we have one zero real eigenvalue the equilibrium point P_5^* called a normal hyperbolic and the stability of these line of equilibrium points will be determined by studying the signs of the remaining two eigenvalues. There are two line of sinks:

- If $c < \frac{1}{2}$ then the part of the line that is a sink is $-2 < y^* < \frac{1}{c}$.

- If $c > \frac{1}{2}$ then the entire line $-\frac{1}{c} < y^* < \frac{1}{c}$ is a sink.

Stability of Equilibrium Point P_6^*

The equilibrium point P_6^* exists when $y^* < \frac{1}{c}$ with $c > 0$. We did similar analysis as we did earlier for $P_{1,2}$: since

$$\triangledown f\Big|_{P_6^*} = \left(-\sqrt{1-c^2y^{*2}}, y^*, 0\right). \tag{8.2.21}$$

Therefore, we can not globally use W substitution to analyze the stability of P_6^*. Thus, we can reduce the 4D system (8.2.1) to a 3D system in a neighbourhood of P_6^* by eliminating the variable y. Evaluating the lineraization of (8.2.13) at P_6^* leads to three eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{2\sqrt{3}(y^*+2)}{3}, \quad \lambda_3 = \left(\sqrt{2}k\sqrt{1-c^2y^{*2}} - \sqrt{3}\right). \tag{8.2.22}$$

Since we have one zero real eigenvalue the equilibrium point P_6^* called a normal hyperbolic and the stability of these line of equilibrium points will be determined by studying the signs of the remaining two eigenvalues. If $y^* > -2$ then we will have several portions that we have sinks:

1. If $k < \sqrt{\frac{3}{2}}$, we have two cases:

   (a) If $c < \frac{1}{2}$ the only line of sink is $-2 < y^* < \frac{1}{c}$.

   (b) If $c > \frac{1}{2}$ the entire line is sink $-\frac{1}{c} < y^* < \frac{1}{c}$.

2. If $k > \sqrt{\frac{3}{2}}$, we have two cases:

   (a) If $c < \frac{1}{2}; 4c^2 + \frac{3}{2k^2} > 1$, then the only line of sink is $-2 < y^* < \frac{1}{c}\sqrt{1-\frac{3}{2k^2}}$

   (b) If $c < \frac{1}{2}; 4c^2 + \frac{3}{2k^2} < 1$ the sink is in the portion $\frac{1}{c}\sqrt{1-\frac{3}{2k^2}} < y^* < \frac{1}{c}$

(c) If $c > \frac{1}{2}$ there are two line of sinks are $-\frac{1}{c} < y^* < -\frac{1}{c}\sqrt{1-\frac{3}{2k^2}}$; $\frac{1}{c}\sqrt{1-\frac{3}{2k^2}} < y^* < \frac{1}{c}$.

Stability of Equilibrium Point P_7

In order for P_7 to exist we need to find the conditions in the parameters where W, x are real and $W \in \left[0, \frac{1}{\sqrt{a_1}}\right]$.

W, x **are real when** $B > 0$

$$B > 0 \implies 3{a_2}^2k^2 + 6a_1 - 4k^2a_1 > 0$$

$$\Rightarrow 3{a_2}^2k^2 + 2a_1(3 - 2k^2) > 0$$

Thus, there are two cases for $B > 0$:

$$\boxed{B > 0 \Longrightarrow \begin{array}{lllll} 1) & \text{If} \quad k < \sqrt{\dfrac{3}{2}} \Rightarrow & B > 0 & \text{always.} & \\ 2) & \text{If} \quad k > \sqrt{\dfrac{3}{2}} \Rightarrow & B > 0 & \text{if} \quad {a_2}^2 > \dfrac{2a_1}{3k^2}(2k^2 - 3). \end{array}}.$$

$W_7 > 0$: $W_7 > 0 \Rightarrow -2a_2k^2 + \sqrt{2B} > 0$, which can be written of the form

$$-N1 + \sqrt{(N1)^2 - M1}, \tag{8.2.23}$$

where

$$N1 := -2a_2k^2, \quad M1 := 2({a_2}^2k^2 + 2a_1)(2k^2 - 3).$$

So,

$$-N1 + \sqrt{(N1)^2 - M1} > 0,$$

in two cases:

- If $a_2 < 0$ then $W_7 > 0$ always.

- If $a_2 > 0$ then $W_7 > 0$ provided $M1 > 0 \Rightarrow k > \sqrt{\frac{3}{2}}$.

Therefore,

$$\boxed{W_7 > 0 \Longrightarrow \begin{array}{c} 1) \quad \text{If} \quad a_1 > 0, \quad a_2 \quad \text{any value} \Rightarrow W_7 > 0 \quad \text{if} \quad k < \sqrt{\frac{3}{2}}. \\ 2) \quad \text{If} \quad a_1 > 0, \quad a_2 < 0 \Rightarrow W_7 > 0 \quad \text{if} \quad k > \sqrt{\frac{3}{2}} \end{array}}.$$

$W_7 \leq \frac{1}{\sqrt{a_1}}$: which always true when P_7 exists. Therefore, P_7 exists with these conditions:

$$\boxed{P_7 \text{ exists} \Longrightarrow \begin{array}{l} 1) \quad \text{If} \quad k < \sqrt{\frac{3}{2}} \\ 2) \quad \text{If} \quad k > \sqrt{\frac{3}{2}}, \quad a_2 < 0 \quad \text{and} \quad {a_2}^2 > \frac{2a_1}{3k^2}(2k^2 - 3). \end{array}}.$$

Evaluating the linearization martix of the system (8.2.5) at $Q = 1, y = 0$ and $x = x_7$ gives us the following three eigenvalues

$$\lambda_1 = \frac{\sqrt{3}}{3\sqrt{a_1}} \left(-3\sqrt{2}a_2 k x_7 \sqrt{1 - x_7^2} + 2\sqrt{a_1}(3x_7^2 - 1) \right), \tag{8.2.24a}$$

$$\lambda_2 = \frac{\sqrt{3}}{2\sqrt{a_1}} \left(-\sqrt{2}a_2 k x_7 \sqrt{1 - x_7^2} + 2\sqrt{a_1}(x_7^2 - 1) \right), \tag{8.2.24b}$$

$$\lambda_3 = \frac{1}{2\sqrt{a_1}} \left(-3\sqrt{6}a_2 k x_7 \sqrt{1 - x_7^2} + 2\sqrt{3a_1}(3x_7^2 - 1) - 4\sqrt{2a_1} k x_7 \right). \tag{8.2.24c}$$

Note that, by plugging $y = 0, Q = 1$ into the equation of x' in (8.2.5) we obtain the following expression:

$$\frac{\sqrt{6}a_2 k \sqrt{1 - x_7^2}}{2\sqrt{a_1}} = \sqrt{3}x_7 - \sqrt{2}k, \tag{8.2.25}$$

which can simplify our expression for the eigenvalues to

$$\lambda_{1,2} = \sqrt{2}kx_7 - \sqrt{3}, \tag{8.2.26a}$$

$$\lambda_3 = 2\left(\sqrt{2}kx_7 - \frac{\sqrt{3}}{3}\right). \tag{8.2.26b}$$

Sink conditions:

1. $\lambda_{1,2} < 0 :\Rightarrow \sqrt{3}a_2k^2\left[-N2 + \sqrt{(N2)^2 - M2}\right] < 0$, where

$$N2 := \tfrac{1}{a_2k^2}(3(a_2{}^2k^2 + 2a1) - 4k^2a_1),$$

$$M2 := \tfrac{1}{a_2{}^2k^4}((a_2{}^2k^2 + 2a1)(3 - 2k^2)(3a_2{}^2k^2 + 2a_1(3 - 2k^2)).$$

   Thus, $\lambda_{1,2} < 0$ in two cases:

   - If $a_2 < 0$ then $\lambda_{1,2}$ always negative.
   - If $a_2 > 0$ then $\lambda_{1,2} < 0$ provided $M2 > 0 \Rightarrow k < \sqrt{\frac{3}{2}}$.

2. $\lambda_3 < 0 \Rightarrow 2\sqrt{3}a_2k^2\left[-N3 + \sqrt{(N3)^2 - M3}\right] < 0$, where

$$N3 := \tfrac{1}{a_2k^2}(a_2{}^2k^2 + 2a1 - 4k^2a_1),$$

$$M3 := \tfrac{1}{a_2{}^2k^4}((a_2{}^2k^2 + 2a1)(a_2{}^2k^2(1 - 6k^2) + 2a_1(1 - 2k^2)^2).$$

   Thus, $\lambda_3 < 0$ in two cases:

   - If $a_2 < 0 \quad \text{and} \quad a_2{}^2k^2 + 2a_1(1 - 2k^2) < 0 \Rightarrow \lambda_3 < 0$ always.
   - If $a_2 > 0$ and $a_2{}^2k^2 + 2a_1(1 - 2k^2) > 0 \Rightarrow M3 > 0 \Rightarrow a_2{}^2k^2(1 - 6k^2) + 2a_1(1 - 2k^2)^2 > 0$.

Hence, P_7 is a sink if there cases:

- If $a_2 > 0$ and $k < \sqrt{\frac{3}{2}}$; ${a_2}^2k^2 + 2a_1(1-2k^2) > 0$ and ${a_2}^2k^2(1-6k^2) + 2a_1(1-2k^2)^2 > 0$.

- If $a_2 < 0$, $k < \sqrt{\frac{3}{2}}$; and ${a_2}^2k^2 + 2a_1(1-2k^2) < 0$.

- If $a_2 < 0$, $k > \sqrt{\frac{3}{2}}$; and ${a_2}^2 > \frac{2a_1}{3k^2}(2k^2-3)$; ${a_2}^2k^2 + 2a_1(1-2k^2) < 0$.

Stability of Equilibrium Point P_8

In order for P_8 to exist we need to find the conditions in the parameters where W, x are real and $W \in \left[0, \frac{1}{\sqrt{a_1}}\right]$.

W, x **are real when** $B > 0$

$$B > 0 \implies 3{a_2}^2k^2 + 6a_1 - 4k^2a_1 > 0$$

$$\Rightarrow 3{a_2}^2k^2 + 2a_1(3-2k^2) > 0$$

Thus, there are two cases for $B > 0$:

$$\boxed{B > 0 \Longrightarrow \begin{array}{lllll} 1) & \text{If} \quad k < \sqrt{\dfrac{3}{2}} \Rightarrow & B > 0 & \text{always.} & \\ 2) & \text{If} \quad k > \sqrt{\dfrac{3}{2}} \Rightarrow & B > 0 & \text{if} & {a_2}^2 > \dfrac{2a_1}{3k^2}(2k^2-3). \end{array}}.$$

$W_8 > 0$: $\quad W_8 > 0 \Rightarrow 2a_2k^2 + \sqrt{2B} < 0$, which can be written of the form

$$N4 + \sqrt{(N4)^2 - M4} < 0, \tag{8.2.27}$$

where

$$N4 := 2a_2k^2, \quad M4 := 2({a_2}^2k^2 + 2a_1)(2k^2-3).$$

So,

$$N4 + \sqrt{(N4)^2 - M4} < 0$$

$$\boxed{W_8 > 0 \quad \text{if} \quad a_2 < 0 \quad \text{and} \quad k > \sqrt{\tfrac{3}{2}}}$$

.

$W_8 \leq \frac{1}{\sqrt{a_1}}$: which always true when P_8 exists. Therefore, P_8 exists with these conditions:

$$\boxed{P_8 \text{ exists } \Rightarrow a_2 < 0 \quad \text{and} \quad k > \sqrt{\tfrac{3}{2}}}.$$

Evaluating the lineraization of the system (8.2.5) at $Q = 1, y = 0$ and $x = x_8$ gives us the following three eigenvalues

$$\lambda_1 = \frac{\sqrt{3}}{3\sqrt{a_1}}\left(-3\sqrt{2}a_2kx_8\sqrt{1-x_8^2} + 2\sqrt{a_1}(3x_8^2 - 1)\right), \tag{8.2.28a}$$

$$\lambda_2 = \frac{\sqrt{3}}{2\sqrt{a_1}}\left(-\sqrt{2}a_2kx_8\sqrt{1-x_8^2} + 2\sqrt{a_1}(x_8^2 - 1)\right), \tag{8.2.28b}$$

$$\lambda_3 = \frac{1}{2\sqrt{a_1}}\left(-3\sqrt{6}a_2kx_8\sqrt{1-x_8^2} + 2\sqrt{3a_1}(3x_8^2 - 1) - 4\sqrt{2a_1}kx_8\right). \tag{8.2.28c}$$

Note that, by plugging $y = 0, Q = 1$ into the equation of x' in (8.2.5) we obtain the following expression:

$$\frac{\sqrt{6}a_2k\sqrt{1-x_8^2}}{2\sqrt{a_1}} = \sqrt{3}x_8 - \sqrt{2}k, \tag{8.2.29}$$

which can simplify our expression for the eigenvalues to

$$\lambda_{1,2} = \sqrt{2}kx_8 - \sqrt{3}, \tag{8.2.30a}$$

$$\lambda_3 = 2\left(\sqrt{2}kx_8 - \frac{\sqrt{3}}{3}\right). \tag{8.2.30b}$$

Sink conditions:

1. $\lambda_{1,2} < 0 :\Rightarrow -\sqrt{3}a_2k^2\left[N5 + \sqrt{(N5)^2 - M5}\right] < 0$, where

$$N5 := \tfrac{1}{a_2k^2}(3({a_2}^2k^2 + 2a1) - 4k^2a_1),$$

$$M5 := \tfrac{1}{{a_2}^2k^4}(({a_2}^2k^2 + 2a1)(3 - 2k^2)(3{a_2}^2k^2 + 2a_1(3 - 2k^2)).$$

Thus, $\lambda_{1,2} < 0$ if

$$a_2 < 0, \quad k > \sqrt{\frac{3}{2}}. \tag{8.2.31}$$

2. $\lambda_3 < 0 \Rightarrow \sqrt{3} - 2\sqrt{3}a_2k^2\left[N6 + \sqrt{(N6)^2 - M6}\right] < 0$, where

$$N6 := \tfrac{1}{a_2k^2}({a_2}^2k^2 + 2a1 - 4k^2a_1),$$

$$M6 := \tfrac{1}{{a_2}^2k^4}(({a_2}^2k^2 + 2a1)({a_2}^2k^2(1 - 6k^2) + 2a_1(1 - 2k^2)^2).$$

Thus, $\lambda_3 < 0$ if

$$a_2 < 0, k > \sqrt{\frac{3}{2}}; {a_2}^2 > \frac{2a_1}{k^2}(2k^2 - 1) \quad \text{and} 2a_1(1 - 2k^2)^2 > {a_2}^2k^2(6k^2 - 1). \tag{8.2.32}$$

Hence, P_8 is a sink if these following conditions hold

$$a_2 < 0, \quad k > \sqrt{\frac{3}{2}}; \quad {a_2}^2 > \frac{2a_1}{k^2}(2k^2 - 1) \quad \text{and} \quad 2a_1(1 - 2k^2)^2 > {a_2}^2k^2(6k^2 - 1).$$

Stability of Equilibrium Point P_9

In order for P_9 to exist we need to find the conditions in the parameters where W, x are real and $W \in \left[0, \frac{1}{\sqrt{a_1}}\right]$.

W, x **are real when** $B > 0$

$$B > 0 \implies 3{a_2}^2k^2 + 6a_1 - 4k^2a_1 > 0$$

$$\Rightarrow 3{a_2}^2k^2 + 2a_1(3 - 2k^2) > 0$$

Thus, there are two cases for $B > 0$:

$$\boxed{B > 0 \implies \begin{array}{l} 1) \quad \text{If} \quad k < \sqrt{\dfrac{3}{2}} \Rightarrow \quad B > 0 \quad \text{always.} \\ 2) \quad \text{If} \quad k > \sqrt{\dfrac{3}{2}} \Rightarrow \quad B > 0 \quad \text{if} \quad {a_2}^2 > \dfrac{2a_1}{3k^2}(2k^2 - 3). \end{array}}.$$

$W_9 > 0$: $W_9 > 0 \Rightarrow 2a_2k^2 - \sqrt{2B} < 0$, which can be written of the form

$$N7 + \sqrt{(N7)^2 - M7} < 0, \tag{8.2.33}$$

where

$$N7 := 2a_2k^2, \quad M7 := 2({a_2}^2k^2 + 2a_1)(2k^2 - 3).$$

So,

$$N7 - \sqrt{(N7)^2 - M7} > 0$$

$$\boxed{W_9 > 0 \quad \text{if} \quad a_2 > 0 \quad \text{and} \quad k > \sqrt{\tfrac{3}{2}}}$$

.

$W_9 \leq \frac{1}{\sqrt{a_1}}$: which always true when P_9 exists. Therefore, P_9 exists with these conditions:

$$\boxed{P_9 \text{ exists } \Rightarrow a_2 > 0; k > \sqrt{\tfrac{3}{2}} \quad \text{and} \quad {a_2}^2 > \tfrac{2a_1}{3k^2}(2k^2 - 3)}.$$

Evaluating the lineraization of the system (8.2.5) at $Q=-1, y=0$ and $x=x_9$ gives us the following three eigenvalues

$$\lambda_1 = \frac{\sqrt{3}}{3\sqrt{a_1}}\left(3\sqrt{2}a_2kx_9\sqrt{1-x_9^2}+2\sqrt{a_1}(1-3x_9^2)\right), \tag{8.2.34a}$$

$$\lambda_2 = \frac{\sqrt{3}}{2\sqrt{a_1}}\left(\sqrt{2}a_2kx_9\sqrt{1-x_9^2}+2\sqrt{a_1}(1-x_9^2)\right), \tag{8.2.34b}$$

$$\lambda_3 = \frac{1}{2\sqrt{a_1}}\left(3\sqrt{6}a_2kx_9\sqrt{1-x_9^2}+2\sqrt{3a_1}(1-3x_9^2)-4\sqrt{2a_1}kx_9\right). \tag{8.2.34c}$$

Note that, by plugging $y=0, Q=-1$ into the equation of x' in (8.2.5) we obtain the following expression:

$$\frac{\sqrt{6}a_2k\sqrt{1-x_9^2}}{2\sqrt{a_1}} = \sqrt{3}x_9-\sqrt{2}k, \tag{8.2.35}$$

which can simplify our expression for the eigenvalues to

$$\lambda_{1,2} = \sqrt{2}kx_9+\sqrt{3}, \tag{8.2.36a}$$

$$\lambda_3 = 2\left(\sqrt{2}kx_9+\frac{\sqrt{3}}{3}\right). \tag{8.2.36b}$$

Sink conditions:

1. $\lambda_{1,2}<0 :\Rightarrow -\sqrt{3}a_2k^2\left[N8+\sqrt{(N8)^2-M8}\right]<0$, where

$$N8 := \tfrac{1}{a_2k^2}(3({a_2}^2k^2+2a1)-4k^2a_1),$$

$$M8 := \tfrac{1}{{a_2}^2k^4}(({a_2}^2k^2+2a1)(3-2k^2)(3{a_2}^2k^2+2a_1(3-2k^2)).$$

   Thus, $\lambda_{1,2}$ always positive since P_9 exists only when $a_2>0$..

2. $\lambda_3<0 \Rightarrow \sqrt{3}-2\sqrt{3}a_2k^2\left[N9+\sqrt{(N9)^2-M9}\right]<0$, where

$$N9 := \tfrac{1}{a_2k^2}({a_2}^2k^2+2a1+4k^2a_1),$$

$$M9 := \tfrac{1}{{a_2}^2k^4}(({a_2}^2k^2+2a1)({a_2}^2k^2(1-6k^2)+2a_1(1-2k^2)^2).$$

Thus, $\lambda_3 > 0$ always positive. Hence, P_9 is a source if

$$a_2 > 0, k > \sqrt{\frac{3}{2}}; {a_2}^2 > \frac{2a_1}{3k^2}(2k^2 - 3).$$

Thus, P_9 is not a sink.

Stability of Equilibrium Point P_{10}

In order for P_{10} to exist we need to find the conditions in the parameters where W, x are real and $W \in \left[0, \frac{1}{\sqrt{a_1}}\right]$.

W, x **are real when** $B > 0$

$$\begin{aligned} B > 0 \ &\Longrightarrow 3{a_2}^2k^2 + 6a_1 - 4k^2a_1 > 0 \\ &\Rightarrow 3{a_2}^2k^2 + 2a_1(3 - 2k^2) > 0 \end{aligned}$$

Thus, there are two cases for $B > 0$:

$$\boxed{B > 0 \Longrightarrow \begin{array}{llll} 1) & \text{If} \quad k < \sqrt{\frac{3}{2}} \Rightarrow & B > 0 & \text{always.} \\ 2) & \text{If} \quad k > \sqrt{\frac{3}{2}} \Rightarrow & B > 0 & \text{if} \quad {a_2}^2 > \frac{2a_1}{3k^2}(2k^2 - 3). \end{array}}.$$

$W_{10} > 0:$ $W_{10} > 0 \Rightarrow 2a_2k^2 + \sqrt{2B} > 0$, which can be written of the form

$$N10 + \sqrt{(N10)^2 - M10}, \tag{8.2.37}$$

where

$$N10 := 2a_2k^2, \quad M10 := 2({a_2}^2k^2 + 2a_1)(2k^2 - 3).$$

So,

$$N1 + \sqrt{(N1)^2 - M1} > 0 \quad \text{if}$$

- If $a_2 > 0$ then $W_{10} > 0$ always positive.

- If $a_2 < 0$ then $W_{10} > 0$ provided $M10 > 0 \Rightarrow k < \sqrt{\frac{3}{2}}$.

Therefore,

$$W_{10} > 0 \Longrightarrow \begin{array}{l} 1) \quad \text{If} \quad a_1 > 0, \quad a_2 \quad \text{any value} \Rightarrow W_{10} > 0 \quad \text{if} \quad k < \sqrt{\dfrac{3}{2}}. \\ 2) \quad \text{If} \quad a_1 > 0, \quad a_2 > 0 \Rightarrow W_{10} > 0 \quad \text{if} \quad k > \sqrt{\dfrac{3}{2}} \end{array}.$$

$W_{10} \leq \frac{1}{\sqrt{a_1}}$: which always true when P_{10} exists. Therefore, P_{10} exists with these conditions:

$$P_{10} \text{ exists} \Longrightarrow \begin{array}{l} 1) \quad \text{If} \quad k < \sqrt{\dfrac{3}{2}} \\ 2) \quad \text{If} \quad k > \sqrt{\dfrac{3}{2}}, a_2 > 0 \quad \text{and} \quad {a_2}^2 > \dfrac{2a_1}{3k^2}(2k^2 - 3). \end{array}.$$

Evaluating the lineraization of the system (8.2.5) at $Q = -1, y = 0$ and $x = x_{10}$ gives us the following three eigenvalues

$$\lambda_1 = \frac{\sqrt{3}}{3\sqrt{a_1}} \left(3\sqrt{2}a_2 k x_{10}\sqrt{1 - x_{10}^2} + 2\sqrt{a_1}(1 - 3x_{10}^2) \right), \tag{8.2.38a}$$

$$\lambda_2 = \frac{\sqrt{3}}{2\sqrt{a_1}} \left(\sqrt{2}a_2 k x_{10}\sqrt{1 - x_{10}^2} + 2\sqrt{a_1}(1 - x_{10}^2) \right), \tag{8.2.38b}$$

$$\lambda_3 = \frac{1}{2\sqrt{a_1}} \left(3\sqrt{6}a_2 k x_{10}\sqrt{1 - x_{10}^2} + 2\sqrt{3a_1}(1 - 3x_{10}^2) - 4\sqrt{2a_1}k x_{10} \right). \tag{8.2.38c}$$

Note that, by plugging $y = 0, Q = -1$ into the equation of x' in (8.2.5) we obtain the following expression:

$$\frac{\sqrt{6}a_2 k\sqrt{1 - x_{10}^2}}{2\sqrt{a_1}} = \sqrt{3}x_{10} - \sqrt{2}k, \tag{8.2.39}$$

which can simplify our expression for the eigenvalues to

$$\lambda_{1,2} = \sqrt{2}kx_{10} + \sqrt{3}, \quad (8.2.40a)$$

$$\lambda_3 = 2\left(\sqrt{2}kx_{10} + \frac{\sqrt{3}}{3}\right). \quad (8.2.40b)$$

Sink conditions:

1. $\lambda_{1,2} < 0 :\Rightarrow -\sqrt{3}a_2k^2\left[N11 + \sqrt{(N11)^2 - M11}\right] < 0$, where

$$N11 := \tfrac{1}{a_2k^2}(3({a_2}^2k^2 + 2a1) - 4k^2a_1),$$

$$M11 := \tfrac{1}{{a_2}^2k^4}(({a_2}^2k^2 + 2a1)(3 - 2k^2)(3{a_2}^2k^2 + 2a_1(3 - 2k^2)).$$

Thus, $\lambda_{1,2} > 0$ always.

2. $\lambda_3 < 0 \Rightarrow \sqrt{3} - 2\sqrt{3}a_2k^2\left[N12 + \sqrt{(N12)^2 - M12}\right] < 0$, where

$$N12 := \tfrac{1}{a_2k^2}({a_2}^2k^2 + 2a1 + 4k^2a_1),$$

$$M12 := \tfrac{1}{{a_2}^2k^4}(({a_2}^2k^2 + 2a1)({a_2}^2k^2(1 - 6k^2) + 2a_1(1 - 2k^2)^2).$$

Thus, $\lambda_3 > 0$ if $a_2 > 0, k > \sqrt{\frac{3}{2}}$, but $a_2 < 0, k < \sqrt{\frac{3}{2}}$, then $\lambda_3 < 0$.

Hence, P_{10} is a source if $a_2 > 0, k > \sqrt{\frac{3}{2}}$, but if $a_2 < 0$; and $k < \sqrt{\frac{3}{2}}$, then P_{10} is saddle. Thus, P_{10} is not a sink.

Stability of Equilibrium Point P_{11}

Recall the 3D system

$$x' = (1 - c^2y^2 - x^2)(\sqrt{2}k - \sqrt{3}xQ) + \frac{\sqrt{3}}{3}xy(1 - Q^2) + \frac{\sqrt{6}}{2}\frac{a_2kQ\sqrt{1 - x^2 - c^2y^2}(1 - x^2)}{\sqrt{a_1}}, \tag{8.2.41a}$$

$$y' = \frac{\sqrt{3}}{3c^2}(1 - Q^2)(c^2y^2 - 1) - \sqrt{3}yQ(1 - c^2y^2 - x^2) - \frac{\sqrt{6}}{2}\frac{a_2kxQy\sqrt{1 - x^2 - c^2y^2}}{\sqrt{a_1}}, \tag{8.2.41b}$$

$$Q' = \frac{\sqrt{3}(1 - Q^2)}{3}\left[Qy - 3c^2y^2 - 3x^2 + 1 + \frac{3\sqrt{2}}{2}\frac{a_2kx\sqrt{1 - c^2y^2 - x^2}}{\sqrt{a_1}}\right], \tag{8.2.41c}$$

which has the equilibrium point P_{11};

$$x_{11} = \frac{\sqrt{6}}{2k}, \quad y_{11} = \frac{\sqrt{4k^2a_1 - 3(a_2{}^2k^2 + 2a_1)}}{2kc\sqrt{a_1}}, \quad Q_{11} = 1 \quad W_{11} = -\frac{\sqrt{3}a_2}{2a_1}. \tag{8.2.42}$$

P_{11} exists when $a_2 < 0$ (since $W_{11} > 0$); $k > \sqrt{\frac{3}{2}}$ and $a_2{}^2k^2 < \frac{2a_1}{3}(2k^2 - 3)$ (because y_{11} is real). Evaluating the linearization of the system (8.2.41) at P_{11} leads to three eigenvalues:

$$\lambda_1 = \frac{a_2\sqrt{6[3(a_2{}^2k^2 + 2a_1) - 4k^2a_1]}}{4a_1}, \quad \text{(complex with real part zero)}, \tag{8.2.43a}$$

$$\lambda_2 = -\frac{a_2\sqrt{6[3(a_2{}^2k^2 + 2a_1) - 4k^2a_1]}}{4a_1}, \quad \text{(complex with real part zero)} \tag{8.2.43b}$$

$$\lambda_3 = -\frac{\sqrt{3}}{3ka_1c}\left(-4a_1kc + \sqrt{a_1[4k^2a_1 - 3(a_2^2k^2 + 2a_1)]}\right) \quad \text{(real)}. \tag{8.2.43c}$$

λ_3 is real and negative if these two conditions hold: $c < \frac{1}{2}$ and $3(a_2{}^2k^2 + 2a_1) < 4a_1k^2(1 - 4c^2)$. Since P_{11} have two purely complex eigenvalues (i,e., P_{11} called a non-hyperbolic equilibrium point), then Hartman Grobman theorem fails to tell the stability of P_{11}. Thus, we will analyze the stability of such equilibrium point using different approach. Since $Q = 1$ is an invariant set, then plugging $Q = 1$ into the 3D system (8.2.41) leads to

$$x' = (1 - c^2y^2 - x^2)(\sqrt{2}k - \sqrt{3}x) + \frac{\sqrt{6}}{2}\frac{a_2k\sqrt{1 - x^2 - c^2y^2}(1 - x^2)}{\sqrt{a_1}}, \tag{8.2.44a}$$

$$y' = -\sqrt{3}y(1 - c^2y^2 - x^2) - \frac{\sqrt{6}}{2}\frac{a_2kxy\sqrt{1 - x^2 - c^2y^2}}{\sqrt{a_1}}. \tag{8.2.44b}$$

Doing the linreaization around P_{11} of the system (8.2.44) to second order terms yields

$$\bar{x}' = A\bar{x} - B\bar{y} + C\bar{x}^2 + D\bar{x}\bar{y} + E\bar{y}^2, \tag{8.2.45a}$$

$$\bar{y}' = F\bar{x} - A\bar{y} + G\bar{x}^2 + H\bar{x}\bar{y} + J\bar{y}^2, \tag{8.2.45b}$$

where $\bar{x}, \bar{y}$ define as

$$\bar{x} = x - x_{11}, \quad \bar{y} = y - y_{11}. \tag{8.2.46}$$

and $A, B, C, D, E, F, G, H, J$ are constants (i.e., arbitrary functions of the parameters $a_1; a_2; k; c$) defined by

$$A := \frac{\sqrt{3}}{4k^2a_1}\left[3({a_2}^2k^2 + 2a_1) - 4k^2a_1\right], \tag{8.2.47a}$$

$$B := \frac{\sqrt{2}c(2k^2 - 3)}{4k^2\sqrt{a_1}}\sqrt{4k^2a_1 - 3({a_2}^2k^2 + 2a_1)}, \tag{8.2.47b}$$

$$C := \frac{\sqrt{2}(3{a_2}^4k^4 + a_1(2k^2 - 3)(2a_1 - {a_2}^2k^2))}{4k^3a_1{a_2}^2}, \tag{8.2.47c}$$

$$D := \frac{\sqrt{3a_1}c(2k^2 - 3)\sqrt{4k^2a_1 - 3({a_2}^2k^2 + 2a_1)}}{3k^3{a_2}^2}, \tag{8.2.47d}$$

$$E := \frac{\sqrt{2}c^2(2k^2 - 3)\left[-3{a_2}^2k^2 + a_1(2k^2 - 3)\right]}{6{a_2}^2k^3}, \tag{8.2.47e}$$

$$F := \frac{3\sqrt{2}({a_2}^2k^2 + 2a_1)\sqrt{4k^2a_1 - 3({a_2}^2k^2 + 2a_1)}}{8ck^2{a_1}^{\frac{3}{2}}}, \tag{8.2.47f}$$

$$G := -\frac{\sqrt{3}({a_2}^2k^2 + 2a_1)\sqrt{4k^2a_1 - 3({a_2}^2k^2 + 2a_1)}}{4k^3c{a_2}^2\sqrt{a_1}}, \tag{8.2.47g}$$

$$H := -\frac{\sqrt{2}({a_2}^2k^2 + 2a_1)(-3{a_2}^2k^2 + 2a_1k^2 - 3a_1)}{2a_1{a_2}^2k^3}, \tag{8.2.48a}$$

$$J := \frac{\sqrt{3}c\sqrt{4k^2a_1 - 3({a_2}^2k^2 + 2a_1)}\,[6{a_2}^2k^2 - a_1(2k^2 - 3)]}{6{a_2}^2k^3\sqrt{a_1}}. \tag{8.2.48b}$$

Rescaling $\bar{x}$ and $\bar{y}$ as follows;

$$\bar{x} = \epsilon X, \quad \bar{y} = \epsilon Y, \quad \text{where} \quad \epsilon \quad \text{small}. \tag{8.2.49}$$

Thus, the system (8.2.45) becomes

$$X' = AX - BY + \epsilon(CX^2 + DXY + EY^2), \tag{8.2.50}$$

$$Y' = FX - AY + \epsilon(GX^2 + HXY + JY^2). \tag{8.2.51}$$

Now, changing the system in equation (8.2.50 - 8.2.51) from two variables to one variable in terms of Y as follow:

First, from (8.2.51) X is given by

$$X = \frac{1}{F}\left(Y' + AY - \epsilon(GX^2 + HXY + JY^2)\right). \tag{8.2.52}$$

Next, differentiating (8.2.51) then using (8.2.50); (8.2.51) and (8.2.52), and keeping only the second order terms of ϵ, yields to

$$Y'' + w^2Y = \epsilon(\alpha_1Y^2 + \alpha_2Y'Y + \alpha_3{Y'}^2) + \epsilon^2(\alpha_4Y^3 + \alpha_5Y'Y^2 + \alpha_6{Y'}^2Y + \alpha_7{Y'}^3), \tag{8.2.53}$$

where $w = \sqrt{FB - A^2}$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ are constants (i.e., arbitrary functions of the parameters $a_1; a_2; k; c$) defined by

$$\alpha_1 := \frac{3ca_2{}^2k\left(-3\,a_2{}^2k^2 + 2\,k^2a_1 - 6\,a_1\right)\sqrt{4k^2a_1 - 3(a_2{}^2k^2 + 2a_1)}}{4a_1{}^{3/2}\left(a_2{}^2k^2 + 2\,a_1\right)}, \tag{8.2.54a}$$

$$\alpha_2 := \frac{2\sqrt{3a_1}c\sqrt{4k^2a_1 - 3(a_2{}^2k^2 + 2a_1)}k}{a_2{}^2k^2 + 2\,a_1}, \tag{8.2.54b}$$

$$\alpha_3 := -\frac{2\sqrt{a_1}ck\left(-3\,a_2{}^2k^2 + a_1(2k^2 - 3)\right)}{\sqrt{-3\,a_2{}^2k^2 + 4\,k^2a_1 - 6\,a_1}\left(a_2{}^2k^2 + 2\,a_1\right)}, \tag{8.2.54c}$$

$$\alpha_4 := \frac{-c^2k^2}{3a_1(a_2{}^2k^2 + 2a_1)^2}\left[a_2{}^2k^4(9a_2{}^4 - 12a_1a_2{}^2 + 4a_1{}^2)\right.$$

$$\left. + 4k^2a_1(9a_2{}^2(a_2{}^2 - a_1) + 4a_1{}^2) + 12a_1{}^2(3a_2{}^2 - 2a_1)\right], \tag{8.2.54d}$$

$$\alpha_5 := \frac{4\sqrt{3}\left(-3\,a_2{}^2k^2 + 2\,k^2a_1 - 6\,a_1\right)k^2a_1\,c^2}{3\left(a_2{}^2k^2 + 2\,a_1\right)^2}, \tag{8.2.54e}$$

$$\alpha_6 := \frac{-8c^2\left(-3\,a_2{}^2k^2 + 2\,k^2a_1 - 6\,a_1\right)^2a_1\,k^2}{3\left(a_2{}^2k^2 + 2\,a_1\right)^2\left(-3\,a_2{}^2k^2 + 4\,k^2a_1 - 6\,a_1\right)}, \tag{8.2.54f}$$

$$\alpha_7 := -\frac{32\sqrt{3}a_1{}^3c^2k^2\left(-3\,a_2{}^2k^2 + 2\,k^2a_1 - 6\,a_1\right)}{27\,a_2{}^2\left(-3\,a_2{}^2k^2 + 4\,k^2a_1 - 6\,a_1\right)\left(a_2{}^2k^2 + 2\,a_1\right)^2}. \tag{8.2.54g}$$

Multiple Scales Method

Here we seek an asymptotic approximation for Y in equation (8.2.53) of the following form

$$Y \equiv \mathcal{Y}(t,\tau,s) \sim \mathcal{Y}_0(t,\tau,s) + \epsilon\mathcal{Y}_1(t,\tau,s) + \epsilon^2\mathcal{Y}_2(t,\tau,s), \tag{8.2.55}$$

for three time scales $t, \tau = \epsilon t, s = \epsilon^2 t$. Substituting (8.2.55) into (8.2.53) and then collecting coefficients of equal power of ϵ gives

$$\mathcal{Y}_{0tt} + \mathcal{Y}_0 + \epsilon\Bigg(\mathcal{Y}_{1tt} + 2\mathcal{Y}_{0t\tau} + w^2\mathcal{Y}_1 - \alpha_1{\mathcal{Y}_0}^2 - \alpha_2\mathcal{Y}_0\mathcal{Y}_{0t} - \alpha_3\mathcal{Y}0t^2\Bigg) + \epsilon\Bigg(2\mathcal{Y}_{0ts}$$

$$+\mathcal{Y}_{0\tau\tau} + 2\mathcal{Y}_{1t\tau} + \mathcal{Y}_{2tt} + w^2\mathcal{Y}_2 - 2\alpha_3\mathcal{Y}_{0t}(\mathcal{Y}_{0\tau} + \mathcal{Y}_{1t}) - \alpha_2\mathcal{Y}_{0t}\mathcal{Y}_1\Bigg) + \epsilon^2\Bigg(-\alpha_2\mathcal{Y}_0$$

$$(\mathcal{Y}_{0\tau} + \mathcal{Y}_{1t}) - 2\alpha_1\mathcal{Y}_0\mathcal{Y}_1 - \alpha_7{\mathcal{Y}_{0t}}^3 - \alpha_6\mathcal{Y}_0{\mathcal{Y}_{0t}}^2 - \alpha_5{\mathcal{Y}_0}^2\mathcal{Y}_{0t} - \alpha_4{\mathcal{Y}_0}^3\Bigg) + O(\epsilon^3) = 0.$$

Note that we use the symbol $\mathcal{Y}_t$ in place of $\frac{\partial \mathcal{Y}}{\partial t}$ (similarly for $\mathcal{Y}_\tau$) for simplification. Equating coefficients of like powers of ϵ to 0, gives the following sequences of linear partial differential equations:

$$O(1): \quad \mathcal{Y}_{0tt} + w^2\mathcal{Y}_0 = 0, \tag{8.2.56a}$$

$$O(\epsilon): \quad \mathcal{Y}_{1tt} + w^2\mathcal{Y}_1 = -2\mathcal{Y}_{ot\tau} + \alpha_1{\mathcal{Y}_0}^2 + \alpha_2\mathcal{Y}_0\mathcal{Y}_{ot} + \alpha_3{\mathcal{Y}_{ot}}^2, \tag{8.2.56b}$$

$$O(\epsilon^2): \quad \mathcal{Y}_{2tt} + w^2\mathcal{Y}_2 = -2\mathcal{Y}_{ots} + 2\alpha_1\mathcal{Y}_0\mathcal{Y}_1 + \alpha_2(\mathcal{Y}_1\mathcal{Y}_0)_t + 2\alpha_3\mathcal{Y}_{0t}\mathcal{Y}_{1t}$$

$$+ \alpha_4{\mathcal{Y}_0}^3 + \alpha_5\mathcal{Y}_{0t}{\mathcal{Y}_0}^2 + \alpha_6\mathcal{Y}_0{\mathcal{Y}_{0t}}^2 + \alpha_7{\mathcal{Y}_{0t}}^3. \tag{8.2.56c}$$

Equation (8.2.56a) has the following solution

$$\mathcal{Y}_0 = A(s)e^{wit} + \overline{A(s)}e^{-wit}. \tag{8.2.57}$$

Then equation (8.2.56b) becomes

$$A(s)^2e^{2wit}(\alpha_1 + \alpha_2 iw - \alpha_3 w^2) + \overline{A(s)}^2 e^{-2wit}(\alpha_1 - i\alpha_2 w - \alpha_3 w^2) \tag{8.2.58}$$

$$+2A(s)\overline{A(s)}(\alpha_1 + \alpha_3 w^2), \tag{8.2.59}$$

which yields to

$$\mathcal{Y}_1 = -\tfrac{1}{3}A(s)^2e^{2iwt}(\alpha_1 + \alpha_2 iw - \alpha_3 w^2) - \tfrac{1}{3}\overline{A(s)}^2 e^{-2iwt}(\alpha_1 - \alpha_2 iw - \alpha_3 w^2)$$

$$-\tfrac{2}{3}A(s)\overline{A(s)}(\alpha_1 + \alpha_3 w^2).$$

Hence, the term of order ϵ^2 leads to

$$\mathcal{Y}_{2tt} + w^2\mathcal{Y}_2 = -2\mathcal{Y}_{ots} + 2\alpha_1\mathcal{Y}_0\mathcal{Y}_1 + \alpha_2(\mathcal{Y}_1\mathcal{Y}_0)_t + 2\alpha_3\mathcal{Y}_{0t}\mathcal{Y}_{1t}$$

$$+\alpha_4{\mathcal{Y}_0}^3 + \alpha_5\mathcal{Y}_{0t}{\mathcal{Y}_0}^2 + \alpha_6\mathcal{Y}_0{\mathcal{Y}_{0t}}^2 + \alpha_7{\mathcal{Y}_{0t}}^3.$$

Substituting $\mathcal{Y}_0$ and $\mathcal{Y}_1$, followed by some calculation yields the following coefficient of e^{wit}

$$iA_s = \tfrac{1}{2}|A(s)|^2\overline{A(s)}\Big[\left(-2{\alpha_1}^2 - 2\alpha_1\alpha_3w^2 + 3\alpha_4 + \tfrac{1}{3}{\alpha_2}^2w^2 + \tfrac{4}{3}w^4{\alpha_3}^2\right)$$

$$+i\left(-\tfrac{5}{3}\alpha_2w(\alpha_1 + \alpha_3w^2) + \alpha_5w + 3\alpha_7w^3\right)\Big]. \qquad (8.2.60)$$

Note that the expression in front of i in the right hand side of equation (8.2.60) is equal to zero (i.e., $-\frac{5}{3}\alpha_2w(\alpha_1 + \alpha_3w^2) + \alpha_5w + 3\alpha_7w^3 = 0$). Suppression of secularity requires the above equation (i.e., 8.2.60) to be zero. The substitution $A = Re^{i\Phi}$, yields

$$\Phi_s = -\tfrac{1}{2}R^2\left(-2{\alpha_1}^2 - 2\alpha_1\alpha_3w^2 + 3\alpha_4 + \tfrac{1}{3}{\alpha_2}^2w^2 + \tfrac{4}{3}w^4{\alpha_3}^2\right). \qquad (8.2.61)$$

$$R_s = 0. \qquad (8.2.62)$$

Therefore, the behaviour of the model near to P_{11} is oscillations. Note that the expression in front of R^2 in equation (8.2.62) can be simplified in terms of a_1, a_2, k, c as follows

$$\beta_0 := -2{\alpha_1}^2 - 2\alpha_1\alpha_3 w^2 + 3\alpha_4 + \tfrac{1}{3}{\alpha_2}^2 w^2 + \tfrac{4}{3} w^4 {\alpha_3}^2$$

$$\beta_0 = -\tfrac{c^2}{4{a_1}^3({a_2}^2k^2+2\,a_1)^2k^2}\Big[(81\,{a_2}^{10}k^6 - 216\,a_1\,{a_2}^8k^6 + 180\,{a_1}^2{a_2}^6k^6$$

$$-48\,{a_1}^3{a_2}^4k^6 + 486\,a_1\,{a_2}^8k^4 - 882\,{a_1}^2{a_2}^6k^4 + 456\,{a_1}^3{a_2}^4k^4$$

$$-80\,{a_1}^4{a_2}^2k^4 + 972\,{a_1}^2{a_2}^6k^2 - 936\,{a_1}^3{a_2}^4k^2 + 144\,{a_1}^4{a_2}^2k^2$$

$$+64\,{a_1}^5k^2 + 648\,{a_1}^3{a_2}^4 - 72\,{a_1}^4{a_2}^2 - 96\,{a_1}^5\Big]$$

When $a_1 = 3, a_2 = -1, c = 1, k = 2$ then $\beta_0 = \frac{24}{25}$. Hence,

$$\mathcal{Y}_0 = A(s)e^{wit} + \overline{A(s)}e^{-wit} = 2R\cos\left(t\left[1 - \frac{24}{25}\epsilon^2 R^2\right]\right). \tag{8.2.63}$$

with the initial condition $\mathcal{Y}_0(0) = 1, \mathcal{Y}_{0t} = 0$ then $R = \frac{1}{2}$. Therefore,

$$\mathcal{Y}_0 = \cos\left(t\left[1 - \frac{6}{25}\epsilon^2\right]\right). \tag{8.2.64}$$

Numerical Plot

In figure 8.1, we plot for the solution of the system with 3 variables (x, y, Q) in (8.2.41)

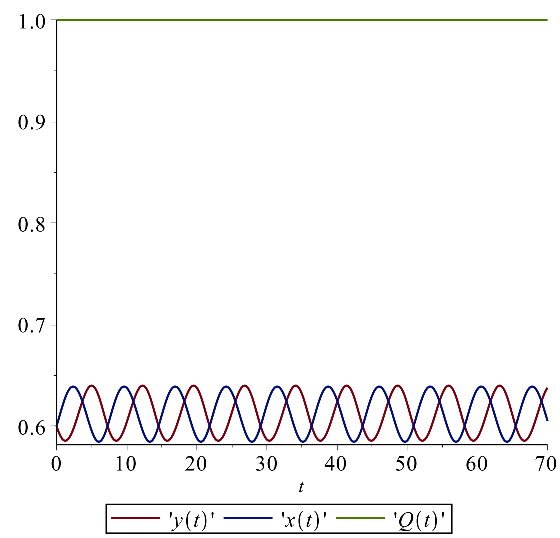

Figure 8.1: Plot of the system (8.2.41) with $a_1 = 3, a_2 = -1, c = 1, k = 2$ with one initial condition: $[x(0) = 0.6, y(0) = 0.6, Q(0) = 0.999999]$

As can be seen from figure 8.1 that the solutions for x, y oscillate and $Q \to 1$.

Note that, when we plot the solution of the system (8.2.53) with $\cos(t)$, without the shift in figure 8.2 it shows the solutions are different. However, when we plot the solution of the system (8.2.53) with the multiple scalar approximation $(\cos\left(t\left(1 - \frac{24}{25}\epsilon^2\right)\right))$ in figure 8.3 with the shift it works.

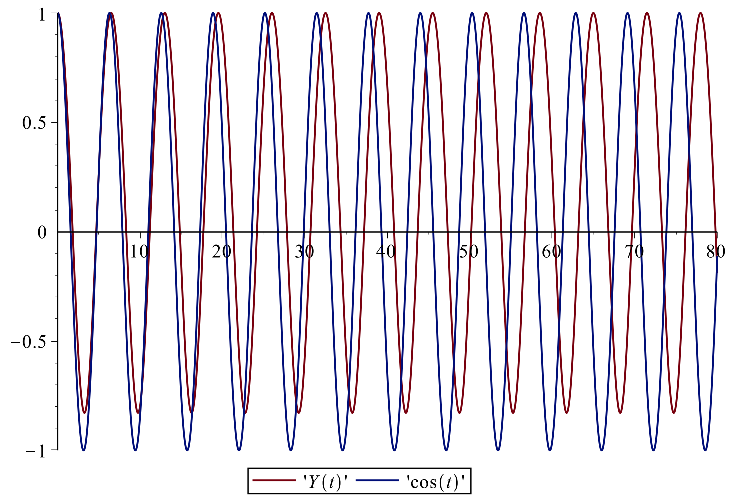

Figure 8.2: Plot of the system (8.2.53) with $a_1 = 3, a_2 = -1, \epsilon = 0.3; c = 1, k = 2$ with $\cos(t)$ without the shift

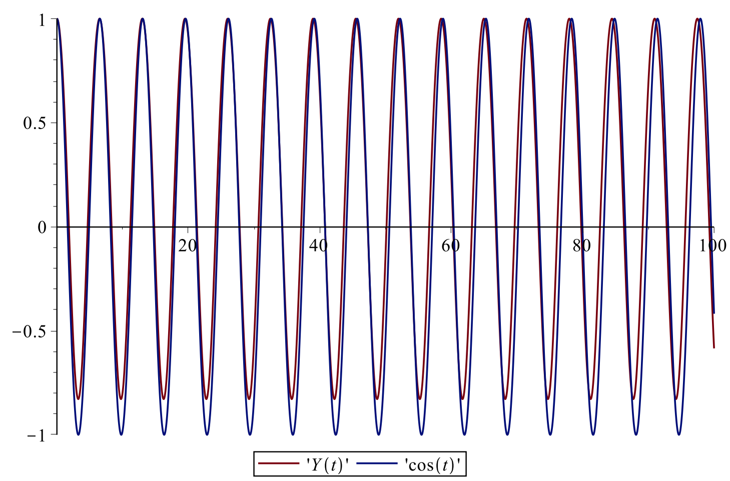

Figure 8.3: Plot of the system (8.2.53) with $a_1 = 3, a_2 = -1, \epsilon = 0.3; c = 1, k = 2$ with $\cos(T) = \cos\left(t\left(1 - \frac{24}{25}\epsilon^2\right)\right)$ with shift.

Stability of Equilibrium Point P_{12}

P_{12} exists when $a_2 < 0$ (since $W_{11} > 0$); $k > \sqrt{\frac{3}{2}}$ and ${a_2}^2k^2 < \frac{2a_1}{3}(2k^2 - 3)$ (because y_{11} is real). Evaluating the linearization of the system (8.2.41) at P_{11} leads to three eigenvalues:

$$\lambda_1 = \frac{a_2\sqrt{6[3({a_2}^2k^2+2a_1)-4k^2a_1]}}{4a_1}, \quad \text{(complex with real part zero)}, \tag{8.2.65a}$$

$$\lambda_2 = -\frac{a_2\sqrt{6[3({a_2}^2k^2+2a_1)-4k^2a_1]}}{4a_1}, \quad \text{(complex with real part zero)} \tag{8.2.65b}$$

$$\lambda_3 = \frac{\sqrt{3}}{3ka_1c}\left(4a_1kc+\sqrt{a_1[4k^2a_1-3(a_2^2k^2+2a_1)]}\right) \quad \text{(real)}, \tag{8.2.65c}$$

and always positive which implies that P_{12} is not attractor.

Stability of Equilibrium Point $P_{13,14}$

We did same analysis as we earlier for P_{11} and we found that there is oscillations which implies that $P_{13,14}$ are center.

In table (8.3) we summarizes of all the eigenvalues and the stability of all the equilibrium points

Pt	Eigenvalues	Sink
P_1	$\left(\frac{\sqrt{3}(2c^2+1)}{3c}, -\frac{\sqrt{3}(4c^2-1)}{3c}, -\frac{\sqrt{3}(4c^2-1)}{3c}\right)$	No Sink
P_2	$\left(-\frac{\sqrt{3}(2c^2+1)}{3c}, \frac{\sqrt{3}(4c^2-1)}{3c}, \frac{\sqrt{3}(4c^2-1)}{3c}\right)$	Sink if $c < \frac{1}{2}$
P_3^*	$\left(0, -\frac{2\sqrt{3}(y^*-2)}{3}, -\sqrt{2}k\sqrt{1-c^2y^{*2}}+\sqrt{3}\right)$	Sink in two cases if $k > \sqrt{\frac{3}{2}},\ c < \frac{1}{2}$ 1) $4c^2 + \frac{3}{2k^2} < 1$ then the sink part is $\frac{1}{c}\sqrt{1-\frac{3}{2k^2}} < y^* < 2$ 1) $4c^2 + \frac{3}{2k^2} > 1$ then the sink part is $-\frac{1}{c}\sqrt{1-\frac{3}{2k^2}} < y^* < \frac{1}{c}\sqrt{1-\frac{3}{2k^2}}$
P_4^*	$\left(0, -\frac{2\sqrt{3}(y^*-2)}{3}, \sqrt{2}k\sqrt{1-c^2y^{*2}}+\sqrt{3}\right)$	No sink
P_5^*	$\left(0, \frac{-2\sqrt{3}(y^*+2)}{3}, -(\sqrt{2}k\sqrt{1-c^2y^{*2}}+\sqrt{3})\right)$	Sink conditions: $y^* > -2, k$ any value sink: (1) $c < \frac{1}{2} \rightarrow -2 < y^* < \frac{1}{c}$, (2) $c > \frac{1}{2} \rightarrow -\frac{1}{c} < y^* < \frac{1}{c}$.
P_6^*	$\left(0, -\frac{2\sqrt{3}(y^*+2)}{3}, \sqrt{2}k\sqrt{1-c^2y^{*2}}-\sqrt{3}\right)$	If $y^* > -2$ two sink cases: 1) If $k < \sqrt{\frac{3}{2}}$: two cases: (a) If $c < \frac{1}{2}$ sink in $-2 < y^* < \frac{1}{c}$ (b) If $c > \frac{1}{2}$ sink in $-\frac{1}{c} < y^* < \frac{1}{c}$ 2) $k > \sqrt{\frac{3}{2}}$: three sink portions: (a) $c < \frac{1}{2}; 4c^2 + \frac{3}{2k^2} > 1$ the only is a sink is: $-2 < y^* < \frac{1}{c}\sqrt{1-\frac{3}{2k^2}}$ (b) $c < \frac{1}{2}$ sink in $\frac{1}{c}\sqrt{1-\frac{3}{2k^2}} < y^* < \frac{1}{c}$ (c) $c > \frac{1}{2}$ sink in two parts: $-\frac{1}{c} < y^* < -\frac{1}{c}\sqrt{1-\frac{3}{2k^2}}$ and $\frac{1}{c}\sqrt{1-\frac{3}{2k^2}} < y^* < \frac{1}{c}$.

Table 8.3: Eigenvalues of the equilibrium points ($P_{1,2}$ and $P^*_{3,4,5,6}$) of the system (8.2.5); the sinks conditions. We use the notation $B = 3(a_2^2k^2 + 2a_1) - 4a_1k^2$.

Pt	Eigenvalues	Sink
P_7	$\left(\sqrt{2}kx_7-\sqrt{3},\sqrt{2}kx_7-\sqrt{3},2\left(\sqrt{2}kx_7-\frac{\sqrt{3}}{3}\right)\right)$	Sink if 3- cases: 1) $a_2>0,\quad k<\sqrt{\frac{2}{3}}$ ${a_2}^2k^2>2a_1(2k^2-1)$ $2a_1(1-2k^2)^2>{a_2}^2k^2(6k^2-1)$. 2) $a_2<0,\quad k<\sqrt{\frac{2}{3}}$ ${a_2}^2k^2<2a_1(2k^2-1)$ 3) $a_2<0,\quad k>\sqrt{\frac{2}{3}}$ ${a_2}^2>\frac{2a_1}{3k^2}(2k^2-3)$.
P_8	$\left(\sqrt{2}kx_8-\sqrt{3},\sqrt{2}kx_8-\sqrt{3},2\left(\sqrt{2}kx_8-\frac{\sqrt{3}}{3}\right)\right)$	Sink if $a_2<0,k>\sqrt{\frac{3}{2}}$ ${a_2}^2k^2>2a_1(2k^2-1)$ $2a_1(1-2k^2)^2>{a_2}^2k^2(6k^2-1)$.
P_9	$\left(\sqrt{2}kx_9+\sqrt{3},\sqrt{2}kx_9+\sqrt{3},2\left(\sqrt{2}kx_9+\frac{\sqrt{3}}{3}\right)\right)$	No Sink
P_{10}	$\left(\sqrt{2}kx_{10}+\sqrt{3},\sqrt{2}kx_{10}+\sqrt{3},2\left(\sqrt{2}kx_{10}+\frac{\sqrt{3}}{3}\right)\right)$	No sink
P_{11}	$\left(\frac{a_2\sqrt{6B}}{4a_1},-\frac{a_2\sqrt{6B}}{4a_1},-\frac{\sqrt{3}}{3ka_1c}(-4a_1kc+\sqrt{-a_1B})\right)$	Center
P_{12}	$\left(\frac{a_2\sqrt{6B}}{4a_1},-\frac{a_2\sqrt{6B}}{4a_1},\frac{\sqrt{3}}{3ka_1c}(4a_1kc+\sqrt{-a_1B})\right)$	Center
P_{13}	$\left(\frac{a_2\sqrt{6B}}{4a_1},-\frac{a_2\sqrt{6B}}{4a_1},-\frac{\sqrt{3}}{3ka_1c}(4a_1kc+\sqrt{-a_1B})\right)$	Center
P_{14}	$\left(\frac{a_2\sqrt{6B}}{4a_1},-\frac{a_2\sqrt{6B}}{4a_1},\frac{\sqrt{3}}{3ka_1c}(-4a_1kc+\sqrt{-a_1B})\right)$	Center

Table 8.4: Eigenvalues of the equilibrium points ($P_{7,8,9,10,11,12,13,14}$) of the system (8.2.5); the sinks conditions. We use the notation $B=3(a_2^2k^2+2a_1)-4a_1k^2$.

Numerical Solutions for the 4D System in (8.2.1) into the Future

We plot the 4D system in equations (8.2.1) for different values of the parameters near each of the sink equilibrium points and we obtain the following figures:

Point P_2: We plot the 4D system in (8.2.1) and we obtain that P_2 is a sink when $c < \frac{1}{2}$.

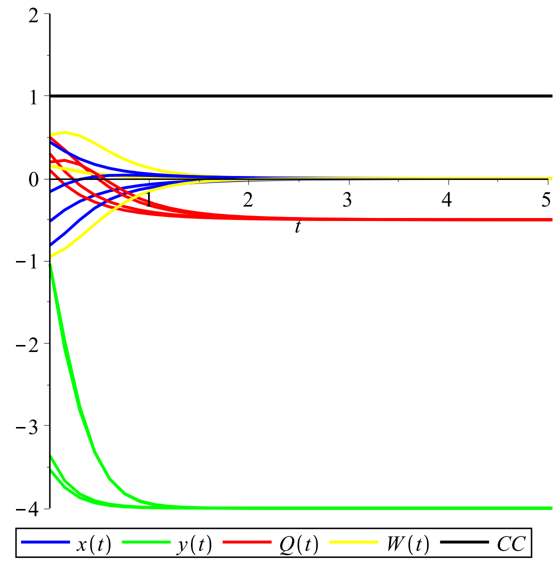

Figure 8.4: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = 1/4, k = 1$ with three different initial conditions: $[[x(0) = \cos(60)\sin(45), y(0) = 4\sin(60)\cos(45), Q(0) = 0.5, W(0) = \cos(45)], [x(0) = \cos(45)\sin(30), y(0) = 4\sin(45)\cos(30), Q(0) = 0.3, W(0) = \cos(30)], [x(0) = \cos(90)\sin(30), y(0) = 4\sin(90)\cos(30), Q(0) = 0.1, W(0) = \cos(30)]]$. Note that $CC = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.4 that $x \to 0, y \to -4; Q \to -\frac{1}{2}$ and $W \to 0$ (which is P_2 when $c = \frac{1}{4}$). Note that, we have the same figure when $a_2 < 0$ and for any value of $k > 0$.

PointP_3^*: We plot the 4D system in (8.2.1) and we obtain that P_3^* is a sink in two cases:

- If $c < \frac{1}{2}; \quad k > \sqrt{\frac{3}{2}}; \quad 4c^2 + \frac{3}{2k^2} > 1$ then the part of the line that is a sink is $-\frac{1}{c}\sqrt{1 - \frac{3}{2k^2}} < y^* < \frac{1}{c}\sqrt{1 - \frac{3}{2k^2}}$.

- If $c < \frac{1}{2}$; $\quad k > \sqrt{\frac{3}{2}}$; $\quad 4c^2 + \frac{3}{2k^2} < 1$ then the part of the line that is a sink is $\frac{1}{c}\sqrt{1 - \frac{3}{2k^2}} < y^* < 2$.

Case(1): $c < \frac{1}{2}$; $\quad k > \sqrt{\frac{3}{2}}$; $\quad 4c^2 + \frac{3}{2k^2} > 1$

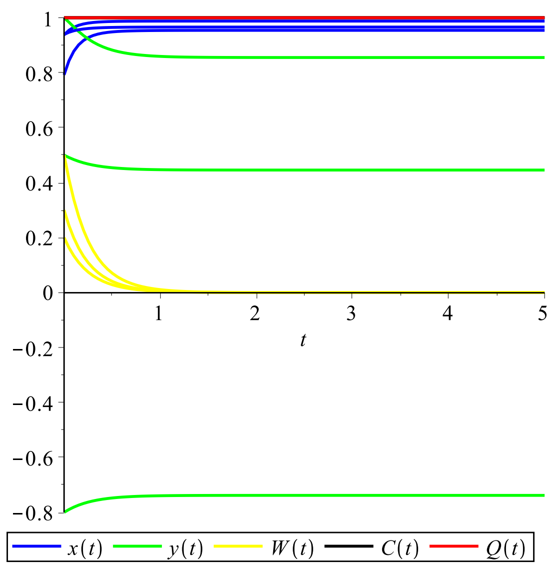

Figure 8.5: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = 0.45, k = 1.5$ with three different initial conditions: $[[x(0) = 0.9377499667, y(0) = 0.5, Q(0) = 0.99, W(0) = 0.3], [x(0) = 0.7921489759, y(0) = 1, Q(0) = 0.99, W(0) = 0.5], [x(0) = 0.9389355675, y(0) = -0.8, Q(0) = 0.99, W(0) = 0.2]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$as $t \to \infty$.

Discussion: We can see from figure 8.5 that $x \to 0.99; -1.6 < y < 1.6; Q \to 1$ and $W \to 0$ which corresponds to P_3^* at $c = 0.45$.

Case(2): $c < \frac{1}{2}; \quad k > \sqrt{\frac{3}{2}}; \quad 4c^2 + \frac{3}{2k^2} < 1$

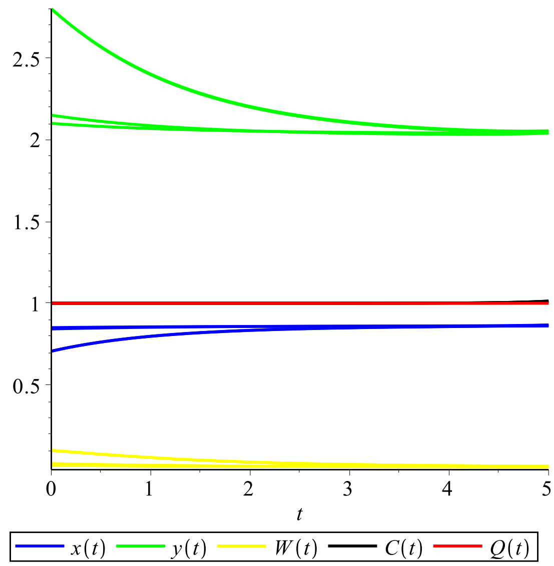

Figure 8.6: Plot of the system (8.2.1) with$a_1 = 1 = a_2, c = \frac{1}{4}, k = 2$ with three different initial conditions: $[[x(0) = 0.7071067812, y(0) = 2.8, Q(0) = 0.99, W(0) = 0.1], [x(0) = 0.84330265417, y(0) = 2.15, Q(0) = 0.99, W(0) = 0.02], [x(0) = 0.8510434772, y(0) = 2.1, Q(0) = 0.99, W(0) = 0.01]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$as $t \to \infty$.

Discussion: We can see from figure 8.6 that $x \to 0.9; 3.13 < y < 2; Q \to 1$ and $W \to 0$ which which corresponds to P_3^* at $c = 0.25$.

Point P_5^*: We plot the 4D system in (8.2.1) and we obtain that P_5^* is a sink in two cases:

1. If $c < \frac{1}{2}$ then $-2 < y^* < \frac{1}{c}$ is the only portion of the line of sink.

2. If $c > \frac{1}{2}$ then P_5^* is a sink in the whole line $-\frac{1}{c} < y^* < \frac{1}{c}$.

Case (1): $c < \frac{1}{2}$**:**

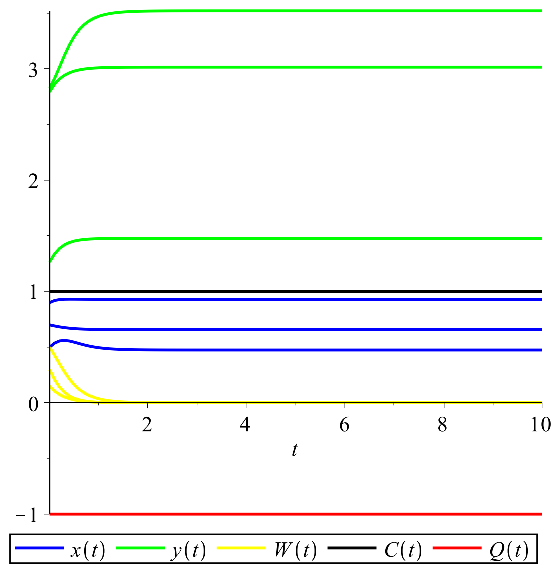

Figure 8.7: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = \frac{1}{4}, k = 2$ with three different initial conditions: $[[x(0) = 0.5, y(0) = 2.828427125, Q(0) = -0.99, W(0) = 0.5], [x(0) = 0.55, y(0) = 1.264911064, Q(0) = -0.99, W(0) = 0.3], [x(0) = 0.7, y(0) = 2.792848009, Q(0) = -0.99, W(0) = 0.15]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.7 that $-4 < y < 4, Q \to -1$ and $W \to 0$ which corresponds to P_5^*. Note that we have the same figure for $a_2 < 0$ and for any value of $k > 0$.

Case(2): $c > \frac{1}{2}$**:**

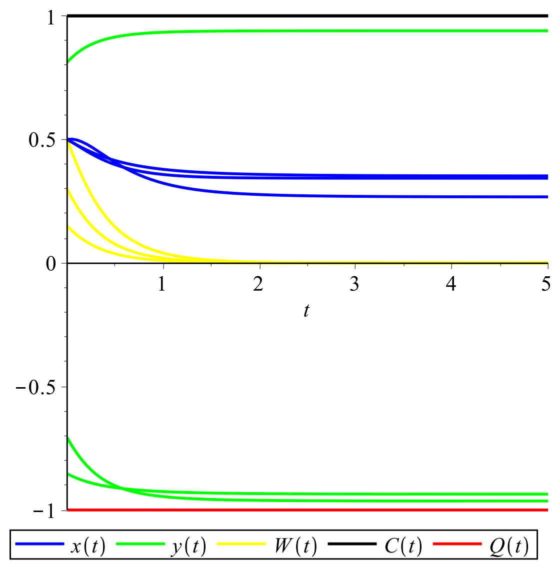

Figure 8.8: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = \frac{1}{4}, k = 2$ with three different initial conditions: $[[x(0) = 0.5, y(0) = -0.7071067812, Q(0) = -0.99, W(0) = 0.5], [x(0) = 0.5, y(0) = 0.8124038405, Q(0) = -0.99, W(0) = 0.3], [x(0) = 0.5, y(0) = -0.8529361055, Q(0) = -0.99, W(0) = 0.15]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.8 that $-1 < y < 1, Q \to -1$ and $W \to 0$ which corresponds to P_5^*.

Point P_6^***:** We plot the 4D system in (8.2.1) and we obtain that P_6^* is a sink in several cases:

1. If $k < \sqrt{\frac{3}{2}}$, we have two cases:

    (a) If $c < \frac{1}{2}$ the sink is in the portion $-2 < y^* < \frac{1}{c}$.

    (b) If $c > \frac{1}{2}$ the sink is in the portion $-\frac{1}{c} < y^* < \frac{1}{c}$.

2. If $k > \sqrt{\frac{3}{2}}$, we have two cases:

(a) If $c < \frac{1}{2}; 4c^2 + \frac{3}{2k^2} > 1$. then the sink is in the portion $-2 < y^* < \frac{1}{c}\sqrt{1 - \frac{3}{2k^2}}$.

(b) If $c < \frac{1}{2}$ the sink is in the portion $\frac{1}{c}\sqrt{1 - \frac{3}{2k^2}} < y^* < \frac{1}{c}$

(c) If $c > \frac{1}{2}$ the sink is in the two portion: $-\frac{1}{c} < y^* < -\frac{1}{c}\sqrt{1 - \frac{3}{2k^2}}; \frac{1}{c}\sqrt{1 - \frac{3}{2k^2}} < y^* < \frac{1}{c}$.

Case (1a): when $k < \sqrt{\frac{3}{2}}$ and $c < \frac{1}{2}$

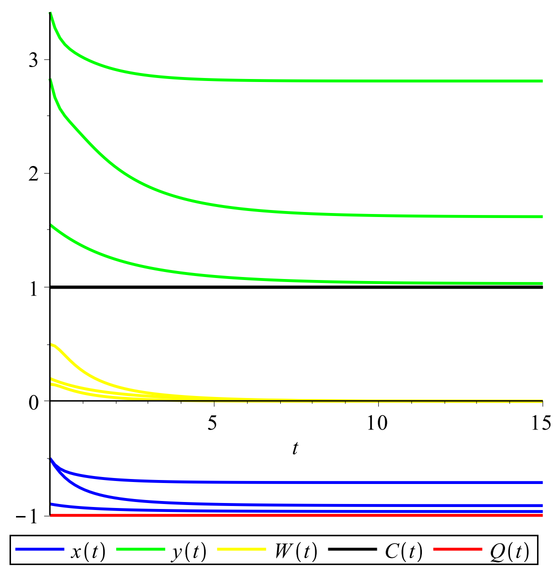

Figure 8.9: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = \frac{1}{4}, k = \frac{1}{2}$ with three different initial conditions: $[[x(0) = -0.5, y(0) = 2.828427125, Q(0) = -0.99, W(0) = 0.5], [x(0) = -0.5, y(0) = 3.411744422, Q(0) = -0.99, W(0) = 0.15], [x(0) = -0.5, y(0) = 1.549193338, Q(0) = -0.999, W(0) = 0.2]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.9 that $-2 < y < 4; Q \to -1$, and $W \to 0$ which corresponds to P_6^*.

Case (1b): when $k < \sqrt{\frac{3}{2}}$ and $c > \frac{1}{2}$

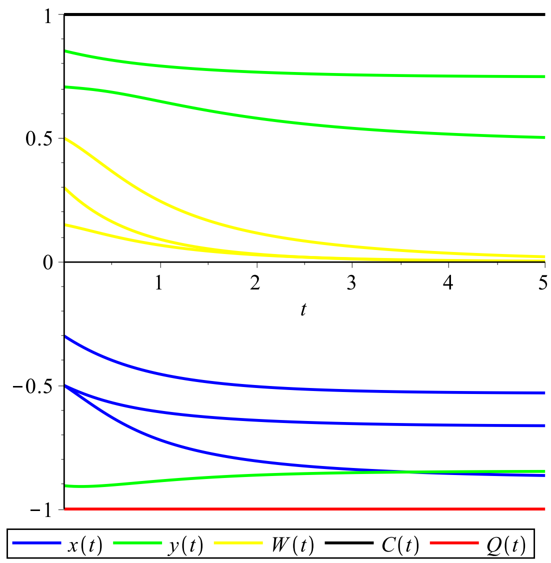

Figure 8.10: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = 1, k = \frac{1}{2}$ with three different initial conditions: $[[x(0) = -0.5, y(0) = -0.7071067812, Q(0) = -0.8, W(0) = 0.5], [x(0) = -0.3, y(0) = -0.9055385138, Q(0) = -0.99, W(0) = 0.3], [x(0) = -0.5, y(0) = 0.8529361055, Q(0) = -0.79, W(0) = 0.15]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.10 that $-1 < y < 1; Q \to -1$ and $W \to 0$ which corresponds to P_6^* at $c = 1$.

Case (2a): when $k > \sqrt{\frac{3}{2}}$; $4c^2 + \frac{3}{2k^2} > 1$ and $c < \frac{1}{2}$

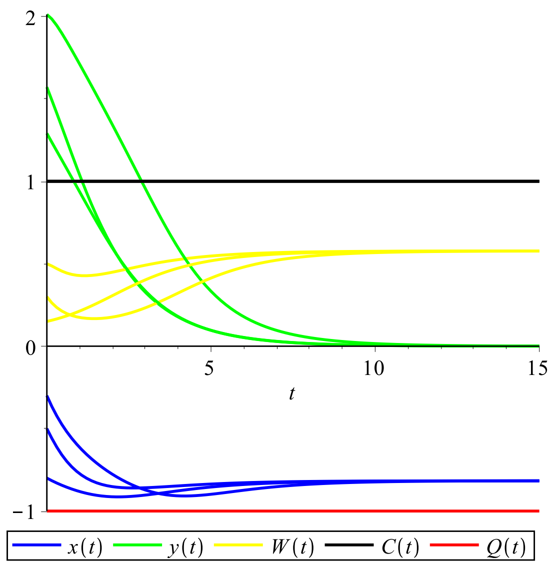

Figure 8.11: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = 0.45, k = 2$ with three different initial conditions: $[[x(0) = 0.5, y(0) = 1.571348403, Q(0) = -0.9, W(0) = 0.5], [x(0) = 0.3, y(0) = 2.012307808, 8, Q(0) = -0.99, W(0) = 0.3], [x(0) = 0.8, y(0) = 1.29094449, Q(0) = -0.9, W(0) = 0.15]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.11 that $-2 < y < 2.3; Q \to -1$ and $W \to 0$ which corresponds to P_6^* at $c = \frac{1}{2}$.

Case (2b): when $k > \sqrt{\frac{3}{2}}$ and $c < \frac{1}{2}$

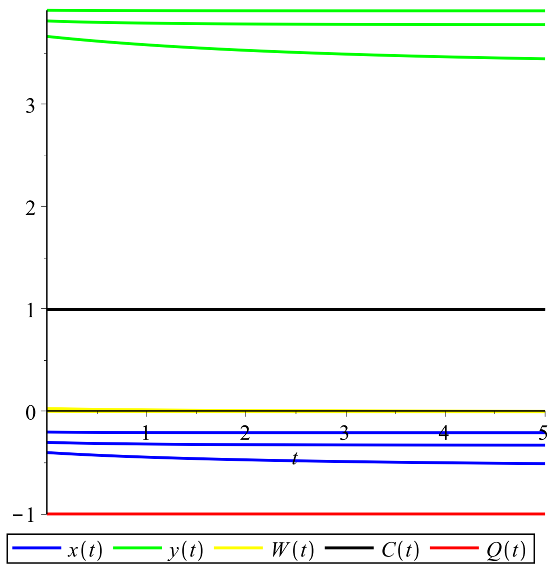

Figure 8.12: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = \frac{1}{4}, k = 2$ with three different initial conditions: $[[y(0) = 3.815547143, Q(0) = -0.99, x(0) = -0.3, W(0) = 0.01], [y(0) = 3.664096069, Q(0) = -0.9999, x(0) = -0.4, W(0) = 0.03], [y(0) = 3.919165217, Q(0) = -0.999, x(0) = -.2, W(0) = 0.003]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.12 that $x \to 0.5$; $2.3 < y < 4; Q \to -1$ and $W \to 0$ which corresponds to P_6^* at $c = \frac{1}{4}$.

Case (2c): when $k > \sqrt{\frac{3}{2}}$ and $c > \frac{1}{2}$

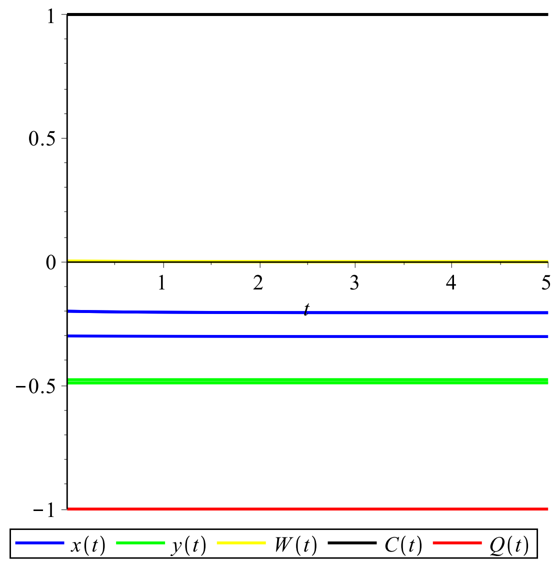

Figure 8.13: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = 2, k = 2$ with three different initial conditions:$[[x(0) = -0.2, y(0) = -.4898956522, Q(0) = -0.99, W(0) = 0.003], [x(0) = -0.3, y(0) = -0.4769693386, Q(0) = -0.99, W(0) = 0.001], [x(0) = -0.25, y(0) = -.4841218855, Q(0) = -0.99, W(0) = 0.002]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.13 that $0.79 < y < 1; Q \to -1$ and $W \to 0$ which corresponds to P_6^* at $c = 1$.

P_7 is a sink if there cases:

- If $a_2 > 0$ and $k < \sqrt{\frac{3}{2}}$; ${a_2}^2k^2 + 2a_1(1 - 2k^2) > 0$ and ${a_2}^2k^2(1 - 6k^2) + 2a_1(1 - 2k^2)^2 > 0$.
- If $a_2 < 0$, $k < \sqrt{\frac{3}{2}}$; and ${a_2}^2k^2 + 2a_1(1 - 2k^2) < 0$.
- If $a_2 < 0$, $k > \sqrt{\frac{3}{2}}$; and ${a_2}^2 > \frac{2a_1}{3k^2}(2k^2 - 3); {a_2}^2k^2 + 2a_1(1 - 2k^2) < 0$

Point P_7**:** We plot the 4D system in (8.2.1) and we obtain that P_7 is a sink in there cases:

Case(1): when $a_1 > 0;\ a_2 > 0, \quad k < \sqrt{\frac{3}{2}}$

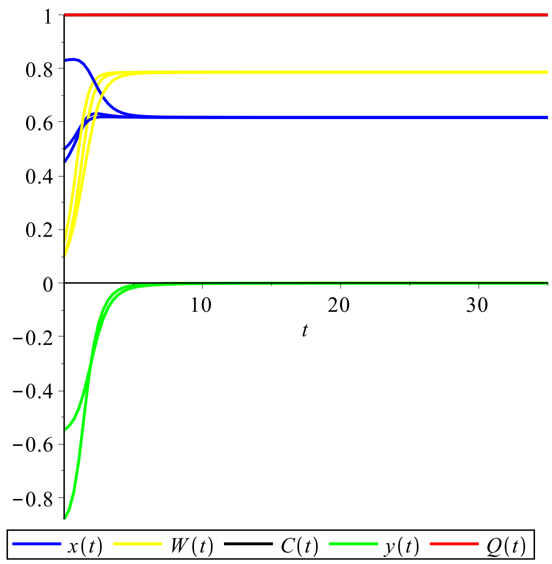

Figure 8.14: Plot of the system (8.2.1) with $a_1 = 1 = a_2, c = 1, k = 0.45$ with three different initial conditions: $[[x(0) = 0.45, y(0) = -0.8803408431, Q(0) = 0.99, W(0) = 0.15], [x(0) = 0.83, y(0) = -0.5487257967, Q(0) = 0.99, W(0) = 0.1], [x(0) = 0.5, y(0) = 0.8660253980, Q(0) = 0.9, W(0) = 0.15]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.14 that $x \to 0.6; y \to 0; Q \to 1$ and $W \to 0.8$ which corresponds to P_7 at $c = 1;\ a_1 = 1 = a_2$.

Case(2): when $a_1 > 0;\ a_2 < 0, \quad k < \sqrt{\frac{3}{2}}$

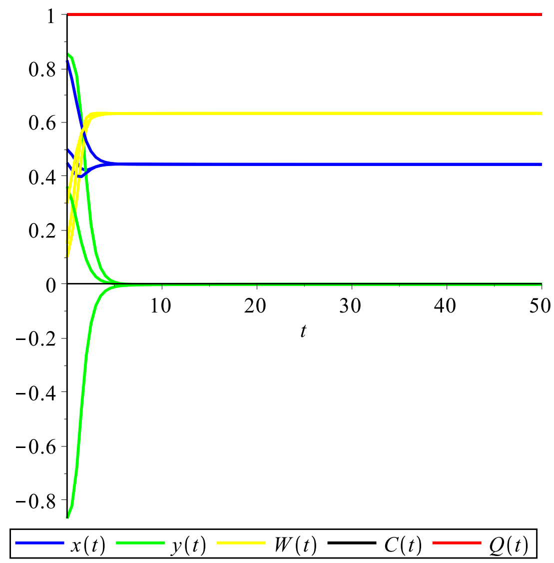

Figure 8.15: Plot of the system (8.2.1) with $a_1 = 1;\ a_2 = -\frac{1}{2}, c = 1, k = 0.75$ with three different initial conditions: $[[x(0) = 0.83, y(0) = -0.5395368384, Q(0) = 0.99, W(0) = 0.1], [x(0) = 0.5, y(0) = 0.8660253922, Q(0) = 0.99, W(0) = 0.0001], [x(0) = 0.45, y(0) = -0.86674675786, Q(0) = 0.9, W(0) = 0.15]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.15 that $x \to 0.4; y \to 0; Q \to 1$ and $W \to 0.6$ which corresponds to P_7 at $c = 1;\ a_1 = 1;\ a_2 = -\frac{1}{2}$.

Case(3): when $a_1 > 0;\ a_2 < 0, \quad k > \sqrt{\frac{3}{2}}$

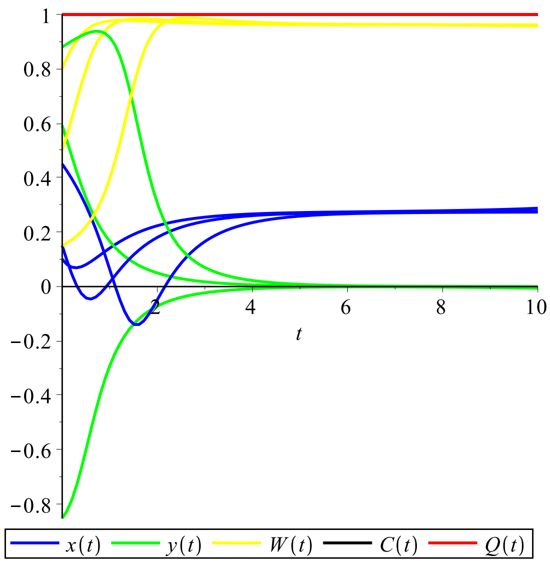

Figure 8.16: Plot of the system (8.2.1) with $a_1 = 1;\ a_2 = -1, c = 1, k = 2$ with three different initial conditions: $[[x(0) = 0.45, y(0) = 0.8803408431, Q(0) = 0.99, W(0) = 0.15], [x(0) = 0.15, y(0) = -0.8529361055, Q(0) = 0.99, W(0) = 0.5], [x(0) = 0.1, y(0) = 0.5916079783, Q(0) = 0.9, W(0) = 0.8]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.16 that $x \to 0.2; y \to 0; Q \to 1$ and $W \to 0.97$ which corresponds to P_7 at $c = 1;\ a_1 = 1;\ a_2 = -\frac{1}{2}$.

Point P_8**:** We plot the 4D system in (8.2.1) and we obtain that P_8 is sink when $a_2 < 0$ and $k > \sqrt{\frac{3}{2}}$.

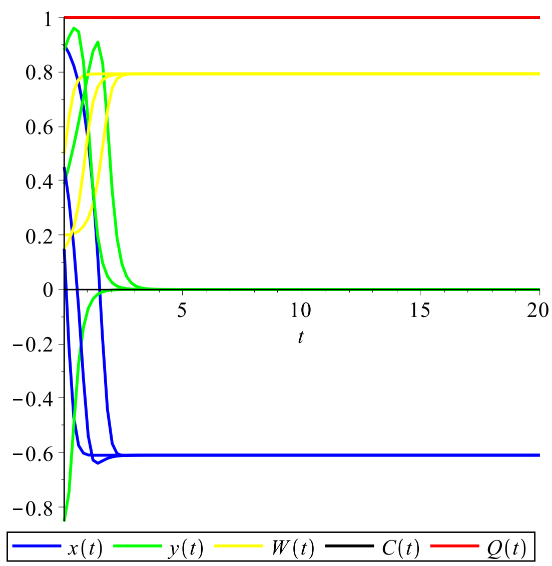

Figure 8.17: Plot of the system (8.2.1) with $a_1 = 1, a_2 = -1, c = 1, k = 1$ with three different initial conditions: $[[x(0) = 0.9, y(0) = 0.3872983346, Q(0) = 0.99, W(0) = 0.2], [x(0) = 0.45, y(0) = 0.8803408431, Q(0) = 0.99, W(0) = 0.15], [x(0) = 0.15, y(0) = -0.8529361055, Q(0) = 0.9, W(0) = 0.5]]$. Note that $C(t) = x^2 + c^2y^2 + a_1W^2$ as $t \to \infty$.

Discussion: We can see from figure 8.17, $y \to 0$; $W \to 0.8$; $x \to -0.6$ and $Q \to 1$. which corresponds to P_8 at $c = 1$; $a_1 = 1$; $a_2 = -1, k = 2$.

Inflation:

Evaluating each of the equilibrium point in table (8.1) at the deceleration parameter in equation (8.1.7) we obtain

1. q at P_1:

$$q\big|_{P_1} = \frac{1}{2c^2} > 0, \tag{8.2.66}$$

2. q at P_2:

$$q\big|_{P_2} = \frac{1}{2c^2} > 0, \tag{8.2.67}$$

3. q at P_3^*:

$$q\big|_{P^*{}_3} = 2 > 0, \tag{8.2.68}$$

4. q at P_4^*:

$$q\big|_{P^*{}_4} = 2 > 0, \tag{8.2.69}$$

5. q at P_5^*:

$$q\big|_{P^*{}_5} = 2 > 0, \tag{8.2.70}$$

6. q at P_6^*:

$$q\big|_{P^*{}_6} = 2 > 0, \tag{8.2.71}$$

7. q at P_7:

$$q\big|_{P_7} = \frac{\sqrt{2}a_2k^2\sqrt{B} - (a^2{}_2k^2 + 2a_1) + 4a_1k^2}{(a^2{}_2k^2 + 2a_1)}. \tag{8.2.72}$$

8. q at P_8:

$$q\big|_{P_8} = \frac{-\sqrt{2}a_2k^2\sqrt{B} - (a^2{}_2k^2 + 2a_1) + 4a_1k^2}{(a^2{}_2k^2 + 2a_1)}. \tag{8.2.73}$$

9. q at P_9:

$$q\big|_{P_9} = \frac{\sqrt{2}a_2k^2\sqrt{B} - (a^2{}_2k^2 + 2a_1) + 4a_1k^2}{(a^2{}_2k^2 + 2a_1)}, \tag{8.2.74}$$

10. q at P_{10}:

$$q\big|_{P_{10}} = \frac{-\sqrt{2}a_2k^2\sqrt{B} - (a^2{}_2k^2 + 2a_1) + 4a_1k^2}{(a^2{}_2k^2 + 2a_1)}, \tag{8.2.75}$$

11. q at P_{11}:

$$q\Big|_{P_{11}} = 2 > 0, \tag{8.2.76}$$

12. q at P_{12}:

$$q\Big|_{P_{12}} = 2 > 0, \tag{8.2.77}$$

13. q at P_{13}:

$$q\Big|_{P_{13}} = 2 > 0, \tag{8.2.78}$$

14. q at P_{14}:

$$q\Big|_{P_{14}} = 2 > 0. \tag{8.2.79}$$

There are three cases where P_7 has inflationary attractor:

- If $a_2 > 0$; $\quad k < \sqrt{\frac{3}{2}}$; $\quad 3{a_2}^2k^2 + 2a_1(1-2k^2) > 0$ and ${a_2}^2k^2(1-6k^2) + 2a_1(1-2k^2)^2 > 0$.

- If $a_2 < 0$; $\quad k < \sqrt{\frac{3}{2}}$; $\quad 3{a_2}^2k^2 + 2a_1(1-2k^2) < 0$ and ${a_2}^2k^2(1-6k^2) + 2a_1(1-2k^2)^2 < 0$.

- If $a_2 < 0$, $k > \sqrt{\frac{3}{2}}$; and ${a_2}^2 > \frac{2a_1}{3k^2}(2k^2-3)$; ${a_2}^2k^2 + 2a_1(1-2k^2) < 0$.

All other equilibrium points do not have an inflationary solution since q positive at each equilibrium points.

Chapter 9

Discussion

In part II of this thesis, we have studied spherically symmetric Einstein-Aether models with a scalar field; whose potential depends on the time-like "Aether" vector field through the expansion and shear scalars. These models are also solutions of the IR limit of Horava gravity [19]. We used the 1+3 frame formalism [5, 44] to write down the evolution equations for non-comoving scalar field spherically symmetric models. We also introduced bounded normalized variables. The formalism is particularly well-suited for numerical and qualitative analysis.

In particular, we considered the "spatially homogeneous Kantowski-Sachs models". We investigated a special case where we assumed

$$c_\theta \equiv c_1 + 3c_2 + c_3 = 0, \quad \text{and} \quad a_3 = 0, \tag{9.0.1}$$

and analyzed the qualitative behaviour. First, we derived the general evolution equations in terms of expansion-normalized variables, which reduce to a five dynamical system with two constraints. Second, one of the constraints allow us to eliminate one variable (z) globally which leads to a four dimensional dynamical system with one constraint. Third, we used the $W-$ substitution, which leads to three dimensional dynamical system. Finally, we have studied the local stability of the equilibrium points of the dynamical system, corresponding to physically realistic solutions. We are especially interested in the possible inflationary behaviour of the models for different ranges of the parameters where $a_1 > 0, c > 0, k > 0$ and a_2 could be either positive or negative signs.

For $P_{1,2}$ and the lines of equilibrium points $P^*_{3,4,5,6}$, we have W equal to zero. Therefore, we can not globally make a $W-$ substitution to analyze the stability of such an equilibrium points. Thus, to study the stability in the neighbourhood of these equilibrium points we eliminate the variable y. We found that when $c < \frac{1}{2}$; $k > \sqrt{\frac{3}{2}}$, P_2 is a sink and P^*_3; P^*_6 are sinks in the portion $-2 < y^* < \frac{1}{c}\sqrt{1 - \frac{3}{2k^2}}$. In addition, for any value of k, P^*_5 is a sink in two portion $c < \frac{1}{2}; \quad -2 < y^* < \frac{1}{c}$ and $c > \frac{1}{2}; \quad -\frac{1}{c} < y^* < \frac{1}{c}$.

For the points $P_{7,8,9,10,11,12,13,14}$, we used the W substitution to study the stability. We found that when $k < \sqrt{\frac{3}{2}}$, there is only the unique shear-free, zero curvature (FLRW) inflationary future attractor at P_7. When $k < \sqrt{\frac{3}{2}}$ point P_8 does not exist. For $k > \sqrt{\frac{3}{2}}$, P_8 is sink but not inflationary.

For $P_{11,12,13,14}$, we found these equilibrium point have a pair of purely complex eigenvalues and the third eigenvalue is real. We analysis these points at the invariant $Q = 1$, which leads to two dynamic system. Then, reduced to dynamic system with one variable. Finally, we used the multiple scale method to analysis the approximation solution of reduce system and we found that there is oscillation. To illustrate this, for instance, P_{11} have a pair of purely complex eigenvalue and the third eigenvalue is negative when these two conditions hold; $c < \frac{1}{2}$ and $3({a_2}^2k^2 + 2a_1) < 4a_1k^2(1 - 4c^2)$. After plugging $Q = 1$, we studied the simplify differential equation

$$Y'' + w^2Y = \epsilon(\alpha_1 Y^2 + \alpha_2 Y'Y + \alpha_3 {Y'}^2) + \epsilon^2(\alpha_4 Y^3 + \alpha_5 Y'Y^2 + \alpha_6 {Y'}^2 Y + \alpha_7 {Y'}^3), \quad (9.0.2)$$

;where the parameters are defined earlier, using the multiple scale method and we found that there is oscillations which implies that P_{11} is center, similarly for P_{13}, P_{14}.

In future work we shall investigate the general Kantowski-Sachs models when $a_3 \neq 0$ and $c_\theta \neq 0$. In particular, it would be of interest to determine whether or not these model will have more inflationary solutions than the cases we have studied in this part of the thesis.

Chapter 10

Conclusion

In this thesis we have investigated two applications of cosmological models in Einstein-Aether models with scalar field, which are also solutions of the IR limit of Horava gravity [32]. We used the 1+3 frame formalism [30, 44, 45] to write down the evolution equations. In each of these two class of models, we derive the evolution equations in terms of expansion-normalized variables, which reduce to a dynamical system. Then, we study the local stability of the equilibrium points of the dynamical system corresponding to physically realistic solutions. We are especially interested in the possible inflationary behaviour of the models.

In the first application, we investigated the qualitative behaviour of isotropic and Anisotropic model. In the isotropic model, we study two sub cases. In case (1) (i.e., $V(\phi,\theta)$, $\mu = 0$), we found there is we found that the equilibrium point P_0 is a sink but not inflationary and $P_{3,4}$ are inflationary saddles. In case (1b) (i.e., $V(\phi,\theta)$, $\mu > 0$), we found that if $\mu > \mu_c$ there is an inflationary attractor at P_4 and there is inflationary sources at P_3 but there is no inflationary saddle and P_0 is attractor when $\mu < \mu_c$.

For the Anisotropic model, we did three sub-cases. For case (2a) (i.e., $V(\phi,\sigma,\theta)$,

$\mu > 0$ and $\nu \neq 0$), we found that P_4 is inflationary attractor when $\sqrt{6}\sqrt{4\,n^2+\nu^2}+\sqrt{6}\nu - 6\,\mu < 0$. In the case (2b) ((i.e., $V(\phi,\sigma,\theta)$, $\mu > 0$ and $\nu = 0$)), we found that P_4 is an inflationary attractor when $\mu > \sqrt{\frac{3}{2}}n$. Finally, in case (2c) ((i.e., $V(\phi,\sigma,\theta)$, $\mu = 0$ and $\nu \neq 0$), we found that there is no any inflationary attractor equilibrium point among the equilibrium point that we looked at it.

Possible future work related to this part is to look at the harmonic potential of the following form

$$V(\theta,\phi,\sigma) = \frac{1}{2}n^2\phi^2 + \mu\theta\phi + \nu\sigma\phi + [a_{20}\theta^2 + a_{11}\theta\sigma + a_{02}\sigma^2], \tag{10.0.1}$$

and investigate the inflationary behaviour of this model.

For the second application, we investigated spherically symmetric cosmological models in Einstein-aether theory with scalar field. In particular, we consider special case of the spatially homogeneous Kantowski-Sachs models using appropriate bounded normalized variables, where we assume $c_\theta = 0, a_3 = 0$. We found that there there is only one inflationary attractor solution at P_7 when $k < \sqrt{\frac{2}{3}}$. Possible future work related to this application is to look at the general Kantowski-Sachs models when $a_3 \neq 0$ and $c_\theta \neq 0$.

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