VOLUME-PRESERVING COORDINATE GAUGES IN LINEAR PERTURBATION THEORY

by

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Submitted in partial fulfillment of the
requirements for the degree of
Master of Science

at

Dalhousie University
Halifax, Nova Scotia
December 2012

DALHOUSIE UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

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Dated: December 21, 2012

Supervisor: ____________________

Reader: ____________________

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DALHOUSIE UNIVERSITY

DATE: December 21, 2012

AUTHOR: D. Leigh Herman

TITLE: VOLUME-PRESERVING COORDINATE GAUGES IN LINEAR PERTURBATION THEORY

DEPARTMENT OR SCHOOL: Department of Mathematics and Statistics

DEGREE: M.Sc. CONVOCATION: May YEAR: 2013

Signature of Author

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Abstract

The main goal of this thesis is to present cosmological perturbation theory (based on the standard Friedmann cosmological model) in volume-preserving coordinates, which then provides a suitable basis for studies in cosmological averaging. We review perturbation theory to second order, allowing for averaging to second order in future research. To solve the averaging problem we need a method of covariantly and gauge invariantly averaging tensorial objects on a background manifold. This is a very difficult problem. However, the definition of an average takes on a particularly simple form when written in a system of volume-preserving coordinates. Therefore, we develop a three dimensional and a four dimensional volume-preserving coordinate gauge in this thesis that can be used for averaging in cosmological perturbation theory.

List of Abbreviations and Symbols Used

$V_{\mathcal{D}}$ .... volume of a domian D

h .... determinant of the 3-metric

$\mathbf{x}$ .... generic 3-vector

h_{ij} .... induced 3-metric

ρ .... energy density and density within a fluid

$a(\eta)$ .... scale factor with repsect to conformal time

U^{μ} .... general four-vector

δ^{ij} .... kronecker delta

$g_{\mu\nu}$ .... metric tensor (unperturbed)

$\delta g_{\mu\nu}$ .... perturbed metric tensor

B_i .... shift vector

C_{ij} .... tensor perturbation

ϕ .... scalar curvature

E .... scalar metric perturbation (component of shear)

F_i .... vector metric perturbations

H_{ij} .... tensor metric perturbations

S_i .... vector metric perturbation

ϕ .... lapse

B .... scalar component of the shift (component of the shear)

∂_i .... partial derivative with regard to space

H .... Hubble expansion rate

$\mathcal{H}$ .... conformal Hubble parameter

$P_{\mu\nu}$ .... projection tensor

Θ .... overall expansion rate of geometrical hypersurfaces

$\sigma_{\mu\nu}$ .... shear

$w_{\mu\nu}$ .... vorticity

a_{μ} .... acceleration

$K_{\mu\nu}$ .... extrinsic curvature of spatial hypersurfaces

${}^{(3)}R$ .... intrinsic curvature of spatial hypersurfaces

$T_{\mu\nu}$ .... energy momentum tensor

P .... isotropic pressure

$\Pi_{\mu\nu}$ .... anisotropic stress tensor

Π .... scalar component of the anisotropic stress tensor

ξ^{μ} .... generating vector field for active approach gauge transformations

$\pounds_{\xi}$ .... Lie derivative with respect to the generating vector field

α .... scalar temporal component of the generating vector field

β .... scalar spatial component of the generating vector field

γ^{i} .... vector spatial component of the generating vector field

∇^{-2} .... inverse Laplacian

τ .... 3D volume-preserving time coordinate

g .... metric tensor determinant

σ .... 4D volume-preserving time coordinate

Ω .... normalised density parameter

π .... the number Pi

G .... universal gravitational constant

V .... total covariant velocity perturbation

c_s^2 .... adiabatic speed of sound

δ .... density contrast

Acknowledgements

I owe a great deal of gratitude to my supervisor Dr. Alan Coley for his guidance and understanding and for being the mentor that I needed to move on to the next step. I would like to thank Dr. Iain Brown for all his help and dedication to helping me to understand perturbation theory and cosmological averaging and for constantly fixing my LaTex code, Dr. Timothy Clifton for pulling me out of the fire when I had over restricted my gauge and Joey Latta for all the late night talks and for always being ready with a drink when I needed it. I would also like to thank Dr. Tina Harriott for getting me interested in GR and Cosmology during my undergraduate degree at MSVU and for being so kind whenever I was in hard times. I must also thank Dr. Robert van den Hoogen, Matthew Stephen, Julien Ross, Chris Levy, Lucas Mol, Dr. David Irons, and Ben Hersey for never agreeing with anything I have ever said. Ever. I also must thank all the faculty and staff at Dalhousie, NSERC, the Patrick Lett Fund and all my students who let me flex my sarcasm muscle when I needed to brighten my mood. Lastly I must thank my family and close friends who have always stood beside me and always were ready with words to uplift and great food to keep me going. All of you deserve and have my sincere thanks and appreciation.

Chapter 1

Introduction

Cosmology is the study of the dynamics of the Universe on the largest of scales. A cosmologist takes the view that the large-scale Universe is a complete, self-contained system. The Universe is not a random clustering of matter arbitrarily distributed throughout space; rather, its constituents, through gravitational interaction, have created a self-contained, dynamical system that we can study. In cosmology, we often take constituents to be galaxies. Compared to the size of the Universe, we consider these galaxies to be extremely small.

Observations show that galaxies are distributed fairly uniformly as there are no regions of the Universe that are either particularly dense nor particularly devoid of galaxies. This has allowed researchers to assume that the galaxies at the present time are spatially homogeneously distributed. The "Cosmological Principle" asserts that the universe is spatially homogeneous everywhere and is also isotropic in every orientation. The Universe, in other words, is the same at every point at a given time, meaning that an observer would be unable distinguish one spatial direction from another. Observations suggest on scales of $200 - 300$ Mpc and larger, the Universe appears to be homogeneous, see [98, 113].

The Wilkinson Microwave Anisotropy Probe (WMAP) was launched into space by NASA to allow measurement of the Cosmic Microwave Background (CMB). The CMB is radiation left over from the big bang at the beginning of the Universe. This CMB radiation has a temperature of 2.7 K which has been measured to an accuracy of 20μK. These precise measurements show us that there are anisotropies and inhomogeneities caused by an inhomogeneous distribution of radiation. Since the standard cosmological model does not account for these inhomogeneities in the CMB it is therefore necessary to study inhomogeneous models.

Also of interest are observations of Type Ia supernovae. These supernovae are considered to be the "standard candles" for measuring the expansion of the Universe since they have similar spectral time series, light curve shapes, and absolute magnitudes [94]. The furthest supernovae can be measured to be approximately seven billion light years away. The light from these supernovae was affected by the expansion of the Universe as it travelled through space. Therefore the light undergoes a Doppler shift. Since the supernovae are moving away from us, the light is red-shifted, meaning it has shifted towards longer wavelengths. As we view objects at greater and greater distances, measurements of the amount of red shift of the light from very distant objects indicate that the expansion of the Universe is accelerating.

The observations of anisotropies in the CMB from WMAP and the recent supernovae Type 1a data are both unable to be predicted and fully explained using a standard model without also introducing exotic fields into the model. Therefore, a new model must be constructed to account for this behaviour within the system.

1.1 Perturbation Theory

The standard cosmological model used to date is the Friedmann-Lemaître-Robertson-Walker (FLRW) model. This model is based on an assumption that the Universe is spatially homogeneous and isotropic. With this model we can evaluate the average expansion of the large scale Universe following Einstein's theory of general relativity (GR) using Einstein's field equations (EFE). Following Einstein's theory and using the FLRW model we have been able to describe the Universe from a very dense, hot, radiation dominated state, to the current, cooler, lower density, matter dominated state.

However, the FLRW model uses the assumption of spatial homogeneity which is unable to describe the complex distribution of matter and energy that we observe in the Universe around us. The aim of perturbation theory is to construct a model that can better describe the actual spatial inhomogeneity and anisotropic distribution of matter and energy. The FLRW model will be used as a background solution within which we will study the inhomogeneous perturbations order by order.

In order to use the FLRW model as a background solution to describe the inhomogeneous distribution of matter and energy, we will be assigning a mapping between the homogeneous background and the inhomogeneous perturbed spacetime. The FLRW model has an obvious time slicing of a four-dimensional (3+1) spacetime. Since GR has no preferred set of coordinates, we must choose a coordinate system that will assign a mapping between spacetime points in the inhomogeneous Universe and the homogeneous background model. However, the process of decomposing variables into perturbations and background is not a covariant procedure. Therefore, within GR we are free to choose any coordinate system with which to make our gauge, but the very process of constructing perturbative variables means that the gauge we choose will necessarily produce quantities which may not be physically interpretable. Bardeen was able to construct gauges while retaining physical interpretability by studying the behaviour of quantities on hypersurfaces [3]. Bardeen showed that by fixing four degrees of freedom within the metric, we are able to reinstate covariance into the theory and construct interpretable quantities [77]. The freedom to choose our coordinate system is known as the gauge freedom, or gauge problem, in GR perturbation theory. A better description of the gauge problem is presented in Chapter 5.

While addressing the gauge issue will be a large part of this thesis, the aim is to also provide a full review of perturbation theory to be used in future research. In particular, we must discuss how to construct a variety of gauge-invariant variables so that we will have a variety of gauges to be used in different cosmological models. The dynamical equations for general scalars, vectors, and tensor perturbations will be reviewed. At linear order, the dynamical equations are relatively simple since at linear order the scalar, vector, and tensor variables decouple from one another. At second order, things become much more difficult. At second order the perturbations involve terms which are quadratic in first order perturbations. Solving the linear order equations in a particular coordinate gauge at first order analytically will be possible, but doing so at second order is far more complicated. Solving the second order equations will be the subject of future research as the second order solutions may provide additional qualitative results.

1.2 Averaging in Cosmology

We have already briefly introduced an issue with the current standard cosmological model; the Universe is not spatially homogeneous or isotropic on local scales. Correcting the governing equations can be done theoretically by averaging the EFE [99]. Averaging the inhomogeneous spacetimes in Einstein's GR can lead to very different dynamical behaviour from the FLRW background. The difference in the dynamical behaviour is caused by the non-linearity of the Einstein tensor. Even if that metric that best describes the Universe is the FLRW metric, we are not guaranteed that the dynamical behaviour of the physical quantities will behave like FLRW quantities. Corrections within the dynamics may arise in the form of an effective fluid known as cosmological *backreaction*. The backreaction may have great effect on the dynamics including the expansion rate of the Universe. Therefore, a solution to the averaging problem is of great consequence in cosmology since the affect on the expansion rate could greatly change how we observe the Universe; i.e., how we interpret Type 1a supernovae distances. In order to solve the averaging problem we need a method for covariantly (and gauge-invariantly) averaging tensorial objects on a background manifold. Unfortunately, this is a very difficult problem.

While many different averaging schemes have been constructed, one technique of particular importance to this thesis was developed by Gasperini, Marozzi, Nugier, and Veneziano [39] in which they define a covariant and gauge-invariant formalism for averaging objects on light-cones. This formalism can analyse the effects of inhomogeneities on objects on a light-like hypersurface and objects on a two-surface embedded in a specific light-cone. However, in this averaging procedure it is uncertain whether a true average over the whole past light-cone can be obtained since averaged quantities are susceptible to inhomogeneities over the whole past-light cone. With more detailed calculations using this technique [39], we may be able to provide a physical surface on which the true average can be calculated. Another averaging scheme has been constructed in terms of bilocal operators which are covariant and linear. The averaged object will have the same tensorial character as the non-averaged object [24]. In any manifold with a volume n-form there exist locally volume-preserving divergence-free operators, in which the bilocal operator takes the simplest possible form, essentially identity maps. The definition of an average consequently takes on a particularly simple form when written in a system of volume-preserving coordinates (VPC). One selects a VPC coordinate system and uses the coordinate directions to define the bilocal operators and therefore the bilocal operators are not unique and are gauge dependent.

1.2.1 Background

The cosmological backreaction in perturbation theory has typically been evaluated in a specific coordinate system – synchronous coordinates. This coordinate system is locked to the cold dark matter (CDM), so that the CDM comoves with it. Synchronous gauge is constructed by using this synchronous coordinate system with gauge conditions and is both numerically useful and convenient since the averages preserve the number density of CDM perturbations; however, it is not necessarily the *best* choice of coordinates theoretically [13]. Another approach would be to instead work with flat gauge or longitudinal gauge, and an appropriate coordinate system. These gauges take on a much more simplistic form than synchronous gauge for the purposes of calculating the backreaction, since the spatial metric is purely diagonal. However, they have their own drawbacks, chiefly, the power spectra of the perturbations is extremely poorly behaved on superhorizon scales and the perturbations are also made problematic by the averaging scheme [13].

Past approaches to averaging within perturbation theory (see, e.g., Brown, Behrend, and Malik [15]) have been based on the "Buchert approach" [22], which involves performing a (3+1) split and defining a three-volume on the spatial hypersurface in which one averages

$$\langle A \rangle = \frac{1}{V_{\mathcal{D}}} \int A \sqrt{h} d^3\mathbf{x} \tag{1.1}$$

where h is the metric determinant of the induced three-metric h_{ij} and $V_{\mathcal{D}}$ is the volume of our domain. The volume, and the averages, within these domains are then dependent on the perturbations, which makes these calculations extremely tricky and conceptually displeasing.

We can instead consider other coordinate (or gauge) choices in which to take the averages. We can choose to work in a gauge that simplifies the numerical calculation of the average. For example, a gauge could be chosen such that to second order the effective energy density of the backreaction is of the form

$$\overline{\rho}_{\text{eff}} \propto \int P(k) A(k) B^*(k) \frac{\mathrm{d}k}{k}, \tag{1.2}$$

where $A(k)$ and $B^*(k)$ are linear perturbations and $P(k)$ is the primordial power spectra, typically assumed to arise from inflation, [14, 77]. In this form only quadratic combinations of linear perturbations contribute and are achieved through the uniform or flat curvature gauge. Averaging in this gauge, however, appears to be exceedingly unwieldy.

1.3 Gauges and Volume-Preserving Coordinates

Different gauges used in perturbation theory to study the backreaction lead to an ambiguity in the definition of the spatial volume which is, of course, of huge importance to the averaging procedure. From previous studies, (see [15]), it also does not appear that there is a gauge in which the backreaction terms take on a particularily simple form. The definition of average is not mathematically well defined in all applications. Therefore, to simplify our averaging procedure we shall work in a gauge with VPC.

In the flat gauge the spatial surfaces align with the surfaces of the FLRW background and the inhomogeneities are embedded in the choice of threading (the choice of spatial coordinates on a given conformal time hypersurface) effectively removing the curvature correction; see [15]. This gauge also reduces the metric determinant for scalar perturbations to a constant comoving volume of a spatial domain, which simplifies the averaging procedure [13].

The longitudinal gauge does not remove any of the specific individual correction terms; rather, the perturbation of the 3-metric is diagonalised, removing the anisotropic stress terms and we choose the shift (Eq. (2.12)) to vanish to complete the gauge. This vanishing shift causes the curvature correction to be considerably simplified; see [27] and [40]. The longitudinal gauge also provides a clear interpretation of the gravitational potential and spatial curvature. These quantities also remain small on all scales studied, see [15]. However, the longitudinal gauge is still quite complex when studying the backreaction, and although it yields the simplest correction term forms, it is not a natural gauge choice for introducing VPC.

There are two options which we will discuss in order to choose an appropriate VPC gauge which we introduce briefly here and then rigorously define these gauges in Chapter 7.

1.The 3D VPC Gauge

We will take a gauge which is already known and use VPC in that gauge. Of particular use will be a 3D gauge that uses the 3+1 foliation, which relies upon a non-covariant (3+1) split, especially when restricting ourselves to the scalar perturbations. We will choose to work in flat gauge even though it is a "comoving" volume-preserving gauge rather than a volume preserving coordinate gauge. Despite the fact that it is comoving, the flat gauge is appropriate since the volume element becomes simply $a^3(\eta)$, which cancels out in the average [13]. This gauge can be easily adapted to VPC.

2.The 4D VPC Gauge

The second option will require us to choose a 4D VPC system which is well motivated on a theoretical level [13]. This 4D VPC system will be, by definition, well suited for unimodular gravity, which we will be turning to in future research. The aim is to average in a 4D region in a VPC gauge at linear and higher orders. Of course, once we have developed the gauge and gauge transformation equations we will be able to transform quantities from any gauge in the VPC gauge, average the quantity, and then transform the quantities back into the original gauge for interpretation.

It deserves clarification that a choice of coordinates is different from a choice of gauge. A gauge choice will construct the mapping between the FLRW background and our "physical" perturbed surface. A gauge transformation will not change the background model but will affect the way the points on the perturbed surface are mapped to points on the background model. In GR we are free to use any coordinate system so we can use appropriate coordinates for the problem at hand. The freedom to make any coordinate choice on the other hand is an important feature arising from the covariance of GR. However, there are gauges which are more natural to VPC. When it comes to cosmology the gauge transformation selects the variables one will be taking as "physical" – that is, it chooses the variables to be calculated. Performing a coordinate transformation to enforce a unit metric determinant does not necessarily change the gauge.

1.4 Overview

In Chapter 2 through Chapter 4 we will discuss the generic equations needed for perturbation theory, following the thorough review of [77]. Chapter 5 will construct the necessary gauge transformations while Chapter 6 will construct the gauge invariant variables in the longitudinal, Chapter 6.1, and flat, Chapter 6.2, gauges. We will rigorously construct and discuss appropriate VPC gauges in Chapter 7 and how these gauges will be used in cosmological averaging. Chapter 8 will look at some of the dynamics of perturbation theory within the different types of perturbations. Finally in Chapter 9 we will conclude this thesis with a discussion of the results and the research prospects for the future.

Chapter 2

Perturbations in Cosmology

In Chapter 1 we discussed how the standard FLRW model is based on the assumption that the Universe is spatially homogeneous and isotropic. This FLRW model gives us an approximate description of the structure and dynamics of the observable Universe. Therefore, a perturbed approach which describes the physical quantities to be composed of background and perturbations is presumed to be valid.

Previous conventions have split the four dimensional FLRW spacetime into a (3+1) decomposition; the same convention will be used here. This chapter will define arbitrary perturbations for tensorial quantities and then start decomposing the vectors and tensors into "time" and "space" components on the spatial hypersurfaces.

For example, a comma is used to denote partial derivatives with respect to comoving spatial coordinates unless otherwise indicated; i.e.,

$$T_{,i} \equiv \frac{\partial T}{\partial x^i} \, . \tag{2.1}$$

A prime is used to denote derivatives with respect to conformal time which is a different convention to [77]. We mainly use the definitions and notation from Malik and Wands [77].

2.1 Defining Perturbations

As we discussed in Chapter 1, we will be splitting quantities into a homogeneous background and inhomogeneous perturbations

$$\mathbf{T}(\eta, x^i) = \mathbf{T}_0(\eta) + \delta\mathbf{T}(\eta, x^i) \, . \tag{2.2}$$

where the subscript zero in this equation indicates the homogeneous background quantities.

As we can see the background is time-dependent only and the inhomogeneous perturbations are dependent on both space and time. We will also be interested in higher orders of perturbations which will be expressed as part of a power series expansion of the inhomogeneous perturbation quantities

$$\delta\mathbf{T}(\eta, x^i) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \mathbf{T}_n(\eta, x^i)\,, \tag{2.3}$$

where the subscript n denotes the order of the perturbations and we include the small parameter ϵ. In the following chapters we shall omit the ϵ in an effort to keep the equations as simple as possible.

2.2 Decomposing Tensorial Quantities

The standard convention for arranging our 4 dimensional spacetime is to split them using a 3+1 foliation of constant time. This 3+1 convention was first introduced by Darmois in 1927 (see Ref. [41]) and popularised by Arnowitt, Deser and Misner [2] (for conditions on the existence of the foliation see Ref. [109]).

2.2.1 Vectors

In order for us to use perturbation theory to its full potential we need to be able to decompose all objects into constituent parts. Therefore, we seperate here an arbitrary four vector, $\mathcal{U}^\mu$, into temporal and spatial parts,

$$\mathcal{U}^\mu = \left[\mathcal{U}^0, \mathcal{U}^i\right]\,. \tag{2.4}$$

Here we identify $\mathcal{U}^0$ as a *scalar* on spatial hypersurfaces. We can further decompose the spatial part of the four-vector, $\mathcal{U}^i$, into a further *scalar* part $\mathcal{U}$ and a *vector* part $\mathcal{U}^i_{\rm vec}$,

$$\mathcal{U}^i \equiv \delta^{ij}\mathcal{U}_{,j} + \mathcal{U}^i_{\rm vec}\,, \tag{2.5}$$

where $\partial\mathcal{U}^i_{\rm vec}/\partial x^i = 0$. The designations *scalar* and *vector* refer back to Bardeen [3] as they are defined by their transformation behaviour on spatial hypersurfaces of $\mathcal{U}$ and $\mathcal{U}^i_{\rm vec}$.

In an isotropic Universe, like the one used in our FLRW background, there are no preferred directions. No preferred direction corresponds to there being no spatial vector part at zeroth order. However, there can be a non-zero temporal part:

$$\mathcal{U}^0_0 \neq 0\,, \qquad \mathcal{U}^i_0 = 0\,. \tag{2.6}$$

2.2.2 Tensors

A rank-two tensor can also be decomposed into time and spatial part, but a tensor will also have combined time and space components.

We require the metric tensor, $g_{\mu\nu}$, to be symmetric;

$$g_{\mu\nu} \equiv g_{\nu\mu} \,. \tag{2.7}$$

The symmetry of the metric tensor means the tensor has only ten independent components in four dimensions. We split the metric tensor into a background and a perturbed part first, using Eq. (2.2). It is useful to split the metric perturbation into different parts namely *scalar*, *vector* or *tensor* according to their transformation properties on spatial hypersurfaces [3, 104]. Each of these components are then expanded into first and higher order parts using Eq. (2.3).

First we can describe our background spacetime by a spatially flat FLRW background metric

$$ds^2 = a^2 \left[-d\eta^2 + \delta_{ij} dx^i dx^j \right] \,, \tag{2.8}$$

where η is conformal time and $a = a(\eta)$ is the scale factor. The cosmic time, measured by observers at fixed comoving spatial coordinates, x^i, is given by $t = \int a(\eta)\, d\eta$.

The perturbed part of the metric tensor can be written as

$$\delta g_{00} = -2a^2\phi \,, \tag{2.9}$$

$$\delta g_{0i} = a^2 B_i \,, \tag{2.10}$$

$$\delta g_{ij} = 2a^2 C_{ij} \,. \tag{2.11}$$

The $0i$ and the ij-components of the metric tensor can be further decomposed into *scalar*, *vector* and *tensor* parts

$$B_i = B_{,i} - S_i \,, \tag{2.12}$$

$$C_{ij} = -\psi\, \delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2} H_{ij} \,, \tag{2.13}$$

where ϕ, B, ψ and E are *scalar* metric perturbations, S_i and F_i are *vector* metric perturbations, and H_{ij} is a *tensor* metric perturbation, which we will now define.

The round brackets surrounding the indices of the spatial derivatives of the vector F_i indicates symmetrization.

The *scalar* metric perturbation, ϕ, lapse function, ψ, and curvature perturbations which make up the scalar shear, B and E, can be constructed from a scalar or scalar derivatives or from background quantities. We should note that any three-vector constructed from a scalar is curl-free; i.e., $B_{,[ij]} = 0$. *Vector* perturbations, S_i and F_i, are divergence-free and the *tensor* perturbation, H_{ij}, is a transverse and traceless tensor. Therefore, the vectors and tensor perturbations follow

$$\partial^i S_i = 0\,, \tag{2.14}$$

$$\partial^i F_i = 0\,, \tag{2.15}$$

$$\partial^j H_{ij} = H_{ij,}{}^{j} = 0\,, \tag{2.16}$$

$$H^i_i = 0. \tag{2.17}$$

When raising and lowering spatial indices of vector and tensor perturbations we use the comoving background spatial metric, δ_{ij}, so that, for instance, $H^j_i \equiv \delta^{jk} H_{ik}$.

We split the metric perturbation into these three types because it is possible to decouple the EFE at linear order; therefore we can solve each perturbation type separately. At higher orders, $n > 1$, the perturbation types no longer decouple within the governing equations [89].

With these different perturbation types we have four scalar functions, two spatial vector valued functions with three components each, and a symmetric spatial tensor with six components. These functions are subject to many constraints due to the construction of these perturbation types. There are four constraints from H_{ij} as it is transverse and traceless and two constraints from F_i and S_i as these vectors are divergence-free. With these constraints we have constructed variables that leave us with ten degrees of freedom. This is the same as the number of independent components of the metric perturbation.

The choice of variables is not unique and we follow the notation of Mukhanov, Feldman, and Brandenberger [86]. It will be useful to define explicitly the trace of the perturbed spatial metric as

$$C = C^i_i = -3\psi + \partial_a \partial^a E. \tag{2.18}$$

At first order the trace coincides with the perturbation of the determinant of the spatial metric. Including terms up to second order we have

$$\begin{aligned}
\det\left(\delta_{ij}+2C_{ij}\right) &= 1+2C+2\left(C^2-C_{ij}C^{ij}\right)\\
&= 1-6\psi+2\partial_a\partial^a E\\
&\quad +12\psi^2-8\psi\partial_a\partial^a E+2(\partial_a\partial^a E)^2-2E_{,ij}E_{,}{}^{ij}-2F_{i,j}F^{i}{}_{,}{}^{j}\\
&\quad -\frac{1}{2}H_{ij}H^{ij}-2E_{,ij}H^{ij}-2F_{i,j}H^{ij}\,.
\end{aligned} \tag{2.19}$$

The metric perturbations in Eqs. (2.9–2.13) include all orders. The complete metric tensor, up to and including second-order, is

$$\begin{aligned}
g_{00} &= -a^2\left(1+2\phi_1+\phi_2\right)\,,\\
g_{0i} &= a^2\left(B_{1i}+\frac{1}{2}B_{2i}\right)\,,\\
g_{ij} &= a^2\left[\delta_{ij}+2C_{1ij}+C_{2ij}\right]\,,
\end{aligned} \tag{2.20}$$

where the subscript numeral indicates the order of the perturbation. Also the first and second order perturbations can be further split according to Eqs. (2.12) and (2.13).

The contravariant metric tensor follows from the constraint (to the required order),

$$g_{\mu\nu}\; g^{\nu\lambda}=\delta_{\mu}{}^{\lambda}\,, \tag{2.21}$$

which up to second-order gives

$$\begin{aligned}
g^{00} &= -a^{-2}\left[1-2\phi_1-\phi_2+4{\phi_1}^2-B_{1k}{B_1}^k\right]\,,\\
g^{0i} &= a^{-2}\left[{B_1}^i+\frac{1}{2}{B_2}^i-2\phi_1 B_1^i-2B_{1k}{C_1}^{ki}\right]\,,\\
g^{ij} &= a^{-2}\left[\delta^{ij}-2{C_1}^{ij}-{C_2}^{ij}+4{C_1}^{ik}{C_{1k}}^{j}-{B_1}^i{B_1}^j\right]\,.
\end{aligned} \tag{2.22}$$

Chapter 3

Geometry of Hypersurface

In this chapter we consider the geometry of the foliation at linear order from [77]. A summary of the second order results can be found in Appendix D.

3.1 Timelike Vector Fields

We are able to use the perturbed metric given in Section 2.2.2 to implicitly define a unit time-like vector field orthogonal to constant η-hypersurfaces,

$$n_\mu \propto \frac{\partial \eta}{\partial x^\mu}\,, \tag{3.1}$$

subject to the constraint

$$n^\mu n_\mu = -1\,. \tag{3.2}$$

This vector field coincides with the four-velocity of matter and the expansion of the velocity field $\theta = 3H$ in the FLRW background, where H is the Hubble expansion rate. The conformal Hubble parameter is defined as

$$\mathcal{H} \equiv aH\,. \tag{3.3}$$

We will use the the vector field n^ν to calculate geometrical quantities defined by the perturbed metric tensor. It should be noted here that the vector field n^μ does not need to coincide to the four-velocity of matter fields at first order and beyond.

Up to and including second order, the covariant vector field is

$$n_\mu = -a\left[1 + \phi_1 + \frac{1}{2}\phi_2 + \frac{1}{2}\left(B_{1k}B_1^k - \phi_1^2\right), \mathbf{0}\right]\,, \tag{3.4}$$

and the contravariant vector field is

$$\begin{aligned} n^0 &= \frac{1}{a}\left[1 - \phi_1 - \frac{1}{2}\phi_2 + \frac{3}{2}\phi_1^2 - \frac{1}{2}B_{1k}B_1^k\right]\,, \\ n^i &= \frac{1}{a}\left[-\left(B_1^i + \frac{1}{2}B_2^i\right) + 2B_{1k}C_1^{ki} + \phi_1 B_1^i\right]\,. \end{aligned} \tag{3.5}$$

3.2 Geometrical Quantities

We can decompose the covariant derivative of a time-like unit vector field n_μ as follows [109]:

$$n_{\mu;\nu} = \frac{1}{3}\theta\,\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu n_\nu \,, \tag{3.6}$$

where the spatial projection tensor $\mathcal{P}_{\mu\nu}$, orthogonal to n^μ, is given by

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu . \tag{3.7}$$

The overall expansion rate, the (trace-free and symmetric) shear, the (antisymmetric) vorticity and the acceleration are

$$\theta = n^\mu{}_{;\mu} \,, \tag{3.8}$$

$$\sigma_{\mu\nu} = \frac{1}{2}\mathcal{P}_\mu{}^\alpha \mathcal{P}_\nu{}^\beta \left(n_{\alpha;\beta} + n_{\beta;\alpha}\right) - \frac{1}{3}\theta\mathcal{P}_{\mu\nu} \,, \tag{3.9}$$

$$\omega_{\mu\nu} = \frac{1}{2}\mathcal{P}_\mu{}^\alpha \mathcal{P}_\nu{}^\beta \left(n_{\alpha;\beta} - n_{\beta;\alpha}\right) \,, \tag{3.10}$$

$$a_\mu = n_{\mu;\nu} n^\nu \,. \tag{3.11}$$

On spatial hypersurfaces the expansion, shear, vorticity and acceleration coincide with their Newtonian counterparts in fluid dynamics [42, 102].

If we take the Lie derivative of the projection tensor we used in Eq. (3.7), $\mathcal{P}_{\mu\nu}$, along the vector field n^μ, we can express the extrinsic curvature of the hypersurface embedded in the higher-dimensional spacetime [109, 26]. The extrinsic curvature of the spatial hypersurfaces defined by n_μ is thus given by

$$K_{\mu\nu} \equiv \frac{1}{2}\pounds_n \mathcal{P}_{\mu\nu} = \mathcal{P}_\nu{}^\lambda n_{\mu;\lambda} = \frac{1}{3}\theta\,\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} \,. \tag{3.12}$$

To first order, the intrinsic curvature of spatial hypersurfaces is

$$^{(3)}R_1 = \frac{4}{a^2}\partial_a\partial^a\psi_1 \,. \tag{3.13}$$

The scalar part of the shear (3.9) up to first order is given by

$$\sigma_{1ij} = \left(\partial_i\partial_j - \frac{1}{3}\partial_a\partial^a\delta_{ij}\right) a\sigma_1 \,, \tag{3.14}$$

where we define the shear potential

$$\sigma_1 \equiv E_1' - B_1 \,. \tag{3.15}$$

The vector part and the tensor part are

$$\sigma^{\rm V}_{1ij} = a\left(F'_{1(i,j)} - B_{1(i,j)}\right)\,, \tag{3.16}$$

$$\sigma^{\rm T}_{1ij} = \frac{a}{2}h'_{1ij}\,. \tag{3.17}$$

To first order the acceleration is

$$a_i = \phi_{,i}\,. \tag{3.18}$$

The expansion rate up to first order is given by

$$\theta_1 = \frac{3}{a}\left[\mathcal{H} - \mathcal{H}\phi - \psi' + \frac{1}{3}\partial_a\partial^a\sigma\right]\,. \tag{3.19}$$

The overall expansion, up to second order is given by

$$\begin{aligned}\theta_2 = \frac{1}{a}\Bigg[\quad & 3\frac{a'}{a} - 3\frac{a'}{a}\phi_1 + {C_{1k}}^{k\prime} - {B_{1k,}}^{k} \\ & -\frac{3}{2}\frac{a'}{a}\left(\phi_2 - 3{\phi_1}^2\right) + \frac{1}{2}\left({C_{2k}}^{k\prime} - {B_{2k,}}^{k}\right) + \phi_1\left({B_{1k,}}^{k} - {C_{1k}}^{k\prime}\right) \\ & -\frac{3}{2}\frac{a'}{a}B_{1k}B_1^k - 2C_1^{kl}C'_{1kl} + 2C_1^{kl}B_{1l,k} + 2B_1^l{C_{1lk,}}^{k} - B_1^kC_1^{l}{}_{l,k}\Bigg]\,. \end{aligned} \tag{3.20}$$

The intrinsic spatial curvature, shear and acceleration of n_μ are given up to second order in Appendix D in a special case where $n_i \equiv \mathbf{0}$.

Chapter 4

Energy-Momentum Tensor for Fluids

GR allows us to describe the geometry of spacetime since spacetime is affected by the matter content of the Universe. The metric tensor is affected by the perturbations of the matter content as described by the energy-momentum tensor. In this chapter we will construct the energy-momentum tensor for a single fluid.

The four-velocity of matter, v^μ, is defined by

$$v^\mu = \frac{dx^\mu}{d\vartheta}\,, \tag{4.1}$$

where ϑ is the proper time comoving with the fluid, subject to the constraint

$$v_\mu v^\mu = -1\,. \tag{4.2}$$

The 4-velocity up to second order is given by

$$\begin{aligned}
v_0 &= -a\left[1+\phi_1+\frac{1}{2}\phi_2-\frac{1}{2}{\phi_1}^2+\frac{1}{2}v_{1k}v_1^k\right]\,,\\
v_i &= a\left[v_{1i}+B_{1i}+\frac{1}{2}\left(v_{2i}+B_{2i}\right)-\phi_1 B_{1i}+2C_{1ik}v_1^k\right]\,,\\
v^0 &= a^{-1}\left[1-\phi_1-\frac{1}{2}\phi_2+\frac{3}{2}{\phi_1}^2+\frac{1}{2}v_{1k}v_1^k+v_{1k}B_1^k\right]\,,\\
v^i &= a^{-1}\left(v_1^i+\frac{1}{2}v_2^i\right)\,.
\end{aligned} \tag{4.3}$$

The spatial part of the velocity can be split following Eq. (2.5) as

$$v^i \equiv \delta^{ij}v_{,j}+v_{\rm v}^i\,, \tag{4.4}$$

where v is the scalar component and $v_{\rm v}^i$ is the transverse vector component, $\partial^i v_i^{\rm v}=0$.

4.1 Single Fluid

The energy-momentum tensor of a fluid with density, ρ, isotropic pressure, P and 4-velocity, v^μ, as given by Eq. (4.3), is defined as

$$T_{\mu\nu} = \left(\rho+P\right)v_\mu v_\nu + P g_{\mu\nu} + \pi_{\mu\nu}\,, \tag{4.5}$$

see [33, 35, 49, 102, 112]. The anisotropic stress tensor $\pi_{\mu\nu}$ is split into first and second order parts as defined in Eq. (2.3),

$$\pi_{\mu\nu} \equiv \pi_{1\mu\nu} + \frac{1}{2}\pi_{2\mu\nu}\,, \tag{4.6}$$

and is subject to the constraints

$$\pi_{\mu\nu}v^{\nu} = 0\,, \qquad \pi^{\mu}{}_{\mu} = 0\,. \tag{4.7}$$

The anisotropic stress vanishes for a perfect fluid.

The anisotropic stress tensor decomposes into a trace-free scalar part, Π, a vector part, Π_i, and a tensor part, Π_{ij}, at each order according to

$$\pi_{ij} = a^2\left[\Pi_{,ij} - \frac{1}{3}\partial_a\partial^a\Pi\delta_{ij} + \frac{1}{2}\left(\Pi_{i,j} + \Pi_{j,i}\right) + \Pi_{ij}\right]\,. \tag{4.8}$$

The components for the stress energy tensor order by order starting with the background are

$$T^0_{\ 0} = -\rho_0\,, \qquad T^0_{\ i} = 0\,, \qquad T^i_{\ j} = \delta^i{}_j P_0\,, \tag{4.9}$$

at first order,

$$\begin{aligned} {}^1\delta T^0_{\ 0} &= -\delta\rho_1\,, && (4.10)\\ {}^1\delta T^0_{\ i} &= \left(\rho_0 + P_0\right)\left(v_{1i} + B_{1i}\right)\,, && (4.11)\\ {}^{(1)}\delta T^i_{\ j} &= \delta P_1\delta^i{}_j + a^{-2}\pi_{(1)}{}^i{}_j\,, && (4.12) \end{aligned}$$

and at second order,

$$\begin{aligned} {}^2\delta T^0_{\ 0} &= -\delta\rho_2 - 2\left(\rho_0 + P_0\right)v_{1k}\left(v_1{}^k + B_1{}^k\right)\,, && (4.13)\\ {}^2\delta T^0_{\ i} &= \left(\rho_0 + P_0\right)\left[v_{2i} + B_{2i} + 4C_{1ik}v_1{}^k - 2\phi_1\left(v_{1i} + 2B_{1i}\right)\right]\\ &\quad + 2\left(\delta\rho_1 + \delta P_1\right)\left(v_{1i} + B_{1i}\right) + \frac{2}{a^2}\left(B_1^k + v_1^k\right)\pi_{1ik}\,, && (4.14)\\ {}^2\delta T^i_{\ j} &= \delta P_2\,\delta^i{}_j + a^{-2}\pi_2{}^i{}_j - \frac{4}{a^2}C_1^{ik}\pi_{1jk} + 2\left(\rho_0 + P_0\right)v_1^i\left(v_{1j} + B_{1j}\right)\,. && (4.15) \end{aligned}$$

Chapter 5

Gauge Transformations

Within this chapter we review how one transforms quantities from one gauge to another. For a detailed review of the construction of gauge transformations see [3, 104, 85, 19, 74].

As was previously discussed in section 1.1, the gauge issue arises in perturbation theory when discussing the gauge transformations. Any time we separate a system into background and perturbations we inevitably break covariance [77]. The non-covariance of the procedure causes a gauge dependence, but since the background is fully covariant, the gauge dependence effects only the perturbations.

We retain as much covariance as possible by eliminating the degrees of freedom. In Chapter 6 we show how the gauge dependencies can be made to cancel out by constructing gauge-invariant variables, which means that the quantities will not lose their qualitative behaviour when transforming from one gauge to another. We will show here the transformation equations for *scalar*, *vector*, and *tensor* fields.

5.1 Active Approach to Gauge Transformations

The active approach to gauge transformations starts with the exponential map, which allows us to write down how a tensor $\mathbf{T}$ transforms up to second order once the vector field generator of the gauge transformation, ξ^μ, has been specified. The exponential map is

$$\widetilde{\mathbf{T}} = e^{\pounds_\xi}\mathbf{T}\,, \tag{5.1}$$

where $\pounds_\xi$ denotes the Lie derivative with respect to ξ^λ defined as

$$\pounds_\xi\varphi = \xi^\lambda\varphi_{,\lambda}, \tag{5.2}$$

where φ is an arbitrary scalar field.

The vector field generating the transformation, ξ^λ, up to second order is

$$\xi^\mu \equiv \epsilon\xi_1^\mu + \frac{1}{2}\epsilon^2\xi_2^\mu + O(\epsilon^3)\,. \tag{5.3}$$

The exponential map can be expanded as

$$\exp(\pounds_{\xi}) = 1 + \epsilon\pounds_{\xi_1} + \frac{1}{2}\epsilon^2\pounds_{\xi_1}^2 + \frac{1}{2}\epsilon^2\pounds_{\xi_2} + \dots \tag{5.4}$$

where we keep terms up to $O(\epsilon^2)$. Splitting the tensor $\mathbf{T}$ up to second order, as given in Eq. (2.3), and collecting terms of like order in ϵ, we find that tensorial quantities transform (see [85, 19]) at zero order as

$$\widetilde{\mathbf{T}_0} = \mathbf{T}_0 \,, \tag{5.5}$$

at first order as

$$\widetilde{\mathbf{T}_1} = \mathbf{T}_1 + \mathcal{L}_{\xi_1}\mathbf{T}_0 \,, \tag{5.6}$$

and at second order as

$$\widetilde{\mathbf{T}_2} = \left(\mathbf{T}_2 + \mathcal{L}_{\xi_2}\mathbf{T}_0 + \mathcal{L}^2_{\xi_1}\mathbf{T}_0 + 2\mathcal{L}_{\xi_1}\mathbf{T}_1\right) . \tag{5.7}$$

As can be seen in Eq.(5.5), the background is not affected by the transformation.

5.2 Four-Scalar Gauge Transformations

We will now apply the active approach by studying the simplest tensorial quantity, the four-scalar.

From Eqs. (2.2) and (2.3) we get the perturbed energy density, an example of a four-scalar, up to second order as

$$\rho = \rho_0 + \rho_1 + \frac{1}{2}\rho_2 \,, \tag{5.8}$$

where, as before, the subscript indicates the order of perturbation.

5.2.1 First Order

Before we can study the transformation behaviour of the perturbations at first order, we split the generating vector ξ_1^μ into a scalar temporal part α_1 and a spatial scalar and vector part, β_1 and ${\gamma_1}^i$. The generating vector field therefore becomes

$$\xi_1^\mu = \left(\alpha_1, {\beta_{1,}}^i + {\gamma_1}^i\right) \,, \tag{5.9}$$

where the vector part is divergence-free: $\partial_i {\gamma_1}^i = 0$.

Under a first-order transformation a four-scalar, such as the energy density, ρ, transforms from Eqs. (5.5) and (5.2)

$$\widetilde{\rho_1} = \rho_1 + \rho_0' \alpha_1 \,. \tag{5.10}$$

Therefore, the first-order density perturbation is fully determined by the time slicing, α_1. For a general scalar ϕ, $\widetilde{\phi_1} = \phi_1 + \phi_0' \alpha_1$ so $\widetilde{\phi_1}$ is gauge-invariant if $\phi_0' = 0$.

5.2.2 Second Order

At second order we split the generating vector ξ_2^μ again into a scalar time and scalar and vector spatial parts as

$$\xi_2^\mu = \left(\alpha_2, {\beta_{2,}}^i + {\gamma_2}^i\right) \,, \tag{5.11}$$

where the vector part is divergence-free $\partial_i {\gamma_2}^i = 0$. We then find from Eqs. (5.7) and (5.2) that a four scalar transforms as

$$\begin{aligned} \widetilde{\rho_2} = \rho_2 + \rho_0' \alpha_2 \quad &+ \quad \alpha_1 \left(\rho_0'' \alpha_1 + \rho_0' {\alpha_1}' + 2{\rho_1}'\right) \\ &+ \quad \left(2\rho_1 + \rho_0' \alpha_1\right)_{,k} \left({\beta_{1,}}^k + {\gamma_1}^k\right) . \end{aligned} \tag{5.12}$$

Thus, at second order, the gauge transformation is fully determined once we have specified the time-slicing at first and second order (α_1 and α_2) and have also specified the threading to first order (β_1 and γ_1^i).

5.3 Tensor Gauge Transformations

5.3.1 First Order

We can now calculate how the first order metric perturbations change under a gauge transformation. The Lie derivative of a covariant tensor $t_{\mu\nu}$ with respect to the generating vector field ξ^λ is given as

$$\pounds_\xi t_{\mu\nu} = t_{\mu\nu,\lambda} \xi^\lambda + t_{\mu\lambda} {\xi^\lambda}_{,\,\nu} + t_{\lambda\nu} {\xi^\lambda}_{,\,\mu} \,. \tag{5.13}$$

At first order, the metric perturbations transform according to Eq. (5.6).

The transformations for each individual metric function can be obtained from Eq. (5.13) using the method from [77]. From the Lie derivative in Eq. (5.13) we can show the transformation behaviour of C_{1ij} as

$$2\widetilde{C}_{1ij} = 2C_{1ij} + 2\mathcal{H}\alpha_1\delta_{ij} + \xi_{1i,j} + \xi_{1j,i}\,. \tag{5.14}$$

Following the method thoroughly discussed in [77] we can find the transformation behaviour of the spatial metric functions. The transformations of the scalar metric perturbations are

$$\widetilde{\phi_1} = \phi_1 + \mathcal{H}\alpha_1 + \alpha_1'\,, \tag{5.15}$$

$$\widetilde{\psi_1} = \psi_1 - \mathcal{H}\alpha_1\,, \tag{5.16}$$

$$\widetilde{B_1} = B_1 - \alpha_1 + \beta_1'\,, \tag{5.17}$$

$$\widetilde{E_1} = E_1 + \beta_1\,, \tag{5.18}$$

and the vector perturbations are

$$\widetilde{B_{1i}} = B_{1i} + \xi_{1i}' - \alpha_{1,i}\,, \tag{5.19}$$

$$\widetilde{S_1{}^i} = S_1{}^i - \gamma_1{}^{i\prime}\,, \tag{5.20}$$

$$\widetilde{F_1{}^i} = F_1{}^i + \gamma_1{}^i\,. \tag{5.21}$$

The first order tensor perturbation is found to be gauge-invariant,

$$\widetilde{h}_{1ij} = h_{1ij}\,, \tag{5.22}$$

by substituting Eqs. (5.15) to (5.21) into Eq. (5.14).

We note that the scalar shear potential, $\sigma_1 = E_1' - B_1$, defined in Eq. (3.15) transforms as

$$\widetilde{\sigma_1} = \sigma_1 + \alpha_1\,. \tag{5.23}$$

5.3.2 Second Order

The metric tensor transforms at second order, from Eqs. (5.7) and (5.13), as

$$\begin{aligned}
\widetilde{g^2_{\mu\nu}} = {} & g^2_{\mu\nu} + g^0_{\mu\nu,\lambda}\xi_2^\lambda + g^0_{\mu\lambda}\xi_2{}^\lambda{}_{,\nu} + g^0_{\lambda\nu}\xi_2{}^\lambda{}_{,\mu} + 2\Big[g^1_{\mu\nu,\lambda}\xi_1^\lambda + g^1_{\mu\lambda}\xi_1{}^\lambda{}_{,\nu} + g^1_{\lambda\nu}\xi_1{}^\lambda{}_{,\mu}\Big] \\
& + g^0_{\mu\nu,\lambda\alpha}\xi_1^\lambda\xi_1^\alpha + g^0_{\mu\nu,\lambda}\xi_1{}^\lambda{}_{,\alpha}\xi_1^\alpha + 2\Big[g^0_{\mu\lambda,\alpha}\xi_1^\alpha\xi_1{}^\lambda{}_{,\nu} + g^0_{\lambda\nu,\alpha}\xi_1^\alpha\xi_1{}^\lambda{}_{,\mu} + g^0_{\lambda\alpha}\xi_1{}^\lambda{}_{,\mu}\xi_1{}^\alpha{}_{,\nu}\Big] \\
& + g^0_{\mu\lambda}\left(\xi_1{}^\lambda{}_{,\nu\alpha}\xi_1^\alpha + \xi_1{}^\lambda{}_{,\alpha}\xi^\alpha_{1,\,\nu}\right) + g^0_{\lambda\nu}\left(\xi_1{}^\lambda{}_{,\mu\alpha}\xi_1^\alpha + \xi_1{}^\lambda{}_{,\alpha}\xi^\alpha_{1,\,\mu}\right)\,.
\end{aligned} \tag{5.24}$$

Following [77], we can again extract the perturbed spatial part of the metric, C_{2ij}, transformation at second order as

$$2\widetilde{C}_{2ij} = 2C_{2ij} + 2\mathcal{H}\alpha_2\delta_{ij} + \xi_{2i,j} + \xi_{2j,i} + \mathcal{X}_{ij}\,, \tag{5.25}$$

where $\mathcal{X}_{ij}$ contains the terms quadratic in the first order perturbations defined below in Eq. (5.30). By following the method in [77] we can extract the transformation equations for each individual component. The scalar metric perturbations transform as

$$\widetilde{\psi}_2 = \psi_2 - \mathcal{H}\alpha_2 - \frac{1}{4}\mathcal{X}^k_{\ k} + \frac{1}{4}\nabla^{-2}\mathcal{X}^{ij}_{\ \ ,ij}\,, \tag{5.26}$$

$$\begin{aligned}\widetilde{\phi}_2 \;=\; & \phi_2 + \mathcal{H}\alpha_2 + \alpha_2{}' + \alpha_1\left[\alpha_1{}'' + 5\mathcal{H}\alpha_1{}' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\alpha_1 + 4\mathcal{H}\phi_1 + 2\phi_1'\right] \\ & + 2\alpha_1{}'\left(\alpha_1{}' + 2\phi_1\right) + \xi_{1k}\left(\alpha_1{}' + \mathcal{H}\alpha_1 + 2\phi_1\right)_{,}^{\ k} \\ & + \xi_{1k}'\left[\alpha_{1,}^{\ \ k} - 2B_{1k} - \xi_1^{k\prime}\right]\,,\end{aligned} \tag{5.27}$$

$$\widetilde{E}_2 = E_2 + \beta_2 + \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathcal{X}^{ij}_{\ \ ,ij} - \frac{1}{4}\nabla^{-2}\mathcal{X}^k_{\ k}\,, \tag{5.28}$$

and

$$\widetilde{B}_2 = B_2 - \alpha_2 + \beta_2' + \nabla^{-2}\mathcal{X}_{\mathrm{B}}{}^k{}_{,k}\,, \tag{5.29}$$

where $\mathcal{X}_{ij}$ and $\mathcal{X}_{Bi}$ contains the terms quadratic in the first order perturbations. These terms are defined as

$$\begin{aligned}\mathcal{X}_{ij} \;\equiv\; & 2\Big[\left(\mathcal{H}^2 + \frac{a''}{a}\right)\alpha_1^2 + \mathcal{H}\left(\alpha_1\alpha_1' + \alpha_{1,k}\xi_1{}^k\right)\Big]\delta_{ij} \\ & + 4\Big[\alpha_1\left(C_{1ij}' + 2\mathcal{H}C_{1ij}\right) + C_{1ij,k}\xi_1{}^k + C_{1ik}\xi_1{}^k{}_{,j} + C_{1kj}\xi_1{}^k{}_{,i}\Big] \\ & + 2\left(B_{1i}\alpha_{1,j} + B_{1j}\alpha_{1,i}\right) + 4\mathcal{H}\alpha_1\left(\xi_{1i,j} + \xi_{1j,i}\right) - 2\alpha_{1,i}\alpha_{1,j} + 2\xi_{1k,i}\xi_1{}^k{}_{,j} \\ & + \alpha_1\left(\xi_{1i,j}' + \xi_{1j,i}'\right) + \left(\xi_{1i,jk} + \xi_{1j,ik}\right)\xi_1{}^k \\ & + \xi_{1i,k}\xi_1{}^k{}_{,j} + \xi_{1j,k}\xi_1{}^k{}_{,i} + \xi_{1i}'\alpha_{1,j} + \xi_{1j}'\alpha_{1,i}\,,\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}\mathcal{X}_{\mathrm{B}i} \equiv\; & 2\Big[\left(2\mathcal{H}B_{1i} + B_{1i}'\right)\alpha_1 + B_{1i,k}\xi_1^k - 2\phi_1\alpha_{1,i} + B_{1k}\xi^k_{1,\ i} + B_{1i}\alpha_1' + 2C_{1ik}\xi_1^{k\prime}\Big] \\ & + 4\mathcal{H}\alpha_1\left(\xi_{1i}' - \alpha_{1,i}\right) + \alpha_1'\left(\xi_{1i}' - 3\alpha_{1,i}\right) + \alpha_1\left(\xi_{1i}'' - \alpha_{1,i}'\right) \\ & + \xi_1^{k\prime}\left(\xi_{1i,k} + 2\xi_{1k,i}\right) + \xi_1^k\left(\xi_{1i,k}' - \alpha_{1,ik}\right) - \alpha_{1,k}\xi^k_{1,\ i}\,,\end{aligned} \tag{5.31}$$

where ∇^{-2} is the inverse Laplacian.

The following vector perturbations are parts of the three-metric perturbation rather than four-vectors and transform as

$$\widetilde{B}_{2i} = B_{2i} + \xi'_{2i} - \alpha_{2,i} + \mathcal{X}_{\mathrm{B}i} \,, \tag{5.32}$$

$$\widetilde{F}_{2i} = F_{2i} + \gamma_{2i} + \nabla^{-2}\mathcal{X}_{ik,}{}^{k} - \nabla^{-2}\nabla^{-2}\mathcal{X}^{kl}{}_{,kli} \,, \tag{5.33}$$

$$\widetilde{S}_{2i} = S_{2i} - {\gamma_2}^{i'} - \mathcal{X}_{\mathrm{B}i} + \nabla^{-2}\mathcal{X}_{\mathrm{B}}{}^{k}{}_{,ki} \,. \tag{5.34}$$

The tensor perturbation is not gauge invariant at second order as it is at first order. Therefore, it transforms at second order as

$$\begin{aligned}\widetilde{h}_{2ij} = {} & h_{2ij} + \mathcal{X}_{ij} + \frac{1}{2}\left(\nabla^{-2}\mathcal{X}^{kl}{}_{,kl} - \mathcal{X}^{k}_{k}\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\mathcal{X}^{kl}{}_{,klij} \\ & + \frac{1}{2}\nabla^{-2}\mathcal{X}^{k}_{k,ij} - \nabla^{-2}\left(\mathcal{X}_{ik,}{}^{k}{}_{j} + \mathcal{X}_{jk,}{}^{k}{}_{i}\right) \,. \end{aligned} \tag{5.35}$$

5.4 Four-Vector Gauge Transformations

To examine the transformation properties of four-vectors we will use the unit four-velocity v^{μ}, which we defined in Eq. (4.3).

5.4.1 First Order

To define the transformation of a four-vector to first order we use Eq. (5.5) and the definition of the Lie derivative. The Lie derivative of a covariant vector v_{μ} with respect to the generating vector field ξ^{λ} is defined as

$$\pounds_{\xi} v_{\mu} = v_{\mu,\alpha}\xi^{\alpha} + v_{\alpha}\xi^{\alpha}_{,\ \mu} \,. \tag{5.36}$$

Now we can explicitly write the four-vector transformation as

$$\widetilde{\delta\mathcal{U}_{1\mu}} = \delta\mathcal{U}_{1\mu} + \mathcal{U}'_{(0)\mu}\alpha_1 + \mathcal{U}_{(0)\lambda}\xi^{\lambda}_{1,\mu} \,, \tag{5.37}$$

where we used the fact that in a FLRW spacetime, background quantities are time dependent only.

For the specific example of the four-velocity, defined in Eq. (4.3), we find

$$\widetilde{v_{1i}} + \widetilde{B_{1i}} = v_{1i} + B_{1i} - \alpha_{1,i} \,. \tag{5.38}$$

In addition, using, the decompositions of vectors given in Chapter (2.2.1), we get the first order transformations for the scalar part and the vector part are respectively

$$\widetilde{v_1} = v_1 - \beta_1' \,, \tag{5.39}$$

$$\widetilde{v^i_{\rm vec1}} = v^i_{\rm vec1} - {\gamma_1}^{i\prime} \,. \tag{5.40}$$

5.4.2 Second Order

At second order we find that a four-vector transforms, using Eqs. (5.7) and (5.36), as

$$\begin{aligned}\widetilde{\mathcal{U}_{2\mu}} \quad = \quad & \mathcal{U}_{2\mu} + \mathcal{U}'_{(0)\mu}\alpha_2 + \mathcal{U}_{(0)0}\alpha_{2,\mu} + \mathcal{U}''_{(0)\mu}\alpha_1^2 + \mathcal{U}'_{(0)\mu}\alpha_{1,\lambda}\xi_1^\lambda \\ & + 2\mathcal{U}'_{(0)0}\alpha_1\alpha_{1,\mu} + \mathcal{U}_{(0)0}\left(\xi_1^\lambda\alpha_{1,\mu\lambda} + \alpha_{1,\lambda}\xi^\lambda_{1,\mu}\right) + 2\left(\delta\mathcal{U}_{1\mu,\lambda}\xi_1^\lambda + \delta\mathcal{U}_{1\lambda}\xi^\lambda_{1,\mu}\right) \,,\end{aligned} \tag{5.41}$$

where, as before, we used $\mathcal{U}_{(0)\mu} \equiv \mathcal{U}_{(0)\mu}(\eta)$ and $\mathcal{U}_{(0)i} = \mathbf{0}$ for the background. Brackets were used in Eq. (5.41) to help with the identification of the order of the quantites and will only be used when the subscripts become overly cluttered.

At second order we have components within quantities which are combined scalar and vector parts. Therefore the transformations have to account for these terms. The four-velocity, Eq. (4.3), transforms as

$$\widetilde{v_{2i}} = v_{2i} - \xi'_{2i} + \mathcal{X}_{{\rm v}i} \,, \tag{5.42}$$

where $\mathcal{X}_{{\rm v}i}$ contains the terms quadratic in the first order perturbations. $\mathcal{X}_{{\rm v}i}$ is given by

$$\begin{aligned}\mathcal{X}_{{\rm v}i} \equiv \quad & \xi'_{1i}\left(2\phi_1 + \alpha_1' + 2\mathcal{H}\alpha_1\right) - \alpha_1\xi''_{1i} \\ & -\xi_1^k\xi'_{1i,k} + \xi_1^{k\prime}\xi_{1i,k} - 2\alpha_1\left(v'_{1i} + \mathcal{H}v_{1i}\right) + 2v_{1i,k}\xi_1^k - 2v_1^k\xi_{1i,k} \,.\end{aligned} \tag{5.43}$$

Using Eq. (5.32), we have already substituted for the transformation of the metric perturbation B_{2i}. To get the transformation at second order for the velocity we decompose the second order transformation, Eq. (5.42), into scalar and vector parts

$$\widetilde{v_2} = v_2 - \beta_2' + \nabla^{-2}{\mathcal{X}_{\rm v}}^k{}_{,k} \,, \tag{5.44}$$

$$\widetilde{v^i_{\rm vec(2)}} = v^i_{\rm vec(2)} - {\gamma_2}^{i\prime} + \mathcal{X}_{{\rm v}i} - \nabla^{-2}{\mathcal{X}_{\rm v}}^k{}_{,ki} \,. \tag{5.45}$$

Chapter 6

Gauge-Invariant Variables

As we briefly discussed at the beginning of Chapter 5, the splitting of the quantities into background and perturbation is not a covariant procedure. The inevitable break in covariance is because the choice of gauge gives each quantity its qualitative behaviour. In order to have covariance of the quantities under gauge transformations we need to cancel out any gauge dependencies in the quantities. By forming *gauge-invariant variables* we can once again establish covariance. These gauge-invariant variables were first introduced and studied by Bardeen [3].

The gauge-invariant variables are formed by studying the degrees of freedom. The metric itself has ten degrees of freedom, of which six are contained in the perturbative terms. The six degrees of freedom within the perturbations are: two scalar freedoms which form under arbitrary coordinates, two more degrees of freedom come from the gradient and divergenceless vector perturbation modes, and the last two degrees of freedom are in the polarisations of the tensor perturbations. That leaves four degrees of freedom in the metric to be fixed in order to ensure that the results are still physically interpretable. Since the observations do not depend on the gauge choice, this decision to fix the last four degrees of freedom is known as the *gauge issue* in perturbation theory.

Here we must state that gauge-invariance is different than *gauge independence*. In Chapter 5.3 the first order tensor metric perturbation, h_{1ij}, is gauge independent. On the other hand, the scalar curvature perturbation, ψ_1, is very different under different time-slicing, making it dependent upon the choice of gauge.

As when we constructed the gauge transformation equations, at first order the equations are simple since all the terms decouple. However, at second order, due to the quadratic terms of first order perturbations, the gauge-invariant variables are much more complicated to construct. However, once we do choose a gauge, we will follow that choice throughout all orders.

6.1 Longitudinal Gauge

6.1.1 First Order

By studying the transformations in Eqs. (5.15–5.18), Bardeen [3] was able to construct quantities that are explicitly invariant under gauge transformations. At linear order we take the generating vector field's temporal component to be on spatial hypersurfaces with vanishing shear. We find from Eqs. (5.17),(5.18) and (3.15) that the shear scalar transforms as $\widetilde{\sigma_1} = \sigma_1 + \alpha_1 = 0$. This transformation implies that we should perform a transformation starting with arbitrary coordinates so that

$$\alpha_{1\ell} = -\sigma_1 = B_1 - E_1' \tag{6.1}$$

where the ℓ denotes the value in the longitudinal gauge. If we make $\widetilde{E_{1\ell}} = 0$, which requires from Eq. (5.18)

$$\beta_{1\ell} = -E_1 \,, \tag{6.2}$$

we now have a fully specified generating vector for the scalar perturbation quantities.

The last two scalar metric perturbations, ϕ_1 and ψ_1, come from Eqs. (5.15) and (5.16) as

$$\widetilde{\phi_{1\ell}} = \phi_1 + \mathcal{H}(B_1 - E_1') + (B_1 - E_1')' \,, \tag{6.3}$$

$$\widetilde{\psi_{1\ell}} = \psi_1 - \mathcal{H}\left(B_1 - E_1'\right) \,, \tag{6.4}$$

These are the Bardeen potentials denoted by $\Phi_A Q^0$ and $-\Phi_H Q^0$, respectively, in Bardeen's notation [3]. They also coincide with the scalar metric perturbations in [86].

The extension to include vector and tensor metric perturbations is called the Poisson gauge [9, 19, 23, 77]. By fixing the spatial component of the generating vector field to be zero, $\widetilde{S_1^i} = 0$, and using Eq. (5.20), we get a fixed vector part of the spatial gauge transformation

$$\gamma_{1l}^i = \int S_1^i d\eta + \hat{\mathcal{C}}_1^i(x^j), \tag{6.5}$$

where $\hat{\mathcal{C}}_1^i$ is an arbitrary three-vector dependent on the choice of spatial coordinates on the hypersurface. The last gauge invariant vector metric perturbation in Poisson gauge is

$$\widetilde{F^i_{1l}} = F_1^{i\prime} + \int S^i_1 d\eta + \hat{\mathcal{C}}^i_1(x^j). \tag{6.6}$$

The EFE for the Poisson gauge can be found in Appendix (A).

6.1.2 Second Order

We can extend the longitudinal, or Poisson, gauge to higher orders using the same principle for constructing gauge invariant variables in first order. We choose the specific vector field ξ^μ that generates our transformations, Eq. (5.1), from an arbitrary gauge choice [75, 87].

In Chapter (6.1.1) we specified the Poisson gauge to fix α_1 (Eq. 6.1), β_1 (Eq. 6.2), and γ_{1i} (Eq. 6.5). Now, we can specify $\mathcal{X}_{ij}$ from Eq. (5.30).

We will use the same gauge conditions as first order $\widetilde{E_{2\ell}} = 0$, so the spatial part of the scalar gauge is fixed using Eq. (5.28), to get

$$\beta_{2\ell} = -E_2 - \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathcal{X}^{ij}{}_{,ij} + \frac{1}{4}\nabla^{-2}\mathcal{X}^k{}_k\,. \tag{6.7}$$

Requiring that the perturbed part of the shift vector is zero, $\widetilde{B_{2\ell}} = 0$, sets the temporal gauge using Eq. (5.29). If we also set the vector part as zero, $F^i_2 = \mathbf{0}$, using Eq. (5.33), then we can specify the vector part of the spatial gauge, $\gamma^i_{2\ell}$, up to a constant of integration.

The gauge invariant definition of Φ, Ψ and other perturbations at second order can now be found using these gauge transformations in Eq. (5.27) and (5.26). The results are

$$\begin{aligned}
\widetilde{\phi_{2\ell}} &= \phi_2 + \mathcal{H}\alpha_{2\ell} + \alpha_{2\ell}{}' + \alpha_{1\ell}\left[\alpha_{1\ell}{}'' + 5\mathcal{H}\alpha_{1\ell}{}' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\alpha_{1\ell} + 4\mathcal{H}\phi_1 + 2\phi_1'\right] \\
&\quad + 2\alpha_{1\ell}{}'\left(\alpha_{1\ell}{}' + 2\phi_1\right) + \xi_{1\ell k}\left(\alpha_{1\ell}{}' + \mathcal{H}\alpha_{1\ell} + 2\phi_1\right)_{,}{}^{k} \\
&\quad + \xi'_{1\ell k}\left[\alpha_{1\ell,}{}^{k} - 2B_{1k} - \xi^{k}_{1\ell}{}'\right], && (6.8) \\
\widetilde{\psi_{2\ell}} &= \psi_2 - \mathcal{H}\alpha_{2\ell} - \frac{1}{4}\mathcal{X}_{\ell k}{}^{k} + \frac{1}{4}\nabla^{-2}\mathcal{X}_{\ell}{}^{ij}{}_{,ij}\,, && (6.9)
\end{aligned}$$

where $\mathcal{X}_{\ell ij}$ denotes the quadratic first order terms in Eq. (5.30) using the longitudinal gauge transforms $\alpha_{1\ell}$ and $\xi_{\ell i}$.

The tensor metric perturbation at second order is independent of the second order gauge transformations α_2 and ξ_{2i}. Yet, the second order tensor metric perturbation is dependent on first order variables, so the second order tensor is dependent on the first order choice of gauge, α_1 and ξ_{1i}. Therefore, our first order gauge choice will determine our second order tensor gauge invariant definitions. Recent work on the second order tensor mode in the Poisson gauge has been undertaken in [82, 88, 1, 5, 4].

By including the tracefree and transverse tensor part of the second order gauge transformation along with Eq. (5.35) we get our gauge invariant definition for the tensor metric perturbation in the Poisson gauge

$$\begin{aligned}\widetilde{h}_{2\ell ij} &= h_{2ij} + \mathcal{X}_{\ell ij} + \frac{1}{2}\left(\nabla^{-2}\mathcal{X}^{\ell kl}{}_{,kl} - \mathcal{X}_{\ell k}{}^{k}\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\mathcal{X}^{\ell kl}{}_{,klij} \\ &\quad + \frac{1}{2}\nabla^{-2}\mathcal{X}_{\ell k,ij}^{k} - \nabla^{-2}\left(\mathcal{X}_{\ell ik,}{}^{k}{}_{j} + \mathcal{X}_{\ell jk,}{}^{k}{}_{i}\right) . \end{aligned} \tag{6.10}$$

6.2 Spatially Flat Gauge

6.2.1 First Order

The spatially flat or uniform curvature gauge [49, 44, 45, 47, 103] is a gauge choice in which the metric is left unperturbed by scalar and vector perturbations. This condition is satisfied by setting $\widetilde{\psi_{\rm flat}} = \widetilde{E_{\rm flat}} = 0$ and $\widetilde{F_{{\rm flat}\,i}} = \mathbf{0}$. We can now use Eqs. (5.16), (5.18) and (5.21) to construct our transformation equations (5.9) as

$$\alpha_{\rm flat} = \frac{\psi}{\mathcal{H}}\,, \qquad \beta_{\rm flat} = -E\,, \qquad \gamma^i_{\rm flat} = -F^i\,. \tag{6.11}$$

The last two scalar degrees of freedom to be made gauge invariant are from Eqs. (5.15) and (5.17), which gives

$$\widetilde{\phi_{\rm flat}} = \phi + \psi + \left(\frac{\psi}{\mathcal{H}}\right)'\,, \tag{6.12}$$

$$\widetilde{B_{\rm flat}} = B - E' - \frac{\psi}{\mathcal{H}}. \tag{6.13}$$

From Eqs. (5.20) we have the definition of the gauge invariant vector as

$$\widetilde{S^i_{\rm flat}} = S^i + F,^i\,. \tag{6.14}$$

The density perturbation, a scalar quantity, has a gauge invariant definition from Eq. (5.10),

$$\widetilde{\delta\rho_{\rm flat}} = \delta\rho + \rho_0'\frac{\psi}{\mathcal{H}}\,. \tag{6.15}$$

The shear perturbation is given by $\widetilde{\sigma_{\rm flat}} = -\widetilde{B_{\rm flat}}$. Gauge-invariant quantities, such as $\widetilde{B_{\rm flat}}$ or $\widetilde{\psi_l}$ are proportional to the displacement between two different choices of spatial hypersurface,

$$\widetilde{B_{\rm flat}} = -\frac{\widetilde{\psi_\ell}}{\mathcal{H}} = \alpha_{\rm flat} - \alpha_\ell \,, \tag{6.16}$$

which would vanish for a homogeneous cosmology.

6.2.2 Second Order

The second order equations for the spatially flat gauge have $\widetilde{\psi} = \widetilde{E} = \widetilde{F_i} = 0$ for both first and second order. First and second order variables will be seperately denoted by a subscript "1" and "2" respectively. Using Eq. (5.26) we have the gauge condition that $\widetilde{\psi_2} = 0$ at second order, allowing us to get

$$\alpha_{2\rm flat} = \frac{\psi_2}{\mathcal{H}} + \frac{1}{4\mathcal{H}}\left[\nabla^{-2}\mathcal{X}^{ij}_{{\rm flat},ij} - \mathcal{X}^k_{{\rm flat}k}\right] \,, \tag{6.17}$$

where we have $\mathcal{X}_{{\rm flat}ij}$ from Eq. (5.30) using the first order gauge generators given above, as

$$\begin{aligned}
\mathcal{X}_{{\rm flat}ij} \;=\;& 2\left[\psi_1\left(\frac{\psi_1'}{\mathcal{H}} + 2\psi_1\right) + \psi_{1,k}\xi^k_{1\rm flat}\right]\delta_{ij} + \frac{4}{\mathcal{H}}\psi_1\left(C'_{1ij} + 2\mathcal{H}C_{1ij}\right) \\
&+4C_{1ij,k}\xi^k_{1\rm flat} + \left(4C_{1ik} + \xi_{1{\rm flat}i,k}\right)\xi^k_{1{\rm flat},j} + \left(4C_{1jk} + \xi_{1{\rm flat}j,k}\right)\xi^k_{1{\rm flat},i} \\
&+\frac{1}{\mathcal{H}}\left[\psi_{1,i}\left(2B_{1j} + \xi'_{1{\rm flat}j}\right) + \psi_{1,j}\left(2B_{1i} + \xi'_{1{\rm flat}i}\right)\right] - \frac{2}{\mathcal{H}^2}\psi_{1,i}\psi_{1,j} \\
&+\frac{2}{\mathcal{H}}\psi_1\left(\xi'_{1{\rm flat}(i,j)} + 4\mathcal{H}\xi_{1{\rm flat}(i,j)}\right) + 2\xi^k_{1\rm flat}\xi_{1{\rm flat}(i,j)k} + 2\xi_{1{\rm flat}k,i}\xi^k_{1{\rm flat},j} \,,
\end{aligned} \tag{6.18}$$

where we define

$$\xi_{1{\rm flat}i} = -\left(E_{1,i} + F_{1i}\right) \,. \tag{6.19}$$

The trace of Eq. (6.18) is then

$$\begin{aligned}
\mathcal{X}^k_{{\rm flat}k} \;=\;& 6\left[\psi_1\left(\frac{\psi_1'}{\mathcal{H}} + 2\psi_1\right) + \psi_{1,k}\xi^k_{1\rm flat}\right] + \frac{4}{\mathcal{H}}\psi_1\left(C^{k\prime}_{1\;k} + 2\mathcal{H}C^k_{1\;k}\right) \\
&+4C^k_{1\;k,l}\xi^l_{1\rm flat} + 4\left(2C^{kl}_1 + \xi^k_{1{\rm flat},}{}^l\right)\xi_{1{\rm flat}(k,l)} - 2\partial_a\partial^a E_{1,k}\xi^k_{1\rm flat} \\
&+\frac{2}{\mathcal{H}}\left(2B_{1k} + \xi'_{1{\rm flat}k} - \frac{1}{\mathcal{H}}\psi_{1,k}\right)\psi_{1,}{}^k - \frac{2}{\mathcal{H}}\left(\psi_1\partial_a\partial^a E_1' + 4\mathcal{H}\partial_a\partial^a E_1\right) \,.
\end{aligned} \tag{6.20}$$

The equation for the second order tensor perturbation in the flat gauge is

$$\begin{aligned}\widetilde{h}_{2\mathrm{flat}ij} &= h_{2ij} + \mathcal{X}_{\mathrm{flat}ij} + \frac{1}{2}\left(\nabla^{-2}\mathcal{X}^{kl}_{\mathrm{flat},kl} - \mathcal{X}^{k}_{\mathrm{flat}k}\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\mathcal{X}^{kl}_{\mathrm{flat},klij} \\ &+ \frac{1}{2}\nabla^{-2}\mathcal{X}^{k}_{\mathrm{flat}k,ij} - \nabla^{-2}\left(\mathcal{X}_{\mathrm{flat}ik,j}{}^{k} + \mathcal{X}_{\mathrm{flat}jk,i}{}^{k}\right) \, . \end{aligned} \tag{6.21}$$

Chapter 7

Volume-Preserving Coordinate Gauges and Spatial Averaging

7.1 Background

As was previously discussed in Chapter 1.2.1, synchronous and longitudinal gauges have typically been used for evaluating the cosmological backreaction. Synchronous gauge is able to provide useful numeric evaluations of the backreaction, but theoretically it is not very useful. Using the synchronous gauge, the volume domain is defined comoving with the CDM. This chosen volume will preserve the number of particles within the domain but the volume itself will be constantly changing, making the average calculation very difficult. As such, the volume-preserving coordinate (VPC) gauge will be developed as it is well motivated theoretically for averaging in cosmology.

There are two different VPC gauges that will be developed in this Chapter. The first is a 3D VPC gauge which will be developed from flat or uniform curvature gauge. The reason for this construction is because flat gauge will be of particular use when we restrict ourselves to averaging scalar perturbations. The second VPC gauge that will be developed will be a 4D VPC gauge. This 4D VPC gauge is motivated on a theoretical level since it will be, by definition, well suited for unimodular gravity. Future research will seek to use a 4D VPC gauge to average a 4D region in a VPC system.

7.2 3D Flat Gauge and Volume-Preserving Coordinates

We choose to work in uniform curvature or flat gauge to develop our 3D VPC. Since the tensor perturbations are gauge-invariant we cannot simply pick a gauge which is volume-preserving in general. However, since linear tensor perturbations are subdominant to the scalar perturbations in the standard model of cosmology, it is reasonable to assume that the tensor perturbations are negligible since we will be restricting ourselves to linear order and we will shortly show that tensor perturbations only contribute at higher orders. Flat gauge is not volume-preserving, per se. Rather, it is a comoving volume-preserving gauge. The volume element becomes simply $a^3(\eta)$, and these factors cancel in the average. Therefore, in principle, we can formally adapt it to a VPC.

We shall refer to this generalisation of the flat gauge calculation, restricting ourselves to scalar perturbations, as a 3D VPC gauge.

7.2.1 3D Flat Gauge

From Eq. (2.20) we know that a gauge-unfixed, flat, perturbed FLRW metric line element is

$$ds^2 = a^2(\eta)\left(-(1+2\phi_1)d\eta^2 + 2B_i d\eta dx^i + (\delta_{ij} + 2C_{ij})\, dx^i dx^j\right). \tag{7.1}$$

In the (3+1) split the spatial three-metric is $h_{ij} = a^2\left(\delta_{ij} + 2C_{ij}\right)$ [15]. This formalism identifies the coordinates for the (3+1) slicing in which the gauge of the system is solved. Simplifying the average then requires enforcing

$$\sqrt{h} = f(\eta) \tag{7.2}$$

for some function $f(\eta)$. The metric determinant to second-order is

$$h = a^6\left(1 + 2C + 2\left(C^2 - C^{ij}C_{ij}\right)\right). \tag{7.3}$$

From this determinant it is immediately clear that to simplify the domain volume such that the average is not affected by the perturbations, we require

$$C_{ij} = 0. \tag{7.4}$$

Remembering that from Eq. (2.13)

$$C_{ij} = -\psi\delta_{ij} + \partial_i\partial_j E + F_{(i,j)} + \frac{1}{2}H_{ij}, \tag{7.5}$$

we can therefore choose $\psi = E = F_i = 0$. This exhausts the two scalar and two vector degrees of freedom (which leaves us with a lapse function, a scalar shift and a vector shift). To reduce our gauge transformations into flat gauge we use Eq. (5.13) and Eq.'s (5.15) to (5.22) to show that

$$\psi_{\rm flat} = 0 \to \alpha = \frac{\psi}{\mathcal{H}}, \tag{7.6}$$

$$E_{\rm flat} = 0 \to \beta = -E, \tag{7.7}$$

$$\phi_{\rm flat} = \phi + \psi + \frac{1}{\mathcal{H}}(\dot{\psi} - \frac{\dot{\mathcal{H}}}{\mathcal{H}}\psi), \tag{7.8}$$

$$B_{\rm flat} = B - \dot{E} - \frac{\psi}{\mathcal{H}}, \tag{7.9}$$

and

$$F_i^{\rm flat} = 0 \to \gamma_i = -F_i \to S_i^{\rm flat} = S_i - F_i. \tag{7.10}$$

Unfortunately, it is impossible to remove the tensor perturbations through a gauge transformation, and we will be forced to accept

$$h = a^6(\eta)\left(1 - 2H^{ij}H_{ij}\right). \tag{7.11}$$

However, in cosmology, the tensor perturbations are suppressed compared to the scalar perturbations. The ratio of tensor perturbations to scalar perturbations immediately after inflation is quantified by a parameter r. Whether you believe in an inflationary epoch or not is irrelevant; the *observed* r is a phenomenological parameter quantifying the power of tensor perturbations compared to scalars. Currently the bounds are $r \lesssim 0.1$, so it is generally safe to neglect the tensor perturbations. Also, the corrections that will arise from an average of the tensors will be second order and therefore a linear VPC is sufficient for this thesis.

Neglecting tensor perturbations, flat gauge gives us an integral within a domain,

$$\mathcal{I} = \int_{\mathcal{D}} A(\mathbf{x})\sqrt{h(\mathbf{x})}d^3\mathbf{x} = a^3(\eta)\int_{\mathcal{D}} A(\mathbf{x})d^3\mathbf{x}. \tag{7.12}$$

Clearly, the volume $V_{\mathcal{D}} = a^3(\eta)\int_{\mathcal{D}} d^3\mathbf{x}$, and the average is

$$\langle A \rangle = \frac{\int_{\mathcal{D}} A(\mathbf{x})d^3\mathbf{x}}{\int_{\mathcal{D}} d^3\mathbf{x}}, \tag{7.13}$$

where $\mathbf{x} = x^i$. In any other gauge, $V_{\mathcal{D}}$ will contain additional perturbations and make the average non-trivially time dependent.

7.2.2 3D Volume-Preserving Coordinates

Within a flat gauge and removing the tensor perturbations, we can find a time dependent coordinate transformation that would set $h = 1$. By setting the spatial 3-metric determinant to unity, the volume of the average will simplify to $V_{\mathcal{D}} = \int_{\mathcal{D}} d^3\mathbf{x}$ which is not time dependent, therefore, $\dot{V}_{\mathcal{D}} = 0$, meaning we have a volume preserving domain to average.

However, we still need to create a coordinate system for our volume preserving domain. By setting the condition that $\sqrt{-g} = 1$ we solve for our new volume-preserving time coordinate change as

$$\tau = \int a^4(\xi)d\xi. \tag{7.14}$$

In practice, we are now free to perform the calculation in any gauge, then transform the results into a form with unit determinant, use VPC for averaging, and then convert the results back to our original gauge. However, since the 3D VPC is a "co-moving" VPC and not a true VPC, this gauge will be useful for averaging scalars. A 4D VPC will be more appropriate for future averaging research.

7.3 Paranjape's 4D Volume-Vreserving Coordinates

4D VPC's are much more general. One of the major benefits of these coordinates is that the use of VPC clarifies the separation between a gauge choice – where the perturbation equations are solved – and a coordinate choice.

An integral across an arbitrary four-volume is

$$\mathcal{I} = \int A(\mathbf{x})\sqrt{-g}d^4\mathbf{x}. \tag{7.15}$$

A volume-preserving gauge is then equivalent to choosing a coordinate system where the metric determinant is equal to unity, (i.e, $g = -1$). It is always possible to cast a metric into such a form.

We can study a field theory explicitly with this restriction on the metric built in, called *Unimodular Gravity*. Unimodular gravity is only invariant under volume-preserving diffeomorphisms and is perhaps a natural theory within which to do averaging [25] since covariance can always be reinstated back into the theory. Unimodular gravity was initially formulated by Einstein [32] in order to eliminate problems with the interpretation of the cosmological constant. More recently, unimodular gravity has been employed to try to explain observational phenomena without introducing exotic fields (i.e. dark energy or quintessence) into the theory, [25].

7.3.1 Volume-Preserving Gauge to Linear Order

In his PhD thesis [92], Paranjape defines a volume-preserving gauge to *linear order*. The metric determinant of a linearly-perturbed FLRW universe with line element

$$ds^2 = a^2(\eta)\left(-(1+2\phi)d\eta^2 + (\delta_{ij} + 2C_{ij})\, dx^i dx^j\right) \tag{7.16}$$

is

$$g = -a^8\left(1 + 2\phi - 6\psi + 2\partial^i\partial_i E\right). \tag{7.17}$$

Paranjape set up what he terms a *volume preserving gauge* in a manner very similar to the 3D approach in Chapter 7.1: he declared that he wanted a *comoving* VP gauge and proceeded to enforce the gauge condition

$$\phi = 3\psi - \partial^i\partial_i E. \tag{7.18}$$

We should state here that Paranjape used a different definition of $\widetilde{\psi} = \psi + \partial_i\partial^i E$ so that his spatial metric was formed in a way that the extra components of the spatial metric are traceless and transverse; i.e. $\partial^i E = \delta^{ij}E_{ij} = 0$. With this choice of gauge, the average of a linear or quadratic perturbation $A(\mathbf{x})$ becomes

$$\langle A\rangle = \frac{1}{V}\int a^4(\eta)A(\mathbf{x})d^4\mathbf{x}, \tag{7.19}$$

with any corrections coming in at higher orders. The condition (7.18) can be fixed employing only one of our two scalar gauge freedoms. A VPC can then be established through a redefinition of the time coordinate. We know from [92] that the average of the temporal component of the metric is

$$\langle g_{tt}\rangle = \langle\frac{-1}{h}\rangle = -f^2(t). \tag{7.20}$$

Therefore, since $f^2(t)$ is a function of time, the following VPC coordinate change can be established as

$$f^2(t) = \langle\frac{1}{h}\rangle = \frac{1}{\langle h\rangle} = \frac{1}{a^6}. \tag{7.21}$$

This VPC leaves Paranjape with a metric similar to a FLRW metric in a volume-preserving gauge,

$$ds^2 = -\frac{dt^2}{a^6(t)} + a^2(t)\delta_{ij}dx^i dx^j \tag{7.22}$$

from which he can calculate the averaged EFE. Paranjape used Zalaletdinov's theory of Macroscopic Gravity [114] to posit the answer of the FLRW metric from averaging.

7.3.2 Paranjape's Metric and Gauge Restrictions

In this section we will apply Paranjape's gauge restrictions to the general metric and then change the general metric into our perturbative variables in order to form a VPC transformation in perturbation theory. In general the metric is

$$ds^2 = a^2(t)\left(-(N^2 - h_{ij}N^iN^j)dt^2 + 2(h_{ij}N^j)dtdx^i + (h_{ij})\,dx^idx^j\right), \tag{7.23}$$

where the $(N^2 - h_{ij}N^iN^j)$ is the lapse, $(h_{ij}N^iN_j)$ is the shift, and h_{ij} is the spatial three metric. The determinant of this metric is

$$g = h(-(N^2 - h_{ij}N^iN^j) - N_ih_{ij}N_j) = -N^2h = -1. \tag{7.24}$$

In his thesis, Paranjape used two gauge restrictions in order to try to make his averaging equations easier to use. One gauge restriction, of course, is from the coordinate system of a volume preserving system. By definition this gives us the restriction of

$$g = -1, \tag{7.25}$$

which gives the constraint

$$N = \frac{1}{\sqrt{h}}. \tag{7.26}$$

For convenience, Paranjape set the shift of the metric to zero

$$N_i = N^i = 0, \tag{7.27}$$

where we can of course raise and lower indices using the 3-metric so $h_{ij}N^i = N_j$. From these restrictions we can generate some equations to allow us to find coordinate restrictions for a VPC system.

Under a coordinate change, $\widetilde{N_i} \to N_i$, by setting the shift to be zero we have a gauge restriction. For convenience, we are going to relabel the lapse function so that $(N^2 - h_{ij}N^iN^j) \equiv A^2$, making the general Paranjape metric

$$ds^2 = a^2(t)\left(-(A^2)dt^2 + 2(h_{ij}N^iN_j)dtdx^i + (h_{ij})\,dx^idx^j\right). \tag{7.28}$$

In this form we can look at the general coordinate transformation equations for each term of the metric easily. Under a general coordinate transformation the lapse would transform as

$$-\widetilde{A^2} = -A^2\left(\frac{dt}{\widetilde{dt}}\right)^2 + 2h_{ki}N^k\left(\frac{dx^i}{\widetilde{dt}}\frac{dt}{\widetilde{dt}}\right) + h_{km}\left(\frac{dx^k}{\widetilde{dt}}\frac{dx^m}{\widetilde{dt}}\right), \tag{7.29}$$

the shift would transform as

$$\widetilde{N}_i = -A^2\left(\frac{dt}{\widetilde{dt}}\frac{dt}{\widetilde{dx^i}}\right) + 2h_{ki}N^k\left(\frac{dx^i}{\widetilde{dx^i}}\frac{dt}{\widetilde{dt}}\right) + h_{km}\left(\frac{dx^k}{\widetilde{dx^i}}\frac{dx^m}{\widetilde{dx^j}}\right), \tag{7.30}$$

and the spatial metric would transform as

$$\widetilde{h}_{ij} = -A^2\left(\frac{dt}{\widetilde{dx^i}}\frac{dt}{\widetilde{dx^j}}\right) + 2h_{ki}N^k\left(\frac{dx^i}{\widetilde{dx^i}}\frac{dt}{\widetilde{dx^j}}\right) + h_{km}\left(\frac{dx^k}{\widetilde{dx^i}}\frac{dx^m}{\widetilde{dx^j}}\right). \tag{7.31}$$

Applying the constraint of a unit determinant, Eqn. (7.26), and a zero shift, Eqn. (7.27), to the three coordinate transformations Eqns. (7.31), (7.29), and (7.30), we get an equation for our coordinate restrictions. The equation we will be looking at in order to find our VPC coordinates is

$$\frac{1}{h}\frac{dt}{\widetilde{dt}}\frac{dt}{\widetilde{dx^a}} = h_{cd}\frac{dx^c}{\widetilde{dt}}\frac{dx^d}{\widetilde{dx^a}}. \tag{7.32}$$

7.3.3 Discussion of Paranjape

There have been many different averaging procedures introduced (e.g, [90], [37], [48], [11]). All of these procedures have the same goal of defining and interpreting an averaging procedure. The Buchert [22] approach to averaging scalars and the Zalaletdinov [114] fully covariant tensor averaging approach are the most widely used averaging operations to date. It is hoped that unimodular gravity will provide an alternative approach to these procedures, and this will be pursued in future research.

In Paranjape's thesis two main approaches to averaging were discussed. First, the 3D method developed by Buchert [22] and, second, a 4D method developed by Zalaletdinov [114]. Buchert defines an average in a model with a pressureless dust matter source (e.g., Lemaître-Tolman-Bondi or LTB solution [12], [57], [106]) with the assumption that the dust is irrotational and the four-velocity is orthogonal to the 3D spatial surfaces. The metric can be written in terms of synchronous and comoving coordinates, but we can only average the scalar quantities of the EFE's. Zalaletdinov's averaging procedure, on the other hand, is able to average all of Einstein's equations and can even average tensorial quantities by introducing additional mathematical structure into the averaging procedure.

In his thesis, Paranjape proceeds by taking a spatial limit of the Zalaletdinov equations in order to construct scalar equations that can be compared to Buchert's averaged scalar equations. Paranjape finds that the structure of the correction terms in each averaging approach is very similar after the spatial limit is applied. Therefore, Paranjape is able to reference an inhomogeneous spacetime whose average leads to the FLRW dynamics. This comparison is crucial since, in modern cosmology, the current observations of the cosmos come from observing the inhomogeneities around us, while ignoring how the inhomogeneities evolve when solving the averaged dynamics. In other words, we do not know what inhomogeneities lead to averaged homogeneous dynamics, if any. In this thesis, our concern is primarily with the explicit definition and dynamics of the VPC and therefore we will not be pursuing a spatial limit. Yet, it is easily seen that a rigorous and more conservative approach to defining quantities *before* and *after* averaging is of great importance to the credibility of the theory of cosmological averaging.

Paranjape also performed an *ensemble* average in his thesis. In ensemble averaging one takes a direct mean across an infinite number of realisations of a system, through which an average value can be constructed. The domain of the ensemble average is never explicitly defined. Once the system from which the average is to be calculated has been determined, we simply generate an infinite number of realizations (copies) of the system to find its average. This type of ensemble averaging comes out of techniques in statistical mechanics and is different from a *volume* average. In a volume average one simply takes the average across the volume in question. The domain of a volume average is usually taken to be larger then 30 Mpc ([59, 107]). Linear cosmological averages are not valid for scales less then 20 Mpc because non-linear effects are more significant on smaller scales. Also, homogeneity is considered to be valid for domains larger then 150 Mpc (see [113]). Therefore, most volume averages use a domain close to the Hubble volume. While at small volumes ensemble and volume averaging will produce very different results, at very large volumes (i.e., close to the Hubble volume) the theory of ergodicity applies [91]. According to this theory, the time average of a system's properties is equal to the average over the entire space once the system's dynamics have relaxed. It is *assumed* in current cosmological observations that once a sufficiently large domain has been selected (of the order of the Hubble volume), the ensemble and volume averages will produce essentially the same results. The benefit of performing an ensemble average of linear perturbations, and Paranjape only averages first order perturbations, is that the ensemble average is vanishing by definition. This ensures that the averaged objects produce FLRW-like quantities.

However, in this thesis we have also discussed second order perturbations. We can always define a coordinate system in which some of the second order terms are defined as a product of the first order terms. In such a coordinate system, the ensemble average is non-vanishing. In particular, at second order

$$V = \int a^4(\eta)\left(1 + 2\phi + 2C + 2\left(C^2 - C^{ij}C_{ij}\right) + 4\phi C\right) d^4x \neq \text{const.} \tag{7.33}$$

For these reasons we are going to follow Paranjape's method for fixing a 4D VPC but we will construct a fully fixed 4D VPC in Chapter 7.4. It is vital for future research that the 4D VPC system developed, if extended to second order, would balance these additional terms within the determinant (employing only two scalar and two vector gauge freedoms) to preserve a comoving unit determinant and make it easier to calculate the average to second order.

7.4 Linear 4D Volume-Preserving Coordinate System

The aim of this section is to follow a similar procedure to Paranjape and recast a linear procedure to construct a VPC. This VPC procedure will hold in 4D rather than just 3D as in Section 7.2.2. With the gauge and coordinate restrictions from the general Paranjape metric formulated in Section 7.3.2, we need to see what happens when the same metric is perturbed to linear order. We would also like to perturb the Paranjape metric in order to write out the gauge restrictions in our original perturbation variables that were introduced in Chapter 2. We start in a gauge independent metric

$$ds^2 = a^2(t)\left(-(1+2\phi)dt^2 + 2B_i dt dx^i + h_{ij}dx^i dx^j\right) , \tag{7.34}$$

where the spatial component is $h_{ij} = \left(\delta_{ij}(1-2\psi) + \partial^i\partial_i E + F_{(i,j)} + \frac{1}{2}H_{ij}\right)$. The determinant of this metric is

$$g = -a^8\left(1 + 2\phi - 6\psi + 2\partial^i\partial iE\right). \tag{7.35}$$

The gauge restriction Eq. (7.26) can be written as $\sqrt{-g} = N\sqrt{h}$ and of course we set the shift to zero $B_i = 0$. Now, we can define all the new variables as

$$N^2 = a^2\left(1 + 2\phi\right), \tag{7.36}$$

$$N_i = a^2\partial_i B_i. \tag{7.37}$$

Taking the determinant of h_{ij} we have

$$h = a^6(1 - 6\psi + 2\partial^i\partial_i E). \tag{7.38}$$

Using the gauge restriction and the determinant from Eq. (7.35) we get

$$\sqrt{-g} = N\sqrt{h} = a^4\left(1 + \phi - 3\psi + \partial^i\partial iE\right). \tag{7.39}$$

By studying this determinant we can see that our VPC gauge will have the condition

$$\phi - 3\psi + \partial^i\partial iE = 0. \tag{7.40}$$

We will construct the VPC gauge out of this condition by using the gauge transformations from Eq's (5.15), (5.16) and (5.18). Using these three gauge transformation equations and our gauge condition, Eq. (7.40), we get

$$\begin{aligned} \phi_V - 3\psi_V + 2\partial_i\partial^i E_V &= \phi + \dot{\alpha} + \mathcal{H}\alpha - 3\psi + 3\mathcal{H}\alpha + \partial_a\partial^a E + \partial_a\partial^a\beta \\ &= 0. \end{aligned} \tag{7.41}$$

Equating the gauge condition, Eq. (7.40) with Eq. (7.41) we obtain

$$\dot{\alpha} + 4\mathcal{H}\alpha + \partial_a\partial^a\beta = -\phi + 3\psi - \partial_a\partial^a E. \tag{7.42}$$

The derivatives in Eq. (7.42) are defined everywhere and we have a degree of freedom to make a gauge choice in order to make things easier to solve. The easiest gauge appears to be

$$\psi_V = 0, \tag{7.43}$$

which we will choose to use for our 4D VPC. However, it should be stated that we could have chosen $E = 0$ or set the density to zero, $\delta = 0$, in order to solve for α and β algebraically.

Using these conditions along with our gauge transformations we can solve for α, β, γ^i in order to be able to transform into our 4D VPC from any other gauge. Starting with the gauge transformation Eq. (5.16) and using the gauge condition from Eq. (7.43) we find that

$$\alpha = \frac{\psi}{\mathcal{H}}. \tag{7.44}$$

We can substitute α back into Eq. (7.42) to find β for our gauge transformation as

$$\partial_a\partial^a\beta = -\left(\psi + \phi + \partial_a\partial^a E + \left(\frac{\psi}{\mathcal{H}}\right)'\right). \tag{7.45}$$

With our definitions of α and β, we can now solve for the variables that we need to make sure that the gauge is volume preserving

$$\phi_V = \phi + \frac{\dot{\psi}}{\mathcal{H}} - \frac{\dot{\mathcal{H}}}{\mathcal{H}^2}\psi + \psi, \tag{7.46}$$

$$\psi_V = \psi - \psi = 0, \tag{7.47}$$

$$\partial_a\partial^a E_V = -\psi - \phi - \frac{d}{d\eta}\left(\frac{\psi}{\mathcal{H}}\right). \tag{7.48}$$

Now that we have the variables for the 4D VPC explicitly defined, we can substitute the ϕ_V, ψ_V, and $\partial_a\partial^a E_V$ into our gauge condition, Eq. (7.40), to show

$$\phi_V - 3\psi_V + \partial_a\partial^a E_V = \phi + \psi + \frac{\dot{\psi}}{\mathcal{H}} - \phi - \psi - \frac{\dot{\psi}}{\mathcal{H}} = 0, \tag{7.49}$$

meaning that the variables defined satisfy our condition for volume preservation.

Since the gauge condition is satisfied we can now show the rest of our gauge transformation equations for the shift, density and velocity respectively as:

$$\begin{aligned} \partial_a\partial^a B_V &= -\frac{\partial_a\partial^a\psi}{\mathcal{H}} - \frac{d}{d\eta}\left(\psi + \phi + \partial_a\partial^a E_V + \frac{d}{d\eta}\left(\frac{\psi}{\mathcal{H}}\right)\right), & (7.50)\\ \delta_V &= \delta - 3(1+w)\psi, & (7.51)\\ \partial_a\partial^a v_V &= \partial_a\partial^a v + \frac{d}{d\eta}\left(\psi + \phi + \frac{d}{d\eta}\left(\frac{\psi}{\mathcal{H}}\right)\right). & (7.52) \end{aligned}$$

Remember that our general metric is

$$ds^2 = a^2(\eta)\left(-(1+2\widetilde{\phi})d\eta^2 + \widetilde{h}_{ij}dx^i dx^j\right). \tag{7.53}$$

This metric has a determinant of $\sqrt{-g_V} = a^4(\eta)$. We want the metric to be volume preserving, $\sqrt{-g_V} = 1$, which requires us to set a new time coordinate along with our gauge conditions. We define this time coordinate as

$$\frac{\sigma^2}{a^6} = a^2 d\eta^2 \tag{7.54}$$

which we can rearrange and integrate to solve for our new time coordinate:

$$\sigma = \int \frac{1}{a^4}dt. \tag{7.55}$$

This gives us a final metric

$$ds^2 = \frac{-(1-2\partial_a\partial^a E_V)}{a^6(\sigma)}d\sigma^2 + a^2(\sigma)(\delta_{ij} + 2\partial_i\partial^i E_V + \frac{1}{2}H^V_{ij})dx^i dx^j. \tag{7.56}$$

which gives us a determinant of

$$-g_V = \frac{1}{a^6}(1-2\partial_a\partial^a E_V)a^6(1+2\partial_a\partial^a E_V) = 1. \tag{7.57}$$

7.4.1 A 4D Averaging Domain

When we perform averages we need to specify the averaging domain. In 3D this is entirely arbitrary, and distinctly non-covariant. In 4D, however, we could utilize the causal structure of spacetime to give us a one-parameter averaging domain. If we let the averaging radius be a proper time τ, then an averaging domain can be defined around an event E by the future and past light cones. To average across past history, one would extend along the past light cone an interval τ. To perform a spacelike average, one would extend along spacelike geodesics an interval τ. An average could be taken around the entire event by extending along past-oriented timelike geodesics, spacelike geodesics and future-oriented timelike geodesics up to an interval τ.

Chapter 8

Dynamics

With the construction of the VPC gauges in Chapter 7, we want to introduce the EFE in this chapter which will allow us to view some key properties of the perturbative quantities within the VPC gauges. Solutions will be shown in longitudinal gauge and then transformed into spatially flat gauge, using a standard gauge transformation. From the flat gauge, a coordinate system has been defined to transform the solutions into the 3D VPC gauge. The solutions in longitudinal gauge will be transformed directly into the 4D VPC gauge without the intermediary transformation into the spatially flat gauge.

The connection coefficients for the construction of the Einstein tensor up to and including second order perturbations for scalars, vectors, and tensors is given in [77].

The theory of GR gives us the EFE which relate the geometry of spacetime with the local energy momentum

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{8.1}$$

where G is the universal gravitational constant.

Our coordinate system is defined by EFE components that are tangent and orthogonal to the time-like four-vector field n^μ defined in Eq. (3.5). The definition of the coordinate system will provide us with constraint equations for the metric perturbations known as the energy and momentum constraint equations. The Bianchi identities, $\nabla_\mu G^\mu_\nu = 0$, imply local energy and momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{8.2}$$

Within this chapter, there is a change in the time coordinate. We will therefore define a prime, for instance ϕ', to indicate a derivative with respect to the time coordinate, η.

8.1 Background

From the EFE (8.1) we have the Friedmann constraint and the evolution equations for the background FLRW model:

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho \,, \tag{8.3}$$

$$\mathcal{H}' = -\frac{4\pi G}{3} a^2 \left(\rho + 3P\right) \,, \tag{8.4}$$

while the energy momentum conservation forms our continuity equation

$$\rho' = -3\mathcal{H}\left(\rho + P\right) \,, \tag{8.5}$$

where ρ and P are the total energy density and the total pressure, η, using the scale factor a, we can form the conformal Hubble parameter, $\mathcal{H} \equiv a'/a$.

8.1.1 Background Matter and Radiation Solutions

Assuming the equation of state equation $P = w\rho$, we can solve Eq.(8.5). We will express the continuity equation with the equation of state as

$$\rho' + 3\mathcal{H}(1 + w)\rho = 0. \tag{8.6}$$

For a matter dominated universe we know that $w = 0$, which gives

$$\rho_m' + 3\mathcal{H}\rho_m = 0. \tag{8.7}$$

Solving this differential equation for ρ gives us

$$\rho_m = \frac{\rho_{0m}}{a_m^3(\eta)} \tag{8.8}$$

where we have normalised the solution such that $a_0 = 1$ and the subscript "m" is for the matter domination epoch. Using this solution for ρ_m along with the Friedmann equation, Eq. (8.3), will allows us to solve for the scale factor, $a_m(\eta)$, for the matter dominated model. We use Eq. (8.3) and (8.8) to show

$$a_m^{-1/2} da = \sqrt{C} d\eta \tag{8.9}$$

where we set $C = \frac{8\pi G\rho_{0m}}{3}$ and ρ_{0m} is a constant. Integrating we find that

$$a_m(\eta) = \left(\frac{\eta}{\eta_0}\right)^2. \tag{8.10}$$

For a radiation dominated universe, we again start with Eq. (8.6) and set $w = 1/3$ to get

$$\rho' + 4\mathcal{H}\rho = 0. \tag{8.11}$$

Solving this differential equation for ρ gives us

$$\rho_r = \frac{C_0}{a_r^4(\eta)} \tag{8.12}$$

where C_0 is a constant and the subscript "r" is for the radiation dominated epoch. We can use the Friedmann equation to solve for the constant C_0 in terms of ρ_{0r} to get

$$\rho_r = \frac{\rho_{0r}}{a_r^4(\eta)}. \tag{8.13}$$

However, we need to consider the period in which the Universe is changing from a radiation dominated epoch into a matter dominated epoch. This period is called the time of *equality*. We can solve for the density at equality which we consider a constant,

$$\rho_{eq} = \frac{C_1}{a_{eq}^4(\eta)}. \tag{8.14}$$

The subscript "*eq*" stands for the value of the scale factor at the point of equality between the matter dominated and radiation dominated epochs.

Solving for the constant from integration we find that ρ evolves as

$$\rho_r = \rho_{eq}\left(\frac{a_{eq}(\eta)}{a(\eta)}\right)^4, \tag{8.15}$$

Using Eq.'s (8.15), (8.8) and the Friedmann equation, Eq. (8.3), we can solve for the scale factor, $a(\eta)$, for the radiation model:

$$\frac{d}{d\eta}a(\eta) = D \tag{8.16}$$

where the $D = \frac{8\pi G\rho_{0r}}{3}$. We integrate this equation and normalise the scale factor to get

$$a_r(\eta) = a_{eq}(\eta)\left(\frac{\eta}{\eta_{eq}}\right). \tag{8.17}$$

The time at which equality happens, η_{eq}, is considered to be constant since the equality happens at a particular time. At the time of equality we know that

$$\rho_m(\eta_{eq}) = \rho_r(\eta_{eq}), \tag{8.18}$$

which gives us

$$\frac{\rho_{0m}}{a_{eq}^3} = \frac{\rho_{0r}}{a_{eq}^4}. \tag{8.19}$$

We can solve for the size of the scale factor at the time of equality as

$$a_{eq}(\eta) = \frac{\rho_{0r}}{\rho_{0m}}. \tag{8.20}$$

We also know that

$$\frac{\rho_{0r}}{h^2} = \Omega_r, \tag{8.21}$$

$$\frac{\rho_{0m}}{h^2} = \Omega_m, \tag{8.22}$$

and we can find the exact values for $\Omega_r = 4.17 \times 10^{-5}$ and $\Omega_m = 0.273$ from [14] and [52] and $h = 0.704 = \frac{H_0}{100km/sec/Mpc}$ where H_0 is the Hubble constant. These values produce a scale factor at the time of equality of

$$a_{eq}(\eta_{eq}) = \frac{\Omega_r}{\Omega_m} \approx 3.096 \times 10^{-4} = \frac{1}{3200}. \tag{8.23}$$

This value of our scale factor at the time of equality is confirmed by the current 7-year WMAP data from [52]. We are therefore left with a scale factor for the radiation dominated epoch which evolves as $a(\eta) \propto \eta$.

8.2 Einstein Field Equations

In this section we are going to construct the first order scalar, vector and tensor perturbation equations from the EFE.

8.2.1 First Order Scalar Perturbations

We can obtain the scalar metric perturbations in an arbitrary gauge via the matter perturbations from the first-order energy and momentum constraints [49, 86]

$$3\mathcal{H}\left(\psi' + \mathcal{H}\phi\right) - \partial_a\partial^a\left[\psi + \mathcal{H}\sigma\right] = -4\pi G a^2 \delta\rho, \tag{8.24}$$

$$\psi' + \mathcal{H}\phi = -4\pi G a^2(\rho + P)V \tag{8.25}$$

where the total covariant velocity perturbation is given by

$$V \equiv v + B\,, \tag{8.26}$$

where v is the total scalar velocity potential as defined in Eq. (4.4).

At first order, the perturbed EFE also gives us two scalar metric perturbation evolution equations

$$\psi'' + 2\mathcal{H}\psi' + \mathcal{H}\phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\phi = 4\pi G a^2 \left(\delta P + \frac{2}{3}\partial_a\partial^a\Pi\right)\,, \tag{8.27}$$

$$\sigma' + 2\mathcal{H}\sigma + \psi - \phi = 8\pi G a^2 \Pi\,, \tag{8.28}$$

where Π is the scalar part of the (tracefree) anisotropic stress, defined in Eq. (4.8).

We restate the equation for the shear potential from Eq. (3.15) as

$$\sigma \equiv E' - B\,. \tag{8.29}$$

8.2.2 First Order Vector Perturbations

The divergence-free part of the three-momentum [see Eqs. (2.12), (4.4) and (4.11)]

$$\delta q_i = (\rho + P)(v_i^{\rm vec} - S_i)\,, \tag{8.30}$$

is constrained by the momentum conservation equation,

$$\delta q_i' + 4\mathcal{H}\delta q_i = -\partial_a\partial^a\Pi_i\,. \tag{8.31}$$

The vector part of the anisotropic stress, Eq. (4.8), is given by $a^2\partial_{(i}\Pi_{j)}$. A gauge-invariant vector metric perturbation is directly related to the divergence-free part of the momentum through the constraint equation

$$\partial_a\partial^a\left(F_i' + S_i\right) = -16\pi G a^2 \delta q_i\,. \tag{8.32}$$

8.2.3 First Order Tensor Perturbations

The tensor perturbations have no constraint equation. The spatial part of the EFE yields

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \partial_a\partial^a h_{ij} = 8\pi G a^2 \Pi_{ij}\,, \tag{8.33}$$

where Π_{ij} is the transverse and tracefree component of the anisotropic stress Eq. (4.8).

8.2.4 Energy and Momentum Conservation

Energy-momentum conservation gives evolution equations for the perturbed energy and momentum:

$$\delta\rho' + 3\mathcal{H}\left(\delta\rho + \delta P\right) - 3\left(\rho + P\right)\psi' + (\rho + P)\partial_a\partial^a\left(V + \sigma\right) = 0\,, \tag{8.34}$$

$$V' + (1 - 3c_{\mathrm{s}}^2)\mathcal{H}V + \phi + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\partial_a\partial^a\Pi\right) = 0\,, \tag{8.35}$$

where c_{s}^2 is the adiabatic speed of sound, defined as

$$c_{\mathrm{s}}^2 \equiv \frac{P'}{\rho'}\,. \tag{8.36}$$

8.2.5 Longitudinal Gauge EFE and Gauge Transformations

In this section we are going to express the first order metric in longitudinal gauge. Since we will be including tensor perturbations, this gauge is technically called the Poisson gauge. For more on the Poisson gauge see Appendix A. By setting $E_\ell = B_\ell = S_i^\ell = 0$, where the super and subscript "ℓ" stands for the longitudinal gauge, we can construct the scalar quantities to resemble Newtonian gravity (and, of course, we also remove the shift). This produces the metric:

$$ds^2 = a^2(\eta)\left(-(1 + 2\phi_\ell)d\eta^2 + (\delta_{ij}(1 - 2\psi_\ell) + 2\partial_i F_j^\ell + H_{ij}^\ell)dx^i dx^j\right). \tag{8.37}$$

The EFE for the longitudinal gauge are

$$3\mathcal{H}\left(\psi_\ell' + \mathcal{H}\phi_\ell\right) - \partial_a\partial^a\psi_\ell = -4\pi G a^2 \delta\rho_\ell, \tag{8.38}$$

$$\psi_\ell' + \mathcal{H}\phi_\ell = -4\pi G a^2(\rho_\ell + P_\ell)v_\ell, \tag{8.39}$$

Eq. (8.28) is the evolution equation for the scalar shear in a general gauge. In the longitudinal gauge, however, Eq. (8.28) becomes a constraint equation for the gauge-invariant perturbations ϕ and ψ,

$$\psi_\ell - \phi_\ell = 8\pi G a^2 \Pi\,. \tag{8.40}$$

This allows us to have the constraint $\psi_\ell = \phi_\ell$ when we set the anisotropic stress to zero as in the context of a perfect fluid.

Eq. (8.27) can then provide a second-order evolution equation for the metric perturbation in the longitudinal gauge driven by isotropic pressure:

$$\psi_\ell'' + 3\mathcal{H}\psi_\ell' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\psi_\ell = 4\pi G a^2 \delta P_\ell\,. \tag{8.41}$$

To produce our gauge transformations from an arbitrary gauge we must use Eq. (5.13) and Eq.'s (5.15) to (5.22) giving

$$\begin{aligned}
\phi_\ell &= \phi + (B' - E'') + \frac{a'}{a}(B - E'), && (8.42)\\
\psi_\ell &= \psi - \frac{a'}{a}(B - E'), && (8.43)\\
B_\ell &= 0 \rightarrow \alpha = B - E', && (8.44)\\
E_\ell &= 0 \rightarrow \beta = -E, && (8.45)\\
\sigma_i^\ell &= 0 \rightarrow \gamma_i' = -\sigma_i \rightarrow \gamma_i = -\int \sigma_i d\eta + C_i(\mathbf{x}), && (8.46)\\
F_i^\ell &= F_i - \int \sigma_i d\eta + C_i(\mathbf{x}). && (8.47)
\end{aligned}$$

8.3 Matter and Radiation Solutions

In this section we will introduce the general matter and radiation solutions. The solutions are presented first in the longitudinal gauge from [27] and then transformed into a VPC gauge followed by a discussion of the results at the end of the chapter.

8.3.1 Solutions in the Longitudinal Gauge

We shall first discuss the matter solutions in the longitudinal gauge. The matter is assumed to have $w = c_s^2 = p = 0$ and $\Pi = 0$, which implies $\phi_\ell = \psi_\ell$ from Eq. (8.39). We use the scale factor from Eq. (8.10) so that $\mathcal{H} = \frac{2}{\eta}$. Eq. (8.41) becomes

$$\phi_\ell'' + 3\mathcal{H}\phi_\ell' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\phi_\ell = 0. \tag{8.48}$$

With the conformal Hubble parameter, $\mathcal{H}$, we know that $2\mathcal{H}' + \mathcal{H}^2 = 0$ to make our dynamical equation

$$\phi_\ell'' + \frac{6}{\eta}\phi_\ell' = 0. \tag{8.49}$$

The general solution of this equation is

$$\phi_\ell = \phi_2 + \frac{\phi_1}{k^2\eta^2}, \tag{8.50}$$

where ϕ_2 and ϕ_1 are constants. For times where η approaches zero the matter solution is unbounded. We set $\phi_1 = 0$, making the bounded matter solution

$$\phi_\ell = \phi_2. \tag{8.51}$$

Next we move to the radiation solution in longitudinal gauge. The radiation dominated epoch has $w = c_s^2 = p = \frac{1}{3}$ and $\Pi = 0$ which implies $\phi_\ell = \psi_\ell$. We use the scale factor from Eq. (8.17) so that $\mathcal{H} = \frac{1}{\eta}$. We can now use Eq. (8.38) which gives us a second order differential equation

$$\eta^2\phi_\ell'' + 4\eta\phi_\ell' + \frac{1}{3}k^2\eta^2\phi_\ell = 0. \tag{8.52}$$

The solution of this differential equation, [13], is

$$\phi_\ell = \phi_3\left(\frac{\sqrt{3}\sin(\frac{k\eta}{\sqrt{3}}) - k\eta\cos(\frac{k\eta}{\sqrt{3}})}{k^3\eta^3}\right) + \phi_4\left(\frac{\sqrt{3}\cos(\frac{k\eta}{\sqrt{3}}) + k\eta\sin(\frac{k\eta}{\sqrt{3}})}{k^3\eta^3}\right), \tag{8.53}$$

where ϕ_3 and ϕ_4 are constants. If we substitute $k = \frac{x}{\eta}$ and expand the solution as a power series for sine and cosine we get

$$\phi_\ell = \frac{\phi_4\sqrt{3}}{x^3} + \frac{\phi_4\sqrt{3}}{6x} - \frac{\phi_3}{9} - \frac{\phi_4 x\sqrt{3}}{72} + \frac{\phi_3 x^2}{270}, \tag{8.54}$$

which we have truncated to first order. When η or $k\eta$ go to zero, such as at a time close to the Big Bang, most of these terms are unbounded. Therefore, some of the modes of inhomogeneous effects would grow much too large. We therefore must bound the radiation solution, Eq. (8.54), since the perturbations were initialized at an early time where $\eta \ll 1$. During this period the two modes of perturbations ϕ_3 and ϕ_4 will be comparable and therefore $\phi_4 < \phi_3$ and so we can neglect ϕ_4 from now on to get a super-horizon solution of

$$\phi_\ell = -\frac{\phi_3}{9}. \tag{8.55}$$

However, for future use we state the full scale solution

$$\phi_\ell = \phi_3\left(\frac{\sqrt{3}\sin(\frac{k\eta}{\sqrt{3}}) - k\eta\cos(\frac{k\eta}{\sqrt{3}})}{k^3\eta^3}\right). \tag{8.56}$$

We can now also express the velocity solutions in longitudinal gauge using Eq. (8.39)

$$v_\ell = \frac{-2\left(\phi_\ell' + \mathcal{H}\phi_\ell\right)}{3\mathcal{H}^2(1+w)} \tag{8.57}$$

which for matter, with $w = 0$ and $\mathcal{H} = \frac{2}{\eta}$, becomes

$$v_\ell = \frac{-\phi_\ell \eta}{3} = \frac{-\phi_2 \eta}{3}. \tag{8.58}$$

For radiation with $w = \frac{1}{3}$ and $\mathcal{H} = \frac{1}{\eta}$, we find the velocity to be

$$v_\ell = -\frac{\eta^2}{2}\left(\phi_\ell' + \frac{\phi_\ell}{\eta}\right). \tag{8.59}$$

Using the super-horizon scale solution, Eq. (8.55) gives the velocity as

$$v_\ell = \frac{\phi_3 \eta}{18} \tag{8.60}$$

and the solution on all scales using Eq. (8.56) is

$$v_\ell = \left(-\frac{\phi_3\eta}{2}\right)\left(\frac{\sqrt{3}\sin(\frac{k\eta}{\sqrt{3}})k^2\eta^2 + 6\cos(\frac{k\eta}{\sqrt{3}})k\eta - 6\sqrt{3}\sin(\frac{k\eta}{\sqrt{3}})}{k^3\eta^4}\right)^2. \tag{8.61}$$

The density solution for longitudinal gauge using Eq. (8.38) is

$$\delta_\ell = -2\phi_\ell - \frac{2\phi_\ell'}{\mathcal{H}} - \frac{k^2\phi_\ell}{3\mathcal{H}} \tag{8.62}$$

which for matter becomes

$$\delta_\ell = -2\phi_2 - \frac{k^2\eta^2\phi_0}{6}. \tag{8.63}$$

For radiation the super-horizon scale solution is

$$\delta_\ell = \frac{2\phi_3}{9} + \frac{2k^2\eta^2\phi_3}{27} \tag{8.64}$$

and the solution on all scales is

$$\delta_\ell = \frac{-2\phi_3}{3k^3\eta^3}\Bigg[-6\sqrt{3}\sin\left(\frac{k\eta}{\sqrt{3}}\right) + 6k\eta\cos\left(\frac{k\eta}{\sqrt{3}}\right) + 2\sqrt{3}k^2\eta^2\sin\left(\frac{k\eta}{\sqrt{3}}\right) - k^3\eta^3\cos\left(\frac{k\eta}{\sqrt{3}}\right)\Bigg]. \tag{8.65}$$

8.4 VPC Solutions

Within this section we are going to transform the longitudinal gauge matter and radiation solutions into our VPC gauges. As demonstrated in Chapter 7 we have a 3D and 4D VPC gauge. Discussion of these findings will be detailed in Section 8.5. These findings show that these gauges will be useful in further work involving cosmological averaging and unimodular gravity.

8.4.1 3D VPC Solutions

In Section 7.2, the 3D VPC gauge was constructed. Before we solve the matter and radiation systems, we must first define the gauge transformations for our 3D VPC gauge using Eq. (5.13) and Eqs. (5.15) to (5.22). These equations show that the 3D VPC gauge has the same gauge restrictions as the spatially flat gauge:

$$\psi_V = E_V = F_i^V = 0 \tag{8.66}$$

where the superscript or subscript “V” stands for the VPC gauge. Since these are the same gauge conditions as spatially flat gauge, we can use the gauge transformation equations for the spatially flat gauge, from Section 6.2, to transform the longitudinal solutions into the 3D VPC.

Solving the gauge transformation equations gives us the generating vector field, defined in Eq. (5.9) to transform into our 3D VPC as

$$\alpha = \frac{\psi_\ell}{\mathcal{H}}\,, \qquad \beta_{,i} = 0\,, \qquad \gamma^i = -F_\ell^i.$$

Now that we have our generating vector field we can transform the matter and radiation solutions from longitudinal gauge in Section 8.3.1 into our 3D VPC.

First we will transform the matter solution to find our lapse, ϕ_V, our velocity, v_V, and our density quantity, σ_V. Remember that our matter solution for ϕ_ℓ from Eq. (8.51) is

$$\phi_\ell = \phi_2, \tag{8.67}$$

where the subscript "ℓ" stands for the longitudinal gauge and ϕ_2 is a constant. Substituting this solution for ϕ_ℓ into the gauge transformation Eq. (5.15) we find that

$$\phi_V = \frac{5\phi_2}{2}. \tag{8.68}$$

For the velocity the transformation equation, Eq. (5.39), gives our new velocity as

$$v_V = v_\ell = \frac{-\phi_2\eta}{3}. \tag{8.69}$$

We will use Eq. (5.10) to transform our energy density as

$$\delta_V = -\phi_V\left(5 + \frac{k^2}{3\mathcal{H}}\right) = -5\phi_2 - \frac{k^2\eta\phi_2}{6}. \tag{8.70}$$

We follow the same general gauge transformations to solve for our lapse, ϕ_V, our velocity, v_V, and our density quantity, σ_V for our radiation solution. As before we start with Eq. (5.15) to find in radiation the super-horizon scale solution is

$$\phi_V = -3\phi_\ell = \frac{\phi_3}{3}, \tag{8.71}$$

and for all scales using Eq. (8.56) we get

$$\phi_V = -3\phi_3\left(\frac{\sqrt{3}\sin(\frac{k\eta}{\sqrt{3}}) - k\eta\cos(\frac{k\eta}{\sqrt{3}})}{k^3\eta^3}\right). \tag{8.72}$$

Using Eq. (5.39) and our longitudinal radiation solution from Eq. (8.57), we find the super-horizon scale solution for our velocity in radiation is

$$v_V = \frac{\eta\phi_3}{18}, \tag{8.73}$$

and for all scales using Eq. (8.56) we find the velocity is

$$v_V = \left(-\frac{\phi_3\eta}{2}\right)\left(\frac{\sqrt{3}\sin(\frac{k\eta}{\sqrt{3}})k^2\eta^2 + 6\cos(\frac{k\eta}{\sqrt{3}})k\eta - 6\sqrt{3}\sin(\frac{k\eta}{\sqrt{3}})}{k^3\eta^4}\right)^2. \tag{8.74}$$

Finally we solve for our energy density solution for radiation using the transformation for the density, Eq. (5.10), and our longitudinal radiation solution, Eq. (8.62), to find

$$\delta_V = \frac{5\phi_3}{9} + \frac{k^2\eta\phi_3}{27} \tag{8.75}$$

and for all scales using Eq. (8.56) we get

$$\delta_V = \frac{-\phi_3}{3k^2\eta^2}\Bigg[-3\sqrt{3}\sin\left(\frac{k\eta}{\sqrt{3}}\right) + 3k\eta\cos\left(\frac{k\eta}{\sqrt{3}}\right)$$
$$+\sqrt{3}k^2\eta\sin\left(\frac{k\eta}{\sqrt{3}}\right) - k^3\eta^2\cos\left(\frac{k\eta}{\sqrt{3}}\right) + 2\sqrt{3}k^2\eta^2\sin\left(\frac{k\eta}{\sqrt{3}}\right)\Bigg]. \tag{8.76}$$

Of course we must also do a coordinate transformation from our current time coordinate into our VPC system using Eq. (7.14). For the matter dominated epoch our new time coordinate is defined as

$$\eta = \left(9\eta_0^8\tau\right)^{\frac{1}{9}}. \tag{8.77}$$

Using our new time coordinate we can find the velocity and density respectively as

$$v_V = \frac{-\phi_2}{3}\left(9\eta_0^8\tau\right)^{\frac{1}{9}} \tag{8.78}$$

$$\delta_V = -5\phi_0 - \frac{k^2\left(9\eta_0^8\tau\right)^{\frac{1}{9}}\phi_0}{6}. \tag{8.79}$$

Eq. (7.14) gives our new time coordinate for the radiation dominated epoch as

$$\eta = \left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}. \tag{8.80}$$

which will make the super-horizon radiation solutions for velocity and density

$$v_V = \frac{\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}\phi_3}{18} \tag{8.81}$$

$$\delta_V = \frac{5\phi_3}{9} + \frac{k^2\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}\phi_3}{27}. \tag{8.82}$$

The full scale solutions for ϕ_V, v_V and δ_V in our new time coordinate are

$$\phi_V = -3\phi_3\left(\frac{\sqrt{3}\sin\left(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}}\right) - k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}\cos\left(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}}\right)}{k^3\left(5\eta_{eq}^4\tau\right)^{\frac{3}{5}}}\right), \tag{8.83}$$

$$v_V = \left(-\frac{\phi_3\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{2}\right)\left[\frac{\sqrt{3}\sin(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}})k^2\left(5\eta_{eq}^4\tau\right)^{\frac{2}{5}}}{k^3\left(5\eta_{eq}^4\tau\right)^{\frac{4}{5}}}\right.$$
$$\left.+\frac{6\cos(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}})k\eta - 6\sqrt{3}\sin(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}})}{k^3\left(5\eta_{eq}^4\tau\right)^{\frac{4}{5}}}\right]^2, \tag{8.84}$$

$$\delta_V = \frac{-\phi_3}{3k^2\left(5\eta_{eq}^4\tau\right)^{\frac{2}{5}}}\left[-3\sqrt{3}\sin\left(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}}\right)+3k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}\cos\left(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}}\right)\right.$$
$$+\sqrt{3}k^2\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}\sin\left(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}}\right)-k^3\left(5\eta_{eq}^4\tau\right)^{\frac{2}{5}}\cos\left(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}}\right)$$
$$\left.+2\sqrt{3}k^2\left(5\eta_{eq}^4\tau\right)^{\frac{2}{5}}\sin\left(\frac{k\left(5\eta_{eq}^4\tau\right)^{\frac{1}{5}}}{\sqrt{3}}\right)\right]. \tag{8.85}$$

8.4.2 4D VPC Solutions

For this section of Chapter 8, we will be using the previously constructed 4D VPC from Section 7.3. We will first show the gauge transformation equations that we need to transform any other gauge into our 4D VPC gauge are

$$\phi_V = \phi_\ell + \frac{\psi_\ell'}{\mathcal{H}} - \frac{\mathcal{H}}{(\mathcal{H}')^2}\psi_\ell + \psi_\ell, \tag{8.86}$$
$$-k^2 v_V = -k^2 v_\ell + \frac{d}{d\eta}\left(\psi_\ell + \phi_\ell + \frac{d}{d\eta}\left(\frac{\psi_\ell}{\mathcal{H}}\right)\right), \tag{8.87}$$
$$\delta_V = \delta_\ell - 3(1+3w)\psi_\ell, \tag{8.88}$$
$$-k^2 B_V = \frac{k^2\psi_\ell}{\mathcal{H}} - \frac{d}{d\eta}\left(\psi_\ell + \phi_\ell + \frac{d}{d\eta}\left(\frac{\psi_\ell}{\mathcal{H}}\right)\right), \tag{8.89}$$
$$-k^2 E_V = -\psi_\ell - \phi_\ell - \frac{d}{d\eta}\left(\frac{\psi_\ell}{\mathcal{H}}\right). \tag{8.90}$$

For our matter solution remember that we have $w = c_s^2 = \Pi = 0$ and $\mathcal{H} = \frac{2}{\eta}$ which implies $\phi_\ell = \psi_\ell$. Using these conditions we can transform our matter solution from longitudinal gauge into our 4D VPC gauge.

Using the longitudinal gauge matter solutions and the transformation equations for our 4D VPC gauge we find that

$$\phi_V = \phi_2\left(2-\frac{\eta^3}{2}\right), \tag{8.91}$$

$$v_V = -\frac{\eta\phi_2}{3}, \tag{8.92}$$

$$\delta_V = \phi_2\left(\frac{2}{k^2}+\frac{\eta^2}{6}-3\right), \tag{8.93}$$

$$B_V = -\frac{\eta\phi_2}{2}, \tag{8.94}$$

$$E_V = \frac{5\phi_2}{2k^2}. \tag{8.95}$$

For our radiation solution we have $w = c_s^2 = \frac{1}{3}$, $\Pi = 0$ and $\mathcal{H} = \frac{1}{\eta}$, which implies $\phi_\ell = \psi_\ell$. We can transform our longitudinal gauge radiation solutions, Eq.'s (8.55) and (8.56), into our 4D VPC gauge using Eq.'s (8.86) to (8.90). The full scale solutions using Eq. (8.56) are

$$\phi_V = \tfrac{\phi_3}{3k^3\eta^3}\Bigg[-3\sqrt{3}\sin\left(\tfrac{k\eta}{\sqrt{3}}\right)+3k\eta\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)+3\sqrt{3}\eta\sin\left(\tfrac{k\eta}{\sqrt{3}}\right) -3k\eta^2\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)+\sqrt{3}k^2\eta^2\sin\left(\tfrac{k\eta}{\sqrt{3}}\right)\Bigg] \tag{8.96}$$

$$v_V = \tfrac{-1}{6\eta^7k^6}\Bigg[9k^4\eta^4-9k^4\eta^4\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)^2-108k^2\eta^2+216k^2\eta^2\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)^2 +36\sqrt{3}k^3\eta^3\sin\left(\tfrac{k\eta}{\sqrt{3}}\right)\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)+324-324\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)^2 -216\sqrt{3}k\eta\sin\left(\tfrac{k\eta}{\sqrt{3}}\right)\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)-2\sqrt{3}k^3\eta^5\sin\left(\tfrac{k\eta}{\sqrt{3}}\right) +2k^4\eta^6\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)\Bigg] \tag{8.97}$$

$$\delta_V = \tfrac{2\phi_3}{3k^5\eta^3}\Bigg[-6\sqrt{3}\sin\left(\tfrac{k\eta}{\sqrt{3}}\right)+6k\eta\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)+2\sqrt{3}k^2\eta^2\sin\left(\tfrac{k\eta}{\sqrt{3}}\right) -k^3\eta^3\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)+4\sqrt{3}\sin\left(\tfrac{k\eta}{\sqrt{3}}\right)-k\eta\cos\left(\tfrac{k\eta}{\sqrt{3}}\right)\Bigg] \tag{8.98}$$

$$B_V = \frac{-4\phi_3\left(\sqrt{3}\sin\left(\frac{k\eta\sqrt{3}}{3}\right) - \cos\left(\frac{\sqrt{3}k\eta}{3}\right)k\eta\right)}{3k^3\eta^2} \tag{8.99}$$

$$E_V = \frac{\phi_3\sqrt{3}\sin\left(\frac{k\eta\sqrt{3}}{3}\right)}{3k^3\eta}. \tag{8.100}$$

Using the bounded radiation solution, Eq. (8.55), we find the super-horizon scale solutions for radiation are

$$\phi_V = -\frac{(2+\eta)\phi_3}{9}, \quad v_V = \frac{\eta\phi_3}{18}$$

$$\delta_V = \left(\frac{-1}{k^2}\right)\left(\frac{2\phi_3}{9} + \frac{2k^2\eta^2\phi_3}{27}\right) - \frac{4\phi_3}{9} \tag{8.101}$$

$$B_V = \frac{-\phi_3}{9}, \quad E_V = \frac{-\phi_3}{3k^2}.$$

For our 4D VPC we also have a time coordinate change which was defined in Eq. (7.55). For our matter solution we find that our new time coordinate is

$$\eta = \left(\frac{-\eta_0^8}{7\sigma}\right)^{\frac{1}{7}}. \tag{8.102}$$

For our matter solutions ϕ_V, v_V, δ_V and B_V are time dependent so these transform into

$$\phi_V = \phi_2\left(2 - \frac{(-\eta_0^8)^{\frac{3}{7}}}{2(7\sigma)^{\frac{3}{7}}}\right), \tag{8.103}$$

$$v_V = -\frac{\phi_2(-\eta_0^8)^{\frac{1}{7}}}{3(7\sigma)^{\frac{1}{7}}}, \tag{8.104}$$

$$\delta_V = \phi_2\left(\frac{2}{k^2} + \frac{(-\eta_0^8)^{\frac{2}{7}}}{6(7\sigma)^{\frac{2}{7}}} - 3\right), \tag{8.105}$$

$$B_V = \frac{(\eta_0^8)^{\frac{1}{7}}\,\phi_2}{2\,(7\sigma)^{\frac{2}{7}}}. \tag{8.106}$$

We will use Eq. (7.55) to also transform our radiation solutions using the new time coordinate. The new time coordinate for radiation is defined as

$$\eta = -\left(\frac{\eta_{eq}^4}{3\sigma}\right)^{\frac{1}{3}}. \tag{8.107}$$

Finally we express our 4D VPC radiation solutions in this new time coordinate system. First we show the super-horizon radiation solutions with the time coordinate transformation. The lapse, velocity and density quantities are the only super-horizon solutions that are time dependent

$$\phi_V = \phi_3\left(\left(\frac{\eta_{eq}^4}{3^7\sigma}\right)^{\frac{1}{3}} - \frac{2}{9}\right), \tag{8.108}$$

$$v_V = -\left(\frac{\eta_{eq}^4}{3\sigma}\right)^{\frac{1}{3}}\frac{\phi_3}{18} \tag{8.109}$$

$$\delta_V = -\left(\frac{\phi_3}{k^2}\right)\left(\frac{2}{9} + \frac{2k^2(-\eta_{eq}^4)^{\frac{2}{3}}}{27(3\sigma)^{\frac{2}{3}}}\right) - \frac{4\phi_3}{9}. \tag{8.110}$$

The full scale radiation solutions using our volume-preserving time coordinate are

$$\begin{aligned}\phi_V = \tfrac{\phi_3(3\sigma)}{3k^3(-\eta_{eq}^4)}\Bigg[&-3\sqrt{3}\sin\left(\tfrac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) + 3k(-\eta_{eq}^4)^{\frac{1}{3}}\cos\left(\tfrac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) \\ &+3\sqrt{3}(-\eta_{eq}^4)^{\frac{1}{3}}\sin\left(\tfrac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) - 3k\left(\tfrac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{2}{3}}\cos\left(\tfrac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) \\ &+\sqrt{3}k^2\left(\tfrac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{2}{3}}\sin\left(\tfrac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right)\Bigg],\end{aligned} \tag{8.111}$$

$$v_V = \frac{-(3\sigma)^{\frac{7}{3}}}{6(-\eta_{eq}^4)^{\frac{7}{3}} k^6} \Bigg[9k^4 \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{4}{3}} - 9k^4 \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{4}{3}} \cos\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right)^2$$
$$-108k^2 \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{2}{3}} + 216k^2 \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{2}{3}} \cos\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right)^2$$
$$+36\sqrt{3}k^3 \left(\frac{-\eta_{eq}^4}{3\sigma}\right) \sin\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) \cos\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) + 324$$
$$-324 \cos\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right)^2 - 2\sqrt{3}k^3 \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{5}{3}} \sin\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right)$$
$$+2k^4 \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^2 \cos\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right)$$
$$-216\sqrt{3}k \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{1}{3}} \sin\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) \cos\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) \Bigg], \tag{8.112}$$

$$\delta_V = \frac{-2(3\sigma)\phi_3}{3k^5(\eta_{eq}^4)} \Bigg[-6\sqrt{3} \sin\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) + 6k \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{1}{3}} \cos\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right)$$
$$+2\sqrt{3}k^2 \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{2}{3}} \sin\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) + k^3 \left(\frac{\eta_{eq}^4}{3\sigma}\right) \cos\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right)$$
$$+4\sqrt{3} \sin\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) - k \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{1}{3}} \cos\left(\frac{k(-\eta_{eq}^4)^{\frac{1}{3}}}{\sqrt{3}(3\sigma)^{\frac{1}{3}}}\right) \Bigg], \tag{8.113}$$

$$B_V = \frac{-4(-3\sigma)^{\frac{2}{3}}}{3k^3(\eta_{eq}^4)} \Bigg[\phi_3 \Bigg[\sqrt{3} \sin\left(\frac{k\left(-\eta_{eq}^4\right)^{\frac{1}{3}}\sqrt{3}}{3\left(3\sigma\right)^{\frac{1}{3}}}\right)$$
$$- \cos\left(\frac{k\left(-\eta_{eq}^4\right)^{\frac{1}{3}}\sqrt{3}}{3\left(3\sigma\right)^{\frac{1}{3}}}\right) k \left(\frac{-\eta_{eq}^4}{3\sigma}\right)^{\frac{1}{3}} \Bigg]\Bigg], \tag{8.114}$$

$$E_V = \frac{\phi_3 \left(3\sigma\right)^{\frac{1}{3}} \sqrt{3} \sin\left(\frac{k\left(-\eta_{eq}^4\right)^{\frac{1}{3}}\sqrt{3}}{3(3\sigma)^{\frac{1}{3}}}\right)}{3k^3 \left(-\eta_{eq}^4\right)^{\frac{1}{3}}}. \tag{8.115}$$

8.5 Discussion

Within Chapter 8 we used the fully developed VPC gauges from Chapter 7 and solved for the matter and radiation dynamical solutions. These gauges were developed in a particular way to try to address some of the issues with perturbative gauges used in cosmological averaging. As discussed at the beginning of Chapter 7, the synchronous gauge is the most practical gauge for using numerics to quantitatively evaluate the backreaction, but the synchronous gauge does not have a constant volume and so averaging in this gauge is not rigorously defined.

We developed the 3D VPC gauge from the flat gauge. The flat gauge allowed us to construct a VPC gauge since the flat gauge leaves the scalar and vector components of the three-metric unperturbed and we assume that the size of the tensor perturbations is so small that the tensor perturbations can be neglected. Applying a proper time coordinate transformation, Eq. (7.14), to the flat gauge to make the metric determinant unity, transforms the flat gauge into a 3D VPC gauge. In practice, the 3D VPC gauge gives us a volume which is not time dependent, $V_{\mathcal{D}} = \int_{\mathcal{D}} d^3\mathbf{x}$; therefore, the volume is constant for all times, $\dot{V}_{\mathcal{D}} = 0$. The 3D VPC gauge will be particularly useful in Buchert's approach to scalar averaging [22]. The 4D VPC gauge is different from the 3D VPC gauge. The 4D VPC gauge can be used for averaging all objects of interest within Zalaletdinov's procedure of averaging [114]. The 4D VPC gauge is a true, well defined, VPC gauge that is theoretically motivated and may be the best gauge in which to evaluate averages.

However, there is a limitation to the 3D VPC gauge within the averaging regime. The 3D VPC gauge can be extended to second order; all the formalism for this extension is provided within this thesis or from [23]. However, the 3D VPC gauge, extended to second order will no longer be volume-preserving since the tensor perturbations will no longer be negligible at this order. At second order some of the second order perturbations will be a product of first order perturbations, meaning the perturbative quantities will be coupled together. The second order perturbations will no longer be (comoving) volume-preserving, meaning that the volume over which the average would be taken would not be constant. The 4D VPC gauge, on the other hand, which does account for all perturbative quantities, including tensors, can be extended to second order and no matter the order at which we truncate the perturbative quantities, the 4D VPC metric will retain its volume preserving characteristic by definition. Therefore, an extension of the 4D VPC gauge to second order and the subsequent averaging should be analysed. The benefit of this second order extension of the 4D VPC gauge is that the higher the order of the perturbations, the better the approximation the model takes of the Universe's true dynamics. While the mechanics of extending these gauges to second order is reviewed in this thesis, following [77], this extension is still quite a difficult and technical issue. For the purpose of this thesis and the subsequent research in cosmological averaging using these gauges, development of the linear order equations is sufficient.

Within Chapter 8 we have fully expressed the solutions to the EFE for the longitudinal gauge according to [27]. We used the gauge transformations for the 3D VPC gauge, Eq.'s (5.15), (5.39), (5.10) and the 4D VPC gauge transformations, Eq.'s (8.86) to (8.90), to convert the longitudinal gauge EFE and solutions into volume-preserving gauges and then the time coordinate transformations fully specified the transformation into the VPC gauges. The results in the VPC gauges were consistent with the longitudinal gauge solutions except for the 4D VPC lapse function solutions. The lapse has been constructed to qualitatively show the time dilation between proper time and the coordinate time. In the Newtonian gauge, the lapse function acts as a Newtonian potential, or Bardeen potentials Eq.'s (6.3) and (6.4), since the foliation of surfaces has been chosen to be constant. The VPC gauges, however, have a more complicated time-slicing and therefore have time dependence within the lapse function. This time dependence means that the lapse is no longer a Newtonian potential but this is not unusual. The 4D VPC time dilation, while small, will grow rapidly (see the matter domination solution Eq. (8.91)). However, with the explicit η^3 term appearing within the lapse function, we can see that perturbation theory will break down since the perturbations will grow very large and eventually distort the VPC construction. Further study into the possible long term dynamics of the VPC gauge should be studied since if perturbation theory breaks down in one well defined gauge then it will break down in all well defined gauges [13]. Note that all of the other quantities besides the lapse in the VPC gauges have the same qualitative behaviour as in the longitudinal gauge.

With the rigorous description of the gauge transformations, the EFE and their solutions in the 3D and 4D VPC gauges, the VPC gauges are now ready to be used for future research including averaging in cosmology and applications of unimodular gravity.

Chapter 9

Conclusions

Chapter 1 of this thesis introduced the possibility that within the standard FLRW cosmological model there is a potential discrepancy between what is currently being observed in our Universe and the physics that defines those observations. Of particular interest for this thesis is the supernovae Type 1a data showing an accelerating Universal expansion rate and the WMAP observations showing the distribution of CMB radiation. The introduction of exotic fields, which have yet to be explicitly observed, to try to account for these discrepancies leaves us with a theory that seems to be incomplete. The main goal of this thesis was to use perturbation theory to introduce a gauge in which averaging could be done more rigorously, namely a VPC gauge.

In Chapter 7 we were able to introduce two volume-preserving coordinate gauges and in Chapter 8 we displayed gauge transformations between these gauges and an arbitrary gauge. The longitudinal gauge solutions were transformed into the VPC gauges and these solutions were shown to have the same qualitative behaviour as the longitudinal gauge solutions except, of course, for the lapse function as discussed in Section 8.5. The lapse in the longitudinal gauge has been constructed to behave as a Newtonian Potential and the variables for this are known as the Bardeen Potentials Eq.'s (6.3) and (6.4). However, in our 4D VPC guage it can be seen that the lapse function is time dependent, Eq. (8.91). The time dependency of the lapse function means that the lapse is no longer a Newtonian potential, but this is not unusual in gauges which use complicated and different time-slicing. The 4D VPC time dilation, while small, will grow rapidly and the VP characteristic of the VPC guage will begin to distort and break the gauge. Further study of the possible long term dynamics of the VPC gauge should be considered since if perturbation theory breaks down in one well defined gauge then it will break down in all well defined gauges [13]. Note that all of the other quantities besides the lapse in the VPC gauges have the same qualitative behaviour as in the longitudinal gauge. The 3D and 4D VPC gauges developed are viable for use in cosmological averaging within perturbation theory.

In Section 7.3.3, we discussed how Buchert developed an averaging procedure using a (3+1) foliation, which is used primarily for averaging scalar quantities. A scalar averaging procedure can be used to test the size of the backreaction and inhomogeneous effects on quantities using the 3D VPC gauge. The 3D VPC gauge is best suited for the Buchert approach to averaging. The averaging procedure developed by Zalaletdinov [114], which is fully covariant and can be used to average any object, should be used with the 4D VPC gauge. The 4D VPC gauge ensures that $\sqrt{-g} = 1$ (within Eq. (7.15)) making the averaging calculation rigorous and much easier to compute. When these perturbative quantities are averaged, it is hoped that the averaged objects will have a backreaction of a size to account for the observational discrepancies. The averaged objects will retain the correct qualitative behaviour since the averaging is done in a suitable gauge. It is anticipated that the size of the backreaction from inhomogeneous effects on the Universal dynamics after averaging, when done rigorously, are of a size that will stimulate further analysis using these gauges to higher orders.

Indeed, as an application, in [16] volume-preserving uniform curvature and uniform density gauges (VPG's) in perturbation theory were analysed using spatial averaging to second order. The VPC gauge formalisms within this thesis will allow an easy transformation of the gauge specific quantities into the VPC system constructed here, facilitate the calculation of the average of any object and transformation of the averaged objects back into the original gauge for interpretation and comparison of effects. The hope is that the 3D and 4D VPC gauges will give more reliable estimates of the size of the backreaction. While the average may provide a proper backreaction size, the VPC gauge was only developed to linear order within this thesis. In the future, development of the VPC to second order is desirable in order to further analyse the average of any object.

Appendix A

Poisson Gauge

In this appendix we present the second order equations in the Poisson gauge, see [23]. The gauge is defined by $\widetilde{E} = 0 = \widetilde{B}$, and then $\widetilde{\phi} = \Phi$ and $\widetilde{\psi} = \Psi$. In the absence of anisotropic stress, as is the case for this work, $\Psi_1 = \Phi_1$ (though note that this does not hold true for the second order variables Φ_2 and Ψ_2). Note also that, in this gauge, $V = v$.

Energy conservation then becomes

$$\begin{aligned}
&{\rho_2}' + 3\mathcal{H}(\rho_2 + P_2) + (\rho_0 + P_0)\Big(\partial_a\partial^a v_2 - 3\Psi_2'\Big) + 2(\rho_1 + P_1)_{,i}{v_1}^i \\
&+ 2(\rho_1 + P_1)\Big(\partial_a\partial^a v_1 - 3\Phi_1'\Big) + 2(\rho_0 + P_0)\Big[2(v_{1,i}' + 4\mathcal{H}v_{1,i}){v_{1,}}^i \\
&+ 3\Phi_1\Phi_1' + \partial_a\partial^a v_1\Phi_1 - {v_{1,}}^i\Phi_{1,i}\Big] = 0\,,
\end{aligned} \tag{A.1}$$

while the momentum conservation equation is

$$\begin{aligned}
&\Big[(\rho_0 + P_0)v_{2,i}\Big]' + (\rho_0 + P_0)\Big(\Phi_2 + 4\mathcal{H}v_2\Big)_{,i} + \delta P_{2,i} + 2\Big[v_{1,i}(\rho_1 + P_1)\Big]' \\
&+ 2(\rho_1 + P_1)\Big(\Phi_1 + 4\mathcal{H}v_1\Big)_{,i} - 6(\rho_0 + P_0)'\Phi_1 v_{1,i} \\
&+ 2(\rho_0 + P_0)\Big[v_{1,i}\Big(\partial_a\partial^a v_1 - 3\Phi_1'\Big) + {v_{1,}}^j v_{1,ij} - \Phi_1\Big(v_1' + 2\Phi_1 + 4\mathcal{H}v_1\Big)_{,i} \\
&- 2\Big(\Phi_1 v_{1,i}\Big)' - 8\mathcal{H}\Phi_1 v_{1,i}\Big] = 0\,.
\end{aligned} \tag{A.2}$$

Then, the EFE are

$$\begin{aligned}
&3\mathcal{H}(\Psi_2' + \mathcal{H}\Phi_2) - \partial_a\partial^a\Psi_2 - 3\Phi_1'\Phi_1' - 3{\Phi_{1,}}^i\Phi_{1,i} - 8\partial_a\partial^a\Phi_1\Phi_1 - 12\mathcal{H}^2\Phi_1^2 \\
&\qquad = -4\pi G a^2\Big(2(\rho_0 + P_0){v_1}^k v_{1k} + \rho_2\Big)\,,
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
&\Psi_{2,i}' + \mathcal{H}\Phi_{2,i} + 4(\Phi_{1,i}\Phi)' - \Phi_{1,i}(8\mathcal{H}\Phi_1 + 2\Phi_1') - 4\Phi_{1,i}'\Phi_1 \\
&\qquad = -4\pi G a^2\Big[(\rho_0 + P_0)(v_{2i} - 6\Phi_1 v_{1i}) + 2(\rho_1 + P_1)v_{1i}\Big]\,,
\end{aligned} \tag{A.4}$$

and

$$\Psi_2'' + \mathcal{H}(2\Psi_2 + \Phi_2)' + \frac{1}{3}\partial_a\partial^a(\Phi_2 - \Psi_2) + \Big(\frac{2a''}{a} - \mathcal{H}^2\Big)\Phi_2$$
$$+ 4\Phi_1^2\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big) - 2\Phi_{1,}{}^{i}\Phi_{1,i} - 8\mathcal{H}\Phi_1\Phi_1' - \frac{8}{3}\partial_a\partial^a\Phi_1\Phi_1 - 3(\Phi_1')^2$$
$$= 4\pi G a^2\Big(P_2 + \frac{2}{3}(\rho_0 + P_0)v_1{}^{i}v_{1i}\Big)\,. \tag{A.5}$$

For completeness, we present the fourth field equation, obtained by applying the operator $\partial_i\partial^j$ to the $i-j$ component of the EFE, Eq. (C.3):

$$\Psi_2'' + \mathcal{H}(2\Psi_2' + \Phi_2') + \Big(\frac{2a''}{a} - \mathcal{H}^2\Big)\Phi_2 = 4\pi G a^2\delta P_2 + 8\pi G a^2(\rho_0 + P_0)\nabla^{-2}\partial_i\partial^j(v_1{}^{i}v_{1j})$$
$$- \nabla^{-2}\Big\{2\Phi_{1,k}\Phi_{1,}{}^{k\prime} + 4\Phi_{1,}{}^{i}{}_{j}\Phi_{1,i}{}^{j} - \partial_a\partial^a\Big[\Phi_1 + \Phi_1'' + 2\Phi_1^2\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big)\Big]$$
$$+ \Phi_1'\Big(4\nabla^4\Phi_1' - 3\partial_a\partial^a\Phi_1' + 2\mathcal{H}\partial_a\partial^a\Phi_1\Big)\Big\}\,, \tag{A.6}$$

where ∇^{-2} is the inverse Laplacian operator. Finally, combining Eqs. (A.5) and (A.6), we obtain

$$\partial_a\partial^a(\Psi_2 - \Phi_2) = 24\pi G a^2(\rho_0 + P_0)\Big[v_1^i v_{1i} - \nabla^{-2}\big(\partial_i\partial^j(v^i v_j)\big)\Big] + 12\Phi_1^2\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big)$$
$$- 18\mathcal{H}\Phi_1\Phi_1' - 3\nabla^{-2}\Big\{2\Phi_{1,k}{}'\Phi_{1,}{}^{k\prime} + 4\Phi_{1,}{}^{i}{}_{j}\Phi_{1,i}{}^{j} + \Phi_1'(4\nabla^4\Phi_1' - 3\partial_a\partial^a\Phi_1' + 2\mathcal{H}\partial_a\partial^a\Phi_1)\Big\}$$
$$- 6\Phi_{1,i}\Phi_{1,}{}^{i} + \Phi_1\Phi_1'' + (\Phi_1')^2 + 2\Phi_1^2\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big)\,. \tag{A.7}$$

which is the second order analogue of the equation which, at first order, tells us that the two Newtonian potentials are identical in the absence of anisotropic stress.

Appendix B

Synchronous Gauge

The synchronous gauge was introduced by Lifshitz in [61](see also Ref. [53]). This gauge was introduced by studying the symmetry arguments in order to eliminate unphysical gauge modes. The gauge conditions for synchronous gauge are defined as $\widetilde{\phi} = \widetilde{B}_i = 0$. These gauge conditions are so that the proper time of observers at fixed spatial coordinates conincides with cosmic time in the FLRW background model, restricting the perturbations to the spatial part of the metric leaving the temporal part unperturbed. This simplifies dynamical equations as the time derivatives can be directly related to proper time derivatives. Synchronous gauge is very popular for use in numerical studies and used in many Boltzmann solvers such as CMBFAST [100]. It is also discussed in detail and compared to the longituidinal gauge in [69].

The gauge condition at first order is $\widetilde{\phi_1} = \widetilde{B_{1i}} = 0$, which from Eqs. (5.15) and (5.17) gives

$$\alpha_{1\text{syn}} = -\frac{1}{a}\left(\int a\phi_1 d\eta - \mathcal{C}_1(x^i)\right) , \tag{B.1}$$

$$\beta_{1\text{syn}} = \int \left(\alpha_{1\text{syn}} - B_1\right) d\eta + \hat{\mathcal{C}}_1(x^i) , \tag{B.2}$$

$$\gamma^i_{1\text{syn}} = \int S^i_1 d\eta + \hat{\mathcal{C}}^i_1(x^i) . \tag{B.3}$$

These transformations equations do not determine the time-slicing definitively since we are left with two arbitrary scalar functions of the spatial coordinates, $\mathcal{C}_1$ and $\hat{\mathcal{C}}_1$. Note that $\mathcal{C}_1$ affects scalar perturbations on spatial hypersurfaces. We are left with two non-zero geometrical scalar perturbations,

$$\widetilde{\psi_{1\text{syn}}} = \psi_1 + \frac{\mathcal{H}}{a}\left(\int a\phi_1 d\eta - \mathcal{C}(x^i)\right) , \tag{B.4}$$

$$\widetilde{\sigma}_{1\text{syn}} = \sigma_1 + \alpha_{1\text{syn}} - B_1 , \tag{B.5}$$

and the matter variables are

$$\widetilde{\delta\rho_{1\rm syn}} = \delta\rho_1 - \frac{\rho_0'}{a}\left(\int a\phi_1 d\eta - \mathcal{C}(x^i)\right) , \qquad \text{(B.6)}$$

$$\widetilde{v_{1\rm syn}} = v_1 + B_1 - \alpha_{1\rm syn} . \qquad \text{(B.7)}$$

Thus it is not possible to define gauge-invariant quantities in general using this gauge condition [79].

To remove the symmetry ambiguity, we can follow Ref. [21] and choose the initial velocity of cold dark matter to be zero, $\widetilde{v_{1\rm cdm}} \equiv 0$, which fixes the residual gauge freedom

$$\mathcal{C}_1(x) = a(v_{1\rm cdm} + B_1) . \qquad \text{(B.8)}$$

Note that for pressureless matter, momentum conservation equation ensures that $a(v_{1\rm cdm} + B_1)$ is a constant.

Appendix C

Second Order Governing Equations

We can extend the governing equations presented in this thesis to beyond linear order by simply not truncating the expansion of each variable after the first term. Doing so, we obtain equations with similar structure to those at linear order, however with new couplings between different type of perturbation. In fact, these couplings will turn out to be the reason for the qualitative difference between the linear and higher order theories. In this chapter, we will present the full second order equations for scalar, vector and tensor perturbations in a gauge dependent format. For a full discussion and decomposition of higher order perturbation terms see [23].

The EFE give the $(0-0)$ component

$$
\begin{aligned}
&\partial_a\partial^a C^{j}_{2j} - C_{2ij,}{}^{ij} + 2\mathcal{H}(-C^{i'}_{2\,i} + B^{i}_{2,i} + 3\mathcal{H}\phi_2) + 2C^{j}_{1\,j,i}(\frac{1}{2}C^{k}_{1\,k,}{}^{i} - 2C^{ik}_{1}{}_{,k}) \\
&+ 2B^{i}_{1}\Big[C^{j'}_{1\,j,i} - C'_{1ij,}{}^{j} + \frac{1}{2}\left(\partial_a\partial^a B_{1i} - B_{1j,i}{}^{j}\right) + 2\mathcal{H}\left(C^{1j}{}_{j,i} - 2C_{1ij,}{}^{j} - \phi_{1,i}\right)\Big] \\
&+ 4C^{ij}_{1}\Big[2C_{1jk,i}{}^{k} - C^{k}_{1\,k,ij} - \partial_a\partial^a C_{1ij} + 2\mathcal{H}(C'_{1ij} - B_{1i,j})\Big] + 2C_{1jk,i}(C^{ik}_{1}{}_{,}{}^{j} - \frac{3}{2}C^{jk}_{1}{}_{,}{}^{i}) \\
&+ 2C^{i'}_{1\,i}(B_{1j,}{}^{j} - \frac{1}{2}C^{j'}_{1\,j} + 4\mathcal{H}\phi) + 4C^{ij}_{1}{}_{,i}C_{1jk,}{}^{k} + 2C'_{1ij}(\frac{1}{2}C^{ij'}_{1} - B^{j}_{1,}{}^{i}) \\
&+ \frac{1}{2}B_{1j,i}(B^{i}_{1,}{}^{j} + B^{j}_{1,}{}^{i}) - 6\mathcal{H}^2(4\phi_1^2 - B_{1i}B^{i}_{1}) - B^{i}_{1,i}B_{1j,}{}^{j} - 8\mathcal{H}B^{i}_{1,i}\phi_1 \\
&= -8\pi G a^2\Big[2(\rho_0 + P_0)V^{k}_{1}v_{1k} + \rho_2\Big], \qquad \text{(C.1)}
\end{aligned}
$$

and the $(0-i)$ component

$$
\begin{aligned}
& C^{k\prime}_{2\,k,i} - C^{\prime}_{2ik,}{}^{k} - \frac{1}{2}\left(B_{2k,i}{}^{k} - \partial_a\partial^a B_{2i}\right) - 2\mathcal{H}\phi_{2,i} + 16\mathcal{H}\phi_{1,i}\phi_1 - 2C^{j\prime}_{1\ j}\phi_{1,i} \\
& + 2C^{\prime}_{1ij}\left(2C^{kj}_{1\ ,k} - C^{k}_{1\,k,}{}^{j} + \phi_{1,}{}^{j}\right) + 4C^{kj}_{1}\left[C^{\prime}_{1ik,j} - C^{\prime}_{1jk,i} + \frac{1}{2}\left(B_{1k,ij} - B_{1i,kj}\right)\right] \\
& + 2B^{j}_{1}\left(C_{1kj,i}{}^{k} - C^{k}_{1\,k,ij} + C_{1ik,k}{}^{j} - \partial_a\partial^a C_{1ij} - 2\mathcal{H}B_{1j,i}\right) - \Big(B_{1i,j} + B_{1j,i}\Big)\phi_{1,}{}^{j} \\
& + 2\left(B_{1i,j} - B_{1j,i}\right)\left(\frac{1}{2}C^{k}_{1\,k,}{}^{j} - C^{jk}_{1\ ,k}\right) - 2C_{1ik,j}\left(B^{k\ j}_{1,} - B^{j\ k}_{1,}\right) + 2B^{j}_{1,j}\phi_{1,i} \\
& + 2\phi_1\left[B_{1j,i}{}^{j} - \partial_a\partial^a B_{1i} + 2\left(C^{\prime}_{1ij,}{}^{j} - C^{1j\prime}{}_{j,i}\right)\right] - 2C^{kj\prime}_{1}C_{kj,i} \\
& = 16\pi G\Big[\frac{1}{2}V_{2i} - \phi_1(V_{1i} + B_{1i}) + 2C_{1ik}v^{k}_{1} + (\rho_1 + P_1)V_{1i}\Big]\,,
\end{aligned}
\tag{C.2}
$$

and the full $(i-j)$ component

$$
\begin{aligned}
& C^{i\prime\prime}_{2\ j} + 2\mathcal{H}C^{i\prime}_{2j} - \tfrac{1}{2}(B^{i\prime}_{2\ ,j} + B_{2j,}{}^{i\prime}) - C^{l}_{2l,j}{}^{i} + C^{i}_{2l,j}{}^{l} - \partial_a\partial^a C^{i}_{2j} + C_{2jl,}{}^{il} - \phi_{2,}{}^{i}{}_{j} - \mathcal{H}\Big(B^{i}_{2,j} + B_{2j,}{}^{i}\Big) \\
& + \delta^{i}{}_{j}\Big\{2\Big(\tfrac{2a''}{a} - \mathcal{H}^2\Big)\phi_2 + 2\mathcal{H}\Big(B^{k}_{2\,,k} - C^{k\prime}_{2\ k} + \phi^{\prime}_{2}\Big) + B^{k\prime}_{2\ k} - C^{kl}_{2\ ,kl} - C^{k\prime\prime}_{2\ k} + \partial_a\partial^a\Big(\phi_2 + C^{l}_{2l}\Big)\Big\} \\
& + B^{k}_{1}\Big[C_{1jk,}{}^{i\prime} + C^{i\prime}_{1\ k,j} - 2C^{i\prime}_{1\ j,k} + 2\mathcal{H}(C_{1jk,}{}^{i} + C^{i}_{1k,j} - C^{i}_{1j,k}) + \tfrac{1}{2}\Big(B_{1j,}{}^{i}{}_{k} + B^{i}_{1,jk} - 2B_{1k,}{}^{i}{}_{j}\Big)\Big] \\
& + (C^{k\prime}_{1\,k} - \phi^{\prime}_{1} - B^{k}_{1\,,k})(C^{i\prime}_{1j} - \tfrac{1}{2}\Big(B^{i}_{1,j} + B_{1j,}{}^{i}\Big)) + C^{ik\prime}_{1}\Big(B_{1j,k} - 2C^{\prime}_{1kj}\Big) + C^{\prime}_{1kj}B^{i}_{1,}{}^{k} + \phi_{1,}{}^{i}\phi_{1,j} \\
& + (B^{k\prime}_{1} - 2C^{kl}_{1\ ,l} + C^{l}_{1l,}{}^{k} + \phi_{1,}{}^{k})(C_{1jk,}{}^{i} + C^{i}_{1k,j} - C^{i}_{1j,k}) + \tfrac{1}{2}B^{i}_{1}(B_{1k,j}{}^{k} - \partial_a\partial^a B_{1j} + 4\mathcal{H}\phi_{1,j} - 2C^{k\prime}_{1\ k,j} + 2C^{\prime}_{1kj,}{}^{k}) \\
& + 2C^{ik}_{1}\Big[\tfrac{1}{2}\Big(B^{\prime}_{1j,k} + B^{\prime}_{1k,j}\Big) - C^{\prime\prime}_{1kj} + \phi_{1,jk} - C_{1kl,j}{}^{l} - C_{1jl,k}{}^{l} + \partial_a\partial^a C_{1kj} + C^{l}_{1l,jk} + \mathcal{H}\Big(B_{1j,k} + B_{1k,j} - 2C^{\prime}_{1kj}\Big)\Big] \\
& - \tfrac{1}{2}\Big(B_{1k,}{}^{i}B^{k}_{1,j} + B_{1j,}{}^{k}B^{i}_{1,k}\Big) + \phi_1\Big[(B_{1j,}{}^{i\prime} + B^{i\prime}_{1,j} + 2\phi_{1,}{}^{i}{}_{j} + 2\mathcal{H}(B_{1j,}{}^{i} + B^{i}_{1,j}) - 2C^{i\prime\prime}_{1j} - 4\mathcal{H}C^{i\prime}_{1j}\Big] \\
& + 2\Big(C^{i}_{1k,l}C^{k}_{1j,}{}^{l} - C^{l}_{1j,}{}^{k}C^{i}_{1k,l} + C^{kl}_{1\ ,j}C_{1kl,}{}^{i}\Big) + 2C^{kl}_{1}\Big[C_{1kl,j}{}^{i} - C_{1jl,}{}^{i}{}_{k} - C^{i}_{1l,jk} + C^{i}_{1j,kl}\Big] \\
& + \delta^{i}{}_{j}\Big\{\Big(\mathcal{H}^2 - \tfrac{2a''}{a}\Big)(4\phi^2_1 - B_{1k}B^{k}_{1}) + 2\phi_1\Big[C^{k\prime\prime}_{1\ k} - B^{k\prime}_{1\ k} - \partial_a\partial^a\phi_1 + 2\mathcal{H}(C^{k\prime}_{1\ k} - 2\phi_1{}^{\prime} - B^{k}_{1,k})\Big] \\
& \quad + B^{k}_{1}\Big[2C^{l\prime}_{1\ l,k} - 2C^{\prime}_{1kl,}{}^{l} + \partial_a\partial^a B_{1k} - B_{1l,k}{}^{l} + 2\mathcal{H}(B^{\prime}_{1k} - \phi_{1,k} - 2C^{l}_{1k,l} + C^{l}_{1l,k})\Big] + C^{kl\prime}_{1}\Big(\tfrac{3}{2}C^{\prime}_{1kl} - B_{1l,k}\Big) \\
& \quad + 2C^{kl}_{1}\Big[C^{\prime\prime}_{1kl} - \partial_a\partial^a C_{1kl} + 2\mathcal{H}C^{\prime}_{1kl} + 2C_{1lm,k}{}^{m} - C^{m}_{1\ m,kl} - 2\mathcal{H}B_{1l,k} - B^{\prime}_{1l,k} - \phi_{1,kl}\Big] + 2B^{k\prime}_{1}(C^{l}_{1l,k} - C_{1kl,}{}^{l}) \\
& \quad + C_1{}^{k\prime}{}_{k}\Big(B^{l}_{1,l} - \tfrac{1}{2}C^{l\prime}_{1\ l}\Big) + 2C^{kl}_{1\ ,k}C_{1lm,}{}^{m} + C_{1lm,k}\Big(C^{km}_{1\ ,}{}^{l} - \tfrac{3}{2}C^{lm}_{1\ ,}{}^{k}\Big) - C^{l}_{1l,k}\Big(2C^{k}_{1m,}{}^{m} - \tfrac{1}{2}C^{m}_{1\ m,}{}^{k}\Big) \\
& \quad + \phi_1{}^{\prime}\Big(C_1{}^{k\prime}{}_{k} - B^{k}_{1,k}\Big) - \tfrac{1}{4}\Big(2B^{k}_{1,k}B_{1l,}{}^{l} - B_{1l,k}B^{k}_{1,}{}^{l} - 3B^{l}_{1,k}B_{1l,}{}^{k}\Big) + \phi_{1,k}\Big(C_1{}^{l}{}_{l,}{}^{k} - 2C_1{}^{lk}{}_{,l} - \phi_{1,}{}^{k}\Big)\Big\} \\
& = 8\pi G a^2\Big\{\delta P_2\delta^{i}{}_{j} + 2(\rho_0 + P_0)v_1{}^{i}(v_{1j} + B_{1j})\Big\}\,.
\end{aligned}
\tag{C.3}
$$

The equations for scalar perturbations only in a gauge dependent form are then obtained by substituting $C_{ij} = -\psi\delta_{ij} + E_{,ij}$ and $B_i = B_{,i}$, at both first and second order, into the above [23]. The energy conservation equation then becomes

$$
\begin{aligned}
&\rho_2{}' + 3\mathcal{H}(\rho_2 + P_2) + (\rho_0 + P_0)\Big(\partial_a\partial^a(E_2{}' + v_2) - 3\psi_2{}'\Big) + 2(\rho_1 + P_1)_{,i}v_1{}^i \\
&+ 2(\rho_1 + P_1)\Big(\partial_a\partial^a(E_1{}' + v_1) - 3\psi_1{}'\Big) + 2(\rho_0 + P_0)\Big[(V'_{1,i} + 4\mathcal{H}v_{1,i})(V_{1,i} + v_{1,i}) \\
&+ 3\psi_1\psi_1{}' + \partial_a\partial^a v_1\phi_1 - (\psi_1\partial_a\partial^a E)' + E'_{1,ij}E_{1,}{}^{ij} + v_{1,}{}^i(2\phi_{1,i} - 3\psi_{1,i} + \partial_a\partial^a E_{1,i})\Big] = 0\,,
\end{aligned}
\tag{C.4}
$$

while the momentum conservation equation is

$$
\begin{aligned}
&\Big[(\rho_0 + P_0)V_{1,i}\Big]' + (\rho_0 + P_0)\Big(\phi_2 + 4\mathcal{H}V_2\Big)_{,i} + \delta P_{2,i} + 2\Big[V_{1,i}(\rho_1 + P_1)\Big]' \\
&+ 2(\rho_1 + P_1)\Big(\phi_1 + 4\mathcal{H}V_1\Big)_{,i} - 2(\rho_0 + P_0)'\Big[(V_1 + B_1)_{,i}\phi_1 - 2(E_{1,ij}v_{1,}{}^j - \psi_1 v_{1,i})\Big] \\
&+ 2(\rho_0 + P_0)\Big[V_{1,i}\Big(\partial_a\partial^a(E_1{}' + v_1) - 3\psi_1{}'\Big) - B_{1,i}(\phi_1{}' + 8\mathcal{H}\phi_1) + v_{1,}{}^j(v_{1,ij} + 8\mathcal{H}E_{1,ij}) \\
&+ 2\Big(v_{1,}{}^jE_{1,ij} - \psi_1 v_{1,i}\Big)' - \phi_1\Big((V_1 + B_1)' + 2\phi_1 + 4\mathcal{H}v_1\Big)_{,1} - 8\mathcal{H}\psi_1 v_{1,i}\Big] = 0\,.
\end{aligned}
\tag{C.5}
$$

Turning now to the EFE, the energy constraint is

$$
\begin{aligned}
&3\mathcal{H}(\psi_2{}' + \mathcal{H}\phi_2) + \partial_a\partial^a\Big(\mathcal{H}(B_2 - E_2{}') - \psi_2\Big) + \partial_a\partial^a B_1\Big(\partial_a\partial^a(E_1{}' - \frac{1}{2}B_1) - 2\psi_1{}'\Big) \\
&+ B_{1,i}\Big(\mathcal{H}(3\mathcal{H}B_{1,}{}^i - 2\partial_a\partial^a E_{1,}{}^i - 2(\psi_1 + \phi_1)_{,}{}^i) - 2\psi_{1,}{}^{i\prime}\Big) + 2E_{1,}{}^{ij}(\psi_1 - 2\mathcal{H}B_1)_{,ij} \\
&+ 4\mathcal{H}(\psi_1 - \phi_1)\Big(3\psi_1{}' - \partial_a\partial^a(E_1{}' - B_1)\Big) + E_{1,}{}^{ij\prime}\Big(4\mathcal{H}E_1 + \frac{1}{2}E_1{}' - B_1\Big)_{,ij} \\
&+ \psi_1{}'\Big(2\partial_a\partial^a(E_1{}' - 2\mathcal{H}E_1) - 3\psi_1{}')\Big) + \psi_{1,}{}^i(2\partial_a\partial^a E_1 - 3\psi_1)_{,i} + 2\partial_a\partial^a\psi_1(\partial_a\partial^a E_1 - 4\psi_1) \\
&- 12\mathcal{H}^2\phi_1{}^2 + \frac{1}{2}\Big(B_{1,ij}B_{1,}{}^{ij} + \partial_a\partial^a E_{1,j}\partial_a\partial^a E_{1,}{}^j - E_{1,ijk}E_{1,}{}^{ijk} - \partial_a\partial^a E_1{}'\partial_a\partial^a E_1{}'\Big) \\
&= -4\pi G a^2\Big(2(\rho_0 + P_0)V_{1,}{}^k v_{1,k} + \rho_2\Big)\,,
\end{aligned}
\tag{C.6}
$$

and the momentum constraint

$$
\begin{aligned}
&\psi'_{2,i} + \mathcal{H}\phi_{2,i} - E'_{1,ij}(\psi_1 + \phi_1 + \partial_a\partial^a E_1)_{,}{}^j + B_{1,ij}(2\mathcal{H}B_1 + \phi_1)_{,}{}^j \\
&- \Big[\psi_{1,i}(\partial_a\partial^a E_1 - 4\psi_1)\Big]' - \phi_{1,i}\Big(8\mathcal{H}\phi_1 + 2\psi_1{}' + \partial_a\partial^a(E_1{}' - B_1)\Big) \\
&- B_{1,j}\psi_{1,i}{}^j + 2\psi_{1,}{}^{j\prime}E_{1,ij} + E_{1,jk}{}'E_{1,i}{}^{jk} - \psi'_{1,i}(\partial_a\partial^a E_1 + 4\phi_1) - \partial_a\partial^a\psi_1 B_{1,i} \\
&= -4\pi G a^2\Big[(\rho_0 + P_0)\Big(V_{2,i} - 2\phi_1(V_1 + B_1)_{,i} - 4(\psi_1 v_{,i} - E_{1,ik}v_{1,}{}^k)\Big) \\
&\qquad + 2(\rho_1 + P_1)V_{1,i}\Big]\,,
\end{aligned}
\tag{C.7}
$$

while, from the trace of the $i-j$ component, we obtain

$$
\begin{aligned}
&3\mathcal{H}(2\psi_2+\phi_2)' + \partial_a\partial^a(E_2{}'' + 2E_2{}' + 2\psi_2 - B_2{}' - \phi_2 + 2\mathcal{H}B_2) - 3\phi_2\Big(\mathcal{H}^2 - 2\frac{a''}{a}\Big) + 3\psi_2{}'' \\
&+ (\psi_1-\phi_1)\Big(12(\psi_1{}'' + 2\mathcal{H}\psi_1{}') + 4\partial_a\partial^a(\phi_1 + (B_1-E_1)') + 8\mathcal{H}\partial_a\partial^a(B_1 - E_1{}')\Big) \\
&+ E_{1,}{}^{ij}\Big(8\mathcal{H}(E_1{}' - B_1)_{,ij} + 2\psi_{1,ij} - 4\phi_{1,ij} - 4B_1{}'_{,ij}\Big) + E_{1,}{}^{ij}\Big(\frac{5}{2}E_1{}'_{,ij} - B_{1,ij}\Big) \\
&+ 2\partial_a\partial^a E_1{}'\Big(4\phi_1{}' - \partial_a\partial^a(E_1{}' - 2B_1)\Big) + \partial_a\partial^a E_{1,}{}^{i}\Big(\partial_a\partial^a E_{1,i} + 2\phi_{1,i} - 4\mathcal{H}B_{1,i} - 2B_1{}'_{,i}\Big) \\
&+ \psi_{1,}{}^{i}\Big(2\partial_a\partial^a E_{1,i} - 4\mathcal{H}B_{1,i} - 2(\psi_1+\phi_1)_{,i} - 2B_1{}'_{,i}\Big) - 2\phi_1{}'(\partial_a\partial^a B_1 + 12\mathcal{H}\phi_1) \\
&- 2\phi_{1,i}\phi_{1,}{}^{i} + 2\partial_a\partial^a\psi_1(\partial_a\partial^a E_1 - 4\psi_1) + \psi_1{}'\Big(3\psi_1{}' - 6\phi_1{}' - 8\mathcal{H}\partial_a\partial^a E_1 - 2\partial_a\partial^a(E_1{}' + B_1)\Big) \\
&+ 2B_{1,}{}^{i}\Big(\mathcal{H}(3B_1{}' - 2\phi_1) - 3\psi_1{}'\Big)_{,i} + \frac{1}{2}\Big(B_{1,ij}B_{1,}{}^{ij} - E_{1,ijk}E_{1,}{}^{ijk} - \partial_a\partial^a B_1\partial_a\partial^a B_1\Big) \\
&+ 4(E_{1,}{}^{ij}E_1{}''_{,ij} - \psi_1{}''\partial_a\partial^a E_1) + 3\Big(\mathcal{H}^2 - 2\frac{a''}{a}\Big)\Big(4\phi_1{}^2 - B_{1,i}B_{1,}{}^{i}\Big) \\
&= 4\pi G a^2\Big(3P_2 + 2(\rho_0 + P_0)v_{1,}{}^{i}V_{1,i}\Big)\,. \qquad \text{(C.8)}
\end{aligned}
$$

Appendix D

Geometry of Spatial Hypersurfaces

D.1 Components at Second Order of Shear, Expansion, and Acceleration

The calculation of the shear, defined in Eq. (3.9), simplifies in case of the unit normal vector field n^μ, that is for $n_i \equiv \mathbf{0}$,

$$\sigma_{ij} = -n_0 \Gamma^0_{ij} - \frac{1}{3}\theta \; g_{ij} \,, \tag{D.1}$$

which gives (including vectors and tensors) at second order

$$\delta\sigma_{00} = 0 \,, \tag{D.2}$$

$$\delta\sigma_{0i} = 2a\Big[B_1^k\left(C'_{1ik} - B_{1(1,k)}\right) - \frac{1}{3}B_{1i}\left({C'_{1k}}^{k} - {B_{1k,}}^{k}\right)\Big] \,, \tag{D.3}$$

$$\begin{aligned}\delta\sigma_{ij} = \; & a\Big[C'_{2ij} - B_{2(i,j)} + 2B_1^k\left(C_{1ki,j} + C_{1kj,i} - C_{1ij,k}\right) + 2\phi_1\left(B_{1(i,j)} - C'_{1ij}\right) \\ & -\frac{4}{3}C_{1ij}\left({C'_{1k}}^{k} - {B_{1k,}}^{k}\right) + \frac{1}{3}\delta_{ij}\Big\{ - {C'_{2k}}^{k} + {B_{2k,}}^{k} + 2\phi_1\left({C'_{1k}}^{k} - {B_{1k,}}^{k}\right) \\ & +4C_1^{kl}\left(C'_{1kl} - B_{1k,l}\right) - 2B_1^l\left(2{C_{1lk,}}^{k} - {C_1}^{k}{}_{k,l}\right)\Big\}\Big] \,. \end{aligned} \tag{D.4}$$

The expansion is given from Eq. (3.8) at second order

$$\begin{aligned}\delta\theta_2 = \frac{1}{a}\Big[& -3\frac{a'}{a}\left(\phi_2 - 3{\phi_1}^2\right) + \left({C_{2k}}^{k\prime} - {B_{2k,}}^{k}\right) + 2\phi_1\left({B_{1k,}}^{k} - {C_{1k}}^{k\prime}\right) \\ & -3\frac{a'}{a}B_{1k}B_1^k - 4C_1^{kl}C'_{1kl} + 4C_1^{kl}B_{1l,k} + 4B_1^l {C_{1lk,}}^{k} - 2B_1^k {C_1}^{l}{}_{l,k}\Big] \,. \end{aligned} \tag{D.5}$$

The acceleration is given from Eq. (3.11) at second order as

$$a_0 = 2B_1^k \phi_{1,k} \,, \qquad a_i = \Big[\phi_{2,i} + \left(B_{1k}B_1^k - 2{\phi_1}^2\right)_{,i}\Big] \,. \tag{D.6}$$

D.2 Curvature of Spatial Three-Hypersurfaces at Second Order

The intrinsic curvature of spatial three-hypersurfaces is given at second order, respectively, by

$$\begin{aligned}\delta^{(3)}R_2 \quad = \quad & \frac{1}{a^2}\Big[4\partial_a\partial^a\psi_2 - 4C_{1km,}{}^{m}C_1^{kn}{}_{,n} + 3C_{1mn,}{}^{k}C_1^{mn}{}_{,k} - C_1^k{}_{k,n}C_1^m{}_{m,}{}^{n} \\ & +4C_1^{mn}\left(C_{1mn,}{}^{k}{}_{k} + C_1^k{}_{k,mn} - C_{1mk,n}{}^{k} - C_{1kn,m}{}^{k}\right) \\ & +2\left(C_1^k{}_{k,j}C_1^{jn}{}_{,n} + C_{1jk,}{}^{j}C_1^m{}_{m,}{}^{k} - C_1^k{}_{n,m}C_1^{mn}{}_{,k}\right) \,, \end{aligned} \tag{D.7}$$

where we used

$$2\left(C^{mn}{}_{,mn} - C^m_{m,}{}^{k}{}_{k}\right) = 4\partial_a\partial^a\psi \,. \tag{D.8}$$

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