ANALYSIS OF CLUSTER AND STRIPE SOLUTIONS IN A ONE-DIMENSIONAL MODEL OF BIOLOGICAL AGGREGATION

by

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "ANALYSIS OF CLUSTER AND STRIPE SOLUTIONS IN A ONE-DIMENSIONAL MODEL OF BIOLOGICAL AGGREGATION" by Hanadi Alzubadi in partial fulfillment of the requirements for the degree of Master of Science.

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Abstract

This thesis studies the model of swarming of organisms in groups or clusters and the solution of the interaction equation in a one dimensional biological swarm model. This process describes the behavior of some animal swarms like birds and fish that attract or repulse each other. We will discuss the formation of point clusters and conditions that relate to the number of clusters and their stability. Expanding the clusters on this analysis we will observe that the formation of stripe solutions is inevitable under certain assumption on the interaction force function. In this thesis we are interested in determining how the number of clusters and stripes depend on the functions, initial conditions and model parameters. While investigating the properties we will experiment with equal and asymmetric distributions of particles and observe that although the analytical calculations become more complicated, the same general dependence on a few simple parameters is observed.

List of Abbreviations and Symbols Used

Symbol	Description
PDE	Partial Differential Equation
ODE	Ordinary Differential Equation
RHS	Right Hand Side
LHS	Lift Hand Side
$X_i(t)$	spatial location of the $i-th$ individual at time t
F	Reflects repulsive-attractive behavior of the particles
N	The total number of jth interacting particles via an interaction force F
d	The distance between clusters

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Chapter 1

Introduction

Swarm analysis is an advanced topic in applied mathematics in which swarms are modeled using differential equations. This thesis will study several swarming models and some of the specific aggregation patterns that are observed to arise in nature and numerical simulations. Specifically we will focus on understanding these models in one-dimension and study the point cluster and stripe pattern solutions.

Aggregation models try to simulate the process of collective behavior of large animal groups such as flocks of birds or schools of fish that are observed to exhibit attraction and repulsion behaviour. We will explore the formation and stability of some of these complex equilibrium patterns, and demonstrate that they often depend on the values of simple parameters. We will use linear stability analysis to study the behaviour of these steady states and verify our analytical findings with numerical results.

Particles, biological organisms or mechanical robots, have been observed to aggregate in two or more clusters. We will start by studying two and three cluster steady state solutions and then generalize our analysis to the arbitrary k-cluster steady state solution. We will then derive conditions that determine the stability of these steady states. The aim of the study of this one-dimensional swarm is to completely understand the steady states and requirements for their stability. In particular we will analyze in detail the dependence of number of clusters on the slope of the force function at the origin.

There is a large body of work done on the subject of aggregation models. While we will use the discrete model that consists of a system of Ordinary Differential Equation (ODEs), we will also introduce a popular continuous Partial Differential Equation (PDEs) model and discuss some of the important results in one dimension that relate to our work.

We will study the model we will be using in detail. We will first discuss formation of point clusters and discuss necessary and essential conditions that relate to the number of clusters and their stability. Expanding on this analysis we will observe that the formation of stripe solutions is inevitable if certain assumptions are used to define the interaction force function; the formation of this steady-state will be computed. While investigating the properties of cluster and stripe solutions, we will experiment with equal and asymmetric distributions of particles and observe that although the analytical calculations become more complicated, the same general dependence on a few simple parameters is observed. The numerical simulations, describing the cluster and stripe shape that we will discuss in this thesis, are illustrated in figures 1.1, 1.2 and 1.3 (indicating the distance d on the x axis during the time t on the vertical axis).

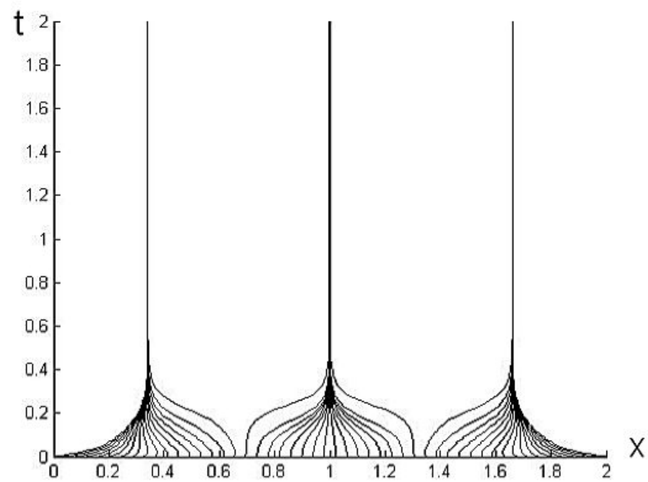

Figure 1.1: Three cluster steady state

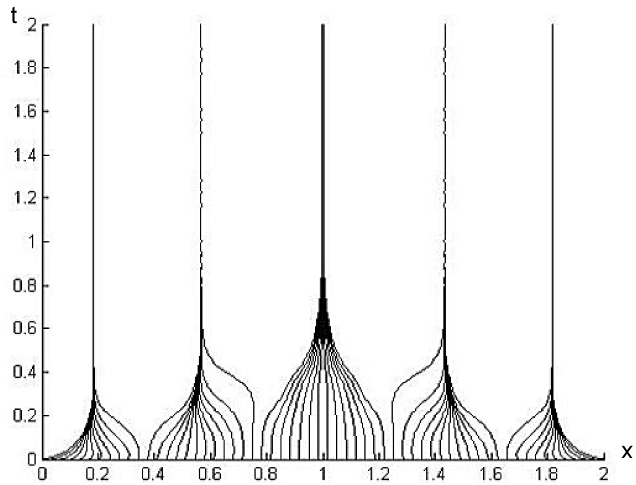

Figure 1.2: cluster steady state

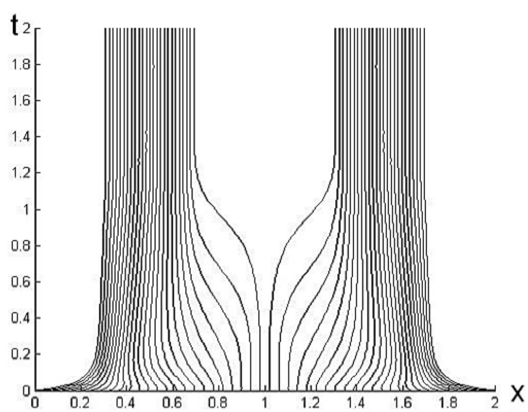

Figure 1.3: Stripe steady state

Chapter 2

The Model

The models for swarming behaviour can become really complex if we try to include all of the biological components. For example, complex aggregation models may include environmental factors such as gravity, wind, sound and light, chemicals and the presence of a predator[16]. This behaviour can be modeled discretely or generalized to a continuous PDE model when the number of particles is assumed to be large. It turns out that even the simplest models yield a rich variety of steady state patterns that are observed in nature and laboratory experiments. We will thus use the simplest discrete model that describes non-local aggregation behaviour and use it to analyze the cluster and stripe patterns, which are the focus of this thesis. In this model the position of the j th particle X_j depends on the position of all the other particles. The contributions of the other particles to the position of the jth particle depend on the distance between the two particles via an interaction force F. This model can be thus described by the following system of ordinary differential equations:

$$\frac{d}{dt}X_i = \sum_{j=1}^{N} F(|X_i - X_j|)\frac{X_i - X_j}{|X_i - X_j|}, \quad i = 1, \ldots, N \tag{2.1}$$

where N is the total number of interacting particles. We assume that the force F reflects repulsive-attractive behaviour of the particles: we have attraction if $F < 0$ and repulsion if $F > 0$. Following biological observations we assume that the particles tend to exhibit repulsive behavior at small distances and attract when the distance is sufficiently large. For convenience in analytical calculations we will often simplify the model by defining $f(r)$ as $\frac{F(r)}{r}$ and thus rewriting the model as follows:

$$\frac{d}{dt}X_i = \sum_{j=1}^{N} f(|X_i - X_j|)(X_i - X_j) \quad i = 1, \ldots, N \tag{2.2}$$

Also note that since all of the analysis in this thesis is done in one dimension, the vector norm $|X_i - X_j|$ can be treated simply as an absolute valued function, and the particle positions can be visualized simply as points on the real axis.

In the following chapters we will carry out analysis for different steady state solutions and introduce the restrictions on interaction forces necessary to yield particular types of solutions.

Chapter 3

Literature Overview

Before we present our study in swarm behavior, we would like to introduce some related studies. One of the first papers on biological swarms appeared in 1954 when Beder [7] suggested a simple mathematical model for the behavour of schools of fish. Since then many different approaches have been used to analyze the swarming phenomenon, and many new mathematical models have been developed. Popular discrete models analyze the position of each particle in the swarm as a function of position and the velocities of other particles. Other researchers [23] tried to model these complex social interaction and aggregation behaviours in a swarm in terms of Newton's equation of motion

$$m^i a^i = F^i$$

where m^i describes the mass of the individual, a^i its acceleration, and F^i gives the total acting force. Many other studies considered different models of group behavior with non-local interactions between particles. There are many studies related to these models such as self assembly of antiparticles [18,19], theory of granular gases [20], invasion models [13], chemotaxic motion [17,14] and molecular dynamics simulations of matter. The aggregation model has been extensively analyzed, see for example [18,5,8,4,6,15,16,20,3,1]. Furthermore, there are studies in self-propelled particle motion which analyze the second order models [21,12,9]. The analysis of swarm models that use the attractive-repulsive force law is covered in [20,1]. Moreover, we recognize many models of group behaviour between the species [2,10,11,22]. We aim to extend the analysis of these models by understanding cluster and stripe steady state solutions. In this thesis we will investigate the problem in one-dimension using linear stability analysis.

3.1 Motivation for the Discrete Model

The motivation for the model we used in our study comes from work in which the following ODE is analyzed [3]:

$$\dot{X}_i = \sum_{j=1}^{N} F(|X_i - X_j|)\frac{X_i - X_j}{|X_i - X_j|}$$

The repulsive - attractive interaction force used corresponds to the Morse potential and is described into the following equation:

$$F(r) = e^{-r} - G\, e^{\frac{-r}{l}}$$

Here $G < 1$ and $l > 1$. In the continuum limit as N goes to infinity the above discrete model can be translated into the following continuous model

$$\rho_t + \nabla.(\rho v) = 0 \tag{3.1}$$

$$v = -\nabla k * \rho$$

where k is the interaction potential and denotes convolution, where ρ is the density at x_i and v the velocity. In our work we will use the discrete model, which will be explained in detail in the following chapter. However, we will specialize to one dimension and demonstrate our results when the gradient of the interaction potential is piece-wise linear. We take $F(r) = min(mr + \delta, 1 - r)$, as illustrated in figure (3.1).

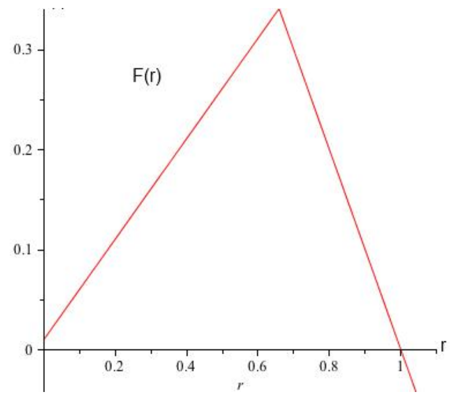

Figure 3.1: This figure illustrate $F(r) = min(mr + \delta, 1 - r)$ where $r = 0 \ldots 1.1$, $\delta = 0.01$ and $m = 0.5$

3.2 Related Results Using Continuous PDE Model

The generalized continuous particle aggregation model:

$$\partial_t \rho = \nabla_x . (\rho \nabla_x [W * \rho + V]) \tag{3.2}$$

has many advantages over the discrete model when studying certain steady state solutions and has been used in many papers. In this model $\rho(t, x)$ describes the density of particles at position $x \in \mathbb{R}^d$. The conditions are time $t \geqslant 0$, the interaction potential $W(x)$ is even and $V(x)$ is an external potential. The detailed study of the cluster steady state solutions in one dimention has been done by Fellner and Raoul [15] using this continuous model with $\int_0^n \rho(t, x)dx = 1$. Observe that (3.1) is a special case of (3.2) with $v = 0$. In one dimension $F(r) = -W'$ corresponds to the interaction force F in the discrete model.

Fellner and Raoul proceed to analyze the quadratic interaction potential $W(x)$, where $W(x) = W(-x)$. They take $W_\epsilon(x)$, which define the family of interaction potentials, in the following way:

$$W_\epsilon(x) = x^2 - |x|_\epsilon$$

$$W'_\epsilon(x) = 2x - sign_\epsilon(x)$$

$$W''_\epsilon(x) = 2 - \delta_\epsilon(x)$$

where

$$\delta_\epsilon(x) = \begin{cases} 1/2\epsilon & |x| < \epsilon \\ 0 & |x| > \epsilon \end{cases}$$

Then

$$sign_\epsilon(x) := \int_0^x \delta_\epsilon(s)\ ds$$

$$|x|_\epsilon := \int_0^x sign_\epsilon(s)\ ds$$

Note that $\int_{-\infty}^{\infty} \delta_\epsilon(x)\ dx = 1$ so that δ is concentrated near zero. Observe that the gradient of this potential is linear when two particles are sufficiently far away. i.e, $-W' = 1 - 2x$ when $x \notin (-\epsilon, \epsilon)$. In addition, $-W'(0) = 2(0) - sign_\epsilon(0) = 0$, which is constant for the fixed value of ϵ. Our work includes the special case of the Fellner and Raoul definition of the interaction potential; the essential assumptions for the potential and its gradient correspond precisely to the assumptions for the potential and the linear intercation force we used in this thesis.

In the following chapters we will study cluster steady states, which correspond to the δ concentrations in the Fellner and Raoul continuous model. Although our approach to these solutions is very different, there are many parallels between the methods used in this thesis and those used by Fellner and Raoul. For example, in the later chapters when the stability of the cluster steady states is analyzed, we observe that applying two different ansatz for the perturbations of the cluster steady states of the solution gives the complete description of the cluster stability. In the first case the center of mass of perturbations of all particles is preserved. In the second all particles are perturbed identically. The work of Fellener and Raoul features the same observation, and these perturbations are referred to as Reallocations and Shifts.

In this thesis we will first develop a method of analyzing cluster steady states for the discrete model. We will then be able to use this method to analyze more complex steady state structures such as stripe solutions.

Chapter 4

Clusters Problem Analysis

In numerical simulations, particular choices of the interaction function lead to formation of point clusters. We observe that this behaviour can only occur when the repulsion at the origin is very weak, and the necessary condition is that $F(0) = 0$. Furthermore, when this condition is satisfied, we observe formation of different numbers of clusters; this appears to depend on the value of certain parameters of the function such as the slope at the origin. Experimentation with various initial conditions lead us to believe that the particles do not always distribute themselves equally between clusters; instead it is possible to have steady states where the particles are not evenly distributed. In this chapter we will carry out a careful analytical analysis of the cluster steady states. We will begin by working out the details for the two and three cluster configurations, and then extend the theory to the general "k-cluster" steady state.

4.1 Two Clusters Steady State

We begin our study of the cluster solutions of the aggregation model by deriving the two-cluster steady state for both equal and asymmetric particle distribution. In this section, we consider a solution of two clusters of size n_1 and n_2, and have the clusters separated by some distance d. We use equation (2.2) in all our analysis.
We will first analyze the two cluster steady state when the distribution of particles is symmetric and both clusters contain an equal number of particles.

Proposition 4.1. *Suppose that $F(r)$ satisfies the conditions $F(0) = 0 = F(d)$. Then the system (2.2) admits a steady state consisting of "two clusters" where N is even:*

$$X_1, \dots, X_{\frac{N}{2}} = 0$$

$$X_{\frac{N}{2}+1}, \dots, X_N = d$$

.

Proof. Let X_i be partitioned in the following way (using indicies x and y):

$$x_i = X_i$$

$$y_i = X_{i+n}$$

where $i = 1, \ldots, n$; $n = \frac{N}{2}$ and N is even. Substituting x_i, y_i into (2.2) gives the following system:

$$\dot{x_i} = \sum_{j=1}^{n} F(|x_i - x_j|)sign(x_i - x_j) + \sum_{j=1}^{n} F(|x_i - y_j|)sign(x_i - y_j) \tag{4.2}$$

$$\dot{y_i} = \sum_{j=1}^{n} F(|y_i - x_j|)sign(y_i - x_j) + \sum_{j=1}^{n} F(|y_i - y_j|)sign(y_i - y_j) \tag{4.3}$$

In the steady state for x_i , y_i the left hand sides of (4.2) and (4.3) vanish, giving

$$0 = \sum_{j=1}^{n} F(|x_i - x_j|)sign(x_i - x_j) + \sum_{j=1}^{n} F(|x_i - y_j|)sign(x_i - y_j) \tag{4.4}$$

$$0 = \sum_{j=1}^{n} F(|y_i - x_j|)sign(y_i - x_j) + \sum_{j=1}^{n} F(|y_i - y_j|)sign(y_i - y_j) \tag{4.5}$$

Equations (4.4) and (4.5) are satisfied when we take the steady state $x_i = 0$ and $y_j = d$. $\square$

We can now consider the case when the distribution of particles is asymmetric, and each cluster contains a different number of particles. Consider the following steady state where $n_1 + n_2 = N$ and n_1, n_2 are not equal.

$$x_1, \ldots, x_{n_1} = 0$$

$$y_1, \ldots, y_{n_2} = d$$

4.2 Stability of the Two-Cluster Solution

We will first consider the symmetric case where the distribution of particles in each of the clusters is equal. Since the $x_i's$ are equal for all i, we observe that $|x_i - x_j| = |0 - 0| = 0$ for all i, j. Similarly, since the $y_i's$ are equal for all i, $|y_i - y_j| = 0$. Also, note that $|x_i - y_i| = |0 - d| = d$. We are using these conditions to illustrate the stability.

Theorem 4.6. *The steady state consisting of "two equal clusters" problem, as constructed in proposition 1, is stable if $F'(0) < -F'(d)$ and $F'(d) < 0$.*

Proof. We begin the steady state analysis by perturbing the steady state x_i, y_i in the following way:

$$x_i(t) = x_i^e + e^{\lambda t}\,\phi_i \tag{4.7}$$

$$y_i(t) = y_i^e + e^{\lambda t}\psi_i \tag{4.8}$$

where ϕ_i, $\psi_i \ll 1$, $x_i^e = 0$, $y_i^e = d$ and $i = 1, \ldots, n$. Differentiating (4.7) and (4.8) with respect to time gives:

$$\dot{x}_i(t) = \dot{x}_i^e + \lambda e^{\lambda t}\,\phi_i$$

$$\dot{y}_i(t) = \dot{y}_i^e + \lambda e^{\lambda t}\psi_i$$

Since $\dot{x}_i^e = 0$, $\dot{y}_i^e = 0$, $x_i^e = 0$ and $y_i^e = d$, we get:

$$x_i(t) = 0 + e^{\lambda t}\,\phi_i$$

$$y_i(t) = d + e^{\lambda t}\psi_i$$

$$\dot{x}_i(t) = \lambda e^{\lambda t}\phi_i$$

$$\dot{y}_i(t) = \lambda e^{\lambda t}\psi_i$$

Substituting these into equation (4.2) gives:

$$\lambda e^{\lambda t}\phi_i = \sum_{j=1}^{\frac{N}{2}} F(\left|e^{\lambda t}(\phi_i - \phi_j)\right|)\,sign(\phi_i - \phi_j)$$

$$+\sum_{j=1}^{N} F(\left|-d + e^{\lambda t}(\phi_i - \psi_j)\right|)sign(-d + e^{\lambda t}(\phi_i - \psi_j))$$

Expanding F by using Taylor series with ϕ_i , $\psi_i \ll 1$ gives:

$$\lambda e^{\lambda t}\phi_i = \sum_{j=1}^{\frac{N}{2}}(F(d) - F'(d)e^{\lambda t}(\phi_i - \psi_j))sign(\phi_i - \psi_j)$$

Since $F(0) = 0$ and $F(d) = 0$ we simplify the above equation to get:

$$\lambda\phi_i = \frac{1}{2n}\left\{\sum_{j=1}^{n} F'(0)\,(\phi_i - \phi_j) - \sum_{j=1}^{n} F'(d)\,(\psi_j - \phi_i)\right\} \tag{4.9}$$

where $n = \frac{N}{2}$ Similarly substituting in equation (4.3) we find:

$$\lambda\psi_i = \frac{1}{2n}\left\{\sum_{j=1}^{n} F'(0)\,(\psi_i - \psi_j) - \sum_{j=1}^{n} F'(d)\,(\phi_j - \psi_i)\right\} \tag{4.10}$$

We observe that this system has $2n$ eigenvalues. The conditions hold for every i (where i is the number of particle in each clusters).

(Case 1) We start the first case by letting $\sum_{j=1}^{n}\phi_j = 0$ and $\sum_{j=1}^{n}\psi_j = 0$.

Applying this condition to equations (4.9), (4.10) gives:

$$\lambda\,\phi_i = \frac{1}{2n}\left\{F'(0)\,n\,\phi_i + F'(d)\,n\,\phi_i\right\}$$

$$\lambda\psi_i = \frac{1}{2n}\left\{F'(0)\,n\,\psi_i + F'(d)\,n\,\psi_i\right\}$$

Simplifying the above equations gives the following expression for the eigenvalues:

$$\lambda = \frac{F'(0) + F'(d)}{2} \tag{4.11}$$

This eigenvalue $\lambda < 0$, provided that $F'(0) < -F'(d)$ in the steady state, corresponding to stability.

Consider the vector space V defined by:

$$V = \left\{\overrightarrow{\phi}, \overrightarrow{\psi} : \phi, \psi \in \mathbb{R}^n,\ \sum_{j=1}^{N}\phi_k = 0\,,\ \sum_{j=1}^{N}\psi_k = 0\right\}$$

Then $dim(V) = 2n - 2$. Therefore the multiplicity of λ is $2n - 2$.

(Case 2) Let $\phi_j = a$, $\psi_j = b$.

Substituting these condition into (4.9) and (4.10) gives:

$$\lambda a = \frac{F'(d)}{2}(a - b)$$

$$\lambda b = \frac{F'(d)}{2}(b-a)$$

Rewriting this in matrix notation gives the following eigenvalues problem:

$$\lambda \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2} \begin{pmatrix} F'(d) & -F'(d) \\ -F'(d) & F'(d) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The eigenvalues can easily be computed for the matrix $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where $A = \frac{F'(d)}{2}$ and $B = -\frac{F'(d)}{2}$:

$$(\frac{A}{2} - \lambda)^2 - \frac{B}{2}^2 = 0$$

$$\lambda = \frac{1}{2}(A \pm B) \tag{4.12}$$

This gives the eigenvalues $\lambda = 0$ and $\lambda = F'(d)$. Together with case 1 we have $2n$ eigenvalues. For stability we require $\lambda < 0$. Therefore the eigenvalues given by (4.12) are stable if $F'(d) < 0$. □

We can now generalize this to the situation when we have an unequal distribution.

Theorem 4.13. *Consider the steady state consisting of two clusters having n_1, n_2 particles, respectively, with $n_1 + n_2 = N$. Then the two clusters is stable if $F'(0)n_1 + F'(d)n_2 < 0$ and $F'(0)n_2 + F'(d)n_1 < 0$.*

Proof. In this case equations (4.9) and (4.10) will change as follows equations:

$$\lambda\phi_i = \frac{1}{N}\left\{\sum_{j=1}^{n_1} F'(0)\,(\phi_i - \phi_j) + \sum_{j=1}^{n_2} F'(d)\,(\phi_i - \psi_j)\right\}, i = 1, \ldots n_1 \tag{4.14}$$

$$\lambda\psi_i = \frac{1}{N}\left\{\sum_{j=1}^{n_2} F'(0)\,(\psi_i - \psi_j) + \sum_{j=1}^{n_1} F'(d)\,(\psi_i - \phi_j)\right\}, i = 1, \ldots n_2 \tag{4.15}$$

where n_1 is the number of particles in first position x_i, n_2 is the number of particles in the second position y_i and $n_1 + n_2 = N$. To compute the eigenvalues we consider two cases:

(case 1) Let $\sum_{j=1}^{n} \phi_j = 0$ and $\sum_{j=1}^{n} \psi_j = 0$.

Substituting into equations (4.14) and (4.15) gives:

$$\lambda\phi_i = \frac{1}{N}\left\{F'(0)\,n_1\phi_i + F'(d)\,n_2\phi_k\right\}$$

$$\lambda\psi_i = \frac{1}{N}\left\{F'(0)\, n_2\psi_i + F'(d)\, n_1\psi_k\right\}$$

Rewriting this gives the following eigenvalues:

$$\lambda_1 = \frac{F'(0)\, n_1 + F'(d)n_2}{N}, \lambda_2 = \frac{F'(0)n_2 + F'(d)n1}{N}$$

(Case 2) In this case we will take $\phi_j = a$, $\psi_j = b$ for every k. Substituting into equations (4.14) and (4.15) gives:

$$a\,\lambda = \frac{1}{N}\left\{-F'(d)n_2 b + F'(d)n_2 a\right\}$$

$$b\lambda = \frac{1}{N}\left\{-F'(d)n_1 a + F'(d)n_1 b\right\}$$

Rewriting in matrix notation gives the following eigenvalue problem:

$$\lambda\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{N}\begin{pmatrix} F'(d)n_2 & -F'(d)n_2 \\ -F'(d)n_1 & F'(d)n_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\left(F'(d)n_2 - \lambda\right)\left(F'(d)n_1 - \lambda\right) - F'(d)n_1 n_2 = 0$$

$$\lambda\left(\lambda - \left(F'(d)n_1 + F'(d)n_2\right)\right) = 0$$

After computing the eigenvalues we have $\lambda_3 = 0$, $\lambda_4 = \frac{n_2+n_1}{N}F'(d) = F'(d)$. □

4.3 Three Clusters Steady State

The previous sections discussed the steady state analysis and stability of the two-clusters solution. In this section we will increase the number of clusters to three and analyze the three-cluster steady state. First, we assume that the particles are equally distributed between the clusters.

Let x_k be the position of the kth particle in the first cluster, y_k be the position of the kth particle of the second cluster and z_k be the position of the kth particle of the third cluster.
Equation (2.2) implies the following system for x_k, y_k and z_k:

$$\dot{x_k} = \frac{1}{3n}\{\sum_{j=1}^{n} F(x_k - y_j)sgn(x_k - y_j) + \sum_{j=1}^{n} F(x_k - y_j)sgn(x_k - y_j) + \sum_{j=1}^{n} F(x_k - z_j)sgn(x_k - z_j)\} \tag{4.16}$$

$$\dot{y_k} = \frac{1}{3n}\{\sum_{j=1}^{n} F(y_k - y_j)sgn(y_k - y_j) + \sum_{j=1}^{n} F(y_k - x_j)sgn(y_k - x_j) + \sum_{j=1}^{n} F(y_k - z_j)sgn(y_k - z_j)\} \tag{4.17}$$

$$\dot{z_k} = \frac{1}{3n}\{\sum_{j=1}^{n} F(z_k - z_j)sgn(z_k - z_j) + \sum_{j=1}^{n} F(z_k - x_j)sgn(z_k - x_j) + \sum_{j=1}^{n} F(z_k - y_j)sgn(z_k - y_j)\} \tag{4.18}$$

Proposition 4.19. *The system (2.2) admits a steady state for three equal clusters if $-F(d) = F(2d)$, where d is the equal distance between the position $y_k - x_k$ and $z_k - y_k$.*

Proof. We assume that $y_k - x_k = d = z_k - y_k$. We apply this in equations (4.16), (4.17) and (4.18) and get the following system:

$$F(d)\,(-1) + F(d+d)\,(-1) = 0$$

$$F(d)\,(+1) + F(d)\,(-1) = 0$$

$$F(d+d)(+1) + F(d) = 0$$

where d is the equal distances between three clusters. By solving these equations we will have.

$$-F(d) = F(2d)$$

$\square$

4.4 Stability Analysis for three Clusters Problem

We now explore the stability analysis for three clusters. We have the following proposition.

Proposition 4.20. *The three equal cluster equilibrium is stable, if*
$F'(0) + F'(d_{12}) + F'(d_{13}) < 0$, $F'(0) + F'(d_{12}) + F'(d_{23}) < 0$,
$F'(d_{13}) + F'(d_{23}) + F'(0) < 0$ *and* $F'(d_{ij}) < 0$
where $d_{12} = d_{23} = d$ *and* $d_{13} = 2d$ *and* d *is given in proposition (4.18) .*

Proof. Let x, y and z be the partition of the clusters on the line such that $|x - y| = d_{12}$, $|y - z| = d_{23}$, and $|x - z| = d_{13}$.
Considering x_k ,y_k ,z_k where ϕ_k ,ψ_k, $\xi_k \ll 1$ we get:

$$x_k(t) = x_k^e + e^{\lambda t}\,\phi_k \tag{4.21}$$

$$y_k(t) = y_k^e + e^{\lambda t}\psi_k \tag{4.22}$$

$$z_k(t) = z_k^e + e^{\lambda t}\xi_k \tag{4.23}$$

where $x_k^e = 0$, $y_k^e = d_{12}$ and $z_k^e = d_{13}$.

Substituting (4.21),(4.22) and (4.23) in equations (4.16), (4.17) and (4.18) gives the following system:

$$\lambda\phi_k = \frac{1}{3n}\left\{\sum_{j=1}^{n} F'(0)(\phi_k - \phi_j) - \sum_{j=1}^{n} F'(d_{12})\,(\psi_j - \phi_k) - \sum_{j=1}^{n} F'(d_{13})\,(\xi_j - \phi_k)\right\} \tag{4.24}$$

$$\lambda\psi_k = \frac{1}{3n}\left\{\sum_{j=1}^{n} F'(0)\,(\psi_k - \psi_j) - \sum_{j=1}^{n} F'(d_{12})\,(\phi_j - \psi_k) - \sum_{j=1}^{n} F'(d_{23})(\xi_j - \psi_k)\right\} \tag{4.25}$$

$$\lambda\xi_k = \frac{1}{3n}\left\{\sum_{j=1}^{n} F'(0)(\xi_k - \xi_j) - \sum_{j=1}^{n} F'(d_{13})\,(\phi_j - \xi_k) - \sum_{j=1}^{n} F'(d_{23})(\psi_j - \xi_k)\right\} \tag{4.26}$$

As in theorem (4.12) we consider two cases.

(case 1) Let $\sum_{j=1}^{n} \phi_k = 0$, $\sum_{j=1}^{n} \psi_k = 0$ and $\sum_{j=1}^{n} \xi_k = 0$.
Substituting these condition into equations (4.24), (4.25) and (4.26) gives:

$$\lambda\,\phi_k = \frac{1}{3}\,\{F'(0)\,\phi_k + F'(d_{12})\,\phi_k + F'(d_{13})\,\phi_k\}$$

$$\lambda\,\psi_k = \frac{1}{3}\,\{F'(0)\,\psi_k + F'(d_{12})\,\psi_k + F'(d_{23})\,\psi_k\}$$

$$\lambda\,\xi_k = \frac{1}{3}\,\{F'(d_{13})\,\xi_k + F'(d_{23})\xi_k + F'(0)\,\xi_k\}$$

Simplifying these equations gives:

$$\lambda = \frac{1}{3}\,\{F'(0) + F'(d_{12}) + F'(d_{13})\,\}$$

$$\lambda = \frac{1}{3}\,\{F'(0) + F'(d_{12}) + F'(d_{23})\,\}$$

$$\lambda = \frac{1}{3}\,\{F'(d_{13})\; + F'(d_{23}) + F'(0)\}$$

Stability requires that:

$F'(0) + \; F'(d_{12}) + \; F'(d_{13}) < \; 0$, $F'(0) + \; F'(d_{12}) + \; F'(d_{23}) < \; 0$ and $F'(d_{13}) \; + \; F'(d_{23}) + \; F'(0) < \; 0$.

(Case 2): Let $\phi_k = a$, $\psi_k = b$, $\xi_k = c$.

Substituting into equations (4.24), (4.25)and (4.26) gives:

$$\lambda a = \frac{1}{3n}\,\{-nF'(d_{12})(b-a) - nF'(d_{13})(c-a)\}$$

$$\lambda b = \frac{1}{3n}\,\{-nF'(d_{12})\,(a-b) - nF'(d_{23})(c-b)\}$$

$$\lambda c = \frac{1}{3n}\,\{-nF'(d_{13})\,(a-c) - nF'(d_{23})(b-c)\}$$

Rewriting this system in matrix notation gives the following eigenvalues problem:

$$\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{3} \begin{pmatrix} (F'(d_{13})\; + F'(d_{12})) & -F'(d_{12}) & -F'(d_{13}) \\ -F'(d_{12}) & F'(d_{12}) + F'(d_{23}) & -F'(d_{23}) \\ -F'(d_{13}) & -F'(d_{23}) & (F'(d_{13})\; + F'(d_{23})) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{4.27}$$

Assuming $F'(d_{ij}) < 0$, the matrix in (4.27) has non-positive eigenvalues by the Gershgorin theorem (see page(24)).

If we take $F(r) = min(mr, 1-r)$ and apply it in equations (4.24), (4.25) and (4.26) we have:

$$\lambda\phi_k = \frac{1}{3n}\{m\sum_{j=1}^{n}(\phi_k - \phi_j) + \sum_{j=1}^{n}(\psi_j - \phi_k) + \sum_{j=1}^{n}(\xi_j - \phi_k)\}$$

$$\lambda\psi_k = \frac{1}{3n}\{m\sum_{j=1}^{n}(\psi_k - \psi_j) + \sum_{j=1}^{n}(\phi_j - \psi_k) + \sum_{j=1}^{n}(\xi_j - \psi_k)\}$$

$$\lambda\xi_k = \frac{1}{3n}\{m\sum_{j=1}^{n}(\xi_k - \xi_j) + \sum_{j=1}^{n}(\xi_j - \phi_k) + \sum_{j=1}^{n}(\psi_j - \xi_k)\}$$

We have from first case $\lambda = \frac{m-2}{3}$, which results in the steady state being stable if $m < 2$. Also if we apply case 2 where $\phi_k = a$, $\psi_k = b$, $\xi_k = c$ we get the following matrix:

$$\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where the eigenvalues are $\lambda = 0$ and $\lambda = -1$. □

Definition 4.28. *A matrix is said to be diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other* $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$.

We can now consider a steady state for the three clusters problem for a asymmetric number of particles.

From the main equation (3.2) we get the following system:

$$0 = n_1F(0) - n_2F(d_{12}) - n_3F(d_{13}) \tag{4.29}$$

$$0 = -n_1F(d_{12}) + n_2F(0) + n_3F(d_{23}) \tag{4.30}$$

$$0 = -n_1F(d_{13}) - n_2F(d_{23}) + n_3F(0) \tag{4.31}$$

where $F(0) = 0$. The above system becomes:

$$F(d_{13}) = -F(d_{12})\frac{n_2}{n_3} = -F(d_{23})\frac{n_2}{n_1}$$

We can compute d_{12} and d_{23} by substituting $F(r) = min(mr, 1-r)$ using equations (4.29), (4.30) and (4,31) :

$$F(d_{13}) = -F(d_{12})\frac{n_2}{n_3}$$

$$d_{12} + d_{23}\frac{n_3}{n_2+n_3} - \frac{n_2+n_3}{n_2+n_3} = 0$$

$$F(d_{13}) = -F(d_{23})\frac{n_2}{n_1}$$

$$d_{12} + d_{23}\frac{n_1+n_2}{n_1} - \frac{n_2+n_1}{n_1} = 0$$

Solving the above equations gives:

$$d_{12} = \frac{n_1+n_2}{n_1+n_2+n_3}$$

$$d_{23} = \frac{n_2+n_3}{n_1+n_2+n_3}$$

In the case $n_1 = n_3$, we have $d_{12} = d_{23} = \frac{n_1+n_2}{2n_1+n_2}$.

Proposition 4.32. *The asymmetrically spaced three clusters equilibrium is stable if:* $n_1F'(0) + n_2F'(d_{12}) + n_3F'(d_{13}) < 0$, $n_2F'(0) + n_1F'(d_{12}) + n_3F'(d_{23}) < 0$, $n_1F'(d_{13}) + n_2F'(d_{23}) + n_3F'(0) < 0$ *and* $F'(d_{ij}) < 0$. *where* $n_1 + n_2 + n_3 = N$.

Proof. When the number of particles in three clusters are asymmetric, equations (4.16), (4.17) and (4.18) result in the following equations, where $n_1 + n_2 + n_3 = N$.

$$\lambda\phi_K = \frac{1}{N}\left\{\sum_{j=1}^{n_1} F'(0)\,(\phi_k - \phi_j) - \sum_{j=1}^{n_2} F'(d_{12})\,(\psi_j - \phi_k) - \sum_{j=1}^{n_3} F'(d_{13})\,(\xi_j - \phi_k)\right\} \tag{4.33}$$

$$\lambda\psi_K = \frac{1}{N}\left\{\sum_{j=1}^{n_2} F'(0)\,(\psi_k - \psi_j) - \sum_{j=1}^{n_1} F'(d_{12})\,(\phi_j - \psi_k) - \sum_{j=1}^{n_3} F'(d_{23})(\xi_j - \psi_k)\right\} \tag{4.34}$$

$$\lambda\xi_K = \frac{1}{N}\left\{\sum_{j=1}^{n_3} F'(0)\,(\xi_k - \xi_j) - \sum_{j=1}^{n_1} F'(d_{13})\,(\phi_j - \xi_k) - \sum_{j=1}^{n_2} F'(d_{23})(\psi_j - \xi_k)\right\} \tag{4.35}$$

First we compute the eigenvalues by considering two cases:

(Case 1) Let $\sum_{j=1}^{n_1}\phi_j = 0$, $\sum_{j=1}^{n_2}\psi_j = 0$ and $\sum_{j=1}^{n_3}\xi_j = 0$

Substituting these conditions in to equations (4.33), (4.34), (4.35) gives:

$$\lambda\,\phi_k = \frac{1}{N}\left\{n_1F'(0)\,\phi_k + n_2F'(d)\,\phi_k + n_3F'(2d)\,\phi_k\right\} \tag{4.36}$$

$$\lambda\,\phi_k = \frac{1}{N}\left\{n_1F'(0)\,\phi_k + n_2F'(d)\,\phi_k + n_3F'(2d)\,\phi_k\right\} \tag{4.37}$$

$$\lambda\,\xi_k = \frac{1}{N}\left\{n_3F'(0)\,\xi_k + n_1F'(d)\xi_k + n_2F'(2d)\,\xi_k\right\} \tag{4.38}$$

For stability we need (4.36), (4.37) and (4.38) to be negative.

(Case 2) Let $\phi_k = a$, $\psi_k = b$ and $\xi_k = c$

Substituting into equations (4.33), (4.34) and (4.35) gives:

$$\lambda a = \frac{1}{N}\left\{-n_2F'(d_{12})(b-a) - n_3F'(d_{13})(c-a)\right\}$$

$$\lambda b = \frac{1}{N}\left\{-n_1F'(d_{12})\,(a-b) - n_3F'(d_{23})(c-b)\right\}$$

$$\lambda c = \frac{1}{N}\left\{-n_1F'(d_{13})\,(a-c) - n_2F'(d_{23})(b-c)\right\}$$

Rewriting it in matrix notation gives the following eigenvalues problem:

$$\lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \tfrac{1}{N}$$

$$\begin{pmatrix} n_2F'(d_{12}) + n_3F'(d_{13}) & -n_2F'(d_{12}) & -n_3F'(d_{13}) \\ -n_1F'(d_{12}) & n_1F'(d_{12}) + n_3F'(d_{23}) & -n_3F'(d_{23}) \\ -n_1F'(d_{13}) & -n_2F'(d_{23}) & n_1F'(d_{13}) + n_2F'(d_{23}) \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The asymmetrically spaced three clusters equilibrium is stable if: $n_2F'(d_{12}) + n_3F'(d_{13}) < 0$, $n_1F'(d_{12}) + n_3F'(d_{23}) < 0$,
$n_1F'(d_{13}) + n_2F'(d_{23}) < 0$ and $F'(d_{ij}) < 0$ where $n_1 + n_2 + n_3 = N$. $\square$

4.5 Existence of k-cluster Solution

We will show the existence of the arbitrary k-cluster steady state using the following energy argument.

Theorem 4.39. *Suppose that $F(r) < c_0$ with $c_0 > 0$ for all sufficiently large $r > r_0$ and suppose that $F(r)$ is continuous for all $r \geq 0$.*
Then for any $k \geq 1$ there exists $y_1 \dots y_k \in \Re$ such that:

$$\sum_{j=1}^{k} F(|y_i - y_j|) \frac{y_i - y_j}{|y_i - y_j|} = 0\,,\; i = 0, \dots, k \tag{4.40}$$

Proof. Let consider the odd function $G(r) = -\int F(r)\, dr$ and we define E by:

$$E(y_1 \dots y_k) = \sum_{i,j}^{k} G(|y_i - y_j|) \tag{4.41}$$

Then equation (4.40) is equivalent to $\frac{\partial}{\partial y_i} E = 0$, $i = 1, \dots, k$. Thus if E has a minimum, then (4.40) is satisfied.
Now we have $G(r) \sim c_0\, r$ as $r \longrightarrow \infty$, to be more precise $G(r) > c_0\, r + c_1$, $\forall\, r > 0$, for some c_1.
Thus $E \longrightarrow \infty$ as $y_j \longrightarrow \infty$ for any j. It follows by a compactness argument that E has a minimum. □

4.6 Steady State Analysis for " k clusters"

In the previous sections we analyzed two and three-cluster steady states. We would like to be able to extend our analysis to the arbitrary number of clusters.

Proposition 4.42. *The system (2.2) admits a steady state for k clusters:*

$$X_1, \dots, X_{\frac{N}{k}} = 0$$

$$X_{\frac{N}{k}+1}, \dots, X_{\frac{2N}{k}} = d_{12}$$

$$X_{\frac{2N}{k}+1}, \dots, X_{\frac{2N}{k}} = d_{13}$$

$$\vdots$$

$$X_{\frac{(k-1)N}{k}+1}, \ldots, X_N = d_{1j}$$

where $X_1, \ldots, X_{\frac{N}{k}}$ *is the partition of the first cluster,* $X_{\frac{N}{k}+1}, \ldots, X_{\frac{2N}{k}}$ *is the partition of the second cluster and* $X_{\frac{(k-1)N}{k}+1}, \ldots, X_N = d_{ij}$ *is the partition of the* N*th cluster. For convenience, we define* x_i^j *to be the* i*th particle in the* j*th cluster:*
$x_i^1 = X_i$, $x_i^2 = X_{i+l} \ldots x_i^n = X_{i+(k-1)l}$, *where* $l = \frac{N}{k}$ *and* $i = 1, \ldots n$.

4.7 Stability Analysis for k Clusters Problem

The following system illustrates the "k clusters" problem for equation (2.2):

$$\dot{x_s}^l = \sum_{j=1}^{n} \sum_{k=1}^{m} F(|x_s^l - x_j^k|)\, sign(x_s^l - x_j^k) \tag{4.43}$$

where n is the number of particles in each hole and k is the number of clusters along the x axis.
Perturbing the steady state:

$$x_s^l = \varphi_s^l + e^{\lambda t}\varphi_s^l$$

where $\varphi_s^l \ll 1$ and $\varphi_s^l = 0, d_{12} \ldots, d_{ij}$. Differentiating gives:

$$\dot{x_s}^l = 0 + \lambda e^{\lambda t}\varphi_s^l$$

Then substituting into the equation (4.43) gives:

$$\lambda\varphi_s^l = \frac{1}{kn} \sum_{j=1}^{n} \sum_{i=1}^{k} F'(d_{ij})(\varphi_i^l - \varphi_j^s) \tag{4.44}$$

where $d_{ij} = x_i^{el} - x_j^{es}$.
Assume that $\varphi_i^l = \alpha_l$ and $\varphi_j^s = \alpha_s$

$$\lambda\alpha_l = \frac{1}{nk} \sum_{j=1}^{n} \sum_{i=1}^{k} F'(d_{ij})(\alpha_l - \alpha_s) \tag{4.45}$$

Proposition 4.46. *The steady state consisting of "k clusters" is stable if* $F'(d_{ij}) < 0$ $\forall i, j = 1 \ldots k$ *where* $i \neq j$ *,and* $\sum_{j=1}^{k} n_j F'(d_{ij}) < 0$ *for* $i = 1 \ldots k$.

Proof. To show that the steady state is stable, we have to show that all eigenvalues in k clusters case satisfy $\lambda < max\,(a_{ii})$ for all diagonal entries of the diagonal matrix $x < 0$.

$$\lambda\alpha_l = \frac{1}{nm}\sum_{j=1}^{n}\sum_{i=1}^{m} F'(d_{ij})\alpha_l - \sum_{j=1}^{n}\sum_{i=1}^{m} F'(d_{ij})\alpha_s$$

The diagonal entries are:

$$\frac{1}{nm}\sum_{j=1}^{n}\sum_{i=1}^{m} F'(d_{ij})\alpha_l = \frac{1}{n}\sum_{j=1}^{n} F'(d_{ij})$$

This will give a matrix which is said to be diagonally dominant if $F'(d_{ij}) < 0$, thus $\sum_{i\neq j} n_j F'(d_{ij}) < 0$ satisfied.

Gershgorin circle theorem: Every eigenvalue of matrix A_{nn} satisfies:

$$|\lambda - A_{ii}| \leq \sum_{j\neq i} |A_{ij}|\,,\ i = 1, 2, \ldots n$$

$\square$

Chapter 5

Stripes Analysis Problem

In numerical simulations we observed that if the interaction force is weak, but non-zero repulsive at the origin, the cluster steady state becomes impossible and is often replaced by a stripe steady state: particles arrange themselves in a thin stripe, where the distance between each particle appears to be uniform.

We will explore this behavior with the particular example of a piecewise-linear interaction force $F(r) = min(mr + \delta, 1 - r)$, with $0 < \delta \ll 1$ where m and δ are positive constants. We observe that using various initial conditions, these stripes may contain an equal or asymmetric number of particles. We will explore this behavior in detail for the two-stripe steady state, examining both symmetric and asymmetric and compute their steady state.

5.1 Symmetric Steady State

We label in equation (2.2) :

$$x_1 \dots x_n = X_1 \dots X_n$$

$$y_1 \dots y_n = X_{n+1} \dots X_N$$

where $n = \frac{N}{2}$. For the stripe we have following equations for $\dot{x}_i$ and $\dot{y}_i$:

$$\dot{x}_i = \sum_{j=1}^{n} F(|x_i - x_j|) \frac{x_i - x_j}{|x_i - x_j|} + \sum_{j=1}^{n} F(|x_i - y_j|) \frac{x_i - y_j}{|x_i - y_j|} \tag{5.1}$$

$$\dot{y}_i = \sum_{j=n+1}^{N} F(|y_i - y_j|) \frac{y_i - y_j}{|y_i - y_j|} + \sum_{j=n+1}^{N} F(|y_i - x_j|) \frac{y_i - x_j}{|y_i - x_j|} \tag{5.2}$$

These equations illustrate the steady state for two stripes where x_i is the first position of the stripe and y_j is the second position of the stripe for $i > j$, $\frac{x_i - x_j}{|x_i - x_j|} = +1$. Then,

$F = \delta + m(x_i - x_j)$. In the other case, $j > i$, $\frac{x_i - x_j}{|x_i - x_j|} = -1$. Then, $F = \delta - m(x_i - x_j)$. Therefore,

$$\dot{x}_i = \sum_{j=1}^{n} (m\,|x_i - x_j| + \delta)\, sign(x_i - x_j) + \sum_{j=1}^{n} (1 - |x_i - y_j|)\, sign(x_i - y_j) \tag{5.3}$$

For convenience we divide equation (5.3) in two parts. We assume that $y_n = -x_n$ and $x_1 < x_2 < \ldots < x_i$. This implies $y_j > x_i$ where $\frac{x_i - y_j}{|x_i - y_j|} = -1$. Then the first part of the equation (5.3):

$$S_1 = \sum_{j=1}^{n} (m\,|x_i - x_j| + \delta)\, sign(x_i - x_j))$$

$$= \sum_{j=1}^{i-1} (m(x_i - x_j) + \delta)(+1) + \sum_{j=i+1}^{n} (m(x_j - x_i) + \delta)(-1)$$

$$= \sum_{j=1}^{i-1} m(x_i - x_j) + (i-1)\delta + \sum_{j=i+1}^{n} m(x_i - x_j) - (n - (i+1))\delta$$

$$S_1 = \sum_{j=1}^{n} m(x_i - x_j) + (2i - n - 1)\delta \tag{5.4}$$

where $x_i - y_j = x_i + x_j$ and $F(x_i - x_j) = 1 - 2x_i$. Thus, the next part of equation (5.3) becomes:

$$S_2 = \sum_{j=1}^{n} (1 - |x_i - y_j|)\, sign(x_i - y_j) = \sum_{i=1}^{n} (1 + (x_i + x_j))(-1) = -n - \sum_{j=1}^{n} (x_i + x_j) \tag{5.5}$$

Substituting equations (5.4) and (5.5) into (5.3) gives:

$$\dot{x}_i = n(m-1)x_i - \sum_{j=1}^{n} (1+m)\, x_j + (2i - n - 1)\delta - n \tag{5.6}$$

To verify the result, we take $m = 1$ and $\delta = 0$ that gives the two cluster steady state where $x_i = -\frac{1}{2}$, which implies the same result we had before where $F(r) = min(mr, 1 - r)$.

5.2 Construction of the Symmetric Solution

In this section we construct a stripe solution. We assume that the the stripe has the following form:

Let $x_j = a + c\,j$ and $x_i = a + c\,i$, the equation (5.6) becomes:

$$n(m-1)(a + c\,i) - \sum_{j=1}^{n}(1+m)\,(a + c\,j) + (2i - n - 1)\delta - n = 0 \tag{5.7}$$

Collecting the ith term in equation (5.7) gives:

$$i[n\,c\,(m-1) + 2\delta] = 0$$
$$c = \frac{2\delta}{n(1-m)}$$

The rest of the term in equation (5.7) then yield:

$$n\,a\,(m-1) - (m+1)[a + c\frac{n(n+1)}{2}] - (n+1)\delta - n = 0$$
$$a[(m-1)n - (m+1)] = [\frac{(m+1)(n+1)}{(m-1)} + (n+1)]\delta + n$$
$$a = [(\frac{(m+1)(n+1)}{(m-1)} + (n+1))\delta + n]/[(m-1)n - (m+1)]$$

Note that we must have $c > 0$, since we assumed that $x_1 < x_2 < x_3 \ldots$, which imposes the restriction $m < 1$. This corresponds precisely to the stability regime of two equal clusters when $\delta = 0$ (see Theorem 4.6).

5.3 Asymmetric Steady State

Stability analysis for two clusters for $F(r) = min(mr + \delta, 1 - r)$ where $\delta \ll 1$:

We suppose that:

$$x_i = x_e + \delta\phi_i$$
$$y_i = y_e + \delta\psi_i$$

If $\delta = 0$, then $x_e = -0.5$ and $y_e = 0.5$ and we will have the same result in chapter 4.

To compute $\dot{\phi}_i$ and $\dot{\psi}_i$ we first find the values for $\dot{x}_i$ and $\dot{y}_i$ from (5.1) and (5.2):

$$\dot{x}_i = \sum_{j=1}^{k} F(|x_i - x_j|)sign(x_i - x_j) + \sum_{j=1}^{l} F(|x_i - y_j|)sign(x_i - y_j)$$

$$\dot{x_i} = \sum_{j=1}^{i-1} F(x_i - x_j)(+1) - \sum_{j=i+1}^{k} F(-(x_i - x_j)) - \sum_{j=1}^{l} F(-(x_i - y_j)) \tag{5.8}$$

where k is the number of particle in the first stripe and l is the number of particle in the second stripe (which these numbers are different).

Next we approximate:

$$F(x_i - x_j) = F(x_e + \phi_i\delta - x_e - \phi_j\delta) = F((\phi_i - \phi_j)\delta) \simeq F(0) + F'(0)(\phi_i - \phi_j)\delta$$

$$F(x_i - y_j) = F(x_e + \phi_i\delta - y_e - \psi_j\delta) = F((x_e - y_e) + (\phi_i - \psi_j)\delta)$$

$$= F(x_e - y_e) + F'(x_e - y_e)(\phi_i - \psi_j)\delta$$

$$F(-(x_i - x_j)) = F(0) - F'(0)(\phi_i - \phi_j)\delta$$

On the right hand side of (5.8) we have $\dot{x_i} = \dot{\phi_i}\delta$, so that the leading order terms in (5.7) yield

$$\dot{\phi_i}\delta = \sum_{j=1}^{i-1}(F(0) + F'(0)(\phi_i - \phi_j)\delta) - \sum_{j=i+1}^{k}(F(0) - F'(0)(\phi_i - \phi_j)\delta) - \sum_{j=1}^{l}(F(x_e - y_e)$$

$$-F'(x_e - y_e)(\phi_i - \psi_j)\delta)$$

Then substituting $F(0) = \delta$,$F(x_e - y_e) = 0$, we obtain the following equation for $\dot{\phi_i}$:

$$\dot{\phi_i}\delta = \sum_{j=1}^{i-1}(\delta + F'(0)(\phi_i - \phi_j)\delta) - \sum_{j=i+1}^{k}(\delta - \delta F'(0)(\phi_i - \phi_j))$$

$$+\delta F'(x_e - y_e)\sum_{j=1}^{l}(\phi_i - \psi_j)$$

$$\dot{\phi_i} = \; i - 1 + F'(0)\sum_{j=1}^{k}(\phi_i - \phi_j) - (k - i) + F'(x_e - y_e)\sum_{j=1}^{l}(\phi_i - \psi_j)$$

$$\dot{\phi_i} = F'(0)\left[k\phi_i - \sum_{j=1}^{k}\phi_j\right] + F'(d)\left[l\phi_i - \sum_{j=1}^{l}\psi_j\right] + 2i - k - 1 \tag{5.9}$$

When $F'(0) = m$ and $F'(d) = -1$, (5.9) becomes:

$$\dot{\phi_i} = m\left[k\phi_i - \sum_{j=1}^{k}\phi_j\right] - \left[l\phi_i - \sum_{j=1}^{l}\psi_j\right] + 2i - k - 1 \tag{5.10}$$

Similarly we find the equation for $\dot{\psi}_i$:

$$\dot{\psi}_i = m\left[l\psi_i - \sum_{j=1}^{l}\psi_j\right] - \left[k\psi_i - \sum_{j=1}^{k}\phi_j\right] + 2i - l - 1 \tag{5.11}$$

5.4 Construction of the Asymmetric Solution

Next we construct the asymmetric equilibrium solution. Let $\phi_i = a+ib$ and $\psi_i = c+id$. Substituting into equation (5.10) we obtain:

$$0 = m\left[k\left(a+ib\right) - \sum_{j=1}^{k}(a+ib)\right] - \left[l(a+ib) - \sum_{j=1}^{l}(c+id)\right] + 2i - k - 1$$

$$0 = m[k(a+ib) - a(k+1) - \frac{1}{2}(k+1)^2 + \frac{1}{2}(k+1)b + a] - l(a+ib) + c(l+1)$$
$$+\frac{1}{2}d\left(l+1\right)^2 - \frac{1}{2}(l+1)d - c + 2i - 1 - k \tag{5.12}$$

Collecting the i th terms in equation (5.12) to solve for b we get:

$$-lb + 2 + mkb = 0$$

$$b = \frac{2}{l-mk}$$

We write the remaining value in equation (5.12) as

$$m(ka+a-a(k+1)-\frac{1}{2}b(k+1)^2+\frac{1}{2}(k+1)b)+\frac{1}{2}d(l+1)^2-la-c+c(l+1)-1-\frac{1}{2}(l+1)d-k=0$$

Now we do a similar substitution for equation (5.11) which gives:

$$m[l(c+id) - c(l+1) - \frac{1}{2}d(l+1)^2 + \frac{1}{2}(l+1)d + c] - k(c+id) + a(k+1)$$
$$+\frac{1}{2}b(k+1)^2 - \frac{1}{2}(k+1)b - a + 2i - 1 - l = 0 \tag{5.13}$$

Collecting the i th terms in equation (5.13) gives:

$$mld - kd + 2 = 0$$

$$d = \frac{2}{k-ml}$$

Then writing the remaining terms in equation (5.13) gives following result:

$$m(lc+c-c(l+1)-(1/2)d(l+1)^2+(1/2(l+1))d)+(1/2)b(k+1)^2-kc-a$$

$$+a(k+1)-l-(1/2(k+1))b-1=0$$

Setting equation (5.12) and (5.13) in Matrix form:

$$M=\begin{bmatrix} -l & m((1/2)k+1/2-(1/2)(k+1)^2) & l & (1/2)(l+1)^2-(1/2)l-1/2 \\ 0 & -l+mk & 0 & 0 \\ k & (1/2)(k+1)^2-(1/2)k-1/2 & -k & m((1/2)l+1/2-(1/2)(l+1)^2) \\ 0 & 0 & 0 & ml-k \end{bmatrix}$$

$$V=\begin{bmatrix} -1-K \\ 2 \\ -1-L \\ 2 \end{bmatrix}$$

Solving $MX=V$ for $X=a,b,c,d$ we obtain:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}=\begin{bmatrix} \frac{-l-k\,l\,c+m\,k-l^2+k+m\,l^2\,c-m^2\,k\,l\,c+m\,k^2c-m\,l+k^2}{-kl-m^2lk+mk^2+ml^2} \\ \frac{-2}{-l+mk} \\ c \\ \frac{-2}{-l+mk} \end{bmatrix}$$

where c is a free parameter. If we take $l=20$, $k=30$, $m=0.5$ and $c=1$ we obtain:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}=\begin{bmatrix} -4.150 \\ 0.4000 \\ 1 \\ 0.100 \end{bmatrix}$$

Note that we must have $b>0$ and $d>0$ since we assumed that $x_1<x_2<x_3\ldots$, which imposes the restriction $m<1$. This corresponds precisely to the stability regime of two non equal clusters when $\delta=0$ (see Theorem 4.13).

$$\Delta_1=(k-1)\,b\,\delta$$

$$\Delta_2 = (l-1)\, d\, \delta$$

Substituting $k = 30$, $l = 20$, $b = 0.4$ and $d = 0.1$ we have:

$$\Delta_1 = (30-1)(0.4)(0.01) = 0.116$$

$$\Delta_2 = (20-1)(0.1)(0.01) = 0.019$$

To verify our theory, we compared this result with the full numerical simulation of equation (5.8). The results of this simulation are shown in Figure 5.1. From the numerical simulation we get that by measuring the width of LHS that $\Delta_1 \approx 0.115$ and of the RHS that $\Delta_2 = 0.019$, which is very close to our theoretical result. This confirms that our theoretical results are accurate.

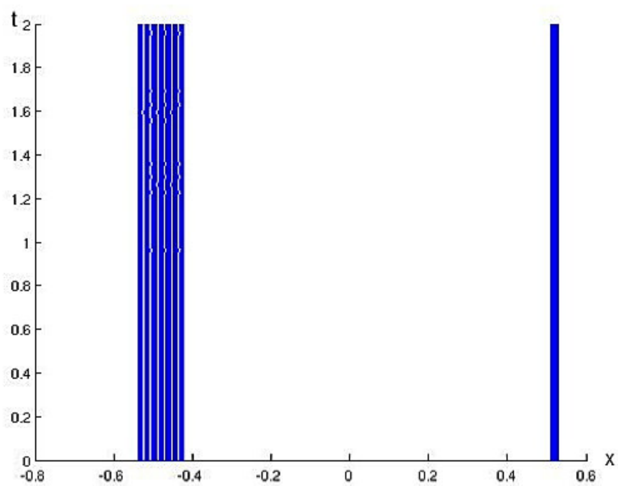

Figure 5.1: Steady states of (5.8) with $F(r) = min(mr + \delta, 1 - r)$. Writing this function using $N = n + l$ where $n = 20$ and $l = 30$ particles with $m = 0.5$ and $\delta = 0.01$ as indicated by the non symmetry condition

Chapter 6

Numerical Simulation

In this chapter we will summarize the numerical simulations we conducted to first understand the cluster and stripe equilibria and then verify numerically that our analysis was indeed yielding the correct results. We will first describe numerical simulations for the cluster steady states, followed by the numerical simulations for the stripe steady state. A sample program is included in the Appendix.

6.1 Cluster Steady State

We do numerical simulations by using the Euler method for the basic equation. The numerics are plotted by using Mat-lab program. We use $n = 50$, then test stability by taking various values of the slope m. The function $F(r) = min(m\,r, 1 - r)$ applies in the two clusters, three and "k clusters". The results we find will confirm the stability in $m = 1$ where the number of clusters are two and $m = 2$ where the number of clusters are three.
To find the distances d_1, d_2 in three clusters, which in this case are equal:

$$F(2d) = -F(d)$$

Substituting in $F(r)$ gives:

$$1 - 2d = -1 + d$$

$$d = \frac{2}{3}$$

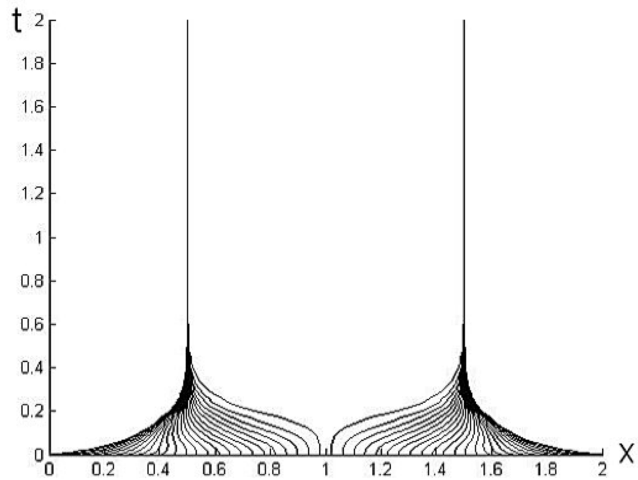

Figure 6.1: steady states of two clusters with $F(r) = min(mr, 1 - r)$, proving the unstable case where $m = 0.5$, using $N = 50$ particles

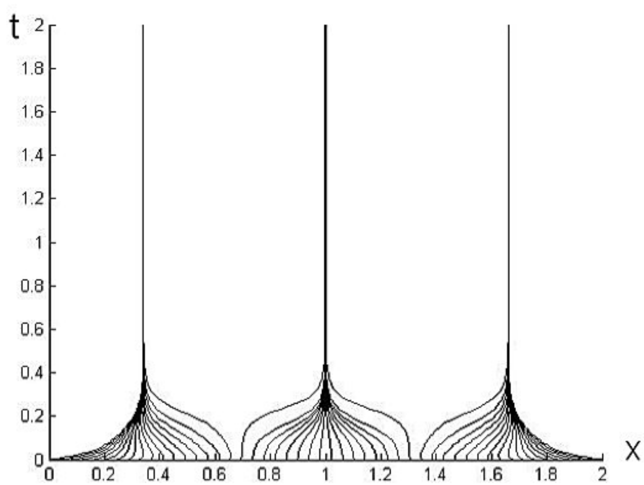

Figure 6.2: Steady states of three clusters with $F(r) = min(mr, 1 - r)$, proving the stability condition where $m = 1$, using $N = 60$ particles. We have this result from Matlab using Euler method with step size 0.5.

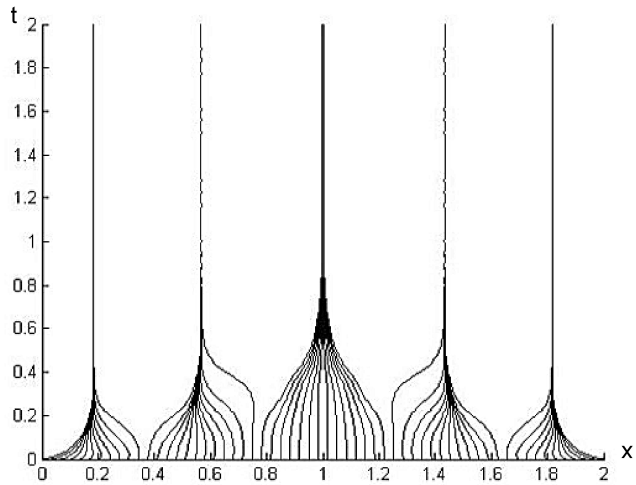

Figure 6.3: Steady states of more than three clusters with $F(r) = min(mr, 1 - r)$ proving the stability condition where $m = 1.5$, using $N = 50$ particles.

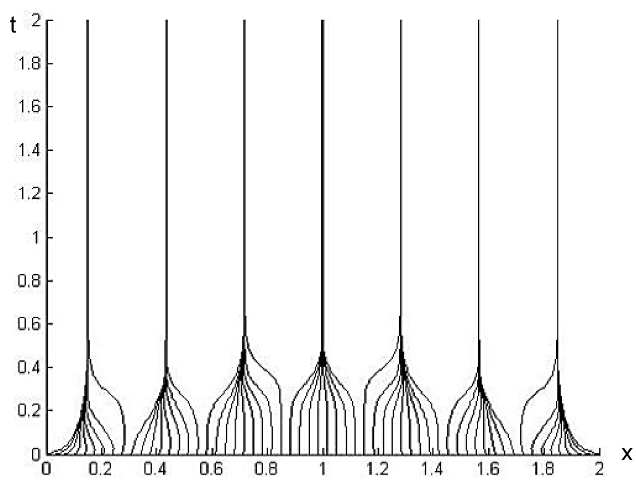

Figure 6.4: Steady states of more than three clusters with $F(r) = min(mr, 1 - r)$, proving the stability condition where $m = 4$ using $N = 50$ particles. We have this result from Matlab using Euler method with step size 0.5.

6.2 Symmetric Stripe Steady State

When we analyze this problem theoretically and numerically we get the same results for $n = 25$, $\delta = 0.1$ and $m = 0.5$ (the result are $c = 0.16$ and $a = 3.6$ for $x_j = a + cj$ where $j = 1 \dots n$). The numerical result confirm these findings. We measured the width of the segment and compared it with the theoretical results:

$$\Delta = (25 - 1)(0.16)(0.1) = 0.384 \sim 0.4$$

We have the same result numerically by measuring the width in the two graphs below.

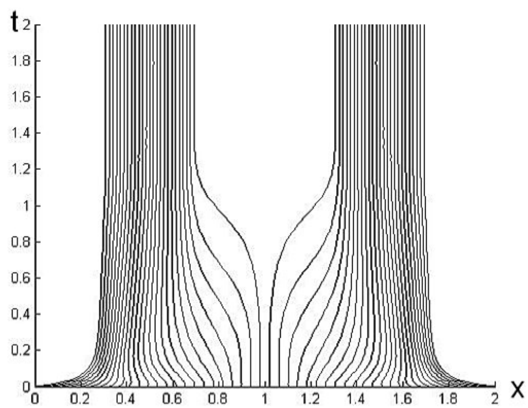

Figure 6.5: Steady states of (5.1) with $F(r) = min(mr + \lambda, 1 - r)$, where r is the distance between two the strips $r = |X_i - X_j|$. We use as an example $N = 50$ particles with $\delta = 0.1$ and $m = 0.5$ as indicated by the symmetry condition.

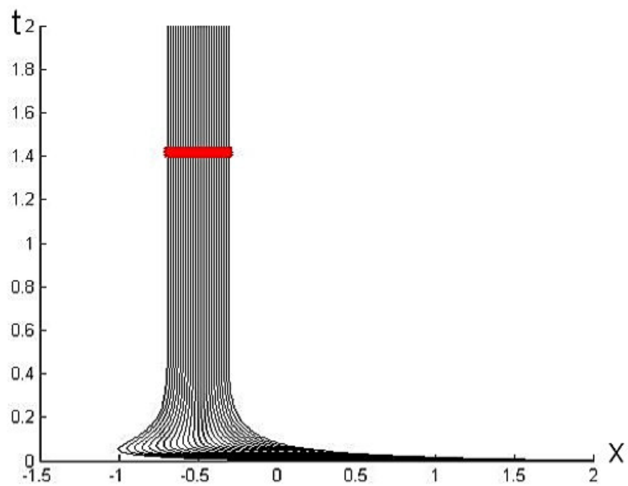

Figure 6.6: Stability for symmetry case. We use equation (5.6) which is the result for stability analysis for the main system; where $m = 0.5; n = 50, \delta = 0.1$ the red line represents the width which is 0.4

6.3 Asymmetric Stripe Steady State

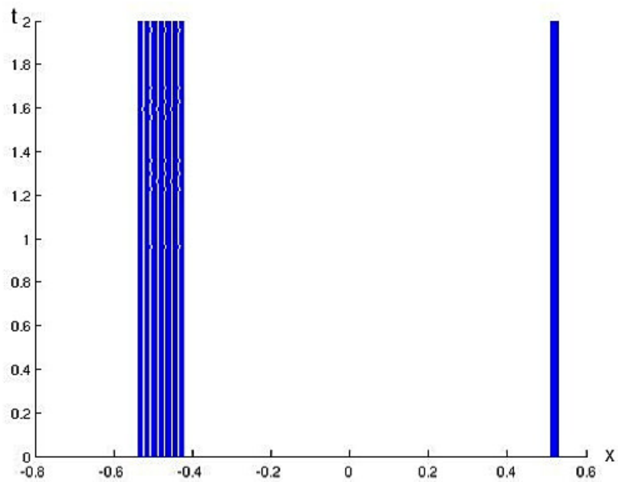

Figure 6.7: Steady states of (5.8) with $F(r) = min(mr + \delta, 1 - r)$. Writing this function using $N = n + l$ where $n = 20$ and $l = 30$ particles with $m = 0.5$ and $\delta = 0.01$ as indicated by the non symmetry condition

Chapter 7

Conclusion

The approach I have used in my work has been used previously for studying various models. Both continuous and discrete models are important in the analysis of swarm equilibria. In this thesis we have presented the stability analysis for the discrete model in one dimension using linear stability analysis. We have demonstrated that the stability depends on the value of the slope at the origin, regardless of whether the particles are evenly distributed among the clusters or not. We first worked out the 2-cluster and 3-cluster steady states, and then generalized it to the arbitrary k-cluster case. To do the analysis we first approached the problem analytically and then confirmed the stability thresholds numerically to confirm our results.

In the last two chapters we have analyzed the stripe steady state, which arises when the function is slightly positive at the origin (unlike the cluster steady state which can only occur if the function is zero at the origin). To generate and study this steady state we have perturbed functions used in the k-cluster analysis by a small positive value δ. This repulsion at the origin causes the clusters to expand and form a stripe. We analyzed this steady state by first assuming equal distribution of particles, and then without assuming any symmetry.

In the future this work can be generalized to particle models in higher dimensions. The difference will be in providing more sets of clusters and small changes in the basic models to be suitable for higher dimensional result. Throughout this thesis we used a simple family of functions $F(r) = min(mr, 1-r)$ to study the model and verify the analytic results by direct numerical simulation. In the future, various functions with more complex properties can be used to explore swarm equilibrium and their stability.

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Appendix

.1 Appendix A

```
cluster matlab code for figures (1.1), (1.2), (6.1) and (6.4)
function dx=hanadi(t,x)
m=0.5;
n=50;
dx=zeros(n,1);
y=zeros(n,n);
for k=1:n
for i=1:n
    y(i,k)=-min(m*abs(x(i)-x(k)),(1-abs(x(i)-x(k))))*sign(x(i)-x(k));
end
dx(k)=sum(y(:,k));
end
end
```

.2 Appendix B

```
Matlab code for figure(6.5)
function dx=firstgraphex2(~,x)
m=0.5;
b=0.1;
n=50;
dx=zeros(n,1);
y=zeros(n,n);
for k=1:n
for i=1:n
   y(i,k)=min((b+m*abs(x(i)-x(k)),(1-abs(x(i)-x(k))))*sign(x(i)-x(k));
end
dx(k)=sum(y(:,k));
end
end
```

.3 Appendix C

Matlab code for figure(6.6)

```
function dx=hanadin(t,x)
m=0.5;
n=25;
b=0.1;
dx=zeros(n,1);
y=zeros(n,n);
for i=1:n
    dx(i)=n*(m-1)*x(i)+(i-1)*b-n-(n-(i+1))*b-(1+m)*sum( x );
end
```

.4 Appendix D

```
matlab code for figure (5.1)
function dx=hanadi(t,x)
m=0.5;
b=0.1
n=20;
l=30;
dx=zeros(n,1);
dy=zeros(l,1);
y=zeros(n,n);
x=zeros(l,l);
for k=1:n
    h=1:l
for i=1:n
    y(i,k)=-min(b+m*abs(x(i)-x(k)),(1-abs(x(i)-x(k))))*sign(x(i)-x(k));
end
dx(k)=sum(y(:,k));
end
end
```

Appendix holes

```
code for view the figure
clear;
global m;
global n;
n=60
m=0.5;
x=zeros(n,1);
%for i=1:(n/2)
    %x(i)=0;
    %x(25+i)=1;
%end;
y=x;
y=x+0.001*(rand(n,1)-0.5*ones(n,1))
for i=1:(n/3)
    x(i)=0;
    x(n/3+i)=2/3;
    x(2*n/3+i)=4/3;
end;
y=x+0.1*(rand(n,1)-0.5*ones(n,1));
y=rand(n,1)*2;
m = 0.5;
y = linspace(0, 2, n)';
[T,Y] = ode45(@hanadi,[0 2],y);
clf;
hold on;
for i=1:n
   plot(Y(:,i), T, 'k')
end;
```