The Daycare Assignment: A Dynamic Matching Problem*†

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Abstract

We study the problem of centralized allocation of children to public daycares, illustrated by the case of Denmark. Our framework applies more broadly to problems of dynamic matching in which there is entry and exit of agents over time; for example, it can be used to study the school choice problem once student mobility is taken into account. First, we show that the Gale-Shapley deferred acceptance mechanism adapted to the dynamic problem always yields a stable matching. However, we show that there does not exist any mechanism that is both stable and strategy-proof. We also show that the well-known Top Trading Cycles mechanism is neither Pareto efficient nor strategy-proof. Finally, a mechanism in which parents sequentially choose menus of schools is both strategy-proof and Pareto efficient.

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1 Introduction

The decision of which daycare center to enroll a child is an important and difficult one, justified by mounting evidence that early childhood care facilities are crucial to the development of critical noncognitive skills.[1] The decision is further complicated by the heterogeneity of these facilities and the fact that there are important risks associated with opting out of a daycare facility in favor of home care.[2]

Many public daycare systems are centrally administered, particularly in European countries. A centralized public daycare system attempts to balance parents' reported preferences for the different daycare centers with the priorities of these daycare centers regarding the various children. These priorities are set by local governments and vary across municipalities. The assignment system currently in place in Denmark, which is our main example, is such that the oldest unassigned child is given high priority in a daycare where no current capacity restriction exists– a concept called "child care guarantee." Another important feature of the Danish system, which is common to other dynamic matching problems, is that children currently allocated to a daycare center have the highest priority in those places in the subsequent period. That is, children currently allocated to a daycare center will not be displaced from that center involuntarily.

In the current paper, we study this problem of centralized assignment of children to daycare centers. Our problem can be seen as a dynamic version of the well-known *school choice problem*, in which children of a specific cohort are assigned to different public schools.[3] Specifically, our problem extends the school choice problem in two fundamental ways. First, we consider a dynamic structure: in our model, each child may attend daycare for two periods, but not necessarily in the same facility. Moreover, in any given period, children of different ages may be allocated to the same daycare. In Denmark, for example, children attending the same daycare range in age from six months to three years. Every month, a new group of young children start daycare while those children who have turned three leave for the next level of preschool. The second defining feature of our problem is that the schools' priorities are history-dependent: a school gives the highest priority to the children allocated to it in the previous period.

In practice, the school choice itself also has dynamic features and it has been documented that there is considerable mobility of children across schools. To illustrate, consider the example of New York City primary schools, where Schwartz et al. (2009) report that students move considerably both within year and across years. In their sample, only 3.4% of 8th graders had attended the same school in the entire period from 1996-97 to 2000-01, while 22.75% of the students had

[1] For example, see Chetty et al. (2011) and Heckman (2008).

[2] See, for example, Goldin (1994).

[3] See Abdulkadiroğlu and Sönmez (2003) for an important paper in the area, and also Pathak (2011) for a recent survey.

had at least one “moving year” (a year in which the student switched schools within the year). This is consistent with a study conducted in 2010 by the U.S. General Accounting Office, where it is reported that “nearly all students change schools at some point before reaching high school.”[4] Hence, while it is true that students from different cohorts do not compete for the same spots, there is considerable entry and exit of students in each school. Many of the students who start, say 9th grade, this year move out of their school district/city and many new students move into. Theoretically, one could allocate the new students and the old students who want to change their school through a centralized mechanism before these students start 10th grade. Thus, we believe that our results could have implications on the school choice problem if mobility is taken into account.

One of the main objectives in the school choice literature has been to identify mechanisms that implement allocations that satisfy one or more well-defined positive properties, such as stability and Pareto efficiency. A stable mechanism is one that leads to an allocation in which no child would (i) prefer a different school to her current one, or rather, prefer to be left unassigned, and (ii) find a student in that preferred school with a lower priority than hers–or an empty seat in that school. Pareto efficiency, on the other hand, is a welfare criterion which considers only the well-being of the students. Abdulkadiroğlu and Sönmez (2003) discuss two important mechanisms that could be used in this allocation problem: the Gale-Shapley Deferred Acceptance (DA) mechanism, which is both stable and strategy-proof; and the Top-Trading Cycles (TTC) mechanism, which is both efficient and strategy-proof. Here, we extend the concepts of stability and Pareto efficiency to our problem and study whether these concepts are compatible with one another in a dynamic environment.

In our model, we show that a stable matching always exists. To find such matching, one can treat our problem as a sequence of separate school choice problems and use the DA mechanism in each period.[5] We also show that this matching is not Pareto dominated by any other stable matching, and that if there exists an efficient and stable matching, it must be the DA one. Importantly, though, the DA mechanism is not strategy-proof: parents might have incentives to misreport their true preferences.

The manipulability of the DA in our dynamic environment raises the question of whether there is *any* mechanism that is both stable and strategy-proof.[6] Here we prove an impossibility result: no mechanism is both stable and strategy-proof.

For most of the paper, we assume that priorities of schools are history-dependent in only a

[4]U.S. General Accounting Office (2010).

[5]Precisely, we use an adaptation of the DA mechanism to our dynamic setting, which we denote by DA-IP (see section 4).

[6]In the school choice framework much attention has been given to stability, and the DA mechanism has since been adopted in the New York and Boston public school systems. (See Abdulkadiroğlu et al. (2009) and Abdulkadiroğlu et al. (2005) for a discussion of the practical considerations in the student assignment mechanisms in these two cities).

rather weak sense: the priority ranking of each school will change only for children previously allocated to it, while for all other children, the priorities will remain the same. We call this condition *independence of previous assignment*. We also consider a restriction on preferences, which we call *rankability*, and a stronger version of it, denoted *strong rankability*. The rankability restriction implies that preferences over schools are stable and consistent over time.[7] In this way, we make our model as close as possible to the static problem. Nevertheless even with only this weak link between periods, our problem is substantially different from the static case, in which the DA mechanism is strategy-proof.

Next we turn the focus to studying Pareto efficiency and strategy-proofness. Unlike the case of stability, extending the concept of *efficiency* in the dynamic assignment problem is straightforward–at least conceptually. However, although in static settings it is impossible to find a Pareto improving matching in which a child trades her placement for a worse one, in a dynamic setting this may be possible as long as the child obtains a better placement in the other period. Hence, as long as there are two or more "willing" participants in such a trade, there is room for Pareto improvement even if none exists by changing only one-period matchings. This possibility of Pareto improving intertemporal trade is the main reason behind our result that the TTC mechanism is not efficient.We also show that TTC is not strategy-proof, and that even a variation of this mechanism, which we call "TTC by cohort," is not strategy-proof. The reason why strategy-proofness is more difficult to achieve in the dynamic environment that we consider here is that there is an additional potential benefit for a player to gain from misreporting her true preferences: the player can affect the priority rankings of schools in the subsequent period.

Finally, the serial dictatorship mechanism adapted to our environment is shown to be strategy-proof and efficient.[8] In this mechanism, children are exogenously ordered by the planner and they choose a menu of schools over time according to their position in the queue. This means that in a dynamic environment like ours, there are mechanisms that are both efficient and strategy-proof.

We should highlight the fact that although our problem is motivated by the assignment of children to daycare centers, it has many other potential applications. The school choice problem itself has dynamic features, as mentioned previously. Other interesting applications are the assignment of teachers to public schools, diplomats to different embassies, or high-level bureaucrats to dif-

[7]Our strong rankability assumption does not rule out preferences with switching costs, i.e., costs for switching schools across periods. However, if these costs are prohibitively large, any student would rather stay in whichever school she is allocated to in the first period, and would not consider moving to other schools. The problem would be very close to the static school choice problem.

[8]Our problem is not part of the literature on multi-unit allocation. Pápai (2001) and Ehlers and Klaus (2003), for example, have obtained negative results concerning strategy-proofness and efficiency; however, the problem here is substantially different and their results do not apply to our setting. Many of the results in that literature depend on the feature that the agents' preferences over bundles of objects vary in a permissive way. In contrast, in our problem the preferences of the agents are restrictive because the children's preferences are rankable.

ferent regions.[9] Another problem related to ours is the market for new physicians in the United Kingdom, where each doctor is allocated to two six-month positions, a medical post and a surgical post.[10]

The theory of market design in dynamic settings is very recent.[11,12] Kurino (2013) studies the centralized housing allocation problem with overlapping generations of agents. The school choice problem differs from the housing allocation problem in the sense that the objects have priorities in the former but not in the latter. Hence, stability – a central issue in our paper– is not considered in Kurino (2013). The second part of our paper, where we consider the compatibility of efficiency and strategy-proofness, is related to Kurino's but with the important difference that the domain of possible mechanisms in our study contains the priorities of the schools.[13]

Bloch and Cantala (2011) study a dynamic matching problem focusing on the long-run properties of different assignment rules. Pereyra (2013) studies the allocation of teachers to public schools, restricting his attention to rankable preferences and seniority-based priorities and shows that the DA is strategy-proof in his setting. In this sense, the first part of our paper, where we consider the compatibility of stability and strategy-proofness, and Pereyra (2013) complement each other. In the second part of our paper, we investigate efficiency and strategy-proofness, which is not studied by Pereyra. Dur (2011) considers a dynamic school choice problem in which the incentives for placing siblings are taken into account. He shows that there is no fair (stable) and strategy-proof mechanism.

Abdulkadiroğlu and Leortscher (2007) study a dynamic house allocation problem in which the set of agents is common in all periods. With a focus on efficiency, they propose a random mechanism that is superior in terms of efficiency to the random serial dictatorship. Finally, Ünver (2010) extends the literature on centralized matching for kidney exchanges to a dynamic environment in which the pool of agents evolves over time.

The structure of this paper is as follows. In Section 2 we present a brief description of the Danish Daycare system. In Section 3, we describe the model in detail. In Section 4, we study stable matchings and their properties. In Section 5, we prove an impossibility result relating stability and

[9]See Bloch and Cantala (2011).

[10]See Roth (1991) and Irving (1998).

[11]Abdulkadiroğlu and Sönmez (1999) and Guillen and Kesten (2012) study the house allocation problem with existing tenants. In their models, the existing tenants have the highest priority for the house (room) they occupied in the preceding period. In this aspect these papers are related to ours, but their models are static while ours is dynamic.

[12]Blum et al. (1997) study two-sided matching in labor markets in which there are vacancy openings over time. In their model, however, preferences are essentially static and their focus is on the decentralized (re)-equilibration of stable matchings.

[13]Thus, the version of TTC used in our paper is Abdulkadiroğlu and Sönmez (2003)'s TTC mechanism for the school choice problem, while Kurino (2013) focuses on Abdulkadiroğlu and Sönmez (1999)'s TTC mechanism for the housing market problem with existing tenants. Because of this, Kurino (2013) obtains that the constant TTC mechanism favoring existing tenants is both strategy-proof and efficient when the agents' preferences are rankable.

strategy-proofness. In Section 6, we discuss a strategy-proof and efficient mechanism. In Section 7, we provide a brief conclusion. Longer proofs are collected in the Appendix.

2 The Danish Daycare System

In Denmark, children are allocated at the different daycare centers by the local municipalities. Below, we highlight the essential features of the allocation rules at Aarhus, which are also common to most municipalities in Denmark, including Copenhagen.

Children can start a daycare at the age of 6 months and when she turns 3 years she must exit, moving to the next level of pre-schooling. The assignment takes place once a month and each parent reports their top 3 choices among all daycare centers. They also report whether they want the option for what is called as a "guaranteed spot," in case the child is currently unassigned.[14] The parents can enroll their child any time after birth. Even if a child has a spot in some daycare she can participate in the assignment algorithm without having to give up her spot, i.e., she may sign up for two different daycare centers and will be placed in a waiting list for these two facilities. It is important to highlight that children currently allocated to a daycare, will not be displaced from that daycare involuntarily.

For specificity, below we present the "placement assignment rules," as stated by the Aarhus municipality.[15] Children are assigned according to the following order.

1. Children with special needs, e.g., children with disabilities.
2. Children with siblings in the same daycare.
3. Immigrant children who after expert evaluations are considered in need of special assistance in daycare.
4. The oldest child who is listed for a *guaranteed place* in his or her own district i.e., not at a particular daycare.

[14]"You can choose a guaranteed place and also a desired place with one or more specific institutions. These requests will be taken into account when we find a place for you. However, we cannot guarantee your desired institution. If your desired institutions does not have an opening, you will be offered a "guaranteed place. A guaranteed place is a place within the district you live in, or at a distance from your home which involves no more than half an hour of extra transport each way to and from work. The municipal placement guarantee is satisfied when you have been offered a place. To be assigned a guaranteed seat at a desired time, the application must be received by the placement guarantee office no later than 3 months before the place is desired." (Translated from https://www.borger.dk)

[15]For the original document see: https://www.borger.dk/

5. The oldest child who is listed for a guaranteed place in the local warranty district. Aarhus Municipality is divided into 8 major warranty districts. A warranty district consists of one to several districts.

6. The oldest child listed for a guaranteed place from a different warranty district.

7. The oldest child from the waiting list of a particular daycare. This offer is also made to a child already in a daycare.

In Section 3.4, after we have adapted the concepts of efficiency and stability to our setting, we show that this assignment mechanism is manipulable, fails efficiency and stability.

3 Model

In Section 3.1 we build our model. Specifically, we define matching for our setting, and we discuss the preference relation of the children over the different profiles of daycare centers and the priority orderings of the daycare centers over the set of children. In Section 3.2 we define the concepts of Pareto efficiency and stability. In Section 3.3 we define a mechanism and its properties, and, in particular, we define strategy-proofness.

3.1 Setup

Time is discrete and $t = 1, \cdots, \infty$. There are a finite number of infinitely lived schools. Let $S = \{s_1, \cdots, s_m\}$ be the set of schools. Each school $s \in S$ has a maximal capacity r_s which we assume is constant.[16] Children can attend school when they are 1 and 2 years old. School attendance is not mandatory. Let h stand for the option of staying home. For technical convenience, we treat h as a school with unbounded capacity. Let $\bar{S} = S \cup \{h\}$ and $r = (r_s)_{s \in \bar{S}}$. In each period $t \geq 1$, a new set of 1-year old children I_t (which is possibly empty) arrives. We use the notation I_0 to denote the set of two year old children in period 1. Consequently, at any period $t \geq 1$ the set of school-age children is $I_{t-1} \cup I_t$. As time passes the set of school-age children evolves in the "overlapping generations" (OLG) fashion. The set of all children is $I = \cup_{t=0,\cdots,\infty} I_t$.

Matching

A period t matching is a correspondence indicating which school-age child in period t attends which school, and a matching is a collection of all period t matchings. First, we define the period 0 matching, μ^0, as a correspondence $\mu^0 : I^0 \cup \bar{S} \to I^0 \cup \bar{S}$ satisfying the following properties: (i) For

[16] One can relax this assumption to allow the possibility that the schools' capacities increase over time.

all $i \in I_0$, $\mu^0(i) \subset \bar{S}$ and $|\mu^0(i)| = 1$; (ii) For all $s \in \bar{S}$, $|\mu^0(s)| \leq r^s$ and $\mu^0(s) \subset I_0$; and (iii) For all $i \in I_0$, $i \in \mu^0(s)$ iff $\{s\} = \mu^0(i)$.

Definition 1 (Matching). *A period t matching μ^t (where $t \geq 1$) is a correspondence*

$$\mu^t : I_{t-1} \cup I_t \cup \bar{S} \to I_{t-1} \cup I_t \cup \bar{S}$$

such that

1. *For all $i \in I_{t-1} \cup I_t$, $\mu^t(i) \subset \bar{S}$ and $|\mu^t(i)| = 1$*
2. *For all $s \in \bar{S}$, $|\mu^t(s)| \leq r^s$ and $\mu^t(s) \subset I_{t-1} \cup I_t$*
3. *For all $i \in I_{t-1} \cup I_t$ and all $s \in \bar{S}$, $i \in \mu^t(s)$ iff $\{s\} = \mu^t(i)$.*

A matching μ is a collection of period matchings: $\mu = (\mu^0, \mu^1, \cdots, \mu^t, \cdots)$. We use the notation $\mu(i)$ to denote $(\mu^t(i), \mu^{t+1}(i))$ where t is the period in which i is one year old.

Requirement (1) ensures that each child is placed at most at one school, while requirement (2) ensures that each school does not house more children than its capacity. Due to requirement (3), a child is matched to a school if and only if the school is matched to the child.

Children's Preferences

Each child is characterized by a strict preference relation $\succ_i$ over $\bar{S}^2$. The notation (s, s') denotes the allocation in which a child is placed at school s at age 1 and at school s' at age 2. We write $(s, s') \succeq_i (\bar{s}, \bar{s}')$ if either $(s, s') \succ_i (\bar{s}, \bar{s}')$ or $(s, s') = (\bar{s}, \bar{s}')$. Throughout the paper, we maintain the following assumption on preferences:

Assumption 1 (Rankability). *Each child i's preferences satisfy the following assumption which we call rankability: if $(s, s) \succ_i (s', s')$ for some $s, s' \in \bar{S}$, then $(s, s'') \succ_i (s', s'')$ and $(s'', s) \succ_i (s'', s')$ for any $s'' \neq s'$.*

A direct consequence of the rankability assumption is the following: whenever $(s, s) \succ_i (s', s')$ for some $s \neq s' \in \bar{S}$, then it must be that $(s, s) \succ_i (s, s')$ and $(s, s) \succ_i (s', s)$. In addition, if $(s', s') \succ_i (s'', s'')$, then $(s, s) \succ_i (s', s'')$. However, it is possible that $(s', s') \succ_i (s, s')$ (and $(s', s') \succ_i (s', s)$). Here also note that for each child there must exist some school such that attending this school for two consecutive periods is the most preferred option for the child.

The reasonings behind the rankability assumption is that i) each parent has rankings of the schools (not the pairs of schools) that is stable over time and that derive her preferences over the pairs of schools, and ii) there is a constant switching cost of schools that parents care about. Based on these reasonings, we think that if a parent ranks school s ahead of s', then she should prefer (s,s'') to (s',s'') for all $s'' \neq s'$. However, a parent could prefer (s',s') to (s,s) in order to save switching costs. These properties are captured in Assumption 1.

Below we present a stronger version of the rankability assumption.

Definition 2 (Strong Rankability). *Child i's preferences satisfy* strong rankability *if, for any $s,s' \in \bar{S}$*

$$(s,s) \succ_i (s',s') \Longleftrightarrow (s,s'') \succ_i (s',s'') \text{ and } (s'',s) \succ_i (s'',s') \text{ for all } s'' \in \bar{S}.$$

Under strongly rankable preferences a child always prefers attending two (weakly) superior schools to attending an inferior school for two periods. Here we note that for all of our positive results we always assume that the preferences are rankable. On the other hand, for our negative results we assume that the preferences are strongly rankable because doing so strengthens these negative results.[17]

Schools' Priorities

At any period $t \geq 1$, each school ranks all the school-age children by priority. Priorities do not represent school preferences but rather, they are imposed by local municipality. For example, children with special needs might be given higher priority by the schools tailored to meet those needs, moreover, in the existing assignment mechanism in Denmark, all schools give priority to their currently enrolled children.

Henceforth, we assume that each institution gives the highest priority to its currently enrolled children, which is a feature of the assignment mechanism currently in place in Denmark. A rationale behind this priority is that no school forces its current enrollee out in order to free a spot for some other child. Because of this assumption, the priority ranking of each school is history dependent, i.e., a school's priority ranking depends on its attendees of the previous period.

One can argue that even in the school choice problem, the schools' priorities are history dependent because a typical school (for example, in Boston) gives priority to children whose siblings are in it. In other words, the matchings of the previous periods affect how the schools rank the new applicants. However, in the school choice literature, this history dependence of the schools' priorities is not modelled explicitly.[18] This omission is justified if the older siblings make decision

[17]Kurino (2013) considers two types of preferences: time-separable and time-invariant. We note here that our assumption of rankable preferences is neither weaker nor stronger than his assumption of time-separable preferences. His time-invariability assumption is equivalent to our strong rankability assumption. Also, strong rankability is closely related to the responsiveness assumption used in many-to-one matching settings.

[18]With the exception of one recent working paper (Dur, 2011) that consider the sibling priorities explicitly (and thus, history-dependence) in the school choice problem.

without caring about the younger ones, i.e., one sibling's well-being is not dependent on another's. However, in our model, the children participate in the assignment mechanism twice and of course, any child's well being depends on the schools she attends in different periods. Therefore, in our model, we have to take the history dependence of the schools' priorities seriously.

We will denote the binary relation which generates the priority ranking of school s at period $t \geq 1$ by $\rhd_s^t(\mu^{t-1})$. That is, if at period t child i has a higher priority than child j at school s given the period $t-1$ matching μ^{t-1}, then we denote $i \rhd_s^t \left(\mu^{t-1}\right) j$. We will assume throughout the paper that priorities are strict. In practice, whenever the school system outlines coarse priorities, as in the Danish system that we described in section 2, the system often designs a tie-breaking rule, so that, effectively priorities are indeed strict. In the Danish case, the tie-breaking rule is based on a first-come first-served basis.

We write $i \trianglerighteq_s^t \left(\mu^{t-1}\right) j$ if either $i \rhd_s^t \left(\mu^{t-1}\right) j$ or $i = j$.

We assume that each school ranks the children in a lexicographical manner in which children's past attendance matters the most and then some criterion based on exogenous characteristics of the child (e.g., proximity to school, medical condition, immigration status and age). Let us now state formally the assumptions we impose on the priorities.

Assumption 2 (Priorities)**.** *For all $i \in I$ and all $t = 1, 2, \ldots$, each school's priorities satisfy:*

1. *(Priority for currently enrolled children) If $i \in I_{t-1}$ and $i \in \mu^{t-1}(s)$ for some $s \in S$, then $i \rhd_s^t (\mu^{t-1}) j$ for all $j \notin \mu^{t-1}(s)$.*

2. *(Weak consistency of different period rankings) If $i \rhd_s^{t-1} (\mu^{t-2}) j$ for some $i, j \in I_{t-1}$, $s \in S$ and μ, then $i \rhd_s^t \left(\mu^{t-1}\right) j$ in any of the following cases:*

    - $\mu^{t-1}(i) = \mu^{t-1}(j) = s$
    - $\mu^{t-1}(i) = s, h$ *and* $\mu^{t-1}(j) = h$
    - $\mu^{t-1}(j) \neq s, h$

3. *(Weak irrelevance of previous assignment) If $i \rhd_s^t (\mu^{t-1}) j$ for some $i, j \in I_{t-1}$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s, h$ and $\mu^{t-1}(j) \neq s, h$, then $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ for any $\bar{\mu}$ satisfying one of the following conditions.*

    - $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t-1}(j) = s$
    - $\bar{\mu}^{t-1}(i) = s, h$ *and* $\bar{\mu}^{t-1}(j) = h$
    - $\bar{\mu}^{t-1}(j) \neq s, h$

4. *(Weak irrelevance of difference in age) If* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $i \in I_{t-1}$, $j \in I_t$, $s \in S$, *and* μ *with* $\mu^{t-1}(i) \neq s,h$, *then* $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ *for all* $\bar{\mu}$. *In addition, if* $j \rhd_s^t (\mu^{t-1}) i$ *for some* $i \in I_{t-1}$, $j \in I_t$, $s \in S$, *and* μ *with* $\mu^{t-1}(i) \neq s,h$, *then* $j \rhd_s^t \left(\bar{\mu}^{t-1}\right) i$ *for all* $\bar{\mu}$ *with* $\bar{\mu}^{t-1}(i) \neq s,h$.

Loosely speaking, the last three assumptions mean that the priorities of any school do not depend on the attendees of other schools (excluding staying home). Specifically, the second one says that if child i has higher priority than child j at school s in period $t-1$, then child i keeps her advantage over child j in the following period unless child j attends school s (h) while child i does not attend s (s or h). The third one says that at any period, school s's relative ranking of any two children is not affected by the fact that one child has attended school $s' \neq s$ and the other $s'' \neq s$. The fourth assumption says that at any period school s's relative ranking of any two children is not affected by the fact that one child has attended school $s' \neq s$ at period $t-1$ while the other is one year old at period t.

Assumption 2 resembles the priorities in the Danish daycare system. For instance, in the Danish system a child's priority at some school can be improved from one period to the next one if (i) she attends the school in the first period or (ii) she stays home in the first period and asks for guaranteed spot in the next period.

Here we remark that Assumption 2 does not rule out the possibility that a school s gives priorities to the children who have not attended any school over the ones who have attended some school other than s in the previous period. This possibility is ruled out if the schools' priorities satisfy the *Independence of Past Attendance* property which we define below.

Definition 3 (*Independence of Past Attendance*)**.** *School s's priorities satisfy the* Independence of Past Attendance *(IPA) property if the conditions below are satisfied:*

1a. *(Consistency of different period rankings) If* $i \rhd_s^{t-1} (\mu^{t-2}) j$ *for some* $i,j \in I_{t-1}$, $s \in S$ *and* μ, *then* $i \rhd_s^t \left(\mu^{t-1}\right) j$ *in any of the following cases:*

- $\mu^{t-1}(i) = \mu^{t-1}(j) = s$
- $\mu^{t-1}(j) \neq s$

2a. *(Irrelevance of previous assignment) If* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $i,j \in I_{t-1}$, $s \in S$, *and* μ *with* $\mu^{t-1}(i) \neq s$ *and* $\mu^{t-1}(j) \neq s$, *then* $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ *for any* $\bar{\mu}$ *satisfying one of the following conditions.*

- $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t-1}(j) = s$
- $\bar{\mu}^{t-1}(j) \neq s$

3a. *(Irrelevance of difference in age) If $i \rhd_s^t (\mu^{t-1}) j$ for some $i \in I_{t-1}$, $j \in I_t$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s$, then $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ for all $\bar{\mu}$. In addition, if $j \rhd_s^t (\mu^{t-1}) i$ for some $i \in I_{t-1}$, $j \in I_t$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s$, then $j \rhd_s^t \left(\bar{\mu}^{t-1}\right) i$ for all $\bar{\mu}$ with $\bar{\mu}^{t-1}(i) \neq s$.*

In practice, *IPA* is often not satisfied: many schools give priority to two year old children who have not attended any school in the previous period over one year old children and the two year old children who have attended school in the previous period. In particular, given a concept called "guaranteed spots," *IPA* is not satisfied in the current Danish daycare assignment mechanism.

3.2 Properties of a Matching: Efficiency and Stability

We first define the concept of a market, which will be used in our other definitions.

Definition 4 (Market)**.** *A market M is $M = (I, \bar{S}, r, \mu^0, \succ, \rhd)$ where μ^0 is a period 0 matching, $\succ = (\succ_i)_{i \in I}$ is a preference profile of the children, and $\rhd = (\rhd_s)_{s \in S}$ is a priority function of the schools. The set of markets is $\mathcal{M}$.*

Here, observe that the period 0 matching is included in the definition of market explicitly. Since our model starts at period 1, the period 0 matching cannot be changed. Thus, all the matchings in a given market $M = (I, \bar{S}, r, \mu^0, \succ, \rhd)$ must have the common period 0 matching, μ^0.

In this section we define the properties of a matching for a fixed market M. The matching literature has identified Pareto efficiency and stability as the two main desirable properties. The main goal of this subsection is to adapt these concepts to our dynamic assignment problem.

For both Pareto efficiency and stability, we start defining a weaker concepts because they will be useful later in our analysis. First, let us define autarkic-efficiency which requires to eliminate all one period "trades" that improve at least one child without hurting others.

Definition 5 (Autarkic Efficiency)**.** *Matching μ is autarkic efficient if for any $t \geq 1$, there does not exist period t matching $\bar{\mu}^t$ such that $(\mu^0, \cdots, \mu^{t-1}, \bar{\mu}^t, \mu^{t+1}, \cdots)$ Pareto dominates μ.*

For autarkic efficiency, one considers the possibilities to improve everyone by altering one period matchings. However, even when this possibility does not exist, one maybe able to (weakly) improve every agent by changing matchings of several periods. Below we present an example in which two children from the same cohort (or generation) improve over an autarkic efficient matching by trading their allocations.

Example 1 (Pareto Improving Trade Within Cohort)**.** *Suppose that $I = \{i_1, i_2, j_1, j_2\}$ and in period 1, i_1 and i_2 are two years old and j_1 and j_2 are one year old. There are 4 schools s_1, s_2, s_3 and s_4 and each school has a capacity of 1 child. The schools' priorities satisfy* IPA *and the* *children's preferences satisfy* strong rankability. *The schools' priorities are given as follows under the assumption that the children have not attended any school in the previous period:*

$$i_1 \rhd_{s_1} i_2 \rhd_{s_1} j_1 \rhd_{s_1} j_2$$
$$i_2 \rhd_{s_2} i_1 \rhd_{s_2} j_2 \rhd_{s_2} j_1$$
$$i_1 \rhd_{s_3} i_2 \rhd_{s_3} j_1 \rhd_{s_3} j_2$$
$$i_1 \rhd_{s_4} i_2 \rhd_{s_4} j_2 \rhd_{s_4} j_1$$

Child i_1's top choice is s_1 while child i_2's is s_2. The other two children's preferences satisfy the following conditions:

$$(s_2,s_2) \succ_{j_1} (s_1,s_1) \succ_{j_1} (s_4,s_2) \succ_{j_1} (s_3,s_1) \succ_{j_1} (s_3,s_3) \succ_{j_1} (s_4,s_4)$$
$$(s_2,s_2) \succ_{j_2} (s_1,s_1) \succ_{j_2} (s_3,s_1) \succ_{j_2} (s_4,s_2) \succ_{j_2} (s_3,s_3) \succ_{j_2} (s_4,s_4)$$

Now consider the following matching μ: $\mu^1(i_1) = s_1$, $\mu^1(i_2) = s_2$, $\mu^1(j_1) = s_3$, $\mu^1(j_2) = s_4$, $\mu^2(j_1) = s_1$, $\mu^2(j_2) = s_2$. Matching μ satisfies autarkic efficiency. However, observe that children j_1 and j_2 strictly improves over μ if they trade their matchings.

Loosely speaking, in Example 1, children j_1 and j_2 are assigned "extreme" allocations under matching μ. Hence, these children j_1 and j_2 improve over the extreme allocations by "trading" their allocations.

In the example above the children from the same cohort strictly improve over an autarkic-efficient matching by trading their matchings. The example illustrates the need to strengthen the Autarkic efficiency concept. We say a matching μ is *Pareto efficient* if no other matching strictly improves at least one child without hurting the others.

Definition 6 (Pareto Efficiency)**.** *A matching $\bar{\mu}$ Pareto dominates μ if*

$$\bar{\mu}(i) \succeq_i \mu(i) \; \forall i \in I \text{ and } \bar{\mu}(j) \succ_j \mu(j) \text{ for some } j \in I.$$

A matching μ is Pareto efficient if no matching $\bar{\mu}$ Pareto dominates μ.

Note here that any Pareto efficient matching is also autarkic efficient.

Now let us consider stability. Adapting the definition of stable matching in our setting is not straightforward as the dynamic nature of our setting presents some challenges, which are absent in the school choice problem. We propose a stability concept based on the idea of *justified envy* freeness.[19] As in the case of efficiency, we first define the concept of *autarkic stability*, which we perceive as a naive version of our main stability concept. A matching is said to satisfy autarkic stability if no child can *justify* her envy of another child at some period t, without considering the effects that her alternative placement would have on the priorities of the schools. That is, if child i improves by moving to school s from her currently matched school only at t while keeping her past/future allocation fixed, then s must not have assigned a seat to any child who has lower priority than i. In a way, for autarkic stability, we are analyzing the problem at fixed period t, assuming that the matching of every other period $t' \neq t$ is fixed.

Definition 7 (Autarkic Stability)**.** *A matching μ satisfies* autarkic stability *if at any period $t \geq 1$, there does not exist a school-child pair (s,i) such that (1) and (2) below hold at the same time*

1. *(a)* $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$, *or*

   (b) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$,

2. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1})\, j$ *for some* $j \in \mu^t(s)$.

Condition (1) above refers to the fact that child i would be strictly better off by switching to some school s rather than the school specified by the matching μ. On top of that, condition (2) implies that either there are unfilled spots at the preferred school s of child i, or the school is in full capacity but some child j placed at this school under the matching μ has lower priority than child i.

In the notion of autarkic stability, each child ignores that switching her school at age 1 could lead to a different matching at the period when she is two. In this sense, justified envy is towards the status quo matching (not against potential matchings that form as a result of some child's school switch). In a marriage market with externality studied in Sasaki and Toda (1996) any pair who is contemplating to form a new match together considers the other agents' potential responses to the pair's action. These potential responses could include the status quo, i.e., the ones in which the agents who were not matched to the blocking pair in the original matching remains matched to the same agent. In this sense, in the autarkic stability concept we consider a fixed potential response, which is the status quo.

In the definition of autarkic stability, one considers only the one period potential deviations, therefore there are two shortcomings in this stability notion: (1) because the children can attend school for two periods, a child could imagine situations in which she changes her match in both periods and (2) the schools' priorities, which have to be considered for stability, evolve depending

[19] In static settings in which one side of the market has priorities but not preferences, stable matchings have been interpreted as matchings that are free of justified envy (see Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003)).

on the past matchings. These shortcomings are magnified if strong rankability or IPA are not satisfied. To illustrate this let us consider the following two examples of matchings that might satisfy autarkic stability, but that nevertheless present a case for justified envy.

Example 2 (Justified Envy under Failure of Strong Rankability)**.** *Consider a matching that places child i at school s' when she is both 1 and 2 years old. However, there is another school s such that child i improves only if she switches to school s in both periods. Observe that child i's preferences are not* strongly rankable. *Moreover, suppose that when child i is 1 year old, at school s she has priority over another child i' who is placed at school s at that time. In addition, suppose that when child i is two years old, there is no child at school s with lower priority than i. With this information, we cannot rule out the possibility that the matching satisfies* autarkic stability *because child i prefers (s',s') to (s,s').*

However, one can argue that child i's envy of i' is justified: she has the right to attend school s ahead of i' at age 1. Then, in the following period, she will be in the highest priority group at school s. This would give her the right to attend school s when she is 2. ◇

Example 3 (Justified Envy under Failure of IPA)**.** *Suppose that there are 2 schools, s and s', with respective capacities of 1 and 2 children. Children i and i' are born at the same period and their preferences satisfy the following property: $(s,s) \succ (s',s) \succ (h,s) \succ (s',s')$. Suppose that school s gives higher priority to child i than i' at period t when the children are 1 year old. However, i' is given higher priority over child i by school s at period $t+1$ if at period t, i' does not attend any school while i attends s'. Observe that school s's priorities do not satisfy* IPA.

Consider a matching which places both children at school s' in period t but places child i at school s and child i' at school s' in period $t+1$. Implicitly, the period t spot of school s is assigned to some other child who has higher priority at school s over both children. With this information only, we cannot prove that the matching does not satisfy autarkic stability.

However, one can argue that child i' envies i in a justified manner: if she stays home at period t and attends school s at period $t+1$, then she would definitely improve. In addition, she would have had priority over i at school s in period $t+1$. ◇

To account for the issues raised in Examples 2 and 3, we strengthen the concept of autarkic stability. Mainly, for our stability concept we will consider children who take into consideration that priorities are history-dependent, so that justified envy is not simply based on the current period's matching. Before formally defining the concept, we need to define the following notation.

For any $i,j \in I_t$, $s \in \bar{S}$ and μ such that $\mu(i) \neq \mu(j)$ and $\mu(j) \in S$, let

$$\bar{M}^t(i,j,\mu) \equiv \left\{ \bar{\mu}^t : \bar{\mu}^t(i) = \mu^t(j), \bar{\mu}^t(j) \neq \mu^t(j) \,\&\, \bar{\mu}^t(i') = \mu^t(i') \,\forall i' \neq i,j \in I_{t-1} \cup I_t \right\}.$$

That is, the set $\bar{M}^t(i,j,\mu)$ is a set of matchings at period t such that j is replaced by i in the allocation specified by the matching μ^t, j is placed at a different school and all other children's placements remain unchanged. One may think of this as the set of all hypothetical matchings at time t such that i replaces j who then finds a school somewhere else — perhaps home, or some other school — and all other children remain in the same school. Under this view, an allocation of a particular period is considered "unfair" (or subject to justified envy) if the child takes the matching of all other children at all other periods as given. In particular, when the child "feels" that she has justified envy over some child in a particular school, for the following period, she imagines that this child over whom she had priority will either stay at home, or be placed in some other school that will not affect the next period's matching and all other children remain matched as originally.

Definition 8 (Stability)**.** *Matching μ is* stable *if it satisfies autarkic stability and at any period $t \geq 1$, there does not exist a triplet (s,s',i) such that*

$$(s,s') \succ_i (\mu^t(i),\mu^{t+1}(i)),$$

for $s \neq \mu^t(i)$, $s' \neq \mu^{t+1}(i)$ and one of the following conditions holds:

1. *$|\mu^t(s)| < r_s$ and $\left|\mu^{t+1}(s')\right| < r_{s'}$,*
2. *$|\mu^t(s)| < r_s$, $\left|\mu^{t+1}(s')\right| = r_{s'}$, and, for some $j' \in \mu^{t+1}(s')$, $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ where $\bar{\mu}^t$ is the period t matching with $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(i') = \mu^t(i')$ for all $i' \neq i \in I_{t-1} \cup I_t$,*
3. *$|\mu^t(s)| = r_s$, $\left|\mu^{t+1}(s')\right| < r_{s'}$, and, for some $j \in \mu^t(s)$, $i \rhd_s^t (\mu^{t-1}) j$,*
4. *$|\mu^t(s)| = r_s$, $\left|\mu^{t+1}(s')\right| = r_{s'}$, for some $j \in \mu^t(s)$, $j' \in \mu^{t+1}(s')$ and for any $\bar{\mu}^t \in \bar{M}(i,j,\mu)$, $i \rhd_s^t (\mu^{t-1}) j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$.*[20]

We interpret justified envy in the dynamic context as the existence of a pair of schools for which a child prefers to its current match and such that in some "reasonable" way it would be "fair" for her to go to the preferred schools. Specifically, a reasonable way may mean one of the following four cases: (1) both of these schools have unassigned spots; (2) in the first period a preferred school has an unassigned spot and in the second, the child has a higher priority over another child allocated at a preferred school; (3) a preferred school in the second period is operating with less than full capacity and in the first period the child is placed on a higher priority in that preferred school than some other child already allocated there, and finally (4) in the first year the child has a higher priority than some other child in a particular school and in the second year, the child has a higher priority than some other child even if there had been a reallocation in the first period,

[20] Observe that $\mu^t(j) = s \neq h$ as h has an unlimited capacity. Hence, $M^t(i,j,\mu)$ is well defined.

in which she replaced some child in year 1, as long as in this new allocation all other children remained in the same school.

In the definition of stability each child ignores the fact that switching her school at age 1 could change the matching at the period when she is two. In fact, the history dependence of the schools' priorities is considered for justified envy carefully but it is always towards the status quo matching.

In static settings it is well known that the stability notion based on the idea of blocking is equivalent to the one based on the idea of elimination of justified envy. However, this is not true in our setting because the schools' priorities are history-dependent in our setting. If one considered a notion of stability based on the idea of blocking groups, it would have been necessary to define "preferences" for schools. Then, due to the history-dependence of the schools' priorities, a school could "prefer" a matching in which it matches to the same child for two periods to another matching in which the child is replaced with another in one period only. In such cases, the stability notions based on the idea of blocking may not be equivalent to ours, which we illustrate through the example below.

Consider a market M and a matching in this market in which school s with a capacity of one child is matched to child i in periods 1 and 2. Suppose that there is a two-year-old child j in period 1 who improves if she attends s at period 1. In addition, let school s give a higher priority to j than to i at period 1. According to our definition of stability, child j has a justified envy; thus, the matching above is not stable. However, this matching could be stable according to the notions of stability based upon the idea of blocking pairs. For instance, s's preferences can be such that it prefers the original matching to the matching in which it matches with j in period 1 and with i in period 2, due to the history-dependence of its priorities. If this is the case, school s will not be a part of a coalition that blocks the original matching.[21]

Stability is a refinement of autarkic stability and we believe that it is a natural concept that captures the meaning of justified envy in our setting. We must remark that the definition of stability is stronger than what Examples 2 and 3 call for. In other words, one can slightly weaken Definition 8 so that a matching is stable if it satisfies autarkic stability and is free of justified envy, as discussed in Examples 2 and 3. However, this does not change any of the results in the next section. Given this, weakening the definition of stability is not beneficial from a technical perspective.

Examples 2 and 3 show that our stability concept is not equivalent to the autarkic stability notion if either strong rankability or IPA are not satisfied. But what if both of them are satisfied? In this case, it turns out that the two concepts of stability are equivalent. Since this is a lengthy result, we refer the interested readers to Appendix A.

[21] Thus, our stability notion is not equivalent to Kurino (2009)'s notion of dynamic pairwise-stability. There are also other notions of stability in two sided dynamic marriage models, such as Kurino (2009)'s dynamic pairwise-stability or Damiano and Lam (2005)'s self-sustaining stability. In these notions agents are farsighted, which is not the case for our notions of stability.

3.3 Mechanism and Its Properties

A mechanism φ is a systematic process that assigns a matching for each market. Here we consider direct mechanisms i.e., each child reports her full preferences and based on these reports the mechanism returns a matching. In each market, no mechanism modifies the period 0 matching corresponding to this market. In other words, a mechanism returns period matchings in periods $t \geq 1$ for each market. Let $\varphi_i(M)$ be the pair of schools to which child i is matched under mechanism φ. We will focus mainly on the strategy-proof mechanisms: i.e., the ones in which reporting a true preference is a weakly dominant strategy for each agent in its associated preference revelation game.

Definition 9 (Strategy-Proofness). *A mechanism* φ *is* strategy-proof *if the following condition is satisfied for all* $M = \left(I, \bar{S}, \mu^0, \succ, r, \rhd\right) \in \mathcal{M}$*,* $i \in I$*, and* $\succ_i'$

$$\varphi_i\left(I, \bar{S}, \mu^0, \succ, r, \rhd\right) \succeq_i \varphi_i\left(I, \bar{S}, \mu^0, \succ_i', \succ_{-i}, r, \rhd\right)$$

where $\succ_{-i}$ *is the preferences of the players except* i*.*

The mechanisms we study in this paper collect the preference reports of one year old children in each period $t \geq 1$ and they produce a period matching for period t based on the reports accumulated in periods $t' \leq t$. For such a mechanism, a child's matching in the period when she is two depends on the reports of the children who are born in that period. This implies that the child must worry about the actions of all the children born in the future periods. In this sense, it is very difficult for children to choose their optimal strategies. For this reason, the class of strategy-proof mechanisms is very important in our setting for practical reasons.

One may worry that reporting preference profiles over pairs of schools is a big burden on the children. However, we will later define the notion of isolated preferences which ranks the schools (not the pairs of schools) depending on the past matchings. Then all the mechanisms we consider in this paper can be adjusted so that each child reports her isolated preferences in each period.

Definition 10 (Stability and Efficiency). *A mechanism* φ *is* efficient (stable)*, if it yields an efficient (stable) matching in each market* $M \in \mathcal{M}$*.*

3.4 Danish Mechanism Revisited

In this subsection, we revisit the Danish mechanism. For specificity, we focus on the Aarhus mechanism presented in Section 2 and show that the mechanism does not satisfy any of the desirable properties discussed in the previous subsection.

Example 4 (Aarhus Mechanism)**.** *Suppose there are 2 schools,* $\{s_1, s_2\}$ *and each school has a capacity of one child. In each period, 1 child is born. Their preferences satisfy the following property:* $(s_1,s_1) \succ (s_2,s_1) \succ (h,s_1) \succ (s_2,s_2)$*. Denote the child born in period t by* i_t*. If all children report truthfully, their allocation in the Aarhus mechanism will be* $\mu(i_1) = (s_1,s_1)$; $\mu(i_2) = (s_2,s_2)$; $\mu(i_3) = (s_1,s_1)$;...$\mu(i_k) = (s_1,s_1)$;*and* $\mu(i_{k+1}) = (s_2,s_2)$*, for k odd.*[22]

Consider the following strategy: each child participates in the Aarhus mechanism when she is 2. Each child also participates in the Aarhus mechanism when she is one if and only if the child from the previous generation attended school s_2 *in the previous period. Whenever a child participates her reported preferences rank the schools as follows:* s_1, h, s_2*.*

The resulting matching from the strategy described above is that (h,s_1) *for each child. It is easy to see that this strategy profile is an (subgame perfect) equilibrium: no child wants to deviate because she cannot attend school* s_1 *when she is 1. If she attends school* s_2 *when she is 1, then she cannot attend* s_1 *when she is 2 because she will lose her priority over the younger child in that period.*

Clearly, the Aarhus mechanism is not efficient as each child matching with (s_2,s_1) *Pareto dominates* (h,s_1)*. Furthermore, in each period, the younger child can attend school* s_2 *as it has an unfilled spot. Consequently, the Aarhus allocation mechanism is not weakly stable. Finally, in the Aarhus mechanism, each child reports that h is preferred to* s_2*. Thus, the mechanism fails strategy-proofness too.*

4 Stable Matchings

Now we turn our attention to the question of whether stable matchings exist. We first show that if the schools' priority rankings do not satisfy *IPA*, then the existence of a stable matching is not guaranteed. Later, we show that *IPA* is a sufficient condition for the existence of stable matchings.

Example 5. *Consider the following market in which* IPA *is violated. There are 2 schools, s and* s' *with respective capacities of 1 and 3. In each period, there are two identical one-year old children. Their preferences are strongly rankable and satisfy the following property:* $(s,s) \succ (h,s) \succ (s',s') \succ (h,h)$*.*

Each period, the schools rank the children in which the highest priority groups are: (1) the previous period's attendees (2) two year old children who have not attended any school in the previous period. (Note that condition (2) violates IPA*).*

Now we show that stable matchings do not exist in this example. By contradiction, suppose that μ *is a stable matching.*

[22] In any given period t, if child $i \in I_{t-1}$ is allocated to s_2 and child $j \in I_t$ asks for a guaranteed place (see section 2), then, when a spot opens at school s_1 in period t, j will have a higher priority at s_1 than child i.

1. *Suppose there exist i and t such that $\mu^t(i) = h$. Then because there are 4 school age children and 4 spots at the two schools, at least one unassigned spot must exist at period t. Let $\bar{s} \in \{s, s'\}$ be a school with an unassigned at period t. If $i \in I_t$, then $(\bar{s}, \mu^{t+1}(i)) \succ_i (h, \mu^{t+1}(i)) = \mu(i)$ due to strong rankability. This means that μ is not stable, leading to a contradiction. If $i \in I_{t-1}$, then we reach a contradiction in a similar fashion.*

2. *Suppose for some i and t, $(\mu^t(i), \mu^{t+1}(i)) = (s, s')$. Clearly, i has the highest priority at school s in period $t+1$. In addition, as $(s, s) \succ_i (s, s')$ by strong rankability, child i can be improved in a justified manner. This is a contradiction.*

3. *Suppose for $i \in I_t$, $\mu^{t+1}(i) = s$. Then one of the following happens: (1) $\mu^{t+2}(s) = j$ for some $j \in I_{t+1}$ or (2) $\mu^{t+2}(s) \neq j$ for all $j \in I_{t+1}$. In the former case, the matching of j is (s', s); otherwise, we are back to case 1. Consequently, the matching of $\bar{j} \neq j \in I_{t+1}$ is (s', s'). If $\bar{j}$ stays home at $t+1$, at $t+2$ she has priority over any one-year old or j (who attended s′ at $t+1$). In addition, $\bar{j}$ prefers (h, s) to (s', s'). Hence, $\bar{j}$ can be improved in a justified manner. In case (2), either we are back to case 1 or both children born at I_{t+1} match with (s', s'). At $t+2$ both of these children have priority over any one year old at school s. In addition, (s', s) is preferred to (s', s'). Hence, both children child can be improved in a justified manner.*

In example 5 the chldren's preferences are strongly rankable preferences. However, one can construct a similar example in which no stable matching exists and the children's preferences *are not strongly rankable*. Hence, we conclude that the existence of stable matchings is not guaranteed without *IPA* regardless of whether *strong rankability* is satisfied or not. But with *IPA*, is the existence guaranteed? The answer to this question is positive, but first let us introduce the algorithm used for the existence result.

The Gale-Shapley Deferred Acceptance Mechanism and Its Properties

The Gale and Shapley deferred acceptance algorithm (DA algorithm) was originally designed to deal with static two-sided matching problems. To run this algorithm at a certain period *t*, one needs to know the schools' priorities over all school-age children as well as the children's preferences over schools. In our setting, the schools' priorities are well defined given the previous period's matching. However, the children's preferences are defined over the pairs of schools. Hence, we propose a version of the DA algorithm, in which we use "one period preferences" for each child at a given period, based on the past matchings and the original preferences of the children over the pairs of schools (we do not want to derive one period preferences based on the future matchings as the current matchings affect next period's priority rankings of the schools).

For now, let us assume that at period $t \geq 1$, we have derived the one period preference relation $\mathcal{P}_i(\mu^{t-1})$ for each $i \in I_{t-1} \cup I_t$ depending on μ^{t-1} matchings. Let $\mathcal{P}(\mu^{t-1}) = \{\mathcal{P}_i(\mu^{t-1})\}_{i \in I_{t-1} \cup I_t}$.

Thus, $s\mathcal{P}_i(\mu^{t-1})s'$ means that at time t, player i prefers school s to s' given the period $t-1$ matching μ^{t-1}. Now we define stability in a static context, which we will use in some of our proofs.

Definition 11 (Static Stability)**.** *Period t matching μ^t is statically stable under preferences $\mathcal{P}(\mu^{t-1})$ and μ^{t-1}, if there exists no school-child pair (s,i) such that*

1. $s\mathcal{P}_i(\mu^{t-1})\mu^t(i)$,
2. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1})\, j$ *for some* $j \in \mu^t(s)$

Now we will define the one-period preferences, called isolated preferences, that will be used in the algorithms we consider in the paper. We construct the concept of isolated preferences with the purpose of having a meaningful one-period preference ranking of the children. It is perhaps not controversial how a two year old child who was matched to some school in the previous period would rank the schools. Specifically, if child i was matched to school s in the previous period, then she ranks school s' ahead of s'' only if $(s,s') \succ_i (s,s'')$. The answer to the question of how one year old children rank the schools is not clear. In our opinion, a one year old child i would rank school s ahead of school s' if $(s,s) \succ_i (s',s')$. Indeed, recall that if $(s,s) \succ_i (s',s')$ then $(s,s'') \succ_i (s',s'')$ for all $s'' \neq s'$. Therefore, as long as a mechanism does not match i with s and s' when she is one and two, respectively, it seems like one year old child i should rank s ahead of s' in this situation. Below we define the isolated preferences formally.

Definition 12 (Isolated Preference Relation)**.** *For given μ^{t-1},*

1. *the* isolated preference relation *for $i \in I_t$ is the preference relation $\succ_i^1$ such that $s' \succ_i^1 s''$ if and only if $(s',s') \succ_i (s'',s'')$ for any $s' \neq s'' \in \bar{S}$,*
2. *the* isolated preference relation *for $i \in I_{t-1}$ is the preference relation $\succ_i^2(\mu^{t-1})$ depending on previous period's matching and such that $s' \succ_i^2 (\mu^{t-1})s''$ if and only if $(\mu^{t-1}(i),s') \succ_i (\mu^{t-1}(i),s'')$ for any $s' \neq s'' \in \bar{S}$.*

Here, we remark that for any child whose preferences satisfy *strong rankability*, the isolated preferences are independent of the previous period's matching. Furthermore, the isolated preferences for one year old child is identical to the ones for the two year old self of the same child.

We stress that in a world in which preferences do not satisfy rankability, the concept of isolated preferences is not useful: it is not plausible to assume that one year old-children rank the schools according to her isolated preferences if there are complementarities between some schools. Furthermore, in such cases it can be shown that a stable matching might not exist.[23] Thus, our assumption that the preferences are rankable is key for our results.

[23] Proof upon request.

Now we will state the formal definition of the Gale and Shapley deferred acceptance algorithm (henceforth, we will refer to it as the DA-IP algorithm). The algorithm is the same in each period, and it only uses the matching of the preceding period. Recall that the matching of period $t = 0$ is fixed, and will not be altered by the algorithm. In any period $t \geq 1$, assume that the previous period's matching is given (if $t \neq 0$, then the previous period's matching was obtained by the DA-IP algorithm). At period t, the schools assign their spots to the all school-age children in finite rounds as follows:

Round 1: Each child proposes to her first choice according to her isolated preferences. Each school tentatively assigns its spots to the proposers according to its priority ranking. If the number of proposers to school s is greater than the number of available spots r_s, then the remaining proposers are rejected.

In general, at:
Round k: Each child who was rejected in the previous round proposes to her next choice according to her isolated preferences. Each school considers the pool of children who it had been holding plus the current proposers. Then it tentatively assigns its spots to this pool of children according to its priority ranking. The remaining proposers are rejected.

The algorithm terminates when no proposal is rejected and each child is assigned her final tentative assignment.

Given that the children's preferences as well as schools' priority rankings are strict, it is easy to see that the DA-IP algorithm yields a unique matching. We refer to this matching as the DA-IP matching and use the notation μ_{DA} for it.

We denote by deferred acceptance with isolated preferences mechanism (*DA-IP*) the revelation mechanism which maps each market M to the matching produced by the DA algorithm for market M, using isolated preferences.

With the next result we show that, when assuming *IPA*, stability is equivalent to static stability under isolated preferences.

Lemma 1. *If μ is stable then for all $t \geq 1$, μ^t is statically stable under isolated preferences and μ^{t-1}. Conversely, if for all $t \geq 1$, μ^t is statically stable under isolated preferences and μ^{t-1}, then $\mu = (\mu^0, \cdots, \mu^t, \cdots)$ satisfies:*

1. *autarkic stability;*

2. *stability if each school's priorities satisfy* IPA.

Proof. See Appendix C. □

Lemma 1 implies that to find a stable matching, it suffices to find a stable matching under isolated preferences in each period, sequentially starting from period 1. In other words, for the purpose of finding a stable matching, one can view the dynamic problem of assigning children to daycare centers as separate school choice problems in different periods. Consequently, the matching obtained from the DA-IP mechanism is stable (Gale and Shapley (1962) shows that the DA algorithm yields a stable matching in static settings). We state the result below.

Theorem 1 (Existence of Stable Matching)**.** *The DA-IP matching satisfies autarkic stability. Furthermore, if the schools' priorities satisfy* IPA*, then the DA-IP matching is stable.*

As we already mentioned, examples 2 and 3 illustrate the need of strengthening the (autarkic) stability concept if *strong rankability* or *IPA* are not satisfied. However, Theorem 1 demonstrates that *IPA* is a sufficient condition for the existence of stable matchings even if *strong rankability* is not satisfied (but assuming rankability). In addition, Theorem 5 shows that with or without *strong rankability*, the existence of stable matchings is not guaranteed without *IPA*.

Remark 1 (Special Case)**.** *An interesting special case of our problem is one in which all the children are born in period* 0*. Then this problem is a static allocation problem in which two year old children are assigned to schools only once. In addition, the schools' priorities are well defined at period 1. Furthermore, each child's preferences can be set to her isolated preferences at period 1. Then one can see that this special case of our dynamic problem is a school choice problem.*[24]

One of the most important results in the matching literature is that the DA-IP matching Pareto dominates all other stable matchings.[25] We study how the DA-IP matching compares to the other stable matchings in a dynamic environment. Our results are presented in detail in Appendix B. We summarize our findings in the following proposition.

Proposition 1. *The DA-IP matching does not necessarily Pareto dominate all other stable matchings. However, it is not Pareto dominated by any stable matching in any market. Moreover, if there exists a stable and efficient matching in some market, then it must be the DA-IP matching.*

5 Strategy-Proofness and Stability

It is well known that in static settings, the DA mechanism is strategy-proof. We show below, that in our dynamic setting this result no longer holds for our version of DA-IP. In fact, the result below is much stronger: there is no mechanism that is strategy-proof and stable.

[24]Recall that the school choice problem is a static allocation problem in which each student has preferences over the schools (not over the pairs of schools), each school prioritizes all the children, and each student obtains at most one seat at some school.

[25]See Gale and Shapley (1962).

Theorem 2 (Impossibility Result)**.** *No mechanism satisfies both autarkic stability and strategy-proofness.*

Proof. Consider the following example: there are 4 schools $\{s,\bar{s},s_1,s_2\}$ and each school has a capacity of one child. There is no school-age child until period $t-1 \geq 1$. Suppose $I_{t-1} = \{i,\bar{\imath}\}$, $I_t = \{i_1,i_2\}$, $I_{t+1} = \{i'\}$ and $I_\tau = \emptyset$ for all $\tau \geq t+2$. The schools' priorities satisfy *IPA*. In addition, any school $s' \in \{s,\bar{s},s_1,s_2\}$ prioritizes the children as follows under the assumption that no child attended s' in the previous period:

$$\begin{array}{ccccccccc}
i & \rhd_s & i' & \rhd_s & i_1 & \rhd_s & i_2 & \rhd_s & \bar{\imath} \\
i & \rhd_{s_1} & i_1 & \rhd_{s_1} & i_2 & \rhd_{s_1} & i' & \rhd_{s_1} & \bar{\imath} \\
i & \rhd_{s_2} & i_1 & \rhd_{s_2} & i' & \rhd_{s_2} & i_2 & \rhd_{s_2} & \bar{\imath} \\
\bar{\imath} & \rhd_{\bar{s}} & i_1 & \rhd_{\bar{s}} & i' & \rhd_{\bar{s}} & i_2 & \rhd_{\bar{s}} & i
\end{array}$$

We consider two preference profiles which differ from each other in child i_1's preferences. Each child's preferences are *strongly rankable*. Child i's top choice is (s,s) while child $\bar{\imath}$'s is $(\bar{s},\bar{s})$. The preferences of children i_2 and i' satisfy the following conditions:

$$\begin{array}{ccccccc}
(s_2,s_2) & \succ_{i_2} & (s_1,s_1) & \succ_{i_2} & (s,s) & \succ_{i_2} & (\bar{s},\bar{s}) \\
(s_2,s_2) & \succ_{i'} & (s,s) & \succ_{i'} & (s_1,s_1) & \succ_{i_2} & (\bar{s},\bar{s})
\end{array}$$

Child i_1's preference ordering is $\succ^1_{i_1}$ under preference profile 1 and is $\succ^2_{i_1}$ under profile 2. These preferences are given as follows:

$$\begin{array}{ccccccc}
(s,s) & \succ^1_{i_1} & (s_1,s_1) & \succ^1_{i_1} & (s_2,s_2) & \succ^1_{i_1} & (\bar{s},\bar{s}) \\
(s,s) & \succ^2_{i_1} & (\bar{s},\bar{s}) & \succ^2_{i_1} & (s_2,s_2) & \succ^2_{i_1} & (s_1,s_1)
\end{array}$$

In addition, suppose $(s_2,s) \succ^1_{i_1} (s_1,s_1)$.

Step 1. Under profile 1, the only matching μ that satisfies autarkic stability is: $\mu^{t-1}(i) = \mu^t(i) = s$, $\mu^{t-1}(\bar{\imath}) = \mu^t(\bar{\imath}) = \bar{s}$, $\mu^t(i_1) = \mu^{t+1}(i_1) = s_1$, $\mu^t(i_2) = \mu^{t+1}(i_2) = s_2$, $\mu^{t+1}(i') = s$ and $\mu^{t+2}(i') = s_2$.

Proof of Step 1. Under profile 1, the DA-IP matching is as follows: $\mu^{t-1}(i) = \mu^t(i) = s$, $\mu^{t-1}(\bar{\imath}) = \mu^t(\bar{\imath}) = \bar{s}$, $\mu^t(i_1) = \mu^{t+1}(i_1) = s_1$, $\mu^t(i_2) = \mu^{t+1}(i_2) = s_2$, $\mu^{t+1}(i') = s$ and $\mu^{t+2}(i') = s_2$. We know that DA-IP matching satisfies autarkic stability. We now show that it is the unique matching that satisfies autarkic stability.

Let $\hat{\mu}$ be a matching that satisfies autarkic stability. It is clear that $\hat{\mu}^{t-1}(i) = \hat{\mu}^t(i) = s$, $\hat{\mu}^{t-1}(\bar{\imath}) = \hat{\mu}^t(\bar{\imath}) = \bar{s}$ and $\hat{\mu}^{t+2}(i') = s_2$. Consequently, we obtain that $\hat{\mu}^t(i_1) = s_1$ because child i_1 has higher priority in school s_1 at period t than anyone but i. However, i must match with s at period t. Hence, $\hat{\mu}^t(i_1) = s_1$. This implies that $\hat{\mu}^t(i_2) = s_2$. Then i_2 has the highest priority at school s_2 at period $t+1$. Since s_2 is the top choice for i_2, $\hat{\mu}^{t+1}(i_2) = s_2$. Consequently, $\hat{\mu}^{t+1}(i') = s$ which means

$\hat{\mu}^{t+1}(i_1) = s_1$. Now we have shown that $\hat{\mu} = \mu$.

Step 2. Under profile 2, the only matching that satisfies autarkic stability, $\bar{\mu}$, is as follows: $\bar{\mu}^{t-1}(i) = \bar{\mu}^t(i) = s$, $\bar{\mu}^{t-1}(\bar{\imath}) = \bar{\mu}^t(\bar{\imath}) = \bar{s}$, $\bar{\mu}^t(i_1) = s_2$, $\bar{\mu}^t(i_2) = s_1$, $\bar{\mu}^{t+1}(i_1) = s$, $\bar{\mu}^{t+1}(i_2) = s_1$, $\bar{\mu}^{t+1}(i') = s_2$ and $\bar{\mu}^{t+2}(i') = s_2$.

Proof of Step 2. Under profile 2, the DA-IP matching $\bar{\mu}$ is as follows: $\bar{\mu}^{t-1}(i) = \bar{\mu}^t(i) = s$, $\bar{\mu}^{t-1}(\bar{\imath}) = \bar{\mu}^t(\bar{\imath}) = \bar{s}$, $\bar{\mu}^t(i_1) = s_2$, $\bar{\mu}^t(i_2) = s_1$, $\bar{\mu}^{t+1}(i_1) = s$, $\bar{\mu}^{t+1}(i_2) = s_1$, $\bar{\mu}^{t+1}(i') = s_2$ and $\bar{\mu}^{t+2}(i') = s_2$. We know that the DA-IP matching $\bar{\mu}$ is a matching that satisfies autarkic stability. We now show that $\bar{\mu}$ is the only one.

Let $\hat{\mu}$ be matching that satisfies autarkic stability. It is clear that $\hat{\mu}^{t-1}(i) = \hat{\mu}^t(i) = s$, $\hat{\mu}^{t-1}(\bar{\imath}) = \hat{\mu}^t(\bar{\imath}) = \bar{s}$ and $\hat{\mu}^{t+2}(i') = s_2$. Consequently, we obtain that $\hat{\mu}^t(i_1) = s_2$ because child i_1 has higher priority in school s_2 at period t than i_2. This means that $\hat{\mu}^t(i_2) = s_1$.

Now let us argue that $\hat{\mu}^{t+1}(i') = s_2$. If not, $\hat{\mu}^{t+1}(i_1) = s_2$; otherwise, child i' has higher priority than child i_2 at school s_2 and s_2 is the top choice of child i'. Hence, this contradicts with $\hat{\mu}$ being a matching that satisfies autarkic stability. Thus, $\hat{\mu}^{t+1}(i_1) = s_2$. But because $(s_2, \bar{s}) \succ^2_{i_1} (s_2, s_2)$ and child i_1 has higher priority at school $\bar{s}$ than anyone but $\bar{\imath}$, $\hat{\mu}$ satisfies autarkic stability. This is a contradiction. Hence, $\hat{\mu}^{t+1}(i') = s_2$.

Because $\hat{\mu}^{t+1}(i') = s_2$, $\hat{\mu}^{t+1}(i_1) = s$ as i_1 has higher priority at school s than i_2. Consequently, $\hat{\mu}^{t+1}(i_2) = s_1$. This means $\hat{\mu} = \bar{\mu}$.

Step 3. If a mechanism yields a matching that satisfies autarkic stability, then this mechanism is not strategy proof.

Proof of Step 3. If a mechanism yields a matching that satisfies autarkic stability, then it must allocate (s_1, s_1) to i_1 under profile 1 and (s_2, s) under profile 2. Now one can easily see that under profile 1 child i_1 has incentive to misreports her preference as if under profile 2. □

Theorem 2 has two important, direct consequences which we present next.

Corollary 1. *1. No mechanism satisfies both strategy-proofness and stability.*

2. The DA-IP mechanism is not strategy-proof.

Proof. Recall that each stable matching satisfies autarkic stability. This and Theorem 2 prove item 1 of the corollary. □

Even when *strong rankability* and *IPA* are satisfied, strategy-proofness is hard to achieve in our dynamic assignment problem. In static problems, a child has a motive to misreport her preferences only if she can obtain a better placement. This motive is also present in our dynamic assignment problem. To be specific, a child will misreport her preferences if she can improve her present placement without hurting her placement in the other period. This motive, as known from the school choice literature, is eliminated if the mechanism is the DA or Top Trading Cycles mechanism. However, in our setting, there is an extra motive absent in the school choice problem: one might misrepresent her preferences to affect the schools' priorities in the subsequent period. This way, she could obtain a better future placement by (weakly) sacrificing her current one.

In the example used for the proof of Theorem 2, type 1 child i_1 likes school s better than any other school, but attending s in period t is impossible for her. Again, in period $t+1$, she cannot attend s because child i' attends s. But observe that child i' wants to attend school s_2 but cannot do so because child i_2 attends s_2. The most important aspect is that child i_2 has higher priority over child i' at school s_2 in period $t+1$ only because she attends school s_2 in period t. Child i_1 can eliminate child i_2's advantage over i' if she attends school s_2 in period t. By doing this, i_1 enables i' to attend s_2 at $t+1$. Ultimately, she frees a spot at school s for herself at $t+1$. This is the reason why type 1 child i_1 has an incentive to misreport her preferences.

Remark 2 (OLG Structure)**.** *For Theorem 2, the OLG structure of our model plays a key role. To illustrate this point, let us consider the following dynamic model in which all the children are born at period 1 and attend school for two periods. Lemma 1 is valid in this modified model; thus, the DA-IP algorithm produces a stable matching in each market. Furthermore, the DA-IP algorithm matches each child to the same school in periods 1 and 2 because the preferences satisfy rankability. Thus, in the modified model, by running the DA-IP algorithm only once in period 1 and then by replicating period 1 matching in period 2, one obtains a stable matching in each market. Observe here that the mechanism corresponding to this process only uses the period 1 isolated preferences of the children. Consequently, the new mechanism is essentially a static mechanism; thus, no child can improve by misreporting her isolated preferences.*

Remark 3 (History-Dependent-Priorities)**.** *The assumption of history-dependent priorities of the schools is indispensable in Theorem 2 if the children's preferences are strongly rankable. To see this point, suppose that the children's preferences are strongly rankable and that the schools' priorities are independent of the previous period's matching–in particular, a child who did not attend a school in the previous period can have higher priority over some other child who did attend that school. In this case, the DA-IP mechanism must be strategy-proof. Let us discuss why this is the case. For the DA-IP mechanism, one has to report her preferences over the pairs of schools. But this, in fact, is equivalent to the case in which the school-age children report their isolated preferences in each period and the algorithm is run sequentially because the DA-IP algorithm uses the isolated preference. As the preferences satisfy* strong rankability *and the schools' preferences are independent of history, any child's reported isolated preferences in one period do not affect her placement in the other period. Now recall that the DA-IP mechanism is strategy-proof in the static settings. Hence, by misreporting one's isolated preferences in some period, she is worse off in that* *period without affecting her placement in the other period. Accordingly, no one misreports her isolated preferences. Thus, the DA-IP mechanism is strategy-proof.*

Remark 4 (Rankable Preferences)**.** *If the children's preferences satisfy rankability but do not satisfy strong rankability, then an impossibility result similar to Theorem 2 arises even if the schools' priorities are independent of history. To see this consider a market in which there are 3 schools and two one-year old children i, j in period t. Each school has a capacity of one child and the preferences of i and j satisfy that $(s_1, s_1) \succ_i (s_3, s_1) \succ_i (s_2, s_2)$ and $(s_1, s_1) \succ_j (s_2, s_2) \succ_j (s_2, s_1)$. Suppose there is another child who is two at period t. Let this child's most preferred option be s_1 and suppose s_1 gives its highest priority to this child. Furthermore, assume that school s_2 gives priority to i over j. Then the DA-IP mechanism matches i with (s_2, s_2) and j with (s_3, s_1) if both children reports their preferences truthfully. However, i can obtain (s_3, s_1) by reporting s_2 as her least preferred school. Hence, the DA-IP mechanism is not strategy-proof if the children's preferences do not satisfy strong rankability even if the schools' priorities are independent of history.*

Remark 5 (Property-Rights)**.** *We argued above that history-dependent priorities of the schools are crucial for Theorem 2. If schools' priorities are not history-dependent, then a strategy-proof and stable mechanism implies that there are markets in which some children will be forced out of the schools that they attended in the previous period. For example, in the example used in the proof of Theorem 2, child i_2 is forced out of school s_2 at period $t+1$. Therefore, under the restriction that no 2-year old child can be forced out of the school she attended in the previous period, Theorem 2 is valid even when the schools' priorities are independent of the previous period's matching.*

Remark 6 (DA-IP mechanism and Strategy-Proofness)**.** *The DA-IP mechanism is strategy-proof under some restrictive set of markets $\mathcal{M}$, i.e., under some restrictive sets of the preferences and priorities. We consider three possibilities here. The first case is when the cost of switching schools is very large for the children, i.e., $(s, s) \succ_i (s', s'')$ for all $i \in I$, $s \in \bar{S}$ and $s' \neq s'' \in \bar{S}$. In this case, each child's goal is to obtain the best possible school when she is one and to stay in the same school when she is two. When the DA-IP is the implementation mechanism, each child achieves this goal by truthfully reporting her preferences. In practice, schools are heterogenous in quality and switching costs might not play such a decisive role in parents' choices: the switching costs might not be prohibitively large. Second, if the preferences are strongly rankable and the priorities of the schools favor the older cohort (or generation). The latter means that whenever $i \in I_{t-1}$ and $j \in I_t$, it must be that $i \rhd_s^t j$ for every s. This result, which is proven in Pereyra (2013), is not valid if the preferences are not strongly rankable (see the example used in Remark 1). Finally, the DA-IP mechanism is strategy-proof if not only the schools' priorities favor the older cohort but also each school ranks the younger children in the exact same way. In this case, the DA-IP mechanism is equivalent to the DA-IP mechanism done by cohorts: in each period the DA-IP mechanism is* *run first among the two year old children only, and after allocating the two year old children and adjusting the schools' capacities accordingly the DA-IP is run among the one year old children. Consequently, in each period the whole set of schools is available for the old children, but only some schools are available for the young children. Since the old cohort in period 1 stays in the system only for one period no child from this cohort has incentive to misreport. Now let us focus on the period 1 young child who has the highest priority in all schools in period 1. In period 2 this child will have the highest priority in the schools that had no open spots for the young children in period 1. As a result, the DA-IP assigns this child to her most preferred pair of schools among those that are available to her cohort if the child reports her preferences truthfully. Hence, this child has no incentive to misreport. Then the second highest ranked child has no incentive to misreport, and so on.*

6 Efficiency and Strategy-Proofness

In this section, we start pointing out that some mechanisms that are known to be efficient in static settings are not efficient in our setting. Then, in Section 6.2 we study the Top-Trading Cycles in detail, and we propose a version of it using isolated preferences (TTC-IP). We show that it is neither Pareto efficient nor strategy-proof. Finally, in Section 6.3 we study a variation of the serial dictatorship mechanism, which is both strategy-proof and efficient.

6.1 Efficient Matchings

We have shown that stability and strategy-proofness may be incompatible for the dynamic assignment problem. In the remaining sections of this paper, we investigate whether strategy-proofness is compatible with efficiency. However, before doing so, let us consider some properties of efficient matchings.

From the school choice literature, we know that the Top Trading Cycles (TTC) or the Serial Dictatorship (SD) mechanisms yield efficient matchings. Hence, one might expect that these mechanisms when run using the isolated preferences of the children yield efficient matchings. In other words, one may expect that a result analogous to the result of Lemma 1 will hold for efficiency as well. However, let us show that this is not the case using Example 1 in which an autarkic efficient matching is not Pareto efficient. However, this autarkic efficient matching is produced by the TTC-IP mechanism using isolated preferences which we will consider in the next subsection.

6.2 The Top Trading Cycles Mechanism

The TTC mechanism was introduced by Abdulkadiroğlu and Sönmez (2003) for the context of the school choice problem.[26] Next we will state the formal definition of the TTC-IP mechanism.

In each period $t \geq 2$, we assume that the preceding period's matching is produced by the TTC-IP mechanism according to the isolated preferences of children. Recall that in period $t = 0$, the matching is exogenously given and is not affected by the TTC-IP mechanism. In period $t \geq 1$:

Round 1: Each child points to her preferred school. Each school $s \in S$ points to its highest ranked child. Then we look for cycles: a cycle is either (i) a set $\{i, h\}$ where h is i's preferred school, or (ii) an ordered set $\{i_1, s_1, i_2, s_2, \cdots, i_k, s_k\}$ such that h is not in this set, and s_j is child i_j's preferred school, whereas child i_l is the highest ranked child in school s_{l-1}, for $l = 2, ..., k$; and child i_1 is the highest ranked child at school s_k. There always must be at least one cycle and each child and school can be a part of only one cycle. Each child in any cycle is allocated to her preferred school.

In general, at:
Round k: All children allocated in the previous rounds as well as all the schools which have filled their capacity in the previous rounds do not participate in step k. Each remaining child points to its preferred school, among the set of schools with remaining spots. Each remaining school $s \in S$ points to the highest priority child among the remaining children. Then we look for cycles and each child in any cycle is allocated to the school that she pointed to.

The process continues until all children are allocated.[27]

As we already hinted, the TTC-IP mechanism is not efficient. Given the importance of the TTC mechanism in the school choice problem, let us state this result in the following proposition.

Proposition 2 (TTC-IP is not Pareto Efficient)**.** *The TTC-IP mechanism is not Pareto efficient.*

Proof. Consider Example 1 and observe that μ is the matching from the TTC-IP mechanism. As we mentioned μ is not efficient. □

[26]TTC mechanism, which is attributed to David Gale, is first considered in Shapley and Scarf (1974).

[27]We point out that the version of the TTC-IP that we use is similar to the one Abdulkadiroğlu and Sönmez (1999) use in the housing allocation problem with existing tenants. In both versions, the object to be assigned will point to its current owner, unless she already obtained another object. In the case of Abdulkadiroğlu and Sönmez (1999), each house points to its current tenant unless she is already assigned a house while in our model, due to the fact that the schools give their highest priorities to its current enrollees, each school points to one of these children unless all of them are assigned to a school. However, the two versions of TTC are different in the sense that in Abdulkadiroğlu and Sönmez (1999), no house prioritizes the (non existing) tenants but in our model, different schools can prioritize the children differently.

Note that in Example 1, not only the TTC-IP mechanism is not efficient, but also a variation of it, done by cohorts. Precisely, consider the following mechanism. At any period $t \geq 1$, the children born in period $t-1$ are allocated according to the TTC-IP mechanism (see Abdulkadiroğlu and Sönmez (2003)). Once every children $i \in I_{t-1}$ is allocated, most schools will have less, if any, spots available. Consider only the schools with open spots and use the TTC-IP mechanism for the generation born in period t, where from the initial number of spots for each school, we have subtracted the number of 2-year-old children already allocated. For this round, consider only the priority of schools over the children of generation t. i.e., a young child cannot replace an already allocated 2-year-old child. This variation of the TTC-IP mechanism is also not Pareto efficient.

In the example below, we show that the TTC-IP mechanism is not strategy-proof.

Example 6 (TTC-IP is not Strategy-Proof)**.** *Assume that there are 4 schools* $\{s, s_1, s_2, s_3\}$*, and 4 children:* $\{i, i_1, i_2, i_3\}$*, with* $i \in I_{-1}$ *and* $\{i_1, i_2, i_3\} \in I_0$*. Assume also that* $I_t = \varnothing$ *for all* $t \geq 1$*. The schools' priorities satisfy* IPA *and the children's preferences are* strongly rankable*. School* $\bar{s} = s, s_1, s_2, s_3$ *prioritizes the children as follows assuming that these children has not attended* $\bar{s}$ *in the previous period:*

$$\begin{array}{lllll} i & \rhd_s & i_2 & \rhd_s & i_1 \\ i_1 & \rhd_{s_1} & j, & \forall j \neq i_1 & \\ i_2 & \rhd_{s_2} & j, & \forall j \neq i_2 & \\ i_1 & \rhd_{s_3} & i_3 & \rhd_{s_3} & j, \quad \forall j \neq i_1, i_3 \end{array}$$

The children's preferences are:

$$\begin{array}{ccccccc} (s,s) & \succ_i & (s_1,s_1) & \succ_i & (s_2,s_2) & \succ_i & (s_3,s_3) \\ (s,s) & \succ_{i_1} & (s_1,s_1) & \succ_{i_1} & (s_2,s_2) & \succ_{i_1} & (s_3,s_3) \\ (s_3,s_3) & \succ_{i_2} & (s,s) & \succ_{i_2} & (s_2,s_2) & \succ_{i_2} & (s_1,s_1) \\ (s_3,s_3) & \succ_{i_3} & (s_1,s_1) & \succ_{i_3} & (s_2,s_2) & \succ_{i_3} & (s,s) \end{array}$$

In addition, child i_1 *prefers* (s', s) *to* (s_1, s_1)*.*

The matching resulting from the TTC-IP is: $\mu^0(i) = s$*,* $\mu^0(i_1) = s_1$*,* $\mu^0(i_2) = s_2$*,* $\mu^0(i_3) = s_3$*,* $\mu^1(i_1) = s_1$*,* $\mu^1(i_2) = s$ *and* $\mu^1(i_3) = s_3$*. However, if* i_1 *misreports its preferences as* $s \succ_{i_1} s_2 \succ_{i_1} s_1 \succ_{i_1} s_3$*, while all others report truthfully. The resulting matching is:* $\bar{\mu}^0(i) = s$*,* $\bar{\mu}^0(i_1) = s_2$ *,* $\bar{\mu}^0(i_2) = s_3$*,* $\bar{\mu}^0(i_3) = s_1$*,* $\bar{\mu}^1(i_1) = s$*,* $\bar{\mu}^1(i_2) = s_3$ *and* $\bar{\mu}^1(i_3) = s_1$*.*

Note that under truth-telling, i_1*'s allocation was:* (s_1, s_1)*, while after misreporting it is* (s_2, s)*. Thus,* i_1 *has improved herself by misreporting.* ◇

Observe that the example above shows that a variation of the TTC-IP which is done by cohorts is not strategy-proof.[28]

[28] Kurino (2013) shows that under strongly rankable preferences, the constant TTC mechanism favoring existing

Remark 7 (TTC-IP: Strategy-Proofness and Efficiency)**.** *One may ask if there is any restriction on the priorities and preferences that would restore the efficiency and strategy-proofness for TTC-IP. One such case is the one in which the switching the schools is prohibitively large for the children i.e., if attending the same school for two periods is preferred by each child to attending any two different schools. Again, in practice schools are heterogenous and switching costs might not be prohibitively large. On the other hand, priorities favoring older generation (cohort) is not a solution as long as the switching cost is "reasonable" because the TTC-IP by cohort is neither strategy-proof nor efficient. Another case in which TTC-IP is both efficient and strategy-proof occurs when the priorities are such that (i) they favors the older generation and (ii) the young children in each period are ranked in the exact same way in each school. In this case, even under rankable preferences our TTC-IP mechanism is both efficient and strategy-proof. To see this, first observe that because each school gives higher priority to the older children, the TTC-IP mechanism is equivalent to the TTC-IP mechanism by cohort: as long as an old child is not assigned under TTC-IP, all schools with open spots point to some old child. Then because TTC-IP is strategy-proof in the static school choice problem, in period 1, no old child has incentive to manipulate the TTC-IP. Now let us focus on the period 1 young child who has the highest priority in all schools in period 1. In period 2 this child will have the highest priority in the schools that had no open spots for the young children in period 1. As a result, the TTC-IP assigns this child to her most preferred pair of schools among those that are available to her cohort if the child reports her preferences truthfully. Hence, this child has no incentive to misreport. Then the second highest ranked child has no incentive to misreport, and so on. This shows that TTC-IP is strategy-proof in this case, and in a similar way one can argue that the TTC-IP is Pareto efficient.*

6.3 Serial Dictatorship Mechanism

To answer the question of whether any mechanism is efficient and strategy-proof we will adopt the well-known SD mechanism in our setting. In our version of the SD mechanism we will utilize the feature of our model that the old children of the current period do not participate in the system next period. This allows us to let each young child choose two schools (one for the period in which she is one and one for the period in which she is two).

Formally, in each period $t \geq 0$ children are exogenously ordered. First, recall that the matching of period $t = 0$ is exogenous. The serial dictatorship algorithm runs as follows: at period 1,

tenants, which is based on Abdulkadiroğlu and Sönmez (1999)'s TTC mechanism in the housing allocation model with existing tenants, is Pareto efficient and strategy-proof. Because the houses do not have priorities Kurino's model, the constant TTC mechanism favoring existing tenants is not dependent on the priorities, but it respects the property rights' of the older generation. Our version of TTC is based Abdulkadiroğlu and Sönmez (2003)'s TTC mechanism in the school choice problem, and it depends on the schools' priorities. In other words, the reason why Kurino (2013) and we obtain seemingly different results for TTC is because these papers consider different versions of TTC.

following the ordering for the period 0 children, the 2-year-old children are allocated sequentially to their preferred schools from the set of schools that have not yet filled their capacity. Once all 2-year-old children are allocated, following the ordering of the period 1 children, each 1-year-old child is allocated sequentially to their most preferred pairs of schools – one for each period– which have not filled their capacities. Here, observe that each 1 year old child finds out her allocation for two periods in period 1. Thus, in period 2, all the two year old children are already matched to schools. Consequently, in period 2, following the ordering of the period 2 children, each 1-year-old child is allocated sequentially to their most preferred pairs of schools which have not filled their capacities. This process is replicated in each period.

At any given period there is a finite number of school-age children, therefore this is a well-defined algorithm that always converges to a unique matching. The serial dictatorship mechanism is the revelation mechanism that implements this algorithm. It is easy to see that the serial dictatorship mechanism is efficient and strategy-proof.[29] It is strategy-proof since each child can be allocated to the best available menu. Moreover, it is efficient since the first child to choose in a given cohort can only improve if there is a school chosen by another child in the previous cohort that would make her better off. No child in the previous cohort would engage in such a trade, since all open schools were available to the older cohort and not chosen by them. The child with an index 2 of the young cohort cannot improve by trading with the first child, since the first child is already choosing the best available option for her. A similar argument holds for any other indexed child.

There is a shortcoming of our SD mechanism: in period 1 some two year old child could be forced out of the school she has attended in period 0.[30] To overcome this, we can modify the SD mechanism so that it differs from the previously considered SD mechanism only in how the two year old children in period 1 are allocated. Specifically, to determine the allocations of the two year old children in period 1 we first run the TTC-IP mechanism among these children. Afterwards, starting with the young children in period 1, we run the SD mechanism. This modified SD mechanism is strategy-proof and efficient.

[29]Kurino (2013) considers the constant serial dictatorship favoring the existing tenants (the older cohorts in our language), and he shows that it is a strategy-proof and efficient mechanism. In this mechanism, all the agents are placed on an ordering in which the older agents appear ahead of the younger agents. Then in each period, following this ordering, the mechanism matches sequentially each agent (who are in the alive in that period) to the house that is available and that is the agent's most preferred according to her *period* preferences. In our setting, period preferences are not well defined when the preferences are not strongly rankable. Thus, to use Kurino (2013)'s constant serial dictatorship mechanism favoring the existing tenants in our setting, one has to modify it so that it uses the isolated preferences of the children. Now it is not complicated to see that Kurino's and our version of the serial dictatorships produce the same matching in each market. However, if the preferences are neither rankable and nor time-separable, it is not clear how one can run the constant serial dictatorship. On the other hand, ours can be run without any adjustment, and it will be Pareto efficient and strategy-proof.

[30]This problem does not arise for the children who are 1 in any period other than 0.

7 Conclusion

In this paper we introduced the daycare assignment problem. This problem is a dynamic version of the school choice problem in which there is entry and exit of students over time and in which the daycare centers' priorities are history-dependent. We showed that the Gale-Shapley deferred-acceptance mechanism and the Top-Trading Cycles mechanism– both commonly used in the school choice problem– are not strategy-proof in the dynamic problem.

In general, we study two main questions in the paper: (i) whether stability and strategy-proofness are compatible with one another in our model, and (ii) whether Pareto efficiency and strategy-proofness are compatible. For the first question, we proved an impossibility result: no stable and strategy-proof mechanism exists for this class of dynamic matching problems. This result is particularly important in the context of the school choice problem, in which much attention has been given to stability and, in particular, to the DA mechanism (which has been adopted in the New York and Boston public school systems). For the second question, we show that a version of the TTC adapted to a dynamic problem is not strategy-proof nor efficient, but that the SD mechanism is both Pareto efficient and strategy-proof.

We offer two practical suggestions for the problem of assigning children to public daycare centers. First, the education authorities might use the DA mechanism period-by-period. We have shown that, under truth-telling, this mechanism is stable and not Pareto dominated by any other stable mechanism. Moreover, it is often the case that non-strategy-proof mechanisms are implemented successfully, provided that the strategic issues are not severe (see Kojima and Pathak (2009) and Budish and Cantillon (2012), for example). When *IPA* is satisfied, the sophistication level needed for a successful manipulation of the DA mechanism is rather high.[31] In addition, one would need to have information about the preference profile of the children born in the succeeding period. All this leads to an important practical question of how the DA-IP mechanism performs in practice. We are planning to explore this question in a laboratory setting. Another approach is to study the performance of the DA-IP mechanism in large markets which is the main concern of a follow-up paper by Monte and Tumennasan (2012a). Our preliminary results indicate that if *IPA* is satisfied, the incentives for manipulation disappear as the market becomes large. This seems to suggest that the DA-IP mechanism could be implemented in practice, successfully.

The second suggestion for the practical problem of designing a centralized allocation in daycare centers is to use the SD mechanism. This mechanism has disadvantages, since it disregards the schools' priorities. However, in addition to being efficient and strategy-proof, in our dynamic problem there is an important, but less obvious, advantage of the SD mechanism which is a no-

[31]When *IPA* is not satisfied (which is the case for the current Danish system), then there may be a simpler manipulation of the DA mechanism: by staying at home at age 1, a child improves her priority ranking in all schools in the next period. This ultimately enables the child to go to her favorite school in the next period.

tion of "fairness." In the standard school choice problem, the SD mechanism is considered unfair because parents listed last are at a clear disadvantage to parents listed first. This problem with serial dictatorship is somewhat mitigated in a dynamic assignment problem. To illustrate this point, consider the case in which the number of children born at every period is the same. The child who chooses last in her cohort will have at least half of the daycare-spots available to her in period 2, whereas in the static problem, the last child to choose in the serial dictatorship mechanism might have only one option.[32] In fact, if each period has only 1 child (or if the number of periods that children attend increases so that there is at most one child born in each period), then the option sets of the children are somewhat similar.

Finally, Monte and Tumennasan (2012b) show in a follow-up paper that for the multi-market allocation problem, the set of nonbossy and strategy-proof rules that implement a Pareto efficient outcome is the set of sequential dictatorships — a slight generalization of the serial dictatorship. This result provides further support for the use of the serial dictatorship mechanism in this dynamic environment.

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[32]This assumes that there is at least the same number of spots as there are children in a given period. Formally, consider the case in which there are $2n$ children at every period (with n children being born every period) and $2n$ daycare spots available. The last child choosing in her cohort, will have $n+1$ options in her second period. In the static case with $2n$ children and $2n$ spots, the last child to choose in the serial dictatorship mechanism might have only one spot available.

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Appendix A: The Relation between Stability and Autarkic Stability

Now we will explore under what conditions, the stability concepts will coincide. From examples 2 and 3, one could conjecture that stable matchings may be equivalent to matchings that satisfy autarkic stability if the children's preferences are *strongly rankable* and the schools' priority rankings satisfy *IPA*. Indeed this is the case, as we will show in the next two lemmas.

Lemma 2. *Suppose that all schools' priorities satisfy* IPA. *If μ satisfies autarkic stability but is not stable, then for some period $t \geq 1$ and some school-child pair (s,i),*

1. $\mu^t(i) = \mu^{t+1}(i)$,
2. $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$,
3. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1})\, j$ *for some* $j \in \mu^t(s)$.

Proof. Since μ is not but satisfies autarkic stability, for some $t \geq 1$, there must exist (s,s',i) such that $(s,s') \succ_i (\mu^t(i), \mu^{t+1}(i))$, $s \neq \mu^t(i)$, $s' \neq \mu^{t+1}(i)$ and one of the following conditions are satisfied:

1. $|\mu^t(s)| < r_s$ and $\left|\mu^{t+1}(s')\right| < r_{s'}$,
2. $|\mu^t(s)| < r_s$, $\left|\mu^{t+1}(s')\right| = r_{s'}$, and, for some $j' \in \mu^{t+1}(s')$, $i \rhd_{s'}^{t+1} (\bar{\mu}^t)\, j'$ where $\bar{\mu}^t$ is the period t matching with $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(i') = \mu^t(i')$ for all $i' \neq i \in I^{t-1} \cup I^t$,
3. $|\mu^t(s)| = r_s$, $\left|\mu^{t+1}(s')\right| < r_{s'}$, and, for some $j \in \mu^t(s)$, $i \rhd_s^t (\mu^{t-1})\, j$,
4. $|\mu^t(s)| = r_s$, $\left|\mu^{t+1}(s')\right| = r_{s'}$, for some $j \in \mu^t(s)$, $j' \in \mu^{t+1}(s')$ and for any $\bar{\mu}^t \in M(i,j,\mu)$, $i \rhd_s^t (\mu^{t-1})\, j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t)\, j'$.

Case 1. $s = s'$. Consequently, $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$. In addition, $|\mu^t(s)| < r_s$ (conditions 1 or 2) or/and $i \rhd_s^t (\mu^{t-1})\, j$ for some $j \in \mu^t(s)$ (conditions 3 or 4). Combining this with μ being such that it satisfies autarkic stability, one obtains that $(\mu^t(i), \mu^{t+1}(i)) \succ_i (s, \mu^{t+1}(i))$. Given weak rankability, this, in turn, implies that if $\mu^t(i) \neq \mu^{t+1}(i)$ then $(\mu^t(i), \mu^t(i)) \succ_i (s,s)$. Then, by transitivity of preferences, $(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$. This implies that μ does not satisfy autarkic stability because child i has the highest priority at school s at period $t+1$, hence, at $t+1$, she has a right to attend school s ahead of any other child. Therefore, $\mu^t(i) = \mu^{t+1}(i)$. This is the condition we seek.
Case 2. $s \neq s'$ and $\mu^t(i) = \mu^{t+1}(i)$. Consequently, $(s,s') \succ_i (\mu^t(i), \mu^t(i))$. In addition, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})\, j$ for some $j \in \mu^t(s)$. Combining this with the fact that μ satisfies autarkic stability, one obtains $(\mu^t(i), \mu^t(i)) \succ_i (s, \mu^t(i))$. Recall that $(s,s') \succ_i (\mu^t(i), \mu^t(i))$. Hence, by transitivity,

$(s,s') \succ_i (s,\mu^t(i))$. Then, by rankability, $(s',s') \succ_i (\mu^t(i),\mu^t(i))$. Suppose $(s,s) \succ_i (s',s')$. Then $(s,s) \succ_i (\mu^t(i),\mu^t(i))$and, by assumption, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})\, j$ for some $j \in \mu^t(s)$. Hence, we have identified a pair (s,i) asked in the lemma.

Now suppose $(s',s') \succ_i (s,s)$. Since μ satisfies autarkic stability, at least one of the two conditions must hold: (a) $(\mu^t(i),\mu^t(i)) \succ_i (\mu^t(i),s')$ or/and (b) $|\mu^{t+1}(s')| = r_{s'}$ and there exists no $j' \in \mu^{t+1}(s')$ such that $i \rhd_{s'}^{t+1} (\mu^t)\, j'$.

Suppose (a) occurs. Recall $(s,s') \succ_i (\mu^t(i),\mu^t(i))$, hence, $(s,s') \succ_i (\mu^t(i),s')$. Then rankability implies that $(s,s) \succ_i (\mu^t(i),\mu^t(i))$ because $s \neq s'$. Observe that the pair (s,i) is the pair asked in the lemma as we already pointed out that $(s,s) \succ_i (\mu^t(i),\mu^t(i))$, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})\, j$ for some $j \in \mu^t(s)$.

Suppose now (b) occurs but not (a). Recall that one of the 4 conditions listed in the beginning of the proof must be satisfied. Since $|\mu^{t+1}(s')| = r_{s'}$, 1 and 3 are ruled out. If condition 2 is satisfied, then $i \rhd_{s'}^{t+1} (\bar{\mu}^t)\, j'$ for some $j' \in \mu^{t+1}(s')$. Furthermore, $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$. Then, by *IPA*, $i \rhd_{s'}^{t+1} (\mu^t)\, j'$. This a contradiction with b occurring. If condition 4 is satisfied, then there must exist j, j' such that, for any $\bar{\mu}^t \in M(i,j,\mu)$, $i \rhd_s^t (\mu^{t-1})\, j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t)\, j'$. In particular, it must be true for $\bar{\mu}^t$ such that $\bar{\mu}^t(j) = h$. Observe that $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(j) = h$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t)\, j'$. This a contradiction with b occurring.

Case 3. $s \neq s'$ and $\mu^t(i) \neq \mu^{t+1}(i)$. Consequently, $(s,s') \succ_i (\mu^t(i),\mu^{t+1}(i))$. Since μ satisfies autarkic stability, one of the two conditions must hold: (a) $(\mu^t(i),\mu^{t+1}(i)) \succ_i (\mu^t(i),s')$ or/and (b) $|\mu^{t+1}(s')| = r_{s'}$ and no $j' \in \mu^{t+1}(s')$ with $i \rhd_{s'}^{t+1} (\mu^t)\, j'$ exists.

Suppose (a) occurs. Recall that by assumption, in case 3, $(s,s') \succ_i (\mu^t(i),\mu^{t+1}(i))$, hence, $(s,s') \succ_i (\mu^t(i),s')$. rankability and this imply $(s,s) \succ_i (\mu^t(i),\mu^t(i))$. Then, $\left(s,\mu^{t+1}(i)\right) \succ_i \left(\mu^t(i),\mu^{t+1}(i)\right)$ by rankability. Consider the pair (s,i). As pointed out earlier, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})\, j$ for some $j \in \mu^t(s)$. This means that μ does not satisfy autarkic stability which is a contradiction.

Suppose now (b) occurs but not (a), therefore $(\mu^t(i),s') \succ_i (\mu^t(i),\mu^{t+1}(i))$. Recall that $(s,s') \succ_i (\mu^t(i),\mu^{t+1}(i))$. In addition, one of the 4 conditions listed in the beginning of the proof must be satisfied. Since $|\mu^{t+1}(s')| = r_{s'}$, 1 and 3 are ruled out. If condition 2 is satisfied, then $i \rhd_{s'}^{t+1} (\bar{\mu}^t)\, j'$ for some $j' \in \mu^{t+1}(s')$. Furthermore, $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t)\, j'$. This is a contradiction with (b) occurring. If condition 4 is satisfied, then there must exist j, j' such that, for any $\bar{\mu}^t \in M(i,j,\mu)$, $i \rhd_s^t (\mu^{t-1})\, j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t)\, j'$. Fix $\bar{\mu}^t$ such that $\bar{\mu}^t(j) = h$. Observe that $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(j) = h$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t)\, j'$. This is a contradiction with (b) occurring. □

Next we show that the stability concept for the our dynamic problem is in fact equivalent to the static concept of stability for a large class of problems. Precisely, if the children's preferences are *strongly rankable* and the schools' priorities satisfy *IPA*, the two concepts are equivalent.

Theorem 3 (Equivalence of Autarkic Stability and Stability)**.** *Suppose every child's preferences satisfy* strong rankability *and every school's priorities satisfy* IPA. *Then matching* μ *is stable if and only if it satisfies autarkic stability..*

Proof. By definition, any stable matching satisfies autarkic stability. Hence, we need to show that any matching that satisfies autarkic stability is stable. Suppose otherwise, i.e., there exists a matching μ which satisfies autarkic stability but is not stable. By Lemma 2, if μ satisfies autarkic stability but is not stable, then for some period $t \geq 1$ and some school-child pair (s,i),

1. $\mu^t(i) = \mu^{t+1}(i)$,
2. $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$,
3. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})\, j$ for some $j \in \mu^t(s)$.

Clearly, $(s,s) \succ_i (\mu^t(i), \mu^t(i))$. Moreover, each child's preferences are *strongly rankable*, hence, $(s, \mu^t(i)) \succ_i (\mu^t(i), \mu^t(i))$. By combining this with the 3rd condition above, one obtains that μ does not satisfy autarkic stability. □

Appendix B: Properties of the Gale and Shapley Matching

First, we show that, in contrast to static problems, there could be multiple stable matchings that do not Pareto dominate one another. The following example illustrates this point.

Example 7 (The DA-IP matching does not Pareto dominate other stable matchings in some markets.)**.** *There are 3 schools* $\{s, s_1, s_2\}$. *All schools have a capacity of one child. There is no school-age child until period* $t-1 \geq 1$. *At period* $t-1$, *only one child i is 1 year old. At period t, there are 2 one-year old children* $\{i_1, i_2\}$. *At period* $t+1$, *child* i' *is 1 year old. If children* $\bar{\imath} \neq \bar{\imath}' \in \{i, i_1, i_2, i'\}$ *have not attended school* $\bar{s} = s, s_1, s_2$ *in the previous period, then school* $\bar{s}$ *ranks child* $\bar{\imath}$ *and child* $\bar{\imath}'$ *according to the following rankings.*

$$
\begin{array}{ccccccc}
i & \rhd_s & i_1 & \rhd_s & i_2 & \rhd_s & i' \\
i & \rhd_{s_1} & i' & \rhd_{s_1} & i_2 & \rhd_{s_1} & i_1 \\
i & \rhd_{s_2} & i_1 & \rhd_{s_2} & i_2 & \rhd_{s_2} & i'
\end{array}
$$

Each child's preferences are strongly rankable. *Child i's top choice is* (s,s). *The preferences of children* i_1, i_2 *and* i' *satisfy the following conditions:*

$$\begin{array}{ccccc} (s_1,s_1) & \succ_{i_1} & (s_2,s_2) & \succ_{i_1} & (s,s), \\ (s,s) & \succ_{i_2} & (s_2,s_2) & \succ_{i_2} & (s_1,s_1), \\ (s_1,s_1) & \succ_{i'} & (s_2,s_2) & \succ_{i'} & (s,s). \end{array}$$

The DA-IP matching μ_{DA} *is as follows:* $\mu_{DA}^{t-1}(i) = \mu_{DA}^{t}(i) = s$, $\mu_{DA}^{t}(i_1) = \mu_{DA}^{t+1}(i_1) = s_1$, $\mu_{DA}^{t}(i_2) = s_2$, $\mu_{DA}^{t+1}(i_2) = s$, $\mu_{DA}^{t+1}(i') = s_2$ *and* $\mu_{DA}^{t+2}(i') = s_1$. *Thanks to Theorem 1,* μ_{DA} *satisfies autarkic stability.*

Now let us consider the following matching $\bar{\mu}$*:* $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t}(i) = s$, $\bar{\mu}^{t}(i_1) = \bar{\mu}^{t+1}(i_1) = s_2$, $\bar{\mu}^{t}(i_2) = s_1$, $\bar{\mu}^{t+1}(i_2) = s$, $\bar{\mu}^{t+1}(i') = s_1$ *and* $\bar{\mu}^{t+2}(i') = s_1$. *It easy to check* $\bar{\mu}$ *is stable.*

Now observe that matching μ_{DA} *does not Pareto dominate matching* $\bar{\mu}$ *because child* i' *prefers* $\bar{\mu}$ *to* μ. *In fact,* $\bar{\mu}$ *is not Pareto dominated by any stable matching. To see this, observe that the only matching that Pareto dominates* $\bar{\mu}$ *is the one in which children 1 and 2 switch their matches in period* t. *But this is not stable because child* i_1 *justifiably envies child* i' *at* $t+1$. ◇

First observe that in Example 7 both *IPA* and *strong rankability* are satisfied. Hence, stability coincides with autarkic stability. The example above shows that there may exist mechanisms that produce stable matchings not Pareto dominated by the DA-IP matching. This is the first main distinction between the matching produced by the DA-IP algorithm in the school choice problem versus the matching produced by the DA-IP algorithm in the dynamic problem of assigning children to daycare centers.

Given the importance of this result when compared to the static case, we state the result below.

Theorem 4. *The DA-IP matching does not necessarily Pareto dominate all stable matchings.*

In light of Theorem 4, one must explore whether any stable matching Pareto dominates the DA-IP matching. This, indeed, is impossible which we show in the following proposition.

Proposition 3 (The DA-IP matching is not Pareto dominated by any stable matching)**.** *If each school's priority rankings satisfy* IPA*, then the DA-IP matching is not Pareto dominated by any other stable matchings.*

Proof of Proposition 3. Fix a market $M = (I, \bar{S}, \mu^0, \succ, \rhd)$, and recall each matching in this market has the common period 0 matching, μ^0.

On contrary to the proposition, suppose that some stable matching μ Pareto dominates matching μ_{DA}.

Step 1. If $i \in I_0$, then $\mu_{DA}^{1}(i) = \mu^{1}(i)$.

Proof of Step 1. For any 2 year old child, her isolated preference is $\succ_i^2(\mu^0)$. From Lemma 1, we have that μ^1_{DA} and μ^1 are stable period 1 matchings under isolated preferences and μ^0. Gale and Shapley (1962) show that μ^1_{DA} Pareto dominates every other statically stable period 1 matchings under isolated preferences and μ^0 in terms of isolated preferences. This means $\mu^1_{DA}(i) \succ^2_i (\mu^0)\mu^1(i)$ if $\mu^1_{DA}(i) \neq \mu^1(i)$. By definition of $\succ^2_i (\mu^0)$, $(\mu^0(i), \mu^1_{DA}(i)) \succ_i (\mu^0(i), \mu^1(i))$ if $\mu^1_{DA}(i) \neq \mu^1(i)$. Hence, if μ Pareto dominates μ_{DA}, then $\mu^1_{DA}(i) = \mu^1(i)$.

Step 2. If $i \in I_1$, then $\mu^1_{DA}(i) = \mu^1(i)$.

Proof of Step 2. Suppose $\mu^1_{DA}(i) \neq \mu^1(i)$ for some $i \in I_1$. Then, as in the proof of step 1, we obtain that $\mu^1_{DA}(i) \succ^1_i \mu^1(i)$ or equivalently,

$$(\mu^1_{DA}(i), \mu^1_{DA}(i)) \succ_i (\mu^1(i), \mu^1(i)). \tag{1}$$

The stability of μ_{DA} implies $(\mu^1_{DA}(i), \mu^2_{DA}(i)) \succeq_i (\mu^1_{DA}(i), \mu^1_{DA}(i))$; otherwise, μ_{DA} does not satisfy autarkic stability as child i is in the highest priority group in period 2. Now weak rankability yields $(\mu^2_{DA}(i), \mu^2_{DA}(i)) \succeq_i (\mu^1_{DA}(i), \mu^1_{DA}(i))$. Now it is easy to see that

$$(\mu^2_{DA}(i), \mu^2_{DA}(i)) \succeq_i (\mu^1_{DA}(i), \mu^2_{DA}(i)) \succeq_i (\mu^1_{DA}(i), \mu^1_{DA}(i)). \tag{2}$$

Similarly, as μ is stable, we obtain

$$(\mu^2(i), \mu^2(i)) \succeq_i (\mu^1(i), \mu^2(i)) \succeq_i (\mu^1(i), \mu^1(i)). \tag{3}$$

Now let us show that $\mu^1(i) \neq \mu^2(i)$. Suppose otherwise. Then relations 1 and 2 yield that $(\mu^1_{DA}(i), \mu^2_{DA}(i)) \succ_i (\mu^1(i), \mu^1(i))$. This contradicts with μ Pareto dominating μ_{DA}. Hence, $\mu^1(i) \neq \mu^2(i)$. Consequently, the preference relations in 3 must be strict. Also observe that $\mu^1(i) \neq \mu^2_{DA}(i)$ thanks to relations 1 and 3.

Now let us show that $(\mu^2(i), \mu^2(i)) \succ_i (\mu^2_{DA}(i), \mu^2_{DA}(i))$. If not, rankability and relation 1 yield that $(\mu^1_{DA}(i), \mu^2_{DA}(i)) \succeq_i (\mu^1(i), \mu^2_{DA}(i))$ and $(\mu^1(i), \mu^2_{DA}(i)) \succeq_i (\mu^1(i), \mu^2(i))$ as $\mu^1(i) \neq \mu^2_{DA}(i)$ and $\mu^1(i) \neq \mu^2(i)$. Consequently, $(\mu^1_{DA}(i), \mu^2_{DA}(i)) \succeq_i (\mu^1(i), \mu^2(i))$ which contradicts that μ Pareto dominates μ_{DA}. Now let us summarize the preference relation we found so far.

$$(\mu^2(i), \mu^2(i)) \succ_i (\mu^2_{DA}(i), \mu^2_{DA}(i)) \succeq_i (\mu^1_{DA}(i), \mu^1_{DA}(i)) \succ_i (\mu^1(i)), \mu^1(i)) \tag{4}$$

From Lemma 1, we know that μ^2 is statically stable under isolated preferences and μ^1. Now suppose we ran the DA-IP algorithm at period 1 under isolated preferences and μ^1. Let us denote the resulting matching $\bar{\mu}^2$. From Gale and Shapley (1962), we know that if $\bar{\mu}^2(i) \neq \mu^2(i)$, then $\bar{\mu}^2(i) \succ^2_i (\mu^1)\mu^2(i)$. In other words, $(\mu^1(i), \bar{\mu}^2(i)) \succeq_i (\mu^1(i), \mu^2(i))$. This along with relation 1 and $\mu^1(i) \neq \mu^2(i)$ implies that $\bar{\mu}^2(i) \neq \mu^1(i)$. Then by rankability, $(\mu^1(i), \bar{\mu}^2(i)) \succ_i (\mu^1(i), \mu^2(i))$ implies

$(\bar{\mu}^2(i),\bar{\mu}^2(i)) \succ_i (\mu^2(i),\mu^2(i))$. Now let us update relation 4.

$$(\bar{\mu}^2(i),\bar{\mu}^2(i)) \succ_i (\mu^2(i),\mu^2(i)) \\ \succ_i (\mu^2_{DA}(i),\mu^2_{DA}(i)) \succ_i (\mu^1_{DA}(i),\mu^1_{DA}(i)) \succ_i (\mu^1(i)),\mu^1(i)) \tag{5}$$

Next we will proceed to show that $\bar{\mu}^2$ is statically stable under isolated preferences and μ^1_{DA}. Let us postpone the proof momentarily to discuss its implications. From Lemma 1, we know that μ^2_{DA} is a stable matching under isolated preferences and μ^1_{DA}. In addition, it must Pareto dominate $\bar{\mu}^2$ in terms of the isolated preferences, since $\bar{\mu}^2$ is statically stable and the μ^2_{DA} must Pareto dominate all stable matchings (see Gale and Shapley (1962)). Hence, if $\mu^2_{DA}(i) \neq \bar{\mu}^2(i)$, then $\mu^2_{DA}(i) \succ^2_i (\mu^1_{DA})\bar{\mu}^2(i)$. By the definition of $\succ^2_i (\mu^1_{DA})$, $(\mu^1_{DA}(i),\mu^2_{DA}(i)) \succ_i (\mu^1_{DA}(i),\bar{\mu}^2(i))$. Recalling that $(\mu^1_{DA}(i),\mu^1_{DA}(i)) \succ_i (\mu^1(i),\mu^1(i))$, we find that $(\mu^1_{DA}(i),\bar{\mu}^2(i)) \succ_i (\mu^1(i),\bar{\mu}^2(i))$. Weak rankability and $(\bar{\mu}^2(i),\bar{\mu}^2(i)) \succ_i (\mu^2(i),\mu^2(i))$ yield $(\mu^1(i),\bar{\mu}^2(i)) \succ_i (\mu^1(i),\mu^2(i))$. The previous three relations yield $(\mu^1_{DA}(i),\mu^2_{DA}(i)) \succ_i (\mu^1(i),\mu^2(i))$. However, recall that μ Pareto dominates μ_{DA}. This is the contradiction we are looking for. Thus, to complete the proof, it is left to show that $\bar{\mu}^2$ is statically stable under isolated preferences and μ^1_{DA}.

We now proceed to show that $\bar{\mu}^2$ is indeed a stable matching under isolated preferences and μ^1_{DA}. We already know from Assumption 1 and (5) that, for all $i \in I_1$, $\bar{\mu}^2(i) \succ^2_i (\mu^1)\mu^2(i)$ if $\bar{\mu}^2(i) \neq \mu^2(i)$. Also, from Gale and Shapley (1962), we know that, for all $i \in I_2$, $\bar{\mu}^2(i) \succ^1_i \mu^2(i)$ if $\bar{\mu}^2(i) \neq \mu^2(i)$. Recall that $\bar{\mu}^2$ is statically stable matching under isolated preferences and μ^1. Now consider the isolated preferences in period 1 from μ^1_{DA} and suppose, under these isolated preferences, $\bar{\mu}^2$ is not stable. Therefore, there must exist a school-child pair (s,i) such that both conditions are satisfied:

I. – if $i \in I_1$, then $s \succ^2_i (\mu^1_{DA})\bar{\mu}^2(i)$, or
   – if $i \in I_2$, then $s \succ^2_i \bar{\mu}^2(i)$;

II. $|\bar{\mu}^2(s)| < |r_s|$ or/and $i \rhd^2_s (\mu^1_{DA})j$ for some $j \in \bar{\mu}^2(s)$.

Because $\bar{\mu}^2$ statically stable under the isolated preferences and μ^1, the conditions 1 and 2 below cannot be satisfied at the same time.

1. (a) if $i \in I_1$, then $s \succ^2_i (\mu^1)\bar{\mu}^2(i)$, or
   (b) if $i \in I_2$, then $s \succ^1_i \bar{\mu}^2(i)$.
2. $|\bar{\mu}^2(s)| < r_s$ or/and $i \rhd^2_s (\mu^1)j$ for some $j \in \bar{\mu}^2(s)$.

Suppose $i \in I_1$. Then $s \succ^2_i (\mu^1_{DA})\bar{\mu}^2(i)$. We show that in this case condition $1\,(a)$ is satisfied. By the definition of $\succ^2_i (\mu^1_{DA})$,

$$(\mu^1_{DA}(i),s) \succ_i (\mu^1_{DA}(i),\bar{\mu}^2(i)).$$

If $\mu^1(i) = \mu^1_{DA}$, then

$$(\mu^1(i), s) \succ_i (\mu^1(i), \bar{\mu}^2(i)).$$

This means that condition 1a is satisfied. Let $\mu^1(i) \neq \mu^1_{DA}$. Then preference relations given in (5), Assumption 1,

$$(\mu^1_{DA}(i), s) \succ_i (\mu^1_{DA}(i), \bar{\mu}^2(i))$$

and the fact that

$$(s, s) \succ_i \left(\bar{\mu}^2(i), \bar{\mu}^2(i)\right)$$

imply that

$$(\mu^1(i), s) \succ_i (\mu^1(i), \bar{\mu}^2(i)).$$

Hence, condition $1\,(a)$ is satisfied. Suppose $i \in I_2$. Then $s \succ^1_i \bar{\mu}^2(i)$. Since $\succ^1$ does not depend on the last period's matching, condition $1\,(b)$ is satisfied. Therefore, we find that either $1\,(a)$ or $1\,(b)$ is satisfied. This means that 2 cannot be satisfied. Clearly, it must be that $|\bar{\mu}^2(s)| = r_s$. This implies that school s's priority ranking must satisfy $i \rhd^2_s (\mu^1_{DA})\, j$ and $j \rhd^2_s (\mu^1) i$, for at least some $j \in \bar{\mu}^2(s)$. There are 2 cases consider:

(i) $i \notin \mu^1_{DA}(s)$, or

(ii) $i \in \mu^1_{DA}(s)$ and $i \in I_1$.

If case (i) happens, this implies that $j \notin \mu^1_{DA}(s)$; otherwise, j would have the highest priority at school s, hence, we reach a contradiction with $i \rhd^2_s (\mu^1_{DA})\, j$. Therefore, $j \notin \mu^1_{DA}(s)$. Since school s's priority ranking satisfies *IPA*, given that $i \rhd^2_s (\mu^1_{DA})\, j$ it must be that $j \in \mu^1(s)$ and $j \in I_1$ to have the required reversal of school s's priority ranking. Then $\mu^1_{DA}(j) \neq \mu^1(j)$. This, as argued earlier in step 1, implies that $(\mu^1_{DA}(j), \mu^1_{DA}(j)) \succ_j (\mu^1(j), \mu^1(j)) = (s, s)$, where the last equality comes from the fact above, that if $j \notin \mu^1_{DA}(s)$, it must be that $j \in \mu^1(s)$. Now recall that $j \in \bar{\mu}^2(s)$. Therefore,

$$(\mu^1_{DA}(j), \mu^1_{DA}(j)) \succ_j (\mu^1(j), \bar{\mu}^2(j))$$

which is a contradiction (see preference relation 5).

Suppose (ii) happens, $i \in \mu^1_{DA}(s)$, i.e., $s = \mu^1_{DA}(i)$. We know $s \succ^2_i (\mu^1_{DA})\bar{\mu}^2(i)$. These conditions yield

$$(\mu^1_{DA}(i), \mu^1_{DA}(i)) \succ_i (\mu^1_{DA}(i), \bar{\mu}^2(i)).$$

This is a contradiction which we are looking for.

This completes the proof of step 2.

Step 3. The DA-IP algorithm yields a stable matching that is not Pareto dominated by any other stable matchings.

Proof of Step 3. Proving step 3 is just a matter of reiterating the arguments of steps 1 and 2 assuming previous periods' matchings are identical with the ones resulted from the DA-IP algorithm. □

Now we study if any stable matching is efficient. The next proposition yields that unless one follows the DA-IP algorithm, then any stable matching is not efficient.

Proposition 4. *Consider any market in which the schools' priorities satisfy* IPA. *Then any stable matching different from the DA-IP matching is not efficient.*

Proof of Proposition 4. Consider any stable matching μ with some period $t \geq 1$ matching that is different from the one that the DA-IP algorithm under isolated preferences and μ^{t-1} yields. Consider any $i \in I_t$. Then $\mu^t(i) = \mu^{t+1}(i)$ or

$$(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i));$$

otherwise, μ is not stable because, in this case, child i would have the higher priority at school $\mu^t(i)$ and

$$(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$$

by Assumption 1.

For each child $i \in I_{t-1} \cup I_t$, define her preference relation to be $\mathcal{P}_i^t$ such that $s\mathcal{P}_i^t s'$ if and only if

$$(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), s') \text{ whenever } i \in I_{t-1}$$

$$(s, \mu^{t+1}(i)) \succ_i (s', \mu^{t+1}(i)) \text{ whenever } i \in I_t$$

Because μ is stable, there cannot exist any school-child pair (s,i) such that

1. $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$ or $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$,
2. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

In terms of $\mathcal{P}$, these conditions mean that there is no school-child pair (s,i) such that

1. $s\mathcal{P}_i^t \mu^t(i)$,
2. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

In other words, μ^t is a statically stable matching under $\mathcal{P}$ and μ^{t-1}.

Consider matching $\bar{\mu}$ such that $\bar{\mu}^\tau = \mu^\tau$ for all $\tau \neq t$ but $\bar{\mu}^t$ is the resulting matching from the DA-IP algorithm under $\mathcal{P}$ and μ^{t-1}.

From Gale and Shapley (1962), we know that $\bar{\mu}^t$ must Pareto dominate every other stable matching under $\mathcal{P}$ and μ^{t-1}. This and that μ^t is a statically stable matching under $\mathcal{P}$ and μ^{t-1} imply that $\bar{\mu}^t(i)\mathcal{P}_i\mu^t(i)$ for all $i \in I_{t-1} \cup I_t$ if $\bar{\mu}^t(i) \neq \mu^t(i)$. Consequently, if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_{t-1}$, then $(\mu^{t-1}(i), \bar{\mu}^t(i)) \succ_i (\mu^{t-1}(i), \mu^t(i))$. Similarly, if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_t$ then

$$(\bar{\mu}^t(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i)).$$

Now consider $\bar{\mu}$ and μ. Clearly, $\bar{\mu}$ Pareto dominates μ if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_{t-1} \cup I_t$. Hence, it must be that $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$.

Consider $\hat{\mu}$ such that $\hat{\mu}^\tau = \mu^\tau$ for all $\tau \neq t$ but $\hat{\mu}^t$ is the resulting matching from the DA-IP algorithm under isolated preferences and $\hat{\mu}^{t-1}$. Clearly, $\bar{\mu}^{t-1} = \hat{\mu}^{t-1}$, hence, the priority rankings of the schools are the same under both $\bar{\mu}$ and $\hat{\mu}$. In addition, for each $j \in I_{t-1}$, the isolated preference relation $\succ_j^2 (\mu^{t-1})$ is equivalent to $\mathcal{P}_j$. Now consider any child $j \in I_t$. Then under $\mathcal{P}$, the relative ranking of $\mu^{t+1}(j)$ weakly improves from the one under $\succ_j^1$. In all other aspects, $\mathcal{P}_j$ and $\succ_j^1$ are the same. Now recall that $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$. In addition, recall that $\mu^t(i) = \mu^{t+1}(i)$ or

$$(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i)).$$

Therefore, under both $\mathcal{P}_j$ and $\succ_j^1$, the set of schools that are strictly preferred to $\mu^t(j)$ is the same. Consequently, we obtain that under $\mathcal{P}$ and isolated preferences, for each $j \in I^{t-1} \cup I^t$, the set of schools that are strictly preferred to $\mu^t(j)$ is the same. In addition, because the DA-IP algorithm is used for both cases and $\bar{\mu}^t(j) = \mu^t(j)$ for all $j \in I_{t-1} \cup I_t$, it must be $\bar{\mu}^t = \hat{\mu}^t$ thanks to Theorem 9 in Dubins and Freedman (1981). Consequently, $\mu^t = \hat{\mu}^t$, which contradicts that μ^t differs from the matching that the DA-IP algorithm yields. □

Proposition 4 means that if any stable matching is efficient, then it must be the DA-IP matching. However, from Roth (1982), it is well known that the DA-IP matching (in static settings) is not necessarily Pareto efficient. This is still the case in our setting.

Appendix C: Proofs

Proof of Lemma 1. Necessity. Assume μ is stable. We need to show that for all $t \geq 1$, μ^t is statically stable under isolated preferences and μ^{t-1}. Suppose otherwise. Then there must exist $t \geq 1$ and a school-child pair (s, i) such that

1. if $i \in I_t$, then $s \succ_i^1 \mu^t(i)$ and at least one of the following is satisfied: $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$,

2. if $i \in I_{t-1}$, then $s \succ_i^2 \left(\mu^{t-1}\right) \mu^t(i)$ and at least one of the following is satisfied: $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Suppose $i \in I_t$. Then we are in case 1. Since μ satisfies autarkic stability, the following 2 conditions cannot be satisfied at the same time: (a) $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$ and (b) $|\mu^t(s)| < r_s$ and/or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. If (b) is not true, then this is a contradiction because (s,i) must satisfy the conditions given in case 1. Hence, assume that (b) is satisfied but (a) is not, i.e., $(\mu^t(i), \mu^{t+1}(i)) \succ_i (s, \mu^{t+1}(i))$. If $\mu^t(i) \neq \mu^{t+1}(i)$, Assumption 1 implies that $(\mu^t(i), \mu^t(i)) \succ_i (s,s)$. By the definition of $\succ^1$, $\mu^t(i) \succ_i^1 s$ which contradicts with the assumption that $s \succ_i^1 \mu^t(i)$. Suppose $\mu^t(i) = \mu^{t+1}(i)$. Recall that $s \succ_i^1 \mu^t(i)$, hence, $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Recall that (b) is satisfied. Thus, by moving to school s in period t, child i would have the highest priority at school s at time $t+1$. Hence, μ is not stable. Hence, $i \notin I_t$.

Suppose $i \in I_{t-1}$. Then we are in case 2. Because μ satisfies autarkic stability, the following 2 conditions cannot be satisfied at the same time: (a) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$ and (b) $|\mu^t(s)| < r_s$ and/or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. If (b) is not true, then this is a contradiction because (s,i) must satisfy the conditions given in case 2. Hence, (b) must be satisfied but (a) is not, i.e., $(\mu^{t-1}(i), \mu^t(i)) \succ_i (\mu^{t-1}(i), s)$. By the definition of $\succ_i^2 (\mu^{t-1})$, we have that $\mu^t(i) \succ_i^2 (\mu^{t-1}) s$ which contradicts with the assumption that $s \succ_i^2 \left(\mu^{t-1}\right) \mu^t(i)$. Hence, $i \notin I_{t-1}$. Therefore, for all t, μ^t is statically stable under isolated preferences and μ^{t-1}.

Sufficiency. For any $t \geq 1$, μ^t is statically stable under isolated preferences and μ^{t-1}. First let us show that μ satisfies autarkic stability. Suppose otherwise. Then, at some period $t \geq 1$, there must exist a pair (s,i) such that one of the two conditions below is satisfied:

1. (a) $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$, and

   (b) $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

   or

2. (a) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$, and

   (b) $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Suppose case 1 occurs. If $s \neq \mu^{t+1}(i)$, then rankability and

$$(s, \mu^{t+1}(i)) \succ_i \left(\mu^t(i), \mu^{t+1}(i)\right)$$

yield $(s,s) \succ_i (\mu^t(i), \mu^t(i))$. By definition of $\succ_i^1$, we have that $s \succ_i^1 \mu^t(i)$. This and $1b$ mean that μ^t is not statically stable under isolated preferences and μ^{t-1}. This is a contradiction. Suppose, on the other hand, that $s = \mu^{t+1}(i)$. If

$$(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i)),$$

then the definition of $\succ_i^1$ yields $\mu^{t+1}(i) \succ_i^1 \mu^t(i)$. This and 1b mean that μ^t is not statically stable under isolated preferences and μ^{t-1}.

Suppose $(\mu^t(i), \mu^t(i)) \succ_i (\mu^{t+1}(i), \mu^{t+1}(i))$. This and Assumption 1 yield

$$(\mu^t(i), \mu^t(i)) \succ_i \left(\mu^t(i), \mu^{t+1}(i)\right).$$

Consider period $t+1$. Then by the definition of $\succ_i^2(\mu^t)$, we have that $\mu^t(i) \succ_i^2 (\mu^t)\mu^{t+1}(i)$. In addition, observe that child i has the highest priority at school $\mu^t(i)$. The last 2 conditions contradict that μ^{t+1} is statically stable under isolated preferences and μ^t.

Suppose case 2 occurs. By the definition of $\succ_i^2(\mu^{t-1})$, we have that $s \succ_i^2 (\mu^{t-1})\mu^t(i)$ since $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$. But this and 2*b* directly imply that μ^t is not statically stable under isolated preferences and μ^{t-1}. This is a contradiction.

We have shown that μ satisfies autarkic stability. Now we are left to show that μ is stable if *IPA* is satisfied. Suppose otherwise. Then by Lemma 2, for some period t and some school-child pair (s,i),

1. $\mu^t(i) = \mu^{t+1}(i)$
2. $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$
3. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})\, j$ for some $j \in \mu^t(s)$

The first 2 conditions and the definition of $\succ_i^1$ yield $s \succ_i^1 \mu^t(i)$. This and the third condition imply that μ^t is not statically stable under isolated preferences and μ^{t-1}. □