THE SELF-REFERENTIAL GAMES MINNIE AND WYNNIE AND SOME VARIANTS

by

Todd Mullen

Submitted in partial fulfillment of the requirements
for the degree of Master of Science

at

Dalhousie University
Halifax, Nova Scotia
August 2016

Table of Contents

List of Tables

List of Figures

Abstract

Several self-referential impartial combinatorial games have been studied and some have been solved. However, by making small changes to the rules we can get vastly different outcomes. We look at many new games, solving some, and coming up short in others, but in particular, we look at MINNIE and WYNNIE. With MINNIE representing a small tweak to the rules of NIM and WYNNIE representing a small tweak to the rules of WYTHOFF, one might expect rather uninteresting, repetitive results. However, that is far from the case.

We solve WYNNIE and misère WYNNIE, and we nearly solve MINNIE up to three stacks. In addition, we solve other new games NIMB and NIMTB, and we look at WYNNIE on graphs.

List of Abbreviations and Symbols Used

S_n	Star graph on $n+1$ vertices
P_n	Path on n vertices
C_n	Cycle on n vertices
K_n	Complete graph on n vertices
W_n	Wheel graph on n vertices
$[a_1, a_2, \ldots, a_n]$	A game played on n stacks containing $a_1, a_2, \ldots, a_n$ and chips.
$\rightarrow$	A move that can be made to an option in a game.

Acknowledgements

Richard and Margaret-Ellen

Chapter 1

Introduction

In this thesis, we will focus on self-referential, impartial, combinatorial games, all of which are variants of the game NIM. We define a self-referential game as a game in which the moves available to a player are dependent on some parameter from the board state. So, a self-referential version of NIM places restrictions on the players depending on the stack sizes. In particular, we look at three new games in increasing order of complexity: NIMB (Chapter 3), WYNNIE (Chapter 4), and MINNIE (Chapter 5).

NIMB is a variant of NIM (see Section 1.1). In NIMB, the board is composed of some number of stacks each containing some number of chips. Let the stacks be of sizes $a_1, a_2, \ldots, a_n$. We denote this game NIMB$[a_1, a_2, \ldots, a_n]$. On a player's turn, the player chooses a stack and removes some number of chips from it. However, every move must cause the game to have a new minimum stack value. Stacks of size 0 are eliminated. The game ends when every stack is empty. With Theorem 3.1, we solve NIMB. We also solve NIMTB, a variant of NIMB in which both the maximum stack value and the minimum stack value must change with every move, with Theorem 3.2.

WYNNIE is a variant of Wythoff's game (see Section 1.3). In WYNNIE, the board is composed of two stacks each containing some number of chips. Let the stacks be of sizes m and n with $n \geq m$. We denote this game $W[m, n]$. On a player's turn, the player chooses an integer $k > 0$ and less than or equal to the smallest stack size (in this case m). That player then either removes k chips from one stack or removes k chips from both stacks. The game ends when every stack is empty. With Theorem 4.5, we solve WYNNIE. We also solve misère WYNNIE with Theorem 4.10.

MINNIE is a variant of NIM (see Section 1.1). In MINNIE, the board is composed of some number of stacks each containing some number of chips. Let the stacks be of sizes $a_1, a_2, \ldots, a_n$. We denote this game $M[a_1, a_2, \ldots, a_n]$. On a player's turn, the player chooses an integer greater than 0 and less than or equal to the smallest stack size (in this case a_1) and removes that number of chips from a single stack. The game ends when every stack is empty. With Theorem 5.2 and Theorem 5.3, we solve MINNIE for 2 stacks and nearly solve it for 3 stacks. We also solve MINE, a variant of

MINNIE in which no two stacks can ever have the same number of chips, for 2 and 3 stacks. In both games, four or more stacks remains unsolved.

All unreferenced work is the author's.

1.1 Combinatorial Game Theory Background

In this section, we give the background necessary for this thesis. For more detail, see [2], [3].

NIM is a two player game played on a board consisting of some number of stacks of chips. Players take turns choosing a stack and removing some number of chips from it. The winner is the player who removes the last chips from the board, or equivalently, the loser is the player who has no legal move on his turn. In Figure 1.1, we see an example of NIM played with three stacks containing 3, 4, and 7 chips. We denote this game NIM$[3, 4, 7]$. In this example, we see that Player Two has won; in his last move the removal of two chips from the center stack left Player One with no legal action on his turn.

(3) (4) (7)

Player 1 moves to

(1) (4) (7)

Player 2 moves to

(1) (4) (6)

Player 1 moves to

(1) (4) (0)

Player 2 moves to

(1) (2) (0)

Player 1 moves to

(0) (2) (0)

Player 2 moves to

(0) (0) (0)

Figure 1.1: A game of NIM on stacks of sizes 3, 4, and 7.

All games considered in this thesis are combinatorial games in that there are two players who move alternately and who have perfect information. Also, there are no chance devices and the game must end within a finite number of moves. Note how popular games like Chess and Checkers fit this definition but other popular games like Battleship and Yahtzee fail due to lack of perfect information and use of chance devices, respectively.

We define a *board-state* as a set of stack values in a variant of NIM.

Let G be a game. We will call the two players Left and Right. The set of all positions that Left can move to in one move is called the set of left options and is represented by $G^{\mathscr{L}}$. Similarly, Right can move to any position in $G^{\mathscr{R}}$. Thus, we can identify G with its options and write $G = \{G^{\mathscr{L}} \mid G^{\mathscr{R}}\}$. A *follower* of G is any position that can be reached in a finite number of moves.

Note that in NIM, $G^{\mathscr{L}} = G^{\mathscr{R}}$. This is because NIM is an *impartial game*, or a game in which the two players' sets of options are always equal and the same is true for every follower. So, from every possible board-state, the active player will have the same set of options, regardless of whether it is Left or Right. On the other hand, Checkers is not an impartial game. This is because Left and Right will be assigned their own colour and their own pieces. So, from a given board-state Left may be allowed to move his king, but Right would not be allowed to move Left's king. For this reason, we call Checkers a *partizan* game. All of these games are games of pure strategy in that one player can force a result, as seen in the next theorem.

Theorem 1.1. *(Fundamental Theorem of Combinatorial Games [2]) Fix a game G played between Albert and Bertha, with Albert moving first. Either Albert can force a win moving first, or Bertha can force a win moving second, but not both.*

Proof. Each of Albert's moves is to a position which, by induction, is either a win for Bertha playing first or a win for Albert playing second. If any of his moves belong to the latter category, then by choosing one of them, Albert can force a win. On the other hand, if all of his moves belong to the first category, then Bertha can force a win by using her winning strategy in the position resulting from any of Albert's moves. □

In partizan games, like Chess, Checkers, and Tic-tac-toe, the possible *outcomes* are $\mathcal{L}$, $\mathcal{R}$, $\mathcal{N}$, and $\mathcal{P}$, respectively, these stand for Left player win, Right player win, Next player win, and Previous player win. If a game is Left or Right player win, that means that that player is capable of forcing a win no matter what the other player does, regardless of who goes first. If a game is Next player win, then the next player to move is capable of forcing a win no matter what the other player does. If a game is Previous player win, then the player that moved previously is capable of forcing a win no matter what the other player does. NIM, and all other impartial games, are different in that there is no concept of Left or Right player win games. Every game must be Next player win or Previous player win since every possible board state has the same move set for both players.

In this thesis, we will focus only on impartial games. From now on, all the theorems are applicable to only impartial games. So, the only outcomes we will be mentioning are $\mathcal{N}$ and $\mathcal{P}$. A *terminal position* is a game in which there are no legal moves. So, in NIM, a game with every stack containing 0 chips is a terminal position and it has outcome $\mathcal{P}$. We define a *misère* game as a game in which the winner is the player who has no options on his turn. So, in a misère game, terminal positions are in $\mathcal{N}$.

Suppose we are playing an impartial game $G = \{G^{\mathscr{L}} | G^{\mathscr{R}}\}$. How do we discover if it is $\mathcal{N}$ or $\mathcal{P}$? Since we know $G^{\mathscr{L}} = G^{\mathscr{R}}$, we will use $\{G^{\mathscr{I}}\}$ for impartial game sets. So, $G = \{G^{\mathscr{I}}\}$. There is a recursive method for finding the outcome class.

Theorem 1.2. *Let $G = \{G^{\mathscr{I}}\}$ be an impartial game. If the outcome of any $G' \in G^{\mathscr{I}}$ is in $\mathcal{P}$, then the outcome of G is $\mathcal{N}$, otherwise the outcome of G is $\mathcal{P}$.*

Proof. If the next player has a move to a $\mathcal{P}$ position, then the next player can force a win, that is, $G \in \mathcal{N}$. If the next player cannot move to a $\mathcal{P}$ position, all options are in $\mathcal{N}$ then the previous player in G can force a win. □

We will use the following theorem often in our proofs and therefore, refer to Theorem 1.3 as the **Partition Theorem for Impartial Games (PT)**.

Theorem 1.3. *(Theorem 2.13 from [2]): Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets A and B with the properties: (i) every option of a position in A is in B; and (ii) every position in B has at least one option in A. Then A is the set of $\mathcal{P}$-positions and B is the set of $\mathcal{N}$-positions.*

We include the proof to give the reader an idea how impartial game proofs are often conducted.

Proof. We begin with the terminal positions which are in $\mathcal{P}$. There are no options from the terminal positions, so they must be in set A. Next, we note that every game with a terminal position as an option is in B and is in $\mathcal{N}$. We now induct on the options, taking turns placing every game with all of its options in B into set A noting that these games must also be in $\mathcal{P}$, and then placing every game with at least one option in A into set B noting that these games must also be in $\mathcal{N}$. Since the two sets are defined to be mutually exhaustive, no game will be excluded from both sets. So, we get that A is the entire set of $\mathcal{P}$ positions and B is the entire set of $\mathcal{N}$ positions. □

Note how this theorem changes in misère play. The terminal positions are in $\mathcal{N}$. So, (ii) becomes every *non-terminal* position in B has at least one option in A, for misère play.

A *disjunctive sum* of games is a single game composed of a number of disjoint games. During a player's turn, he may act on any of the disjoint games. A player wins not when his opponent cannot play on a single game, but when his opponent cannot play on any game. Formally, for impartial games $G = \{G^{\mathscr{I}}\}$ and $H = \{H^{\mathscr{I}}\}$, then $G + H = \{G^{\mathscr{I}} + H, G + H^{\mathscr{I}}\}$. Intuitively, this operation is commutative and associative (it doesn't matter how you orient the boards, you still must complete both and can always act on either).

1.2 Sprague-Grundy Theory and NIM

Discovered independently by R.P. Sprague and P.M. Grundy, the Sprague-Grundy Theorem states that every impartial game is equivalent to an instance of NIM played on a single stack of some size (see [2] for more on Sprague-Grundy Theory). The size of this stack is called that game's equivalent *nimber*. The Sprague-Grundy Theorem also gives an algorithm for finding this number even in the case of a disjunctive sum. Note that Player One has the obvious winning strategy in any game of NIM consisting of only one stack of removing the entire stack. So, any game with nimber $n > 0$ is in $\mathcal{N}$. However, conversely, any game with nimber $n = 0$ is in $\mathcal{P}$.

As defined in [2], the *nim-sum* of numbers $a, b, \ldots, k$ written $a \oplus b \oplus \cdots \oplus k$ is obtained by adding the numbers in binary without carrying. For example, to compute the nim-sum of $3 \oplus 31 \oplus 21$:

$$\begin{aligned} 3 &= 00011 \\ \oplus 31 &= 11111 \\ \oplus 21 &= 10101 \\ &= 01001 \\ &= 9. \end{aligned} \tag{1.1}$$

Theorem 1.4. *[2] Let $G = \text{NIM}(a, b, \ldots, k)$. Then G is a $\mathcal{P}$-position if and only if $a \oplus b \oplus \cdots \oplus k = 0$.*

The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set (Definition 7.6 from [2]). This is denoted by $mex\{a, b, c, \ldots, k\}$. We use the minimum excluded value to determine the equivalent nimber of a combinatorial game, denoted $\mathscr{G}(G)$, by examining the set of nimbers of possible moves from a given board-state. For example, if an impartial game G has options with nimbers 0, 0, 2, and 5, then G cannot be equivalent to any nim stack with size greater than 1, because Player One does not have the option to move to a game with nimber 1. So, the equivalent nimber of G, or the $mex\{G^{\mathscr{I}}\}$, or $\mathscr{G}(G)$, is 1. Note that $\mathscr{G}(\emptyset) = mex\{\emptyset\} = 0$.

Theorem 1.5. *[2] Let G and H be impartial games.*

1. $\mathscr{G}(G) = mex\{\mathscr{G}(G') : G' \text{is an option of } G\}$.

2. $\mathscr{G}(G + H) = \mathscr{G}(G) \oplus \mathscr{G}(H)$.

3. $\mathscr{G}(G) = 0 \Leftrightarrow G \in \mathcal{P}$.

1.3 Wythoff's Game

In the NIM variant, Wythoff's game, the board-state is equivalent to that of NIM played on two stacks, but the list of legal moves now includes the removal of the same number of chips from both stacks. So, in Wythoff's game played on stacks of sizes m, n with $m \leq n$, the active player may remove up to m chips from one or both stacks or remove up to n chips from the stack of size n. In Figure 1.2, we see an example of Wythoff's game played with stacks of sizes 6 and 12. In this example, we see that Player One has won since his removal of one chip from both stacks left Player Two with no legal action on his turn.

(6) (12)

Player 1 moves to

(2) (8)

Player 2 moves to

(2) (5)

Player 1 moves to

(1) (5)

Player 2 moves to

(1) (1)

Player 1 moves to

(0) (0)

Figure 1.2: Wythoff's game on stacks of sizes 6 and 12.

The $\mathcal{P}$-positions of Wythoff's game are known [9]. Firstly, $[0,0]$ is a terminal position and in $\mathcal{P}$. With this knowledge, we can determine that all games of the form $[0,n]$, $[n,0]$, and $[n,n]$ are in $\mathcal{N}$ since $[0,0]$ is an option. So, any game G with a stack size of 0 is in $\mathcal{N}$ since G is a NIM move away from $[0,0]$. Also, any game G in which the two stacks have a difference of 0 is in $\mathcal{N}$ since G is a WYTHOFF diagonal move away from $[0,0]$. These relations between games can be equated to rows, columns, and diagonals in a table. In Table 1.1, we can start to see how much easier it can be determine the outcome of a game with a graphic representation. In our tables, we will denote Next Player win games with $\circ$ and Previous Player win games with $\mathcal{P}$.

8	$\circ$								$\circ$
7	$\circ$							$\circ$	
6	$\circ$						$\circ$		
5	$\circ$					$\circ$			
4	$\circ$				$\circ$				
3	$\circ$			$\circ$					
2	$\circ$		$\circ$						
1	$\circ$	$\circ$							
0	$\mathcal{P}$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
	0	1	2	3	4	5	6	7	8

Table 1.1: Partial Table of WYTHOFF outcomes.

Since, $[1, 2]$ (and equivalently, $[2, 1]$) does not share a row, column or diagonal with a $\mathcal{P}$, it must be in $\mathcal{P}$. Thus, any other game G with a stack of size 1 or 2 must be in $\mathcal{N}$, since G is a NIM move from $[1, 2]$. Also, any game G in which the stack sizes have difference 1 must be in $\mathcal{N}$, since G is a WYTHOFF diagonal move away from $[1, 2]$. Continuing in this way, if we let $[a_n, b_n]$ be the n^{th} $\mathcal{P}$ position, then a_n is the mex of all a_i and b_i, for $i < n$ (because $[a_n, b_n]$ cannot have a NIM move to a previous $\mathcal{P}$ position), and b_n is equal to $a_n + n$ (because $[a_n, b_n]$ cannot have a WYTHOFF diagonal move to a previous $\mathcal{P}$ position). It is known that $a_n = \lfloor n\tau \rfloor$ and $b_n = \lfloor n\tau^2 \rfloor$ where τ is the golden ratio. It is conjectured that the nim-values, i.e. $\mathscr{G}([a, b])$, are eventually periodic for fixed a, but in general little is known. In Table 1.2, we see the $\mathcal{P}$ positions up to stacks of size 8.

8	○	○	○	○	○	○	○	○	○
7	○	○	○	○	$\mathcal{P}$	○	○	○	○
6	○	○	○	○	○	○	○	○	○
5	○	○	○	$\mathcal{P}$	○	○	○	○	○
4	○	○	○	○	○	○	○	$\mathcal{P}$	○
3	○	○	○	○	○	$\mathcal{P}$	○	○	○
2	○	$\mathcal{P}$	○	○	○	○	○	○	○
1	○	○	$\mathcal{P}$	○	○	○	○	○	○
0	$\mathcal{P}$	○	○	○	○	○	○	○	○
	0	1	2	3	4	5	6	7	8

Table 1.2: Complete table of Wythoff outcomes up to stack size 8.

Chapter 2

Other Self-Referential Games

There have been some well-studied self-referential games. Some have complete solutions, others have partial solutions at best. Here we present a number of them. Of the games we mention in this chapter, only EUCLID has a simple solution. This shows that even making the tiniest alteration to NIM can create a difficult problem.

2.1 Euclid

EUCLID is a game played on two stacks. Suppose the stacks have sizes a and b with $0 \leq b \leq a$. Then the only legal moves are those of the form $[a, b] \rightarrow [a - kb, b]$ for some positive integer k.

Theorem 2.1. *[4] The $\mathcal{P}$-positions in* EUCLID *are known; $[a, b] \in \mathcal{P} \Leftrightarrow \frac{a}{b} < \tau$ where τ is the golden ratio.*

In fact, any position in which the next player has two or more options is in $\mathcal{N}$. For an example, we look at EUCLID played on stacks of sizes 51 and 14. We can see that $[51, 14]$ has options $[37, 14]$, $[23, 14]$, and $[14, 9]$. So, we either find $[14, 9] \in \mathcal{P}$ (and thus, $[51, 14] \in \mathcal{N}$) or we find $[14, 9] \in \mathcal{N}$ (and thus, $[23, 14] \in \mathcal{P}$ since $[14, 9]$ is its only option). So, in either case, $[51, 14] \in \mathcal{N}$.

There also exists a partizan version of EUCLID in which each player's distinct moves are based on the Euclidean algorithm [8]. Since it is partizan, we mention it for completeness. Partizan EUCLID is played on two stacks, say p and q, with $p \geq q$. So, $p = kq + t$ where $0 \leq t < q$. Left's option is (q, t) and Right's option is $(q, q - t)$. The game ends when $t = 0$. For example, let $p = 31$ and $q = 12$. Then, $31 = (2)12 + 7$. Suppose Left moves first. Then the game would play out as follows: $(31, 12) \rightarrow (12, 7) \rightarrow (7, 2) \rightarrow (2, 1)$ and then the game ends since $2 = (2)1 + 0$. So, since Right has no legal moves, Left wins.

2.2 Restricted NIM

Fractal has many definitions, but for our purposes we are referring to an infinite sequence that, upon the removal of the first instance of each integer in order, generates itself.

Restricted NIM is a variant of NIM in which the number of chips a player may remove on his turn is restricted by an upper or lower bound (Maximum NIM and Minimum NIM). Results from Levine in 2006 [7] show that some restrictions result in a fractal structure in the sequence of nimbers as the size of the stack increases. One interesting result in particular shows that if players are restricted to removing strictly less than half of the stack size, then the only games on a single stack that are in $\mathcal{P}$ are those of size 2^n for some n. Using this ruleset, we now look at the sequence of nim values starting with a stack size of $n = 1$:

$$\mathbf{0}, 0, \mathbf{1}, 0, \mathbf{2}, 1, \mathbf{3}, 0, \mathbf{4}, 2, \mathbf{5}, 1, \mathbf{6}, 3, \mathbf{7}, 0, \mathbf{8}, 4, \mathbf{9}, 2, \mathbf{10}, \ldots.$$

Note how the sequence remains when the boldface sequence of increasing integers is removed.

$$0, 0, 1, 0, 2, 1, 3, 0, 4, 2, \ldots$$

A *weakly increasing sequence* is a sequence (a_n) with $a_{n+1} \geq a_n$ for all n.

A *rule* is a maximum number of chips that are allowed to be removed from a stack in Maximum NIM (often a function of the stack size).

A *weakly increasing rule sequence* is a sequence (a_n) of rule values for the game of Maximum NIM with stack size n as n increases, with $a_{n+1} \geq a_n$ for all n.

A *Grundy sequence* of a game played on a single stack is the sequence (a_n) of equivalent nimbers of the game as the size of the stack, n, increases.

Theorem 2.2. *[7] Let $(g_n)_{n \geq 0}$ be an infinitive sequence. The following are equivalent:*

(i) g is a fractal sequence;

(ii) g is the Grundy sequence for Maximum Nim for some weakly increasing rule sequence f.

2.3 Greedy NIM

Greedy NIM is a variant of NIM in which players can only remove chips from the largest stack on their respective turns.

Theorem 2.3. *[1] The $\mathcal{P}$-positions of Greedy NIM occur precisely in those games where there are an even number of stacks with an equal largest amount of chips.*

While Greedy NIM is solved for $\mathcal{P}$-positions, there are no solutions for the $\mathscr{G}$ values.

2.4 SUSEN

Define a *subtraction set* as the set of values that can legally be removed from a stack in a variant of NIM. So, in NIM, the subtraction set for a given stack is $\{1, 2, 3, ..., k\}$ where k is the size of that stack.

SUSEN is a variant of NIM in which the stack sizes of a board-state are the subtraction set for that board-state. It is important to note in SUSEN that once a stack reaches 0, it ceases to exist rather than allowing future turns to be "passed" by removing 0 chips. In Figure 2.1, we see an example of SUSEN played with stacks of sizes 7, 21, and 39. We refer to this game as $S[7, 21, 39]$. In this example, we see that Player Two has won since his removal of all 18 remaining chips from the right stack left Player One with no legal action on his turn.

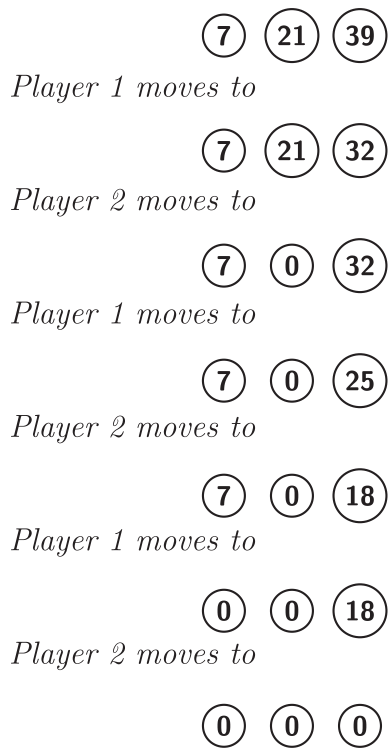


Figure 2.1: SUSEN on stacks of sizes 7, 21, and 39.

We now take a brief detour to define the Length function which will be useful in proving results for SUSEN. We define $L(0, 0) = 0$, $L(a, 0) = 1$, $L(0, b) = 1$, $L(a, b) = L(b, a)$ for all $a, b > 0$, and $L(a, b) = 1 + L(a - b, b)$ where $a \geq b > 0$. With this definition, we are able to generate a table of Length values.

8	1	9	5	6	3	6	5	9	2
7	1	8	6	6	6	6	8	2	9
6	1	7	4	3	4	7	2	8	5
5	1	6	5	5	6	2	7	6	6
4	1	5	3	5	2	6	4	6	3
3	1	4	4	2	5	5	3	6	6
2	1	3	2	4	3	5	4	6	5
1	1	2	3	4	5	6	7	8	9
0	0	1	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8

Table 2.1: Complete table of Length outcomes up to stack size 8.

The $\mathcal{P}$-positions for SUSEN are known for 3 or less stacks. All games of SUSEN with 1 stack are in $\mathcal{N}$ since the only legal move is for Player One to remove the entire stack and win.

Theorem 2.4. *[5] Suppose $a \geq b$. Then, $S[a,b] \in \mathcal{P} \Leftrightarrow L(a,b)$ is even.*

Proof. We know $S[0,0] \in \mathcal{P}$ and $L(0,0) = 0$. Now, $S[a,b]$ has 3 options: $S[0,b] \in \mathcal{N}$, $S[a,0] \in \mathcal{N}$, and $S[a-b,b]$. So, from every position with $a, b > 0$, any option other than $S[a-b,b]$ will cause the active player to lose. Thus, the outcome of $S[a,b]$ can be determined entirely by the parity of the number of these moves from $S[a,b]$ to $S[0,0]$. The length function, $L(a,b)$, measures the number of these moves from $S[a,b]$ to $S[0,0]$. So, $S[a,b] \in \mathcal{P} \Leftrightarrow L(a,b)$ is even. □

Theorem 2.5. *[5] For $a, b, c > 0$, the SUSEN position [a,b,c] is in $\mathcal{P}$ iff [a,b] and [a,c] and [b,c] are all in $\mathcal{N}$.*

Proof. From such an $[a,b,c]$, removing any stack is answered by subtracting the smaller of the remaining two from the larger, and vice versa. From all other three-stack positions, removing one of the stacks wins. □

2.5 JENNIFER

JENNIFER is a variant of SUSEN in which the subtraction set is every value except for a stack size. Obviously, JENNIFER differs from our other impartial games in that the game will not end when all the stacks are empty because a player will be unable to make a legal action before this occurs. The $\mathcal{P}$-positions for JENNIFER are only known up to 2 stacks. JENNIFER on one stack is easy. The winning strategy is to always remove all but one chip. Several participants from Games @ Dal 2014 claim to have solved the 2 stack game but no one has written it down. In Figure 2.2, we see an example of JENNIFER played with stacks of sizes 7, 21, and 39. We refer to this game as $J[7, 21, 39]$. In this example, we see that Player Two has won since his removal of 2 chips from the right stack left Player One with no legal action on his turn.

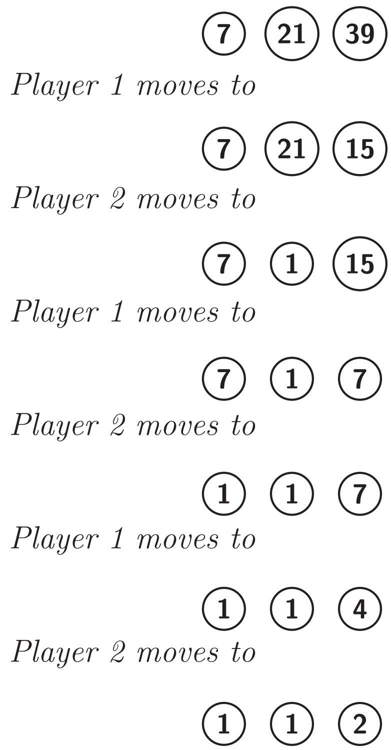


Figure 2.2: JENNIFER on stacks of sizes 7, 21, and 39.

Chapter 3

NIMB

In this chapter, we solve two simple variants of NIM. We call them NIMB (short for NIM bottom) and NIMTB (short for NIM top bottom). In these games, the maximum and minimum heap sizes dictate what moves can and cannot be made. We show that the nim-values of these two games are equal.

In NIMB, the rules are identical to those of NIM except that when comparing a board state to an option, that option is only a legal move if the minimum non-zero stack size of the option is not equal to the minimum non-zero stack size of the board-state. So, for example, if NIMB were being played on stacks of sizes 3, 4, and 6, the only legal moves would be those that leave the board in such a state that the smallest stack value is not 3. So, the legal options in this instance would be $[2, 4, 6], [1, 4, 6], [0, 4, 6], [3, 2, 6], [3, 1, 6], [3, 4, 2]$, and $[3, 4, 1]$. Thus, the legal moves of NIMB can be broken into two groups: those that remove some number of chips from the smallest stack and those that remove enough chips from another stack such that it becomes the new smallest stack.

Theorem 3.1. *Suppose* NIMB *is being played on n stacks of sizes $a_1, a_2, a_3, ...a_n$, with $a_1 \leq a_2 \leq a_3 \leq ... \leq a_n$, $n \geq 2$, and $a_1 \geq 1$. Then, $\mathscr{G}([a_1, a_2, a_3, ..., a_n]) = a_1 - 1$.*

Proof. We begin by noting that $\mathscr{G}([1, 1]) = 0$ since there are no legal moves. Then, $\mathscr{G}([1, 1+i]) = 0$ for all $i > 0$ since the only legal move is to remove the stack with only 1 chip and lose next turn. Also, $\mathscr{G}([2, 2 + i]) = 1$ for all i since the only legal moves are to reduce either stack to 1 chip and in that case, we know $\mathscr{G}([1, 1 + j]) = 0$ for all $j > 0$. Now, suppose that $k > 2$ and that for all $0 < n < k$, $\mathscr{G}([n, n + i]) = n - 1$ for all i. So, now consider NIMB played on two stacks of sizes k and $k + m$ for some $m \geq 0$. Player One must begin the game by reducing a stack to a size less than k (or remove a stack completely). We know that for all $0 < n < k$ that $\mathscr{G}([n, n+i]) = n-1$ for all i. So, $\mathscr{G}([k, k + m]) = mex(0, 1, ..., k - 2, k + m - 1) = k - 1$.

We now induct on the number of stacks and the size of the smallest stack. Suppose that for some $k > 2$, that for all $1 < n < k$, $\mathscr{G}(a_1, a_2, a_3, ..., a_n) = a_1 - 1$ for all $a_1 \leq a_2 \leq a_3 \leq ... \leq a_n$. So, now suppose NIMB is being played on k stacks of sizes $a_1 \leq a_2 \leq a_3 \leq ... \leq a_k$. We first suppose $a_1 = 1$ as a base case. Player One has a maximum of one legal move: remove the stack with a_1 chips (this move is only legal if $a_1 < a_2$). By our induction, we know that $\mathscr{G}(a_2, a_3, ..., a_k) = a_2 - 1$. So, either Player One has no legal option and loses, or Player One has one legal option, but that option is in $\mathcal{N}$, so he loses. In both cases, $\mathscr{G}(a_1, a_2, a_3, \ldots, a_k) = 0$. When we induct on the size of a_1, we get a game with up to two legal moves: remove the stack with a_1 chips (this move is only legal if $a_1 < a_2$), or reduce a stack to $0 < i < a_1$ chips. If Player One removes the stack with a_1 chips, then we are left with $\mathscr{G}(a_2, a_3, ...a_k) = a_2 - 1$. Otherwise, we are left with stacks of sizes $i \leq x_2 \leq x_3 \leq ... \leq x_k$. However, by our induction on the size of the smallest stack, we know $\mathscr{G}(i, x_2, x_3, \ldots, x_k) = i - 1$. So, $\mathscr{G}(a_1, a_2, a_3, \ldots, a_k) = mex(0, 1, \ldots, a_1 - 3, a_1 - 2, a_2 - 1) = a_1 - 1$. □

In NIMTB (TopBottom), the rules are identical to those of NIM except that when comparing a board state to an option, that option is only a legal move if it fulfills three criteria:

1. The minimum non-zero stack size of the option is not equal to the minimum non-zero stack size of the board-state.

2. The maximum stack size of the option is not equal to the maximum stack size of the board-state.

3. No two stacks in the option can have the same number of chips and no two stacks in the board-state can have the same number of chips. (We include this third criterion because of how it simplifies the proof and gives results similar to those of NIMB. The game that excludes this third criterion remains an open problem.)

So, for example, if NIMTB were being played on stacks of sizes 3, 4, and 6, the only legal moves would be those that leave the board in such a state that the smallest stack value is not 3 and the largest stack value is not 6. So, the legal options in this instance would be $[2, 3, 4]$ and $[1, 3, 4]$.

Theorem 3.2. *Suppose* NIMTB *is being played on n stacks of sizes $a_1, a_2, a_3, ...a_n$, with $a_1 < a_2 < a_3 < ... < a_n$, $n \geq 2$, and $a_1 \geq 1$. Then, $\mathscr{G}(a_1, a_2, a_3, ...a_n) = a_1 - 1$.*

Proof. We start by looking at games with $n = 2$. If $a_1 = 1$, then there are no legal moves. So, $\mathscr{G}(1, 1 + i) = 0$ for all i. If $a_1 = 2$, then the only legal move is to reduce the game to $\mathscr{G}(1, 2) = 0$. So, $\mathscr{G}(2, 2 + i) = 1$. Now, suppose that for some $k > 2$ that for all $0 < n < k, \mathscr{G}(n, n + i) = n - 1$ for all i. So, now suppose NIMTB is being played on two stacks of sizes k and $k+m$ for some $m \geq 1$. Player One's only option is to create a game of the form $\mathscr{G}(k-i,k) = k-i-1$. So, we get that $\mathscr{G}(k,k+m) = mex(0,1,...,k-i-1,...,k-2) = k-1$.

We now induct on the number of stacks and the size of the smallest stack. Suppose that for some $k > 2$, that for all $1 < n < k$, $\mathscr{G}(a_1,a_2,a_3,...a_n) = a_1 - 1$ for all $a_1 < a_2 < a_3 < ... < a_n$. So, now suppose NIMTB is being played on k stacks of sizes $a_1 < a_2 < a_3 < ... < a_k$. We first suppose $a_1 = 2$ as a base case ($a_1 = 1$ is trivial). Player One's only option is to reduce the largest stack to 1. This option is in $\mathcal{P}$ since it has no legal moves. Since this is Player One's only option, $\mathscr{G}(a_1,a_2,a_3,\ldots,a_k) = mex\{0\} = 1$. When we increase the size of the smallest stack, we get a game where all legal options are of the form $[i,a_1,a_2,a_3,\ldots,a_{k-1}]$ where $i < a_1$. By our induction on the size of the smallest stack, we know that $\mathscr{G}(i,a_1,a_2,a_3,\ldots,a_{k-1}) = i-1$. So, $\mathscr{G}(a_1,a_2,a_3,\ldots,a_k) = mex\{0,1,2,\ldots,a_1-2\} = a_1 - 1$. □

Chapter 4

WYNNIE and Variants

In this chapter, we look at a new self-referential, impartial game: WYNNIE. We completely solve the $\mathcal{P}$ positions of WYNNIE and misère WYNNIE in Theorem 4.5 and Theorem 4.10, respectively. Then, we look into some ways to play WYNNIE on graphs. Due to the complexity of the problem, we will break it into several cases.

4.1 WYNNIE

WYNNIE is a variant of Wythoff's game in which the subtraction set contains all natural numbers up to and including the smallest stack size. So, in a game of WYNNIE played on stacks of sizes m, n with $m \leq n$, denoted $W[m, n]$ or equivalently $W[n, m]$, the active player can remove up to m chips from one or both stacks. It is important to note that once a stack is reduced to 0 chips, it ceases to exist. So, the game may continue with the remaining stacks rather than forcing players to somehow remove up to 0 on their turn. In Figure 5, we see an example of WYNNIE played with stacks of sizes 17, and 23. In this example, we see that Player Two has won since his removal of all 4 remaining chips from the left stack left Player One with no legal action on his turn.

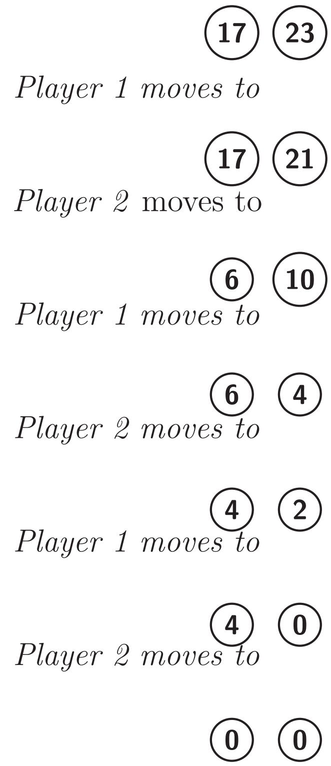


Figure 4.1: WYNNIE on stacks of sizes 17 and 23.

Before diving into WINNIE, first we prove a lemma about integers that we will use later.

Lemma 4.1. *For all* $k \in \mathbb{N}$, $k = 2^n a + 2^{n-1} - 1$ *for some* $n \in \mathbb{N}$ *and some* $a \in \mathbb{N} \cup \{0\}$.

Proof. Proof by induction. Observe $k = 1 = 2^2(0) + 2^{2-1} - 1$ and $k = 2 = 2^1(1) + 2^{1-1} - 1$. Now we suppose for some $m \geq 2$, that $m = 2^n a + 2^{n-1} - 1$ for some $n \in \mathbb{N}$ and some $a \in \mathbb{N} \cup \{0\}$. We seek to show that $m + 1 = 2^k b + 2^{k-1} - 1$ for some $k \in \mathbb{N}$ and some $b \in \mathbb{N} \cup \{0\}$.

Case 1: m is odd. Then $m + 1$ is even and thus, equal to $2a$ for some $a \in \mathbb{N}$. Then the result holds as $m + 1 = 2a = 2^1(a) + 2^{1-1} - 1$ for some $a \in \mathbb{N}$.

Case 2: m is even. So, $m = 2\ell$ and $m + 2 = 2(l + 1)$ for some $\ell \in \mathbb{N}$. Let 2^r be the greatest power of 2 that divides $\ell + 1$. Then, $m + 2 = 2(l + 1) = 2^{r+1}(2q + 1)$ for some $q \in \mathbb{Z} \cup 0$. So, $m + 1 = 2(l + 1) - 1 = 2^{r+2}(q) + 2^{r+1} - 1$. If we let $b = r + 2$, then we get $m + 1 = 2^b(q) + 2^{b-1} - 1$.

□

Lemma 4.2. *Suppose* $k > 0$. *Then,* $W[1, k] \in \mathcal{P}$ *iff* k *is even.*

Proof. Proof by induction. For base cases, note that $W[1,1]$ is Next Player win and $W[1,2]$ is Previous Player win. Now suppose that $W[1,k]$ is Next Player win (case 1). Then, in the game $W[1,k+1]$, Player One has two options: make a move involving the removal of the lone chip in the first stack (and lose next turn when Player Two removes the remaining stack), or remove a chip from only the stack with $k+1$ chips (leaving the game $W[1,k]$ which he will lose by the earlier assumption). So, $W[1,k+1]$ is Previous Player win. Alternatively, we could suppose that $W[1,k]$ is Previous Player win (case 2). Then $W[1,k+1]$ is obviously Next Player win since Player One, on his first turn, can create the Previous Player win game $W[1,k]$. □

Lemma 4.3. *$W[r,c] \in \mathcal{N}$ if $c > r$ and r is even.*

Proof. Suppose r is even and $c > r$.
Case 1: $r = 0$. Then, $W[r,c]$ is Next Player win (player one can remove all c chips). So, $W[r,c] \in \mathcal{N}$.
Case 2: c is even and $r \neq 0$. Then $W[r,c]$ yields $W[1,c]$ which is in $\mathcal{P}$ by Lemma 4.2. So, $W[r,c] \in \mathcal{N}$.
Case 3: c is odd and $r \neq 0$. Then $W[r,c]$ yields $W[r-(r-1), c-(r-1)] = W[1, c-r+1] \in \mathcal{P}$ by Lemma 4.2. So, $W[r,c] \in \mathcal{N}$. □

Lemma 4.4. *$W[r,c] \in \mathcal{N}$ if $c > r > 1$ and c is even.*

Proof. Suppose c is even and $c > r > 1$.
Case 1: r is even. Then $W[r,c] \in \mathcal{N}$ by Lemma 4.3.
Case 2: r is odd. Then, $W[r,c]$ yields $W[r-(r-1), c-(r-1)] = W[1, c-r+1] \in \mathcal{P}$. So, $W[r,c] \in \mathcal{N}$. □

Theorem 4.5. *If $c > r$, then $W[r,c] \in \mathcal{P}$ if and only if $r = 2^n - 1$ and $c = 2^n a + 2^{n-1} - 1$ for some $n, a \in \mathbb{N}$.*

Proof. We proceed by inducting on n. For $n = 1$, $W[2^n - 1, 2^n a + 2^{n-1} - 1] = W[1, 2a] \in \mathcal{P}$ by Lemma 4.2. Also, by Lemma 4.2, we know that for any odd value of a, that $W[1,a] \in \mathcal{N}$. Next, we note that, by Lemma 4.3, $W[2,a] \in \mathcal{N}$ for all $a > 2$.

For all values of r from 1 to 2^n, we suppose that for $c > r$, $W[r,c] \in \mathcal{P}$ if and only if $r = 2^m - 1$ and $c = 2^m d + 2^{m-1} - 1$ for some $m, d \in \mathbb{N}$. (Note that $m \leq n$.)

We seek to show that for all values of r from $2^n + 1$ to 2^{n+1} that if $c > r$, then $W[r,c] \in \mathcal{P}$ if and only if $r = 2^m - 1$ and $c = 2^m d + 2^{m-1} - 1$ for some $m, d \in \mathbb{N}$. To prove this, we examine all r values of the form $2^n + b$ for $0 < b < 2^n$ (b is odd by Lemma 4.3). By our induction, we know which values of c produce Previous Player win games for $r < 2^n + 1$. Those are c values of the form $2^m d + 2^{m-1} - 1$ for $m \leq n$, and $m, d \in \mathbb{N}$. If we exclude those values, we are left only with c values of the form $2^n d - 1$, $d \in \mathbb{N}$, by Lemma 4.1. So, these become our only choices when searching for a Previous Player win game with r value $2^n + b$. Since c must be greater than r, we further restrict to only $c = 2^n d - 1$ with $d > 1$. We now rewrite b in the form $2^k j + 2^{k-1} - 1$ for some $k \in \mathbb{N}$ and $j \in 0 \cup \mathbb{N}$ (which is possible by Lemma 4.1). We now break into cases depending on how k is related to n. Note that $k \leq n + 1$ and $j < 2^{n-k}$ since $b < 2^n$.

Case 1: $k < n$

Then,

$$\begin{aligned} W[2^n + b, 2^n d - 1] &= W[2^n + 2^k j + 2^{k-1} - 1, 2^n d - 1] \\ &\to W[2^k - 1, 2^n(d-1) - 2^k(j-1) - 2^{k-1} - 1] \\ &= W[2^k - 1, 2^k(2^{n-k}d - 2^{n-k} - j + 1) - 2^{k-1} - 1] \\ &= W[2^k - 1, 2^k(2^{n-k}d - 2^{n-k} - j) + 2^{k-1} - 1] \\ &= W[2^k - 1, 2^k(a) + 2^{k-1} - 1] \in \mathcal{P}. \end{aligned} \tag{4.1}$$

So, since $W[2^n + b, 2^n d - 1]$ can be moved to a Previous Player win game, $W[2^n + b, 2^n d - 1] \in \mathcal{N}$.

Case 2: $k = n$

Then, $W[2^n + b, 2^n d - 1] = W[2^n + 2^n j + 2^{n-1} - 1, 2^n d - 1]$

But $j = 0$ since $b < 2^n$.

Then,

$$\begin{aligned} W[2^n + b, 2^n d - 1] &= W[2^n + 2^{n-1} - 1, 2^n d - 1] \\ &\to W[2^n - 1, 2^n d - 2^{n-1} - 1] \\ &= W[2^n - 1, 2^n(d-1) + 2^{n-1} - 1] \in \mathcal{P}. \end{aligned} \tag{4.2}$$

Since $W[2^n + b, 2^n d - 1]$ can be moved to a Previous Player win game, $W[2^n + b, 2^n d - 1] \in \mathcal{N}$.

Case 3: $k = n + 1$

Then, $W[2^n + b, 2^n d - 1] = W[2^n + 2^k j + 2^{k-1} - 1, 2^n d - 1]$.

Now, $j = 0$ since $b < 2^n$. So, $W[2^n + b, 2^n d - 1] = W[2^n + 2^{k-1} - 1, 2^n d - 1]$.

Also note that $2^{k-1} = 2^n$ because $k = n + 1$.

Observe,

$$\begin{aligned} W[2^n+b,2^nd-1] &= W[2^n+2^n-1,2^nd-1] \\ &= W[2^{n+1}-1,2^nd-1]. \end{aligned} \tag{4.3}$$

We begin by proving our statement for the smallest possible value of d. Thus, we seek to show that $W[2^{n+1}-1,2^n(3)-1] = W[2^{n+1}-1,2^{n+1}+2^n-1] \in \mathcal{P}$. To do this, we must show that every option is in $\mathcal{N}$. We know every move involving the removal of chips from just the smaller stack will yield a Next player win game by our induction. By our induction, every game with a largest stack of the form 2^a-1 is in $\mathcal{N}$. So, we also know that the removal of chips from just the larger stack will yield a Next player win game. The only remaining options to consider are $W[2^{n+1}-1-j,2^n(3)-1-j]$ for all $j \leq 2^{n+1}-1$. By our induction, we must have $2^{n+1}-1-j = 2^{n-i}-1$ for some $i < n$. These games take on the form $W[2^{n-i}-1,2^{n-i}(2^i(3)-2^{i+1}+1)-1]$.

When we put this larger stack in the appropriate form to be a $\mathcal{P}$ position, we get

$$\begin{aligned} 2^{n-i}(2^i(3)-2^{i+1}+1)-1 &= 2^{n-i}(2^i(3)-2^{i+1}-2^{-1}+1)+2^{n-i-1}-1 \\ &= 2^{n-i}(a)+2^{n-i-1}-1 \end{aligned} \tag{4.4}$$

Note that the number multiplying 2^{n-i} is not an integer. We conclude these games are not of the correct form to be in $\mathcal{P}$ by our induction. Thus, $W[2^{n+1}-1,2^n(3)-1] \in \mathcal{P}$. However, not all games of the form $W[2^{n+1}-1,2^n(d)-1]$ are in $\mathcal{P}$. Consider $W[2^{n+1}-1,2^n(4)-1]$ which has $W[2^{n+1}-1,2^n(3)-1]$ as an option. So, $W[2^{n+1}-1,2^n(4)-1] \in \mathcal{N}$. Thus, we get that $W[2^{n+1}-1,2^nd-1] \in \mathcal{P}$ iff $d > 2$ is odd. So, $W[2^{n+1}-1,2^{n+1}(\frac{d-1}{2})+2^n-1] \in \mathcal{P}$ for all odd integers d such that $d > 2$. So, $W[2^{n+1}-1,2^{n+1}(f)+2^n-1] \in \mathcal{P}$ for all $f > 0$.

So, we conclude that for all values of r from 2^n+1 to 2^{n+1} that if $c > r$, then $W[r,c] \in \mathcal{P}$ if and only if $r = 2^m-1$ and $c = 2^md+2^{m-1}-1$ for some $m,d \in \mathbb{N}$. So, if $c > r$, then $W[r,c] \in \mathcal{P}$ if and only if $r = 2^n-1$ and $c = 2^na+2^{n-1}-1$ for some $r,a \in \mathbb{N}$. □

4.2 Misère WYNNIE

We define misère WYNNIE to be the same as the game of WYNNIE except that the winner is the player who does not act last. That is to say that a player will lose if his is the last move. We use $W^{-}[m,n]$ to denote misère WYNNIE played on stacks of sizes m and n. We begin by proving several lemmas before getting to our main result about the location of $\mathcal{P}$-positions.

Lemma 4.6. *$W^{-}[0,1] \in \mathcal{P}$ and $W^{-}[0,n] \in \mathcal{N}$ if $n > 1$.*

Proof. If the board only contains one chip, Player One has no choice but to remove it and lose. Otherwise, Player One will opt to remove $n-1$ chips on the first turn. Player Two must then remove that last remaining chip on his turn losing the game. □

Lemma 4.7. *$W^{-}[1,n] \in \mathcal{P}$ if and only if n is odd.*

Proof. Suppose misère WYNNIE is being played with two stacks of chips of sizes 1 and n. Note that any player who reduces a stack that contains only one chip will lose unless they have left the other stack with only one chip. So, the optimal play is always to remove from only the larger stack. However, since the smaller stack contains only one chip, a maximum of one chip can be removed per stack per turn. So, each turn, the active player will reduce the larger stack by one chip. This pattern will only be stopped when a player gets the opportunity to play on the game $W^{-}[1,2]$. Then, the active player will reduce the game to $W^{-}[0,1]$ and win on the next turn. So, if n is even, then Player One will act on $W^{-}[1,2k]$ while Player Two will act on $W^{-}[1,(2k+1)]$. Thus, Player One will play on $W^{-}[1,2]$. Conversely, if n is odd, Player Two will play on $W^{-}[1,2]$. So, Player One has a winning strategy if and only if n is even. □

Lemma 4.8. *Suppose $1 < m \leq n$ and either m is even or n is odd. Then, $W^{-}[m,n] \in \mathcal{N}$.*

Proof. Suppose misère WYNNIE is being played with two stacks of chips of sizes m and n with $1 < m \leq n$.

Case 1: m and n are either both even or both odd. By removing $m-1$ chips from both stacks on turn one, Player One can move the game to $W^{-}[1,2k+1]$ (since $n-m+1$ is odd). Player One will win this game since he gets to play second and it is a Previous Player win game.

Case 2: m is even and n is odd. By removing $m-1$ chips from the stack with m chips on turn one, Player One can move the game to $W^{-}[1,(2k+1)]$ (since n is odd). Player One will win this game since he gets to play second and it is a Previous Player win game. □

Lemma 4.9. *Suppose $m > 0$, then $W^{-}[m,m+1] \in \mathcal{N}$.*

Proof. Suppose misère WYNNIE is being played with two stacks of chips of sizes m and $m+1$. On turn one, Player One will remove m chips from both stacks. This will leave the [0,1] game which Player One will win next turn. $\square$

Theorem 4.10. *Suppose $m \leq n$. Then, $W^-[m,n] \in \mathcal{P}$ if and only if $m = 2^j - 1$ and $n = 2^j k + 2^{j-1}$, $j, k \in \mathbb{N}$.*

Proof. (The proof proceeds in a structure similar to that of Theorem 4.5.) Suppose misère WYNNIE is being played with two stacks of chips of sizes m and n with $m \leq n$. We know from Lemma 4.8 that if Player Two has a winning strategy, then m is odd and n is even. So, we suppose m is odd and n is even. (We now look at base cases with $m = 1, 2,$ and 3.)

Games with $m = 1$ are characterized by Lemma 4.7 (and satisfy the form we seek with $j = 1$). Suppose $m = 2$. Then, $m \neq 2^j - 1$ and, by Lemma 4.8, $W^-[m,n] \in \mathcal{N}$. Now, suppose $m = 3$. Then, $m = 2^2 - 1$. By Lemma 4.8, $W^-[3,n] \in \mathcal{N}$ for all odd n. By Lemma 4.9, $W^-[3,4] \in \mathcal{N}$. Next we inspect $W^-[3,6]$. In searching for a winning strategy, Player One need only look for ways to reduce this game to one of the form $W^-[1,k]$ for some odd k on his first turn. This, however, is impossible since it is not a legal move to remove an even number from 3 in the same turn that you remove an odd number from 6. So, $W^-[3,6] \in \mathcal{P}$. Next, we note that $W^-[3,8] \in \mathcal{N}$ since $W^-[3,6]$ on his first turn. So, we get that $W^-[3,a] = 0$ if and only if $a = 4k+2$. So, $W^-[2^2-1,a] = 0$ if and only if $a = 2^2k + 2^{2-1}$.

Now, let us suppose that for all values of m from 1 to 2^ℓ, $W^-[m,n]$ is in $\mathcal{P}$ if and only if $m = 2^j - 1$ and $n = 2^j k + 2^{j-1}$, for some $j, k \in \mathbb{N}$. We now examine all m values of the form $2^\ell + b$ for some $b < 2^\ell$ (b is odd by Lemma 4.8). We know which values of n produce Previous Player win games for $m < 2^\ell + 1$. Those are n values of the form $2^i d + 2^{i-1}$ for $i \leq \ell$, and $i, d \in \mathbb{N}$. If we exclude those values, we are left only with n values of the form $2^\ell d$, $d \in \mathbb{N}$. So, these become our only choices when searching for a Previous Player win game with m value $2^\ell + b$. We now put b in $2^s t + 2^{s-1} - 1$ form for some $s \in \mathbb{N}$ and $t \in 0 \cup \mathbb{N}$. We now break into cases depending on how s is related to ℓ. Note that $s \leq \ell + 1$ and $t < 2^{\ell - s}$ since $b < 2^\ell$.

Case 1: $s < \ell$

Then,

$$\begin{aligned} W^-[2^\ell + b, 2^\ell d] &= W^-[2^\ell + 2^s t + 2^{s-1} - 1, 2^\ell d] \\ &\to W^-[2^s - 1, 2^\ell(d-1) - 2^s(t-1) - 2^{s-1}] \\ &= W^-[2^s - 1, 2^s(2^{\ell-s}d - 2^{\ell-s} - t + 1) - 2^{s-1}] \\ &= W^-[2^s - 1, 2^s(2^{\ell-s}d - 2^{\ell-s} - t) + 2^{s-1}] \in P \end{aligned} \tag{4.5}$$

So, since $W^-[2^\ell+b, 2^\ell d]$ can be moved to a Previous Player win game, $W[2^\ell+b, 2^\ell d] \in \mathcal{N}$.

Case 2: $s = \ell$

Then, $W^-[2^\ell + b, 2^\ell d] = W^-[2^\ell + 2^\ell t + 2^{\ell-1} - 1, 2^\ell d]$

But, $t = 0$ since $b < 2^\ell$.

So,

$$\begin{aligned} W^-[2^\ell + b, 2^\ell d] &= W^-[2^\ell + 2^{\ell-1} - 1, 2^\ell d] \\ &\to W^-[2^\ell - 1, 2^\ell d - 2^{\ell-1}] \\ &= W^-[2^\ell - 1, 2^\ell(d-1) + 2^{\ell-1}] \in P \end{aligned} \tag{4.6}$$

Since $W^-[2^\ell+b, 2^\ell d]$ can be moved to a Previous Player win game, $W^-[2^\ell+b, 2^\ell d] \in \mathcal{N}$.

Case 3: $s = \ell + 1$

Then, $W^-[2^\ell + b, 2^\ell d] = W^-[2^\ell + 2^s t + 2^{s-1} - 1, 2^\ell d]$.

Now, $t = 0$ since $b < 2^n$. So, $W^-[2^\ell + b, 2^\ell d] = W^-[2^\ell + 2^{s-1} - 1, 2^\ell d]$.

Also, $2^{s-1} = 2^\ell$ since $s = \ell + 1$.

So,

$$\begin{aligned} W^-[2^\ell + b, 2^\ell d] &= W^-[2^\ell + 2^\ell - 1, 2^\ell d]. \\ &= W^-[2^{\ell+1} - 1, 2^\ell d] \end{aligned} \tag{4.7}$$

We begin by proving our statement for the smallest possible value of d. So, we seek to show that $W[2^{\ell+1} - 1, 2^\ell(3)] = W[2^{\ell+1} - 1, 2^{\ell+1} + 2^\ell] \in \mathcal{P}$. So, we must show that every option is in $\mathcal{N}$. We know every move involving the removal of chips from just the smaller stack will yield a Next player win game by our induction. By our induction, every game with an odd largest stack is in $\mathcal{N}$. So, we also know that the removal of chips from just the larger stack will yield a Next player win game. The only remaining options to consider are $W[2^{\ell+1} - 1 - j, 2^\ell(3) - j]$ for all $j \leq 2^{\ell+1} - 1$. By our induction, we must have $2^{\ell+1} - 1 - j = 2^{\ell-i} - 1$ for some $i < \ell$. These games take on the form $W[2^{\ell-i}-1, 2^{\ell-i}(2^i(3)-2^{i+1}+1)]$.

When we put this larger stack in the appropriate form to be a $\mathcal{P}$ position, we get

$$\begin{aligned} 2^{\ell-i}(2^i(3)-2^{i+1}+1) &= 2^{\ell-i}(2^i(3)-2^{i+1}-2^{-1}+1)+2^{\ell-i-1} \\ &= 2^{\ell-i}(a)+2^{\ell-i-1} \end{aligned} \tag{4.8}$$

for some $a \notin \mathbb{N}$.

So, these games are not of the correct form to be in $\mathcal{P}$ by our induction. So, $W[2^{\ell+1}-1, 2^{\ell}(3)] \in \mathcal{P}$. However, not all games of the form $W[2^{\ell+1}-1, 2^{\ell}(d)]$ are in $\mathcal{P}$. For example, we look at $W[2^{\ell+1}-1, 2^{\ell}(4)]$ which has $W[2^{\ell+1}-1, 2^{\ell}(3)]$ as an option. So, $W[2^{\ell+1}-1, 2^{\ell}(4)] \in \mathcal{N}$. So, we get that $W[2^{\ell+1}-1, 2^{\ell}d] \in \mathcal{P}$ iff $d > 2$ is odd. So, $W[2^{\ell+1}-1, 2^{\ell+1}(\frac{d-1}{2})+2^{\ell}] \in \mathcal{P}$ for all odd integers d such that $d > 2$. So, $W[2^{\ell+1}-1, 2^{\ell+1}(f)+2^{\ell}] \in \mathcal{P}$ for all $f > 0$.

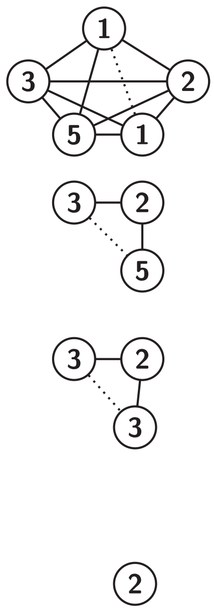
So, we conclude that for all values of m from $2^{\ell}+1$ to 2^{l+1} that if $n \geq m$, then $W^{-}[m,n] \in \mathcal{P}$ if and only if $m = 2^j - 1$ and $n = 2^j k + 2^{j-1}$ for some $j, k \in \mathbb{N}$. So, if $m \leq n$, then $W^{-}[m,n] \in \mathcal{P}$ if and only if $m = 2^j - 1$ and $n = 2^j k + 2^{j-1}$, $j, k \in \mathbb{N}$. $\square$

4.3 WYG

We will also explore using a graph as the board for WYNNIE.

Rules of WYG (WYNNIE on graphs):

Let G be a graph with a nonnegative integral number of chips initially on each vertex (i.e. each vertex contains a NIM stack). Player 1 chooses an edge of G and then beginning with Player 1, the players alternate moves on the endpoints of that edge until the edge is deleted (i.e. until one of the endpoints has NIM stack size 0). A move for a player is to remove up to k chips from one or both endpoints of the edge, where k is the smallest nonnegative NIM stack of the two endpoints. Once a vertex has NIM stack size 0, it cannot be used and so is effectively deleted from the graph. If a player's move results in a vertex being deleted from the graph (and hence the edge that had been played upon), then the other player now chooses the next edge to play upon and moves first. See Figure 4.5 for a sample game.



Player One opts to remove both vertices.

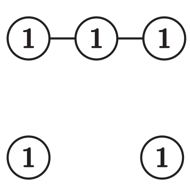
Player Two removes 2 chips from a single stack.

Player One removes both vertices from the dotted edge.

Player Two has no edge to play on and loses.

Figure 4.2: WYG on K_5 with various stack sizes.

Let $W_1(G)$ be an instance of WYG on graph G in which each vertex initially contains 1 chip. Let $W_{dg}(G)$ be an instance of WYG on graph G in which each vertex initially contains degree-many chips. Let $W_n(G)$ be an instance of WYG on graph G in which each vertex initially contains n chips (so, each vertex has the same number of chips). As an example, consider the game $W_1(P_3)$. Player One's first choice of edge does not matter due to the symmetry of the graph (see the top of Figure 4.3). He will choose one of them and opt to remove the chip from the center vertex leaving no edges remaining in the graph, winning the game.



Player One removes the center vertex

Player Two has no legal moves and loses

Figure 4.3: The game $W_1(P_3)$.

Next, we consider the game $W_{dg}(P_3)$ as illustrated in Figure 4.4. Player One's first choice of edge again does not matter due to the symmetry of the graph. Player One should not choose to remove a chip from just the center vertex for fear of yielding the Next Player win game $W_1(P_3)$. Alternatively, Player One may choose to remove a chip from the center vertex and from a leaf yielding $W_1(P_2)$. However, $W_1(P_2)$ is obviously Next Player win since every possible play on Turn 1 will win the game. Player One's final option is to remove just a leaf from the graph. Player Two will then remove the other leaf, leaving a single vertex with two chips and no edges, winning the game. So, $W_{dg}(P_3)$ is Previous Player win.

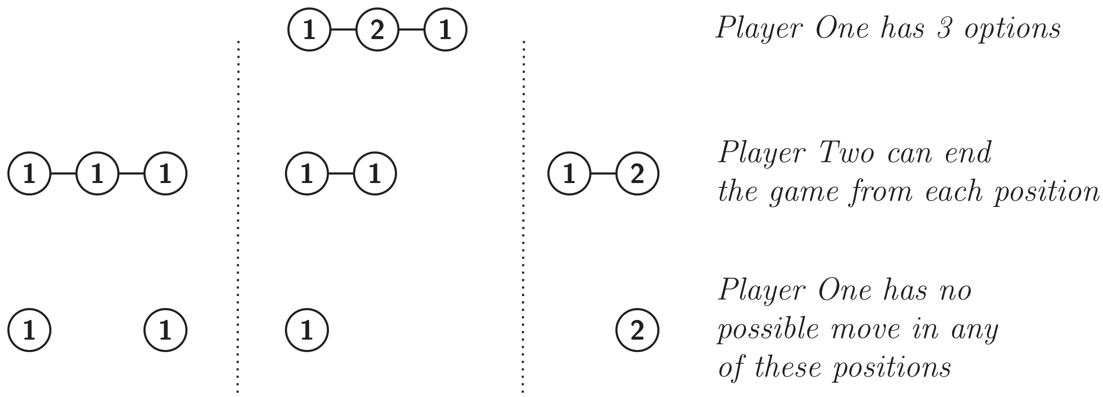


Figure 4.4: Degree WYG on P_3

Note that in this version of WYNNIE on graphs, once an edge has been chosen, players move on that edge until it is deleted (i.e. until an endpoint is deleted). An alternate version would be where on a player's turn, he may choose to play on *any* edge and then moves on that edge (i.e. the subsequent player may choose to play on the same edge if it still exists, or may choose to play on a different edge). Certainly in the case where each vertex of G initially contains 1 chip, the two versions are equivalent.

Let $e = uv$ be an edge in graph G. If $V(G) \setminus \{u, v\}$ is disconnected, then e is called a *cut-edge neighbourhood* of G.

Theorem 4.11. *Let G be a graph that contains a cut vertex or cut edge neighbourhood that, upon its removal, yields two isomorphic components. Then $W_n(G) \in \mathcal{N}$.*

Proof. Suppose WYG is being played on a graph G that contains a cut vertex or cut edge neighbourhood that, upon its removal, yields two isomorphic components, and each vertex of G has n chips.

Case 1: Suppose G contains a cut vertex v whose removal yields two isomorphic components. Then Player One will opt to play WYNNIE on an edge incident with v

and remove only the chips on v. This will break G into two isomorphic components. Then by mirroring Player 2's moves, Player 1 will win.

Case 2: Suppose G contains a cut edge neighbourhood e whose removal yields two isomorphic components. Then Player One will opt to play WYNNIE on e and remove all chips from both incident vertices. This will break G into two isomorphic components. Then by mirroring Player 2's moves, Player 1 will win. □

In fact, this strategy will prove successful for Player One for any distribution of chips on the graph as long as he is able to break the game into two isomorphic disjoint games on his first turn. However, the converse to this theorem is not true. Not all games of the form $W_1(G)$ that are in $\mathcal{N}$ have a cut vertex or cut edge-neighbourhood as evidenced by the example on K_5 in Figure 4.5.

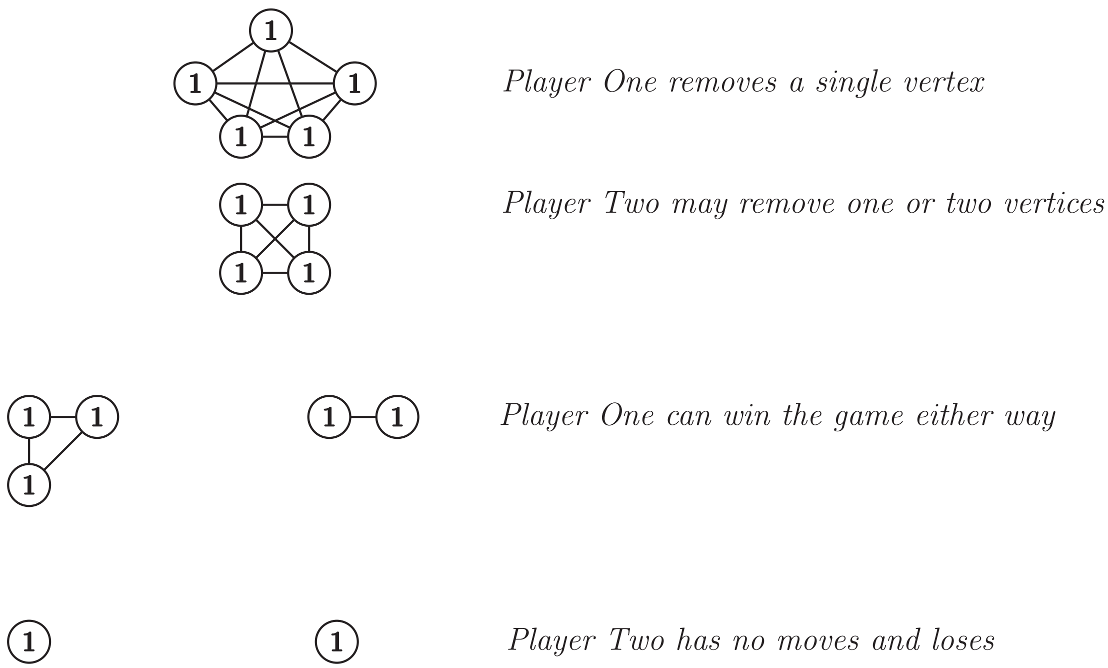


Figure 4.5: The game $W_1(K_5)$.

WYG with a single chip on each vertex closely resembles the "edge-delete game" proposed by Gallant et al. [6]. In the edge-delete game, players take turns removing edges from a graph without creating isolated vertices. The winner is the last player to make a legal move.

Theorem 4.12. *[6] Suppose $G = (V, E)$ has an automorphism $f : V \to V$ with the properties: (1) f has no fixed points; (2) $f(f(x)) = x$ for every vertex x in V; and (3)*

x is not adjacent to $f(x)$ for any x in V. Then Player Two has a winning strategy on G.

This theorem is proven using a mirroring strategy that calls for Player Two to respond to the removal of edge xy by removing edge $f(x)f(y)$. It is as yet unclear how and if this theorem can be adapted for WYG.

Theorem 4.13. *$W_1(K_n) \in \mathcal{P}$ if and only if $n \equiv 1$ (mod 3).*

Proof. (By induction) Suppose WYG is played on a complete graph K_n and each vertex has 1 chip. Observe that K_1 is Previous Player win and K_2,K_3 are Next Player win.

We now assume that $W_1(K_n) \in \mathcal{P}$ if and only if $n \equiv 1 \pmod 3$ for all values of n from 1 to k for some $k \in \mathbb{N}$. Note that Player One's first move on a complete graph, K_{k+1}, must simplify the game to a smaller complete graph, namely, K_k or K_{k-1}.

Case 1: K_k and K_{k-1} are Next Player win (So, $k \equiv 0 \pmod 3$). We are playing on K_{k+1}. Then, Player One's only option on Turn one is to create a Next Player win game. Thus, K_{k+1} is Previous Player win.

Case 2: K_k is Next Player win and K_{k-1} is Previous Player win (So, $k \equiv 2 \pmod 3$). We are playing on K_{k+1}. Then, Player One can create a Previous Player win game on Turn One, namely, K_{k-1}. So, K_{k+1} is Next Player win.

Case 3: K_k is Previous Player win (So, K_{k-1} is Next Player win and $k \equiv 1 \pmod 3$). Then, Player One can create a Previous Player win game on Turn One, namely, K_k. So, K_{k+1} is Next Player win.

So, for $n \geq 1$, $W_1(K_n) \in \mathcal{P}$ if and only if $n \equiv 1 \pmod 3$. □

Corollary 4.14. *$W_1(K_n)$ has nimber value*

$$\begin{cases} 0 & \textit{if and only if } n \equiv 1 \textit{ (mod 3)} \\ 1 & \textit{if and only if } n \equiv 2 \textit{ (mod 3)} \\ 2 & \textit{if and only if } n \equiv 0 \textit{ (mod 3).} \end{cases}$$

Proof. We know $\mathcal{G}(W_1(K_{3k+1})) = 0$ since $W_1(K_{3k+1}) \in \mathcal{P}$. Since $\mathcal{G}(W_1(K_1)) = 0$, we know $\mathcal{G}(W_1(K_2)) = 1$, and consequently, $\mathcal{G}(W_1(K_3)) = 2$. This pattern holds by the same induction used in the previous theorem. □

Theorem 4.15. *For $n > 3$, $W_1(C_n) \in \mathcal{P}$.*

Proof. Suppose WYG is being played on a cycle C_n, $n > 3$, and each vertex has 1 chip. Player One's initial move will remove one or two vertices from the graph. So, the result will be a path, which is Next Player win by Theorem 4.11. So, Player Two has a winning strategy on C_n, $n > 3$, when each vertex has 1 chip. □

Theorem 4.16. *$W_1(W_n) \in \mathcal{N}$ for all $n > 4$.*

Proof. Suppose WYG is being played on a wheel W_n, $n > 4$, and each vertex has 1 chip. Player One can remove the hub vertex, leaving C_{n-1} which is Previous Player win by Theorem 4.15. Thus, Player One has a winning strategy on the wheel W_n, $n > 4$, when each vertex has 1 chip. □

The following lemma is used to prove the main result for complete bipartite graphs, Theorem 4.18.

Lemma 4.17. *(a) $W_1(K_{m,1}) \in \mathcal{N}$ for all $m \in \mathbb{Z}^+$ and*
(b) $W_1(K_{m,2}) \in \mathcal{P}$ for all even positive integers m.

Proof. In $K_{m,1}$ where each vertex initially has one chip, Player One can win by removing the chip from the vertex with degree m, leaving no legal moves for Player Two.

In $K_{m,2}$ where m is even and every vertex initially has one chip, observe that Player One's options on his first turn are to remove a single vertex from either part or remove a vertex from both. If Player One finishes his turn with only a single vertex in either part, then he will lose by (a). So, Player One must remove a single vertex from the part containing m vertices. Player Two's winning strategy will repeat Player One's move, removing a single vertex from the part that now contains $m-1$ vertices. So, we now have a game where Player One is acting on $K_{2k,2}$ and Player Two is acting on $K_{2k+1,2}$ as the order of the larger part is decreasing. Since Player One will lose if he ends his turn with only a single vertex in either part, his action on $K_{2,2}$ will lead to a Player Two win. □

Theorem 4.18. *For $m, n \in \mathbb{Z}^+$, $W_1(K_{m,n}) \in \mathcal{P}$ if and only if both m, n are even.*

Proof. We fix m_0 and seek to inductively prove that $W_1(K_{m_0,n}) \in \mathcal{P}$ if and only if m_0, n even for all $n \in \mathbb{Z}^+$. To do so, we assume our result is true for $m_0 - 1$ for all n (inductive hypothesis 1). By Lemma 4.17, our result holds for $n = 1$, $n = 2$. Suppose the result holds for $n = n_0 - 1$ (inductive hypothesis 2). On Player One's first turn, he moves from $W_1(K_{m_0,n_0})$ to either $W_1(K_{m_0-1,n_0})$, $W_1(K_{m_0,n_0-1})$, or $W_1(K_{m_0-1,n_0-1})$. By inductive hypothesis 1, we know $W_1(K_{m_0-1,n_0})$ and $W_1(K_{m_0-1,n_0-1})$ are in $\mathcal{P}$ if and only if both parts contain an even number of vertices. By inductive hypothesis 2, we know $W_1(K_{m_0,n_0-1})$ is in $\mathcal{P}$ if and only if both parts contain an even number of vertices. If $W_1(K_{m_0,n_0}) \in \mathcal{P}$, then every one of Player One's first turn options must yield a game that is in $\mathcal{N}$. This implies that both m and n are even since no pair of $m_0 - 1$ and n_0, $m_0 - 1$ and $n_0 - 1$, nor m_0 and $n_0 - 1$ can both be even. So, for all $m, n \in \mathbb{Z}^+$, $W_1(K_{m,n}) \in \mathcal{P}$ if and only if m, n even. □

The following lemma will be used to prove $W_{dg}(C_n) \in \mathcal{N}$ in Theorem 4.20.

Lemma 4.19. *Suppose* WYG *is being played on a path* P_n*, and each vertex has* 2 *chips except for one of the endpoints which has* 1 *chip (see Figure 4.6). Then Player One has a winning strategy.*

Figure 4.6: WYG on P_n.

Proof. On Player One's first turn, he will choose to play WYNNIE on the edge incident with the vertex with only one chip. He will remove one chip from the vertex adjacent to the endpoint with only 1 chip.

Figure 4.7: Player Two must respond on the same edge.

Player Two must respond by continuing the game on the same edge (see Figure 4.7). Player Two must either create a path of length $n - 2$ with each vertex containing 2 chips (Player One would be able to break this path into two isomorphic pieces as in Theorem 4.11 to win the game) or create a path of length $n - 1$ with each vertex containing 2 chips except for one endpoint with only 1 chip. We now seek to show that if Player One repeats this same move on each of his turns that the continued use of this second strategy by Player Two will be ineffective.

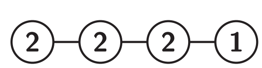


Figure 4.8: Player One has a winning strategy in this game

Player One can finish this game in Figure 4.8 by removing both vertices incident with the center edge. Thus, this game is in $\mathcal{N}$. Thus, Player One has a winning strategy of forcing Player Two to either shorten the path each turn or remove the vertex with only 1 chip. □

Theorem 4.20. *Suppose* WYG *is being played on a cycle C_n, and each vertex has 2 chips. Then Player One has a winning strategy.*

Proof. Initially, Player One will remove one chip from two adjacent vertices. Player Two must then continue the game of WYNNIE on the same edge. After Player Two's move, the resulting game will either be the one given in the previous lemma and Player One will win or $W_2(P_{n-2})$ which is Next Player win by Theorem 4.11. □

Theorem 4.21. *Let G be the graph composed of two stars S_m and S_n connected by a single central vertex as shown in Figure 4.9. Then, $W_1(G) \in \mathcal{N}$ if and only if $n+m$ is even.*

Proof. Suppose WYG is being played on a graph G composed of two stars S_m and S_n connected by a single central vertex and each vertex initially has one chip.

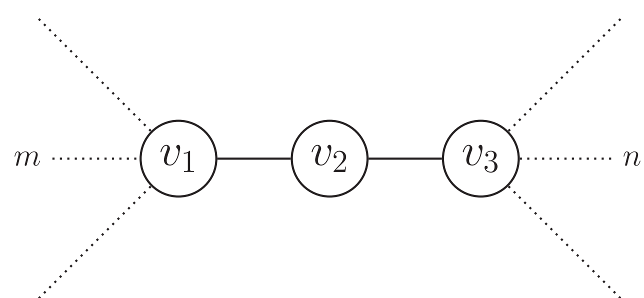


Figure 4.9: The graph G described in Theorem 4.21.

If at any point, a player removes the chip from either v_1 or v_3, then the other player will remove the chip from the other next turn and win. So, neither player should ever remove two vertices during his turn. Therefore, the optimal strategy for each player is to remove chips one at a time leading to a parity-based result. So, Player One will win if there is an odd number of total vertices and Player Two will win if there is an even number of total vertices. □

Theorem 4.22. *Let G be the graph composed of two stars S_m and S_n connected by a path of length 3 as shown in Figure 4.10. Then, Player One has a winning strategy for $W_1(G)$.*

Proof. Let G be the graph composed of two stars S_m and S_n connected by a path of length 3 and initially place one chip at each vertex of G.

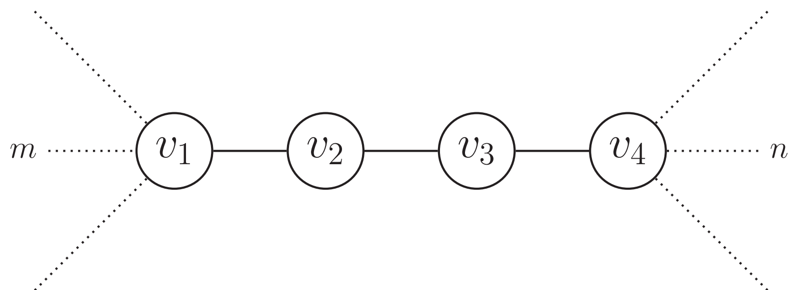


Figure 4.10: The graph described in Theorem 4.22.

Case 1: $n + m$ is even. Then, Player One will remove chips from both v_2 and v_3 leaving S_m disconnected from S_n. What remains is a game where removing two chips at a time is obviously not the optimal move for either player. So, the chips will be removed one at a time leading to a parity-based result (Lemma 4.17. Since this game is Previous Player win (even number of total vertices), Player One has a winning strategy.

Case 2: $n + m$ is odd. Then, Player One will remove v_2 leaving S_m disconnected from S_{n+1}. What remains is a game where removing two chips at a time is obviously not the optimal move for either player. So, the chips will be removed one at a time leading to a parity-based result. Since this game is Previous Player win (even number of total vertices), Player One has a winning strategy. □

Theorem 4.23. *Let G be the graph composed of two stars S_m and S_n connected by a path of length 4 as shown in Figure 4.11. Then, Player One has a winning strategy for $W_1(G)$.*

Proof. Let G be the graph composed of two stars S_m and S_n connected by a path of length 4 and initially place one chip at each vertex of G.

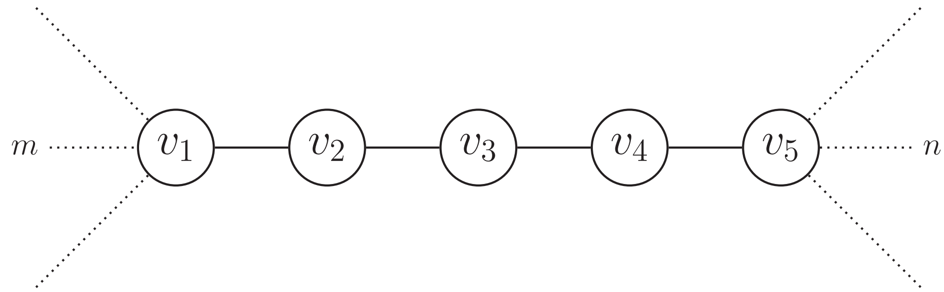


Figure 4.11: The graph described in Theorem 4.23.

Case 1: $n + m$ is even. Then, Player One will remove the chip from v_3 leaving disjoint stars S_{m+1} and S_{n+1}. Since this game is Previous Player win, Player One has a winning strategy.

Case 2: $n + m$ is odd. Then, Player One will remove the chips from v_2 and v_3 leaving disjoint stars S_m and S_{n+1}. Since this game is Previous Player win, Player One has a winning strategy. □

Theorem 4.24. *Let G be the graph composed of two stars S_m and S_n connected by a path of length 5 as shown in Figure 4.12. Then, Player One has a winning strategy for $W_1(G)$.*

Proof. Let G be the graph composed of two stars S_m and S_n connected by a path of length 5 and initially place one chip at each vertex of G.

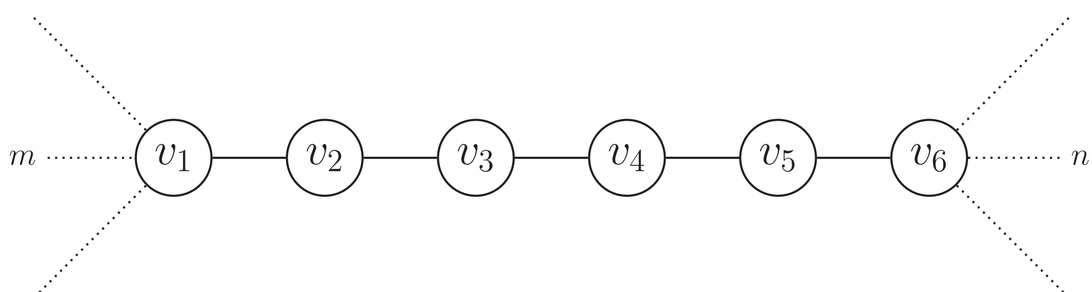


Figure 4.12: The graph described in Theorem 4.24.

Case 1: $n+m$ is even. Then, Player One will remove chips from v_3 and v_4 leaving disjoint stars S_{m+1} and S_{n+1}. Since this game is Previous Player win, Player One has a winning strategy.

Case 2: $n + m$ is odd. Suppose, without loss of generality, that n is even and m is odd. Then, Player One will remove chips from v_1 and v_2 leaving S_1 connected to S_n which is Previous Player win by Theorem 4.21. Since this game is Previous Player win, Player One has a winning strategy. □

As the paths get longer, the proof method changes because referencing previous theorems becomes less useful. So, there seems to be no reason to assume that every game of this form is Next Player Win. Solving this problem for a path of length n remains an open question, but we conjecture that this pattern of always being Next Player Win does not continue as the path grows longer.

Chapter 5

MINNIE

MINNIE is a variant of NIM in which the subtraction set contains all natural numbers up to and including the smallest stack size. It is important to note that once a stack is reduced to 0 chips, it ceases to exist. So, the game may continue with the remaining stacks rather than forcing players to somehow make 0 not the smallest stack size on their turn. In Figure 5.1, we see an example of MINNIE played with stacks of sizes 7, 9, and 20. In this example, we see that Player One has won since his removal of all 6 remaining chips from the center stack left Player Two with no legal action on his turn.

In this chapter we solve MINNIE for 2 stacks and nearly solve it for 3 stacks. We also solve a similar game MINE for 2 and 3 stacks. MINE is equivalent to MINNIE except that no legal move leaves multiple stacks with the same number of chips. With Theorem 5.2, we show that all the $\mathcal{P}$-positions of MINNIE on two stacks occur when the smaller stack has at most 4 chips. With Theorem 5.4, we are able to locate most $\mathcal{P}$-positions of MINNIE on three stacks.

7 9 20

Player 1 moves to

7 9 14

Player 2 moves to

5 9 14

Player 1 moves to

0 9 14

Player 2 moves to

0 9 5

Player 1 moves to

0 6 5

Player 2 moves to

0 6 0

Player 1 moves to

0 0 0

Figure 5.1: MINNIE on stacks of sizes 7, 9, and 20.

5.1 MINNIE on 1 and 2 stacks

Since the case of Minnie on 3 stacks is complicated one, we will consider the problem on 1 and 2 stacks first. However, the problem on 1 stack is quite simple.

Lemma 5.1. *Let a be a positive integer then $\mathscr{G}(M[a]) = a$.*

Proof. This game is the same as 1 stack NIM. □

Theorem 5.2. *Let $a \leq b$ be positive integers: $M[a, b] \in \mathcal{P}$ iff*

$$M[a,b] = \begin{cases} M[1, 2i+1], i \geq 0 \\ M[2, 4i+2], i \geq 0 \\ M[3, 4i], i \geq 2 \\ M[4, 4] \end{cases}$$

Proof. **Case 1:** $a = 1$

Then, $M[1,1] \in \mathcal{P}$, and $M[1,2] \in \mathcal{N}$ since $M[1,1]$ is an option. Inducting on the options, we get $M[1,2k] \in N$ since $M[1,2k-1]$ is an option; $M[1,2k+1] \in \mathcal{P}$ since the only two options are $M[0,2k+1]$ and $M[1,2k]$, both of which are in $\mathcal{N}$.

Case 2: $a = 2$

We know $M[2,1] \in \mathcal{N}$ since $M[1,1]$ is an option; $M[2,2] \in \mathcal{P}$ since the only options are $M[1,2]$ and $M[0,2]$, both of which are in $\mathcal{N}$; $M[2,3] \in \mathcal{N}$ since $M[2,2]$ is an option; $M[2,4] \in \mathcal{N}$ since $M[2,2]$ is an option. We have the base cases. We now consider the general case.

- Both $M[2,4k+1] \in \mathcal{N}$ and $M[2,4k+3] \in \mathcal{N}$ since $M[1,4k+1]$ and $M[1,4k+3]$ are options, respectively.

- $M[2,4k] \in \mathcal{N}$ since $M[2,4k-2] = M[2,4(k-1)+2]$ is an option and is in $\mathcal{P}$ by the inductive hypothesis.

- $M[2,4k+2] \in \mathcal{P}$ since the only options are $M[0,4k+2]$, $M[1,4k+2]$, $M[2,4k]$, and $M[2,4k+1]$, all of which are in $\mathcal{N}$.

Case 3: $a = 3$

We know $M[3,1]$ and $M[3,2]$ are in $\mathcal{P}$ and $\mathcal{N}$, respectively by cases 1 and 2.

- $M[3,3] \in \mathcal{N}$ since $M[1,3]$ is an option.

- $M[3,4] \in \mathcal{N}$ since $M[1,3]$ is an option.

- $M[3,5] \in \mathcal{N}$ since $M[1,5]$ is an option.

- $M[3,6] \in \mathcal{N}$ since $M[2,6]$ is an option.

- $M[3,7] \in \mathcal{N}$ since $M[1,7]$ is an option.

- $M[3,8] \in \mathcal{P}$ since the only options are $M[0,8] \in \mathcal{N}$ (Lemma 5.1), $M[1,8] \in \mathcal{N}$ (case 1) , $M[2,8] \in \mathcal{N}$ (case 2), $M[3,7] \in \mathcal{N}$, $M[3,6] \in \mathcal{N}$ and $M[3,5] \in \mathcal{N}$.

We have the base cases. We now consider the general case.

- Both $M[3,4k+1]$ and $M[3,4k+3]$ are in $\mathcal{N}$ since $M[1,4k+1] \in \mathcal{P}$ (case 1), and $M[1,4k+3] \in \mathcal{P}$ (case 1) are options, respectively.

- $M[3,4k+2] \in \mathcal{N}$ since $M[2,4k+2] \in \mathcal{P}$ (case 2) is an option.

- $M[3,4k] \in \mathcal{P}$ since the only options are $M[0,4k] \in \mathcal{N}$ (Lemma 5.1), $M[1,4k] \in \mathcal{N}$ (case 1), $M[2,4k] \in \mathcal{N}$ (case 2), $M[3,4k-1] \in \mathcal{N}$ (induction), $M[3,4k-2] \in \mathcal{N}$ (induction) and $M[3,4k-3] \in \mathcal{N}$ (induction).

Now suppose MINNIE is being played on two stacks of sizes m and n for some $n \geq m \geq 4$.

- $M[4,4] \in \mathcal{P}$ since the only options are $M[0,4] \in \mathcal{N}$ (Lemma 5.1), $M[1,4] \in \mathcal{N}$ (case 1), $M[2,4] \in \mathcal{N}$ (case 2), and $M[3,4] \in \mathcal{N}$ (case 3).

Aside from the case of $M[4,4]$ (which is an exception since $M[3,4k] \in \mathcal{P}$ for all $k > 1$ instead of being for all $k \geq 1$), we have that of the following three options: $M[1,n]$, $M[2,n]$, or $M[3,n]$, one of them must be Previous Player win since $n = 0, 1, 2,$ or 3 (mod 4). So, $M[m,n] \in \mathcal{N}$. □

5.2 MINNIE on 3 stacks

We have a nearly complete solution for Minnie on 3 stacks, but the proof is long. As usual, our strategy involves inducting on the options.

Theorem 5.3. *Suppose* MINNIE *is being played on three stacks of sizes ℓ, m, and n with $n \geq m \geq \ell$ and $1 \leq \ell \leq 3$. Then, Player Two has a winning strategy if and only if one of the following conditions is met:*

$$M[\ell,m,n] = \begin{cases} M[1,m,n], m+n \equiv 1 \ (mod\ 2), m \neq 3 \\ M[1,3,4] \\ M[1,3,n], n \equiv 2 \ (mod\ 4) \\ M[2,m,n], m \neq 3,\ m+n \equiv 2 \ (mod\ 4) \\ M[3,3,n], n \equiv 3 \ (mod\ 4) \\ M[3,m,n], m > 3,\ m \neq 5,\ n > 4,\ and\ m+n \equiv 0 \ (mod\ 4). \end{cases}$$

Proof. We consider the cases in order.

Case 1: $\ell = 1$

Case 1.1: $\ell = 1$ and $m = 1$

- $M[1,1,2k+1] \in \mathcal{N}$ since $M[0,1,2k+1] \in \mathcal{P}$ (Theorem 5.2) is an option.

- $M[1,1,2k] \in \mathcal{P}$ since the only options are $M[0,1,2k] \in \mathcal{N}$ (Theorem 5.2) and $M[1,1,2k-1] \in \mathcal{N}$ (induction).

Case 1.2: $\ell = 1$ and $m = 2$

- $M[1,2,2k] \in \mathcal{N}$ since $M[1,1,2k] \in \mathcal{P}$ (case 1.1) is an option.
- $M[1,2,2k+1] \in \mathcal{P}$ since the only options are $M[0,2,2k+1] \in \mathcal{N}$ (Theorem 5.2), $M[1,1,2k+1] \in \mathcal{N}$ (case 1.1), and $M[1,2,2k] \in \mathcal{N}$.

Case 1.3: $\ell = 1$ and $m = 3$

- $M[1,3,3] \in \mathcal{N}$ since $M[1,2,3] \in \mathcal{P}$ (case 1.2) is an option.
- $M[1,3,4] \in \mathcal{P}$ since the only options are $M[0,3,4] \in \mathcal{N}$ (Theorem 5.2), $M[1,2,4] \in \mathcal{N}$ (case 1.2), and $M[1,3,3]$.
- $M[1,3,5] \in \mathcal{N}$ since $M[1,3,4] \in \mathcal{P}$ is an option.
- $M[1,3,6] \in \mathcal{P}$ since the only options are $M[0,3,6] \in \mathcal{N}$ (Theorem 5.2), $M[1,2,6] \in \mathcal{N}$ (case 1.2), and $M[1,3,5] \in \mathcal{N}$.

We have the base cases. We now consider the general case on larger stacks.

- $M[1,3,4k+3] \in \mathcal{N}$ since $M[1,3,4k+2] \in \mathcal{P}$ (induction) is an option.
- $M[1,3,4k] \in \mathcal{N}$ since $M[0,3,4k] \in \mathcal{P}$ (Theorem 5.2) is an option.
- $M[1,3,4k+1] \in \mathcal{N}$ since $M[1,2,4k+1] \in \mathcal{P}$ (case 1.2) is an option.
- $M[1,3,4k+2] \in \mathcal{P}$ since the only options are $M[0,3,4k+2] \in \mathcal{N}$ (Theorem 5.2), $M[1,2,4k+2] \in \mathcal{N}$ (case 1.2), and $M[1,3,4k+1] \in \mathcal{N}$ (induction).

Case 1.4: $\ell = 1$ and $m = 4$

- $M[1,4,4] \in \mathcal{N}$ since $M[1,3,4] \in \mathcal{P}$ (case 1.3) is an option.
- $M[1,4,5] \in \mathcal{P}$ since the only options are $M[0,4,5] \in \mathcal{N}$ (Theorem 5.2), $M[1,3,5] \in \mathcal{N}$ (case 1.3), and $M[1,4,4] \in \mathcal{N}$.

We have the base cases. We now consider the general case on larger stacks.

- $M[1,4,2k] \in \mathcal{N}$ since $M[1,4,2k-1] \in \mathcal{P}$ (induction) is an option.

- $M[1,4,2k+1] \in \mathcal{P}$ since the only options are $M[0,4,2k+1] \in \mathcal{N}$ (Theorem 5.2), $M[1,3,2k+1] \in \mathcal{N}$ (case 1.3), and $M[1,4,2k] \in \mathcal{N}$.

We have the base cases. We now consider the general case on larger stacks.
Case 1.5: $\ell = 1$ and $m > 4$

- $M[1,2j+1,2k+1] \in \mathcal{N}$ since $M[1,2j+1,2k] \in \mathcal{P}$ (induction) is an option.
- $M[1,2j+1,2k] \in \mathcal{P}$ since the only options are $M[0,2j+1,2k] \in \mathcal{N}$ (Theorem 5.2), $M[1,2j,2k] \in \mathcal{N}$ (induction), and $M[1,2j+1,2k-1] \in \mathcal{N}$ (induction).
- $M[1,2j,2k+1] \in \mathcal{P}$ since the only options are $M[0,2j,2k+1] \in \mathcal{N}$ (Theorem 5.2), $M[1,2j-1,2k+1] \in \mathcal{N}$ (induction), and $M[1,2j,2k] \in \mathcal{N}$ (induction).
- $M[1,2j,2k] \in \mathcal{N}$ since $M[1,2j,2k-1] \in \mathcal{P}$ (induction) is an option.

So, we can conclude that for $m \neq 3$, $M[1,m,n] \in \mathcal{P}$ iff $m+n$ is odd.
Case 2: $\ell = 2$
Case 2.1: $\ell = 2$ and $m = 2$

- $M[2,2,4k+1]$ and $M[2,2,4k+3]$ are in $\mathcal{N}$ since $M[1,2,4k+1] \in \mathcal{P}$ (case 1.2), and $M[1,2,4k+3] \in \mathcal{P}$ (case 1.2) are options, respectively.
- $M[2,2,4k+2] \in \mathcal{N}$ since $M[0,2,4k+2] \in \mathcal{P}$ (Theorem 5.2) is an option.
- $M[2,2,4k] \in \mathcal{P}$ since the only options are $M[0,2,4k] \in \mathcal{N}$ (Theorem 5.2), $M[1,2,4k] \in \mathcal{N}$ (case 1.2), $M[2,2,4k-2] \in \mathcal{N}$ (induction), and $M[2,2,4k-1] \in \mathcal{N}$ (induction).

Case 2.2: $\ell = 2$ and $m = 3$

- $M[2,3,4k] \in \mathcal{N}$ since $M[2,2,4k] \in \mathcal{P}$ (case 2.2) is an option.
- $M[2,3,4k+1]$ and $M[2,3,4k+3]$ are in $\mathcal{N}$ since $M[1,2,4k+1] \in \mathcal{P}$ (case 1.2) and $M[1,2,4k+3] \in \mathcal{P}$ (case 1.2) are options, respectively.
- $M[2,3,4k+2] \in \mathcal{N}$ since $M[1,3,4k+2] \in \mathcal{P}$ (case 1.3) is an option.

Case 2.3: $\ell = 2$ and $m = 4$

- $M[2,4,4k+1]$ and $M[2,4,4k+3]$ are in $\mathcal{N}$ since $M[1,4,4k+1] \in \mathcal{P}$ (case 1.4), and $M[1,4,4k+3] \in \mathcal{P}$ (case 1.4) are options, respectively.

- $M[2,4,4k] \in \mathcal{N}$ since $M[2,2,4k] \in \mathcal{P}$ (case 2.2) is an option.

- $M[2,4,4k+2] \in \mathcal{P}$ since the only options are $M[0,4,4k+2] \in \mathcal{N}$ (Theorem 5.2), $M[1,4,4k+2] \in \mathcal{N}$ (case 1.4), $M[2,3,4k+2] \in \mathcal{N}$ (case 2.3), $M[2,2,4k+2] \in \mathcal{N}$ (case 2.2), $M[2,4,4k+1] \in \mathcal{N}$ (induction),
and $M[2,4,4k] \in \mathcal{N}$ (induction).

Case 2.4: $\ell = 2$ and $m > 4$

- If $m+n = 4k$, then $M[2,m,n] \in \mathcal{N}$ since $M[2,m,n-2] \in \mathcal{P}$ (induction) is an option.

- If $m+n = 4k+1$ or $m+n = 4k+3$, then $M[2,m,n] \in \mathcal{N}$ since $M[1,m,n] \in \mathcal{P}$ (case 1.5) is an option.

- If $m+n = 4k+2$, then $M[2,m,n] \in \mathcal{P}$ since the only options are $M[0,m,n] \in \mathcal{N}$ (Theorem 5.2), $M[1,m,n] \in \mathcal{N}$ (case 1.5), $M[2,m-2,n] \in \mathcal{N}$ (induction), $M[2,m-1,n] \in \mathcal{N}$ (induction), $M[2,m,n-2] \in \mathcal{N}$ (induction), and $M[2,m,n-1] \in \mathcal{N}$ (induction).

So, we conclude that for $m \neq 3$, $M[2,m,n] \in \mathcal{P}$ iff $m+n \equiv 2 \pmod 4$.
Case 3: $\ell = 3$
Case 3.1: $\ell = 3$ and $m = 3$

- $M[3,3,3] \in \mathcal{P}$ since the only options are $M[0,3,3]$, $M[1,3,3]$, or $M[2,3,3]$, all of which are in $\mathcal{N}$ (next player win by Theorem 5.2, Case 1, and Case 2, respectively).

- $M[3,3,4], M[3,3,5]$ and $M[3,3,6]$ are in $\mathcal{N}$ since $M[3,3,3] \in \mathcal{P}$ is an option (previous player win by earlier statement).

We have the base cases. We now consider the general case on larger stacks.

- $M[3,3,4k]$, $M[3,3,4k+1]$, and $M[3,3,4k+2]$ are in $\mathcal{N}$ since $M[3,3,4k-1] \in \mathcal{P}$ (induction) is an option.

- $M[3,3,4k+3] \in \mathcal{P}$ since the only options are $M[0,3,4k+3] \in \mathcal{N}$ (Theorem 5.2), $M[1,3,4k+3] \in \mathcal{N}$ (case 1.3), $M[2,3,4k+3] \in \mathcal{N}$ (case 2.3), $M[3,3,4k] \in \mathcal{N}$ (induction), $M[3,3,4k+1] \in \mathcal{N}$ (induction), and $M[3,3,4k+2] \in \mathcal{N}$ (induction).

Case 3.2: $\ell = 3$ and $m = 4$

- $M[3, 4, 4k+1]$ and $M[3, 4, 4k+3]$ are both in $\mathcal{N}$ since $M[1, 4, 4k+1] \in \mathcal{P}$ (case 1.4), and $M[1, 4, 4k+3] \in \mathcal{P}$ (case 1.4) are options, respectively.

- $M[3, 4, 4] \in \mathcal{N}$ since $M[1, 3, 4] \in \mathcal{P}$ (case 1.3) is an option.

- $M[3, 4, 6] \in \mathcal{N}$ since $M[2, 4, 6] \in \mathcal{P}$ (case 2.4) is an option.

- $M[3, 4, 8] \in \mathcal{P}$ since $M[0, 4, 8] \in \mathcal{N}$ (Theorem 5.2), $M[1, 4, 8] \in \mathcal{N}$ (case 1.4), $M[2, 4, 8] \in \mathcal{N}$ (case 2.4), $M[1, 3, 8] \in \mathcal{N}$ (case 1.3), $M[2, 3, 8] \in \mathcal{N}$ (case 2.3), $M[3, 3, 8] \in \mathcal{N}$ (case 3.3), $M[3, 4, 5] \in \mathcal{N}$, $M[3, 4, 6] \in \mathcal{N}$, and $M[3, 4, 7] \in \mathcal{N}$ (induction).

We have the base cases. We now consider the general case on larger stacks.

- $M[3, 4, 4k+1]$ and $M[3, 4, 4k+2]$ and $M[3, 4, 4k+3]$ are all in $\mathcal{N}$ since $M[3, 4, 4k] \in \mathcal{P}$ (induction) is an option.

- $M[3, 4, 4k]$ is in $\mathcal{P}$ since the only options are $M[0, 4, 4k] \in \mathcal{N}$ (Theorem 5.2), $M[1, 4, 4k] \in \mathcal{N}$ (case 1.4), $M[2, 4, 4k] \in \mathcal{N}$ (case 2.4), $M[1, 3, 4k] \in \mathcal{N}$ (case 1.3), $M[2, 3, 4k] \in \mathcal{N}$ (case 2.3), $M[3, 3, 4k] \in \mathcal{N}$ (case 3.3), $M[3, 4, 4k-3] \in \mathcal{N}$, $M[3, 4, 4k-2] \in \mathcal{N}$, and $M[3, 4, 4k-1] \in \mathcal{N}$ (induction).

Case 3.3: $\ell = 3$ and $m = 5$

- $M[3, 5, 4k]$ and $M[3, 5, 4k+2]$ are in $\mathcal{N}$ since $M[1, 5, 4k] \in \mathcal{P}$ (case 1.5), and $M[1, 5, 4k+2] \in \mathcal{P}$ (case 1.5) are options, respectively.

- $M[3, 5, 4k+1] \in \mathcal{N}$ since $M[2, 5, 4k+1] \in \mathcal{P}$ (case 2.5) is an option.

- $M[3, 5, 4k+3] \in \mathcal{N}$ since $M[3, 3, 4k+3] \in \mathcal{P}$ (case 3.3) is an option.

Case 3.4: $\ell = 3$ and $m = 6$

- $M[3, 6, 4k+1]$ and $M[3, 6, 4k+3]$ are in $\mathcal{N}$ since $M[1, 6, 4k+1] \in \mathcal{P}$ (case 1.5), and $M[1, 6, 4k+3] \in \mathcal{P}$ (case 1.5) are options, respectively.

- $M[3, 6, 4k] \in \mathcal{N}$ since $M[2, 6, 4k] \in \mathcal{P}$ (case 2.5) is an option.

- $M[3,6,4k+2] \in \mathcal{P}$ since the only options are $M[0,6,4k+2] \in \mathcal{N}$ (Theorem 5.2), $M[1,6,4k+2] \in \mathcal{N}$ (case 1.5), $M[2,6,4k+2] \in \mathcal{N}$ (case 2.5), $M[3,3,4k+2] \in \mathcal{N}$ (case 3.3), $M[3,4,4k+2] \in \mathcal{N}$ (case 3.4), $M[3,5,4k+2] \in \mathcal{N}$ (case 3.5), $M[3,6,4k-1] \in \mathcal{N}$ (induction), $M[3,6,4k] \in \mathcal{N}$ (induction), and $M[3,6,4k+1] \in \mathcal{N}$ (induction).

Case 3.5: $\ell = 3$ and $m = 7$

- $M[3,7,4k]$ and $M[3,7,4k+2]$ are in $\mathcal{N}$ since $M[1,7,4k] \in \mathcal{P}$ (case 1.5), and $M[1,7,4k+2] \in \mathcal{P}$ (case 1.5) are options, respectively.

- $M[3,7,4k+3] \in \mathcal{N}$ since $M[2,7,4k+3] \in \mathcal{P}$ (case 2.5) is an option.

- $M[3,7,4k+1] \in \mathcal{P}$ since the only options are $M[0,7,4k+1] \in \mathcal{N}$ (Theorem 5.2), $M[1,7,4k+1] \in \mathcal{N}$ (case 1.5), $M[2,7,4k+1] \in \mathcal{N}$ (case 2.5), $M[3,4,4k+1] \in \mathcal{N}$ (case 3.4), $M[3,5,4k+1] \in \mathcal{N}$ (case 3.5), $M[3,6,4k+1] \in \mathcal{N}$ (case 3.6), $M[3,7,4k-2] \in \mathcal{N}$ (induction), $M[3,7,4k-1] \in \mathcal{N}$ (induction), and $M[3,7,4k] \in \mathcal{N}$ (induction).

Case 3.6: $\ell = 3$ and $m > 7$

- If $m+n = 4k+3$ or $m+n = 4k+1$, then $M[3,m,n] \in \mathcal{N}$ since $M[1,m,n] \in \mathcal{P}$ (case 1.5) is an option.

- If $m+n = 4k+2$, then $M[3,m,n] \in \mathcal{N}$ since $M[2,m,n] \in \mathcal{P}$ (case 2.5) is an option.

- If $m+n = 4k$, then $M[3,m,n] \in \mathcal{P}$ since the only options are $M[0,m,n] \in \mathcal{N}$ (Theorem 5.2), $M[1,m,n] \in \mathcal{N}$ (case 1.5), $M[2,m,n] \in \mathcal{N}$ (case 2.5), $M[3,m-3,n] \in \mathcal{N}$ (induction), $M[3,m-2,n] \in \mathcal{N}$ (induction), $M[3,m-1,n] \in \mathcal{N}$ (induction), $M[3,m,n-3] \in \mathcal{N}$ (induction), $M[3,m,n-2] \in \mathcal{N}$ (induction), and $M[3,m,n-1] \in \mathcal{N}$ (induction).

So, we conclude that if $m > 5$, $M[3,m,n] \in \mathcal{P}$ iff $m+n \equiv 0 \pmod 4$. We have now covered all cases, and the proof is complete. □

Theorem 5.4. *Suppose* MINNIE *is being played on three stacks of sizes* 4*,* m*, and* n *with* $6 \le m \le n$*. Then, this game is in* $\mathcal{N}$*.*

Proof. In Theorem 5.3, we showed that for m and n greater than 5, $M[1,m,n] \in \mathcal{P}$ if and only if $m+n$ is odd, $M[2,m,n] \in \mathcal{P}$ if and only if $m+n = 4k+2$, and $M[3,m,n] \in \mathcal{P}$ if and only if $m+n = 4k$. So, $M[4,m,n] \in \mathcal{N}$ for m and n greater than 5. □

Question 5.5. *Few board-states are not covered by one of the previous two theorems. In particular, we ask: What are the $\mathcal{P}$ positions when $4 \leq \ell \leq 5$?*

Conjecture 5.6. *Suppose* MINNIE *is being played on k stacks, the smallest of which having greater than 3 chips and the second smallest having greater than 5 chips. Then, Player One has a winning strategy.*

5.3 MINE

Let the rules of MINE be the same as MINNIE except that at no point can any two stacks have the same number of chips. We denote a game of MINE on n stacks by $M_n[a_1, a_2, \ldots, a_n]$. We solve MINE for up to 3 stacks and conclude with a strategy for Player One on any game on 3 stacks with every stack containing at least 5 chips.

5.4 MINE on 1 and 2 stacks

We break our results into separate sections due to the game on 3 stacks requiring a lengthy proof. Note that MINE on one stack is identical to MINNIE on 1 stack.

Lemma 5.7. *Let a be a positive integer then $\mathscr{G}(M_n[a]) = a$.*

Proof. This game is the same as 1 stack NIM. □

Theorem 5.8. *Suppose Mine is being played on two stacks of sizes a, b, $a < b$. Then the $\mathcal{P}$-positions are $M_n[1, 2k], M_n[2, 4j+1], M_n[3, 4i+3]$ for all $k, j, i > 0$.*

Proof. We consider the cases in order.

Case 1: $a = 1$

- $M_n[1,2] \in \mathcal{P}$ since the only option is $M_n[0,2] \in \mathcal{N}$ by Theorem 5.7.

We have the base cases. We now consider the general case on larger stacks.

- $M_n[1, 2k+1] \in \mathcal{N}$ since $M_n[1, 2k] \in \mathcal{P}$ (induction), is an option.

- $M_n[1, 2k] \in \mathcal{P}$ since the only options are $M_n[0, 2k] \in \mathcal{N}$ (Theorem 5.7) and $M_n[1, 2k-1] \in \mathcal{N}$ (induction).

Case 2: $a = 2$

- $M_n[2, 3] \in \mathcal{N}$ since $M_n[1, 2] \in \mathcal{P}$ (Case 1), is an option.

We have the base case. We now consider the general case on larger stacks.

- $M_n[2, 4k]$ and $M_n[2, 4k+2]$ are in $\mathcal{N}$ since $M_n[1, 4k] \in \mathcal{P}$ (Case 1), and $M_n[1, 4k+2] \in \mathcal{P}$ (Case 1), are options, respectively.

- $M_n[2, 4k+1] \in \mathcal{P}$ since the only options are $M_n[0, 4k+1] \in \mathcal{N}$ (Theorem 5.7), $M_n[1, 4k+1] \in \mathcal{N}$ (Case 1), $M_n[2, 4k-1] \in \mathcal{N}$ (induction), and $M_n[2, 4k] \in \mathcal{N}$ (induction).

- $M_n[2, 4k+3] \in \mathcal{N}$ since $M_n[2, 4k+1] \in \mathcal{P}$ (induction) is an option.

Case 3: $a = 3$

- $M_n[3, 4k]$ and $M_n[3, 4k+2]$ are in $\mathcal{N}$ since $M_n[1, 4k] \in \mathcal{P}$ (Case 1), and $M_n[1, 4k+2] \in \mathcal{P}$ (Case 1) are options, respectively.

- $M_n[3, 4k+1] \in \mathcal{N}$ since $M_n[2, 4k+1] \in \mathcal{P}$ (Case 2) is an option.

- $M_n[3, 4k+3] \in \mathcal{P}$ since the only options are $M_n[0, 4k+3] \in \mathcal{N}$ (Theorem 5.7), $M_n[1, 4k+3] \in \mathcal{N}$ (Case 1), $M_n[2, 4k+3] \in \mathcal{N}$ (Case 2), $M_n[3, 4k] \in \mathcal{N}$ (induction), $M_n[3, 4k+1] \in \mathcal{N}$ (induction), and $M_n[3, 4k+2] \in \mathcal{N}$ (induction), all of which are in $\mathcal{N}$.

Case 4: $a > 3$

We now inspect Mine played on two stacks of sizes m and n with $n \geq m \geq 4$. So, $M_n[1, n]$, $M_n[2, n]$, and $M_n[3, n]$ are all options. If $n = 4k$ or $n = 4k+2$, then $[1, n] \in \mathcal{P}$ (Case 1). If $n = 4k+1$, then $[2, n] \in \mathcal{P}$ (Case 2). If $n = 4k+3$, then $[3, n] \in \mathcal{P}$ (Case 3). So, no matter what n is, there exists an option in $\mathcal{P}$. Thus, $M_n[m, n] \in \mathcal{N}$ for all $4 \leq m \leq n$. □

5.5 MINE on 3 stacks

We now look at MINE on 3 stacks. Since this problem is solved through such a lengthy process, we do not solve this problem in a single theorem. We instead break it into smaller theorems and cases and reference them throughout. We conclude this section with a strategy for Player One when playing on 3 stacks of sizes greater than 4.

Theorem 5.9. *Suppose Mine is being played on three stacks of sizes* $1, 2, c$ *with* $c > 4$. *Then,* $M_n[1, 2, c] \in \mathcal{P}$ *if and only if* $c \equiv 0 \ (mod\ 2)$.

Proof. We consider the cases in order.

- $M_n[1, 2, 3] \in \mathcal{P}$ since $M_n[0, 2, 3] \in \mathcal{N}$ (Theorem 5.8) is the only option.
- $M_n[1, 2, 4] \in \mathcal{N}$ since $M_n[1, 2, 3] \in \mathcal{P}$ is an option.
- $M_n[1, 2, 5] \in \mathcal{N}$ since $M_n[0, 2, 5] \in \mathcal{P}$ (Theorem 5.8) is an option.
- $M_n[1, 2, 2k] \in \mathcal{P}$ since the only options are $M_n[0, 2, 2k] \in \mathcal{N}$ (Theorem 5.8) and $M_n[1, 2, 2k - 1] \in \mathcal{N}$ (induction).
- $M_n[1, 2, 2k + 1] \in \mathcal{N}$ since $M_n[1, 2, 2k] \in \mathcal{P}$ (induction) is an option.

□

Theorem 5.10. *Suppose Mine is being played on three stacks of sizes* $1, 3, c$ *with* $c > 5$. *Then,* $M_n[1, 3, c] \in \mathcal{P}$ *if and only if* $c \equiv 1 \ (mod\ 4)$.

Proof. We consider the cases in order.

- $M_n[1, 3, 4] \in \mathcal{P}$ since the only options are $M_n[0, 3, 4] \in \mathcal{N}$ (Theorem 5.8) and $M_n[1, 2, 4] \in \mathcal{N}$ (by the proof of Theorem 5.9).
- $M_n[1, 3, 5] \in \mathcal{N}$ since $M_n[1, 3, 4] \in \mathcal{P}$ is an option.
- $M_n[1, 3, 4k + 2] \in \mathcal{N}$ since $M_n[1, 2, 4k + 2] \in \mathcal{P}$ (Theorem 5.9) is an option.
- $M_n[1, 3, 4k + 3] \in \mathcal{N}$ since $M_n[0, 3, 4k + 3] \in \mathcal{P}$ (Theorem 5.8) is an option.
- $M_n[1, 3, 4k] \in \mathcal{N}$ since $M_n[1, 2, 4k] \in \mathcal{P}$ (Theorem 5.9) is an option.
- $M_n[1, 3, 4k + 1] \in \mathcal{P}$ since the only options are $M_n[0, 3, 4k + 1] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, 2, 4k+1] \in \mathcal{N}$ (Theorem 5.9), and $M_n[1, 3, 4k] \in \mathcal{N}$ (induction).

□

Theorem 5.11. *Suppose Mine is being played on three stacks of sizes $1, b, c$ with $3 < b < c$. Then, $M_n[1, b, c] \in \mathcal{P}$ if and only if*

$$M_n[1,b,c] = \begin{cases} c = b+1 \\ c = b+3 \\ c > b+4 \text{ and } b+c \equiv 0 \ (mod\ 2) \end{cases}$$

Proof. We consider the cases in order.

- $M_n[1, 4, 5] \in \mathcal{P}$ since the only options are $M_n[0, 4, 5] \in \mathcal{N}$ (Theorem 5.8), and $M_n[1, 3, 5] \in \mathcal{N}$ (by the proof of Theorem 5.10).
- $M_n[1, 4, 6] \in \mathcal{N}$ since $M_n[1, 4, 5] \in \mathcal{P}$ is an option.
- $M_n[1, 4, 7] \in \mathcal{P}$ since the only options are $M_n[0, 4, 7] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, 3, 7] \in \mathcal{N}$ (Theorem 5.10), and $M_n[1, 4, 6] \in \mathcal{N}$.
- $M_n[1, 4, 8] \in \mathcal{N}$ since $M_n[1, 4, 7] \in \mathcal{P}$ is an option.
- $M_n[1, 4, 4k + 1] \in \mathcal{N}$ since $M_n[1, 3, 4k + 1] \in \mathcal{P}$ (Theorem 5.10) is an option.
- $M_n[1, 4, 4k + 2] \in \mathcal{P}$ since the only options are $M_n[0, 4, 4k + 2] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, 3, 4k + 2] \in \mathcal{N}$ (Theorem 5.10), and $M_n[1, 4, 4k + 1] \in \mathcal{N}$ (induction).
- $M_n[1, 4, 4k + 3] \in \mathcal{N}$ since $M_n[1, 4, 4k + 2] \in \mathcal{P}$ (induction) is an option.
- $M_n[1, 4, 4k] \in \mathcal{P}$ since the only options are $M_n[0, 4, 4k] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, 3, 4k] \in \mathcal{N}$ (Theorem 5.10), and $M_n[1, 4, 4k - 1] \in \mathcal{N}$ (induction).
- $M_n[1, j, j + 1] \in \mathcal{P}$ since the only options are $M_n[0, j, j + 1] \in \mathcal{N}$ (Theorem 5.8) and $M_n[1, j - 1, j + 1] \in \mathcal{N}$ (induction).
- $M_n[1, j, j + 2] \in \mathcal{N}$ since $M_n[1, j, j + 1] \in \mathcal{P}$ (induction) is an option.
- $M_n[1, j, j + 3] \in \mathcal{P}$ since the only options are $M_n[0, j, j + 3] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, j - 1, j + 3] \in \mathcal{N}$ (induction), or $M_n[1, j, j + 2] \in \mathcal{N}$ (induction).
- $M_n[1, j, j + 4] \in \mathcal{N}$ since $M_n[1, j, j + 3] \in \mathcal{P}$ (induction) is an option.
- $M_n[1, j, j + 5] \in \mathcal{N}$ since $M_n[1, j - 1, j + 5] \in \mathcal{P}$ (induction) is an option.

- If $j + i = 2k$, then $M_n[1, j, i] \in \mathcal{P}$ since the only options are $M_n[0, j, i] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, j-1, i] \in \mathcal{N}$ (induction), $M_n[1, j, i-1] \in \mathcal{N}$ (induction).
- If $j + i = 2k + 1$, then $M_n[1, j, i] \in \mathcal{N}$ since $M_n[1, j, i-1] \in \mathcal{P}$ (induction) is an option.

□

Theorem 5.12. *Suppose Mine is being played on three stacks of sizes $2, 3, c$ with $c > 3$. Then, $M_n[2, 3, c] \in \mathcal{P}$ if and only if $c = 5$.*

Proof. We consider the cases in order.

- $M_n[2, 3, 4] \in \mathcal{N}$ since $M_n[1, 3, 4] \in \mathcal{P}$ (by the proof of Theorem 5.10) is an option.
- $M_n[2, 3, 5] \in \mathcal{P}$ since the only options are $M_n[0, 3, 5] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, 3, 5] \in \mathcal{N}$ (by the proof of Theorem 5.10), $M_n[1, 2, 5] \in \mathcal{N}$ (Theorem 5.9), and $M_n[2, 3, 4] \in \mathcal{N}$.
- $M_n[2, 3, 4k + 2] \in \mathcal{N}$ since $M_n[1, 2, 4k + 2] \in \mathcal{P}$ (Theorem 5.9) is an option.
- $M_n[2, 3, 4k + 3] \in \mathcal{N}$ since $M_n[0, 3, 4k + 3] \in \mathcal{P}$ (Theorem 5.8) is an option.
- $M_n[2, 3, 4k] \in \mathcal{N}$ since $M_n[1, 2, 4k] \in \mathcal{P}$ (Theorem 5.9) is an option.
- $M_n[2, 3, 4k + 1] \in \mathcal{N}$ since $M_n[1, 3, 4k + 1] \in \mathcal{P}$ (Theorem 5.10) is an option.

□

Theorem 5.13. *Suppose Mine is being played on three stacks of sizes $2, b, c$ with $3 < b < c$. Then, $M_n[2, b, c] \in \mathcal{P}$ if and only if*

$$M(2, b, c) = \begin{cases} c = b + 2 \\ c > b + 2 \text{ and } b \equiv 0 \pmod 2 \text{ and } b + c \equiv 1 \pmod 4 \\ c > b + 2 \text{ and } b \equiv 1 \pmod 2 \text{ and } b + c \equiv 3 \pmod 4). \end{cases}$$

Proof. We consider the cases in order.

- $M_n[2, 4, 5] \in \mathcal{N}$ since $M_n[1, 4, 5] \in \mathcal{P}$ (Theorem 5.11) is an option.

- $M_n[2,4,6] \in \mathcal{P}$ since the only options are $M_n[0,4,6] \in \mathcal{N}$ (Theorem 5.8), $M_n[1,4,6] \in \mathcal{N}$ (Theorem 5.11), $M_n[2,3,6] \in \mathcal{N}$ (Theorem 5.12), and $M_n[2,4,5] \in \mathcal{N}$.

- $M_n[2,4,7] \in \mathcal{N}$ since $M_n[1,4,7] \in \mathcal{P}$ (Theorem 5.11) is an option.

We have the base cases. We now consider the general case on larger stacks.

- $M_n[2,4,4k] \in \mathcal{N}$ since $M_n[1,4,4k] \in \mathcal{P}$ (Theorem 5.11) is an option.

- $M_n[2,4,4k+1] \in \mathcal{P}$ since the only options are $M_n[0,4,4k+1] \in \mathcal{N}$ (Theorem 5.8), $M_n[1,4,4k+1] \in \mathcal{N}$ (Theorem 5.11), $M_n[2,3,4k+1] \in \mathcal{N}$ (Theorem 5.12), $M_n[2,4,4k-1] \in \mathcal{N}$ (induction), and $M_n[2,4,4k] \in \mathcal{N}$ (induction).

- $M_n[2,4,4k+2] \in \mathcal{N}$ since $M_n[1,4,4k+2] \in \mathcal{P}$ (Theorem 5.11) is an option.

- $M_n[2,4,4k+3] \in \mathcal{N}$ since $M_n[2,4,4k+1] \in \mathcal{P}$ (induction) is an option.

We have the base case. We now consider the general case on larger stacks.

- $M_n[2,j,j+1] \in \mathcal{N}$ since $M_n[1,j,j+1] \in \mathcal{P}$ (Theorem 5.11) is an option.

- $M_n[2,j,j+2] \in \mathcal{P}$ since the only options are $M_n[0,j,j+2] \in \mathcal{N}$ (Theorem 5.8), $M_n[1,j,j+2] \in \mathcal{N}$ (Theorem 5.11), $M_n[2,j-2,j+2] \in \mathcal{N}$ (This is true since $M_n[2,j-2,j] \in \mathcal{P}$ by induction and Theorem 5.12), $M_n[2,j-1,j+2] \in \mathcal{N}$ (This is true since $M_n[2,j-1,j+1] \in \mathcal{P}$ by induction), and $M_n[2,j,j+1] \in \mathcal{N}$ (induction).

- $M_n[2,j,j+3] \in \mathcal{N}$ since $M_n[2,j,j+2] \in \mathcal{P}$ (induction) is an option.

- $M_n[2,j,j+4] \in \mathcal{N}$ since $M_n[2,j,j+2] \in \mathcal{P}$ (induction) is an option.

- If $j = 2k+1$ and $i+j = 4\ell+1$, then $M_n[2,j,i] \in \mathcal{N}$ since $M_n[2,j,i-2] \in \mathcal{P}$ (induction) is an option.

- If $j = 2k+1$ and $i+j = 4\ell+2$, then $M_n[2,j,i] \in \mathcal{N}$ since $M_n[1,j,i] \in \mathcal{P}$ (Theorem 5.11) is an option.

- If $j = 2k+1$ and $i+j = 4\ell+3$, then $M_n[2,j,i] \in \mathcal{P}$ since the only options are $M_n[0,j,i] \in \mathcal{N}$ (Theorem 5.8), $M_n[1,j,i] \in \mathcal{N}$ (Theorem 5.11), $M_n[2,j-2,i] \in \mathcal{N}$ (induction), $M_n[2,j-1,i] \in \mathcal{N}$ (induction), $M_n[2,j,i-2] \in \mathcal{N}$ (induction), and $M_n[2,j,i-2] \in \mathcal{N}$ (induction).

- If $j = 2k+1$ and $i+j = 4\ell$, then $M_n[2,j,i] \in \mathcal{N}$ since $M_n[1,j,i] \in \mathcal{P}$ (Theorem 5.11) is an option.

- If $j = 2k$ and $i+j = 4\ell+1$, then $M_n[2,j,i] \in \mathcal{P}$ since the only options are $M_n[0,j,i] \in \mathcal{N}$ (Theorem 5.8), $M_n[1,j,i] \in \mathcal{N}$ (Theorem 5.11), $M_n[2,j-2,i] \in \mathcal{N}$ (induction), $M_n[2,j-1,i] \in \mathcal{N}$ (induction), $M_n[2,j,i-2] \in \mathcal{N}$ (induction), and $M_n[2,j,i-1] \in \mathcal{N}$ (induction).

- If $j = 2k$ and $i+j = 4\ell+2$, then $M_n[2,j,i] \in \mathcal{N}$ since $M_n[1,j,i] \in \mathcal{P}$ (Theorem 5.11) is an option.

- If $j = 2k$ and $i+j = 4\ell+3$, then $M_n[2,j,i] \in \mathcal{N}$ since $M_n[2,j,i-2] \in \mathcal{P}$ (induction) is an option.

- If $j = 2k$ and $i+j = 4\ell$, then $M_n[2,j,i] \in \mathcal{N}$ since $M_n[1,j,i] \in \mathcal{P}$ (Theorem 5.11) is an option.

□

Theorem 5.14. *Suppose Mine is being played on three stacks of sizes $3, b, c$ with $3 < b < c$. Then, $M_n[3,b,c] \in \mathcal{P}$ if and only if*

$$M(3,b,c) = \begin{cases} c = b+4 \\ c \neq b+3 \text{ and } c \neq b+7 \text{ and } b \equiv 1 \ (mod\ 2) \text{ and } b+c \equiv 1 \ (mod\ 4) \\ c \neq b+3 \text{ and } c \neq b+7 \text{ and } b \equiv 0 \ (mod\ 2) \text{ and } b+c \equiv 3 \ (mod\ 4) \end{cases}$$

Proof. We consider the cases in order.

- $M_n[3,4,5] \in \mathcal{N}$ since $M_n[1,4,5] \in \mathcal{P}$ (Theorem 5.11) is an option.

- $M_n[3,4,6] \in \mathcal{N}$ since $M_n[2,4,6] \in \mathcal{P}$ (Theorem 5.13) is an option.

- $M_n[3,4,7] \in \mathcal{N}$ since $M_n[1,4,7] \in \mathcal{P}$ (Theorem 5.11) is an option.

- $M_n[3,4,8] \in \mathcal{P}$ since the only options are
  $M_n[0,4,8] \in \mathcal{N}$ (Theorem 5.8), $M_n[1,4,8] \in \mathcal{N}$ (Theorem 5.11),
  $M_n[2,4,8] \in \mathcal{N}$ (Theorem 5.13), $M_n[1,3,8] \in \mathcal{N}$ (Theorem 5.10),
  $M_n[2,3,8] \in \mathcal{N}$ (Theorem 5.12), $M_n[3,4,5] \in \mathcal{N}$,
  $M_n[3,4,6] \in \mathcal{N}$, and $M_n[3,4,7] \in \mathcal{N}$.

- $M_n[3,4,9] \in \mathcal{N}$ since $M_n[3,4,8] \in \mathcal{P}$ is an option.
- $M_n[3,4,10] \in \mathcal{N}$ since $M_n[3,4,8] \in \mathcal{P}$ is an option.
- $M_n[3,4,11] \in \mathcal{N}$ since $M_n[3,4,8] \in \mathcal{P}$ is an option.

We have the base cases. We now consider the general case on larger stacks.

- $M_n[3,4,4k] \in \mathcal{N}$ since $M_n[1,4,4k] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[3,4,4k+1] \in \mathcal{N}$ since $M_n[1,3,4k+1] \in \mathcal{P}$ (Theorem 5.10) is an option.
- $M_n[3,4,4k+2] \in \mathcal{N}$ since $M_n[1,4,4k+2] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[3,4,4k+3] \in \mathcal{P}$ since the only options are $M_n[0,4,4k+3] \in \mathcal{N}$ (Theorem 5.8), $M_n[1,4,4k+3] \in \mathcal{N}$ (Theorem 5.11), $M_n[2,4,4k+3] \in \mathcal{N}$ (Theorem 5.13), $M_n[1,3,4k+3] \in \mathcal{N}$ (Theorem 5.10), $M_n[2,3,4k+3] \in \mathcal{N}$ (Theorem 5.12), $M_n[3,4,4k] \in \mathcal{N}$ (induction), $M_n[3,4,4k+1] \in \mathcal{N}$ (induction), and $M_n[3,4,4k+2] \in \mathcal{N}$ (induction).
- $M_n[3,j,j+1] \in \mathcal{N}$ since $M_n[1,j,j+1] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[3,j,j+2] \in \mathcal{N}$ since $M_n[2,j,j+2] \in \mathcal{P}$ (Theorem 5.13) is an option.
- $M_n[3,j,j+3] \in \mathcal{N}$ since $M_n[1,j,j+3] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[3,j,j+4] \in \mathcal{P}$ since the only options are $M_n[0,j,j+4] \in \mathcal{N}$ (Theorem 5.8), $M_n[1,j,j+4] \in \mathcal{N}$ (Theorem 5.11), $M_n[2,j,j+4] \in \mathcal{N}$ (Theorem 5.13), $M_n[3,j-3,j+4] \in \mathcal{N}$ (induction), $M_n[3,j-2,j+4] \in \mathcal{N}$ (induction), $M_n[3,j-1,j+4] \in \mathcal{N}$ (This is true since $M_n[2,j-1,j+4] \in \mathcal{P}$ by Theorem 5.13 and by induction), $M_n[3,j,j+1] \in \mathcal{N}$ (induction), $M_n[3,j,j+2] \in \mathcal{N}$ (induction), and $M_n[3,j,j+3] \in \mathcal{N}$ (induction).
- $M_n[3,j,j+5] \in \mathcal{N}$ since $M_n[3,j,j+4] \in \mathcal{P}$ (induction) is an option.
- $M_n[3,j,j+6] \in \mathcal{N}$ since $M_n[3,j,j+4] \in \mathcal{P}$ (induction) is an option.
- $M_n[3,j,j+7] \in \mathcal{N}$ since $M_n[3,j,j+4] \in \mathcal{P}$ (induction) is an option.

We have the base cases. We now consider the general case on larger stacks.

- If $j = 2k$ and $i+j = 4\ell$, then $M_n[3,j,i] \in \mathcal{N}$ since $M_n[1,j,i] \in \mathcal{P}$ (Theorem 5.11) is an option.

- If $j = 2k$ and $i + j = 4\ell + 1$, then $M_n[3, j, i] \in \mathcal{N}$ since $M_n[2, j, i] \in \mathcal{P}$ (Theorem 5.13) is an option.

- If $j = 2k$ and $i + j = 4\ell + 2$, then $M_n[3, j, i] \in \mathcal{N}$ since $M_n[1, j, i] \in \mathcal{P}$ (Theorem 5.11) is an option.

- If $j = 2k$ and $i + j = 4\ell + 3$, then $M_n[3, j, i] \in \mathcal{P}$ since the only options are $M_n[0, j, i] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, j, i] \in \mathcal{N}$ (Theorem 5.11), $M_n[2, j, i] \in \mathcal{N}$ (Theorem 5.13), $M_n[3, j-3, i] \in \mathcal{N}$ (induction), $M_n[3, j-2, i] \in \mathcal{N}$ (induction), $M_n[3, j-1, i] \in \mathcal{N}$ (induction), $M_n[3, j, i-3] \in \mathcal{N}$ (induction), $M_n[3, j, i-2] \in \mathcal{N}$ (induction), and $M_n[3, j, i-1] \in \mathcal{N}$ (induction).

- If $j = 2k + 1$ and $i + j = 4\ell + 1$, then $M_n[3, j, i] \in \mathcal{P}$ since the only options are $M_n[0, j, i] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, j, i] \in \mathcal{N}$ (Theorem 5.11), $M_n[2, j, i] \in \mathcal{N}$ (Theorem 5.13), $M_n[3, j-3, i] \in \mathcal{N}$ (induction), $M_n[3, j-2, i] \in \mathcal{N}$ (induction), $M_n[3, j-1, i] \in \mathcal{N}$ (induction), $M_n[3, j, i-3] \in \mathcal{N}$ (induction), $M_n[3, j, i-2] \in \mathcal{N}$ (induction), and $M_n[3, j, i-1] \in \mathcal{N}$ (induction).

- If $j = 2k + 1$ and $i + j = 4\ell + 2$, then $M_n[3, j, i] \in \mathcal{N}$ since $M_n[1, j, i] \in \mathcal{P}$ (Theorem 5.11) is an option.

- If $j = 2k + 1$ and $i + j = 4\ell + 3$, then $M_n[3, j, i] \in \mathcal{N}$ since $M_n[2, j, i] \in \mathcal{P}$ (Theorem 5.13) is an option.

- If $j = 2k + 1$ and $i + j = 4\ell$, then $M_n[3, j, i] \in \mathcal{N}$ since $M_n[1, j, i] \in \mathcal{P}$ (Theorem 5.11) is an option.

□

Theorem 5.15. *Suppose Mine is being played on three stacks of sizes $4, b, c$ with $4 < b < c$. Then, $M_n[4, b, c] \in \mathcal{P}$ if and only if $c = b + 7$ and $b \neq 5$ and $b \neq 6$.*

Proof. We consider the cases in order.

- $M_n[4, 5, 6] \in \mathcal{N}$ since $M_n[2, 4, 6] \in \mathcal{P}$ (Theorem 5.13) is an option.
- $M_n[4, 5, 7] \in \mathcal{N}$ since $M_n[1, 4, 7] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[4, 5, 8] \in \mathcal{N}$ since $M_n[1, 5, 8] \in \mathcal{P}$ (Theorem 5.11) is an option.

We have the base cases. We now consider the general case on larger stacks.

- $M_n[4, 5, 2k + 1] \in \mathcal{N}$ since $M_n[1, 5, 2k + 1] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[4, 5, 2k] \in \mathcal{N}$ since $M_n[1, 4, 2k] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[4, 6, 7] \in \mathcal{N}$ since $M_n[1, 6, 7] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[4, 6, 8] \in \mathcal{N}$ since $M_n[3, 4, 8] \in \mathcal{P}$ (Theorem 5.14 is an option.

We have the base cases. We now ocnsider the general case on larger stacks.

- $M_n[4, 6, 4k + 1] \in \mathcal{N}$ since $M_n[2, 4, 4k + 1] \in \mathcal{P}$ (Theorem 5.13) is an option.
- $M_n[4, 6, 4k + 2] \in \mathcal{N}$ since $M_n[1, 6, 4k + 2] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[4, 6, 4k + 3] \in \mathcal{N}$ since $M_n[2, 6, 4k + 3] \in \mathcal{P}$ (Theorem 5.13) is an option.
- $M_n[4, 6, 4k] \in \mathcal{N}$ since $M_n[1, 6, 4k] \in \mathcal{P}$ (Theorem 5.11) is an option.

We have the base cases. We now consider the general case on larger stacks.

- $M_n[4, j, j + 1] \in \mathcal{N}$ since $M_n[1, j, j + 1] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[4, j, j + 2] \in \mathcal{N}$ since $M_n[2, j, j + 2] \in \mathcal{P}$ (Theorem 5.13) is an option.
- $M_n[4, j, j + 3] \in \mathcal{N}$ since $M_n[1, j, j + 3] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[4, j, j + 4] \in \mathcal{N}$ since $M_n[3, j, j + 4] \in \mathcal{P}$ (Theorem 5.14) is an option.
- $M_n[4, j, j + 5] \in \mathcal{N}$ since $M_n[2, j, j + 5] \in \mathcal{P}$ (Theorem 5.13) is an option.
- $M_n[4, j, j + 6] \in \mathcal{N}$ since $M_n[1, j, j + 6] \in \mathcal{P}$ (Theorem 5.11) is an option.
- $M_n[4, j, j + 7] \in \mathcal{P}$ since the only options are $M_n[0, j, j + 7] \in \mathcal{N}$ (Theorem 5.8), $M_n[1, j, j + 7] \in \mathcal{N}$ (Theorem 5.11), $M_n[2, j, j + 7] \in \mathcal{N}$ (This true since $M_n[2, j, j + 5] \in \mathcal{P}$ by Theorem 5.13), $M_n[3, j, j + 7] \in \mathcal{N}$ (Theorem 5.14), $M_n[4, j - 4, j + 7] \in \mathcal{N}$ (induction), $M_n[4, j - 3, j + 7] \in \mathcal{N}$ (induction), $M_n[4, j - 2, j + 7] \in \mathcal{N}$ (induction),$M_n[4, j - 1, j + 7] \in \mathcal{N}$ (induction), $M_n[4, j, j + 3] \in \mathcal{N}$ (induction), $M_n[4, j, j + 4] \in \mathcal{N}$ (induction), $M_n[4, j, j + 5] \in \mathcal{N}$ (induction), and $M_n[4, j, j + 6] \in \mathcal{N}$ (induction).
- If $i = j + 4k$, then $M_n[4, j, i] \in \mathcal{N}$ since $M_n[1, j, i] \in \mathcal{P}$ (Theorem 5.11) is an option.

- If $i = j + 4k + 1$, then $M_n[4, j, i] \in \mathcal{N}$ since $M_n[2, j, i] \in \mathcal{P}$ (Theorem 5.13) is an option.

- If $i = j + 4k + 2$, then $M_n[4, j, i] \in \mathcal{N}$ since $M_n[1, j, i] \in \mathcal{P}$ (Theorem 5.11) is an option.

- If $i = j + 4k + 3$, then $M_n[4, j, i] \in \mathcal{N}$ since $M_n[3, j, i] \in \mathcal{P}$ (Theorem 5.14) is an option.

□

So, we get the following winning strategy for Player One in $M_n[a, b, c]$ with $4 < a < b < c$:

If $c = b + 1 \rightarrow M_n[1, b, c]$
If $c = b + 2 \rightarrow M_n[2, b, c]$
If $c = b + 3 \rightarrow M_n[1, b, c]$
If $c = b + 4 \rightarrow M_n[3, b, c]$
If $c = b + 5 \rightarrow M_n[2, b, c]$
If $c = b + 6 \rightarrow M_n[1, b, c]$
If $c = b + 7 \rightarrow M_n[4, b, c]$
else if $b \equiv 1 \pmod 2$ and $c \equiv 1 \pmod 2 \rightarrow M_n[1, b, c]$
else if $b \equiv 0 \pmod 2$ and $c \equiv 0 \pmod 2 \rightarrow M_n[1, b, c]$
else if $c = b + m, m \equiv 1 \pmod 4 \rightarrow M_n[2, b, c]$
else if $c = b + m, m \equiv 3 \pmod 4 \rightarrow M_n[3, b, c]$.

Chapter 6

Conclusion and Open Questions

We have mostly solved the outcomes of MINNIE for up to 3 stacks. Beyond completely solving the game for 3 stacks, the most obvious next step is to expand that result to n stacks. This seems difficult. We have conjectured that any game with a minimum stack value of at least 4 and a second least stack value of at least 6 is in $\mathcal{N}$. So, we have an amount of confidence that any game of that form has an option in $\mathcal{P}$. However, we do not yet know where all of those options are. What we have noticed is that the $\mathcal{P}$-positions often follow a simple pattern (similar to the one used to solve MINNIE on 3 stacks) except for games with multiple stacks of value 5. Games of this form seem to follow a pattern all their own and have evaded our grasp to this point. Also, we did not look into misère MINNIE or MINNIE on graphs. We have no reason to believe those results are significantly more difficult that those of MINNIE and WYG, so these may be some of the more reasonable open problems that arise from this thesis. The best way to define MINNIE on graphs may be as a game in which each turn the active player chooses a vertex and removes up to the minimum stack size in that vertex's closed neighbourhood from that vertex. We call this game MIG and we note that any game played with every vertex starting with 1 chip is trivial, with the winner being decided through parity.

Question 6.1. *What are the $\mathcal{P}$ positions for* MINNIE *on n stacks for $n \geq 4$?*

Conjecture 6.2. *Suppose* MINNIE *is played on n stacks of sizes $a_1, a_2, \dots, a_n$ with $a_1 \leq a_2 \leq \cdots \leq a_n$, $a_1 \geq 4$, $a_2 \geq 6$. Then $M[a_1, a_2, \dots, a_n] \in \mathcal{N}$.*

Question 6.3. *What graphs and board states are $\mathcal{P}$ positions in* MIG*?*

We have solved NIMB, NIMTB, WINNIE, and misère WINNIE for outcomes. So, there are not any obvious open questions. However, it seems a reasonable next step would be to find some results for these games when played on graphs, or, in the case of WINNIE and misère WINNIE, one could look into developing a rule set that can accommodate more than 2 stacks. WINNIE's expansion into n stacks is not as obvious as many other games. While it seems obvious to maintain the option of removing some number of chips from a single stack, the option of removing chips from both stacks may have different interpretations on n stacks. Perhaps this rule becomes that the active player may remove chips from exactly two stacks, or that the active player may remove chips from up to n stacks, or that the active player may remove chips form exactly n stacks. We have not looked into any of the variants and thus, do not know which, if any, will yield simple or intuitive results.

To say a little more about WYG in particular, there is still much work to be done. We looked at many obvious graph classes and we have some solutions, but some seemingly simple problems still linger. In particular, $W_{dg}(K_n)$ and the general statement from Theorem 4.21, Theorem 4.22, Theorem 4.23, and Theorem 4.24. If G_n represents a graph of two stars connected by a path of length n, we conjecture that $W_1(G)$ does not continue the pattern that we showed of always being in $\mathcal{N}$.

Question 6.4. *What are the $\mathcal{P}$ positions of $W_{dg}(K_n)$?*

Here, we quickly introduce another way of playing WYNNIE on graphs. We call this game WYGTHOFF due to its resemblance to the game WYTHOFF on stacks. The rules are equivalent to those of WYG with the exception that after the edge is chosen, it is legal to remove up to all of the chips from the vertex with the larger stack. So, if WYGTHOFF were being played and an edge was chosen with vertices a and b having stacks of sizes 2 and 3, respectively, the legal moves are to remove up to 2 chips from a, up to 3 chips from b, or up to 2 chips from both.

Also, although we did not cover them in this thesis, SUSEN, JENNIFER, and WYGTHOFF are still unsolved for most values.

Question 6.5. *What are the $\mathcal{P}$ positions for* SUSEN *on n stacks, $n \geq 4$?*

Question 6.6. *What are the $\mathcal{P}$ positions for* JENNIFER *on n stacks, $n \geq 3$?*

Question 6.7. *What are the $\mathcal{P}$ positions for* WYGTHOFF *played on K_3? What are the $\mathcal{P}$ positions for* WYGTHOFF *played on a disjunctive sum of K_3's? What are the $\mathcal{P}$ positions for* WYGTHOFF *played on K_n?*

Bibliography

[1] M. H. Albert and R. J. Nowakowski, Nim restrictions, INTEGERS, 2004, **4**, #G1, 10pp..

[2] M. H. Albert, R. J. Nowakowski, and D. Wolfe, Lessons in Play, A K Peters, Wellesley, MA, 2007.

[3] E. R. Berlekamp, J. H. Conway and R. K. Guy, Winning Ways for your Mathematical Plays, A K Peters, Wellesley, MA, 2001-2004, 1-4.

[4] A. J. Cole and A. J. T. Davie, A game based on the euclidean algorithm and a winning strategy for it, The Mathematical Gazette, 1969, Vol. 35, No. 386, 354–357.

[5] A. Fink, personal communication with R.J. Nowakowski.

[6] R.P. Gallant, G. Gunther, B.L. Hartnell, D.F. Rall, A game of edge removal on graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **57** (2006) 75–82.

[7] L. Levine, Fractal Sequences and Restricted Nim, *Ars Combinatoria*, 2006, LXXX, 113–128.

[8] N. A. McKay and R. J. Nowakowksi, Outcomes of partizan euclid, Combinatorial Number Theory, 2013, pp. 123–137.

[9] W. A. Wythoff, A modification of the game of Nim *Nieuw Arch. Wisk.*, 1907, 7, 199-202.