ON THE COMBINATORICS OF RESOLUTIONS OF MONOMIAL IDEALS

by

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Abstract

In this thesis, we investigate the relation between invariants of minimal free resolutions of monomial ideals and combinatorial properties of simplicial complexes. We provide a sufficient combinatorial condition for monomial ideals to have nonzero Betti numbers and show that such a condition completely characterizes Betti numbers of facet ideals of simplicial forests. We also present a new approach to computing Betti numbers of path ideals of certain graph classes.

List of Abbreviations and Symbols Used

Notation	Description
$\alpha_0'(\Gamma)$	maximum cardinality of a minimal vertex cover of Γ
$\{\emptyset\}$	the irrelevant complex
$\{\}$	the void complex
$\Bbbk$	field
$b_i(I)$	the ith total Betti number of I
$b_{i,j}(I)$	graded Betti number of I
$b_{i,\mathbf{m}}(I)$	multigraded Betti number of I
C_n	cycle of order n
$\mathrm{del}_\Gamma(\sigma)$	the deletion of Γ with respect to σ
$\delta_{i,j}$	Kronecker delta function
$E(G)$	edge set of a graph G
$\mathrm{Facets}(\Gamma)$	the set of facets of Γ
$F(\mathcal{B})$	the flower set of a bouquet $\mathcal{B}$
$\mathcal{F}(\Gamma)$	facet ideal of Γ
Γ_A	the induced subcollection of Γ on A
$\Gamma_{\mathbf{m}}$	the induced subcollection of Γ on the set of vertices which divide the monomial $\mathbf{m}$
Γ_P	the facet complex of $\mathcal{F}(\Gamma)_P$
$\Gamma \setminus \langle F_1, \ldots, F_p \rangle$	simplicial complex whose facet set is $\mathrm{Facets}(\Gamma) \setminus \{F_1, \ldots, F_p\}$
$\tilde{H}_i(\Gamma, \Bbbk)$	the ith reduced homology of Γ over $\Bbbk$
$I(G)$	edge ideal of G
$I_t(G)$	path ideal of G

Notation	**Description**
$im(G)$	the induced matching number of G
I_P	localization of I at P
$\langle F_1, \ldots, F_q \rangle$	simplicial complex generated by the facets $F_1, \ldots, F_q$
$\Lambda_{I,<}$	Lyubeznik simplicial complex associated to I and $<$
lcm	least common multiple
P_n	path of order n
$\mathbb{N}$	the natural numbers $\{0, 1, 2, \ldots\}$
$\mathrm{pd}(I)$	projective dimension of I
$\mathrm{reg}(I)$	regularity of I
S	$\Bbbk[x_1, \ldots, x_n]$, the polynomial ring
$\mathcal{S}_n$	star of size n
$\mathrm{Taylor}(I)$	the Taylor simplex of I
$V(\Gamma)$	vertex set of a simplicial complex Γ
$V(G)$	vertex set of a graph G

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Chapter 1

Introduction

A central topic in commutative algebra is minimal free resolutions over polynomial rings. A particularly interesting case of this topic is the minimal free resolutions of monomial ideals. Although it is possible to give inductive procedures to construct minimal free resolutions of monomial ideals, no explicit description is known in general. By virtue of the polarization method [18, 45] the study of resolutions of monomial ideals can be restricted to squarefree monomial ideals [39, Theorem 21.10]. In the squarefree case one can set a one-to-one correspondence between monomial ideals and simplicial complexes thus connecting algebra and combinatorics. Two common ways of setting such a correspondence has been traditionally through Stanley-Reisner ideals of Stanley [41] and Reisner [40] and facet ideals of Faridi [17] (or, edge ideals of Villarreal [43] for the quadratic monomial ideals). In this thesis we take the second view and study squarefree monomial ideals via facet ideals.

The main theme of this thesis is to investigate the relation between combinatorial properties of simplicial complexes and invariants of minimal free resolutions of their facet ideals. In Chapter 2 we start with the necessary background. Chapter 3 is devoted to some technical preliminary results on minimal vertex covers which will be used in the subsequent two chapters. In Chapter 4 we generalize the methods developed by Kimura [31] and we introduce the notion of well ordered facet cover to give a sufficient condition (Corollary 4.2.6) for nonvanishing Betti numbers of facet ideals. It was proved by Katzman [28] that the regularity of an edge ideal is strictly greater than the induced matching number of the associated graph. As an application of our condition we are able to extend Katzman's regularity bound to facet ideals.

In Chapter 5 we focus on facet ideals of simplicial forests. Theorem 5.1.3 shows that multigraded Betti numbers of such ideals are either 0 or 1; moreover a multidegree does not appear at two different homological positions in the resolution. This generalizes a result of Bouchat [7] on edge ideals of graph forests. Using well ordered facet covers we give a combinatorial description in Theorem 5.2.2 for the Betti numbers of facet ideals of simplicial forests, and this generalizes the previously given description of Betti numbers of edge ideals of forests by Kimura [31]. As a consequence of our description we express the regularity of these ideals in terms of facet covers (Corollary 5.2.3) and this extends a result of Zheng [46] which linked the induced matching number of a graph forest to the regularity of its edge ideal.

In [7] Bouchat showed that the homological degree of a multigraded Betti number of the edge ideal of a forest can be determined by the projective dimension of the edge ideal of the induced subgraph corresponding to the given multidegree. In Theorem 5.3.7 we extend this result to some simplicial forests which include graph forests. In Section 5.4 we use the combinatorial description of Betti numbers of edge ideals of forests to give more detailed features of their multigraded Betti numbers.

In Chapter 6 we study path ideals. The path ideal of a graph is generated by monomials which correspond to paths of a certain order. The graded Betti numbers of path ideals of paths and cycles were computed by Alilooee and Faridi [1, 2] using Hochster's formula ([24, Theorem 8.1.1]). Also Bouchat, Hà and A. O'Keefe [8] studied path ideals of paths in a different setting and they obtained formulas for projective dimension and regularity of these ideals. In Sections 6.2 and 6.3 we go on to give formulas for multigraded Betti numbers of those ideals except the top degree ones for cycles using a different technique. We also compute graded Betti numbers of path ideals of star graphs in Section 6.4 and extend the work of Jacques [26], Hà and Van Tuyl [22] who gave formulas for Betti numbers of edge ideals of stars.

The results of Section 4.1 and Section 5.1 were published in [14]. Sections 4.2, 4.3 and 5.2 are included in [13] and submitted for publication.

Chapter 2

Background

2.1 Simplicial Complexes and Homology

An **abstract simplicial complex** Δ on a set of **vertices** $V(\Delta) = \{v_1, \ldots, v_n\}$ is a collection of subsets of $V(\Delta)$ such that $\{v_i\} \in \Delta$ for all i, and $F \in \Delta$ implies that all subsets of F are also in Δ. The elements of Δ are called **faces** and the maximal faces under inclusion are called **facets**. If the facets $F_1, \ldots, F_q$ generate Δ, we write $\Delta = \langle F_1, \ldots, F_q \rangle$ or $\text{Facets}(\Delta) = \{F_1, \ldots, F_q\}$.

The **dimension** of a face F is equal to $|F| - 1$. The dimension of Δ is the maximum of the dimensions of its faces.

A face $\{v_1, v_2, \ldots, v_n\} \setminus \{v_{i_1}, \ldots, v_{i_s}\}$ will be denoted by $\{v_1, \ldots, \widehat{v}_{i_1}, \ldots, \widehat{v}_{i_s}, \ldots, v_n\}$ for $i_1 < i_2 < \cdots < i_s$.

A **subcollection** of Δ is a simplicial complex Γ such that every facet of Γ is also a facet of Δ. If A is a set of vertices of Δ, then the simplicial complex Δ_A is the **induced subcollection** of Δ on A and it is equal to $\langle F \in \text{Facets}(\Delta) \mid F \subseteq A \rangle$. For a monomial m, the simplicial complex Δ_m stands for the induced subcollection of Δ on the set of vertices which divide m. A simplicial complex Γ is called a **subcomplex** of Δ if $\Gamma \subseteq \Delta$. Note that every subcollection of Δ is a subcomplex of Δ.

A set D of facets of Δ is called a **matching** if the facets in D are pairwise disjoint. We say D is an **induced matching** if D is a matching, and moreover the induced subcollection of Δ on $\cup_{F \in D} F$ is generated by the facets in D.

Facet removal: If $F_1, \ldots, F_p$ are facets of Δ, then $\Delta \setminus \langle F_1, \ldots, F_p \rangle$ denotes the simplicial complex whose facet set is $\text{Facets}(\Delta) \setminus \{F_1, \ldots, F_p\}$. Note that if Δ_1

and Δ_2 are simplicial complexes, then $\Delta_1 \setminus \Delta_2$ is not necessarily a simplicial complex. Therefore this notation should not be confused with taking the difference of simplicial complexes.

A simplicial complex Δ is **connected** if for any two facets F and G of Δ there exists faces $F_0 = F, F_1, \ldots, F_k = G$ of Δ such that $F_i \cap F_{i+1} \neq \emptyset$ for every $i = 0, \ldots, k-1$. The maximal connected subcomplexes of Δ are called the **connected components** of Δ.

A facet F of Δ is called a **leaf** if either F is the only facet of Δ, or there exists a facet $G \neq F \in \Delta$, called a **joint** of F, such that $F \cap H \subseteq G$ for every facet $H \neq F$. By definition, every leaf F of Δ has to contain a **free vertex**, i.e., a vertex v such that $v \notin H$ for every facet $H \in \mathrm{Facets}(\Delta) \setminus \{F\}$. A connected simplicial complex Δ is called a **simplicial tree** if every nonempty subcollection of Δ has a leaf. If every connected component of a simplicial complex Δ is a simplicial tree, then Δ is called a **simplicial forest**. Simplicial forests were defined by Faridi [17] and they generalize the notion of forest in graph theory to simplicial complexes.

Suppose that $\Bbbk$ is a fixed field. Given a simplicial complex Γ on the vertices $x_1, \ldots, x_n$ the facet ideal of Γ is the squarefree monomial ideal

$$\mathcal{F}(\Gamma) = (x_1 \ldots x_n \mid \{x_1, \ldots, x_n\} \text{ is a facet of } \Gamma)$$

of $S = \Bbbk[x_1, \ldots, x_n]$. Conversely, every squarefree monomial ideal of S is associated in this way to the facet ideal of a unique simplicial complex since every monomial ideal has a unique minimal monomial set of generators [24, Proposition 1.1.6], thus establishing a bijection between facet ideals and squarefree monomial ideals.

If I is a squarefree monomial ideal which is minimally generated by the monomials $m_1, \ldots, m_q$, then the **facet complex** of I is the simplicial complex whose facets correspond to the minimal monomial generators of I. Note that if Γ is a simplicial complex, then the facet complex of $\mathcal{F}(\Gamma)$ is Γ itself.

Example 2.1.1. The simplicial complex Γ in Figure 2.1 is a simplicial tree with leaves K and F. The joint of K is H and the joint of F is G. Also the free vertices are x_2 and x_6. However the simplicial complex in Figure 2.2 is not a simplicial tree.

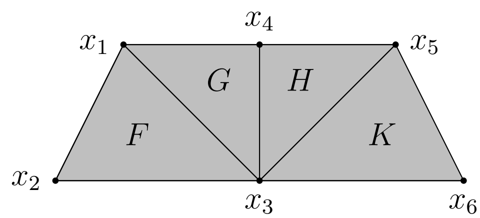

Figure 2.1: A simplicial tree $\Gamma = \langle F, G, H, K \rangle$ which has the facet ideal $\mathcal{F}(\Gamma) = (x_1x_2x_3, x_1x_3x_4, x_3x_4x_5, x_3x_5x_6)$.

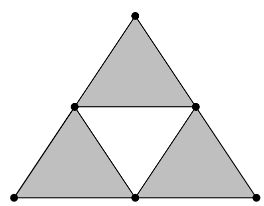

Figure 2.2: A simplicial complex which is not a tree.

A set of vertices C of a simplicial complex Γ is called a **vertex cover** if $F \cap C \neq \emptyset$ for every $F \in \text{Facets}(\Gamma)$. A set $D \subseteq \text{Facets}(\Gamma)$ is called a **facet cover** of Γ if every vertex v of Γ belongs to some F in D. A facet cover (respectively vertex cover) is called **minimal** if no proper subset of it is a facet cover (respectively vertex cover) of Γ.

Localization: Suppose that Γ is a simplicial complex on vertices $x_1, \ldots, x_n$ and $\{x_{i_1}, \ldots, x_{i_k}\}$ is a vertex cover of Γ. Then $P = (x_{i_1}, \ldots, x_{i_k})$ is a prime ideal containing the ideal $\mathcal{F}(\Gamma)$ and by $\mathcal{F}(\Gamma)_P$ we mean the ideal of $\Bbbk[x_1, \ldots, x_n]$ whose monomial generators are identified with those of the localized ideal of $\mathcal{F}(\Gamma)$ at P. We denote by Γ_P the facet complex of $\mathcal{F}(\Gamma)_P$. We will make use of the fact that if Γ is a simplicial forest, then Γ_P is also a simplicial forest [17, Lemma 1]. For example, if Γ is the simplicial complex in Figure 2.1, $P_1 = (x_1, x_3, x_5)$ and $P_2 = (x_2, x_4, x_6)$, then $\Gamma_{P_1} = \langle \{x_1, x_3\}, \{x_3, x_5\} \rangle$ and $\Gamma_{P_2} = \langle \{x_2\}, \{x_4\}, \{x_6\} \rangle$.

Two simplicial complexes Δ and Γ are **isomorphic** if there is a bijection $\varphi : V(\Delta) \to V(\Gamma)$ between their vertex sets such that F is a face of Δ if and only if $\varphi(F)$ is a face of Γ.

Let Δ and Γ be simplicial complexes which have no common vertices. Then the **join** of Δ and Γ is the simplicial complex given by

$$\Delta * \Gamma = \{\delta \cup \gamma : \delta \in \Delta, \gamma \in \Gamma\}.$$

A **cone** with **apex** v is a special join obtained by joining a simplicial complex Δ with $\{\emptyset, v\}$ where v is not in the vertex set of Δ. Equivalently, a simplicial complex is a cone with apex v if v is a member of every facet.

If σ is a face of Δ, then the **deletion** of Δ with respect to σ is the subcomplex

$$\mathrm{del}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset\}.$$

For further definitions from combinatorial topology we refer to Björner [6].

We briefly review some concepts and results regarding simplicial homology, see Hatcher [20] for a comprehensive treatment of this topic. Let Δ be a simplicial complex with vertex set $\{1, \dots, n\}$. For each integer i, let $C_i(\Delta)$ be a vector space over $\Bbbk$ whose basis elements e_σ correspond to i-dimensional faces σ of Δ. Note that if $i < -1$ or $i > n-1$, then $C_i(\Delta) = 0$ by definition. Also, as $\emptyset$ is the only (-1)-dimensional face of Δ we have $C_{-1}(\Delta) \cong \Bbbk$. The boundary maps

$$\partial_i : C_i(\Delta) \longrightarrow C_{i-1}(\Delta)$$

are defined on the basis elements e_σ where $\sigma = \{t_0, t_1, \ldots, t_i\}$ with $t_0 < t_1 < \cdots < t_i$ by

$$\partial_i(e_\sigma) = \sum_{j=0}^{i} (-1)^j e_{\sigma \setminus \{t_j\}}$$

and extended linearly to all of $C_i(\Delta)$. It is easy to check that $\partial_i \partial_{i+1} = 0$. This means that the image of ∂_{i+1} is contained in the kernel of ∂_i. For each integer i, the $\Bbbk$ -vector space

$$\tilde{H}_i(\Delta, \Bbbk) = \operatorname{Ker}(\partial_i)/\operatorname{Im}(\partial_{i+1})$$

is called the ith **reduced homology** of Δ over $\Bbbk$. For the sake of simplicity, we drop $\Bbbk$ and write $\tilde{H}_i(\Delta)$ whenever we work on a fixed ground field $\Bbbk$.

A **simplex** is a simplicial complex that contains all subsets of its nonempty vertex set. The **boundary** Σ of a simplex $\Delta = \langle \{v_1, \ldots, v_n\} \rangle$ is obtained from Δ by removing the maximal face of Δ. And, the homology groups of Σ are given by

$$\tilde{H}_p(\Sigma, \Bbbk) \cong \begin{cases} \Bbbk, & \text{if } p = n-2 \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.1}$$

The **irrelevant complex** $\{\emptyset\}$ has the homology groups

$$\tilde{H}_p(\{\emptyset\}, \Bbbk) \cong \begin{cases} \Bbbk, & \text{if } p = -1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.2}$$

whereas the **void complex** $\{\}$ has trivial reduced homology in all degrees.

A simplicial complex Δ is **acyclic** (over $\Bbbk$) if $\tilde{H}_i(\Delta, \Bbbk)$ is trivial for all i. Examples of acyclic complexes include cones and simplices, see page 1853 of Björner [6].

The homology of two simplicial complexes is related to homology of their union and intersection by the **Mayer-Vietoris long exact sequence**.

Theorem 2.1.2. [20] *Let Δ_1 and Δ_2 be two simplicial complexes. Then there is a long exact sequence*

$$\cdots \to \tilde{H}_p(\Delta_1) \oplus \tilde{H}_p(\Delta_2) \to \tilde{H}_p(\Delta_1 \cup \Delta_2) \to \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2) \to \tilde{H}_{p-1}(\Delta_1) \oplus \tilde{H}_{p-1}(\Delta_2) \to \cdots \tag{2.1.3}$$

provided that $\Delta_1 \cap \Delta_2 \neq \{\}$.

A particular case of Theorem 2.1.2 occurs when a simplicial complex $\Delta = \Delta_1 \cup \Delta_2$ is a union of two acyclic subcomplexes Δ_1 and Δ_2. In that case, the sequence (2.1.3) becomes

$$\cdots \to 0 \to \tilde{H}_p(\Delta_1 \cup \Delta_2) \to \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2) \to 0 \to \cdots$$

whence $\tilde{H}_p(\Delta_1 \cup \Delta_2)$ and $\tilde{H}_{p-1}(\Delta_1 \cap \Delta_2)$ are isomorphic for all p. Since we will make frequent use of this specific case, we state it separately as an immediate Corollary.

Corollary 2.1.3. *If Δ_1 and Δ_2 are acyclic simplicial complexes over $\Bbbk$, then*

$$\tilde{H}_p(\Delta_1 \cup \Delta_2, \Bbbk) \cong \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2, \Bbbk)$$

for every p, provided that $\Delta_1 \cap \Delta_2 \neq \{\}$.

2.2 Graph Theory

A **graph** G is an ordered pair $(V(G), E(G))$ consisting of a finite **vertex set** $V(G)$ and an **edge set** $E(G)$ such that every edge $e \in E(G)$ is a subset of $V(G)$ of cardinality 2. We use the standard terminology on graph theory, see [44], for instance. We say G is a graph of **order** n and **size** m if $|V(G)| = n$ and $|E(G)| = m$. Two vertices u and v are called **adjacent** if $\{u, v\}$ is an edge of G. A vertex v is an **isolated vertex** of G if no vertex of G is adjacent to v.

We say H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If A is a set of vertices of G, then the **induced subgraph** G_A of G on A is the graph whose vertex set is A and edge set is $\{e \in E(G) \mid e \subseteq A\}$.

If $\mathcal{E}$ is a set of edges of G, then $G - \mathcal{E}$ denotes the graph which is obtained by G by **removing the edges** in $\mathcal{E}$.

A set D of pairwise disjoint edges of G is called a **matching** of G. If D is a matching and moreover, the induced subgraph of G on $\bigcup\limits_{e \in D} e$ has no edges outside D, then D is called an **induced matching** of G. The maximum cardinality of an induced matching of G is called its **induced matching number** and is denoted by $im(G)$.

Example 2.2.1. Suppose that G is the graph given in Figure 2.3. Then $\{\{a, b\}, \{c, d\}\}$ is a matching but not induced matching of G. Also $\{\{a, b\}, \{d, e\}\}$ is an induced matching of G.

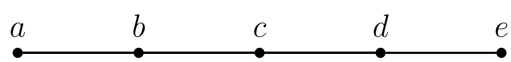

Figure 2.3: A graph G with $im(G) = 2$.

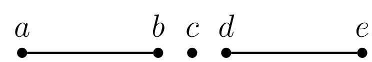

Figure 2.4: $G - \{\{b, c\}, \{c, d\}\}$

A set C of vertices of G is called a **vertex cover** if every edge of G contains a vertex from C. A vertex cover is called **minimal** if no proper subset of it is a vertex cover of G. A set D of edges of G is called an **edge cover** if $\bigcup\limits_{e \in D} e = V(G)$. If no proper subset of D is an edge cover of G, then D is called a **minimal edge cover**.

A **path** of order n is a graph, denoted by P_n, with vertex set $V(P_n) = \{v_1, \dots, v_n\}$ and edge set $E(P_n) = \{\{v_i, v_{i+1}\} \mid i = 1, \dots, n-1\}$. A **cycle** of order $n \geq 3$, denoted by C_n, is a graph with edges

$$\{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_{n-1}, u_n\}, \{u_n, u_1\}.$$

A **star** of size n, denoted by $\mathcal{S}_n$, is a graph with vertices $z_0, \ldots, z_n$ and edges $\{z_0, z_1\}, \ldots, \{z_0, z_n\}$. Star graphs are also known as bouquet graphs (see, Zheng [46]) in the combinatorial commutative algebra literature. We shall use these terms separately because with bouquet graphs we also choose some vertices as flowers (see, Definition 3.1.3).

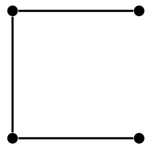

Figure 2.5: P_4

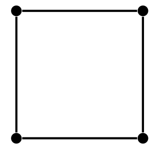

Figure 2.6: C_4

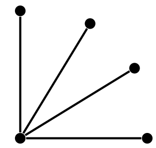

Figure 2.7: $\mathcal{S}_4$

Theorem 2.2.2. [44, Theorem 2.1.4] *If G is a connected graph on n vertices, then the following conditions are equivalent.*

(1) *G has no cycles.*

(2) *G has $n-1$ edges.*

A connected graph which satisfies one of the equivalent conditions of Theorem 2.2.2 is called a **tree**. If every connected component of G is a tree, then G is called a **forest**. When G is considered as a 1-dimensional or 0-dimensional simplicial complex the definition of tree matches that of simplicial tree.

Let G be a graph with $V(G) = \{x_1, \ldots, x_n\}$. The **edge ideal** of G is defined as

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \text{ is an edge of } G).$$

When a graph G without isolated vertices is considered as a simplicial complex, we have $I(G) = \mathcal{F}(G)$. For instance, if the adjacent vertices of P_5 are labeled respectively with $x_1, \ldots, x_5$, then $I(P_5) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5)$.

2.3 Minimal Free Resolutions

Let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\Bbbk$. A **monomial** is a polynomial of the form $x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$ with $a_i \in \mathbb{N}$. An ideal I is called **monomial** if it is generated by monomials. A **free resolution** of a monomial ideal

I is an exact sequence of free S-modules

$$\mathbf{F} : 0 \longrightarrow F_r \xrightarrow{d_r} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} I \longrightarrow 0. \tag{2.3.1}$$

If the differential maps satisfy $d_{i+1}(F_{i+1}) \subseteq (x_1, \dots, x_n)F_i$ for all $i \geq 0$, then the resolution is called **minimal**. The minimality of the resolution is alternatively equivalent to each of the modules F_i having minimum possible rank. Up to an isomorphism, there exists a unique minimal free resolution of I ([39, Theorem 7.5]) and therefore the ranks of the free modules are independent of the choice of the minimal free resolution. The rank of F_i is called the ith **total Betti number** of I and is denoted by $b_i^S(I)$.

We consider a **standard grading** on S by setting the **degree** of a monomial $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ (denoted by $\deg(x_1^{a_1}x_2^{a_2}\dots x_n^{a_n})$) equal to $a_1 + \cdots + a_n$. If S_i is the $\Bbbk$-vector space generated by all monomials of degree i, then $S = \oplus_{i\in\mathbb{N}} S_i$ is a direct sum decomposition of S as a $\Bbbk$-vector space such that

$$S_iS_j \subseteq S_{i+j} \text{ for all } i, j \in \mathbb{N}.$$

If a polynomial $u \in S$ belongs to some S_i, then we say that u has (standard) degree i. Similarly, S has a **multigrading** where the **multidegree** of a monomial $m = x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ is equal to $\mathbf{m} = (a_1, \dots, a_n)$. For simplicity, we shall use a monomial m and its multidegree $\mathbf{m}$ interchangeably.

For $a \in \mathbb{N}$ the module $S(-a)$ denotes the graded free S-module which is generated by one element in degree a. Likewise, if $\mathbf{m} \in \mathbb{N}^n$, then the module $S(-\mathbf{m})$ denotes the multigraded free S-module which is generated by one element in degree $\mathbf{m}$.

If the differential maps of a minimal free resolution preserve the standard degrees, then the resolution is called a **minimal graded free resolution**. In this case, the resolution takes the form

$$0 \longrightarrow \bigoplus_{j\in\mathbb{N}} S(-j)^{b^S_{r,j}(I)} \xrightarrow{d_r} \cdots \longrightarrow \bigoplus_{j\in\mathbb{N}} S(-j)^{b^S_{1,j}(I)} \xrightarrow{d_1} \bigoplus_{j\in\mathbb{N}} S(-j)^{b^S_{0,j}(I)} \xrightarrow{d_0} I \longrightarrow 0$$

where the integers $b^S_{i,j}(I)$ are the **graded Betti numbers** of I. Similarly, a **minimal multigraded free resolution** of I has the form

$$0 \longrightarrow \bigoplus_{\mathbf{m}\in\mathbb{N}^n} S(-\mathbf{m})^{b^S_{r,\mathbf{m}}(I)} \xrightarrow{d_r} \cdots \longrightarrow \bigoplus_{\mathbf{m}\in\mathbb{N}^n} S(-\mathbf{m})^{b^S_{1,\mathbf{m}}(I)} \xrightarrow{d_1} \bigoplus_{\mathbf{m}\in\mathbb{N}^n} S(-\mathbf{m})^{b^S_{0,\mathbf{m}}(I)} \xrightarrow{d_0} I \longrightarrow 0$$

The associated ranks $b^S_{i,\mathbf{m}}(I)$ are called **multigraded Betti numbers** of I. We say that $b^S_{i,\mathbf{m}}(I)$ has **homological degree** i and **multidegree m**. Then the Betti numbers are related with the following equations.

$$b^S_i(I) = \sum_{j\in\mathbb{N}} b^S_{i,j}(I)$$

$$b^S_{i,j}(I) = \sum_{\deg(\mathbf{m})=j} b^S_{i,\mathbf{m}}(I). \tag{2.3.2}$$

The **projective dimension** of I is defined by

$$\mathrm{pd}(I) = \max\{i \mid b^S_i(I) \neq 0\}.$$

Note that the projective dimension of a monomial ideal is always a finite number because of Hilbert's Syzygy Theorem ([39, Theorem 15.2]) which states that every graded finitely generated S-module has a finite graded free resolution. The **(Castelnuovo-Mumford) regularity** of I is

$$\mathrm{reg}(I) = \max\{j - i \mid b^S_{i,j}(I) \neq 0\}.$$

Remark 2.3.1. The minimal free resolution of the S-module S/I is similar to that of I. If (2.3.1) is the minimal free resolution of I, then

$$\mathbf{F}' : 0 \longrightarrow F_r \xrightarrow{d_r} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} S \longrightarrow S/I \longrightarrow 0$$

is the minimal free resolution of S/I where the last differential map is the natural projection. Therefore we have

- $b^S_{i,\mathbf{m}}(I) = b^S_{i+1,\mathbf{m}}(S/I)$ for all i, j and $\mathbf{m}$
- $\mathrm{pd}(I) = \mathrm{pd}(S/I) - 1$
- $\mathrm{reg}(I) = \mathrm{reg}(S/I) + 1$

provided that $S \neq I \neq 0$.

While Betti numbers are invariants of minimal free resolutions, one can get information about them without finding minimal free resolutions. For example, if $\mathbf{G} : 0 \longrightarrow G_q \xrightarrow{\partial_q} \cdots \longrightarrow G_1 \xrightarrow{\partial_1} G_0 \xrightarrow{\partial_0} I \longrightarrow 0$ is any free resolution of I, then

$$b^S_{i,j}(I) = \dim_{\Bbbk}(H_i(\mathbf{G} \otimes_S \Bbbk)_j) \tag{2.3.3}$$

since $b^S_{i,j}(I)$ is equal to $\dim_{\Bbbk}(\mathrm{Tor}^S_i(I, \Bbbk)_j)$, see [39, Theorem 11.2].

Example 2.3.2. Let $I = (abc, acd, cde, cef)$ be the facet ideal of Γ in Figure 2.1 with the $x_1, x_2, \ldots$ changed to $a, b, \ldots$. Using Macaulay2 [35] and taking $\Bbbk$ as the field of rational numbers, we compute a minimal multigraded free resolution of I as

$$0 \to S(-abcdef) \xrightarrow{\begin{pmatrix} -ef \\ -bf \\ -ab \\ d \end{pmatrix}} S(-abcd) \oplus S(-acde) \oplus S(-cdef) \oplus S(-abcef)$$

$$\xrightarrow{\begin{pmatrix} -d & 0 & 0 & -ef \\ b & -e & 0 & 0 \\ 0 & a & -f & 0 \\ 0 & 0 & d & ab \end{pmatrix}} S(-abc) \oplus S(-acd) \oplus S(-cde) \oplus S(-cef)$$

$$\xrightarrow{\begin{pmatrix} abc & acd & cde & cef \end{pmatrix}} I \to 0.$$

Therefore

$$b^S_{0,abc}(I) = b^S_{0,acd}(I) = b^S_{0,cde}(I) = b^S_{0,cef}(I) = 1 \text{ and } b^S_{0,3}(I) = 4,$$
$$b^S_{1,abcd}(I) = b^S_{1,acde}(I) = b^S_{1,cdef}(I) = b^S_{1,abcef}(I) = 1 \text{ and } b^S_{1,4}(I) = 3, b^S_{1,5}(I) = 1,$$
$$b_{2,abcdef}(I) = 1 \text{ and } b^S_{2,6}(I) = 1.$$

The projective dimension of I is 2 and the regularity is 4.

Recall that the study of minimal free resolutions of monomial ideals can be restricted to squarefree monomial ideals [39, Theorem 21.10] thanks to the polarization method [18, 45].

2.3.1 Simplicial Resolutions

The first explicit (multigraded) free resolution of a monomial ideal was constructed by Taylor [42]. Taylor's resolution is similar to a chain complex associated to a simplicial complex in algebraic topology. Inspired by Taylor's resolution, Bayer, Peeva and Sturmfels [5] introduced a general approach to construct free resolutions which is **simplicial resolutions**.

We now describe this construction and refer the reader for further details to Mermin [37] from where we adopted this presentation.

Suppose that I is a monomial ideal. Let Γ be an $(r-1)$-dimensional simplicial complex whose vertices are labeled with s monomials, say $m_1, \ldots, m_s$. For every face F of Γ we set

$$\operatorname{lcm}(F) = \operatorname{lcm}(m_i \mid m_i \in F).$$

For each $i \geq 0$, we let H_i be the free S-module generated by $\{[F] : F \in \Gamma \text{ and } |F| = i\}$, where $[F]$ is a symbol for the generator corresponding to the face F. The differential map $\phi_i : H_i \to H_{i-1}$ with $1 \leq i \leq r$ is defined by

$$\phi_i([F]) = \sum_{j=1}^{i} (-1)^{j+1} \frac{\operatorname{lcm}(F)}{\operatorname{lcm}(F \setminus \{m_{t_j}\})} [F \setminus \{m_{t_j}\}]$$

for $F = \{m_{t_1}, m_{t_2}, \ldots, m_{t_i}\}$ and $t_1 < t_2 < \cdots < t_i$. Also ϕ_0 is the natural projection map. If the algebraic chain complex

$$\mathbf{H}_\Gamma : 0 \longrightarrow H_r \xrightarrow{\phi_r} \cdots \longrightarrow H_1 \xrightarrow{\phi_1} H_0 \xrightarrow{\phi_0} S/I \longrightarrow 0$$

is exact, then it is called a **simplicial resolution supported on** Γ.

2.3.2 The Taylor Resolution

Although Taylor's resolution is usually non-minimal, it has been a useful tool to investigate the invariants of minimal free resolutions. Suppose that I is a monomial ideal which is minimally generated by the monomials $m_1, \ldots, m_s$. Let Taylor(I) (called the **Taylor simplex**) be the simplex whose vertices are labeled with $m_1, \ldots, m_s$. Then the free resolution constructed by Taylor is the same as the simplicial resolution supported on Taylor(I).

Since the minimal free resolution of I is a summand of any free resolution of I [39, Theorem 7.5] we can get immediate bounds on Betti numbers via Taylor's resolution.

For instance, for each $i \geq 0$ we have

$$b^S_{i,\mathbf{m}}(I) \leq \text{ the number of } i \text{ dimensional faces of } \operatorname{Taylor}(I) \text{ whose label is } \mathbf{m}\,.$$

In fact, the Betti numbers of I can be computed by the dimensions of reduced homologies of certain subcomplexes of the Taylor simplex. Before stating this precisely, we fix the following notation. For any monomial m in S the simplicial subcomplex $\operatorname{Taylor}(I)_{<m}$ stands for

$$\operatorname{Taylor}(I)_{<m} = \{\tau \in \operatorname{Taylor}(I) \mid \operatorname{lcm}(\tau) \text{ strictly divides } m\}.$$

Example 2.3.3. For $I = (x_1x_2, x_1x_3, x_1x_4, x_3x_4)$ the Taylor simplex of I and a subcomplex $\operatorname{Taylor}(I)_{<x_1x_2x_3x_4}$ are illustrated in Figures 2.8 and 2.9 respectively.

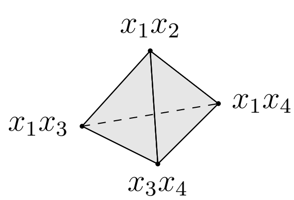

Figure 2.8: $\operatorname{Taylor}(I)$

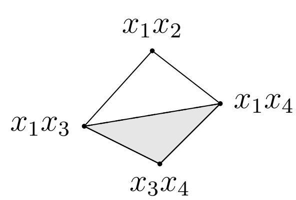

Figure 2.9: $\operatorname{Taylor}(I)_{<x_1x_2x_3x_4}$

Theorem 2.3.4 ([5])**.** *Let I be a monomial ideal of S which is minimally generated by the monomials $m_1, \dots, m_s$ and such that $I \neq 0$ and $I \neq S$. For $i \geq 0$, the multigraded Betti numbers of I are given by*

$$b^S_{i,\mathbf{m}}(I) = \begin{cases} \dim_{\Bbbk} \tilde{H}_{i-1}(\operatorname{Taylor}(I)_{<m}; \Bbbk), & \textit{if } m \textit{ divides } \operatorname{lcm}(m_1, \dots, m_s) \\ 0, & \textit{otherwise.} \end{cases} \tag{2.3.4}$$

Remark 2.3.5. By Theorem 2.3.4, we are allowed not to specify a polynomial ring S when we deal with Betti numbers. We can think of a facet ideal $\mathcal{F}(\Gamma)$ lying in a polynomial ring over $\Bbbk$ that contains at least as many variables as the vertices of Γ. Therefore we drop S and write $b_{i,\mathbf{m}}(\mathcal{F}(\Gamma))$ and $b_{i,j}(\mathcal{F}(\Gamma))$ for the Betti numbers.

Remark 2.3.6. If $I = (m_1, \ldots, m_s)$ and $q = \deg(\mathrm{lcm}(m_1, \ldots, m_s))$, then for any $r > q$ we have $b_{i,r}(I) = 0$ for all i. That is, q is the largest possible degree in which the Betti number can be nonzero. Therefore we call the numbers $b_{i,q}(I)$ with $i \in \mathbb{N}$ the **top degree Betti numbers**. If Γ is a simplicial complex, then its facet ideal $\mathcal{F}(\Gamma)$ is squarefree monomial ideal, and hence the highest possible degree for a Betti number is the number of vertices of Γ.

2.3.3 The Lyubeznik Resolution

Another tool of which we will make use is a refinement of Taylor resolution due to Lyubeznik [34]. Let J be a monomial ideal and let M be the set of minimal generators of J. Suppose that $<$ is a total ordering on the elements of M. For a monomial $u \in J$, define

$$\min(u) = \min_{<}\{m \in M \mid m \text{ divides } u\}$$

and for a face $F \in \mathrm{Taylor}(J)$ define

$$\min(F) = \min(\mathrm{lcm}(F)).$$

A face $F \in \mathrm{Taylor}(J)$ is called **rooted** (or **L-admissable**) if for every $\emptyset \neq G \subseteq F$, the property $\min(G) \in G$ holds. The rooted faces of $\mathrm{Taylor}(J)$ form a simplicial complex $\Lambda_{J,<}$ which is called the **Lyubeznik simplicial complex** associated to J and $<$. This simplicial complex always supports a resolution of S/J and is called the Lyubeznik resolution of S/J.

Example 2.3.7. Let Γ be the simplicial complex in Example 2.3.2 whose facets are $F = \{a, b, c\}$, $G = \{a, c, d\}$, $H = \{c, d, e\}$ and $K = \{c, e, f\}$. Let $H < K < F < G$

be an ordering on the facets of Γ. Observe that $\min(\{K, G\}) = H$, so $\{K, G\}$ is not a face of $\Lambda_{\mathcal{F}(\Gamma),<}$. However $\{H, K, F\}$ and $\{H, F, G\}$ are rooted and therefore they are the facets of $\Lambda_{\mathcal{F}(\Gamma),<}$. The simplicial complexes $\mathrm{Taylor}(\mathcal{F}(\Gamma))$ and $\Lambda_{\mathcal{F}(\Gamma),<}$ are illustrated in Figures 2.10 and 2.11 respectively.

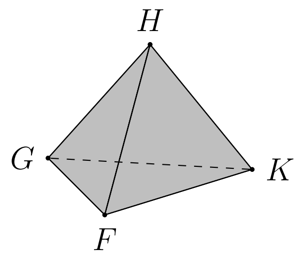

Figure 2.10: $\mathrm{Taylor}(\mathcal{F}(\Gamma))$

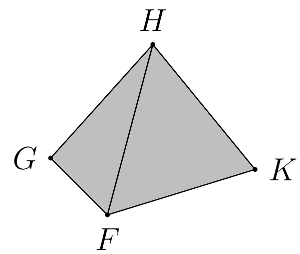

Figure 2.11: $\Lambda_{\mathcal{F}(\Gamma),<}$

The Taylor resolution of $S/\mathcal{F}(\Gamma)$ is

$$0 \to S(-abcdef) \xrightarrow{\begin{pmatrix} 1 \\ b \\ -1 \\ -f \end{pmatrix}} S(-abcdef) \oplus S(-acdef) \oplus S(-abcdef) \oplus S(-abcde)$$

$$\xrightarrow{\begin{pmatrix} 0 & -f & 0 & -b \\ ef & 0 & 0 & e \\ 0 & 0 & -f & 1 \\ 0 & a & ab & 0 \\ d & 0 & d & 0 \\ -b & 1 & 0 & 0 \end{pmatrix}} S(-acde) \oplus S(-abcd) \oplus S(-abcde) \oplus S(-cdef) \oplus S(-abcef)$$

$$\oplus S(-acdef) \xrightarrow{\begin{pmatrix} -a & 0 & -ab & -f & 0 & 0 \\ 0 & 0 & 0 & d & -ab & -ad \\ 0 & -d & de & 0 & ef & 0 \\ e & b & 0 & 0 & 0 & ef \end{pmatrix}} S(-cde) \oplus S(-cef) \oplus S(-abc)$$

$$\oplus S(-acd) \xrightarrow{\begin{pmatrix} cde & cef & abc & acd \end{pmatrix}} S \longrightarrow S/\mathcal{F}(\Gamma) \longrightarrow 0.$$

The Lyubeznik resolution of $S/\mathcal{F}(\Gamma)$ is

$$0 \to S(-abcdef) \oplus S(-abcde) \xrightarrow{\begin{pmatrix} 0 & -b \\ 0 & e \\ -f & 1 \\ ab & 0 \\ d & 0 \end{pmatrix}} S(-acde) \oplus S(-abcd) \oplus S(-abcde)$$

$$\oplus S(-cdef) \oplus S(-abcef) \xrightarrow{\begin{pmatrix} -a & 0 & -ab & -f & 0 \\ 0 & 0 & 0 & d & -ab \\ 0 & -d & de & 0 & ef \\ e & b & 0 & 0 & 0 \end{pmatrix}} S(-cde) \oplus S(-cef)$$

$$\oplus S(-abc) \oplus S(-acd) \xrightarrow{\begin{pmatrix} cde & cef & abc & acd \end{pmatrix}} S \longrightarrow S/\mathcal{F}(\Gamma) \longrightarrow 0.$$

Chapter 3

Bouquets, Vertex Covers and Facet Ideals

In this chapter, we will investigate the relation between minimal vertex covers of simplicial complexes and projective dimension of squarefree monomial ideals. We will associate bouquet structures to minimal vertex covers and use them to bound the projective dimension of facet ideals in Corollary 3.1.12. While our bound for projective dimension is equivalent to a previously known bound, we will take technical advantage of using bouquets in the subsequent chapters.

3.1 Bouquets and Minimal Vertex Covers

Let Δ be a simplicial complex on the vertices $x_1, \ldots, x_n$. Following the notation used by Morey and Villarreal [38] we write $\alpha_0'(\Delta)$ for the maximum possible cardinality of a minimal vertex cover of Δ. There is a one-to-one correspondence between the minimal vertex covers of Δ and the **minimal prime ideals** of $\mathcal{F}(\Delta)$ given by

$$C \text{ is a minimal vertex cover of } \Delta \Leftrightarrow (x_i : x_i \in C) \text{ is a minimal prime ideal of } \mathcal{F}(\Delta).$$

Therefore the parameter $\alpha_0'(\Delta)$ coincides with the **big height** of $\mathcal{F}(\Delta)$, which is the maximum height of the minimal prime ideals of $\mathcal{F}(\Delta)$. This invariant is linked to the projective dimension of facet ideals as follows.

Theorem 3.1.1. [38, Corollary 3.33] *If Δ is a simplicial complex, then*

$$\mathrm{pd}(S/\mathcal{F}(\Delta)) \geq \alpha_0'(\Delta)$$

with equality if $S/\mathcal{F}(\Delta)$ *is sequentially Cohen-Macaulay.*

Corollary 3.1.2 (**Projective dimension of simplicial forests**)**.** *If* Γ *is a simplicial forest, then* $\mathrm{pd}(S/\mathcal{F}(\Gamma)) = \alpha_0'(\Gamma)$.

Proof. By [16, Corollary 5.6] if Γ is a simplicial forest, then $S/\mathcal{F}(\Gamma)$ is sequentially Cohen-Macaulay. □

Given a simplicial complex Δ and a minimal vertex cover A of size i, since $\alpha_0'(\Delta)$ bounds $\mathrm{pd}(S/\mathcal{F}(\Delta))$, there exists a non-zero Betti number $b_i(S/\mathcal{F}(\Delta))$. It is interesting to know what conditions on A give rise to non-zero Betti numbers of $S/\mathcal{F}(\Delta)$ in homological degree i. A partial answer to this question was given by Kimura in [31, Theorem 3.1] where the author developed the concept of strongly disjoint bouquets to give sufficient conditions for non-vanishing Betti numbers of edge ideals of simple graphs. In Theorem 3.1.10 we will give an alternative description of minimal vertex covers which is closely aligned with the positioning of facets with respect to each other.

Definition 3.1.3 (**Bouquet**)**.** (Compare to [46, Definition 1.7]) A **bouquet** is a simplicial complex $\mathcal{B} = \langle F_1, \dots, F_d \rangle$ together with an assigned set of vertices called **flowers** $F(\mathcal{B}) = \{u_1, \dots, u_d\}$ such that

(1) $\bigcap_{i=1}^{d} F_i \neq \emptyset$

(2) $u_i \in F_j \Leftrightarrow i = j$ for all $i, j \in \{1, \dots, d\}$.

The notion of bouquet for simple graphs was defined and used by Zheng [46] to study resolutions of edge ideals of forests. Our definition generalizes this to arbitrary simplicial complexes. Observe that one can assign flowers to a simplicial complex in different ways to make it a bouquet. However if the bouquet $\mathcal{B}$ is a simple graph with at least two edges, then its flowers are automatically determined and one can think of it as a star graph which was defined in Section 2.2.

Example 3.1.4. Figures 3.1 and 3.2 respectively illustrate 1-dimensional and 2-dimensional bouquets.

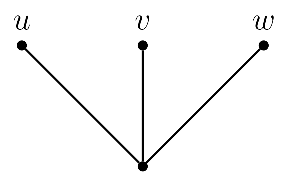

Figure 3.1: A bouquet with flowers u, v, w.

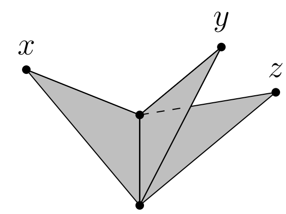

Figure 3.2: A bouquet with flowers x, y, z.

Remark 3.1.5. In [25] Hoefel and Mermin defined **supernovas** to characterize Gotzmann squarefree monomial ideals. A d-dimensional simplicial complex Δ is called a **supernova** if there exists a chain of faces $\emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ such that every i-dimensional facet of Δ contains the $(i-1)$-dimensional face F_{i-1}. Observe that the facets of the supernova Δ have a nonempty intersection as F_0 is contained in every facet of Δ. Moreover, if G is an i-dimensional facet, then the vertex in $G \setminus F_{i-1}$ is a free vertex of Δ. Therefore every supernova is a bouquet.

On the other hand, not every bouquet is a supernova. For example, let $\mathcal{B}$ be a bouquet with facets $\{a, b, d\}, \{a, b, c, f\}, \{b, c, e\}$ and flowers d, e, f. Then $\mathcal{B}$ is not a supernova. To see this, notice that if G and H are distinct facets of a supernova with the same dimension, then $G \setminus F$ is a 0-dimensional face of the supernova. But we see that $\{a, b, d\} \setminus \{b, c, e\} = \{a, d\}$ is a 1-dimensional face of $\mathcal{B}$.

If a subcollection $\mathcal{B}$ of Δ is a bouquet, then we simply say that $\mathcal{B}$ **is a bouquet of** Δ. Suppose that $\boldsymbol{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_j\}$ is a set of bouquets of Δ. Then we call

$$F(\boldsymbol{B}) := \bigcup_{i=1}^{j} F(\mathcal{B}_i), \text{ the } \textbf{flower set} \text{ of } \boldsymbol{B}$$

$$\text{Facets}(\boldsymbol{B}) := \bigcup_{i=1}^{j} \text{Facets}\,(\mathcal{B}_i), \text{ the } \textbf{facet set} \text{ of } \boldsymbol{B}, \text{ and}$$

$$V(\boldsymbol{B}) := \bigcup_{i=1}^{j} V(\mathcal{B}_i), \text{ the } \textbf{vertex set} \text{ of } \boldsymbol{B}.$$

In [31] Kimura made use of bouquets to study resolutions of edge ideals. She introduced the notion of semi-strongly disjoint bouquets [31, Definition 5.1] which we broaden to simplicial complexes.

Definition 3.1.6 (**Semi-strongly disjoint bouquets**). (Compare to Kimura [31, Definition 5.1]) A set $\boldsymbol{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_j\}$ of bouquets of a simplicial complex Δ is said to be **semi-strongly disjoint** in Δ if the following conditions hold.

(1) If $\sigma \in \text{Facets}\,(\mathcal{B}_p)$, then $\sigma \cap F(\mathcal{B}_q) = \emptyset$ for all $q \neq p$.

(2) $V(\boldsymbol{B}) \setminus F(\boldsymbol{B})$ does not contain any facet of Δ.

We define

$$d'_\Delta = \max\{|\,\text{Facets}(\boldsymbol{B})| : \boldsymbol{B} \text{ is a semi-strongly disjoint set of bouquets of } \Delta\}.$$

Example 3.1.7. Suppose that $\Delta = \langle\{a,b,c\},\{b,c,d\},\{b,d,e\},\{d,e,f\},\{e,f,g\}\rangle$ Let $\mathcal{B}_1 = \langle\{a,b,c\},\{b,c,d\}\rangle$ with $F(\mathcal{B}_1) = \{a,d\}$ and $\mathcal{B}_2 = \langle\{e,f,g\}\rangle$ with $F(\mathcal{B}_2) = \{g\}$. Then $\{\mathcal{B}_1, \mathcal{B}_2\}$ is a set of semi-strongly disjoint bouquets in Δ.

For a simple graph G, Khosh-Ahang and Moradi proved that $\alpha'_0(G) \geq d'_G$ ([30, Proposition 2.7]). They proved $\alpha'_0(G) = d'_G$ for the special case of vertex decomposable graphs ([29, Theorem 3.8]). We will see that in fact, the parameters $\alpha'_0(\Delta)$ and d'_Δ are the same for any simplicial complex Δ and in particular for a simple graph G.

Lemma 3.1.8. *Let Δ be a simplicial complex. If Γ is an induced subcollection of Δ, then any minimal vertex cover of Γ can be extended to a minimal vertex cover of Δ. In particular, $\alpha'_0(\Gamma) \leq \alpha'_0(\Delta)$.*

Proof. Suppose that $V(\Delta) \setminus V(\Gamma) = A$ and C is a minimal vertex cover of Γ. If $A = \emptyset$, then $\Gamma = \Delta$ and there is nothing to prove. So we assume that $A \neq \emptyset$. Then $A \cap C = \emptyset$ and $A \cup C$ covers Δ. By removing the redundant elements from $A \cup C$, one can get a minimal vertex cover $C' \subseteq A \cup C$ of Δ. But then $C' \setminus A$ is a vertex cover of Γ. Since $C' \setminus A \subseteq C$ we get $C' \setminus A = C$ by minimality of C. Thus $C \subseteq C'$ is the desired extension. □

Remark 3.1.9. The Lemma above is not necessarily true if Γ is an arbitrary subcollection of Δ. See for example Figures 3.3 and 3.4.

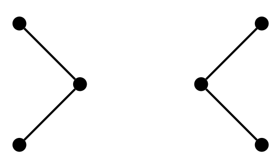

Figure 3.3: A simple graph K with $\alpha_0'(K) = 4$

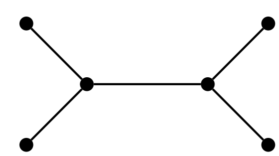

Figure 3.4: A simple graph H with $\alpha_0'(H) = 3$

We now prove the main result of this section which shows the equivalence of the parameters $\alpha_0'(\Delta)$ and d_Δ'.

Theorem 3.1.10. *For any simplicial complex Δ, the flower set of a semi-strongly disjoint set of bouquets of Δ can be extended to a minimal vertex cover of Δ. Also, for any minimal vertex cover C of Δ, there exists a semi-strongly disjoint set of bouquets of Δ with the flower set C. In particular, the equality $\alpha_0'(\Delta) = d_\Delta'$ holds.*

Proof. First suppose that $\boldsymbol{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_j\}$ is a semi-strongly disjoint set of bouquets of Δ. Consider $\Delta_{V(\boldsymbol{B})}$, the induced subcollection of Δ on $V(\boldsymbol{B})$. Then $F(\boldsymbol{B})$ is a vertex cover of $\Delta_{V(\boldsymbol{B})}$ by condition (2) of Definition 3.1.6. To see the minimality of $F(\boldsymbol{B})$, assume for a contradiction $F(\boldsymbol{B}) \setminus \{v\}$ covers $\Delta_{V(\boldsymbol{B})}$ for some $v \in F(\mathcal{B}_p)$ with $p \in \{1, \ldots, j\}$. Let σ be a facet of $\mathcal{B}_p$ containing v. Then by Definition 3.1.6(1) and Definition 3.1.3(2) we get $\sigma \cap (F(\boldsymbol{B}) \setminus \{v\}) = \emptyset$. This contradicts the initial assumption that $F(\boldsymbol{B}) \setminus \{v\}$ covers $\Delta_{V(\boldsymbol{B})}$. Thus, by Lemma 3.1.8 the first part of the given statement is verified.

Next, suppose that C is a minimal vertex cover of Δ. We will construct a set $\boldsymbol{B}$ of semi-strongly disjoint bouquets of Δ such that $F(\boldsymbol{B}) = C$. Observe that if Δ has any facets of the form $\{u\}$ for some vertex u, then C must contain u, and every semi-strongly disjoint bouquet can have $\{u\}$ added to it as a bouquet with one facet and u as the flower of that facet. Therefore we may assume that Δ has no facets of cardinality 1.

Note that by Lemma 3.2.1(1), for every $v \in C$ there exists a facet σ_v of Δ such that $\sigma_v \cap C = \{v\}$. Pick an element $u_1^1 \in C$. Then there exists a facet σ_1^1 of Δ such that $C \cap \sigma_1^1 = \{u_1^1\}$. As $\sigma_1^1 \neq \{u_1^1\}$ there exists $r_1 \in \sigma_1^1 \setminus \{u_1^1\}$. Suppose that $u_1^1, u_2^1, \ldots, u_{d_1}^1$ are the elements of C that satisfy the property

$$\text{there exists } \sigma_i^1 \in \text{Facets}(\Delta) \text{ such that } \sigma_i^1 \cap C = \{u_i^1\} \text{ and } r_1 \in \sigma_i^1 \tag{3.1.1}$$

for every $1 \leq i \leq d_1$. Let $\sigma_1^1, \ldots, \sigma_{d_1}^1$ be chosen fixed facets that satisfy the property above. Consider the subcollection $\mathcal{B}_1 = \langle \sigma_1^1, \ldots, \sigma_{d_1}^1 \rangle$ of Δ with the assigned flowers $u_1^1, \ldots, u_{d_1}^1$.

Now if $F(\mathcal{B}_1) = C$, then $\boldsymbol{B} = \{\mathcal{B}_1\}$ and we are done. Otherwise we keep constructing new bouquets inductively as follows. Suppose that we have semi-strongly disjoint bouquets $\{\mathcal{B}_1, \ldots, \mathcal{B}_t\}$ such that $\cup_{i=1}^t F(\mathcal{B}_i)$ is a proper subset of C. Pick an element $u_1^{t+1} \in C \setminus \cup_{i=1}^t F(\mathcal{B}_i)$ and a facet σ_1^{t+1} of Δ such that $\sigma_1^{t+1} \cap C = \{u_1^{t+1}\}$. Fix $r_{t+1} \in \sigma_1^{t+1} \setminus \{u_1^{t+1}\}$ and let $u_1^{t+1}, u_2^{t+1}, \ldots, u_{d_{t+1}}^{t+1}$ be the elements of $C \setminus \cup_{i=1}^t F(\mathcal{B}_i)$ that satisfy the property

$$\text{there exists } \sigma_i^{t+1} \in E(\Delta) \text{ such that } \sigma_i^{t+1} \cap C = \{u_i^{t+1}\} \text{ and } r_{t+1} \in \sigma_i^{t+1} \tag{3.1.2}$$

for every $1 \leq i \leq d_{t+1}$. Let $\sigma_1^{t+1}, \ldots, \sigma_{d_{t+1}}^{t+1}$ be chosen fixed facets that satisfy the property above. Consider the subcollection $\mathcal{B}_{t+1} = \langle \sigma_1^{t+1}, \ldots, \sigma_{d_{t+1}}^{t+1} \rangle$ of Δ as a bouquet with flowers $u_1^{t+1}, u_2^{t+1}, \ldots, u_{d_{t+1}}^{t+1}$. Now we show that $\{\mathcal{B}_1, \ldots, \mathcal{B}_{t+1}\}$ is semi-strongly disjoint. Condition (1) of Definition 3.1.6 clearly holds by construction. To see that the second condition holds, observe that

$$V(\{\mathcal{B}_1, \ldots, \mathcal{B}_{t+1}\}) \setminus F(\{\mathcal{B}_1, \ldots, \mathcal{B}_{t+1}\}) = V(\{\mathcal{B}_1, \ldots, \mathcal{B}_{t+1}\}) \setminus C$$

does not contain any facet of Δ since C is a vertex cover, so every facet intersects C.

Having verified that this construction yields semi-strongly disjoint bouquets at every step, we know that it will terminate as Δ has finitely many vertices. In that case, $C = \cup_{i=1}^{p} F(\mathcal{B}_i)$ for some $p \geq 1$ and $\boldsymbol{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_p\}$ is as desired. $\square$

The following is an immediate consequence of Theorem 3.1.10.

Corollary 3.1.11. *Given a simplicial complex Δ, we have the following.*

(1) *If $\boldsymbol{B}$ is a semi-strongly disjoint set of bouquets of Δ such that $|F(\boldsymbol{B})| = d'_\Delta$, then $F(\boldsymbol{B})$ is a minimal vertex cover of Δ of maximum cardinality.*

(2) *If C is a minimal vertex cover of Δ of maximum cardinality, then there exist a semi-strongly disjoint set $\boldsymbol{B}$ of bouquets of Δ such that $F(\boldsymbol{B}) = C$ and $|F(\boldsymbol{B})| = d'_\Delta$.*

We conclude with a statement regarding projective dimension of facet ideals.

Corollary 3.1.12 (A bound on projective dimension). *Let Δ be a simplicial complex. Then* $\operatorname{pd}(S/\mathcal{F}(\Delta)) \geq d'_\Delta$.

Proof. By Theorem 3.1.1, we have $\operatorname{pd}(S/\mathcal{F}(\Delta)) \geq \alpha'_0(\Delta)$. Hence the proof is immediate from Theorem 3.1.10. $\square$

3.2 Properties of Minimal Vertex Covers

In this section, we will investigate how minimal vertex covers change under facet removal and localization. As we have seen in Corollary 3.1.2, when Γ is a simplicial forest $\alpha_0'(\Gamma)$ determines the projective dimension of the facet ideal of Γ. Therefore the results of this section will be used in the sequel.

Lemma 3.2.1. *Suppose that Γ is a simplicial complex. Let C be a minimal vertex cover of Γ and let F be a facet of Γ. Then the following properties hold.*

(1) *For every $a \in C$ there exists a facet H such that $C \cap H = \{a\}$. In particular $|\mathrm{Facets}(\Gamma)| \geq \alpha_0'(\Gamma)$.*

(2) *If $|C \cap F| \geq 2$, then C is also a minimal vertex cover of $\Gamma \setminus \langle F \rangle$.*

(3) *Either C is a minimal vertex cover of $\Gamma \setminus \langle F \rangle$ or there exists $u \in F$ such that $C \cap F = \{u\}$ and $C \setminus \{u\}$ is a minimal vertex cover of $\Gamma \setminus \langle F \rangle$.*

(4) $\alpha_0'(\Gamma \setminus \langle F \rangle) + 1 \geq \alpha_0'(\Gamma)$.

(5) *If F contains a free vertex of Γ, then $\alpha_0'(\Gamma) \geq \alpha_0'(\Gamma \setminus \langle F \rangle)$.*

Proof. (1) Assume for a contradiction that for every facet H of Γ we have $a \notin H$ or $u \in H \cap C$ for some $u \neq a$. But then a is redundant in C which contradicts minimality of C.

(2) Suppose that C contains at least 2 vertices from F. Then C does not contain a free vertex which belongs to F as it is a minimal vertex cover of Γ. This implies that C covers $\Gamma \setminus \langle F \rangle$. To see that it is minimal, let $C = \{u_1, \ldots u_n\}$ and observe that by part (1) there exists facets $G_1, \ldots, G_n$ of Γ such that $C \cap G_i = \{u_i\}$ for all $i = 1, \ldots, n$. As $F \notin \{G_1, \ldots, G_n\}$ no element in C is redundant as a cover $\Gamma \setminus \langle F \rangle$.

(3) Suppose that C is not a minimal vertex cover of $\Gamma \setminus \langle F \rangle$. Then by (2) we have $C \cap F = \{u\}$ for some u. Note that by (1) no element of $C \setminus \{u\}$ is redundant in C as a cover of $\Gamma \setminus \langle F \rangle$. Therefore u is redundant in C as a cover $\Gamma \setminus \langle F \rangle$.

(4) The proof is immediate by applying part (3) to a minimal vertex cover of Γ of maximum size.

(5) It is straightforward to show that every minimal vertex cover of $\Gamma \setminus \langle F \rangle$ is either a minimal vertex cover of Γ or it can be extended to a minimal vertex cover of Γ by adding the free vertex of F to it. $\square$

Remark 3.2.2. In statement (5) of Lemma 3.2.1 the assumption of F containing a free vertex is crucial. For instance, if H is the graph in Figure 3.4 and F is the edge which does not contain a free vertex, then we have $\alpha_0'(H) = 3 < 4 = \alpha_0'(H \setminus \langle F \rangle)$.

Remark 3.2.3 (Localized Complex). Let F be a facet of a simplicial complex Γ. Let Δ be the localized complex $(\Gamma \setminus \langle F \rangle)_{(x_i | x_i \notin F)}$. Then the set of facets of Δ is given by

$$\{(G \setminus F) \mid G \neq F \text{ and } (G \setminus F) \not\supseteq (H \setminus F) \text{ for all } H \in \text{Facets}(\Gamma) \setminus \{G, F\}\}.$$

For instance, if $\Gamma = \langle \{a, b, c\}, \{a, c, d\}, \{c, d, e\}, \{c, e, f\} \rangle$ and $F = \{a, b, c\}$, then $\Delta = \langle \{d\}, \{e, f\} \rangle$.

Note that every vertex of Δ belongs to $V(\Gamma) \setminus F$. However $V(\Delta)$ is not necessarily equal to $V(\Gamma) \setminus F$. For example, if Γ is the same simplicial complex as above and $F = \{a, c, d\}$, then $\Delta = \langle \{b\}, \{e\} \rangle$.

Lemma 3.2.4. *Let F be a facet of a simplicial complex Γ which contains a free vertex u. Let Δ be the localized complex $(\Gamma \setminus \langle F \rangle)_{(x_i | \, x_i \notin F)}$. Then the following conditions hold.*

(1) *If C is a vertex cover of Δ, then $C \cup \{u\}$ is a vertex cover of Γ. Moreover, if C is a minimal vertex cover of Δ, then $C \cup \{u\}$ is a minimal vertex cover of Γ. In particular,*

$$\alpha_0'(\Gamma) \geq \alpha_0'(\Delta) + 1.$$

(2) *If* $V(\Delta) = V(\Gamma) \setminus F$ *and* C *is a minimal vertex cover of* Γ *that contains* u*, then* $C \setminus \{u\}$ *is a minimal vertex cover of* Δ*.*

Proof. (1) Suppose that C is a vertex cover of Δ. We first show that $C \cup \{u\}$ covers Γ. We need to verify that for every facet G of Γ such that $G \neq F$ we have $G \cap C \neq \emptyset$. To this end, we consider two cases. First, if $G \setminus F$ is a facet of Δ, then we have nothing to show as $G \cap C \neq \emptyset$ is clear. Second, suppose that $G \setminus F$ is not a facet of Δ. Then $K \setminus F \subseteq G \setminus F$ for some facet $K \setminus F$ of Δ. But since C is a vertex cover of Δ, there exists a vertex $v \in C \cap (K \setminus F)$. Thus $v \in C \cap (G \setminus F)$ and $v \in G \cap C \neq \emptyset$ as we claimed.

Now, suppose that C is a minimal vertex of Δ. We claim that $C \cup \{u\}$ is minimal as a vertex cover of Γ. First, observe that u is not redundant in $C \cup \{u\}$ because $C \cap F = \emptyset$. Assume for a contradiction that v is redundant in $C \cup \{u\}$ for some $v \in C$ so that $(C \cup \{u\}) \setminus \{v\}$ covers Γ. Since C is minimal vertex cover of Δ, by Lemma 3.2.1(1) there exists a facet $H \setminus F$ of Δ such that $C \cap (H \setminus F) = \{v\}$. But then as u is a free vertex, we get $((C \cup \{u\}) \setminus \{v\}) \cap H = \emptyset$ which is a contradiction.

(2) First note that by minimality of C we have $C \cap F = \{u\}$ and $C \setminus \{u\} \subseteq V(\Gamma) \setminus F = V(\Delta)$. If $H \setminus F$ is a facet of Δ for some facet H of Γ, then $(H \setminus F) \cap (C \setminus \{u\}) \neq \emptyset$ since $H \cap (C \setminus \{u\}) \neq \emptyset$. So $C \setminus \{u\}$ is a vertex cover of Δ. To see that $C \setminus \{u\}$ minimally covers Δ, assume for a contradiction that $C \setminus \{u, v\}$ covers Δ for some $v \in C$ such that $v \neq u$. By Lemma 3.2.1(1) there exists a facet $G \neq F$ of Γ such that $G \cap C = \{v\}$. Now if $G \setminus F$ is a facet of Δ, then $(G \setminus F) \cap (C \setminus \{u, v\}) = \emptyset$ because $G \cap C = \{v\}$. But this contradicts $C \setminus \{u, v\}$ being a vertex cover of Δ. Therefore $G \setminus F$ is not a facet of Δ. Then there exists a facet $H \setminus F$ of Δ such that $H \setminus F \subseteq G \setminus F$. Since $C \setminus \{u, v\}$ covers Δ there exists $w \in C \setminus \{u, v\}$ such that $w \in H \setminus F \subseteq G \setminus F$. But this is again a contradiction as $G \cap C = \{v\}$. $\square$

Chapter 4

Well Ordered Facet Covers and Betti Numbers

Let $I = (m_1, \ldots, m_s)$ be a squarefree monomial ideal in $S = \Bbbk[x_1, \ldots, x_n]$. As seen in Section 2.3.2, Taylor's resolution gives some restrictions on possible Betti numbers of I. Suppose that $b_{i,\mathbf{m}}(S/I) \neq 0$ for some i and m. Since any minimal multigraded free resolution of S/I is a direct summand of Taylor's resolution, the nonzero Betti number $b_{i,\mathbf{m}}(S/I)$ implies that I has i monomial generators whose least common multiple is $\mathbf{m}$. Let us rephrase this in terms of facet ideals: If I is the facet ideal of the simplicial complex Γ, then the nonzero $b_{i,\mathbf{m}}(S/I)$ tells us that the induced subcollection $\Gamma_{\mathbf{m}}$ has a facet cover of cardinality i.

This observation raises the natural question of conversely which facet covers give nonzero Betti numbers. Now, suppose that

$$\mathbf{F} : 0 \longrightarrow F_r \xrightarrow{d_r} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} S/I \longrightarrow 0$$

is a simplicial resolution supported on a simplicial complex Θ. For simplicity, we use the minimal monomial generators of I with the facets of Γ interchangeably. A minimal facet cover $\{m_{t_1}, \ldots, m_{t_i}\}$ of $\Gamma_{\mathbf{m}}$ satisfies

$$\operatorname{lcm}(m_{t_1}, \ldots, \widehat{m_{t_j}}, \ldots, m_{t_i}) \neq \operatorname{lcm}(m_{t_1}, \ldots, m_{t_i}) \text{ for all } j = 1, \ldots, i$$

and thus corresponds to an element in $\mathrm{Ker}(d_i \otimes 1_{\Bbbk})$ as in Section 2.3. To see this, observe that

$$\begin{aligned}(d_i \otimes 1_{\Bbbk})([\{m_{t_1}, \ldots, m_{t_i}\}] \otimes 1) &= \sum_{j=1}^{i} (-1)^{j+1} \frac{\mathrm{lcm}(m_{t_1}, \ldots, m_{t_i})}{\mathrm{lcm}(m_{t_1}, \ldots, \widehat{m_{t_j}}, \ldots, m_{t_i})} [\{m_{t_1}, \ldots, \widehat{m_{t_j}}, \ldots, m_{t_i}\}] \otimes 1 \\ &= \sum_{j=1}^{i} (-1)^{j+1} [\{m_{t_1}, \ldots, \widehat{m_{t_j}}, \ldots, m_{t_i}\}] \otimes \underbrace{\frac{\mathrm{lcm}(m_{t_1}, \ldots, m_{t_i})}{\mathrm{lcm}(m_{t_1}, \ldots, \widehat{m_{t_j}}, \ldots, m_{t_i})}}_{\in (x_1, \ldots, x_n)} \\ &= \sum_{j=1}^{i} (-1)^{j+1} [\{m_{t_1}, \ldots, \widehat{m_{t_j}}, \ldots, m_{t_i}\}] \otimes 0.\end{aligned}$$

Moreover, if $\{m_{t_1}, \ldots, m_{t_i}\}$ is a facet of Θ, then $[\{m_{t_1}, \ldots, m_{t_i}\}] \otimes 1 \notin \mathrm{Im}(d_{i+1} \otimes 1_{\Bbbk})$. In that case, the ith homology of $\mathbf{F} \otimes \Bbbk$ in degree m does not vanish and we have $b_{i,\mathbf{m}}(S/I) \neq 0$ by the multigraded version of (2.3.3).

In [4] Barile noted this nonvanishing condition for Lyubeznik resolutions (Subsection 2.3.3).

Theorem 4.0.5 ([4])**.** *Let $I = (m_1, \ldots, m_s)$ be a monomial ideal and let $<$ be a total order on the set of minimal generators of I. If there exists a facet $F = \{m_{t_1}, m_{t_2}, \ldots, m_{t_i}\}$ of $\Lambda_{I,<}$ such that*

$$\mathrm{lcm}(m_{t_1}, \ldots, \widehat{m_{t_j}}, \ldots, m_{t_i}) \neq \mathrm{lcm}(m_{t_1}, \ldots, m_{t_i})$$

for all $j = 1, 2, \ldots, i$, then $b_{i,\mathbf{u}}(S/I) \neq 0$ where $u = \mathrm{lcm}(m_{t_1}, \ldots, m_{t_i})$.

Kimura [31] used this condition to give sufficient conditions for nonvanishing Betti numbers of edge ideals in terms of bouquet subgraphs. In this chapter, we will define the concept of well ordered facet covers to generalize Kimura's results to facet ideals. In Section 4.3 we will see that well ordered edge covers of graphs are the same as strongly disjoint bouquets (Definition 4.3.1).

4.1 Betti Numbers of Induced Subcollections and Connected Components

Lemma 4.1.1. *Suppose that Γ is a simplicial complex on vertices labeled with the variables $x_1, \dots, x_n$ of the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$. If $\mathbf{m}$ is a squarefree monomial of degree j, then $b_{i,\mathbf{m}}(\mathcal{F}(\Gamma)) = b_{i,j}(\mathcal{F}(\Gamma_{\mathbf{m}}))$.*

Proof. Observe that $\text{Taylor}_{<\mathbf{m}}(\mathcal{F}(\Gamma)) = \text{Taylor}_{<\mathbf{m}}(\mathcal{F}(\Gamma_{\mathbf{m}}))$. So by Theorem 2.3.4, we have $b_{i,\mathbf{m}}(\mathcal{F}(\Gamma)) = b_{i,\mathbf{m}}(\mathcal{F}(\Gamma_{\mathbf{m}}))$. But by Equation (2.3.2) we get $b_{i,\mathbf{m}}(\mathcal{F}(\Gamma_{\mathbf{m}})) = b_{i,j}(\mathcal{F}(\Gamma_{\mathbf{m}}))$ since $\mathbf{m}$ is the only possible squarefree monomial of degree j that can divide the lcm of the generators of $\mathcal{F}(\Gamma_{\mathbf{m}})$. □

Lemma 4.1.2. *If $I_1, I_2, \dots, I_N$ are squarefree monomial ideals whose minimal generators contain no common variable, then for $i, j \geq 0$*

$$b_{i,j}(S/(I_1 + I_2 + \cdots + I_N)) = \sum_{\substack{u_1+\cdots+u_N=i \\ v_1+\cdots+v_N=j}} b_{u_1,v_1}(S/I_1) \dots b_{u_N,v_N}(S/I_N). \tag{4.1.1}$$

Moreover, for each $k = 1, \dots, N$ if q_k is the degree of the least common multiple of the entire minimal monomial generating set of I_k, then

$$b_{i,q_1+\cdots+q_N}(S/(I_1 + I_2 + \cdots + I_N)) = \sum_{u_1+\cdots+u_N=i} b_{u_1,q_1}(S/I_1) \dots b_{u_N,q_N}(S/I_N). \tag{4.1.2}$$

Proof. The case $N = 2$ of Equation (4.1.1) is Corollary 2.2 of [27], and the general case follows from an easy induction on N. To see (4.1.2), note that we have

$$b_{i,q_1+\cdots+q_N}(S/(I_1 + I_2 + \cdots + I_N)) = \sum_{\substack{u_1+\cdots+u_N=i \\ v_1+\cdots+v_N=q_1+\cdots+q_N}} b_{u_1,v_1}(S/I_1) \dots b_{u_N,v_N}(S/I_N).$$

by Equation (4.1.1). Suppose that $v_1 + \cdots + v_N = q_1 + \cdots + q_N$. If $v_\ell \neq q_\ell$ for some ℓ, then there exists a j such that $v_j > q_j$ whence $b_{u_j,v_j}(S/I_j) = 0$ since b_{u_j,q_j} is a top degree Betti number. In this case the term $b_{u_1,v_1}(S/I_1)\dots b_{u_N,v_N}(S/I_N)$ vanishes. So we can rewrite the sum above as

$$\sum_{\substack{u_1+\cdots+u_N=i \\ v_1=q_1,\dots,v_N=q_N}} b_{u_1,v_1}(S/I_1)\dots b_{u_N,v_N}(S/I_N)$$

and this completes the proof. $\square$

4.2 Resolutions via well ordered facet covers

Definition 4.2.1. A sequence $F_1,\dots,F_k$ of facets of a simplicial complex Γ is called a **well ordered facet cover** if $\{F_1,\dots,F_k\}$ is a minimal facet cover of Γ and for every facet $H \notin \{F_1,\dots,F_k\}$ of Γ there exists $i \leq k-1$ such that $F_i \subseteq H \cup F_{i+1} \cup F_{i+2} \cup \ldots \cup F_k$.

Example 4.2.2. Suppose that Γ is the simplicial complex given in Figure 4.1. Then F_1, F_2, F_3 is a well ordered facet cover of Γ. Indeed $\{F_1, F_2, F_3\}$ is a minimal facet cover of Γ and $F_1 \subseteq F_4 \cup F_2$.

If we consider the subcomplex $\Delta = \langle F_1, F_2, F_4\rangle$ of Γ, then Δ has no well ordered facet cover. To confirm this, observe that $\{F_2, F_4\}$ is the only minimal facet cover of Δ. But neither F_2, F_4 nor F_4, F_2 is a well ordered facet cover of Δ.

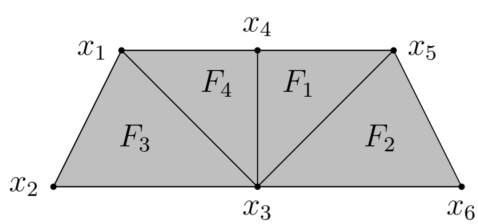

Figure 4.1: A simplicial tree Γ with a well ordered facet cover F_1, F_2, F_3.

Remark 4.2.3. Observe that if $\{F_1,\dots,F_k\}$ is a minimal facet cover of a simplicial complex Γ, then there exist $a_1,\dots,a_k$ such that $a_i \in F_j$ if and only if $i = j$ for all $i, j \in$

$\{1, \ldots, k\}$. Moreover, if $F_1, \ldots, F_k$ is a well ordered facet cover, then $\{a_1, \ldots, a_k\}$ is a minimal vertex cover of Γ.

Example 4.2.4. Let D be an induced matching of Γ. Then any ordering on the elements of D forms a well ordered facet cover for the induced subcollection Γ_A where $A = \bigcup_{F \in D} F$.

Theorem 4.2.5. *Let $F_1, \ldots, F_i$ be a well ordered facet cover of Γ. Then there is a total order $<$ on the facets of Γ such that $\{F_1, \ldots, F_i\}$ is a facet of the Lyubeznik simplicial complex $\Lambda_{\mathcal{F}(\Gamma),<}$.*

Proof. First note that since $\{F_1, \ldots, F_i\}$ is a minimal facet cover of Γ, we have

$$F_1 \cup \ldots \cup \widehat{F_\ell} \cup \ldots \cup F_i \neq F_1 \cup \ldots \cup F_i \tag{4.2.1}$$

for all $\ell = 1, \ldots, i$ by Remark 4.2.3. Consider the order

$$F_1 < F_2 < \cdots < F_i < \text{Facets}(\Gamma) \setminus \{F_1, \ldots, F_i\}$$

on the facets of Γ where the facets in Facets$(\Gamma) \setminus \{F_1, \ldots, F_i\}$ have any fixed order. Observe that for any $\{F_{j_1}, \ldots, F_{j_t}\} \subseteq \{F_1, \ldots, F_i\}$ with $j_1 < \cdots < j_t$ we have $\min(\{F_{j_1}, \ldots, F_{j_t}\}) = F_{j_1}$ because of Eq. (4.2.1). Therefore $\{F_1, \ldots, F_i\}$ is rooted. To see that the face $\{F_1, \ldots, F_i\}$ is a facet, assume for a contradiction that $\{F_1, \ldots, F_i, H\}$ is rooted for some $H \notin \{F_1, \ldots, F_i\}$. But then since $F_1, \ldots, F_i$ is a well ordered facet cover of Γ there exists $k \leq i-1$ such that $F_k \subseteq H \cup F_{k+1} \cup F_{k+2} \cup \ldots \cup F_i$. Thus $\min(\{H, F_{k+1}, F_{k+2}, \ldots, F_i\}) \leq F_k$ by the given order and $\min(\{H, F_{k+1}, \ldots, F_i\}) \notin \{H, F_{k+1}, \ldots, F_i\}$. This contradicts rootedness of $\{F_1, \ldots, F_i, H\}$. $\square$

Corollary 4.2.6 (**Betti numbers from facet covers**). *Let Γ be a simplicial complex and let $\mathbf{m}$ be a squarefree monomial. Suppose that $\Gamma_{\mathbf{m}}$ has a well ordered facet cover of cardinality i. Then $b_{i,\mathbf{m}}(S/\mathcal{F}(\Gamma)) \neq 0$.*

Proof. Since any minimal facet cover satisfies Eq.(4.2.1) the proof follows from applying Theorem 4.2.5 to Theorem 4.0.5. $\square$

Note that in Proposition 4.3.3 we show that well ordered facet covers of 1-dimensional simplicial complexes correspond to strongly disjoint bouquets (Definition 4.3.1). Therefore the corollary above generalizes Theorem 3.1 of Kimura [31] from quadratic to all squarefree monomial ideals.

In [38, Corollary 3.9] Morey and Villarreal proved that if Γ is a simplicial complex and $\{F_1, \ldots, F_s\}$ is an induced matching in Γ, then

$$\operatorname{reg}(S/\mathcal{F}(\Gamma)) \geq \left| \bigcup_{i=1}^{s} F_i \right| - s$$

which improved the previously known bounds in [28, Lemma 2.2] and [21, Theorem 6.5]. Using Corollary 4.2.6 we give the following bound for regularity which is an improvement of the bound mentioned above because of Example 4.2.4.

Corollary 4.2.7 (**Combinatorial bound for regularity**). *Let Γ be a simplicial complex and let $F_1, \ldots, F_s$ be a well ordered facet cover of some induced subcollection of Γ. Then*

$$\operatorname{reg}(S/\mathcal{F}(\Gamma)) \geq \left| \bigcup_{i=1}^{s} F_i \right| - s.$$

Remark 4.2.8. Let $\tau(\Gamma)$ be the maximum cardinality of a well ordered facet cover of an induced subcollection of Γ. Then we have

$$\operatorname{pd}(S/\mathcal{F}(\Gamma)) \geq \alpha_0'(\Gamma) \geq \tau(\Gamma)$$

where the first inequality comes from Theorem 3.1.1 and the second one holds by Remark 4.2.3 and Lemma 3.1.8. Note that for a simplicial complex Γ, we may have $\alpha_0'(\Gamma) > \tau(\Gamma)$. For instance, $\alpha_0'(C_5) = 3$, but no induced subcollection of C_5 has a well ordered facet cover of cardinality greater than 2. Therefore Corollary 4.2.6 gives a weaker bound than Morey and Villarreal's result mentioned in Theorem 3.1.1 for the projective dimension.

4.3 Well ordered edge covers of graphs

In this section we will show that well ordered facet covers of graphs correspond to certain bouquet subgraphs.

Definition 4.3.1 (Kimura [31, Definitions 2.1, 2.3])**.** A set $\boldsymbol{B} = \{B_1, \ldots, B_q\}$ of bouquets of a graph G is called **disjoint** in G if $V(B_k) \cap V(B_\ell) = \emptyset$ for all $k \neq \ell$. Moreover, if for every $k = 1, \ldots, q$ there exists $e_k \in E(B_k)$ such that $\{e_1, \ldots, e_q\}$ is an induced matching in G, then $\boldsymbol{B}$ is called a **strongly disjoint** set of bouquets in G. We say that G **contains a strongly disjoint set of bouquets** if there exists a strongly disjoint set of bouquets $\boldsymbol{B}$ of G such that $V(G) = V(\boldsymbol{B})$.

Notice that if G contains a strongly disjoint set $\boldsymbol{B}$ of bouquets of G, then $\boldsymbol{B}$ is semi-strongly disjoint.

Remark 4.3.2. It is a well known fact in graph theory that if D is a minimal facet cover of a graph G which has no isolated vertices, then there is a set $\boldsymbol{B}$ of disjoint bouquets in G such that Facets($\boldsymbol{B}$) $= D$. We refer to [44, Theorem 3.1.22] for a proof of this fact.

Proposition 4.3.3. *Let G be a graph with no isolated vertices, and let $D = \{d_1, \ldots, d_n\}$ be a set of facets in G. Then the following are equivalent.*

(1) *D is a well ordered facet cover of G.*

(2) $D = \text{Facets}(\boldsymbol{B})$ *where* $\boldsymbol{B}$ *is a strongly disjoint set of bouquets in* G.

Proof. First suppose that G has a well ordered facet cover $d_1, \ldots, d_n$. Then by Remark 4.3.2 there exists a set $\boldsymbol{T} = \{T_1, \ldots, T_q\}$ of disjoint bouquets with $\text{Facets}(\boldsymbol{T}) = \{d_1, \ldots, d_n\}$. We claim that $\boldsymbol{T}$ is strongly disjoint. Let $s_p = d_{\max\{i | d_i \in \text{Facets}(T_p)\}}$ for every $p = 1, \ldots, q$. We will show that $\{s_1, \ldots, s_q\}$ is an induced matching. Let $h \notin \{d_1, \ldots, d_n\}$ be a facet of G. Then $d_\ell \subseteq h \cup d_{\ell+1} \cup \cdots \cup d_n$ for some $\ell \leq n-1$ because $d_1, \ldots, d_n$ is a well ordered facet cover. As $\{d_1, \ldots, d_n\}$ is a minimal facet cover, $d_\ell \nsubseteq d_{\ell+1} \cup \cdots \cup d_n$. Since each facet has exactly 2 vertices, we have $d_\ell \subseteq h \cup d_k$ for some $k > \ell$. This implies that $d_k \cap d_\ell \neq \emptyset$. Therefore d_ℓ and d_k belong to the same bouquet of $\boldsymbol{T}$, say T_m. Let u be the vertex of d_ℓ which does not belong to d_k. Since $d_\ell \subseteq h \cup d_k$, the vertex u is also in h. But $u \notin s_m$ because $k > \ell$. Therefore h contains a vertex, namely u, which does not belong to s_p for each $p = 1, \ldots, q$. Then we conclude that there is no pair i, j $(i \neq j)$ such that h intersects both of s_i and s_j.

Conversely, suppose that G contains a strongly disjoint set of bouquets $\boldsymbol{B} = \{B_1, \ldots, B_q\}$ where $\text{Facets}(B_p) = \{e_1^p, \ldots, e_{t_p}^p, s_p\}$ for every $p = 1, \ldots, q$, and $\{s_1, \ldots, s_q\}$ is an induced matching in G. Then $\text{Facets}(\boldsymbol{B})$ is a minimal facet cover of G since $V(G) = V(\boldsymbol{B})$. We claim that $\text{Facets}(G) \setminus \{s_1, \ldots, s_q\}, s_1, \ldots, s_q$ is a well ordered facet cover of G where the facets in $\text{Facets}(G) \setminus \{s_1, \ldots, s_q\}$ are listed in any fixed order.

Let $h \in \text{Facets}(G) \setminus \text{Facets}(\boldsymbol{B})$. Observe that since $\{s_1, \ldots, s_q\}$ is an induced matching, $h \cap (\cup_{r=1}^q s_r)$ has cardinality at most one. Therefore h contains at least one vertex which do not belong to $\cup_{r=1}^q s_r$. Then there exists $p \in \{1, \ldots, q\}$ and $1 \leq j \leq t_p$ such that $e_j^p \cap h \neq \emptyset$ and $e_j^p \cap h \neq e_j^p \cap s_p$. Hence $e_j^p \subseteq h \cup s_p$ as desired. $\square$

Chapter 5

Facet Ideals of Simplicial Forests

In this chapter we will generalize many results about Betti numbers of edge ideals of forests to facet ideals of simplicial forests. Along the way we prove some new properties of edge ideals.

5.1 Multigraded Betti Numbers

Faridi [15] proved that if Γ is a simplicial tree, then one can order its facets as $F_0, F_1, \ldots, F_q$ so that each facet F_u is a leaf of the simplicial tree $\Gamma_u = \langle F_0, \ldots, F_u \rangle$ for $0 \leq u \leq q$ and F_0 is a **good leaf** of Γ. Based on such an order, she gave a refinement ([15, Proposition 4.9]) of the recursive formula for graded Betti numbers of simplicial forests ([21, Theorem 5.8]) of Hà and Van Tuyl. As pointed out in [15], the proof of [15, Proposition 4.9] does not use the fact that F_0 is a good leaf. Therefore this refinement can be stated in the following form.

Theorem 5.1.1. *Let Γ be a simplicial tree whose facets $F_0, F_1, \ldots, F_q$ are ordered such that each facet F_u is a leaf of the simplicial tree $\Gamma_u = \langle F_0, \ldots, F_u \rangle$ for $0 \leq u \leq q$. Let $x_1, \ldots, x_n$ be the vertices of Γ and for each u let P_u be the ideal $(x_t \mid x_t \notin F_u)$. Then for all $i \geq 1$ and $j \geq 0$*

$$b_{i,j}(\mathcal{F}(\Gamma)) = b_{i,j}(\mathcal{F}(\langle F_0 \rangle)) + \sum_{u=1}^{q} b_{i-1,j-|F_u|}\big(\mathcal{F}(\Gamma_{u-1})_{P_u}\big)$$

where we adopt the convention that $b_{-1,j}(I)$ is 1 if $j = 0$ and is 0 otherwise for any ideal I.

Proof. The proof is identical to that of [15, Proposition 4.9]. $\square$

Note that by [17, Lemma 1], if Γ is a simplicial forest, then its localization Γ_P is also a simplicial forest for any prime ideal P of $\mathbb{k}[x_1, \dots, x_n]$. Therefore the formula given in Theorem 5.1.1 is recursive.

Remark 5.1.2. If $\mathbf{m}$ is one of the minimal monomial generators of I, then $b_{i,\mathbf{m}}(I) = 1$ when $i = 0$ and is 0 otherwise. If I is generated by a single monomial $\mathbf{m}$, then its multigraded resolution is

$$0 \to S(-\mathbf{m}) \to I \to 0.$$

We now prove the main result of this section.

Theorem 5.1.3. *Let Γ be a simplicial forest. Then the multigraded Betti numbers of $\mathcal{F}(\Gamma)$ are either 0 or 1. Also, if for some monomial $\mathbf{m}$ we have $b_{i,\mathbf{m}}(\mathcal{F}(\Gamma)) \neq 0$, then $b_{h,\mathbf{m}}(\mathcal{F}(\Gamma)) = 0$ for all $h \neq i$.*

Proof. We prove the given statements by induction on the number of vertices of Γ. The cases when Γ has only one vertex or $i = 0$ follow from Remark 5.1.2. Suppose that the given statements hold for any simplicial forest whose number of vertices is s or less. Now let Γ be a simplicial forest on $s+1$ vertices and take a monomial $\mathbf{m}$ which divides the product of vertices of Γ. Then the induced subcollection $\Gamma_{\mathbf{m}}$ is also a forest by definition. Note that we have $b_{i,\mathbf{m}}(\mathcal{F}(\Gamma)) = b_{i,j}(\mathcal{F}(\Gamma_{\mathbf{m}}))$ by Lemma 4.1.1 where $j = \deg(\mathbf{m})$. If j is greater than the number of vertices of $\Gamma_{\mathbf{m}}$, then by Remark 2.3.6 we have $b_{i,\mathbf{m}}(\mathcal{F}(\Gamma)) = b_{i,j}(\mathcal{F}(\Gamma_{\mathbf{m}})) = 0$. So we assume $|V(\Gamma_{\mathbf{m}})| = j = \deg(\mathbf{m})$.

If $\Gamma_{\mathbf{m}}$ is not a tree, then its connected components $\Upsilon_1, \dots, \Upsilon_t$ satisfy the induction hypothesis. If $\mathcal{F}(\Gamma_{\mathbf{m}}) = 0$ we have nothing to prove. So we assume that $\mathcal{F}(\Gamma_{\mathbf{m}}) \neq 0$, and using Remark 2.3.1 and Lemma 4.1.2 we get

$$b_{i,j}(\mathcal{F}(\Gamma_{\mathbf{m}})) = b_{i+1,j}(S/\mathcal{F}(\Gamma_{\mathbf{m}})) = \sum_{\gamma_1+\cdots+\gamma_t=i+1} b_{\gamma_1,l_1}(S/\mathcal{F}(\Upsilon_1)) \dots b_{\gamma_t,l_t}(S/\mathcal{F}(\Upsilon_t))$$

where l_v is the number of vertices of Υ_v for each $1 \leq v \leq t$. As each connected component has at least one vertex, $\mathcal{F}(\Upsilon_v) \neq 0$ for each v. By induction hypothesis for each l_v there exists at most one γ_v such that $b_{\gamma_v,l_v}(\mathcal{F}(\Upsilon_v)) \neq 0$. Therefore for each l_v there exists at most one γ_v such that $b_{\gamma_v,l_v}(S/\mathcal{F}(\Upsilon_v)) \neq 0$.

Hence we see that there must be at most one i such that $b_{i,j}(\mathcal{F}(\Gamma_{\mathbf{m}})) \neq 0$. And, in such a case

$$b_{i,j}(\mathcal{F}(\Gamma_{\mathbf{m}})) = \prod_{v=1}^{t} b_{\gamma_v,l_v}(S/\mathcal{F}(\Upsilon_v)) = \prod_{v=1}^{t} b_{\gamma_v-1,l_v}(\mathcal{F}(\Upsilon_v)) = \prod_{v=1}^{t} 1 = 1$$

as desired. Therefore we assume that $\Gamma_{\mathbf{m}}$ is a tree and $j = |V(\Gamma_{\mathbf{m}})|$.

Suppose that the facets $F_0, F_1, \ldots, F_q$ of $\Gamma_{\mathbf{m}}$ are ordered as in Theorem 5.1.1. Then we have $j = |\bigcup_{r=0}^{q} F_r|$ as $\Gamma_{\mathbf{m}}$ is a simplicial complex on j vertices. Now we have

$$b_{i,j}(\mathcal{F}(\Gamma_{\mathbf{m}})) = b_{i,j}\big(\mathcal{F}(\langle F_0\rangle)\big) + \sum_{u=1}^{q} b_{i-1,j-|F_u|}\big(\mathcal{F}((\Gamma_{\mathbf{m}})_{u-1})_{P_u}\big) \tag{5.1.1}$$

by Theorem 5.1.1. If F_0 is the only facet of $\Gamma_{\mathbf{m}}$, then we are done by Remark 5.1.2. So assume that $q \geq 1$ and let Δ_u be the facet complex of $\mathcal{F}((\Gamma_{\mathbf{m}})_{u-1})_{P_u}$ for every $1 \leq u \leq q$. Note that the set of facets of Δ_u is a subset of $\{F_0 \setminus F_u, \ldots, F_{u-1} \setminus F_u\}$ for every $1 \leq u \leq q$.

Since F_q has a free vertex in $\Gamma_{\mathbf{m}}$, $|V((\Gamma_{\mathbf{m}})_u)| < j$ for $u < q$. In particular, $|F_0| < j$ and $|V(\Delta_u)| < j - |F_u|$ when $u < q$. Hence by Remark 2.3.6, Equation (5.1.1) turns into

$$b_{i,\mathbf{m}}(\mathcal{F}(\Gamma)) = b_{i-1,j-|F_q|}\big(\mathcal{F}((\Gamma_{\mathbf{m}})_{q-1})_{P_q}\big).$$

Observe that by definition of Δ_q some of $F_0 \setminus F_q, \ldots, F_{q-1} \setminus F_q$ might have already been omitted when forming the facet set of Δ_q. So, $j - |F_q|$ is greater than or equal to the number of vertices of Δ_q. If it is greater, then

$$b_{i,\mathbf{m}}(\mathcal{F}(\Gamma)) = b_{i-1,j-|F_q|}\big(\mathcal{F}((\Gamma_m)_{q-1})_{P_q}\big) = 0$$

and nothing is left to prove. Otherwise, Δ_q is a simplicial forest on $j - |F_q|$ vertices by [17, Lemma 1]. Since $j \leq s+1$, Δ_q satisfies the induction hypothesis. The proof follows from observing that $b_{i-1,j-|F_q|}\big(\mathcal{F}((\Gamma_{\mathbf{m}})_{q-1})_{P_q}\big)$ is also a multigraded Betti number. □

Observe that in the proof of Theorem 5.1.3 when we focused on the top degree Betti numbers of a simplicial tree Γ, the recursive formula of Theorem 5.1.1 came down to

$$b_{i,n}(\mathcal{F}(\Gamma)) = b_{i-1,n-|F|}(\mathcal{F}(\Gamma \setminus \langle F \rangle)_{(x_i:\ x_i \notin F)}) \tag{5.1.2}$$

where F is a leaf of Γ, $i \geq 1$ and n is the number of vertices of Γ. By the virtue of Lemma 4.1.1 we will only need this shortened version of the recursive formula given in Theorem 5.1.1.

5.2 Combinatorial Description for the Betti Numbers, Projective Dimension and Regularity

The Betti numbers of facet ideals of simplicial forests are completely characterized by the well ordered facet covers. We set to prove this next by observing how such facet covers behave under localization.

Lemma 5.2.1. *Let Γ be a simplicial complex and let F be a facet of Γ that contains a free vertex. Let $P = (x_i \mid x_i \notin F)$ be an ideal of $\Bbbk[x_1, \ldots, x_n]$. Suppose that $\mathcal{F}(\Gamma \setminus \langle F \rangle)_P$ is the facet ideal of Δ and $V(\Delta) = V(\Gamma) \setminus F$. Then the following hold.*

1. *If $\{F_1 \setminus F, \ldots, F_k \setminus F\}$ is a minimal facet cover of Δ, then $\{F_1, \ldots, F_k, F\}$ is a minimal facet cover of Γ.*

(2) *If* $F_1 \setminus F, \dots, F_k \setminus F$ *is a well ordered facet cover of* Δ*, then* $F_1, \dots, F_k, F$ *is a well ordered facet cover of* Γ*.*

Proof. (1) Assume that $\{F_1 \setminus F, \dots, F_k \setminus F\}$ is a minimal facet cover of Δ. Observe that $\{F_1, \dots, F_k, F\}$ covers Γ since $V(\Delta) = V(\Gamma) \setminus F$. To see the minimality of $\{F_1, \dots, F_k, F\}$, assume for a contradiction one of its elements is redundant. Note that the redundant facet cannot be F since it contains a free vertex of Γ. So say F_s is redundant for some $s = 1, \dots, k$. Then we obtain $F_s \subseteq F \cup (\bigcup_{i \neq s} F_i)$ or equivalently $F_s \setminus F \subseteq \bigcup_{i \neq s}(F_i \setminus F)$ which contradicts the minimality of $\{F_1 \setminus F, \dots, F_k \setminus F\}$.

(2) Assume that $F_1 \setminus F, \dots, F_k \setminus F$ is a well ordered facet cover of Δ. By part (1), $\{F_1, \dots, F_k, F\}$ is a minimal facet cover of Γ. Let $H \notin \{F_1, \dots, F_k, F\}$ be a facet of Γ. Then we consider two cases:

Case 1: If $H \setminus F$ is a facet of Δ, then by assumption

$$F_s \setminus F \subseteq (H \setminus F) \cup (F_{s+1} \setminus F) \cup \ldots \cup (F_k \setminus F)$$

for some $s \leq k-1$ and hence $F_s \subseteq H \cup F_{s+1} \cup \ldots \cup F_k \cup F$ as desired.

Case 2: If $H \setminus F$ is not a facet of Δ, then $K \setminus F \subseteq H \setminus F$ for some facet $K \setminus F$ of Δ. Now if $K \setminus F = F_t \setminus F$ for some $1 \leq t \leq k$, then $F_t \setminus F \subseteq H \setminus F$ and $F_t \subseteq H \cup F \subseteq H \cup F_{t+1} \cup \ldots \cup F_k \cup F$ as desired. Therefore we assume that $K \setminus F \neq F_t \setminus F$ for all $1 \leq t \leq k$. Then by assumption we have

$$F_\ell \setminus F \subseteq (K \setminus F) \cup (F_{\ell+1} \setminus F) \cup \ldots \cup (F_k \setminus F)$$

for some $\ell \leq k-1$. Thus we get $F_\ell \subseteq K \cup F_{\ell+1} \cup \ldots \cup F_k \cup F \subseteq H \cup F_{\ell+1} \cup \ldots \cup F_k \cup F$ which completes the proof. $\square$

We are now ready to prove the main result of this section.

Theorem 5.2.2 (Combinatorial description for Betti numbers of simplicial forests). *Let Γ be a simplicial forest. Suppose that $\mathbf{m}$ is a monomial and $i \geq 1$. Then the following are equivalent.*

(1) $b_{i,\mathbf{m}}(S/\mathcal{F}(\Gamma)) \neq 0$.

(2) $b_{i,\mathbf{m}}(S/\mathcal{F}(\Gamma)) = 1$.

(3) *The induced subcollection $\Gamma_{\mathbf{m}}$ has a well ordered facet cover of cardinality i.*

In particular, $b_{i,j}(S/\mathcal{F}(\Gamma))$ is the number of induced subcollections of Γ which have j vertices and which have well ordered facet covers of cardinality i.

Proof. The equivalence of (1) and (2) follows from Theorem 5.1.3. So we only need to prove that (1) and (3) are equivalent. We may assume that $i \geq 2$ since the statement is trivial for $i = 1$. By Lemma 4.1.1, it suffices to prove that $b_{i,n}(S/\mathcal{F}(\Gamma)) \neq 0$ if and only if Γ has a well ordered facet cover of cardinality i where n is the number of vertices of Γ. First observe that if Γ has a well ordered facet cover of cardinality i, then $b_{i,n}(S/\mathcal{F}(\Gamma)) \neq 0$ follows from Corollary 4.2.6. Therefore we assume that $b_{i,n}(S/\mathcal{F}(\Gamma)) \neq 0$. We will proceed by induction on the number of vertices of Γ. If Γ is not connected, then we can apply Lemma 4.1.2 to $\mathcal{F}(\Upsilon_1), \ldots, \mathcal{F}(\Upsilon_k)$ where $\Upsilon_1, \ldots, \Upsilon_k$ are the connected components of Γ. Then by using (4.1.2) and Theorem 5.1.3 we get $0 \neq b_{i,n}(S/\mathcal{F}(\Gamma)) = b_{u_1,q_1}(S/\mathcal{F}(\Upsilon_1)) \ldots b_{u_k,q_k}(S/\mathcal{F}(\Upsilon_k))$ for some $u_1, \ldots, u_k$ where $q_1, \ldots, q_k$ are the number of vertices of $\Upsilon_1, \ldots, \Upsilon_k$ respectively and $u_1, \ldots, u_k$ satisfy $u_1 + \cdots + u_k = i$. But then by induction hypothesis for each $t = 1, \ldots, k$ the simplicial tree Υ_t has a well ordered facet cover $F_1^t, \ldots, F_{u_t}^t$. It is easy then to check that $F_1^1, \ldots, F_{u_1}^1, \ldots, F_1^k, \ldots, F_{u_k}^k$ is a well ordered facet cover of Γ of cardinality i.

Now we may assume that Γ is a simplicial tree on the vertices $x_1, \ldots, x_n$. Let F be a leaf of Γ and $\mathcal{F}(\Gamma \setminus \langle F \rangle)_{(x_i | x_i \notin F)}$ be the facet ideal of Δ. Observe that Δ has at most $n - |F|$ vertices. By Eq. (5.1.2) we have $1 = b_{i,n}(S/\mathcal{F}(\Gamma)) = b_{i-1,n-|F|}(S/\mathcal{F}(\Delta))$. Hence the nonvanishing Betti number and Remark 2.3.6 require Δ to have exactly

$n - |F|$ vertices. By induction hypothesis Δ has a well ordered facet cover of size $i-1$. Thus Lemma 5.2.1 gives that Γ has a well ordered facet cover of cardinality i which completes the proof. □

Corollary 5.2.3 (**Combinatorial description for projective dimension and regularity of simplicial forests**). *If* Γ *is a simplicial forest, then*

(1) $\mathrm{pd}(S/\mathcal{F}(\Gamma))$ *is the maximum cardinality of a well ordered facet cover of an induced subcollection of* Γ.

(2) $\mathrm{reg}(S/\mathcal{F}(\Gamma))$ *is equal to*

$$\max\left\{ \deg(m) - s_m \mid F_1, \dots, F_{s_m} \text{ is a well ordered facet cover of } \Gamma_{\mathbf{m}} \right\}.$$

(3) *All well ordered facet covers of a simplicial forest have the same cardinality.*

Proof. Immediately follows from Theorem 5.2.2 and Theorem 5.1.3. □

Recall that Corollary 3.1.2 already provided a combinatorial formula for the projective dimension of facet ideals of simplicial forests. However, our formula for projective dimension in Corollary 4.2.7 is different since it is expressed in terms of minimal facet covers instead of minimal vertex covers.

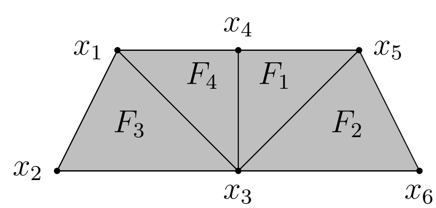

Figure 5.1: A simplicial tree $\Gamma = \langle F_1, F_2, F_3, F_4 \rangle$

Example 5.2.4. Suppose that Γ is the simplicial tree in Figure 5.1. We wish to apply Theorem 5.2.2 to find the Betti numbers of $S/\mathcal{F}(\Gamma)$. Observe that F_1, F_2, F_3 is a well ordered facet cover of Γ since $F_1 \subseteq F_2 \cup F_3 \cup F_4$. Therefore $b_{3,6} = 1$. Since the multigraded Betti numbers come from the induced subcollections we check which subcollections give Betti numbers. We see that Γ has two induced subcollections which are generated by 3 facets, namely $\Gamma_{x_1x_2x_3x_4x_5} = \langle F_1, F_3, F_4\rangle$ and $\Gamma_{x_1x_3x_4x_5x_6} = \langle F_1, F_2, F_4\rangle$. These two simplicial complexes are isomorphic and have no well ordered facet covers, so they do not give Betti numbers. Next, we see that Γ has four induced subcollections which are generated by 2 facets, namely $\Gamma_{x_3x_4x_5x_6} = \langle F_1, F_2\rangle$, $\Gamma_{x_1x_3x_4x_5} = \langle F_1, F_4\rangle$, $\Gamma_{x_1x_2x_3x_5x_6} = \langle F_2, F_3\rangle$ and $\Gamma_{x_1x_2x_3x_4} = \langle F_3, F_4\rangle$. For each of these subcollections the facet set is the same as the unique minimal facet cover. Therefore they all have well ordered facet covers of cardinality 2 and $b_{2,x_3x_4x_5x_6} = b_{2,x_1x_3x_4x_5} = b_{2,x_1x_2x_3x_5x_6} = b_{2,x_1x_2x_3x_4} = 1$. Finally every facet of Γ generates an induced subcollection with a well ordered facet cover. Thus $b_{1,x_1x_2x_3} = b_{1,x_1x_3x_4} = b_{1,x_3x_4x_5} = b_{1,x_3x_5x_6} = 1$. The Betti diagram of $S/\mathcal{F}(\Gamma)$ is

	0	1	2	3
Total	1	4	4	1
0	1	–	–	–
1	–	–	–	–
2	–	4	3	–
3	–	–	1	1

where ith column and jth row is $b_{i,i+j}$.

Remark 5.2.5. Although the Betti numbers of facet ideals of simplicial forests can be described as in Theorem 5.2.2, it is not possible in general to minimally resolve such ideals by Lyubeznik resolution. In fact, the ideal given in Example 5.2.4 has no simplicial resolution. To see this, assume for a contradiction that Θ supports a minimal free resolution of $S/\mathcal{F}(\Gamma)$. Looking at the multigraded Betti number in the third homological degree, we can see that either $\{F_2, F_3, F_4\}$ or $\{F_1, F_2, F_3\}$ must be a face of Θ. If $\{F_2, F_3, F_4\}$ is a face, then $\{F_2, F_4\}$ is a face as well. But then $b_{2,x_1x_3x_4x_5x_6} \neq 0$ which is not true. Similarly if $\{F_1, F_2, F_3\}$ is a face, then $\{F_1, F_3\}$ is a face and we get $b_{2,x_1x_2x_3x_4x_5} \neq 0$, a contradiction.

5.3 A Homological Characterization for the Betti Numbers of Some Facet Ideals

In [7] Bouchat provided the following homological description for the Betti numbers of edge ideals of forests.

Theorem 5.3.1. [7, Theorem 2.2.7] *Let the graph G be a forest and $\mathbf{m}$ be a monomial. Then the following are equivalent.*

(1) $b_{i,\mathbf{m}}(S/I(G)) \neq 0$ *for some i.*

(2) $b_{\mathrm{pd}(S/I(G_{\mathbf{m}})),\mathbf{m}}(S/I(G)) \neq 0$.

Note that this characterization does not involve any combinatorial properties of the graph. The theorem asserts that appearance of a monomial $\mathbf{m}$ as a multidegree in the resolution of $S/I(G)$ is equivalent to the fact that $\mathbf{m}$ appears as a multidegree in the last homological position of the resolution of $S/I(G_{\mathbf{m}})$. Notice that since the statement (2) of the theorem trivially implies the first statement, we can rephrase it in a plainer form.

Theorem 5.3.2. [7, Theorem 2.2.7] *Let the graph G be a forest and $\mathbf{m}$ be a monomial. If $b_{i,\mathbf{m}}(S/I(G)) \neq 0$ for some i, then $i = \mathrm{pd}(S/I(G_{\mathbf{m}}))$.*

Remark 5.3.3. The converse of Theorem 5.3.2 is not true. For example, consider the graph G with $I(G) = (ab, bc, cd)$ and $\mathbf{m} = abcd$. The projective dimension of $S/I(G)$ is 2 but we have $b_{i,\mathbf{m}}(S/I(G)) = 0$ for all i.

The goal of this section is to show that Theorem 5.3.1 holds for a larger class of ideals, namely for facet ideals of those simplicial forests which satisfy a certain condition.

We will first prove some results concerning minimal vertex covers and localizations which will be essential in the proof of the main result (Theorem 5.3.7) of this section.

Proposition 5.3.4. *If* Γ *is a simplicial forest which satisfies*

$$F \cap G \cap H \neq \emptyset \Longrightarrow F \cap G = G \cap H = H \cap F \tag{5.3.1}$$

for all facets F, G, H *(in particular, if* Γ *is any graph forest), then* Γ *has a minimal vertex cover of maximum cardinality which contains a free vertex.*

Proof. We use induction on the number of facets of the simplicial forest. If Γ has only one facet, then for any vertex v the set $\{v\}$ is a minimal vertex cover of maximum cardinality, and v is a free vertex of Γ.

So we assume that Γ has at least two facets. If Γ is not connected, then its connected components satisfy the induction hypothesis. Since every minimal vertex cover of Γ is a union of minimal vertex covers of its connected components, we are done.

We assume that Γ is connected. Let C be a minimal vertex cover of Γ with $|C| = \alpha'_0(\Gamma)$. Let F be a leaf of Γ with a free vertex u. If C is not a minimal vertex cover of $\Gamma \setminus \langle F \rangle$, then by Lemma 3.2.1(3) we have $C \cap F = \{a\}$ for some vertex a, and $C \setminus \{a\}$ is a minimal vertex cover of $\Gamma \setminus \langle F \rangle$. But since u is a free vertex $(C \setminus \{a\}) \cup \{u\}$ is a minimal vertex cover of Γ and we are done.

So, we assume that C is a minimal vertex cover of $\Gamma \setminus \langle F \rangle$. Then we have $\alpha'_0(\Gamma) \leq \alpha'_0(\Gamma \setminus \langle F \rangle)$ and combining this with Lemma 3.2.1(5) yields $\alpha'_0(\Gamma) = \alpha'_0(\Gamma \setminus \langle F \rangle)$. Since any subcollection of Γ satisfies (5.3.1), the simplicial forest $\Gamma \setminus \langle F \rangle$ satisfies the induction hypothesis. Therefore there exists a minimal vertex cover D of $\Gamma \setminus \langle F \rangle$ of maximum cardinality such that D contains a free vertex v of $\Gamma \setminus \langle F \rangle$. Let F_1 be a joint of F. We will consider two cases:

Case 1: There are at least two facets of Γ, say, F_1 and F_2 that intersect F. Since F_1 is a joint, $F \cap F_1 \cap F_2 \neq \emptyset$ and $F \cap F_1 = F \cap F_2$ by (5.3.1). But then $v \notin F$ as otherwise $v \in F_1 \cap F_2$ is not a free vertex of $\Gamma \setminus \langle F \rangle$. Thus D is a minimal vertex cover of Γ which contains the free vertex v of Γ.

Case 2: Suppose that F_1 is the only facet of Γ that intersects F. If $v \notin F$, then D is a minimal vertex cover of Γ which contains the free vertex v of Γ and we the proof is complete. So we assume that $v \in F \cap F_1$. But then F and F_1 are the only facets of Γ that contain v and $D \setminus \{v\}$ covers $\Gamma \setminus \langle F, F_1 \rangle$. Observe that by Lemma 3.2.1(1), $D \setminus \{v\}$ is a minimal vertex cover of $\Gamma \setminus \langle F, F_1 \rangle$. Therefore, $\alpha_0'(\Gamma \setminus \langle F, F_1 \rangle) \geq \alpha_0'(\Gamma) - 1$. Note that any minimal vertex cover of $\Gamma \setminus \langle F, F_1 \rangle$ can be extended to that of Γ by adding at least one vertex. Thus we have in fact $\alpha_0'(\Gamma \setminus \langle F, F_1 \rangle) = \alpha_0'(\Gamma) - 1$. By induction assumption, $\Gamma \setminus \langle F, F_1 \rangle$ has a minimal vertex cover E of cardinality $\alpha_0'(\Gamma) - 1$ which contains a free vertex w of $\Gamma \setminus \langle F, F_1 \rangle$. Recall that F_1 is the only facet of Γ that intersects F and thus $w \notin F$. We will consider two cases.

Case 2.1: Suppose that $w \in F_1$. Then, it is straightforward to check that $E \cup \{u\}$ is a minimal vertex cover of Γ of cardinality $\alpha_0'(\Gamma)$, and we are done.

Case 2.2: Suppose that $w \notin F_1$. Then w is a free vertex of Γ. Now we claim that E can be extended to a minimal vertex cover of Γ by adding exactly one vertex. To this end, if $E \cap F_1 \neq \emptyset$, then $E \cup \{u\}$ is a minimal vertex cover of Γ as desired. Otherwise pick $z \in F \cap F_1$ and observe that $E \cup \{z\}$ is a minimal vertex cover of Γ. □

Remark 5.3.5. Note that if the assumption (5.3.1) is omitted, Proposition 5.3.4 is no longer true. For example, for the simplicial tree Γ of Figure 5.2 we have $\alpha_0'(\Gamma) = 4$. Indeed, $\{b, e, f, h\}$ is a minimal vertex cover, and no minimal vertex cover has cardinality greater than 4. However if a minimal vertex cover contains a free vertex of Γ, then it has cardinality 3.

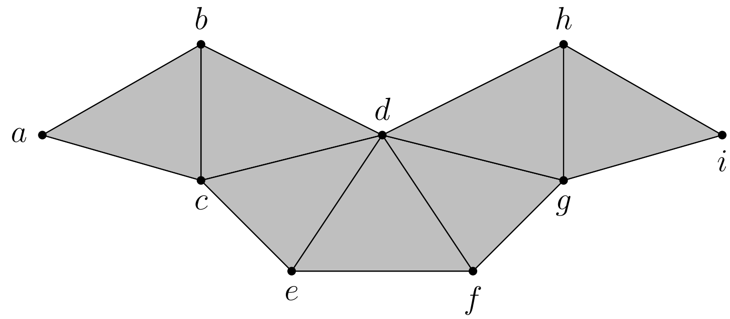

Figure 5.2: A simplicial tree Γ with $\alpha_0'(\Gamma) = 4$.

Lemma 5.3.6. *Let* Γ *be a simplicial tree which satisfies* (5.3.1) *for all facets* F, G, H *of* Γ. *Let* K *be a facet of* Γ *and let* Δ *be the localized complex* $(\Gamma \setminus \langle K \rangle)_{(x_i | x_i \notin K)}$. *Then for any* $F, G, H \in \operatorname{Facets}(\Delta)$ (5.3.1) *holds.*

Proof. Let σ, τ and κ be facets of Δ. Then,

$$\begin{aligned}
\sigma \cap \tau \cap \kappa \neq \emptyset &\Rightarrow \sigma = F \setminus K,\ \tau = G \setminus K,\ \kappa = H \setminus K \text{ for some } F, G, H \in \operatorname{Facets}(\Gamma) \\
&\quad \text{and } (F \setminus K) \cap (G \setminus K) \cap (H \setminus K) \neq \emptyset \\
&\Rightarrow F \cap G \cap H \neq \emptyset \\
&\Rightarrow F \cap G = G \cap H = H \cap F \\
&\Rightarrow (F \cap G) \setminus K = (G \cap H) \setminus K = (H \cap F) \setminus K \\
&\Rightarrow (F \setminus K) \cap (G \setminus K) = (G \setminus K) \cap (H \setminus K) = (H \setminus K) \cap (F \setminus K) \\
&\Rightarrow \sigma \cap \tau = \tau \cap \kappa = \kappa \cap \sigma.
\end{aligned}$$

□

We are now ready to prove the main result of this section.

Theorem 5.3.7. *Suppose that Γ is a simplicial forest which satisfies*

$$F \cap G \cap H \neq \emptyset \Longrightarrow F \cap G = G \cap H = H \cap F \tag{5.3.2}$$

for any facets F, G, H of Γ. Then for any monomial $\mathbf{m}$ the following statements are equivalent.

(1) $b_{i,\mathbf{m}}(S/\mathcal{F}(\Gamma)) \neq 0$ *for some i.*

(2) $b_{\mathrm{pd}(S/\mathcal{F}(\Gamma_{\mathbf{m}})),\mathbf{m}}(S/\mathcal{F}(\Gamma)) \neq 0$.

Proof. Since the statement (2) trivially implies the statement (1), we only need to show that (1) implies (2). We use induction on the number of facets of Γ. When Γ is generated by a single facet, the given statement follows from Remark 5.1.2. Suppose that $b_{i,\mathbf{m}}(S/\mathcal{F}(\Gamma)) \neq 0$ for some i. By virtue of Theorem 3.1.1 it suffices to show that $\alpha_0'(\Gamma_{\mathbf{m}}) = i$. Since we already know that $\alpha_0'(\Gamma_{\mathbf{m}}) \geq i$ we only want to show that $\alpha_0'(\Gamma_{\mathbf{m}}) \leq i$. Note that by Lemma 4.1.1 we have $b_{i,\mathbf{m}}(S/\mathcal{F}(\Gamma)) = b_{i,j}(S/\mathcal{F}(\Gamma_{\mathbf{m}}))$ where j is the degree of $\mathbf{m}$. We will consider two cases.

Case 1: Suppose that $\Gamma_{\mathbf{m}}$ is not connected. Let $\Upsilon_1, \ldots, \Upsilon_N$ be the connected components of $\Gamma_{\mathbf{m}}$. By Equation (4.1.2) and Theorem 5.1.3 we have

$$b_{i,j}(S/\mathcal{F}(\Gamma_{\mathbf{m}})) = b_{u_1,q_1}(S/\mathcal{F}(\Upsilon_1)) \ \ldots \ b_{u_N,q_N}(S/\mathcal{F}(\Upsilon_N))$$

for some $u_1, \ldots, u_N$ such that $u_1 + \cdots + u_N = i$ and where $q_1, \ldots, q_N$ are respectively the number of vertices of $\Upsilon_1, \ldots, \Upsilon_N$ and $q_1 + \cdots + q_N = j$. Observe that for each $k = 1, \ldots, N$, we have

$$b_{u_k,q_k}(S/\mathcal{F}(\Upsilon_k)) = b_{u_k,\mathbf{q_k}}(S/\mathcal{F}(\Upsilon_k))$$

where $\mathbf{q_k}$ is the monomial corresponding to the product of vertices of Υ_k. Therefore by induction hypothesis, $\alpha_0'(\Upsilon_k) = u_k$ for each $k = 1, \ldots, N$. Since any minimal vertex cover of a simplicial complex is a disjoint union of minimal vertex covers of its connected components, we have

$$\alpha_0'(\Gamma_{\mathbf{m}}) = \alpha_0'(\Upsilon_1) + \cdots + \alpha_0'(\Upsilon_N) = u_1 + \cdots + u_N = i$$

which completes the proof for this case.

Case 2: Suppose that $\Gamma_{\mathbf{m}}$ is a simplicial tree and let F be a leaf of $\Gamma_{\mathbf{m}}$. Let Δ be the localized complex $(\Gamma_{\mathbf{m}} \setminus \langle F \rangle)_{(x_i | x_i \notin F)}$. By Equation (5.1.2) we have

$$b_{i,j}(S/\mathcal{F}(\Gamma_{\mathbf{m}})) = b_{i-1,j-|F|}(S/\mathcal{F}(\Delta)) \neq 0.$$

But then by Remark 2.3.6, Δ must have $j - |F|$ vertices and $b_{i-1,j-|F|}(S/\mathcal{F}(\Delta)) = b_{i-1,\mathbf{q}}(S/\mathcal{F}(\Delta))$ where $\mathbf{q}$ is the product of vertices of Δ. Observe that Δ satisfies the induction hypothesis by Lemma 5.3.6. Therefore by Theorem 5.1.3 we get

$$i - 1 = \mathrm{pd}(S/\mathcal{F}(\Delta_{\mathbf{q}})) = \mathrm{pd}(S/\mathcal{F}(\Delta)) = \alpha_0'(\Delta).$$

By Proposition 5.3.4 there exists a minimal vertex cover C of $\Gamma_{\mathbf{m}}$ of cardinality $\alpha_0'(\Gamma_{\mathbf{m}})$ which contains a free vertex u of $\Gamma_{\mathbf{m}}$. But then by Lemma 3.2.4(2), $C \setminus \{u\}$ is a minimal vertex cover of Δ. Hence $\alpha_0'(\Delta) = i - 1 \geq \alpha_0'(\Gamma_{\mathbf{m}}) - 1$ which completes the proof. $\square$

We do not know if Theorem 5.3.7 holds for all simplicial forests. In particular, by Lemma 4.1.1 this is equivalent to ask whether the top degree Betti numbers occur at the projective dimension.

Although Proposition 5.3.4 cannot be extended to all simplicial complexes, Theorem 5.3.7 might be true for all simplicial complexes. Going over the proof of this theorem, we can see that it requires a simplicial forest to have a minimal vertex cover of maximum cardinality with free vertex only if there is a nonzero Betti number at the top degree. Therefore the following questions arise naturally.

Question 5.3.8. Does Theorem 5.3.7 still hold when Γ is an arbitrary simplicial forest?

Question 5.3.9. Suppose that Γ is a simplicial forest which has a well ordered facet cover. Then does Γ have a minimal vertex cover of maximum cardinality which contains a free vertex?

As we have discussed above, an affirmative answer to Question 5.3.9 would settle Question 5.3.8 in the positive as well.

5.4 Edge Ideals of Forests

In this section we will focus on the Betti numbers of edge ideals of forests. When G is a graph, unless it has isolated vertices, its edge ideal $I(G)$ is the same as its facet ideal $\mathcal{F}(G)$ where G is considered as a simplicial complex in the latter case. The following observation will be useful in the sequel.

Remark 5.4.1. Let G be a graph such that the induced subgraph $G_{\mathbf{m}}$ contains an isolated vertex. Then $\text{Taylor}(I(G))_{<\mathbf{m}}$ is a simplex and thus Theorem 2.3.4 gives $b_{i,\mathbf{m}}(S/I(G)) = 0$. Therefore if $b_{i,\mathbf{m}}(S/I(G)) \neq 0$, then the induced subgraph $G_{\mathbf{m}}$ does not contain any isolated vertices and

$$G_{\mathbf{m}} \text{ (induced subgraph of } G) = G_{\mathbf{m}} \text{ (induced subcollection of } G)$$

where G is considered as a simplicial complex in the latter one.

If we restrict Theorem 5.2.2 to edge ideals, we get the following description for Betti numbers of edge ideals of forests which is originally due to Kimura [31].

Corollary 5.4.2. [31, Theorem 4.1] *If the graph G is a forest and $\mathbf{m}$ is a monomial, then the following are equivalent.*

(1) $b_{i,\mathbf{m}}(S/I(G)) \neq 0$.

(2) $b_{i,\mathbf{m}}(S/I(G)) = 1$.

(3) *The induced subgraph $G_{\mathbf{m}}$ contains a strongly disjoint set of $\deg(\mathbf{m}) - i$ bouquets.*

Proof. Note that by Remark 5.4.1 we may assume that G has no isolated vertices. Since G has no isolated vertices, $\mathcal{F}(G) = I(G)$. The equivalence of the given statements then follows by Theorem 5.2.2 and Proposition 4.3.3. □

As an immediate consequence of the result above we recover a result of Zheng [46] which relates the induced matching number of a forest to the regularity of its edge ideal.

Corollary 5.4.3. [46, Theorem 2.18] *If G is a forest, then $reg(S/I(G)) = im(G)$.*

Using Corollary 5.4.2 we will establish further properties of edge ideals of forests. To this end, we will first prove some combinatorial results concerning graph forests.

Lemma 5.4.4. *Let the graph G be a forest on q vertices. Let A be a subset of $V(G) \cup E(G)$ such that for every $u, v \in V(G)$*

$$\{u, v\} \in A \Longrightarrow u, v \notin A. \tag{5.4.1}$$

Then $|A| \leq q$. Also if $A \cap E(G) \neq \emptyset$, then $|A| \leq q - 1$.

Proof. If $A \cap E(G) = \emptyset$, then $A \subseteq V(G)$ and $|A| \leq q$ is clear.

Suppose that $A \cap E(G) \neq \emptyset$. Let H be the subgraph of G which is obtained by taking the edges of G which appear in A. More precisely, H is the subgraph of G such that

$$E(H) := A \cap E(G) \text{ and } V(H) := \bigcup_{e \in A \cap E(G)} e.$$

Let $H_1, \ldots, H_r$ be the connected components of H. Observe that by (5.4.1) we have $A \cap V(H_i) = \emptyset$ for every $i = 1, \ldots, r$ and thus $\bigcup_{i=1}^{r} V(H_i) \subseteq V(G) \setminus A$. Then we get

$$\begin{aligned}
q = |V(G)| &= |A \cap V(G)| + |V(G) \setminus A| \\
&\geq |A \cap V(G)| + |\bigcup_{i=1}^{r} V(H_i)| \\
&= |A \cap V(G)| + \sum_{i=1}^{r} |V(H_i)| \\
&= |A \cap V(G)| + \sum_{i=1}^{r} |E(H_i) + 1|, \quad \text{since each } H_i \text{ is a tree} \\
&= |A \cap V(G)| + |E(H)| + r \\
&= |A \cap V(G)| + |A \cap E(G)| + r \\
&= |A| + r
\end{aligned}$$

and this gives $|A| \leq q - 1$ as $r \geq 1$. $\square$

Lemma 5.4.5. *Let the graph G be a forest. Suppose that G contains s strongly disjoint bouquets $\mathcal{B}_1, \ldots, \mathcal{B}_s$. Then the following statements hold.*

(1) *Given any two distinct bouquets $\mathcal{B}_j$ and $\mathcal{B}_k$ there exists at most one edge of G which contains a vertex of each.*

(2) *$im(G) = s$.*

(3) *$im(G) > im(G - E(\mathcal{B}_i))$ for each $i = 1, \ldots, s$.*

Proof. (1) Assume for a contradiction that there exist exactly $r \geq 2$ edges of G which contain a vertex of each of the bouquets $\mathcal{B}_j$ and $\mathcal{B}_k$. Then the induced subgraph K

of G on $V(\mathcal{B}_j) \cup V(\mathcal{B}_k)$ is a connected graph of order $|V(\mathcal{B}_j)| + |V(\mathcal{B}_k)|$ and of size $|V(\mathcal{B}_j)| + |V(\mathcal{B}_k)| + r - 2$. Therefore K is not a tree by Theorem 2.2.2(2). But this is absurd because every connected subgraph of a forest must be a tree.

Before proving the next statement, we define a new graph H whose vertices are the bouquets and two vertices are adjacent if there exists an edge of G which contains a vertex from each of these bouquets. In other words,

$$V(H) := \{\mathcal{B}_1, \ldots, \mathcal{B}_s\},$$
$$E(H) := \big\{\{\mathcal{B}_j, \mathcal{B}_k\} \mid \ e \cap V(\mathcal{B}_j) \neq \emptyset, e \cap V(\mathcal{B}_k) \neq \emptyset \text{ for some } j \neq k \text{ and } e \in E(G)\big\}.$$

This will allow us to identify some set of edges of G with some subsets of $V(H) \cup E(H)$. To make this precise, we will need the function $f : E(G) \to V(H) \cup E(H)$ such that for every $e \in E(G)$

$$f(e) = \begin{cases} \{\mathcal{B}_j, \mathcal{B}_k\} & \text{if } e \cap V(\mathcal{B}_j) \neq \emptyset \text{ and } e \cap V(\mathcal{B}_k) \neq \emptyset \text{ for some } j \neq k \\ \mathcal{B}_j & \text{if } e \in E(\mathcal{B}_j) \text{ for some } j. \end{cases}$$

First, we will prove the following.

Claim: H is a forest.

Proof of the claim: Assume for a contradiction H contains a cycle on the vertices $\mathcal{B}_{i_1}, \ldots, \mathcal{B}_{i_t}$ with $t \geq 3$ where $\{\mathcal{B}_{i_u}, \mathcal{B}_{i_{u+1}}\} \in E(H)$ for $1 \leq u \leq t-1$ and $\{\mathcal{B}_{i_1}, \mathcal{B}_{i_t}\} \in E(H)$. By definition of H and since the bouquets are pairwise disjoint, there exist t edges $e_1, e_2, \ldots, e_t$ of G such that

$$e_u \cap V(\mathcal{B}_{i_u}) \neq \emptyset \text{ and } e_u \cap V(\mathcal{B}_{i_{u+1}}) \neq \emptyset, \text{ for all } 1 \leq u \leq t-1,$$
$$e_t \cap V(\mathcal{B}_{i_t}) \neq \emptyset \text{ and } e_t \cap V(\mathcal{B}_{i_1}) \neq \emptyset.$$

Note that since each bouquet $\mathcal{B}_{i_u}$ for $1 \leq u \leq t$ intersects exactly two of the edges $e_1, e_2, \ldots, e_t$, the induced subgraph of G on $\cup_{i=1}^{t} e_i$ cannot contain a star subgraph of size 3 or more. Now if $e_u \cap e_{u+1} \neq \emptyset$ for all $1 \leq u \leq t$ (we use the convention

$e_{t+1} := e_1$), then

$$e_1, e_2, \dots, e_t \tag{5.4.2}$$

is the list of edges of a cycle of G and we are done. Otherwise, the list in (5.4.2) can be completed to the list of edges of a cycle of G by inserting new edges. To see this, let $e_p \cap e_{p+1} = \emptyset$ for some $1 \leq p \leq t$. Let r_p and r_{p+1} be respectively the roots of $\mathcal{B}_{i_p}$ and $B_{i_{p+1}}$ (by roots we mean the vertices which are not flowers). If e_p does not contain r_{p+1}, then in the list we insert the edge $\{r_{p+1}, V(\mathcal{B}_{i_{p+1}}) \cap e_p\}$ right after e_p. Similarly, if e_{p+1} does not contain r_{p+1}, then in the list we insert the edge $\{r_{p+1}, V(\mathcal{B}_{i_{p+1}}) \cap e_{p+1}\}$ right before e_{p+1}. Thus, H is indeed a forest.

(2) Next, we will show that $im(G) = s$. We already know $im(G) \geq s$ and so, we only need to show that $im(G) \leq s$. Let D be an induced matching in G. We claim that $|D| \leq s$. Notice that if $f(e_1) = f(e_2) = \{\mathcal{B}_j, \mathcal{B}_k\}$ for some $e_1, e_2 \in D$, then $e_1 = e_2$ by part (1). Also since D is a matching, we have $|D \cap E(\mathcal{B}_k)| \leq 1$ for every $k = 1, \dots, s$. Therefore if $f(e_1) = f(e_2) = \mathcal{B}_j$, then $e_1 = e_2$. Hence the function f is one-to-one on D and we get $|D| = |f(D)|$. Notice that since D is an induced matching we have

$$\{\mathcal{B}_j, \mathcal{B}_k\} \in f(D) \Longrightarrow \mathcal{B}_j, \mathcal{B}_\ell \notin f(D).$$

But then as H is a forest, we can apply Lemma 5.4.4 to get $|f(D)| \leq s$ as desired.

(3) To prove the last statement, let i be fixed and let A be an induced matching of $G - E(\mathcal{B}_i)$. By part (2) it suffices to show that $s > |A|$. Observe that since A is a matching, we have $|A \cap E(\mathcal{B}_k)| \leq 1$ for every $k = 1, \dots, s$. Therefore by part (1), the function f is one-to-one on A and we get $|A| = |f(A)|$. Notice that since A is an induced matching of $G - E(\mathcal{B}_i)$ we have

$$\{\mathcal{B}_j, \mathcal{B}_k\} \in f(A) \Longrightarrow \mathcal{B}_j, \mathcal{B}_\ell \notin f(A). \tag{5.4.3}$$

Now we consider two cases.

Case 1: Suppose that $f(A)$ does not contain any edges of H so that $f(A) \subseteq V(H)$. But since we know that $\mathcal{B}_i \notin f(A)$ we must have $f(A) \subseteq V(H) \setminus \{\mathcal{B}_i\}$. This yields $|f(A)| \leq |V(H)| - 1 = s - 1$ as desired.

Case 2: Suppose that $f(A)$ contains at least one edge from $E(H)$. Then since H is a forest, by Lemma 5.4.4 and (5.4.3) we obtain $|f(A)| = |A| \leq s - 1$. □

As a consequence of this lemma, we obtain a result that has a similar flavor to Theorem 5.3.2.

Corollary 5.4.6. *Suppose that the graph G is a forest and $\mathbf{m}$ a monomial. If $b_{i,\mathbf{m}}(S/I(G)) \neq 0$ for some i, then $\deg(\mathbf{m}) - i = \operatorname{reg}(S/I(G_{\mathbf{m}}))$.*

Proof. Follows from combining Lemma 5.4.5, Corollary 5.4.2 and Lemma 4.1.1. □

Theorem 5.4.7. *Let G be a forest and $\mathbf{m}$ a monomial such that $b_{i,\mathbf{m}}(S/I(G)) = b_{i,\mathbf{m}'}(S/I(G)) = 1$ for some monomial $\mathbf{m}' \neq \mathbf{m}$. Then $\mathbf{m}' \nmid \mathbf{m}$.*

Proof. Assume for a contradiction $\mathbf{m}' \mid \mathbf{m}$. Suppose that $G_{\mathbf{m}'}$ is a graph on $i + r$ vertices and $G_{\mathbf{m}}$ is a graph on $i + k + r$ vertices for some $k, r > 0$. Note that by Lemma 4.1.1 and Remark 5.4.1 we have $b_{i,r+i+k}(S/I(G_{\mathbf{m}})) = b_{i,r+i}(S/I(G_{\mathbf{m}'})) = 1$. By Theorem 5.3.1 and Corollary 3.1.2 we have $\alpha_0'(G_{\mathbf{m}}) = \alpha_0'(G_{\mathbf{m}'}) = i$. By Corollary 5.4.2, the graph $G_{\mathbf{m}'}$ contains r strongly disjoint bouquets, say $\boldsymbol{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_r\}$. Since $F(\boldsymbol{B})$ (the flowers of $\boldsymbol{B}$) is a minimal vertex cover of $G_{\mathbf{m}'}$ of cardinality of i and since $G_{\mathbf{m}'}$ is an induced subgraph of $G_{\mathbf{m}}$, we can extend $F(\boldsymbol{B})$ to a minimal vertex cover of $G_{\mathbf{m}}$ by Lemma 3.1.8. But we have $\alpha_0'(G_{\mathbf{m}}) = i$ and so, $F(\boldsymbol{B})$ must be a minimal vertex cover of $G_{\mathbf{m}}$ as well. Note that by Lemma 5.4.5 we also know that $im(G_{\mathbf{m}}) = r + k$ and $im(G_{\mathbf{m}'}) = r$.

Let D be an induced matching of $G_{\mathbf{m}}$ of cardinality $r + k$. We claim that the following statements hold.

Claim 1: If $e \in E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'})$, then $e = \{u_e, v_e\}$ for some $u_e \in F(\boldsymbol{B})$ and $v_e \in V(G_{\mathbf{m}}) \setminus V(G_{\mathbf{m}'})$.

Claim 2: $|D \cap E(G_{\mathbf{m}'})| = r$ and $|D \cap (E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'}))| = k$.

Claim 3: There exists $t \in \{1, \ldots, s\}$ such that $D \cap (E(G_{\mathbf{m}'}) \setminus E(\mathcal{B}_t))$ is an induced matching of $G_{\mathbf{m}'} - E(\mathcal{B}_t)$ and $|D \cap (E(G_{\mathbf{m}'}) \setminus E(\mathcal{B}_t))| = r$.

Proof of Claim 1: Suppose that $e \in E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'})$. Since $G_{\mathbf{m}'}$ is an induced subgraph of $G_{\mathbf{m}}$, one of the vertices in e, call it v_e, do not belong to $V(G_{\mathbf{m}'})$. Since $F(\boldsymbol{B})$ is a minimal vertex cover of $G_{\mathbf{m}}$ we have $F(\boldsymbol{B}) \cap e \neq \emptyset$. But $v_e \notin F(\boldsymbol{B}) \cap e$ as $F(\boldsymbol{B}) \subseteq V(G_{\mathbf{m}'})$. Therefore $F(\boldsymbol{B}) \cap e = e \setminus \{v_e\}$.

Proof of Claim 2: Observe that $D \cap E(G_{\mathbf{m}'})$ is an induced matching of $G_{\mathbf{m}'}$ because D is an induced matching of $G_{\mathbf{m}}$ and $G_{\mathbf{m}'}$ is a subgraph of $G_{\mathbf{m}}$. Therefore $|D \cap E(G_{\mathbf{m}'})| \leq r$ is clear. Now we want to show that $|D \cap E(G_{\mathbf{m}'})| \geq r$. Since

$$r + k = |D| = |D \cap (E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'}))| + |D \cap E(G_{\mathbf{m}'})| \tag{5.4.4}$$

it suffices to show that $|D \cap (E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'}))| \leq k$. Moreover as D is a matching, for $e, e' \in D \cap (E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'}))$ and $e \neq e'$ we have $v_e \neq v_{e'}$. Therefore the assignment $D \cap (E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'})) \to V(G_{\mathbf{m}}) \setminus V(G_{\mathbf{m}'})$ given by $e \mapsto v_e$ is one-to-one. This implies that

$$|D \cap (E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'}))| \leq |V(G_{\mathbf{m}}) \setminus V(G_{\mathbf{m}'})| = k$$

as desired. Hence we obtained $|D \cap (E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'}))| = k$ and $|D \cap E(G_{\mathbf{m}'})| = r$ as we claimed.

Proof of Claim 3: Since $k > 0$ and $|D \cap (E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'}))| = k$ there exists $e \in D \cap (E(G_{\mathbf{m}}) \setminus E(G_{\mathbf{m}'}))$. Let $e = \{u_e, v_e\}$ as in Claim 1. Since $u_e \in F(\boldsymbol{B})$ there exists $t \in \{1, \ldots, s\}$ such that $u_e \in F(\mathcal{B}_t)$. Recall that $D \cap E(G_{\mathbf{m}'})$ is an induced matching of $G_{\mathbf{m}'}$. Therefore $(D \cap E(G_{\mathbf{m}'})) \setminus E(\mathcal{B}_t)$ must be an induced matching of $G_{\mathbf{m}'} - E(\mathcal{B}_t)$. To complete the proof of Claim (3), we need to show that

$|D \cap (E(G_{\mathbf{m}'}) \setminus E(\mathcal{B}_i))| = r$. By Claim (2) it suffices to show that $D \cap E(\mathcal{B}_t) = \emptyset$. But this is clear because D is an induced matching of G and e contains a vertex of $\mathcal{B}_t$.

Having verified all of the claims, we complete the proof as follows. By Claim 3 there exists $t \in \{1, \dots, s\}$ such that $im(G_{\mathbf{m}'} - E(\mathcal{B}_t)) \geq r$. But then by Lemma 5.4.5, we get $r = im(G_{\mathbf{m}'}) > im(G_{\mathbf{m}'} - E(\mathcal{B}_t)) \geq r$ which is a contradiction. $\square$

Clark and Mapes [10] called a monomial ideal **rigid** if all its multigraded Betti numbers are 0 or 1, and if two multidegrees appear in the same homological degree, then the monomials corresponding to those multidegrees do not divide each other. Theorem 5.4.7 along with Corollary 5.4.2 shows that edge ideals of forests are rigid. The authors in [10] gave a construction which produces the minimal resolution for rigid monomial ideals [36]. Therefore such a construction can be applied to edge ideals of forests. We do not know if the facet ideals of simplicial forests are rigid monomial ideals as well. In order to prove they are rigid, thanks to Theorem 5.1.3 and Theorem 5.2.2, it suffices to show that if a simplicial tree Γ has a well ordered facet cover of cardinality i, then no induced subcollection of Γ has a well ordered facet cover of cardinality i.

Chapter 6

Betti Numbers of Path Ideals

The path ideal of a directed graph was introduced by Conca and De Negri [12] and since then these ideals and their generalizations have been studied by many authors, see [1, 2, 3, 8, 9, 23, 32, 33].

If G is a graph with vertex set $V = \{x_1, \ldots, x_n\}$, then its **path ideal** $I_t(G)$ is the monomial ideal defined by

$$I_t(G) = (x_{i_1} \ldots x_{i_t} \mid G \text{ has a path subgraph with vertices } x_{i_1}, \ldots, x_{i_t}).$$

Note that when $t = 2$ the path ideal $I_t(G)$ is the same as the edge ideal of G. Therefore path ideals generalize edge ideals. Formulas for Betti numbers of edge ideals of paths, cycles and stars were given by Jacques in [26] using Hochster's formula ([24, Theorem 8.1.1]). Alilooee and Faridi [1, 2] generalized the techniques of Jacques to find Betti numbers of path ideals of paths and cycles.

In this section we will use a different method, namely Theorem 2.3.4 to find multigraded Betti numbers of path ideals of paths, cycles and stars.

Example 6.0.8. Consider the graph G in Figure 6.1. Then the path ideals of G are $I_4(G) = (x_2x_1x_4x_3)$, $I_3(G) = (x_2x_1x_4, x_2x_1x_3, x_1x_3x_4)$ and $I_2(G) = (x_1x_2, x_1x_3, x_1x_4, x_3x_4)$.

Definition 6.0.9. For any $n \geq t \geq 1$ the simplicial complex Ω_t^n on the set of vertices $\{1, \ldots, n\}$ is defined by

$$\Omega_t^n = \langle \{1, \ldots, \hat{i}, \widehat{i+1}, \ldots, \widehat{i+t-1}, i+t, \ldots, n\} \mid i = 1, \ldots, n-t+1 \rangle.$$

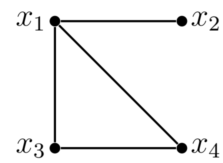

Figure 6.1: A graph G of order 4.

Example 6.0.10. For $n = 5$ and $t = 2$ the simplicial complex Ω_2^5 has facets $\{\hat{1}, \hat{2}, 3, 4, 5\}$, $\{1, \hat{2}, \hat{3}, 4, 5\}$, $\{1, 2, \hat{3}, \hat{4}, 5\}$ and $\{1, 2, 3, \hat{4}, \hat{5}\}$.

Remark 6.0.11. For $n = t$ the simplicial complex Ω_t^n is the irrelevant complex $\{\emptyset\}$. If $t = 1$, then Ω_1^n coincides with the boundary of an $n-1$ dimensional simplex.

6.1 Homology computation for Ω_t^n

In Section 6.2 when we calculate the Betti numbers of path ideals of paths we will come across the simplicial complex Ω_t^n. This section is devoted to computing the homology groups of Ω_t^n as we will need them later on.

Lemma 6.1.1. *For $n \geq 2t+1$ we have $\tilde{H}_p(\Omega_t^n) \cong \tilde{H}_{p-2}(\Omega_t^{n-t-1})$ for each integer p. Otherwise,*

$$\tilde{H}_p(\Omega_t^n) \cong \begin{cases} \tilde{H}_p(\{\emptyset\}), & \text{if } n = t \\ \tilde{H}_{p-1}(\{\emptyset\}), & \text{if } n = t+1 \\ 0, & \text{if } t+2 \leq n \leq 2t. \end{cases} \tag{6.1.1}$$

Proof. The case $n = t$ is clear as $\Omega_t^t = \{\emptyset\}$. So we assume that $n > t$ and fix an index p. Note that

$$\Omega_t^n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-t+1} \rangle$$

where for each i, $\sigma_i = \{1, \ldots, n\} \setminus \{i, i+1, \ldots, i+t-1\}$. Then $\Omega_t^n = S \cup C$ where $S = \langle \sigma_1 \rangle$ is the simplex on vertices $\{t+1, \ldots, n\}$ and $C = \langle \sigma_2, \ldots, \sigma_{n-t+1} \rangle$ is the cone generated by the facets of Ω_t^n that contain the vertex 1. Note that by Corollary 2.1.3

we have

$$\tilde{H}_p(\Omega_t^n) \cong \tilde{H}_{p-1}(S \cap C).$$

We consider the three possible cases left.

Case 1: Suppose that $n = t+1$. Then $S = \langle \{t+1\} \rangle$ and $C = \langle \{1\} \rangle$. Thus $S \cap C$ is the irrelevant complex and we are done.

Case 2: Suppose that $t+2 \leq n \leq 2t$. Observe that since $n \leq 2t$ we have $\sigma_1 \cap \sigma_i \subseteq \sigma_1 \cap \sigma_2$ for all $i = 2, 3, \ldots, n-t+1$. Therefore $S \cap C = \langle \sigma_1 \cap \sigma_2 \rangle$ is a simplex whose maximal face is $\{t+2, \ldots, n\}$ as $t+2 \leq n$.

Case 3: Suppose that $n \geq 2t+1$. In this case, we have $\sigma_1 \cap \sigma_i \subseteq \sigma_1 \cap \sigma_2$ for all $i = 2, 3, \ldots, t+1$. Thus we obtain

$$S \cap C = \langle \sigma_1 \cap \sigma_2 \rangle \bigcup \langle \sigma_1 \cap \sigma_{t+2}, \ldots, \sigma_1 \cap \sigma_{n-t+1} \rangle.$$

Let us set $S_1 := \langle \sigma_1 \cap \sigma_2 \rangle$ and $C_1 := \langle \sigma_1 \cap \sigma_{t+2}, \ldots, \sigma_1 \cap \sigma_n \rangle$. Then S_1 is a simplex with vertex set $\{t+2, \ldots, n\}$, and C_1 is a cone with apex $t+1$ such that

$$\text{Facets}(C_1) = \big\{ \{t+1, \ldots, n\} \setminus \{i, i+1, \ldots, i+t-1\} \mid i = t+2, \ldots, n-t+1 \big\}.$$

Hence we get $S_1 \cap C_1 \cong \Omega_t^{n-t-1}$. Since both S_1 and C_1 are acyclic, by Corollary 2.1.3 we get $\tilde{H}_{p-1}(S \cap C) \cong \tilde{H}_{p-2}(S_1 \cap C_1) \cong \tilde{H}_{p-2}(\Omega_t^{n-t-1})$ which completes the proof. $\square$

Notation 6.1.2. For two integers i, j the symbol $\delta_{i,j}$ is 1 if $i = j$, and is 0 otherwise.

Corollary 6.1.3. *The dimensions of reduced homology modules of Ω_t^n are independent of the ground field. And they are given by*

$$\dim \tilde{H}_p(\Omega_t^n) = \begin{cases} \delta_{p+2,\frac{2n}{t+1}}, & \text{if } n \equiv 0 \bmod t+1 \\ \delta_{p+3,\frac{2(n+1)}{t+1}}, & \text{if } n \equiv t \bmod t+1 \\ 0, & \text{otherwise.} \end{cases} \tag{6.1.2}$$

Proof. Follows from a straightforward induction using Lemma 6.1.1 and Eq. (2.1.2). □

6.2 Paths

The purpose of this section is to calculate (multi)graded Betti numbers of path ideals of paths. Before proving our results we need to fix some notations. Let I be a monomial ideal. For any monomial u let us define the simplicial complex

$$\mathrm{Taylor}(I)_{\leq u} = \{\tau \in \mathrm{Taylor}(I) \mid \mathrm{lcm}(\tau) \text{ divides } u\}.$$

Then we can write

$$\mathrm{Taylor}(I)_{<u} = \bigcup_{x_i \text{ divides } u} \mathrm{Taylor}(I)_{\leq \frac{u}{x_i}} \tag{6.2.1}$$

Recall that by Theorem 2.3.4 we need to find the homology of $\mathrm{Taylor}(I)_{<u}$. Throughout this section let Δ be the Taylor simplex of $I_t(P_n)$ where P_n is a path on vertices $x_1, \ldots, x_n$. If $n < t$, then there is no path on P_n of order t, and therefore $I_t(P_n) = 0$. Let us assume $n \geq t$ then we have

$$\Delta = \langle \{x_i x_{i+1} \ldots x_{i+t-1} \mid i = 1, \ldots, n-t+1\} \rangle.$$

For simplicity, we replace the label of a vertex $x_i x_{i+1} \dots x_{i+t-1}$ of Δ with τ_i for all $i = 1, \dots, n-t+1$. Hence Δ is the simplex given by

$$\Delta = \langle \{\tau_1, \tau_2, \dots, \tau_{n-t+1}\} \rangle.$$

Now we want to find $\Delta_{<m}$ where $m = x_1 \dots x_n$. Observe that we have

- $\Delta_{\leq \frac{m}{x_i}} \subseteq \Delta_{\leq \frac{m}{x_1}}$ for all $1 \leq i \leq t$
- $\Delta_{\leq \frac{m}{x_i}} = \langle \{\tau_1, \dots, \tau_{n-t+1}\} \setminus \{\tau_{(i-t)+1}, \dots, \tau_{(i-t)+t}\} \rangle$ for all $t+1 \leq i \leq n-t$
- $\Delta_{\leq \frac{m}{x_i}} \subseteq \Delta_{\leq \frac{m}{x_n}}$ for all $n-t+1 \leq i \leq n$

Therefore,

$$\text{Facets}(\Delta_{<m}) = \text{Facets}(\Delta_{\leq \frac{m}{x_1}}) \bigcup \text{Facets}(\Delta_{\leq \frac{m}{x_n}}) \bigcup \bigcup_{i=t+1}^{n-t} \text{Facets}(\Delta_{\leq \frac{m}{x_i}}).$$

For future reference, we explicitly list the facets of $\Delta_{<m}$.

$$\begin{aligned} \text{Facets}(\Delta_{<m}) = \big\{ & \{\widehat{\tau_1}, \tau_2, \dots, \tau_{n-t+1}\}, \{\tau_1, \dots, \tau_{n-t}, \widehat{\tau_{n-t+1}}\} \big\} \\ & \bigcup \big\{ \{\tau_1, \dots, \tau_{n-t+1}\} \setminus \{\tau_i, \dots, \tau_{i+t-1}\} \mid i = 2, \dots, n-2t+1 \big\}. \end{aligned} \tag{6.2.2}$$

Note that when $n < 2t+1$ there is no i such that $t+1 \leq i \leq n-t$. Therefore the list of items above become

- $\Delta_{\leq \frac{m}{x_i}} \subseteq \Delta_{\leq \frac{m}{x_1}}$ for all $1 \leq i \leq t$
- $\Delta_{\leq \frac{m}{x_i}} \subseteq \Delta_{\leq \frac{m}{x_n}}$ for all $n-t+1 \leq i \leq n$

for $n < 2t+1$. Thus we get

$$\Delta_{<m} = \langle \{\widehat{\tau_1}, \tau_2, \dots, \tau_{n-t+1}\}, \{\tau_1, \dots, \tau_{n-t}, \widehat{\tau_{n-t+1}}\} \rangle. \tag{6.2.3}$$

for $n < 2t+1$.

We will first determine the top degree Betti numbers of path ideals of paths. The following result was also proved by Alilooee and Faridi [1].

Theorem 6.2.1 (Top degree Betti numbers of path ideals of paths). [1, Theorem 4.13] *For all $i \geq 1$, $n \geq t$ and $n \geq 1$, we have*

$$b_{i,n}(S/I_t(P_n)) = \begin{cases} \delta_{i,\frac{2n}{t+1}}, & \text{if } n \equiv 0 \bmod t+1 \\ \delta_{i+1,\frac{2n+2}{t+1}}, & \text{if } n \equiv t \bmod t+1 \\ 0, & \text{otherwise.} \end{cases} \tag{6.2.4}$$

Proof. Let $\mathbf{m}$ be the product of vertices of P_n. By Equation (2.3.2) and Theorem 2.3.4 we have

$$b_{i,n}(S/I_t(P_n)) = b_{i,\mathbf{m}}(S/I_t(P_n)) = \dim_{\Bbbk} \tilde{H}_{i-2}(\Delta_{<m}, \Bbbk).$$

We consider two cases:

Case 1: Suppose that $n < 2t+1$. By Equation (6.2.3) we have

$$\Delta_{<m} = \mathrm{del}_{\Delta}(\{\tau_1\}) \cup \mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}).$$

Observe that if $n = t$, then $\Delta_{<m} = \{\emptyset\}$ and so that

$$\dim_{\Bbbk} \tilde{H}_{i-2}(\Delta_{<m}, \Bbbk) = \dim_{\Bbbk} \tilde{H}_{i-2}(\{\emptyset\}, \Bbbk) = \delta_{i-2,-1}.$$

But for $n = t$ we have $\delta_{i-2,-1} = \delta_{i+1,\frac{2n+2}{t+1}}$ which proves the formula given in (6.2.4) for this case. So, we may assume that $n > t$. Note that by Corollary 2.1.3 we have

$$\dim_{\Bbbk} \tilde{H}_{i-2}(\Delta_{<m}, \Bbbk) \cong \dim_{\Bbbk} \tilde{H}_{i-3}(\mathrm{del}_{\Delta}(\{\tau_1\}) \cap \mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}), \Bbbk)$$

as both $\mathrm{del}_\Delta(\{\tau_1\})$ and $\mathrm{del}_\Delta(\{\tau_{n-t+1}\})$ are simplices. We consider two cases.

Case 1.1: Suppose that $n = t+1$. Then $\Delta = \langle\{\tau_1, \tau_2\}\rangle$ and $\mathrm{del}_\Delta(\{\tau_1\}) \cap \mathrm{del}_\Delta(\{\tau_{n-t+1}\})$ is the irrelevant complex. Therefore we have

$$\dim_{\Bbbk} \tilde{H}_{i-3}(\mathrm{del}_\Delta(\{\tau_1\}) \cap \mathrm{del}_\Delta(\{\tau_{n-t+1}\}), \Bbbk) \cong \dim_{\Bbbk} \tilde{H}_{i-3}(\{\emptyset\}, \Bbbk) = \delta_{i-3,-1}.$$

Check that if $n = t+1$, then $\delta_{i-3,-1} = \delta_{i,\frac{2n}{t+1}}$ and the proof is complete for this case.

Case 1.2: Next, suppose that $n \geq t+1$. Then $\mathrm{del}_\Delta(\{\tau_1\}) \cap \mathrm{del}_\Delta(\{\tau_{n-t+1}\}) = \mathrm{del}_\Delta(\{\tau_1, \tau_{n-t+1}\})$ is a simplex and has trivial reduced homology in all degrees.

Case 2: Suppose that $n \geq 2t+1$. Then by Equation (6.2.2) we have

$$\Delta_{<m} = \mathrm{del}_\Delta(\{\tau_1\}) \cup \mathrm{del}_\Delta(\{\tau_{n-t+1}\}) \cup \Upsilon$$

where $\Upsilon = \langle\{\tau_1, \ldots, \tau_{n-t+1}\} \setminus \{\tau_i, \ldots, \tau_{i+t-1}\} \mid i = 2, \ldots, n-2t+1\rangle$. Now we can see that $\Delta_{<m}$ is a union of $\mathrm{del}_\Delta(\{\tau_1\})$ and $\mathrm{del}_\Delta(\{\tau_{n-t+1}\}) \cup \Upsilon$. But since $\mathrm{del}_\Delta(\{\tau_1\})$ is a simplex and $\mathrm{del}_\Delta(\{\tau_{n-t+1}\}) \cup \Upsilon$ is a cone with apex τ_1, by virtue of Corollary 2.1.3 we have

$$\dim_{\Bbbk} \tilde{H}_{i-2}(\Delta_{<m}, \Bbbk) = \dim_{\Bbbk} \tilde{H}_{i-3}(\mathrm{del}_\Delta(\{\tau_1\}) \cap \big(\mathrm{del}_\Delta(\{\tau_{n-t+1}\}) \cup \Upsilon\big), \Bbbk).$$

Now observe that $\mathrm{del}_\Delta(\{\tau_1\}) \cap \big(\mathrm{del}_\Delta(\{\tau_{n-t+1}\}) \cup \Upsilon\big) = C \cup \mathrm{del}_\Delta(\{\tau_1, \tau_{n-t+1}\})$ where C is the cone generated by the facets of $\mathrm{del}_\Delta(\{\tau_1\}) \cap \big(\mathrm{del}_\Delta(\{\tau_{n-t+1}\}) \cup \Upsilon\big)$ that contain the vertex τ_{n-t+1}. Again, as $\mathrm{del}_\Delta(\{\tau_1, \tau_{n-t+1}\})$ is a simplex, by Corollary 2.1.3 we get

$$\dim_{\Bbbk} \tilde{H}_{i-3}(\mathrm{del}_\Delta(\{\tau_1\}) \cap \big(\mathrm{del}_\Delta(\{\tau_{n-t+1}\}) \cup \Upsilon\big), \Bbbk) = \dim_{\Bbbk} \tilde{H}_{i-4}(C \cap \mathrm{del}_\Delta(\{\tau_1, \tau_{n-t+1}\}), \Bbbk).$$

Finally, observe that $C \cap \mathrm{del}_\Delta(\{\tau_1, \tau_{n-t+1}\})$ is isomorphic to the simplicial complex Ω_t^{n-t-1} (recall Definition 6.0.9) and by Corollary 6.1.3 we have

$$\dim \tilde{H}_{i-4}(\Omega_t^{n-t-1}) = \begin{cases} \delta_{i-2, \frac{2(n-t-1)}{t+1}}, & \text{if } n \equiv 0 \bmod t+1 \\ \delta_{i-1, \frac{2(n-t)}{t+1}}, & \text{if } n \equiv t \bmod t+1 \\ 0, & \text{otherwise} \end{cases} \tag{6.2.5}$$

which agrees with the formula given in Equation (6.2.4). □

Remark 6.2.2. In [23, Corollary 2.9] He and Van Tuyl proved that the facet complex of the path ideal of a rooted tree is a simplicial tree. This in particular implies that the facet complex of the path ideal of a path is a simplicial tree. Therefore all results in this section can be stated in terms of facet ideals of simplicial forests. In [1, Corollary 4.14] Alilooee and Faridi provided a formula for projective dimension of path ideals of paths. Using their formula and Theorem 6.2.1 one can see that for all $i \geq 1$ and $n \geq t$

$$b_{i,n}(S/I_t(P_n)) \neq 0 \Longrightarrow i = \mathrm{pd}(S/I_t(P_n)).$$

Therefore using a straightforward induction and [27, Corollary 2.3] one can show that

$$b_{i,\mathbf{m}}(S/I_t(P_n)) \neq 0 \Longrightarrow i = \mathrm{pd}(S/I_t((P_n)_\mathbf{m})) \tag{6.2.6}$$

answering Question 5.3.8 in the affirmative for the class of simplicial trees which correspond to path complexes of path ideals of paths.

Theorem 6.2.3 (**Multigraded Betti numbers of path ideals of paths**). *Let $t \geq 2$ and $\mathbf{m}$ be a squarefree monomial of degree j. Then the multigraded Betti number $b_{i,\mathbf{m}}(S/I_t(P_n)) = 1$ if the induced graph $(P_n)_\mathbf{m}$ consists of a collection of disjoint paths that satisfy the following conditions:*

(1) *Each path is of order* 0 *or* $t \bmod t+1$

(2) *The number of paths of order* $t \bmod t+1$ *is equal to* $\frac{i(t+1)-2j}{1-t}$.

Otherwise, $b_{i,\mathbf{m}}(S/I_t(P_n)) = 0$.

Proof. Let $Q_1, \dots, Q_p$ be the connected components of $(P_n)_{\mathbf{m}}$ where each Q_ℓ is a path of order v_ℓ for $1 \le \ell \le p$. We have

$$\begin{aligned} b_{i,\mathbf{m}}(S/I_t(P_n)) &= b_{i,j}(S/I_t((P_n)_{\mathbf{m}})) \text{ by Lemma 4.1.1} \\ &= \sum_{u_1+\cdots+u_p=i} b_{u_1,v_1}(S/I_t(Q_1)) \dots b_{u_p,v_p}(S/I_t(Q_p)) \text{ by Equation (4.1.2).} \end{aligned}$$

By Theorem 6.2.1 if one of Q_ℓ is not of order 0 or $t \bmod t+1$, then the sum above is 0. So without loss of generality let us assume that $Q_1, \dots, Q_z$ are of order $0 \bmod t+1$ and $Q_{z+1}, \dots, Q_p$ are of order $t \bmod t+1$ for some $0 \le z \le p$. Again by Theorem 6.2.1, the sum above is equal to 1 if

$$\sum_{\ell=1}^{z} \frac{2v_\ell}{t+1} + \sum_{\ell=z+1}^{p} \Big(\frac{2v_\ell + 2}{t+1} - 1 \Big) = i \tag{6.2.7}$$

and 0 otherwise. Observe that (6.2.7) holds if and only if $p - z = \frac{i(t+1)-2j}{1-t}$ since $v_1 + \cdots + v_p = j$. Hence the result follows. $\square$

Remark 6.2.4. It is worth noting that for a path P_n (or a cycle C_n) on the vertex set $\{x_1, \dots, x_n\}$, every induced subcollection on a subset W of $\{x_1, \dots, x_n\}$ is a disjoint union of paths. For simplicity of arguments one can represent such an induced subcollection as disjoint paths $Q_1, \dots, Q_p$ where for each $i \in \{1, \dots, p\}$ there are integers a_i and ℓ_i with

$$V(Q_i) = \{x_{a_i}, x_{a_i+1}, \dots, x_{a_i+\ell_i}\} \text{ (mod } n \text{ in the case of } C_n\text{)},$$

where $1 \le a_1 < a_2 < \cdots < a_p \le n$.

The fact that the Q_i are disjoint in the induced subollections implies that for each i,

$$a_i + \ell_i + 1 < a_{i+1};$$

in other words, between each two consecutive Q_i and Q_{i+1} there is a vertex of P_n (or C_n) not in W.

We are now ready to give a combinatorial description for the graded Betti numbers of path ideals of paths. Alilooee and Faridi [2] also gave a combinatorial description for the graded Betti numbers of path ideals of paths which has a different form than ours. They defined eligible subcollections ([2, Definition 4.2.1]) to state these formulas.

Corollary 6.2.5. *If P is a path, $b_{i,j}(S/I_t(P))$ is the number of ways of choosing a collection of disjoint paths of P that correspond to an induced subcollection of P and that satisfy the following conditions:*

(1) *The orders of the paths add up to j*

(2) *Each path is of order 0 or $t \bmod t+1$*

(3) *The number of paths of order $t \bmod t+1$ is equal to $\frac{i(t+1)-2j}{1-t}$.*

Proof. Immediately follows from Theorem 6.2.3 and Equation (2.3.2). □

Using the description above, we will obtain a closed formula for graded Betti numbers thus recovering a result of Alilooee and Faridi given in [1, Theorem 4.13]. First, we will need some combinatorial lemmas. The lemma below is well known but we shall provide a proof for completeness.

Lemma 6.2.6. *The number of integer solutions of the equation*

$$A_1 + A_2 + \cdots + A_r = N$$

with $A_i \geq 0$ *is* $\binom{N+r-1}{r-1}$.

Proof. Every solution of the given equation corresponds to a partition of N objects in a row into r parts using $r-1$ bars. Then we count the number of ways of partitioning N objects in a row into r parts using $r-1$ bars. But this is simply ordering N objects and $r-1$ bars in a row. $\square$

Lemma 6.2.7. *The number of ways of fitting* α *red balls and* β *green balls in* $\gamma \geq 1$ *bags in a row such that each bag contains at most* 1 *red ball is*

$$\binom{\gamma}{\alpha}\binom{\gamma-1+\beta}{\gamma-1}$$

provided that $\gamma \geq \alpha$.

Proof. Since every bag contains at most 1 red ball, there must be at least α bags, i.e., $\gamma \geq \alpha$. To count the number of solutions to this problem we spread the α bags which contain a red ball in a row as below. We denote the bags which contain a red ball by B_*.

$$B_*\ B_*\ B_*\ \cdots\ B_*$$

Next, we will insert the remaining $\gamma - \alpha$ bags in the row. Note that there are $\alpha + 1$ places to insert these bags. We denote these places by $-$ as below.

$$-\ B_*\ -\ B_* -\ B_*\ - \cdots -\ B_*\ -$$

This is equivalent to finding the number of integer solutions to the equation

$$A_1 + A_2 + \cdots + A_{\alpha+1} = \gamma - \alpha$$

with $A_i \geq 0$, which is $\binom{\gamma}{\alpha}$ by Lemma 6.2.6.

Finally we fit the β green balls in the γ bags. But the number of ways of doing this is equal to the number of integer solutions of the equation

$$A_1 + A_2 + \cdots + A_\gamma = \beta$$

with $A_i \geq 0$, which is $\binom{\gamma-1+\beta}{\gamma-1}$ by Lemma 6.2.6. Hence the total number of ways of fitting α red balls and β green balls in γ bags such that each bag contains at most 1 red ball is

$$\binom{\gamma}{\alpha}\binom{\gamma-1+\beta}{\gamma-1}$$

as desired. $\square$

Theorem 6.2.8 (Graded Betti numbers of path ideals of paths). [1, Theorem 4.13] *For $t \geq 2$, the nonzero graded Betti numbers of $S/I_t(P_n)$ are given by*

$$b_{i,j}(S/I_t(P_n)) = \binom{n-j+1}{\frac{i(t+1)-2j}{1-t}}\binom{n-j+\frac{j-ti}{1-t}}{n-j}$$

provided that n, i and j satisfy the following relations.

(1) $n \geq j$

(2) $j \geq t\left(\frac{i(t+1)-2j}{1-t}\right) \geq 0$

(3) $n - j \geq \frac{i(t+1)-2j}{1-t} - 1$

Otherwise, the graded Betti numbers $b_{i,j}(S/I_t(P_n))$ are 0.

Proof. By Corollary 6.2.5 we count the number of ways of choosing induced subcollections of P_n that satisfy the conditions given in Corollary 6.2.5. Note that there might be no way to choose such a subcollection, and in that case the Betti number $b_{i,j}(S/I_t(P_n))$ is 0.

Assume for a moment that $Q_1, \ldots, Q_p$ are disjoint induced paths of P_n as in Remark 6.2.4 and satisfying the conditions in Corollary 6.2.5. Since the orders of

$Q_1, \ldots, Q_p$ add up to j, we have

$$j = |\bigcup_{k=1}^{p} V(Q_k)|.$$

But this requires $n \geq j$ which is the necessary condition in (1). Also as the number of paths of order t mod $t+1$ is equal to $u = \frac{i(t+1)-2j}{1-t}$, at least tu vertices in $\bigcup_{k=1}^{p} V(Q_k)$ belong to a path of order t mod $t+1$. This in particular requires

$$j \geq t\left(\frac{i(t+1)-2j}{1-t}\right) \geq 0$$

which is condition (2).

There are $j - tu = (1+t)(\frac{j-ti}{1-t})$ remaining vertices of $\bigcup_{k=1}^{p} V(Q_k)$. Let $v = \frac{j-ti}{1-t}$.

Since each of the Q_i has order 0 or t mod $t+1$, we can say each Q_i consists of some blocks of $t+1$ vertices, and at most one block of t vertices.

By Remark 6.2.4, between every two of the Q_i there must be a vertex of P_n not in any of the Q_i, and we have $n-j$ such vertices. We call these $n-j$ vertices the "gold vertices".

Putting all this together, we have to try to fit between each of the gold vertices of P_n at most one block of t vertices and some (or no) blocks of $t+1$ vertices. We have available to us u blocks of t verices and v blocks of $t+1$ vertices. This is the same as fitting u red balls and v green balls in $n-j+1$ bags such that each bag contains at most 1 red ball. According to Lemma 6.2.7 there are

$$\binom{n-j+1}{u}\binom{n-j+v}{n-j}$$

ways of doing that provided that $n-j \geq u-1$ which is the necessary condition in (3). Having proved our claim, the proof follows. $\square$

6.3 Cycles

Corollary 6.3.1 (Multigraded Betti numbers of path ideals of cycles). *Let $t \geq 2$ and $\mathbf{m}$ be a squarefree monomial of degree $j < n$. Then the multigraded Betti number $b_{i,\mathbf{m}}(S/I_t(C_n)) = 1$ if the induced graph $(C_n)_{\mathbf{m}}$ consists of a collection of disjoint paths that satisfy the following conditions:*

(1) *Each path is of order 0 or $t \bmod t+1$*

(2) *The number of paths of order $t \bmod t+1$ is equal to $\frac{i(t+1)-2j}{1-t}$.*

Otherwise, $b_{i,\mathbf{m}}(S/I_t(C_n)) = 0$.

Proof. By Lemma 4.1.1 we have $b_{i,\mathbf{m}}(S/I_t(C_n)) = b_{i,j}(S/I_t((C_n)_{\mathbf{m}}))$. Since $(C_n)_{\mathbf{m}}$ is a disjoint union of paths the proof follows from Theorem 6.2.3. $\square$

6.4 Stars

Throughout this section $\mathcal{S}_n$ will be a star graph of size n.

Lemma 6.4.1. *Let G be a connected graph with no isolated vertices. Then G is a star graph if and only if every edge of G contains a free vertex.*

Proof. If G is a star graph, then every edge contains a free vertex by definition. So, suppose that every edge of G contains a free vertex. Then G has no path subgraph of order at least 4. Indeed, if e_1, e_2, e_3 are distinct edges of G, then $e_1 \not\subseteq e_2 \cup e_3$ because e_1 has a free vertex. This implies that the longest path of G has 2 edges. In particular, since G is connected every pair of edges intersect. $\square$

Lemma 6.4.2. *Let G be a connected graph with no isolated vertices. If m is the product of the vertices of G, then the simplicial complex* $\mathrm{Taylor}(I(G))_{<m}$ *is the boundary of* $\mathrm{Taylor}(I(G))$ *if and only if G is a star.*

Proof. Suppose that $e_1, \ldots, e_q$ are the edges of the graph G. Then, $\text{Taylor}(I(G))_{<m}$ is the boundary of $\text{Taylor}(I(G))$ if and only if $F_i := \{e_1, \ldots, e_q\} \setminus \{e_i\}$ is a facet of $\text{Taylor}(I)_{<m}$ for each $i = 1, \ldots, q$. But observe that

$$\begin{aligned} F_i \text{ is a facet of } \text{Taylor}(I(G))_{<m} &\Leftrightarrow F_i \text{ is a face of } \text{Taylor}(I(G))_{<m} \\ &\Leftrightarrow \operatorname{lcm}(e_1, \ldots, \hat{e_i}, \ldots, e_q) \neq m, \text{ for each } i = 1, \ldots, q \\ &\Leftrightarrow e_i \text{ contains a free vertex, for each } i = 1, \ldots, q. \end{aligned}$$

Since G is a connected graph, our claim follows from Lemma 6.4.1. $\square$

Using the lemma above, we will determine the graded Betti numbers of edge ideals of stars recovering results of Jacques [26], Hà and Van Tuyl [22].

Corollary 6.4.3 (**Graded Betti numbers of edge ideals of stars**). [26, Theorem 5.4.11], [22, Theorem 2.7] *Let $\mathcal{S}_n$ be a star on $n+1$ vertices. Then*

$$b_{i,j}(S/I(\mathcal{S}_n)) = \begin{cases} \binom{n}{j-1}, & \textit{if } i = j-1 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. First, we find the top degree Betti numbers. Combining Lemma 6.4.2, Theorem 2.3.4 and Equation (2.1.1) we get

$$b_{i,n+1}(S/I(\mathcal{S}_n)) = \delta_{i,n} \text{ for all } i. \tag{6.4.1}$$

Fix j and recall Equation (2.3.2) and Lemma 4.1.1. Any induced subgraph of $\mathcal{S}_n$ is either a star or contains an isolated vertex. If it contains an isolated vertex, then by Remark 5.4.1 the multigraded Betti number for such an induced subgraph is 0. Hence by (6.4.1) we see that $b_{i,j}(S/I(\mathcal{S}_n))$ is the number of induced star subgraphs of $\mathcal{S}_n$ of order j if $i = j - 1$ and is 0 otherwise. $\square$

We will extend the result above to path ideals. To this end, we will need the following proposition.

Proposition 6.4.4. *Let Γ be a simplicial complex which is not a cone. Suppose that $\langle F_1, \ldots, F_q\rangle = \Gamma$ and there exists a sequence of distinct vertices $v_1, \ldots, v_q$ of Γ such that $v_i \notin F_j$ if and only if $i = j$. Then $\tilde{H}_p(\Gamma, \Bbbk) \cong \tilde{H}_{p-q+1}(\{\emptyset\}, \Bbbk)$ for any field $\Bbbk$.*

Proof. We use induction on q, the number of facets. Since there is no simplex which satisfies the assumptions of the given proposition, the base case starts at $q = 2$.

Suppose that $\Gamma = \langle F_1, F_2\rangle$ is not a cone and it has two vertices v_1, v_2 such that $v_i \notin F_j \Leftrightarrow i = j$. Since Γ is not a cone, $\langle F_1\rangle \cap \langle F_2\rangle = \{\emptyset\}$. Now $\Gamma = \langle F_1\rangle \cup \langle F_2\rangle$, and the simplicial complexes $\langle F_1\rangle, \langle F_2\rangle$ are acyclic. By virtue of Corollary 2.1.3 we obtain

$$\tilde{H}_p(\Gamma) \cong \tilde{H}_{p-1}(\langle F_1\rangle \cap \langle F_2\rangle) = \tilde{H}_{p-1}(\{\emptyset\})$$

as desired.

Now let $\Gamma = \langle F_1, \ldots, F_q\rangle, q \geq 3$ be a simplicial complex as in the statement of the Proposition. We write

$$\Gamma = \langle F_1, \ldots, F_{q-1}\rangle \cup \langle F_q\rangle$$

where $\langle F_q\rangle$ is a simplex and $\langle F_1, \ldots, F_{q-1}\rangle$ is a cone with apex v_q. By Corollary 2.1.3 we have $\tilde{H}_p(\Gamma) \cong \tilde{H}_{p-1}(\langle F_1, \ldots, F_{q-1}\rangle \cap \langle F_q\rangle)$. But observe that

$$\langle F_1, \ldots, F_{q-1}\rangle \cap \langle F_q\rangle = \langle F_1 \cap F_q, \ldots, F_{q-1} \cap F_q\rangle$$

as for all $1 \leq i \neq j \leq q-1$, $v_j \in F_i \cap F_q, v_j \notin F_j \cap F_q$ so that $F_i \cap F_q \nsubseteq F_j \cap F_q$. Observe that the simplicial complex $\langle F_1 \cap F_q, \ldots, F_{q-1} \cap F_q\rangle$ is not a cone since Γ is not a cone. Moreover for all $1 \leq i, j \leq q-1$

$$v_i \notin F_j \cap F_q \Longleftrightarrow i = j.$$

Hence it satisfies the inductive hypothesis and we get

$$\tilde{H}_{p-1}(\langle F_1 \cap F_q, \ldots, F_{q-1} \cap F_q \rangle) \cong \tilde{H}_{p-q+1}(\{\emptyset\})$$

which completes the proof. □

Theorem 6.4.5. *Let $\mathcal{S}_n$ be a star graph of size $n \geq 2$. Then for all $i \geq 1$*

$$b_{i,n+1}(S/I_3(\mathcal{S}_n)) = \begin{cases} i, & \text{if } i = n-1 \\ 0, & \text{otherwise.} \end{cases} \tag{6.4.2}$$

Proof. Let $\mathcal{S}_n$ be a star graph of size n with the edge set $E(\mathcal{S}_n) = \{\{x_0, x_i\} \mid i = 1, \ldots, n\}$. Suppose that Θ is the Taylor simplex of $I_3(\mathcal{S}_n)$. We prove the given statement by induction on n. We will use Theorem 2.3.4 so that for all $i \geq 1$ we have

$$b_{i,n+1}(S/I_3(\mathcal{S}_n)) = \dim_{\Bbbk} \tilde{H}_{i-2}(\Theta_{<m}, \Bbbk)$$

where $m = x_0 \ldots x_n$ is the product of vertices of $\mathcal{S}_n$. For $n = 2$, we have $\Theta_{<m} = \{\emptyset\}$ so the base case is settled by (2.1.2).

Now suppose that $n \geq 3$ is fixed. Recall that by (6.2.1) we have a decomposition

$$\Theta_{<m} = \Theta_{\leq \frac{m}{x_n}} \bigcup \left(\bigcup_{i=1}^{n-1} \Theta_{\leq \frac{m}{x_i}} \right) \tag{6.4.3}$$

since $\Theta_{\leq \frac{m}{x_0}} = \{\emptyset\}$. For $i \geq 1$ we set

$$F_i := \{x_0 x_j x_k \mid j, k \in \{1, \ldots, n\} \setminus \{i\} \text{ and } j < k\}$$

Because of the symmetry of star graphs, every element of $\{F_1, \ldots, F_n\}$ is maximal with respect to inclusion. Consequently, we get $\Theta_{<m} = \langle F_1, \ldots, F_n \rangle$ and (6.4.3)

becomes

$$\Theta_{<m} = \langle F_n \rangle \cup \langle F_1, \ldots, F_{n-1} \rangle \tag{6.4.4}$$

Let $H = \mathcal{S}_n - \{\{x_0, x_n\}\}$ be the star obtained by removing the edge $\{x_0, x_n\}$ from $\mathcal{S}_n$. We claim the following:

Claim 1: $\langle F_n \rangle \cap \langle F_1, \ldots, F_{n-1} \rangle \cong \mathrm{Taylor}(I_3(H))_{<x_0 \ldots x_{n-1}}$.

Claim 2: $\tilde{H}_p(\langle F_1, \ldots, F_{n-1} \rangle) \cong \tilde{H}_{p-n+2}(\{\emptyset\})$.

Observe that K is a facet of $\langle F_n \rangle \cap \langle F_1, \ldots, F_{n-1} \rangle$ if and only if $K = F_n \cap F_i$ for some $1 \leq i \leq n-1$. But the latter means that K consists of all monomials corresponding to paths of the form $x_0 x_j x_k$ where $j, k \in \{x_1, \ldots, x_n\} \setminus \{x_i, x_n\}$ and $j \neq k$. And this proves Claim 1.

For Claim 2 we show that Proposition 6.4.4 applies to the simplicial complex $\langle F_1, \ldots, F_{n-1} \rangle$. To this end, we first check that $\langle F_1, \ldots, F_{n-1} \rangle$ is not a cone. Assume for a contradiction it is a cone with apex $x_0 x_i x_j$. Then $x_0 x_i x_j \in F_1 \cap \cdots \cap F_{n-1}$ which is only possible if $i = j = n$ which is a contradiction. Now consider $v_1 := x_0 x_1 x_n, \ldots, v_{n-1} := x_0 x_{n-1} x_n$, a sequence of vertices of $\langle F_1, \ldots, F_{n-1} \rangle$. By definition of $F_1, \ldots, F_n$ we have $v_i \notin F_j \Leftrightarrow i = j$, and this proves Claim 2.

Therefore (2.1.2) yields

$$\dim \tilde{H}_p(\langle F_1, \ldots, F_{n-1} \rangle) = \begin{cases} 1, & \text{if } p = n-3 \\ 0, & \text{otherwise.} \end{cases} \tag{6.4.5}$$

Also by our induction hypothesis and Claim 1 we have

$$\dim \tilde{H}_p(\langle F_n \rangle \cap \langle F_1, \ldots, F_{n-1} \rangle) = \begin{cases} p+2, & \text{if } p = n-4 \\ 0, & \text{otherwise.} \end{cases} \tag{6.4.6}$$

Therefore the Mayer-Vietoris sequence for (6.4.4) is

$$\cdots \to 0 \to 0 \to \tilde{H}_n(\Theta_{<m}) \to 0 \to 0 \to \tilde{H}_{n-1}(\Theta_{<m}) \to 0 \to 0 \to \tilde{H}_{n-2}(\Theta_{<m}) \to 0$$
$$\to \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1}\rangle) \to \tilde{H}_{n-3}(\Theta_{<m}) \to \tilde{H}_{n-4}(\langle F_n\rangle \cap \langle F_1, \dots, F_{n-1}\rangle) \to 0 \to$$
$$\tilde{H}_{n-4}(\Theta_{<m}) \to 0 \to 0 \to \tilde{H}_{n-5}(\Theta_{<m}) \to 0 \to 0 \to \tilde{H}_{n-6}(\Theta_{<m}) \to 0 \to 0 \to \cdots .$$

It follows $\tilde{H}_i(\Theta_{<m}) = 0$ for $i \geq n-2$ and $i \leq n-4$. Hence we have

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1}\rangle) \longrightarrow \tilde{H}_{n-3}(\Theta_{<x_0 \dots x_n})$$
$$\longrightarrow \tilde{H}_{n-4}(\langle F_n\rangle \cap \langle F_1, \dots, F_{n-1}\rangle) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

which gives that

$$\begin{aligned} \dim \tilde{H}_{n-3}(\Theta_{<m}) &= \dim \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1}\rangle) + \dim \tilde{H}_{n-4}(\langle F_n\rangle \cap \langle F_1, \dots, F_{n-1}\rangle) \\ &= 1 + (n-2) \text{ by (6.4.5) and (6.4.6)} \\ &= n-1 \end{aligned}$$

and the proof is completed. ☐

Finally, we extend Corollary 6.4.3 to path ideals.

Theorem 6.4.6 (Graded Betti numbers of path ideals of stars). *Let $\mathcal{S}_n$ be a star graph of size $n \geq 2$. For all $i \geq 1$ and $j \leq n+1$*

$$b_{i,j}(S/I_3(\mathcal{S}_n)) = \begin{cases} i\binom{n}{j-1}, & \text{if } i = j-2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Similar to proof of Corollary 6.4.3. ☐

Chapter 7

Conclusion

In Corollary 4.2.6 we gave a sufficient combinatorial condition for nonvanishing Betti numbers of squarefree monomial ideals in terms of facet covers of simplicial complexes. While this condition turned out to be a necessary one for facet ideals of simplicial forests, it is far from being a necessary condition in general. For instance, if we consider the class of graphs whose complement has no induced C_4, our condition can help us to detect only nonvanishing of Betti numbers which lie in the linear strand of the minimal free resolution. This is because when G is a such graph, its induced matching number is 1 and we cannot draw any conclusions about $b_{i,i+j}(I(G))$ unless $j = 2$.

It would be worthwhile to find generalizations of our nonvanishing condition which would yield complete answers for classes of ideals that are associated to interesting simplicial complexes or graphs. We should note that it is not possible to find a necessary and sufficient combinatorial condition which applies to all squarefree monomial ideals because the Betti numbers also depend on the characteristic of the ground field (see examples in [28]). However as we discussed in Remark 4.2.8 and Section 3.1 we think that improvements of Corollary 4.2.6 using minimal vertex covers might be possible.

There are some questions which were partially answered in Chapter 5. For instance, we would like to know if Theorem 5.3.1 generalizes to simplicial forests. As we discussed after the proof of Theorem 5.3.7, it is sufficient to give a positive answer to Question 5.3.9 in order to achieve such a generalization. We also do not know if the results in Section 5.4 can be extended to simplicial forests. The proofs of these results heavily relied on the fact that both minimal vertex covers and minimal edge covers of graphs can be managed with bouquet subgraphs. For higher dimensional simplicial complexes minimal vertex and facet covers seem to have more complicated structure.

In [8] Bouchat, Hà and O'Keefe studied path ideals of rooted trees. Using the mapping cone construction they obtained numerical formulas for the invariants of such ideals. Since the path ideal of a rooted tree is the facet ideal of a simplicial tree [23, Corollary 2.9], our results provide a new combinatorial method to study path ideals of rooted trees. Note that not every facet ideal of a simplicial tree is the path ideal of a rooted tree. Therefore our approach is more general in this setting.

In conclusion, we think that our results in Chapter 5 help to better understand minimal free resolutions of facet ideals of simplicial forests and they will also find applications in future research.

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