ON HOPF ALGEBRA EXTENSIONS AND COHOMOLOGIES

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To my wife Jessica, who did not type this manuscript, but did remain cheerful throughout and promised to type the next one.

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Abstract

The purpose of this work is to study Hopf algebra extensions and their relation to cohomology. We introduce various cohomology theories for Hopf algebras, explore their relations to each other and how they classify different kinds of extensions. An exact sequence connecting these cohomology theories is obtained, vastly generalizing those of Kac, Tahara and Masuoka. The morphisms in the low degree part of this sequence are given explicitly, which enables us to use them for concrete computations.

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Chapter 0

Introduction

In this thesis we discuss various cohomology theories for Hopf algebras and their relation to extension theory.

It is natural to think of building new algebraic objects from simpler structures, or to get information about the structure of complicated objects by decomposing them into simpler parts. Algebraic extension theories serve exactly that purpose, and the classification problem of such extensions is usually related to cohomology theories.

In the case of Hopf algebras, extension theories are proving to be invaluable tools for the construction of new examples of Hopf algebras, as well as in the efforts to classify finite dimensional Hopf algebras.

Hopf algebras, which occur for example as group algebras, as universal envelopes of Lie algebras, as algebras of representative functions on Lie groups, as coordinate algebras of algebraic groups and as Quantum groups, have many 'group like' properties. In particular, cocommutative Hopf algebras are group objects in the category of cocommutative coalgebras, and are very much related to ordinary groups and Lie algebras. In fact, over an algebraically closed field of characteristic zero, such a Hopf algebra is a semi-direct product of a group algebra by a universal envelope of a Lie algebra, hence just a group algebra if finite dimensional (see [MM, Ca, Ko] for the connected case, [Gr, Sw2] for the general case).

In view of these facts it appears natural to try to relate the cohomology of Hopf algebras to that of groups and Lie algebras. The first work in this direction was done by M.E. Sweedler [Sw1] and by G.I. Kac [Kac] in the late 1960's. Sweedler introduced a cohomology theory of algebras that are modules over a Hopf algebra (now called Sweedler cohomology). He compared it to group cohomology, to Lie algebra cohomology and to Amitsur cohomology. In that paper he also shows how the second cohomology group classifies cleft comodule algebra extensions. Kac considered Hopf algebra extensions of a group algebra by the dual of a group algebra obtained from a matched pair of groups (N, T), $k \longrightarrow k^N \longrightarrow H \longrightarrow kT \longrightarrow k$, and found an exact sequence connecting the cohomology of the groups involved and the group of Hopf algebra extensions $\mathrm{Opext}(k^N, kT)$

$$\begin{aligned} 1 \quad &\longrightarrow \quad \mathrm{H}^1(N \bowtie T, k^\bullet) \longrightarrow \mathrm{H}^1(T, k^\bullet) \oplus \mathrm{H}^1(N, k^\bullet) \longrightarrow \mathrm{Aut}(k^N \# kT) \\ &\longrightarrow \quad \mathrm{H}^2(N \bowtie T, k^\bullet) \longrightarrow \mathrm{H}^2(T, k^\bullet) \oplus \mathrm{H}^2(N, k^\bullet) \longrightarrow \mathrm{Opext}(k^N, kT) \\ &\longrightarrow \quad \mathrm{H}^3(N \bowtie T, k^\bullet) \longrightarrow \ldots \end{aligned}$$

which is now known as the Kac sequence. In the work of Kac all Hopf algebras are over the field of complex numbers and also carry the structure of a C^*-algebra. Such structures are now called Kac algebras. The generalization to arbitrary fields appears in recent work by A. Masuoka [Ma1], where it is also used to show that certain groups of Hopf algebra extensions are trivial. Masuoka also obtained a version of the Kac sequence for matched pairs of Lie bialgebras [Ma3] and in [Ma4] he describes a Kac type sequence that connects extensions of quasi Hopf algebras and extensions of ordinary Hopf algebras. In this thesis we obtain a significant generalization of the Kac sequence, namely that for a general abelian matched pair of Hopf algebras (N, T, μ, ρ), consisting of two cocommutative Hopf algebras acting compatibly on each other

$$\begin{aligned} 1 \quad &\longrightarrow \quad \mathrm{H}^1(N \bowtie T, A) \longrightarrow \mathrm{H}^1(T, A) \oplus \mathrm{H}^1(N, A) \longrightarrow \mathcal{H}^1(T, N, A) \\ &\longrightarrow \quad \mathrm{H}^2(N \bowtie T, A) \longrightarrow \mathrm{H}^2(T, A) \oplus \mathrm{H}^2(N, A) \longrightarrow \mathcal{H}^2(T, N, A) \\ &\longrightarrow \quad \mathrm{H}^3(N \bowtie T, A) \longrightarrow \ldots \end{aligned}$$

Even more, in low degrees, we get an explicit description of the differentials in the sequence. Schauenburg [Sch] obtains a Kac sequence for arbitrary matched pairs of finite dimensional Hopf algebras, but most terms of the sequence are not groups. We also obtain a five term exact sequence for a smash product of Hopf algebras $N \rtimes T$, generalizing that of K. Tahara for a semi-direct product of groups

$$\begin{aligned} 1 \quad \longrightarrow \quad & \mathrm{H}^1_{\mathrm{meas}}(T, \mathrm{Hom}(N, A)) \longrightarrow \tilde{\mathrm{H}}^2(H, A) \longrightarrow \mathrm{H}^2(N, A)^T \\ \longrightarrow \quad & \mathrm{H}^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A)) \longrightarrow \tilde{\mathrm{H}}^3(H, A). \end{aligned}$$

These sequences give information about extensions of cocommutative Hopf algebras by commutative ones. They can also be used in certain cases to compute the (low degree) Sweedler cohomology groups of Hopf algebras.

The thesis is composed of six chapters, and an appendix. In the appendix some results from homological algebra used in the main body are presented, including some results from the cohomology of groups and Lie algebras.

In Chapter 1, some preliminaries are introduced. In particular we talk about Sweedler cohomology, Hopf algebra extensions and the cohomology of an abelian Singer pair of Hopf algebras [Si], [Ho].

In the second chapter matched pairs of Hopf algebras are discussed. They are compared to Singer pairs. We introduce a cohomology theory for such a matched pair of Hopf algebras with coefficients in a commutative algebra, and talk about how it compares to the cohomology of a Singer pair.

The generalized Kac sequence for an abelian matched pair of Hopf algebras, connecting Sweedler cohomology and the cohomology of the matched pair, is presented in Chapter 3. The homomorphisms in the sequence are given explicitly, so as to make it possible to use them in explicit calculations of groups of Hopf algebra extensions and low degree Sweedler cohomology groups.

In Chapter four, we are constructing a five term exact sequence for a smash products of cocommutative Hopf algebras, generalizing that of Tahara for a semi-direct product of groups. It is in fact a special case of the Kac sequence, but again explicit presentation of the homomorphisms in the sequence make it suitable for computations. In this spirit we use the sequence to compute Sweedler cohomology of smash products of group algebras and universal envelopes of Lie algebras in terms of the group and Lie algebra cohomologies.

Chapter five examines how the tools introduced in the previous chapters, combined with some new observations, help to describe explicitly some extensions of the dual of a group algebra k^N by a group algebra kH. Here the groups N and H are finite and H acts on N to give a semi-direct product $N \rtimes H$.

We conclude by indicating the possible avenues for some future work in this area.

Chapter 1

Preliminaries

1.1 Notation and conventions

- All vector spaces are over a fixed ground field k.
- The identity map is denoted by id.
- Given an algebra A, we denote its multiplication by $\mathrm{m}: A\otimes A \longrightarrow A$, $\mathrm{m}(a\otimes b) = ab$ and unit by $\eta: k \longrightarrow A$, $\eta(x) = x1_A$. Frequently we identify $x \equiv \eta(x)$.
- The comultiplication and counit for a coalgebra C are denoted by $\Delta: C \longrightarrow C \otimes C$ and $\varepsilon: C \longrightarrow k$, respectively. We use Sweedler's sigma notation for the comultiplication: $\Delta c = \sum c_1 \otimes c_2$, $(\mathrm{id} \otimes \Delta)\Delta c = \sum c_1 \otimes c_2 \otimes c_3$ and so forth; or sometimes just $\Delta c = c_1 \otimes c_2$. If the coalgebra in question is cocommutative, then the numbering of the subscripts is not necessary, and so we sometimes omit them and write $\Delta(a) = \sum_{(a)} a \otimes a$, or even just $\Delta a = a \otimes a$.

  Similarly if M is a right C-comodule, $\rho: M \longrightarrow M \otimes C$, we use the notation $\rho(m) = m_0 \otimes m_1$, $(\rho \otimes \mathrm{id})\rho(m) = (\mathrm{id} \otimes \Delta_C)\rho(m) = m_0 \otimes m_1 \otimes m_2$ and so on. In case M also has a structure of coalgebra, we will write $\rho(m) = m_M \otimes m_C$, to avoid ambiguity.
- The antipode for a given Hopf algebra H, is denoted by $S: H \longrightarrow H$.

- If A is an algebra (bialgebra, Hopf algebra), then A° denotes its finite dual coalgebra (bialgebra, Hopf algebra).
- If V is a vector space (algebra, coalgebra,...), then $\otimes^n V$, $V^{\otimes n}$, or sometimes just V^n denotes the n-fold tensor product $V \otimes \ldots \otimes V$.
- If U and W are vector spaces, then the vector space of linear maps from U to V is denoted by $\mathrm{Hom}(U, V)$.
- If C is a coalgebra and A an algebra, then we denote the convolution product in $\mathrm{Hom}(C, A)$ by $*\colon \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(C, A)$.
- The categories of vector spaces, algebras and coalgebras are denoted by $\mathcal{V}$, $\mathcal{A}$ and $\mathcal{C}$, respectively. If H is a Hopf algebra, then the categories of left H-modules, left H-module algebras and left H-module coalgebras are denoted by ${}_H\mathcal{V}$, ${}_H\mathcal{A}$ and ${}_H\mathcal{C}$, respectively.
- When dealing with quotients, we denote the induced equivalence relation by $f \sim g$. In particular, if f and g are cocycles, then $f \sim g$ means that they are cohomologous.

1.2 Sweedler cohomology

The Sweedler cohomology [Sw1] is a cohomology theory for module algebras over a given cocommutative Hopf algebra. If H is a cocommutative Hopf algebra then the category ${}_H\mathcal{V}$ of H-modules is monoidal closed. The tensor product of two H-modules V and W is the tensor product of the underlying vector spaces $V \otimes W$ together with the diagonal action

$$\mu_{V\otimes W} = (\mu_V \otimes \mu_W)\sigma_{23}(\Delta \otimes \mathrm{id} \otimes \mathrm{id})\colon H \otimes V \otimes W \longrightarrow V \otimes W,$$

i.e: $h(v \otimes w) = \sum h_1 v \otimes h_2 w$. The H-module of internal homomorphisms $\mathrm{Hom}(V, W)$ is the vector space of linear maps $\mathcal{V}(V, W)$ together with the 'diagonal' action

$$\mu_{\mathrm{Hom}(V,W)}; H \otimes \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(V, W)$$

given by $(hf)(v) = \sum h_1 f(S(h_2)v)$.

An H-module algebra A is an algebra in the category of H-modules ${}_H\mathcal{V}$, that is an H-module which is also an ordinary algebra such that the multiplication and unit are H-module maps, so that $h(ab) = \sum (h_1(a))(h_2(b))$ and $h(1) = \varepsilon(h)1$, i.e: so that the diagrams

$$\begin{array}{ccc} H \otimes A \otimes A & \xrightarrow{\mu_{A\otimes A}} & A \otimes A \\ \downarrow{\scriptstyle \mathrm{id}\otimes \mathrm{m}} & & \downarrow{\scriptstyle \mathrm{m}} \\ H \otimes A & \xrightarrow[\mu_A]{} & A \end{array} \qquad \begin{array}{ccc} H \otimes k & \xrightarrow{\mu_k} & k \\ \downarrow{\scriptstyle \mathrm{id}\otimes \eta} & & \downarrow{\scriptstyle \eta} \\ H \otimes A & \xrightarrow[\mu_A]{} & A \end{array}$$

commute.

An H-module coalgebra C is a coalgebra in the category of H-modules ${}_H\mathcal{V}$, that is an H-module with an ordinary coalgebra structure such that comultiplication and counit are H-module maps, so that $\Delta(h(c)) = \sum h_1(c_1) \otimes h_2(c_2)$ (that is μ is a coalgebra map) and $\varepsilon_C(h(c)) = \varepsilon_H(h)\varepsilon_C(c)$, i.e: so that the diagrams

$$\begin{array}{ccc} H \otimes C & \xrightarrow{\mu_C} & C \\ \downarrow{\scriptstyle \mathrm{id}\otimes \Delta} & & \downarrow{\scriptstyle \Delta} \\ H \otimes C \otimes C & \xrightarrow[\mu_{C\otimes C}]{} & C \otimes C \end{array} \qquad \begin{array}{ccc} H \otimes C & \xrightarrow{\mu_C} & C \\ \downarrow{\scriptstyle \mathrm{id}\otimes \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ H \otimes k & \xrightarrow[\mu_k]{} & k \end{array}$$

commute.

If A is a commutative H-module algebra and C is a cocommutative H-module coalgebra then the vector space of H-module maps ${}_H\mathrm{Hom}(A, C)$ carries the associative and commutative convolution algebra structure with multiplication and unit defined by

$$f * g = \mathrm{m}_A(f \otimes g)\Delta_C : C \xrightarrow{\Delta} C \otimes C \xrightarrow{f\otimes g} A \otimes A \xrightarrow{\mathrm{m}_A} A$$

and unit $\eta_A\varepsilon_C : C \xrightarrow{\varepsilon} k \xrightarrow{\eta} A$. Then the convolution invertible elements of ${}_H\mathrm{Hom}(C, A)$

form an abelian group $U({}_H\mathrm{Hom}(C,A)) = {}_H\mathrm{Reg}(C,A)$. This gives a functor

$$ {}_H\mathrm{Reg}(-,A): {}_H\mathcal{C}^{op} \longrightarrow \mathrm{Ab}. $$

The free H-module coalgebra functor $F: \mathcal{C} \longrightarrow {}_H\mathcal{C}$, defined by $F(C) = H \otimes C$ with H-action on the first factor and tensor product coalgebra structure, is left adjoint to the forgetful functor $U: {}_H\mathcal{C} \longrightarrow \mathcal{C}$. The natural isomorphism

$$ \theta_{C,D}: {}_H\mathcal{C}(H \otimes C, D) \longrightarrow \mathcal{C}(C, UD) $$

is given by $\theta(f)(c) = f(1 \otimes c)$ and $\theta^{-1}(g)(h \otimes c) = hg(c)$, i.e: $\theta(f) = f(\eta_H \otimes \mathrm{id})$ and $\theta^{-1}(g) = \mu_D(\mathrm{id} \otimes g)$. The unit $\eta_C: C \longrightarrow UFC$ and the counit $\epsilon_D: FUD \longrightarrow D$ of the adjunction are given by $\eta(c) = 1 \otimes c$ and $\epsilon(h \otimes d) = hd$, i.e: $\eta = \eta_H \otimes \mathrm{id}$ and $\epsilon = \mu_D$, respectively. The resulting cotriple $\mathbf{G} = (FU, \epsilon, \delta = F\eta U)$ on ${}_H\mathcal{C}$ is then used to define the Sweedler cohomology. Every object D of ${}_H\mathcal{C}$ has a simplicial resolution with $X_n = \mathbf{G}^{n+1}D$, faces

$$ \partial_i: X_{n+1} = \mathbf{G}^{n+2}D \longrightarrow \mathbf{G}^{n+1}D = X_n $$

given by $\partial_{n+1} = \mathrm{id} \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes \mu_D$ and $\partial_i = \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes m_H \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}$ for $0 \le i \le n$, and degeneracies

$$ s_i: X_n = \mathbf{G}^{n+1}D \longrightarrow \mathbf{G}^{n+2}D = X_{n+1} $$

by $s_i = \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes \eta_H \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}$ for $0 \le i \le n$. Applying the functor ${}_H\mathrm{Reg}(-,A): {}_H\mathcal{C}^{op} \longrightarrow \mathrm{Ab}$ to this simplicial resolution gives a cosimplicial complex

$$ ({}_H\mathrm{Reg}(X_n, A), \partial_i^*, s_j^*) $$

and the associated cochain complex of abelian groups

$$ {}_H\mathrm{Reg}(X_0, A) \xrightarrow{d^0} {}_H\mathrm{Reg}(X_1, A) \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} {}_H\mathrm{Reg}(X_n, A) \xrightarrow{d^n} \ldots $$

with differential $d_n = \partial_0 * \partial_1^{-1} * \ldots * \partial_{n+1}^{(-1)^{n+1}}$. The homology of this complex is the sequence of right derived functors

$$ R^*{}_H\mathrm{Reg}(-,A): {}_H\mathcal{C}^{op} \longrightarrow \mathrm{Ab} $$

of ${}_H\mathrm{Reg}(-,A)\colon {}_H\mathcal{C}^{op} \longrightarrow \mathrm{Ab}$. The Sweedler cohomology is obtained by evaluating this sequence of right derived functors at the trivial H-module coalgebra k

$$H^*(H,A) = R^*{}_H\mathrm{Reg}(-,A)(k) = H^*({}_H\mathrm{Reg}(G^{*+1}k,A),d^*).$$

Observe that the natural isomorphism

$$u\colon {}_H\mathrm{Reg}(H\otimes C,A) \longrightarrow \mathrm{Reg}(C,A)$$

defined by $u(f)(c) = f(1\otimes c)$ induces a natural isomorphism of cosimplicial complexes

$$u^*\colon ({}_H\mathrm{Reg}(X_*,A),\partial^i,s^j) \longrightarrow (\mathrm{Reg}(\otimes^* H\otimes D,A),\delta^i,\sigma^j)$$

with coface operators $\delta^0(f) = \mu_{H\otimes(H^n\otimes A)}(\mathrm{id}\otimes f)$, $\delta^i(f) = f(\mathrm{id}\otimes\ldots\otimes m\ldots\otimes \mathrm{id})$ for $0 < i < n+1$, $\delta^{n+1}(f) = f(\mathrm{id}\otimes\ldots\otimes \mathrm{id}\otimes\mu_D)$ and codegeneracies $\sigma^i(g) = g(\mathrm{id}^i\otimes\eta\otimes\mathrm{id}^{n+2-i})$ for $0\le i\le n$. In terms of the associated cochain complexes this means that the diagrams

$$\begin{array}{ccc}
{}_H\mathrm{Reg}(X_n,A) & \xrightarrow{d^i} & {}_H\mathrm{Reg}(X_{n+1},A) \\
\Big\downarrow{\scriptstyle u_n} & & \Big\downarrow{\scriptstyle u_{n+1}} \\
\mathrm{Reg}(\otimes^n H\otimes D,A) & \xrightarrow[\delta^i]{} & \mathrm{Reg}(\otimes^{n+1} H\otimes D,A)
\end{array}$$

commute, where

$$\delta^n(f) = (u(1\otimes f))*(f^{-1}(m\otimes^n 1))*\ldots*f^{(-1)^{n+1}}(\otimes^{n-1}1\otimes m\otimes 1)*(f^{(-1)^{n+2}}(\otimes^n 1\otimes\mu_D)).$$

The inclusion of the normalized complex

$$(\mathrm{Reg}_+(\otimes^* H\otimes D,A),\delta^*)$$

into $(\mathrm{Reg}(\otimes^* H\otimes D,A),\delta^*)$ with $\mathrm{Reg}_+(\otimes^n H\otimes D,A) = \cap_{i=0}^n \ker\sigma^i$ induces an isomorphism in homology [Mc], in particular an isomorphism

$$H^*(H,A) \cong H^*(\mathrm{Reg}_+(\otimes^* H,A),\delta^*)$$

suitable to compute Sweedler cohomology in terms of the normalized standard complex $(\mathrm{Reg}_+(\otimes^* H, A), \delta^*)$.

The dual version of the theory described above can be used to get a cohomology theory of comodule coalgebras C over a commutative Hopf algebra K. The category of right K-comodules $\mathcal{V}^K$ is symmetric monoidal. The tensor product of two K-comodules is the tensor product of the underlying vector spaces $V \otimes W$ together with the diagonal coaction

$$\delta_{V\otimes W} = (1 \otimes 1 \otimes m)\sigma_{23}(\delta_V \otimes \delta_W) \colon V \otimes W \longrightarrow V \otimes W \otimes K.$$

A K-comodule coalgebra C is a coalgebra and a K-comodule algebra A is an algebra in $\mathcal{V}^K$. The cofree K-comodule algebra functor

$$G \colon \mathcal{A} \longrightarrow \mathcal{A}^K,$$

defined by $G(A) = A \otimes K$ with K-coaction on the second factor and tensor product coalgebra structure, is right adjoint to the forgetful functor $U \colon \mathcal{A}^K \longrightarrow \mathcal{A}$. The natural isomorphism

$$\theta_{B,A} \colon \mathcal{A}(UB, A) \longrightarrow \mathcal{A}^K(B, GA)$$

is given by $\theta(f) = (1 \otimes f)\delta_B$ and $\theta^{-1}(g) = (1 \otimes \varepsilon_K)g$. The unit $\eta \colon 1 \longrightarrow GU$ and the counit $\epsilon \colon UG \longrightarrow 1$ are given by $\eta_B = \delta_B$ and $\epsilon_A = \varepsilon_K \otimes \mathrm{id}_A$, respectively. The resulting triple is $\mathbf{T} = (GU, \eta, \mu = G\epsilon U)$ on $\mathcal{A}^K$. Every object B of $\mathcal{A}^K$ has a cosimplicial resolution with $Y^n = \mathbf{T}^{n+1}B$, faces

$$\partial^i \colon Y^n = \mathbf{T}^{n+1}B \longrightarrow \mathbf{T}^{n+2}B = Y^{n+1}$$

given by $\partial^0 = \delta_B \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}$ and $\partial^i = \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes \Delta_K \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}$ for $1 \leq i \leq n+1$, and degeneracies

$$s^i \colon Y^{n+1} = \mathbf{T}^{n+2}B \longrightarrow \mathbf{T}^{n+1}B = Y^n$$

by $s_i = \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes \varepsilon_K \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}$ for $0 \leq i \leq n$. Applying the functor $\mathrm{Reg}^K(C, -) \colon \mathcal{A}^K \longrightarrow \mathrm{Ab}$ to this cosimplicial resolution gives a cosimplicial complex

$$(\mathrm{Reg}^K(C, Y^*), \partial^i_*, s^j_*)$$

and the associated cochain complex of abelian groups

$$\mathrm{Reg}^K(C,Y^0) \overset{d^0}{\leftarrow} \mathrm{Reg}^K(C,Y^1) \overset{d^1}{\leftarrow} \dots \overset{d^{n-1}}{\leftarrow} \mathrm{Reg}^K(C,Y^{n-1}) \overset{d^n}{\leftarrow} \dots$$

with differential $d^n = \partial_0\partial_1^{-1} * \dots * \partial_{n+1}^{(-1)^{n+1}}$. The homology of this complex is the sequence of right derived functors

$$R^*\mathrm{Reg}^K(C,-)\colon \mathcal{A}^K \longrightarrow Ab$$

of $\mathrm{Reg}^K(C,-)\colon \mathcal{A}^K \longrightarrow Ab$. The cohomology of C is obtained by evaluating this sequence of right derived functors at the trivial K-comodule algebra k

$$H^*(C,H) = R^*\mathrm{Reg}^K(C,-)(k) = H^*(\mathrm{Reg}^K(C,T^{*+1}k),d^*).$$

Observe that the natural isomorphism

$$v\colon \mathrm{Reg}(C,A) \longrightarrow \mathrm{Reg}^K(C,A\otimes K)$$

defined by $v(g) = (g\otimes 1)\delta_C$ with inverse $v^{-1}(f) = (1\otimes\varepsilon_K)f$ induces a natural isomorphism of cosimplicial complexes

$$v^*\colon (\mathrm{Reg}(C,A\otimes^* K),\delta^i,s^j) \longrightarrow (\mathrm{Reg}^K(C,Y^*),\partial^i,\sigma^j)$$

with coface operators $\delta^0(f) = (f\otimes 1)(\delta_C)$, $\delta^i(f) = (\mathrm{id}\otimes\dots\otimes\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\dots\otimes\mathrm{id})f$ for $0 < i < n+1$, $\delta^{n+1}(f) = (\delta_A\otimes 1)f$ and codegeneracies $\sigma^i(g) = (\mathrm{id}^i\otimes\varepsilon_K\otimes\mathrm{id}^{n+2-i})g$ for $0\le i\le n+1$. In terms of the associated cochain complexes this means that the diagrams

$$\begin{array}{ccc} \mathrm{Reg}(C,A\otimes K^n) & \xrightarrow{\delta^n} & \mathrm{Reg}(C,A\otimes K^{n+1}) \\ \downarrow{\scriptstyle v_n} & & \downarrow{\scriptstyle v_{n+1}} \\ \mathrm{Reg}^K(C,Y^n) & \xrightarrow[d^n]{} & \mathrm{Reg}^K(C,Y^{n+1}) \end{array}$$

commute, where

$$\delta^n(f) = (\delta_A\otimes\mathrm{id}^n)f * (\mathrm{id}\otimes\Delta\otimes\mathrm{id}^{n-1})f^{-1} * \dots * (\mathrm{id}^n\otimes m)f^{(-1)^n} * (f^{(-1)^{n+1}}\otimes\mathrm{id})\Delta_C.$$

The inclusion of the normalized complex

$$(\mathrm{Reg}_{+}(C, A \otimes^{*} K), \delta^{*})$$

into $\mathrm{Reg}(C, A \otimes^{*} K), \delta^{*})$ with $\mathrm{Reg}_{+}(C, A \otimes^{n} K) = \cap_{i=0}^{n} \ker \sigma^{i}$ induces an isomorphism in homology [Mc], in particular an isomorphism

$$H^{*}(C, K) \cong H^{*}(\mathrm{Reg}_{+}(C, \otimes^{*} K), \delta^{*})$$

suitable to compute the cohomology of C in terms of the normalized standard complex $(\mathrm{Reg}_{+}(C, \otimes^{*} K), \delta^{*})$.

1.2.1 Standard complex

Here we review in more detail the standard (normalized) complex for computing the Sweedler cohomology of a cocommutative Hopf algebra H with coefficients in a commutative H-module algebra A, where the action of H on A is denoted by Ψ. The complex in question is given as follows.

$$\ldots \longrightarrow \mathrm{Reg}_{+}(H^{\otimes q-1}, A) \overset{\delta^{q-1}}{\longrightarrow} \mathrm{Reg}_{+}(H^{\otimes q}, A) \longrightarrow \ldots,$$

where $\delta^{q-1}(f) = (\Psi(\mathrm{id} \otimes f)) * (f^{-1}(\mathrm{m} \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}) * (f(\mathrm{id} \otimes \mathrm{m} \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}) * \ldots * (f^{\pm 1}(\mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes \mathrm{m})) * (f^{\mp 1} \otimes \varepsilon)$. Here $\mathrm{Reg}_{+}(H^{\otimes q}, A)$ denotes the abelian group of convolution invertible normal linear maps $f \colon H^{\otimes q} \longrightarrow A$ (by normal we mean that $f(h_1 \otimes \ldots h_q) = \varepsilon(h_1) \ldots \varepsilon(h_q) 1_A$ whenever some $h_i \in k$). The cocycles and coboundaries for the degrees 1 and 2 cohomology groups are described as follows:

$$\begin{aligned}
\mathrm{Z}^1(H, A) &= \{f \in \mathrm{Reg}_{+}(H, A) | f(gh) = \Psi(g_1 \otimes f(h)) f(g_2)\}, \\
\mathrm{B}^1(H, A) &= \{f \in \mathrm{Reg}_{+}(H, A) | \exists a \in \mathcal{U}(A), \text{ s.t. } f(h) = \psi(h \otimes a) a^{-1}\}, \\
\mathrm{Z}^2(H, A) &= \{f \in \mathrm{Reg}_{+}(H \otimes H, A) | \Psi(g_1 \otimes f(h_1 \otimes k_1)) f(g_2 \otimes h_2 k_2) \\
&\quad = f(g_1 h_1 \otimes k) f(g_2 \otimes h_2)\}, \\
\mathrm{B}^2(H, A) &= \{f \in \mathrm{Reg}_{+}(H \otimes H, A) | \exists t \in \mathrm{Reg}(H, A), \text{ s.t.} \\
&\quad f(h \otimes g) = \sum \Psi(h_1 \otimes f(g_1)) f^{-1}(h_2 g_2) f(h_3)\}.
\end{aligned}$$

1.2.2 Comparison with group and Lie algebra cohomologies

Suppose G is a group and $H = kG$ is a group algebra and A is a commutative kG-module algebra. The elements of G act as automorphisms of A, so they preserve $\mathcal{U}(A)$ (the abelian group of units). Hence we can consider the group cohomology $\mathrm{H}^*(G, \mathcal{U}(A))$.

Theorem 1.2.1 (Sw1, Theorem 3.1) *The cohomology groups* $\mathrm{H}^q(kG, A)$ *and* $\mathrm{H}^q(G, \mathcal{U}(A))$ *are canonically isomorphic for all positive* q. *The isomorphism is induced by a canonical isomorphism between the standard complex to compute* $\mathrm{H}^q(kG, A)$ *and the standard Hochschild complex (see appendix) to compute* $\mathrm{H}^q(G, \mathcal{U}(A))$.

Now let $\mathbf{g}$ be a Lie algebra and let $H = U\mathbf{g}$ be its universal envelope. Denote the underlying vector space of A by A^+.

Theorem 1.2.2 (Sw1, Theorem 4.3) *The cohomology groups* $\mathrm{H}^q(U\mathbf{g}, A)$ *and* $\mathrm{H}^q(\mathbf{g}, A^+)$ *are canonically isomorphic for* $q \geq 2$.

The isomorphism is given by the so called exponential map. In characteristic 0 this is the map $exp\colon \mathrm{Hom}(U\mathbf{g}^{\otimes n}, A) \longrightarrow \mathrm{Reg}(U\mathbf{g}^{\otimes n}, A)$, $exp(f) = \varepsilon + \sum_{i=1}^{\infty} f^i/i!$. The inverse is given by $\log(f) = \sum_1^{\infty}(-1)^{i-1}(f-\varepsilon)^i/i!$.

1.2.3 Interpretation of the second cohomology group

As expected the second cohomology classifies some sort of extensions. In this case the extensions in question are cleft comodule algebra exact sequences. Let H be a Hopf algebra and let C be an H-comodule algebra (denote the comodule structure by $\rho\colon C \longrightarrow C \otimes H$). We say that the sequence $A \longrightarrow C \longrightarrow H$ is exact (here A is an algebra) if $A = C^{coH} = \{c \in C | \rho(c) = c \otimes 1_A\}$ and it is cleft if there exists a convolution invertible H-comodule map $\chi\colon H \longrightarrow C$. If A is commutative and H cocommutative then χ defines an H-module structure on A (via conjugation in C) and a Sweedler 2-cocycle $\alpha\colon H \otimes H \longrightarrow A$ ($\alpha(h \otimes k) = \chi(h_1)\chi(k_1)\chi^{-1}(h_2k_2)$). This way we get an isomorphism between equivalence classes of extensions and the second cohomology group $\mathrm{H}^2(H, A)$.

1.3 Singer pairs, cohomology and extensions

Let (B, A) be a pair of Hopf algebras together with an action $\mu: B \otimes A \longrightarrow A$ and a coaction $\rho: B \longrightarrow B \otimes A$. Then $A \otimes B$ can be equipped with the cross product algebra structure, as well as the cross product coalgebra structure. To ensure compatibility of these structures, further conditions on (B, A, μ, ρ) are necessary. We give them in terms of the action of B on A, twisted by the coaction of A on B,

$$\mu_2 = (\mu \otimes \mathrm{m}_A(\mathrm{id} \otimes \mu))(14235)((\rho \otimes \mathrm{id})\Delta_B \otimes \mathrm{id} \otimes \mathrm{id}): B \otimes A \otimes A \longrightarrow A \otimes A,$$

i.e. $b(a \otimes a') = b_{1B}(a) \otimes b_{1A} \cdot b_2(a')$, and the coaction, twisted by the action,

$$\rho_2 = (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{m}_A(\mathrm{id} \otimes \mu))(14235)((\rho \otimes \mathrm{id})\Delta_B \otimes \rho): B \otimes B \longrightarrow B \otimes B \otimes A,$$

$\rho_2(b \otimes b') = b_{1B} \otimes b'_B \otimes b_{1A} \cdot b_2(b'_A)$.

Definition 1.3.1 *The quadruple (B, A, μ, ρ) is called a* **Singer pair** *if*

1. *A is a B-***module algebra***, i.e.:*

   (a)

$$\begin{array}{ccc} B \otimes A \otimes A & \xrightarrow{\mu_{A\otimes A}} & A \otimes A \\ \Big\downarrow {\scriptstyle \mathrm{id} \otimes \mathrm{m}_A} & & \Big\downarrow {\scriptstyle \mathrm{m}_A} \\ B \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

   commutes, where $\mu_{A\otimes A} = (\mu \otimes \mu)\sigma_{2,3}(\Delta_B \otimes \mathrm{id}_A \otimes \mathrm{id}_A)$; i.e.

$$b \cdot aa' = \sum (b_1 \cdot a)(b_2 \cdot a'),$$

(b)

$$\begin{CD} B\otimes k @>{\mu_k=\eta_B}>> k \\ @V{\mathrm{id}\otimes\eta_A}VV @VV{\eta_A}V \\ B\otimes A @>>{\mu}> A \end{CD}$$

commutes, i.e.

$$b\cdot 1_A=\varepsilon_B(b)1_A.$$

2. *B is an A*-**comodule coalgebra**:

(a)

$$\begin{CD} B @>{\rho}>> B\otimes A \\ @V{\Delta_B}VV @VV{\Delta_B\otimes\mathrm{id}_A}V \\ B\otimes B @>>{\rho_{B\otimes B}}> B\otimes B\otimes A \end{CD}$$

commutes, where $\rho_{B\otimes B}=(\mathrm{id}_B\otimes\mathrm{id}_B\otimes \mathrm{m}_A)\sigma_{2,3}(\rho\otimes\rho)$; *i.e.*

$$\sum\Delta_B(b_B)\otimes b_A=\sum b_{1B}\otimes b_{2B}\otimes b_{1A}b_{2A},$$

(b)

$$\begin{CD} B @>{\rho}>> B\otimes A \\ @V{\varepsilon_B}VV @VV{\varepsilon_B\otimes\mathrm{id}_A}V \\ k @>>> k\otimes A \end{CD}$$

commutes, i.e.

$$\sum\varepsilon_B(b_B)b_A=\varepsilon_B(b)1_A.$$

3.

$$\rho \mathrm{m}_B = (\mathrm{m}_B \otimes \mathrm{id})\rho_2$$

i.e.:

$$\rho(bb') = \sum b_{1B} b'_B \otimes b_{1A}(b_2 \cdot b'_A).$$

4.

$$\Delta_A \mu = \mu_2(\mathrm{id} \otimes \Delta_A)$$

i.e.:

$$\Delta_A(b \cdot a) = \sum b_{1B} \cdot a_1 \otimes b_{1A}(b_2 \cdot a_2).$$

The Singer pair (B, A, μ, ρ) is called **abelian** *if A is commutative and B is cocommutative.*

The twisted action of B on A^n and the twisted coaction of A on B^n can now be defined inductively:

$$\mu_1 = \mu$$

and

$$\begin{array}{ccc}
B \otimes A^n \otimes A & \xrightarrow{\mu_{n+1}} & A^n \otimes A \\
\downarrow \Delta_B \otimes \mathrm{id}_{A^n} \otimes \mathrm{id}_A & & \uparrow \mathrm{id}_{A^n} \otimes \mathrm{m}_A \\
B \otimes B \otimes A^n \otimes A & & A^n \otimes A \otimes A \\
\downarrow \rho \otimes \mathrm{id}_B \otimes \mathrm{id}_{A^n} \otimes \mathrm{id}_A & & \uparrow \mu_n \otimes \mathrm{id}_A \otimes \mu \\
B \otimes A \otimes B \otimes A^n \otimes A & \xrightarrow[(14235)]{} & B \otimes A^n \otimes A \otimes B \otimes A
\end{array}$$

Dually the coaction of A on B^m is twisted by the action and is defined inductively as follows

$$\rho_1 = \rho$$

and

$$\begin{array}{ccc}
B \otimes B^m & \xrightarrow{\quad \rho_{m+1} \quad} & B \otimes B^m \otimes A \\
{\scriptstyle \Delta_B \otimes \mathrm{id}_{B^m}} \downarrow & & \uparrow {\scriptstyle \mathrm{id}_B \otimes \mathrm{id}_{B^m} \otimes \mathrm{m}_A} \\
B \otimes B \otimes B^m & & B \otimes B^m \otimes A \otimes A \\
{\scriptstyle \rho \otimes \mathrm{id}_B \otimes \rho_m} \downarrow & & \uparrow {\scriptstyle \mathrm{id}_B \otimes \mathrm{id}_{B^m} \otimes \mathrm{id}_A \otimes \mu} \\
B \otimes A \otimes B \otimes B^m \otimes A & \xrightarrow[(14235)]{\qquad} & B \otimes B^m \otimes A \otimes B \otimes A
\end{array}$$

1.3.1 Hopf algebra extensions

A sequence of Hopf algebra maps $A \xrightarrow{\iota} C \xrightarrow{\pi} B$ is called an extension provided that it is exact, i.e. $A = C^{\mathrm{co}\,B} = \{c \in C | (\mathrm{id} \otimes \pi)\Delta(c) = c \otimes 1\}$, and that it is cleft, i.e. there exists a B-comodule map $\chi \in \mathrm{Reg}(B, C)$. From now on assume that A is commutative and B cocommutative. Such an extension gives rise to a unique abelian Singer pair (B, A, μ, ρ).

The set of equivalence classes of extensions giving rise to the same abelian Singer pair can be equipped with an intrinsic 'Baer-type' abelian group structure given by the bitensor product construction (see [Ho]). The abelian group of equivalence classes of extensions giving rise to (B, A, μ, ρ) is denoted by $\mathrm{Opext}(B, A, \mu, \rho)$ (or just by $\mathrm{Opext}(B, A)$ if the choice of μ and ρ is obvious). Similarly as in the group case extensions can be classified by the second cohomology group.

1.3.2 Cohomology of a Singer pair of Hopf algebras

Let (B, A, μ, ρ) be an abelian Singer pair. It is convenient to introduce the abelian category ${}_B\mathcal{V}^A$, whose objects are triples (V, ω, λ), such that V is a left B-module, via $\omega\colon B \otimes V \longrightarrow V$, $\omega(b \otimes v) = b(v)$, a right A-comodule via $\lambda\colon V \longrightarrow V \otimes A$, $\lambda(v) = v_0 \otimes v_1$ and that the following compatibility condition holds:

$$\lambda(b(v)) = b_{1B}(v_0) \otimes b_{1A} \cdot b_2(v_1).$$

The morphisms are B-linear and A-collinear maps. Observe that (B, m_B, ρ), (A, μ, Δ_A) and $(k, \varepsilon_B \otimes \mathrm{id}, \mathrm{id} \otimes \eta_A)$ are objects of ${}_B\mathcal{V}^A$. Moreover ${}_B\mathcal{V}^A$ is a symmetric monoidal category, so that commutative algebras and cocommutative coalgebras are defined in $({}_B\mathcal{V}^A, \otimes, k)$.

The free functor $F: \mathcal{V}^A \longrightarrow {}_B\mathcal{V}^A$, defined by $F(X,\alpha) = (B \otimes X, \alpha_{B\otimes X})$ with twisted coaction $\alpha_{B\otimes X} = (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{m}_A(\mathrm{id} \otimes \mu))(14235)((\rho \otimes \mathrm{id})\Delta_B \otimes \alpha)$, is left adjoint to the forgetful functor $U: {}_B\mathcal{V}^A \longrightarrow \mathcal{V}^A$.

Similarly the cofree functor $L: {}_B\mathcal{V} \longrightarrow {}_B\mathcal{V}^A$, defined by $L(Y, \beta) = (Y \otimes A, \beta_{Y\otimes A})$, with twisted action $\beta_{Y\otimes A} = (\beta \otimes \mathrm{m}_A(\mathrm{id} \otimes \mu))(14235)((\rho \otimes \mathrm{id})\Delta_B \otimes \mathrm{id} \otimes \mathrm{id})$, is right adjoint to the forgetful functor $U: {}_B\mathcal{V}^A \longrightarrow {}_B\mathcal{V}$.

These adjunctions give rise to a comonad $\mathbf{G} = (FU, \varepsilon, \delta)$ and a monad $\mathbf{T} = (LU, \eta, \mu)$ on ${}_B\mathcal{V}^A$. Moreover ${}_B\mathcal{V}^A(FU(M), LU(N)) \simeq \mathcal{V}(M, N)$.

The cohomology is now defined by means of simplicial G-resolutions $G_* M$ and the cosimplicial T-resolutions $T^* N$. We get a bi-complex

$$(X^{m,n}) = ({}_B\mathrm{Reg}^A(G_{m+1}(k), T^{n+1}(k)), \delta', \delta) = ({}_B\mathrm{Reg}^A(B^{m+1}, A^{n+1}), \delta', \delta)_{m,n\geq 0}.$$

Use the isomorphism ${}_B\mathcal{V}^A(FU(M), LU(N)) \simeq \mathcal{V}(M, N)$ to get the double complex $(Y^{m,n}) = (\mathrm{Reg}(B^m, A^n), \delta', \delta)$ and then define $Y_0^{m,n}$ to be the double complex obtained from Y by replacing the 0^{th} column and the 0^{th} row by zeroes.

The Singer pair cohomology is defined to be the cohomology of $\mathrm{Tot}(Y_0)$. For computing the cohomology we use the normalized complex, that is, we replace $Y_0^{m,n} = \mathrm{Reg}(B^m, A^n)$, $(m, n \geq 1)$ by $\mathrm{Reg}_+(B^m, A^n)$, the intersections of the degeneracies. The group $\mathrm{Reg}_+(B^m, A^n)$ consists of all convolution invertible maps $f: B^m \longrightarrow A^n$, which have the property that $f(\mathrm{id}\otimes \ldots \otimes \eta \otimes \ldots \otimes \mathrm{id}) = \eta\varepsilon$ and $(\mathrm{id} \otimes \ldots \otimes \eta\varepsilon \otimes \ldots \otimes \mathrm{id})f = \eta\varepsilon$.

We write this out in more detail. Given an abelian Singer pair (B, A, μ, ρ) we can construct a bicomplex

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Reg}_+(B, A) & \xrightarrow[\partial_{1,1}]{} & \mathrm{Reg}_+(B^2, A) & \xrightarrow[\partial_{2,1}]{} & \mathrm{Reg}_+(B^3, A) & & \cdots \\
\downarrow & & \downarrow \partial^{1,1} & & \downarrow \partial^{2,1} & & \vdots & & \\
0 & \longrightarrow & \mathrm{Reg}_+(B, A^2) & \xrightarrow[\partial_{1,2}]{} & \mathrm{Reg}_+(B^2, A^2) & & \cdots & & \\
\downarrow & & \downarrow \partial^{1,2} & & \vdots & & & & \\
0 & \longrightarrow & \mathrm{Reg}_+(B, A^3) & & \cdots & & & & \\
\vdots & & \vdots & & & & & &
\end{array}
$$

The coboundary maps

$$d^i_{n,m}: \mathrm{Reg}_+(B^n, A^m) \longrightarrow \mathrm{Reg}_+(B^{n+1}, A^m)$$

defined by

$$
\begin{aligned}
d^0_{n,m}\alpha &= \mu_m(\mathrm{id}_B \otimes \alpha) \\
d^i_{n,m}\alpha &= \alpha(\mathrm{id}_B \otimes \mathrm{id}_B \otimes \ldots \otimes \mathrm{m}_B \otimes \ldots \otimes \mathrm{id}_B) = \alpha(\mathrm{id}_{B^{i-1}} \otimes \mathrm{m}_B \otimes \mathrm{id}_{B^{n-i}}) \\
d^{n+1}_{n,m}\alpha &= \alpha \otimes \varepsilon
\end{aligned}
$$

are used to construct the horizontal differentials

$$\partial_{n,m}: \mathrm{Reg}_+(B^n, A^m) \longrightarrow \mathrm{Reg}_+(B^{n+1}, A^m),$$

given by the 'alternating' convolution product

$$\partial_{n,m}\alpha = d^0_{n,m}\alpha * d^1_{n,m}\alpha^{-1} * d^2_{n,m}\alpha * \ldots .$$

Dually the coboundaries

$$d'^i_{n,m}: \mathrm{Reg}_+(B^n, A^m) \longrightarrow \mathrm{Reg}_+(B^n, A^{m+1})$$

defined by

$$
\begin{aligned}
d'^{0}_{n,m}\beta &= (\beta \otimes \mathrm{id}_A)\rho_n \\
d'^{i}_{n,m}\beta &= (\mathrm{id}_A \otimes \mathrm{id}_A \otimes \ldots \otimes \Delta_A \otimes \ldots \otimes \mathrm{id}_A)\beta = (\mathrm{id}_{A^{i-1}} \otimes \Delta_A \otimes \mathrm{id}_{A^{n-i}})\beta \\
d'^{n+1}_{n,m}\beta &= \eta \otimes \beta
\end{aligned}
$$

determine the vertical differentials

$$\partial^{n,m}\colon \mathrm{Reg}_+(B^n, A^m) \longrightarrow \mathrm{Reg}_+(B^n, A^{m+1}),$$

where

$$\partial^{n,m}\beta = d'^{0}_{n,m}\beta * d'^{1}_{n,m}\beta^{-1} * d'^{2}_{n,m}\beta * \ldots$$

The cohomology of the abelian Singer pair (B, A, μ, ρ) is by definition the cohomology of the total complex.

$$
\begin{aligned}
0 \longrightarrow \mathrm{Reg}_+(B, A) &\longrightarrow \mathrm{Reg}_+(B^2, A) \oplus \mathrm{Reg}_+(B, A^2) \longrightarrow \\
&\longrightarrow \ldots \longrightarrow \bigoplus_{i=1}^{n} \mathrm{Reg}_+(B^{n+1-i}, A^i) \longrightarrow \ldots
\end{aligned}
$$

In low dimension the cocycles and coboundaries are described as follows [Ho]:

$$
\begin{aligned}
Z^0(B, A, \mu, \rho) &= 0, \\
Z^1(B, A, \mu, \rho) &= \{f \in \mathrm{Reg}_+(B, A) | \\
&\quad f\mathrm{m}_B = (f \otimes \varepsilon) * \mu(\mathrm{id} \otimes f), \\
&\quad \Delta f = (f \otimes \mathrm{id})\rho * (\eta \otimes f)\}, \\
B^1(B, A, \mu, \rho) &= 0, \\
Z^2(B, A, \mu, \rho) &= \{(\alpha, \beta) \in \mathrm{Reg}_+(B^2, A) \oplus \mathrm{Reg}_+(B, A^2) | \\
&\quad (\alpha \otimes \varepsilon) * \alpha(\mathrm{m} \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \alpha) * \alpha(\mathrm{id} \otimes \mathrm{m}) \\
&\quad (\mathrm{id} \otimes \Delta)\beta * (\eta \otimes \beta) = (\Delta \otimes \mathrm{id})\beta * (\beta \otimes \mathrm{id})\rho \\
&\quad \Delta\alpha * \beta\mathrm{m} = (\beta \otimes \varepsilon) * \mu_2(\mathrm{id} \otimes \beta) * (\alpha \otimes \mathrm{id})\rho_2 * (\eta \otimes \alpha)\},
\end{aligned}
$$

$$\begin{aligned} B^2(B, A, \mu, \rho) &= \{(\alpha, \beta) \in \mathrm{Reg}_+(B^2, A) \oplus \mathrm{Reg}_+(B, A^2) | \\ & \exists \gamma \in \mathrm{Reg}_+(B, A) : \\ & \alpha = (\gamma \otimes \varepsilon) * \gamma^{-1} \mathrm{m}_B * \mu(\mathrm{id}_B \otimes \gamma), \\ & \beta = (\gamma^{-1} \otimes \mathrm{id}_A)\rho \otimes \Delta_A \gamma * (\eta \otimes \gamma^{-1})\}. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{H}^1(B, A, \mu, \rho) &= \mathrm{Z}^1(B, A)/\mathrm{B}^1(B, A) \simeq \mathrm{Aut}(A\#B), \\ \mathrm{H}^2(B, A, \mu, \rho) &= \mathrm{Z}^2(B, A)/\mathrm{B}^2(B, A) \simeq \mathrm{Opext}(B, A). \end{aligned}$$

Chapter 2

Abelian matched pairs of Hopf algebras

Here we consider pairs of Hopf algebras (T, N) together with a left action $\mu: T \otimes N \longrightarrow N$, $\mu(t \otimes n) = t(n)$, and a right action $\nu: T \otimes N \longrightarrow T$, $\nu(t \otimes n) = t^n$. Then we have the twisted switch

$$\tilde{\sigma} = (\mu \otimes \nu)\Delta_{T \otimes N}: T \otimes N \longrightarrow N \otimes T$$

or in shorthand $\tilde{\sigma}(t \otimes n) = t_1(n_1) \otimes t_2^{n_2}$, which in case of trivial actions reduces to the ordinary switch $\sigma: T \otimes N \longrightarrow N \otimes T$, $\sigma(t \otimes n) = n \otimes t$.

Definition 2.0.2 (Kas, IX.2.2) *Such a configuration* (T, N, μ, ν) *is called a* **matched pair** *if*

1. *N is a left T-module coalgebra, i.e. μ is a coalgebra map,*

2. *T is an right N-module coalgebra, i.e. ν is a coalgebra map,*

3. *N is a left T-module algebra with respect to the twisted left action $\mu_2 = (\mathrm{id} \otimes \mu)(\tilde{\sigma} \otimes \mathrm{id}): T \otimes N \otimes N \longrightarrow N$ ($\mu_2(t \otimes n \otimes m) = t_1(n_1) \otimes t_2^{n_2}(m)$), in the sense that the following diagrams*

$$\begin{array}{ccc} T\otimes N\otimes N & \xrightarrow{\mathrm{id}\otimes \mathrm{m}_N} & T\otimes N \\ \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu} \\ N\otimes N & \xrightarrow[\mathrm{m}_N]{} & N \end{array} \qquad \begin{array}{ccc} T\otimes k & \xrightarrow{\mathrm{id}\otimes \eta_N} & T\otimes N \\ \downarrow{\scriptstyle \varepsilon_T\otimes \mathrm{id}} & & \downarrow{\scriptstyle \mu} \\ k & \xrightarrow[\eta_N]{} & N \end{array}$$

commute, i.e.: $t(mn) = t_1(n_1)t_2^{n_2}(m)$, $t(1_N) = \varepsilon(t)1_N$.

4. *T is a right N-module algebra with respect to the twisted right action* $\nu_2 = (\nu\otimes \mathrm{id})(\mathrm{id}\otimes\tilde{\sigma}): T\otimes T\otimes N \longrightarrow T\otimes T$, *in a sense that the diagrams*

$$\begin{array}{ccc} T\otimes T\otimes N & \xrightarrow{\mathrm{m}_T\otimes \mathrm{id}} & T\otimes N \\ \downarrow{\scriptstyle \nu_2} & & \downarrow{\scriptstyle \nu} \\ T\otimes T & \xrightarrow[\mathrm{m}_T]{} & T \end{array} \qquad \begin{array}{ccc} k\otimes N & \xrightarrow{\eta\otimes \mathrm{id}} & T\otimes N \\ \downarrow{\scriptstyle \mathrm{id}\otimes \varepsilon} & & \downarrow{\scriptstyle \nu} \\ k & \xrightarrow[\eta_T]{} & T \end{array}$$

commute, i.e. $(ts)^n = t^{s_2(n_2)}s_1^{n_1}$, $(1_T)^n = \varepsilon(n)1_T$.

5. $\nu(t_1\otimes n_1)\otimes \mu(t_2\otimes n_2) = \nu(t_2\otimes n_2)\otimes\mu(t_1\otimes n_1)$

The last condition is needed to guarantee compatibility of multiplication and comultiplication in the bismash product $N \bowtie T$. Note that it is automatically satisfied when both N and T are cocommutative, in which case (T, N, μ, ν) is called an **abelian matched** pair. We will be considering abelian matched pairs exclusively.

Remark. *The matched pairs of finite groups and of Lie groups are considered in [Tk] and [Mj] respectively.*

The bicross product $(N \bowtie T, \mathrm{m}, \Delta, \eta, \varepsilon, S)$ is the tensor product coalgebra $N\otimes T$, with unit $\eta_{N\otimes T}: k \longrightarrow N\otimes T$, multiplication

$$\mathrm{m} = (\mathrm{m}\otimes\mathrm{m})(\mathrm{id}\otimes\tilde{\sigma}\otimes\mathrm{id}): N\otimes T\otimes N\otimes T \longrightarrow N\otimes T,$$

in short $(n \otimes t)(m \otimes s) = nt_1(m_1) \otimes t_2^{m_2} s$, and antipode

$$S = \tilde{\sigma}(S \otimes S)\sigma : N \otimes T \longrightarrow N \otimes T,$$

i.e $S(n \otimes t) = S(t_2)(S(n_2)) \otimes S(t_1)^{S(n_1)}$.

For a proof that $N \bowtie T$ is a Hopf algebra we refer to [Kas].

To avoid ambiguity we write $n \bowtie t$ for $n \otimes t$. We also identify N and T with the Hopf subalgebras $N \bowtie k$ and $k \bowtie T$ of $N \bowtie T$, i.e. $n \equiv n \bowtie 1$ and $t \equiv 1 \bowtie t$. In this sense we have $n \bowtie t = nt$ and $tn = t_1(n_1) \cdot t_2^{n_2}$.

Remark. *If the action $\nu: T \otimes N \longrightarrow T$ is trivial, then the bismash product $N \bowtie T$ becomes a* **smash** *(or semi-direct) product and is denoted by $N \rtimes T$.*

An action $\mu: T \otimes N \longrightarrow N$, is compatible with the trivial action, i.e. $(T, N, \mu, \mathrm{id} \otimes \varepsilon)$ is a matched pair, if and only if N is a T-module bialgebra and $\mu(t_1 \otimes n) \otimes t_2 = \mu(t_2 \otimes n) \otimes t_1$. Note that the last condition is trivially satisfied if T is cocommutative.

The twisted actions can be extended by induction to higher tensor powers. We define $\mu_p: T \otimes N^{\otimes p} \longrightarrow N^{\otimes p}$ and $\nu_q: T^{\otimes q} \otimes N \longrightarrow T^{\otimes q}$ as follows

$$\mu_{p+1}(t \otimes n \otimes \mathbf{m}) = \sum \mu(t_1 \otimes n_1) \otimes \mu_p(\nu(t_2 \otimes n_2) \otimes \mathbf{m}),$$
$$\nu_{q+1}(\mathbf{t} \otimes s \otimes n) = \sum \nu_q(\mathbf{t} \otimes \mu(s_2 \otimes n_2) \otimes \nu(s_1 \otimes n_1)),$$

and $\mu_1 = \mu$, $\nu_1 = \nu$.

Observe $F\mu_{p+1} = \mu_p(\mathrm{id} \otimes F)$, when $F = \mathrm{id} \otimes \ldots \otimes \mathrm{m} \otimes \ldots \otimes \mathrm{id}$ and similarly $G\nu_{q+1} = \nu_{q+1}(\mathrm{id} \otimes G)$, when $G = \mathrm{id} \otimes \ldots \otimes \mathrm{m} \otimes \ldots \otimes \mathrm{id}$.

From now on, as indicated in 1.1, we will usually omit subscripts when using the Sweedler notation for cocommutative coalgebras.

Lemma 2.0.3 (Ma3,Proposition 2.3) *Let (T, N, μ, ν) be an abelian matched pair.*

1. *A left T-module, left N-module V is a left $N \bowtie T$-module if and only if $t(n(v)) = t(n)(t^n(v))$.*

2. *A right T module, right N-module W is a right $N \bowtie T$-module if and only if $(v^t)^n = (v^{t(n)})^{t^n}$.*

3. Let V be a left T-module and W be a right N-module. Then

(a) $N \otimes V$ is a left $N \bowtie T$ module with N-action on the first factor and T action given by $t(n \otimes v) = t_1(n_1) \otimes t_2^{n_2}(v)$.

(b) $W \otimes T$ is a right $N \bowtie T$-module with T-action on the right factor and N-action given by $(w \otimes t)^n = w^{t_2(n_2)} \otimes t_1^{n_1}$. Moreover $W \otimes T$ is a left $N \bowtie T$-module by twisting the action via the antipode of $N \bowtie T$.

(c) Regard $(W \otimes T) \otimes (N \otimes V)$ as a left $N \bowtie T$-module via the diagonal action, i.e.

$$(nt)(w \otimes s \otimes m \otimes v) = w^{(sS(t))(S(n))} \otimes (sS(t))^{S(n)} \otimes nt(m) \otimes t^m(v).$$

and let $(N \bowtie T) \otimes (W \otimes V)$ be a left $N \bowtie T$ module with action on the first factor. Then the map $\psi: (N \bowtie T) \otimes W \otimes V \longrightarrow (W \otimes T) \otimes (N \otimes V)$ defined by

$$\psi((n \bowtie t) \otimes w \otimes v) = w^{S(t)(S(n))} \otimes S(t)^{S(n)} \otimes n \otimes t(v)$$

is an isomorphism of $N \bowtie T$-modules. In particular $(W \otimes T) \otimes (N \otimes V)$ is a free left $N \bowtie T$-module in which any basis of the vector space $(W \otimes k) \otimes (k \otimes V)$ is a $N \bowtie T$-free basis.

Remark. Note that the inverse of ψ is given by

$$\psi^{-1}((w \otimes t) \otimes (n \otimes v)) = (n \bowtie S(t^n)) \otimes (w^{t(n)} \otimes t^n(v)).$$

Consider T^i a right T-module via ν_i and N^j a T-module via μ_j. Then we can equip $T^{i+i} \otimes N^{j+1}$ with an $N \bowtie T$-module structure in accordance with part 3(c) of the Lemma above, i.e. $(nt)(\mathbf{r} \otimes \mathbf{k}) = \mathbf{r}^{(S(t))(S(n))} \otimes S(t)^{S(n)} \otimes n \cdot t(m) \otimes t^m(\mathbf{k})$.

Corollary 2.0.4 *The map $\psi: (N \bowtie T) \otimes T^i \otimes N^j \longrightarrow T^{i+1} \otimes N^{j+1}$, defined by $\psi((nt) \otimes (\mathbf{r} \otimes s \otimes m \otimes \mathbf{k})) = \mathbf{r}^{(sS(t))(S(n))} \otimes (sS(t))^{S(n)} \otimes n \otimes t(\mathbf{k})$, is an isomorphism of $N \bowtie T$-modules.*

According to Lemma 2.0.3 we have a square of 'free' functors between monoidal categories

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{F_T} & {}_T\mathcal{V} \\ {\scriptstyle F_N}\downarrow & & \downarrow{\scriptstyle \tilde{F}_N} \\ {}_N\mathcal{V} & \xrightarrow[\tilde{F}_T]{} & {}_{N\bowtie T}\mathcal{V} \end{array}$$

each with a corresponding tensor preserving right adjoint forgetful functor.

The two resulting comonads on ${}_{N\bowtie T}\mathcal{V}$

$$\tilde{\mathbf{G}}_T = (\tilde{G}_T, \delta_T, \epsilon_T)$$

with $\tilde{G}_T = \tilde{F}_T\tilde{U}_T$, $\delta_T(t \otimes x) = t \otimes 1 \otimes x$, $\epsilon_T(t \otimes x) = tx$, and

$$\tilde{\mathbf{G}}_N = (\tilde{G}_N, \delta_N, \epsilon_N)$$

with $\tilde{G}_N = \tilde{F}_N\tilde{U}_N$, $\delta_N(n \otimes x) = n \otimes 1 \otimes x$, $\epsilon_N(n \otimes x) = nx$, satisfy a distributive law [Barr]

$$\tilde{\sigma} : \tilde{G}_T\tilde{G}_N \longrightarrow \tilde{G}_N\tilde{G}_T$$

given by $\tilde{\sigma}(t \otimes n \otimes -) = \tilde{\sigma}(t \otimes n) \otimes - = t_1(n_1) \otimes t_2^{n_2} \otimes -$. It is easy to see that the equations for a distributive law

$$\tilde{G}_N\delta_T \cdot \tilde{\sigma} = \tilde{\sigma}\tilde{G}_T \cdot \tilde{G}_T\tilde{\sigma} \cdot \delta_T\tilde{G}_N \quad , \quad \delta_N\tilde{G}_T \cdot \tilde{\sigma} = \tilde{G}_N\tilde{\sigma} \cdot \tilde{\sigma}\tilde{G}_N \cdot \tilde{G}_T\delta_N,$$

and

$$\epsilon_N\tilde{G}_T \cdot \tilde{\sigma} = \tilde{G}_T\epsilon_N \quad , \quad \tilde{G}_N\epsilon_T \cdot \tilde{\sigma} = \epsilon_T\tilde{G}_N$$

are satisfied.

Then [Barr, Th. 2.2] the composite

$$\mathbf{G} = \mathbf{G}_N \circ_{\tilde{\sigma}} \mathbf{G}_T$$

with $G = (G_NG_T$, $\delta = G_N\tilde{\sigma}G_T \cdot \delta_N\delta_T$ and $\epsilon = \epsilon_N\epsilon_T)$ is again a comonad. Moreover, $\mathbf{G} = \mathbf{G}_{N\bowtie T}$.

2.1 Dual matched pairs

Let (T, N, μ, ν) be an abelian matched pair. Our aim is to define a dual matched pair (N, T, ν_S, μ_S).

Define $\mu_S = S\mu(S \otimes S)\sigma\colon N \otimes T \longrightarrow N$ and $\nu_S = S\nu(S \otimes S)\sigma\colon N \otimes T \longrightarrow T$. We abbreviate

$$\begin{aligned} \mu(t \otimes n) &= t(n), \\ \nu(t \otimes n) &= t^n, \\ \nu_S(n \otimes t) &= n[t], \\ \mu_S(n \otimes t) &= n^t, \end{aligned}$$

for $n \in N$, $t \in T$. Note that in our cocommutative setting $S \circ S = \mathrm{id}$ and hence $\mu_{S\,S} = \mu$ and $\nu_{S\,S} = \nu$.

Proposition 2.1.1 *Let (T, N, μ, ν) be an abelian matched pair and let μ_S and ν_S be as above. Then*

- *(N, T, ν_S, μ_S) is a matched pair,*
- *$\mu_S(\mu * \nu) = \varepsilon \otimes \mathrm{id}$, i.e. $t(m)[t^m] = \varepsilon(m)t$,*
- *$\nu_S(\mu * \nu) = \mathrm{id} \otimes \varepsilon$, i.e. $(t(m))^{t^m} = \varepsilon(t)m$.*

We will need the following Lemma in the proof of this Proposition.

Lemma 2.1.2

$$\begin{aligned} S(t(n)) &= t^{n_1}(S(n_2)), \\ S(t^n) &= S(t_2)^{t_1(n)}. \end{aligned}$$

Proof.

$$
\begin{aligned}
t^{n_1}(S(n_2)) &= S(t_1(n_1))t_2(n_2)t_3^{n_3}(S(n_4)) \\
&= S(t_1(n_1))t_2(n_2S(n_3)) \\
&= S(t(n)),
\end{aligned}
$$

$$
\begin{aligned}
S(t_2)^{t_1(n)} &= S(t_4)^{t_3(n_3)}t_2^{n_2}(S(t_1^{n_1})) \\
&= (S(t_3)t_2)^{n_2}S(t_1^{n_1}) \\
&= S(t^n). \blacksquare
\end{aligned}
$$

Proof (of the Proposition 2.1.1). By the Lemma above we have

$$
\begin{aligned}
\mu_S(n \otimes t) &= n^t = S(S(n)^{S(t)}) \\
&= S(t)^{S(n)}(SS(n)) \\
&= S(t)^{S(n)}(n)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_S(n \otimes t) &= n[t] = S(S(t)^{S(n)}) \\
&= SS(t)^{S(t)(S(n))} \\
&= t^{S(t)(S(n))}.
\end{aligned}
$$

It is clear that μ_S and ν_S are coalgebra maps. We have to show that they preserve the algebra structures in the twisted sense. This is done as follows.

- $$
\begin{aligned}
n[ts] &= (ts)^{S(ts)(S(n))} \\
&= t^{s(S(ts)(S(n)))}s^{S(ts)(S(n))} \\
&= t^{(sS(s)S(t))(S(n))}s^{S(s)(S(t)(S(n)))} \\
&= t^{S(t)(S(n))}s^{S(s)(S(n^t))} \\
&= n[t]n^t[s],
\end{aligned}
$$

•

$$\begin{aligned}(nm)^t &= S(t)^{S(nm)}(nm)\\ &= S(t)^{S(nm)}(n)(S(t)^{S(nm)})^n(m)\\ &= S(m[t])^{S(n)}(n)(S(t)^{S(m)S(n)n})(m)\\ &= n^{m[t]}m^t.\end{aligned}$$

We conclude the proof by the following computations.

•

$$\begin{aligned}t(m)[t^m] &= (t^m)^{S(t^m)(S(t(m))}\\ &= (t^m)^{(S(t)^{t(m)})(t^m(S(m)))}\\ &= (t^m)((S(t)^{t(m)}t^m)(S(m))))\\ &= (t^m)^{(S(t)t)^m(S(m))}\\ &= (t^m)^{S(m)} = \varepsilon(m)t,\end{aligned}$$

•

$$\begin{aligned}t(m)^{t^m} &= S(t^m)^{S(t(m))}(t(m))\\ &= (S(t)^{t(m)})^{t^m(S(m))}(t(m))\\ &= S(t)^{t(m)t^m(S(m))}(t(m))\\ &= S(t)^{t(mS(m))}(t(m))\\ &= S(t)(t(m)) = \varepsilon(t)m.\end{aligned}$$

■

Remark. The inverse of the twisted switch is

$$\tilde{\sigma}^{-1} = (\nu_S \otimes \mu_S)\Delta_{N\otimes T} : N \otimes T \longrightarrow T \otimes N$$

given by $\tilde{\sigma}^{-1}(n \otimes t) = n_1[t_1] \otimes n_2^{t_2}$, and induces the inverse distributive law

$$\tilde{\sigma}^{-1} : G_N G_T \longrightarrow G_T G_N.$$

We conclude the section by some equalities, which are about the interplay of actions (μ, ν) and their duals (ν_S, μ_S) when products are involved. These equalities are very helpful when doing explicit computations.

Lemma 2.1.3

$$\begin{aligned}
(tn[s])^{n^s m} &= t^{n \cdot s(m)} s^m && (2.1)\\
(tn[s])(n^s m) &= t(n) \cdot (t^n s)(m) && (2.2)\\
(nt(m))^{t^m s} &= n^{t \cdot m[s]} m^s && (2.3)\\
(nt(m))[t^m s] &= n[t] \cdot (n^t m)[s] && (2.4)
\end{aligned}$$

Proof. It is sufficient to prove (2.1) and (2.2), since (2.3) and (2.4) can be obtained by applying (2.1) and (2.2) to (ν_S, μ_S) and $(\mu_{SS}, \nu_{SS}) = (\mu, \nu)$. We do the proof by computations:

$$\begin{aligned}
(tn[s])^{n^s m} &= t^{n[s](n^s m)} n[s]^{n^s m}\\
&= t^{n[s](n^s) \cdot (n[s]^{n^s})(m)} (n[s]^{n^s})^m\\
&= t^{n(s) \cdot m} s^m
\end{aligned}$$

and

$$\begin{aligned}
(tn[s])(n^s m) &= (tn[s])(n^s) \cdot (tn[s])^{n^s}(m)\\
&= t(n[s](n^s)) \cdot (t^{n[s](n^s)} \cdot n[s]^{n^s})(m)\\
&= t(n) \cdot (t^n \cdot s)(m). \blacksquare
\end{aligned}$$

2.2 Matched pair cohomology

Let $H = N \bowtie T$ be a bismash product of an abelian matched pair of Hopf algebras and let the algebra A be a left N and a right T-module such that it is a left H-module via $nt(a) = n(a^{S(t)})$, i.e. $(n(a))^{S(t)} = (t(n))(a^{S(t^n)})$.

Note that $\mathrm{Hom}(T^p, A)$ becomes a left N-module via $n(f)(\mathbf{t}) = n(f(\nu_p(\mathbf{t}, n)))$ and $\mathrm{Hom}(N^q, A)$ becomes a right T-module via $f^t(\mathbf{n}) = (f(\mu_q(t, \mathbf{n})))^t = S(t)(f(\mu_q(t, \mathbf{n})))$.

We have a double simplicial complex $G_T^p G_N^q(k) = (T^p \otimes N^q)_{p,q}$, $p, q \geq 1$ of free H-modules, with horizontal face operators $\mathrm{id} \otimes d_{N*}: T^p \otimes N^{q+1} \longrightarrow T^p \otimes N^q$, vertical face operators $d_{T*} \otimes \mathrm{id}: T^{p+1} \otimes N^q \longrightarrow T^p \otimes N^q$, horizontal degeneracies $\mathrm{id} \otimes s_{N*}: T^p \otimes N^q \longrightarrow T^p \otimes N^{q+1}$ and vertical degeneracies $s_{N*}: T^p \otimes N^q \longrightarrow T^p \otimes N^{q+1}$. More precisely

$$\begin{aligned}
d_{Ni} &= \begin{cases} \mathrm{id}^i \otimes \mathrm{m} \otimes \mathrm{id}^{q-i-1} & , 0 \leq i \leq q-1 \\ \mathrm{id}^q \otimes \varepsilon & , i = q \end{cases}, \\
s_{Ni} &= \mathrm{id}^i \otimes \eta \otimes \mathrm{id}^{q-i},\ 0 \leq i \leq q-1, \\
d_{Tj} &= \begin{cases} \mathrm{id}^{p-j-1} \otimes \mathrm{m} \otimes \mathrm{id}^j & , 0 \leq j \leq p-1 \\ \varepsilon \otimes \mathrm{id}^p & , j = p \end{cases}, \\
s_{Tj} &= \mathrm{id}^{p-j} \otimes \eta \otimes \mathrm{id}^j,\ 0 \leq j \leq p-1.
\end{aligned}$$

Note that these preserve the H-module structure on $T^p \otimes N^q$. Apply the functor ${}_H\mathrm{Reg}(\_, A): {}_H\mathcal{C}^{op} \longrightarrow \mathrm{Ab}$ to get a cosimplicial double complex of abelian groups, $\mathbf{B} = (B^{p,q})_{p,q \geq 1}$, where $B^{p,q} = {}_H\mathrm{Reg}(T^p \otimes N^q, A)$, coface operators are ${}_H\mathrm{Reg}(d_{N*}, A)$, ${}_H\mathrm{Reg}(d_{T*}, A)$ and codegeneracies are ${}_H\mathrm{Reg}(s_{N*}, A)$, ${}_H\mathrm{Reg}(s_{T*}, A)$.

We have an isomorphism

$$\begin{aligned}
{}_H\mathrm{Reg}(T^p \otimes N^q, A) &\overset{{}_H\mathrm{Reg}(\psi, A)}{\longrightarrow} {}_H\mathrm{Reg}(H \otimes T^{p-1} \otimes N^{q-1}, A) \\
&\overset{\theta}{\longrightarrow} \mathrm{Reg}(T^{p-1} \otimes N^{q-1}, A).
\end{aligned}$$

The first isomorphism comes from Corollary 2.0.4 and the second is obvious. It induces an isomorphism of double complexes $(B^{p,q})_{p,q \geq 1} \xrightarrow{\sim} (C^{p,q})_{p,q \geq 0}$, where $C^{p,q} = C^{p,q}(N, T, A)$ is the abelian group of convolution invertible linear maps $f: N^{\otimes p} \otimes T^{\otimes q} \longrightarrow A$.

The vertical differentials $\delta_N: C^{p,q} \longrightarrow C^{p+1,q}$ and the horizontal differentials $\delta_T: C^{p,q} \longrightarrow C^{p,q+1}$ are transported from $\mathbf{B}$ and turn out to be the twisted Sweedler differentials on the N and T parts respectively. In the following paragraph we review these in more detail. We have the coface operators

$$\delta_{Ni} f(\mathbf{t} \otimes \mathbf{n}) = f(\mathbf{t} \otimes n_1 \otimes \ldots \otimes n_i n_{i+1} \otimes \ldots \otimes n_{q+1}),$$

for $i = 1, \ldots, q$,

$$\delta_{N0} f(\mathbf{t} \otimes \mathbf{n}) = n_1 \left(f(\nu_q(\mathbf{t} \otimes n_1) \otimes n_2 \otimes \ldots \otimes n_{p+1}) \right),$$

$$\delta_{Nq+1}(\mathbf{t} \otimes \mathbf{n}) = f(\mathbf{t} \otimes n_1 \otimes \ldots \otimes n_q)\varepsilon(n_{q+1}),$$

where $\mathbf{t} \in T^{\otimes p}$ and $\mathbf{n} = n_1 \otimes \ldots \otimes n_{q+1} \in N^{\otimes q+1}$,

$$\delta_{Tj} f(\mathbf{t} \otimes \mathbf{n}) = f(t_{p+1} \otimes \ldots \otimes t_{j+1}t_j \otimes \ldots \otimes t_1 \otimes \mathbf{n}),$$

for $j = 1, \ldots, p$,

$$\delta_{T0} f(\mathbf{t} \otimes \mathbf{n}) = \left(f(t_{p+1} \otimes \ldots \otimes t_2 \otimes \mu_p(t_1 \otimes \mathbf{n}))\right)^{t_1},$$

$$\delta_{Tp+1} f(\mathbf{t} \otimes \mathbf{n}) = \varepsilon(t_{p+1}) f(t_p \otimes \ldots \otimes t_1 \otimes \mathbf{n}),$$

where $\mathbf{t} = t_1 \otimes \ldots t_{q+1} \in T^{\otimes q+1}$ and $\mathbf{n} \in N^q$. And then the differentials in the associated double cochain complex are the alternating convolution products

$$\delta_N f = \delta_{N0} f * \delta_{N1} f^{-1} * \ldots * \delta_{Nq+1} f^{\pm 1}$$

and

$$\delta_T f = \delta_{T0} f * \delta_{T1} f^{-1} * \ldots * \delta_{Tp+1} f^{\pm 1}.$$

Consider the associated normalized double complex, whose (p, q)th term, denoted by $C_+^{p,q} = \mathrm{Reg}_+(T^p \otimes N^q, A)$, is the intersection of the degeneracy operators, that is, it is the abelian group of convolution invertible maps $f\colon T^p \otimes N^q \longrightarrow A$, with the property that $f(t_p \otimes \ldots \otimes t_1 \otimes n_1 \otimes \ldots n_q) = \varepsilon(t_p) \ldots \varepsilon(n_q)$, whenever one of $t_i \in k$ or one of $n_j \in k$.

Furthermore, replace the edges of $(C_+^{p,q})_{p,q \geq 0}$ by zeroes to obtain $\mathbf{D}$, i.e. $D^{p,q} = C^{p,q}$ if $p, q \geq 1$ and $D^{p,q} = 0$ if either $p = 0$ or $q = 0$.

We denote the cohomology of the total complex $\mathrm{H}^{*+1}(\mathrm{Tot}\mathbf{D})$ by $\mathcal{H}^*(N, T, A)$ and call it the cohomology of the matched pair (T, N, μ, ν) with coefficients in the algebra A.

The cocycles shall be denoted by $\mathcal{Z}^i(T,N,A)$ and coboundaries by $\mathcal{B}^i(T,N,A)$. Note that these are i-tuples of maps $(f_j)_{1\leq j\leq i}$, $f_j\colon T^{\otimes(i+1-j)}\otimes N^{\otimes j}\longrightarrow A$ that satisfy certain conditions.

We introduce the subgroups $\mathcal{Z}^i_p(T,N,A)\leq \mathcal{Z}^i(T,N,A)$, that are spanned by i-tuples in which the f_j's are trivial for $j\neq p$ and subgroups $\mathcal{B}^i_p=\mathcal{Z}^i_p\cap\mathcal{B}^i\subset\mathcal{B}_i$. These give rise to subgroups of cohomology groups $\mathcal{H}^i_p=\mathcal{Z}^i_p/\mathcal{B}^i_p\simeq(\mathcal{Z}^i_p+\mathcal{B}^i)/\mathcal{B}^i\subseteq\mathcal{H}^i$ which have a nice interpretation when $i=2$ and $p=1,2$, see Section 2.4.1.

2.2.1 Interpretation of degree 2 cohomology

The 2-cocycles are pairs of maps (α,β), $\alpha\colon T\otimes T\otimes N\longrightarrow A$, $\beta\colon T\otimes N\otimes N\longrightarrow A$ that satisfy certain conditions. These conditions are

- α is a Sweedler 2-cocycle as a map $\alpha\colon T\otimes T\longrightarrow \mathrm{Hom}(N,A)$, hence gives rise to a cleft T-comodule algebra extension $E_\alpha=\mathrm{Hom}(N,A)\longrightarrow\mathrm{Hom}(N,A)\#_\alpha T\longrightarrow T$
- β is a Sweedler 2-cocycle as a map from $N\otimes N$ to $\mathrm{Hom}(T,A)$ and hence gives rise to the extension $E_\beta=\mathrm{Hom}(T,A)\longrightarrow\mathrm{Hom}(T,A)\#_\beta N\longrightarrow N$
- α and β are compatible in the sense that $\delta_N\alpha*\delta_T\beta=\varepsilon$.

Note also that there is a map $\Psi\colon\mathrm{Hom}(N,A)\#_\alpha T\otimes\mathrm{Hom}(T,A)\#_\beta N\longrightarrow A$, given by $\Psi(f\#t\otimes g\#n)=f(n)g(t)$.

Our aim is to represent cocycle pairs as pairs of compatible cleft comodule algebra extensions. Given extensions $(X)\colon\mathrm{Hom}(N,A)\longrightarrow X\longrightarrow T$ and $(Y)\colon\mathrm{Hom}(T,A)\longrightarrow Y\longrightarrow N$, we would like to express this compatibility in terms of a map $\Psi\colon X\otimes Y\longrightarrow A$.

One way to do this is as follows:

Suppose we have a pair of extensions as above and let $\chi_T\colon T\longrightarrow X$ and $\chi_N\colon N\longrightarrow Y$ be the cleftness maps. Then $x_0\chi_T^{-1}(x_1)\in\mathrm{Hom}(N,A)\subseteq X$ and $y_0\chi_N^{-1}(y_1)\in\mathrm{Hom}(T,A)\subseteq Y$. Get a map $\Psi\colon X\otimes Y\longrightarrow A$ by $\Psi(x,y)=(x_0\chi_T^{-1}(x_1))(y_2)\cdot(y_0\chi_N^{-1}(y_1))(x_2)$. Then define a pair of maps

$$\Psi_N=\Psi(\mathrm{id}\otimes \mathrm{m}_Y(\chi_N\otimes\chi_N))\colon X\otimes N\otimes N\longrightarrow A,$$

i.e $\Psi(x, n, m) = \Psi(x, \chi_N(n)\chi_N(m))$ and

$$\Psi_T = \Psi(\mathrm{m}_X(\chi_T \otimes \chi_T) \otimes \mathrm{id}): T \otimes T \otimes Y \longrightarrow A.$$

Since T acts on $N \otimes N$, via the twisted action, we can define $\Psi_N^t(s \otimes n \otimes m) = \Psi_N(s \otimes \mu_2(t \otimes n \otimes m))$ and similarly $n(\Psi_T)(t, s, m) = \Psi_T(\nu_2(t \otimes s \otimes n) \otimes m)$.

Definition 2.2.1 *We say the extensions (X) and (Y) are compatibly cleft, if the following is satisfied:*

$$\Psi(xx' \otimes yy') = \Psi_N(x_0 \otimes y_1 \otimes y_1')\Psi_N^{x_1}(x_0' \otimes y_2 \otimes y_2')y_3'(\Psi_T)(x_2 \otimes x_1' \otimes y_0)\Psi_T(x_3 \otimes x_2' \otimes y_0').$$

We say that two pairs of compatibly cleft extensions $((X), (Y))$ and $((X'), (Y'))$ are equivalent, if (X) is equivalent to (X') via $f_x: X \longrightarrow X'$, (Y) is equivalent to (Y') via $f_y: Y \longrightarrow Y'$ and $\Phi = \Phi'(f_x \otimes f_y): X \otimes Y \overset{f_x \otimes f_y}{\longrightarrow} X' \otimes Y' \overset{\Phi'}{\longrightarrow} A$.

Proposition 2.2.2 *The group $\mathcal{H}^2(T, N, A)$ classifies pairs of compatibly cleft extensions $((X), (Y))$.*

Proof. The result follows from the way the definition of compatible cleftness was chosen. If (α, β) is a 2-cocycle, then the pair of extensions E_α and E_β (see the beginning of this section) are compatibly cleft (the compatible cleftness condition is just the condition $\delta_N\alpha * \delta_T\beta = \varepsilon$ in disguise).

On the other hand, if we are given extensions (X, χ_T), (Y, χ_N), then define a cocycle pair by $\alpha(t \otimes t') = \chi_T(t_1)\chi_T(t_1')\chi_T^{-1}(t_2t_2')$, $\beta(n \otimes n') = \chi_N(n_1)\chi_N(n_1')\chi_N^{-1}(n_2n_2')$ and note that $((X), (Y)) \sim ((E_\alpha), (E_\beta))$.

Similarly as above we also see that there is a bijective correspondence between trivial pairs of extensions and coboundaries. ■

2.3 Singer pairs vs. matched pairs

Definition 2.3.1 *We say that an action $\mu: A \otimes M \longrightarrow M$ is locally finite, if every orbit $A(m) = \{a(m) | a \in A\}$ is finite dimensional.*

Lemma 2.3.2 (Mo, Lemma 1.6.4) *Let A be an algebra and C a coalgebra.*

1. *If M is a right C-comodule via $\rho: M \longrightarrow M \otimes C$, $\rho(m) = m_0 \otimes m_1$, then M is a left C^*-module via $\mu: C^* \otimes M \longrightarrow M$, $\mu(f \otimes m) = f(m_1)m_0$.*

2. *Let M be a left A-module via $\mu: A \otimes M \longrightarrow M$. Then M is a right A° comodule if the action μ is locally finite. The coaction $\rho: M \longrightarrow M \otimes A^\circ$ is given by $\rho(m) = \sum f_i \otimes m_i$, where $\{m_i\}$ is a basis for $A(m)$ and $f_i \in A^\circ \subseteq A^*$ are coordinate functions of $a(m)$, i.e. $a(m) = \sum f_i(a)m_i$.*

Let (T, N, μ, ν) be an abelian matched pair and suppose μ is locally finite. Then the Lemma above gives a coaction $\rho: N \longrightarrow N \otimes T^\circ$, $\rho(n) = n_N \otimes n_{T^\circ}$, such that $t(n) = \sum n_N \cdot n_{T^\circ}(t)$.

There is a left action $\nu': N \otimes T^* \longrightarrow T^*$ given by pre-composition, i.e. $\nu'(n \otimes f)(t) = f(t^n)$. If μ is locally finite, it is easy to see that ν' restricts to $T^\circ \subseteq T^*$.

Lemma 2.3.3 (Ma3, Lemma 4.1, Corollary 4.2) *The quadruple (N, T°, ν', ρ) forms an abelian Singer pair and the category ${}_N\mathcal{V}^{T^\circ}$ is a full subcategory of ${}_{N \bowtie T}\mathcal{V}$, consisting of the objects which are locally finite T-modules.*

There is a natural correspondence from the set of the structures (μ, ν) of a matched pair on (T, N) such that μ is locally finite, to the set of structures (α, ρ) of a Singer pair on (N, T°). The correspondence is injective if T is proper (i.e. T° separates points).

Remark. *The correspondence is clearly bijective if T is finite dimensional.*

On the other hand, if we start with an abelian Singer pair (B, A, ω, ρ), we can obtain a matched pair (A°, B).

Observe that there is a right action $\nu: A^* \otimes B \longrightarrow A^*$ given by pre-composition, i.e. $f^b(a) = f(b(a))$ and note that since A is a B-module algebra this action restricts to A° ($f^b(aa') = f(b(aa')) = f(b_1(a))f(b_2(a')) = f^{b_1}(a)f^{b_2})(a')$).

There is also an action $A^* \otimes B \longrightarrow B$, induced by ρ, i.e. $f(b) = f(b_A)b_B$ (see Lemma 2.3.2). Observe also that the action is locally finite, since $A^*(b)$ is spanned by $\{b_B\}$.

Proposition 2.3.4 *The pair of Hopf algebras (A°, B) together with a pair of actions (μ, ν) as described above is a matched pair.*

Proof. The condition that A is a B-module algebra shows that A° is a B-module coalgebra, i.e. ν is a coalgebra map: $\Delta(f^b)(a \otimes a') = f(b(aa')) = f(b_1(a)b_2(a')) = (f_1^{b_1} \otimes f_2^{b_2})(a \otimes a')$. Similarly the condition that A is a B-module coalgebra in the twisted sense, shows that A° is a B-module algebra via the twisted action.

The condition that ρ makes B an A-comodule coalgebra shows that B is an A° module coalgebra: $\Delta(b^f) = \delta(f(b_A)b_B) = f(b_A)b_{B1} \otimes b_{B2} = f(b_{1A}b_{2A}) \otimes b_{1B} \otimes b_{2B} = (\Delta f)(b_{1A}) \otimes b_{2A}) = b_1^{f_1} \otimes b_2^{f_2}$. Similarly we see that B is A°-module algebra in a twisted sense. ■

2.4 Comparison of Singer and matched pair cohomologies

Let (T, N, μ, ν) be an abelian matched pair of Hopf algebras, with μ locally finite. Let (N, T°, ν', ρ) be the Singer pair associated to it (see the section above).

Via the embedding $T^{\circ j} = (T^j)^\circ \subseteq (T^j)^*$ we can embed $\mathrm{Hom}(N^i, (T^\circ)^j) \subseteq \mathrm{Hom}(N^i, (T^j)^*) \simeq \mathrm{Hom}(T^j \otimes N^i, k)$. This induces an embedding $\mathrm{Reg}_{\eta,\varepsilon}(N^i, (T^\circ)^j) \subseteq \mathrm{Reg}_+(N^i \otimes T^j, k)$. A routine calculation shows that this preserves the differentials, i.e. that this gives an imbedding of double complexes. This embedding is an isomorphism in case T is finite dimensional.

There is no apparent reason for the embedding of complexes to induce an isomorphism of cohomology groups in general. It is our conjecture that this is not always the case.

In some cases we can compare the multiplication part of $\mathrm{H}^2(N, T^\circ)$ (see the following section) and $\mathcal{H}_2^2(N, T, k)$. We use the following lemma for this purpose.

Lemma 2.4.1 *Let (T, N, μ, ν) be an abelian matched pair with the action μ locally finite. If $f: T \otimes N^{\otimes i} \longrightarrow k$ is a convolution invertible map, such that $\delta_T f = \varepsilon$, then* *for each* $\mathbf{n} \in N^i$*, the map* $f_{\mathbf{n}} = f(\_, \mathbf{n}): T \longrightarrow k$ *lies in the finite dual* $T^\circ \subseteq T^*$.

We need the following description of finite duals.

Proposition 2.4.2 (DNR, Proposition 1.5.6) *Let* $A = (A, \mathrm{m}, \eta)$ *be an algebra and* $f \in A^*$*. Then the following assertions are equivalent.*

1. $f \in A^\circ$
2. $\mathrm{m}^*(f) \in A^\circ \otimes A^\circ$
3. $\mathrm{m}^*(f) \in A^* \otimes A^*$
4. $A(f)$ *is finite dimensional.*
5. f^A *is finite dimensional.*
6. $A(f)^A$ *is finite dimensional.*

Remark. *Here we are considering the left action* $\rightharpoonup: A \otimes A^* \longrightarrow A^*$, $a(f)(b) = f(ba)$ *and the right action* $\leftharpoonup: A^* \otimes A \longrightarrow A^*$, $f^a(b) = f(ab)$.

Proof (of the Lemma 2.4.1). By the proposition above it suffices to show that $T(f_{\mathbf{n}})$ is finite dimensional. Using the fact that $\delta_T f = \varepsilon$ we get $s(f_{\mathbf{n}})(t) = f_{\mathbf{n}}(ts) = \sum f_{\mathbf{n}_1}(s_1) f_{\mu_i(s_2 \otimes \mathbf{n}_2)}(t)$.

Let $\Delta(\mathbf{n}) = \sum_j \mathbf{n}'_j \otimes \mathbf{n}''_j$. The action $\mu_i: T \otimes N^i \longrightarrow N^i$ is locally finite, since $\mu: T \otimes N \longrightarrow N$ is, and hence we can choose a finite basis $\{\mathbf{m}_p\}$ for $\mathrm{Span}\{\mu_i(s \otimes \mathbf{n}''_j) | s \in T\}$. Now note that $\{f_{\mathbf{m}_p}\}$ is a finite set which spans $T(f_{\mathbf{n}})$. ■

Corollary 2.4.3 *If* (T, N, μ, ν) *is an abelian matched pair, with* μ *locally finite and* $(N, T^\circ, \omega, \rho)$ *is the corresponding Singer pair, then* $\mathcal{H}^1(T, N, k) = \mathrm{H}^1(N, T^\circ)$.

2.4.1 The multiplication and comultiplication parts of the second cohomology group of a Singer pair

Here we discuss in more detail the Hopf algebra extensions that have an "unperturbed" multiplication and those that have an "unperturbed" comultiplication, more precisely we look at two subgroups $\mathrm{H}^2_{\mathrm{m}}(B,A)$ and $\mathrm{H}^2_{\mathrm{c}}(B,A)$ of $\mathrm{H}^2(B,A)$, one generated by the cocycles with a trivial multiplication part and the other generated by the cocycles with a trivial comultiplication part. Let

$$\mathrm{Z}^2_{\mathrm{c}}(B,A) = \{\beta \in \mathrm{Reg}(B, A \otimes A) | (\eta\varepsilon, \beta) \in \mathrm{Z}^2(B,A)\}.$$

We shall identify $\mathrm{Z}^2_{\mathrm{c}}(B,A)$ with a subgroup of $\mathrm{Z}^2(B,A)$ via the injection $\beta \mapsto (\eta\varepsilon, \beta)$. Similarly let

$$\mathrm{Z}^2_{\mathrm{m}}(B,A) = \{\alpha \in \mathrm{Reg}(B \otimes B, A) | (\alpha, \eta\varepsilon) \in \mathrm{Z}^2(B,A)\}.$$

If

$$\mathrm{B}^2_{\mathrm{c}}(B,A) = \mathrm{B}^2(B,A) \cap \mathrm{Z}^2_{\mathrm{c}}(B,A) \text{ and } \mathrm{B}^2_{\mathrm{m}}(B,A) = \mathrm{B}^2(B,A) \cap \mathrm{Z}^2_{\mathrm{m}}(B,A).$$

then we define

$$\mathrm{H}^2_{\mathrm{c}}(B,A) = \mathrm{Z}^2_{\mathrm{c}}(B,A)/\mathrm{B}^2_{\mathrm{c}}(B,A)$$

and

$$\mathrm{H}^2_{\mathrm{m}}(B,A) = \mathrm{Z}^2_{\mathrm{m}}(B,A)/\mathrm{B}^2_{\mathrm{m}}(B,A).$$

The identification of $\mathrm{H}^2_{\mathrm{c}}(B,A)$ with a subgroup of $\mathrm{H}^2(B,A)$ is given by

$$\mathrm{H}^2_{\mathrm{c}}(B,A) \xrightarrow{\sim} (\mathrm{Z}^2_{\mathrm{c}}(B,A) + \mathrm{B}^2(B,A))/\mathrm{B}^2(B,A)) \leq \mathrm{H}^2(B,A),$$

and similarly for $\mathrm{H}^2_{\mathrm{m}} \leq \mathrm{H}^2$.

Note that in case T is finite dimensional $\mathrm{H}^2_{\mathrm{c}}(N,T^*) \simeq \mathcal{H}^2_2(T,N,k)$ and $\mathrm{H}^2_{\mathrm{m}}(N,T^*) \simeq \mathcal{H}^2_1(T,N,k)$.

Proposition 2.4.4 *Let (T,N,μ,ν) be an abelian matched pair, with μ locally finite and let (N,T°,ω,ρ) be the corresponding Singer pair. Then*

$$\mathrm{H}^2_{\mathrm{m}}(N,T^\circ) \simeq \mathcal{H}^2_1(T,N,k).$$

Proof. Observe that we have an inclusion $\mathrm{Z}^2_{\mathrm{m}}(N, T^\circ) = \{\alpha : N \otimes N \longrightarrow T^\circ | \partial\alpha = \varepsilon, \partial'\alpha = \varepsilon\} \subseteq \{\alpha : T \otimes N \otimes N \longrightarrow k | \delta_T\alpha = \varepsilon, \delta_N\alpha = \varepsilon\} = \mathcal{Z}^2_{\mathrm{l}}(T, N, k)$. The inclusion is in fact an equality by Lemma 2.4.1. Similarly the inclusion $\mathrm{B}^2_{\mathrm{m}}(N, T^\circ) \subseteq \mathcal{B}^2_{\mathrm{l}}(T, N, k)$ is an equality as well. ∎

Chapter 3

Generalized Kac sequence

We start by sketching a conceptual way to obtain a version of the Kac sequence for an arbitrary abelian matched pair of Hopf algebras. Since the homomorphisms involved cannot be explicitly described in this manner, we then proceed in the next section to give an explicit version of the low degree part of this sequence.

Theorem 3.0.5 *Let $H = N \bowtie T$, where (T, N) is an abelian matched pair, and let A be a commutative left H-module algebra. Then we have a long exact sequence*

$$\begin{aligned} 0 \;\longrightarrow\;& \mathrm{H}^1(H, A) \longrightarrow \mathrm{H}^1(T, A) \oplus \mathrm{H}^1(N, A) \longrightarrow \mathcal{H}^1(T, N, A) \\ \longrightarrow\;& \mathrm{H}^2(H, A) \longrightarrow \mathrm{H}^2(T, A) \oplus \mathrm{H}^2(N, A) \longrightarrow \mathcal{H}^2(T, N, A) \\ \longrightarrow\;& \mathrm{H}^3(H, A) \longrightarrow \mathrm{H}^3(T, A) \oplus \mathrm{H}^3(N, A) \longrightarrow \dots \end{aligned}$$

Proof. If $H = N \bowtie T$, then $\mathbf{Y} = (G_T^i G_N^j(k)) = (T^i \otimes N^j)$ is an H-free simplicial double complex and $\mathbf{B} = (B^{p,q}) = {}_H\mathrm{Reg}(\mathbf{Y}, A) = ({}_H\mathrm{Reg}(T^p \otimes N^q, A))$ is a cosimplicial double complex of abelian groups (see Section 2.2). Obtain a double complex $\mathbf{B}_0$ from $\mathbf{B}$ by replacing the 0^{th} column and the 0^{th} row by zeroes. Let $\mathbf{E}$ denote the edge sub complex of $\mathbf{B}$. Then the short exact sequence of total cochain complexes

$$0 \longrightarrow \mathrm{Tot}(\mathbf{B}_0) \longrightarrow \mathrm{Tot}(\mathbf{B}) \longrightarrow \mathrm{Tot}(\mathbf{E}) \longrightarrow 0$$

gives rise to a long exact sequence of cohomologies

$$0 \;\longrightarrow\; \mathrm{H}^1(\mathrm{Tot}\mathbf{B}) \longrightarrow \mathrm{H}^1(\mathrm{Tot}\mathbf{E}) \longrightarrow \mathrm{H}^2(\mathrm{Tot}\mathbf{B}_0) \longrightarrow \mathrm{H}^2(\mathrm{Tot}\mathbf{B}) \longrightarrow \mathrm{H}^2(\mathrm{Tot}\mathbf{E})$$

$$\longrightarrow \mathrm{H}^3(\mathrm{Tot}\mathbf{B}_0) \longrightarrow \mathrm{H}^3(\mathrm{Tot}\mathbf{B}) \longrightarrow \mathrm{H}^3(\mathrm{Tot}\mathbf{E}) \longrightarrow \dots$$

where $\mathrm{H}^0(\mathrm{Tot}\mathbf{B}_0) = 0 = \mathrm{H}^1(\mathrm{Tot}\mathbf{B}_0)$ and $\mathrm{H}^0(\mathrm{Tot}\mathbf{B}) = \mathrm{H}^0(\mathrm{Tot}\mathbf{E})$ have already been taken into account. Recall from Section 2.2 that by definition $\mathcal{H}^*(T, N, A) = \mathrm{H}^{*+1}(\mathrm{Tot}\mathbf{B}_0)$ is the cohomology of the matched pair (N, T) with coefficients in A, and note that $\mathrm{H}^*(\mathrm{Tot}\mathbf{E}) = \mathrm{H}^*(T, A) \oplus \mathrm{H}^*(N, A)$.

By the cosimplicial version of the Eilenberg-Zilber theorem (A.1.2) we have $\mathrm{H}^*(\mathrm{Tot}(C\mathbf{B})) = \mathrm{H}^*(C\mathrm{Diag}(\mathbf{B}))$. Moreover $\mathrm{Diag}G_T^*G_N^*(k) \simeq (G_TG_N)^*(k) \simeq G_H^*(k)$ by Barr's theorem [Barr], so that ${}_H\mathrm{Reg}(G_H^*(k), A) \simeq {}_H\mathrm{Reg}(\mathrm{Diag}(G_T^*G_N^*(k)), A) \simeq \mathrm{Diag}({}_H\mathrm{Reg}(G_T^*G_N^*(k), A), A) = \mathrm{Diag}\mathbf{B}$. Thus

$$\mathrm{H}^*(H, A) = \mathrm{H}^*(C_H\mathrm{Reg}(G_H^*(k), A)) \simeq \mathrm{H}^*(C\mathrm{Diag}\mathbf{B}) \simeq \mathrm{H}^*(\mathrm{Tot}(C\mathbf{B})). \blacksquare$$

In the next section we do the low terms of that sequence explicitly.

3.1 Generalized Kac explicitly

The aim of this section is to define explicitly homomorphisms that make the following sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1(H, A) &\xrightarrow{\mathrm{res}_2} \mathrm{H}^1(T, A) \oplus \mathrm{H}^1(N, A) \xrightarrow{\delta_N * \delta_T} \mathcal{H}^1(T, N, A) \xrightarrow{\phi} \mathrm{H}^2(H, A) \\ &\xrightarrow{\mathrm{res}_2} \mathrm{H}^2(T, A) \oplus \mathrm{H}^2(N, A) \xrightarrow{\delta_N * \delta_T^{-1}} \mathcal{H}^2(T, N, A) \xrightarrow{\psi} \mathrm{H}^3(H, A). \end{aligned}$$

exact. This is the low degree part of the generalized Kac sequence. Here $H = N \bowtie T$ is the bismash product arising from a matched pair $\mu: T \otimes N \longrightarrow N$, $\nu: T \otimes N \longrightarrow T$. Recall that we abbreviate $\mu(t, n) = t(n)$, $\nu(t, n) = t^n$. For the sake of simplicity we shall assume that A is a trivial H-module.

We define $\mathrm{res}_2 = \mathrm{res}_2^i: \mathrm{H}^i(H, A) \longrightarrow \mathrm{H}^i(T, A) \oplus \mathrm{H}^i(N, A)$ to be the map $(\mathrm{res}_T, \mathrm{res}_N)\Delta$, more precisely if $f: \mathrm{H}^{\otimes i} \longrightarrow A$ is a cocycle, then it gets sent to a pair of cocycles $(f|_{T^{\otimes i}}, f|_{N^{\otimes i}})$. We will abbreviate $f_T = f|_{T^{\otimes i}}$, $f_N = f|_{N^{\otimes i}}$.

By $\delta_N * \delta_T{}^{(-1)^{i+1}}$, we mean the map

$$\begin{aligned} \mathrm{H}^i(T, A) \oplus \mathrm{H}^i(N, A) &\xrightarrow{\delta_N \oplus \delta_T{}^{(-1)^{i+1}}} \mathcal{H}_i^i(T, N, A) \oplus \mathcal{H}_1^i(T, N, A) \\ &\xrightarrow{\iota \oplus \iota} \mathcal{H}^i(T, N, A) \oplus \mathcal{H}^i(T, N, A) \xrightarrow{*} \mathcal{H}^i(T, N, A). \end{aligned}$$

If $i = 1$, this sends a pair of cocycles $a \in Z^1(T, A)$, $b \in Z^1(N, A)$ to a map $\delta_N a * \delta_T b: T \otimes N \longrightarrow A$ and if $i = 2$ a pair of cocycles $a \in Z^2(T, A)$, $b \in Z^2(N, A)$ becomes a cocycle pair $(\delta_N a, \varepsilon)*(\varepsilon, \delta_T b^{-1}) = (\delta_N a, \delta_T b^{-1}): (T\otimes T\otimes N)\oplus(T\otimes N\otimes N) \longrightarrow A$, where δ_N and δ_T are as given in Section 2.2.

The map $\phi: \mathcal{H}^1(T, N, A) \longrightarrow H^2(H, A)$ assigns to a cocycle $\gamma: T \otimes N \longrightarrow A$, a map $\phi(\gamma): H \otimes H \longrightarrow A$, that is characterized by $\phi(\gamma)(nt, n't') = \gamma(t, n')$.

Finally the homomorphism $\psi: \mathcal{H}^2(T, N, A) \longrightarrow H^3(H, A)$ maps a cocycle pair $(\alpha, \beta) \in \mathcal{Z}^2(T, N, A)$ to a cocycle $f = f_{\alpha,\beta} = \psi(\alpha, \beta): H \otimes H \otimes H \longrightarrow A$ given by

$$f(nt, n't', n''t'') = \varepsilon(n)\varepsilon(t'')\alpha(t^{n'}, t', n'')\beta(t, n', t'(n'')).$$

The following shows that f is indeed a 3-cocycle.

Start by computing:

$$\begin{aligned}
\delta f(nt, n't', n''t'', n'''t''') &= f(n't', n''t'', n'''t''')f^{-1}(nt(n') \cdot t^{n'}t', n''t'', n'''t''') \\
&\cdot f(nt, n't'(n'') \cdot t'^{n''}t'', n'''t''')f^{-1}(nt, n't', n''t''(n''') \cdot t''^{n'''}t''') \\
&\cdot f(nt, n't', n''t'') \\
= \quad & \varepsilon(n)\varepsilon(t''')\alpha(t'^{n''}, t'', n''')\beta(t', n'', t''(n''')) \\
&\cdot \alpha^{-1}(t^{n' \cdot t'(n'')}t'^{n'}, t'', n''')\beta^{-1}(t^{n'}t', n'', t''(n''')) \\
&\cdot \alpha(t^{n' \cdot t'(n'')}, t'^{n'}t'', n''')\beta(t, n' \cdot t'(n''), (t'^{n''}t'')(n''')) \\
&\cdot \alpha^{-1}(t^{n'}, t', n'' \cdot t''(n'''))\beta^{-1}(t, n', t'(n'') \cdot (t'^{n''}t'')(n''')) \\
&\cdot \alpha(t^{n'}, t', n'')\beta(t, n', t'(n'')).
\end{aligned}$$

In the following we collect the α terms and then insert

$$\alpha^{-1}(t^{n't'(n'')}, t'^{n''}, t''(n'''))\alpha(t^{n't'(n'')}, t'^{n''}, t''(n'''))$$

between the third and fourth term.

$$\begin{aligned}
&\alpha(t'^{n''}, t'', n''')\alpha^{-1}(t^{n' \cdot t'(n'')}t'^{n'}, t'', n''')\alpha(t^{n' \cdot t'(n'')}, t'^{n'}t'', n''') \\
&\cdot \alpha^{-1}(t^{n'}, t', n'' \cdot t''(n'''))\alpha(t^{n'}, t', n'')
\end{aligned}$$

$$
\begin{aligned}
&= \alpha(t'^{n''}, t'', n''')\alpha^{-1}(t^{n'\cdot t'(n'')}t'^{n'}, t'', n''')\alpha(t^{n'\cdot t'(n'')}, t'^{n'}t'', n''') \\
&\quad \cdot\alpha^{-1}(t^{n't'(n'')}, t'^{n''}, t''(n'''))\alpha(t^{n't'(n'')}, t'^{n''}, t''(n''')) \\
&\quad \cdot\alpha^{-1}(t^{n'}, t', n'' \cdot t''(n'''))\alpha(t^{n'}, t', n'') \\
&= (\delta_T\alpha)(t^{n'\cdot t'(n'')}, t'^{n''}, t'', n'''))(\delta_N\alpha)(t^{n'}, t', n'', t''(n''')) \\
&= (\delta_N\alpha)(t^{n'}, t', n'', t''(n''')).
\end{aligned}
$$

Similarly we collect the β terms and then insert

$$\beta(t^{n'}, t'(n''), (t'^{n'}t'')(n'''))\beta^{-1}(t^{n'}, t'(n''), (t'^{n'}t'')(n'''))$$

between the second and third term.

$$
\begin{aligned}
&\beta(t', n'', t''(n'''))\beta^{-1}(t^{n'}t', n'', t''(n''')) \\
&\quad \cdot\beta(t, n' \cdot t'(n''), (t'^{n''}t'')(n'''))\beta^{-1}(t, n', t'(n'') \cdot (t'^{n''}t'')(n''')) \\
&\quad \cdot\beta(t, n', t'(n'')) \\
&= \beta(t', n'', t''(n'''))\beta^{-1}(t^{n'}t', n'', t''(n''')) \\
&\quad \cdot\beta(t^{n'}, t'(n''), (t'^{n'}t'')(n'''))\beta^{-1}(t^{n'}, t'(n''), (t'^{n'}t'')(n''')) \\
&\quad \cdot\beta(t, n' \cdot t'(n''), (t'^{n''}t'')(n'''))\beta^{-1}(t, n', t'(n'') \cdot (t'^{n''}t'')(n''')) \\
&\quad \cdot\beta(t, n', t'(n'')) \\
&= (\delta_T\beta)(t^{n'}, t', n'', t''(n'''))(\delta_N\beta)(t, n', t'(n''), (t'^{n'}t'')(n''')) \\
&= (\delta_T\beta)(t^{n'}, t', n'', t''(n''')).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
(\delta f)(nt, n't', n''t'', n'''t''') &= \varepsilon(n)\varepsilon(t''')(\delta_N\alpha * \delta_T\beta)(t^{n'}, t', n'', t''(n''')) \\
&= \varepsilon(nt)\varepsilon(n't')\varepsilon(n''t'')\varepsilon(n'''t''').
\end{aligned}
$$

■

Now we show the exactness at $\mathcal{H}^2(T, N, A)$.

Suppose a cocycle pair $(\alpha, \beta) \in \mathcal{Z}^2(T, N, A)$ is such that $f = f_{\alpha,\beta} = \psi(\alpha, \beta)$ is a coboundary. We would like to show that there exist $(a, b) \in Z^2(T, A) \oplus Z^2(N, A)$ such that $(\delta_N a, \delta_T b^{-1}) \sim (\alpha, \beta)$.

Assume the 2-cochain $g: H \otimes H \longrightarrow A$ is such that $f = \delta g$.

Without loss of generality, we can assume that $g|_{N\otimes T}$ is trivial, i.e $g(n,t) = \varepsilon(n)\varepsilon(t)$ for $n \in N$, $t \in T$. If g does not satisfy that condition, then define a 1-cochain $h: H \longrightarrow A$, by $h(nt) = g(n,t)$ and replace g by $g' = g * \delta h$ (note that $\delta g' = \delta g * \delta\delta h = \delta g$ and that g' has the desired property).

Note that

$$(\delta g)(n, x, x') = \varepsilon(n)\varepsilon(x)\varepsilon(x'), \tag{3.1}$$

$$(\delta g)(x, x', t) = \varepsilon(x)\varepsilon(x')\varepsilon(t), \tag{3.2}$$

for $n \in N$, $t \in T$ and $x, x' \in H$.

In the computations we use the following equalities.

Lemma 3.1.1 *Assume that $g|_{N\otimes T}$ is trivial, and that the conditions (3.1) and (3.2) are satisfied. We have the following equalities:*

$$g(nt, s) = \varepsilon(n)g(t, s), \tag{3.3}$$

$$g(t, ns) = g(t, n)g(t^n, s), \tag{3.4}$$

$$g(n, mt) = \varepsilon(t)g(n, m), \tag{3.5}$$

$$g(nt, m) = g(t, m)g(n, t(m)), \tag{3.6}$$

for $n, m \in N$, $t, s \in T$.

Proof. Apply the condition (3.1) to $(\delta g)(n, t, s)$ to obtain (3.3) and then use this result, combined with the condition (3.2) applied to $(\delta g)(t, n, s)$, to obtain (3.4) (all along we assume $g|_{N\otimes T}$ is trivial).

Similarly (3.5) is obtained by applying (3.2) to $(\delta g)(n, m, t)$, and (3.6) by using (3.5) and applying (3.1) to $(\delta g)(n, t, m)$. ■

We proceed by defining $\gamma = g|_{T\otimes N}: T \otimes N \longrightarrow A$, $a = g|_{T\otimes T}: T \otimes T \longrightarrow A$, $b = g|_{N\otimes N}: N \otimes N \longrightarrow A$. The equations (3.1) and (3.2) show that a and b are cocycles. Now note:

$$\alpha(t, s, n) = (\delta g)(t, s, n)$$

$$\begin{aligned} &= g(s,n)g^{-1}(ts,n)g(t,s(n)\cdot s^n)g^{-1}(t,s) \qquad \text{use (3.4)} \\ &= g(s,n)g^{-1}(ts,n)g(t,s(n))g(t^{s(n)},s^n)g^{-1}(t,s) \\ &= (\delta_T\gamma)(t,s,n)\cdot(\delta_N a)(t,s,n), \end{aligned}$$

and

$$\begin{aligned} \beta(t,n,m) &= (\delta g)(t,n,m) \\ &= g(n,m)g^{-1}(t(n)\cdot t^n,m)g(t,nm)g^{-1}(t,n) \qquad \text{use (3.6)} \\ &= g(n,m)g^{-1}(t(n),t^n(m))g^{-1}(t^n,m)g(t,nm)g^{-1}(t,n) \\ &= (\delta_T b^{-1})(t,n,m)\cdot(\delta_N\gamma^{-1})(t,n,m). \end{aligned}$$

Hence $(\delta_N a, \delta_T b^{-1}) \sim (\alpha,\beta)$, since $(\alpha,\beta)(\delta_N a,\delta_T b^{-1})^{-1} = (\delta_T\gamma,\delta_N\gamma^{-1}) \in \mathcal{B}^2(T,N,A)$.

Now we show that the composite $\psi(\delta_N * \delta_T{}^{-1})$ is trivial, i.e. that if $(a,b) \in Z^2(T,A) \oplus Z^2(N,A)$, then $f = f_{\delta_N a,\delta_T b^{-1}}: H\otimes H \longrightarrow A$ is a coboundary. Define $g: H\otimes H \longrightarrow A$ by $g(nt,n't') = a(t^{n'},t')b(n,t(n'))$ and observe that $f = \delta g$.■

From now on assume that $f: H\otimes H \longrightarrow A$ is a Sweedler 2-cocycle. Define a 1-cochain $g_f: H \longrightarrow A$ by the rule $g_f(nt) = f(n,t)$. Furthermore let $h = f*\delta g_f: H\otimes H \longrightarrow A$. Let $f_T = f|_{T\otimes T}: T\otimes T \longrightarrow A$, $f_N = f|_{N\otimes N}: N\otimes N \longrightarrow A$, and define $f_c: T\otimes N \longrightarrow A$ by $f_c(t,n) = f(t,n)f^{-1}(t(n),t^n)$.

The Lemma below is the most important tool used to establish the exactness of the generalized Kac sequence.

Lemma 3.1.2 *Let $f: H\otimes H \longrightarrow A$ be a cocycle and let the g_f, f_T, f_N, f_c, $h = f*\delta g_f$ be as above. Then*

1. $h(nt,n't') = f_T(t^{n'},t')f_N(n,t'(n'))f_c(t,n')$
2. $h_T = f_T$, $h_N = f_N$, $h|_{N\otimes T} = \varepsilon$, $h|_{T\otimes N} = h_c = f_c$, $g_h = \varepsilon$
3. *the maps f_T and f_N are cocycles and $\delta_N f_T = \delta_T f_c^{-1}$, $\delta_T f_N = \delta_N f_c^{-1}$*

4. If $a: T \otimes T \longrightarrow A$*,* $b: N \otimes N \longrightarrow A$ *are cocycles and* $\gamma: T \otimes N \longrightarrow A$ *is a convolution invertible map, such that* $\delta_N a = \delta_T \gamma$ *and* $\delta_T b = \delta_N \gamma$*, then the map* $f = f_{a,b,\gamma}: H \otimes H \longrightarrow A$*, defined by*

$$f(nt, n't') = a(t^{n'}, t')b(n, t(n'))\gamma^{-1}(t, n')$$

is a cocycle and $f_T = a$*,* $f_N = b$*,* $f_c = f|_{T \otimes N} = \gamma^{-1}$ *and* $f|_{N \otimes T} = \varepsilon$*.*

Proof (of 1). We enlist the help of the cocycle condition $f(x, y)f(xy, z) = f(y, z)f(x, yz)$. We use it in the following forms:

$$f(xy, z) = f^{-1}(x, y)f(y, z)f(x, yz) \tag{3.7}$$

$$f(x, yz) = f(x, y)f(x, yz)f^{-1}(y, z) \tag{3.8}$$

Apply the first equation to $x = n$, $y = t$, $z = n't'$ to get the first of the equalities below.

$$f(nt, n't') = f^{-1}(n, t) \cdot f(t, n't') \cdot f(n, tn't').$$

Now

$$f(t, n't') = f(t, n')f(tn', t')f^{-1}(n', t'),$$

$$\begin{aligned} f(n, tn't') &= f(n, t(n')t^{n'}t') \\ &= f(n, t(n'))f(n \cdot t(n'), t^{n'}t')f^{-1}(t(n'), t^{n'}t'), \end{aligned}$$

$$\begin{aligned} f(tn', t') &= f(t(n') \cdot t^{n'}, t'n') \\ &= f^{-1}(t(n'), t^{n'})f(t^{n'}, t')f(t(n'), t^{n'}t'), \end{aligned}$$

and hence

$$\begin{aligned} f(nt, n't') &= \\ &= f^{-1}(n, t) \end{aligned}$$

$$
\begin{aligned}
& \cdot f(t,n')f(tn',t')f^{-1}(n',t') \\
& \cdot f(n,t(n'))f(n\cdot t(n'),t^{n'}t')f^{-1}(t(n'),t^{n'}t') \\
= \;& f^{-1}(n,t) \\
& \cdot f(t,n') \\
& \cdot f^{-1}(t(n'),t^{n'})f(t^{n'},t')f(t(n'),t^{n'}t') \\
& \cdot f^{-1}(n',t') \\
& \cdot f(n,t(n'))f(n\cdot t(n'),t^{n'}t')f^{-1}(t(n'),t^{n'}t') \\
= \;& f(t^{n'},t') \\
& \cdot f(n,t(n')) \\
& \cdot f(t,n')f^{-1}(t(n'),t^{n'}) \\
& \cdot f^{-1}(n,t)f^{-1}(n',t')f(n\cdot t(n'),t^{n'}t')) \\
= \;& f_T(t^{n'},t') \\
& \cdot f_N(n,t(n')) \\
& \cdot f_c(t,n') \\
& \cdot(\delta g_f)^{-1}(nt,n't').\ \blacksquare
\end{aligned}
$$

Proof (of 2). Clear. ■

Proof (of 3). Since f satisfies the cocycle condition, so do f_T and f_N. Compute:

$$
\begin{aligned}
\delta_T f_c(t,t',n) &= \delta_T h_c(t,t',n) = h_c(t',n)h_c(t,t'(n))h_c^{-1}(tt',n) \\
&= h(t',n)h(t,t'(n))h^{-1}(tt',n) \\
&= h(t',n)h(t,t'(n)) \\
&\quad \cdot h(t,t')h^{-1}(t,t'n)h^{-1}(t',n) \\
&= h(t,t'(n))h(t,t') \\
&\quad \cdot h^{-1}(t,t'(n))h^{-1}(t\cdot t'(n),t'^{n})h(t'(n),t'^{n}) \\
&\quad \cdot h((tt')(n),t^{t'(n)})h^{-1}(t^{t'(n)},t'^{n})h^{-1}((tt')(n),t^{t'(n)}t'^{n}) \\
&= h(t,t')h^{-1}(t^{t'(n)},t'^{n}) \\
&= \delta_N h_t^{-1}(t,t',n) = \delta_N f_t^{-1}(t,t',n),
\end{aligned}
$$

and

$$\begin{aligned}
\delta_N f_c(t,n,n') &= \delta_N h_c(t,n,n') = h_c(t,n)h_c(t^n,n')h_c^{-1}(t,nn') \\
&= h(t,n)h(t^n,n')h^{-1}(t,nn') \\
&= h(t,n)h(t^n,n') \\
&\quad \cdot h^{-1}(t,n)h^{-1}(tn,n')h(n,n') \\
&= h(t^n,n') \\
&\quad \cdot h(t(n),t^n)h^{-1}(t^n,n')h^{-1}(t(n),t^n n') \\
&\quad \cdot h(n,n') \\
&= h^{-1}(t(n),t^n(n'))h^{-1}(t(n)\cdot t^n(n'),t^{nn'})h(t^n(n'),t^{nn'}) \\
&\quad \cdot h(n,n') \\
&= h(n,n')h^{-1}(t(n),t^n(n')) \\
&= \delta_T h_N^{-1}(t,n,n') = \delta_T f_N^{-1}(t,n,n'). \blacksquare
\end{aligned}$$

Proof (of 4). The equalities $f_T = a$, $f_N = b$, $f_c = f|_{T\otimes N} = \gamma$ and $f|_{N\otimes T} = \varepsilon$ are apparent from the formula for f. We only have to check, that f is indeed a cocycle. This is done by the following monstrosity.

$$\begin{aligned}
&(\delta f)(nt,n't',n''t'') = \\
&\quad = f(nt,n't')f^{-1}(nt,n't'(n'')t^{n''}t'')f(nt(n')t^{n'}t',n''t'')f^{-1}(n't',n''t'') \\
&\quad = a(t^{n'},t')b(n,t(n'))\gamma^{-1}(t,n') \\
&\qquad \cdot a^{-1}(t^{n'\cdot t'(n'')},t'^{n''}t'') \\
&\qquad \cdot b^{-1}(n,t(n')\cdot(t^{n'}t')(n'')) \\
&\qquad \cdot \gamma(t,n'\cdot t'(n'')) \\
&\qquad \cdot a(t^{n'\cdot t'(n'')}t'^{n''},t'') \\
&\qquad \cdot b(n\cdot t(n'),(t^{n'}t')(n'')) \\
&\qquad \cdot \gamma^{-1}(t^{n'}t',n'') \\
&\qquad \cdot a^{-1}(t'^{n''},t'')b^{-1}(n',t'(n''))\gamma(t',n'')
\end{aligned}$$

We compute the product of the a terms:

$$
\begin{aligned}
& a(t^{n'},t')a^{-1}(t^{n'\cdot t'(n'')},t'^{n''}t'')a(t^{n'\cdot t'(n'')}t'^{n''},t'')a^{-1}(t'^{n''},t'') \\
&= a(t^{n'},t')a^{-1}(t^{n'\cdot t'(n'')},t'^{n''}t'') \\
&\quad \cdot a(t^{n'\cdot t'(n'')},t'^{n''})a^{-1}(t^{n'\cdot t'(n'')},t'^{n''}) \\
&\quad a(t^{n'\cdot t'(n'')}t'^{n''},t'')a^{-1}(t'^{n''},t'') \\
&= (\delta_N a^{-1})(t^{n'},t',n'')\cdot(\delta_T a)(t^{n'\cdot t'(n'')},t'^{n''},t'') \\
&= (\delta_N a^{-1})(t^{n'},t',n'') \\
&= \delta_T\gamma^{-1}(t^{n'},t',n'').
\end{aligned}
$$

Similarly we compute the product of the b terms:

$$
\begin{aligned}
& b(n,t(n'))b^{-1}(n,t(n')\cdot(t^{n'}t')(n''))b(n\cdot t(n'),(t^{n'}t')(n''))b^{-1}(n',t'(n'')) \\
&= b(n,t(n'))b^{-1}(n,t(n')\cdot(t^{n'}t')(n''))b(n\cdot t(n'),(t^{n'}t')(n'')) \\
&\quad \cdot b^{-1}(t(n'),(t^{n'}t')(n''))b(t(n'),(t^{n'}t')(n'')) \\
&\quad \cdot b^{-1}(n',t'(n'')) \\
&= (\delta_N b)(n,t(n'),(t^{n'}t')(n''))\cdot(\delta_T b)(t,n',t'(n'')) \\
&= (\delta_T b)(t,n',t'(n'')) \\
&= \delta_N\gamma(t,n',t'(n'')).
\end{aligned}
$$

We conclude the proof by observing that the inverse of the products of the a and b terms equals the product of the γ terms.

$$
\begin{aligned}
& (\delta_T\gamma)(t^{n'},t',n'')\cdot(\delta_N\gamma^{-1})(t,n',t'(n'')) = \\
&= \gamma(t',n'')\gamma(t^{n'},t'(n''))\gamma^{-1}(t^{n'}t',n'') \\
&\quad \cdot\gamma^{-1}(t,n')\gamma^{-1}(t^{n'},t'(n''))\gamma(t,n'\cdot t'(n'')) \\
&= \gamma(t',n'')\gamma^{-1}(t^{n'}t',n'')\gamma^{-1}(t,n')\gamma(t,n'\cdot t'(n'')).\ \blacksquare
\end{aligned}
$$

Now we proceed by showing the exactness of the remaining part of the sequence defined at the beginning of this section.

- Exactness at $\mathrm{H}^2(T, A) \oplus \mathrm{H}^2(N, A)$:

  The composite $(\delta_N * \delta_T{}^{-1})\mathrm{res}_2$, $f \mapsto (f_T, f_N) \mapsto (\delta_N f_T, \delta_T f_N^{-1})$, is trivial, since $(\delta_N f_T, \delta_T f_N^{-1}) = (\delta_T f_c, \delta_N f_c^{-1})^{-1} \in \mathcal{B}^2(T, N, A)$, by Lemma 3.1.2, part 3.

  If a pair of cocycles $(a, b) \in \mathrm{Z}^2(T, A) \oplus \mathrm{Z}^2(N, A)$ is such that $(\delta_N a, \delta_T b^{-1}) \in \mathcal{B}^2(T, N, A)$, then there is a $\gamma: T \otimes N \longrightarrow A$, such that $\delta_N a = \delta_T \gamma$ and $\delta_T b = \delta_N \gamma$. Use Lemma 3.1.2, part 4, to define a cocycle $f: H \otimes H \longrightarrow A$ by the rule

$$f(nt, n't') = a(t^{n'}, t')b(n, t(n'))\gamma^{-1}(t, n')$$

  and note that $\mathrm{res}_2(f) = (a, b)$. ∎

- Exactness at $\mathrm{H}^2(H, A)$:

  First we have to check that the map $\phi: \mathcal{H}^1(T, N, A) \longrightarrow \mathrm{H}^2(H, A)$, defined at the beginning of the section is well defined. This is verified by applying Lemma 3.1.2, part 4, to a pair of trivial cocycles $a: T \otimes T \longrightarrow A$, $b: N \otimes N \longrightarrow A$ and a cocycle $\gamma: T \otimes N \longrightarrow A$ from $\mathcal{H}^1(T, N, A)$ (i.e. $\delta_N \gamma = \varepsilon$ and $\delta_T \gamma = \varepsilon$).

  Clearly $(\phi(\gamma)_T, \phi(\gamma)_N) = (\varepsilon_{T\otimes T}, \varepsilon_{N\otimes N})$ and hence the composite $\mathrm{res}_2\phi$ is trivial.

  Now suppose a cocycle $f: H \otimes H \longrightarrow A$ is such that $a = f_T: T \otimes T \longrightarrow A$ and $b = f_N: N \otimes N \longrightarrow A$ are coboundaries. Let $\overline{a}: T \longrightarrow A$ and $\overline{b}: N \longrightarrow A$ be such that $a = \delta_T \overline{a}$ and $b = \delta_N \overline{b}$ and define $\gamma = (\delta_T \overline{a} * \delta_N \overline{b})^{-1}: T \otimes N \longrightarrow A$, that is $\gamma(t, n) = \overline{a}(t)\overline{a}^{-1}(t^n)\overline{b}(n)\overline{b}^{-1}(t(n))$.

  Define $g: T \longrightarrow A$ by $g(nt) = \overline{a}(t)\overline{b}(n)$. Note

$$\begin{aligned}
\delta g(nt, n't') &= g(nt)g(n't')g^{-1}(nt(n')t^{n'}t') \\
&= \overline{a}(t)\overline{a}(t')\overline{a}^{-1}(t^{n'}t') \\
&\quad \cdot \overline{b}(n')\overline{b}(n)\overline{b}^{-1}(n \cdot t(n')) \\
&= \overline{a}(t)\overline{a}^{-1}(t^{n'})\overline{a}(t^{n'})\overline{a}(t')\overline{a}^{-1}(t^{n'}t') \\
&\quad \cdot \overline{b}(n')\overline{b}^{-1}(t(n'))\overline{b}(t(n'))\overline{b}(n)\overline{b}^{-1}(n \cdot t(n')) \\
&= (\delta_N \overline{a})(t, n') \cdot (\delta_T \overline{a})(t^{n'}, t')
\end{aligned}$$

$$\cdot(\delta_T\bar{b})(t,n')\cdot(\delta_N\bar{b})(n,t(n'))$$
$$= a(t^{n'},t')b(n,t(n'))\gamma(t,n').$$

Remember that by Lemma 3.1.2, part 1, there is $h \sim f$ (cohomologous), such that $h_T = f_T = a$, $h_N = f_N = b$ and

$$h(nt,n't') = a(t^{n'},t')b(n,t(n'))h_c(t,n').$$

Let $h' = h * \delta g^{-1}$ and note that $h'(nt,n't') = (h * \delta g^{-1})(nt,n't') = (\gamma^{-1} * h_c)(t,n')$, hence $h'(nt,n't') = h'_c(t,n')$, with $\delta_N h'_c = \varepsilon$ and $\delta_T h'_c = \varepsilon$, that is $h'_c \in \mathcal{H}^1(T,N,A)$ and $h' = \phi(h'_c)$. We are done, since f is cohomologous to h'. ∎

- Exactness at $\mathcal{H}^1(T,N,A)$:

  First we show that the composite $\phi(\delta_N * \delta_T)$ is trivial. Let $a \in \mathrm{H}^1(T,A)$, $b \in \mathrm{H}^1(N,A)$ and $\gamma = \delta_N a * \delta_T b : T \otimes N \longrightarrow A$. Define a 1-cochain $g: H \longrightarrow A$ by $g(nt) = a(t)b(n)$ and note that $\phi(\gamma) = \delta g \in \mathrm{B}^2(H,A)$.

  Now assume that $\gamma \in \mathcal{H}^1(T,N,A)$ and suppose that $\phi(\gamma): H \otimes H \longrightarrow A$ is a coboundary; that is $\phi(\gamma) = \delta g$, for some $g: H \longrightarrow A$. Note that $g(tt') = g(t)g(t')$, $g(nn') = g(n)g(n')$ and $g(nt') = g(n)g(t')$, since the maps $(\delta g)|_{T\otimes T}$, $(\delta g)|_{N\otimes N}$ and $(\delta g)|_{N\otimes T}$ are trivial. Hence $g|_T \in \mathrm{H}^1(T,A)$, $g|_N \in H^1(N,A)$ and $\gamma = \delta_N g|_T * \delta_T g|_N$. ∎

- Exactness at $\mathrm{H}^1(T,A) \oplus \mathrm{H}^1(N,A)$:

  Let $f: H \longrightarrow A$ be a cocycle, i.e. an algebra map. Then

$$\begin{aligned}\varepsilon(t)\varepsilon(n) &= g(tn)g^{-1}(tn) = g(tn)g^{-1}(t(n)t^n)\\ &= g(t)g(n)g^{-1}(t(n))g^{-1}(t^n)\\ &= (\delta_N g|_T * \delta_T g|_N)^{-1}(t,n).\ \blacksquare\end{aligned}$$

On the other hand, if $\delta_N a * \delta_T b = \varepsilon \in \mathcal{H}^1(N,T,A)$, for some $a \in \mathrm{H}^1(T,A)$, $b \in \mathrm{H}^1(T,A)$, then define $f: H \longrightarrow A$ by the rule $g(tn) = a(t)b(n)$ and note that g is an algebra map that restricts to a and b. ∎

- Exactness at $\mathrm{H}^1(H, A)$: Since $f(nt) = f_T(t)f_N(n)$, the homomorphism res_2, given by $f \mapsto (f_T, f_N)$, is injective. ■

Remark. *Suppose that the action* $\mu: T \otimes N \longrightarrow N$ *is locally finite and let* $(N, T^\circ, \omega, \rho)$ *be the Singer pair corresponding to the matched pair* (T, N, μ, ν).

By Corollary 2.4.3 we have $\mathcal{H}^1(T, N, k) = \mathrm{H}^1(N, T^\circ)$. *Recall also* $\mathrm{H}^1(N, T^\circ) \simeq \mathrm{Aut}(T^\circ \# N)$, *[Ho].*

From the explicit description of the generalized Kac sequence, we see that we have $(\delta_N * \delta_T{}^{-1})|_{\mathrm{H}^2(T,A)} = \delta_N: \mathrm{H}^2(T, A) \longrightarrow \mathcal{H}^2_2(N, T, A)$ *and* $(\delta_N * \delta_T{}^{-1})|_{\mathrm{H}^2(N,A)} = \delta_T{}^{-1}: \mathrm{H}^2(N, A) \longrightarrow \mathcal{H}^2_1(N, T, A)$. *By Proposition 2.4.4 we also have* $\mathcal{H}^2_1(T, N, k) = \mathrm{H}^2_{\mathrm{m}}(N, T^\circ)$. *Recall that* $\mathrm{H}^2_{\mathrm{m}}(N, T^\circ) \subseteq \mathrm{H}^2(N, T^\circ) \simeq \mathrm{Opext}(N, T^\circ)$.

If the action ν *is locally finite as well, then there is also a (right) Singer pair* $(T, N^\circ, \omega', \rho')$. *By 'right' we mean that we have a right action* $\omega': N^\circ \otimes T \longrightarrow N^\circ$ *and a right coaction* $\rho': T \otimes N^\circ \otimes T$. *In this case we get that* $\mathcal{H}^2_2(T, N, k) \simeq \mathrm{H}^2_{\mathrm{m}}{}'(T, N^\circ) \subseteq \mathrm{Opext}'(T, N^\circ)$. *The dash refers to the fact that we have a right Singer pair.*

Also note $\mathrm{H}^2_{\mathrm{c}}(N, T^\circ) \cap \mathrm{H}^2_{\mathrm{m}}(N, T^\circ) \simeq \mathcal{H}^2_2(N, T, k) \cap \mathcal{H}^2_1(N, T, k) \simeq \mathrm{H}^2_{\mathrm{m}}{}'(T, N^\circ) \cap \mathrm{H}^{2\prime}_{\mathrm{c}}(T, N^\circ)$. *Hence*

$$\begin{aligned} \mathrm{Im}(\delta_N * \delta_T{}^{-1}) &\subseteq \mathcal{H}^2_1(T, N, k) + \mathcal{H}^2_2(T, N, k) \simeq \frac{\mathcal{H}^2_1(T, N, k) \oplus \mathcal{H}^2_2(T, N, k)}{\mathcal{H}^2_1(T, N, k) \cap \mathcal{H}^2_2(T, N, k)} \\ &= \frac{\mathrm{H}^2_{\mathrm{m}}(N, T^\circ) \oplus \mathrm{H}^2_{\mathrm{m}}{}'(T, N^\circ)}{\langle \mathrm{H}^2_{\mathrm{mc}}(N, T^\circ) \equiv \mathrm{H}^2_{\mathrm{mc}}{}'(T, N^\circ) \rangle}, \end{aligned}$$

where $\mathrm{H}^2_{\mathrm{mc}} = \mathrm{H}^2_{\mathrm{m}} \cap \mathrm{H}^2_{\mathrm{c}}$ *and* $\mathrm{H}^2_{\mathrm{mc}}{}' = \mathrm{H}^2_{\mathrm{m}}{}' \cap \mathrm{H}^{2\prime}_{\mathrm{c}}$. *In other words,* $\mathrm{Im}(\delta_N * \delta_T{}^{-1})$ *is contained in a subgroup of* $\mathcal{H}^2(T, N, k)$, *that is isomorphic to the pushout*

$$\begin{array}{ccc} \mathrm{H}^2_{\mathrm{mc}}{}'(T, N^\circ) \simeq \mathrm{H}^2_{\mathrm{mc}}(N, T^\circ) & \longrightarrow & \mathrm{H}^2_{\mathrm{m}}(N, T^\circ) \\ \downarrow & & \downarrow \\ \mathrm{H}^2_{\mathrm{m}}{}'(T, N^\circ) & \longrightarrow & X \end{array}$$

Chapter 4

Sweedler cohomology of smash products

The aim in this chapter is to obtain a sequence for the low degree cohomology of a smash product of (cocommutative) Hopf algebras that generalizes that of Tahara for a semi-direct product of groups (in A.3.2).

4.1 Measuring cohomology

Let $H = N \rtimes T$, more precisely let (T, N, μ, ν) be an abelian matched pair, with ν trivial. Furthermore let T act on $\mathrm{Hom}(N, A)$ via pre-composition: $\Psi\colon \mathrm{Hom}(N, A) \otimes T \longrightarrow \mathrm{Hom}(N, A)$, $f \otimes t \longrightarrow f^t$, where $f^t(n) = f(t(n))$. Let $\mathrm{Reg}_{\mathrm{meas}}(T^{\otimes q}, \mathrm{Hom}(N, A))$ denote the subgroup of $\mathrm{Reg}(T^{\otimes q}, \mathrm{Hom}(N, A))$ consisting of the maps $f\colon T^{\otimes q} \longrightarrow \mathrm{Hom}(N, A)$ that make $T^{\otimes q}$ measure N to A, i.e. $f(\mathbf{t})(nn') = \sum f(\mathbf{t}_1)(n) f(\mathbf{t}_2)(n')$ and $f(\mathbf{t})(1_N) = \varepsilon(\mathbf{t}) 1_A$ for $\mathbf{t} \in T^{\otimes q}$ and $n, n' \in N$. The differential

$$\mathrm{Reg}(T^{\otimes q-1}, \mathrm{Hom}(N, A)) \xrightarrow{\delta^{q-1}} \mathrm{Reg}(T^{\otimes q}, \mathrm{Hom}(N, A)),$$

described in Section 1.2 restricts to

$$\mathrm{Reg}_{\mathrm{meas}}(T^{\otimes q-1}, \mathrm{Hom}(N, A)) \xrightarrow{\delta^{q-1}} \mathrm{Reg}_{\mathrm{meas}}(T^{\otimes q}, \mathrm{Hom}(N, A)),$$

thus giving rise to a sub complex of the complex given in Section 1.2. We name the cohomology it produces the "measuring cohomology" and denote it by $\mathrm{H}^q_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$. We denote the groups of measuring cocycles and coboundaries by $\mathrm{Z}^q_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$ and $\mathrm{B}^q_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$, respectively. In the case $q = 1$ they are as follows:

$$\begin{aligned} \mathrm{Z}^1_{\mathrm{meas}}(T, \mathrm{Hom}(N, A)) &= \{f \in \mathrm{Reg}(T, \mathrm{Hom}(N, A))| \\ &\quad f(tt')(n) = \sum f(t'_1)(t_1(n_1))f(t_2)(n_2) \text{ and} \\ &\quad f(t)(nn') = \sum f(t_1)(n)f(t_2)(n')\}, \\ \mathrm{B}^1_{\mathrm{meas}}(T, \mathrm{Hom}(N, A)) &= \mathrm{B}^1(T, \mathrm{Hom}(N, A)) \cap \mathrm{Z}^1_{\mathrm{meas}}(T, \mathrm{Hom}(N, A)). \end{aligned}$$

4.1.1 Measuring cohomology of group algebras

In case $T = kG$ is a group algebra, then we can identify

$$\mathrm{Reg}_{\mathrm{meas}}((kG)^{\otimes i}, \mathrm{Hom}(N, A)) \simeq \mathrm{Map}(G^{\times i}, \mathrm{Alg}(N, A)),$$

where $\mathrm{Alg}(N, A)$ denotes the group of algebra maps from N to A (with convolution product) and $\mathrm{Map}(H, B)$ denotes the abelian group of unital maps from H to B (with pointwise multiplication). Note that this isomorphism induces an isomorphism of complexes, i.e. preserves the differentials; hence we have:

Theorem 4.1.1 *There is an isomorphism*

$$\mathrm{H}^i_{\mathrm{meas}}(kG, \mathrm{Hom}(N, A)) \simeq \mathrm{H}^i(G, \mathrm{Alg}(N, A)).$$

If $N = U\mathbf{g}$ is the universal envelope of a Lie algebra then we have a natural isomorphism $\mathrm{Alg}(U\mathbf{g}, A) \simeq \mathrm{Lie}(\mathbf{g}, \mathit{Lie}(A))$ [CE], where $\mathrm{Lie}(\mathbf{g}, \mathbf{h})$ denotes the group of Lie algebra maps $\mathbf{g} \longrightarrow \mathbf{h}$ (with pointwise addition) and $\mathit{Lie}(A)$ denotes the underlying Lie algebra of the algebra A, that is the Lie algebra where the Lie bracket is given by $[x, y] = xy - yx$. In our case $\mathit{Lie}(A)$ is an abelian Lie algebra (since the algebra A is commutative) and hence $\mathrm{Lie}(\mathbf{g}, \mathit{Lie}(A)) \simeq \mathrm{Lie}(\mathbf{g}/[\mathbf{g}, \mathbf{g}], \mathit{Lie}(A)) \simeq \mathrm{Vect}((\mathbf{g}/[\mathbf{g}, \mathbf{g}])^+, A^+)$. Here $\mathbf{g}/[\mathbf{g}, \mathbf{g}]$ is the abelianization of the Lie algebra $\mathbf{g}$, $\_^+$ is the underlying vector space functor and $\mathrm{Vect}(V, W)$ is the abelian group of linear maps from V to W (with pointwise addition). Hence we have:

Theorem 4.1.2 *Let G be a finite group, and $\mathbf{g}$ a Lie algebra. Then*

$$\mathrm{H}^i_{\mathrm{meas}}(kG, \mathrm{Hom}(U\mathbf{g}, A)) \simeq \mathrm{H}^i(G, \mathrm{Vect}((\mathbf{g}/[\mathbf{g},\mathbf{g}])^+, A^+)).$$

If $|G|^{-1} \in k$ then $\mathrm{H}^i_{\mathrm{meas}}(kG, \mathrm{Hom}(U\mathbf{g}, A)) = 0$.

Proof. The first equality was already explained in the paragraph preceding the theorem. Note that if $|G|^{-1} \in k$, then $\mathrm{Vect}((\mathbf{g}/[\mathbf{g},\mathbf{g}])^+, A^+)$ is uniquely $|G|$ divisible and hence $\mathrm{H}^i_{\mathrm{meas}}(kG, \mathrm{Hom}(U\mathbf{g}, A)) = 0$. ■

4.2 Universal measuring coalgebras and Hopf algebras

4.2.1 Universal measuring coalgebra

Here we introduce a functor

$$\mathrm{M}(\_, A): \mathcal{A}^{op} \longrightarrow \mathcal{C},$$

that is adjoint to

$$\mathrm{Hom}(\_, A): \mathcal{C} \longrightarrow \mathcal{A}^{op}.$$

For more details on this topic, we refer to [Sw2, Chapter VII] and [GP].

Let A, B, be algebras, C a coalgebra.

Proposition 4.2.1 (Sw2, Proposition 7.0.1) *A map $\psi: C \otimes B \longrightarrow A$ corresponds to an algebra map $\rho: B \longrightarrow \mathrm{Hom}(C, A)$, $\rho(b)(c) = \psi(c \otimes b)$ if and only if*

1. $\psi(c \otimes bb') = \psi(c_1 \otimes b)\psi(c_2 \otimes b')$,

2. $\psi(c \otimes 1_B) = \varepsilon(c)1_A$

If the equivalent conditions from the Proposition above are satisfied, we say that (ψ, C) **measures** B to A.

Given algebras B and A there is a measuring $(\theta, \mathrm{M}(B, A))$ with the following universal property.

Theorem 4.2.2 (Sw2, 7.0.4) *The universal measuring $\theta\colon \mathrm{M}(B, A) \otimes B \longrightarrow A$ has the following universal property:*

for any measuring $f\colon C \otimes B \longrightarrow A$ there exists a unique coalgebra map $\overline{f}\colon C \longrightarrow \mathrm{M}(B, A)$, s.t. $f = \theta(\overline{f} \otimes \mathrm{id})$.

The construction is as follows. Let $E \longrightarrow \mathrm{Hom}(B, A)$ be a cofree coalgebra over $\mathrm{Hom}(B, A)$. Then define $\mathrm{M}(B, A)$, to be the largest subcoalgebra of E, s.t. the canonical map $\theta\colon M(B, A) \otimes B \longrightarrow E \otimes B \longrightarrow \mathrm{Hom}(B, A) \otimes B \longrightarrow A$ measures.

4.2.2 Measuring Hopf algebra

If N is a Hopf algebra and A a commutative algebra, then it is possible to endow $\mathrm{M}(N, A)$ with a Hopf algebra structure.

This is done as follows:

- the multiplication $\mathrm{m}\colon \mathrm{M}(N, A) \otimes \mathrm{M}(N, A) \longrightarrow \mathrm{M}(N, A)$ is the unique coalgebra map s.t. $\theta(\mathrm{m} \otimes \mathrm{id}) = \omega$, where $\omega = \mathrm{m}_A(\theta \otimes \theta)\sigma_{2,3}(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_N)\colon \mathrm{M}(N, A) \otimes \mathrm{M}(N, A) \otimes N \longrightarrow A$

- the unit $\eta\colon k \longrightarrow \mathrm{M}(N, A)$ is the unique coalgebra map s.t. $\theta(\eta \otimes \mathrm{id}) = \eta_A \varepsilon_N \tau$, where $\tau\colon k \otimes N \longrightarrow N$ is the natural isomorphism $(x \otimes n \mapsto xn)$

- the antipode S is the unique coalgebra map $\mathrm{M}(N, A)^{co-op} \longrightarrow \mathrm{M}(N, A)$, s.t. $\theta(S \otimes \mathrm{id}) = \theta(\mathrm{id} \otimes S_N)$

Proposition 4.2.3 *Let N be a Hopf algebra and A a commutative algebra. Then $(\mathrm{M}(N, A), \mathrm{m}, \Delta, \eta, \varepsilon, S)$, where m, η and S are the maps defined above, is a Hopf algebra.*

Proof. We have to show the following:

- ω is a measuring:

  $\omega(f \otimes g \otimes nm) = \theta(f \otimes n_1 m_1)\theta(g \otimes n_2 m_2) = \theta(f_1 \otimes n_1)\theta(f_2 \otimes m_1)\theta(g_1 \otimes n_2)\theta(g_2 \otimes m_2) = \omega((f_1 \otimes g_1 \otimes n)\omega(f_2 \otimes g_2 \otimes m)$.

- m is associative:

  Define a measuring $\omega_3\colon \mathrm{M}(N, A) \otimes \mathrm{M}(N, A) \otimes \mathrm{M}(N, A) \otimes N \longrightarrow A$ by the rule $\omega_3(f \otimes g \otimes h \otimes n) = \theta(f \otimes n_1)\theta(g \otimes n_2)\theta(h \otimes n_3)$ and note that $\theta(\mathrm{m}(\mathrm{id} \otimes \mathrm{m}) \otimes \mathrm{id}) = \omega_3 = \theta(\mathrm{m}(\mathrm{m} \otimes \mathrm{id}) \otimes \mathrm{id})$ and hence by the uniqueness $\mathrm{m}(\mathrm{id} \otimes m) = \mathrm{m}(\mathrm{m} \otimes \mathrm{id})$.

- $\eta\varepsilon\tau$ is a measuring: clear

- η is the unit for multiplication, i.e. $\mathrm{m}(\eta \otimes \mathrm{id}) = \tau_{left}$ and $\mathrm{m}(\mathrm{id} \otimes \eta) = \tau_{right}$:

  by the universal property it is sufficient to see $\theta(\mathrm{m}(\eta \otimes \mathrm{id}) \otimes \mathrm{id}) = \theta(\tau_{left} \otimes \mathrm{id})$ and $\theta(\mathrm{m}(\mathrm{id} \otimes \eta) \otimes \mathrm{id}) = \theta(\tau_{right} \otimes \mathrm{id})$. We show the second of the equalities (the argument for the first is symmetric): $LHS(f \otimes x \otimes n) = \theta(\mathrm{m}(f \otimes \eta(x)) \otimes n) = \omega(f \otimes \eta(x) \otimes n) = \theta(f \otimes n_1)\theta(\eta(x) \otimes n_2) = \theta(f \otimes n_1)x\varepsilon(n_2)1_A = \theta(xf \otimes n) = RHS(f \otimes x \otimes n)$.

- $\theta(\mathrm{id} \otimes S)\colon \mathrm{M}(N, A)^{co-op} \otimes N \longrightarrow A$ is a measuring: clear

- S is an antipode, i.e. $S * \mathrm{id} = \mathrm{id} * S = \eta\varepsilon$:

  it is sufficient to show $\theta(S * \mathrm{id} \otimes \mathrm{id}) = \theta(\eta\varepsilon \otimes \mathrm{id}) = \theta(\mathrm{id} * S \otimes \mathrm{id})$. This is proven by the following computation: $\theta(S * \mathrm{id} \otimes \mathrm{id})(f \otimes n) = \theta(S(f_1)f_2 \otimes n) = \omega(S(f_1) \otimes f_2 \otimes n) = \theta(S(f_1) \otimes n_1)\theta(f_2 \otimes n_2) = \theta(f_1 \otimes S(n_1))\theta(f_2 \otimes n_2) = \theta(f \otimes S(n_1)n_2) = \theta(f \otimes \varepsilon(n)1_N) = \varepsilon(f)\varepsilon(n)1_A = \theta(\eta\varepsilon(f) \otimes n)$; symmetrically for $\theta(\eta\varepsilon \otimes \mathrm{id}) = \theta(\mathrm{id} * S \otimes \mathrm{id})$. ■

Proposition 4.2.4 *If N is cocommutative, then* $\mathrm{M}(N, A)$ *is commutative.*

Proof. $\theta(\mathrm{m} \otimes \mathrm{id})(f \otimes g \otimes n) = \theta(f \otimes n_1)\theta(g \otimes n_2) = \theta(f \otimes n_2)\theta(g \otimes n_1) = \theta(gf \otimes n) = \theta((\sigma\mathrm{m}) \otimes \mathrm{id})(f \otimes g \otimes n)$. ■

4.3 Measuring cohomology vs Singer cohomology

In this section we interpret $\mathrm{H}^2_{\mathrm{meas}}$ as $\mathrm{H}^2_{\mathrm{m}}$, the multiplication part of H^2.

Proposition 4.3.1 *If N is a T-module bialgebra via $\mu: N \otimes T \longrightarrow N$ $(n \otimes t \mapsto n^t)$, then so is $\mathrm{M}(N, A)$ via $\overline{\mu}: T \otimes \mathrm{M}(N, A) \longrightarrow \mathrm{M}(N, A)$ $(t \otimes f \mapsto t(f))$, which is the unique map, such that $\theta(\overline{\mu} \otimes \mathrm{id}) = \theta(\mathrm{id} \otimes \mu)(2, 3, 1)$.*

Proof. By the universal property $\overline{\mu}$ is a coalgebra map. The following computation proves that $\mathrm{M}(N, A)$ is also a T-module algebra, i.e. $\overline{\mu}(\mathrm{id} \otimes \mathrm{m}) = \mathrm{m}(\overline{\mu} \otimes \overline{\mu})\sigma_{2,3}(\Delta \otimes \mathrm{id} \otimes \mathrm{id})$: $\theta(t(fg) \otimes n) = \theta(fg \otimes n^t) = \theta(f \otimes n_1^{t_1})\theta(g \otimes n_2^{t_2}) = \theta(t_1(f) \otimes n_1)\theta(t_2(g) \otimes n_2) = \theta(t_1(f)t_2(g) \otimes n)$. ■

Remark1. *Note that $(T, \mathrm{M}(N, A), \overline{\mu}, \rho)$, where $\overline{\mu}$ is as above and ρ is the trivial coaction $\rho = \mathrm{id} \otimes \eta: T \longrightarrow T \otimes \mathrm{M}(N, A)$ is a Singer pair.*

Remark2. *If instead of a right action $\mu: N \otimes T \longrightarrow N$, we are given a left action $\mu': T \otimes N \longrightarrow N$, we can "switch" sides by applying the antipode, i.e. $\mu'(S \otimes \mathrm{id})\sigma: N \otimes T \longrightarrow N$ is a right action.*

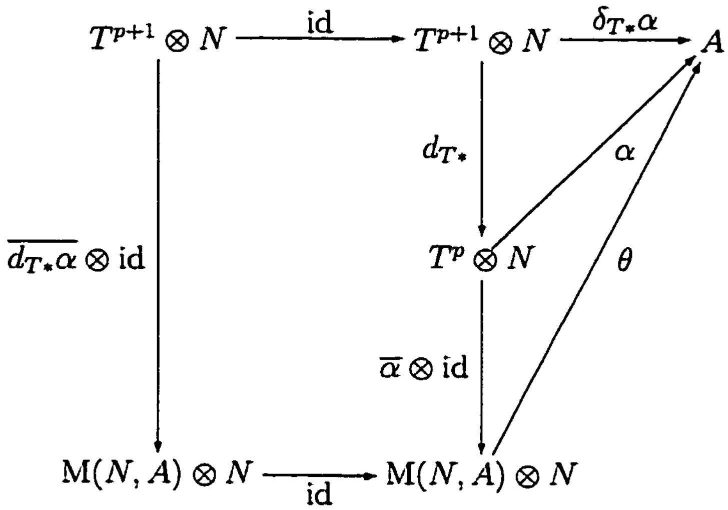
From now on assume T and N are both cocommutative. In this case we can talk about the differentials $\delta: \mathrm{Reg}(T^{\otimes p}, \mathrm{M}(N, A)) \longrightarrow \mathrm{Reg}(T^{\otimes p+1}, \mathrm{M}(N, A))$ for computing Sweedler cohomology $\mathrm{H}^p(T, \mathrm{M}(N, A))$.

Lemma 4.3.2 *If $\overline{\alpha}: T^{\otimes p} \longrightarrow \mathrm{M}(N, A)$ is the coalgebra map corresponding to the measuring $\alpha: T^{\otimes p} \otimes N \longrightarrow A$ then, $\delta\overline{\alpha} = \overline{\delta_T \alpha}$ i.e. the coalgebra map corresponding to the measuring $\delta_T\alpha: T^{\otimes p+1} \otimes N \longrightarrow A$ is $\delta\overline{\alpha}: T^{\otimes p+1} \longrightarrow \mathrm{M}(N, A)$.*

Proof. We have to show $\theta(\delta\overline{\alpha} \otimes \mathrm{id}) = \delta_T\alpha$. Since $\theta(\overline{\alpha} * \overline{\beta} \otimes \mathrm{id}) = \overline{\alpha * \beta}$ and also $\overline{\alpha^{-1}} = \overline{\alpha}(\mathrm{id}_{T^{\otimes p}} \otimes S) = S\overline{\alpha} = \overline{\alpha}^{-1}$ (it is a coalgebra map, since T is cocommutative), it is sufficient to see that

$$\begin{aligned}
\theta(\overline{\alpha}(\mathrm{id} \otimes \ldots \otimes \mathrm{m} \otimes \ldots \otimes \mathrm{id}) \otimes \mathrm{id}_N) &= \alpha(\mathrm{id} \otimes \ldots \otimes \mathrm{m} \otimes \ldots \otimes \mathrm{id} \otimes \mathrm{id}_N) \\
\theta(\overline{\mu}(\mathrm{id} \otimes \overline{\alpha}) \otimes \mathrm{id}) &= \mu'(\mathrm{id} \otimes \alpha)
\end{aligned}$$

These equalities are observed by the commutativity of the following diagram.



■

Observe that $\overline{\eta_A \varepsilon_{T^{\otimes p} \otimes N}} = \eta_{\mathrm{M}(N,A)} \varepsilon_{T^{\otimes p}}$ and hence

$$\overline{(\_)}: \{\alpha: T^{\otimes p} \otimes N \longrightarrow A | \alpha \text{ measures}\} \longrightarrow \mathrm{Coalg}(T^{\otimes p}, \mathrm{M}(N, A))$$

gives an isomorphism of complexes

$$(\mathrm{Reg}_{meas}(T^{\otimes p}, \mathrm{Hom}(N, A)), \delta_T) \longrightarrow (\mathrm{Coalg}(T^{\otimes p}, \mathrm{M}(N, A)), \delta).$$

Theorem 4.3.3 $\mathrm{H}^2_{\mathrm{m}}(T, \mathrm{M}(N, A)) \simeq \mathrm{H}^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$.

Proof. Note that in case of the trivial coaction, the condition, that $\alpha: T \otimes T \longrightarrow \mathrm{M}(N, A)$ is compatible with the trivial map $\eta\varepsilon: T \longrightarrow \mathrm{M}(N, A) \otimes \mathrm{M}(N, A)$, is equivalent to α being a coalgebra map. ■

Remark1. *Hence the degree two measuring cohomology characterizes those Hopf algebra extensions* $\mathrm{M}(N, A) \longrightarrow H \longrightarrow T$, *for which there exists an* $\mathrm{M}(N, A)$*-module coalgebra map* $\xi: H \longrightarrow \mathrm{M}(N, A)$.

Remark2. *Direct comparison of equations shows also that*

$$\mathrm{H}^i_{\mathrm{meas}}(T, \mathrm{Hom}(N, A)) \simeq \mathcal{H}^i_i(T, N, A).$$

4.4 Five term exact sequence for a smash product

The purpose of this section is to prove the following theorem by explicitly describing the maps involved.

Theorem 4.4.1 *Let $H = N \rtimes T$ be a smash product of cocommutative Hopf algebras (more precisely, we are given an action $\mu: T \otimes N \longrightarrow N$, that makes N into a T-module bialgebra) and let the commutative algebra A be a trivial H-module. Then we have the following exact sequence:*

$$\begin{aligned} 0 \quad &\longrightarrow \quad \mathrm{H}^1_{\mathrm{meas}}(T, \mathrm{Hom}(N, A)) \xrightarrow{\iota} \tilde{\mathrm{H}}^2(H, A) \xrightarrow{\mathrm{res}} \mathrm{H}^2(N, A)^T \\ &\xrightarrow{d} \quad \mathrm{H}^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A)) \xrightarrow{j} \tilde{\mathrm{H}}^3(H, A). \end{aligned}$$

We prove the above theorem by transporting some arguments from [Ta] into our more general setting.

Remark. *Even though Hopf algebras N and T are cocommutative, we usually do not omit subscripts, when using Sweedler notation in this section.*

First we have to define the Hopf algebra analog of the stable part of cohomology.

Definition 4.4.2 *Let N, T, μ, A be as above. We say that a cohomology class $[f] \in \mathrm{H}^i(N, A)$, where $f \in \mathrm{Z}^i(N, A)$, is T-stable if there exists a convolution invertible linear map $g: T \otimes N^{i-1} \longrightarrow A$, such that $f * (f^{-1})^t = \delta_N g(t \otimes \_)$. The subgroup of $\mathrm{H}^i(N, A)$ consisting of all T-stable elements is called the T-stable part of cohomology and is denoted by $\mathrm{H}^i(N, A)^T$.*

Note that if $T = kG$ then $\mathrm{H}^i(N, A)^T = \mathrm{H}^i(N, A)^G$.

The following lemma is the main tool in establishing this result. It is a generalization of the essential part of Proposition 1 from [Ta]:

Lemma 4.4.3 *Let the Hopf algebra H be a smash product of cocommutative Hopf algebras N and T. Furthermore assume that H acts trivially on a commutative algebra A. Then every cocycle $f: H \otimes H \longrightarrow A$ is cohomologous to a cocycle $f': H \otimes H \longrightarrow A$, which is trivial on $N \otimes T$, i.e. $f'(n \otimes t) = \varepsilon(n)\varepsilon(t)1_A$.*

Remark. *The result of the lemma above also follows from Lemma 3.1.2. Here we present a different proof, which is an adaptation of Tahara's proof. Note also, that the proof works also in the case H is a general bismash product (that is neither of the actions is needed to be trivial). There are also other results stated in this chapter that are consequences of the results in the previous chapter.*

Proof (of Lemma 4.4.3). Let the extension

$$A \xrightarrow{\iota} K \underset{\chi}{\overset{\pi}{\rightleftarrows}} H$$

be an H-comodule algebra extension with associated 2-cocycle f (see Section 1.2.3). We shall denote the H-comodule structure on K by $\rho: K \longrightarrow K \otimes H$. We will show that χ can be "repaired" into a $\chi': H \longrightarrow K$ that satisfies the equality $\chi'(nt) = \chi'(n)\chi'(t)$, for $n \in N$, $t \in T$. Then it is easy to see that the cocycle $f': H \otimes H \longrightarrow A$ associated to χ' satisfies the desired condition.

Let $\{u_i\}_{i \in I}$ be a basis for N and let $\{v_j\}_{j \in J}$ be a basis for T. Then $\{u_i \rtimes v_j\}_{(i,j) \in I \times J}$ is a basis for H. Define a linear map $\chi': H \longrightarrow K$ by the rule $\chi'(u_i \rtimes v_j) = \mathrm{m}_K(\chi(u_i) \otimes \chi(v_j))$. The following calculation shows that χ' is an H-comodule map, i.e. $\rho\chi' = (\chi' \otimes \mathrm{id})\Delta_H$:

$$\begin{aligned}
(\chi' \otimes \mathrm{id})\Delta(u_i \rtimes v_j) &= \sum \chi'((u_i)_1 \rtimes (v_j)_1) \otimes (u_i)_2 \rtimes (v_j)_2 \\
&= \sum \chi(((u_i)_1)\chi((v_j)_1) \otimes (u_i)_2(v_j)_2 \\
&= \sum \mathrm{m}_{K\otimes H}((\chi((u_i)_1) \otimes (u_i)_2) \otimes (\chi((v_j)_1) \otimes (v_j)_2)) \\
&= \mathrm{m}_{K\otimes H}(\rho\chi(u_i) \otimes \rho\chi(v_j)) = \rho\mathrm{m}_K(\chi(u_i) \otimes \chi(v_j)) \\
&= \rho\chi'(u_i \rtimes v_j).
\end{aligned}$$

Now observe $\chi'(u_i) = \chi(u_i)$ and $\chi'(v_j) = \chi(v_j)$ and hence $\chi'(nt) = \chi'(n \rtimes t) = \chi'(n)\chi'(t)$ for $n \in N$ and $t \in T$. ∎

A cocycle f' that satisfies the condition of Lemma 4.4.3 will be called a normalized cocycle.

Corollary 4.4.4 *Let $f: H \otimes H \longrightarrow A$ be a normalized cocycle , where H is a smash product of N and T acting trivially on the commutative algebra A. Then f satisfies* *the following equations:*

$$f(nt \otimes h') = \sum f(t_1 \otimes h'_1) f(n \otimes t_2 h'_2) \tag{4.1}$$

$$f(nt \otimes t') = \varepsilon(n) f(t \otimes t') \tag{4.2}$$

$$f(h \otimes n't') = \sum f(h_1 \otimes n'_1) f(h_2 n'_2 \otimes t') \tag{4.3}$$

$$f(n \otimes n't') = f(n \otimes n')\varepsilon(t') \tag{4.4}$$

$$f(nt \otimes n't') = \sum f(t_1 \otimes t') f(t_2 \otimes n'_1) f(n \otimes t_3(n'_2)) \tag{4.5}$$

for $n, n' \in N$, $t, t' \in T$ and $h, h' \in H$.

Proof. The equations (4.1) and (4.3) are just special cases of the cocycle condition. Equations (4.2) are (4.4) are special cases of (4.1) and (4.3) respectively and (4.5) follows from (4.1)-(4.4). ■

Corollary 4.4.5 *A map $f: H \otimes H \longrightarrow A$ is a normalized 2-cocycle if and only if the following are satisfied:*

1. $f|_{N\otimes T} = \varepsilon$
2. $f|_{N\otimes N}$ *is a 2-cocycle on N*
3. $f|_{T\otimes T}$ *is a 2-cocycle on T*
4. $f(tt' \otimes n') = \sum f(t'_1 \otimes n'_1) f(t \otimes t'_2(n'_2))$, *where $n' \in N$ and $t, t' \in T$*
5. $\sum f(n_1 \otimes n'_1) f^{-1}(t_1(n_2) \otimes t_2(n'_2)) = \sum f(t_1 \otimes n_1) f^{-1}(t_2 \otimes n_2 n'_1) f(t_3 \otimes n'_2)$, *where $n, n' \in N$ and $t \in T$.*

Moreover, the data $f|_{N\otimes N}$, $f|_{T\otimes T}$, $f|_{T\otimes N}$ satisfying the conditions above determine a unique normalized cocycle.

Proof. First assume that f is a normalized 2-cocycle on H. Then clearly f is also a 2-cocycle on both N and T. The equations in conditions 4.3 and 4.5 are obtained from the cocycle condition together with the equations (4.2) and (4.3) from the previous corollary.

Now suppose that we have the data from conditions 4.1-4.5. Then the Equation (4.5) of the Corollary 4.4.4 gives a formula for a map $f: H\otimes H \longrightarrow A$ (it is well defined, since it is linear in each of the variables). An elementary (but lengthy) computation shows that the cocycle condition is satisfied. ■

Let $\tilde{\mathrm{H}}^i(H,A)$ be the kernel of the restriction homomorphism $\mathrm{H}^i(H,A) \stackrel{\mathrm{res}}{\longrightarrow} \mathrm{H}^i(T,A)$. Since the inclusion $T \longrightarrow H$ splits we can conclude that

Proposition 4.4.6

$$\mathrm{H}^i(H,A) \simeq \mathrm{H}^i(T,A) \oplus \tilde{\mathrm{H}}^i(H,A).$$

We shall denote the group of normalized cocycles $H \otimes H \longrightarrow A$ that are trivial when restricted to T by $\mathrm{Z}'^2(H,A)$, i.e. $\mathrm{Z}'^2(H,A) = \{f \in \mathrm{Z}^2(H,A) | f(n\otimes t) = \varepsilon(n)\varepsilon(t)$ and $f(t\otimes t') = \varepsilon(t)\varepsilon(t')$, $n \in N,\ t,t' \in T\}$. Furthermore, let $\mathrm{B}'^2(H,A) = \mathrm{B}^2(H,A) \cap \mathrm{Z}'^2(H,A)$. Using the canonical map $H \longrightarrow T$ together with Corollaries 4.4.4 and 4.4.5 we can show that there is an injective map $\mathrm{B}^2(T,A) \longrightarrow \mathrm{B}^2(H,A)$ and hence

Proposition 4.4.7 $\mathrm{H}'^2(H,A) = \mathrm{Z}'^2(H,A)/\mathrm{B}'^2(H,A) \simeq \tilde{\mathrm{H}}^2(H,A)$.

We proceed by defining the homomorphisms involved in the generalized Tahara sequence, and also prove exactness at the same time.

The injective homomorphism $\mathrm{H}^1_{\mathrm{meas}}(T, \mathrm{Hom}(N,A)) \stackrel{\iota}{\longrightarrow} \tilde{\mathrm{H}}^2(H,A)$:

First define a homomorphism $\iota: \mathrm{Z}^1_{\mathrm{meas}}(H, \mathrm{Hom}(N,A)) \longrightarrow \mathrm{Z}'^2(H,A)$ by the rule $\iota(f)(nt\otimes n't') = f(t)(n')\varepsilon(n)\varepsilon(t')$ (it is well defined, since the map $N\times T\times N\times T \longrightarrow A$, given by $(n,t,n',t') \mapsto f(t)(n')$ is bilinear). This homomorphism induces an injective homomorphism $\mathrm{H}^1_{\mathrm{meas}}(T, \mathrm{Hom}(N,A)) \longrightarrow \tilde{\mathrm{H}}^2(H,A)$. ■

The exactness at $\tilde{\mathrm{H}}^2(H,A)$:

We claim, that the image of the homomorphism just described equals the kernel of the restriction homomorphism $\tilde{\mathrm{H}}^2(H,A) \stackrel{\mathrm{res}}{\longrightarrow} \mathrm{H}^2(N,A)^T$.

Clearly $\operatorname{res}\iota = 0$. Suppose the cocycle $f \in \mathrm{Z}'^2(H, A)$ is such that $f|_{N\otimes N} \in \mathrm{B}^2(N, A)$, that is there exists $g: N \longrightarrow A$ such that $f(n \otimes n') = \delta g(n \otimes n')$. Extend g to a linear map $g: H \longrightarrow A$ by the rule $g(n \rtimes t) = g(n)\varepsilon(t)$. Now define $f' \in \mathrm{Z}^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$ by the rule $f'(t)(n') = \sum f(t_1 \otimes n'_1)g^{-1}(n'_2)g(t_2(n'_3))$ A routine calculation shows that $f * \delta g^{-1} = \iota(f')$ (we use Equation (5) of Corollary 4.4.5 to expand $f(nt \otimes n't')$ and take into account that $f|_{N\otimes N} = \delta g$ and that $f|_{T\otimes T} = \varepsilon$) and hence $[f] = \iota([f'])$. ∎

The homomorphism $\mathrm{H}^2(N, A)^T \xrightarrow{d} \mathrm{H}^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$ and the exactness at $\mathrm{H}^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$:

Take $[f] \in H^2(N, A)^T$, for $f \in \mathrm{Z}^2(N, A)$. Then there is a $g: T \otimes N \longrightarrow A$ s.t. $f * (f^{-1})^t = \delta_N g(t \otimes \_)$, i.e. $\sum f(n_1 \otimes n'_1)f^{-1}(t_1(n_2) \otimes t_2(n'_2)) = \sum g(t_1 \otimes n_1)g^{-1}(t_2 \otimes n_2n'_1)g(t_3 \otimes n'_2)$. Now define d by $(df)(t \otimes t' \otimes n) = \sum g(t'_1 \otimes n_1)g^{-1}(t_1t'_2 \otimes n_2)g(t_2 \otimes t'_3(n_3))$. Clearly $df \in \mathrm{Z}^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$. We claim that the class $[df]$ is independent of the choice of g, which in turn also implies that $d(\mathrm{B}^2(N, A)) \subseteq \mathrm{B}^2(T, \mathrm{Hom}(N, A))$ and hence d gives rise to a homomorphism $\mathrm{H}^2(N, A)^T \longrightarrow \mathrm{H}^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$. So suppose there is a $g': T \otimes N \longrightarrow A$ such that $\sum f(n_1 \otimes n'_1)f^{-1}(t_1(n_2) \otimes t_2(n'_2)) = \sum g'(t_1 \otimes n_1)g'^{-1}(t_2 \otimes n_2n'_1)g'(t_3 \otimes n'_2)$. We need to show that there exists $w \in \mathrm{Reg}_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$ such that $\sum g(t'_1 \otimes n_1)g^{-1}(t_1t'_2 \otimes n_2)g(t_2 \otimes t'_3(n_3))g'^{-1}(t'_4 \otimes n_4)g'(t_4t'_5 \otimes n_5)g'^{-1}(t_5 \otimes t'_6(n_6)) = \sum w(t'_1)(n_1)w^{-1}(t_1t'_2)(n_2)w(t_2)(t'_3(n_3))$. Define w by $w(t \otimes n) = \sum g(t_1 \otimes n_1)g'^{-1}(t_2 \otimes n_2)$ and observe that it does the trick.

It is clear that $d\iota = 0$. Suppose $df \in \mathrm{B}^2(T, \mathrm{Hom}(N, A))$. Then there exists a $w \in \mathrm{Reg}_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$ s.t. $(df)(t \otimes t' \otimes n) = \sum w(t'_1 \otimes n_1)w^{-1}(t_1t'_2 \otimes n_2)w(t_2 \otimes t'_3(n_3))$. Define $z: T \otimes N \longrightarrow A$ by $z(t \otimes n) = \sum h(t_1 \otimes n_1)u^{-1}(t_2 \otimes n_2)$ and note that this gives rise to a normalized cocycle $z \in \mathrm{Z}'^2(H, A)$ given by $z(nt \otimes n't') = z(t \otimes n)$ (see Lemma 4.4.5). ∎

The homomorphism $\mathrm{H}^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A)) \longrightarrow \tilde{\mathrm{H}}^3(H, A)$ and the exactness at $H^2_{\mathrm{meas}}(T, \mathrm{Hom}(N, A))$

Let $f \in Z^2_{\text{meas}}(T, \text{Hom}(N, A))$. Define a map $jf: H \otimes H \otimes H \longrightarrow A$ by $jf(nt \otimes n't' \otimes n''t'') = f(t \otimes t' \otimes n'')$. A straightforward calculation shows that jf is a 3-cocycle on H. Suppose that f is a measuring 2-coboundary. Then there exists $v \in \text{Reg}_{\text{meas}}(T, \text{Hom}(N, A))$ s.t. $f(t \otimes t' \otimes n'') = \sum v(t'_1 \otimes n''_1) v^{-1}(t_1 t'_2 \otimes n''_2) v(t_2 \otimes t'_3(n''_3))$. Now let $v': H \otimes H \longrightarrow A$ be defined by $v'(nt \otimes n't') = v(t \otimes n')$ and show $jf = \delta^2 v'$. This proves that the homomorphism j is well defined.

Suppose $[h] \in H^2(N, A)^T$ and let $u: T \otimes N \longrightarrow A$ be s.t. $h * (h^{-1})^t = \delta(u(t))$. Define $v: H \otimes H \longrightarrow A$ by $v(tn \otimes t'n') = \sum u(t_1 \otimes n'_1) h(n \otimes t_2(n'_2))$ and observe that $jdh = \delta^2 v \in B^3(H, A)$. This shows that $jd = 0$.

Now suppose the measuring 2-cocycle f is such that jf is a 3-coboundary. Then there exists $v \in \text{Reg}(H \otimes H, A)$ s.t. $jf = \delta^2 v$. Define a $u: T \otimes N \longrightarrow A$ by $u(t \otimes n) = \sum v(t_1 \otimes n_1) v^{-1}(t_2(n_2) \otimes t_3)$ and $h: N \otimes N \longrightarrow A$ by $h(n \otimes n') = v(n \otimes n')$ and note that $\delta^1(h) = \eta\varepsilon$ (hence $h \in Z^2(N, A)$) and that $h * (h^{-1})^t = \delta_N u(t \otimes \_)$, so that $[h] \in H^2(N, A)^T$. Observe also that $f(t \otimes t') = \sum u(t'_1) u^{-1}(t_1 t'_2) u^{t'_3}(t_2)$, i.e $[f] = d[h]$.■

Corollary 4.4.8 *If $H = N \otimes T$, i.e. if the action of T on N is trivial, then we have a canonical isomorphism*

$$H^2(H, A) \simeq H^2(T, A) \oplus H^1_{\text{meas}}(T, \text{Hom}(N, A)) \oplus H^2(N, A).$$

Proposition 4.4.9 *If the action of T on N is trivial, then*

$$H^1_{\text{meas}}(T, \text{Hom}(N, A)) \simeq P(T, N, A),$$

where $P(T, N, A)$ denotes the abelian group of maps $f: T \otimes N \longrightarrow A$ that measure in both variables, i.e correspond to algebra maps $T \longrightarrow \text{Hom}(N, A)$ and $N \longrightarrow \text{Hom}(T, A)$.

Remark. *The isomorphism $H^2(N \otimes T, A) \xrightarrow{\sim} H^2(T, A) \oplus H^2(N, A) \oplus P(T, N, A)$ has a description similar to that in the case of group cohomology (for the group cohomology case see for instance [Kar]): $[f] \mapsto ([f|_{T \otimes T}], [f|_{N \otimes N}], \tilde{f})$, where $\tilde{f}(t \otimes n) = \sum f(t_1 \otimes n_1) f^{-1}(n_2 \otimes t_2)$.*

4.5 Sweedler cohomology of a smash product of a group algebra and a universal envelope of a Lie algebra

Here we illustrate how the generalized Tahara sequence sheds some light on the Sweedler cohomology, when the cocommutative Hopf algebra in question is a smash product of a group algebra $T = kG$ and the universal envelope of a Lie algebra $N = U\mathbf{g}$.

Theorem 4.5.1 *Let G be a finite group acting on a Lie algebra $\mathbf{g}$, furthermore let A be a commutative algebra which is also a trivial $U\mathbf{g} \rtimes kG$ module and assume the ground field k contains $|G|^{-1}$. Then*

$$\mathrm{H}^2(U\mathbf{g} \rtimes kG) \simeq \mathrm{H}^2(\mathbf{g}, A^+)^G \oplus \mathrm{H}^2(G, A).$$

Proof. If $|G|^{-1} \in k$ then $\mathrm{H}^i_{\mathrm{meas}}(kG, \mathrm{Hom}(U\mathbf{g}, A))$ is trivial and hence the restriction homomorphism res: $\tilde{\mathrm{H}}^2(N \rtimes T) \longrightarrow \mathrm{H}^2(N, A)^T$ is an isomorphism. So we get the isomorphism $\mathrm{H}^2(N \rtimes T, A) \simeq \mathrm{H}^2(T, A) \oplus \mathrm{H}^2(N, A)^T$. Now $H^2(T, A) = H^2(G, \mathcal{U}(A))$, $\mathrm{H}^2(N, A) \simeq \mathrm{H}^2(\mathbf{g}, A^+)$ (see [Sw]) and $\mathrm{H}^2(N, A)^T \simeq \mathrm{H}^2(\mathbf{g}, A^+)^G$. ∎

Example 4.5.2 *Assume the ground field k has characteristic 0, let $\mathbf{g} = sl_n(k)$ be a Lie algebra consisting of trace zero $n \times n$ matrices (with Lie bracket given by a commutator) and let $G \simeq C_n \leq Gl_n(k)$ be a group generated by the standard n-cycle permutation matrix acting on $\mathbf{g}$ by conjugation. Then $\mathrm{H}^2(U\mathbf{g} \rtimes kG, k) \simeq k^\bullet/(k^\bullet)^n$*

Proof. Apply Theorem 4.5.1 and note that $\mathrm{H}^2(\mathbf{g}, A^+)$ is trivial by the Whitehead's second lemma and that $\mathrm{H}^2(C_n, k^\bullet) = k^\bullet/(k^\bullet)^n$. (example A.3.6). ∎

Chapter 5

On extensions arising from a smash product of Hopf algebras

We start with a smash product $H = N \rtimes T$ of cocommutative Hopf algebras N and T, i.e. with an action $\nu: N \otimes T \longrightarrow N$, that makes N into a T-module bialgebra.

Combining the generalized Kac sequence and generalized Tahara sequence we see that there is an exact sequence

$$\mathrm{H}^2(N \rtimes T, A) \xrightarrow{\mathrm{res}} \mathrm{H}^2(N, A) \xrightarrow{\Phi} \mathcal{H}_2^2(N, T, A),$$

where the homomorphism Φ is given by $[a] \mapsto [\delta_T a^{-1}]$. We call Φ the connecting homomorphism.

5.1 Cohomology of Singer pairs arising from semi-direct products of finite groups

Here we restrict ourselves to the case of a smash product of group algebras kN and kT and the algebra of coefficients is the ground field k. More precisely we are given finite groups N and T and a **right** action $@: N \times T \longrightarrow N$. Note that $k(N \rtimes T) = kN \rtimes kT$. The Singer pair (kT, k^N, μ, ρ) is given as follows:

- The action $\mu: kT \otimes k^N \longrightarrow k^N$ is given by pre-composition, i.e $\mu(t \otimes f)(n) = f(n^t)$.
- The coaction $\rho: kT \longrightarrow kT \otimes k^N$ is trivial.

In this case the double complex for computing the cohomology of a Singer pair kT, k^N is canonically isomorphic to the following complex:

For groups T and N, $\mathrm{Map}(H^{\times m} \times N^{\times n}, k^\bullet)$ shall denote the abelian group of all maps $f: H^{\times m} \times N^{\times n} \longrightarrow k^\bullet$, that satisfy the condition $f(h_1, \dots, h_m; x_1, \dots x_n) = 1$ whenever one of the elements $h_1, \dots, h_m; x_1, \dots x_n$ is a unit element. The differentials $\delta_H: \mathrm{Map}(H^{\times m} \times N^{\times n}, k^\bullet) \longrightarrow \mathrm{Map}(H^{m+1} \times N^{\times n}, k^\bullet)$ and $\delta_N: \mathrm{Map}(H^{\times m} \times N^{\times n}, k^\bullet) \longrightarrow \mathrm{Map}(H^{\times m} \times N^{n+1}, k^\bullet)$, are defined by

$$\begin{aligned}
&\delta_H(f)(x_1, \dots, x_{m+1}; \mathbf{u}) = (x_1(f))(x_2, \dots, x_{m+1}; \mathbf{u}) \\
&\quad \left(\prod_{i=1}^{m} f(x_1, \dots, x_i x_{i+1}, \dots, x_{m+1}; \mathbf{u})^{(-1)^i} \right) f(x_1, \dots x_m; \mathbf{u})^{(-1)^{m+1}} \\
&\delta_N(f)(\mathbf{x}; u_1, \dots, u_{n+1}) = f(\mathbf{x}, u_2, \dots, u_{n+1}) \\
&\quad \left(\prod_{j=1}^{n} f(\mathbf{x}; u_1, \dots, u_j u_{j+1}, \dots, u_{n+1})^{(-1)^j} \right) f(\mathbf{x}, u_1, \dots u_n)^{(-1)^{m+1}},
\end{aligned}$$

where

$$x(f)(\mathbf{y}, \mathbf{u}) = f(\mathbf{y}; x^{-1}(\mathbf{u})) = f(\mathbf{y}; x^{-1}(u_1), \dots, x^{-1}(u_n)).$$

5.1.1 The connecting homomorphism

In this special case the connecting homomorphism $\Phi: \mathrm{H}^2(N, k^\bullet) \longrightarrow \mathrm{H}^2_c(kT, k^N)$ is given by $\Phi(c)(t; u, v) = c(u, v)c^{-1}(u^t, v^t)$. The results from above give us:

Theorem 5.1.1 *The sequence*

$$\mathrm{H}^2(N \rtimes T, k^\bullet) \xrightarrow{\mathrm{res}} \mathrm{H}^2(N, k^\bullet) \xrightarrow{\Phi} \mathrm{H}^2_c(kT, k^N)$$

is exact.

Corollary 5.1.2 $\ker \Phi \leq \mathrm{H}^2(N, k^\bullet)^T$

Proof. This follows from the above theorem. It is also possible to see this directly: Let $[c] \in \ker \Phi$, i.e. $\beta_c \in \mathrm{B}^2_c(kT, k^N)$. Then $\forall x \in T$:

$$(cx(c)^{-1})(u,v) = \beta_c(x)(u,v) = \gamma(x)(u)^{-1}\gamma(x)(uv)\gamma^{-1}(x)(v), \ u,v \in N$$

which means that $cx(c)^{-1} \in \mathrm{B}^2(N, k^\bullet)$. Hence $\forall x \in T$, $[c] = [x(c)]$, i.e. $[c] \in \mathrm{H}^2(N, k^\bullet)^T$. ■

Remark. *In the following example from [Ta] the inequality in the above Corollary is a strict one. Let* $N = C_9 = \langle u \rangle$, $T = C_3 = \langle x \rangle$, $k = \mathbb{F}_4$ *(hence* $k^\bullet \simeq C_3$*) and let the action of* T *on* N *be given by* $u^x = u^4$*. Then* $\mathrm{H}^2(N, k^\bullet)^T = \mathrm{H}^2(N, k^\bullet) = C_3$ *and* $\ker \Phi = 0$.

The following proposition about group cohomology will help to establish the equality $\ker \Phi = \mathrm{H}^2(N, k^\bullet)^T$ in case $\gcd(|N|, |T|) = 1$.

Proposition 5.1.3 *If* $\gcd(|T|, |N|) = 1$*, then each cohomology class in* $\mathrm{H}^2(N, k^\bullet)^T$ *has a* T*-invariant representative in* $\mathrm{Z}^2(N, k^\bullet)$*. Moreover for each* $[\alpha] \in \mathrm{H}^2(N, k^\bullet)^T$ *there exists a* $\beta \in \mathrm{Z}^2(N, k^\bullet)$ *s.t.* $\alpha \sim \prod_{x \in T} x(\beta)$.

Proof: Let $h, n \in \mathbb{N}$ be such that $h|T| + n|N| = 1$. Define $\beta = \alpha^h$. Note $\beta^{|T|} \sim \alpha$, since

$$\beta^{|T|} = \alpha^{h|T|} = \alpha^{1-n|N|} = \alpha\alpha^{-n|N|} \sim \alpha.$$

Since $[\beta] \in \mathrm{H}^2(N, k^\bullet)^T$ we have

$$\beta^{|T|} = \prod_{x \in T} \beta \sim \prod_{x \in T} x(\beta).$$

■

Corollary 5.1.4 *If* $\gcd(|T|, |N|) = 1$ *then* $\ker \Phi = H^2(N, k^\bullet)^T$.

Proof: We've already seen that $\ker \Phi \leq H^2(N, k^\bullet)^T$. Let $[c] \in \mathrm{H}^2(N, k^\bullet)^T$. By the Proposition above there exists $c' \in \mathrm{Z}^2(N, k^\bullet)$, s.t. $x(c') = c'$ and $c \sim c'$. Now $\beta_c \sim \beta_{c'}$, but $\beta_{c'}(x) = 1_{k^N}$. ■

5.1.2 The coconnecting homomorphism

For $\beta \in Z^2_c(kT, k^N)$ we define a map $c_\beta: N \times N \longrightarrow k$ by the rule $c_\beta(u,v) = \prod_{x\in T}\beta(x)(u,v)$. Since each $\beta(x)$ satisfies the group cocycle condition, so will their product. Hence $c_\beta \in Z^2(N, k^\bullet)$. If $\beta \in B^2_c(kT, k^N)$ then in particular for each $x \in T$ there is $\gamma(x): N \longrightarrow k$ s.t. $\beta(x) = \delta(\gamma(x))$ and hence $c_\beta = \delta(\prod_{x\in T}\gamma(x))$, i.e. $c_\beta \in B^2(N, k^\bullet)$. Note that the map $\beta \mapsto c_\beta$ induces a homomorphism $\Psi: H^2_c(kT, k^N) \longrightarrow H^2(N, k^\bullet)$. We shall call it the coconnecting homomorphism.

Theorem 5.1.5 *Let $\Phi: H^2(N, k^\bullet) \longrightarrow H^2_c(kT, k^N)$ be as defined in the previous section and let Ψ be as above. Then $\Phi\Psi = \_^{|T|}$, i.e. $\Phi(\Psi([\beta])) = [\beta^{|T|}]$.*

Proof: We calculate:

$$\begin{aligned}
\beta_{c_\beta}(x) &= c_\beta x(c_\beta)^{-1} = \prod_{y\in T}\beta(y)\left(\prod_{y\in T}x(\beta(y))\right)^{-1} \\
&= \prod_{z\in T}\beta(xz)\prod_{y\in T}x(\beta(y))^{-1} \\
&= \prod_{y\in T}\beta(xy)x(\beta(y))^{-1} \\
&= \prod_{y\in T}\beta(x) = \beta(x)^{|T|}.
\end{aligned}$$

■

Corollary 5.1.6 *If $\gcd(|N|, |T|) = 1$ then $\Phi\Psi$ is an isomorphism. So in this case Φ is an epimorphism and Ψ is a monomorphism and*

$$H^2(N, k^\bullet) = H^2(N, k^\bullet)^T \oplus H^2_c(kT, k^N).$$

Proof: Let $n|N| + h|T| = 1$. Observe that $\_^{-h}$ is the inverse of $\_^{|T|}$. ■

5.1.3 The map $H^2(kT, k^N) \xrightarrow{\pi} H^1(T, H^2(N, k^\bullet))$

Now we shall provide a description of $H^1(T, H^2(N, k^\bullet))$. Let $B \leq A$ be abelian groups, we write the group operation multiplicatively. Recall ([Kar]) that

$$Z^1(T, A/B) = \mathrm{Der}(T, A/B) = \frac{\{f: T \longrightarrow A \mid f(xy)f(x)^{-1}x(f(y))^{-1} \in B\}}{\{f: T \longrightarrow B\}}$$

and

$$B^1(T, A/B) = \mathrm{IDer}(T, A/B) = \frac{\{f: T \longrightarrow A | \exists a \in A s.t. f(x)x(a)^{-1}a \in B\}}{\{f: T \longrightarrow B\}}$$

So

$$\mathrm{H}^1(T, A/B) = \frac{\{f: T \longrightarrow A | f(xy)f(x)^{-1}x(f(y))^{-1} \in B\}}{\{f: T \longrightarrow A | \exists a \in A s.t. f(x)x(a)^{-1}a \in B\}}.$$

Let

$$\mathrm{Z}'^1(T, A/B) = \{f: T \longrightarrow A | f(xy)f(x)^{-1}x(f(y))^{-1} \in B\}$$

and

$$\mathrm{B}'^1(T, A/B) = \{f: T \longrightarrow A | \exists a \in A s.t. f(x)x(a)^{-1}a \in B\}$$

and observe that $\mathrm{H}^1(T, A/B) = \mathrm{Z}'^1(T, A/B)/\mathrm{B}'^1(T, A/B)$. Now let $A/B = H^2(N, k^\bullet)$, more precisely $A = \mathrm{Z}^2(N, k^\bullet)$, $B = \mathrm{B}^2(N, k^\bullet)$. With this in mind we have

$$\mathrm{Z}'^1(T, \mathrm{H}^2(N, k^\bullet)) = \mathrm{Z}'^1(T, \mathrm{Z}^2(N, k^\bullet)/\mathrm{B}^2(N, k^\bullet)) =$$
$$\{f \in \mathrm{Map}(T \times N^2, k^\bullet) | \delta_N f \equiv 1 \text{ and } \exists g \in \mathrm{Map}(H^2 \times N, k^\bullet) \text{ s.t. } \delta_H f = \delta_N g\}$$

and

$$\mathrm{B}'^1(T, \mathrm{H}^2(N, k^\bullet)) = \mathrm{B}'^1(T, \mathrm{Z}^2(N, k^\bullet)/\mathrm{B}^2(N, k^\bullet)) =$$
$$\{f \in \mathrm{Map}(T \times N^2, k^\bullet) | \exists g \in \mathrm{Map}(T \times N, k^\bullet) \text{ and } \exists h \in \mathrm{Map}(N \times N, k^\bullet) \text{ s.t. }$$
$$f(\delta_H h)^{-1} = \delta_N g \text{ and } \delta_N h \equiv 1\}$$

Now note that there is a homomorphism

$$\mathrm{Z}^2(kT, k^N) \longrightarrow \mathrm{Z}'^1(T, \mathrm{H}^2(N, k^\bullet))$$

given by

$$(\alpha, \beta) \mapsto \beta,$$

which in turn induces a homomorphism

$$\pi: \mathrm{H}^2(kT, k^N) \longrightarrow \mathrm{H}^1(T, H^2(N, k^\bullet)).$$

Theorem 5.1.7 *The sequence*

$$\mathrm{H}^2(N,k^\bullet)\oplus \mathrm{H}^2_{\mathrm{m}}(kT,k^N)\xrightarrow{\Phi+\iota}\mathrm{H}^2(kT,k^N)\xrightarrow{\pi}\mathrm{H}^1(T,\mathrm{H}^2(N,k^\bullet)).$$

is exact.

Proof. It is apparent that $\pi\Phi = 0$ and obviously also $\pi(\mathrm{H}^2_{\mathrm{m}}) = 0$. Hence $\Phi(\mathrm{H}^2(N,k^\bullet)) + \mathrm{H}^2_{\mathrm{m}}(kT,k^N) \subseteq \ker\pi$.

Suppose $(\alpha,\beta) \in \mathrm{Z}^2(kT,k^N)$ and $\beta \in \mathrm{B}'^1(T,\mathrm{H}^2(N,k^\bullet))$. Then $\exists\gamma \in \mathrm{Map}(T\times N,k^\bullet)$ and $\exists c \in \mathrm{Z}^2(N,k^\bullet)$ s.t. $\beta\delta_H(c)^{-1} = \delta_N(\gamma)$, but that means $(\alpha,\beta) = (\eta\varepsilon,\delta_H(c))*(\alpha,\delta_N(\gamma)) \sim (\eta\varepsilon,\delta_H(c))*(\alpha\delta_H(\gamma),\eta\varepsilon)$. ■

Corollary 5.1.8 *If the orders of N and T are relatively prime, then* $\mathrm{H}^2(kT,k^N) = \mathrm{H}^2_{\mathrm{c}}(kT,k^N) \simeq \mathrm{H}^2(N,k^\bullet)/\mathrm{H}^2(N,k^\bullet)^T$.

Proof. Note that if $\gcd(|N|,|T|) = 1$, then $\mathrm{H}^2_{\mathrm{m}}(kT,k^N) = \mathrm{H}^2(T,\mathrm{H}^1(N,k^\bullet)) = 0$ and $\mathrm{H}^1(T,\mathrm{H}^2(N,k^\bullet)) = 0$. ■

5.2 Examples

5.2.1 Dihedral groups

Let $N \rtimes T$ be a dihedral group of order $2n$, i.e. let $N = C_n = \langle u\rangle$, $T = C_2 = \langle x\rangle$, and let T act on N by $x(u) = u^{-1}$. Remember that the isomorphism $k^\bullet/k^{\bullet n} \xrightarrow{\sim} \mathrm{H}^2(C_n,k^\bullet)$ is defined by $ak^{\bullet n} \mapsto [c_a]$, where $a \in k^\bullet$ and $c_a \in \mathrm{Z}^2(C_n,k^\bullet)$ is given by $c_a(u^i,u^j) = a^{\lfloor\frac{i+j}{n}\rfloor}, i,j \in \{0,1,\ldots,n-1\}$.

For $a \in k^\bullet$ define $t_a \in \mathrm{Z}^1(N,k^\bullet)$ by

$$t_a(u^i) = \begin{cases} 1 & ;u^i = 1 \\ a & ;\text{otherwise} \end{cases}.$$

Now note the

$$c_a x(c_a) = \delta(t_a) \in \mathrm{B}^2(N,k^\bullet).$$

Hence if n is odd then $\mathrm{H}^2(N,k^\bullet)^T$ is trivial, since by Proposition 5.1.3 every cohomology class in $\mathrm{H}^2(N,k^\bullet)^T$ has a representative of the form $cx(c)$.

In case n is even, $\mathrm{H}^2(N,k^\bullet)^T$ will not be trivial. This is seen by observing that

$$c_a x(c_a)^{-1}\delta(t_a) = c_a x(c_a)^{-1} c_a x(c_a) = c_a^2 = c_{a^2}.$$

Hence $c_a \sim x(c_a)$ if and only if $a^2 \in k^{\bullet n}$. This happens when either $a \in k^{\bullet \frac{n}{2}}$ or $a \in -k^{\bullet \frac{n}{2}}$. Hence in case n is even

$$\mathrm{H}^2(N,k^\bullet)^T \simeq k^\bullet / k^{\bullet \frac{n}{2}} \cup -k^{\bullet \frac{n}{2}}.$$

Combining the connecting homomorphism $k^\bullet/k^{\bullet n} \simeq \mathrm{H}^2(N,k^\bullet) \xrightarrow{\Phi} \mathrm{H}^2_c(kT,k^N)$ and the homomorphism $\mathrm{H}^2_c(kT,k^N) \leq \mathrm{H}^2(kT,k^N) \simeq \mathrm{Opext}(kT,k^N)$ we now get a homomorphism

$$\Phi' \colon k^\bullet/k^{\bullet n} \longrightarrow \mathrm{Opext}(k^N,kT):$$

Define $\beta_a = \Phi(c_a)$ and abbreviate $a_{i,j} = \beta_a(x)(u^i,u^j)$, i.e.

$$\begin{aligned} a_{i,j} &= \beta_a(x)(u^i,u^j) = (c_a x(c_a)^{-1})(u^i,u^j) \\ &= \begin{cases} 1 & ; i=0 \text{ or } j=0 \text{ or } i+j=n \\ a^{-1} & ; 0<i; \text{and } 0<j \text{ and} i+j<n \\ a & ; n<i+j \end{cases} . \end{aligned}$$

So

$$\beta_a(x) = \sum_{i,j} a_{i,j} p_i \otimes p_j,$$

where $p_i \in k^N$ denotes the characteristic function of $u^i \in N$, i.e. $p_i(u^j) = \delta_{i,j}$.

Now define C_a to be a free k^N module, with basis $\{1,\hat{x}\}$. We define multiplication on C_a by the rule

$$\hat{x}^2 = 1 \text{ and } \hat{x}p_i = p_{-i}\hat{x},$$

and comultiplication by the rule

$$\Delta\hat{x} = \sum_{i,j} a_{i,j} p_i\hat{x} \otimes p_j\hat{x}.$$

This gives a Hopf algebra extension

$$(C_a): k^N \longrightarrow C_a \longrightarrow kT,$$

where the projection $C_a \longrightarrow kT$ is induced by $\hat{x} \mapsto x$. Now define $\Phi'([a]) = [(C_a)]$. If n is odd, then Φ' is an isomorphism by Theorem 5.1.7

Remark. In case k has "few" n-th roots, Opext is "very big". In particular if $k = \mathbf{Q}$, we get infinitely many of non isomorphic extensions.

5.2.2 The cyclic group of order 2 acting on $G \times G$

Let G be a finite group, $N = G \times G$ and $T = C_2 = \langle x \rangle$, and let T act on N by $x(u,v) = (v,u)$, $u,v \in G$. It follows from Theorem A.3.1 (Appendix) that $\mathrm{H}^2(N,k^\bullet) \simeq \mathrm{H}^2(G,k^\bullet) \times \mathrm{H}^2(G,k^\bullet) \times \mathrm{P}(G,G,k^\bullet)$. The comultiplication Hopf algebra cocycle $\beta_{c,c',f} \in \mathrm{H}^2_c(kT,k^N)$ associated to (c,c',f) is defined as follows:

$$\begin{aligned}\beta_{c,c',f}(x;(u,v),(u',v')) &= c(u,v)c'(u',v')f(u,v')c(u',v')^{-1}c'(u,v)^{-1}f(v,u') \\ &= \beta_{c/c',1,f}(x;(u,v),(u',v')).\end{aligned}$$

It depends on the quotient c/c' only and thus induces a homomorphism $\mathrm{H}^2(G,k^\bullet) \times \mathrm{P}(G,G,k^\bullet) \xrightarrow{\Phi'} \mathrm{H}^2_c(kT,k^N)$. Direct computation shows, that $\mathrm{P}(G,G,k^\bullet)$ is invariant under the action of T and that the action is given by $x(f)(u,v) = f^{-1}(v,u)$. Using this we can deduce that $\ker\Phi'$ is contained in $\mathrm{P}(G,G,k^\bullet)^T$.

5.2.3 Example revisited

In case G is a cyclic group the example above becomes an example already mentioned in [Kac] (if $k = \mathbf{C}$) and also in [Ma1] (if $k^\bullet = k^{\bullet n}$). So let $G = C_n$, more precisely $N = C_n \times C_n = \langle u,v \rangle$. Recall that $\mathrm{H}^2(C_n,k^\bullet) \simeq k^\bullet/k^{\bullet n}$ and $\mathrm{P}(C_n,C_n,k^\bullet) = \Omega_n$, where Ω_n denotes the subgroup of n-th roots of unity in $k^\bullet$. Direct computation shows

$$\Omega_n^T = \begin{cases} \{1\} & ; n \text{ odd} \\ \{1,-1\} & ; n \text{ even} \end{cases}$$

So in case n is an odd integer

$$\mathrm{H}^2_c(kT, k^N) \simeq k^\bullet/k^{\bullet n} \times \Omega_n.$$

Now let us examine the homomorphism $\Phi': k^\bullet/k^{\bullet n} \times \Omega_n \longrightarrow \mathrm{H}^2_c(kT, k^N)$ more closely:

$$\beta_{a,t}(x)(u^i v^j, u^k v^l) \;=\; a^{\lfloor (i+k)/n \rfloor - \lfloor (j+l)/n \rfloor} t^{il-jk}.$$

If n is odd, Φ' is a monomorphism.

Now assume n is even. The following paragraph will show that $\ker \Phi' = 1 \times \{-1, 1\}$:

Define $\gamma \in \mathrm{Reg}(kT, k^N)$, by the rule

$$\gamma(x; u^i v^j) = (-1)^{ij} \text{ and } \gamma(1_N; u^i v^j) = 1.$$

We can routinely verify that $\gamma(x)\gamma(xy)^{-1}x(\gamma(y)) = 1_{k^N}$ and hence $\delta_N\gamma \in \mathrm{B}^2_c(kT, k^N)$. A simple computation also shows that

$$\delta_N\gamma(x; u^i v^j, u^k v^l) = (-1)^{ij+kl} = (-1)^{ij-kl}. \blacksquare$$

Hence in case n is even

$$\beta_{a,t} \sim \beta_{a,t}\delta\gamma = \beta_{a,-t}.$$

The extensions associated with these cocycles are the following:

$$(C_{a,t}): k^N \longrightarrow C_{a,t} \longrightarrow kT$$

where $C_{a,t}$ is a free k^N module with basis $\{1, \hat{x}\}$, with multiplication given by

$$\hat{x}^2 = 1$$

and comultiplication given by

$$\Delta\hat{x} = \sum_{i,j,k,l} a^{\lfloor \frac{i+j}{n} \rfloor - \lfloor \frac{l+k}{n} \rfloor} t^{il-jk} p_{i,j}\hat{x} \otimes p_{k,l}\hat{x},$$

where $p_{i,j} \in k^N$ denotes a characteristic function of $u^i v^j \in N$.

Remark. *The following paragraph will describe the explicit isomorphism between extensions as above and the extensions described in [Ma1]. Assume that n is odd. For $t \in \Omega_n$ define $\gamma_t \in \mathrm{Reg}(kT, k^N)$ by the rule:*

$$\gamma_t(x; u^i v^j) = t^{ij}, \ \gamma_t(1; u^i v^j) = 1.$$

Let $(\alpha_t, \beta_t) = ((\delta_N \gamma)^{-1}, \delta_H \gamma) \in \mathrm{B}^2(kT, k^N)$. A routine calculation shows

$$\alpha_t(x,x)(u^i v^j) = t^{2ij} \text{ and } \beta_t(x)(u^i v^j, u^k v^l) = t^{il+jk}.$$

Let

$$(D_t) \colon k^N \longrightarrow D_t \longrightarrow kT$$

be an extension as in [Ma1], i.e. D_t is a free k^N module with basis $\{1, \hat{x}\}$, where multiplication is given by

$$\hat{x}^2 = \sum_{i,j} t^{ij} p_{i,j},$$

and comultiplication is given by

$$\Delta \hat{x} = \sum_{i,j,k,l} t^{jk} p_{i,j} \hat{x} \otimes p_{j,k} \hat{x}.$$

We have shown above that $(C_{1,t}) \sim (D_{t^{-2}})$.

5.2.4 A noncommutative example

Let now $G = G_{k,m,r}$ as in A.3.7. The cohomology groups $\mathrm{H}^2(G_{k,m,r}, k^\bullet)$ are already explicitly described in [Ta]. So the only ingredient missing is $\mathrm{P}(G_{k,m,r}, G_{k,m,r}, k^\bullet)$. First we need to calculate the abelianization of the group $G_{k,m,r}$:

$$\begin{aligned}
(G_{k,m,r})_{ab} &= \langle u, v | u^k = v^m = 1, vuv^{-1} = u^r, uv = vu \rangle \\
&= \langle u, v | u^k = v^m = 1, u = u^r, uv = uv \rangle \\
&= \langle u, v | u^{\gcd(k,r-1)} = v^m = 1, uv = uv \rangle \\
&= \langle u | u^{\gcd(k,r-1)} = 1 \rangle \times \langle v | v^m = 1 \rangle \\
&= C_{\gcd(k,r-1)} \times C_m.
\end{aligned}$$

If $l = \gcd(k, r-1)$ and $s = \gcd(k, m, r-1)$ then

$$\mathrm{P}(G_{k,m,r}, G_{k,m,r}, k^\bullet) \simeq \Omega_l \times \Omega_s \times \Omega_s \times \Omega_m,$$

where the paring f_{t_1,t_2,t_3,t_4} corresponding to $(t_1, t_2, t_3, t_4) \in \Omega_l \times \Omega_s \times \Omega_s \times \Omega_m$ is given by

$$f_{t_1,t_2,t_3,t_4}(u^i v^j, u^p v^q) = t_1^{ip} t_2^{iq} t_3^{jp} t_4^{pq}.$$

While it is possible to describe this example in this generality we shall restrict ourselves to the simpler case when $\gcd(k, r-1) = 1$. In this case we have an isomorphism $\mathrm{P}(G_{k,m,r}, G_{k,m,r}, k^\bullet) \simeq \Omega_m$, where the pairing $f_t \in \mathrm{P}(G_{k,m,r}, G_{k,m,r}, k^\bullet)$, corresponding to $t \in \Omega_m$ is given by $f_t(u^i v^j, u^p v^q) = t^{jq}$. A Hopf algebra cocycle $\beta_{a,b,t} \in \mathrm{H}^2_{\mathrm{c}}(kT, k^N)$ corresponding to a pair $(f_{a,b}, f_t) \in \mathrm{Z}^2(G_{k,m,r}, k^\bullet) \times \mathrm{P}(G_{k,m,r}, G_{k,m,r}, k^\bullet)$ is given by

$$\begin{aligned} &\beta_{a,b,t}(x; (u^i v^j, u^p v^q), (u^{i'} v^{j'}, u^{p'} v^{q'})) \\ &= \; f_{a,b}(u^i v^j, u^{i'} v^{j'}) f_{a,b}(u^p v^q, u^{p'} v^{q'})^{-1} t^{jq' - qj'}, \end{aligned}$$

(for the definition of $f_{a,b}$ consult Example A.3.8) and if m is odd then

$$\Phi' \colon \tilde{\mathrm{H}}^2(G_{k,m,r}, k^\bullet) \times \Omega_m \longrightarrow \mathrm{H}^2_{\mathrm{c}}(kT, k^N),$$

induced by $(a, b, t) \mapsto \beta_{a,b,t}$, is a monomorphism (if m is even $\ker \phi' = \{-1, 1\} \leq \Omega_m$). If k is odd as well then Φ' is an isomorphism (and also $\mathrm{H}^2_{\mathrm{c}} \simeq \mathrm{Opext}$).

5.3 Cocyclic forms

The aim of this section is to extend the class of examples given in Section 5.2.2.

Let the finite group G be a direct product of subgroups N and T and let A be a trivial $\mathbb{Z}G$ module. A classical result on Hochschild cohomology of finite groups tells us that there is an isomorphism $\phi \colon \mathrm{H}^2(G, A) \xrightarrow{\sim} \mathrm{H}^2(N, A) \oplus \mathrm{H}^2(T, A) \oplus \mathrm{P}(N, T, A)$ (appendix).

This generalizes to G being a direct product of any finite number of its subgroups. Let $G = \prod N_i$.

We will take a close look at the isomorphisms

$$\mathrm{H}^2(G,A) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \prod_i \mathrm{H}^2(N_i,A) \times \prod_{i<j} \mathrm{P}(N_i,N_j,A).$$

We shall denote the elements of

$$\prod_i \mathrm{H}^2(N_i,A) \times \prod_{i<j} \mathrm{P}(N_i,N_j,A)$$

by boldface letters. The subscript will denote the projection to the appropriate component. Hence if $\mathbf{u} = (c_i)_{i=1}^n \times (f_{i,j})_{1\le i<j\le n}$, where $c_i \in \mathrm{H}^2(N_i,A)$ and $f_{i,j} \in \mathrm{P}(N_i,N_j,A)$ then $\mathbf{u}_{i,i} = c_i$ and $\mathbf{u}_{i,j} = f_{i,j}$. We can view $\mathbf{u}$ as a matrix

$$\mathbf{u} = \begin{pmatrix} c_1 & f_{1,2} & \cdots & f_{1,n} \\ & c_2 & \cdots & f_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix}.$$

Note that $\psi(\mathbf{u})$ becomes a "quadratic form", i.e. for column "vectors" $\mathbf{g}, \mathbf{h} \in G$ we have $\psi(\mathbf{u})(\mathbf{g},\mathbf{h}) = \mathbf{g}^{\mathrm{T}}\mathbf{u}\mathbf{h}$, more precisely:

$$\psi(\mathbf{u})(\mathbf{g},\mathbf{h}) = (g_1, g_2, \ldots, g_n) \cdot \begin{pmatrix} c_1 & f_{1,2} & \cdots & f_{1,n} \\ & c_2 & \cdots & f_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix};$$

in future we shall identify $\mathbf{u}$ and $\psi(\mathbf{u})$.

More notation

A cocyclic form (described above) works as follows (also keep in mind that our abelian group operation is written multiplicatively):

For maps $f\colon G \longrightarrow K$, $\lambda\colon K \times L \longrightarrow A$, $g\colon T \longrightarrow L$, the map $\beta = f \cdot \lambda \cdot g$ shall denote a "product" $\beta\colon G \times T \longrightarrow A$ given by $\beta(x,y) = \lambda(f(x), g(y))$. We shall identify

$f \cdot \lambda = f \cdot \lambda \cdot \mathrm{id}$ and $\lambda \cdot g = \mathrm{id} \cdot \lambda \cdot g$. Elements $x \in K$ and $y \in L$ shall be identified by maps $1 \longrightarrow K$ and $1 \longrightarrow L$ and thus $x\lambda = \lambda(x, \_)$, $\lambda y = \lambda(\_, y)$ and $x\lambda y = \lambda(x, y)$.

We shall say that a map $\lambda: N \times N \longrightarrow A$ has a matrix representation if there exists a matrix $\Lambda = (\lambda_{i,j})_{i,j=1}^{n}$, $\lambda_{i,j}: G_i \times G_j \longrightarrow A$ s.t.

$$\lambda(\mathbf{x}, \mathbf{y}) = \mathbf{x}^{\mathrm{T}} \Lambda \mathbf{y} = \prod_{i,j} \lambda_{i,j}(x_i, y_j).$$

We have seen above that every cocycle is cohomologous to one that has a matrix representation. In fact there exists a canonical upper triangular matrix, we shall denote it by

$$Q_\lambda = \phi(\lambda),$$

that has cocycle diagonal entries and whose strictly upper triangular entries are pairings. The matrix is unique up to cohomology classes of its cocycle entries.

Moreover if we have a matrix form Λ of a cocycle λ, it is easy to obtain its triangularization $Q_\lambda = (q_{i,j})$ in the following way:

$$\begin{aligned} q_{i,i} &= \lambda_{i,i} \\ q_{i,j}(x_i, y_j) &= \lambda_{i,j}(x_i, y_j)\lambda_{j,i}(y_j, x_i)^{-1}, \ i < j. \end{aligned}$$

Remark. *If G is a finite abelian group, then we can write $G = C_{m_1} \times \ldots \times C_{m_n}$. If k is algebraically closed (it suffices that $(k^\bullet)^{\mathrm{lcm}(m_1,\ldots,m_n)} = k^\bullet$), then all cohomology groups $\mathrm{H}^2(C_{m_i}, k^\bullet)$ are trivial and for every cocycle $\lambda \in \mathrm{H}^2(G, k^\bullet)$ there exists a unique strictly upper triangular matrix Q_λ, whose entries are pairings. Hence in this case we are really dealing with a bilinear transformation.*

Acting on cocyclic forms

Assume now that a group T acts on the group G and let $f: G \longrightarrow G$ be a homomorphism. Then f has a matrix form $F = (f_{i,j})_{i,j}$, $f_{i,j} = \iota_i f \pi_j: N_i \longrightarrow N_j$, i.e.

$$f(\mathbf{x}) = F\mathbf{x} = (\prod_i f_{i,k}(x_i))_k.$$

Now F acts on Λ via the action of f on λ.

Lemma 5.3.1

$$F(\Lambda) = F^{\mathrm{T}}\Lambda F.$$

Proof.

$$f(\lambda)(\mathbf{x},\mathbf{y}) = \lambda(F\mathbf{x}, F\mathbf{y}) = (F\mathbf{x})^{\mathrm{T}}A(F\mathbf{y}) = \mathbf{x}^{\mathrm{T}}(F^{\mathrm{T}}\Lambda F)\mathbf{y}. \blacksquare$$

Proposition 5.3.2 *If $N_1 = N_2 = \ldots = N_n = H$ and $f \in S_n$, i.e. F is a permutation matrix, then*

$$(F(A))_{i,j} = \begin{cases} \lambda_{f(i),f(j)} & ; f(i) \le f(j) \\ 1 & ; \text{otherwise} \end{cases}.$$

Observe that S_n acts on $\mathrm{H}^2(H,k^\bullet)^{\times n} \times \mathrm{P}(H,H,k^\bullet)^{\times\binom{n}{2}}$ via ϕ and ψ, i.e. by $f(\mathbf{u}) = \phi(f(\psi(\mathbf{u})))$. We are able to describe this action more precisely:

Proposition 5.3.3 *For $f \in S_n$, the action*

$$f\colon \mathrm{H}^2(H,k^\bullet)^{\times n} \times \mathrm{P}(H,H,k^\bullet)^{\times\binom{n}{2}} \longrightarrow \mathrm{H}^2(H,k^\bullet)^{\times n} \times \mathrm{P}(H,H,k^\bullet)^{\times\binom{n}{2}}$$

is given by

$$\begin{aligned} f(\mathbf{u})_{k,k} &= \mathbf{u}_{f(k)}, \\ f(\mathbf{u})_{k,l} &= \begin{cases} \mathbf{u}_{f(k),f(l)} & ; f(k) < f(l) \\ \mathbf{u}^{-1}_{f(l),f(k)} & ; f(k) > f(l) \end{cases}, \end{aligned}$$

where $\mathbf{u}$ is a cocyclic form (upper triangular matrix) corresponding to a cocycle in $H^2(G,k^\bullet) \simeq \mathrm{H}^2(H,k^\bullet)^{\times n} \times \mathrm{P}(H,H,k^\bullet)^{\times\binom{n}{2}}$.

Proof. The result above is a direct application of Lemma 5.3.1. $\blacksquare$

Corollary 5.3.4 *Subgroups $\mathrm{H}^2(H,k^\bullet)^{\times n}$ and $\mathrm{P}(H,H,k^\bullet)^{\times\binom{n}{2}}$ of $\mathrm{H}^2(H,k^\bullet)^{\times n} \times \mathrm{P}(H,H,k^\bullet)^{\times\binom{n}{2}}$ are invariant under the action of S_n.*

Corollary 5.3.5 *If the action of T on the set $\mathbb{N}_n = \{1,2,\ldots,n\}$ is transitive, then the T-invariant subgroup of $\mathrm{H}^2(H,k^\bullet)^{\times n}$ is the diagonal subgroup, i.e. $(\mathrm{H}^2(H,k^\bullet)^{\times n})^T = \{(c,c,\ldots,c) | c \in \mathrm{H}^2(G,A)\}$.*

Corollary 5.3.6 *If the action of T on the set $\mathbb{N}_n$ is doubly transitive then $\mathrm{P}(H,H,k^\bullet)^{\times\binom{n}{2}}$ has no T-invariant elements.*

5.3.1 Examples

Here we illustrate how the theory above could be used to produce a multitude of examples of Hopf algebra extensions. Let R be a commutative ring. Write $\mathcal{M}_n(R)$ for the algebra of matrices with entries in R and let $B = (R, +)$ be the additive group of our ring. Then any group $T \leq \mathcal{U}(\mathcal{M}_n(R))$ acts on $N = B^n$ in the obvious way. The machinery above then enables us to explicitly describe some extensions of the form $k^N \longrightarrow X \longrightarrow kT$.

In particular, if $R = \mathbb{F}_p$ is a finite field of prime order, then $B = (\mathbb{F}_p, +) \simeq C_p$, $T \subseteq GL_n(\mathbb{F}_p)$ and $\mathrm{H}^2(N, k^\bullet) = \mathrm{H}^2(B, k^\bullet)^n \times \mathrm{P}(B, B, k^\bullet)^{\binom{n}{2}}$. If furthermore T consists of permutation matrices and T acts doubly transitively, then we can explicitly describe the following subgroup $(k^\bullet/(k^\bullet)^p)^{n-1} \oplus \Omega_p^{\binom{n}{2}} \leq \mathrm{H}^2_c(kT, k^N)$, via the connecting homomorphism Φ.

Chapter 6

Epilogue

Here we list a few research topics that arose during the composition of this work.

1. Generalize some results to non abelian matched pairs.

2. Describe a very general example of an abelian matched pair of Hopf algebras (as general as possible, i.e. both actions non trivial, both Hopf algebra infinite dimensional, neither of the actions locally finite, etc.) and apply the general Kac sequence to compute the cohomology (hopefully also for algebras different from the ground field, maybe even infinite dimensional).

3. Find nice (workable) conditions for the properties of a matched pair (T, N, μ, ν), that would force the local finiteness of μ.

4. The interpretation of the second cohomology of a matched pair is rather artificial. It would be nice to find a more natural definition of compatible cleftness or something that would replace that.

   The matched pair conditions come from a distributive law. Can we find a distributive law such that the compatibility of pairs of extensions would come from it?

5. Find a cotriple, such that the cohomology of a matched pair comes directly from it (i.e. without truncating a double complex).

6. The universal measuring coalgebra $\mathrm{M}(B, A)$ is a rather mysterious object. If $A = k$, then $\mathrm{M}(B, k) = B^{\circ}$, which can be described as a certain vector subspace of B^*. Can we do something similar for $\mathrm{M}(B, A)$. We conjecture, this is possible in the case $A = K$ is a finite field extension of k. How about a broader class of algebras?

7. Suppose N is a commutative Hopf algebra. Then Section 4.2.2 describes a Hopf algebra structure on $\mathrm{M}(N, N)$. There is also a bialgebra structure on $\mathrm{M}(N, N)$ in which the multiplication comes from composition in $\mathrm{Hom}(N, N)$, [Sw2, GP]. Is it possible to compare the two structures?

Appendix A

Homological algebra

A.1 Simplicial homological algebra

This is a collection of notions and results from simplicial homological algebra used in the main text. The emphasis is on the cohomology of cosimplicial objects.

A.1.1 Simplicial and cosimplicial objects

Let $\mathbf{\Delta}$ denote the simplicial category [Mc]. If $\mathcal{A}$ is a category then the functor category $\mathcal{A}^{\mathbf{\Delta}^{op}}$ is the category of simplicial objects while $\mathcal{A}^{\mathbf{\Delta}}$ is the category of cosimplicial objects in $\mathcal{A}$. Thus a simplicial object in $\mathcal{A}$ is given by a sequence of objects $\{X_n\}$ together with, for each $n \geq 0$, face maps $\partial_i: X_{n+1} \longrightarrow X_n$ for $0 \leq i \leq n+1$ and degeneracies $\sigma_j : X_n \longrightarrow X_{n+1}$ for $0 \leq j \leq n$ such that the equations

$$\partial_i\partial_j = \partial_{j-1}\partial_i \text{ for } i < j,$$

$$\sigma_i\sigma_j = \sigma_{j+1}\sigma_i \text{ for } i \leq j,$$

$$\partial_i\sigma_j = \begin{cases} \sigma_{j-1}\partial_i, & \text{if } i < j; \\ 1, & \text{if } i = j, j+1; \\ \sigma_j\partial_{i-1}, & \text{if } i > j+1 \end{cases}$$

are satisfied.

A cosimplicial object in $\mathcal{A}$ is a sequence of objects $\{X^n\}$ together with, for each

$n \geq 0$, coface maps $\partial^i : X^n \longrightarrow X^{n+1}$ for $0 \leq i \leq n+1$ and degeneracies $\sigma^j : X^{n+1} \longrightarrow X^n$ such that

$$\partial^j \partial^i = \partial^i \partial^{j-1} \text{ for } i < j,$$
$$\sigma^j \sigma^i = \sigma^i \sigma^{j+1} \text{ for } i \leq j,$$
$$\sigma^j \partial^i = \begin{cases} \partial^i \sigma^{j-1}, & \text{if } i < j; \\ 1, & \text{if } i = j, j+1; \\ \partial^{i-1} \sigma^j, & \text{if } i > j+1. \end{cases}$$

Two cosimplicial maps $f, g : \mathbf{X} \longrightarrow \mathbf{Y}$ are homotopic if for each $n \geq 0$ there is a family of maps $\{h^i : X^{n+1} \longrightarrow Y^n | 0 \leq i \leq n\}$ in $\mathcal{A}$ such that

$$h^0 \partial^0 = f,\ h^n \partial^{n+1} = g,$$
$$h^j \partial^i = \begin{cases} \partial^i h^{j-1}, & \text{if } i < j; \\ h^{i-1} \partial^i, & \text{if } i = j \neq 0; \\ \partial^{i-1} h^j, & \text{if } i > j+1, \end{cases}$$
$$h^j \sigma^i = \begin{cases} \sigma^i h^{j+1}, & \text{if } i \leq j; \\ \sigma^{i-1} h^j, & \text{if } i > j. \end{cases}$$

Clearly, homotopy of cosimplicial maps is an equivalence relation.

For a cosimplicial object $\mathbf{X}$ in the abelian category $\mathbf{A}$ let $N^n(\mathbf{X}) = \cap_{i=0}^{n-1} \ker \sigma^i$ and $D^n(\mathbf{X}) = \sum_{j=0}^{n-1} \mathrm{Im} \partial^j$. Then $C(\mathbf{X}) \cong N(\mathbf{X}) \oplus D(\mathbf{X})$. Moreover, $\mathbf{X}/D(\mathbf{X}) \cong N(\mathbf{X})$ is a (normalized) cochain complex with differentials $\partial^n : X^n/D^n \longrightarrow X^{n+1}/D^{n+1}$, and $\pi^*(\mathbf{X}) = H^*(N^*(\mathbf{X}))$ is the sequence of cohomotopy objects of $\mathbf{X}$.

Theorem A.1.1 (Cosimplicial Dold-Kan correspondence [We, 8.4.3]) *If $\mathcal{A}$ is an abelian category then*

1. *$N : \mathcal{A}^\Delta \longrightarrow \mathrm{Coch}(\mathcal{A})$ is an equivalence and $N(\mathbf{X})$ is a summand of $C(\mathbf{X})$;*

2. *$\pi^*(\mathbf{X}) = H^*(N(\mathbf{X})) \cong H^*(C(\mathbf{X}))$.*

3. *If $\mathcal{A}$ has enough injectives, then $\pi^* = H^*N : \mathcal{A}^\Delta \longrightarrow \mathrm{Coch}(\mathcal{A})$ and $H^*C : \mathcal{A}^\Delta \longrightarrow \mathrm{Coch}(\mathcal{A})$ are the sequences of right derived functors of $\pi^0 = H^0 N : \mathcal{A}^\Delta \longrightarrow \mathcal{A}$ and $H^0 C : \mathcal{A}^\Delta \longrightarrow \mathcal{A}$, respectively.*

The inverse equivalence $K : \mathrm{Coch}(\mathcal{A}) \longrightarrow \mathcal{A}^\Delta$ has a description, similar to that for the simplicial case [We, 8.4.4].

A.1.2 Cosimplicial bicomplexes

The category of cosimplicial bicomplexes in the abelian category $\mathcal{A}$ is the functor category $\mathcal{A}^{\Delta\times\Delta} = (\mathcal{A}^{\Delta})^{\Delta}$. In particular, in a cosimplicial bicomplex $X = \{X^{p,q}\}$ in $\mathcal{A}$

1. Horizontal and vertical cosimplicial identities are satisfied;

2. Horizontal and vertical cosimplicial operators commute.

The associated (unnormalized) cochain bicomplex $C(X)$ with $C(X)^{p,q} = X^{p,q}$ has horizontal differentials

$$d_h = \sum_{i=0}^{p+1} (-1)^i \partial_h^i : X_{p,q} \longrightarrow X^{p+1,q}$$

and vertical differentials

$$d_v = \sum_{j=0}^{q+1} (-1)^{p+j} \partial_v^j : X^{p,q} \longrightarrow X^{p,q+1},$$

so that $d_h d_v = d_v d_h$. The normalized cochain bicomplex $N(X)$ is obtained from X by taking the normalized cochain complex of each row and each column. It is a summand of CX. The cosimplicial Dold-Kan theorem then says that $H^{**}(CX) \cong H^{**}(NX)$ for every cosimplicial bicomplex.

The diagonal diag: $\Delta \longrightarrow \Delta\times\Delta$ induces the diagonalization functor $\mathrm{Diag} = \mathcal{A}^{\mathrm{diag}} : \mathcal{A}^{\Delta\times\Delta} \longrightarrow \mathcal{A}^{\Delta}$, where $\mathrm{Diag}^p(X) = X^{p,p}$ with coface maps $\partial^i = \partial_h^i \partial_v^i : X^{p,p} \longrightarrow X^{p+1,p+1}$ and codegeneracies $\sigma^j = \sigma_h^j \sigma_v^j : X^{p+1,p+1} \longrightarrow X^{p,p}$ for $0 \le i \le p+1$ and $0 \le j \le p$, respectively.

Theorem A.1.2 (The cosimplicial Eilenberg-Zilber Theorem) *Let $\mathcal{A}$ be an abelian category with enough injectives. There is a natural isomorphism*

$$\pi^*(\mathrm{Diag}X) = H^*(C\mathrm{Diag}(X)) \cong H^*(\mathrm{Tot}(CX)).$$

Moreover, there is a convergent first quadrant cohomological spectral sequence

$$E_1^{p,q} = \pi_v^q(X^{p,*}) \quad , \quad E_2^{p,q} = \pi_h^p \pi_v^q(X) \Rightarrow \pi^{p+q}(\mathrm{Diag}X).$$

Remark. *The isomorphism in the theorem is induced by the cosimplicial version of the Alexander-Whitney map.*

A.2 Hochschild cohomology of algebras

All algebras and modules in this section are assumed to be over a fixed commutative ring k.

Let R be a k algebra and M and $R-R$ bimodule. We obtain a cosimplicial k-module, with $[n] \mapsto \mathrm{Hom}_k(R^{\otimes n})$ by declaring

- $(\partial^0 f)(r_0, \ldots, r_n) = r_0(f(r_1, \ldots, r_n))$,
- $(\partial^i f)(r_0, \ldots, r_n) = f(r_0, \ldots, r_i r_{i+1}, \ldots, r_n)$, $0 < i < n$
- $(\partial^n f)(r_0, \ldots, r_n) = f(r_0, \ldots, r_{n-1}) r_n$

The **Hochschild cohomology** $H^*(R, M)$ of R with coefficients in M is defined to be k-modules $\mathrm{H}^n(R, M) = \mathrm{H}^n(\mathrm{Hom}_k(R^{\otimes *}, M), \partial)$.

A.3 Group cohomology

We compute group cohomology as $H^*(G, A) = H^*_{Hochschild}(\mathbb{Z}G, A)$. The standard normalized (Hochschild) chain complex for computing group cohomology is given by

$$A \longrightarrow \mathrm{Map}(G, A) \longrightarrow \mathrm{Map}(G \times G, A) \longrightarrow \mathrm{Map}(G \times G \times G, A) \longrightarrow \ldots$$

where $\mathrm{Map}(G^{\times n}, A)$ denotes the abelian groups of all maps, that satisfy the condition that $f(g_1, \ldots, g_n) = 1$, whenever one of $g_i = 1$.

A.3.1 Second cohomology of direct and semidirect products of groups

If G_1 and G_2 are groups and A is an abelian group then $\mathrm{P}(G_1, G_2, A)$ denotes the abelian group of pairings $G_1 \times G_2 \longrightarrow A$ (maps that are homomorphisms in both variables). It is easy to see that every pairing $f: G_1 \times G_2 \longrightarrow A$ is constant on each equivalence class in $G_1/[G_1, G_1] \times G_2/[G_2, G_2]$. So we can observe that

$$\mathrm{P}(G_1, G_2, A) \simeq \mathrm{P}(G_{1ab}, G_{2ab}, A) \simeq \mathrm{Hom}(G_{1ab} \otimes_{\mathbb{Z}} G_{2ab}, A).$$

In the above expression Hom means the abelian group of group homomorphisms and G_{ab} denotes the abelianization of G, i.e. $G_{ab} = G/[G, G]$. In future we will abbreviate $G_1 \otimes G_2 = G_{1ab} \otimes_{\mathbb{Z}} G_{2ab}$.

We use the following classical result which can be found for example in [Kar]:

Theorem A.3.1 *Let A be an abelian group and G_1 and G_2 be finite groups. If the direct product $G_1 \times G_2$ acts trivially on A then there exists an isomorphism*

$$\phi: \mathrm{H}^2(G_1 \times G_2, A) \xrightarrow{\sim} \mathrm{H}^2(G_1, A) \times \mathrm{H}^2(G_2, A) \times \mathrm{P}(G_1, G_2, A).$$

The isomorphism ϕ is given by $\phi(c) = (c_1, c_2, f_c)$, where $c \in \mathrm{Z}^2(G_1 \times G_2, A)$, $c_i = \mathrm{res}_{G_i} c$, for $i = 1, 2$ and $f_c \in \mathrm{P}(G_1, G_2, A)$ is given by

$$f_c(g_1, g_2) = c(g_1, g_2)c(g_2, g_1)^{-1}.$$

The inverse of this isomorphism ψ is given by

$$\psi(c_1, c_2, f)(g_1g_2, g_1'g_2') = c_1(g_1, g_1')c_2(g_2, g_2')f(g_1, g_2')$$

for $g_i, g_i' \in G_i$; where $c_i \in \mathrm{Z}^2(G_i, A)$ and $f \in \mathrm{P}(G_1, G_2, A)$.

Remark: The isomorphism

$$\mathrm{H}^2(G_1 \times G_2, A) \simeq \mathrm{H}^2(G_1, A) \times \mathrm{H}^2(G_2, A) \times \mathrm{P}(G_1, G_2, A)$$

The following beautiful result is due to Tahara ([Ta]).

Theorem A.3.2 *Let G be a group which acts trivially on an abelian group A and let G be a semidirect product of a normal subgroup N and a subgroup T. Let $\tilde{\mathrm{H}}^i(G, A)$ be the kernel of the restriction map*

$$\mathrm{res}: \mathrm{H}^i(G, A) \longrightarrow \mathrm{H}^i(T, A).$$

Let the cohomology groups

$$\mathrm{H}^1(T, \mathrm{H}^1(N, A)) \text{ and } \mathrm{H}^2(H, \mathrm{H}^1(N, A))$$

be defined with respect to the action by conjugation of T *on* $\mathrm{Hom}(N, A)$*. Then*

1. $\mathrm{H}^2(G, A) \simeq \tilde{\mathrm{H}}^2(G, A) \times \mathrm{H}^2(H, A)$

2. *There is an exact sequence*

$$\begin{aligned} 1 \;&\longrightarrow\; \mathrm{H}^1(H, \mathrm{H}^1(N, A)) \longrightarrow \tilde{\mathrm{H}}^2(G, A) \xrightarrow{\mathrm{res}} \mathrm{H}^2(N, A)^T \longrightarrow \\ &\xrightarrow{d_2}\; \mathrm{H}^2(H, \mathrm{H}^1(N, A)) \longrightarrow \tilde{\mathrm{H}}^3(G, A). \end{aligned}$$

Remark. *The homomorphisms involved in the Tahara sequence are explicitly described in [Ta].*

Corollary A.3.3 *If* $\gcd(|T|, |N|) = 1$ *then* $\mathrm{H}^2(G, A) \simeq \mathrm{H}^2(N, A)^T \times \mathrm{H}^2(H, A)$.

We also need the following tool that was used to establish the above results:

Proposition A.3.4 ([Ta], Proposition 1.) *Let* G *be the semidirect product of a normal subgroup* N *and a subgroup* T *and let* A *be a commutative group and let* G *act trivially on* A.

1. *Let* $f: G \times G \longrightarrow A$ *be a 2-cocycle on* G*. Then* f *can be normalized up to coboundaries as follows:*

$$f(N, T) \equiv 1_A,$$

and hence

$$f(nt, n't') = f(t, t')f(t, n')f(n, t(n')),$$

where $n, n' \in N$ *and* $t, t' \in T$*. We shall call such a 2-cocycle* f *a normal 2-cocycle. Thus a normal 2-cocycle* f *on* G *is determined uniquely by* $f|_{N\times N}$, $f|_{T\times T}$, $f|_{T\times N}$.

2. *The data* $f|_{N\times N}$*,* $f|_{T\times T}$ *and* $f|_{T\times N}$ *determine a normal* 2*-cocycle on* G *if and only if they satisfy the following conditions:*

   (a) f *is a* 2*-cocycle on* N*,*

   (b) f *is a* 2*-cocycle on* T*,*

   (c) $f(tt', n) = f(t', n)f(t, t'(n))$,

   (d) $f(n, n')f(t(n), t(n'))^{-1} = f(t, n)f(t, nn')^{-1}f(t, n')$,

   where $n, n' \in N$ *and* $t, t' \in T$.

A.3.2 Examples

Example A.3.5 *If* $G_{ab} = \mathrm{C}_n = \langle x\rangle$ *then there is an isomorphism* $\Omega_n \longrightarrow \mathrm{P}(G, G, k^{\bullet n})$*, given by* $t \mapsto f_t$*, where* Ω_n *is the group of n-th roots of unity in* $k^\bullet$ *and where* $f_t(x^i u, x^j v) = t^{ij}$ *for* $i, j \in \mathbb{Z}$ *and* $u, v \in [G, G]$*. The inverse is given by* $f \mapsto f(x, x)$.

Example A.3.6 *If* $G = \mathrm{C}_n = \langle x\rangle$ *then there is an isomorphism* $A/A^n \longrightarrow \mathrm{H}^2(G, A)$*, given by* $a \mapsto c_a$*, where* $a \in A$ *and* $c_a \in \mathrm{Z}^2(G, A)$ *is the cocycle defined by* $c_a(x^i, x^j) = a^{\lfloor\frac{i+j}{n}\rfloor}$*, for* $i, j \in \{0, 1, \ldots n-1\}$*. The inverse of the isomorphism is given by* $c \mapsto c(x, x^{-1})$*, for* $c \in \mathrm{Z}^2(G, A)$.

Remark. We can drop the condition $i, j \in \{0, 1, \ldots n-1\}$ in the definition of c_a if we replace $\lfloor\frac{i+j}{n}\rfloor$ by $\lfloor\frac{i+j}{n}\rfloor - \lfloor\frac{i}{n}\rfloor - \lfloor\frac{j}{n}\rfloor$.

Definition A.3.7 *We adopt the notation of [Ta] for a semidirect product of two cyclic groups,* $G_{k,m} := C_k \ltimes C_m$*. That is*

$$G_{k,m} = \langle u, v | u^k = 1, v^m = 1, vuv^{-1} = u^r\rangle,$$

where $r^m \equiv 1 (\mathrm{mod}\ k)$*. In case the choice of* r *is not clear, we shall write* $G_{k,m,r}$.

In [Ta] the second cohomology of $G_{k,m}$ is described explicitly. We shall limit ourselves to the particularly nice case when k and $r-1$ are relatively prime (we also assume that the semidirect product is not a direct product, i.e. $r \neq 1$). In this case (A denotes an abelian group and $G_{k,m,r}$ acts trivially on A) we have the following description.

Example A.3.8

$$\mathrm{H}^2(G_{k,m,r}, A) = \frac{\{(a,b) \in A \times A | a^{r-1}b^k = 1_A\}}{\{(c^k, c^{1-r}) | c \in A\}}.$$

A cocycle $f_{a,b}$ *that corresponds to* $(a,b) \in A \times A$ *is given by*

$$\begin{aligned} f_{a,b}(u^i, u^j) &= a^{\lfloor (i+j)/n \rfloor},\ 0 \leq i,j < k \\ f_{a,b}(v,u) &= b, \end{aligned}$$

more precisely:

$$f_{a,b}(u^i v^j, u^{i'} v^{j'}) = a^{\lfloor (i' r^j)/k \rfloor} b^{i'(r^{i'}-1)/(r-1)} f_{a,b}(u^i, u^{ri'})$$

where $0 \leq i, i' < k,\ 0 \leq j, j' < m$.

A.4 Lie algebra cohomology

We compute the Lie algebra cohomology by $\mathrm{H}^*(\mathbf{g}, V) = \mathrm{H}^*_{Hochschild}(U\mathbf{g}, V)$, where $\mathbf{g}$ is a Lie algebra, $U\mathbf{g}$ is its universal envelope and V is a $\mathbf{g}$-module.

Proposition A.4.1 (Whitehead's first and second lemma, [We]) *Let* $\mathbf{g}$ *be a semisimple Lie algebra over a field of characteristic* 0. *If* M *is any finite dimensional* $\mathbf{g}$ *module, then the cohomology groups* $\mathrm{H}^1(\mathbf{g}, M)$ *and* $\mathrm{H}^2(\mathbf{g}, M)$ *are trivial.*

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