CONSTRAINED COPS AND ROBBER

By

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Signature of Author

For my grandmother,

Eliza Blanche Francis

whom I still miss

every day.

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Abstract

We study variations of the pursuit and evasion game Cops and Robber [23, 25] in which one or both of the opposing sides play with constraints.

Both the cops and the robber traditionally play with perfect information. We consider the game when the cops play with only partial information. This partial information is provided first via selected edges of a graph and then via selected vertices. When the partial information includes the robber's direction in addition to his position, we are able to bound the amount of information required by a cop to win on a copwin graph. When the partial information includes only the robber's position, we give bounds on the amount of information required by a cop to win on a tree.

We take steps toward the characterization of graphs with copnumber 2. We consider tandem-win graphs in an attempt to generalize the notion of a copwin graph. We present a recognition theorem for tandem-win graphs, and a characterization of triangle-free tandem-win graphs.

We also consider the game when the cops are restricted to moving on assigned subgraphs. We bound the copnumbers of powers of graphs under a variety of products, and show that in many cases, our results are asymptotically exact. Finally we translate several problems into games where the movements of both the cops and the robber are restricted, and the cop side is reduced to a single cop.

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Chapter 1

Pursuit and Evasion Games

1.1 Introduction

In this thesis, the pursuit and evasion game *Cops and Robber* is considered. When the game was first introduced, the opposing sides, the cops and the robber, were free to move among all of the vertices of the graph as the cops attempted to apprehend the robber. In addition, both sides played with perfect information and thus were always aware of the position of the opponent. Several variations of the game will be introduced in which one or both of the opposing sides play with constraints.

In this first chapter, we look at the historical development of the Cops and Robber game, as well as some strategies that prove useful in the remainder of the thesis. We also introduce the related game, *searching* which will be considered as a variation of Cops and Robber where the cops take on the role of the searchers.

The motivation for introducing the searching game is that the cops in this game play with no information which is in sharp contrast to the Cops and Robber game. In the second chapter, we propose several variations of Cops and Robber in which the cops play with partial information. The robber continues to play with perfect information. In the first variation, the partial information is provided by sensing devices called *photo radar* which are placed on *edges* of the graph. Photo radar provide the cops with information regarding the robber's position, as well as the direction in which he is moving. In the second variation, the partial information is provided by *video cameras* which are placed on *vertices* of the graph. Like photo radar, video cameras provide the cops with information regarding the robber's position and the direction in which he is moving. However video cameras are more powerful than photo radar. In the third variation, the partial information is provided by *alarms*. Like video cameras, alarms are placed on *vertices* of the graph, and provide the cops with information regarding the robber's position. They do not, however give any indication of the direction in which the robber is moving. Finally in the fourth variation, the partial information is provided by *alarms* placed on *edges* of the graph. Again the alarms provide the cops with information regarding the robber's position, but do give any indication of the direction in which the robber is moving.

In the third chapter, the cops are restricted to moving on assigned subgraphs. If each such subgraph is a copwin graph and also a retract, then the number of subgraphs needed to cover the vertices of the graph on which the game is being played bounds the copnumber of this graph. In particular, we consider covering the vertices of a graph with isometric paths, complete graphs and finally, special copwin graphs. The graphs considered in this chapter are graph products.

The cops' movements are again restricted in the fourth chapter. The cops are required to move in *tandems*, meaning that the cops are partnered and the cops in every pair, or tandem, must be located on adjacent vertices after every move. If the cop side is composed of just one tandem and these cops can win on a particular graph, then the graph has copnumber at most 2. Outerplanar graphs are also considered in this chapter as another step toward the characterization of copnumber 2 graphs.

The final chapter presents problems for further research, and translates several problems into games where the movements of both the robber and the cops are restricted, and the cop side is reduced to a single cop.

All graphs considered are finite and simple unless otherwise noted. If two distinct vertices, x and y of a graph G are joined by an edge, the vertices are said to be **adjacent**. This is denoted $x \sim y$. If x is either adjacent or equal to y, then we write $x \simeq y$. Otherwise, x and y are said to be **non-adjacent**. This is denoted $x \perp y$. A **walk** in G is a sequence of vertices $\{v_0, v_1, \ldots, v_k\}$ such that $v_i \sim v_{i+1}$ for $i = 0, 1, \ldots, k-1$. A **path** is a walk with no repeated vertex. A path with n vertices is denoted P_n, and has length $n-1$. A **cycle** is a set of vertices $\{v_0, v_1, \ldots, v_k\}$ such that $v_i \sim v_{i+1}$ for $i = 0, 1, \ldots, k-1$ and $v_0 \sim v_k$. A cycle with n vertices is denoted C_n. A **complete** graph is a graph in which every pair of vertices forms an edge. A complete graph with n vertices is denoted K_n. For other terms see [29].

All results included in the thesis and not referenced are those of the author.

1.2 Cops and Robber

1.2.1 Rules of the Game

This is a standard introduction as can be found in [7]. The game of Cops and Robber is a pursuit game played on a reflexive graph G; that is, a graph with loops at every vertex. This game was introduced by Nowakowski and Winkler [23] and independently, by Quilliot [25]. The game is played by two opposing sides; the cop side is composed of a set of $k > 0$ cops and the robber side is composed of a single robber. Both sides play with perfect information; that is, each side is aware of the position of the other at each stage of the game. The rules require that the cops begin the game by each choosing a vertex to occupy. These vertices do not have to be distinct. The robber must then also choose a vertex to occupy. The opponents move alternately where a move is to slide along an edge. The loops are a technical device which allows any subset of the cops and the robber to pass and remain stationary during a turn. The cops win if at least one of them occupies the same vertex as the robber after a finite number of moves. The robber wins if this situation can be avoided forever. We note that in this game, unlike searching [24] discussed in the next section, the players are always assumed to be located on vertices.

Suppose the game is played on a loopless graph. The **passive** game allows players on both sides to pass during a turn. In the **active** game, the robber and a nonempty subset of the cops must move during their respective turns. It has been shown by Neufeld [17] that if k cops have a winning strategy on a graph G in the passive game, then the number of cops, k' needed in the active game must satisfy $k' \leq k \leq k' + 1$.

When all graphs considered are reflexive graphs, the passive and active versions are equivalent. Throughout the thesis, we assume that the passive game is being played on simple graphs.

1.2.2 Characterization of Copwin Graphs

When the game was originally proposed, it was played with a single cop and a robber. Any graph could be characterized as either **copwin** or robber-win depending on the outcome of the game. Copwin graphs were completely characterized in [23] and [25].

Example. Each member of the set $\{T_i : T_i \text{ is a finite tree}\}$ is copwin. To see this, consider the vertex occupied by the cop at any stage. The robber is unable to move past the cop because there is just a single path joining any two vertices. Hence the tree is partitioned into two parts by the cop, and the robber is restricted to moving within one of those parts. As the cop moves toward the robber, that part of the graph that is inaccessible to the robber strictly increases as the robber's portion becomes smaller. Hence after a finite number of moves, the robber is apprehended.

Example. Let C be the family of cycles of length greater than three. Each member of this family is robber-win since after his move, the robber can always stay at least two vertices away from the cop.

Definition 1.2.1 *Let G and H be reflexive graphs. A mapping $f : V(G) \to V(H)$ is said to be **edge preserving** (or a homomorphism) if it preserves adjacencies; that is, if $x, y \in V(G)$ and $x \sim y$, then $f(x) \simeq f(y)$.*

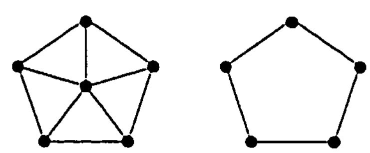
Definition 1.2.2 *Let G be a reflexive graph and let H be a (labelled) induced subgraph of G. It is said that H is a **retract** of G if there is an edge preserving map f from G to H such that the restriction of f to H is the identity map on H.*

The style of argument used in the proof of the theorem that follows is used many times and so is reproduced here.

Theorem 1.2.1 (Nowakowski and Winkler [23]) *Any retract H of a copwin graph G is also a copwin graph.*

Proof. Let G be a copwin graph and let H be a retract of G. Further let f be a retraction map from G to H. Since G is copwin, the cop has a winning strategy on G. This strategy can be modified and used on the subgraph H. The cop simply plays the image under f of his winning strategy on G. Using this strategy, the cop captures the image of the robber on H. Since the robber is actually playing on H and f is the identity map on H, the robber's image coincides with his actual position. Hence the robber is apprehended on H and therefore, H is a copwin graph. □

Notice that it is not necessarily true that an induced subgraph H of a copwin graph G is copwin. To see this, consider the graphs G and H shown in Figure 1.1. The graph G is copwin. The cop begins on the central vertex and is able to win after the robber's first move. However the graph H is robber-win since after his move, the robber can always remain two vertices away from the cop.



G H

Figure 1.1: The induced subgraph H of the graph G is robber-win even though G is copwin.

Definition 1.2.3 *Let G be a graph and let $v \in V(G)$. The* ***neighborhood*** *of v, denoted $N(v)$, is the set of vertices adjacent to v in G. The* ***closed*** *neighborhood of v, denoted $N[v]$, is defined as $N(v) \cup \{v\}$.*

Definition 1.2.4 *A vertex d of a graph G is said to* ***dominate*** *another vertex v if d is adjacent to each of the vertices in the closed neighborhood of v.*

Suppose a given graph G is copwin. To determine the properties that characterize such a graph, it is useful to consider the last move made by the robber before he is apprehended. Let the position of the robber before this last move be denoted v. There are three options open to the robber. He can choose to pass and remain on vertex v, he can move onto the vertex occupied by the cop, or he can move to a vertex adjacent to the cop's position. Since all of these options must lead to the immediate capture of the robber, it must be true that the vertex u occupied by the cop is adjacent to v and also to every vertex that is adjacent to v; that is, u dominates v. The vertex v will be referred to as a **corner** since the robber has no means of escape once he is forced to move onto this vertex.

Clearly a graph without a corner cannot be copwin. Suppose a graph G has a corner. The robber will only move onto the corner if he is forced to do so. This is because any robber-win strategy that uses v can be modified to use u. Hence the question becomes whether or not the cop can force the robber onto the corner. This can be determined by removing the corner and determining if the resulting graph is copwin. Intuitively, the successive removal of corners from a copwin graph will result in a single vertex. This is the idea used by Nowakowski and Winkler [23] to characterize copwin graphs.

Again, approaches similar to the one taken in the proof of the next theorem are used with *tandem-win* graphs in Chapter 4, and so it is useful to reproduce the argument here. Tandem-win graphs use a variation on the notion of corner.

Theorem 1.2.2 (Nowakowski and Winkler [23]) *Let G be a graph and let c be a corner of G. Let $G' = G \setminus \{c\}$. Then G is copwin if and only if G' is copwin.*

Proof. Let G be a graph and let c be a corner of G. Let $G' = G \setminus \{c\}$. Further let d be a vertex that dominates the corner c. Now G' is a retract of G with a retraction map f defined as follows: $f(c) = d$ and $\forall v \in V(G')$, $f(v) = v$. Suppose G is copwin. By Theorem 1.2.1, G' is copwin.

Conversely suppose G' is copwin, and thus the cop has a winning strategy on G'. Since the game is actually being played on G, the cop's winning strategy on G' can be thought of as catching the image of the robber. Now suppose this image is caught on vertex u. If $u \neq d$, then the robber's image on G' corresponds to his actual position on G as f is the identity map on G'. Hence the robber is apprehended. Otherwise, the robber's image is apprehended on vertex d. Since it is known that $f(c) = f(d) = d$, the robber is on vertex c or vertex d in the graph G. If he is on d, his actual position corresponds to his image and he is caught. If he is on c then he will be caught on the cop's next move. This is because the cop is on vertex d and it is known that d dominates c. □

We proceed with some needed definitions.

Definition 1.2.5 *Let G be a graph and let $v \in V(G)$. Suppose there exists a vertex $u \in V(G)$ such that $N[v] \subseteq N[u]$. Then v is said to be **irreducible**. The vertex v is also known as a corner or pitfall.*

Definition 1.2.6 *A graph G is said to be **dismantlable** if there is an ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertices of G such that for each $i < n$, v_i is irreducible in the subgraph induced by $\{v_i, v_{i+1}, \ldots, v_n\}$.*

We now give the the main theorem in this section which is due to Nowakowski and Winkler [23].

A finite graph is copwin if and only if it is dismantlable.

The ordering of the vertices of the graph G referred to in Definition 1.2.6 is known as a **copwin ordering**.

Example. This example refers to Figure 1.2. The circled vertices represent corners at each of the stages. Also, at each stage it does not matter in which order the corners are removed. The original graph is copwin.

There are copwin graphs that are not finite, so Nowakowski and Winkler [23] extended their characterization to obtain a complete characterization of copwin graphs.

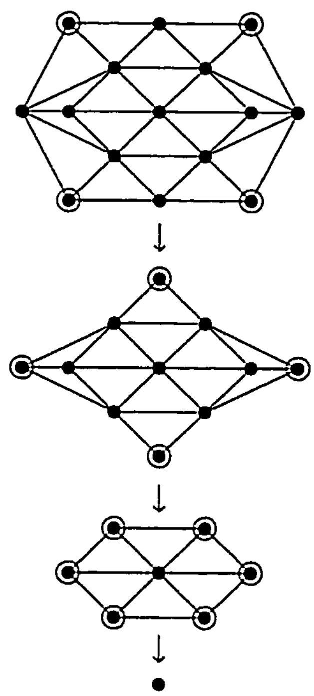


Figure 1.2: An illustration of dismantling. The original graph is copwin.

Define for each ordinal α a binary relation $\leq_\alpha$ on the set $V(G)$ of an arbitrary graph G. Let $x \leq_0 y$ if and only if $x = y$, and for each $\alpha > 0$ set $x \leq_\alpha y$ if and only if for all $u \in N(x)$, there exists a $v \in N(y)$ such that $u \leq_\rho v$ for some $\rho < \alpha$. Finally let α' be the least ordinal such that $\leq_{\alpha'} = \leq_{\alpha'+1}$ and define $\leq$ to be $\leq_{\alpha'}$.

G is a copwin graph if and only if the relation $\leq$ described above is trivial; that is, $x \leq y$ for every $x, y \in V(G)$.

1.2.3 Strategy for a Copwin Graph

Suppose $\{x_1, x_2, \ldots, x_n\}$ is a copwin ordering of the vertices of a graph G. We know that the cop must have a winning strategy on G. But this strategy has not been made explicit. The goal of this section is to describe a strategy that can be used by the cop to win, and to prove that this strategy is effective in capturing the robber.

Copwin Strategy (Clarke and Nowakowski [7, 9]) Let $\{x_1, x_2, \ldots, x_n\}$ be a copwin ordering of the vertices of a graph G. Define the induced subgraphs $G_i = G_{i-1} \setminus \{x_{i-1}\}$ where $G_1 = G$, and let $f_i : G_i \rightarrow G_{i+1}$ be the retraction map from G_i to G_{i+1} which maps x_i onto a vertex that dominates x_i in G_i. Further if the robber is on vertex x, define $F_i(x) = f_{i-1} \circ f_{i-2} \circ \cdots \circ f_2 \circ f_1(x)$ so that $F_i(x)$ can be considered as the robber's image or shadow on G_i. The robber is always thought to be playing on the graph G. However the cop initially moves on the subgraph G_n, beginning on vertex x_n, the vertex on which the cop's position coincides with the robber's image under the mapping $f_{n-1} \circ f_{n-2} \circ \cdots \circ f_2 \circ f_1(x)$. Now suppose the cop is occupying the robber's image in the subgraph G_i under the mapping $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_2 \circ f_1(x)$. The cop then moves onto the image of the robber in G_{i-1}.

After at most n moves, the robber is apprehended, and this is proven in the next theorem.

Theorem 1.2.3 (Clarke and Nowakowski [9]) *Let G be a copwin graph with $|V(G)| = n$. Playing the Copwin Strategy, a cop will capture the robber on G after at most n moves.*

Proof. First note that for all i, f_i is an edge preserving map. Hence when the robber moves from x to y it follows that for all j, $F_j(x) = F_j(y)$ or $F_j(x) \sim F_j(y)$. Also note that since the retraction maps are one-point retractions then for any i, $F_i(x)$ and $F_{i+1}(x)$ are either on the same vertex or are on adjacent vertices.

We prove the result by induction and consider the situation after the cop has moved. The cop begins on vertex x_n, the vertex on which the cop's position coincides with the robber's image under the mapping $f_{n-1} \circ f_{n-2} \circ \cdots \circ f_2 \circ f_1(x)$; that is, the cop begins on the same vertex as $F_n(x)$ and so the cop can pass. Suppose for some $i \leq n$ the cop has captured $F_i(y)$, where y is the robber's position on G, and it is the robber's turn to move. Suppose he moves to vertex z. We need to show that $F_i(y) \simeq F_{i-1}(z)$ for then the cop can move to immediately capture the image in G_{i-1}. There are two cases. If $F_i(y) = F_{i-1}(y)$ then $F_{i-1}(z) \simeq F_i(y)$. Otherwise $F_i(y) \sim F_{i-1}(y)$. But then $F_{i-1}(y)$ is the corner that is removed from G_{i-1} to obtain

G_i and so $N[F_{i-1}(y)] \subset N[F_i(y)]$ and therefore $F_i(y) \simeq F_{i-1}(z)$. Thus in all cases, the robber's image can be caught in one move in the larger graph.

Since there are only a finite number n of graphs G_i, the robber's image will coincide with his actual position after at most n moves. □

It has been shown that if the cop is playing in the subgraph G_i, and is occupying the robber's image under the mapping $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_2 \circ f_1$, then the cop is able to move onto the robber's image in G_{i-1} under $f_{i-2} \circ f_{i-3} \circ \cdots \circ f_2 \circ f_1$. If the cop is playing in the subgraph G_i, the robber can never move to a vertex in this subgraph without being apprehended by the cop; that is, the robber cannot get 'behind' the cop. This is the reason that in [9], the Copwin Strategy is known as a **no-backtrack strategy**.

Theorem 1.2.4 (Clarke and Nowakowski [7]) *Suppose the cop is playing the Copwin Strategy in the subgraph G_i, and is occupying the robber's image under the mapping $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_2 \circ f_1$. The robber can never move to a vertex of G_i without the cop immediately landing on the same vertex.*

Proof. Suppose the cop is playing in the subgraph G_i, and is occupying the robber's image under the mapping $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_2 \circ f_1$. The cop is able to move so as to always stay with the image of the robber on this subgraph. Now the mapping $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_2 \circ f_1$ is the identity on G_i. Hence if the robber moves to a vertex of G_i, his image will correspond to his actual position and he will be apprehended. □

1.2.4 A Variety

Now that copwin graphs have been characterized, a property of such graphs will be explored. This is a foretaste of Chapter 3.

Definition 1.2.7 *The* ***strong product*** *of a set of graphs $\{G_i : i = 1, 2, \ldots, k\}$ is the graph $\boxtimes_{i=1}^{k} G_i$ whose vertex set is the Cartesian product of the sets $\{V(G_i) : i = 1, 2, \ldots, k\}$, and there is an edge between $\overline{a} = (a_1, a_2, \ldots, a_k)$ and $\overline{b} = (b_1, b_2, \ldots, b_k)$ if and only if a_i is adjacent or equal to b_i for all $i = 1, 2, \ldots, k$.*

Definition 1.2.8 *A* ***variety*** *of graphs is a class of graphs which is closed under finite products and retracts.*

The technique used in the proof of the next theorem is useful in Chapter 4 when we consider the strong product of a copwin graph and a tandem-win graph, and also the strong product of two *clique-win* graphs, a generalization of tandem-win graphs.

Theorem 1.2.5 (Nowakowski and Winkler [23]) *Let* $\{G_i : i = 1, 2, \ldots, k\}$ *be a finite collection of copwin graphs. The strong product of these graphs is also copwin.*

Proof. Let $\{G_i : i = 1, 2, \ldots, k\}$ be a finite collection of copwin graphs. Let $G = \boxtimes_{i=1}^{k} G_i$ be the strong product of these graphs. We wish to show that G is copwin.

There is an edge-preserving projection of G onto each of the graphs G_i. Hence the cop and robber can be projected onto each of the original graphs and a game can take place there. Consider one such projection onto the graph G_i. Now G_i is copwin, and so the cop has a winning strategy and is able to apprehend the robber. In terms of the larger graph G, the cop has apprehended the projection of the robber on G_i. The cop stays with this projection for the remainder of the game as he similarly captures the other projections of the robber on the graphs $G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_k$. Since the collection of graphs is finite, the robber will be apprehended on each of the projections after a finite number of moves. At this time, the robber is apprehended on G. It should be noted that on G the cop has played the composition of his winning strategies on each of the graphs G_i. □

Example. The previous theorem tells us that the product of a finite collection of copwin graphs is copwin. This example illustrates why the theorem cannot be extended to infinite collections of copwin graphs.

Define a path $P_n = \{0, 1, 2, \ldots, n-1\}$. Now P_n is copwin. Consider the product of an infinite collection of such paths $\boxtimes_{i=1}^{\infty} P_i$. This graph is not copwin since the vertices $(0, 0, 0, \ldots)$ and $(0, 1, 2, \ldots)$ are not connected by a finite path.

The next theorem can be found in papers by Aigner and Fromme [1], and by Nowakowski and Winkler [23]. It follows immediately from Theorem 1.2.1 and Theorem 1.2.5.

The class of copwin graphs is a variety.

1.2.5 Bridged Graphs

Let G be a graph and let H be a subgraph of G. The graph H is said to be **isometric** if the distance between any pair of vertices in H is the same as that in G. Clearly an isometric subgraph of a graph G must be an induced subgraph. Isometric subgraphs will be considered in detail in Chapter 3.

Definition 1.2.9 *A graph G is said to be* ***bridged*** *if all isometric cycles of G have length 3.*

Example. See Figure 1.3 for examples of these graphs. The graph H is an isometric subgraph of G but not of I. The graph J is bridged.

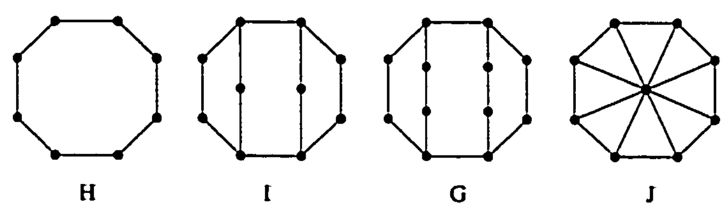


Figure 1.3: The graph H is an isometric subgraph of G but not of I.

Retracts are necessarily isometric since walks are mapped to walks but note that in Figure 1.1, H is an isometric subgraph of G but not a retract.

In effect, Definition 1.2.9 says that if C is any cycle with length greater than three in a bridged graph G, then there is a 'shortcut' between a pair of vertices on the cycle.

Anstee and Farber [2] published a paper concerning bridged graphs and copwin graphs. The paper begins by proving that every nontrivial bridged graph contains a corner, and goes on to prove the next theorem.

Theorem 1.2.6 (Anstee and Farber [2]) *Let G be a bridged graph. There exists a vertex $u \in V(G)$ such that $G \setminus \{u\}$ is bridged.*

Proof. Let G be a bridged graph. Choose a pair of vertices $u, v \in V(G)$ such that $N[u] \subseteq N[v]$. Let P be a shortest path in G that contains u but in which u is not a leaf. Now u can be replaced by v in this path. Hence $G \setminus \{u\}$ is an isometric subgraph of G. It is noted that a cycle C is isometric in $G \setminus \{u\}$ if and only if it is isometric in G. Therefore $G \setminus \{u\}$ is also a bridged graph. □

In the proof of the theorem, the vertex u was taken to be a corner. Hence the theorem actually tells us that the removal of a corner from a bridged graph results in another bridged graph.

Let us review the information given by several theorems. First, it is known that every bridged graph contains a corner, and that its removal results in another bridged graph. It is also known that a copwin graph is one in which the successive removal of corners results in a single vertex. Hence we are able to conclude, as Anstee and Farber [2] did, that every bridged graph is copwin. This result follows.

Let G be a bridged graph. Then G is copwin.

An algorithmic proof of this theorem has been given by Chepoi [5]. He has shown that every ordering of the vertices of a bridged graph produced by a breadth first search is a copwin ordering as defined by Nowakowski and Winkler [23].

The relationship between generalizations of both bridged graphs and copwin graphs is considered in Chapter 4.

1.2.6 Cops and Robber with k Cops

It is evident that there are many graphs which are not copwin. A natural question to pose when considering such a graph G is how many cops are needed to apprehend the robber.

Definition 1.2.10 *Let G be a graph. The minimum number of cops needed to apprehend a robber on G is known as the* **copnumber** *of G and is denoted $c(G)$.*

It is clear that for a finite graph this number exists since a cop on every vertex would suffice.

Shortly after the introduction of the game by Nowakowski and Winkler, Aigner and Fromme [1] introduced the notion of copnumber, along with several results important to further study.

Aigner and Fromme [1] were able to prove that there exists an n-regular graph without 3- or 4-cycles for every natural number n. Using this result, they showed that there are graphs which require an arbitrary number of cops as stated below.

> *Let G be a graph with minimum degree $\delta(G) \geq n$ which has no 3- or 4-cycles. Then $c(G) \geq n$.*

The most interesting result presented here involves planar graphs. The previous theorem shows that there are graphs which require an arbitrary number of cops to apprehend a robber. The next theorem addresses an opposing question. It is desirable to identify a class of graphs for which a bound can be placed on the copnumber. It is known that a 3-cycle or a 4-cycle is contained in every planar graph whose minimum degree is greater than or equal to four. This relation led Aigner and Fromme [1] to prove the following result.

> *Let G be a planar graph. Then $c(G) \leq 3$.*

Example. This example refers to Figure 1.4. The graph G shown has minimum degree $\delta(G) = 3$. The smallest cycle contained in G is of length 5. Hence $c(G) \geq 3$. Since G is a planar graph, $c(G) \leq 3$ and hence $c(G) = 3$.

Copnumbers of graphs have also been considered by Berarducci and Intrigila [3] using retracts. The first result is an easy extension of Theorem 1.2.1.

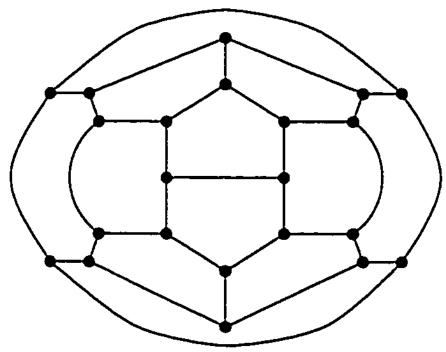


Figure 1.4: An illustration of theorems due to Aigner and Fromme [1].

Theorem 1.2.7 (Berarducci and Intrigila [3]) *Let G be a graph and let H be a retract of G. Then $c(H) \leq c(G)$.*

Proof. Let H be a retract of the graph G, and let f be a retraction map from G onto H. Now $c(G)$ cops have a winning strategy on G. Through the map f, the cops are able to translate this winning strategy onto H. Hence the copnumber of H is at most $c(G)$. □

Berarducci and Intrigila pursued the idea used in the previous theorem, and proved the next result.

Theorem 1.2.8 (Berarducci and Intrigila [3]) *Let G be a graph and let H be a retract of G. Suppose $c(H)$ cops are playing on H. After a finite number of moves, the robber will be immediately apprehended if he moves onto H.*

Theorem 3.2.1 proven in Section 3.2 is a special case of this result, and Theorem 3.2.1 is useful in motivating Chapter 3.

1.3 Searching

Again what follows is a standard introduction as can be found in [7]. A pursuit and evasion game known as **searching** was introduced by T. D. Parsons [24]. This game can be thought of as a modification of Cops and Robber in which the cops have no information about the robber's position, both the cops and the robber move continuously and the robber moves infinitely fast. In keeping with the remainder of the thesis, we will think of this game in terms of the robber. To differentiate this infinitely fast robber from the robber we have been considering up to now, this robber will be referred to as an f-robber. We note that the f-robber can be located on edges of the graph. This is because the original idea of the game was to search for people in caverns.

The game is played on a finite, connected graph G that may be assumed to be embedded in $\mathbb{R}^3$. Thus the vertices of G are represented by distinct points, and the edges of G intersect only at vertices of G.

Given such a graph G, the main objective arising out of the searching game is to determine the minimum number of searchers required to apprehend the f-robber. This number will be referred to as the **search number** of G and is denoted $s(G)$. Suppose $s(G) = k$. The k searchers have an efficient strategy or way of searching the graph for the f-robber. Define searcher i's strategy as the path he follows on the graph G. It will be useful to think of this strategy as a continuous function $f_i : [0, \infty) \to G$ where $f_i(t)$ is the position of the ith searcher at time t. The set $\{f_i : 1 \leq i \leq k\}$ is the **collective strategy** of the k cops. Similarly, define the f-robber's position at time t as $e(t)$. Clearly the search is over when $f_i(t^*) = e(t^*)$ for some $i \in \{1, 2, \ldots, k\}$ and some $t^* \in [0, \infty)$. Here the search number can be thought of as the minimum cardinality of all such collective strategies.

We begin with an intuitive and useful result due to Parsons [24] that bounds the search number of a subgraph in terms of the search number of the larger graph.

> *Let G be a graph and let H be a connected subgraph of G. Then $s(H) \leq s(G)$.*

Recall from Section 1.2.2 that a similar relationship does not hold for copnumbers.

It is straightforward to obtain an upper bound for the required number of searchers of any graph. Suppose the graph G_n has n vertices. Then n searchers can position themselves on the vertices of G_n, one searcher to a vertex. One additional searcher is needed to search the edges of G_n. Hence $s(G_n) \leq n+1$. It is often possible to do much better than that which is suggested by this particular upper bound as shown in the next example.

Example: Consider the graph, G_{17} shown in Figure 1.5.

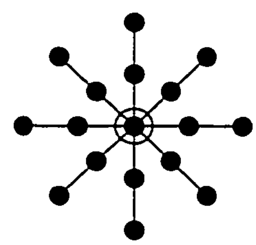


Figure 1.5: The graph G_{17}.

Clearly two searchers are sufficient to search this graph. One searcher remains stationary on the root (indicated by a double circle) while the second searches each of the branches of the tree. The stationary searcher prevents the f-robber from moving into a previously searched area. The upper bound obtained previously gives $s(G_{17}) \leq 18$.

Although particular searching strategies are not considered in depth here, this example can help provide insight into the kinds of strategies that are needed by the searchers. The example indicates that the search number of a graph depends upon the searchers being able to prevent the f-robber from moving into an area that has already been searched. A similar notion arises as a consequence of the Copwin Strategy. See Theorem 1.2.4. These ideas prove useful in Chapter 2.

For a simple graph $G = (V, E)$ with n vertices let $f\colon V \rightarrow \{1, \ldots, n\}$ be a one-to-one function on V—a **linear layout**—and let

$$S_i = |\{v\colon v \in V, f(v) \leq i, \text{ and } \exists \text{ an edge } (v,u) \text{ such that } f(u) > i\}|$$

for $1 \leq i \leq n$. For a given $G = (V, E)$, with $|V| = n$, and given $f \in F$, where F is the set of all possible linear layouts on V, the **vertex separation** number of a graph is $vs(G) = \min_{f \in F} \max_{1 \leq i \leq n} S_i(G, f)$. Ellis, Sudborough, and Turner [11] bound the search number of a graph in terms of the vertex separation number.

Let G be a graph with search number $s(G)$ and vertex separation number $vs(G)$. Then $s(G) \leq vs(G) + 1$.

This result also places a bound on the copnumber of a graph G. This is because the games of Cops and Robber and searching differ in the amount of information available to the cops or searchers. Surely cops with information concerning the robber's position will be able to apprehend the robber at least as efficiently as those with no information, and therefore $c(G) \leq s(G)$. Hence $vs(G)+1$ is also an upper bound for the copnumber of G.

Chapter 2

Cops with only Partial Information

In the game of Cops and Robber, the cops play with perfect information, and in the game of searching, the cops (searchers) play with no information. In this chapter, we propose four variations of the Cops and Robber game in which the cops play with partial information. The robber continues to play with perfect information.

In the first variation, the cops have partial information provided by sensing devices called photo radar. Photo radar units are located on selected edges of a graph. If the robber moves along an edge with a photo radar unit, the unit alerts the cops to the robber's position as well as the direction in which he is moving. One metaphor here is for the edges to be thought to represent roads, with the cops and the robber traveling in cars.

In the second variation, the cops get partial information from video cameras. Video cameras are located on selected vertices of a graph. If the robber moves onto a vertex with a video camera, the camera provides the cops with the robber's position as well as his direction when he leaves the vertex.

In the third variation, the partial information is provided to the cops through alarms located on selected vertices of a graph. If the robber moves onto a vertex with an alarm, his position is known to the cops. Unlike photo radar and video cameras, alarms do not give the cops an indication of the robber's direction when he leaves a vertex with an alarm. Video cameras and alarms can be thought to be located in buildings, with the vertices representing particular rooms.

A fourth variation has the partial information being provided to the cops through alarms located on selected edges of a graph. If the robber moves along an edge with an alarm, the cops are aware of the movement of the robber but they do not know the direction in which he is moving. In this variation, the vertices can be thought to represent rooms in a building, and the edges the doors between rooms. If the robber moves along an edge with an alarm, the cops are aware that he has gone through the corresponding door, but they do not know which of the adjoining rooms he has entered and which he has exited. This problem is considered briefly at the end of the chapter, but is left largely as an open question.

2.1 Photo Radar

In [7] and [8], the photo radar version of the Cops and Robber game is introduced. The cops have partial information provided by sensing devices called photo radar. Suppose the game is being played on a graph G. Photo radar units are placed on the edges of G. These units alert the cops if the robber moves along an edge equipped with a photo radar unit. The units also indicate the direction in which the robber is moving. The minimum number of photo radar units required by a single cop to guarantee the capture of the robber on G will be referred to as the **photo radar number** of G, and will be denoted $pr(G)$. In general, one can ask for the least number of photo radar units, $pr_k(G)$ needed if there are k cops.

It would appear that all but one of the edges incident with any vertex v of a copwin graph G require photo radar for a single cop to win. It will be shown that this is not always the case. Note however that the subgraphs of G without photo radar must have search number 1.

Lemma 2.1.1 (Clarke and Nowakowski [8]) *Let T_1 be the tree shown in Figure 2.1. A single cop playing without photo radar cannot guarantee the capture of a robber on T_1.*

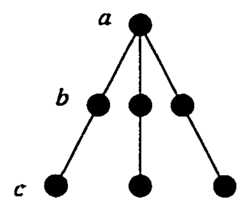


Figure 2.1: The tree T_1.

Proof. The tree T_1, given in Figure 2.1, has three branches. Once the cop has searched down a branch and returned to the root, he has a choice of taking one of two branches. In the four moves that it takes to search this branch and return to the root the robber has the time to move between the other two branches. The cop is again faced with deciding which of two branches to search. Thus there is no guaranteed win by the cop. □

Note that a single cop playing without video cameras or alternately, without alarms cannot guarantee the capture of a robber on T_1.

In Sections 2.2 and 2.3, theorems similar to Theorem 2.1.1 are proven in like fashion.

Theorem 2.1.1 (Clarke and Nowakowski [8]) *For all finite, positive integers n, there exists a graph G such that $pr(G) > n$.*

Proof. Consider a tree similar to T_1 shown in Figure 2.1 with $n+3$ branches rather than 3. No matter how n photo radar units are placed there is still a subgraph with three branches and no photo radar units. The robber restricts himself to playing on this subgraph and by Lemma 2.1.1 there is no guaranteed win by the cop. □

Consider a tree T with n vertices. Now, $pr(T) \leq n-1$ since the photo radar units can be placed one to an edge. If the robber doesn't move then visiting all the vertices ensures a win by the cop. If the robber moves, the game is equivalent to Cop and Robber since the cop will always know the location of the robber. This bound can be improved.

Let G be a graph. An edge is *free* if it has no photo radar and a path P of G is said to be a *freepath* if every edge of P is free. Analogous concepts for video cameras and alarms are defined in Sections 2.2, 2.3 and 2.4.

Let T be a tree. Let T_a be the tree T rooted at vertex a. An *a-branch* of T_a is a path of T with a as one end vertex. We define $k(T_a)$ as the minimum number of edges having photo radar such that the free edges form freepaths and each maximal freepath is on an a-branch. Let $T' = T \setminus \{l \in V(T) : l \text{ is a leaf}\}$ and set $k_T = \min\{k(T'_a) : a \in V(T')\}$. (See Figures 2.2 and 2.3, and the accompanying example.)

Theorem 2.1.2 (Clarke and Nowakowski [8]) *Let T be a finite tree. Then $pr(T) \leq k_T$.*

Proof. Let T be a tree and let a be a vertex for which $k(T'_a) = k_T$. Draw a planar representation of T with a as the root vertex at the top and all edges directed downward. Also since every vertex is incident with at most two edges of a freepath in T', we can assume that any edge of a freepath of T' emanating from a vertex is the leftmost edge. We can also place any leaves so as to be the next edges (in a counterclockwise direction) incident with the same vertex as the free edge. We refer to a freepath together with adjacent leaves as a *free area*.

There are two phases to the strategy. Firstly, suppose the robber never moves along an edge with a photo radar unit. The cop does a depth first search of T' except when he comes to a vertex v, he visits any leaves adjacent to v. The robber cannot move to v at this stage without being caught on the cop's next move. The cop always enters at one end of a freepath (never in the middle) and exits at the bottom without leaving the freepath, except for leaves. The robber can never move past the cop, thus once a free area has been searched by the cop, he is assured that the only way a robber could be on that free area is if he has used an edge with a photo radar unit. Thus if the robber stays on the free area then he will be caught in this phase. If he does move off then he will be detected by a photo radar unit and the cop will always know the free area in which the robber is located.

The second phase starts when the robber is detected by a photo radar unit. The cop moves up the tree until he is on a vertex which lies above the free area on which the robber is currently located. Assuming that the robber is not caught in this maneuver, the cop then starts down the a-branch that contains the robber until he enters the same free area as the robber. (Note that the robber can move to a different free area but this move will be detected by the photo radar units and the cop will always move so as to be above the robber.) By moving down the freepath and visiting adjacent leaves, the cop will either catch the robber or force him to leave the freepath moving down the tree and below the cop. The robber will eventually be caught on a leaf if not sooner. □

Example. Consider the graph T_2 shown in Figure 2.2. If there is only one photo radar unit and it is not on the dashed edge then there will be a subtree isomorphic to T_1 so one cop will not suffice. If the unit is on the dashed edge then the cop can force the robber to move across this edge. So the cop knows in which portion of T_2 the robber is located. However, the cop still has a choice of two branches to search. In the four moves it takes to search one branch, the robber has time to move to the other side of T_2 along the edge with a photo radar unit. This can continue indefinitely with the result that the robber is not caught. So $pr(T_2) > 1$.

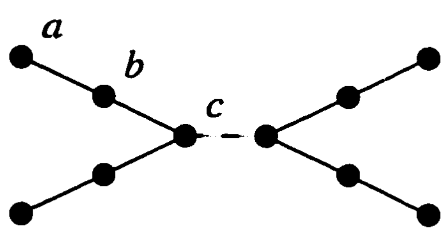


Figure 2.2: The graph T_2.

Note: $k(T_b') = 2$, and $k(T_c') = 3$ as shown in Figure 2.3, and therefore $k_T = 2$; so indeed $pr(T_2) = 2$.

The strategy used in the proof of Theorem 2.1.2 is used again in Sections 2.2, 2.3 and 2.4 to prove similar results concerning video cameras and alarms.

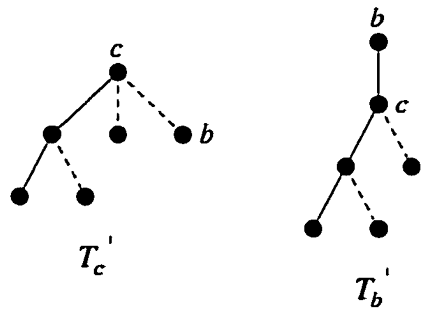


Figure 2.3: The edges with photo radar units are indicated by dashed lines.

We wish to extend this strategy so that it can be used on a copwin graph. This is the subject of Theorem 2.1.3. However, first recall the characterization of copwin graphs given in Chapter 1.

Consider a finite copwin graph G with copwin ordering $\{v_1, v_2, \ldots, v_n\}$. Define the induced subgraphs $G_{i+1} = G_i \setminus \{v_i\}$ where $G_1 = G$, and let $f_i : G_i \to G_{i+1}$ be the retraction map from G_i to G_{i+1}. We note that f_i is a one-point retraction. We define $F_i = f_{i-1} \circ f_{i-2} \circ \cdots \circ f_1$.

Fix a copwin ordering of G, and construct a spanning tree S of G as follows. The root of the spanning tree is the start vertex of the copwin ordering, and for vertices $x_1, x_2 \in V(G)$, $x_1x_2 \in E(S)$ if and only if $f_j(x_1) = x_2$ or $f_j(x_2) = x_1$ for some j. We say that $x_1 \succeq x_2$ if $F_i(x_2) = x_1$ for some i and $x_1 \succ x_2$ if $x_1 \neq x_2$. (See Figure 2.4.) This spanning tree shall be referred to as a **copwin spanning tree**.

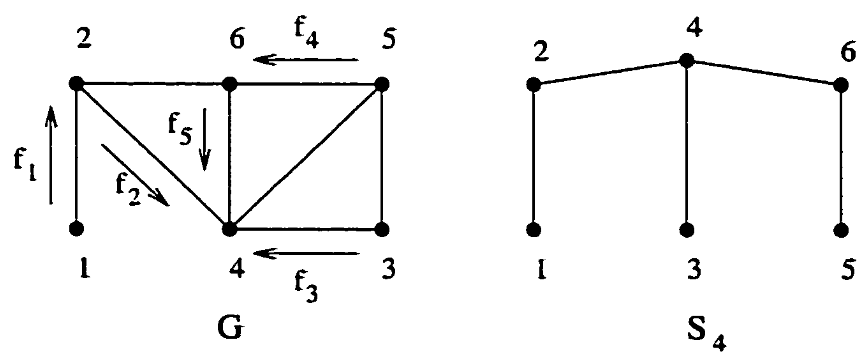


Figure 2.4: A copwin ordering with corresponding copwin spanning tree S_4.

The problem with mimicing the proof of Theorem 2.1.2 here is that the robber can move from one a-branch to another by using an edge not in the tree. The next two lemmas allow us to deal with that problem when it arises in Theorem 2.1.3, and again in Theorem 2.2.5.

Lemma 2.1.2 (Clarke and Nowakowski [8]) *Let G be a copwin graph with copwin spanning tree S_v, and let A and B be two v-branches of S_v. If there exists vertices $x \in A$ and $y \in B$, $x \simeq y$ then for all $a \succeq x$ there exists $b \succeq y$ such that $a \simeq b$.*

Proof. For every f_i, a vertex and its image lie on the same v-branch of S_v. Let j be the least index such that $F_j(x) = a$, note that $F_j(y)$ is still on B and so $a = F_j(x) \simeq F_j(y)$, proving the lemma. □

Let A and B be two distinct v-branches of a copwin spanning tree S_v. Suppose $a \in A$ and a is adjacent to some vertices of B. We take $b \in B$ to be the lowest vertex (with respect to the $\preceq$ ordering of S_v) in B that is adjacent to a and write $a \longrightarrow b$. Under most circumstances, the cop will move from a to b.

Lemma 2.1.3 (Clarke and Nowakowski [8]) *Let G be a copwin graph with copwin spanning tree S_v, and let A and B be two v-branches of S_v. Let $a, x \in A$ and $b, y \in B$ with $x \prec a$, $x \sim y$ and $a \longrightarrow b$, then either $y \prec b$ or $y \sim a$.*

Proof. If $y \prec b$ then we are finished. Suppose now $y \succeq b$. Let $a' \in A$ such that $a' \sim a$ in S_v and $a' \prec a$. Let k be the greatest index such that $F_k(x) = a'$; then $F_{k+1}(x) = a$. If $F_{k+1}(b) \preceq y$ then $F_{k+1}(y) = y$ and so $y = F_{k+1}(y) \sim F_{k+1}(x) = a$, that is $y \sim a$. If $F_{k+1}(b) \succ y$, then since $F_{k+1}(a) = a, a \sim F_i(b)$ for all $i = 1, 2, \ldots, k+1$; that is, $a \sim y = F_j(b)$ for some $j \leq k+1$. □

Corollary 2.1.1 (Clarke and Nowakowski [8]) *Let G be a copwin graph with copwin spanning tree S_v. If both the cop and the robber are on a v-branch A with the cop above the robber in the $\prec$ ordering, and the robber moves to another v-branch then the cop can move to the same v-branch still above or on the same vertex as the robber.*

Proof. Suppose that the cop is above the robber on the same v-branch. Lemma 2.1.2 shows that if the robber moves from one v-branch, A to another, B the cop can also move to the same v-branch. Lemma 2.1.3 shows that the cop will either capture the robber when moving from v-branch to v-branch or stay above him on the new v-branch. □

Hence we need only consider the robber moving from x to y with the cop on a where $a \succ x$, $b \succ y$ and $a \to b$. Let G be a copwin graph, and let $\mathcal{S} = \{S_v : S_v \text{ is a copwin spanning tree with root } v\}$. We define $K_G = \min_{S_v}\{k(S_v) : S_v \in \mathcal{S}\}$.

Theorem 2.1.3 (Clarke and Nowakowski [8]) *Let G be a finite copwin graph. Then*

$$pr(G) \leq |E(G)| - [(n-1) - K_G].$$

Proof. Let S_v be a copwin spanning tree at which K_G is attained. The cop begins on the start vertex v. Draw the tree as in Theorem 2.1.2, however we do not worry about the leaves. Notice that photo radar units are placed on all of the edges of $G \setminus S_v$.

The cop traverses the tree in a depth first search so as to visit all vertices of G as in the proof of Theorem 2.1.2. If the robber never moves off a freepath then he will be caught during this phase. If the robber moves off a freepath, he will be detected by a photo radar unit. The cop moves to the lowest vertex in S_v which is above the freepath containing the robber. (Since the robber can still move off this freepath, the cop may end up at v.)

The cop descends the v-branch leading to the freepath containing the robber. If the robber does not leave this path, he will be caught. If he does leave, then either he descends down the tree and the cop continues his descent toward the robber, or the robber moves to another v-branch and the cop, by Corollary 2.1.1, can always move to the same v-branch. Lemma 2.1.3 shows that the robber can never get above the cop so it remains to show that the robber cannot force repetitions of positions. Note that $F_i(x) = F_{i+1}(x)$ except for the vertex $v_i \in G_i \setminus G_{i+1}$.

Let A_i, $i = 1, 2, \ldots, n$ be distinct v-branches of S_v. Assume that the cop has moved above the robber and the cop knows which freepath the robber is on. Consider the consecutive corresponding moves $x_1, x_2, \ldots, x_{n+1}$ and $c_1, c_2, \ldots, c_{n+1}$ by the robber and cop, respectively where $c_i, x_i \in A_i$ for $i = 1, 2, \ldots, n$, $c_{n+1}, x_{n+1} \in A_1$, and $c_i \succ x_i$ for all i. Also note that $c_i \to c_{i+1}$ and $x_i \sim x_{i+1}$, $i = 1, 2, \ldots, n$. (See Figure 2.5.) Since $x_i \prec c_i$ then for all i, there exists $c_i' \sim c_i$ with $c_i' \prec c_i$. Let j_i be the greatest index such that $F_{j_i}(x_i) = c_i'$. Since $c_i \to c_{i+1}$, it follows that $F_{j_i}(x_{i+1}) \succeq c_{i+1}$ since $F_{j_i+1}(x_i) = c_i$. Thus $F_{j_i+1}(x_{i+1}) = F_{j_i}(x_{i+1})$ since f_{j_i} is a one-point retraction on branch A_i. Consequently $j_i > j_{i+1}$ so $j_n < j_1$. (Recall $x_n \prec c_n$ by Lemma 2.1.2.)

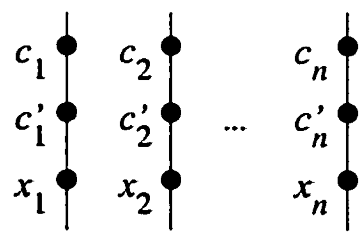


Figure 2.5: The consecutive moves.

Suppose $c_{n+1} \succeq c_1$ and $x_{n+1} \preceq c_1'$. Then $F_{j_n}(x_{n+1}) \preceq F_{j_1}(x_1) = c_1'$. But then $F_{j_n+1}(x_n) = c_n$ by the definition of j_n, $F_{j_n+1}(x_{n+1}) = F_{j_n}(x_{n+1}) \preceq c_1'$ and $F_{j_n}(x_{n+1}) \sim c_n$ since $F_{j_n+1}(x_{n+1}) \sim F_{j_n+1}(x_n)$. This contradicts $c_n \to c_{n+1}$.

The one remaining case is $c_1 \preceq x_{n+1} \preceq c_{n+1}$. (If $x_{n+1} \succeq c_{n+1}$, the robber will be caught on the next move by Lemma 2.1.3.) Since $F_{i_1}(x_1) = c_1'$ it follows that $F_{i_n}(x_{n+1}) = x_{n+1}$. But we have $F_{i_n+1}(x_n) = c_n$ since $F_{i_n}(x_n) = c_n'$, and $F_{i_n+1}(x_{n+1}) = F_{i_n}(x_{n+1}) \sim F_{i_n+1}(x_n) = c_n$. This contradicts $c_n \to c_{n+1}$.

So whenever the cop moves to a v-branch, he is on the same v-branch as the robber, but he is strictly lower than his last position on the v-branch. Therefore since the graph is finite, the robber is eventually caught. □

A similar result is proven for video cameras in the next section. See Theorem 2.2.5. However as shown in Section 2.3, a similar result does not hold for alarms.

2.2 Video Cameras

Suppose again that the cops have only partial information, but that this information comes from video cameras rather than photo radar. Video cameras are placed on the vertices of G. If the robber moves onto a vertex v equipped with a video camera, his position is known to the cops. In addition, the video cameras show the direction taken by the robber when he moves off such a vertex; that is, the edge by which the robber leaves vertex v is known to the cops. The minimum number of video cameras required by a single cop to guarantee the capture of the robber on G will be referred to as the **video camera number** of G, and will be denoted $vc(G)$. In general, one can ask for the least number of video cameras, $vc_k(G)$ needed if there are k cops.

Video cameras are more powerful than photo radar since one video camera at a vertex x could replace any photo radar emanating from that vertex. This observation gives the following result since a cop needs at most perfect information to win on a copwin graph.

Lemma 2.2.1 *Let G be a copwin graph, let $VC = \{x \in V(G) | x \text{ has a video camera}\}$, and let $PR = \{e \in E(G) | e \text{ has a photo radar unit }\}$. Then*

$$\sum_{x \in VC} deg(x) + \sum_{e \in PR} 1 \leq |E(G)|.$$

Again note that each subgraph without video cameras must have search number 1.

We know from Lemma 2.1.1 that a cop playing with no information on the tree T_1 in Figure 2.1 cannot guarantee the capture of the robber. However a single video camera placed on the root vertex a is enough to guarantee a win for the cop. To see this suppose the cop begins on vertex c and then proceeds to search each of the other two branches. If the video camera does not capture footage of the robber, the cop wins during this maneuver. Otherwise, the cop knows on which branch the robber is located and will capture him in at most four moves. The next theorem follows easily.

Theorem 2.2.1 *Let G be a star with at least three branches of length at least two. Then $vc(G) = 1$.*

Proof. The cop places a video camera on the central vertex of the star. He then proceeds to search down each of the branches and return to the root. If the video camera does not capture footage of the robber, the cop wins during this maneuver. Otherwise, the video camera captures footage of the robber and thus the cop knows on which branch the robber is located. The robber will be apprehended in a finite number of moves. □

Theorem 2.2.2 *For all finite, positive integers n, there exists a graph G such that $vc(G) > n$.*

Proof. Consider the tree containing $n+1$ copies of T_1 (shown in Figure 2.1) as subtrees as shown in Figure 2.6. We note that there are $n+1$ vertices corresponding to the root a of T_1. If a subtree has no camera, then the robber can restrict himself to that subtree and win. Hence each subtree requires a camera. □

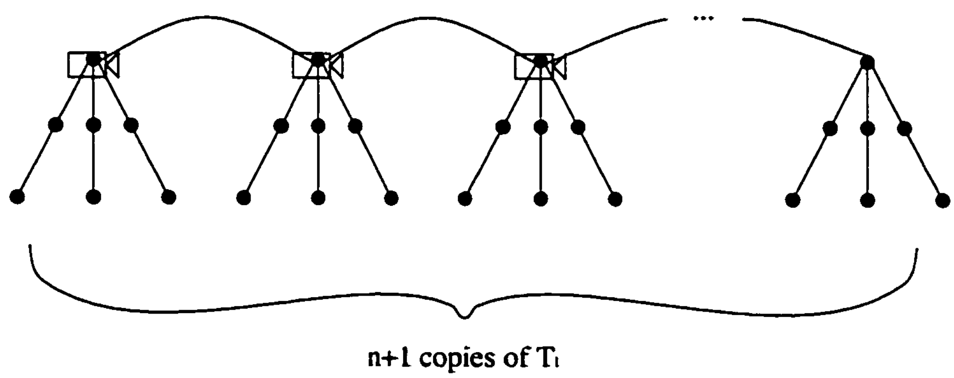


Figure 2.6: The vertices with video cameras are indicated.

Consider a copwin graph G with n vertices. Now $vc(G) \leq n$ since the cameras can be placed one to a vertex making the game equivalent to Cop and Robber. This bound can be improved. We begin with a result for trees that follows directly from Theorem 2.1.2, and then consider two results for general copwin graphs.

Let T be a tree, and T_a the tree rooted at vertex a. Recall from Section 2.1 that $k_T = \min\{k(T'_a) : a \in V(T')\}$ where $k(T_a)$ is the minimum number of edges having photo radar such that the free edges form freepaths and each maximal freepath is on an a-branch, and $T' = T \setminus \{a \in V(T) : a \text{ is a leaf}\}$.

Theorem 2.2.3 *Let T be a finite tree. Then $vc(T) \leq k_T$.*

Proof. Let T be a tree and let a be a vertex for which $k(T'_a) = k_T$. Let $xy \in E(T)$ with $x \preceq y$ be an edge of T with photo radar. This photo radar unit can be replaced by a video camera on vertex y. The cop wins by following the strategy given in the proof of Theorem 2.1.2. □

An **independent set** of a graph G is a set of pairwise nonadjacent vertices of G. The **independence number** of G, denoted $\alpha(G)$ is the maximum size of an independent set in G.

Theorem 2.2.4 *Let G be a copwin graph. Then $vc(G) \leq |V(G)| - \alpha(G)$.*

Proof. Let S be an independent set of G that realizes $\alpha(G)$. Video cameras are placed on the vertices of $G - S$, a vertex cover of G. We will show that the cop has a winning strategy.

If the robber moves entirely on vertices of $G - S$, the cop wins by following the Copwin Strategy since the robber's position is always known.

If the robber never moves onto a vertex of $G - S$, the cop wins by searching the vertices of S. Since S is an independent set, the robber cannot move among the vertices in S, and hence he cannot move onto vertices previously searched by the cop.

So suppose the robber uses vertices both in S and in $G - S$. Once the robber moves onto a vertex in $G - S$, his position is known to the cop for the remainder of the game. This is because when the robber moves from a vertex v in $G - S$ to a vertex in S, the video camera located on v indicates to the cop to which of the vertices in S the robber has moved. We again note that the robber cannot move among vertices in S, and so his next move must be to return to a vertex in $G - S$. Such a move will be known to the cop since all vertices in $G - S$ have video cameras. □

Let G be a copwin graph. A path P of G is said to be a *freepath* if no vertex of P has a video camera. Two freepaths P and Q are said to be *independent freepaths* if for all $x \in V(P)$ and for all $y \in V(Q)$, $x \perp y$.

Let S_v be a copwin spanning tree of G rooted at vertex v, the start vertex of a copwin ordering of G, and let $\{P_i\}_{i=1}^{n}$ be a set of independent freepaths of S_v. Define $ik(S_v) = \max_{\{P_i\}}\{|\cup_{i=1}^{n} P_i| : P_i \text{ are independent freepaths}\}$, and $IK_G = \max_{S_v}\{ik(S_v) : S_v \text{ is a copwin spanning tree with root } v\}$.

Theorem 2.2.5 *Let G be a finite copwin graph. Then $vc(G) \leq |V(G)| - IK_G$.*

Proof. Let S_v be a copwin spanning tree at which IK_G is attained. The cop begins on the start vertex v. Draw the tree as in the proof of Theorem 2.1.3. The proof here follows the proof of Theorem 2.1.3.

The cop traverses the tree in a depth first search so as to visit all vertices of G without video cameras and force the robber to move to avoid capture. Once the robber moves off a freepath, he will be detected by a video camera. The cop moves to the lowest vertex in S_v which is above the freepath containing the robber. (Since the robber can still move off this freepath, the cop may end up at v.)

The cop descends the v-branch leading to the freepath containing the robber. If the robber does not leave this path, he will be caught. If he does leave, then either he descends down the tree and the cop continues his descent toward the robber, or the robber moves to another v-branch and the cop, by Corollary 2.1.1, can always move to the same v-branch. Lemma 2.1.3 shows that the robber can never get above the cop. Note that because the cop knows which edge the robber has taken, the proof of Theorem 2.1.3 shows that the robber cannot force repetitions of positions. □

Example. Consider the graph $P_4 \boxtimes P_4$ as shown in Figure 2.7. A copwin spanning tree of $P_4 \boxtimes P_4$ is indicated by bold lines in the figure. Now $\alpha(P_4 \boxtimes P_4) = 4$ and hence Theorem 2.2.4 gives $vc(P_4 \boxtimes P_4) \leq 12$. As indicated by the circled vertices in the Figure, $IK(P_4 \boxtimes P_4) \geq 6$, and hence $vc(P_4 \boxtimes P_4) \leq 10$.

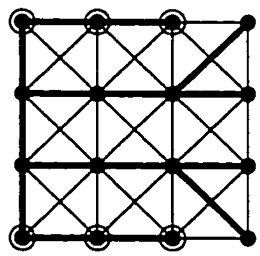


Figure 2.7: Two independent freepaths of $P_4 \boxtimes P_4$ are indicated by circled vertices.

2.3 Alarms on Vertices

Suppose again that the cops have only partial information, but that this information comes from alarms rather than photo radar or video cameras. Alarms are placed on the vertices of G. These units alert the cops if the robber moves onto a vertex equipped with an alarm. However, the alarms do not indicate the direction in which the robber is moving. Since there is no directional signal, the strategies presented here will often require alarms on adjacent vertices as a method of indicating direction. The minimum number of alarms required by a single cop to guarantee the capture of the robber on G will be referred to as the **alarm number** of G, and will be denoted $A(G)$. In general, one can ask for the least number of alarms, $A_k(G)$ needed if there are k cops. Again, for one cop, each unalarmed subgraph S cannot have $s(S) > 1$, but even this does not guarantee a win for the cop. For example, consider the graph T_2 shown in Figure 2.2. If an alarm is placed on vertex c, then for each unalarmed subgraph S, $s(S) = 1$. However there is no guaranteed win by the cop.

We know from Lemma 2.1.1 that a cop playing with no information on the tree T_1 in Figure 2.1 cannot guarantee the capture of the robber. However a single alarm placed on vertex b is enough to guarantee a win for the cop. (Note: an alarm placed on vertex a is not to the cop's benefit.) To see this, suppose the cop begins on vertex c and then proceeds to search each of the other two branches. If the alarm does not sound, the cop wins during this maneuver. If the alarm sounds, the robber is one move away from vertex a, the cop is at most two moves from a and it is the cop's move. Hence the cop either captures the robber on a or else arrives at a before the robber, and the cop can proceed down the branch with the alarm on which the robber is located.

Theorem 2.3.1 *For all finite, positive integers n, there exists a graph G such that $A(G) > n$.*

Proof. Consider a tree similar to T_1 shown in Figure 2.1 with $n+3$ branches rather than 3. Even if the cop places an alarm on each of n branches (and not on the root), there still remains a subgraph with three branches and no alarms. The robber restricts himself to playing on this subgraph and there is no guaranteed win by the cop. □

Consider a copwin graph G with n vertices. Now $A(G) < n$ since an alarm can be placed on all but one of the vertices of G making the game equivalent to Cop and Robber. This bound can be improved for trees.

Theorem 2.3.2 *Let G be a star with k branches of length $l_i \geq 4$, $i = 1, 2, \ldots, k$. Then $A(G) = k - 1$.*

Proof. Suppose $A(G) < k - 1$. If there is a subtree isomorphic to T_1 with either no alarms or one alarm placed on the root, then the robber can restrict himself to playing on that subtree and win. Otherwise there is a rooted subtree isomorphic to T_1', shown in Figure 2.8, with at most one alarm.

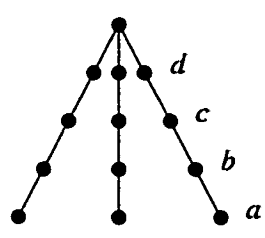


Figure 2.8: The tree T_1'.

If the alarm is on a or b, there is a subtree isomorphic to T_1 with no alarms and the robber can win. So suppose the alarm is on c or d. The cop can force the robber to sound the alarm to avoid capture. However the cop may require up to four moves to reach the root while the robber requires at most two moves. Thus once the cop reaches the root, he has a choice of two branches to search. In the eight moves it takes to search one branch, the robber has time to move between the other two.

We must show that the cop can win with $k-1$ alarms. The cop places an alarm on the root and on each of $k-2$ adjacent vertices so that each of $k-2$ branches has an alarm in addition to the one on the root. (See Figure 2.9 when $k = 8$ and $l = 4$.) These two consecutive alarms provide a directional signal to the cop. The cop begins on a leaf of a branch with a second alarm, then proceeds to search each of the other branches with a second alarm, and finally searches each of the remaining two branches. If none of the alarms sound, the robber is captured during this phase of the search. Otherwise an alarm sounds. There are two possibilities.

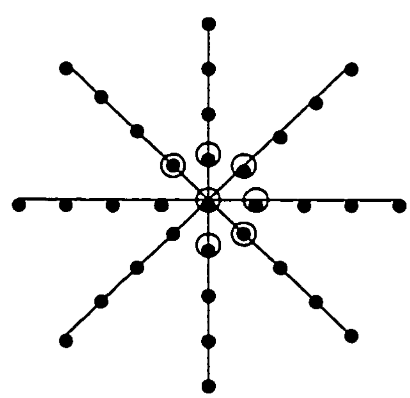


Figure 2.9: The case $k = 8$, $l = 4$. The alarms are represented by the circled vertices.

Suppose first that the cop is located on one of the two branches without a second alarm. The alarm that has sounded must be the alarm on the root since the cop has already searched all of the branches with a second alarm. Once the robber moves again, the cop either knows on which branch he is located, or the cop can deduce that the robber is on one of the two branches without a second alarm. In this case, the cop moves up the branch he is on and then searches down the other branch without a second alarm.

Alternately, suppose the cop is located on a branch with a second alarm. After two moves by the robber, the cop knows the branch on which he is located or can deduce that the robber is on one of the two branches without a second alarm. The cop proceeds to search one such branch and then the other. If no alarms sound, the robber is caught in this portion of the search. Otherwise an alarm sounds and we return to the first case. □

We consider a result similar to Theorem 2.1.2, and then give two theorems that improve upon this result.

Let T be a tree. A path P of T is said to be a *freepath* if no vertex of P has an alarm. A set of freepaths $\{P_j\}_{j=1}^{n}$ is a *packing of freepaths* if for all $a, b, c \in V(T)$ with $a \sim b \sim c$, if $a \in P_i$ and $b, c \notin P_i$ then b and c are alarmed. We would like to be able to mimic Theorems 2.1.3 and 2.2.5 here, and conclude that the alarm number of a copwin graph G is at most the cardinality of a packing of freepaths of a copwin spanning tree S_v of G. However this is not the case. Let us examine the problem that arises here. Suppose the spanning tree S_v is rooted at vertex v, and the cop is above the robber on a particular v-branch. The robber can choose to move from one v-branch to another during his turn. Although the cop can always move to the same v-branch, remaining above the robber as in the proofs of Lemmas 2.1.2 and 2.1.3, the cop is unable to prevent the robber from forcing repetitions of positions when moving to a freepath.

As an example, consider the copwin graph shown in Figure 2.10. The diagonal lines between two branches indicate that every vertex of one of the branches is adjacent to each of the vertices of the other. The freepaths are indicated by dashed lines. The first and last branches (1 and 6) are connected in the same way as branches 2 and 3, 3 and 4, and 5 and 6. Every time the robber moves onto a branch that has a freepath, the cop must move to the topmost unalarmed vertex of that freepath if he is to ensure that he remains above the robber. The game can continue indefinitely in this way. If the cop tries to vary this strategy, the robber moves from branch 1 to 2 to 3 to 4 to 5 to 6, and every time he moves to 2 and 5, it may be assumed that he has moved to the bigger subpath formed when the freepath is bisected by the cop.

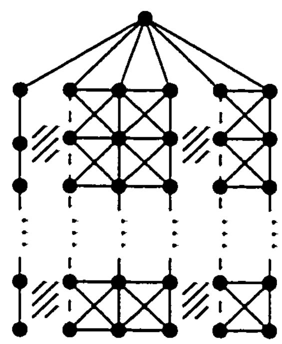


Figure 2.10: A copwin graph which illustrates how the robber is able to force repetitions of positions.

If the robber is prevented from moving between any two v-branches of G, then a result similar to Theorem 2.1.2 holds. Let T be a tree, and T_a the tree rooted at vertex a. Define $pk(T_a)$ as the maximum cardinality of a packing of freepaths on T_a. Let $T' = T \setminus \{a : a \text{ is a leaf}\}$ and set $pk_T = \max\{pk(T'_a) : a \in V(T)\}$. It can be shown as in the proof of Theorem 2.1.2 that $A(T) \leq |V(T')| - pk_T$.

For full, complete binary trees, a packing of freepaths requires that for every vertex on a freepath, one child along with its children are alarmed. So essentially three of every four vertices are alarmed, disregarding the leaves which remain unalarmed. Thus if T is a binary tree then the ratio of alarms to vertices is

$$\frac{A(T)}{|V(T)|} \approx \frac{3}{4}\left(\frac{\sum_{i=0}^{n} 2^i - 2^n}{\sum_{i=0}^{n} 2^i}\right) \approx \frac{3}{8}.$$

This result can be improved.

Let T be a tree. A vertex is *free* if it has no alarm. Again let T_a be the tree rooted at vertex a. A vertex $v \neq a$ is *potentially free* if it has distance at most four from any of its descendents that are leaves. We define $l(T_a)$ as the minimum number of vertices having alarms such that exactly one potentially free child of each vertex is not alarmed, and all other vertices are alarmed. Let $T' = T \setminus \{l \in V(T) : l \text{ is a leaf}\}$ and set $l_T = \min\{l(T'_a) : a \in V(T')\}$.

Theorem 2.3.3 *Let T be a finite tree. Then $A(T) \leq l_T$.*

Proof. Let T be a tree and let a be a vertex for which $l(T'_a) = l_T$. Draw a planar representation of T with a as the root vertex at the top and all edges directed downward. Also since every vertex has at most one child that is free (and not a leaf), we can assume that any free vertex of a parent is the leftmost vertex.

There are two phases to the strategy. The cop begins on the rightmost leaf and does a depth first search of T' except that he travels from right to left and when he comes to a stem v, he visits any leaves adjacent to v after searching all other subtrees rooted at v. At any stage in this phase, the robber cannot move onto a vertex previously searched by the cop without sounding an alarm. Hence if no alarm sounds, the robber will be caught in this phase.

The second phase begins when the robber sounds an alarm. If the robber is below the cop on the same branch, the cop moves toward him. So we assume that the robber must be located on a branch to the left of the cop, or above the cop on the same branch. The cop moves up the tree until he is located above the robber on the same branch. We must consider the cop's strategy for choosing a direction when he comes to a parent, v from a child, u. There are three potentially troublesome cases.

(1) There are two unalarmed vertices in $N(v)$ but the locations of other alarms that have sounded indicate the direction in which the cop should move, and he does so.

(2) There are two unalarmed vertices but the robber hasn't had enough moves to reach this juncture since an alarm last sounded. For example, suppose in Figure 2.11 the cop is located on C and the robber sounds the alarm on R. The cop moves up the tree to v which takes two moves. Assuming no further alarms have sounded (and thus we have returned to a previous case), the cop must choose whether to move onto v_1 or v_2. But the robber could not have reached v and then v_1 in two moves, and so the cop moves to v_2.

(3) There are two unalarmed vertices, v_1 and v_2 but all vertices in the set $N(v_1) - v$ have alarms. Hence the cop visits v_1. If the robber is not caught there and no further alarms have sounded, the cop returns to v and then moves to v_2. If another alarm

has sounded, the cop moves in the direction of that alarm.

Once the cop is above the robber on the same branch, the cop will know which direction to choose at any juncture since all but one of the vertices representing possible directions has an alarm. If one of these alarms has sounded, the cop moves to that vertex. Otherwise he moves to the unalarmed vertex. The robber will eventually be caught on a leaf if not sooner. □

Example. Consider the binary tree shown in Figure 2.11. Theorem 2.3.3 gives $A(T) \leq 16$.

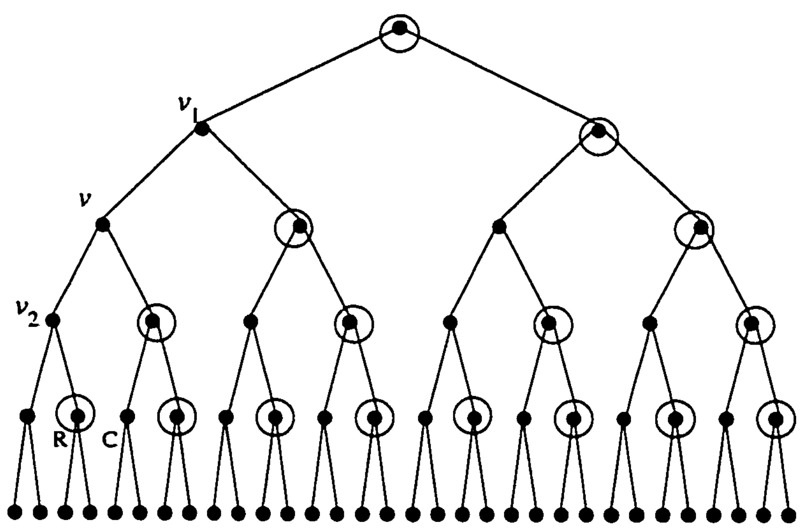


Figure 2.11: The alarms are represented by the circled vertices.

Corollary 2.3.1 *Let T_a be a full, complete k-ary tree rooted at vertex a. If for all leaves v, $d(a,v) = n \geq 5$, a constant, then the ratio of alarms to vertices on T_a that will guarantee a win for the cop is approximately $(k^2 - 2k + 1)/k^2$.*

Proof. Using the strategy of Theorem 2.3.3, the number of vertices without alarms is $k^n + k^{n-2} + k^{n-3} + k^{n-4} + k^{n-5}$ since none of the k^n leaves receive alarms, and $1/k$ of the $k^{n-1} + k^{n-2} + k^{n-3} + k^{n-4}$ vertices at distances one through four from a leaf do not receive alarms. So the number of alarmed vertices is

$$\begin{aligned} A(T_a) &\leq \sum_{i=0}^{n} k^i - (k^n + k^{n-2} + k^{n-3} + k^{n-4} + k^{n-5}) \\ &= \sum_{i=0}^{n-5} k^i + \frac{k-1}{k}(k^{n-4} + k^{n-3} + k^{n-2} + k^{n-1}) \\ &= k^{n-5}(k^5 - k^4 + 1) - 1. \end{aligned}$$

Hence the ratio of alarms to vertices is given by

$$\begin{aligned} \frac{A(T_a)}{|V(T_a)|} &\leq \frac{k^{n-5}(k^5 - k^4 + 1) - 1}{\frac{k^{n+1}-1}{k-1}} \\ &= \frac{k^{n-1}(k^2 - 2k + 1)}{k^{n+1} - 1} \\ &\approx \frac{k^2 - 2k + 1}{k^2}. \end{aligned}$$

□

Example. Consider the binary tree shown in Figure 2.11. Theorem 2.3.1 gives an approximate upper bound of 1/4 for $A(G)/|V(G)|$.

The bound given by Theorem 2.3.3 becomes increasingly inefficient as n becomes larger. The condition that every vertex at distance at least 5 from the leaves be alarmed is very strong, and can be relaxed for k-ary trees. We know that vertices at distance 5 must be alarmed to guarantee a win for the cop by preventing the robber from moving onto unalarmed vertices that have been previously searched by the cop. But it is unnecessary to alarm all vertices above these.

Scheme. By alarming all vertices at distance $d \equiv 0 (\text{mod } 5)$ from the leaves (with the exception of the leaves which receive no alarms), and requiring that vertices at distance $d \equiv i (\text{mod } 5), i \neq 1$ from the leaves have exactly one unalarmed child (again with the exception of vertices at distance 1 from the leaves whose children all remain unalarmed), we obtain a tighter bound.

Theorem 2.3.4 *If G is a full, complete k-ary tree rooted at vertex a, then*

$$A(G) \leq k^{n-1} - \frac{k^{n-1}(k^{-5(\lfloor \frac{n}{5} \rfloor - 1)} - 1)}{k^5 - 1}$$

where n is such that for all leaves l, $d(a,l) = n$.

Proof. Let G be an k-ary tree rooted at vertex a such that for all leaves l, $d(a,l) = n$. We alarm vertices as indicated in the scheme given in the preamble to the theorem and show that the cop has a winning strategy. Thus all vertices at distance $d \equiv 0 (\text{mod } 5)$ from the leaves receive alarms (with the exception of the leaves), and vertices at distance $d \equiv i (\text{mod } 5), i \neq 1$ from the leaves have exactly one unalarmed child (again with the exception of vertices at distance 1 from the leaves).

There are two phases to the strategy. The cop begins on the rightmost leaf and does a depth first search of G except that he travels from right to left. The robber cannot move onto a vertex previously searched by the cop without sounding an alarm. Hence if no alarm sounds, the robber will be caught in this phase.

The second phase begins when the robber sounds an alarm. Suppose the alarm that has sounded is located on a branch to the left of the cop, below the cop on the same branch, or above the cop on the same branch. If the alarm is below the cop on the same branch, the cop moves toward it. Otherwise, the cop moves up the tree until he is above the robber on the same branch. As in the proof of Theorem 2.3.3, we must consider the cop's strategy for choosing a direction when he comes to a parent, v' from a child, u'. There are three potentially troublesome cases.

(1) There are two unalarmed vertices but the locations of other alarms that have sounded indicate the direction in which the cop should move, and he does so.

(2) There are two unalarmed vertices but the robber hasn't had enough moves to reach this juncture since an alarm last sounded. Hence the cop knows to which of these unalarmed vertices he should move, and he does so.

(3) There are two unalarmed vertices, v_1 and v_2 but all vertices in the set $N(v_1) - v'$ have alarms. Hence the cop visits v_1. If the robber is not caught there and no further alarms have sounded, the cop returns to v' and then moves to v_2. If another alarm has sounded, the cop moves in the direction of that alarm.

Once the cop is above the robber on the same branch, the cop will know which direction to choose at any juncture since all but one of the vertices representing possible directions has an alarm. If one of these alarms has sounded, the cop moves to that vertex. Otherwise he moves to the unalarmed vertex. The robber will eventually be caught on a leaf if not sooner in this case.

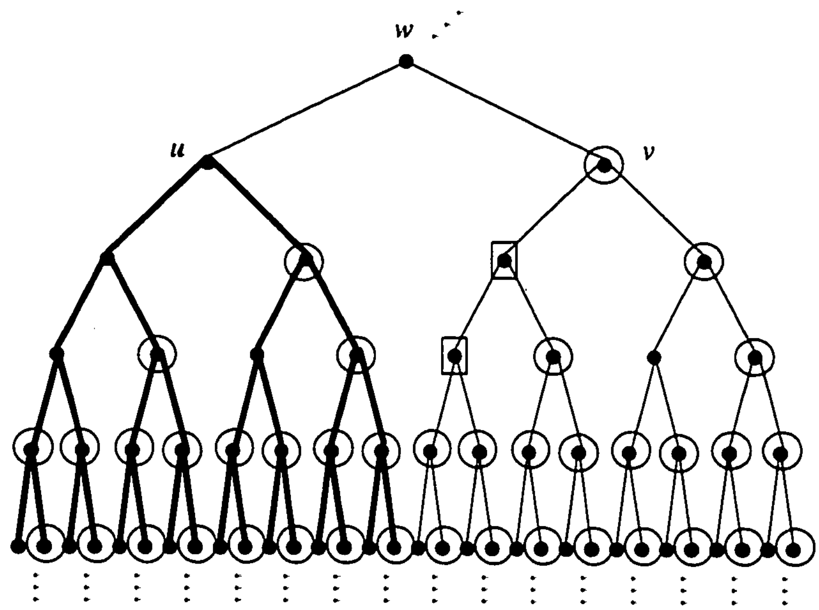


Figure 2.12: A portion of a binary tree. The alarms are represented by the circled vertices.

Now suppose the alarm that has sounded is located on a branch to the right of the cop; that is, the alarm is located on a vertex that has been previously searched by the cop. See vertex v in Figure 2.12 when $k = 2$ for example. The cop is located on the bold portion of the tree, and we assume that no further alarms sound. Otherwise the cop moves until he is above the robber on the same branch and then proceeds as in the previous case. When the cop reaches the leftmost branch, he has two choices. The robber could be located on a vertex previously unsearched by the cop (i.e. the robber moved to w and then onto the bold portion of the tree) or he could be located on one of at most four vertices that have been previously searched as indicated by the boxed vertices in Figure 2.12 when $k = 2$. (Note that in this case there are only two boxed vertices. It is possible to have four since only every fifth level has all vertices alarmed.) Assuming no further alarms sound, the cop continues his search except that he doesn't search any vertices on or below the nearest row in which all vertices are alarmed. If the robber is not caught in this phase then the cop proceeds to search the boxed vertices where the robber must be located. Note that if an alarm sounds, we have again returned to a previous case. Hence the robber is apprehended after a finite number of moves.

It only remains to be shown that the number of alarms used in this strategy is

$$k^{n-1} - \frac{k^{n-1}(k^{-5(\lfloor \frac{n}{5} \rfloor - 1)} - 1)}{k^5 - 1}.$$

Ignoring leaves, the number of vertices with alarms if all but one child of a stem are alarmed is $\frac{k-1}{k}\sum_{i=1}^{n-1} k^i$. But this doesn't count $\frac{1}{k}$ of the vertices that are also alarmed in rows at distance $d \equiv 0 \pmod 5), d \neq 0$ from the leaves, nor does it count the root vertex. Thus we have $1 + \frac{1}{k}\sum_{j=1}^{\lfloor \frac{n}{5} \rfloor - 1} k^{n-5j}$ additional alarmed vertices. Hence the total number of alarmed vertices used in the strategy presented here is

$$\begin{aligned} A(G) &\leq 1 + \frac{k-1}{k}\sum_{i=1}^{n-1} k^i + \frac{1}{k}\sum_{j=1}^{\lfloor \frac{n}{5} \rfloor - 1} k^{n-5j} \\ &= k^{n-1} + \frac{k^n}{k}\sum_{j=1}^{\lfloor \frac{n}{5} \rfloor - 1}\left(\frac{1}{k^5}\right)^j \\ &= k^{n-1} + k^{n-1}\left(\frac{(\frac{1}{k^5})^{\lfloor \frac{n}{5} \rfloor} - 1}{\frac{1}{k^5} - 1} - 1\right) \\ &= k^{n-1} - \frac{k^{n-1}(k^{-5(\lfloor \frac{n}{5} \rfloor - 1)} - 1)}{k^5 - 1}. \end{aligned}$$

□

Corollary 2.3.2 *If G is a full, complete k-ary tree rooted at vertex a, then the ratio of alarms to vertices that will guarantee a win for the cop is approximately*

$$\frac{1}{k} - \frac{1}{k^2} - \frac{(k-1)(k^{-5(\lfloor \frac{n}{5} \rfloor - 1)} - 1)}{k^7}$$

where n is such that for all leaves l, d(a, l) = n.

Proof. Using the strategy of Theorem 2.3.4, the ratio of alarms to vertices that will guarantee a win for the cop is given by

$$\begin{aligned} \frac{A(G)}{|V(G)|} &\leq \frac{k^{n-1} - \frac{k^{n-1}(k^{-5(\lfloor \frac{n}{5} \rfloor - 1)} - 1)}{k^5 - 1}}{\frac{k^{n+1}-1}{k-1}} \\ &\approx \frac{(k-1)(k^{n-1} - k^{n-6}(k^{-5(\lfloor \frac{n}{5} \rfloor - 1)} - 1)}{k^{n+1}} \\ &= \frac{1}{k} - \frac{1}{k^2} - \frac{(k-1)(k^{-5(\lfloor \frac{n}{5} \rfloor - 1)} - 1)}{k^7}. \end{aligned}$$

□

Example. Consider the binary tree shown in Figure 2.11. Theorem 2.3.4 gives an approximate upper bound of 1/4 as did Theorem 2.3.1. As n gets larger however, the bound given by Theorem 2.3.4 becomes increasingly more efficient as compared with the bound given by Theorem 2.3.1.

We would like to improve upon the bound given for the alarm number of an arbitrary copwin graph. However a scheme very different from those used in the proofs of Theorems 2.1.3 and 2.2.5 is needed because of the lack of a directional signal given by the alarms.

2.4 Alarms on Edges

Suppose again that the cops have partial information provided by alarms. However suppose now that the alarms are located on selected edges of a graph G. As before, the alarms alert the cops if the robber moves along an edge equipped with an alarm, but do not indicate the direction in which the robber is moving. The minimum number of alarms on edges required by a single cop to guarantee the capture of the robber on G will be referred to as the **edge-alarm number** of G, and will be denoted $A^*(G)$.

In general, one can ask for the least number of alarms, $A_k^*(G)$ needed if there are k cops. Again for one cop, each unalarmed subgraph, S cannot have $s(S) > 1$.

Since photo radar are more powerful than alarms on edges, the proof of Theorem 2.1.1 shows that for all finite, positive integers n, there exists a graph G such that $A^*(G) > n$. We will prove a result similar to Theorem 2.1.2.

Let T be a tree, and T_a the tree rooted at vertex a. Recall that an a-branch of T_a is a path of T with a as one end vertex. A path P of T is said to be a *freepath* if no edge of P has an alarm. Define $k^*(T_a)$ as the minimum number of edges having alarms such that the unalarmed edges form freepaths and each maximal freepath is on an a-branch. Again let $T' = T \setminus \{l \in V(T) : l \text{ is a leaf}\}$ and set $k_T^* = \min\{k^*(T_a') : a \in V(T')\}$.

Theorem 2.4.1 *Let T be a finite tree. Then $A^*(T) \leq k_T^*$.*

Proof. Let a be a vertex for which $k^*(T_a') = k_T^*$. Draw the tree T_a as in the proof of Theorem 2.1.2. As in the proof of Theorem 2.1.2, there are two phases to the cop's strategy. Firstly, suppose the robber never moves along an edge with an alarm. The cop does a depth first search of T' except when he comes to a vertex v, he visits any leaves adjacent to v. The cop always enters at one end of a freepath and exits at the bottom without leaving the freepath, except for leaves. This phase of the strategy forces the robber to move to avoid capture.

Once the robber is detected by an alarm, the second phase of the cop's strategy begins. The cop moves up the tree until he is on a vertex which lies above the freepath on which the robber is currently located. Assuming that the robber is not caught in this maneuver, the cop then starts down the a-branch that contains the robber until he enters the same freepath as the robber. (Note that the robber can move to a different freepath but this move will be detected by the alarms and the cop will always move so as to be above the robber.) By moving down the freepath and visiting adjacent leaves, the cop will either catch the robber or force him to leave the freepath moving down the tree and below the cop. The robber will eventually be caught on a leaf if not sooner. □

Chapter 3

Cops Restricted to Subgraphs

3.1 Graph Products

A **graph product** is a binary operation such that the vertex set of the resulting graph is the Cartesian product of the vertex sets of the factors, and the edges of the resulting graph are determined only by the adjacency relations of the factors. We will use $\otimes$ to denote an arbitrary graph product.

Graph products with two factors can be represented by 3×3 matrices called **edge matrices** as introduced by Imrich & Izbicki [16]. The rows and columns correspond to the first and second factors respectively. The rows and columns each receive one of three labels: E indicating adjacency of the vertices of the corresponding factor, N indicating nonadjacency, and Δ indicating that the vertex is the same. The entries of the matrix are also E, N, and Δ representing the adjacency relations between the vertices of the product. It should be noted that if the relationship in both factors is Δ the corresponding matrix entry is also Δ since the two vertices are the same.

$$\begin{array}{c} \\ E \\ \Delta \\ N \end{array} \begin{array}{c} \begin{array}{ccc} E & \Delta & N \end{array} \\ \begin{pmatrix} - & - & - \\ - & \Delta & - \\ - & - & - \end{pmatrix} \end{array}$$

We can also consider complementary products since a graph can be defined in terms of its non-edges. Let G^c be the complement of the graph G. Then the **complementary product** $\otimes^c$ to the product $\otimes$ is $G \otimes^c H = (G^c \otimes H^c)^c$.

We now define ten graph products in terms of their edge matrices. These products are presented as complementary pairs.

$$\text{Categorical: } G \times H \quad \begin{pmatrix} E & N & N \\ N & \Delta & N \\ N & N & N \end{pmatrix} \; ; \quad \text{Co-Categorical: } G \times^c H \quad \begin{pmatrix} E & E & E \\ E & \Delta & E \\ E & E & N \end{pmatrix}$$

$$\text{Cartesian: } G \,\Box\, H \quad \begin{pmatrix} N & E & N \\ E & \Delta & N \\ N & N & N \end{pmatrix} \; ; \quad \text{Co-Cartesian: } G \,\Box\,^c H \quad \begin{pmatrix} E & E & E \\ E & \Delta & N \\ E & N & E \end{pmatrix}$$

$$\text{Strong: } G \boxtimes H \quad \begin{pmatrix} E & E & N \\ E & \Delta & N \\ N & N & N \end{pmatrix} \; ; \quad \text{Disjunction: } G \boxtimes^c H \quad \begin{pmatrix} E & E & E \\ E & \Delta & N \\ E & N & N \end{pmatrix}$$

$$\text{Equivalence: } G \textcircled{$\cong$} H \quad \begin{pmatrix} E & E & N \\ E & \Delta & N \\ N & N & E \end{pmatrix} \; ; \quad \text{Symmetric Difference: } G \nabla H \quad \begin{pmatrix} N & E & E \\ E & \Delta & N \\ E & N & N \end{pmatrix}$$

$$\text{Lexicographic: } G \bullet H \quad \begin{pmatrix} E & E & E \\ E & \Delta & N \\ N & N & N \end{pmatrix} \; ; \quad \text{Co-Lexicographic: } G \bullet^c H \quad \begin{pmatrix} E & E & N \\ E & \Delta & N \\ E & N & N \end{pmatrix}$$

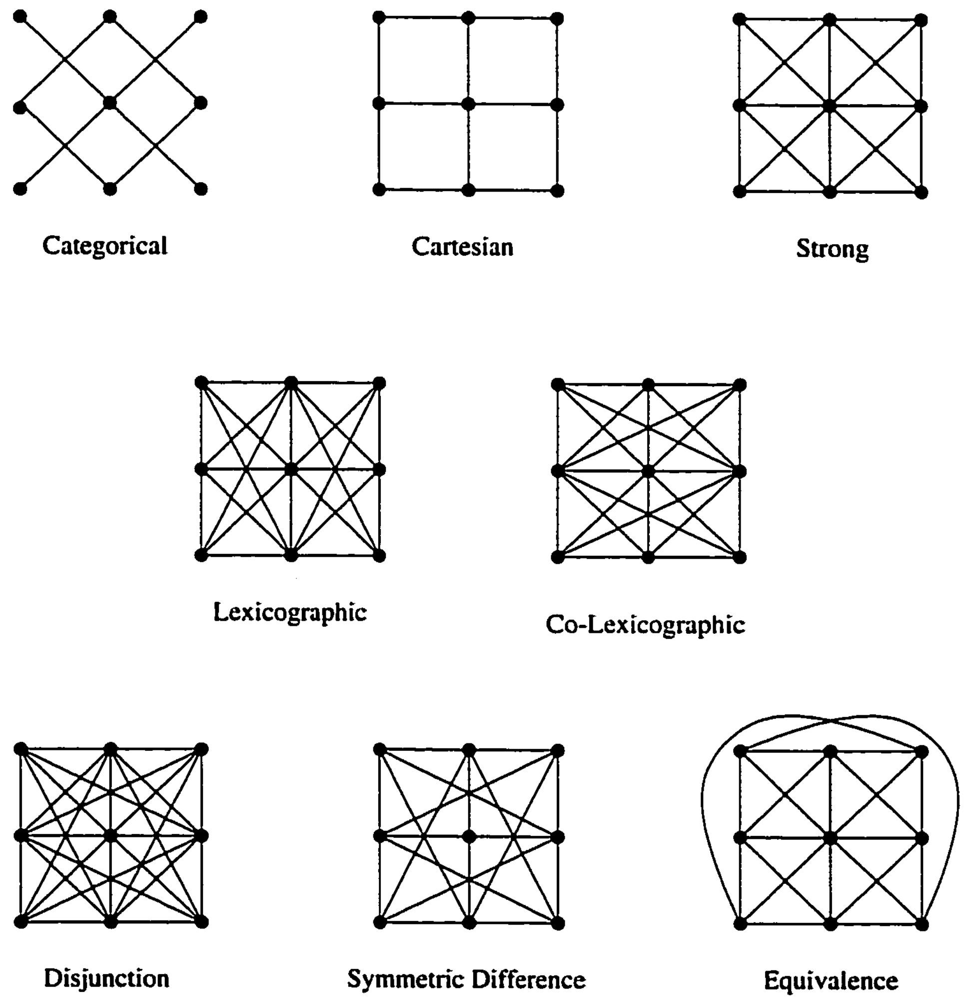


Figure 3.1: The edges of the product of P_3 with itself.

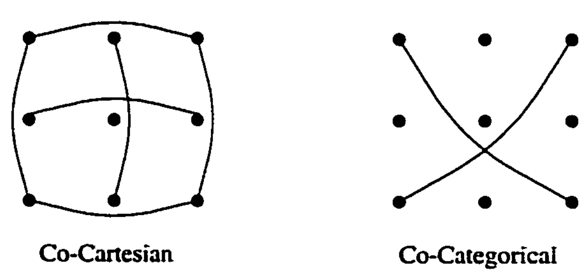


Figure 3.2: The non-edges of the product of P_3 with itself.

These products can be ordered by inclusion; that is $\oplus \leq \otimes$ if $E(G\oplus H) \subseteq E(G\otimes H)$ for all graphs G and H. The suborder of the ten products we consider is shown in Figure 3.3.

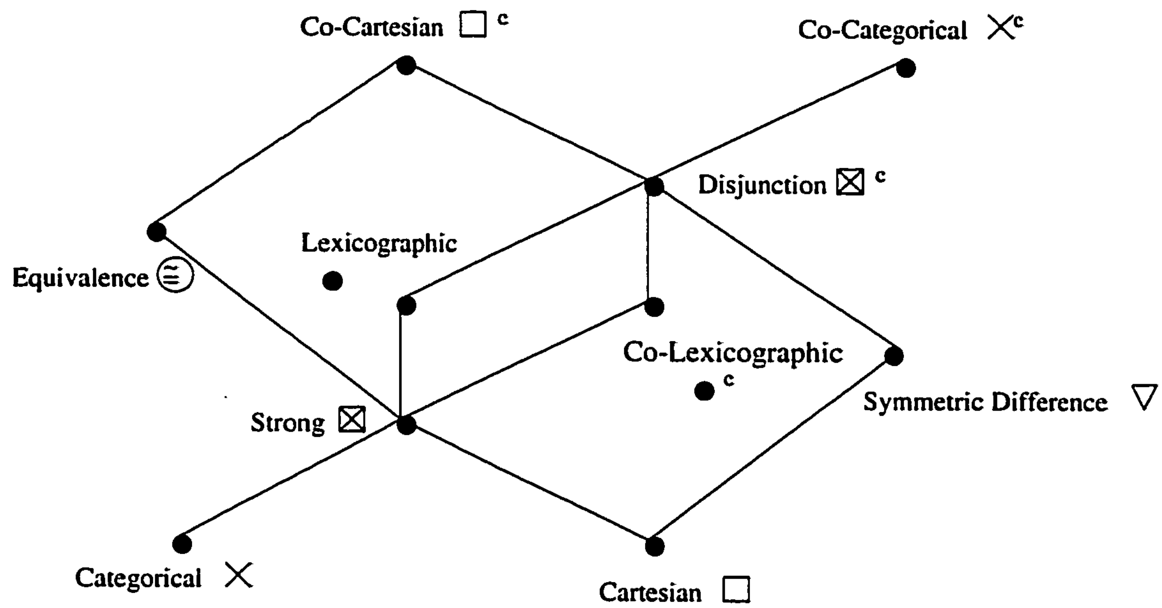


Figure 3.3: A partial order of products under inclusion of the edge sets.

The **eccentricity** of a vertex is the maximum of the distances to other vertices. The **diameter** of a graph G, denoted $\operatorname{diam}(G)$, is the maximum of the eccentricities of its vertices. Table 3.1 gives the diameter of the product of two arbitrary connected graphs G and H under many of the products defined here and can be found in [21].

A **dominating set** in a graph G is a set S such that every vertex in G belongs to S or is adjacent to a vertex in S. The minimum cardinality of a dominating set in a graph G is the **domination number** of G and is denoted $\gamma(G)$.

The equivalence and symmetric difference products have not been included in Table 3.1 because the diameters were given incorrectly in [21]. Results needed for these products follow. We begin with the equivalence product.

Theorem 3.1.1 *Let G and H be connected graphs with $\gamma(G) = 1$ and $\gamma(H) = 1$. Then* $\operatorname{diam}(G \circledcirc H) \leq 2$.

$\otimes$	$\text{diam}(G \otimes H)$
$\square$	$= \text{diam}(G) + \text{diam}(H)$
$\boxtimes$	$= \max\{\text{diam}(G), \text{diam}(H)\}$
$\times$	$\geq \max\{\text{diam}(G), \text{diam}(H)\}$
$\bullet$	≤ 2 if $G \cong K_n$ $\text{diam}(G)$ otherwise
$\boxtimes^c$	≤ 2
$\times^c$	≤ 2
$\square^c$	≤ 2

Table 3.1: The diameters of the products of two arbitrary connected graphs.

Proof. Consider vertices $a, c \in V(G)$ and $b, d \in V(H)$. If $a \simeq c$ and $b \simeq d$ or if $a \perp c$ and $b \perp d$ then $(a,b) \simeq (c,d)$ and so $d((a,b),(c,d)) \leq 1$.

Suppose $a \simeq c$ and $b \perp d$. Since $\gamma(H) = 1$, $\text{diam}(H) \leq 2$ and so there exists a vertex $e \in V(H)$ such that $b \sim e \sim d$. Hence $(a,b) \sim (a,e) \sim (c,d)$ and so $d((a,b),(c,d)) \leq 2$.

Similarly suppose $a \perp c$ and $b \simeq d$. Since $\gamma(G) = 1$, $\text{diam}(G) \leq 2$ and so there exists a vertex $f \in V(G)$ such that $a \sim f \sim c$. Hence $(a,b) \sim (f,d) \sim (c,d)$ and so $d((a,b),(c,d)) \leq 2$.

Since $d((a,b),(c,d)) \leq 2$ for all vertices $a, c \in V(G)$ and $b, d \in V(H)$, $\text{diam}(G \circledcirc H) \leq 2$. $\square$

Theorem 3.1.2 *Let G and H be connected graphs with $\gamma(G) > 1$ and $\gamma(H) > 1$. Then* $\text{diam}(G \circledcirc H) \leq 3$.

Proof. Consider vertices $a, c \in V(G)$ and $b, d \in V(H)$. If $a \simeq c$ and $b \simeq d$ or if $a \perp c$ and $b \perp d$ then $(a,b) \simeq (c,d)$ and so $d((a,b),(c,d)) \leq 1$.

Suppose $a \simeq c$ and $b \perp d$. If $d(b,d) = 2$ in H then there exists a vertex $e \in V(H)$ such that $b \sim e \sim d$. So $(a,b) \sim (a,e) \sim (c,d)$ and so $d((a,b),(c,d)) \leq 2$.

If $d(b,d) = 3$ in H then there exist vertices $e, f \in V(H)$ such that $b \sim e \sim f \sim d$. So $(a,b) \sim (c,e) \sim (c,f) \sim (c,d)$ and so $d((a,b),(c,d)) \leq 3$.

So suppose $d(b,d) = k > 3$ in H. Then there exists an isometric path of length k, $\{b, v_1, v_2, \ldots, v_{k-1}, d\}$ in H. If there exists a vertex $v \in V(G)$ such that $a \perp v$ and $c \perp v$ then $(a,b) \sim (v, v_{\lceil \frac{k}{2} \rceil}) \sim (c,d)$ and so $d((a,b),(c,d)) \leq 2$. Otherwise since $\gamma(G) > 1$, there exists a vertex $u \in V(G)$ such that u is adjacent to exactly one of a and c. We assume $a \perp u$ and $c \sim u$. The other case is similar. So $(a,b) \sim (u, v_{k-1}) \sim (c,d)$ and so $d((a,b),(c,d)) \leq 2$.

The case when $a \perp c$ and $b \simeq d$ is similar.

Since $d((a,b),(c,d)) \leq 3$ for all vertices $a, c \in V(G)$ and $b, d \in V(H)$, $\text{diam}(G \textcircled{\cong} H) \leq 3$. □

Theorem 3.1.3 *Let G and H be connected graphs with $\gamma(G) > 1$ and $\gamma(H) = 1$. Then* $\text{diam}(G \textcircled{\cong} H) \leq \text{diam}(G)$.

Proof. Since $\textcircled{\cong} \geq \boxtimes$ in the partial order of products, $E(G \boxtimes H) \subseteq E(G \textcircled{\cong} H)$ and hence $\text{diam}(G \textcircled{\cong} H) \leq \text{diam}(G \boxtimes H) = \max\{\text{diam}(G), \text{diam}(H)\} = \text{diam}(G)$. □

Note. This upper bound is achieved whenever $H \cong K_m$, $m \geq 1$. Consider $P_5 \textcircled{\cong} K_3$. Now $\text{diam}(P_5 \textcircled{\cong} K_3) = 4 = \text{diam}(P_5)$. See Figure 3.4.

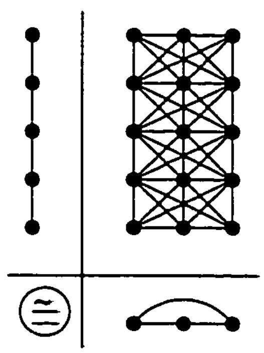


Figure 3.4: The upper bound given in Theorem 3.1.3 is achieved.

We now consider the symmetric difference of two connected graphs G and H. Note that if $|V(G)| = 1$, then $\text{diam}(G \nabla H) = \text{diam}(H)$ since $G \nabla H \cong H$.

Theorem 3.1.4 *Let G and H be connected graphs such that $|V(G)| \geq 2$ and $|V(H)| \geq 2$. Then* $\text{diam}(G \nabla H) \leq 2$.

Proof. Consider vertices $a, c \in V(G)$ and $b, d \in V(H)$. If $a \sim c$ and $b \perp d$, then $(a,b) \sim (c,d)$ by definition. Suppose $a \simeq c$ and $b \simeq d$. Then $(a,b) \simeq (a,d) \simeq (c,d)$ and so $d((a,b),(c,d)) \leq 2$.

Suppose $a = c$ and $b \perp d$. Since $|V(G)| \geq 2$ and G is connected, there exists a vertex $v \in V(G)$ such that $a \sim v$. Then $(a,b) \simeq (v,d) \simeq (a,d) = (c,d)$ and so $d((a,b),(c,d)) \leq 2$.

Finally suppose $a \perp c$ and $b \perp d$. If $d(a,c) = 2$ then there exists a vertex $e \in V(G)$ such that $a \sim e \sim c$. Hence $(a,b) \sim (e,b) \sim (c,d)$ and so $d((a,b),(c,d)) \leq 2$. The case when $d(b,d) = 2$ is similar.

If $d(a,c) = k \geq 3$ and $d(b,d) = l \geq 3$ then there exists an isometric path of length k, $\{a, u_1, u_2, \ldots, u_{k-1}, c\}$ in G and there exists an isometric path of length l, $\{b, v_1, v_2, \ldots, v_{l-1}, d\}$ in H. Hence $(a,b) \sim (u_1, v_{l-l}) \sim (c,d)$ and so $d((a,b),(c,d)) \leq 2$.

Since $d((a,b),(c,d)) \leq 2$ for all vertices $a, c \in V(G)$ and $b, d \in V(H)$, $\text{diam}(G \nabla H) \leq 2$. □

We denote the nth power of a graph G under the product $\otimes$ by $G^n_\otimes$. We will now give an upper bound for $\text{diam}(G^2_\times)$ and hence for $\text{diam}(G^n_\times)$. This may be known but is not common knowledge. We note that if all of the constituent graphs of a categorical product are connected and at least one of them is not bipartite, then the product is also connected. We begin with several lemmas.

Let G be a graph and let $W = \{v_0, v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n\}$ be a walk of length n in G. We say that there is a **shortcut** between vertices v_i and v_j if there exist vertices $u_0, u_1, \ldots, u_m \in V(G)$ such that $\{v_i, u_0, u_1, \ldots, u_m, v_j\}$ is a walk joining v_i and v_j and $m \leq j - i - 1$.

Lemma 3.1.1 *Let G be a connected graph with* $\text{diam}(G) = d$. *Then each walk W of G of length $2d+1$ has an even shortcut.*

Proof. Let $W = \{a_0, a_1, \ldots, a_{2d+1}\}$ be a walk of length $2d+1$ in G. If for any indices j, k with $j \neq k$, $a_j = a_k$, then there is a shortcut of length 0. So assume this does not occur.

Since $d(a_0, a_{d+1}) \leq d$, there is a shortcut, S from a_0 to a_{d+1} of length at most d. If S is even, we are done. So assume S is odd. Then $a_{d+1} \cup S$ is an even shortcut from a_0 to a_{d+2} of length less than $d+2$. □

Theorem 3.1.5 *Let G be a connected graph such that $\chi(G) \geq 3$ and* $\mathrm{diam}(G) = d$*. Then* $\mathrm{diam}(G^2_\times) \leq 2\mathrm{diam}(G)$.

Proof. Let $\{(a_0, b_0), (a_1, b_1), \ldots, (a_{2d+1}, b_{2d+1})\}$ be a path of length $2d+1$ between vertices (a_0, b_0) and (a_{2d+1}, b_{2d+1}) in $G^2_\times$. We will show that this path contains a shorter path between (a_0, b_0) and (a_{2d+1}, b_{2d+1}), and hence $\mathrm{diam}(G^2_\times) \leq 2d$.

Notice that $W_1 = \{a_0, a_1, \ldots, a_{2d+1}\}$ and $W_2 = \{b_0, b_1, \ldots, b_{2d+1}\}$ are walks of length $2d+1$ in G. Lemma 3.1.1 tells us that these walks have even shortcuts. Suppose W_1 has shortcut $\{a_i, c_1, c_2, \ldots, c_k, a_j\}$ between a_i and a_j, and W_2 has shortcut $\{b_m, d_1, d_2, \ldots, d_l, b_n\}$ between b_m and b_n with $l - k = 2p$. So we have walks $W_1' = \{a_0, a_1, \ldots, a_i, c_1, \ldots, c_k, a_j, \ldots, a_{2d+1}\} = \{u_0, u_1, \ldots, u_s\}$ with $s < 2d+1$, and $W_2' = \{b_0, b_1, \ldots, b_m, d_1, \ldots, d_l, b_n, \ldots, b_{2d+1}\} = \{v_0, v_1, \ldots, v_t\}$ with $t < 2d+1$ and $t - s = 2p$. Then $\{(a_0, b_0) = (u_0, v_0), (u_1, v_1), \ldots, (u_s, v_{t-2p}), (u_{s-1}, v_{t-2p+1}), (u_s, v_{t-2p+2}), (u_{s-1}, v_{t-2p+3}), (u_s, v_{t-2p+4}), \ldots, (u_{s-1}, v_{t-1}), (u_s, v_t) = (a_{2d+1}, b_{2d+1})\}$ is a walk of length $t < 2d+1$ between vertices (a_0, b_0) and (a_{2d+1}, b_{2d+1}) in $G^2_\times$. Hence $\mathrm{diam}(G^2_\times) \leq 2\mathrm{diam}(G)$. □

Corollary 3.1.1 *Let G be a connected graph such that $\chi(G) \geq 3$ and* $\mathrm{diam}(G) = d$*. Then* $\mathrm{diam}(G^n_\times) \leq 2\mathrm{diam}(G)$.

Proof. Let $W = \{v_0, v_1, \ldots, v_{2d+1}\}$, $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$ be a path of length $2d+1$ in $G^n_\times$. We will show that this path contains a shorter path between v_0 and v_{2d+1}, and hence $\mathrm{diam}(G^n_\times) \leq 2\mathrm{diam}(G)$.

Notice that for all $j, j = 1, 2, \ldots, n$, $W_j = \{v_{0j}, v_{1j}, \ldots, v_{(2d+1)j}\}$ is a walk of length $2d+1$ in G. Lemma 3.1.1 tells us that these n walks have even shortcuts, $S_j, j = 1, 2, \ldots, n$. So we have walks W_j' obtained from W_j by incorporating these shortcuts. Let $W'_j = \{v_{0j} = c_{j0}, c_{j1}, \ldots, c_{jk_j} = v_{(2d+1)j}\}$ where $k_j = 2d+1-|S_j|+1 = 2(d+1) - |S_j|$.

Suppose $|S_m| \geq |S_j|$ for all j so that W_m has the even shortcut of longest length. We construct a walk $W' = \{u_0, u_1, \ldots, u_{k_m}\}$, $u_i = (u_{i1}, u_{i2}, \ldots, u_{in})$ of length $k_m < 2d+1$ as follows. The mth coordinates of the vertices u_i are (in order) $c_{m0}, c_{m1}, \ldots, c_{mk_m}$. For all $l \neq m$, the lth coordinates of the vertices u_i are (in order) $c_{l0}, c_{l1}, \ldots, c_{l(k_l-1)}, c_{lk_l}, c_{l(k_l-1)}, c_{lk_l}, c_{l(k_l-1)}, \ldots, c_{lk_l}$. □

We update Table 3.1 below.

$\otimes$	$\mathrm{diam}(G \otimes H)$	
$\square$	$= \mathrm{diam}(G) + \mathrm{diam}(H)$	
$\boxtimes$	$= \max\{\mathrm{diam}(G), \mathrm{diam}(H)\}$	
$\times$	$\geq \max\{\mathrm{diam}(G), \mathrm{diam}(H)\}$ $\leq 2\mathrm{diam}(G)$ if $H \cong G$	
$\bullet$	≤ 2 $\mathrm{diam}(G)$	if $G \cong K_n$ otherwise
$\circledcong$	≤ 2 ≤ 3 $\leq \max\{\mathrm{diam}(G), \mathrm{diam}(H)\}$	if both $\gamma(G), \gamma(H) = 1$ if both $\gamma(G), \gamma(H) > 1$ otherwise
∇	≤ 2 $= \mathrm{diam}(H)$ $= \mathrm{diam}(G)$	if $\|V(G)\| \geq 2, \|V(H)\| \geq 2$ if $\|V(G)\| = 1$ if $\|V(H)\| = 1$
$\boxtimes^c$	≤ 2	
$\times^c$	≤ 2	
$\square^c$	≤ 2	

Table 3.2: The diameters of the products of two arbitrary connected graphs (updated).

3.2 Introduction

In [13], Fitzpatrick introduces a variation of Cops and Robber known as the *precinct* version of the game. Each cop's movements are restricted to an assigned "beat" or subgraph. If each subgraph is a copwin graph and a retract, then the minimum number of cops needed to capture the robber on a graph G is bounded by the minimum number of subgraphs needed to cover the vertices of G. A set of subgraphs is said to **cover** G if every vertex of G lies in at least one subgraph of the set.

The following theorem is a special case of Theorem 1.2.8.

Theorem 3.2.1 *Let $\{G_i, i = 1, 2, \ldots, k\}$ be a set of copwin subgraphs of a graph G. If $\cup_{i=1}^{k} V(G_i) = V(G)$ and each G_i is a retract of G then $c(G) \leq k$.*

Proof. For all i, let $f_i : G \rightarrow G_i$ be a retraction map from G to the copwin subgraph G_i. Suppose a single cop C_i is playing on G_i. Using the Copwin Strategy, the cop is able to capture the robber's image under f_i on G_i in a finite number of moves. Since f_i is the identity map on G_i, should the robber move onto G_i, he would be immediately apprehended. The result follows since the copwin subgraphs cover G and thus the robber is on one of the subgraphs. □

If H is a retract of G then it follows that H is an isometric subgraph of G. The converse holds only for some specific classes of graphs. Graphs which are retracts whenever they are isometric subgraphs are said to be **absolute retracts.** Nowakowski and Rival [22] showed that paths are absolute retracts.

If P is an isometric path of a graph G then P is a retract of G.

To see this, let $P = \{0, 1, \ldots k\}$ be an isometric path and consider $x \in V(G)$ with $d(x, 0) = j$. The function $f : G \rightarrow P$ which maps x to j whenever $j \leq k$ and to k otherwise is a retraction map from G to P.

Let $x \in V(G)$. It is also true that the subgraph S of G induced by $N[x]$ is a retract of G. The retraction map $f : G \rightarrow S$ takes all vertices $y \in V(G) \setminus N(x)$ to the central vertex x. In particular,

Cliques are absolute retracts.

Cliques are discussed in Section 3.4.

3.3 Isometric Paths

Recall that an *isometric path* P of a graph G is a subgraph such that for all $x, y \in V(P)$, $d_P(x,y) = d_G(x,y)$. The length of an isometric path is the number of edges in the path, and hence one less than the number of vertices in the path.

If each beat in the precinct game is an isometric path, then the minimum number of cops needed to capture the robber on a graph G is at most the minimum number of isometric paths needed to cover the vertices of G. The minimum number of isometric paths required to cover the vertices of a graph G is the **isometric path number** of G and will be denoted $p(G)$.

Example. Any cycle C_n can be covered by two isometric paths as shown in Figure 3.5 when $n = 6$. Hence $p(C_n) = 2$.

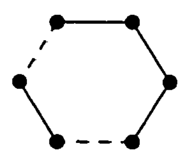


Figure 3.5: A graph C_6 with isometric path number 2. The isometric paths are indicated by solid lines. The dashed edges are those of C_6 not included in either of the isometric paths.

We would like to consider the isometric path number of $G^n_\otimes$ as n gets large. Since this parameter will likely go to infinity, we must consider a normalization. Define

$$\rho(G,\otimes) = \lim_{n\to\infty} \frac{p(G^n_\otimes)}{|V(G^n_\otimes)|},$$

provided the limit exists. In subsequent sections, we are able to determine $\rho(G,\otimes)$ exactly in most cases, and so the desired limit clearly exists in these cases. In other cases we show that $p(G^{n+1}_\otimes) \leq vp(G^n_\otimes)$, $v = |V(G)|$ and thus that the desired limit exists. Finally when $\otimes \in \{\times, \bullet, \bullet^c\}$, it is not clear that the limit exists. Difficulties with the lexicographic and co-lexicographic products are encountered in the next section as well as in Section 3.3.8. The categorical product is problematic since the product of two connected graphs may not even be connected. Consider the categorical product of two bipartite graphs, for example. However we conjecture that the limit, and hence $\rho(G, \otimes)$ exists for each of these three products.

Lemma 3.3.1 (Fitzpatrick [13]) *Let G be any connected graph with vertex set $V(G)$. Then $p(G) \geq \lceil \frac{|V(G)|}{\operatorname{diam}(G)+1} \rceil$.*

Proof. The diameter of a graph G is simply the length of the longest isometric path in G. Hence an isometric path in G has at most $\operatorname{diam}(G) + 1$ vertices. □

Theorem 3.3.1 *Let G be any connected graph with vertex set $V(G)$. Then $\rho(G, \otimes) \geq \frac{1}{\operatorname{diam}(G^n_\otimes)+1}$.*

Proof. From Lemma 3.3.1, we have that $p(G) \geq \lceil \frac{|V(G)|}{\operatorname{diam}(G)+1} \rceil$. Hence

$$\rho(G, \otimes) \geq \frac{\lceil \frac{|V(G^n_\otimes)|}{\operatorname{diam}(G^n_\otimes)+1} \rceil}{|V(G^n_\otimes)|} \geq \frac{\frac{|V(G^n_\otimes)|}{\operatorname{diam}(G^n_\otimes)+1}}{|V(G^n_\otimes)|} = \frac{1}{\operatorname{diam}(G^n_\otimes) + 1}.$$

□

3.3.1 Fractional Approach

Introduction

This section serves as an introduction to fractional graph theory which attempts to modify concepts from graph theory so that parameters may assume rational values rather than just integer values, and thereby efficiency is increased.

As an example, let us consider the chromatic number of a graph G. A similar example is given in [26]. An **independent set** of a graph G is a set of pairwise nonadjacent vertices of G. A k-**coloring** of G is a partition of the vertex set of G into k independent sets. If G has a k-coloring then G is said to be k-**colorable**. The **chromatic number**, $\chi(G)$ is the smallest k such that G is k-colorable.

The notion of graph coloring can be applied to scheduling problems. Suppose committee meetings must be scheduled for a variety of groups. Clearly any two groups with a common member cannot meet at the same time. The objective is to minimize the total time required for the meetings. A graph G can be drawn with a vertex representing each group, and with an edge joining two vertices if the corresponding groups have a common member. Thus groups corresponding to vertices that receive the same color can meet at the same time, and so the chromatic number of G would appear to be the solution to this scheduling problem.

However a coloring of G with $\chi(G)$ colors is optimal only if the meeting times cannot be broken into smaller intervals. Consider the graph C_7 with $\chi(C_7) = 3$ as shown in Figure 3.6.

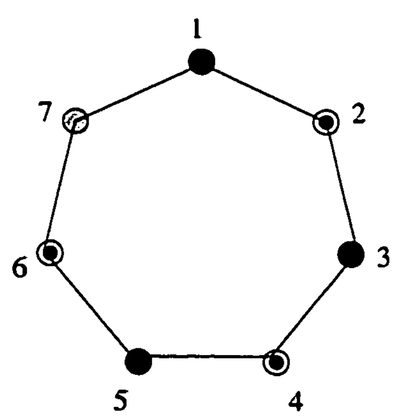


Figure 3.6: A 3-coloring of C_7.

If each meeting is scheduled for an hour, then the 7 committee meetings can be scheduled in $\chi(C_7) = 3$ hours as shown in Figure 3.7.

However if the committees are willing to meet for two one half hour periods rather than a single one hour period, the schedule can be improved so that all 7 meetings take place in a 2 1/2 hour period as shown in Figure 3.8.

A graph concept can be thought of as an integer program. The fractional form of the concept can then be thought of as the linear program relaxation, or LP relaxation of this integer program.

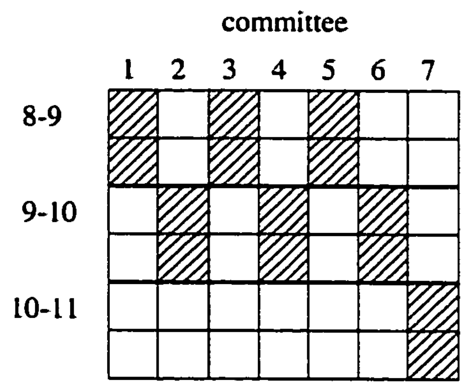


Figure 3.7: A schedule for the 7 committee meetings in $\chi(C_7) = 3$ hours.

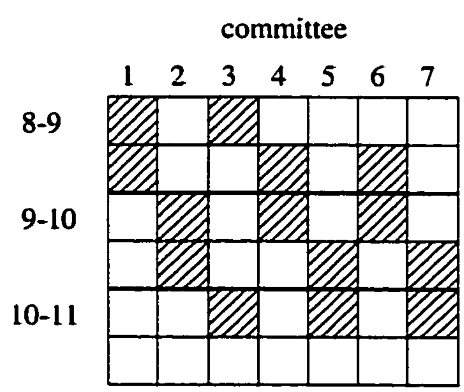


Figure 3.8: A schedule for the 7 committee meetings in 2.5 hours.

An integer program is an optimization problem of the form

maximize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$.

Here $\mathbf{b}$ is a vector of length m, $\mathbf{c}$ a vector of length n, A an $m \times n$ matrix, and $\mathbf{x}$ varies over vectors of length n with integer values. The LP relaxation of an integer program allows the constraint that the entries of $\mathbf{x}$ be integers to be relaxed so that the entries are simply nonnegative.

The real numbers resulting from the LP relaxation of an integer program representing a graph concept are almost always rational. In the following section, we assume rational solutions. Otherwise a rational solution arbitrarily close to the real solution can be considered.

Return to Isometric Paths

We can refer to the problem of covering a graph G with isometric paths as the 0-1 integer programming problem since an isometric path P of G receives one of two weights: 1 if P is included in the covering and 0 otherwise. In this section, we consider the LP relaxation of the 0-1 problem. In this relaxation of the problem, P can receive a weighting w whose value is a real number which lies in the interval $[0, 1]$. Thus for all isometric paths P of G, we assign a weight $w : P \longrightarrow \mathbb{R}^{\geq 0}$. This is known as a fractional weighting of the isometric paths P of G. Just as we wish to minimize the number of isometric paths included in a covering of G in the 0-1 problem, we now wish to minimize the sum of the weights w subject to the restriction that for any $x \in V(G)$, the sum of the weights of the paths to which x belongs is at least 1. That is, if $\mathcal{P}$ is a set of isometric paths of G then we must find a weighting $w : \mathcal{P} \longrightarrow \mathbb{R}^{\geq 0}$ of these isometric paths such that

$\sum_{P \in \mathcal{P}} w(P)$ is minimized subject to the constraint that for all $x \in V(G)$, $\sum_{P | x \in V(P)} w(P) \geq 1$.

Such a weighting will be a feasible solution to the fractional problem. It should be noted that such a weighting exists since a solution to the 0-1 problem is a solution to the LP relaxation. See Figure 3.12 for a fractional weighting of the isometric paths of the graph G shown in Figure 3.9.

We can pose the fractional problem for a graph G in terms of the 0-1 problem for a graph $G \bullet \overline{K}_n$. Let $\mathcal{Q} = \{Q_j, j \in J\}$ be a set of isometric paths that cover $G \bullet \overline{K}_n$ such that no induced subpath of type $\{(x,a),(y,b),(x,c)\}$ is used, where $x, y \in V(G)$, $a, b, c \in V(\overline{K}_n)$, and $x \neq y, a \neq c$. See Figure 3.14 for an example of this type of excluded path. The same problem regarding paths of length 2 will appear again in Section 3.3.8. Define a projection map $f : \mathcal{Q} \to \mathcal{P}$ such that if $f(Q) = P$ then $|Q| = |P|$ and we write $Q \downarrow P$. We define a weighting $w : \mathcal{P} \to \mathbb{R}^{\geq 0}$ of the isometric paths P of G as follows:

$$w(P) = \frac{|\{Q|f(Q) = P\}|}{n}.$$

It follows that

1. if $x \in V(G)$ and $a \in V(\overline{K}_n)$ so that $(x,a) \in G \bullet \overline{K}_n$, then there exists $Q_a \in \mathcal{Q}$ such that $(x,a) \in Q_a$ (that is, every vertex in $G \bullet \overline{K}_n$ belongs to an isometric path), and

2. if $a, b \in V(\overline{K}_n)$ and $b \neq a$, then for each $x \in V(G)$, there exist paths $Q_a, Q_b \in \mathcal{Q}$ such that Q_a and Q_b are distinct (that is, there are distinct paths Q_a and Q_b for all $a \neq b$).

Hence

$$\sum_{P|x \in V(P)} w(P) = \sum_{x \in P} \frac{|\{Q|f(Q) = P\}|}{n} \geq \frac{1}{n} \sum_{(x,a), a \in V(\overline{K}_n)} 1 = \frac{1}{n}(n) = 1.$$

We wish to minimize $\sum_{P \in \mathcal{P}} w(P)$.

Consider first the fractional problem on G. Let $w = \{w_i, i \in I\}$ be a fractional weighting of the isometric paths $\mathcal{P} = \{P_i, i \in I\}$ of G. Define $F_w = \sum_i w_i(P_i)$ and $\mathcal{F}_G = \min_w F_w$. Finally given such a solution, let d be the lowest common denominator of the weights w_i of the isometric paths P_i of G. For any integer $k \geq 1$, consider now the problem of covering $G \bullet \overline{K}_k$ with isometric paths. (Recall that paths of type $\{(x,a),(y,b),(x,c)\}$ are not permitted.) This is a modified 0-1 problem. Let $\mathcal{Q} = \{Q_j, j \in J\}$ be the isometric paths of $G \bullet \overline{K}_k$, and define $I_k(G|\mathcal{Q}) = \sum_j w'(Q_j)$ where $w'(Q_j) = 1$ if Q_j is included in the isometric path cover of $G \bullet \overline{K}_k$ and $w'(Q_j) = 0$ otherwise, and

> $I_k = \min_{w'}\{I_k(G|\mathcal{Q})\}$ so that I_k is the minimum number of isometric paths required to cover $G \bullet \overline{K}_k$ provided paths of type $\{(x,a),(y,b),(x,c)\}$ are not permitted.

Notice that $I_k \geq p(G \bullet \overline{K}_k)$, the isometric path number of $G \bullet \overline{K}_k$, and that $I_k = p(G \bullet \overline{K}_k)$ if no $\{(x,a),(y,b),(x,c)\}$ paths exist.

Comment. It appears that graphs which are largely a clique are the problematic graphs because they must use paths of length two to be efficient. (See Example $*$ at the end of this section.) A similar problem arises in Subsection 3.3.8.

Theorem 3.3.2 *Let G be a graph. Then $d \cdot \mathcal{F}_G = I_d$.*

Proof. For fixed k and an isometric path cover, C of $G \bullet \overline{K}_k$, let $\mathcal{Q} = \{Q_j, j \in J\}$ be the isometric paths of $G \bullet \overline{K}_k$ with $w'(Q_j) = 1$ if Q_j is included in C and $w'(Q_j) = 0$ otherwise. Let $\mathcal{P} = \{P_i, i \in I\}$ be the isometric paths of G. Given the weighting w' of Q, define a weighting w of P as follows:

$$w(P_i) = \frac{1}{k} \sum_{Q_j \downarrow P_i} w'(Q_j);$$

that is, P_i is assigned a weight of $\frac{1}{k}$ of the number of paths Q_j that project onto P_i. Now

$$\sum_i w(P_i) = \frac{1}{k} \sum_i \sum_{Q_j \downarrow P_i} w'(Q_j) = \frac{1}{k} I_k(G|\mathcal{Q}).$$

And so $\frac{1}{k} I_k(G|\mathcal{Q})$ is a feasible solution to the fractional problem. Since $\mathcal{F}_G$ is the minimum of all fractional solutions, $\mathcal{F}_G \leq \frac{I_k(G|\mathcal{Q})}{k}$, and so $\mathcal{F}_G \leq \frac{I_k}{k}$. Since this inequality holds for all k, $\mathcal{F}_G \leq \min_k \frac{I_k}{k}$.

Consider now $d \cdot F_w$ where F_w is a solution to the fractional problem. We will show that this quantity $d \cdot F_w$ is a solution to the 0-1 problem of covering the graph $G \bullet \overline{K}_d$ with isometric paths. For each isometric path P_i, $i \in I$, take $dw(P_i)$ copies of P_i in $G \bullet \overline{K}_d$, taking new uncovered vertices if there are any remaining in $G \bullet \overline{K}_d$. Let $V(\overline{K}_d) = \{1, 2, \ldots, d\}$. If $P_i = \{x_1, x_2, \ldots, x_l\}$ then the next copy of P_i in $G \bullet \overline{K}_d$ is $\{(x_i, i_{x_i}) | i = 1, 2, \ldots, d\}$ where i_{x_i} is the least numbered vertex such that (x_i, i_{x_i}) is not in a previously defined path. If no such vertex exists then take (x_i, d).

We claim that every vertex of $\overline{K}_d \bullet G$ is covered by a path. Consider $x \in V(G)$ and the set $\{P_j | x \in P_j\}$. Now

$$\sum_{x \in P_j} w(P_j) \geq 1$$

and so

$$d \sum_{x \in P_j} w(P_j) \geq d;$$

that is

$$\sum_{x \in P_j} dw(P_j) \geq d$$

and so any vertex (x, i), $i = 1, 2, \ldots, d$ is used at least once. Equivalently each path P_j uses $dw(P_j)$ new copies of x in $G \bullet \overline{K}_d$. Since $\mathcal{F}_G = \min_w F_w$, $d \cdot \mathcal{F}_G$ is a feasible solution. Hence $d \cdot \mathcal{F}_G \geq I_d(G)$ and so $\mathcal{F}_G \geq \frac{I_d(G)}{d} \geq \frac{I_d}{d} \geq \min_k \frac{I_k}{k}$.

We have thus shown that $\mathcal{F}_G = \min_k \frac{I_k}{k}$ and $\min_k \frac{I_k}{k} = \frac{I_d}{d}$. Hence $\mathcal{F}_G = \frac{I_d}{d}$ or equivalently $d \cdot \mathcal{F}_G = I_d$. □

Example. We wish to find $\mathcal{F}_G$ where G is the graph shown in Figure 3.9.

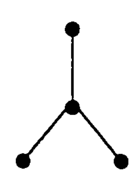


Figure 3.9: The graph G.

Let us consider the graph $G \bullet \overline{K}_2$ as shown in Figure 3.10.

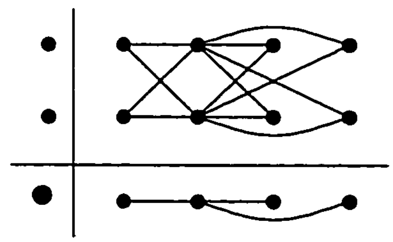


Figure 3.10: The graph $G \bullet \overline{K}_2$.

This graph can be covered with three isometric paths as shown in Figure 3.11. Notice that for all $x \in V(G)$, there exists an isometric path Q_a such that $(x, a) \in Q_a$ and an isometric path Q_b such that $(x, b) \in Q_b$, and $Q_a \neq Q_b$.

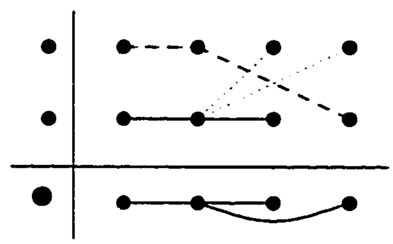


Figure 3.11: 0-1 problem for $G \bullet \overline{K}_2$.

The weighting w defined in this section assigns weight 1/2 to three isometric paths of G as shown in Figure 3.12.

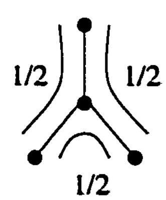


Figure 3.12: A fractional weighting of the isometric paths of G.

Hence $F_G = 3(\frac{1}{2}) = \frac{3}{2}$ and $I_2(G) = 3 = 2(\frac{3}{2}) = d \cdot F_G$, and so $\mathcal{F}_G = \frac{3}{2}$.

$*$ **Example.** Consider the graph C_3. As shown in Figure 3.13, $C_3 \bullet \overline{K}_2$ can be covered by three isometric paths corresponding to a fractional solution for C_3 of $\frac{3}{2}$.

Yet $C_3 \bullet \overline{K_2}$ can be covered by two isometric paths if paths of type $\{(x,a),(y,b),(x,c)\}$ are permitted. See Figure 3.14.

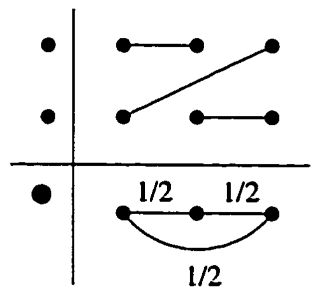


Figure 3.13: A covering of $C_3 \bullet \overline{K_2}$ with isometric paths.

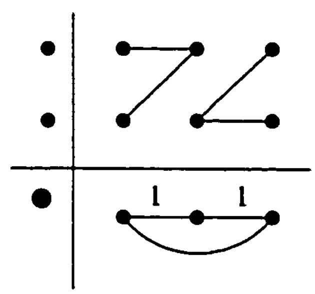


Figure 3.14: An alternate covering of $C_3 \bullet \overline{K_2}$ with isometric paths.

3.3.2 Strong Product

Theorem 3.3.3 *Let G be a graph. Then $\rho(G, \boxtimes) \geq \frac{1}{\text{diam}(G)+1}$.*

Proof. The result follows directly from Theorem 3.3.1 since $\text{diam}(G^n) = \text{diam}(G)$. □

Consider a graph G and suppose an isometric path cover of G includes $m(G)$ disjoint paths of length $\text{diam}(G)$. Further suppose that k vertices of G are not covered by the $m(G)$ disjoint paths so that $k = |V(G)| - m(G)(\text{diam}(G)+1)$. It should be noted that the proof of Lemma 3.3.2 does not depend on the isometric paths of length $\text{diam}(G)$ being disjoint. However disjointness is necessary in the proof of the main theorem.

Lemma 3.3.2 *Let G be a connected graph with v vertices and an isometric path cover that includes $m(G)$ paths of length* diam(G) *and k singletons. Then $\rho(G, \boxtimes) \leq \frac{m(G)}{v-k}$.*

Proof. For ease of notation, we denote the nth power of a graph G under the strong product by G^n. Consider G^n as $G \boxtimes G^{n-1}$. Intuitively G^n is a copy of G with each of its vertices replaced by a copy of G^{n-1}. These copies of G^{n-1} will be denoted $x_i \cdot G^{n-1}$ for $x_i \in V(G), i = 1, 2, \ldots, v$. For $x_i \sim x_j$ in G, if $\mathbf{a}$ and $\mathbf{b} \in V(G^{n-1})$, then vertices $(x_i, \mathbf{a})$ and $(x_j, \mathbf{b})$ in $x_i \cdot G^{n-1}$ and $x_j \cdot G^{n-1}$, respectively are adjacent whenever $\mathbf{a} \simeq \mathbf{b}$. We denote the vertices of $x_i \cdot G^{n-1}$ by $y_{i1}, y_{i2}, \ldots, y_{iv}$, and we let f_{ik} be the isomorphism from $x_i \cdot G^{n-1}$ to $x_k \cdot G^{n-1}$ defined by $f_{ik}(y_{ij}) = y_{kj}$, $j = 1, 2, \ldots, v$.

We consider two types of isometric paths: those within $x_i \cdot G^{n-1}$, and those that include vertices from more than one copy of G^{n-1}. The former shall be called *internal paths*, and the latter *external paths*.

We define a function $f(n)$ recursively by describing a method of covering the vertices of the graph G^n with $f(n)$ disjoint isometric paths. These paths include only singletons and paths of length diam(G). Clearly $f(n)$ will serve as an upper bound for $p(G^n)$.

By definition, $m(G)$ paths of length diam(G) and k singletons can cover G. So $f(1) = m(G) + k$.

Consider now G^n, and suppose that G^{n-1} has been covered by $f(n-1)$ isometric paths, where k^{n-1} of these paths are singletons; that is, in G^{n-1} there is a set of paths $\{P_j | j = 1, 2, \ldots, f(n-1)\}$ which are disjoint and cover G^{n-1}. For each i, $x_i \cdot G^{n-1}$ is covered by the paths $\{x_i \cdot P_j | j = 1, 2, \ldots, f(n-1)\}$. Note that $f(n-1) - k^{n-1}$ of these paths are of length diam(G). This gives in total $v(f(n-1) - k^{n-1})$ internal paths of length diam(G). In each copy of G^{n-1}, there are k^{n-1} vertices remaining that have not been covered. Let the set of these vertices be $Z_i = \{z_{ij}, j = 1, 2, \ldots, k^{n-1}\}$. These vertices will be covered by external paths. For all i and for each $z_{ij} \in Z_i$, the subgraph induced by $\{(x_i, z_{ij}) | i = 1, 2, \ldots v\}$ is isomorphic to G. Each of these k^{n-1} copies of G can be covered by $m(G)$ paths of length diam(G) and k singletons. So we have $k^{n-1}m(G)$ external paths of length diam(G) and k^n singletons. Hence the total number of paths covering G^n is $f(n) = v(f(n-1) - k^{n-1}) + k^{n-1}m(G) + k^n$, k^n of which are singletons.

So recursively we have shown that

$$\begin{aligned} f(n) &= v(f(n-1) - k^{n-1}) + k^{n-1}m(G) + k^n \\ &= vf(n-1) + k^{n-1}(m(G) - v) + k^n, \end{aligned}$$

where $f(1) = m(G) + k$. Solving this recurrence, we find that

$$f(n) = (m(G) + k)v^{n-1} + \frac{k(m(G) + k - v)v^{n-1}}{v - k} + \frac{(v - m(G) - k)k^n}{v - k}.$$

Now $p(G^n) \leq f(n)$ and so

$$\frac{p(G^n)}{v^n} \leq \frac{m(G) + k}{v} + \frac{k(m(G) + k - v)}{v(v - k)} + \frac{v - m(G) - k}{v - k}\left(\frac{k}{v}\right)^n.$$

As $n \to \infty$, $(\frac{k}{v})^n \to 0$ since $k < v$. Hence $\rho(G, \boxtimes) \leq \frac{m(G)+k}{v} + \frac{k(m(G)+k-v)}{v(v-k)} = \frac{m(G)}{v-k}$. □

Theorem 3.3.4 *Let G be a connected graph with v vertices. Then* $\rho(G, \boxtimes) = \frac{1}{\text{diam}(G)+1}$.

Proof. Suppose an isometric path cover of G includes $m(G)$ disjoint paths of length $\text{diam}(G)$ and k singletons. By Lemma 3.3.2, $\rho(G, \boxtimes) \leq \frac{m(G)}{v-k}$. By definition, $v = (\text{diam}(G) + 1)m(G) + k$, or equivalently $m(G) = \frac{v-k}{\text{diam}(G)+1}$. Hence $\rho(G, \boxtimes) \leq \frac{1}{\text{diam}(G)+1}$. Now Theorem 3.3.3 gives the reverse inequality, and so $\rho(G, \boxtimes) = \frac{1}{\text{diam}(G)+1}$. □

3.3.3 Co-Cartesian Product

Theorem 3.3.5 *Suppose G is a connected graph that is not complete. Then* $\rho(G, \Box^c) = \frac{1}{3}$.

Proof. Table 3.1 shows for $n \geq 2$ that $\text{diam}(G^n_{\Box^c}) \leq 2$. Hence the isometric paths covering $G^n_{\Box^c}$ are singletons and paths of length one and two. Now $G^n_{\Box^c} = ((G^c)^n_{\Box})^c$. So for ease of description, we consider the non-edges; that is, we consider $(G^c)^n_{\Box}$.

Singletons in $G^{n}_{\square^c}$ correspond to singletons in $(G^c)^n_{\square}$. Similarly, a path of length one corresponds to a pair of singletons, and a path of length two corresponds to an edge and a singleton. So we can think of covering $G^{n}_{\square^c}$ with isometric paths in terms of covering $(G^c)^n_{\square}$ with singletons, S^* referred to as type 1 subgraphs; pairs of singletons, N^* referred to as type 2 subgraphs; and subgraphs composed of an edge and a singleton, T^* referred to as type 3 subgraphs. Since we wish to minimize the number of subgraphs, we first select subgraphs of type 3, then of type 2, and finally of type 1. Note that we choose these subgraphs to be disjoint.

Let $T(n)$ be the number of subgraphs of type 3 in $(G^c)^n_{\square}$, $N(n)$ the number of subgraphs of type 2, and $S(n)$ the number of subgraphs of type 1. We assume $T(1) > 0$ since otherwise G is isomorphic to K_v, and also $S(1) < v$ since otherwise G is completely disconnected. We now consider what happens when these substructures are multiplied using the Cartesian product. Clearly a subgraph of type $i, i \in \{1, 2, 3\}$ multiplied with a singleton will result in a single subgraph of type i. Now let's consider a subgraph of type 2. When multiplied with a subgraph of type 2 or 3, the result is two copies of the subgraph of type 2 or 3 respectively as shown, vertices grouped by the ovals, in Figure 3.15.

Figure 3.15: (a) The product of two type 2 subgraphs, (b) the product of a type 2 subgraph and a type 3 subgraph.

Finally we consider multiplying two subgraphs of type 3. The resulting subgraph can be covered by three subgraphs of type 3 as shown in Figure 3.16.

A summary of the subgraphs that result when the three types of subgraphs are multiplied is given in Table 3.3.

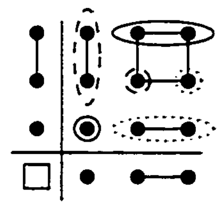


Figure 3.16: The product of two type 3 subgraphs.

	S^*	N^*	T^*
S^*	S^*	N^*	T^*
N^*	N^*	$2N^*$	$2T^*$
T^*	T^*	$2T^*$	$3T^*$

Table 3.3: Summary.

Suppose $(G^c)^{n-1}_{\square}$ is covered by $T(n-1)+N(n-1)+S(n-1)$ subgraphs, and consider $(G^c)^n_{\square} = G^c \square (G^c)^{n-1}_{\square}$. The number of subgraphs covering $(G^c)^n_{\square}$ is $T(n)+N(n)+S(n)$ where

$$\begin{aligned} S(n) &= S(1)S(n-1), \\ N(n) &= 2N(1)N(n-1)+N(1)S(n-1)+S(1)N(n-1), \\ T(n) &= 3T(1)T(n-1)+2T(1)N(n-1)+2N(1)T(n-1)+ \\ &\quad T(1)S(n-1)+S(1)T(n-1). \end{aligned}$$

Solving this system of recurrence relations, noting that $v^n = 3T(n)+2N(n)+S(n)$, we find that

$$S(n) = S(1)^n,$$

$$N(n) = \frac{1}{2}(v-3T(1))^n - \frac{1}{2}(v-2N(1)-3T(1))^n,$$

and

$$T(n) = \frac{1}{3}v^n + \frac{1}{3}(v-3T(1))^n.$$

So

$$p(G^n_{\square^c}) \le S(1)^n + \frac{1}{6}(v-3T(1))^n - \frac{1}{2}(v-2N(1)-3T(1))^n + \frac{1}{3}v^n.$$

Hence

$$\frac{p(G^n_{\square^c})}{v^n} \leq \left(\frac{S(1)}{v}\right)^n + \frac{1}{6}\left(\frac{v-3T(1)}{v}\right)^n - \frac{1}{2}\left(\frac{v-2N(1)-3T(1)}{v}\right)^n + \frac{1}{3}. \quad (3.1)$$

As we let n approach ∞, we find that the first three terms on the right of the inequality go to 0. To see this, recall that $S(1) < v$ since G is connected, and $T(1) > 0$ since otherwise either $S(1) = v$ or G is complete. Thus $\rho(G, \square^c) \leq \frac{1}{3}$.

Now $\rho(G, \square^c) \geq 1/(\text{diam}(G^n_{\square^c}) + 1) \geq 1/(2+1) = 1/3$. Hence $\rho(G, \square^c) = \frac{1}{3}$. □

Corollary 3.3.1 *If $G = K_v$, then $\rho(G, \square^c) = \frac{1}{2}$.*

Proof. If $G = K_v$, then $T(1) = 0$ and $N(1) > 0$. Hence

$$\frac{p(G^n_{\square^c})}{v^n} \leq (\frac{S(1)}{v})^n - \frac{1}{2}(\frac{v-2N(1)}{v})^n + \frac{1}{2}.$$

As we let n approach ∞, we find that the first two terms on the right of the inequality go to 0. Thus $\rho(G, \square^c) \leq \frac{1}{2}$. Now $\rho(G, \square^c) \geq 1/(\text{diam}(G^n_{\square^c}) + 1) \geq 1/(1+1) = 1/2$ since $\text{diam}(K^n_v) = 1$. Hence $\rho(G, \square^c) = \frac{1}{2}$. □

3.3.4 Disjunctive Product

Theorem 3.3.6 *Suppose G is a connected graph that is not complete. Then $\rho(G, \boxtimes^c) = \frac{1}{3}$.*

Proof. The proof is similar to that of Theorem 3.3.5 since for $n \geq 2$, $\text{diam}(G^n) \leq 2$. Again for ease of description we consider the structure of the non-edges. We need only consider what happens when the three types of substructures, S^*, N^*, and T^* are multiplied using the strong product. Clearly a subgraph of type $i, i \in \{1,2,3\}$ multiplied with a singleton will result in a single subgraph of type i. Now let's consider a subgraph of type 2. When multiplied with a subgraph of type 2 or 3, the result is two copies of the subgraph of type 2 or 3 respectively as shown in Figure 3.17.

Finally we consider multiplying two subgraphs of type 3. The resulting graph can be covered by three subgraphs of type 3 as shown in Figure 3.18.

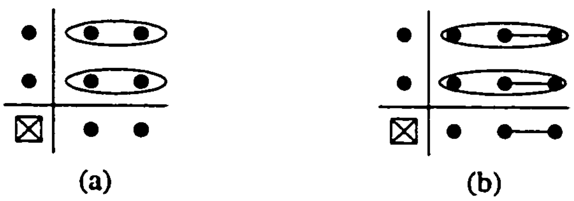


Figure 3.17: (a) The product of two type 2 subgraphs, (b) the product of a type 2 subgraph and a type 3 subgraph.

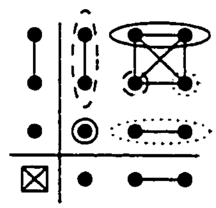


Figure 3.18: The product of two type 3 subgraphs.

A summary of the subgraphs that result when the three types of subgraphs are multiplied is given in Table 3.4.

The computation is now the same as in the proof of Theorem 3.3.5. □

Corollary 3.3.2 *If* $G = K_v$ *then* $\rho(G, \boxtimes^c) = \frac{1}{2}$.

Proof. Similar to that of Corollary 3.3.1. □

3.3.5 Co-Categorical Product

Theorem 3.3.7 *Suppose* G *is a connected graph. Then* $\rho(G, \times^c) = \frac{1}{2}$.

Proof. The proof is similar to that of Theorem 3.3.5 since for $n \geq 2$, $\mathrm{diam}(G^n_{\times^c}) \leq 2$. Again for ease of description we consider the structure of the non-edges. We consider what happens when the three types of substructures, S^*, N^*, and T^* are multiplied using the categorical product. Clearly a subgraph of type $i, i \in \{1, 2\}$ multiplied with a singleton will result in a single subgraph of type i. When a subgraph of type 3 is

	S^*	N^*	T^*
S^*	S^*	N^*	T^*
N^*	N^*	$2N^*$	$2T^*$
T^*	T^*	$2T^*$	$3T^*$

Table 3.4: Summary.

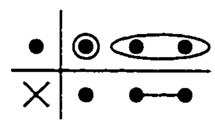


Figure 3.19: The product of a type 1 subgraph and a type 3 subgraph.

multiplied with a singleton, the result is one subgraph of type 1 and one of type 2 as shown in Figure 3.19.

Now let's consider a subgraph of type 2. When multiplied with a subgraph of type $i, i \in \{2, 3\}$, the result is i copies of the subgraph of type 2 as shown in Figure 3.20.

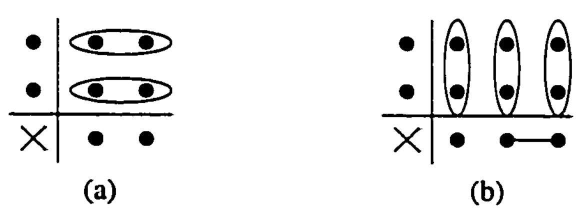


Figure 3.20: (a) The product of two type 2 subgraphs, (b) the product of a type 2 subgraph and a type 3 subgraph.

Finally we consider multiplying two subgraphs of type 3. The resulting graph can be covered by two subgraphs of type 3, one of type 2 and one of type 1 as shown in Figure 3.21.

A summary of the subgraphs that result when the three types of subgraphs are multiplied is given in Table 3.5.

Suppose $(G^c)^{n-1}_{\times}$ is covered by $T(n-1) + N(n-1) + S(n-1)$ subgraphs, and consider $(G^c)^n_{\times} = G^c \times (G^c)^{n-1}_{\times}$. The number of subgraphs covering $(G^c)^n_{\times}$ is $T(n) +$

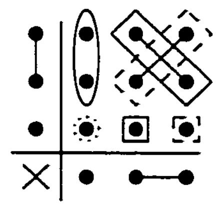


Figure 3.21: The product of two type 3 subgraphs.

	S^*	N^*	T^*
S^*	S^*	N^*	$S^* + N^*$
N^*	N^*	$2N^*$	$3N^*$
T^*	$S^* + N^*$	$3N^*$	$2T^* + N^* + S^*$

Table 3.5: Summary.

$N(n) + S(n)$ where

$$\begin{aligned} S(n) &= (S(1) + T(1))(S(n-1) + T(n-1)), \\ N(n) &= N(1)(S(n-1) + 2N(n-1) + 3T(n-1)) + T(1)T(n-1) \\ &\quad N(n-1)(S(1) + 3T(1)) + S(1)T(n-1) + T(1)S(n-1), \\ T(n) &= 2T(1)T(n-1). \end{aligned}$$

Solving this system of recurrence relations, noting that $v^n = 3T(n) + 2N(n) + S(n)$, we find that

$$T(n) = 2^{n-1}T(1)^n,$$

$$S(n) = (S(1) + T(1))^n \frac{S(1)^2 + T(1)^2}{S(1)^2 - T(1)^2} + (2T(1))^n \frac{T(1) - S(1)}{2(S(1) + T(1))},$$

and

$$N(n) = \frac{k_1(v - 2N(1))^n}{k} + \frac{k_2(S(1) + T(1))^n}{k} + \frac{k_3(2T(1))^2}{k} + \frac{k_4 v^n}{k}$$

where

$$\begin{aligned} k &= (S(1) + T(1)(v - 2N(1))(v - 2N(1) - 2T(1))((S(1) + T(1))^2 \\ &\quad + 2S(1)N(1) + 2N(1)T(1) - vS(1) - vT(1)), \end{aligned}$$

$$k_1 = (S(1)^2 - T(1)^2)[v - S(1) - 2N(1) - 3T(1)]((v - 2N(1))^2 - 2S(1)T(1)),$$

$$k_2 = T(1)(S(1)^2 + T(1)^2)(v - 2N(1))(v - 2N(1) - 2T(1)),$$

$$\begin{aligned} k_3 = {} & \frac{1}{2}v^2S(1)^3 - \frac{3}{2}v^2S(1)T(1)^2 - v^2T(1)^3 - \frac{1}{2}vS(1)^4 - 2vS(1)^3N(1) - \frac{1}{2}vS(1)^3T(1) \\ & + \frac{3}{2}vS(1)^2T(1)^2 + 6vS(1)N(1)T(1)^2 + \frac{5}{2}vS(1)T(1)^3 + 4vN(1)T(1)^3 + 2vT(1)^4 \\ & - 3S(1)^2N(1)T(1)^2 - 6S(1)N(1)^2T(1)^2 - 5S(1)N(1)T(1)^3 - 4N(1)^2T(1)^3 \\ & + S(1)^4N(1) + 2S(1)^3N(1)^2 + S(1)^3N(1)T(1) - 2N(1)T(1)^4, \end{aligned}$$

and

$$k_4 = \frac{1}{2}k.$$

So

$$\begin{aligned} p(G^n_{\times^c}) \leq {} & \frac{v^n}{2} + \frac{k_1(v - 2N(1))^n}{k} + (2T(1))^n\frac{(2k_3 - k)S(1) + (2k_3 + k)T(1)}{2k(S(1) + T(1))} \\ & + (S(1) + T(1))^n\frac{(k + k_2)S(1)^2 - (k - k_2)T(1)^2}{k(S(1)^2 - T(1)^2)}. \end{aligned}$$

Notice however, that the expression for k_1 includes the factor $v - S(1) - 2N(1) - 3T(1)$, which equals zero. Hence

$$\begin{aligned} \frac{p(G^n_{\times^c})}{v^n} \leq {} & \frac{1}{2} + \left(\frac{2T(1)}{v}\right)^n\frac{(2k_3 - k)S(1) + (2k_3 + k)T(1)}{2k(S(1) + T(1))} \\ & + \left(\frac{S(1) + T(1)}{v}\right)^n\frac{(k + k_2)S(1)^2 - (k - k_2)T(1)^2}{k(S(1)^2 - T(1)^2)}. \end{aligned}$$

As we let n approach ∞, we find that $\left(\frac{2T(1)}{v}\right)^n$ and $\left(\frac{S(1)+T(1)}{v}\right)^n$ go to 0. If G is complete, then $T(1) = 0$ and $S(1) < v$. Otherwise $T(1) > 0$ and notice that $2T(1) < v$ whenever $T(1) > 0$, and $S(1) + T(1) < v$ since $S(1) + 2N(1) + 3T(1) = v$. Thus $\rho(G, \times^c) \leq \frac{1}{2}$.

It has been shown that in $G^n_\times$, subgraphs of type 3 occur only as the result of multiplying subgraphs of type 3 in G and $G^{n-1}_\times$. Hence $T(n) = 2^{n-1}T(1)^n$. (Recall Figure 3.21.) The isometric path number will therefore be smallest when the remaining $v^n - 3T(n)$ vertices are covered by subgraphs of type 2. Hence

$$\begin{aligned} p(G^n_{\times^c}) &\geq T(n) + \frac{1}{2}(v^n - 3T(n)) \\ &= \frac{1}{2}v^n - 2^{n-2}T(1)^n, \end{aligned}$$

and so

$$\frac{p(G^n_{\times^c})}{v^n} \geq \frac{1}{2} - \frac{1}{4}\left(\frac{2T(1)}{v}\right)^n.$$

As we let n approach ∞, we find that $\left(\frac{2T(1)}{v}\right)^n$ goes to 0 as shown previously. Hence $\rho(G, \times^c) \geq \frac{1}{2}$, and so $\rho(G, \times^c) = \frac{1}{2}$. □

3.3.6 Equivalence

Theorem 3.3.8 *Suppose G is a connected graph that is not complete. Then $\rho(G, ⊜) = \frac{1}{3}$ if* diam$(G) = 2$ *and $\rho(G, ⊜) = \frac{1}{4}$ otherwise.*

Proof. Theorems 3.1.1 and 3.1.2 show for $n \geq 2$ that diam$(G^n_⊜) \leq 2$ if $\gamma(G) = 1$ and diam$(G^n_⊜) \leq 3$ otherwise. If diam$(G^n_⊜) \leq 3$, then the isometric paths S^*, N^*, T^* and F^* covering $G^n_⊜$ are respectively singletons, and paths of length one, two, and three. We will refer to these as the four 'types' of isometric paths with which to cover $G^n_⊜$. Note that if $\gamma(G) = 1$ then diam$(G) = 2$ and diam$(G^n_⊜) \leq 2$. There are no isometric paths of length three.

Let $F(n)$ be the number of isometric paths of length 3 in $G^n_⊜$, $T(n)$ the number of paths of length 2, $N(n)$ the number of edges or paths of length 1, and $S(n)$ the number of singletons or paths of length 0. We assume $S(1) < v$ since otherwise G is completely disconnected. We now consider what happens when these isometric paths are multiplied. When a path of length $i, i \in \{0, 1, 2, 3\}$ is multiplied with a path of length $j, j \in \{0, 1, 2, 3\}$, $i \leq j$ the result can be covered by $i + 1$ disjoint copies of the path of length j. The $i + 1$ paths of length j can be taken to be the $i + 1$ rows of

$j+1$ vertices in the product of a path of length i with a path of length j. This is shown in Figure 3.22 for $i=2$ and $j=3$.

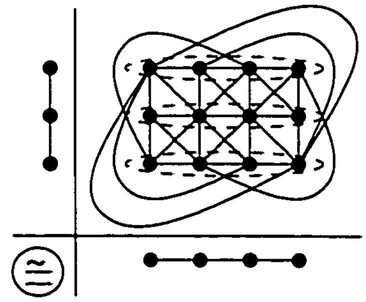


Figure 3.22: The product of a path of length 2 and one of length 3.

A summary of the subgraphs that result when the four types of subgraphs are multiplied is given in Table 3.6.

	S^*	N^*	T^*	F^*
S^*	S^*	N^*	T^*	F^*
N^*	N^*	$2N^*$	$2T^*$	$2F^*$
T^*	T^*	$2T^*$	$3T^*$	$3F^*$
F^*	F^*	$2F^*$	$3F^*$	$4F^*$

Table 3.6: Summary.

Suppose $G^{n-1}_{\circledequal}$ is covered by $F(n-1)+T(n-1)+N(n-1)+S(n-1)$ isometric paths, and consider $G^{n}_{\circledequal} = G \circledequal G^{n-1}_{\circledequal}$. The number of subgraphs covering $G^{n}_{\circledequal}$ is $F(n)+T(n)+N(n)+S(n)$ where

$$\begin{aligned} S(n) &= S(1)S(n-1), \\ N(n) &= S(1)N(n-1)+N(1)S(n-1)+2N(1)N(n-1), \\ T(n) &= T(n-1)(S(1)+2N(1)+3T(1))+T(1)(S(n-1)+2N(n-1)), \end{aligned}$$

and

$$\begin{aligned} F(n) = \quad & F(1)(S(n-1)+2N(n-1)+3T(n-1)) \\ & +F(n-1)(S(1)+2N(1)+3T(1)+4F(1)). \end{aligned}$$

Solving this system of recurrence relations, noting that $v^n = 4F(n) + 3T(n) + 2N(n) + S(n)$, we find that

$$S(n) = S(1)^n,$$
$$N(n) = \frac{1}{2}(S(1) + 2N(1))^n - \frac{1}{2}S(1)^n,$$
$$T(n) = \frac{1}{3}(v - 4F(1))^n - \frac{1}{3}(v - 4F(1) - 3T(1))^n,$$

and

$$F(n) = \frac{1}{4}v^n - \frac{1}{4}(v - 4F(1))^n.$$

So

$$p(G^n_{\circledcirc}) \leq \frac{1}{4}v^n + \frac{1}{2}S(1)^n + \frac{1}{2}(S(1)+2N(1))^n + \frac{1}{12}(v-4F(1))^n - \frac{1}{3}(v-4F(1)-3T(1))^n.$$

Hence

$$\frac{p(G^n_{\circledcirc})}{v^n} \leq \frac{1}{4} + \frac{1}{2}\Big(\frac{S(1)}{v}\Big)^n + \frac{1}{2}\Big(\frac{S(1)+2N(1)}{v}\Big)^n + \frac{1}{12}\Big(\frac{v-4F(1)}{v}\Big)^n - \frac{1}{3}\Big(\frac{v-4F(1)-3T(1)}{v}\Big)^n.$$

Suppose $\gamma(G) > 1$ and $F(1) > 0$. As we let n approach ∞, we find that the last four terms on the right of the inequality go to 0. To see this, recall that $S(1) < v$ and so $\left(\frac{S(1)}{v}\right)^n$ goes to 0. Since $F(1) > 0$, $\left(\frac{v-4F(1)}{v}\right)^n$ and $\left(\frac{v-4F(1)-3T(1)}{v}\right)^n$ go to 0. Finally since $v = S(1) + 2N(1) + 3T(1) + 4F(1)$ and $F(1) > 0$, $S(1) + 2N(1) < v$ and so $\left(\frac{S(1)+2N(1)}{v}\right)^n$ goes to 0. Thus $\rho(G, \circledcirc) \leq \frac{1}{4}$. Now $\rho(G, \circledcirc) \geq 1/(\mathrm{diam}(G^n_{\circledcirc}) + 1) \geq 1/(3+1) = 1/4$ since $\mathrm{diam}(G^n_{\circledcirc}) \leq 3$. Hence $\rho(G, \circledcirc) = \frac{1}{4}$.

Suppose $\gamma(G) > 1$ and $F(1) = 0$, or $\gamma(G) = 1$ and hence $\mathrm{diam}(G) = 2$. Since $F(1) = 0$, recall that $T(1) > 0$ since otherwise G is isomorphic to K_v. The expression for $\frac{p(G^n_{\circledcirc})}{v^n}$ becomes

$$\frac{p(G^n_{\circledcirc})}{v^n} \leq \frac{1}{4} + \frac{1}{12} + \frac{1}{2}\Big(\frac{S(1)}{v}\Big)^n + \frac{1}{2}\Big(\frac{S(1)+2N(1)}{v}\Big)^n - \frac{1}{3}\Big(\frac{v-3T(1)}{v}\Big)^n.$$

As we let n approach ∞, we find that the last three terms on the right of the inequality go to 0. Thus $\rho(G, \circledcirc) \leq \frac{1}{4} + \frac{1}{12} = \frac{1}{3}$.

Now $\rho(G, \circledcirc) \geq 1/(\mathrm{diam}(G^n_{\circledcirc}) + 1) \geq 1/(2+1) = 1/3$ since $\mathrm{diam}(G^n_{\circledcirc}) \leq 2$. Hence $\rho(G, \circledcirc) = \frac{1}{3}$. □

Corollary 3.3.3 *If $G = K_v$ then $\rho(G, \textcircled{\cong}) = \frac{1}{2}$.*

Proof. If $G = K_v$ then $F(1) = 0$, $T(1) = 0$, and $N(1) > 0$. Hence

$$\frac{p(G^n_{\textcircled{\cong}})}{v^n} \leq \frac{1}{4} + \frac{1}{12} + \frac{1}{2} - \frac{1}{3} + \frac{1}{2}\left(\frac{S(1)}{v}\right)^n.$$

If $S(1) > 0$ then as we let n approach ∞, we find that the last term on the right of the inequality goes to 0. Thus $\rho(G, \textcircled{\cong}) \leq \frac{1}{4} + \frac{1}{12} + \frac{1}{2} - \frac{1}{3} = \frac{1}{2}$. Now $\rho(G, \textcircled{\cong}) \geq 1/(\text{diam}(G^n_{\textcircled{\cong}}) + 1) \geq 1/(1+1) = 1/2$ since $\text{diam}(K^n_v) = 1$. Hence $\rho(G, \textcircled{\cong}) = \frac{1}{2}$. □

3.3.7 Symmetric Difference

Theorem 3.3.9 *Suppose G is a connected graph with $|V(G)| \geq 2$ that is not complete. Then $\rho(G, \nabla) = \frac{1}{3}$.*

Proof. Theorem 3.1.4 shows for $n \geq 2$ that $\text{diam}(G^n_\nabla) \leq 2$. Thus we need only consider the structures that result when isometric paths of length at most 2, S^*, N^* and T^* are multiplied. Clearly a path of length $i, i \in \{0, 1, 2\}$ multiplied with a singleton will result in a single path of length i. Now let's consider a path of length 1. When multiplied with a path of length 1 or 2, the result is two copies of the path of length 1 or 2 respectively as shown in Figure 3.23.

(a) (b)

Figure 3.23: (a) The product of two paths of length 1, (b) the product of a path of length 1 and a path of length 2.

Now let's consider a path of length 2. When two paths of length 2 are multiplied, the result is three copies of the path of length 2 as shown in Figure 3.24.

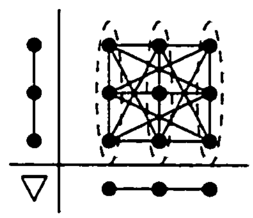


Figure 3.24: The product of two paths of length 2.

	S^*	N^*	T^*
S^*	S^*	N^*	T^*
N^*	N^*	$2N^*$	$2T^*$
T^*	T^*	$2T^*$	$3T^*$

Table 3.7: Summary.

A summary of the subgraphs that result when the three types of subgraphs are multiplied is given in Table 3.7.

Suppose G_∇^{n-1} is covered by $T(n-1)+N(n-1)+S(n-1)$ isometric paths, and consider $G_\nabla^n = G\nabla G_\nabla^{n-1}$. The number of paths covering G_∇^n is $T(n)+N(n)+S(n)$ where

$$\begin{aligned} S(n) &= S(1)S(n-1), \\ N(n) &= S(1)N(n-1)+N(1)S(n-1)+2N(1)N(n-1), \\ T(n) &= S(1)T(n-1)+T(1)S(n-1)+2N(1)T(n-1) \\ &\quad +2T(1)N(n-1)+3T(1)T(n-1). \end{aligned}$$

Solving this system of recurrence relations, noting that $v^n = 3T(n)+2N(n)+S(n)$, we find that

$$S(n) = S(1)^n,$$

$$N(n) = \frac{1}{2}(v-3T(1))^n - \frac{1}{2}(v-2N(1)-3T(1))^n,$$

and

$$T(n) = \frac{1}{3}v^n + \frac{1}{3}(v-3T(1))^n.$$

So

$$p(G_\nabla^n) \le S(1)^n + \frac{1}{6}(v-3T(1))^n - \frac{1}{2}(v-2N(1)-3T(1))^n + \frac{1}{3}v^n.$$

Hence

$$\frac{p(G^n_\nabla)}{v^n} \le \Big(\frac{S(1)}{v}\Big)^n + \frac{1}{6}\Big(\frac{v-3T(1)}{v}\Big)^n - \frac{1}{2}\Big(\frac{v-2N(1)-3T(1)}{v}\Big)^n + \frac{1}{3}.$$

As we let n approach ∞, we find that the first three terms on the right of the inequality go to 0. To see this, recall that $S(1) < v$ since G is connected, and $T(1) > 0$ since otherwise either $S(1) = v$ or G is complete. Thus $\rho(G, \nabla) \le \frac{1}{3}$.

Now $\rho(G, \nabla) \ge \frac{1}{\text{diam}(G^n_\nabla)+1} \ge \frac{1}{2+1} = \frac{1}{3}$ since $\text{diam}(G^n_\nabla) \le 2$. Hence $\rho(G, \nabla) = \frac{1}{3}$.

□

Corollary 3.3.4 *If $G = K_v$ then $\frac{1}{3} \le \rho(G, \nabla) \le \frac{1}{2}$.*

Proof. If $G = K_v$ then $T(1) = 0$, and $N(1) > 0$. Hence

$$\frac{p(G^n_\nabla)}{v^n} \le (\frac{S(1)}{v})^n - \frac{1}{2}(\frac{v-2N(1)}{v})^n + \frac{1}{2}.$$

As we let n approach ∞, we find that the first two terms on the right of the inequality go to 0. Thus $\rho(G, \nabla) \le \frac{1}{2}$. Note that $\rho(G, \nabla) \ge \frac{1}{3}$ as shown in the proof of the main theorem. □

Note. Suppose K^n_v can be covered by $p(K^n_v)$ isometric paths. For each $x \in V(K_v)$, $x \cdot K^n_v$ can be covered by $p(K^n_v)$ isometric paths. Hence $p(K^{n+1}_v) \le vp(K^n_v)$, and so $\frac{p(K^{n+1}_v)}{v^{n+1}} \le \frac{p(K^n_v)}{v^n}$. Thus $\rho(K_v, \nabla)$ exists.

3.3.8 Lexicographic Product

Notice first that $G^n_\bullet \cong G^n_{\bullet^c}$ for all graphs G, and so results for the lexicographic product hold for the co-lexicographic product as well.

Let G be a graph. Let $\mathcal{P} = \{P_i, i \in I\}$ be the set of isometric paths of G. Define $p_f(G)$ as the minimum of all feasible fractional weightings of the isometric paths of $\mathcal{P}$. Compare with Subsection 3.3.1.

Theorem 3.3.10 *Let G be a graph with $|V(G)| = v$. Then $\rho(G, \bullet) \le \frac{p_f(G)}{v}$.*

Proof. Consider $G^n_\bullet$ as a copy of G with each of its vertices replaced by a copy of $G^{n-1}_\bullet$. These copies of $G^{n-1}_\bullet$ will be denoted $x_i \cdot G^{n-1}_\bullet$ for $x_i \in V(G)$. Notice that whenever $x_i \sim x_j$ in G, each of the vertices of $x_i \cdot G^{n-1}_\bullet$ is adjacent to each of the vertices of $x_j \cdot G^{n-1}_\bullet$.

Suppose we have a fractional weighting, $w : \mathcal{P} \to \mathbb{R}^{\geq 0}$ of the isometric paths $\mathcal{P}$ of G which realizes $p_f(G)$. We will show that $\lfloor \frac{v^{n-1}}{k} \rfloor kp_f(G) + lp(G)$ isometric paths will cover $G^n_\bullet$, where k is the lowest common denominator of the fractions in our weighting w of the paths in G and $l < k$.

For $x_i \in V(G)$, let the vertices of $x_i \cdot G^{n-1}_\bullet$ be $x_{ij}, j = 1, 2, \ldots, v^{n-1}$. For each $j, 1 \leq j \leq v^{n-1}$, let $G_j = \{x_{ij} | i = 1, 2, \ldots, v\}$. Then each G_j is isomorphic to G. Let $v^{n-1} = Ak + l, 0 \leq l < k$ and let $H_a = \cup_{(a-1)k+1 \leq j \leq ak} G_j$ for $1 \leq a \leq A-1$. Note that H_a is isomorphic to H_b for all $1 \leq a, b \leq A-1$. subgraphs

Consider one such subgraph H_a. For each isometric path $P_i, i \in I$, take $kw(P_i)$ copies of P_i in H_a. As in the proof of Theorem 3.3.2, we take new uncovered vertices if there are any remaining in H_a. Every vertex of H_a is covered by a path in this way. (Again recall the proof of Theorem 3.3.2.) Hence $kp_f(G)$ isometric paths cover the vertices of H_a. So for every k vertices in $V(G^{n-1}_\bullet)$ we have $kp_f(G)$ paths, and we have $\lfloor \frac{v^{n-1}}{k} \rfloor$ groups of k vertices.

This leaves $l < k$ vertices remaining in each of the copies of $G^{n-1}_\bullet$; that is, l vertices remain in $x_i \cdot G^{n-1}_\bullet$ for each x_i. For each i, we select one vertex from $x_i \cdot G^{n-1}_\bullet$ to obtain a set of vertices which induce a graph isomorphic to G. Now G can be covered by $p(G)$ isometric paths. For each i, we can select a single vertex, x_{is} from $x_i \cdot G^{n-1}_\bullet$ to obtain a graph isomorphic to G in this way l times. Hence these remaining vertices can be covered by $lp(G)$ paths. Hence

$$\begin{aligned} p(G^n_\bullet) &\leq \lfloor \tfrac{v^{n-1}}{k} \rfloor kp_f(G) + lp(G) \\ &\leq \tfrac{v^{n-1}}{k} kp_f(G) + lp(G) \\ &= v^{n-1}p_f(G) + lp(G). \end{aligned}$$

And so $\frac{p(G^n_\bullet)}{v^n} \leq \frac{p_f(G)}{v} + \frac{lp(G)}{v^n}$. Now letting n approach infinity, we find that the last term goes to zero, and hence $\rho(G, \bullet) \leq \frac{p_f(G)}{v}$. □

Consider a graph G with v vertices. Let $D-1$ be the length of a longest isometric path in a cover of G, and let n_i be the number of paths that include $D-i+1$ vertices not covered by other paths for $i = 1, 2, \ldots, D-2$. Let $q = \sum_{i=1}^{D-2} n_i$ so that q is the total number of paths of length at least 2, and define $Q = \sum_{i=1}^{D-2} n_i(D-i+1)$ so that Q is the number of vertices covered in G.

Let $T_3(n)$ be the number of isometric paths of length at most 2 in $G_\bullet^n$.

Lemma 3.3.3 *Let G be a graph. Then $T_3(n) \geq v^{n-1}T_3(1)$.*

Proof. Consider $G_\bullet^n$ as a copy of G with each of its vertices, x replaced by a copy of $G_\bullet^{n-1}$, denoted $x \cdot G_\bullet^{n-1}$. Each of these v copies of $G_\bullet^{n-1}$ contain $T_3(n-1)$ paths of length 2, and these are still isometric in $G_\bullet^n$. Hence the total number of such paths in $G_\bullet^n$ is at least $vT_3(n-1)$. The result follows recursively. □

Theorem 3.3.11 *Let G be a graph with $|V(G)| = v$. Then $\rho(G, \bullet) \leq \frac{q}{v} + \frac{1}{v^2}(v-Q)T_3(1)$.*

Proof. We consider $G_\bullet^n$ as a copy of G with each of its vertices replaced by a copy of $G_\bullet^{n-1}$. These copies of $G_\bullet^{n-1}$ will be denoted $x_i \cdot G_\bullet^{n-1}$ for $x_i \in V(G), i = 1, 2, \ldots, v$. We denote the vertices of $x_i \cdot G_\bullet^{n-1}$ by $y_{i1}, y_{i2}, \ldots, y_{iv^{n-1}}$, and we let f_{ik} be the isomorphism from $x_i \cdot G_\bullet^{n-1}$ to $x_k \cdot G_\bullet^{n-1}$ defined by $f_{ik}(y_{ij}) = y_{kj}$, $j = 1, 2, \ldots, v^{n-1}$.

We describe a method of covering the vertices of $G_\bullet^n$ with isometric paths. For each $i \in \{1, 2, \ldots, v\}$, we choose a vertex y_{ij} from $x_i \cdot G_\bullet^{n-1}$ to obtain a 'set' of v vertices $\{y_{1j}, y_{2j}, \ldots, y_{vj}\}$. We obtain v^{n-1} sets in this way, each of which is the vertex set of a subgraph isomorphic to G. We cover each of these copies of G with the q isometric paths described in the preamble to Lemma 3.3.3. This gives qv^{n-1} disjoint paths.

Since $v-Q$ vertices in each copy of G are not covered by these paths, there remain $v-Q$ copies of $G_\bullet^{n-1}$, $x_i \cdot G_\bullet^{n-1}$ with all v^{n-1} vertices uncovered. However, we know that there are $T_3(n-1)$ paths covering vertices in each of these copies of $G_\bullet^{n-1}$. This gives an additional $(v-Q)T_3(n-1)$ paths.

Hence the total number of paths covering $G^n_\bullet$ is $qv^{n-1} + (v-Q)T_3(n-1)$, and hence

$$p(G^n_\bullet) \leq qv^{n-1} + (v-Q)T_3(n-1).$$

Applying Lemma 3.3.3, we have

$$p(G^n_\bullet) \leq qv^{n-1} + v^{n-2}(v-Q)T_3(1).$$

And so

$$\rho(G,\bullet) \leq \frac{q}{v} + \frac{1}{v^2}(v-Q)T_3(1).$$

□

Example. Consider the graph G shown in Figure 3.25 with vertices x_1, x_2, x_3, and x_4.

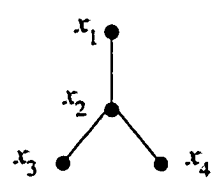


Figure 3.25: A graph G with $\rho(G,\bullet) = \frac{1}{3}$.

The graph $G^n_\bullet$ can be thought of as a copy of G with each of its vertices, x_i replaced by a copy of $G^{n-1}_\bullet$, denoted $x_i \cdot G^{n-1}_\bullet$. For each i, let the vertices of $x_i \cdot G^{n-1}_\bullet$ be x_{ij}, $j = 1, 2, \ldots, 4^{n-1}$. For $i \in \{1,2,3\}$ and fixed k, we select a vertex v_{ik} to obtain a set of three vertices that induce a subgraph isomorphic to a path of length 2, which is itself isometric. Hence the vertices of $x_1 \cdot G^{n-1}_\bullet, x_2 \cdot G^{n-1}_\bullet$, and $x_3 \cdot G^{n-1}_\bullet$ are covered by 4^{n-1} isometric paths of length 2. There remains a copy of $G^{n-1}_\bullet$, $x_4 \cdot G^{n-1}_\bullet$ whose vertices have not yet been covered.

Now $x_4 \cdot G^{n-1}_\bullet$ can be thought of as a copy of G with each of its vertices, x_i replaced by a copy of $G^{n-2}_\bullet$, denoted $x_i \cdot G^{n-2}_\bullet$. As before, the vertices of $x_1 \cdot G^{n-2}_\bullet, x_2 \cdot G^{n-2}_\bullet$, and $x_3 \cdot G^{n-2}_\bullet$ can be covered by 4^{n-2} isometric paths of length 2. There remains a copy of $G^{n-2}_\bullet$, $x_4 \cdot G^{n-2}_\bullet$ whose vertices have not yet been covered.

This recursive process continues until $\sum_{i=1}^{n-1} 4^i = \frac{4^n-4}{3}$ isometric paths of length 2 cover all but one of the vertices of $G_\bullet^n$. So

$$p(G_\bullet^n) \leq \frac{4^n-4}{3} + 1.$$

It is clear that this covering of $G_\bullet^n$ by isometric paths is best possible since $\text{diam}(G^n) = 2$. So

$$p(G_\bullet^n) = \frac{4^n-4}{3} + 1,$$

and hence $\rho(G, \bullet) = \frac{1}{3}$.

Let us compare with the bounds given by Theorems 3.3.10 and 3.3.11. It was shown in Subsection 3.3.1 that $p_f(G) = \frac{3}{2}$. Hence by Theorem 3.3.10, $\rho(G, \bullet) \leq \frac{3}{8}$. Theorem 3.3.11 also gives $\rho(G, \bullet) \leq \frac{3}{8}$.

3.3.9 Cartesian Product

A **hypercube** Q_n is a graph on 2^n vertices whose vertices can be thought of as the set of binary vectors of length n. Two vertices are adjacent if and only if the corresponding vectors differ in exactly one position.

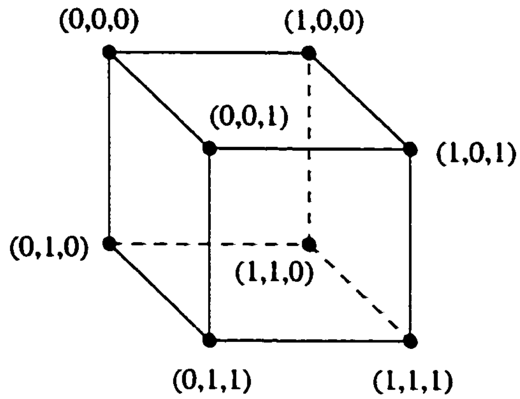


Figure 3.26: The hypercube Q_3.

Lemma 3.3.4 [Fitzpatrick, Nowakowski, Holton, and Caines [14]] *For any integer $n \geq 0$, $p(Q_n) \geq \lceil \frac{2^n}{n+1} \rceil$.*

Proof. Consider an isometric path P_m of Q_n. The vertices of such a path are binary vectors such that any two consecutive vectors differ in exactly one position, and the vectors corresponding to the first and final vertices differ in exactly m positions. It follows that $m \leq n$, and thus an isometric path in Q_n has at most $n+1$ vertices.

$\square$

If $n = 2^t - 1$ for some positive integer t then $p(Q_n) \geq \lceil \frac{2^{2^t-1}}{2^t} \rceil = \lceil 2^{2^t-t-1} \rceil = 2^{2^t-t-1}$. In [14], the authors were able to show that this lower bound is achieved for all positive integers t.

If $n+1 = 2^t$ for some integer $t \geq 1$ then $p(Q_n) = 2^{2^t-t-1}$.

Theorem 3.3.12 *For a connected graph H with $|V(H)| = v_H$ and a perfect matching, $\rho(H, \square) = 0$.*

Proof. Consider a connected graph H with a perfect matching. The number of edges, α needed to cover H is $\frac{1}{2}v_H$. Thus $H^d_\square$ can be covered by α^d hypercubes Q_d since $e^d_\square = Q_d$ where e is an edge of H. And so $H^d_\square$ can be covered by $\alpha^d p(Q_d)$ paths since each of the α^d hypercubes can be covered by $p(Q_d)$ paths. Now

$$\begin{aligned} \rho(H,\square) &\leq \limsup_{d\to\infty} \frac{\alpha^d p(Q_d)}{v_H^d} \\ &= \limsup_{d\to\infty} \frac{\left(\frac{v_H}{2}\right)^d p(Q_d)}{v_H^d} \\ &= \limsup_{d\to\infty} \left(\frac{1}{2^d}\right) p(Q_d). \end{aligned}$$

Suppose $d = 2^n - 1$. Then $|V(Q_d)| = 2^{2^n-1}$ and $\text{diam}(Q_d) = 2^n - 1$. And so $p(Q_d) = \frac{2^{2^n-1}}{(2^n-1)+1} = 2^{2^n-n-1}$. Let $d' = 2^n + j - 1$, $0 \leq j < 2^n$ so that $Q_{d'} = Q_d \square Q_j$ and $p(Q_{d'}) = 2^{2^n-n+1}2^j$. Hence

$$\begin{aligned} \rho(H,\square) &\leq \lim_{n\to\infty} \frac{2^{2^n-n+j-1}}{2^{2^n+j-1}} \\ &= \lim_{n\to\infty} \frac{1}{2^n}. \end{aligned}$$

Since $\frac{1}{2^n} \to 0$ as $n \to \infty$, $\rho(H, \square) = 0$. $\square$

We conjecture that $\rho(H, \square) = 0$ for any connected graph H.

3.3.10 Categorical Product

Theorem 3.3.13 *Let G be a connected graph such that $\chi(G) \geq 3$ and $|V(G)| = v$. Then $\rho(G, \times) \geq \frac{1}{2\text{diam}(G)+1}$.*

Proof. Follows directly from Corollary 3.1.1 and Theorem 3.3.1. □

For an arbitrary graph G, let m_G be the size of a maximum matching in G.

Lemma 3.3.5 *Let G be a connected graph with $|V(G)| = v$. Then $p(G^2_\times) \leq p(G)(2v - 2m_G)$.*

Proof. Consider an isometric path P of G and a matching edge e of G. The graph $P \times e$ can be covered by two isometric paths of length $|P|$ as shown in Figure 3.27.

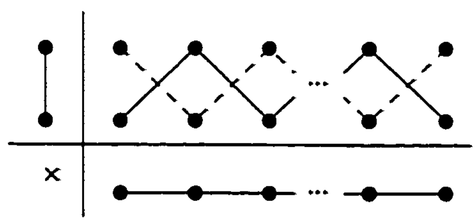


Figure 3.27: Two isometric paths cover $P \times e$.

Consider now a vertex v of G not included in a maximum matching of G. Since G is connected, there exists an edge e' with $v \in V(e')$. As previously shown, $P \times e'$ can be covered by two isometric paths.

Hence for each isometric path P of G, $P \times G$ can be covered by $2m_G + 2(v - 2m_G)$ isometric paths, two paths for each of the m_G matching edges and two for each of the $v - 2m_G$ remaining vertices of G. We note that these isometric paths are not necessarily disjoint (unless $2m_G = v$).

Since G can be covered by $p(G)$ isometric paths, $G^2_\times$ can be covered by $p(G)(2m_G + 2(v - 2m_G)) = p(G)(2v - 2m_G)$ isometric paths. □

Lemma 3.3.6 *Let G be a connected graph with $|V(G)| = v$. Then* $p(G^n_\times) \le p(G)(2v^{n-1} - 2m_{G^{n-1}})$.

Proof. As shown in the proof of Lemma 3.3.5, for each of the isometric paths P of $G^{n-1}_\times$, $P \times G$ can be covered by $2m_G + 2(v - 2m_G) = 2v - 2m_G$ isometric paths, and so $G^{n-1}_\times \times G$ can be covered by $p(G^{n-1}_\times)(2v - 2m_G)$ isometric paths. Similarly for each of the isometric paths P' of G, $P' \times G^{n-1}_\times$ can be covered by $2m_{G^{n-1}} + 2(v^{n-1} - 2m_{G^{n-1}}) = 2v^{n-1} - 2m_{G^{n-1}}$ isometric paths, and so $G \times G^{n-1}_\times$ can be covered by $p(G)(2v^{n-1} - 2m_{G^{n-1}})$ isometric paths.
Recall $\times$ is commutative, and so

$$\begin{aligned} p(G^n_\times) &\le \min\{p(G)(2v^{n-1} - 2m_{G^{n-1}}), p(G^{n-1}_\times)(2v - 2m_G)\} \\ &\le p(G)(2v^{n-1} - 2m_{G^{n-1}}). \end{aligned}$$

□

Theorem 3.3.14 *Let G be a connected graph with $|V(G)| = v$. Then*

$$\rho(G, \times) \le \frac{2p(G)}{v}.$$

Proof. From Lemma 3.3.6, $p(G^n_\times) \le p(G)(2v^{n-1} - 2m_{G^{n-1}})$, and so

$$\begin{aligned} \frac{p(G^n_\times)}{v^n} &\le \frac{p(G)(2v^{n-1} - 2m_{G^{n-1}})}{v^n} \\ &\le \frac{p(G)(2v^{n-1})}{v^n} \\ &= \frac{2p(G)}{v} \end{aligned}$$

which does not depend upon n. □

Example. Suppose $G \cong P_n$. Then $\frac{1}{2n+1} \le \rho(G, \times) \le \frac{2}{n+1}$. Suppose $G \cong K_{2n}$. Then $\frac{1}{3} \le \rho(G, \times) \le 1$. Finally suppose $G \cong C_{2n+1}$. Then $\frac{1}{2n+1} \le \rho(G, \times) \le \frac{4}{2n+1}$.

3.4 Complete Graphs

A **clique** of a graph G is a set of vertices that induce a complete subgraph of G; that is, a subgraph in which every pair of vertices is joined by an edge.

If each beat in the precinct game is a clique, then the minimum number of cops needed to capture the robber on a graph G is bounded by the minimum number of cliques needed to cover the vertices of G. The minimum number of cliques required to cover the vertices of G is the **clique cover number** of G, and is denoted $\theta(G)$. We note that $\theta(G) = \chi(\overline{G})$.

Example. A complete bipartite graph $K_{m,n}$, $m \leq n$ will have clique cover number $\theta(K_{m,n}) = n$.

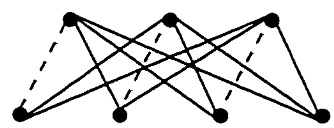


Figure 3.28: A graph $K_{3,5}$ with clique cover number $\theta(K_{3,5}) = 5$.

The maximum size of a clique is two, with one vertex coming from each of the two independent sets. Included in the clique cover of $K_{m,n}$ will be m disjoint cliques of this size, leaving $n - m$ vertices, each of which must be covered by an additional clique.

Define $\rho'(G, \otimes) = \lim_{n\to\infty} \frac{\theta(G^n_\otimes)}{|V(G^n_\otimes)|}$.

Lemma 3.4.1 *If* $\otimes \in \{\times^c, \square^c, \boxtimes, \boxtimes^c, \circledcirc, \bullet, \bullet^c\}$ *then the product* $\otimes$ *of two or more complete graphs is complete.*

Proof. Consider the product of two complete graphs K_m and K_n, $m \leq n$. Since non-edges do not have to be considered, we are interested in just three entries in the edge matrix of this product. If the entry in row i and column j of the edge matrix is a_{ij}, then we need only consider the entries a_{11}, a_{12} and a_{21}. If $a_{11} = a_{12} = a_{21} = E$, then every pair of vertices in the product is joined by an edge, and hence the product is complete on mn vertices. Inductively the result holds for products of two or more complete graphs. □

Theorem 3.4.1 *Let G be a graph that is not completely disconnected, and let $\otimes \in \{\times^c, \square^c, \boxtimes, \boxtimes^c, \circledcirc, \bullet, \bullet^c\}$. Then $\rho'(G, \otimes) = 0$.*

Proof. Let G be any graph with $\theta(G) = m$, $1 \leq m < |V(G)|$ and let $\otimes \in \{\times^c, \square^c, \boxtimes, \boxtimes^c, \circledcirc, \bullet, \bullet^c\}$. From Lemma 3.4.1, we know that $\theta(G^n_\otimes) \leq m^n$. Thus

$$\rho'(G, \otimes) = \lim_{n\to\infty} \frac{\theta(G^n_\otimes)}{|V(G^n_\otimes)|} \leq \lim_{n\to\infty} \frac{m^n}{|V(G^n_\otimes)|} = \lim_{n\to\infty} \left(\frac{m}{|V(G)|}\right)^n = 0$$

since $m < |V(G)|$. □

Corollary 3.4.1 *Let G be a graph that is completely disconnected, and let $\otimes \in \{\times^c, \square^c, \boxtimes, \boxtimes^c, \circledcirc, \bullet, \bullet^c\}$. Then $\rho'(G, \otimes) = 1$.*

Proof. If G is completely disconnected, then $\theta(G^n_\otimes) = |V(G^n_\otimes)|$ and the result follows. □

3.4.1 Symmetric Difference and Cartesian Products

Lemma 3.4.2 *The symmetric difference and Cartesian products of two complete graphs K_m and K_n with $m \leq n$ have clique cover numbers at most m.*

Proof. Let $\otimes \in \{\square, \nabla\}$ and note that $\otimes$ is commutative. We proceed by giving a construction for a clique cover of $K_m \otimes K_n$ that includes m cliques and we may assume $m \leq n$.

Again because we are considering the product of two complete graphs $K_m \otimes K_n$, it is not necessary to consider the non-edges. If the entry in row i and column j of the edge matrix is a_{ij}, then we need only consider the entries a_{11}, a_{12} and a_{21}. For both the symmetric difference and Cartesian products, $a_{11} = N$, and $a_{12} = a_{21} = E$. The entry $a_{12} = E$ indicates that m copies of K_n will be present in the product, one corresponding to each vertex of K_m. Similarly the entry $a_{21} = E$ indicates that n copies of K_m will be present in the product, one corresponding to each vertex of K_n. Thus the product can be covered with m cliques K_n or alternately, n cliques K_m.

Since the clique cover number is the minimum number of cliques required to cover a graph, $\theta(K_m \otimes K_n) \leq m$. □

Lemma 3.4.2 says that if two complete graphs are multiplied using the symmetric difference or Cartesian product, the resulting graph can be covered with a number of copies of the larger complete graph equal to the order of the smaller complete graph. The proof gives a construction for covering the product of two complete graphs with cliques in this way.

Example. Shown with solid edges is the clique cover of the product of $K_3 \otimes K_4$, $\otimes \in \{\square, \nabla\}$ resulting from the construction given in the proof of Lemma 3.4.2.

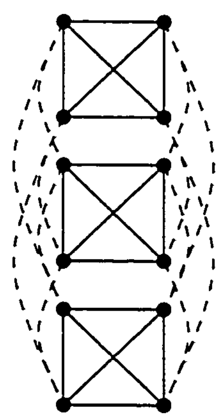


Figure 3.29: $K_3 \otimes K_4$, $\otimes \in \{\square, \nabla\}$.

This construction can be easily generalized to products of finite numbers of complete graphs.

Lemma 3.4.3 *The symmetric difference and Cartesian products of complete graphs* $K_{n_1}, K_{n_2}, \ldots, K_{n_m}$, $n_1 \leq n_2 \leq \cdots \leq n_m$ *have clique cover numbers at most* $n_1 n_2 \cdots n_{m-1}$.

Proof. Repeated applications of Lemma 3.4.2. □

The upper bound given in the previous lemma is in fact best possible when considering the Cartesian product. This is the subject of the next theorem.

Theorem 3.4.2 *Let $K_{n_1}, K_{n_2}, \dots, K_{n_m}$ be complete graphs with $n_1 \le n_2 \le \cdots \le n_m$. Then $\theta(K_{n_1} \,\square\, K_{n_2} \,\square\, \cdots \,\square\, K_{n_m}) = n_1 n_2 \cdots n_{m-1}$.*

Proof. Let $G = K_{n_1} \,\square\, K_{n_2} \,\square\, \cdots \,\square\, K_{n_m}$. Because of Lemma 3.4.3, we need only show that the clique cover number $\theta(G)$ can be no smaller than $n_1 n_2 \cdots n_{m-1}$. But if there are fewer than $n_1 n_2 \cdots n_{m-1}$ cliques then at least one clique must be larger than K_{n_m} which is impossible. Hence the construction given in the proofs of Lemmas 3.4.2 and 3.4.3 is best possible and $\theta(K_{n_1} \,\square\, K_{n_2} \,\square\, \cdots \,\square\, K_{n_m}) = n_1 n_2 \cdots n_{m-1}$. □

We now present the main theorem in this section.

Theorem 3.4.3 *Suppose G is a graph with v vertices and $\otimes \in \{\square, \nabla\}$. Further suppose a clique cover of G includes cliques of order at most k, where k is the smallest value such that this holds true. Then $\rho'(G, \otimes) \le \frac{1}{k}$.*

Proof. Let $c_i(n)$ be the number of cliques K_i of order i, $i = 1, 2, \dots, k$ included in a clique cover of $G^n_\otimes$. A summary of the subgraphs that result when the cliques are multiplied is given in Table 3.8.

Then $G^n_\otimes$ is covered by $\sum_{i=1}^{k} c_i(n)$ cliques where

$$\begin{aligned} c_i(n) &= c_1(1)c_i(n-1) + c_i(1)c_1(n-1) + 2c_2(1)c_i(n-1) + 2c_i(1)c_2(n-1) \\ &\quad + \cdots (i-1)c_{i-1}(1)c_i(n-1) + (i-1)c_i(1)c_{i-1}(n-1) + ic_i(1)c_i(n-1) \\ &= c_i(1)(c_1(n-1) + 2c_2(n-1) + \cdots + (i-1)c_{i-1}(n-1)) \\ &\quad + c_i(n-1)(c_1(1) + 2c_2(1) + \cdots + (i-1)c_{i-1}(1) + ic_i(1) \\ &= c_i(1)(v^{n-1} - \sum_{j=i}^{k} jc_j(n-1)) + c_i(n-1)(v - \sum_{j=i+1}^{k} jc_j(1)) \end{aligned}$$

for $i = 1, 2, \dots, k$. To see this, note that $v^n = \sum_{i=1}^{k} jc_j(n)$ and recall the proofs of Lemmas 3.4.2 and 3.4.3 which tell us that a clique K_i is included in the clique cover only when multiplied by cliques of smaller order.

Solving the system of recurrence relations, we find that

$$c_i(n) = \frac{1}{i}(v - \sum_{j=i+1}^{k} jc_j(1))^n - \frac{1}{i}(v - \sum_{j=i}^{k} jc_j(1))^n.$$

	K_1	K_2	K_3	$\dots$	K_{k-1}	K_k
K_1	K_1	K_2	K_3	$\dots$	K_{k-1}	K_k
K_2	K_2	$2K_2$	$2K_3$	$\dots$	$2K_{k-1}$	$2K_k$
K_3	K_3	$2K_3$	$3K_3$	$\dots$	$3K_{k-1}$	$3K_k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
K_{k-1}	K_{k-1}	$2K_{k-1}$	$3K_{k-1}$	$\dots$	$(k-1)K_{k-1}$	$(k-1)K_k$
K_k	K_k	$2K_k$	$3K_k$	$\dots$	$(k-1)K_k$	kK_k

Table 3.8: Summary.

Hence

$$\theta(G_{\otimes}^n) \le \sum_{i=1}^{k} \Big(\frac{1}{i}(v - \sum_{j=i+1}^{k} jc_j(1))^n - \frac{1}{i}(v - \sum_{j=i}^{k} jc_j(1))^n\Big)$$

and

$$\frac{\theta(G_{\otimes}^n)}{v^n} \le \sum_{i=1}^{k} \Big(\frac{1}{i}\Big(\frac{v - \sum_{j=i+1}^{k} jc_j(1)}{v}\Big)^n - \frac{1}{i}\Big(\frac{v - \sum_{j=i}^{k} jc_j(1)}{v}\Big)^n\Big).$$

Note that $0 < \sum_{j=i+1}^{k} jc_j(1) < v$ and also $0 < \sum_{j=i}^{k} jc_j(1) \le v$ except when $i = k$ and $\sum_{j=i+1}^{k} jc_j(1) = 0$.

Hence as we let n approach infinity, the only nonzero term in the sum on the right side of the expression for $\frac{\theta(G_{\otimes}^n)}{v^n}$ occurs when $i = k$. And so $\rho'(G, \otimes) \le \frac{1}{k}$. □

Note. Let G be a graph with $|V(G)| = v$, and suppose $G_{\otimes}^n$ can be covered by $\theta(G_{\otimes}^n)$ cliques. For each $x \in V(G)$, $x \cdot G_{\otimes}^n$ can be covered by $\theta(G_{\otimes}^n)$ cliques. Hence $\theta(G_{\otimes}^{n+1}) \le v\theta(G_{\otimes}^n)$, and so $\frac{\theta(G_{\otimes}^{n+1})}{v^{n+1}} \le \frac{\theta(G_{\otimes}^n)}{v^n}$. Thus $\rho'(G, \otimes)$ exists.

Corollary 3.4.2 *Suppose G is a graph with v vertices and any clique cover of G includes cliques of order at most k. Further suppose k is the smallest value such that this holds true, so that k is the maximum order of a clique in G. Then $\rho'(G, \square) = \frac{1}{k}$.*

Proof. Theorem 3.4.2 guarantees that the upper bounds for $\theta(G_{\square}^n)$ and $\frac{\theta(G_{\square}^n)}{v^n}$ given by inequalities 3.3 and 3.4 in the proof of Theorem 3.4.3 are in fact best possible.

□

3.4.2 Categorical Product

Lemma 3.4.4 *The categorical product of complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_m}$, $n_1 \leq n_2 \leq \cdots, n_m$ has clique cover number $\theta(K_{n_1} \times K_{n_2} \times \cdots \times K_{n_m}) = n_2 \cdot n_3 \cdots n_m$.*

Proof. We assume $n_1 \leq n_2 \leq \cdots \leq n_m$ since the categorical product is both commutative and associative. The categorical product of the complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_m}$ has edge set $E(K_{n_1} \times K_{n_2} \times \cdots \times K_{n_m}) = \{(v_{1i}, v_{2i}, \ldots, v_{mi})(v_{1j}, v_{2j}, \ldots, v_{mj}) | v_{li} \sim v_{lj}$ for all $l \in \{1, 2, \ldots, m\}\}$. A clique in the product $K_{n_1} \times K_{n_2} \times \cdots \times K_{n_m}$ can include at most one vertex whose m-tuple has ith coordinate $v_{ik}, k \in \{1, 2, \ldots, n_i\}$. Hence a clique in the product can have at most n_1 vertices. So a best possible clique cover would include only cliques of order n_1 and would include each vertex of the product in the cover exactly once. So a best possible clique cover would be composed of $n_2 \cdot n_3 \cdots n_m$ cliques of order n_1.

We proceed by showing that it is always possible to arrange the n_i possible ith coordinates such that each appears at most once in a clique, and any m-tuple of coordinates appears at most once in the set of cliques.

Clearly the vertices in the following set form a clique in the product: $\{(v_{11}, v_{22}, v_{33}, \ldots, v_{mm}), \quad (v_{12}, v_{23}, v_{34}, \ldots, v_{m,m+1}), \quad (v_{13}, v_{24}, v_{35}, \ldots, v_{m,m+2}), \ldots, (v_{1n_1}, v_{2,n_1+1}, v_{3,n_1+2}, \ldots, v_{m,n_1+m-1})\}$. We can obtain $(n_2 \cdot n_3 \cdots n_m) - 1$ additional cliques in the following way. For each coordinate i, $i \in \{2, 3, \ldots, m\}$ of the m-tuples corresponding to vertices in the given clique, add j modulo n_i, $j \in \{1, 2, \ldots, n_i\}$ for each vertex in the given clique. This will produce $(n_2 n_3 \cdots n_m) - 1$ cliques. Of course when all subscripts are left fixed, the original clique is obtained for a total of $n_2 n_3 \cdots n_m$ cliques. Hence $\theta(K_{n_1} \times K_{n_2} \times \cdots \times K_{n_m}) = n_2 \cdot n_3 \cdots n_m$. □

When $m = 2$, Lemma 3.4.4 says that if two complete graphs are multiplied using the categorical product, the resulting graph can be covered with a number of copies of the smaller complete graph equal to the order of the larger complete graph. The proof gives a construction for covering the product of complete graphs with cliques in this way.

Example. Let K_3 have vertices v_{11}, v_{12}, v_{13}, and let K_4 have vertices $v_{21}, v_{22}, v_{23}, v_{24}$. The clique cover construction given in the proof of Lemma 3.4.4 gives the following clique cover of size 4 for $K_3 \times K_4$: $\{(v_{11}, v_{22}), (v_{12}, v_{23}), (v_{13}, v_{24})\}$, $\{(v_{11}, v_{23}), (v_{12}, v_{24}), (v_{13}, v_{21})\}$, $\{(v_{11}, v_{24}), (v_{12}, v_{21}), (v_{13}, v_{22})\}$, $\{(v_{11}, v_{21}), (v_{12}, v_{22}), (v_{13}, v_{23})\}$. The clique cover is also shown in Figure 3.30.

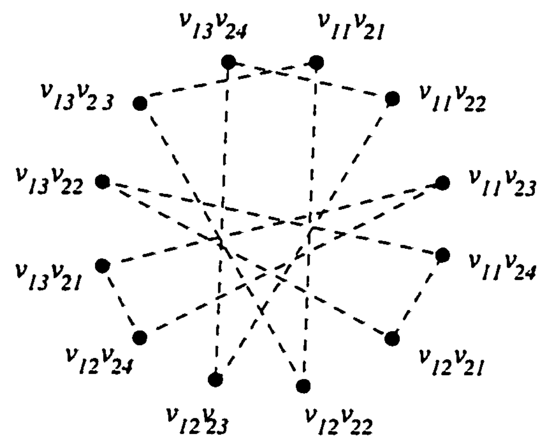


Figure 3.30: A clique cover of size 4 for $K_3 \times K_4$.

We now present the main theorem in this section.

Theorem 3.4.4 *Suppose G is a graph with v vertices, and let $\mathcal{C} = \{C_i, i \in I\}$ be the set of all clique covers of G. Let k_i be the minimum order of a clique in C_i, and define $k = \max_{i \in I} k_i$. Then $\rho'(G, \times) = \frac{1}{k}$.*

Proof. Let $c_i(n)$ be the number of cliques K_i of order i, $i = k, k+1, \ldots, l$ included in a clique cover of $G^n_{\otimes}$. Then $G^n_{\otimes}$ is covered by $\sum_{i=k}^{l} c_i(n)$ cliques where

$$c_i(n) = c_i(1)(v^{n-1} - \sum_{j=k}^{i-1} j c_j(n-1)) + c_i(n-1)(v - \sum_{j=k}^{i} j c_j(1))$$

for $i = k, k+1, \ldots, l$. To see this, note that $v^n = \sum_{i=k}^{l} j c_j(n)$ and recall the proof of Lemma 3.4.4 which tells us that a clique K_i is included in the clique cover only when multiplied by cliques of larger order.

Solving the system of recurrence relations, we find that

$$c_i(n) = \frac{1}{i}(v - \sum_{j=k}^{i-1} jc_j(1))^n - \frac{1}{i}(v - \sum_{j=k}^{i} jc_j(1))^n.$$

Hence

$$\theta(G_\otimes^n) = \sum_{i=k}^{l} \Big(\frac{1}{i}(v - \sum_{j=k}^{i-1} jc_j(1))^n - \frac{1}{i}(v - \sum_{j=k}^{i} jc_j(1))^n \Big)$$

and

$$\frac{\theta(G_\otimes^n)}{v^n} = \sum_{i=k}^{l} \Big(\frac{1}{i}\Big(\frac{v - \sum_{j=k}^{i-1} jc_j(1)}{v}\Big)^n - \frac{1}{i}\Big(\frac{v - \sum_{j=k}^{i} jc_j(1)}{v}\Big)^n \Big).$$

Note that $0 < \sum_{j=k}^{i} jc_j(1) \leq v$ and also $0 < \sum_{j=k}^{i-1} jc_j(1) < v$ except when $i = k$ and $\sum_{j=k}^{i-1} jc_j(1) = 0$.

Hence as we let n approach infinity, the only nonzero term in the sum on the right side of the expression for $\frac{\theta(G_\otimes^n)}{v^n}$ occurs when $i = k$. And so $\rho'(G, \otimes) = \frac{1}{k}$. □

3.5 Copwin Graphs

The number of copwin graphs required to cover the vertices of a graph G is the **copwin number** of G and will be denoted $cpw(G)$.

Lemma 3.5.1 *Let G be a graph. Then $cpw(G) \leq p(G)$.*

Proof. An isometric path is a copwin graph. □

Lemma 3.5.2 *Let G be a graph. Then $cpw(G) \leq \theta(G)$.*

Proof. A complete graph is a copwin graph. □

Theorem 3.5.1 *Let G be a graph and let $\otimes \in \{\times^c, \square^c, \boxtimes, \boxtimes^c, ⊜, \bullet, \bullet^c\}$. Then $cpw(G_\otimes^n) \leq \theta(G)^n$.*

Proof. Lemma 3.4.1 guarantees that the product of two or more complete graphs is complete. □

Recall that a *dominating set* in a graph G is a set S such that every vertex in G belongs to S or is adjacent to a vertex in S. The minimum cardinality of a dominating set in a graph G is the *domination number* of G and is denoted $\gamma(G)$.

Theorem 3.5.2 *Let G be an arbitrary graph. Then $cpw(G) \leq \gamma(G)$.*

Proof. Let G be a graph and suppose $\gamma(G) = 1$. Then there exists $v \in V(G)$ such that $v \sim u$ for all $u \in V(G)$. So for $u \in V(G) - v$, u can be retracted onto v. Hence G is copwin and $cpw(G) = 1 = \gamma(G)$. Now suppose $\gamma(G) > 1$, and let D be a dominating set of G that realizes $\gamma(G)$. For each $x \in D$, the subgraph induced by $N[x]$ is copwin, and $\cup_{x \in D} N[x] = G$. It follows that $cpw(G) \leq \gamma(G)$. □

A graph product $\otimes$ is said to be **dominating multiplicative** if for any two graphs G and H and any two dominating sets $A \subseteq V(G)$ and $B \subseteq V(H)$, the set $A \times B$ is a dominating set in $G \otimes H$.

Lemma 3.5.3 **[Nowakowski and Rall [20]]** *Let $\otimes$ be a graph product. If $\otimes \geq \boxtimes$ then $\otimes$ is dominating multiplicative.*

Proof. Let G and H be any two graphs with dominating sets $A \subseteq V(G)$ and $B \subseteq V(H)$. We must show that the set $A \times B$ is a dominating set in $G \otimes H$ where $\otimes \geq \boxtimes$.

Consider vertices $v \in V(G)$ and $u \in V(H)$. Suppose $v \in A$ and $u \in B$. Then $(v, u) \in A \times B$. Suppose $v \in A$ and $u \in V(H) \setminus B$. Since B is a dominating set in H, there must exist $y \in B$ such that $y \sim u$. Hence $(v, u) \sim (v, y)$. Similarly suppose $v \in V(G) \setminus A$ and $u \in B$. Since A is a dominating set in G, there must exist $x \in A$ such that $x \sim v$. Hence $(v, u) \sim (x, u)$. Finally suppose $v \in V(G) \setminus A$ and $u \in V(H) \setminus B$. Then $(v, u) \sim (x, y)$. Hence $A \times B$ is a dominating set in $G \otimes H$ where $\otimes \geq \boxtimes$. □

Theorem 3.5.3 *Let G and H be arbitrary graphs and let $\otimes \in \{\bullet, \bullet^c, \boxtimes, \boxtimes^c, \circledcirc, \square^c, \times^c\}$. The $cpw(G \otimes H) \leq \gamma(G)\gamma(H)$.*

Proof. By Theorem 3.5.2, $cpw(G \otimes H) \leq \gamma(G \otimes H)$. By Lemma 3.5.3, $\gamma(G \otimes H) \leq \gamma(G)\gamma(H)$. □

For the disjunction and co-categorical products, the results of this section are the best available.

3.5.1 Lexicographic Product

Notice that $G \bullet H \cong H \bullet^c G$ for graphs G and H, and so results similar to those given here for the lexicographic product hold for the co-lexicographic product.

Let G and H be copwin graphs and consider $G \bullet H$. We begin by showing that if x is a corner in H with dominating vertex y and a is a corner in G with dominating vertex b, then it is not necessarily true that (a, x) is a corner in $G \bullet H$ with dominating vertex (b, y).

Consider $P_4 \bullet P_4$ as shown in Figure 3.31.

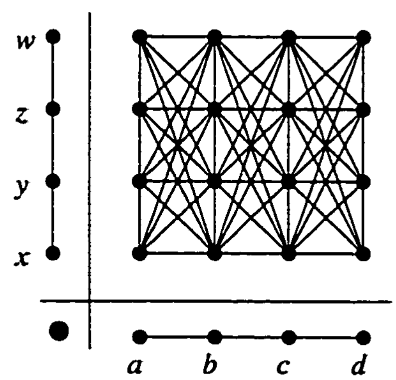


Figure 3.31: The graph $P_4 \bullet P_4$.

Now $(a, x) \sim (b, w)$ but $(b, y) \perp (b, w)$ and so it is not true that $N[(a, x)] \subseteq N[(b, w)]$. Hence (a, x) is not a corner in $P_4 \bullet P_4$ with dominating vertex (b, y). It is also clear that no other vertex becomes a corner in this way.

Hence to ensure that the product of a corner x in H (with dominating vertex y) and a corner in G will be a corner in $G \bullet H$, we require that y is adjacent to all vertices of H. Since it is the vertices of $G \bullet H$ that we are interested in covering, we can take y to be the central vertex of a star with all leaves at distance 1 from y.

Let $N_G[a]$ be the closed neighbourhood of vertex a in the graph G.

Lemma 3.5.4 *Let G be a copwin graph and let H be a star with all leaves at distance 1 from the central vertex y. Suppose $N_G[a] \subseteq N_G[b]$. Then $N_{G\bullet H}[(a,x)] \subseteq N_{G\bullet H}[(b,y)]$.*

Proof. Define $(x, N[y]) = \{(x,w) | w \in N[y]\}$ and $(N(x), N[y]) = \{(z,w) | z \in N(x), w \in N[y]\}$. Now

$$\begin{aligned}
N_{G\bullet H}[(a,x)] &= (a, N[x]) \cup (N(a), V(H)) \\
&= \{(a,x),(a,y)\} \cup (N(a), N[y]) \\
&\quad \text{since } N[y] = V(H) \\
&= \{(a,x),(a,y)\} \cup (b, N[y]) \cup (N(a) - b, N[y]) \\
&\subseteq (a, N[y]) \cup (b, N[y]) \cup (N(b) - a, N[y]) \\
&\quad \text{since } y \in N[y] \text{ and } N(a) - b \subseteq N(b) - a \\
&= (N[b], N[y]) \\
&= N_{G\bullet H}[(b,y)].
\end{aligned}$$

□

Lemma 3.5.5 *The lexicographic product of a copwin graph G and a star H with all leaves at distance 1 from the central vertex is copwin.*

Proof. Let y be the central vertex of H. By Lemma 3.5.4, if a is a corner in G with dominating vertex b, then for all $x \in V(H)$, (a,x) is dominated by (b,y). Therefore $G \bullet H$ is dismantlable. □

Theorem 3.5.4 *Let G and H be arbitrary graphs. Then $cpw(G \bullet H) \leq \gamma(H) \cdot cpw(G)$.*

Proof. The graph H can be covered by $\gamma(H)$ stars (such that each star has all of its leaves at distance 1 from the central vertex), and the graph G can be covered by $cpw(G)$ copwin graphs. □

Theorem 3.5.4 is an improvement over Theorem 3.5.3 since $cpw(G) \leq \gamma(G)$ for all graphs G.

3.5.2 Strong Product

Theorem 3.5.5 *Let G_i, $i = 1, 2, \ldots, n$ be a finite collection of arbitrary graphs. Then $cpw(\boxtimes_{i=1}^{n} G_i) \leq \prod_{i=1}^{n} cpw(G_i)$.*

Proof. By Theorem 1.2.5, the strong product of a finite collection of copwin graphs is copwin. □

3.5.3 Equivalence

Lemma 3.5.6 *Let G be a graph with $\gamma(G) > 1$. Then there exists a subset $A \subseteq V(G)$ such that*

(1) for all $a \in A$ there exists $b \in A$ such that $a \perp b$, and

(2) for all $x \in N(A) \setminus A$ there exists $y \in A$ such that $y \perp x$.

Proof. Since $\gamma(G) > 1$, G does not have a dominating vertex. Hence $V(G)$ is the required set. □

In this subsection, we are interested in defining a subset B of $V(G)$ such that $B \circledcirc B$ dominates $G \circledcirc G$. We will show that any set satisfying conditions (1) and (2) of Lemma 3.5.6 is such a subset. Since Lemma 3.5.6 shows that such a subset exists provided $\gamma(G) > 1$, we will be interested in finding the smallest such subset. We make a more formal definition below.

Let G be a graph and let $S_G \subseteq V(G)$ such that

for all $a \in S_G$ there exists $b \in S_G$ such that $a \perp b$, and

for all $x \in N(S_G) \setminus S_G$ there exists $y \in S_G$ such that $y \perp x$.
Define A_G to be such a subset S_G of smallest cardinality.

Example. Consider C_6 as shown in Figure 3.32. A set S_{C_6} is $\{1, 4\}$ and $A_{C_6} = 2$.

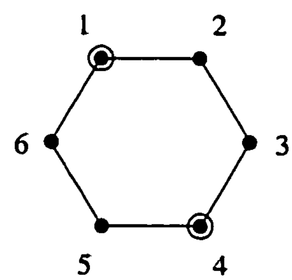


Figure 3.32: The graph C_6 with $A_{C_6} = 2$.

Theorem 3.5.6 *Let G be a graph with $\gamma(G) > 1$. Then $cpw(G \textcircled{$\cong$} G) \leq |A_G|^2$.*

Proof. Partition $V(G)$ into three sets of vertices: A_G, $N(A_G) \setminus A_G$, and $V(G) \setminus N[A_G]$ as shown in Figure 3.33. We must show that $A_G \textcircled{$\cong$} A_G$ dominates $G \textcircled{$\cong$} G$.

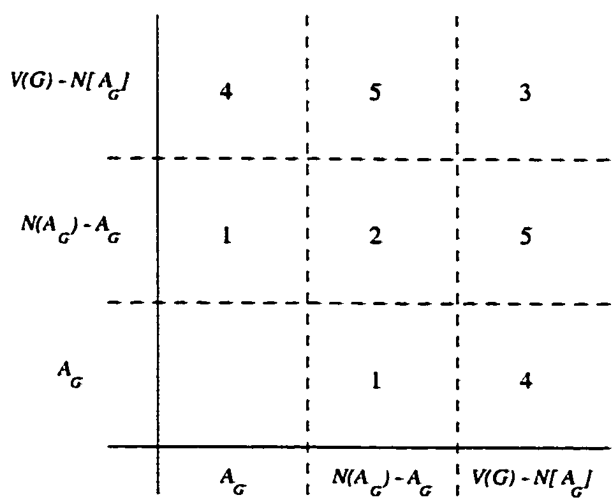


Figure 3.33: An illustration to accompany the proof of Theorem 3.5.6.

We will show that for all $(x,y) \in G \circledcirc G, (x,y) \sim (a,b)$ for some $(a,b) \in A_G \circledcirc A_G$. Suppose $x \in A_G$ and $y \in N(A_G) \setminus A_G$. Since $y \in N(A_G) \setminus A_G$, $y \sim z$ for some $z \in A_G$. Hence $(x,y) \sim (x,z)$. Similarly $(y,x) \sim (z,x)$. The groups of vertices dominated by $A_G \circledcirc A_G$ in this way are indicated by 1's in Figure 3.33.

Suppose $x, y \in N(A_G) \setminus A_G$. Then $x \sim w$ for some $w \in A_G$ and $y \sim z$ for some $z \in A_G$. Hence $(x,y) \sim (w,z)$. The vertices dominated by $A_G \circledcirc A_G$ in this way are indicated by a 2 in Figure 3.33.

Suppose $x, y \in V(G) \setminus N[A_G]$. Then $(x,y) \sim (w,z)$ for all $w, z \in A_G$. The vertices dominated by $A_G \circledcirc A_G$ in this way are indicated by a 3 in Figure 3.33.

Suppose $x \in A_G$ and $y \in V(G) \setminus N[A_G]$. Then by the definition of A_G there exists $z \in A_G$ such that $x \perp z$. Hence $(x,y) \sim (z,w)$ for all $w \in A_G$. Similarly $(y,x) \sim (w,z)$ for all $w \in A_G$. The vertices dominated by $A_G \circledcirc A_G$ in this way are indicated by 4's in Figure 3.33.

Finally suppose $x \in N(A_G) \setminus A_G$ and $y \in V(G) \setminus N[A_G]$. Then there exists $z \in A_G$ such that $x \perp z$. Hence $(x,y) \sim (z,w)$ for all $w \in A_G$. Similarly $(y,x) \sim (w,z)$ for all $w \in A_G$. The vertices dominated by $A_G \circledcirc A_G$ in this way are indicated by 5's in Figure 3.33.

Since all vertices of $G \circledcirc G$ are adjacent to at least one vertex of $A_G \circledcirc A_G$, $A_G \circledcirc A_G$ dominates $G \circledcirc G$ and so $\gamma(G \circledcirc G) \leq |A_G|^2$. Hence $cpw(G \circledcirc G) \leq |A_G|^2$. □

Corollary 3.5.1 *Let G be a graph with* $\operatorname{diam}(G) > 2$. *Then $cpw(G \circledcirc G) \leq 4$.*

Proof. Since $\operatorname{diam}(G) > 2$, there exist vertices x and y in G such that $d(x,y) \geq 3$. Thus $|A_G| \leq |\{x,y\}| = 2$. □

3.5.4 Co-Cartesian Product

Lemma 3.5.7 *Let G be a graph that is not complete. Then there exists a subset $A \subseteq V(G)$ such that for all $a \in A$ there exists $b \in A$ such that $a \perp b$.*

Proof. If G is not complete, then G has an independent set of size 2 which is the required set. □

In this subsection, we are interested in finding a subset B of $V(G)$ such that $B\square {}^cB$ dominates $G \square {}^cG$. We will show that any set as defined in Lemma 3.5.7 is such a subset. Since we have shown that such a subset B exists provided G is not complete, we will be interested in finding the smallest such subset. We make a more formal definition below.

Let G be a graph and let $S_G \subseteq V(G)$ such that for all $a \in S_G$ there exists $b \in S_G$ such that $a \perp b$. Define A_G to be such a subset S_G of smallest cardinality.

Notice that if G is not complete, then $A_G = 2$.

Theorem 3.5.7 *Let G be a graph. Then $cpw(G \square {}^cG) \leq 4$.*

Proof. If G is complete, then G is copwin. So suppose G is not complete. Partition $V(G)$ into three sets of vertices: $A_G, N(A_G) \setminus A_G$, and $V(G) \setminus N[A_G]$ as shown in Figure 3.34. We must show that $A_G \square {}^cA_G$ dominates $G \square {}^cG$.

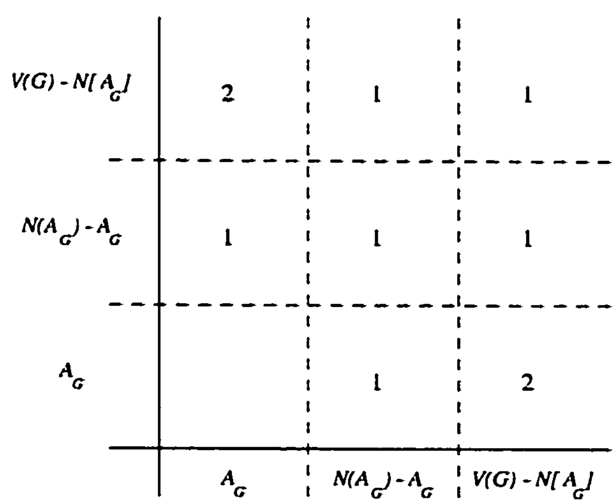


Figure 3.34: An illustration to accompany the proof of Theorem 3.5.7.

We will show that for all $(x, y) \in G \square {}^cG, (x, y) \sim (a, b)$ for some $(a, b) \in A_G \square {}^cA_G$. Suppose $x \in A_G$ and $y \in N(A_G) \setminus A_G$. Since $y \in N(A_G) \setminus A_G$, $y \sim z$ for some $z \in A_G$. Hence $(x, y) \sim (x, z)$. Similarly $(y, x) \sim (z, x)$.

Suppose $x, y \in N(A_G) \setminus A_G$. Then $x \sim w$ for some $w \in A_G$ and $y \sim z$ for some $z \in A_G$. Hence $(x, y) \sim (w, z)$.

Suppose $x, y \in V(G) \setminus N[A_G]$. Then $(x, y) \sim (w, z)$ for all $w, z \in A_G$.

Suppose $x \in N(A_G) \setminus A_G$ and $y \in V(G) \setminus N[A_G]$. Then there exists $z \in A_G$ such that $x \sim z$. Hence $(x, y) \sim (z, w)$ for all $w \in A_G$. Similarly $(y, x) \sim (w, z)$ for all $w \in A_G$.

Notice that the cases considered thus far are indicated by 1's in Figure 3.34. The fact that the vertices considered are dominated by at least one vertex of $A_G \,\square\, {}^cA_G$ follows from the definition of the co-Cartesian product. The definition of the set A_G is needed only for the remaining case that follows.

Suppose $x \in A_G$ and $y \in V(G) \setminus N[A_G]$. Then by the definition of A_G there exists $z \in A_G$ such that $x \perp z$. Hence $(x, y) \sim (z, w)$ for all $w \in A_G$. Similarly $(y, x) \sim (w, z)$ for all $w \in A_G$.

Since all vertices of $G \square {}^cG$ are adjacent to at least one vertex of $A_G \square {}^cA_G$, $A_G \square {}^cA_G$ dominates $G \,\square\, {}^cG$ and so $\gamma(G \,\square\, {}^cG) \leq |A_G|^2$. Hence $cpw(G \,\square\, {}^cG) \leq |A_G|^2 = 4$. $\square$

3.5.5 Symmetric Difference

Let G be a graph. Define $\overline{\gamma}(G)$ as a set of smallest cardinality which dominates both G and G^c.

Theorem 3.5.8 *Let G and H be graphs. Then $cpw(G \nabla H) \leq \min\{\gamma(G)\overline{\gamma}(H), \gamma(H)\overline{\gamma}(G)\}$.*

Proof. Let $A \subseteq V(G)$ be a set which realizes $\overline{\gamma}(G)$ and let $B \subseteq V(H)$ be a set which realizes $\gamma(H)$. We must show that $A \nabla B$ dominates $G \nabla H$; that is, we will show that for all $(x, y) \in V(G \nabla H)$, there exists $(a, b) \in V(A \nabla B)$ such that $(x, y) \sim (a, b)$.

Suppose $x \in A$ and $y \in B^c$. Since B dominates H, there exists $b \in B$ such that $y \sim b$. Hence $(x, y) \sim (x, b)$. Similarly suppose $x \in A^c$ and $y \in B$. There exists $a \in A$ such that $x \sim a$. Hence $(x, y) \sim (a, y)$.

Finally suppose $x \in A^c$ and $y \in B^c$. Since B dominates H, there exists $b \in B$ such that $y \sim b$. Since A dominates G^c, there exists $a \in A$ such that $x \sim a$ in G^c. But then $x \perp a$ in G. Hence $(x,y) \sim (a,b)$. □

A **total dominating set** [10] in a graph G is a set of vertices, S such that every vertex in G is adjacent to a vertex in S. The minimum cardinality of a total dominating set is denoted $\gamma_t(G)$. Notice that a total dominating set is a dominating set S such that each of the vertices of S is adjacent to another vertex of S.

Theorem 3.5.9 *Let G and H be graphs. Then $cpw(G\nabla H) \leq \min\{\gamma_t(G)\gamma_t(H^c), \gamma_t(H)\gamma_t(G^c)\}$.*

Proof. Let $A \subseteq V(G)$ be a set which realizes $\gamma_t(G)$ and let $B \subseteq V(H)$ be a set which realizes $\gamma_t(H^c)$. We must show that $A\nabla B$ dominates $G\nabla H$; that is, we must show that for all $(x,y) \in V(G\nabla H)$, there exists $(a,b) \in V(A\nabla B)$ such that $(x,y) \sim (a,b)$.

Suppose $x \in A^c$ and $y \in B$. Since A is a total dominating set in G, there exists $a \in A$ such that $x \sim a$. Hence $(x,y) \sim (a,y)$.

Suppose $x \in A$ and $y \in B^c$. Since A is a total dominating set in G, there exists $a \in A$ such that $x \sim a$. Since B is a total dominating set in H^c, there exists $b \in B$ such that $y \sim b$ in H^c. But then $y \perp b$ in H. Hence $(x,y) \sim (a,b)$.

Finally suppose $x \in A^c$ and $y \in B^c$. Since A is a total dominating set in G, there exists $a \in A$ such that $x \sim a$. Since B is a total dominating set in H^c, there exists $b \in B$ such that $y \sim b$ in H^c. But then $y \perp b$ in H. Hence $(x,y) \sim (a,b)$. □

3.5.6 Categorical Product

Let G and H be copwin graphs and consider $G \times H$. We begin by showing that if x is a corner in H with dominating vertex y and a is a corner in G with dominating vertex b, then it is not necessarily true that (a,x) is a corner in $G \times H$ with dominating vertex (b,y). If $c \in N(a)$ then $(a,x) \sim (c,y)$ but $(b,y) \perp (c,y)$. Similarly if $z \in N(x)$ then $(a,x) \sim (b,z)$ but $(b,y) \perp (b,z)$. Hence it is not true that $N[(a,x)] \subseteq N[(b,y)]$. And so (a,x) is not a corner in $G \times H$ with dominating vertex (b,y).

We have shown that the categorical product of two copwin graphs need not be copwin. In fact such a product of graphs need not even be connected.

Example. Consider $P_3 \times P_3$ as shown in Figure 3.35.

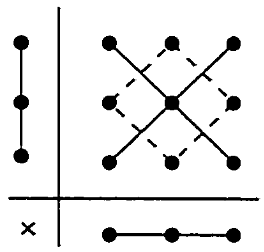


Figure 3.35: The graph $P_3 \times P_3$.

The product has two components, the edges of the first indicated by solid lines and the edges of the second indicated by dashed lines.

However recall from Lemma 3.4.4 that the graph $K_n \times K_m$, $m \leq n$ can be covered by n complete graphs of order m.

Theorem 3.5.10 *Let G be a graph with v vertices and a clique cover that includes c_i cliques K_i of order i, $i \leq m$. Then*

$$cpw(G_{\times}^n) \leq \sum_{i=1}^{m} \Big(\frac{1}{i}(v - \sum_{j=1}^{i-1} jc_j)^n - \frac{1}{i}(v - \sum_{j=1}^{i} jc_j)^n\Big).$$

Proof. The proof of Theorem 3.4.4 gives

$$\theta(G_{\times}^n) \leq \sum_{i=1}^{m} \Big(\frac{1}{i}(v - \sum_{j=1}^{i-1} jc_j)^n - \frac{1}{i}(v - \sum_{j=1}^{i} jc_j)^n\Big).$$

Lemma 3.5.2 gives $cpw(G_{\times}^n) \leq \theta(G_{\times}^n)$. □

Recall the notion of a total dominating set.

Theorem 3.5.11 *Let G and H be graphs. Then $cpw(G \times H) \leq \gamma_t(G)\gamma_t(H)$.*

Proof. Let $A \subseteq V(G)$ be a set which realizes $\gamma_t(G)$ and let $B \subseteq V(H)$ be a set which realizes $\gamma_t(H)$. We must show that $A \times B$ dominates $G \times H$; that is, we will show that for all $(x, y) \in V(G \times H)$, there exists $(a, b) \in V(A \times B)$ such that $(x, y) \sim (a, b)$.

Suppose $x \in A$ and $y \in B^c$. Since A is a total dominating set in G, there exists $a \in A$ such that $x \sim a$. Since B is a total dominating set in H, there exists $b \in B$ such that $y \sim b$. Hence $(x, y) \sim (a, b)$. The cases when $x \in A^c$ and $y \in B$, and when $x \in A^c$ and $y \in B^c$ are similar. □

If the product $\otimes \in \{\times, \bullet, \bullet^c, \boxtimes, \boxtimes^c, ⓔ, \Box^c, \times^c\}$ then a similar result holds for $cpw(G \otimes H)$; that is, $cpw(G \otimes H) \leq \gamma_t(G)\gamma_t(H)$ for arbitrary graphs G and H. This follows from Theorem 3.5.2 because a total dominating set is a dominating set, and Nowakowski and Rall [20] proved that if $\otimes \geq \times$ in the partial order given by Figure 3.3, then $\gamma_t(G \otimes H) \leq \gamma_t(G)\gamma_t(H)$, or equivalently that $\otimes$ is **total dominating multiplicative**. This result is not included at the beginning of Section 3.5 because for products $\otimes \geq \times$ other than the categorical product, the result is not an improvement over Theorem 3.5.3.

3.5.7 Cartesian Product

Let G and H be copwin graphs and consider $G \Box H$. Suppose x is a corner in H with dominating vertex y and a is a corner in G with dominating vertex b. Now $(a, x) \perp (b, y)$ in $G \Box H$. Hence it is not true that $N[(a, x)] \subseteq N[(b, y)]$. And so (a, x) is not a corner in $G \Box H$ with dominating vertex (b, y).

We have shown that Cartesian product of two copwin graphs need not be copwin. However recall from Theorem 3.4.2 that the graph $K_n \Box K_m$, $m \leq n$ can be covered by m complete graphs of order n.

Theorem 3.5.12 *Let G be a graph with v vertices and a clique cover that includes c_i cliques K_i of order i, $i \leq m$. Then*

$$cpw(G^n_{\Box}) \leq \sum_{i=1}^{m} \Big(\frac{1}{i}(v - \sum_{j=i+1}^{m} jc_j)^n - \frac{1}{i}(v - \sum_{j=i}^{m} jc_j)^n \Big).$$

Proof. The proof of Corollary 3.4.2 gives

$$\theta(G^n_\square) = \sum_{i=1}^{m} \Big(\frac{1}{i}(v - \sum_{j=i+1}^{m} jc_j)^n - \frac{1}{i}(v - \sum_{j=i}^{m} jc_j)^n \Big).$$

Lemma 3.5.2 gives $cpw(G^n_\square) \leq \theta(G^n_\square)$. □

A grid $G_{m,n}$ is the Cartesian product of a path P_m of length $m-1$ and a path P_n of length $n-1$. Recall that $p(G_{m,n})$ is the isometric path number of such a grid. In [12], Fisher and Fitzpatrick prove the following result which gives $p(G_{m,n})$ exactly for all integers $m, n \geq 2$.

If $G_{m,n}$ is an $m \times n$ grid for some integers $m, n \geq 2$ then

$$p(G_{m,n}) = \left\lceil \frac{2}{3}(m + n - \sqrt{m^2 - mn + n^2}) \right\rceil.$$

Lemma 3.5.8 *Let G be a graph with v vertices that can be covered by $\sum_{i=1}^{m} c_i$ isometric paths, where c_i is the number of paths P_i of length $i-1$, $i \leq m$. Then*

$$p(G^2_\square) \leq \sum_{i=1}^{m} \sum_{j=1}^{m} c_i c_j \left\lceil \frac{2}{3}(i + j - \sqrt{i^2 - ij + j^2}) \right\rceil.$$

Proof. The product of an isometric path P_n and an isometric path P_m can be covered by $p(G_{m,n})$ isometric paths. Hence if a graph G can be covered by $\sum_{i=1}^{m} c_i$ isometric paths P_i, $i \leq m$, then $G^2_\square$ can be covered by $\sum_{i=1}^{m} \sum_{j=1}^{m} c_i c_j p(G_{i,j})$ isometric paths. Equivalently

$$p(G^2_\square) \leq \sum_{i=1}^{m} \sum_{j=1}^{m} c_i c_j \left\lceil \frac{2}{3}(i + j - \sqrt{i^2 - ij + j^2}) \right\rceil.$$

□

Theorem 3.5.13 *Let G be a graph with v vertices that can be covered by $\sum_{i=1}^{m} c_i$ isometric paths, where c_i is the number of paths P_i of length $i-1$, $i \leq m$. Then*

$$cpw(G^2_\square) \leq \sum_{i=1}^{m} \sum_{j=1}^{m} c_i c_j \left\lceil \frac{2}{3}(i + j - \sqrt{i^2 - ij + j^2}) \right\rceil.$$

Proof. Lemma 3.5.1 gives $cpw(G^2_\square) \leq p(G^2_\square)$. □

Example. Consider the complete graph K_{2n}. By Theorem 3.5.13, $cpw(K_{2n} \square K_{2n}) \leq 2n^2$. However by Theorems 3.4.2 and 3.5.12, $cpw(K_{2n} \square K_{2n}) = 2n$.

Chapter 4

Graphs with Copnumber 2

Recall from Chapter 1 that copwin graphs have been completely characterized [23] and that there is an explicit strategy that can be used by the cops to win on such graphs. (See the Copwin Strategy in Section 1.5.) No such characterization is known for graphs with copnumber 2 (or copnumber k, $k \geq 2$). In this chapter, we discuss outerplanar graphs and tandem-win graphs, both of which are included in the class of graphs with copnumber ≤ 2. Hence we take steps toward the characterization of copnumber 2 graphs.

4.1 Outerplanar Graphs

Recall that Aigner and Fromme [1] were able to show that if G is a planar graph then the copnumber of G is at most 3. One question is to characterize graphs with copnumber 2. So let us consider outerplanar graphs.

An *embedding* of a graph G is a mapping of G into a surface such that the images of its edges do not intersect except for shared endpoints. A *face* is a closed, connected region of an embedding. A graph G is said to be **outerplanar** if it has an embedding in the plane such that every vertex lies on the unbounded face. We may assume that the embedding has all the vertices being on a circle. We can label the vertices clockwise around the circle $a_0, a_1, \ldots, a_n$. Also we may assume that all of the edges lie outside the circle.

Theorem 4.1.1 *If G is a connected outerplanar graph, then $c(G) \leq 2$.*

Proof. Let G be a connected outerplanar graph. Suppose cops C_1 and C_2 are on the vertices x and y of an edge xy as shown in Figure 4.1. (A) Further suppose the robber is covered by this edge; that is, the robber is located on a vertex between x and y as indicated by R_1 in the Figure. We assume that the subgraph H induced by the vertices strictly between x and y in a clockwise direction is connected to both x and y, and is itself connected. Otherwise if H is connected only at y then C_1 moves to the anticlockwise most vertex z of H, and is still anticlockwise from the robber at the end of this series of moves. This is because C_1 moves along the shortest path from x to z. The robber cannot move past C_2. Henceforth we will assume that all such subgraphs H are connected at x and y. If not, then the cops can perform this maneuver. Now C_2 moves clockwise until he is on an edge that covers the robber, but is no longer with C_1. Then C_1 moves to the other end of this covering edge. This procedure repeats until no further such edges exist, at which time the robber is on a cycle or path and C_1 moves to capture him.

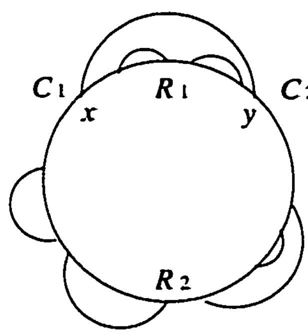


Figure 4.1: An outerplanar graph G.

(B) Otherwise the robber is outside as indicated by R_2 in the Figure. Now C_1 remains stationary and C_2 moves clockwise along the edges of the unbounded face towards the robber, moving in this way until he is on an edge that covers the robber if such an edge exists. Otherwise C_2 moves until he is adjacent to the robber. Notice that the robber cannot pass C_2. Now C_2 remains stationary and C_1 moves along the edges of the unbounded face towards the robber until he is on an edge that covers the robber or until he is adjacent to the robber. The cops continue in this way until situation (A) occurs or the robber is captured. □

The converse does not hold. Consider the graph G shown in Figure 4.2. Since for all $u \in V(G)$, u can be retracted onto v, $c(G) = 1$. However G is not outerplanar.

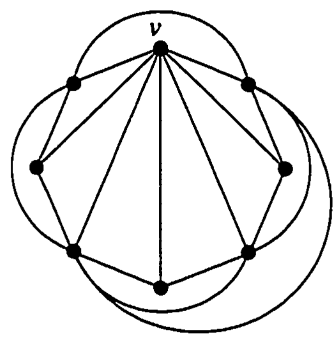


Figure 4.2: A planar copwin graph G that is not outerplanar.

4.2 Tandem-win Graphs

In this section, we propose a variation of Cops and Robber. Recall the discussion regarding copwin graphs in Chapter 1. A vertex $v \in V(G)$ is irreducible in a graph G if there exists a vertex $u \in V(G)$ such that $N[v] \subseteq N[u]$. This is the notion that led to the characterization of copwin graphs. Suppose we substitute the concept of a closed neighborhood in this definition with the concept of an open neighborhood and say that a vertex $v \in V(G)$ is **nearly irreducible** in a graph G if there exists a vertex $u \in V(G)$ such that $N(v) \subseteq N(u)$. The vertex u **nearly dominates** the vertex v.

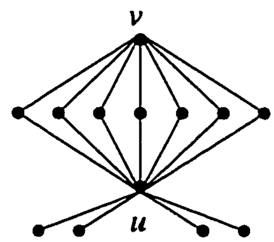


Figure 4.3: A nearly irreducible vertex v and its nearly dominating vertex u.

Let us consider what happens if the cop side consists of two cops, cop 1 and cop 2 say, and the robber moves onto a nearly irreducible vertex v as shown in Figure 4.3. The cops move onto the nearly dominating vertex u. There are two options available to the robber. If the robber moves to an adjacent vertex then the cops can immediately move onto the same vertex and capture the robber since $N(v) \subseteq N(u)$. So let's assume the robber passes. Cop 1 then moves to a vertex in the common neighborhood of u and v and cop 2 remains on vertex u. If the robber then passes, cop 1 can move onto v and capture the robber while cop 2 moves to a vertex in the common neighbourhood of u and v. If the robber moves to an adjacent vertex then cop 2 (who is on vertex u) can immediately move onto the same vertex and capture the robber. Cop 1 moves to one of u and v. Hence two cops can guarantee the capture of the robber once he moves onto v in at most two moves. Notice that the cops were able to remain at distance at most one from each other after every move.

This leads us to propose a variation of the game in which the cop side consists of two cops. The cops must stay within distance one of each other during the game. A graph on which two cops playing in tandem in this way can win is said to be **tandem-win**. Since the movements of the two cops have been restricted in this version, the class of tandem-win graphs will be contained in the class of graphs with copnumber ≤ 2. Hence we are taking a step toward the characterization of graphs with copnumber 2. Note that although the cops must be on adjacent vertices after any move, they are permitted to be at distance greater than one from each other during a move. For example, consider the graph C_4. Cops C_1 and C_2 are permitted to move to vertices x and y, respectively as indicated in Figure 4.4.

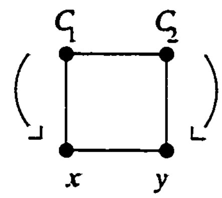


Figure 4.4: A permissible move.

We begin with some useful theorems.

Lemma 4.2.1 *A copwin graph G is tandem-win.*

Proof. One cop follows the Copwin Strategy; the other follows along. □

Theorem 4.2.1 *Any retract H of a tandem-win graph G is also tandem-win.*

Proof. Let G be a tandem-win graph and let H be a retract of G. Further let f be a retraction map from G to H. Since G is tandem-win, the cops have a winning strategy on G. We will indicate how this strategy can be modified and used on the subgraph H. The cops simply play the image under f of their winning strategy on G. Note that since f is a retraction map, if $a, b \in V(G)$ and $a \simeq b$ then $f(a) \simeq f(b)$ where $f(a), f(b) \in V(H)$ so that the cops will be within distance one on H as well. Using this strategy, the cops capture the image of the robber on H. Since the robber is actually playing on H and f is the identity map on H, the robber's image coincides with his actual position. Hence the robber is apprehended on H and therefore, H is a tandem-win graph. □

Notice that it is not necessarily true that an isometric subgraph H of a tandem-win graph G is tandem-win. To see this, consider the graphs G and H shown in Figure 4.5. The graph G is copwin and thus tandem-win. However the graph H is not tandem-win since after his move, the robber can always remain two vertices away from both cops.

From the analysis at the beginning of this section, we know that two cops playing in tandem can guarantee a win if the robber moves onto a nearly irreducible vertex.

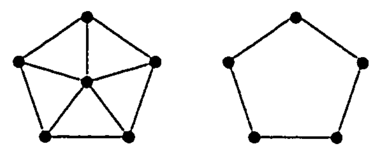


Figure 4.5: The isometric subgraph H of the graph G is not tandem-win even though G is tandem-win.

Therefore we conclude that the robber will not move onto such a vertex unless he is forced to do so. This is because any robber-win strategy that uses v can be modified to use u. Hence we must determine if the cops can force the robber to move onto a nearly irreducible vertex. To do this, we remove this vertex from the graph and play on the resulting subgraph. The next theorem tells us that this subgraph will be tandem-win if and only if the original graph is tandem-win.

Theorem 4.2.2 *Let G be a graph and let c be a nearly irreducible vertex of G. Let $G' = G \setminus \{c\}$. Then G is tandem-win if and only if G' is tandem-win.*

Proof. Let d be a vertex that nearly dominates c. Now G' is a retract of G with a retraction map f defined as follows: $f(c) = d$ and $\forall v \in V(G')$, $f(v) = v$. Suppose G is tandem-win. Then G' is tandem-win by Theorem 4.2.1.

Conversely suppose G' is tandem-win, and thus the cops have a winning strategy on G'. Since the game is actually being played on G, the winning strategy on G' can be thought of as catching the image of the robber. Now suppose this image is caught on vertex u. If $u \neq d$, then the robber's image on G' corresponds to his actual position on G since f is the identity map on G'. Hence the robber is apprehended. Otherwise, the robber's image is apprehended on vertex d. Since it is known that $f(c) = f(d) = d$, the robber is on vertex c or vertex d in the graph G. If he is on d, his actual position corresponds to his image and he is caught. If he is on c then he can be caught in at most two moves by the cops. This is because at least one of the cops is on vertex d and d nearly dominates c. □

This leads us to a main theorem in this section.

Theorem 4.2.3 *Let G be a graph. Suppose there is an ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertices of G such that for each $i < n$, v_i is irreducible or nearly irreducible in the subgraph induced by the vertices in the set $\{v_i, v_{i+1}, \ldots, v_n\}$. Then G is tandem-win.*

Proof. Let G be a graph. It has been shown in Theorem 4.2.2 that if c is a nearly irreducible vertex of G then $G \setminus \{c\}$ is tandem-win if and only if G is tandem-win. A similar result holds for irreducible vertices by Theorem 1.2.2. Inductively then, if $m-1$ irreducible vertices and $n-m$ nearly irreducible vertices can be removed from G, then G is tandem-win. □

This decomposition is not a characterization of tandem-win graphs. Consider the graph shown in Figure 4.6. None of the seven vertices are nearly irreducible or irreducible, yet the graph is tandem-win. The cops begin on the vertices indicated by 1's. The robber begins on vertex R (or is captured on the cops' next move). The cops move to the vertices indicated by 2's, and the robber is captured on the cops' next move.

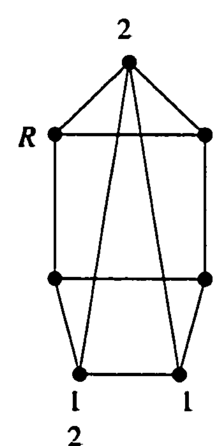


Figure 4.6: A tandem-win graph with no nearly irreducible vertices.

For a graph G, the ordering $\{v_1, v_2, \ldots, v_n\}$, if it exists, in Theorem 4.2.3 will be referred to as a **tandem-win ordering** with **start vertex** v_n.

Fix a tandem-win ordering of G, and construct a tree S of G such that $V(S) = V(G)$, the root of the tree is the start vertex of the tandem-win ordering, and for vertices x_1 and $x_2 \in V(G)$, $x_1x_2 \in E(S)$ if and only if $f_j(x_1) = x_2$ or $f_j(x_2) = x_1$ for some j. This tree shall be referred to as a **tandem-win decomposition tree.** We say that $x_1 \succeq x_2$ if x_1 is eventually retracted onto x_2 and $x_1 \succ x_2$ if $x_1 \neq x_2$. This concept is analogous to that of a copwin spanning tree introduced in the previous chapter. Note that here a vertex may be retracted onto a nonadjacent vertex, and hence the decomposition tree may contain edges that are not in the original graph.

Example. This example refers to Figure 4.7. In (a), the circled vertices represent nearly irreducible or irreducible vertices at each of the stages. Also, at each stage, it does not matter in which order the nearly irreducible vertices are removed. The original graph is tandem-win.

If the vertices are labeled as shown in (b), then one tandem-win ordering is {9, 12, 10, 13, 8, 11, 2, 3, 4, 5, 6, 7, 1}. The corresponding tandem-win decomposition tree is shown.

Suppose we have a tandem-win ordering of the vertices of a graph G. We know that two cops playing in tandem have a winning strategy on G, but this strategy has not yet been made explicit. We now describe a strategy that can be used by the cops to win, and prove that this strategy is effective in capturing the robber.

Tandem-win Strategy. Let $\{x_1, x_2, \ldots, x_n\}$ be a tandem-win ordering of the vertices of a graph G. Define the induced subgraphs $G_i = G_{i-1} \setminus \{x_{i-1}\}$ where $G_1 = G$, and let $f_i : G_i \rightarrow G_{i+1}$ be the retraction map from G_i to G_{i+1}. Further if the robber is on vertex x, define $F_i(x) = f_{i-1} \circ f_{i-2} \circ \cdots \circ f_2 \circ f_1(x)$ so that $F_i(x)$ is the robber's image or shadow on G_i. The robber is always thought to be playing on the graph G. However, the cops initially move on the subgraph G_n. The cops begin on vertex x_n, the vertex on which the cops' position coincides with the robber's image under the mapping $f_{n-1} \circ f_{n-2} \circ \cdots \circ f_2 \circ f_1(x)$. Now suppose at least one of the cops is occupying the robber's image in the subgraph G_i under the mapping $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_2 \circ f_1(x)$ (and the second cop is at most one move away). The cops move so as to capture the image of the robber in G_{i-1}.

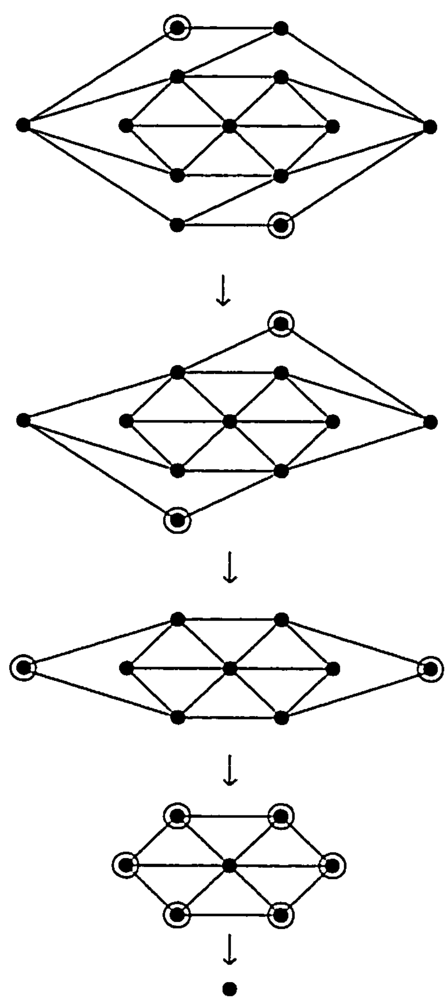


Figure 4.7: (a) An illustration of Theorem 4.2.3. The original graph is tandem-win.

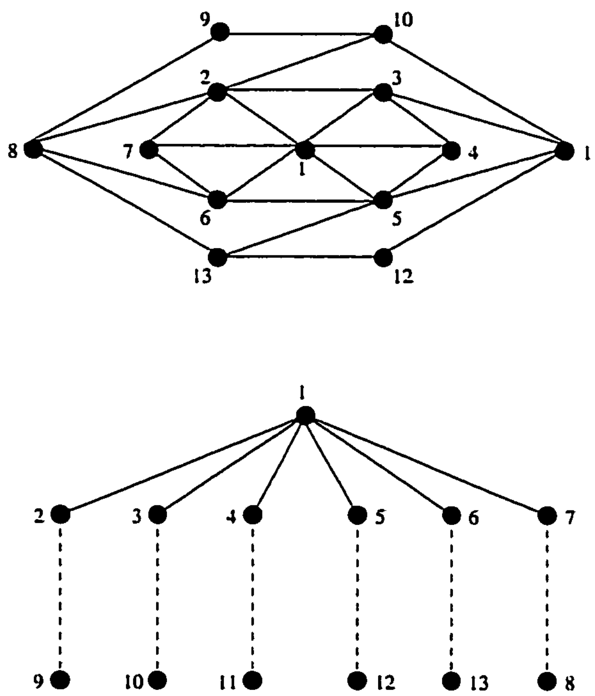


Figure 4.8: (b) The original graph and its tandem-win decomposition tree.

Theorem 4.2.4 *The Tandem-win Strategy is effective in capturing the robber.*

Proof. First note that for all i, f_i is an edge preserving map. Hence when the robber moves from x to y it follows that for all j, either $F_j(x) = F_j(y)$ or $F_j(x) \sim F_j(y)$. Also note that if the robber is on vertex x then for any i, $F_i(x)$ and $F_{i+1}(x)$ are either on the same vertex, adjacent vertices, or vertices distance two apart.

We prove the result by induction and consider the situation after the cops have moved. Cop 1 begins on vertex x_n, the vertex on which the cops' position coincides with the robber's image under the mapping $f_{n-1} \circ f_{n-2} \circ \cdots \circ f_2 \circ f_1$; that is, cop 1 begins on the same vertex as $F_n(x)$ (cop 2 begins on any adjacent vertex) and so the cops can pass. Suppose for some $i \leq n$ the cop has captured $F_i(y)$, where y is the robber's position, and it is the robber's turn to move. Suppose he moves to vertex z. (Assume cop 1 is on $F_i(y)$ and cop 2 is on an adjacent vertex.)

If $F_i(y) = F_{i-1}(y)$ then $F_{i-1}(z) \simeq F_i(y)$ and cop 1 can move immediately to capture the image in G_{i-1}. Cop 2 moves to $F_i(y) = F_{i-1}(y)$.

If $F_i(y) \sim F_{i-1}(y)$ then $F_{i-1}(y)$ is an irreducible vertex that is removed from G_{i-1} to obtain G_i and so $N[F_{i-1}(y)] \subset N[F_i(y)]$ and therefore $F_i(y) \simeq F_{i-1}(z)$. Again cop 1 can move immediately to capture the image in G_{i-1}. Cop 2 can move to $F_i(y)$ to maintain distance at most one from cop 1.

If $d(F_i(y), F_{i-1}(y)) = 2$ then $F_{i-1}(y)$ is a nearly irreducible vertex that is removed from G_{i-1} to obtain G_i and so $N(F_{i-1}(y)) \subset N(F_i(y))$. If $F_{i-1}(y)$ and $F_{i-1}(z)$ are distinct then $F_i(y) \sim F_{i-1}(z)$. Again cop 1 can move immediately to capture the image in G_{i-1}. Cop 2 can move to $F_i(y)$ to maintain distance at most one from cop 1. If $F_{i-1}(y) = F_{i-1}(z)$ then the robber's image is on a nearly irreducible vertex in G_{i-1}, cop 1 is on $F_i(y)$ (a nearly dominating vertex here), cop 2 is adjacent to cop 1, and it is the cops' move. Cop 1 moves to any $w \in N(F_i(y)) \cap N(F_{i-1}(y))$ and cop 2 moves to $F_i(y)$ (cop 1's previous position). As shown in the preamble at the beginning of this section, the cops will capture the robber's image F_{i-1} after the robber's next move.

Thus, in all cases, the robber's image can be caught in at most two moves in the larger graph.

Since there are only a finite number of subgraphs G_i, the robber's image will coincide with his actual position after a finite number of moves. Hence the strategy presented here will result in a win for the cops in a finite number of moves. □

It has been shown that if the cops are playing on the subgraph G_i, and are occupying the robber's image under the mapping F_i, then the cops are able to move onto the robber's image in G_{i-1} under F_{i-1}. If the cops are playing on the subgraph G_i, the robber can never move to a vertex in this subgraph without being immediately apprehended or apprehended on the next move by the cops. Equivalently, the robber cannot avoid capture by moving onto vertices used previously by the cops.

Theorem 4.2.5 *Suppose the cops are playing the Tandem-win Strategy in the subgraph G_i, and are occupying the robber's image under the mapping F_i. The robber can never move to a vertex of G_i without one of the cops being on or immediately landing on the same vertex.*

Proof. Suppose the cops are playing on the subgraph G_i, and are occupying the robber's image under the mapping F_i. The cops are able to move so as to stay with the image of the robber on this subgraph. Now the mapping F_i is the identity on G_i. Hence if the robber moves to a vertex of G_i, his image will correspond to his actual position and he will be apprehended. □

As mentioned earlier, not all tandem-win graphs have a tandem-win tree decomposition. We now investigate a class of graphs that does.

A graph G is said to be **triangle-free** if it does not have K_3 as a subgraph. We have one characterization of triangle-free tandem-win graphs.

Theorem 4.2.6 *Let G be a triangle-free graph. Then G is tandem-win if and only if there is an ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertices of G such that for each $i < n$, v_i is irreducible or nearly irreducible in the subgraph induced by the vertices in the set $\{v_i, v_{i+1}, \ldots, v_n\}$.*

Proof. If there is a tandem-win ordering of G then G is tandem-win by Theorem 4.2.3. So we need only prove the converse. Note that no retraction introduces a triangle. Consider the final move in a game played on a tandem-win graph G as shown in Figure 4.9. Since the robber is captured during the cops' next move if he passes, the robber's position R must be adjacent to exactly one of cop 1's position, C_1 and cop 2's position, C_2. Note that if R is adjacent to both C_1 and C_2, then C_1, C_2, and R form a triangle in G. We assume R is adjacent to C_1. Since the robber is captured during the cops' next move if he moves, $N(R) \subseteq N(C_2)$. Note that $N(C_2)$ could be $\{C_1\}$ so that R is on a leaf and C_2 nearly dominates R. (If the robber moves to C_1, he is immediately captured.) Note also that if $x \in N(R) - C_1$ then $x \perp C_1$ since otherwise C_1, R, and x form a triangle in G. Hence R is a nearly irreducible vertex with nearly dominating vertex C_2. Theorem 4.2.2 guarantees that $G - R$ is tandem-win. Again note that $G - R$ is triangle-free. The result follows recursively. Notice that all of the vertices v_i of G are nearly irreducible in the subgraph induced by $\{v_i, v_{i+1}, \ldots, v_n\}$ with the exception of v_{n-1} which is adjacent to v_n and irreducible in the subgraph induced by $\{v_{n-1}, v_n\}$. □

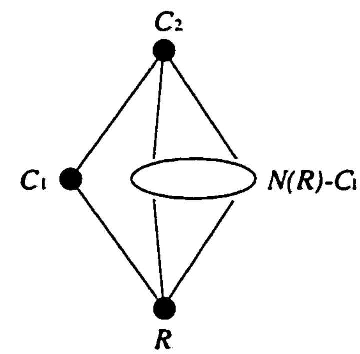


Figure 4.9: The final move in a game played on a triangle-free tandem-win graph.

More can be deduced about the structure of triangle-free tandem-win graphs.

Lemma 4.2.2 *Let G be a triangle-free graph. If there is an odd cycle in G then retraction of a nearly irreducible vertex onto a nearly dominating vertex leaves an odd cycle.*

Proof. Let u be a nearly irreducible vertex in G with nearly dominating vertex v. Suppose an odd cycle C includes u but not v, $C = \{c_1, c_2, \ldots, c_i, u, c_{i+1}, \ldots, c_n, c_1\}$ say. Then after retracting u onto v, there remains an odd cycle $C' = \{c_1, c_2, \ldots, c_i, v, c_{i+1}, \ldots, c_n, c_1\}$ since $N(u) \subseteq N(v)$. See Figure 4.10.

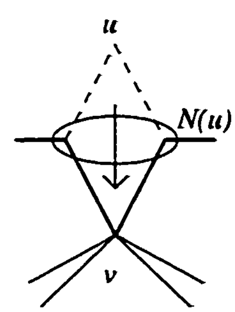


Figure 4.10: A portion of the cycle C' is indicated by bold lines. The dashed lines indicate the change from the cycle C after retraction of u onto v.

Suppose now an odd cycle C includes both u and v, $C = C_1 \cup C_2 \cup \{u\}$ where C_1 is a cycle including both v and x, C_2 is a cycle including both v and y, and $x, y \in N(u)$. (Recall that $N(u) = N(u) \cap N(v)$.) Since C is odd, one of C_1 and C_2, say C_1 must be odd. Then after retracting u onto v, there remains an odd cycle C_1. Note that this cycle cannot be a triangle. See Figure 4.11.

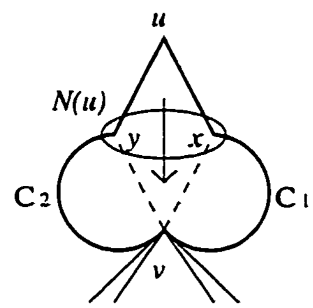


Figure 4.11: The cycle C is indicated by bold lines. The potential shortcuts are indicated by dashed lines.

□

Theorem 4.2.7 *Let G be a triangle-free tandem-win graph. Then G is bipartite.*

Proof. Consider the last subgraph in the tandem-win decomposition tree of G with an odd cycle. But if there exists an odd cycle, then retraction of a nearly irreducible vertex leaves an odd cycle. Hence G contains no odd cycles and is therefore bipartite.

□

The ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set of a graph G is an **elimination ordering** [4, 6] satisfying property P if for all $i \in \{1, 2, \ldots, n\}$, the vertex v_i has property P in the subgraph induced by the vertices $\{v_i, v_{i+1}, \ldots, v_n\}$. A vertex ordering $(v_1, v_2, \ldots, v_n)$ is a **domination elimination ordering** [4, 6] if for all $i \in \{1, 2, \ldots, n-1\}$, there is a $j < i$ such that the vertex v_i is dominated by v_j in G_i; that is, $N_i(v_i) \subseteq N_i[v_j]$.

Notice that tandem-win orderings are domination-elimination orderings, as are copwin orderings with the added restriction that $v_i \sim v_j$ in G_i.

There is considerable discussion of elimination orderings in the literature. See for example [4] and [6]. However none of these orderings provide a characterization of tandem-win graphs.

Recall that a graph is *bridged* if all isometric cycles have length 3. A **chord** in a graph is an edge joining two nonconsecutive vertices of a path or cycle. A graph is said to be **chordal** if it has no induced cycles of length at least 4. Finally a graph is said to be **chordal bipartite** if it is bipartite and any induced cycle of length at least 6 has a chord.

Recall from Section 1.2.5 that bridged graphs are copwin, but the converse does not hold. Because of the close relationship between bridged graphs, chordal graphs and chordal bipartite graphs, it is natural to question whether tandem-win graphs are chordal bipartite. However this is not the case. Consider the graph G shown in Figure 4.12 with tandem-win ordering $\{1,3,2,8,9,7,10,4,5,11,6\}$. Now G is not chordal bipartite since the outer 8-cycle does not have a chord. We have not yet determined if the converse holds.

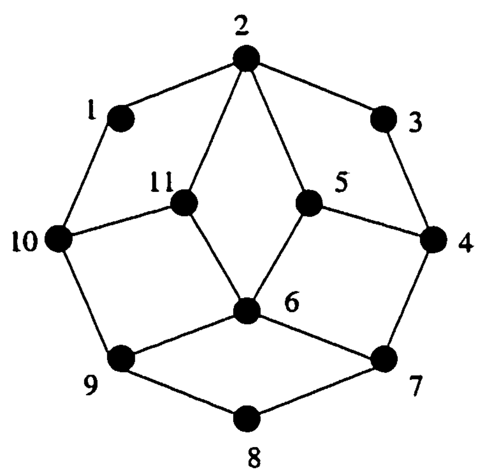


Figure 4.12: A tandem-win graph that is not chordal bipartite.

A house H is a 5-cycle with a chord as shown in Figure 4.13. A graph is **house-free** is if contains no subgraph isomorphic to a house.

Many elimination orderings considered in the literature are for house-free graphs.

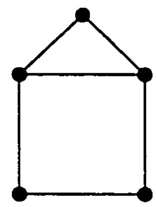


Figure 4.13: A house.

See for example [6]. But a house is itself tandem-win. The graph shown in Figure 4.9 has also been shown to be tandem-win, yet is not house-free.

We define a *tandem* of cops as a pair of cops playing in tandem, and the **tandem number** of a graph G, $T(G)$ as the number of tandems required to guarantee a win on G. Hence if H is a tandem-win graph, then $T(H) = 1$.

In [18] the authors give several results concerning the strong, Cartesian and categorical products of copwin graphs. We present some results regarding tandem-win graphs and these three products. Compare with the results of Neufeld and Nowakowski [18].

Recall from Theorem 1.2.5 that the strong product of a finite number of copwin graphs is copwin.

Theorem 4.2.8 *The strong product of a copwin graph G and a tandem-win graph H is tandem-win.*

Proof. Let G be a copwin graph and let H be a tandem-win graph. We will show that $G \boxtimes H$ is tandem-win. Let h be the projection map from $G \boxtimes H$ onto H, and let g be the projection map from $G \boxtimes H$ onto G. For all $x \in V(G)$, let $x \cdot H$ be the subgraph of $G \boxtimes H$ whose vertices have first coordinate x. Thus if both cops are located on $x \cdot H$, then they project to the same image, x under the map g. So the cops first play on $G \boxtimes H$ so that after each move their positions project to the same image on G. Since G is copwin, the image of the robber is captured by both cops on G using the Copwin Strategy. The cops then play a composition of moves so that they stay with the image of the robber under g and play the Tandem-win Strategy on H under h. Since two cops playing in tandem have a winning strategy, $G \boxtimes H$ is tandem-win. $\square$

Example. This example refers to Figure 4.14. For $j \in \{0,1,2,3\}$, the corner $(0,j)$ can be retracted onto the dominating vertex $(1,j)$, and then the corner $(1,j)$ can be retracted onto the dominating vertex $(2,j)$. This leaves the 4-cycle given by the vertices $(2,k), k \in \{0,1,2,3\}$ which we know is tandem-win.

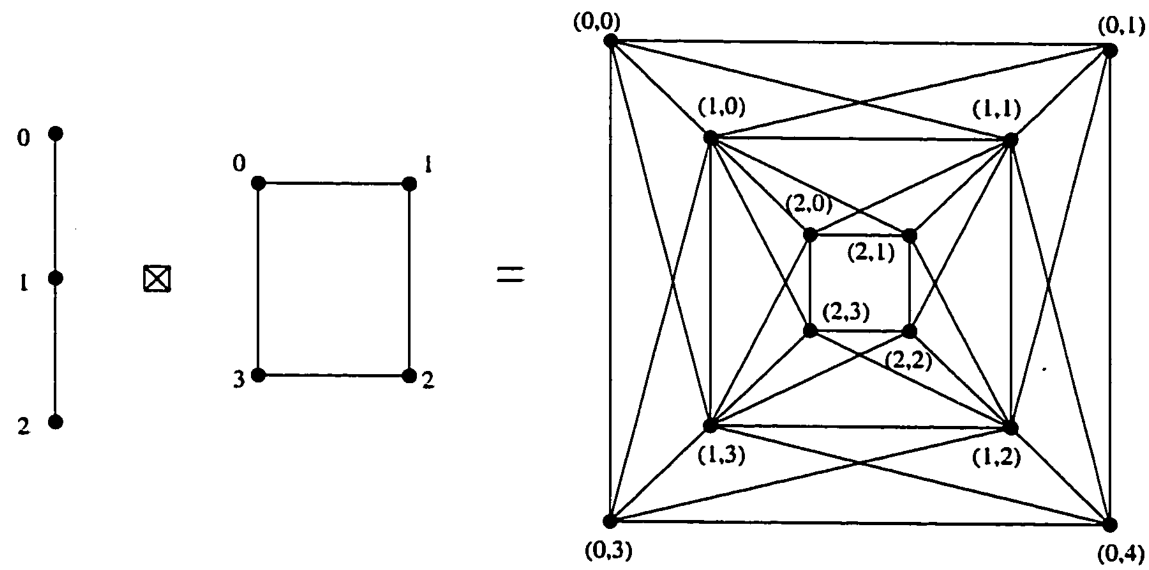


Figure 4.14: An illustration of Theorem 4.2.8.

Note that the strong product of a tandem-win graph and a finite collection of copwin graphs is tandem-win due to Theorem 1.2.5.

Tŏsić [28] proved that if G and H are graphs with copnumbers $c(G)$ and $c(H)$ then $c(G \square H) \leq c(G) + c(H)$. It follows that $c(\square_{i=1}^{n} G_i) \leq \sum_{i=1}^{n} c(G_i)$. Neufeld and Nowakowski [18] proved that if $G = \square_{i=1}^{n} C_i$ and $H = \square_{i=1}^{m} T_i$ where $C_1, C_2, \ldots, C_n$ are cycles of length at least 4 and $T_1, T_2, \ldots, T_m$ are trees, then $c(G \square H) = n + \lceil \frac{m+1}{2} \rceil$.

Theorem 4.2.9 *The Cartesian product of a tree T and a copwin graph G is tandem-win.*

Proof. We show that two cops playing in tandem have a winning strategy on $T \square G$. This strategy has two phases. During the first phase, the cops move to capture the robber's image on $T \cdot v$, for some $v \in V(G)$. Consider the second phase. (a) If the robber moves from a vertex in $T \cdot u$ to a vertex in $T \cdot u$, $u \in V(G)$ and the result of this move is that at least one of the cops is no longer on the robber's image, then the cops move so as to capture the image. (b) If the robber moves from a vertex in $T \cdot u$ to a vertex in $T \cdot u$ and at least one of the cops remains on the robber's image, then the cops are able to move according to their winning strategy in G. (c) Finally if the robber moves from $T \cdot u$ to $T \cdot w$, $u, w \in V(G)$ then the cops remain on the robber's image and are then able to move according to their winning strategy in G.

Note that the robber can play a move of type (a) only a finite number of consecutive times since he will eventually come to a leaf. If the robber only plays moves of type (c), he will be captured by the Copwin Strategy on G. If the robber plays a move of type (b), he has passed on G and given the cops a free move on G. The robber can then play finitely many moves of type (a), all passes in G for the cops, before having to play a move of type (b) or (c), a resumption of the game on G. □

In [18] it is proven that the copnumber of the categorical product of two copwin graphs is at most 3, and more generally, if G and H are connected, non-bipartite graphs with $c(H) \geq c(G)$ and $c(H) \geq 2$, then $c(G \times H) \leq 2c(G) + c(H) - 1$.

Let G and H be triangle-free tandem-win graphs, each having at least one cycle. Suppose there exists a 'special' tandem-win decomposition by nearly-irreducible vertices such that (1) any leaves appear at the beginning of the tandem-win ordering, and only then (2) the nearly-irreducible vertices are retracted.

Note that (a) this decomposition does not introduce any leaves and (b) the subgraph formed by the last four vertices in the decomposition is a 4-cycle.

Lemma 4.2.3 *Let G be a triangle-free tandem-win graph with at least one cycle. If G has a 'special' tandem-win decomposition then the subgraph formed by the last four vertices in this decomposition is a 4-cycle.*

Proof. If not, consider the retraction from a subgraph G' in the tandem-win decomposition to a subgraph G'' in the decomposition which resulted in a tree. Let the nearly irreducible vertex be x with nearly dominating vertex y. By (1) and (a), there are no leaves. So G' contains a cycle but G'' does not. Now any cycle in G' must include x since G'' is a tree. Therefore G' is isomorphic to $K_{2,m}$, $m \geq 2$. Suppose the independent sets are $\{1, 2\}$ and $\{c_1, c_2, \ldots, c_m\}$. We can alter the decomposition to first eliminate members of the set $\{c_1, c_2, \ldots, c_m\}$ until a subgraph isomorphic to $K_{2,2}$ remains. (See Figure 4.15 when $m = 6$.) □

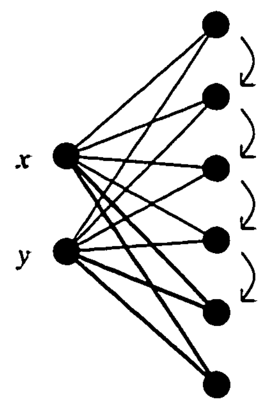


Figure 4.15: $K_{2,6}$ reduces to $K_{2,2}$.

Let $V(G) = \{a_1, a_2, \ldots, a_{g+4}\}$ have the special decomposition ending in a 4-cycle so that $G = a_1 \cup a_2 \cup \cdots \cup a_g \cup C_4$. Similarly $H = b_1 \cup b_2 \cup \cdots \cup b_h \cup C_4$ where $V(H) = \{b_1, b_2, \ldots, b_{h+4}\}$ has the special decomposition ending in a 4-cycle.

Lemma 4.2.4 *Let G and H be triangle-free tandem-win graphs, each having at least one cycle. If y nearly dominates x in H ($y \perp x$) and G has the special tandem-win decomposition, then $G \times H$ reduces to $G \times (H - x)$ by retraction of nearly-irreducible vertices.*

Proof. Inductively suppose a_i is nearly dominated by $c_i \in G \setminus \cup_{j<i} a_j$. If $(b, z) \in N((a_i, x))$, then $b \sim a_i$ and $z \sim x$. It follows that $b \sim c_i$ and $z \sim y$. Thus $(b, z) \sim (c_i, y)$ or equivalently $N((a_i, x)) \subseteq N((c_i, y))$ in $(G \setminus \cup_{j<i} a_j) \times H$. □

Theorem 4.2.10 *Let G and H be triangle-free tandem-win graphs, each having at least one cycle. If G and H have special tandem-win decompositions then $T(G \times H) = 2$.*

Proof. Apply Lemma 4.2.4 to the graph $G \times H$ to obtain $G \times C_4$, and again to obtain $C_4 \times C_4$. Now $C_4 \times C_4$ reduces to $e \times C_4$, where $e \in E(G)$ with another application. Finally notice $e \times C_4 \cong 2C_4$. □

It has been shown in Theorem 1.2.5 that the strong product of a finite collection of copwin graphs is also copwin. An analogous result for the copnumbers of graphs due to Neufeld and Nowakowski [18] bounds the copnumber of the strong product of two graphs in terms of the individual copnumbers of these graphs.

Let G and H be graphs. Then $c(G \boxtimes H) \leq c(G) + c(H) - 1$.

Theorem 4.2.11 *Let G_i, $i = 1, 2, \ldots, n$ be a finite collection of graphs with $T(G_i) = 1$ for all i. Then $T(\boxtimes_{i=1}^{n} G_i) \leq 2^{n-1}$.*

Proof. Consider the projections of $\boxtimes_{i=1}^{n} G_i$ onto the G_i, $i = 1, 2, \ldots, n$ and we will assign the cops' positions so that the projections of all the cops lie on a single edge in each G_i.

Consider the following assignments. The cops' positions in $\boxtimes_{i=1}^{n} G_i$ are (c_{2i}, c_{2i+1}), $i = 0, 1, \ldots, 2^{n-1} - 1$, the 2^{n-1} tandems. We will represent (c_{2i}, c_{2i+1}) by $((2i)_2, \overline{(2i)_2})$ where $(2i)_2 = b_1 b_2 \cdots b_n$ is the base 2 representation of $2i$, $\overline{(2i)_2}$ is the complement of $(2i)_2$, and leading zeros are permitted. Note that the $(2i)_2$ are distinct as are the $\overline{(2i)_2}$. Hence $(2i)_2$ and $\overline{(2i)_2}$, $i = 1, \ldots, 2^{n-1} - 1$ exhaust all of the integers m where $1 \leq m < 2^n - 1$. We consider $(c_0, c_1) = (00 \cdots 0, 11 \cdots 1)$ as a reference pair.

Thus if in $(2i)_2 = b_1 b_2 \cdots b_n$, $b_j = 0$ then c_{2i} is projected to the same position on G_j as c_0. Otherwise c_{2i} is projected to the same position on G_j as c_1.

The cops follow the robber on each projection until he is captured on all n projections. It must be shown that the robber has been captured on $\boxtimes_{i=1}^{n} G_i$.

Consider the binary number $R = x_1x_2\cdots x_n$, again with leading zeros permitted, with $x_i = 0$ if the projection of the robber on G_i is captured by cop c_0 and $x_i = 1$ otherwise. Consider now the pair (c_{2i}, c_{2i+1}) which has $(2i)_2 = x_1x_2\ldots x_n$ or $\overline{(2i)_2} = x_1x_2\cdots x_n$.

Suppose first $(2i)_2 = x_1x_2\cdots x_n$. If $x_j = 0$ then R projects onto c_0 but c_{2i} is on c_0 in G_j. If $x_j = 1$ then R projects onto c_1 but c_{2i} is on c_1 in G_j. Hence c_{2i} captures the robber on $\boxtimes_{i=1}^{n} G_i$. Similarly c_{2i+1} captures the robber on $\boxtimes_{i=1}^{n} G_i$ if instead $\overline{(2i)_2} = x_1x_2\cdots x_n$. □

Example. When $n = 3$, $T(\boxtimes_{i=1}^{n} G_i) \leq 2^{n-1} = 4$. In Table 4.1, the cops' positions are shown in pairs, or tandems along with the corresponding base 2 representations. The projections onto each of G_1, G_2 and G_3 are also indicated.

	projection onto		
	G_1 ↓	G_2 ↓	G_3 ↓
c_0	0	0	0
c_1	1	1	1
c_2	0	1	0
c_3	1	0	1
c_4	1	0	0
c_5	0	1	1
c_6	1	1	0
c_7	0	0	1

Table 4.1: $T(\boxtimes_{i=1}^{3} G_i) \leq 4$.

Consider $G_j \cong C_4$ for $j = 1, 2, \ldots, 2n$. We will show $T(\boxtimes_{i=1}^{2n} C_4) > n$. Suppose n tandems of cops choose their vertices. The robber then chooses a vertex so that on G_j, the robber's projection is two away from the projection of cop c_j. In one move, no cop can capture the robber on all the projections and thus not on $\boxtimes_{i=1}^{2n} C_4$. Thereafter, the robber moves to maintain these distances. Hence $n < T(\boxtimes_{i=1}^{2n} C_4) \leq 2^{2n-1}$. See Figure 4.16.

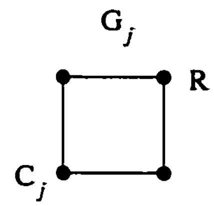


Figure 4.16: The robber is not captured on $\boxtimes_{i=1}^{2n} C_4$.

The notion of a tandem of cops can be generalized to a clique of cops so that instead of two cops who are required to be adjacent, or on a clique K_2 after each move, the cops are required to be on a clique after each move.

Define a **clique of cops** on a graph G as a group of k cops occupying a subgraph K_n of G where $k \geq n$. The cops can change position from a clique C_1 to a clique C_2 if for all $x \in C_1$, there exists $y \in C_2$ such that $x \simeq y$. That is, the cops move from one clique to another and expand to cover the vertices of the new clique.

A graph on which a clique of cops is sufficient to win is said to be **clique-win**.

If one wishes to generalize the notion of a copwin graph, what is needed is the notion of a clique-win graph. The proof of Theorem 1.2.1 can then be generalized to show that a retract of a clique-win graph is clique-win, and the proof of Theorem 1.2.5 can be generalized to show that the strong product of a finite number of clique-win graphs is clique-win. It then follows that the class of clique-win graphs is a variety.

Chapter 5

Restrictions on the Robber and a Summary of Open Questions

In the first several sections of this chapter, we are again considering graphs with copnumber greater than 1. Here the movements of the robber are restricted. Compare with Chapter 3 where the cops are restricted to moving on assigned beats or subgraphs. The final section discusses future work.

Let $c_R(G; H, K)$ denote the copnumber of the graph G when the cops are restricted to moving on a subgraph H of G and the robber is restricted to moving on a subgraph K of G. As mentioned, this restricted version of the game has a flavour similar to that of the precinct version introduced previously. As well, in [19], Neufeld and Nowakowski investigate the game of Cops and Robber when the opposing sides move on disjoint sets of edges. They consider two situations. The first has the cops playing on the edges of a given graph G and the robber playing on the complementary edges; that is $H = G$ and $K = \overline{G}$. The second has the opponents restricted to playing on disjoint subsets of edges which are defined in terms of Cartesian and categorical products of graphs.

This chapter has two main sections and a section of open questions at the end. In the first, we consider a translation of the problem of determining if a graph has copnumber k for $k \geq 2$. In the second, we consider a translation of the problem of determining if a graph is tandem-win. Both translations reduce the cop side to a single cop.

5.1 Graphs with Copnumber at Least 2

Suppose a game is played on an arbitrary graph G with $|V(G)| = v$. Now suppose $c(G) = 2$ so that two cops, cop 1 and cop 2 say, playing on G have a winning strategy in n moves. We would like to be able to translate this strategy so that a single cop playing on $G^2_\otimes$ can use it to guarantee a win. If the positions of cops 1 and 2 on G are x_i and y_i, $i \in \{1, 2, \ldots, n\}$ then the position of the cop on $G^2_\otimes$ is (x_i, y_i). We must first determine which edges must be included in the product $\otimes$.

Consider the robber's sequence of moves on G, $r_1, r_2, \ldots, r_{n-1}$, say. We associate these moves with the moves $(r_1, r_1), (r_2, r_2), \ldots, (r_{n-1}, r_{n-1})$ on $G^2_\otimes$. Hence to ensure that the robber is able to play this sequence of moves in $G^2_\otimes$ under this association, we require that $\otimes \geq \times$. Recall the partial order on the products from Section 3.1. See Figure 3.3.

Suppose that on G, cop 1 moves from x_i to x_j and cop 2 from y_i to y_j. On $G^2_\otimes$, we would like the cop to be able to move from (x_i, y_i) to (x_j, y_j). Note that since a cop is permitted to pass during a turn, x_i and x_j, and y_i and y_j need not be distinct. Hence to guarantee that such a move is possible on $G^2_\otimes$, we require that $\otimes \geq \boxtimes$.

Consider now the final moves made by cops 1 and 2 on G. Recall that only one of the cops must occupy the same vertex r_{n-1} as the robber to win. Suppose cop 1 captures the robber so that cop 1's final move is from x_{n-1} to r_{n-1}. To ensure that the cop playing on $G^2_\otimes$ is able to capture the robber there, we require that (x_{n-1}, y_{n-1}) is adjacent to (r_{n-1}, r_{n-1}). Hence we require that $\otimes \geq \boxtimes^c$. But notice that the cop playing on $G^2_\otimes$ only uses edges $e \in E[(G \boxtimes^c G) \setminus (G \boxtimes G)]$ during his final move. All other moves are made along edges of $G \boxtimes G$. This leads to the following definition of a restricted game.

Restrictions R**:** Suppose a game is played on $G \boxtimes^c G$ where G is an arbitrary graph with $|V(G)| = v$. The robber is restricted to moving on the subgraph induced by the diagonal or equivalently the vertices (x_i, x_i), $i = 1, 2, \dots, v$. The cops are free to move among any of the vertices. However they can only move along edges included in $G \boxtimes G$ except during a move that immediately results in the robber's capture.

Recall that $c_R(G)$ is the copnumber of the graph G under the restrictions R.

Theorem 5.1.1 *Let G be a graph. Then $c(G) = 2$ if and only if $c_R(G \boxtimes^c G) = 1$ where R is the set of restrictions indicated in the preamble to the theorem.*

Proof. Let $f_1 : V(G \boxtimes^c G) \to V(G)$ be the map defined by $f_1((x, y)) = x$, and $f_2 : V(G \boxtimes^c G) \to V(G)$ by $f_2((x, y)) = y$; that is, f_1 and f_2 are the projection maps.

Suppose $c(G) = 2$. Then two cops have a winning strategy on G. The positions of the two cops on move i are x_i for cop 1 and y_i for cop 2. The robber is on r_{i-1}. The single cop on $G \boxtimes^c G$ is on (x_i, y_i). When the robber moves on the diagonal on $G \boxtimes^c G$ to (r_i, r_i), this translates to moving on G from r_{i-1} to r_i. A move for the cops on G to x_{i+1} and y_{i+1} respectively translates on $G \boxtimes^c G$ to moving to (x_{i+1}, y_{i+1}) on the edges of $G \boxtimes G$. When one of the cops actually captures the robber on G, say x_j and $y_j = r_{j-1}$, then the winning move on $G \boxtimes^c G$ is from (x_{j-1}, y_{j-1}) to (y_j, y_j).

Suppose $c_R(G \boxtimes^c G) = 1$. Then the projection maps give the moves for the two cops on G except for the winning move. Let this move be (x_{j-1}, y_{j-1}) to (y_j, y_j), $y_j = r_{j-1}$, $y_j \sim y_{j-1}$. Then the moves on G are x_{j-1} to x_{j-1} and y_{j-1} to $y_j = r_{j-1}$. □

It is clear that this result can be generalized so that if G is a graph then $c(G) = k$ if and only if $c_R(\boxtimes^c G^k) = 1$ where R is the set of restrictions indicated in the preamble to Theorem 5.1.1.

5.2 Tandem-win Graphs

Suppose a game is played on an arbitrary graph G and consider the question of whether the given graph is tandem-win. We introduce a translation of the problem so that we need only consider whether one cop, whose movements are restricted, can win on a related graph G^*.

The idea here is that since two cops playing in tandem must always be located on adjacent vertices, we can transform the graph G to a graph G^* where these edges become vertices. Then a single cop can replace the tandem of cops, and is restricted to playing on these vertices except to move to capture the robber. The new graph also retains the vertices of G, and the robber is restricted to playing on these vertices of G in G^*.

More formally, consider a graph G and define a new graph $G^* = (V(G^*), E(G^*))$ where $V(G^*) = V(G) \cup E(G)$ and $(p, q) \in E(G^*)$ if any of the following hold:

(a) $p, q \in V(G)$ and $p \sim q$,

(b) $p, q \in E(G)$ and p and q share a common endpoint,

(c) $p, q \in E(G)$ and the subgraph induced by the endpoints of p and q is a 4-cycle,

(d) $p = p_1p_2 \in E(G)$, $q \in V(G)$, and $q \sim p_1$ or $q \sim p_2$.

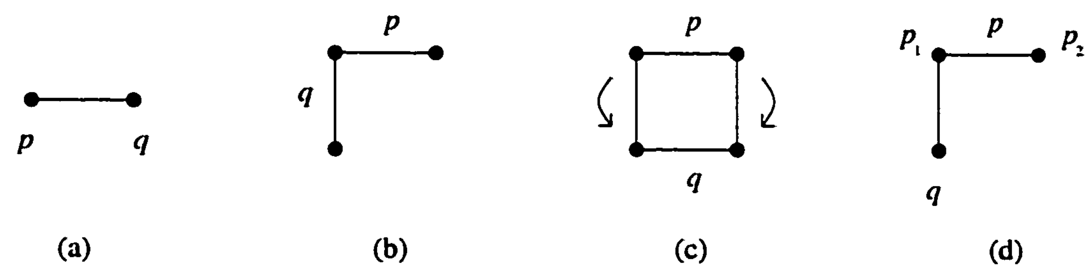


Figure 5.1: Illustrations of conditions (a), (b), (c), and (d).

Restrictions R: On the graph G^*, the robber is restricted to moving on $V(G)$ and the cop is restricted to moving on $E(G)$ except for a move that immediately results in the robber's capture. The graph G^* is shown in Figure 5.2. We have proven the following result.

Theorem 5.2.1 *Let G be a graph with corresponding transformed graph G^*. Then $T(G) = 1$ if and only if $c_R(G^*) = 1$.*

Example. A graph G and the transformed graph G^* are shown in Figure 5.3.

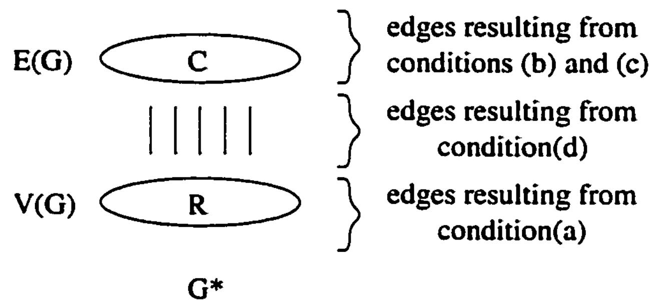


Figure 5.2: The transformed graph G^*.

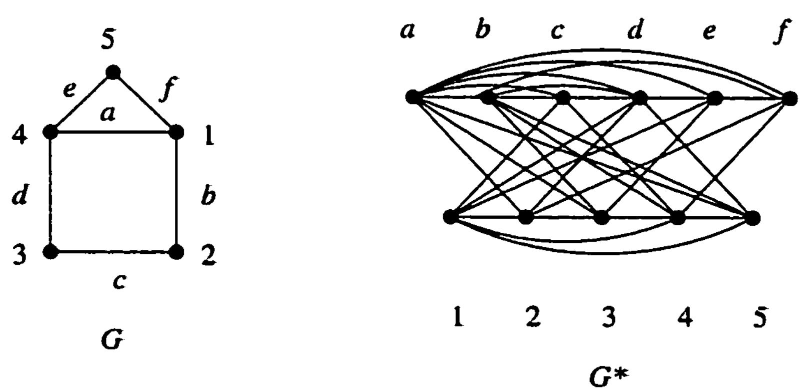


Figure 5.3: A graph G and the transformed graph G^*.

5.3 Open Questions

1) In Chapter 2, bounds were placed on the photo radar number and video camera number of a copwin graph. Bounds were also placed on the alarm number and edge-alarm number of a tree. As mentioned at the end of Section 2.3, we would like to bound the alarm number, as well as the edge-alarm number of an arbitrary copwin graph. However, schemes very different from those used in the proofs of Theorems 2.1.3 and 2.2.5 are needed because of the lack of a directional signal given by the alarms.

Find similar bounds for tandem-win graphs and, in general, for graphs with copnumber $k \geq 2$.

2) Suppose the number of photo radar units, video cameras, or alarms is fixed. How many cops are required to guarantee a win on an arbitrary graph G?

3) In Chapter 3, we have attempted to determine $\rho(G, \otimes)$ for $\otimes \in \{\times, \times^c, \square, \square^c, \bullet, \bullet^c, \boxtimes, \boxtimes^c, ⓔ, \nabla\}$. For the products $\oplus \in \{\times, \bullet, \bullet^c\}$, we have only bounded $\rho(G, \oplus)$ and thus the problem of determining exact values, if they exist, remains open. As well for all v, the value of $\rho(K_v, \nabla)$ has not yet been determined exactly.

For arbitrary graphs G and H, we have bounded the copwin number of $G \otimes H$ for $\otimes \in \{\times, \times^c, \square, \square^c, \bullet, \bullet^c, \boxtimes, \boxtimes^c, ⓔ, \nabla\}$. The problem of determining exact values for $G \otimes H$ remains open.

4) Although we have taken steps toward the characterization of copnumber 2 graphs, the problem remains open. With regard to tandem-win graphs, we have been able to characterize triangle-free tandem-win graphs. However there is more work to be done, in particular with generalizations of chordal bipartite graphs and with elimination orderings on graphs with houses.

5) It is known from the Tandem-win Strategy that a finite number of moves are required to catch the robber on a triangle-free tandem-win graph. How many tandems/ cops are required to catch the robber in time t? (There would be a fixed resource,

such as gas, for the chase.) We would like to answer this question for an arbitrary graph G.

6) Determine the copnumber $c_R(G; H, K)$ of a graph G when the cops are restricted to moving on a subgraph H of G and the robber is restricted to moving on a subgraph K of G. Two such restricted games are introduced in this chapter. Some results when the opposing sides are restricted to moving on disjoint sets of edges are given in [19].

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