LINEARLY CONSTRAINED CONVEX PROGRAMMING OF ENTROPY TYPE: CONVERGENCE AND ALGORITHMS

By

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In memory of

my Mother,

who had devoted all her life

to her

work and family.

Abstract

The problem of estimating a (non-negative) density function, given a finite number of its moments, arises in numerous practical applications. By introducing an entropy-like objective function, we are able to treat this problem as an infinite-dimensional convex programming problem.

The convergence of our estimate to the underlying density is dependent on the choice of the objective. In this thesis, I studied the most commonly used classes of objectives, which include the Boltzmann-Shannon entropy, the Fermi-Dirac entropy, the truncated L_p-entropy. First, I discussed the duality properties of the convex program (P_n), which involves only n moments, and gave theorems to estimate the bounds of the dual gaps. After proving a general necessary optimality condition and giving rates of norm convergence, I set up a set of uniform convergence theorems for certain choices of entropies, provided that the moment functions are algebraic or trigonometric polynomials.

In Chapter 4, I used Newton's method to solve the dual problem. I compared the computational results of the problem with various choices of entropies. For the problem with the Boltzmann-Shannon entropy, using a special structure among the moments, I have developed a set of very efficient algorithm. By using some additional moments, within much less time, we can find a very good estimate function to the underlying density by solving just a couple of linear systems. The algorithms have been implemented in Fortran. Some 2- and 3- dimensional examples have been tested. Since the algorithm is heuristic instead of iterative, some related error analysis has also been performed.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

We study **moment problems** which estimate an unknown density function $\bar{x}$, typically nonnegative, on the basis of a series of known moments. Such problems arise in a wide variety of settings. In constrained approximation, we need to reconstruct an unknown function from a set of known values of certain linear functionals (see, for example, [61], [83]). In spectral estimation, which has a lot of applications in speech processing, geophysics, radio astronomy, sonar and radar and many other areas, we are asked to estimate a power spectral density from certain known correlations (see, for instance, [6], [43], [44], [56], [71], [74], [75], [91], [92], [100]). An interesting class of problems in crystallography is to find out the electronic density of a given crystal on the basis of finitely many known measurements (see, for example, [41]). Many other applications in physics and engineering (such as tomography, signal process and restoration) can be found, for instance, in [31], [45], [57], [58], [70], [76], [82]. For a survey of the wide range of approaches to moment problem and its application, see [2], [73].

Mathematically, the problem we will study in this thesis can be stated as: find a function $\bar{x}$ (usually nonnegative or between some lower and/or upper bounds) in

$L_1(T,\mu)$ defined on a set T, satisfying

$$\int_T a_i(t)\bar{x}(t)d\mu(t) = b_i, \qquad i = 1, 2, \cdots, n, \tag{1.1}$$

where the a_i's are moment functions, normally in $L_\infty(T,\mu)$. For given (usually finitely many) moments, this problem is an underdetermined inverse problem (for existence of a solution see, for example, [3], [8], [111]). When a solution does exist, it is by no means unique. Introducing an objective function, the maximum entropy method seeks an optimal solution of a mathematical program with linear constraints ([72], [101], [103], [104]):

$$\begin{cases} \max & \int_T -\phi(x(t))d\mu(t), \\ s.t. & \int_T a_i(t)\bar{x}(t)d\mu(t) = b_i, \qquad i = 1, 2, \cdots, n, \\ & x(t) \geq 0, \qquad \text{for all } t \in T, \end{cases} \tag{1.2}$$

where ϕ is the **Boltzmann-Shannon entropy** defined by

$$\phi(u) = \begin{cases} \log u - u, & u > 0, \\ 0, & u = 0, \\ +\infty, & u < 0. \end{cases} \tag{1.3}$$

Under reasonable conditions on the moments, the optimal solution of problem (1.2) exists and is unique. In this paper, however, we prefer to minimize the corresponding information measure, which has a general integral form of

$$\int_T \phi(x(t))d\mu(t)$$

for a convex integrand function ϕ. This approach has been widely used in areas such as parameter spectral estimation (see, for example, [32], [80]).

In using this approach, various entropy-like objectives ϕ have been tried. Among them, the most popular ones are the Boltzmann-Shannon entropy (1.3) (suggested by Jaynes in [64], also discussed in [67]), Burg's entropy (suggested by Burg in [33] and [34]), the Fermi-Dirac entropy (see, for example, [24]), and the L_2 or L_p-entropy (see [5], [55], [69]). We will discuss them in the later chapters. We will also try some other entropies. For some other choices, such as the cross entropy, see [85], [90], [94].

In most practical sciences, the moment functions a_i's are typically trigonometric (Fourier) and algebraic (Hausdorff, power) polynomials, usually multidimensional.

A very important question arising in this optimization approach is: how will the estimates converge to the underlying measure as the number of given moments grows? This question has been discussed in many papers (see, for example, [40], [50], [52], [54], [82], [107], for a recent survey see [21]). Several concepts of convergence have then been used, such as, weak*-convergence and weak convergence ([18]), convergence in measure ([22], [79]), norm convergence ([16], [23], [79], [107]), and uniform convergence ([16], [28]), which will be further studied later in this thesis.

To numerically solve the problem, which is infinite dimensional, convex duality theory plays an important role. Using duality theorems, instead of considering the primal problem, we study its dual, which is a finite dimensional (often unconstrained) maximization problem. For a complete study of this duality theory, see Borwein and Lewis' papers (for example, [17], [19], [20]). Dual algorithms seem to be the most popular methods in published papers (see, for example, [4], [29], [41] and [102]).

In Chapter 2, we will discuss a class of entropy-like objectives. We will first give some lower and upper bounds on duality gaps, which will help us to prove norm convergence results. The main results proved in Chapter 3 are general versions of uniform convergence theorems for moment problems with entropy-like objectives under our assumptions, which include many well known entropies as special cases.

In Chapter 4, we will study numerical methods for moment problems. We first implement Newton's method with line search (for a detailed method description, see [42]), and compute the numerical solutions for several test problems in 2 and 3 dimensions with algebraic and trigonometric moment functions. We also compare the numerical results for various choices of entropies.

In the second part of Chapter 4, we will establish a class of heuristic algorithms for problems with the Boltzmann-Shannon entropy and algebraic or trigonometric polynomial moments. These algorithms provide surprisingly good estimates to $\bar{x}$ by just solving a set of linear systems. Numerical computations show our heuristics to be accurate and very fast although theoretical convergence is still an open problem.

In the rest of this chapter, some preliminary definitions are recalled and known results in convex analysis, mathematical programming and approximation theory are stated in the precise form which we are going to use in later chapters. The standard functional analytic terminology used throughout the thesis can mostly be found in [99].

1.2 Convex functions and normal convex integrands

Most of the statements in this section can be found in [39], [47], [60], [93], [95], [96], [97] and [98].

Let X be a real Banach space. A functional $f : X \to (-\infty, +\infty]$ is said to be **convex**, if for all $x, y \in X$, $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y). \tag{1.4}$$

The **domain** of f, $\mathrm{dom}(f)$, is the set of all points $x \in X$ where f is finite. We say f is **proper** if its domain is nonempty. When $X = I\!R^n$, we know that a convex function f is continuous on the interior of its domain (see [97]).

Let X^* be the topological dual space of X. The set of **subgradients** of f at $x_0 \in \mathrm{dom}(f)$ is defined to be

$$\partial f(x_0) \triangleq \{x^* \in X^* \mid \langle x^*, x - x_0 \rangle \leq f(x) - f(x_0), \quad \text{for all } x \in X\}. \tag{1.5}$$

It has the following properties.

Proposition 1.2.1 (Phelps, [93]) *If a convex function f is continuous at $x_0 \in \mathrm{dom}(f)$, then $\partial f(x_0)$ is a nonempty, convex, and weak*-compact subset in X^*.*

Proposition 1.2.2 (Phelps, [93]) *A continuous convex function f on a nonempty open subset $D \subseteq X$ has a global minimum at $x_0 \in D$ if and only if $0 \in \partial f(x_0)$.*

For a proper convex function f, the **convex conjugate** of f is defined as the functional $f^* : X^* \to (-\infty, +\infty]$ given by

$$f^*(x^*) = \sup_{x \in X}\{\langle x, x^*\rangle - f(x)\}. \tag{1.6}$$

Note that each item inside the "sup" is continuous linear functional on X, and hence f^* is always convex and lower semicontinuous. If we similarly write

$$f^{**}(x) = \sup_{x^* \in X^*}\{\langle x, x^*\rangle - f^*(x^*)\}, \tag{1.7}$$

for $x \in X$, then it is easy to see that

$$f^{**} \leq f. \tag{1.8}$$

Moreover, we have

Proposition 1.2.3 (Ekeland and Turnbull, [47]) *The conjugate function f^* is always convex and lower semicontinuous on X^*. If f is proper convex and lower semicontinuous, then we have*

$$f^{**} = f. \tag{1.9}$$

The **Fenchel-Young inequality** states that for any $x \in X$ and $x^* \in X^*$,

$$f^*(x^*) + f(x) \geq \langle x, x^*\rangle, \tag{1.10}$$

and the equality holds exactly for $x^* \in \partial f(x)$.

For $x \in \mathrm{dom}(f)$, $h \in X$, if the limit

$$\delta F(x; h) = \lim_{\alpha \to 0} \frac{1}{\alpha}[F(x + \alpha h) - F(x)] \tag{1.11}$$

exists, it is called the **Gateaux differential of F at x with increment h**. If the above limit exists for each $h \in X$, then F is said to be **Gateaux differentiable at x** [81].

If $\delta F(x; \cdot) : X \to I\!R$ is linear and continuous such that

$$\lim_{\|h\| \to 0} \frac{\|F(x+h) - F(x) - \delta F(x; h)\|}{\|h\|} = 0, \tag{1.12}$$

then F is said to be **Fréchet differentiable at** x.

It is obvious that the Fréchet differentiability implies the Gateaux differentiability but not usually vice versa. For a lower semicontinuous convex function on a Banach space, the Gateaux differential is always linear and continuous. We denote it by $F'(x)$ since it lies in X^*. Also we have

$$\partial F(x) = \{F'(x)\}. \tag{1.13}$$

In this case, the Fenchel-Young inequality becomes

$$\langle F'(x), x\rangle = F(x) + F^*(F'(x)). \tag{1.14}$$

We will use this property frequently in proving theorems in later chapters. When we consider a convex integrand ϕ defined on $I\!R^m$, we will be interested in having these functions be "smooth" and "convex" enough. A proper convex function $f : I\!R^n \to (-\infty, +\infty]$ is said to be **essentially smooth** if it satisfies the following three conditions:

(a) $int(\operatorname{dom}(f)) \neq \emptyset$;

(b) f is continuously differentiable on $int(\operatorname{dom}(f))$;

(c) $\lim_{i\to\infty} | \nabla f(x_i)| = +\infty$, whenever $\{x_i\}$ converges to a boundary point of $\operatorname{dom}(f)$.

Dually, f is said to be **essentially strictly convex** if f is strictly convex on every convex subset of

$$\operatorname{dom}(\partial f) \triangleq \{x \in X \mid \partial f(x) \neq \emptyset\}. \tag{1.15}$$

Proposition 1.2.4 (Rockafellar, [97]) *A proper convex and lower semicontinuous function $f : I\!R^m \to (-\infty, +\infty]$ is essentially strictly convex if and only if its conjugate f^* is essentially smooth.*

We now introduce the normal convex integral defined and studied by Rockafellar in [96] and [98]. Let T denote a complete measure space with a σ-finite measure dt, and let L be a particular space of measurable functions u from T to $I\!R$. In later use, we often assume L to be $L_p(T)$, for $1 \leq p \leq \infty$. By a **convex integrand**

$f : T \times \mathbb{R} \to (-\infty, +\infty]$, we mean the function $f(t, \cdot)$ to be convex for each $t \in T$. We can then define

$$I_f(u) = \int_T f(t, u(t))dt, \qquad \text{for } u \in L. \tag{1.16}$$

The function f is called a **normal convex integrand** if

(a) for each fixed t, the function f is proper convex and lower semicontinuous in the second variable;

(b) there exists a countable collection U of measurable functions u on L having the following properties:

- for each $u \in U$, $f(\cdot, u(\cdot))$ is measurable;
- for each t, $U_t \cup D_t$ is dense in D_t, where

$$U_t \triangleq \{u(t) \mid u \in U\}, \qquad D_t \triangleq \{x \in \mathbb{R}^n \mid f(t, x) < +\infty\}.$$

Some easier-to-check conditions for f to be a normal convex integrand have also been given in [96] and [98], such as:

Theorem 1.2.5 (Rockafellar, 1968, [96]) *A function f is a normal convex integrand if one of the following is true:*

(a) For a lower semicontinuous proper convex function F on $\mathbb{R}$, $f(t, x) \equiv F(x)$.

(b) For each fixed x, $f(\cdot, x)$ is measurable. For each fixed t, $f(t, \cdot)$ is lower semicontinuous, convex and the interior of its domain is nonempty.

In discussing Fenchel duality, we need to know the expression of the conjugate of I_f defined in (1.16). For a function space L, we say it is **decomposable** if it satisfies the following conditions:

(a) for each bounded measurable function u on T which vanishes outside a set of finite measure, we have $u \in L$;

(b) if $u \in L$ and E is a set of finite measure in T, then $\iota_E \cdot u \in L$, where ι_E is the characteristic function of E.

It has already been proved in [96] that $C(T)$ and $L_p(T)$, for $1 \le p \le +\infty$ are decomposable, where T is a σ-finite measure space. Now the following theorem gives the form of the convex conjugate of I_f.

Theorem 1.2.6 (Rockafellar, 1968, [96]) *Suppose L and L^* are decomposable. Let f be a normal convex integrand such that $f(\cdot, u(\cdot))$ is summable for at least one $u \in L$, and $f^*(\cdot, u^*(\cdot))$ is summable for at least one $u^* \in L^*$. Alternatively, f is of the form $f(t,x) \equiv F(x)$, where F is a lower semicontinuous proper convex function on $I\!R^n$. Then I_f on L and I_{f^*} on L^* are proper convex functions conjugate to each other.*

In this thesis, we will study the convex program in the following form:

$$(CP) \qquad \begin{cases} \min & F(x), \\ s.t. & Ax = b, \\ & x \in C \subset X, \end{cases} \tag{1.17}$$

where X is a real normed space, $F : X \to (-\infty, +\infty]$ is a convex functional, $A : X \to I\!R^n$ is linear, $b \in I\!R^n$, and C is a convex subset in X.

We will be particularly interested in cases where:

- $X = L_1(T, \mu)$ for a complete finite measure space (T, μ);
- F takes the form of
$$F(x) = \int_T f(t, x(t)) d\mu(t)$$
  for a normal convex integrand f;
- $A : X \to I\!R^n$ is of the form
$$(Ax)_k = \int_T x(t) a_k(t) d\mu(t), \qquad k = 1, 2, \cdots, n$$
  for $a_k \in L_\infty(T, \mu)$;
- C is a convex set in X.

1.3 Results in approximation theory

In proving our uniform convergence theorems in Chapter 3, we will need some uniform/best approximation results of a function f on $\mathbb{R}^m$ by algebraic or trigonometric polynomials. Most of the results recalled below can be found in the recent survey paper [108].

The **modulus of continuity** of a function $f(x)$ defined on $[a, b]$ is the function

$$\omega(f, \delta) \triangleq \sup_{x', x'' \in [a,b], |x'-x''| \le \delta} |f(x') - f(x'')|. \tag{1.18}$$

It is obvious that if f is continuous on a finite interval $[a, b]$, then

$$\omega(f, \delta) \to 0, \qquad \text{as } \delta \to 0. \tag{1.19}$$

Moreover, if f is α-**Lipschitz**, $0 < \alpha < 1$, i.e. f satisfies the Lipschitzian (or Hölder) condition of order α:

$$|f(x') - f(x'')| \le L|x' - x''|^\alpha, \qquad \text{for all } x', x'' \in [a, b], \tag{1.20}$$

where L is called the α-**Lipschitz constant**, then

$$\omega(f, \delta) \le L\delta^\alpha. \tag{1.21}$$

The **modulus of continuity of order** k of a function $f(x)$ defined on $\mathbb{R}$ or [a,b] is denoted by

$$\omega_k(f, \delta) \triangleq \sup_{x \in [a,b], |h| \le \delta, x+mh \in [a,b]} \Big| \sum_{m=0}^{k} (-1)^m C_k^m f(x + mh) \Big|, \tag{1.22}$$

where

$$C_k^m \triangleq \frac{k!}{m!(k-m)!}. \tag{1.23}$$

We can see that for $k = 2$, $f \in C^1[a.b]$,

$$\begin{aligned} \omega_2(f, \delta) &= \sup_{x \in [a,b], |h| \le \delta, x+2h \in [a,b]} \big| f(x) - 2f(x+h) + f(x+2h) \big| \\ &\le \sup_{x \in [a,b], |h| \le \delta, x+2h \in [a,b]} \big| f'(x') - f'(x'') \big| |h| \\ &\quad \text{(for some } x' \in [x, x+h] \text{ and } x'' \in [x+h, x+2h] \\ &\qquad\qquad \text{by the mean value theorem)}, \end{aligned} \tag{1.24}$$

and hence

$$\omega_2(f,\delta) \le \omega(f',\delta)\delta = o(\delta). \tag{1.25}$$

Furthermore, if $f \in C^2[a,b]$, using the mean value theorem once again in the above inequality, we then have

$$\omega_2(f,\delta) \ \le \ \sup_{x\in[a,b]} |f''(x)||h|^2 = O(\delta^2). \tag{1.26}$$

First introduced by Tchebycheff [106], the **best approximation** of a function $f(x)$, continuous on $[a,b]$, in the metric of $C[a,b]$ by algebraic polynomials of degree at most n is defined as

$$E_n(f) \triangleq \min \|f(x) - p_n(x)\|_\infty, \tag{1.27}$$

where the minimum is take over all polynomials $p_n(x) = \sum_{k=0}^n \lambda_k x^k$, for $\lambda_k \in I\!R$, $k = 0,1,\cdots,n$. Weierstrass's theorem states that if f is continuous on $[a,b]$ then $E_n(f) \to 0$.

More generally, we consider a sequence of sets of functions $\big\{\{a_i(t);\ i \in I_n\}, n = 0,1,\cdots\big\}$, where $a_i \in C[a,b]$ for each i, I_n's are finite index sets satisfying $I_n \subset I_{n+1}$ for $n = 0,1,\cdots$. Usually, we also suppose that $\{a_i, i \in \bigcup_{n=0}^\infty I_n\}$ is taken to be dense in $C[a,b]$. In the one dimensional algebraic polynomial case, we take $I_n = \{0,1,\cdots,n\}$ and $a_i(t) = t^i$. In the trigonometric polynomial case, we take $I_n = \{0,1,\cdots,2n\}$ and $a_0(t) = 1$, $a_{2k-1}(t) = \cos(kt)$, $a_{2k} = \sin(kt)$, for $k = 1,2,\cdots,n$.

Using the same notation, We define for each n,

$$E_n(f) \triangleq \min\big\{\|f - \sum_{i\in I_n} \lambda_i a_i\|_\infty \ \big| \ \lambda_k \in I\!R,\ k \in I_n\big\}. \tag{1.28}$$

We now give some upper bounds for $E_n(f)$ when f is smooth enough.

Theorem 1.3.1 (Jackson,1911, [62]) *Let $[a,b]$ be a bounded interval on $I\!R$, and let $\{a_i, i \in I_n\}$ be algebraic polynomials of the form: $1,\ t,\ t^2,\ \cdots,\ t^n$, or trigonometric polynomials of the form: $1,\ \cos t,\ \sin t,\ \cdots,\ \cos nt,\ \sin nt$. If $f \in C^r[a,b]$, for $r \ge 0$*

(while in the trigonometric case, we also assume that f is periodic with period 2π), then for some constant A independent of n, we have

$$E_n(f) \leq \frac{A}{n^r}\omega(f^{(r)}, \frac{1}{n}). \tag{1.29}$$

A corollary follows directly.

Corollary 1.3.2 *Under the assumptions of Theorem 1.3.1, we have*

1. *If $f \in C^r[a,b], r \geq 1$, then*

$$E_n(f) = o(\frac{1}{n^r}). \tag{1.30}$$

2. *If $f \in C^r[a,b], r \geq 0$, and $f^{(r)}$ is α-Lipschitz on $[a,b]$, for $\alpha > 0$, then*

$$E_n(f) = O(\frac{1}{n^{r+\alpha}}). \tag{1.31}$$

For an analytic function f on $[a,b]$, the corresponding result becomes (proved by Bernstein, 1911, in [9]):

$$E_n(f) \leq Aq^n, \tag{1.32}$$

for some constants $0 < q < 1$, $A > 0$, independent of n. Some similar inequalities related to higher order moduli of continuity are stated in [1] and [105]. For multi-dimensional functions, the best approximation defined analogously has the following properties.

Theorem 1.3.3 (Bernstein, [10],[11], Nikol'skii, [86],[87]) *Let $f(x_1, x_2, \cdots, x_m)$ be a periodic function defined on $\mathbb{R}^m$ with period 2π in each variable. For some integer ρ, we suppose $\partial^\rho f/\partial x_k^\rho$ exists. Further suppose for each $k = 1, 2, \cdots, m$, $\partial^\rho f/\partial x_k^\rho$ is continuous in variable x_k. Then the best approximation of f by trigonometric polynomials of degree at most n in each variable satisfies:*

$$E_n(f) = o(\frac{1}{n^\rho}). \tag{1.33}$$

Theorem 1.3.4 (Timan, 1963,[109]) *Let $f(x_1, x_2, \cdots, x_m)$ be an m-variable function defined on a closed bounded parallelpiped T. Assume for some integer ρ and each $k = 1, 2, \cdots, m$, $\partial^\rho f / \partial x_k^\rho$ exists and is continuous in variable x_k. Then the best approximation of f by algebraic polynomials of degree at most n in each variable satisfies:*

$$E_n(f) = o(\frac{1}{n^\rho}). \tag{1.34}$$

Since our uniform convergence results in Chapter 3 will be mostly based on L_p-norm convergence theorems, we need to investigate the relationships between the different norms of an algebraic or trigonometric polynomial.

Theorem 1.3.5 (Jackson, Nikol'skii, [63],[87], [89]) *Let $T = [-\pi, \pi]^m, 1 \le p < q \le \infty$. Let $p_{n_1, n_2, \cdots, n_m}(x_1, x_2, \cdots, x_m)$ be a trigonometric polynomial of degree at most n_1 in x_1, n_2 in x_2, $\cdots$, n_m in x_m. Then the following inequality is true:*

$$\|p_{n_1, n_2, \cdots, n_m}\|_q \le A(n_1 n_2 \cdots n_m)^{\frac{1}{p} - \frac{1}{q}} \|p_{n_1, n_2, \cdots, n_m}\|_p, \tag{1.35}$$

for some constant A independent of $n_1, n_2, \cdots, n_m$.

In particular, if p_n is a trigonometric polynomial of degree at most n in each variable, that is $n_1 = n_2 = \cdots = n_m = n$, then

$$\|p_n\|_q \le A n^{m(\frac{1}{p} - \frac{1}{q})} \|p_n\|_p. \tag{1.36}$$

The most interesting case is when $q = \infty$, and $p = 2$, where we have

$$\|p_n\|_\infty \le A n^{\frac{m}{2}} \|p_n\|_2. \tag{1.37}$$

If instead we consider p_n to be an algebraic polynomial of degree at most n in each variable, then we have

Theorem 1.3.6 ([77], [88]) *Let T be a bounded domain in $I\!R^m$ with piecewise C^1 boundary, and let $1 \le p < q \le \infty$. Assume that p_n is an algebraic polynomial of total degree at most n :*

$$p_n(x_1, x_2, \cdots, x_m) = \sum_{|s| \le n} a_s x_1^{s_1} x_2^{s_2} \cdots x_m^{s_m},$$

where $s = (s_1, s_2, \cdots, s_m) \in \mathbf{Z}_+^m$, $s_i \geq 0$, $i = 1, 2, \cdots m$, $|s| = s_1 + s_2 + \cdots + s_m$. *Then*

$$\|p_n\|_q \leq An^{2(\frac{1}{p}-\frac{1}{q})}\|p_n\|_p, \tag{1.38}$$

for some constant A *independent of* n. *In particular, for* $q = +\infty$ *and* $p = 2$, *we have*

$$\|p_n\|_\infty \leq An\|p_n\|_2. \tag{1.39}$$

For a polynomial p_n of degree at most n in each variable, the total degree is at most mn. So the inequality (1.38) and hence (1.39) holds except we take a different constant A'.

We will also need some Remez-type inequalities of the following forms in later discussions.

Theorem 1.3.7 (Remez, [27]) *Let* p_n *be an arbitrary trigonometric polynomial of degree at most* n *on* $T = [-\pi, \pi] \subset \mathbb{R}$, dt *be a Lebesgue measure,* $p > 0$. *Let* A *be an arbitrary subset of* T, *with* $\mu(A) \geq \pi$. *Then there exists a constant* C, *such that*

$$\int_{-\pi}^{\pi} |p_n(t)|^p dt \leq (\,1 + e^{Cpn(\,2\pi - \mu(A)\,)}\,) \int_A |p_n(t)|^p dt. \tag{1.40}$$

Theorem 1.3.8 (Remez, [27]) *Let* p_n *be an arbitrary algebraic polynomial of degree at most* n *on* $[a, b] \subset \mathbb{R}$, dt *be a Lebesgue measure,* $p > 0$. *Let* A *be an arbitrary subset of* $[a, b]$, *with* $\mu(A) \geq (b - a)/2$. *Then there exists a constant* C, *such that*

$$\int_a^b |p_n(t)|^p dt \leq (1 + e^{Cpn\sqrt{(b-a)-\mu(A)}}) \int_A |p_n(t)|^p dt. \tag{1.41}$$

Chapter 2

Some Estimation Theorems for Sequential Convex Programs with Linear Constraints

2.1 Introduction

In this chapter, we will define our class of entropy-like objectives, which include many frequently used entropies. For a general choice of the objective, we first estimate the upper bound of duality gaps. Then we will apply the results to some important entropies. Although we always have the strong duality theorem (see, for example, [12], [15], [17], [19], [20], [23], [46]) which guarantees that the duality gap is zero under some reasonable constraint qualification (see, for instance, [26], [65], [66], [78]), it is useful to get an upper bound on how well our estimated density, a solution of the convex program with finitely many moments, approximates to the underlying density in the entropic value. Using these inequalities, we will be able to prove some theorems on norm convergence. The results proved in Sections 2.3., 2.4. and 2.5. will let us establish uniform convergence results in Chapter 3.

We will introduce assumptions that will be used throughout the thesis. An index of these assumptions can be found in Appendix A.

2.2 Problems and examples

We need to estimate an unknown density $\bar{x}$ on T, admitting a finite number of linear constraints (moments), and bounded by two functions $\alpha(\cdot)$ and $\beta(\cdot)$ defined also on T (the problem first proposed in [24]). Introducing an entropy-like objective function, we may consider the following program:

$$(P_n) \qquad \begin{cases} \inf & \int_T \phi(\,x(t)\,)d\mu(t), \\ s.t. & \int_T a_i(t)x(t)d\mu(t) = b_i, \quad i \in I_n, \\ & x \in L_1(T,\mu), \\ & \alpha(t) \le x(t) \le \beta(t), \text{ a.e. on } T. \end{cases} \tag{2.1}$$

where b_i can be considered as given by $b_i = \int_T a_i(t)\bar{x}(t)d\mu(t)$ for $i \in I_n$.

We make the following assumptions:

(A1): (T,μ) is a complete finite measure space;

(A2): a_i linearly independent functions in $L_\infty(T,\mu)$, for all $i \in I_n$. I_n's are finite index sets satisfying:

$$I_n \subset I_{n+1}, \quad n = 0, 1, \cdots,$$

and we denote by $k(n)$ the number of indices in I_n;

(A3): $\phi : I\!R \to (-\infty, +\infty]$ is a proper, lower semicontinuous, convex function with its domain $\mathrm{dom}(\phi)$ satisfying:

$$(a,b) \subseteq \mathrm{dom}(\phi) \subseteq [a,b], \qquad \text{for some } \; -\infty \le a < b \le +\infty,$$
$$\text{i.e.} \qquad a = \inf(\,\mathrm{dom}(\phi)\,), \qquad b = \sup(\,\mathrm{dom}(\phi)\,);$$

(A4): ϕ is essentially smooth and essentially strictly convex on (a,b);

(A5): $\alpha(\cdot)$, $\beta(\cdot)$ are extended real valued μ-measurable functions defined on T, with

$$a \le \alpha(t) < \beta(t) \le b, \qquad \text{a.e. on } T; \tag{2.2}$$

(A6): α, β, ϕ are chosen such that

$$\operatorname{essinf}(\alpha) = a \qquad \text{implies} \qquad \phi'(a) \triangleq \lim_{u \to a^+} \phi'(u) = -\infty, \tag{2.3}$$

and

$$\operatorname{esssup}(\beta) = b \qquad \text{implies} \qquad \phi'(b) \triangleq \lim_{u \to b^-} \phi'(u) = +\infty. \tag{2.4}$$

At the end of this section, we will give some concrete examples, where we choose ϕ to satisfy these conditions.

Extending the function ϕ, we define $\tilde{\phi} : T \times I\!R \to (-\infty, +\infty]$ as

$$\tilde{\phi}(t,u) \triangleq \begin{cases} \phi(u), & \alpha(t) \le u \le \beta(t), \\ +\infty, & \textit{otherwise}. \end{cases} \tag{2.5}$$

We write

$$I_\phi(x) \triangleq \int_T \phi(\, x(t)\,) d\mu(t), \tag{2.6}$$

and

$$I_{\tilde{\phi}}(x) \triangleq \int_T \tilde{\phi}(\, t, x(t)\,) d\mu(t). \tag{2.7}$$

Then (P_n) in (2.1) is equivalent to

$$(\widetilde{P_n}) \qquad \begin{cases} \inf & I_{\tilde{\phi}}(x), \\ s.t. & \int_T a_i(t)x(t)d\mu(t) = b_i, \quad i \in I_n, \\ & x \in L_1(T,\mu). \end{cases} \tag{2.8}$$

We now compute the convex conjugate of $\tilde{\phi}$ and then $I_{\tilde{\phi}}$.

Proposition 2.2.1 *Let (A1)-(A6) hold. Then for almost all $t \in T$,*

1. *the convex conjugate function of $\tilde{\phi}$ is finite everywhere and of the form:*

$$\begin{aligned} \tilde{\phi}^*(t,v) &\triangleq \sup\{\langle u,v\rangle - \tilde{\phi}(t,u)\} \\ &= \begin{cases} \alpha(t)v - \phi(\,\alpha(t)\,), & v \le \phi'(\,\alpha(t)\,), \\ \phi^*(v), & \phi'(\,\alpha(t)\,) < v < \phi'(\,\beta(t)\,), \\ \beta(t)v - \phi(\,\beta(t)\,), & v \ge \phi'(\,\beta(t)\,), \end{cases} \end{aligned} \tag{2.9}$$

(at the boundary points of dom(ϕ)*, we agree to use the left or right directional derivatives);*

2. $\tilde{\phi}^*(t,v)$ *is continuously differentiable in variable* v *on* $\mathbb{R}$*, with*

$$(\tilde{\phi}^*)'_2(t,v) = \begin{cases} \alpha(t), & v \le \phi'(\,\alpha(t)\,), \\ \phi^{*\prime}(v), & \phi'(\,\alpha(t)\,) < v < \phi'(\,\beta(t)\,), \\ \beta(t), & v \ge \phi'(\,\beta(t)\,); \end{cases} \tag{2.10}$$

3. $(\tilde{\phi}^*)_2(t,v)$ *is strictly convex in variable* v *on* $(\,\phi'(\,\alpha(t)\,), \phi'(\,\beta(t)\,)\,)$*;*

4. *the convex conjugate function of* $\tilde{\phi}$ *and the derivative of* $\tilde{\phi}$ *are inverse functions of each other in the sense of*

$$\begin{aligned}
\tilde{\phi}_2'(\,t, (\tilde{\phi}^*)_2'(t,v)\,) &= \phi'((\tilde{\phi}^*)_2'(t,v)) \\
&= \begin{cases} \phi'(\,\alpha(t)\,), & v \leq \phi'(\,\alpha(t)\,), \\ v, & \phi'(\,\alpha(t)\,) < v < \phi'(\,\beta(t)\,), \\ \phi'(\,\beta(t)\,), & v \geq \phi'(\,\beta(t)\,), \end{cases} \qquad (2.11)
\end{aligned}$$

and

$$(\tilde{\phi}^*)_2'(\,t, \tilde{\phi}_2'(t,u)\,) = \phi^{*\prime}(\,\phi'(u)\,) = u, \qquad (2.12)$$

whenever $\alpha(t) \leq u \leq \beta(t)$.

Proof: See [24] or [25]. Note that (A5) and (A6) are required to guarantee that $\tilde{\phi}^*$ is everywhere finite. ∎

Proposition 2.2.2 *Under the assumptions (A1)-(A6), the extended function $\tilde{\phi}$ is a normal convex integrand. Hence the convex conjugate of $I_{\tilde{\phi}}$ is given by: $(I_{\tilde{\phi}})^* = I_{\tilde{\phi}^*}$.*

Proof: See [24] or [25]. ∎

Although the functional $I_{\tilde{\phi}}$ is everywhere nondifferentiable, in the next proposition we will see that the conjugate $I_{\tilde{\phi}^*}$ is very well behaved.

For two extended real numbers a and b, we write

$$a \vee b \triangleq \max\{a,b\}, \qquad a \wedge b \triangleq \min\{a,b\}.$$

Proposition 2.2.3 *Under Assumptions (A1)-(A6), the conjugate $(I_{\tilde{\phi}})^*$ of $I_{\tilde{\phi}}$ is Fréchet differentiable at every $x \in L_\infty(T)$. In fact, for $x \in L_\infty(T)$,*

$$\begin{aligned}
(I_{\tilde{\phi}}^*)'(x) &= \max\left\{\,\alpha(t), \min\left(\,\beta(t), (\phi^*)'(\,x(t)\,)\,\right)\,\right\} \\
&= \alpha(t) \vee (\phi^*)'(x(t)) \wedge \beta(t). \qquad (2.13)
\end{aligned}$$

Proof: See [24] or [25]. ∎

We can write the dual problem of $(\widetilde{P_n})$ (or equivalently, (P_n)) to be of the form:

$$(D_n)\begin{cases} \max & \Phi(\lambda) \triangleq \int_T[\bar{x}(t)\sum_{i\in I_n}\lambda_i a_i(t) - \tilde{\phi}^*(t,\sum_{i\in I_n}\lambda_i a_i(t))]d\mu(t), \\ s.t. & \lambda \in I\!R^{k(n)}. \end{cases} \tag{2.14}$$

A **constraint qualification** condition (CQ) (first given in [19]) which guarantees the strong duality result is of the form (see also [25]):

$$\begin{cases} \text{there exists } \hat{x} \in L_1(T,\mu), \text{ such that} \\ \alpha(t) < \hat{x} < \beta(t), \text{ a.e. on } T, \\ \int_T \phi(t,\hat{x}(t))d\mu(t) < +\infty, \text{ and} \\ \int_T a_i(t)\hat{x}(t)d\mu(t) = b_i, \quad i \in I_n. \end{cases} \tag{2.15}$$

It has been proved that under this constraint qualification the following strong duality theorem holds (see also [17], [20], [24] or [78]).

Theorem 2.2.4 (Borwein and Lewis, 1990) *Under (CQ) of the form in (2.15), the optimal values of $(\widetilde{P_n})$ and (D_n) are equal with dual attainment. Moreover, if $\lambda \in I\!R^{k(n)}$ is an optimal solution of (D_n), then the unique solution of (P_n) is of the form:*

$$x_n(t) = (\tilde{\phi}^*)'_2(t, \sum_{i\in I_n}\lambda_i a_i(t)). \tag{2.16}$$

Proof: See [17]. Note that the uniqueness follows from the strict convexity of ϕ, given in (A4). ∎

The following are some typical choices of ϕ, α and β. All except Burg's entropy satisfy Assumptions (A1)-(A6). We will give ϕ and $\tilde{\phi}$ and then compute the conjugates ϕ^* and $\tilde{\phi}^*$.

Example 2.2.5 (Boltzmann-Shannon entropy)

$$\tilde{\phi}(t,u) \equiv \phi(u) = \begin{cases} u\log u - u, & u > 0, \\ 0, & u = 0, \\ +\infty, & u < 0, \end{cases} \tag{2.17}$$

with

$$\alpha(t) \equiv a = 0, \quad \beta(t) \equiv b = +\infty.$$

Then

$$\tilde{\phi}^*(t, v) \equiv \phi^*(v) = e^v. \tag{2.18}$$

Example 2.2.6 (Generalized Fermi-Dirac entropy)

$$\tilde{\phi}(t, u) \equiv \phi(u) = \begin{cases} (u - \alpha_0)\log(u - \alpha_0) & \\ \qquad +(\beta_0 - u)\log(\beta_0 - u), & \alpha_0 < u < \beta_0, \\ (\beta_0 - \alpha_0)\log(\beta_0 - \alpha_0), & u = \alpha_0, or\ \beta_0, \\ +\infty, & otherwise, \end{cases} \tag{2.19}$$

$$\alpha(t) \equiv a = \alpha_0, \quad \beta(t) \equiv b = \beta_0 \ \ (where\ -\infty < \alpha_0 < \beta_0 < +\infty).$$

Then

$$\tilde{\phi}^*(t, v) \equiv \phi^*(v) = \alpha_0 v + (\beta_0 - \alpha_0)\log(\frac{1 + e^v}{\beta_0 - \alpha_0}). \tag{2.20}$$

In particular, $\alpha_0 = 0, \beta_0 = 1$ *gives the classical Fermi-Dirac entropy.*

Example 2.2.7 (L_p-entropy, $1 < p < +\infty$)

$$\phi(u) = \frac{1}{p}|u|^p, \tag{2.21}$$

where we can see $a = -\infty$, $b = +\infty$. *In most applications we set* $\alpha(t) \equiv 0$, $\beta(t) \equiv +\infty$, *and consider the truncated* L_p*-entropy:*

$$\tilde{\phi}(t, u) = \begin{cases} \frac{1}{p}u^p, & u \geq 0, \\ +\infty, & u < 0. \end{cases} \tag{2.22}$$

In this case

$$\tilde{\phi}^*(t, v) = \begin{cases} \frac{1}{q}v^q, & v \geq 0, \\ 0, & v < 0, \end{cases} \tag{2.23}$$

where $1/p + 1/q = 1$. *Note that without the truncation, we would have*

$$\phi^*(v) = \frac{1}{q}|v|^q. \tag{2.24}$$

Example 2.2.8 (Burg entropy)

Burg's entropy considers:

$$\tilde{\phi}(t,u) = \phi(u) = \begin{cases} -\log u, & u > 0, \\ +\infty, & u \le 0, \end{cases} \tag{2.25}$$

with $a = 0$, $b = +\infty$. *The conjugate function is*

$$\tilde{\phi}^*(t,v) = \phi^*(v) = \begin{cases} -1 - \log(-v), & v < 0, \\ +\infty, & u \ge 0. \end{cases} \tag{2.26}$$

This is a very important entropy and has received a lot of attention. But for this entropy, (A6) does not hold and hence the conjugate function is not everywhere finite as we saw in (2.26). We may set a truncation $\beta(t) \equiv \beta_0 > 0$, *and consider*

$$\tilde{\phi}(t,u) = \begin{cases} -\log u, & 0 < u \le \beta_0, \\ +\infty, & otherwise. \end{cases} \tag{2.27}$$

Now the conjugate function is

$$\tilde{\phi}^*(t,v) = \begin{cases} -1 - \log(-v), & v \le -\frac{1}{\beta_0}, \\ v\beta_0 + \log \beta_0, & v > -\frac{1}{\beta_0}, \end{cases} \tag{2.28}$$

which is finite everywhere on $\mathbb{R}$.

Example 2.2.9 (Burg-type entropy)

In the spirit of Burg's entropy, we consider

$$\phi(u) = \begin{cases} -\log u - \log(1-u), & 0 < u < 1, \\ +\infty, & otherwise, \end{cases} \tag{2.29}$$

with $\alpha \equiv 0 = a$, $\beta \equiv 1 = b$. *Then after some computation, we obtain*

$$\tilde{\phi}^*(t,v) \equiv \phi^*(v) = \frac{1}{2}(v - 2 + \sqrt{v^2+4}) + \log(\frac{\sqrt{v^2+4}-2}{v^2}). \tag{2.30}$$

Example 2.2.10 (Hellinger-type entropy)

$$\phi(u) = \begin{cases} -\sqrt{2uK - u^2}, & 0 \le u \le 2K, \\ +\infty, & otherwise. \end{cases} \tag{2.31}$$

Here $\alpha \equiv 0 = a$, $\beta \equiv 2K = b$. *Then*

$$\tilde{\phi}^*(t,v) \equiv \phi^*(v) = K(v + \sqrt{v^2+1}). \tag{2.32}$$

2.3 Estimates on duality gaps

In this section, we will impose Assumptions (A1)-(A6) throughout. We also assume that $\bar{x}$ is **feasible** for (P_n) in the sense that

$$\begin{cases} \bar{x} \in L_1(T,\mu), \\ \int_T a_i(t)\bar{x}(t)d\mu(t) = b_i, \quad i \in I_n, \\ \alpha(t) \leq \bar{x}(x) \leq \beta(t), \text{ a.e. on } T, \\ \int_T \phi(\,\bar{x}(t)\,)d\mu(t) < +\infty. \end{cases} \tag{2.33}$$

Given each n, $f \in L_\infty(T,\mu)$, for our convenience, the best approximation of f by $\{a_i, i \in I_n\}$ is defined by

$$E_n(f) \triangleq \inf \{ \, \| \sum_{i \in I_n} \lambda_i a_i - f \|_\infty \mid \lambda \in I\!R^{k(n)} \, \}. \tag{2.34}$$

Note that the number of elements in I_n may not be exactly n. For a given choice of ϕ, and a feasible $\bar{x}$, we write

$$E_n \triangleq \begin{cases} E_n(\,\phi'(\bar{x})\,), & \text{if } \phi'(\bar{x}) \in L_\infty(T,\mu), \\ +\infty, & otherwise. \end{cases} \tag{2.35}$$

For problems $(\widetilde{P_n})$ and (D_n) given in (2.8) and (2.13), we can directly prove the following inequalities.

Lemma 2.3.1 (Weak duality) *Denote by $V(\widetilde{P_n})$, $V(D_n)$ the optimal values of problems $(\widetilde{P_n})$ and (D_n), respectively. Then*

$$V(\widetilde{P_n}) \geq V(D_n). \tag{2.36}$$

Proof: This follows directly from the convexity of $\tilde{\phi}$ and the Fenchel-Young inequality (1.10). ∎

Theorem 2.3.2 *Let $E_n < +\infty$, then*

$$\begin{aligned} I_\phi(\bar{x}) &\geq V(\widetilde{P_n}) \geq V(D_n) \geq I_\phi(\bar{x}) - \\ &(\|\bar{x}\|_1 + \|\alpha\|_1 \vee \|\phi^{*\prime}(\phi'(\bar{x}) - E_n)\|_1 \\ &+\|\beta\|_1 \vee \|\phi^{*\prime}(\phi'(\bar{x}) + E_n)\|_1)E_n. \end{aligned} \tag{2.37}$$

Moreover, if both $\alpha, \beta \in L_1(T, \mu)$, then

$$I_\phi(\bar{x}) \geq V(\widetilde{P_n}) \geq V(D_n) \geq I_\phi(\bar{x}) - (\|\bar{x}\|_1 + \|\alpha\|_1 + \|\beta\|_1)E_n. \tag{2.38}$$

Proof: Since $E_n < +\infty$, $\phi'(\bar{x})$ is almost everywhere finite. From the Fenchel-Young inequality (1.14), for almost all $t \in T$ where $\phi'(\,\bar{x}(t)\,)$ is finite, we have

$$\phi(\,\bar{x}(t)\,) + \phi^*(\,\phi'(\,\bar{x}(t)\,)\,) = \bar{x}(t)\phi'(\,\bar{x}(t)\,). \tag{2.39}$$

Also, the feasibility of $\bar{x}$ and the monotonicity and continuous differentiability of ϕ' imply

$$\phi'(\,\alpha(t)\,) \leq \phi'(\,\bar{x}(t)\,) \leq \phi'(\,\beta(t)\,), \tag{2.40}$$

By the continuity of ϕ^* and $\tilde{\phi}^*$, we then have

$$\phi^*(\,\phi'(\,\bar{x}(t)\,)\,) = \tilde{\phi}^*(\,t, \phi'(\,\bar{x}(t)\,)\,). \tag{2.41}$$

By the convexity of $\tilde{\phi}^*(t,\cdot)$, for any $\lambda \in I\!R^{k(n)}$,

$$\tilde{\phi}^*(\,t, \phi'(\,\bar{x}(t)\,)\,) - \tilde{\phi}^*(t, \sum_{i\in I_n} \lambda_i a_i(t)) \geq \tilde{\phi}^{*\prime}(t, \sum_{i\in I_n} \lambda_i a_i(t))\big(\phi'(\,\bar{x}(t)\,) - \sum_{i\in I_n} \lambda_i a_i(t)\big),$$

and hence by (2.41),

$$\begin{aligned} &\phi^*(\,\phi'(\,\bar{x}(t)\,)\,) - \tilde{\phi}^*(t, \sum_{i\in I_n} \lambda_i a_i(t)) \\ &\geq \tilde{\phi}^{*\prime}(t, \sum_{i\in I_n} \lambda_i a_i(t))(\phi'(\,\bar{x}(t)\,) - \sum_{i\in I_n} \lambda_i a_i(t)). \end{aligned} \tag{2.42}$$

Now for each n, we can find $\lambda^n \in I\!R^{k(n)}$, such that

$$\|\sum_{i\in I_n} \lambda_i^n a_i - \phi'(\bar{x})\|_\infty = E_n. \tag{2.43}$$

By weak duality (Lemma 2.3.1), we only need to check the last inequality. For λ^n given in (2.43), using (2.39), (2.42), and (2.43), we have

$$\begin{aligned}
I_\phi(\bar{x}) - V(D_n) &\leq \int_T [\phi(\bar{x}(t)) - \bar{x}(t)\sum_{i\in I_n}\lambda_i^n a_i(t) + \tilde{\phi}^*(t, \sum_{i\in I_n}\lambda_i^n a_i(t))]d\mu(t) \\
&= \int_T [\bar{x}(t)\phi'(\bar{x}(t)) - \phi^*(\phi'(\bar{x}(t))) - \bar{x}(t)\sum_{i\in I_n}\lambda_i^n a_i(t) \\
&\quad + \tilde{\phi}^*(t, \sum_{i\in I_n}\lambda_i^n a_i(t))]d\mu(t) \quad (\text{ by } (2.39)\ ) \\
&\leq \int_T (\bar{x}(t) - (\tilde{\phi}^*)_2'(t, \sum_{i\in I_n}\lambda_i^n a_i(t)))(\phi'(\bar{x}(t)) - \sum_{i\in I_n}\lambda_i^n a_i(t))d\mu(t) \\
&\qquad (\text{ by } (2.42)\ ) \\
&\leq E_n\|\bar{x}(t) - (\tilde{\phi}^*)_2'(t, \sum_{i\in I_n}\lambda_i^n a_i(t))\|_1, \qquad (\text{ by } (2.43)\ ). \qquad (2.44)
\end{aligned}$$

Also from (2.43), we have

$$\phi'(\bar{x}(t)) - E_n \leq \sum_{i\in I_n}\lambda_i^n a_i(t) \leq \phi'(\bar{x}(t)) + E_n, \quad a.e.\ on\ T. \qquad (2.45)$$

Then using the expression for $(\tilde{\phi}^*)_2'$ in (2.10) and the convexity of $\tilde{\phi}^*$, we can see that

$$\begin{aligned}
&\max\{\alpha(t), \phi^{*\prime}(\phi'(\bar{x}(t)) - E_n)\} \\
&\leq (\tilde{\phi}^*)_2'(t, \sum_{i\in I_n}\lambda_i^n a_i(t)) \\
&\leq \min\{\beta(t), \phi^{*\prime}(\phi'(\bar{x}(t)) + E_n)\}, \qquad \text{a.e. on } T. \qquad (2.46)
\end{aligned}$$

This implies

$$\begin{aligned}
&\|(\tilde{\phi}^*)_2'(t, \sum_{i\in I_n}\lambda_i^n a_i(t))\|_1 \\
&\leq \|\alpha\|_1 \vee \|\phi^{*\prime}(\phi'(\bar{x}) - E_n)\|_1 + \|\beta\|_1 \vee \|\phi^{*\prime}(\phi'(\bar{x}) + E_n)\|_1.
\end{aligned}$$

Then we deduce

$$\begin{aligned}
&\|\bar{x} - (\tilde{\phi}^*)_2'(t, \sum_{i\in I_n}\lambda_i^n a_i)\|_1 \\
&\leq \|\bar{x}\|_1 + \|(\tilde{\phi}^*)_2'(t, \sum_{i\in I_n}\lambda_i^n a_i)\|_1 \\
&\leq \|\bar{x}\|_1 + \|\alpha\|_1 \vee \|\phi^{*\prime}(\phi'(\bar{x}) - E_n)\|_1 + \|\beta\|_1 \vee \|\phi^{*\prime}(\phi'(\bar{x}) + E_n)\|_1. \qquad (2.47)
\end{aligned}$$

The inequality (2.37) follows.

Furthermore, if both $\alpha, \beta \in L_1(T, \mu)$, simplifying (2.46), we have

$$\alpha(t) \leq (\tilde{\phi}^*)'_2(t, \sum_{i \in I_n} \lambda_i^n a_i(t)) \leq \beta(t) \qquad \text{a.e. on } T.$$

Then

$$\|(\tilde{\phi}^*)'_2(t, \sum_{i \in I_n} \lambda_i^n a_i(t))\|_1 \leq \|\alpha\|_1 + \|\beta\|_1,$$

which leads to our simplified inequality (2.38). ■

We can see that we made an overestimation in deducing (2.47). The results can be easily improved by assuming the differentiability of $\tilde{\phi}^*$. In the previous section, we saw that $\tilde{\phi}^*(t, \cdot)$ is continuously differentiable for almost all $t \in T$, hence $(\tilde{\phi}^*)'_2(t, \cdot)$ is continuous for those $t \in T$. We now assume that $(\tilde{\phi}^*)'_2(t, \cdot)$ is locally γ- Lipschitzian for fixed t, where the Lipschitzian "constant" can be a function of t.

We will say that $(\tilde{\phi}^*)'_2(t, v)$ is **locally γ-Lipschitzian** ($\gamma > 0$) with respect to v on T, if for small $\eta > 0$, there exists a measurable function $Lip(\cdot, \eta, \bar{x})$ such that for each $t \in T$,

$$|(\tilde{\phi}^*)'_2(t, v_1) - (\tilde{\phi}^*)'_2(t, v_2)| \;\; \leq \;\; Lip(t, \eta, \bar{x})|v_1 - v_2|^\gamma, \tag{2.48}$$

whenever

$$v_1, v_2 \in [\phi'(\,\bar{x}(t)\,) - \eta, \phi'(\,\bar{x}(t)\,) + \eta].$$

Note that we have defined this concept only for small η. The reason is that we will explicitly assume $E_n \to 0$ when we prove uniform convergence. The inequality (2.48) only needs to be checked around $\phi'(\bar{x}(t))$ for fixed t.

Corollary 2.3.3 *Let $(\tilde{\phi}^*)'_2(t, v)$ be locally γ-Lipschitzian in v. Suppose that E_n is small enough and that $Lip(\cdot, E_n, \bar{x}) \in L_1(T, \mu)$. We have*

$$\begin{aligned} I_\phi(\bar{x}) \;\; &\geq \;\; V(\widetilde{P_n}) \geq V(D_n) \\ &\geq \;\; I_\phi(\bar{x}) - \|Lip(t, E_n, \bar{x})\|_1 E_n^{1+\gamma}. \end{aligned} \tag{2.49}$$

Proof: Continuing from (2.44) in the proof of Theorem 2.3.2, for some $\lambda^n \in I\!R^{k(n)}$, we have

$$\begin{aligned}
I_\phi(\bar{x}) - V(D_n) &\leq \int_T \big(\,\bar{x}(t) - (\tilde{\phi}^*)_2'(t, \sum_{i\in I_n} \lambda_i^n a_i(t))\big)\big(\phi'(\,\bar{x}(t)\,) - \sum_{i\in I_n} \lambda_i^n a_i(t)\big)d\mu(t) \\
&\leq E_n \int_T \big|\,\bar{x}(t) - (\tilde{\phi}^*)_2'(t, \sum_{i\in I_n} \lambda_i^n a_i(t))\,\big|d\mu(t) \\
&\leq E_n \int_T Lip(t, E_n, \bar{x})\big|\phi'(\,\bar{x}(t)\,) - \sum_{i\in I_n} \lambda_i^n a_i(t)\big|^\gamma d\mu(t) \\
&\qquad \text{(by assumption (2.48) and Proposition 2.2.1)} \\
&\leq \|Lip(t, E_n, \bar{x})\|_1 E_n^{1+\gamma}.
\end{aligned}$$

■

We can further improve the above estimate by using a bound for $\tilde{\phi}^{*\prime\prime}$. When $\tilde{\phi}^*$ is twice differentiable, we will say $\tilde{\phi}^*(t, v)$ has a **bounded second derivative** in variable v, if for small η, there exists a measurable function $J(\cdot, \eta, \bar{x})$, such that for each $t \in T$ and $v \in [\phi'(\bar{x}(t)) - \eta, \phi'(\bar{x}(t)) + \eta]$,

$$(\tilde{\phi}^*)_2''(t, v) \;\leq\; J(t, \eta, \bar{x}). \tag{2.50}$$

Corollary 2.3.4 *Suppose $\tilde{\phi}^*$ has a bounded second derivative with respect to v in the sense of (2.50). Also assume E_n is small enough that*

$$J(t, E_n, \bar{x}) \in L_1(T, \mu). \tag{2.51}$$

Then

$$I_\phi(\bar{x}) \;\geq\; V(\widetilde{P_n}) \geq V(D_n) \tag{2.52}$$

$$\qquad\quad \geq\; I_\phi(\bar{x}) - \|J(t, E_n, \bar{x})\|_1 E_n^2. \tag{2.53}$$

In particular, when $E_n \to 0$, as $n \to \infty$, (2.52) is true for large n.

Proof: As in the proof of Corollary 2.3.3, for some $\lambda^n \in I\!R^{k(n)}$, we have

$$I_\phi(\bar{x}) - V(D_n) \;\leq\; \int_T \big(\,\bar{x}(t) - (\tilde{\phi}^*)_2'(t, \sum_{i\in I_n} \lambda_i^n a_i(t))\big)\big(\phi'(\,\bar{x}(t)\,) - \sum_{i\in I_n} \lambda_i^n a_i(t)\big)d\mu(t)$$

$$\begin{aligned}
&\leq E_n \int_T \left| \bar{x}(t) - (\tilde{\phi}^*)_2'(t, \sum_{i\in I_n} \lambda_i^n a_i(t)) \right| d\mu(t) \\
&= E_n \int_T \left| (\tilde{\phi}^*)_2'(t, \phi'(\bar{x}(t)) - (\tilde{\phi}^*)_2'(t, \sum_{i\in I_n} \lambda_i^n a_i(t)) \right| d\mu(t) \\
&\qquad \text{(by Proposition 2.2.1)} \\
&= E_n \int_T \left| (\tilde{\phi}^*)_2'(t, v(t))(\phi'(\bar{x}(t)) - \sum_{i\in I_n} \lambda_i^n a_i(t)) \right| d\mu(t) \\
&\qquad \text{(for some } v(t) \in (\phi'(\bar{x}(t)), \textstyle\sum_{i\in I_n} \lambda_i^n a_i(t)), \\
&\qquad \text{using the mean value theorem)} \\
&\leq E_n \int_T J(t, E_n, \bar{x}) \left| \phi'(\bar{x}(t)) - \sum_{i\in I_n} \lambda_i^n a_i(t) \right| d\mu(t) \\
&\qquad \text{(by assumption (2.50))} \\
&\leq \|J(t, E_n, \bar{x})\|_1 E_n^2.
\end{aligned}$$

■

Note that in Corollaries 2.3.3 and 2.3.4, we do not require α or β to be in $L_1(T)$. We now apply the above estimates to some choices of $\tilde{\phi}$. Remember that we have assumed $E_n < +\infty$, and the interesting case is when $E_n \to 0$.

Proposition 2.3 5 *For the Boltzmann-Shannon entropy ϕ defined in (2.17), we have*

$$I_\phi(\bar{x}) \geq V(\widetilde{P_n}) \geq V(D_n) \geq I_\phi(\bar{x}) - \|\bar{x}\|_1 e^{E_n} E_n^2.$$

Proof: Noting that

$$(\tilde{\phi}^*)_2''(t, v) = \phi^{*\prime\prime}(v) = e^v,$$

we may take

$$J(t, \eta, \bar{x}) = \sup\{e^u \mid \phi'(\bar{x}(t)) - \eta \leq u \leq \phi'(\bar{x}(t)) + \eta\} = e^{\phi'(\bar{x}(t))+\eta} = \bar{x}(t)e^\eta,$$

since $\phi'(\bar{x}(t)) = \log \bar{x}(t)$. Then apply Corollary 2.3.4. ■

This recovers a result that was proved in [16]. We note that $e^{E_n} E_n^2 \to 0$ with E_n and that asymptotically it behaves like E_n^2.

Proposition 2.3.6 *For the Fermi-Dirac entropy with arbitrary constant bounds given in (2.19), we have*

$$I_\phi(\bar{x}) \geq V(\widetilde{P_n}) \geq V(D_n) \geq I_\phi(\bar{x}) - \frac{1}{4}(\beta_0 - \alpha_0)\mu(T)E_n^2. \tag{2.54}$$

In particular, when $\alpha_0 \equiv 0, \beta_0 \equiv 1$, we have

$$I_\phi(\bar{x}) \geq V(\widetilde{P_n}) \geq V(D_n) \geq I_\phi(\bar{x}) - \frac{1}{4}\mu(T)E_n^2. \tag{2.55}$$

Proof: Again, we use Corollary 2.3.4, noting that for $t \in T$,

$$\phi_2^{*\prime\prime}(t, v) = \frac{(\beta_0 - \alpha_0)e^v}{(1 + e^v)^2} \leq \frac{1}{4}(\beta_0 - \alpha_0). \tag{2.56}$$

■

The following lemma is needed to deal with the truncated L_p-entropy case.

Lemma 2.3.7 *For any real numbers $A, B \geq 0$, $0 \leq \alpha \leq 1$, we have*

$$|A^\alpha - B^\alpha| \leq |A - B|^\alpha. \tag{2.57}$$

Proof: Define a function

$$f(t) = (1 - t)^\alpha - 1 + t^\alpha, \qquad 0 \leq t \leq 1. \tag{2.58}$$

Since

$$f''(t) = \alpha(\alpha - 1)\big[(1 - t)^{\alpha-2} + t^{\alpha-2}\big] \leq 0, \tag{2.59}$$

f is concave on $[0, 1]$. Then

$$f(0) = f(1) = 0, \text{ implies } f(t) \geq 0, \text{ for all } t \in [0, 1].$$

Now setting $t = A/B$ if $A < B$ or $t = B/A$, if $A > B$, we obtain (2.57). ■

Proposition 2.3.8 *For the truncated L_p-entropy defined in (2.22), we have:*

1. *When $p \geq 2$,*

$$I_\phi(\bar{x}) \geq V(\widetilde{P_n}) \geq V(D_n) \geq I_\phi(\bar{x}) - \mu(T)E_n^q. \tag{2.60}$$

2. *When $1 < p < 2$,*

$$\begin{aligned} I_\phi(\bar{x}) &\geq V(\widetilde{P_n}) \geq V(D_n) \\ &\geq I_\phi(\bar{x}) - (q-1)(\|\bar{x}\|_\infty^{p-1} + E_n)^{q-2}\mu(T)E_n^2, \end{aligned} \tag{2.61}$$

where $1/p + 1/q = 1$.

Proof:

1. Note that $p \geq 2$ implies $q \leq 2$. Using Lemma 2.3.7, we obtain

$$\begin{aligned} |(\tilde{\phi}^*)_2'(t, v_1) - (\tilde{\phi}^*)_2'(t, v_2)| &\leq |v_{1+}^{q-1} - v_{2+}^{q-1}| \\ &\leq |v_{1+} - v_{2+}|^{q-1} \leq |v_1 - v_2|^{q-1}. \end{aligned} \tag{2.62}$$

   Then we may apply Corollary 2.3.3 for $\gamma = q - 1$ and $Lip(t, \eta, \bar{x}) \equiv 1$.

2. Note that we have

$$(\tilde{\phi}^*)_2''(t, v) = (q-1)v_+^{q-2} \leq (q-1)(\bar{x}(t)^{p-1} + \eta)^{q-2}, \tag{2.63}$$

   for $v \in [\bar{x}(t)^{p-1} - \eta, \bar{x}(t)^{p-1} + \eta]$. Corollary 2.3.4 applies for $J(t, \eta, \bar{x}) = (q-1)(\bar{x}(t)^{p-1} + \eta)^{q-2}$.

■

Proposition 2.3.9 *For the bounded Burg entropy of the form in (2.27) for $\beta_0 > 0$, we have*

$$I_\phi(\bar{x}) \geq V(\widetilde{P_n}) \geq V(D_n) \geq I_\phi(\bar{x}) - (\|\bar{x}\|_1 + \beta_0)E_n. \tag{2.64}$$

Proof: Apply Theorem 2.3.2, (2.38), for $\alpha \equiv 0, \beta \equiv \beta_0$. ■

Proposition 2.3.10 *For the Burg-type entropy defined in (2.29), we have*

$$I_\phi(\bar{x}) \geq V(\widetilde{P_n}) \geq V(D_n) \geq I_\phi(\bar{x}) - \frac{1}{8}\mu(T)E_n^2. \tag{2.65}$$

Proof: Apply Corollary 2.3.4 for

$$(\tilde{\phi}^*)_2''(t,v) = \phi^{*\prime\prime}(t,v) = \frac{\sqrt{v^2+4}-2}{v^2\sqrt{v^2+4}} \leq \frac{1}{8}.$$

■

Proposition 2.3.11 *For the Hellinger-type entropy defined in (2.31), we have*

$$I_\phi(\bar{x}) \geq V(\widetilde{P_n}) \geq V(D_n) \geq I_\phi(\bar{x}) - K\mu(T)E_n^2. \tag{2.66}$$

Proof: Apply Corollary 2.3.4 for

$$\phi^{*\prime\prime}(v) = \frac{K}{(v^2+1)^{\frac{3}{2}}} \leq K.$$

■

Remark: The strong duality theorem guarantees that for problems (P_n) (or $(\widetilde{P}_n)$) and (D_n), the duality gap is zero under reasonable constraint qualifications. Even without a constraint qualification, the above theorems give us the explicit bounds not only for

$$V(P_n) - V(D_n)$$

but also more importantly for

$$V(P_n) - I_\phi(\bar{x}) \qquad \text{and} \qquad V(D_n) - I_\phi(\bar{x}),$$

which tell us how well $V(P_n)$ (or $V(D_n)$) approximates to $I_\phi(\bar{x})$.

2.4 Necessary optimality conditions

Consider the following general convex programming problem:

$$(CP) \qquad \inf\{F(x), x \in C\}, \tag{2.67}$$

where X is a Banach space, $F : X \to (-\infty, +\infty]$ is convex, $C \subset X$ is a closed convex set, as discussed in Section 1.4.

A classical necessary condition for $x_0 \in C$ to be an optimal solution of (CP) is

$$\langle g, x - x_0 \rangle \geq 0, \tag{2.68}$$

for some $g \in \partial F(x_0)$ and all $x \in C$.

Now in our problems, we can take $X = L_1(T, \mu)$, $F = I_\phi$, and

$$\begin{aligned} C = \Big\{x \in L_1(T,\mu) \mid \int_T a_i(t)(\,x(t) - \bar{x}(t)\,)d\mu(t) = 0, i \in I_n, \\ \alpha(t) \leq x(t) \leq \beta(t), \text{ a.e. on } T\Big\}. \end{aligned} \tag{2.69}$$

Then the necessary optimality condition for our problem can be stated as:

Theorem 2.4.1 *Suppose x_n is the optimal solution of (P_n). Let $\tilde{x}$ be any feasible solution for (P_n), then*

$$\int_T \phi'(\,x_n(t)\,)(\,\tilde{x}(t) - x_n(t)\,)d\mu(t) \geq 0. \tag{2.70}$$

In particular, we have

$$\int_T \phi'(\,x_n(t)\,)(\,\bar{x}(t) - x_n(t)\,)d\mu(t) \geq 0. \tag{2.71}$$

If we further assume $\alpha(t) \equiv a$, $\beta(t) \equiv b$, and (CQ) holds, i.e.

$$\begin{cases} \textit{there exists } \hat{x} \in L_1(T,\mu), \textit{ such that} \\ a < \hat{x} < b, \textit{ a.e. on } T, \\ \int_T \phi(\,t, \hat{x}(t)\,)d\mu(t) < +\infty, \textit{ and} \\ \int_T a_i(t)\hat{x}(t)d\mu(t) = b_i, \quad i \in I_n, \end{cases} \tag{2.72}$$

then equality holds in (2.70) and (2.71).

Proof: The feasibility of $\tilde{x}$ and x_n implies the feasibility of $x_n + \lambda(\tilde{x} - x_n)$ for any $\lambda \in [0,1]$.

Since x_n is the optimal solution of (P_n) and $I_\phi(x_n) < \infty$, we have

$$I_\phi(x_n + \lambda(\tilde{x} - x_n)) - I_\phi(x_n) \geq 0, \qquad \text{for all } \lambda \in [0,1], \tag{2.73}$$

and hence

$$\frac{1}{\lambda}\int_T [\phi(x_n(t) + \lambda(\tilde{x}(t) - x_n(t))) - \phi(x_n(t))]d\mu(t) \geq 0, \tag{2.74}$$

for all $\lambda \in (0,1]$. For each fixed n, by the convexity of ϕ, we have

$$\frac{\phi(x_n(t) + \lambda(\tilde{x}(t) - x_n(t))) - \phi(x_n(t))}{\lambda} \downarrow \phi'(x_n(t))(\tilde{x}(t) - x_n(t)), \tag{2.75}$$

as $\lambda \downarrow 0$, for all $t \in T$. Note that we agree to use one-sided derivatives when necessary. Hence for almost all $t \in T$ (where both $\phi(\tilde{x}(t))$ and $\phi(x_n(t))$ are finite),

$$\begin{aligned} 0 &\leq \phi(\tilde{x}(t)) - \phi(x_n(t)) - \frac{\phi(x_n(t) + \lambda(\tilde{x}(t) - x_n(t))) - \phi(x_n(t))}{\lambda} \\ &\uparrow \phi(\tilde{x}(t)) - \phi(x_n(t)) - \phi'(x_n(t))(\tilde{x}(t) - x_n(t)). \end{aligned} \tag{2.76}$$

Let

$$f_\lambda(t) = \phi(\tilde{x}(t)) - \phi(x_n(t)) - \frac{\phi(x_n(t) + \lambda(\tilde{x}(t) - x_n(t))) - \phi(x_n(t))}{\lambda}. \tag{2.77}$$

Then

$$f_\lambda(t) \uparrow f(t) \triangleq \phi(\tilde{x}(t)) - \phi(x_n(t)) - \phi'(x_n(t))(\tilde{x}(t) - x_n(t)), \tag{2.78}$$

with $\lambda \to 0$. Noting that $f_1(t) \equiv 0$, using Levi's theorem (see [59]), we obtain

$$\int_T \phi'(x_n(t))(\tilde{x}(t) - x_n(t))d\mu(t) \geq 0. \tag{2.79}$$

In particular, by the feasibility of $\bar{x}$, we have

$$\int_T \phi'(x_n(t))(\bar{x}(t) - x_n(t))d\mu(t) \geq 0. \tag{2.80}$$

We now assume that (CQ) of (2.15) holds, and $\alpha(t) \equiv a$, $\beta(t) \equiv b$. By the duality results, we have

$$x_n(t) = (\tilde{\phi}^*)'_2(t, \sum_{i\in I_n} \lambda_i a_i(t)) \equiv \phi^{*\prime}(\sum_{i\in I_n} \lambda_i a_i(t)), \text{ for some } \lambda \in \mathbb{R}^{k(n)},$$

and

$$\begin{aligned} I_\phi(x_n) &= V(P_n) = V(D_n) \\ &= \int_T \big[\bar{x}(t) \sum_{i\in I_n} \lambda_i a_i(t) - \phi^*(\sum_{i\in I_n} \lambda_i a_i(t))\big] d\mu(t), \end{aligned}$$

and

$$\phi'(x_n) = \sum_{i\in I_n} \lambda_i a_i,$$

which is finite everywhere. Then

$$\begin{aligned} &\int_T \phi'(\,x_n(t)\,)(\,\tilde{x}(t) - x_n(t)\,) d\mu(t) \\ &= \int_T \big[\tilde{x}(t)\phi'(\,x_n(t)\,) - x_n(t)\phi'(\,x_n(t)\,)\big] d\mu(t) \\ &= \int_T \big[\tilde{x}(t)\phi'(\,x_n(t)\,) - \phi(\,x_n(t)\,) - \phi^*(\,\phi'(\,x_n(t)\,)\,)\big] d\mu(t) \\ &\qquad\qquad \text{(convexity of } \phi \text{ and (1.14))} \\ &= \int_T \big[\tilde{x}(t)\phi'(\,x_n(t)\,) - \bar{x}(t) \sum_{i\in I_n} \lambda_i a_i(t) + \phi^*(\sum_{i\in I_n} \lambda_i a_i(t)) - \phi^*(\,\phi'(\,x_n(t)\,)\,)\big] d\mu(t) \\ &= \int_T \big[\tilde{x}(t)(\phi'(\,x_n(t)\,) - \sum_{i\in I_n} \lambda_i a_i(t)) + \phi^*(\sum_{i\in I_n} \lambda_i a_i(t)) - \phi^*(\,\phi'(\,x_n(t)\,)\,)\big] d\mu(t) \\ &\qquad\qquad \text{(feasibility of } \bar{x} \text{ and } \tilde{x}) \\ &= 0, \qquad (\text{ since } \phi'(x_n) = \sum_{i\in I_n} \lambda_i a_i). \end{aligned}$$

In particular,

$$\int_T \phi'(\,x_n(t)\,)(\,\bar{x}(t) - x_n(t)\,) d\mu(t) = 0. \tag{2.81}$$

■

Corollary 2.4.2 *Let x_n be the optimal solution of the problem (P_n), let $\tilde{x}$ be any feasible solution for (P_n). Then:*

1. *For the Boltzmann-Shannon entropy, we have*

$$\int_T \log x_n(t)(\,\tilde{x}(t) - x_n(t)\,)d\mu(t) \geq 0. \tag{2.82}$$

2. *For the Fermi-Dirac entropy, we have*

$$\int_T \log(\frac{x_n(t) - \alpha_0}{\beta_0 - x_n(t)})(\,\tilde{x}(t) - x_n(t)\,)d\mu(t) \geq 0. \tag{2.83}$$

3. *For Burg's entropy, we have*

$$\int_T -\frac{\tilde{x}(t) - x_n(t)}{x_n(t)} d\mu(t) \geq 0. \tag{2.84}$$

4. *For the truncated L_p-entropy, we have*

$$\int_T x_n^{p-1}(t)(\,\tilde{x}(t) - x_n(t)\,)d\mu(t) \geq 0. \tag{2.85}$$

5. *For the Burg-type entropy defined in (2.29),we have*

$$\int_T \frac{2x_n(t) - 1}{x_n(t)(1 - x_n(t))}(\,\tilde{x}(t) - x_n(t)\,)d\mu(t) \geq 0. \tag{2.86}$$

6. *For the Hellinger-type entropy defined in (2.31),we have*

$$\int_T \frac{x_n(t) - K}{\sqrt{x_n(t)(2K - x_n(t))}}(\,\tilde{x}(t) - x_n(t)\,)d\mu(t) \geq 0. \tag{2.87}$$

Moreover, the equality holds in 1, 2, 3, 5 and 6 under (CQ).

Proof: Simply compute the derivatives of the corresponding ϕ. ∎

Note that in the truncated L_p-entropy case, the equality fails since $\alpha(t) \equiv 0$, while $a = -\infty$.

2.5 Norm convergence

In Theorem 2.3.2, Corollaries 2.3.3 and 2.3.4, we gave error bounds for $V(P_n)-V(D_n)$ and $I_{\tilde{\phi}}(\bar{x})-V(P_n)$. If x_n is the optimal solution of (P_n), we thus have the error bounds for $I_{\tilde{\phi}}(\bar{x}) - I_{\tilde{\phi}}(x_n)$.

When

$$\phi'(\bar{x}) \in \overline{span}\{a_i, i \in \bigcup_{n=1}^{\infty} I_n\}, \tag{2.88}$$

(where the closure is taken in the sense of supreme norm), it is clear that

$$E_n \to 0, \qquad \text{as } n \to \infty, \tag{2.89}$$

which implies

$$I_{\tilde{\phi}}(x_n) \to I_{\tilde{\phi}}(\bar{x}), \qquad \text{as } n \to \infty. \tag{2.90}$$

Our next goal is to consider norm convergence in the sense that

$$\|x_n - \bar{x}\|_p \triangleq \Big(\int_T |x_n(t) - \bar{x}(t)|^p d\mu(t)\Big)^{\frac{1}{p}} \to 0, \tag{2.91}$$

for $p \geq 1$. Theorems on norm convergence have been proved in [16], [23], [79] and [107] assuming the strict convexity of $I_{\tilde{\phi}}$. Here we wish to relate the rate of norm convergence to E_n recalled in (2.35). We begin by giving lower bounds (in terms of the norm error $\|x_n - \bar{x}\|_p$) for the difference between the primal values of x_n and $\bar{x}$. We will then be able to use the results given in Section 2.3 and deduce the norm convergence rate, or more precisely estimates.

First, we impose a strong convexity assumption $(A7)$ for ϕ :

There exists $\delta > 0, r > 1$, such that

$$\phi(u_1) - \phi(u_2) - \phi'(u_2)(u_1 - u_2) \geq \delta |u_1 - u_2|^r, \tag{2.92}$$

for all $u_1, u_2 \in (\text{essinf}(\alpha), \text{esssup}(\beta))$.

Theorem 2.5.1 *Let $(A1)-(A7)$ be true for some $r>1$ and $\delta>0$, x_n be the optimal solution of (P_n). Then*

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq \delta\|\bar{x} - x_n\|_r^r. \tag{2.93}$$

Proof: By Theorem 2.4.1,

$$\begin{aligned} I_\phi(\bar{x}) - I_\phi(x_n) &= \int_T \big[\phi(\,\bar{x}(t)\,) - \phi(\,x_n(t)\,)\big]d\mu(t) \\ &\geq \int_T \big[\phi(\,\bar{x}(t)\,) - \phi(\,x_n(t)\,) + \phi'(\,x_n(t))(\,x_n(t) - \bar{x}(t))\big]d\mu(t) \\ &\geq \int_T \delta\big|\bar{x}(t) - x_n(t)\big|^r d\mu(t) = \delta\|\bar{x} - x_n\|_r^r. \end{aligned}$$

■

Combining with Theorem 2.3.2 or its corollaries, it follows that

$$\|\bar{x} - x_n\|_r \to 0, \text{ as } n \to \infty, \tag{2.94}$$

provided that $E_n \to 0$.

Symmetrically, we can require the strong convexity of the conjugate function ϕ^*.
$(A7')$: There exists $\delta > 0, r > 1$, such that

$$\phi^*(v_1) - \phi^*(v_2) - \phi^{*\prime}(v_2)(v_1 - v_2) \geq \delta|v_1 - v_2|^r, \tag{2.95}$$

for all $v_1, v_2 \in \big(\text{essinf}(\,\phi'(\alpha)\,), \text{esssup}(\,\phi'(\beta)\,)\big)$.

Theorem 2.5.2 *Let $(A1)-(A6)$ and $(A7')$ hold. Let x_n be the optimal solution of (P_n). Then*

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq \delta\|\phi'(\bar{x}) - \phi'(x_n)\|_r^r. \tag{2.96}$$

Proof: Again from Theorem 2.4.1 and the conjugate property (1.14),

$$\begin{aligned} &I_\phi(\bar{x}) - I_\phi(x_n) \\ &= \int_T \big[\phi(\,\bar{x}(t)\,) - \phi(\,x_n(t)\,)\big]d\mu(t) \end{aligned}$$

$$
\begin{aligned}
&\geq \int_T \left[\phi(\bar{x}(t)) - \phi(x_n(t)) + \phi'(x_n(t))(x_n(t) - \bar{x}(t))\right] d\mu(t) \\
&= \int_T \left[\phi(\bar{x}(t)) - \phi(x_n(t)) + \phi'(x_n(t))x_n(t) - \phi'(x_n(t))\bar{x}(t)\right] d\mu(t) \\
&= \int_T \left[\phi(\bar{x}(t)) + \phi^*(\phi'(x_n(t))) - \bar{x}(t)\phi'(x_n(t))\right] d\mu(t) \\
&= \int_T \left[\bar{x}(t)\phi(\bar{x}(t)) - \phi^*(\phi'(\bar{x}(t))) + \phi^*(\phi'(x_n)(t)) - \bar{x}(t)\phi'(x_n)(t)\right] d\mu(t) \\
&= \int_T \left[\phi^*(\phi'(x_n(t))) - \phi^*(\phi'(\bar{x}(t))) - \phi^{*\prime}(\phi'(\bar{x}(t)))(\phi'(x_n(t)) - \phi'(\bar{x}(t)))\right] d\mu(t) \\
&\geq \int_T \delta\left|\phi'(x_n)(t) - \phi'(\bar{x}(t))\right|^r d\mu(t) \\
&= \delta\|\phi'(\bar{x}) - \phi'(x_n)\|_r^r.
\end{aligned}
$$

■

Clearly, Assumption $(A7)$ and $(A7')$ are implied by the strict positivity of the second derivatives of ϕ and ϕ^*, respectively. We then have the following corollaries. Note that the conditions we will invoke in the corollaries below are slightly weaker than the uniform boundedness of ϕ'' or $\phi^{*\prime\prime}$.

Corollary 2.5.3 *Let ϕ be twice continuously differentiable on (a, b). Define for each $\eta > 0$,*

$$
S(\eta) \triangleq \{u \in \mathbb{R} \mid \phi''(u) \geq \eta\}. \tag{2.97}
$$

Let x_n be the optimal solution of (P_n), and suppose there exists $\eta_0 > 0$ such that for almost all $t \in T$,

$$
\overline{co}\{x_n(t), \bar{x}(t)\} \subseteq S(\eta_0), \tag{2.98}
$$

then

$$
I_\phi(\bar{x}) - I_\phi(x_n) \geq \eta_0\|\bar{x} - x_n\|_2^2. \tag{2.99}
$$

Proof: As in the proof of Theorem 2.5.1, by the mean value theorem,

$$
I_\phi(\bar{x}) - I_\phi(x_n)
$$

$$
\begin{aligned}
&\geq \int_T \left[\phi(\bar{x}(t)) - \phi(x_n(t)) + \phi'(x_n(t))\big(x_n(t) - \bar{x}(t)\big)\right] d\mu(t) \\
&= \int_T \left[\phi''(\xi(t))(x_n(t) - \bar{x}(t))^2\right] d\mu(t) \\
&\qquad \text{(for some } \xi(t) \in \bar{co}\{x_n(t), \bar{x}(t)\}) \\
&\geq \eta_0 \int_T (x_n(t) - \bar{x}(t))^2 d\mu(t) \qquad \text{(by (2.98))} \\
&= \eta_0 \|x_n - \bar{x}\|_2^2.
\end{aligned}
$$

■

Corollary 2.5.4 *Let ϕ^* be twice continuously differentiable on $(\phi'(a), \phi'(b))$. For each $\eta > 0$, we write*

$$S^*(\eta) \triangleq \{v \in I\!R \mid \phi^{*\prime\prime}(v) \geq \eta\}. \tag{2.100}$$

Let x_n be the optimal solution of (P_n), and suppose there exists $\eta_0 > 0$ such that for almost all $t \in T$,

$$\overline{co}\{\phi'(x_n(t)), \phi'(\bar{x}(t))\} \subseteq S^*(\eta_0). \tag{2.101}$$

Then

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq \eta_0 \|\phi'(\bar{x}) - \phi'(x_n)\|_2^2. \tag{2.102}$$

Proof: This is analogous to the proof of Corollary 2.5.3. ■

In combination with the estimates given in Section 2.3, we have not only proved $\|\bar{x} - x_n\| \to 0$ or $\|\phi'(\bar{x}) - \phi'(x_n)\| \to 0$, but also have compared the "rate" of norm convergence with that of $E_n \to 0$. We can further weaken the conditions (A7) and (A7') just to guarantee $\|\bar{x} - x_n\| \to 0$ or $\|\phi'(\bar{x}) - \phi'(x_n)\| \to 0$. We will say that ϕ satisfies (A8), if for some $r > 0$,

$$\phi(u_1) - \phi(u_2) - \phi'(u_2)(u_1 - u_2) \geq H(u_1, |u_1 - u_2|^r), \tag{2.103}$$

where $H : (\text{dom}\phi) \times I\!R_+ \to I\!R_+$ is monotonic in the first variable, strictly increasing and convex in the second variable, and $H(u_1, 0) \equiv 0$.

Theorem 2.5.5 *Let (A8) hold for some function H and $r > 0$, x_n be the optimal solution of the problem (P_n). Then if H is nondecreasing in the first variable,*

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq H\big(\inf_{t\in T} \bar{x}(t), \|\bar{x} - x_n\|_r^r\big). \tag{2.104}$$

Correspondingly, if H is nonincreasing in the first variable,

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq H\big(\sup_{t\in T} \bar{x}(t), \|\bar{x} - x_n\|_r^r\big). \tag{2.105}$$

Proof: Without the loss of generality, we only prove the "inf" case. By Theorem 2.4.1,

$$\begin{aligned}
& I_\phi(\bar{x}) - I_\phi(x_n) \\
\geq\ & \int_T \big[\phi(\bar{x}(t)) - \phi(x_n(t)) + \phi'(x_n(t))(x_n(t) - \bar{x}(t))\big] d\mu(t) \\
\geq\ & \int_T \big[H\big(\bar{x}(t), |\bar{x}(t) - x_n(t)|^r\big) d\mu(t) \\
\geq\ & \int_T \big[H\big(\inf_{t\in T} \bar{x}(t), |\bar{x}(t) - x_n(t)|^r\big) d\mu(t) \\
& \qquad \text{(by the monotonicity of } H \text{ in the first variable)} \\
\geq\ & H\big(\inf_{t\in T} \bar{x}(t), \int_T |\bar{x}(t) - x_n(t)|^r d\mu(t)\big) \\
& \qquad \text{(by the convexity of } H \text{ in the second variable)} \\
=\ & H\big(\inf_{t\in T} \bar{x}(t), \|\bar{x}(t) - x_n(t)\|_r^r\big).
\end{aligned}$$

∎

Together with the theorems in Section 2.3 and strict monotonicity of H in the second variable, the inverse of $H(\inf_{t\in T} \bar{x}(t), \cdot)$ exists, and noting that $H(u_1, 0) = 0$, we have

$$\|\bar{x} - x_n\|_r \to 0 \qquad \text{when} \qquad I_\phi(x_n) - I_\phi(\bar{x}) \to 0. \tag{2.106}$$

Analogously, we will say ϕ^* satisfy (A8'), if for some $r > 0$,

$$\phi^*(v_1) - \phi^*(v_2) - \phi^{*\prime}(v_2)(v_1 - v_2) \geq H^*(v_2, |v_1 - v_2|^r), \tag{2.107}$$

where $H^* : (\mathrm{dom}\phi^*) \times I\!R_+ \to I\!R_+$ is monotonic in the first variable, strictly increasing and convex in the second variable, and $H^*(v_1, 0) \equiv 0$.

Theorem 2.5.6 *Let (A8') hold for some function H^* and $r > 0$, and let x_n be the optimal solution of the problem (P_n). Then if H^* is nondecreasing in the first variable,*

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq H^*\big(\inf_{t\in T} \phi'(\bar{x}(t)), \|\phi'(\bar{x}) - \phi'(x_n)\|_r^r\big). \tag{2.108}$$

Correspondingly, if H^* is nonincreasing in the first variable,

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq H^*\big(\sup_{t\in T} \phi'(\bar{x}(t)), \|\phi'(\bar{x}) - \phi'(x_n)\|_r^r\big). \tag{2.109}$$

Proof: Similar to the proof of Theorem 2.5.2 and 2.5.5. ■

Hence when $I_\phi(\bar{x}) - I_\phi(x_n) \to 0$, as has been discussed before, we will have $\|\phi'(\bar{x}) - \phi'(x_n)\|_r^r \to 0$, as $n \to 0$. We now come back to the examples of Section 2.2, and combine our lower bounds with the upper bounds given in Section 2.3.

Proposition 2.5.7 *Consider the problem (P_n) in (2.1), where ϕ is the Fermi-Dirac entropy defined in (2.19). Let x_n be the optimal solution of (P_n). Then*

$$\|\bar{x} - x_n\|_2 \leq \frac{\beta_0 - \alpha_0}{4}\sqrt{\mu(T)}E_n. \tag{2.110}$$

Proof: Since

$$\phi''(u) = \frac{1}{u-\alpha_0} + \frac{1}{\beta_0 - u} = \frac{\beta_0 - \alpha_0}{(u-\alpha_0)(\beta_0-u)} \geq \frac{4}{\beta_0 - \alpha_0},$$

we apply Corollary 2.5.3 and get

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq \frac{4}{\beta_0 - \alpha_0}\|\bar{x} - x_n\|_2^2. \tag{2.111}$$

Together with (2.54) in Proposition 2.3.6, we have

$$\frac{4}{\beta_0 - \alpha_0}\|\bar{x} - x_n\|_2^2 \leq \frac{\beta_0 - \alpha_0}{4}\mu(T)E_n^2, \tag{2.112}$$

and the desired inequality follows. ■

Proposition 2.5.8 *Consider the problem (P_n), where ϕ is the Burg-type entropy defined in (2.29). Let x_n be the optimal solution of (P_n). Then*

$$\|\bar{x} - x_n\|_2 \leq \frac{1}{8}E_n. \tag{2.113}$$

Proof: Since

$$\phi''(u) = \frac{1}{u^2} + \frac{1}{(1-u)^2} \geq 8,$$

we apply Corollary 2.5.3 and get

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq 8\|\bar{x} - x_n\|_2^2.$$

On the other hand, we can see

$$\phi^{*\prime\prime}(v) = \frac{\sqrt{v^2+4}-2}{v^2\sqrt{v^2+4}} \leq \frac{1}{8}.$$

Then Corollary 2.3.4 tells us

$$I_\phi(\bar{x}) - I_\phi(x_n) \leq \frac{1}{8}E_n^2, \tag{2.114}$$

and we have

$$\|\bar{x} - x_n\|_2 \leq \frac{1}{8}E_n. \tag{2.115}$$

■

To obtain a corresponding result for the truncated L_p-entropy, we need the following inequalities.

Lemma 2.5.9 *For any real numbers $A, B \geq 0$, $\alpha \geq 1$, we have*

$$(A+B)^\alpha \geq A^\alpha + B^\alpha.$$

Proof: This is obvious when A or B is zero. We assume $A, B > 0$. Define a function

$$f(t) = (1+t)^\alpha - 1 - t^\alpha, \qquad t \geq 0. \tag{2.116}$$

Then

$$f'(t) = \alpha(1+t)^{\alpha-1} - \alpha t^{\alpha-1} = \alpha((1+t)^{\alpha-1} - t^{\alpha-1}) \geq 0.$$

This implies

$$f(t) \geq f(0) = 0,$$

and the inequality (2.116) follows when we set $t = B/A$. ■

Lemma 2.5.10 *For any $A, B \geq 0$, $q \geq 2$, we have*

$$\frac{1}{p}A^q + \frac{1}{q}B^q - A^{q-1}B \geq \frac{1}{q}|A-B|^q. \tag{2.117}$$

Proof: This is trivial when $A = 0$, so we assume $A > 0$. Let

$$f(t) = \frac{1}{p} + \frac{1}{q}t^q - t - \frac{1}{q}|1-t|^q, \qquad t \geq 0. \tag{2.118}$$

For $t \leq 1$, using Lemma 2.5.9, we have

$$f'(t) = t^{q-1} - 1 + (1-t)^{q-1} \leq (t+1-t)^{q-1} - 1 = 0.$$

For $t \geq 1$, we again use Lemma 2.5.9 and get

$$\begin{aligned} f'(t) &= t^{q-1} - 1 - (t-1)^{q-1} = (t-1+1)^{q-1} - 1 - (t-1)^{q-1} \\ &\geq (t-1)^{q-1} + 1^{q-1} - 1 - (t-1)^{q-1} = 0. \end{aligned}$$

So f attains its minimum at $t = 1$, i.e.

$$f(t) \geq f(1) = 0, \; t \geq 0.$$

Then (2.117) follows when we set $t = B/A$. ■

Now we have

Proposition 2.5.11 *Consider problem (P_n), where ϕ is the truncated L_p-entropy defined in (2.22). Let x_n be the optimal solution of (P_n). Then for $p \geq 2$,*

$$\|\bar{x} - x_n\|_p \leq \left(p\mu(T)\right)^{\frac{1}{p}} E_n^{q-1}. \tag{2.119}$$

and for $p \leq 2$, or equivalently, $q \geq 2$,

$$\|\bar{x}^{p-1} - x_n^{p-1}\|_q \leq \left(q(q-1)(\|\bar{x}\|_\infty^{p-1} + E_n)^{q-2}\mu(T)\right)^{\frac{1}{q}} E_n^{\frac{2}{q}}. \tag{2.120}$$

Proof: First let $p \geq 2$ and hence $q \leq 2$. From Proposition 2.3.8 (1), we have

$$I_\phi(\bar{x}) - I_\phi(x_n) \leq \mu(T)E_n^q. \tag{2.121}$$

From Lemma 2.5.10, we have

$$\begin{aligned}
&\phi(u_1) - \phi(u_2) - \phi'(u_2)(u_1 - u_2) \\
&= \frac{1}{p}u_1^p - \frac{1}{p}u_2^p - u_2^{p-1}(u_1 - u_2) \\
&= \frac{1}{q}u_2^p + \frac{1}{p}u_1^p - u_2^{p-1}u_1 \\
&\geq \frac{1}{p}|u_1 - u_2|^p.
\end{aligned}$$

Then using Theorem 2.5.1, we obtain

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq \frac{1}{p}\|\bar{x} - x_n\|_p^p. \tag{2.122}$$

Combining (2.121) and (2.122), we have

$$\|\bar{x} - x_n\|_p \leq (p\mu(T))^{\frac{1}{p}} E_n^{q-1}. \tag{2.123}$$

Now we let $p \leq 2$ which implies $q \geq 2$. From Proposition 2.3.8 (2), we have

$$I_\phi(\bar{x}) - I_\phi(x_n) \leq (q-1)(\|\bar{x}\|_\infty^{p-1} + E_n)^{q-2}\mu(T)E_n^2. \tag{2.124}$$

Applying Lemma 2.5.10 in the following way, we see that

$$\begin{aligned}
&\phi^*(v_1) - \phi^*(v_2) - \phi^{*\prime}(v_2)(v_1 - v_2) \\
&= \frac{1}{q}v_1^q - \frac{1}{q}v_2^q - v_2^{q-1}(v_1 - v_2) \\
&= \frac{1}{q}v_1^q + \frac{1}{p}v_2^q - v_1 v_2^{q-1} \\
&\geq \frac{1}{q}|v_1 - v_2|^q.
\end{aligned}$$

We then apply Theorem 2.5.2 and get

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq \frac{1}{q}\|\bar{x}^{p-1} - x_n^{p-1}\|_q^q. \tag{2.125}$$

Combining (2.124) and (2.125), we obtain

$$\|\bar{x}^{p-1} - x_n^{p-1}\|_q^q \leq q(q-1)(\|\bar{x}\|_\infty^{p-1} + E_n)^{q-2}\mu(T)E_n^2,$$

and (2.120) follows. ∎

Proposition 2.5.12 *Consider problem (P_n), where ϕ is the Boltzmann-Shannon entropy defined in (2.17). Let x_n be the optimal solution of (P_n), and for some $M > 0$, $\|x_n\|_\infty \leq M$ for all n. Then*

$$\|\bar{x} - x_n\|_2 \leq (\max\{M, \|\bar{x}\|_\infty\}\|\bar{x}\|_1)^{\frac{1}{2}} e^{\frac{1}{2}E_n} E_n. \tag{2.126}$$

Hence for $\bar{x} \in L_\infty(T, \mu)$, we have $\|\bar{x} - x_n\|_2 = O(E_n)$ as $E_n \to 0$.

Proof: Since for some $M > 0$, $\|x_n\|_\infty \leq M$, for all $n \in \mathbb{N}$. We may take

$$\eta_0 = \min\{\frac{1}{M}, \frac{1}{\|\bar{x}\|_\infty}\}.$$

Then, applying Corollary 2.5.3 and Proposition 2.3.5, we have

$$\eta_0 \|\bar{x} - x_n\|_2^2 \leq \|\bar{x}\|_1 e^{E_n} E_n^2,$$

and (2.126) follows. ∎

Note that $\|\bar{x}\|_\infty = \infty$ might occur. Also, to avoid needing the uniform boundedness of $\{x_n\}$, we may check condition (A8) defined in (2.103).

Lemma 2.5.13 *For the Boltzmann-Shannon entropy defined in (2.17), (A8) holds.*

Proof: For $u_1, u_2 \geq 0$, we first claim

$$\begin{aligned} &\phi(u_1) - \phi(u_2) - \phi'(u_2)(u_1 - u_2) \\ &\quad \geq -u_1 \log(1 + \frac{|u_2 - u_1|}{u_1}) + |u_2 - u_1|. \end{aligned} \tag{2.127}$$

To show (2.127), we only need to check the inequality for $u_2 - u_1 < 0$, since the other case is clear when we write out ϕ and ϕ' explicitly.

Define a function for $0 \leq s \leq u_1$,

$$\begin{aligned} f(s) &= -u_1 \log(1 - \frac{s}{u_1}) - s + u_1 \log(1 + \frac{s}{u_1}) - s \\ &= -u_1 \log(u_1 - s) + u_1 \log(u_1 + s) - 2s. \end{aligned} \tag{2.128}$$

Then

$$f'(s) = \frac{u_1}{u_1 - s} + \frac{u_1}{u_1 + s} - 2 = \frac{2u_1^2}{u_1^2 - s^2} - 2 \geq 0,$$

since $|s| < |u_1|$. So $f(s) \geq f(0) = 0$, and hence (2.127) holds.

Now in (2.103), for $r = 1$, we take

$$H(x, y) = -x \log(1 + \frac{y}{x}) + y, \qquad x, y \geq 0. \tag{2.129}$$

Noting that $\log x \leq x - 1$ for $x > 0$, we have

$$H'_x(x, y) = \log(\frac{x}{x + y}) + 1 - \frac{x}{x + y} \leq 0.$$

Hence H is nonincreasing in the first variable. On the other hand we see that

$$H'_y(x, y) = \frac{y}{x + y} \geq 0,$$

for $x, y \geq 0$, and equality holds only if $y = 0$. Also

$$H''_y(x, y) = \frac{x}{(x + y)^2} \geq 0,$$

and hence H is strictly increasing and convex in the second variable on $y \geq 0$. Thus (A8) holds. ∎

Combining Proposition 2.3.5, Theorem 2.5.5, and Lemma 2.5.13, we have

Proposition 2.5.14 *Consider problem (P_n), where ϕ is the Boltzmann-Shannon entropy defined in (2.17). Let x_n be the optimal solution of (P_n). Further suppose $\sup_{t \in T} \bar{x}(t) \leq 1/\delta$, for some $\delta > 0$. Then*

$$\delta \|\bar{x} - x_n\|_1 - \log(1 + \delta \|\bar{x} - x_n\|_1) \leq \delta \|\bar{x}\|_1 e^{E_n} E_n^2. \tag{2.130}$$

Therefore if $E_n \to 0$, we have

$$\|\bar{x} - x_n\|_1 \to 0$$

∎

This again recovers the result obtained by Borwein and Lewis in [16]. Using (A8'), we can even get the similar result for Burg's entropy.

Lemma 2.5.15 *For Burg's entropy defined in (2.25), (A8') holds.*

Proof: For $v_1, v_2 < 0$, we claim

$$\begin{aligned} & \phi^*(v_1) - \phi^*(v_2) - \phi^{*\prime}(v_2)(v_1 - v_2) \\ \geq \; & -\log(1 + \frac{|v_1 - v_2|}{-v_2}) + \frac{1}{-v_2}|v_1 - v_2|. \end{aligned} \tag{2.131}$$

For the same reason as in Lemma 2.5.13, we only need to check the last inequality for $v_2 - v_1 < 0$.

Define for $0 \leq s \leq -v_2$,

$$\begin{aligned} f(s) &= -\log(1 + \frac{s}{v_2}) + \frac{s}{v_2} + \log(1 + \frac{s}{-v_2}) - \frac{s}{-v_2} \\ &= \log(\frac{-v_2 + s}{-v_2 - s}) + \frac{2s}{v_2}. \end{aligned} \tag{2.132}$$

Then

$$\begin{aligned} f'(s) &= \frac{1}{-v_2 + s} + \frac{1}{-v_2 - s} + \frac{2}{v_2} \\ &= \frac{-2v_2}{v_2^2 - s^2} + \frac{2}{v_2} = \frac{2s^2}{v_2(s^2 - v_2^2)} > 0, \end{aligned} \tag{2.133}$$

since $|s| \leq |v_2|$ and $v_2 < 0$. This implies the inequality (2.131).

Now in order to check (A8'), for $r = 1$, we may take

$$H^*(x, y) = -\log(1 + \frac{y}{x}) + \frac{y}{x} = -\log(x + y) + \log(x) + \frac{y}{x}, \tag{2.134}$$

for $x, y > 0$. Noting that

$$\begin{aligned} H_x^{*\prime}(x, y) &= -\frac{1}{x + y} + \frac{1}{x} - \frac{y}{x^2} = \frac{y}{x(x + y)} - \frac{y}{x^2} \\ &= \frac{y}{x}(\frac{1}{x + y} - \frac{1}{x}) < 0, \end{aligned}$$

we see that H is nonincreasing in the first variable. On the other hand, we also have

$$H_y^{*\prime}(x,y) = -\frac{1}{x+y} + \frac{1}{x} > 0, \qquad H_y^{*\prime\prime}(x,y) = \frac{1}{(x+y)^2} > 0.$$

Thus H is strictly increasing and convex in the second variable on $y > 0$. So (A8') holds. ∎

Proposition 2.5.16 *Consider the problem (P_n), where ϕ is Burg's entropy defined in (2.25). Let x_n be the optimal solution of (P_n). Further suppose $E_n \to 0$, and*

$$0 < \frac{1}{M} \le \bar{x}(t) \le M < +\infty, \tag{2.135}$$

for some $M > 0$. Then

$$-\frac{1}{M}\|\frac{1}{\bar{x}} - \frac{1}{x_n}\|_1 - \log\left(1 - \frac{1}{M}\|\frac{1}{\bar{x}} - \frac{1}{x_n}\|_1\right) \le 4M^2E_n^2, \tag{2.136}$$

for large n. This implies

$$\|\frac{1}{\bar{x}} - \frac{1}{x_n}\|_1 \to 0, \text{ as } n \to \infty. \tag{2.137}$$

Proof: Since $\bar{x}(t) \le M$ implies $1/\bar{x}(t) \ge 1/M$, we have

$$\phi'(\bar{x}(t)) = -\frac{1}{\bar{x}(t)} \le -\frac{1}{M} < 0.$$

For n large enough such that $E_n \le 1/2M$, there exists $\lambda^n \in \mathbb{R}^{k(n)}$ with the property

$$\|\sum_{i\in I_n} \lambda_i^n a_i - \phi'(\bar{x})\|_\infty = E_n \le \frac{1}{2M}.$$

Then

$$\sum_{i\in I_n} \lambda_i^n a_i \le \phi'(\bar{x}) + \frac{1}{2M} \le -\frac{1}{M} + \frac{1}{2M} < 0,$$

which implies that $\phi^*(\sum_{i\in I_n} \lambda_i^n a_i)$ and $\phi^{*\prime}(\sum_{i\in I_n} \lambda_i^n a_i)$ are finite. Noting that

$$\phi^{*\prime\prime}(\sum_{i\in I_n} \lambda_i^n a_i) = \frac{1}{(\sum_{i\in I_n} \lambda_i^n a_i)^2} \le 4M^2,$$

applying Corollary 2.3.4, we obtain

$$I_\phi(\bar{x}) - I_\phi(x_n) \leq 4M^2\mu(T)E_n^2. \tag{2.138}$$

On the other hand, using Theorem 2.5.6 and $\bar{x} \geq 1/M$,

$$\begin{aligned}
& I_\phi(\bar{x}) - I_\phi(x_n) \\
& \geq \; H^*(\inf_{t\in T} \phi'(\bar{x}), \|\phi'(\bar{x}) - \phi'(x_n)\|_1) \\
& \geq \; H^*(-M, \|\frac{1}{\bar{x}} - \frac{1}{x_n}\|_1) \\
& = \; -\log(1 - \frac{1}{M}\|\frac{1}{\bar{x}} - \frac{1}{x_n}\|_1) - \frac{1}{M}\|\frac{1}{\bar{x}} - \frac{1}{x_n}\|_1.
\end{aligned} \tag{2.139}$$

Hence, combining (2.138) and (2.139), we have

$$-\log(1 - \frac{1}{M}\|\frac{1}{\bar{x}} - \frac{1}{x_n}\|_1) - \frac{1}{M}\|\frac{1}{\bar{x}} - \frac{1}{x_n}\|_1 \leq 4M^2\mu(T)E_n^2, \tag{2.140}$$

and the result follows. ∎

Chapter 3

Uniform Convergence Theorems

3.1 Introduction

Using a maximum entropy method to solve moment problems requires minimizing some measure of entropy/information, a convex integral functional of the density, subject to the given moment constraints. In doing this we hope that the estimates will converge to the unknown density as the number of known moments increases. As proved or stated in many recent papers (see, for example, [21], [18]), we know that weak-star convergence hold almost unconditionally ([50], [82], [79]), and weak convergence can be guaranteed if the level sets of the objective function are weakly compact ([18], [23], [50], [79]). To obtain norm convergence, we require more assumptions such as strict convexity ([16], [79], [110]). Also, a uniform convergence theorem has been proved for the Boltzmann-Shannon entropy in [16] and generally for analytic underlying functions in [28].

The main results which will be proved in this chapter are some uniform convergence theorems for moment problems with entropy-like objectives. We will specialize these theorems to many well known entropies.

As observed in Section 2.1, we know that because of the bound constraints $\alpha(\cdot)$ and $\beta(\cdot)$, ϕ^* is a piecewise defined function. Under the given (CQ), the optimal solution x_n of (P_n) has been seen to be truncated at α and β. However, for entropies such as the Fermi-Dirac entropy, where α and β are simply constants, we see that $\phi'(\alpha) = -\infty$ and $\phi'(\beta) = +\infty$, and then the truncation in the expression of x_n disappears. So the bound constraints are automatically fulfilled. In the next section, we first discuss uniform convergence for this case, which is easier to deal with. In Section 3.6, we return to the truncation-type entropy, where we will need to use Remez-type inequalities.

The reader is reminded that an index of assumptions is given in Appendix A.

3.2 Uniform convergence theorems for FD-type entropies

First we are going to deal with the FD-type entropies, named after the Fermi-Dirac entropy which has the form in (2.19). Using this entropy, the bound constraints are automatically included in the objective function. The function ϕ and its conjugate function ϕ^* are both essentially smooth and essentially strictly convex on $\mathbb{R}$. The optimal solution of the problem (P_n) has the form

$$x_n(t) = \phi^{*\prime}(\sum_{i\in I_n} \lambda_i a_i(t)). \tag{3.1}$$

Knowing this makes uniform convergence theorems much easier to prove.

More generally, we consider the following FD-type problem

$$(FDP_n) \qquad \begin{cases} \inf & \int_T \phi(x(t))d\mu(t), \\ s.t. & \int_T a_i(t)(x(t) - \bar{x}(t))d\mu(t) = 0, \quad i \in I_n, \\ & x \in L_1(T, \mu), \end{cases} \tag{3.2}$$

where (T, μ) is a complete finite measure space, $a_i \in L_\infty(T, \mu)$, for $i \in I_n$, and the function $\phi : \mathbb{R} \to (-\infty, +\infty]$ is a convex integrand with domain $\mathrm{dom}(\phi)$ satisfying $(a, b) \subseteq \mathrm{dom}(\phi) \subseteq [a, b]$ for finite numbers a and b. We also assume: ϕ is essentially strict convex and essentially smooth on the $\mathrm{dom}(\phi)$, which implies

$$\lim_{u\to a^+} \phi'(u) = -\infty, \qquad \lim_{u\to b^-} \phi'(u) = +\infty. \tag{3.3}$$

Then the conjugate function ϕ^* is everywhere finite. It is also essentially strictly convex and essentially smooth on $I\!R$, with the useful property: $\phi^{*\prime} \equiv (\phi')^{-1}$.

The dual problem of (FDP_n) is

$$(FDD_n) \qquad \begin{cases} \sup \ \int_T [\bar{x}(t) \sum_{i \in I_n} \lambda_i a_i(t) - \phi^*(\sum_{i \in I_n} \lambda_i a_i(t))] d\mu(t), \\ s.t. \ \ \lambda \in I\!R^{k(n)}. \end{cases} \tag{3.4}$$

The constraint qualification (CQ) takes the form:

$$\begin{cases} \text{there exists } \hat{x} \in L_1(T, \mu), \text{ such that} \\ \int_T \left[a_i(t)(\hat{x}(t) - \bar{x}(t))\right] d\mu(t) = 0, \ \ i \in I_n, \\ a < \hat{x} < b, \qquad \text{a.e. on } \ T, \ \text{ and} \\ \int_T \phi(\hat{x}(t)) d\mu(t) < +\infty. \end{cases} \tag{3.5}$$

By the duality results, if (CQ) of the form in (3.5) holds, then $V(FDP_n) = V(FDD_n)$, and both optima are attained. Moreover, if $\bar{\lambda} \in I\!R^{k(n)}$ is an optimal solution of (FDD_n), then the unique solution of (FDP_n) is given by

$$x_n = \phi^{*\prime}(\sum_{i \in I_n} \bar{\lambda}_i a_i). \tag{3.6}$$

Also we have $\phi'(x_n) \in span\{a_i, i \in I_n\}$, which implies

$$a < x_n(t) < b, \quad \text{a.e. on } \ T. \tag{3.7}$$

For given $\{a_i, i \in I_n\}$ and each $n \in I\!N$, $p \geq 1$, we define a **renorming constant**

$$\Delta_{n,p} \triangleq \sup\left\{\frac{\|f\|_\infty}{\|f\|_p}, \ f \in span\{a_i, i \in I_n\}, f \neq 0\right\}. \tag{3.8}$$

Noting that

$$\|f\|_p \triangleq \left(\int_T |f(t)|^p d\mu(t)\right)^{\frac{1}{p}} \leq \|f\|_\infty(\mu(T))^{\frac{1}{p}} > 0,$$

it is always true that

$$\Delta_{n,p} \geq (\mu(T))^{-\frac{1}{p}} > 0. \tag{3.9}$$

To obtain uniform bounds for $\{\bar{x} - x_n\}$, we also require ϕ^* to satisfy the following two more assumptions.

(AF1): For some $\delta > 0$, $r > 1$,

$$\phi^*(v_1) - \phi^*(v_2) - \phi^{*\prime}(v_2)(v_1 - v_2) \geq \delta |v_1 - v_2|^r, \tag{3.10}$$

for all $v_1, v_2 \in I\!R$.

(AF2): ϕ^* is γ-Lipschitzian on $I\!R$ for some $\gamma > 0$, i.e. there exists a constant L, such that

$$|\phi^{*\prime}(v_1) - \phi^{*\prime}(v_2)| \leq L|v_1 - v_2|^\gamma, \tag{3.11}$$

for all $v_1, v_2 \in I\!R$.

It is obvious that the strict positivity and boundedness of the second derivative of ϕ^* will imply (AF1) and (AF2) for $r = 2$ and $\gamma = 1$ if we take

$$\delta = \min_{v \in I\!R}\{\phi^{*\prime\prime}(v)\}, \qquad L = \max_{v \in I\!R}\{\phi^{*\prime\prime}(v)\}.$$

Theorem 3.2.1 *Consider the FD-type problem (FDP_n) defined in (3.2). Suppose that ϕ satisfies (AF1)-(AF2) for some $r > 1$, $\delta > 0$, $L > 0$, and $\gamma > 0$. Let $\{x_n\}$ be the optimal solution of (FDP_n), and (CQ) hold. Further assume $\bar{x} \in L_1(T, \mu)$ and $\phi'(\bar{x}) \in L_\infty(T, \mu)$. Then*

$$\begin{aligned} &\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty \leq \\ &\quad E_n + (\mu(T))^{\frac{1}{r}} \Delta_{n,r} E_n + (\tfrac{1}{\delta}\mu(T)L)^{\frac{1}{r}} \Delta_{n,r} E_n^{\frac{1+\gamma}{r}}. \end{aligned} \tag{3.12}$$

In the most interesting case, when $\gamma = 1$ and $r = 2$, we have

$$\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty = O(\Delta_{n,2} E_n).$$

Proof: For each n , we can choose $\lambda^n \in I\!R^{k(n)}$, so that

$$\|\phi'(\bar{x}) - \sum_{i \in I_n} \lambda_i^n a_i\|_\infty = E_n. \tag{3.13}$$

Then by (3.13) and the definition of $\Delta_{n,r}$ in (3.8), we have

$$\begin{aligned}
&\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty \\
&\leq \|\phi'(\bar{x}) - \sum_{i\in I_n} \lambda_i^n a_i\|_\infty + \|\sum_{i\in I_n} \lambda_i^n a_i - \phi'(x_n)\|_\infty \\
&\leq \|\sum_{i\in I_n} \lambda_i^n a_i - \phi'(x_n)\|_\infty + E_n \\
&\leq E_n + \Delta_{n,r}\|\sum_{i\in I_n} \lambda_i^n a_i - \phi'(x_n)\|_r \\
&\leq E_n + \Delta_{n,r}\big(\|\phi'(\bar{x}) - \sum_{i\in I_n} \lambda_i^n a_i\|_r + \|\phi'(\bar{x}) - \phi'(x_n)\|_r\big) \\
&\leq E_n + \Delta_{n,r}\big(E_n(\mu(T))^{\frac{1}{r}} + \|\phi'(\bar{x}) - \phi'(x_n)\|_r\big) \\
&\leq E_n + \Delta_{n,r}\big(E_n(\mu(T))^{\frac{1}{r}} + (\frac{1}{\delta})^{\frac{1}{r}}(I_\phi(\bar{x}) - I_\phi(x_n))^{\frac{1}{r}}\big) \\
&\leq E_n + (\mu(T))^{\frac{1}{r}}\Delta_{n,r}E_n + (\frac{1}{\delta}\mu(T)L)^{\frac{1}{r}}\Delta_{n,r}E_n^{\frac{1+\gamma}{r}}.
\end{aligned}$$

■

Again from Assimption (AF2) and $\phi^{*\prime} \equiv (\phi')^{-1}$, we have

$$\|\bar{x} - x_n\|_\infty = \|\phi^{*\prime}(\phi'(\bar{x})) - \phi^{*\prime}(\phi'(x_n))\|_\infty \leq L\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty^\gamma. \tag{3.14}$$

Hence $\|\bar{x} - x_n\|_\infty \to 0$ is implied by $\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty \to 0$.

In the above theorem, we can see that in order to ensure

$$\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty \to 0, \qquad \text{as } n \to \infty, \tag{3.15}$$

we first must require $E_n \to 0$, which is true if $\overline{span}\{a_i, i \in \bigcup_{n=0}^\infty I_n\} \supset L_1(T)$. More than that, we also need

$$\Delta_{n,r}E_n \to 0, \qquad \text{and} \qquad \Delta_{n,r}E_n^{\frac{1+\gamma}{r}} \to 0. \tag{3.16}$$

In the case where $\{a_i\}$ are algebraic or trigonometric polynomials, (3.16) can be fulfilled for smooth enough $\bar{x}$. We will give detailed conditions later for these to be true.

When ϕ^* is twice continuously differentiable, we have a direct corollary.

Corollary 3.2.2 *Consider the FD-type problem (FDP_n) defined in (3.2). Assume that ϕ^* is twice continuously differentiable on $I\!R$ and for some $\delta_0 > 0$,*

$$\delta_0 \leq \phi^{*\prime\prime}(v) \leq \frac{1}{\delta_0}, \text{ for all } v \in I\!R. \tag{3.17}$$

Let x_n be the optimal solution of (FDP_n), then

$$\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty \leq E_n + (1 + \frac{\sqrt{2}}{\delta_0})(\mu(T))^{\frac{1}{2}}\Delta_{n,2}E_n. \tag{3.18}$$

Proof: Apply Theorem 3.2.1 where $r = 2$, $\gamma = 1$, $L = 1/\delta_0$ and $\delta = \delta_0/2$. ∎

In case when this corollary applies, we only need

$$\Delta_{n,2}E_n \to 0, \quad \text{as } n \to 0 \tag{3.19}$$

to guarantee uniform convergence. For algebraic or trigonometric polynomial moment functions in $I\!R^m$ and a smooth enough underlying function $\bar{x}$, this limit can be ensured by the theorems recalled in Section 1.5. This was first done in [16] for Boltzmann-Shannon entropy case, which could not be covered here since $a = 0$ and $b = \infty$.

In the next theorem, we will weaken the conditions (3.17) and even (AF1) or (AF2) for twice continuously differentiable ϕ^*.

Theorem 3.2.3 *Consider the FD-type problem (FDP_n) defined in (3.2). Assume that ϕ^* is twice continuously differentiable on $I\!R$ and for any $M > 0$,*

$$J_0(M) \triangleq \inf\left\{\phi^{*\prime\prime}(v), \;\; v \in [-M, M]\right\} > 0, \tag{3.20}$$

and

$$L_0(M) \triangleq \sup\left\{\phi^{*\prime\prime}(v), \;\; v \in [-M, M]\right\} < +\infty. \tag{3.21}$$

Then we have

$$\begin{aligned} &\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty \\ \leq\;& (1 + \sqrt{\mu(T)}\Delta_{n,2}(1 + (\frac{L_0(\|\phi'(\bar{x})\|_\infty + E_n)}{J_0(\|\phi'(\bar{x})\|_\infty + \|\phi'(\bar{x}) - \phi'(x_n)\|_\infty)})^{\frac{1}{2}}))E_n. \end{aligned} \tag{3.22}$$

Proof: Using Corollary 2.3.4 by choosing

$$J(t, E_n, \bar{x}) = L_0(\|\phi'(\bar{x})\|_\infty + E_n), \tag{3.23}$$

we have

$$I_\phi(\bar{x}) - I_\phi(x_n) \leq L_0(\|\phi'(\bar{x})\|_\infty + E_n)\mu(T)E_n^2. \tag{3.24}$$

On the other hand, applying Corollary 2.5.4 by choosing

$$\eta_0 = J_0(\|\phi'(\bar{x})\|_\infty + \|\phi'(\bar{x}) - \phi'(x_n)\|_\infty), \tag{3.25}$$

we obtain

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq J_0(\|\phi'(\bar{x})\|_\infty + \|\phi'(\bar{x}) - \phi'(x_n)\|_\infty)\|\phi'(\bar{x}) - \phi'(x_n)\|_2^2. \tag{3.26}$$

Hence

$$\|\phi'(\bar{x}) - \phi'(x_n)\|_2 \leq [\frac{L_0(\|\phi'(\bar{x})\|_\infty + E_n)\mu(T)}{J_0(\|\phi'(\bar{x})\|_\infty + \|\phi'(\bar{x}) - \phi'(x_n)\|_\infty)}]^{\frac{1}{2}} E_n. \tag{3.27}$$

Then

$$\begin{aligned}
\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty &\leq E_n + \|\sum_{i\in I_n} \lambda_i a_i - \phi'(x_n)\|_\infty \\
&\leq E_n + \Delta_{n,2}\|\sum_{i\in I_n} \lambda_i a_i - \phi'(x_n)\|_2 \\
&\leq E_n + \Delta_{n,2}\big(\mu(T)^{\frac{1}{2}} E_n + \|\phi'(\bar{x}) - \phi'(x_n)\|_2\big) \\
&\leq E_n + \Delta_{n,2}(E_n\mu(T)^{\frac{1}{2}} \\
&\quad +(\frac{L_0(\|\phi'(\bar{x})\|_\infty + E_n)\mu(T)}{J_0(\|\phi'(\bar{x})\|_\infty + \|\phi'(\bar{x}) - \phi'(x_n)\|_\infty)})^{\frac{1}{2}} E_n) \\
&= [1 + \Delta_{n,2}\mu(T)^{\frac{1}{2}}(1 + (\frac{L_0(\|\phi'(\bar{x})\|_\infty + E_n)}{J_0(\|\phi'(\bar{x})\|_\infty + \|\phi'(\bar{x}) - \phi'(x_n)\|_\infty)})^{\frac{1}{2}})]E_n.
\end{aligned}$$

■

We can see that this theorem works even if a and/or b is not finite.

In this theorem we can see, if $\{\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty\}$ is bounded, and $\Delta_{n,2}E_n \to 0$, then

$$\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty \to 0. \tag{3.28}$$

We now give another version of a uniform convergence theorem, in which the condition (AF3) looks more complicated but actually is easier to satisfy.

We will say ϕ satisfy (AF1'), if there exist $\delta > 0$ and $r > 1$, such that

$$\phi(u_1) - \phi(u_2) - \phi'(u_2)(u_1 - u_2) \geq \delta |u_1 - u_2|^r, \tag{3.29}$$

for all $u_1, u_2 \in (a, b)$.

We will say ϕ^* satisfy (AF3), if for some $M > 0$, there exists a strictly positive and nonincreasing function $\Gamma_M : \mathbb{R}^+ \to \mathbb{R}^+$, with

$$\liminf_{\xi \to \infty} \Gamma_M(\xi)\xi > 0, \tag{3.30}$$

such that

$$|\phi^{*\prime}(u) - \phi^{*\prime}(v)| \geq \Gamma_M(|v|)|u - v|, \tag{3.31}$$

for any $u, v \in \mathbb{R}, |u| \leq M$.

Before establishing the next theorem, we prove a lemma.

Lemma 3.2.4 *Let ϕ^* satisfy (AF3), let $u_n, v_n \in span\{a_i, \in I_n\}$, and $\|u_n\|_\infty \leq M$, for large enough n. Further suppose for some $p > 0$,*

$$\Delta_{n,p} \|\phi^{*\prime}(u_n) - \phi^{*\prime}(v_n)\|_p \to 0, \ \ as\ n \to +\infty. \tag{3.32}$$

Then

$$\Delta_{n,p} \|u_n - v_n\|_p \to 0, \ \ as\ n \to +\infty. \tag{3.33}$$

Proof: By (3.31), for almost all t in T, we have

$$\begin{aligned}
&|\phi^{*\prime}(u_n(t)) - \phi^{*\prime}(v_n(t))| \\
&\geq \ \Gamma_M(|v_n(t)|)|u_n(t) - v_n(t)| \\
&\geq \ \Gamma_M(|u_n(t)| + \|u_n - v_n\|_\infty)|u_n(t) - v_n(t)| \\
&\qquad \text{(since } \Gamma_M \text{ is nonincreasing)} \\
&\geq \ \Gamma_M(M + \Delta_{n,p}\|u_n - v_n\|_p)|u_n(t) - v_n(t)| \\
&\qquad \text{(by the definition of } \Delta_{n,p}\text{)}.
\end{aligned}$$

Then

$$\Delta_{n,p}\|\phi^{*\prime}(u_n) - \phi^{*\prime}(v_n)\|_p \geq \Gamma_M(M + \Delta_{n,p}\|u_n - v_n\|_p)\Delta_{n,p}\|u_n - v_n\|_p. \tag{3.34}$$

Now we claim that $\{\Delta_{n,p}\|u_n - v_n\|_p\}$ is bounded. If not, for some subsequence $\{n_i\}$, we have

$$\Delta_{n_i,p}\|u_{n_i} - v_{n_i}\|_p \to +\infty, \text{ as } n \to +\infty, \tag{3.35}$$

so does

$$\{M + \Delta_{n_i,p}\|u_{n_i} - v_{n_i}\|_p\}. \tag{3.36}$$

By (3.30),

$$\begin{aligned}\liminf_{i\to+\infty} \Gamma_M(M + \Delta_{n_i,p}\|u_{n_i} - v_{n_i}\|_p)(M + \Delta_{n_i,p}\|u_{n_i} - v_{n_i}\|_p) \\ \geq \liminf_{\xi\to+\infty} \Gamma_M(\xi)\xi > 0.\end{aligned} \tag{3.37}$$

Noting that (3.30) and the monotonicity of Γ_M imply

$$\lim \Gamma_M(\xi) \downarrow 0, \tag{3.38}$$

we have

$$\liminf_{i\to+\infty} \Gamma_M(M + \Delta_{n_i,p}\|u_{n_i} - v_{n_i}\|_p)\Delta_{n_i,p}\|u_{n_i} - v_{n_i}\|_p > 0, \tag{3.39}$$

which is in contradiction with (3.32) and (3.34). So $\{\Delta_{n,p}\|u_n - v_n\|_p\}$ is bounded, say by $\overline{M}$. Hence

$$\Gamma_M(M + \Delta_{n,p}\|u_n - v_n\|_p) \geq \Gamma_M(M + \overline{M}) > 0. \tag{3.40}$$

Then in (3.34), we deduce

$$\Delta_{n,p}\|u_n - v_n\|_p \to 0. \tag{3.41}$$

∎

Theorem 3.2.5 *For the problem given in (3.2), suppose that (AF1'), (AF2) and (AF3) are true for some $\delta, L, \gamma > 0$, $r > 1$, and $\Gamma_M : \mathbb{R}^+ \to \mathbb{R}^+$. Assume $\bar{x} \in L_1(T,\mu)$, $\phi'(\bar{x}) \in L_\infty(T,\mu)$, and*

$$\Delta_{n,r} E_n^{\frac{1+\gamma}{r}} \to 0, \qquad \Delta_{n,r} E_n^{\frac{1}{r-1}} \to 0, \quad \text{as } n \to \infty. \tag{3.42}$$

Let x_n be the optimal solutions for (FDP_n), then

$$\|\phi'(\bar{x}) - \phi'(x_n)\|_\infty \to 0, \ \ as \ n \to \infty, \tag{3.43}$$

and also

$$\|\bar{x} - x_n\|_\infty \to 0, \ \ as \ n \to \infty. \tag{3.44}$$

Proof: We choose $\lambda^n \in I\!R^{k(n)}$, so that

$$\|\phi'(\bar{x}) - \sum_{i \in I_n} \lambda_i^n a_i\|_\infty \leq E_n, \tag{3.45}$$

then

$$\|\sum_{i \in I_n} \lambda_i^n a_i\|_\infty \leq \|\phi'(\bar{x})\|_\infty + E_n. \tag{3.46}$$

From (AF1') and Theorem 2.5.1

$$I_\phi(\bar{x}) - I_\phi(x_n) \geq \delta \|\bar{x} - x_n\|_r^r. \tag{3.47}$$

From (AF2) and Corollary 2.3.3, we have

$$I_\phi(\bar{x}) - I_\phi(x_n) \leq L\mu(T)E_n^{1+\gamma}. \tag{3.48}$$

Thus from (3.42),

$$\Delta_{n,r}\|\bar{x} - x_n\|_r \leq \Delta_{n,r}(\frac{1}{\delta}L\mu(T))^{\frac{1}{r}} E_n^{\frac{1+\gamma}{r}} \to 0. \tag{3.49}$$

Using (3.42), (3.49) and $\phi^{*\prime} = (\phi')^{-1}$, we now have

$$\begin{aligned}
&\Delta_{n,r}\|\phi^{*\prime}(\phi'(x_n)) - \phi^{*\prime}(\sum_{i \in I_n} \lambda_i^n a_i)\|_r \\
&= \ \Delta_{n,r}\|x_n - \phi^{*\prime}(\sum_{i \in I_n} \lambda_i^n a_i)\|_r \\
&\leq \ \Delta_{n,r}\big(\|x_n - \bar{x}\|_r + \|\bar{x} - \phi^{*\prime}(\sum_{i \in I_n} \lambda_i^n a_i)\|_r\big) \\
&\leq \ \Delta_{n,r}\big(\|x_n - \bar{x}\|_r + (\frac{1}{\delta}\|\phi'(\bar{x}) - \sum_{i \in I_n} \lambda_i^n a_i\|_{\frac{r}{r-1}})^{\frac{1}{r-1}}\big) \\
&\qquad \text{(for same } \delta > 0 \text{ as in (3.47))} \\
&\leq \ \Delta_{n,r}\big(\|x_n - \bar{x}\|_r + (\frac{1}{\delta}(\mu(T))^{\frac{r-1}{r}} E_n)^{\frac{1}{r-1}}\big) \\
&= \ \Delta_{n,r}\|x_n - \bar{x}\|_r + \delta^{\frac{1}{1-r}}(\mu(T))^{\frac{1}{r}} E_n^{\frac{1}{r-1}} \Delta_{n,r} \\
&\to \ 0.
\end{aligned}$$

Since $\phi'(x_n)$, $\sum_{i\in I_n}\lambda_i^n a_i \in span\{a_i, \in I_n\}$, and

$$\|\sum_{i\in I_n}\lambda_i^n a_i\|_\infty \leq \|\phi'(\bar{x})\|_\infty + E_n < +\infty, \tag{3.50}$$

using Lemma 3.2.4, we have

$$\Delta_{n,r}\|\phi'(x_n) - \sum_{i\in I_n}\lambda_i^n a_i\|_r \to 0, \text{ as } n \to \infty. \tag{3.51}$$

Hence

$$\|\phi'(x_n) - \sum_{i\in I_n}\lambda_i^n a_i\|_\infty \leq \Delta_{n,r}\|\phi'(x_n) - \sum_{i\in I_n}\lambda_i^n a_i\|_r \to 0. \tag{3.52}$$

So we have

$$\|\phi'(x_n) - \phi'(\bar{x})\|_\infty \leq \|\phi'(x_n) - \sum_{i\in I_n}\lambda_i^n a_i\|_\infty + \|\sum_{i\in I_n}\lambda_i^n a_i - \phi'(\bar{x})\|_\infty \to 0, \tag{3.53}$$

and also from (AF1'),

$$\|x_n - \bar{x}\|_\infty \leq \delta^{\frac{1}{1-r}}\|\phi'(x_n) - \phi'(\bar{x})\|_\infty^{\frac{1}{r-1}} \to 0, \tag{3.54}$$

as $n \to 0$. ■

The most interesting case is when $r = 2$ and $\gamma = 1$. Then we require $\Delta_{n,2}E_n \to 0$ in (3.[illegible]2). We will now apply the theorems to important entropies.

3.3 Application 1: generalized Fermi-Dirac entropy

As the first application, we consider the Fermi-Dirac entropy

$$\phi(u) = \begin{cases} (u-\alpha)\log(u-\alpha) + (\beta - u)\log(\beta - u), & \alpha < u < \beta, \\ (\beta-\alpha)\log(\beta-\alpha), & u = \alpha \text{ or } \beta, \\ +\infty, & \text{otherwise}, \end{cases} \tag{3.55}$$

where α and β are constants.

We can easily check that all conditions in (3.2) hold. Now

$$\begin{aligned}
\phi^*(v) &= \alpha v + (\beta-\alpha)\log(1+e^v) - (\beta-\alpha)\log(\beta-\alpha), \\
\phi^{*\prime}(v) &= \frac{\alpha+\beta e^v}{1+e^v}, \\
\text{and} & \\
\phi^{*\prime\prime}(v) &= \frac{(\beta-\alpha)e^v}{(1+e^v)^2} \le \frac{\beta-\alpha}{4},
\end{aligned}$$

which implies (AF2) with $L = (\beta-\alpha)/4$ and $\gamma = 1$. Also we can see that

$$\phi''(u) = \frac{1}{u-\alpha} + \frac{1}{\beta-u} = \frac{\beta-\alpha}{(u-\alpha)(\beta-u)} \ge \frac{4}{\beta-\alpha}, \tag{3.56}$$

and (AF1') follows for $\delta = 4/(\beta-\alpha)$ and $r = 2$. We now need to check (AF3).

Lemma 3.3.1 *For any $C > 0$, $T_0 > 0$, the following inequality is true for $|t| \le T_0$:*

$$\frac{|1-e^t|}{C+e^t} \ge \frac{|t|}{(C+1)(T_0+1)}. \tag{3.57}$$

Proof: Recall that

$$e^t \ge 1+t, \qquad \text{for all } t \in I\!R. \tag{3.58}$$

For $t \ge 0$,

$$\begin{aligned}
\frac{|1-e^t|}{C+e^t} &= \frac{e^t-1}{C+e^t} = 1 - \frac{C+1}{C+e^t} \\
&\ge 1 - \frac{C+1}{C+1+t} = \frac{t}{C+1+t} \ge \frac{t}{(C+1)(T_0+1)}.
\end{aligned}$$

For $t \le 0$,

$$\begin{aligned}
\frac{|1-e^t|}{C+e^t} &= \frac{1-e^t}{C+e^t} = \frac{C+1}{C+e^t} - 1 \\
&\ge \frac{C+1}{C+\frac{1}{1-t}} - 1 = \frac{-t}{C(1-t)+1} \ge \frac{-t}{(C+1)(T_0+1)}.
\end{aligned}$$

■

Lemma 3.3.2 *For ϕ defined in (3.55), (AF3) holds.*

Proof: For any $u, v \in I\!R$, and $|u| \leq M$, we have

$$\begin{aligned}
|\phi^{*\prime}(u) - \phi^{*\prime}(v)| &= |\frac{\alpha + \beta e^u}{1 + e^u} - \frac{\alpha + \beta e^v}{1 + e^v}| = \frac{\beta - \alpha}{1 + e^u} |\frac{e^v - e^u}{1 + e^v}| \\
&= \frac{\beta - \alpha}{1 + e^u} |\frac{e^{v-u} - 1}{e^{-u} + e^{v-u}}| \geq \frac{\beta - \alpha}{1 + e^M} \frac{|u - v|}{(e^M + 1)(M + |v| + 1)} \\
&\qquad \text{(by Lemma 3.3.1 and } |u - v| \leq M + |v|) \\
&\triangleq \Gamma_M(|v|)|u - v|.
\end{aligned}$$

It is easy to see that Γ_M is a nonnegative and decreasing function and

$$\lim_{\xi \to +\infty} \Gamma_M(\xi)\xi = \frac{\beta - \alpha}{(1 + e^M)^2} > 0, \tag{3.59}$$

and (AF3) follows. ∎

Now we obtain a uniform convergence theorem.

Theorem 3.3.3 *Consider the problem (P_n), with ϕ defined in (3.55). Suppose (T, μ) is a complete finite measure space, $a_i \in L_\infty(T, \mu), i \in I_n$, $\bar{x} \in L_1(T, \mu)$, $\log(\bar{x} - \alpha)/(\beta - \bar{x}) \in L_\infty(T, \mu)$, (or equivalently, there exists $\varepsilon > 0$, such that $\alpha + \varepsilon \leq \bar{x} \leq \beta - \varepsilon$, a.e. on T). We also suppose that $\Delta_{n,2} E_n \to 0$, as $n \to \infty$. Let x_n's be the optimal solutions for (P_n), then*

$$\|x_n - \bar{x}\|_\infty \to 0, \qquad \text{as } n \to \infty. \tag{3.60}$$

Proof: Simply apply Theorem 3.2.5 and Lemma 3.3.2 with $r = 2$ and $\gamma = 1$. ∎

If we recall the theorems stated in Section 1.5, we can get uniform convergence results for algebraic and trigonometric moments.

Consider the case where $\{a_i, i \in I_n\}$ are algebraic polynomials on $T = [A, B]^m \subset I\!R^m$ of degree at most n in each variable. Let $\bar{x} \in C^1[A, B]^m$. Then by Theorem 1.3.6 and 1.3.4, we have

$$\Delta_{n,2} = O(n), \quad \text{and} \quad E_n = o(\frac{1}{n}), \tag{3.61}$$

which implies

$$\Delta_{n,2}E_n \to 0, \quad as\ n \to 0. \tag{3.62}$$

If we consider the case where $T = [-\pi,\pi]^m \in I\!R^m$, and $\{a_i, i \in I_n\}$ are trigonometric polynomials of degree at most n in each variable, then we require the periodic function $\bar{x}$ to be in $C^r[-\pi,\pi]^m$, with $r \geq m/2$, to ensure (3.62).

3.4 Application 2: Burg-type entropy

We now consider the problem (P_n) where ϕ is the Burg-type entropy defined by

$$\phi(u) = \begin{cases} -\log u - \log(1-u), & 0 < u < 1, \\ +\infty, & otherwise. \end{cases} \tag{3.63}$$

Then

$$\begin{aligned} \phi'(u) &= -\frac{1}{u} + \frac{1}{1-u}, \\ \phi''(u) &= \frac{1}{u^2} + \frac{1}{(1-u)^2} \geq 8, \end{aligned}$$

which implies (AF1'). Through calculation, we obtain

$$\begin{aligned} \phi^*(v) &= \frac{1}{2}(v - 2 + \sqrt{v^2+4}) + \log(\frac{\sqrt{v^2+4}-2}{v^2}), \\ \phi^{*\prime}(v) &= \frac{v + \sqrt{v^2+4} - 2}{2v}, \\ \phi^{*\prime\prime}(v) &= \frac{\sqrt{v^2+4}-2}{v^2\sqrt{v^2+4}} \leq \frac{1}{8}, \end{aligned}$$

which implies (AF2). Again we need to check (AF3).

Lemma 3.4.1 *For ϕ defined in (3.63), $\phi^{*\prime}$ satisfies (AF3).*

Proof: For any $u, v \in I\!R$, $|u| \leq M$, we have

$$|\phi^{*\prime}(u) - \phi^{*\prime}(v)|$$

$$
\begin{aligned}
&= \left|\frac{u+\sqrt{u^2+4}-2}{2u} - \frac{v+\sqrt{v^2+4}-2}{2v}\right| \\
&= \left|\frac{v\sqrt{u^2+4}-2v-u\sqrt{v^2+4}+2u}{2uv}\right| \\
&= \left|\frac{\sqrt{u^2+4}\sqrt{v^2+4}-2\sqrt{v^2+4}-2\sqrt{u^2+4}+4-uv}{2uv(\sqrt{u^2+4}+\sqrt{v^2+4})}\right| |u-v| \\
&= \left|\frac{(\sqrt{u^2+4}-2)(\sqrt{v^2+4}-2)-uv}{2uv(\sqrt{u^2+4}+\sqrt{v^2+4})}\right| |u-v| \\
&\geq \frac{|u|\,|v|-(\sqrt{u^2+4}-2)(\sqrt{v^2+4}-2)}{2|u|\,|v|(\sqrt{u^2+4}+\sqrt{v^2+4})} |u-v| \\
&\qquad (\text{ since } 0 \leq \sqrt{u^2+4}-2 \leq |u|).
\end{aligned}
\tag{3.64}
$$

We can actually check that the function

$$
\frac{uv-(\sqrt{u^2+4}-2)(\sqrt{v^2+4}-2)}{uv(\sqrt{u^2+4}+\sqrt{v^2+4})}
\tag{3.65}
$$

is nonnegative and monotonic decreasing in each variable u or v for $u \geq 0$, $v \geq 0$. Hence

$$
\begin{aligned}
&|\phi^{*\prime}(u)-\phi^{*\prime}(v)| \\
&\geq \frac{M\,|v|-(\sqrt{M^2+4}-2)(\sqrt{v^2+4}-2)}{2M\,|v|(\sqrt{M^2+4}+\sqrt{v^2+4})} |u-v| \\
&\qquad (\text{ by } |u| \leq M) \\
&\triangleq \Gamma_M(|v|)\,|u-v|,
\end{aligned}
\tag{3.66}
$$

where $\Gamma_M(\cdot)$ is nonnegative nonincreasing on $I\!R^+$ and

$$
\liminf_{\xi\to+\infty} \Gamma_M(\xi)\xi = \frac{M+2-\sqrt{M^2+4}}{2M} > 0.
\tag{3.67}
$$

Thus (AF3) follows. ∎

We now have the exactly same convergence result for the Burg-type entropy as in Theorem 3.2.5. This case can also be generalized to arbitrary bounds α and β.

3.5 Application 3: Hellinger-type entropy

To estimate an unknown density $\bar{x}$, bounded by 0 and $2K$, we can also consider the problem (P_n) with the Hellinger-type entropy given by

$$\phi(u) = \begin{cases} -\sqrt{2uK - u^2}, & 0 \le u \le 2K, \\ +\infty, & otherwise. \end{cases} \tag{3.68}$$

It is easy to see that

$$\phi'(u) = \frac{u-K}{\sqrt{2uK-u^2}}, \qquad 0 < u < 2K, \tag{3.69}$$

and

$$\phi''(u) = \frac{K^2}{(u(2K-u))^{\frac{3}{2}}} \ge \frac{1}{K} > 0, \tag{3.70}$$

which implies (AF1').

We obtain through calculation

$$\begin{aligned} \phi^*(v) &= K(v + \sqrt{v^2+1}), \\ \phi^{*\prime}(v) &= K(1 + \frac{v}{\sqrt{v^2+1}}), \\ \phi^{*\prime\prime}(v) &= \frac{K}{(v^2+1)^{\frac{3}{2}}} \le K, \end{aligned}$$

and hence (AF2) follows. As to (AF3), we have

Lemma 3.5.1 *For ϕ defined in (3.68), $\phi^{*\prime}$ satisfies (AF3).*

Proof: For $u, v \in \mathbb{R}$, $|u| \le M$, consider

$$|\phi^{*\prime}(u) - \phi^{*\prime}(v)| = K\Big|\frac{u}{\sqrt{u^2+1}} - \frac{v}{\sqrt{v^2+1}}\Big|. \tag{3.71}$$

For $uv \le 0$, we have

$$\Big|\frac{u}{\sqrt{u^2+1}} - \frac{v}{\sqrt{v^2+1}}\Big| = \frac{|u|}{\sqrt{u^2+1}} + \frac{|v|}{\sqrt{v^2+1}}$$

$$
\begin{aligned}
&= \frac{|u|\sqrt{v^2+1}+|v|\sqrt{u^2+1}}{\sqrt{u^2+1}\sqrt{v^2+1}} \\
&\geq \frac{|u|+|v|}{\sqrt{u^2+1}\sqrt{v^2+1}} \\
&= \frac{1}{\sqrt{u^2+1}\sqrt{v^2+1}}|u-v| \\
&\geq \frac{1}{\sqrt{M^2+1}\sqrt{v^2+1}}|u-v|. \qquad (3.72)
\end{aligned}
$$

For $uv \geq 0$, $|u| \geq |v|$,

$$
\begin{aligned}
\left|\frac{u}{\sqrt{u^2+1}} - \frac{v}{\sqrt{v^2+1}}\right| &= \frac{|u|}{\sqrt{u^2+1}} - \frac{|v|}{\sqrt{v^2+1}} \\
&\geq \frac{1}{(u^2+1)^{\frac{3}{2}}}|u-v| \\
&\qquad (\text{ by the concavity of } |x|/\sqrt{x^2+1}) \\
&\geq \frac{1}{(M^2+1)^{\frac{3}{2}}}|u-v|. \qquad (3.73)
\end{aligned}
$$

For $uv \geq 0$, $|u| \leq |v|$,

$$
\begin{aligned}
&\left|\frac{u}{\sqrt{u^2+1}} - \frac{v}{\sqrt{v^2+1}}\right| \\
&= \frac{|v|}{\sqrt{v^2+1}} - \frac{|u|}{\sqrt{u^2+1}} \\
&= \frac{(|v|-|u|)\sqrt{v^2+1} - |v|(\sqrt{v^2+1}-\sqrt{u^2+1})}{\sqrt{v^2+1}\sqrt{u^2+1}} \\
&= \frac{|v|-|u|}{\sqrt{u^2+1}} - \frac{|v|(|u|+|v|)(|v|-|u|)}{\sqrt{u^2+1}\sqrt{v^2+1}(\sqrt{v^2+1}+\sqrt{u^2+1})} \\
&\geq \frac{1}{\sqrt{M^2+1}}\left(1 - \frac{|v|}{\sqrt{v^2+1}} \cdot \frac{|v|+M}{(\sqrt{v^2+1}+\sqrt{M^2+1})}\right)|u-v| \qquad (3.74) \\
&\quad (\text{since } (|v|+|x|)/(\sqrt{v^2+1}+\sqrt{x^2+1}) \text{ is increasing in } |x| \text{ for fixed } v).
\end{aligned}
$$

Combining (3.72), (3.73) and (3.74),

$$
|\phi^{*\prime}(u) - \phi^{*\prime}(v)| \geq \Gamma_M(|v|)|u-v|, \qquad \text{for } |u| \leq M, \qquad (3.75)
$$

where

$$
\Gamma_M(\xi) = \min\left\{\frac{1}{\sqrt{M^2+1}\sqrt{v^2+1}}, \frac{1}{(M^2+1)^{\frac{3}{2}}},\right.
$$

$$\frac{1}{\sqrt{M^2+1}}\left(1-\frac{\xi(\xi+M)}{\sqrt{\xi^2+1}(\sqrt{\xi^2+1}+\sqrt{M^2+1})}\right)\},$$

which is decreasing since each of the items inside the "min" is decreasing. From

$$\lim_{\xi\to\infty}\frac{1}{\sqrt{M^2+1}\sqrt{\xi^2+1}}\xi=\frac{1}{\sqrt{M^2+1}},$$

$$\lim_{\xi\to\infty}\frac{1}{(M^2+1)^{\frac{3}{2}}}\xi=\infty,$$

and

$$\begin{aligned}
&\lim_{\xi\to\infty}\left(1-\frac{\xi(\xi+M)}{\sqrt{\xi^2+1}(\sqrt{\xi^2+1}+\sqrt{M^2+1})}\right)\xi \\
=\;& \lim_{\xi\to\infty}\frac{\xi^2+1+\sqrt{(\xi^2+1)(M^2+1)}-\xi^2-\xi M}{\sqrt{\xi^2+1}(\sqrt{\xi^2+1}+\sqrt{M^2+1})}\xi \\
=\;& \lim_{\xi\to\infty}\frac{\left(\sqrt{\xi^2+1)(M^2+1)}-\xi M+1\right)\left(\sqrt{\xi^2+1)(M^2+1)}+\xi M-1\right)\xi}{\sqrt{\xi^2+1}\left(\sqrt{\xi^2+1}+\sqrt{M^2+1}\right)\left(\sqrt{(\xi^2+1)(M^2+1)}+\xi M-1\right)} \\
=\;& \lim_{\xi\to\infty}\frac{\left[(\xi^2+1)(M^2+1)-(\xi M-1)^2\right]\xi}{\left(\xi^2+1+\sqrt{(\xi^2+1)(M^2+1)}\right)\left(\sqrt{(\xi^2+1)(M^2+1)}+\xi M-1\right)} \\
=\;& \lim_{\xi\to\infty}\frac{(\xi+M)^2\xi}{\left(\xi^2+1+\sqrt{(\xi^2+1)(M^2+1)}\right)\left(\sqrt{(\xi^2+1)(M^2+1)}+\xi M-1\right)} \\
=\;& \frac{1}{2M},
\end{aligned}$$

we get

$$\liminf_{\xi\to\infty}\Gamma_M(\xi)\xi>0. \tag{3.76}$$

∎

Then the analogue result to theorem 3.2.5 for the Hellinger-type entropy follows directly.

3.6 Uniform convergence theorems for truncation-type entropies

In this section, we will consider the problem

$$(TP_n) \qquad \begin{cases} \inf & \int_T \phi(x(t))d\mu(t), \\ s.t. & \int_T a_i(t)(x(t)-\bar{x}(t))d\mu(t)=0, \quad i \in I_n, \\ & x \in L_1(T,\mu), \\ & \alpha(t) \le x(t) \le \beta(t), \qquad \text{a.e. on } T, \end{cases} \tag{3.77}$$

where (T,μ) is a complete finite measure space, $\phi : I\!R \to (-\infty,+\infty]$, α, β are functions which satisfy:

(AT1): $-\infty \le a < \alpha(t) < \beta(t) < b \le +\infty$ for all $t \in T$ with

$$(a,b) \subseteq dom(\phi) \subseteq [a,b]; \tag{3.78}$$

(AT2): ϕ is strictly convex and continuously differentiable on (a,b).

Using the same notations we used before,

$$\tilde{\phi}(t,u) \triangleq \begin{cases} \phi(u), & \alpha(t) \le u \le \beta(t), \\ +\infty, & otherwise, \end{cases} \tag{3.79}$$

and hence (TP_n) is equivalent to

$$(\widetilde{TP_n}) \qquad \begin{cases} \inf & \int_T \tilde{\phi}(t,x(t))d\mu(t), \\ s.t. & \int_T a_i(t)(x(t)-\bar{x}(t))d\mu(t)=0, \qquad i \in I_n, \\ & x \in L_1(T,\mu). \end{cases} \tag{3.80}$$

The dual problem is then

$$(TD_n) \qquad \begin{cases} \max & \int_T[\bar{x}(t)\sum_{i\in I_n}\lambda_i a_i(t) - \tilde{\phi}^*(t,\sum_{i\in I_n}\lambda_i a_i(t))]d\mu(t), \\ s.t. & \lambda \in I\!R^{k(n)}, \end{cases} \tag{3.81}$$

where $\tilde{\phi}^*$ is finite everywhere and given as in (2.9). We know that for almost all $t \in T$, $\tilde{\phi}^*(t,\cdot)$ is continuously differentiable on $I\!R$, with the derivative given in (2.10), and is strictly convex on $(\phi'(\alpha(t)), \phi'(\beta(t)))$, linear outside this interval. Moreover we will frequently use the expressions (2.10) and (2.11). Under (CQ) given in (2.15), the duality results tell that, if $\lambda \in I\!R^{k(n)}$ is an optimal solution of (TD_n), then the optimal solution of (TP_n) is of the form

$$x_n(t) = \tilde{\phi}_2^{*\prime}(t, \sum_{i \in I_n} \lambda_i a_i(t)). \tag{3.82}$$

By (2.10) and (2.11), we know

$$\begin{aligned} & \tilde{\phi}_2'(t, x_n(t)) \\ = \; & \phi'\big(\max\{\alpha(t), \min\{\beta(t), \sum_{i \in I_n} \lambda_i a_i(t)\}\}\big) \\ \triangleq \; & \phi'\big(\alpha(t) \vee \sum_{i \in I_n} \lambda_i a_i(t) \wedge \beta(t)\big). \end{aligned} \tag{3.83}$$

Now we give uniform convergence theorems for these truncation-type entropies. We assume $T = [A, B] \subset I\!R$ (or $[-\pi, \pi]$ in the trigonometric case) and μ is Lebesgue measure. We also assume that $\{a_i, i \in I_n\}$ are algebraic (or trigonometric) polynomials so that we can apply Remez' inequalities given in Theorems 1.3.7 and 1.3.8. In the next section we apply these results to the L_p-entropy.

Theorem 3.6.1 *Let $T = [A, B]$, $\{a_i, i \in I_n\}$ be algebraic polynomials of degree at most n. Suppose for some $\lambda^n \in I\!R^{k(n)}$,*

$$x_n = \tilde{\phi}_2^{*\prime}\big(t, \sum_{i \in I_n} \lambda_i^n a_i(t)\big) \tag{3.84}$$

is the optimal solution for (TP_n). We further assume

(AT3): for some $C > 0$, $q > 1$, $\gamma > 1$,

$$\|\phi'(\bar{x}) - \phi'(x_n)\|_q^q \leq C E_n^{\gamma}, \qquad \textit{for large } n, \tag{3.85}$$

(AT4): for some $K > 0$, $r > 0$,

$$\mu\{t \in [A, B] \mid \phi'(\bar{x}(t)) - \phi'(\alpha(t)) \leq \varepsilon\} \leq K \varepsilon^r, \tag{3.86}$$

and

$$\mu\{t \in [A,B] \mid \phi'(\beta(t)) - \phi'(\bar{x}(t)) \leq \varepsilon\} \leq K\varepsilon^r, \tag{3.87}$$

for small $\varepsilon > 0$.

(AT5): $\phi'(\bar{x}) \in C^d(T,\mu)$, *with*

$$d \geq \max\{\frac{2(r+q)}{\gamma r}, \frac{2}{q}\}. \tag{3.88}$$

Then $\phi'(x_n) \to \phi'(\bar{x})$, *uniformly on* $[A,B]$.

Proof: For x_n given in the form of (3.84) and some $\lambda^n \in \mathbb{R}^{k(n)}$, let

$$N_n = \{t \in [A,B] \mid \sum_{i \in I_n} \lambda_i^n a_i(t) \leq \alpha(t)\}, \tag{3.89}$$

$$M_n = \{t \in [A,B] \mid \sum_{i \in J_n} \lambda_i^n a_i(t) \geq \beta(t)\}, \tag{3.90}$$

and

$$A_n = [A,B] \setminus (N_n \cup M_n). \tag{3.91}$$

From (AT1) and (3.85), for some $C > 0, q > 1, \gamma > 1$, large n and small $\varepsilon > 0$, we have

$$\begin{aligned}
CE_n^\gamma &\geq \|\phi'(\bar{x}) - \phi'(x_n)\|_q^q && (3.92)\\
&= \int_T |\phi'(\bar{x}(t)) - \phi'(x_n(t))|^q d\mu(t)\\
&\geq \int_{N_n} |\phi'(\bar{x}(t)) - \phi'(x_n(t))|^q d\mu(t)\\
&\geq \int_{N_n} |\phi'(\bar{x}(t)) - \phi'(\alpha(t))|^q d\mu(t)\\
&\geq \int_{N_n \setminus \{t \in T \mid \phi'(\bar{x}(t)) - \phi'(\alpha(t)) \leq \varepsilon\}} |\phi'(\bar{x}(t)) - \phi'(\alpha(t))|^q d\mu(t)\\
&\geq \varepsilon^q \mu\{N_n \setminus \{t \in T \mid \phi'(\bar{x}(t)) - \phi'(\alpha(t)) \leq \varepsilon\}\}\\
&\geq \varepsilon^q\big(\mu(N_n) - \mu(\{t \in T \mid \phi'(\bar{x}(t)) - \phi'(\alpha(t)) \leq \varepsilon\})\big)\\
&\geq \varepsilon^q\big(\mu(N_n) - K\varepsilon^r\big), && (3.93)
\end{aligned}$$

and hence

$$\mu(N_n) \le \varepsilon^{-q} C E_n^{\gamma} + K\varepsilon^{r}. \tag{3.94}$$

Since $\phi'(\bar{x}) \in C[A,B]$, $E_n \to 0$, as $n \to \infty$, we may choose

$$\varepsilon_n = E_n^{\frac{\gamma}{r+q}} \to 0, \qquad as\ n \to \infty \tag{3.95}$$

in (3.94), and then get

$$\mu(N_n) \le (C+K)E_n^{\frac{\gamma r}{r+q}}, \qquad \text{for large } n. \tag{3.96}$$

In the same way, we can obtain

$$\mu(M_n) \le (C+K)E_n^{\frac{\gamma r}{r+q}}, \qquad \text{for large } n. \tag{3.97}$$

In particular, when n is large enough, we have

$$\mu(N_n \cup M_n) < \frac{1}{2}(B-A). \tag{3.98}$$

We will use this inequality to apply Remez' Theorems.

Now let $q_n(t)$ be an algebraic polynomial of degree at most n, such that

$$\|\phi'(\bar{x}) - q_n\|_\infty = E_n. \tag{3.99}$$

Then by Remez' inequality (Theorem 1.3.8), and (3.98), for large n, we have

$$\begin{aligned} \|\sum_{i\in I_n} \lambda_i^n a_i - q_n\|_q^q &= \int_T |\sum_{i\in I_n} \lambda_i^n a_i(t) - q_n(t)|^q d\mu(t) \\ &\le \ (1+e^{C_0 q n\sqrt{\mu(N_n\cup M_n)}}) \int_{A_n} |\sum_{i\in I_n} \lambda_i^n a_i(t) - q_n(t)|^q d\mu(t), \end{aligned} \tag{3.100}$$

for some constant $C_0 > 0$.

Noting that in our assumptions, $d \ge 2(r+q)/\gamma r$, using (3.96), (3.97) and Jackson's Theorem (Corollary 1.3.2), we obtain

$$\begin{aligned} n\sqrt{\mu(N_n \cup M_n)} &\le (2(C+K)E_n^{\frac{\gamma r}{r+q}})^{\frac{1}{2}} n \\ &= \sqrt{2(C+K)}(n^d E_n)^{\frac{\gamma r}{2(r+q)}} n^{1-\frac{d\gamma r}{2(r+q)}} \to 0. \end{aligned} \tag{3.101}$$

So for large n,

$$1 + e^{C_0 q n \sqrt{\mu(N_n \cup M_n)}} < 3. \tag{3.102}$$

Hence in (3.100)

$$\begin{aligned} \|\sum_{i\in I_n} \lambda_i^n a_i - q_n\|_q^q &\leq 3\int_{A_n} \big|\sum_{i\in I_n} \lambda_i^n a_i(t) - q_n(t)\big|^q d\mu(t) \\ &= 3\int_{A_n} \big|\phi'(x_n(t)) - q_n(t)\big|^q d\mu(t) \\ &\leq 3\|\phi'(x_n) - q_n\|_q^q. \end{aligned} \tag{3.103}$$

So

$$\begin{aligned} \|\sum_{i\in I_n} \lambda_i^n a_i - q_n\|_q &\leq 3^{\frac{1}{q}} \|\phi'(x_n) - q_n\|_q \\ &\leq 3^{\frac{1}{q}} (\|\phi'(x_n) - \phi'(\bar{x})\|_q + \|\phi'(\bar{x}) - q_n\|_q) \\ &\leq 3^{\frac{1}{q}} \big(C^{\frac{1}{q}} E_n^{\frac{2}{q}} + E_n (B-A)^{\frac{1}{q}}\big). \end{aligned} \tag{3.104}$$

Now we have

$$\begin{aligned} &\|\phi'(x_n) - \phi'(\bar{x})\|_\infty \\ &\leq \|\sum_{i\in I_n} \lambda_i^n a_i - \phi'(\bar{x})\|_\infty \qquad \text{(since } \bar{x} \text{ and } x_n \text{ are feasible)} \\ &\leq \|\sum_{i\in I_n} \lambda_i^n a_i - q_n\|_\infty + \|q_n - \phi'(\bar{x})\|_\infty \\ &\leq C_1 n^{\frac{2}{q}} \|\sum_{i\in I_n} \lambda_i^n a_i - q_n\|_q + E_n \qquad \text{(by Theorem 1.3.6, } \Delta_{n,q} \leq C_1 n^{2/q}) \\ &\leq C_1 3^{\frac{1}{q}} \big(C^{\frac{1}{q}} n^{\frac{2}{q}} E_n^{\frac{2}{q}} + 2(B-A)^{\frac{1}{q}} n^{\frac{2}{q}} E_n\big) + E_n. \end{aligned} \tag{3.105}$$

We note that

$$d \geq \frac{2(r+q)}{\gamma r} > \frac{2}{\gamma}, \text{ implies } \frac{2}{q} - \frac{\gamma d}{q} < 0,$$

and (AT5) implies $n^d E_n \to \infty$ from Corollary 1.3.2. Then we have

$$n^{\frac{2}{q}} E_n^{\frac{2}{q}} = (n^d E_n)^{\frac{2}{q}} \cdot n^{\frac{2}{q} - \frac{2d}{q}} \to 0, \;\; as\ n \to \infty, \tag{3.106}$$

Also from $d \geq 2/q$ in (AT5), we have

$$n^{\frac{2}{q}} E_n = (n^d E_n) n^{\frac{2}{q}-d} \to 0, \quad as\ n \to \infty. \tag{3.107}$$

Then in (3.105)

$$\|\phi'(x_n) - \phi'(\bar{x})\|_\infty \to 0, \quad as\ n \to \infty. \tag{3.108}$$

■

By using Theorems 1.3.3, 1.3.5 and 1.3.7, we can get a similar result for the trigonometric case.

Theorem 3.6.2 *Let $\{a_i, i \in I_n\}$ be trigonometric polynomials on $T = [-\pi, \pi]$ of degree at most n. Let λ^n, x_n be the optimal solutions of (D_n) and (P_n), respectively. Suppose Assumptions (AT1)-(AT4) hold. Let $\bar{x}$ be a periodic function on $[-\pi, \pi]$ with period 2π. Also we assume (AT5'): $\phi'(\bar{x}) \in C^d[-\pi, \pi]$, with*

$$d \geq \max\{\frac{(r+q)}{\gamma r}, \frac{1}{q}\}. \tag{3.109}$$

Then $\phi'(x_n) \to \phi'(\bar{x})$ uniformly on $[-\pi, \pi]$.

Proof: Following the proof of Theorem 3.6.1, we make the following changes:

1. Instead of (3.98), for large n, we have

$$\mu(N_n \bigcup M_n) < \pi. \tag{3.110}$$

2. In (3.100), we use Theorem 1.3.7 and obtain

$$\|\sum_{i\in I_n} \lambda_i^n a_i - q_n\|_q^q \leq (1 + e^{C_0 q n(2\pi - \mu(N_n \cup M_n))}) \int_{A_n} |\sum_{i\in I_n} \lambda_i^n a_i(t) - q_n(t)|^q d\mu(t),$$

for some constant $C_0 > 0$.

3. Instead of (3.101) in Theorem 3.6.1, we have

$$\begin{aligned} n\mu(N_n \cup M_n) &\leq 2(C+K) E_n^{\frac{\gamma r}{r+q}} n \\ &= 2(C+K)(n^d E_n)^{\frac{\gamma r}{r+q}} n^{1-\frac{\gamma r d}{r+q}} \to 0, \end{aligned} \tag{3.111}$$

which is implied by $d \geq (r+q)/\gamma r$ in (AT5').

4. Then in (3.105), using Theorem 1.3.5, we can similarly deduce

$$\begin{aligned} & \|\phi'(x_n) - \phi'(\bar{x})\|_\infty \\ \leq\ & C_2 3^{\frac{1}{q}} \left(C^{\frac{1}{q}} n^{\frac{1}{q}} E_n^{\frac{\gamma}{q}} + 2(2\pi)^{\frac{1}{q}} n^{\frac{1}{q}} E_n \right) + E_n. \end{aligned} \tag{3.112}$$

Thus we need

$$n^{\frac{1}{q}} E_n^{\frac{\gamma}{q}} = (n^d E_n)^{\frac{\gamma}{q}} n^{\frac{1}{q} - \frac{\gamma d}{q}} \to 0, \tag{3.113}$$

and

$$n^{\frac{1}{q}} E_n = (n^d E_n) n^{\frac{1}{q} - d} \to 0, \tag{3.114}$$

which is guaranteed by (AT5').

∎

We know condition (AT3) can be obtained from the results in Sections 2.3 and 2.5. An interesting case is when $q = 2$ and $\gamma = 2$. We then require $d \geq 1 + 2/r$ in Theorem 3.6.1 and $d \geq 1/2 + 1/r$ in Theorem 3.6.2. We now discuss condition (AT4).

It is easy to see that if for some $\varepsilon > 0$,

$$\phi'(\alpha(t)) + \varepsilon \leq \phi'(\bar{x}(t)) \leq \phi'(\beta(t)) - \varepsilon, \quad \text{a.e. on } T, \tag{3.115}$$

then (AT4) is true, since we can co· sider r to be ∞. Another sufficient condition for (AT4) arises from the following lemma.

Lemma 3.6.3 *Let $f > 0$, a.e. on $[A, B]$, let $0 \leq k < +\infty$ be the highest order of any zero of f on $[A, B]$, and $f \in C^k[A, B]$. Then for some $M > 0$,*

$$\mu\big(\{t \in [A, B] \mid f(t) \leq \varepsilon\}\big) \leq M \varepsilon^{\frac{1}{k}}, \tag{3.116}$$

for small $\varepsilon > 0$.

Proof: Let

$$Z = \{t \in [A, B] \mid f(t) = 0\}, \tag{3.117}$$

which is compact. If $Z = \emptyset$, then for some $\delta > 0$, $f(t) \geq \delta$, for all $t \in [A, B]$. Hence

$$\mu\big(\{t \in [A, B] \mid f(t) \leq \varepsilon\}\big) = 0, \tag{3.118}$$

for $\varepsilon < \delta$. The equation (3.116) is trivially true.

Suppose $Z \neq \emptyset$. For each fixed $t_0 \in Z$, we first assume $t_0 \in (A, B)$. Let k_0 be the order of the zero t_0. Noting that $f \geq 0$, we have

$$f(t_0) = 0, \ \cdots, \ f^{(k_0-1)}(t_0) = 0, \ f^{(k_0)}(t_0) > 0. \tag{3.119}$$

From the Taylor expansion formula around t_0, we have

$$f(t) = \frac{1}{k_0!} f^{(k_0)}(t_0)(t - t_0)^{k_0} + o((t - t_0)^{k_0}). \tag{3.120}$$

There exists an open interval $I(t_0)$, with $t_0 \in I(t_0)$, such that

$$f(t) > \frac{1}{2k_0!} f^{(k_0)}(t_0)(t - t_0)^{k_0}, \qquad \text{for all } t \in I(t_0).$$

Hence

$$f(t) \leq \varepsilon \qquad \text{implies} \qquad |t - t_0| \leq \Big(\frac{2k_0!\varepsilon}{f^{(k_0)}(t_0)}\Big)^{\frac{1}{k_0}}.$$

So

$$\mu(I(t_0) \cap \{f(t) \leq \varepsilon\}) \leq 2\Big(\frac{2k_0!}{f^{(k_0)}(t_0)}\Big)^{\frac{1}{k_0}} \varepsilon^{\frac{1}{k_0}} \triangleq M(t_0)\varepsilon^{\frac{1}{k_0}}. \tag{3.121}$$

If $t_0 = A$ or B, using one-sided derivatives, we get the same inequality as in (3.121). Now since

$$\bigcup_{t \in Z} I(t) \supseteq Z, \tag{3.122}$$

and Z is compact, there are only finitely many points, say, $t_1, t_2, \cdots, t_m \in Z$, such that

$$\hat{Z} = \bigcup_{i=1}^{m} I(t_i) \supseteq Z. \tag{3.123}$$

Since f is strictly positive on the compact set $[A,B]\setminus\hat{Z}$, for some $\varepsilon_0 > 0$, we have $f(t) > \varepsilon_0$, for all $t \in [A,B]\setminus\hat{Z}$.

Thus for $\varepsilon < \varepsilon_0$, we have

$$\{t \in [A,B] \mid f(t) \le \varepsilon\} = \{t \in \hat{Z} \mid f(t) \le \varepsilon\} \subseteq \bigcup_{i=1}^{m} \{t \in I(t_i) \mid f(t) \le \varepsilon\}.$$

Then for small enough $\varepsilon > 0$ (in particular we require $\varepsilon < 1$),

$$\begin{aligned}\mu\big(\{t \in [A,B] \mid f(t) \le \varepsilon\}\big) &\le \sum_{i=1}^{m} \mu(\{t \in I(t_i) \mid f(t) \le \varepsilon\}) \\ &\le \sum_{i=1}^{m} M(t_i)\varepsilon^{\frac{1}{k_i}} \le \varepsilon^{\frac{1}{k}} \sum_{i=1}^{m} M(t_i) \triangleq M\varepsilon^{\frac{1}{k}}.\end{aligned}$$

Hence we are done. ∎

Corollary 3.6.4 *Let $T = [A,B]$ (or $[-\pi,\pi]$ in the trigonometric case). Suppose that $\alpha(t) \le \bar{x}(t) \le \beta(t)$ on T. Let k_0 be the highest order of the zeroes of*

$$\lambda(t) \triangleq \big(\phi'(\bar{x}(t)) - \phi'(\alpha(t))\big)\big(\phi'(\bar{x}(t)) - \phi'(\beta(t))\big) = 0. \tag{3.124}$$

Further we assume

$$\phi'(\bar{x}),\ \phi'(\alpha),\ \phi'(\beta) \in C^{k_0}(T,\mu). \tag{3.125}$$

Then (AT4) holds.

Proof: Note that a zero of λ is a point at which either $\phi'(\bar{x}(t)) = \phi'(\alpha(t))$ or $\phi'(\bar{x}(t)) = \phi'(\beta(t))$. Using Lemma 3.6.3 we obtain (AT4). ∎

Corollary 3.6.5 *Let $T = [A,B]$ (or $[-\pi,\pi]$). Assume ϕ is real-analytic on $[a,b]$, α, β are constants, with $a < \alpha < \beta < b$, and $\bar{x}$ is real-analytic on $[A,B]$, $\alpha \le \bar{x}(t) \le \beta$, but neither $\bar{x} \equiv \alpha$, nor $\bar{x} \equiv \beta$. Then (AT4) holds for some $r > 0$.*

In [28], a similar uniform convergence theorem has been proved for an analytic density $\bar{x}$. Finally we will see that, without the condition (AT4), we can still prove uniform convergence on the subset on which $\bar{x}$ stays strictly away from the constraint boundary.

Theorem 3.6.6 *Let $T = [A, B] \in I\!R$, $\alpha(\cdot)$ be lower semicontinuous, $\beta(\cdot)$ be upper semicontinuous on T, $\{a_i, i \in I_n\}$ be algebraic polynomials of degree at most n. Let λ^n, x_n be defined as in (3.84). Assume for some $C > 0, q > 1, \gamma > 1$,*

$$\|\phi'(x_n) - \phi'(\bar{x})\|_q^q \leq C E_n^\gamma, \tag{3.126}$$

for large n. Also assume that $\bar{x} \in C[A, B]$ and $\phi'(\bar{x}) \in C^d[A, B]$, with $d \geq \max\{1/\gamma, 1/q\}$. Then for any $\varepsilon > 0$, $\phi'(x_n) \to \phi'(\bar{x})$ uniformly on the set

$$\{t \in T \mid \alpha(t) + \varepsilon \leq \bar{x}(t) \leq \beta(t) - \varepsilon\}. \tag{3.127}$$

Proof: Let I be any subinterval on $[A, B]$ on which

$$\alpha(t) + \varepsilon \leq \bar{x}(t) \leq \beta(t) - \varepsilon. \tag{3.128}$$

Define

$$N_n(I) \triangleq \{t \in I \mid \sum_{i \in I_n} \lambda_i^n a_i(t) \leq \alpha(t)\}, \tag{3.129}$$

$$M_n(I) \triangleq \{t \in I \mid \sum_{i \in I_n} \lambda_i^n a_i(t) \geq \beta(t)\}, \tag{3.130}$$

and

$$A_n(I) \triangleq I \setminus (N_n(I) \cup M_n(I)). \tag{3.131}$$

Then for same reason as we deduce (3.92),

$$\begin{aligned} C E_n^\gamma &\geq \|\phi'(\bar{x}) - \phi'(x_n)\|_q^q = \int_T |\phi'(\bar{x}(t)) - \phi'(x_n(t))|^q d\mu(t) \\ &\geq \int_{N_n(I)} |\phi'(\bar{x}(t)) - \phi'(\alpha(t))|^q d\mu(t) \geq \varepsilon^q \mu(N_n(I)). \end{aligned}$$

Hence we have

$$\mu(N_n(I)) \leq \varepsilon^{-q} C E_n^\gamma. \tag{3.132}$$

Similarly, we also have

$$\mu(M_n(I)) \leq \varepsilon^{-q} C E_n^{\gamma}, \tag{3.133}$$

so

$$\mu(N_n(I) \cup M_n(I)) \leq 2\varepsilon^{-q} C E_n^{\gamma}. \tag{3.134}$$

For large enough n, we have

$$\mu(N_n(I) \cup M_n(I)) \leq \frac{1}{2}\mu(I). \tag{3.135}$$

Now we may choose $q_n \in span\{a_i, i \in I_n\}$ such that

$$\|q_n - \phi'(\bar{x})\|_\infty = E_n, \tag{3.136}$$

then

$$\begin{aligned}
\|q_n - \sum_{i \in I_n} \lambda_i^n a_i\|_{L_q(A_n)} &= \|q_n - \phi'(x_n)\|_{L_q(A_n)} \\
&\leq \|q_n - \phi'(\bar{x})\|_{L_q(A_n)} + \|\phi'(\bar{x}) - \phi'(x_n)\|_{L_q(A_n)} \\
&\leq \|q_n - \phi'(\bar{x})\|_\infty \mu(A_n)^{\frac{1}{q}} + \|\phi'(\bar{x}) - \phi'(x_n)\|_q \\
&\leq \mu(A_n)^{\frac{1}{q}} E_n + C E_n^{\frac{\gamma}{q}} \\
&\leq \mu(I)^{\frac{1}{q}} E_n + C E_n^{\frac{\gamma}{q}}.
\end{aligned} \tag{3.137}$$

Now using Remez' inequality and (3.135), for some constant $C_0 > 0$,we have

$$\| \sum_{i \in I_n} \lambda_i^n a_i - q_n\|_{L_q(I)}^q \leq \left(1 + e^{C_0 n q \sqrt{\mu(N_n(I) \cup M_n(I))}}\right) \| \sum_{i \in I_n} \lambda_i^n a_i - q_n\|_{L_q(A_n)}^q. \tag{3.138}$$

Note that we have

$$\begin{aligned}
n\sqrt{\mu(N_n(I) \cup M_n(I))} &\leq n(2\varepsilon^{-q} C E_n^{\gamma})^{\frac{1}{2}} \\
&= (2\varepsilon^{-q} C)^{\frac{1}{2}} (n^d E_n)^{\frac{\gamma}{2}} n^{1-\frac{d\gamma}{2}} \\
&\to 0, \qquad \text{(since } d \geq 2/\gamma \text{ and } n^d E_n \to 0 \text{)}
\end{aligned}$$

with n. This implies

$$1 + e^{C_0 n q \sqrt{\mu(N_n(I) \cup M_n(I))}} \leq 3, \tag{3.139}$$

for n large enough. Then in (3.138)

$$\begin{aligned} \| \sum_{i \in I_n} \lambda_i^n a_i - q_n \|_{L_q(I)}^q &\leq 3 \| \sum_{i \in I_n} \lambda_i^n a_i - q_n \|_{L_q(A_n)}^q \\ &\leq 3(\mu(I)^{\frac{1}{q}} E_n + C E_n^{\frac{\gamma}{q}})^q, \end{aligned} \tag{3.140}$$

and hence

$$\begin{aligned} &\|\phi'(x_n) - \phi'(\bar{x})\|_{L_\infty(I)} \\ &\leq \| \sum_{i \in I_n} \lambda_i^n a_i - \phi'(\bar{x}) \|_{L_\infty(I)} \\ &\leq \|\phi'(\bar{x}) - q_n\|_{L_\infty(I)} + \|q_n - \sum_{i \in I_n} \lambda_i^n a_i\|_{L_\infty(I)} \\ &\leq E_n + A(I) n^{\frac{2}{q}} \|q_n - \sum_{i \in I_n} \lambda_i^n a_i\|_{L_q(I)} \\ &\qquad \text{(where } A(I) \text{ is a constant introduced in Theorem 1.3.6, dependent of } I\text{)} \\ &\leq E_n + A(I) n^{\frac{2}{q}} 3^{\frac{1}{q}} \big(\mu(I)^{\frac{1}{q}} E_n + C E_n^{\frac{\gamma}{q}}\big). \end{aligned}$$

By Corollary 1.3.2, $d \geq 2/q$ implies

$$n^{\frac{2}{q}} E_n = (n^d E_n) n^{\frac{2}{q} - d} \to 0, \tag{3.141}$$

and $d \geq 2/\gamma$ implies

$$n^{\frac{2}{q}} E_n^{\frac{\gamma}{q}} = (n^d E_n)^{\frac{\gamma}{q}} n^{\frac{2}{q} - \frac{\gamma d}{q}} \to 0. \tag{3.142}$$

It follows that

$$\|\phi'(x_n) - \phi'(\bar{x})\|_{L_\infty(I)} \to 0, \quad as \quad n \to \infty. \tag{3.143}$$

Now if α is lower semicontinuous on T, β is upper semicontinuous on T, then for any $\varepsilon > 0$, the set

$$\big\{ t \in T \;\big|\; \alpha(t) + \varepsilon \leq \bar{x}(t) \leq \beta(t) - \varepsilon \big\} \tag{3.144}$$

is closed, since $\bar{x}$ is continuous on T. Hence the set

$$Z = \{t \in T \mid \alpha(t) + \varepsilon \leq \bar{x}(t) \leq \beta(t) - \varepsilon\} \tag{3.145}$$

is compact. Now for all $t \in Z$, we can find an interval $I(t)$, such that

$$\alpha(t') + \frac{1}{2}\varepsilon \leq \bar{x}(t') \leq \beta(t') - \frac{1}{2}\varepsilon, \quad \text{for all } t' \in I(t). \tag{3.146}$$

By the compactness of Z, there exist a finite number of points $t_1, t_2, \cdots, t_m$, such that

$$Z \subseteq \sum_{i=1}^{m} I(t_i). \tag{3.147}$$

By what has been proved above, we know that

$$\phi'(x_n) \to \phi'(\bar{x}) \tag{3.148}$$

uniformly on each $I(t_i)$, $i = 1, 2, \cdots, m$, and hence

$$\phi'(x_n) \to \phi'(\bar{x}) \tag{3.149}$$

uniformly on Z. ■

For trigonometric polynomials, we have a similar result:

Theorem 3.6.7 *Let $T = [-\pi, \pi]$,$\{a_i, i \in I_n\}$ be trigonometric polynomials of degree at most n. Let λ^n, x_n be defined as in (3.84). Assume $\alpha(\cdot)$ is lower semicontinuous and $\beta(\cdot)$ is upper semicontinuous on T. For some $C > 0, q > 1, \gamma > 1$, and large enough n, assume that*

$$\|\phi'(x_n) - \phi'(\bar{x})\|_q^q \leq CE_n^\gamma. \tag{3.150}$$

Further suppose that $\bar{x} \in C[-\pi, \pi]$ and is periodic with period 2π, $\phi'(\bar{x}) \in C^d[-\pi, \pi]$, with $d \geq \max\{\frac{1}{\gamma}, \frac{1}{q}\}$. Then for any $\varepsilon > 0$,

$$\phi'(x_n) \to \phi'(\bar{x}) \tag{3.151}$$

uniformly on

$$\{t \in T \mid \alpha(t) + \varepsilon \leq \bar{x}(t) \leq \beta(t) - \varepsilon\}. \tag{3.152}$$

From the theorems we have obtained above, it is interesting to note that uniform convergence is only guaranteed in the interior of $\{t \mid \bar{x}(t) > 0\}$. It seems likely that uniform convergence will take place on the interior of $\{t \mid \bar{x}(t) = 0\}$ (it is true for the trival case $\bar{x} \equiv 0$), but we have been unable to prove this.

We can easily release one bound, that is to set $\alpha(t) \equiv -\infty$ or $\beta(t) \equiv +\infty$, and get the analogous results. Remember that in any case, (A6) must be satisfied.

3.7 Application 4: truncated L_p-entropy

As a typical example, we consider the problem (P_n), where $T = [A, B]$ (or $[-\pi, \pi]$) $\in \mathbb{R}$, $\phi(u) = (1/p)|u|^p$, $1 < p \leq 2$. Define

$$\tilde{\phi}(u) = \begin{cases} \frac{1}{p}u^p, & u \geq 0, \\ +\infty, & otherwise, \end{cases} \tag{3.153}$$

The conjugate function of $\tilde{\phi}$ is

$$\tilde{\phi}^*(v) = \begin{cases} 0, & v \leq 0, \\ \frac{1}{q}v^q, & v > 0, \end{cases} \tag{3.154}$$

where $1/p + 1/q = 1$. Hence

$$\tilde{\phi}^{*\prime}(v) = \begin{cases} 0, & v \leq 0, \\ v^{q-1}, & v > 0, \end{cases} \tag{3.155}$$

and

$$\tilde{\phi}^{*\prime\prime}(v) = \begin{cases} 0, & v < 0, \\ (q-1)v^{q-2}, & v > 0. \end{cases} \tag{3.156}$$

If x_n is the optimal solution for $(\widetilde{TP}_n)$ where $\tilde{\phi}$ is defined in (3.153), from Proposition 2.5.11, we know that for $1 < p \leq 2$, or equivalently, $q \geq 2$,

$$\|\phi'(x_n) - \phi'(\bar{x})\|_q^q \leq q(q-1)(\|\bar{x}\|_\infty^{p-1} + E_n)^{q-2}\mu(T)E_n^2, \tag{3.157}$$

and hence

$$\|x_n^{p-1} - \bar{x}^{p-1}\|_q^q \le q(q-1)(\|\bar{x}\|_\infty^{p-1} + E_n)^{q-2}\mu(T)E_n^2. \tag{3.158}$$

Now we give the final set of uniform convergence theorems for our truncated L_p-entropy.

Theorem 3.7.1 *Let $T = [A, B] \in I\!R$, $\{a_i, i \in I_n\}$ be algebraic polynomials of degree at most n. Assume $\bar{x} \in L_1(T, \mu)$, and for some $M > 0, r > 0$, small enough $\varepsilon > 0$,*

$$\mu\big(\{t \in [A, B] \mid \bar{x}^{p-1} \le \varepsilon\}\big) \le M\varepsilon^r. \tag{3.159}$$

Also suppose that x_n's are the optimal solutions of (TP_n) . Let $\bar{x}^{p-1} \in C^d[A, B]$, with $d \ge 1 + q/r$. Then $x_n^{p-1} \to \bar{x}^{p-1}$ uniformly on $[A, B]$. In particular, if for some $\varepsilon > 0$, $\bar{x}(t) \ge \varepsilon$ for all $t \in T$, (i.e. $r = \infty$ in (3.159)) then $x_n^{p-1} \to \bar{x}^{p-1}$ uniformly on $[A, B]$, whenever $\bar{x}^{p-1} \in C^1[A, B]$.

Proof: By (3.158), (AT3) holds for $\gamma = 2$. Then Theorem 3.6.1 applies since $q \ge 2$. ■

Theorem 3.7.2 *Let $T = [-\pi, \pi] \in I\!R$, and $\tilde{\phi}$ defined in (3.153). Let $\{a_i, i \in I_n\}$ be trigonometric polynomials of degree at most n, $\bar{x}$ be a periodic function on $[-\pi, \pi]$ with period 2π, and $\bar{x}^{p-1} \in C^d[-\pi, \pi]$, with $d \ge (1 + q/r)/2$. For some $M > 0, r > 0$, we also assume*

$$\mu\{t \in [-\pi, \pi] \mid \bar{x}^{p-1} \le \varepsilon\} \le M\varepsilon^r,$$

for small enough $\varepsilon > 0$. Suppose x_n's are the optimal solutions of (TP_n) . Then $x_n^{p-1} \to \bar{x}^{p-1}$ uniformly on $[-\pi, \pi]$.

Theorem 3.7.3 *Let $T = [A, B]$ (or $[-\pi, \pi]$) $\in I\!R$, $\tilde{\phi}$ defined in (3.153), $\{a_i, i \in I_n\}$ be algebraic (or trigonometric) polynomials of degree at most n, x_n's are the optimal solutions of (TP_n) . Suppose $\bar{x}^{p-1} \in C^1[A, B]$ (or $C^1[-\pi, \pi]$ and periodic in the trigonometric case). Then for all $\varepsilon > 0$, $x_n^{p-1} \to \bar{x}^{p-1}$ uniformly on the set*

$$\{t \in T \mid \bar{x}(t) \ge \varepsilon\}. \tag{3.160}$$

In the most interesting case when $p = q = 2$, we have $x_n \to \bar{x}$ uniformly if $0 < \bar{x} \in C^1[A, B]$.

Chapter 4

Numerical Methods

4.1 Introduction

Since the moment problem has applications in so many settings, numerical methods to solve the problem have been discussed repeatedly (see, for example, [37], [36], [50], [57], [73]).

The maximum entropy method, widely used in spectral estimation and other areas, introduces the Boltzmann-Shannon entropy as the objective function and solves a constrained convex programming problem. The use of more general entropy-like functions is now widespread (see, for instance, [6], [7], [32], [35], [38], [40], [45], [48], [49], [51], [53], [68], [72]).

As we saw in the previous chapters, the problem we study is an infinite dimensional one. The variable is a function defined on some function space. We can solve the primal problem directly by discretizing the unknown density function $\bar{x}$ into an unknown vector. It is more interesting to consider the dual problem, which is a finite dimensional, unconstrained, concave maximization problem as we saw in Chapter 2. By solving it, we can obtain the optimum of the dual problem, and then the optimum of the primal problem can be simply reconstructed. The dual method has been discussed in many papers (see, [5], [13],[29], [30], and [41]). As pointed out in [14], there is no difference whether we discretize the primal problem and solve the corresponding dual, or consider the continuous dual problem and (when we numerically solve it) discretize the integral in the objective function, if we use a fixed integration scheme. But the dual structure leads to an appropriate discretization. We also note that from the expression of the objective function in the dual problem, the first and second derivatives can usually be calculated explicitly, and because of the concavity of the objective function, Newton's method (see, for instance, [42], [84]) behaves well in this case.

In the next section, we implement Newton's method in Fortran 77 and examine test problems for various choices of entropies.

For certain entropy functions, such as the Burg entropy and the Boltzmann-Shannon entropy, using special structure in the integration formula, we can determine a finite system of linear equations whose solution produces the dual optimum, or an approximation. In [34], such an algorithm for the Burg entropy moment problem with one dimensional trigonometric moments has been discussed in detail. In Section 4.3, we will give heuristic algorithms of this kind to "solve" algebraic or trigonometric polynomial moment problems in several variables with the Boltzmann-Shannon entropy.

In the final section, we will discuss how the number of nodes in the integration scheme interferes the computational errors and time.

4.2 Dual method – numerical tests

The dual method solves the dual problem:

$$(D_n) \qquad \begin{cases} \max & \sum_{i\in I_n} \lambda_i b_i - \int_T \tilde{\phi}^*(\sum_{i\in I_n} \lambda_i a_i(t))d\mu(t) \\ s.t. & \lambda \in I\!R^{k(n)}, \end{cases} \tag{4.1}$$

which is a finite dimensional, unconstrained, concave maximization problem.

We write

$$\Psi(\lambda) \triangleq \sum_{i\in I_n} \lambda_i b_i - \int_T \tilde{\phi}^*(\sum_{i\in I_n} \lambda_i a_i(t))d\mu(t). \tag{4.2}$$

From Proposition 2.2.3, we know that $I^*_{\tilde{\phi}} = I_{\tilde{\phi}^*}$ is Fréchet differentiable at each $x \in L_\infty(T,\mu)$. Now for each $\lambda \in \mathbb{R}^{k(n)}$, we have $\sum_{i\in I_n} \lambda_i a_i \in L_\infty(T,\mu)$, and hence $I_{\tilde{\phi}^*}$ is Fréchet differentiable at $\sum_{i\in I_n} \lambda_i a_i$. Then by (2.13) and the chain rule, the gradients of Ψ at each $\lambda \in \mathbb{R}^{k(n)}$ are of the form

$$\begin{aligned}\frac{\partial\Psi(\lambda)}{\partial\lambda_j} &= b_j - \langle (I^*_{\tilde{\phi}})'(\sum_{i\in I_n} \lambda_i a_i), a_j\rangle \\ &= b_j - \int_T \tilde{\phi}^{*\prime}(\sum_{i\in I_n} \lambda_i a_i(t)) a_j(t) d\mu(t), \qquad (4.3)\end{aligned}$$

for any $j \in I_n$. Moreover, if $\tilde{\phi}^*$ is twice continuously differentiable and if differentiating through integration poses no problem, we can write for $k, l \in I_n$,

$$\frac{\partial^2\Psi(\lambda)}{\partial\lambda_k\partial\lambda_l} = -\int_T \tilde{\phi}^{*\prime\prime}(\sum_{i\in I_n} \lambda_i a_i(t)) a_k(t) a_l(t) d\mu(t). \qquad (4.4)$$

Let

$$G(\lambda) \triangleq \Big(\frac{\partial\Psi(\lambda)}{\partial\lambda_j},\ j \in I_n \Big)^T \in \mathbb{R}^{k(n)}, \qquad (4.5)$$

and

$$J(\lambda) \triangleq \Big(\frac{\partial^2\Psi(\lambda)}{\partial\lambda_k\partial\lambda_l},\ \ k, l \in I_n \Big) \in \mathbb{R}^{k(n)\times k(n)}. \qquad (4.6)$$

Newton's method with line search gives the following iteration formula:

$$\lambda_{NEW} = \lambda_{OLD} - tJ(\lambda_{OLD})^{-1}\big(b - G(\lambda_{OLD})\big). \qquad (4.7)$$

Note that the matrix J is always negative definite provided that (CQ) holds and $\{a_i, i \in I_n\}$ are linearly independent in the sense that: for any positive measure set $A \subseteq T$, $\lambda \in \mathbb{R}^{k(n)}$,

$$\int_A (\sum_{i\in I_n} \lambda_i a_i(t))^2 d\mu(t) = 0 \qquad \text{implies} \qquad \lambda = 0 \in \mathbb{R}^{k(n)}.$$

Then the objective function Ψ is everywhere strictly concave, and hence Newton's method works very well.

We will try the following six different choices of entropy:

BS-entropy:	$\phi_1(u) = u \log u - u,$	$u \geq 0,$
FD-entropy:	$\phi_2(u) = u \log u + (1-u)\log(1-u),$	$0 \leq u \leq 1,$
L_2-entropy:	$\phi_3(u) = \frac{1}{2}u^2,$	$u \geq 0,$
Burg entropy:	$\phi_4(u) = -\log u,$	$u > 0,$
Burg-type entropy :	$\phi_5(u) = -\log u - \log(1-u),$	$0 < u < 1,$
Hellinger-type entropy:	$\phi_6(u) = -\sqrt{u - u^2},$	$0 \leq u \leq 1.$

The reason we include Burg's entropy here is to make a comparison. From the numerical results we will give below, we will see that Burg's entropy gives us the worst outputs in almost all examples. Also theoretically, we meet the greatest difficulty when we deal with this entropy.

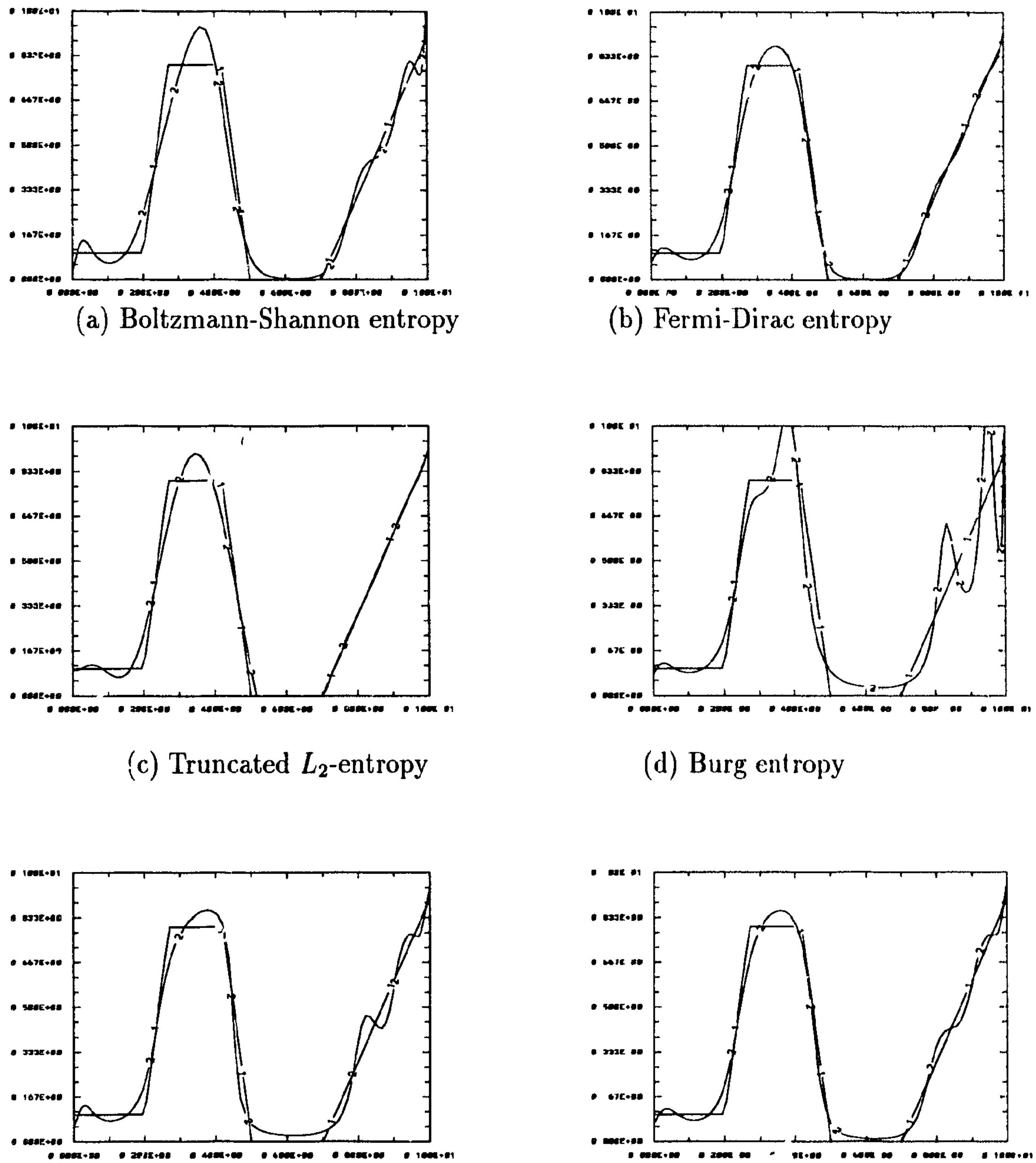
As the first example, we consider the underlying density function $\bar{x}_1$ defined in [0,1] as follows:

$$\bar{x}_1(t) = \begin{cases} 0.1, & 0 \leq t \leq 0.2, \\ 0.1 + 10(t - 0.2), & 0.2 < t \leq 0.27, \\ 0.8, & 0.27 < t \leq 0.42, \\ 0.8 - 10(t - 0.42), & 0.42 < t \leq 0.5, \\ 0, & 0.5 < t \leq 1. \end{cases} \tag{4.8}$$

We build this function $\bar{x}_1$ to be nonsmooth. Although all our convergence theorems require the underlying function $\bar{x}$ to be smooth enough, we can still get a pretty good estimate for a nonsmooth function (even for a discontinuous function).

We first use Newton's method (with line search guarding techniques) to solve the dual problem by using 15 algebraic moments. We discretize the integral in the dual objective function using a Gauss quadrature integration scheme. In Figure 4.1, we give a visual display of the reconstructions when we use six different choices of entropies. The curve labeled "1" is the underlying test function $\bar{x}_1$ and the curves labeled "2" are estimate functions.

We can see that the Fermi-Dirac entropy produces a better estimate than some other entropies do, as well as the Hellinger-type entropy. Also, the Burg-type entropy



(a) Boltzmann-Shannon entropy

(b) Fermi-Dirac entropy

(c) Truncated L_2-entropy

(d) Burg entropy

(e) Burg-type entropy

(f) Hellinger-type Entropy

Figure 4.1: Visual display to reconstruct $\bar{x}_1$ using 15 algebraic moments.

works better than the pure Burg entropy. Remember that all these entropies are belong to the class we called "FD-type entropies". In Section 3.2, we saw that the uniform convergence is much easier to prove for this class of entropies. It is reasonable to use one of these entropies if we know that the value of the underlying function is between certain upper and lower bounds.

The truncated L_2-entropy is one of the truncation-type entropies we defined in Section 3.6. When the underlying function vanishes on one or several subintervals (or a positive measure subset of T), this class of entropies will do a better job than those untruncated ones, as the truncated L_2-entropy does in Figure 4.1 (c).

L_1-error	ϕ_1 [Bol-Shan]	ϕ_2 [Fer-Dir]	ϕ_3 [Trun-L_2]	ϕ_4 [Burg]	ϕ_5 [B-type]	ϕ_6 [Hellinger]
m=3	0.26662	0.26855	0.26999	0.26379	0.26615	0.26786
m=6	0.09136	0.08219	0.09473	0.14952	0.07744	0.07807
m=10	0.06203	0.03005	0.03391	0.13541	0.05720	0.03782
m=18	0.03793	0.02122	0.02654	0.08268	0.04044	0.02465
m=30	0.01911	0.02045	0.03561	0.06997	0.02184	0.01538

Table 4.1: L_1-error of the estimate to $\bar{x}_1$.

L_∞-error	ϕ_1 [Bol-Shan]	ϕ_2 [Fer-Dir]	ϕ_3 [Trun-L_2]	ϕ_4 [Burg]	ϕ_5 [B-type]	ϕ_6 [Hellinger]
m=3	0.51974	0.52231	0.52449	0.51599	0.51893	0.52130
m=6	0.47817	0.31419	0.29297	0.71715	0.45416	0.35280
m=10	0.25242	0.12530	0.14738	0.87777	0.25060	0.12067
m=18	0.16396	0.11455	0.15247	1.35101	0.15020	0.09949
m=30	0.08535	0.08474	0.19738	0.46666	0.11199	0.07467

Table 4.2: L_∞-error of the estimate to $\bar{x}_1$.

In Table 4.1 and Table 4.2, L_1-norm errors and L_∞-norm errors of our estimates to the underlying function $\bar{x}_1$ are given. As the number of moments m increases, we can see how fast they decrease for each choice of entropy. Unsurprisingly, we will find that the Boltzmann-Shannon entropy, the Fermi-Dirac entropy, the Hellinger-type entropy, and the truncated L_2-entropy all behave better than Burg's entropy, especially in the L_∞-norm.

Since our convergence theorems require $\bar{x}$ to be smooth enough and away from the boundaries (which are 0 and 1 in our example here), we now consider a perfectly smooth function:

$$\bar{x}_2(t) = 0.1 + 0.8\sin^2(8t), \tag{4.9}$$

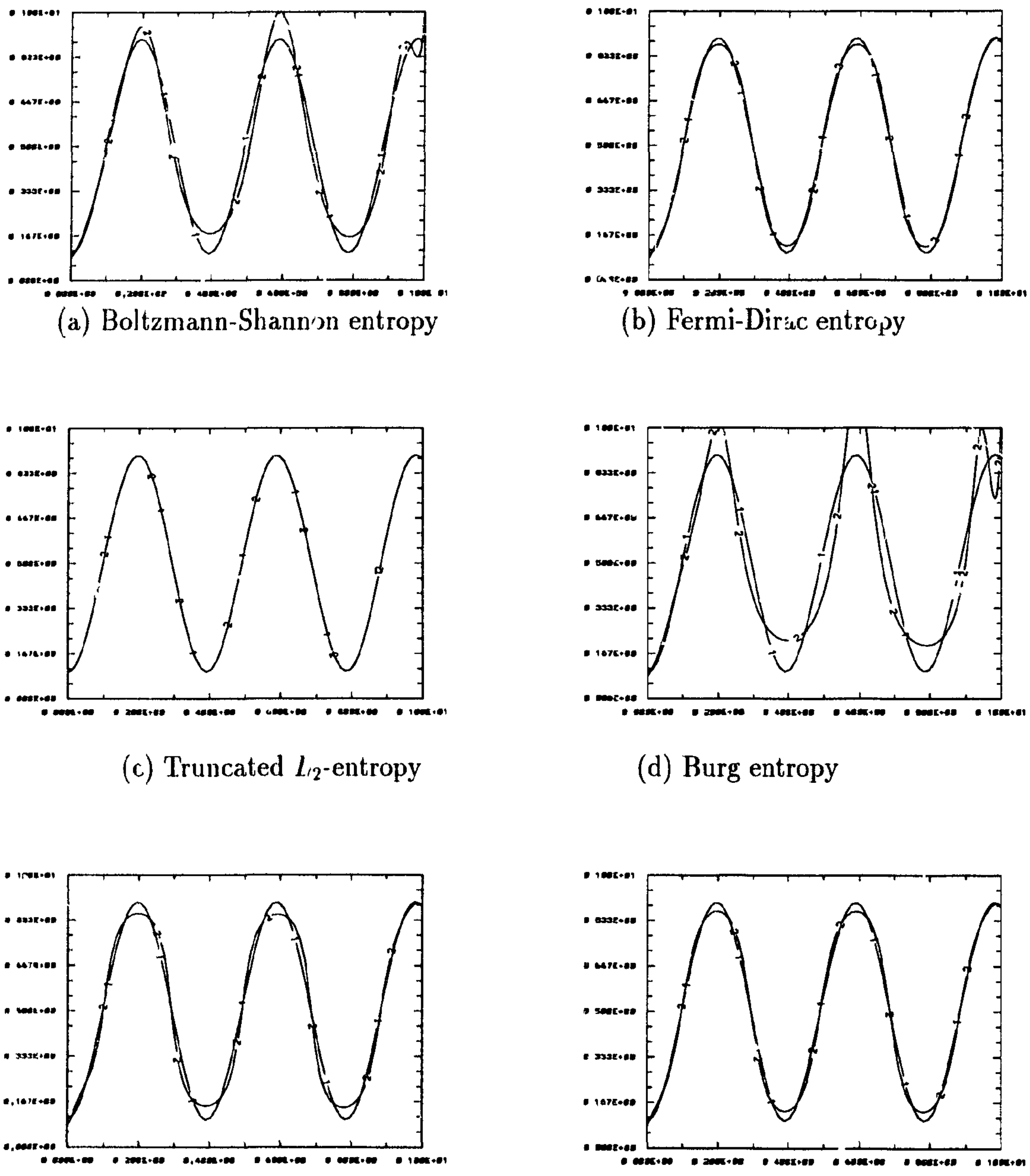
and give corresponding numerical results in Figure 4.2, Table 4.3 and 4.4. Our theoretical results proved in Chapter 3 have been verified numerically once again in this example.

L_1-error	ϕ_1 [Bol-Shan]	ϕ_2 [Fer-Dir]	ϕ_3 [Trun-L_2]	ϕ_4 [Burg]	ϕ_5 [B-type]	ϕ_6 [Hellinger]
m=3	0.25534	0.25555	0.25553	0.25516	0.25560	0.25556
m=6	0.17585	0.14109	0.17328	0.18255	0.11540	0.13238
m=10	0.04460	0.01672	0.01231	0.08391	0.03349	0.04968
m=18	0.03197	0.01339	0.00028	0.07485	0.03044	0.04555
m=30	0.02204	0.01333	0.00157	0.05088	0.02995	0.04194

Table 4.3: L_1-error of the estimate to $\bar{x}_2$.

4.3 Heuristic algorithms for polynomial moment problems with Boltzmann-Shannon entropy

When we use the Boltzmann-Shannon entropy as the objective function in the problem (P_n) to estimate a nonnegative density $\bar{x}$ on $I\!R^m$, given some of its algebraic or trigonometric moments, we will find that there is a special structure in the integration formula. From it, we will derive a useful linear relationship among the moments. A



(a) Boltzmann-Shannon entropy

(b) Fermi-Dirac entropy

(c) Truncated L_2-entropy

(d) Burg entropy

(e) Burg-type entropy

(f) Hellinger-type Entropy

Figure 4.2: Visual display to reconstruct $\bar{x}_2$ using 15 algebraic moments.

L_∞-error	ϕ_1 [Bol-Shan]	ϕ_2 [Fer-Dir]	ϕ_3 [Trun-L_2]	ϕ_4 [Burg]	ϕ_5 [B-type]	ϕ_6 [Hellinger]
m=3	0.42777	0.42877	0.42854	0.42698	0.42800	0.42866
m=6	1.10057	0.26015	0.57017	3.80299	0.22301	0.24860
m=10	0.09958	0.05756	0.07324	0.31243	0.07170	0.02210
m=18	0.08496	0.03332	0.00144	0.20547	0.07418	0.01902
m=30	0.09535	0.02873	0.01361	0.68714	0.07190	0.01867

Table 4.4: L_∞-error of the estimate to $\bar{x}_2$.

simple algorithm then provides a fairly good estimate of $\bar{x}$ by just solving a couple of linear systems.

4.3.1 Algebraic polynomial case on [0,1]

We first consider a problem of the simplest form:

$$(BSP_n) \qquad \begin{cases} \inf & \int_0^1 [x(t)\log(x(t)) - x(t)]dt, \\ s.t. & \int_0^1 t^i x(t)dt = b_i, \quad i = 0, 1, \cdots, n, \\ & 0 \leq x \in L_1[0,1]. \end{cases}$$

By the duality results, the optimal solution x_n of (BSP_n) can be expressed as

$$x_n(t) = \exp(\sum_{i=0}^{n} \lambda_i t^i), \tag{4.10}$$

where the $\{\lambda_i, i = 0, 1, \cdots, n\}$ can be determined by the nonlinear system

$$\int_0^1 \exp(\sum_{i=0}^{n} \lambda_i t^i) t^k dt = b_k, \qquad k = 0, 1, \cdots, n.$$

We now SUPPOSE that the underlying density $\bar{x}$ is exactly of the form:

$$\bar{x}(t) = \exp(\sum_{i=0}^{n} \lambda_i t^i), \tag{4.11}$$

for some n, and we need to find out the arguments λ_i, $i = 0, 1, \cdots n$. If we are lucky enough to have known $2n+1$ moments given by

$$b_k = \int_0^1 \exp(\sum_{i=0}^{n} \lambda_i t^i) t^k dt, \qquad k = 0, 1, \cdots, 2n,$$

integrating by parts, we obtain for $k = 0, 1, \cdots, n$,

$$\begin{aligned} b_k &\triangleq \int_0^1 \exp(\sum_{i=0}^{n} \lambda_i t^i) t^k dt \\ &= \frac{1}{k+1} \exp(\sum_{i=0}^{n} \lambda_i t^i) t^{k+1} \Big|_0^1 \\ &\quad - \frac{1}{k+1} \int_0^1 t^{k+1} \exp(\sum_{i=0}^{n} \lambda_i a_i(t)) \sum_{i=1}^{n} i \lambda_i t^{i-1} dt \\ &= \frac{1}{k+1} \exp(\sum_{i=0}^{n} \lambda_i) - \frac{1}{k+1} \sum_{i=1}^{n} i \lambda_i b_{k+i}, \end{aligned}$$

or

$$(k+1) b_k = \exp(\sum_{i=0}^{n} \lambda_i) - \sum_{i=1}^{n} i \lambda_i b_{k+i}. \tag{4.12}$$

Thus $\lambda_0, \lambda_1, \cdots, \lambda_n$ in (4.11) can be obtained by solving a linear system

$$b = Br, \tag{4.13}$$

where

$$b = \begin{bmatrix} b_0 \\ 2b_1 \\ \vdots \\ (n+1)b_n \end{bmatrix}, \quad B = \begin{bmatrix} 1 & b_1 & b_2 & \cdots & b_n \\ 1 & b_2 & b_3 & \cdots & b_{n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & b_{n+1} & b_{n+2} & \cdots & b_{2n} \end{bmatrix}, \quad r = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_n \end{bmatrix},$$

and

$$\begin{cases} r_0 = \exp(\sum_{i=0}^{n} \lambda_i) \\ r_k = -k\lambda_k, \qquad k = 1, 2, \cdots, n. \end{cases} \tag{4.14}$$

It is not difficult to show that under a mild condition which is implied by (CQ) the linear system (4.13) is solvable.

Lemma 4.3.1 *If there exists a nonzero density $\hat{x}$ on $[0,1]$, such that $b_0, b_1, \cdots, b_{2n}$ are given by*

$$b_k = \int_0^1 \hat{x}(t)t^k dt, \qquad k = 0, 1, \cdots, 2n,$$

then B is nonsingular.

Proof: Note that

$$\begin{aligned}
|B| &= \begin{vmatrix} 1 & b_1 & b_2 & \cdots & b_n \\ 1 & b_2 & b_3 & \cdots & b_{n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & b_{n+1} & b_{n+2} & \cdots & b_{2n} \end{vmatrix} \\
&= \begin{vmatrix} b_1 - b_2 & b_2 - b_3 & \cdots & b_n - b_{n+1} \\ b_2 - b_3 & b_3 - b_4 & \cdots & b_{n+1} - b_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ b_n - b_{n+1} & b_{n+1} - b_{n+2} & \cdots & b_{2n-1} - b_{2n} \end{vmatrix} \triangleq |D|
\end{aligned}$$

For any $v = (v_1, v_2, \cdots, v_n)^T \in I\!R^n$, $v \neq 0$, we have

$$\begin{aligned}
v^T D v &= \sum_{j=1}^n \sum_{i=1}^n v_i v_j (b_{i+j-1} - b_{i+j}) \\
&= \sum_{j=1}^n \sum_{i=1}^n v_i v_j \int_0^1 \hat{x}(t)(t^{i+j-1} - t^{i+j}) dt \\
&= \int_0^1 \hat{x}(t) (\sum_{j=1}^n \sum_{i=1}^n v_i v_j t^{i+j-2}) t(1-t) dt \\
&= \int_0^1 \hat{x}(t) (\sum_{i=1}^n v_i t^{i-1})^2 t(1-t) dt \\
&> 0.
\end{aligned}$$

Hence D is positive definite, $|D| \neq 0$, and so $|B|$ is nonzero. ∎

From Lemma 4.3.1 we see that if $\{b_k\}$ are consistent then there is a unique solution for the linear system (4.13). Thus the parameters $\lambda_0, \lambda_1, \cdots, \lambda_n$ can be obtained from (4.14).

For a density $\bar{x}$ other than of the form in (4.11), we may use this simple method to get a heuristic estimate of $\bar{x}$. We can see that in (4.14), r_0 is required to be positive, which may not be true all the time. But from the first moment

$$b_0 = \int_0^1 \exp(\sum_{i=0}^{n} \lambda_i t^i)dt = e^{\lambda_0} \int_0^1 \exp(\sum_{i=1}^{n} \lambda_i t^i)dt,$$

we can still "determine" λ_0 when $\lambda_1, \lambda_2, \cdots, \lambda_n$ are known.

Algorithm 4.3.2 *Let $2n+1$ moments $b_0, b_1, \cdots, b_{2n}$ be given.*

Step 1. Construct:

$$B_n = \begin{bmatrix} 1 & b_1 & b_2 & \cdots & b_n \\ 1 & b_2 & b_3 & \cdots & b_{n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & b_n & b_{n+1} & \cdots & b_{2n} \end{bmatrix}, \qquad b^n = \begin{bmatrix} b_0 \\ 2b_1 \\ \vdots \\ (n+1)b_n \end{bmatrix}.$$

Step 2. Compute $r^n \in I\!R^{n+1}$ which solves the linear system

$$B_n r^n = b^n.$$

Step 3. Compute $\lambda^n \in I\!R^{n+1}$ as follows:

$$\begin{aligned} \lambda_k^n &= -\frac{r_k^n}{k}, \qquad k = 1, 2, \cdots, n, \\ \lambda_0^n &= \log\left(\frac{b_0}{\int_0^1 \exp(\sum_{i=1}^{n} \lambda_i^n t^i)dt}\right). \end{aligned}$$

Step 4. Construct

$$x_n(t) = \exp(\sum_{i=0}^{n} \lambda_i^n t^i).$$

The following fact is now obvious.

Theorem 4.3.3 *If the prior density $\bar{x}$ is of the form (4.11) for some $\lambda \in I\!R^n$, and the first $2n+1$ moments are given, then the estimate density constructed by Algorithm 4.3.2 is exactly $\bar{x}$ itself.*

In Figures 4.3 and 4.4, we can see that for general positive underlying functions, our algorithm also gives fairly good estimates. We will try to reconstruct the functions

$$\bar{x}_3(t) = 2|t - 0.5|, \tag{4.15}$$

$$\bar{x}_4(t) = |(1-t)\sin 12t|. \tag{4.16}$$

Figures (a), (c), and (e) are our heuristic solutions using 7, 17, and 27 moments, respectively. This means we are approximating x_3, x_8, x_{13}, the optimal solutions of (P_3), (P_8) and (P_{13}). Figures (b), (d), and (f) are the optimal solutions of (P_6), (P_{16}), and (P_{26}), respectively, by using Newton's method to solve the dual problems starting from the initial point

$$\lambda = (\log(b_0), 0, \cdots, 0),$$

(suggested in [24]) and iterating until the stopping criteria

$$\|\nabla G(\lambda)\|_\infty \leq 0.0001$$

is satisfied. That is to say, we have used the same number of moments to get each pair of estimates: (a) and (b), (c) and (d), (e) and (f). We can also see that there is a greater advantage to using our heuristic algorithm when we have enough moments (or observations). This is a consequence of the cost of computation versus the cost of observation. When we have enough data from the moments available, our heuristic algorithm can produce a good estimate (almost as good as the optimal solution of (P_n)) in much less time (for a time comparison, see Section 4.2.5).

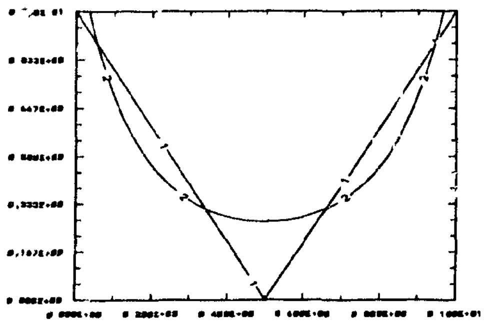
4.3.2 Algorithm generalized to $[0,1]^2$

We now consider $T = [0,1]^2$. For $n = (n_1, n_2) \in \mathbf{Z}_+^2$, let $\{a_i,\ i \in I_n\}$ be algebraic polynomials of degree at most n_1 in t_1 and n_2 in t_2, of the form

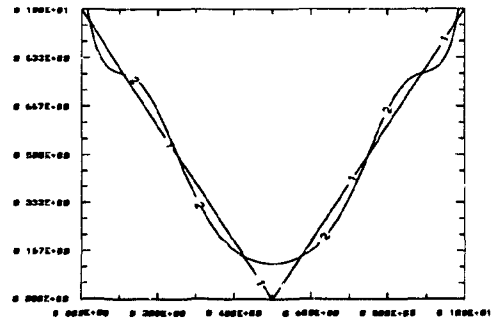
$$t_1^i t_2^j, \qquad i = 0, 1, \cdots, n_1, \qquad j = 0, 1, \cdots, n_2.$$

If we assume that

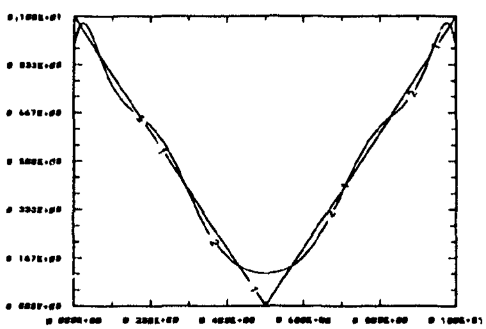
$$\bar{x}(t_1, t_2) = \exp\left(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \lambda_{i,j} t_1^i t_2^j\right), \tag{4.17}$$



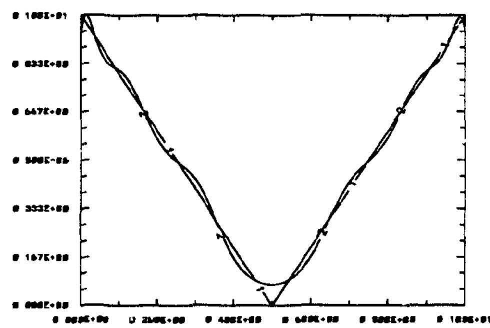
(a) Heuristic solution using 7 moments.



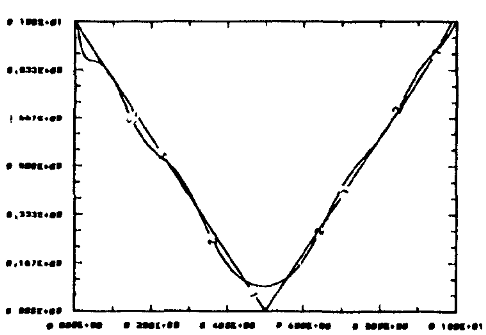
(b) Optimal solution of (P_6).



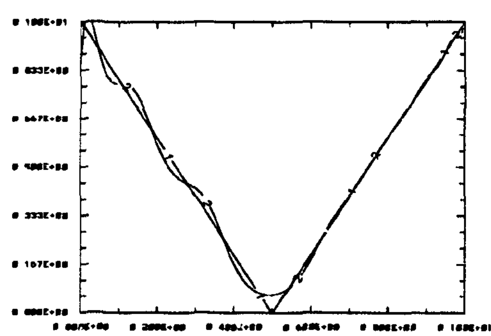
(c) Heuristic solution using 17 moments.



(d) Optimal solution of (P_{16}).

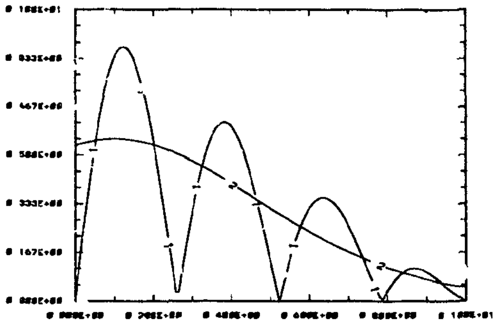


(e) Heuristic solution using 27 moments.

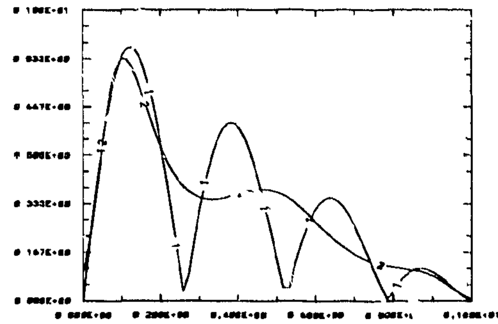


(f) Optimal solution of (P_{26}).

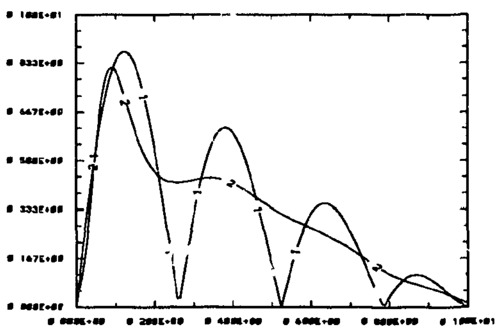
Figure 4.3: Heuristic estimates to $\bar{x}_3$, compared with the optimal solution of (P_n).



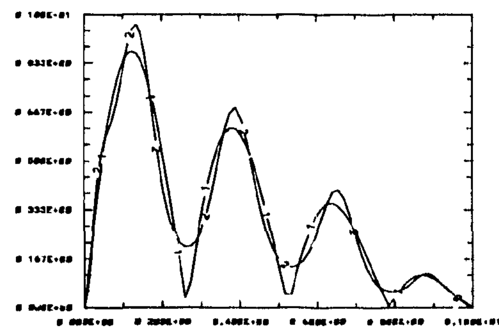
(a) Heuristic solution using 7 moments.



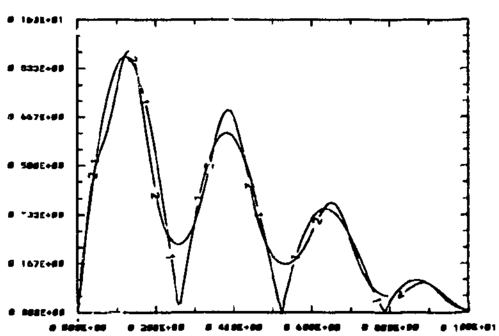
(b) Optimal solution of (P_6).



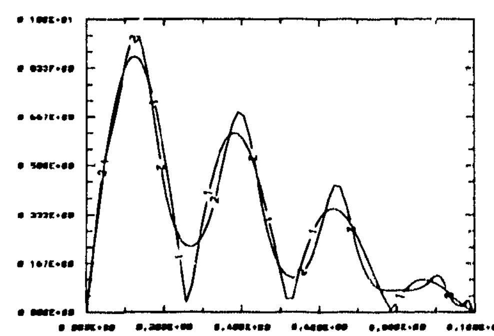
(c) Heuristic solution using 17 moments.



(d) Optimal solution of (P_{16}).



(e) Heuristic solution using 27 moments.



(f) Optimal solution of (P_{26}).

Figure 4.4: Heuristic estimates to $\bar{x}_4$, compared with the optimal solution of (P_n).

and we know the moments given by

$$b_{l_1,l_2} = \int_0^1 \int_0^1 \bar{x}(t_1,t_2) t_1^{l_1} t_2^{l_2} dt_2 dt_1, \tag{4.18}$$

for $l_1 = 0, 1, \cdots, 2n_1$, $l_2 = 0, 1, \cdots, 2n_2$. Then the analogous formula to (4.12) is

$$\begin{aligned}
b_{l_1,l_2} &= \int_0^1 \int_0^1 \exp(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \lambda_{i,j} t_1^i t_2^j) t_1^{l_1} t_2^{l_2} dt_2 dt_1 \\
&= \int_0^1 (\int_0^1 \exp(\sum_{i=0}^{n_1} (t_1^i \sum_{j=0}^{n_2} \lambda_{i,j} t_2^j)) t_2^{l_2} dt_2) t_1^{l_1} dt_1 \\
&= \frac{1}{l_1+1} t_1^{l_1+1} \int_0^1 \exp(\sum_{i=0}^{n_1} t_1^i (\sum_{j=0}^{n_2} \lambda_{i,j} t_2^j)) t_2^{l_2} dt_2 \Big|_{t_1=0}^{t_1=1} \\
&\quad - \frac{1}{l_1+1} \int_0^1 t_1^{l_1+1} (\int_0^1 \exp(\sum_{i=0}^{n_1} t_1^i (\sum_{j=0}^{n_2} \lambda_{i,j} t_2^j)) \sum_{i=1}^{n_1} (i t_1^{i-1} \sum_{j=0}^{n_2} \lambda_{i,j} t_2^j) t_2^{l_2} dt_2) dt_1 \\
&= \frac{1}{l_1+1} \int_0^1 \exp(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \lambda_{i,j} t_2^j) t_2^{l_2} dt_2 - \frac{1}{l_1+1} \sum_{i=1}^{n_1}\sum_{j=0}^{n_2} i \lambda_{i,j} b_{l_1+i,l_2+j},
\end{aligned} \tag{4.19}$$

or

$$(l_1+1) b_{l_1,l_2} = \int_0^1 \exp(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \lambda_{i,j} t_2^j) t_2^{l_2} dt_2 - \sum_{i=1}^{n_1}\sum_{j=0}^{n_2} i \lambda_{i,j} b_{l_1+i,l_2+j},$$

for $l_1 = 0, 1, \cdots, n_1$, $l_2 = 0, 1, \cdots, n_2$. Now let

$$r_{0,l_2} = \int_0^1 \exp(\sum_{i=0}^{n_1}\sum_{i=0}^{n_2} \lambda_{i,j} t_2^j) t_2^{l_2} dt_2, \quad l_2 = 0, 1, \cdots, n_2,$$

and

$$r_{l_1,l_2} = -l_1 \lambda_{l_1,l_2}, \quad l_1 = 1, 2, \cdots, n_1, \ l_2 = 0, 1, \cdots, n_2.$$

We now obtain a linear system:

$$d = Du, \tag{4.20}$$

where

$$d = \begin{bmatrix} b_{0,0} \\ \vdots \\ b_{0,n_2} \\ \vdots \\ \vdots \\ (n_1+1)b_{n_1,0} \\ \vdots \\ (n_1+1)b_{n_1,n_2} \end{bmatrix}, \qquad u = \begin{bmatrix} r_{0,0} \\ \vdots \\ r_{0,n_2} \\ \vdots \\ \vdots \\ r_{n_1,0} \\ \vdots \\ r_{n_1,n_2} \end{bmatrix}, \tag{4.21}$$

and

$$D = \begin{bmatrix} 1 & \cdots & 0 & b_{1,0} & \cdots & b_{1,n_2} & \cdots & \cdots & b_{n_1,0} & \cdots & b_{n_1,n_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & b_{1,n_2} & \cdots & b_{1,2n_2} & \cdots & \cdots & b_{n_1,n_2} & \cdots & b_{n_1,2n_2} \\ 1 & \cdots & 0 & b_{2,0} & \cdots & b_{2,n_2} & \cdots & \cdots & b_{n_1+1,0} & \cdots & b_{n_1+1,n_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & b_{2,n_2} & \cdots & b_{2,2n_2} & \cdots & \cdots & b_{n_1+1,n_2} & \cdots & b_{n_1+1,2n_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & b_{n_1+1,0} & \cdots & b_{n_1+1,n_2} & \cdots & \cdots & b_{2n_1,0} & \cdots & b_{2n_1,n_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & b_{n_1+1,n_2} & \cdots & b_{n_1+1,2n_2} & \cdots & \cdots & b_{2n_1,n_2} & \cdots & b_{2n_1,2n_2} \end{bmatrix}.$$

wzhuang Solving it, we can obtain $\lambda_{i,j}$, for $i = 1, 2, \cdots, n_1$, $j = 0, 1, \cdots, n_2$. Switching the order of t_1 and t_2, and integrating by parts in (4.19), we have

$$\begin{aligned} b_{l_1,l_2} &= \frac{1}{l_2+1}\int_0^1 \exp\left(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\lambda_{i,j}t_1^i\right)t_1^{l_1}\,dt_1 \\ &\quad -\frac{1}{l_2+1}\sum_{i=0}^{n_1}\sum_{j=1}^{n_2} j\lambda_{i,j}b_{l_1+i,l_2+j}. \end{aligned} \tag{4.22}$$

Set $l_1 = 0$, we have

$$\begin{aligned} b_{0,l_2} &= \frac{1}{l_2+1}\int_0^1 \exp(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\lambda_{i,j}t_1^i)dt_1 \\ &-\frac{1}{l_2+1}\sum_{i=0}^{n_1}\sum_{j=1}^{n_2} j\lambda_{i,j}b_{i,l_2+j}, \end{aligned} \tag{4.23}$$

which can be used to find $\lambda_{0,j}$, for $j = 1, 2, \cdots, n_2$. Finally, from the first moment

$$b_{0,0} = \int_0^1\int_0^1 \exp(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} t_1^i t_2^j)dt_2 dt_1, \tag{4.24}$$

we can determine $\lambda_{0,0}$. We then have the following detailed algorithm.

Algorithm 4.3.4 *Let $b_{i,j}$, $i = 0, 1, \cdots, 2n_1$, $j = 0, 1, \cdots, 2n_2$ be given moments in (4.18).*

Step 1. Construct

$$d_k = \begin{bmatrix} b_{k,0} \\ b_{k,1} \\ \vdots \\ b_{k,n_2} \end{bmatrix}, \quad u_k = \begin{bmatrix} r_{k,0} \\ r_{k,1} \\ \vdots \\ r_{k,n_2} \end{bmatrix}, \qquad k = 0, 1, \cdots, n_1,$$

$$D_k = \begin{bmatrix} b_{k,0} & b_{k,1} & \cdots & b_{k,n_2} \\ b_{k,1} & b_{k,2} & \cdots & b_{k,n_2+1} \\ \vdots & \vdots & \cdots & \vdots \\ b_{k,n_2} & b_{k,n_2+1} & \cdots & b_{k,2n_2} \end{bmatrix}, \qquad k = 1, 2, \cdots, 2n_1,$$

Step 2. Solve the following linear system with $(n_1+1)(n_2+1)$ variables

$$d = Du,$$

where

$$d = \begin{bmatrix} d_0 \\ 2d_1 \\ \vdots \\ (n_1+1)d_{n_1} \end{bmatrix}, \quad u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n_1} \end{bmatrix},$$

$$D = \begin{bmatrix} I & D_1 & D_2 & \cdots & D_{n_1} \\ I & D_2 & D_3 & \cdots & D_{n_1+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ I & D_{n_1+1} & D_{n_1+2} & \cdots & D_{2n_1} \end{bmatrix}.$$

Step 3. *Compute:*

$$\lambda_{i,j} = -\frac{1}{l_1} r_{l_1,l_2}, \ \ l_1 = 1, 2, \cdots, n_1, \ l_2 = 0, 1, \cdots, n_2.$$

Step 4. *Compute:*

$$b'_{l_2} = (l_2+1)b_{0,l_2} + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} j\lambda_{i,j} b_{i,l_2+j}, \ \ l_2 = 0, 1, \cdots, n_2,$$

and solve the linear system

$$b' = B'u',$$

where

$$b' = \begin{bmatrix} b'_0 \\ b'_1 \\ \vdots \\ b'_{n_2} \end{bmatrix}, u' = \begin{bmatrix} r'_0 \\ r'_2 \\ \vdots \\ r'_{n_2} \end{bmatrix},$$

and

$$B' = \begin{bmatrix} 1 & b_{0,1} & b_{0,2} & \cdots & b_{0,n_2} \\ 1 & b_{0,2} & b_{0,3} & \cdots & b_{0,n_2+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & b_{0,n_2+1} & b_{0,n_2+2} & \cdots & b_{0,2n_2} \end{bmatrix}.$$

Step 5. *Compute:*

$$\lambda_{0,j} = -\frac{1}{j} r'_j, \qquad j = 1, 2, \cdots, n_2.$$

Step 6. *Finally we have*

$$\lambda_{0,0} = \log \Big[b_{0,0}\Big(\int_0^1\int_0^1 \exp\big(\sum_{(i,j)\neq(0,0)} \lambda_{i,j} t_1^i t_2^j\big) dt_2 dt_1\Big)^{-1}\Big],$$

and

$$x_n(t_1, t_2) = \exp(\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \lambda_{i,j} t_1^i t_2^j)$$

is the estimate density.

Analogously to the Theorem 4.3.11, we have

Theorem 4.3.5 *If the prior density $\bar{x}$ is of the form (4.17) for some n_1, $n_2 \in \mathbf{Z}_+$, and we know the first $(2n_1 + 1)(2n_2 + 1)$ moments given by (4.18), then the estimate density x_n constructed by Algorithm 4.3.4 is exactly $\bar{x}$ itself.*

We now give some numerical test results of our algorithm in $[0, 1]^2 \subset I\!R^2$. We will try to reconstruct some two variable underlying density functions. In the figures given below, pictures labeled (a) give the underlying density functions, pictures labeled (b), (c), or (d) give the heuristic estimations generated by Algorithm 4.3.4 using $49(= 7 \times 7)$, $121(= 11 \times 11)$, or $289(= 17 \times 17)$ algebraic moments, respectively.

In Figure 4.5, the underlying function $\bar{x}_5$ is like a helmet, which is smooth but with sharp derivative at the center. In Figure 4.6, we give a pyramid function $\bar{x}_6$, which is continuous but nonsmooth. In Figure 4.7, we will reconstruct a forest-like function $\bar{x}_7$, which is perfectly smooth but with a lot of peaks. In Figure 4.8, a discontinuous stairway function $\bar{x}_8$ is to be reconstructed.

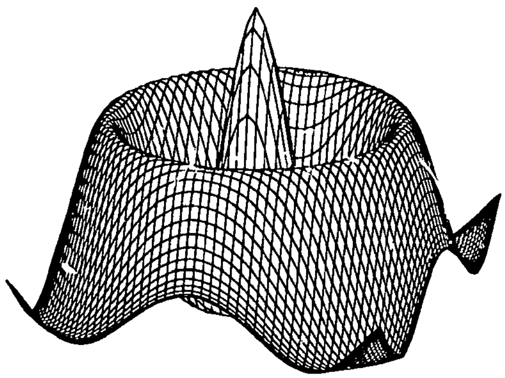
Note that our estimate is always a smooth function and is strictly positive. So for a nonsmooth or discontinuous underlying function, we should not be surprised to see that the reconstruction looks like a melting ice cube (as in Figure 4.6(d)), or a muddy path (as in Figure 4.8(d)).

In the last example, although the estimates do not look as cute as the original stairway function, we can still see the steps (especially in (d)) climbing to the top.

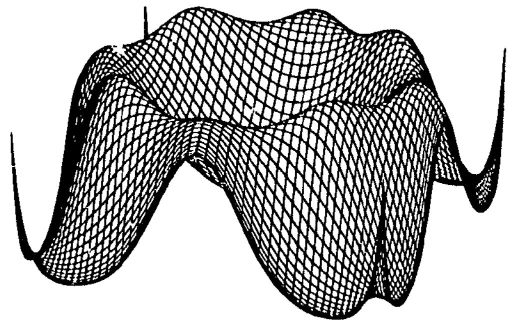
4.3.3 Generalization to $[0, 1]^m$

Now we generalized the algorithm to $[0, 1]^m$. Let $T = [0, 1]^m$, and write

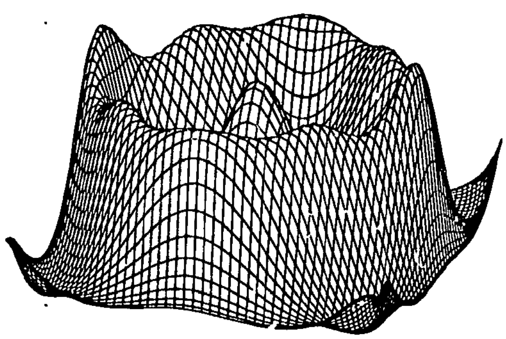
$$n \triangleq (n_1, n_2, \cdots, n_m)^T \in \mathbf{Z}_+^m,$$



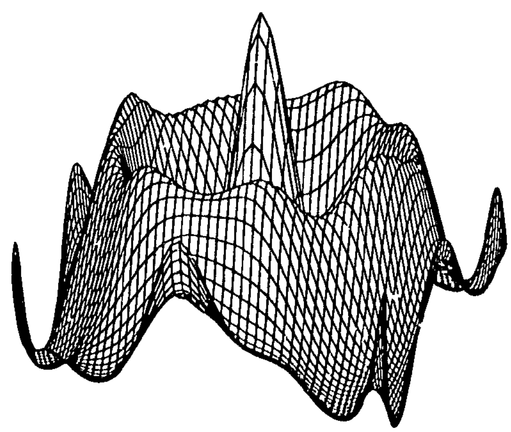
(a) The underlying function.



(b) Using 49 moments.

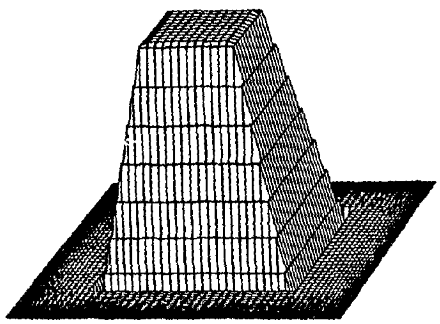


(c) Using 121 moments.

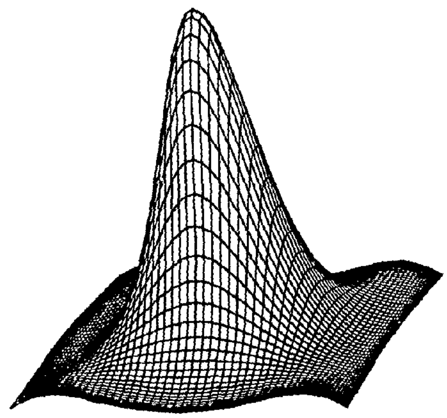


(d) Using 289 moments.

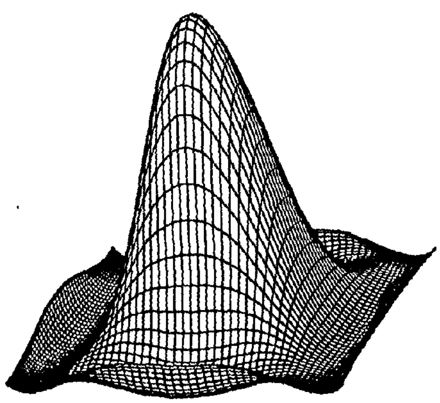
Figure 4.5: Heuristic estimates to the function $\bar{x}_5$.



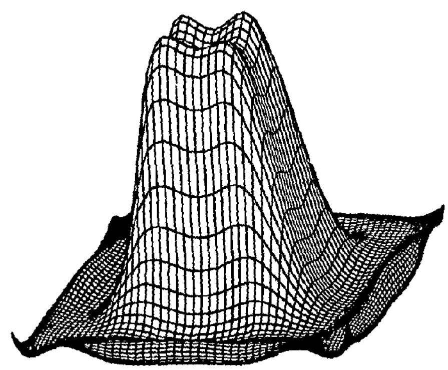
(a) The underlying function.



(b) Using 49 moments.

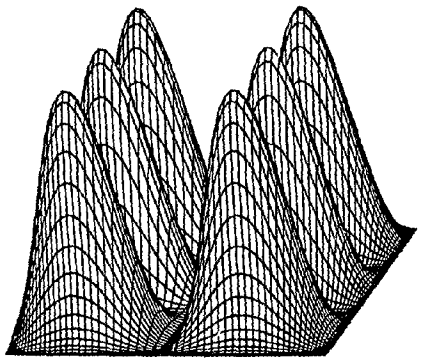


(c) Using 121 moments.

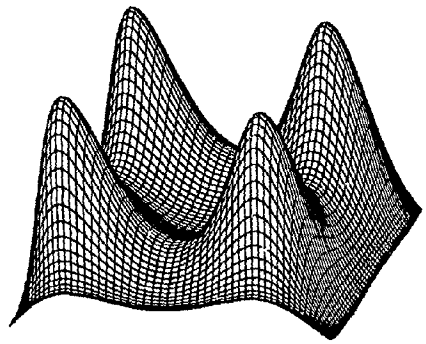


(d) Using 289 moments.

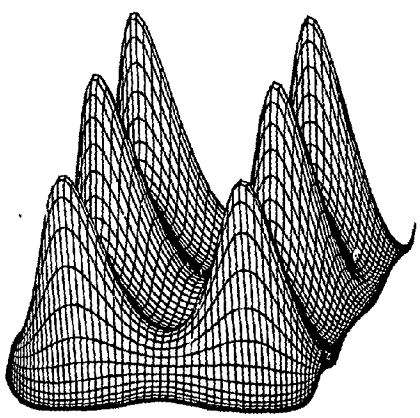
Figure 4.6: Heuristic estimates to the function $\bar{x}_6$.



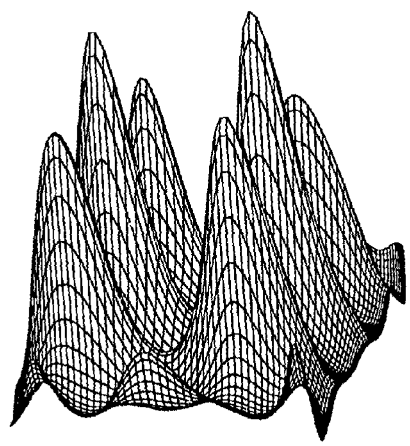
(a) The underlying function.



(b) Using 49 moments.

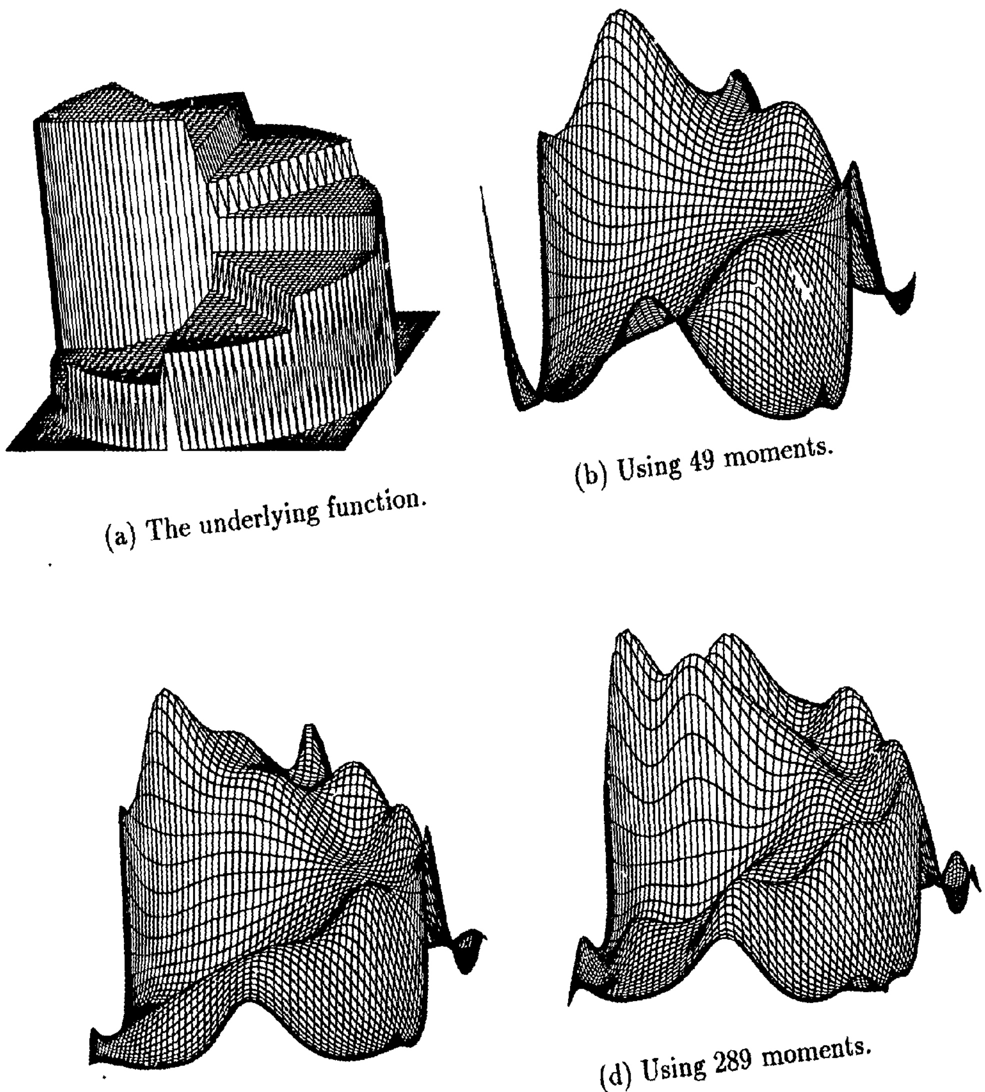


(c) Using 121 moments.



(d) Using 289 moments.

Figure 4.7: Heuristic estimates to the function $\bar{x}_7$.



(a) The underlying function.

(b) Using 49 moments.

(c) Using 121 moments.

(d) Using 289 moments.

Figure 4.8: Heuristic estimates to the function $\bar{x}_8$.

$$\begin{aligned} I_n &\triangleq \{(i_1, i_2, \cdots, i_m)^T \in \mathbf{Z}_+^m \mid i_j = 0, 1, \cdots, n_j,\ j = 1, 2, \cdots, m\} \\ &\equiv \{(i \in \mathbf{Z}_+^m \mid 1 \le i \le n\}, \end{aligned}$$

$\{a_i, i \in I_n\}$ be algebraic polynomials of the form

$$t_1^{i_1} t_2^{i_2} \cdots t_m^{i_m},\ \ i_j = 0, 1, \cdots, n_j,\ j = 1, 2, \cdots, m.$$

Then we have

$$k(n) = \prod_{j=1}^{m} (n_j + 1). \tag{4.25}$$

Again we assume $\bar{x}$ is of the form

$$\bar{x}(t_1, t_2, \cdots, t_m) = \exp(\sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_m=0}^{n_m} \lambda_{i_1, i_2, \cdots, i_m} t_1^{i_1} t_2^{i_2} \cdots t_m^{i_m}),$$

or

$$\bar{x}(t) = \exp(\sum_{i \in I_n} \lambda_i t^i),$$

here we denote

$$\begin{aligned} t &\triangleq (t_1, t_2, \cdots, t_m)^T \in I\!R^m, \\ i &\triangleq (i_1, i_2, \cdots, i_m)^T \in \mathbf{Z}_+^m, \\ t^i &\triangleq t_1^{i_1} t_2^{i_2} \cdots t_m^{i_m}, \end{aligned}$$

and

$$\lambda_i \triangleq \lambda_{i_1, i_2, \cdots, i_m},\ \ i \in I_n,$$

hence

$$\lambda \triangleq (\lambda_i\ i \in I_n) \in I\!R^{k(n)}.$$

Then for $l_i = 0, 1, \cdots, n_i,\ i = 1, 2, \cdots, m$, the moments are given by

$$b_{l_1, l_2, \cdots, l_m} = \int_0^1 \int_0^1 \cdots \int_0^1 \bar{x}(t_1, t_2, \cdots, t_m) t_1^{l_1} t_2^{l_2} \cdots t_m^{l_m} dt_1 dt_2 \cdots dt_m.$$

or

$$b_l = \int_{[0,1]^m} \bar{x}(t)t^l dt \qquad l \in I_n,$$

where

$$\begin{aligned} l &= (l_1, l_2, \cdots, l_m) \in \mathbf{Z}_+^m, \\ dt &= dt_1 dt_2 \cdots dt_m. \end{aligned}$$

Note that for each $j = 1, 2, \cdots, m$, integrating by parts, we have for each $l \in I_n$,

$$\begin{aligned} b_l &= \frac{1}{l_j+1} \int_{[0,1]^{m-1}} \exp(\sum_{i \in I_n(j)} \lambda_i t(j)^{i(j)}) t(j)^{l(j)} dt(j) \\ &\quad - \frac{1}{l_j+1} \sum_{i \in I_n(j)} i_j \lambda_i b_{i+l}, \end{aligned}$$

where

$$\begin{aligned} I_n(j) &\triangleq \{i \in I_n \mid i_j \neq 0\} \\ t(j) &\triangleq (t_1, \cdots, t_{j-1}, t_{j+1}, \cdots, t_m)^T \in I\!R^{m-1} \\ i(j) &\triangleq (i_1, \cdots, i_{j-1}, i_{j+1}, \cdots, i_m)^T \in \mathbf{Z}_+^{m-1} \\ dt(j) &\triangleq dt_1 \cdots dt_{j-1} dt_{j+1} \cdots dt_m. \end{aligned}$$

The algorithm can be stated as follows.

Algorithm 4.3.6 *Let $b_i, i \in I_{2n}$ be the $\prod_{k=1}^m (2n_k+1)$ moments given in (4.18).*

Step 1. Construct linear system with $\prod_{k=1}^m (n_k+1)$ unknowns:

$$d_{l_1,l_2,\cdots,l_m} = r_{0,l_2,\cdots,l_m} + \sum_{i_1=1}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_m=0}^{n_m} r_{i_1,\cdots,i_m} b_{i_1+l_1,\cdots,i_m+l_m},$$

where

$$d_{l_1,l_2,\cdots,l_m} = (l_1+1) b_{l_1,l_2,\cdots,l_m},$$

and solve it. Let

$$\lambda_{i_1,\cdots,i_m} = -\frac{r_{i_1,\cdots,i_m}}{i_1},$$

for

$$i_1 = 1, \cdots, n_1,\ i_2 = 0, \cdots, n_2, \cdots, i_m = 0, \cdots, n_m.$$

Set $j = 1$.

Step 2. If $j \leq m - 1$, *construct the linear system with* $\prod_{k=j+1}^{m}(n_k + 1)$ *unknowns:*

$$d_{l_{j+1},\cdots,l_m} = r_{0,l_{j+2},\cdots,l_m} + \sum_{i_{j+1}=1}^{n_{j+1}} \sum_{i_{j+2}=0}^{n_{j+2}} \cdots \sum_{i_m=0}^{n_m} r_{i_{j+1},\cdots,i_m} b_{0,\cdots,0,i_{j+1}+l_{j+1},\cdots,i_m+l_m},$$

for

$$l_{j+1} = 1, \cdots, n_{j+1},\ l_{j+2} = 0, \cdots, n_{j+2}, \cdots, l_m = 0, \cdots, n_m,$$

where

$$d_{l_{j+1},\cdots,l_m} = (l_{j+1} + 1) b_{0,\cdots,0,l_{j+1},\cdots,l_m} + \sum_{i_1+\cdots+i_j>0} r_{i_{j+1},\cdots,i_m} b_{i_1,\cdots,i_j,i_{j+1}+l_{j+1},\cdots,i_m+l_m}.$$

Solve it and let

$$\lambda_{0,\cdots,0,l_{j+1},\cdots,l_m} = -\frac{r_{l_{j+1},\cdots,l_m}}{l_{j+1}},$$

for

$$l_{j+1} = 1, \cdots, n_{j+1},\ l_{j+2} = 0, \cdots, n_{j+2}, \cdots, l_m = 0, \cdots, n_m.$$

Set $j = j + 1$, *repeat Step 2.*

Step 3. When $j = m$, *compute:*

$$\lambda_{0,\cdots,0} = \log[b_{0,\cdots,0}(\int_0^1 \int_0^1 \cdots \int_0^1 \exp(\sum_{i_1+i_2+\cdots+i_m>0} \lambda_{i_1,\cdots,i_m} t_1^{i_1} \cdots t_m^{i_m} dt_1 \cdots dt_m)^{-1}].$$

Then the estimate density is

$$x_n(t) = \exp(\sum_{i \in I_n} \lambda_i t^i).$$

4.3.4 Trigonometric polynomial cases

We first consider the trigonometrical case on the interval $[-\pi, \pi]$.

Let $I_n = \{-n, \cdots, 0, \cdots, n\}$, $a_k(t) = e^{ikt}$, $k \in I_n$, where $i = \sqrt{-1}$. Then the problem becomes:

$$(P_n)\left\{\begin{array}{ll} \min & \int_{-\pi}^{\pi}[x(t)\log(x(t)) - x(t)]dt \\ s.t. & \int_{-\pi}^{\pi} x(t)e^{ikt}dt = b_k, \qquad k = -n, \cdots, 0, \cdots, n \\ & 0 \le x(t) \in L_1[-\pi, \pi]. \end{array}\right.$$

Note that in the one-variable trigonometric case, $k(n) = 2n + 1$. We consider only the case where $x(t)$ is real. In this case we have for all k,

$$b_{-k} = \bar{b}_k. \tag{4.26}$$

Moreover we may again assume $\bar{x}(t)$ is of the form

$$\bar{x}(t) = \exp(\sum_{k=-n}^{n} \lambda_k e^{ikt}), \tag{4.27}$$

and where we have for each k,

$$\lambda_{-k} = -\bar{\lambda}_k, \tag{4.28}$$

since $\bar{x}$ is assumed to be real.

As to the integration property, for $k \neq 0$ we have

$$\begin{aligned} b_k &= \int_{-\pi}^{\pi} \exp(\sum_{l=-n}^{n} \lambda_l e^{ilt}) e^{ikt} dt \\ &= \frac{1}{ik}\exp(\sum_{l=-n}^{n} \lambda_l e^{ilt}) e^{ikt}|_{-\pi}^{\pi} - \frac{1}{ik}\int_{-\pi}^{\pi} e^{ikt}\exp(\sum_{l=-n}^{n} \lambda_l e^{ilt}) \sum_{l=-n}^{n} il\lambda_l e^{ilt} dt \\ &= -\frac{1}{k}\sum_{l=-n}^{n} l\lambda_l b_{k+l}. \end{aligned}$$

Using the property that $\lambda_{-k} = -\bar{\lambda}_k$, we have the linear system:

$$b = C\bar{r} + Br,$$

where

$$b = \begin{bmatrix} -b_1 \\ -2b_2 \\ \vdots \\ -nb_n \end{bmatrix}, \qquad r = \begin{bmatrix} \lambda_1 \\ 2\lambda_2 \\ \vdots \\ n\lambda_n \end{bmatrix},$$

$$C = \begin{bmatrix} b_0 & \bar{b}_1 & \cdots & \bar{b}_{n-1} \\ b_1 & b_0 & \cdots & \bar{b}_{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n-1} & b_{n-2} & \cdots & b_0 \end{bmatrix}, \ B = \begin{bmatrix} b_2 & b_3 & \cdots & b_{n+1} \\ b_3 & b_4 & \cdots & b_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n+1} & b_{n+2} & \cdots & b_{2n} \end{bmatrix}.$$

Solving this system, we can determine all λ_k, $k \neq 0$. Finally λ_0 can be obtained from

$$\begin{aligned} b_0 &= \int_{-\pi}^{\pi} \exp(\sum_{k=-n}^{n} \lambda_k e^{ikt})dt \\ &= e^{\lambda_0} \int_{-\pi}^{\pi} \exp(\sum_{k\neq 0} \lambda_k e^{ikt})dt. \end{aligned}$$

We can also express everything above in real form. Let

$$\bar{x}(t) = \exp\big(\lambda_0 + \sum_{k=1}^{n} (\lambda_k \cos kt + \mu_k \sin kt)\big),$$

and the moments:

$$a_0 = \int_{-\pi}^{\pi} \bar{x}(t)dt, \tag{4.29}$$

$$a_k = \int_{-\pi}^{\pi} \bar{x}(t) \cos ktdt, \quad k = 1, 2, \cdots, n \tag{4.30}$$

$$b_k = \int_{-\pi}^{\pi} \bar{x}(t) \sin ktdt, \quad k = 1, 2, \cdots, n. \tag{4.31}$$

Then for $l = 1, 2, \cdots, n$, using trigonometric angle formulae, we have

$$\begin{aligned} a_l &= \int_{-\pi}^{\pi} \exp(\lambda_0 + \sum_{k=1}^{n} (\lambda_k \cos kt + \mu_k \sin kt)) \cos ltdt \\ &= \frac{1}{2l} \sum_{k=1}^{n} k[\lambda_k(a_{l-k} - a_{l+k}) - \mu_k(b_{l-k} + b_{l+k})], \end{aligned}$$

and

$$b_l = \frac{1}{2l} \sum_{k=1}^{n} k[-\lambda_k(b_{l+k} - b_{l-k}) + \mu_k(a_{l-k} + a_{l+k})].$$

Note that $\bar{x}$ is real, thus for each k, we have

$$a_{-k} = a_k,$$

and

$$b_{-k} = -b_k,$$

The next algorithm then follows after some arithmetic calculation.

Algorithm 4.3.7 *let $a_k, k = 0, 1, \cdots, 2n$, $b_k, k = 1, 2, \cdots, 2n$, be given moments in (4.29).*

Step 1. Construct:

$$a = \begin{bmatrix} 2a_1 \\ 4a_2 \\ \vdots \\ 2na_n \end{bmatrix}, \quad b = \begin{bmatrix} 2b_1 \\ 4b_2 \\ \vdots \\ 2nb_n \end{bmatrix},$$

$$A_1 = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & a_3 & \cdots & a_{n+1} \\ a_3 & a_4 & \cdots & a_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & -b_1 & \cdots & -b_{n-1} \\ b_1 & 0 & \cdots & -b_{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n-1} & b_{n-2} & \cdots & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_2 & b_3 & \cdots & b_{n+1} \\ b_3 & b_4 & \cdots & b_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n+1} & b_{n+2} & \cdots & b_{2n} \end{bmatrix}.$$

Solve the linear system

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} A_1 - A_2 & -B_1 - B_2 \\ B_1 - B_2 & A_1 + A_2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}. \tag{4.32}$$

Step 2. For $i = 1, 2, \cdots, n$, let

$$\lambda_i = \frac{r_i}{i},$$

and

$$\mu_i = \frac{s_i}{i}.$$

Step 3. Compute:

$$\lambda_0 = \log[a_0(\int_{-\pi}^{\pi} \exp(\sum_{i=1}^{n}(\lambda_i \cos it + \mu_k \sin it))dt)^{-1}].$$

We now give some numerical test results of Algorithm 4.3.7 for trigonometric polynomial moments. We will try to reconstruct the following two functions defined on $[0, 2\pi]$:

$$\bar{x}_9(t) = \begin{cases} 0.5, & 0 \le t < 1, \\ 0.5 + 0.5(t-1), & 1 \le t < 2, \\ 3 - t, & 2 \le t < 3, \\ 0.125(t-3)^2, & 3 \le t < 4, \\ 0.5, & 4 \le t \le 2\pi, \end{cases} \tag{4.33}$$

and

$$\bar{x}_{10}(t) = 0.8\cos^{10}(10^{0.2t}) + 0.1. \tag{4.34}$$

We give a visual display of our numerical results in Figure 4.9.

Notice that the function $\bar{x}_{10}$ behaves really badly, though it is perfectly smooth. By using enough moments, we can get a very accurate reconstruction. (For a possibility to use that many moments, see comments at the end of Section 4.3.6.)

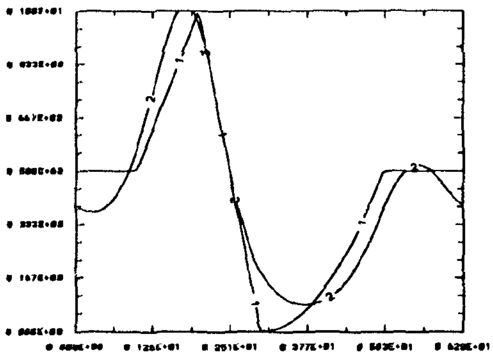
In a similar way, we can generalize this to m-dimensional space $I\!R^m$.

Let $T = [-\pi, \pi]^m$, $\{a_l(t), l \in I_n\}$ be trigonometric polynomials of the form

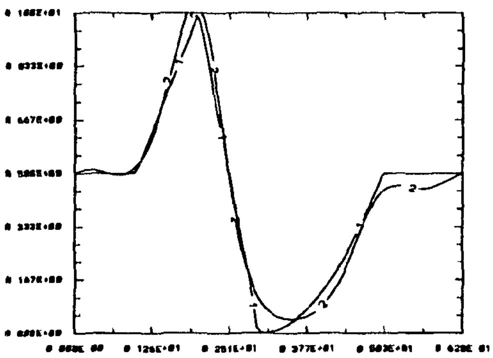
$$e^{i(k_1 t_1 + \cdots + k_m t_m)}, \qquad k_j = -n_j, \cdots, 0, \cdots, n_j,\ j = 1, 2, \cdots, m.$$

Let

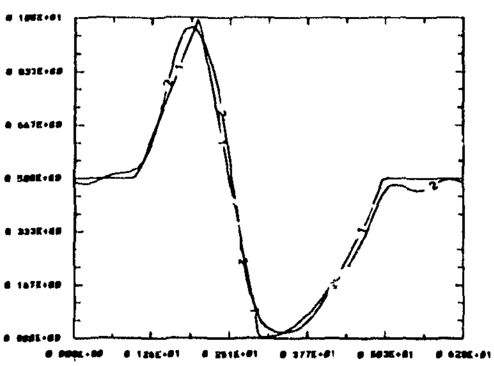
$$\begin{aligned} n &\triangleq (n_1, n_2, \cdots, n_m)^T \in \mathbf{Z}_+^m, \\ k &\triangleq (k_1, k_2, \cdots, k_m)^T \in \mathbf{Z}_+^m, \\ I_n &\triangleq \{k \in \mathbf{Z}_+^m \mid -n \le k \le n\}, \\ I_n(j) &\triangleq \{i \in I_n,\ i_j \neq 0\}, \end{aligned}$$



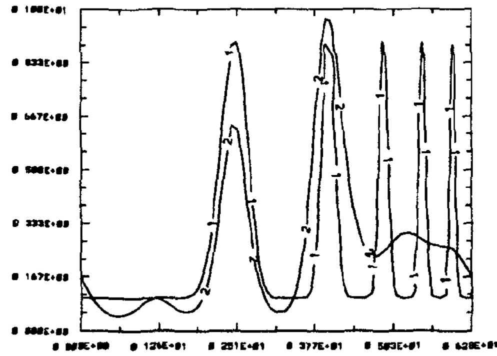
(a) Using 9 moments.



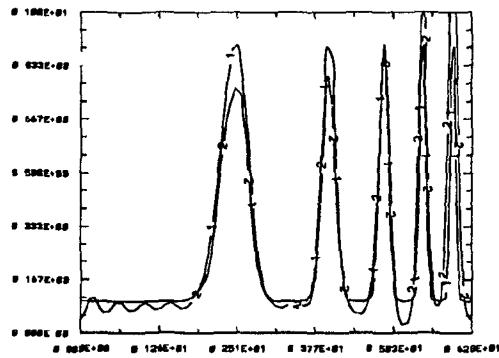
(b) Using 13 moments.



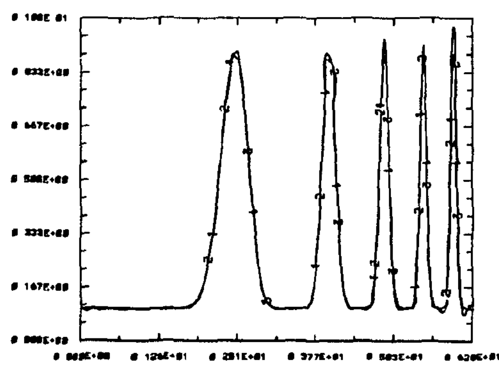
(c) Using 25 moments.



(d) Using 25 moments.



(e) Using 61 moments.



(f) Using 121 moments.

Figure 4.9: Heuristic reconstructions to functions $\bar{x}_9$ and $\bar{x}_{10}$.

then

$$k(n) = \prod_{j=1}^{m}(2n_j + 1).$$

Assume

$$\bar{x}(t) = \exp(\sum_{k\in I_n} \lambda_k e^{ikt}),$$

where

$$t \triangleq (t_1, t_2, \cdots, t_m)^T \in \mathbb{R}^m,$$
$$\lambda \triangleq \{\lambda_k,\ k \in I_n\} \in \mathbf{C}^{k(n)},$$

and the moments are

$$b_l = \int_{[-\pi,\pi]^m} \bar{x}(t)e^{ilt}dt,\ \ l \in I_n,$$

where

$$dt = dt_1 dt_2 \cdots dt_m.$$

By the integration procedure, we have

$$b_l = \frac{1}{l_j}\sum_{k\in I_n}, k_j\lambda_k b_{l+k},\ \ l \in I_n(l_j)\ j = 1, 2, \cdots, m.$$

Suppose we know all the moments b_l, $l \in I_{2n}$, we can get all λ_k, $k \in I_n$, using the following algorithm.

Algorithm 4.3.8 *Let b_l, $l \in I_{2n}$ be given moments.*

Step 1. Solve the linear equations:

$$b_l = \frac{1}{l_1}\sum_{k\in I_n(1)} k_1\lambda_k b_{l+k},\ \ l \in I_n(1),$$

Set $j = 1$.

Step 2. If $j < m$, solve

$$l_{j+1}b_l - \sum_{i\in I_n(j+1)\cap\{i_1=\cdots=i_j=0\}} i_{j+1}\lambda_i b_{i+l} = \sum_{i\in I_n\cap\{i_1=\cdot=i_{j+1}=0\}} i_{j+1}\lambda_i b_{i+l}.$$

Set $j = j + 1$, repeat Step 2.

Step 3. When $j = m$, compute

$$\lambda_0 = \log[a_0(\int_{[-\pi,\pi]^m} \exp(\sum_{k\in I_n\setminus\{0\}} \lambda_k e^{ikt})dt)^{-1}].$$

4.3.5 Further comparison with the dual iteration method

We have proposed several algorithms in the previous sections and also implemented them to solve one and two dimensional best entropy moment problems with algebraic or trigonometric moment functions.

In order to make further comparisons, we will take our heuristic estimate as an initial solution and then use the Newton method combined with the Armijo's step length search technique to iterate for some more steps.

The following notations are helpful in reading the tables and figures below.

- $\bar{x}(t)$ (or $\bar{x}(t_1, t_2)$): the prior density function.
- sup-ERR: the supremum norm of $\bar{x} - x_n$, where x_n is the estimate density function constructed by the corresponding algorithm.
- L_1-ERR: the L_1-norm of $\bar{x} - x_n$.
- d-GAP: the duality gap defined by $V(P_n) - V(D_n)$.
- TIME: execution time (in seconds) used to compute the dual solution λ only.

We first consider a step function

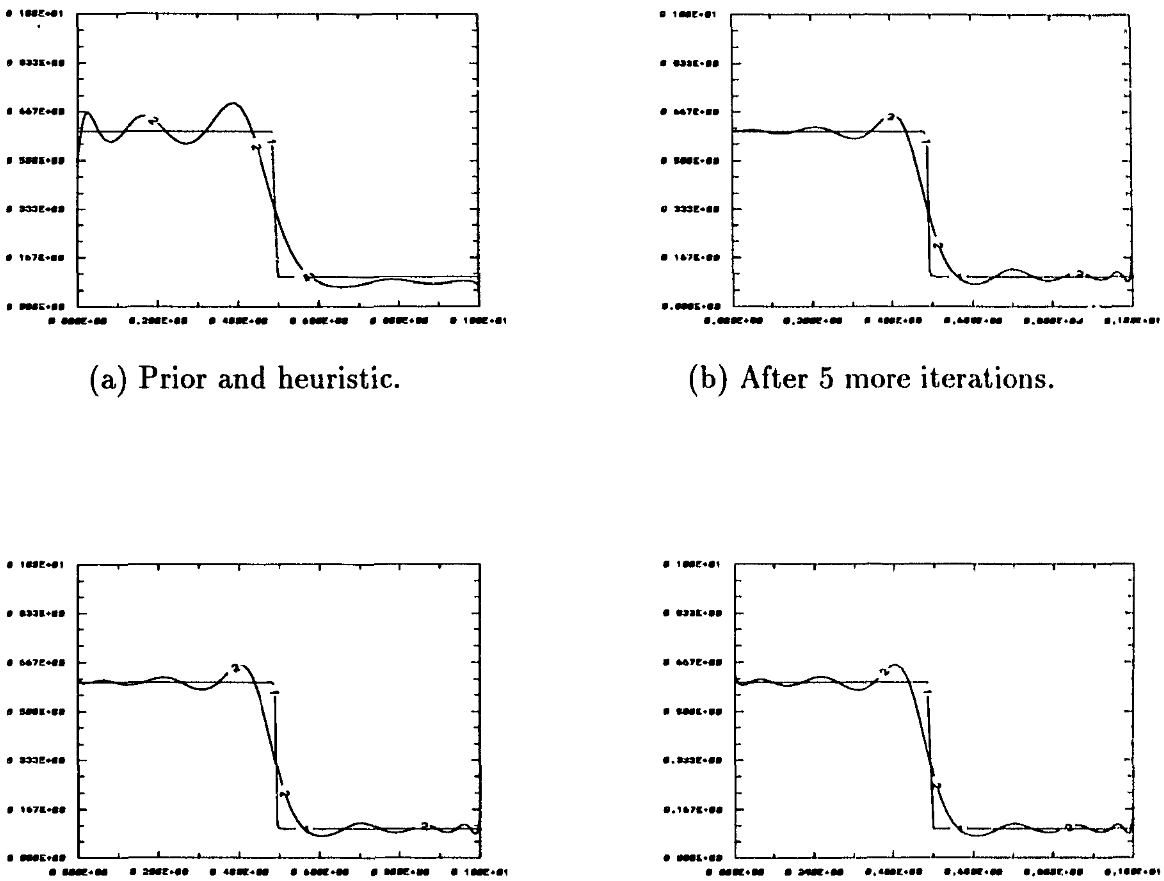
$$\bar{x}_{11}(t) = 0.5\chi_{[0,0.5]} + 0.1 \tag{4.35}$$

on the interval $[0, 1]$, and use the first 25 algebraic moments to reconstruct $\bar{x}_{11}$. In each table below, ALG1 means the Algorithm 4.3.2 given in Section 4.3.1, NEWTON(k) is Newton's method starting from our heuristic solution and making k more iterations, OPTIMAL means we use Newton's method to solve the problem (P_{24}) and iterate until some termination criterion is satisfied, in our case we use

$$\|\nabla\Phi(\lambda)\|_\infty < \varepsilon(= 0.0001).$$

The optimal solution is then of the form

$$\sum_{i \in I_n} \lambda_i^n a_i(t),$$



(a) Prior and heuristic.

(b) After 5 more iterations.

(c) After 15 more iterations.

(d) Prior and optimum.

Figure 4.10: Comparison results for the step function $\bar{x}_{11}$.

	sup-ERR	L_1-ERR	d-GAP	TIME
ALG 1.	0.23105	0.04020	0.01286	0.0199
NEWTON(5)	0.23332	0.02481	0.00407	0.5099
NEWTON(15)	0.23356	0.02472	0.00122	0.8598
OPTIMAL	0.23444	0.02498	0.00008	

Table 4.5: Numerical results for the step function $\bar{x}_{11}$.

where $\lambda^n \in I\!R^{k(n)}$ is obtained from OPTIMAL.

We also consider a smooth density

$$\bar{x}_{12}(t) = t\sin^2(10t) \tag{4.36}$$

on the interval $[0,1]$, and again use the first 25 algebraic moments and give the numerical results in Figure 4.11 and Table 4.6.

	sup-ERR	L_1-ERR	d-GAP	TIME
ALG 1.	0.12194	0.04390	0.00841	0.0299
NEWTON(5)	0.10858	0.02669	0.00656	0.2100
NEWTON(15)	0.11467	0.02676	0.00255	0.5100
OPTIMAL	0.11419	0.02681	0.00047	

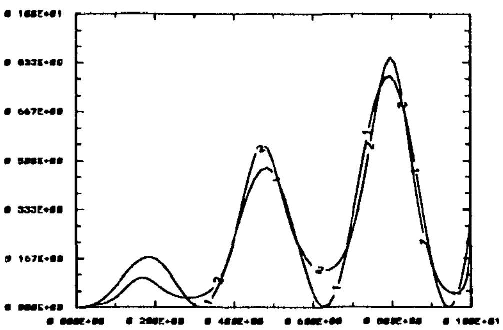
Table 4.6: Numerical results for continuous function $\bar{x}_{12}$.

Note that the objective function we used here is the Boltzmann-Shannon entropy, it is neither the supremum norm nor the L_1-norm. We use these norms here just to compare the results and to measure the goodness of our reconstructions. Actually, it is the d-GAP which measures our success in getting our numerical estimates close to the optimal solution of (P_n).

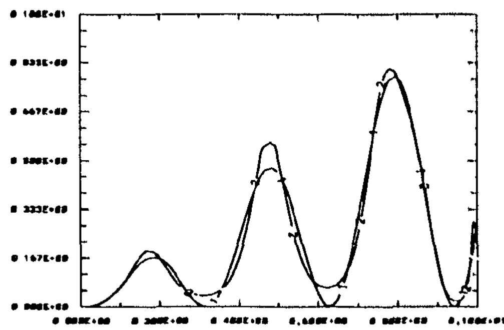
We now deal with 2-dimensional functions: first consider a smooth function,

$$\bar{x}_{13}(t_1,t_2) = 0.8t_1t_2(\sin(6t_1)\cos(8t_2))^2 + 0.1 \tag{4.37}$$

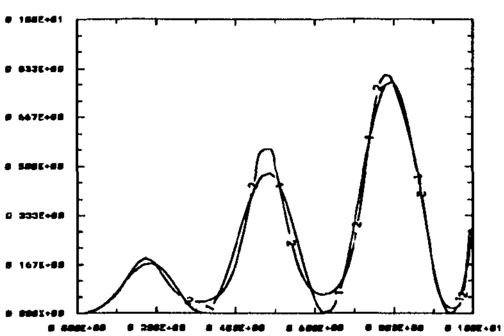
on $[0,1]^2$ (see Figure 4.12(a)), and use $225(= 15 \times 15)$ algebraic moments. We again use the estimate density generated from our heuristic algorithm as the initial solution



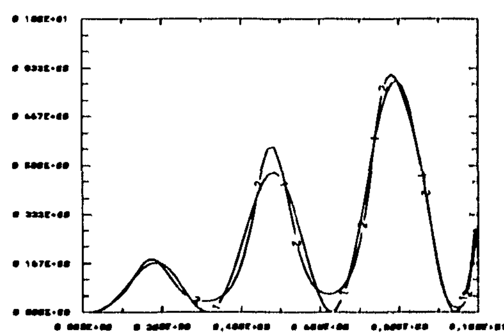
(a) Prior and estimate.



(b) After 5 more iterations.



(c) After 15 more iterations.



(d) Prior and optimum.

Figure 4.11: Comparison results for continuous function $\bar{x}_{12}$.

of the Newton method. This saves a lot of time especially in multidimensional cases. So NEWTON(k) means that we use the Newton method to make k more iterations from the heuristic solution of Algorithm 4.3.4.

	sup-ERR	L_1-ERR	TIME	estimat.	sup-error
ALG 2.	0.08851	0.01319	1.01979	Fig.4.12(b)	Fig.4.12(d)
NEWTON(5)	0.11897	0.00792	221.895	Fig.4.12(c)	Fig.4.12(e)

Table 4.7: Numerical results for 2-dimensional smooth function $\bar{x}_{13}$.

As the final example, we consider a maple-leaf function $\bar{x}_{14}$ on $[0,1]^2$ (see Fig.4.13(a)), which is very discontinuous. For both the heuristic algorithm and Newton's method, we use $121(= 11 \times 11)$ algebraic moments.

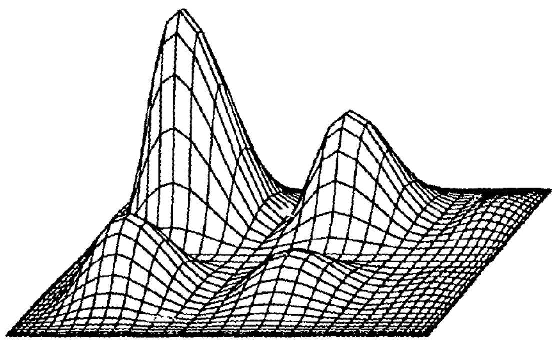
	sup-ERR	L_1-ERR	d-GAP	TIME	estimat.	sup-error
ALG 2.	0.67450	0.05402	0.01383	0.42991	Fig.4.13(b)	Fig.4.13(d)
NEWTON(4)	0.71720	0.04367	0.00035	105.248	Fig.4.13(c)	Fig.4.13(e)

Table 4.8: Numerical results for the maple-leaf function $\bar{x}_{14}$.

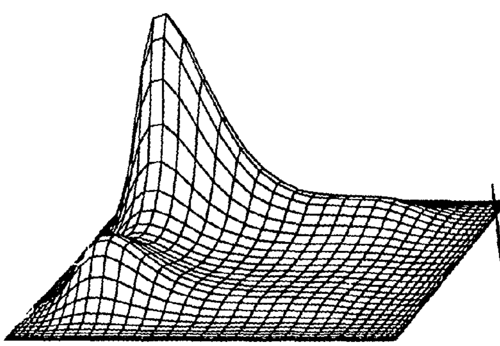
Note that these tables allow us to deduce the very steep cost of each Newton step as compared to our heuristic. Moreover, earlier Newton steps are even more costly because greater work is needed in the line search.

4.3.6 Notes about error analysis in $\mathbb{R}^1$

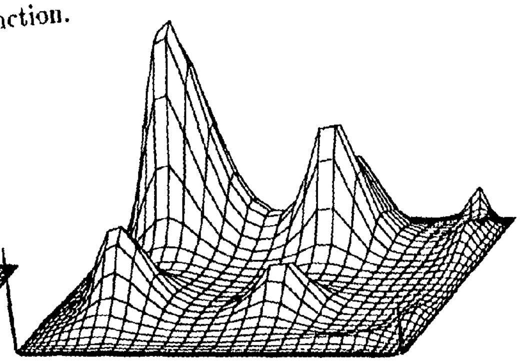
In this section, we give some error estimates in 1-dimensional cases. We consider $T = [0,1]$ or $[-\pi,\pi]$, and $\{a_i(t)\}$ be algebraic or trigonometric polynomials in only one variable. As we know, our algorithms 4.3.2 - 4.3.8 are exact when the underlying density $\bar{x}$ can be expressed as an exponential of a polynomial of $\{a_i, i \in I_n\}$. Now



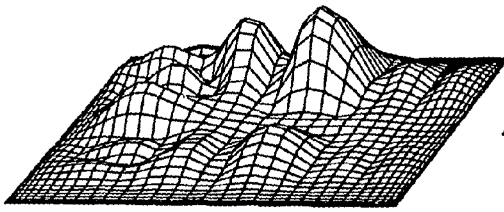
(a) Prior function.



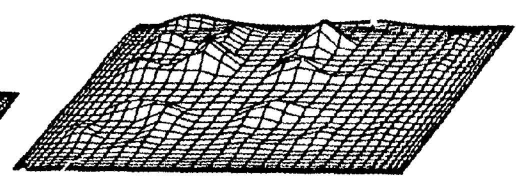
(b) The heuristic estimate.



(c) After 5 more iterations.

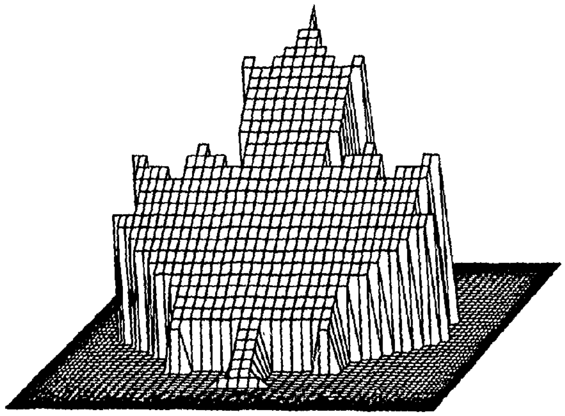


(d) Sup-error for heuristic.

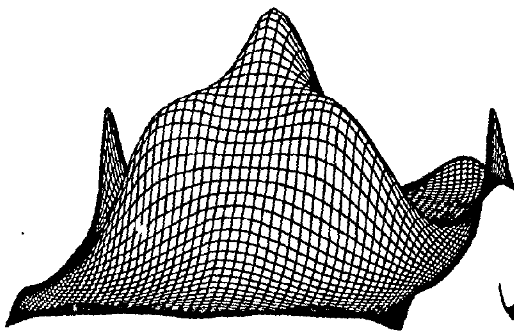


(e) Sup-error after 5 iterations

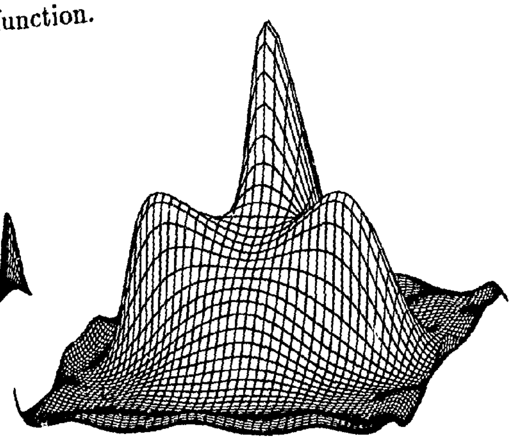
Figure 4.12: Visual results for 2-dimensional smooth function $\bar{x}_{13}$.



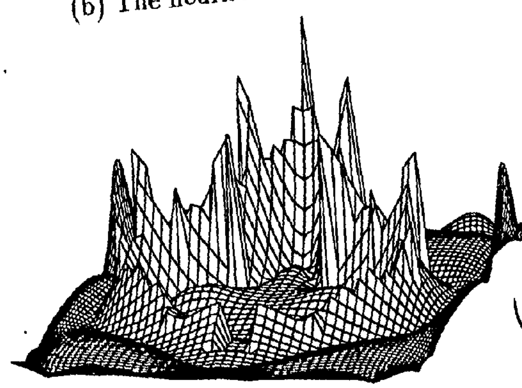
(a) Prior function.



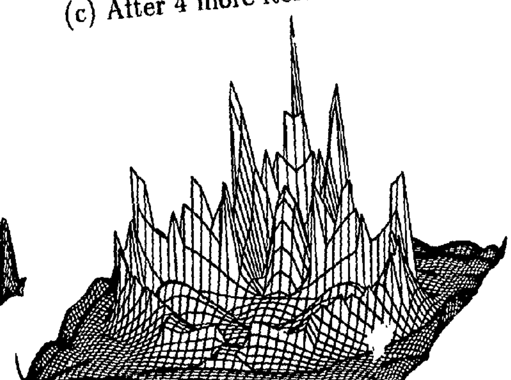
(b) The heuristic estimate.



(c) After 4 more iterations.



(d) Sup-error for heuristic.



(e) Sup-error after 4 iterations

Figure 4.13: Visual results for the maple-leaf function $\bar{x}_{14}$.

we assume that $\bar{x}$ is almost of this form, that is

$$\bar{x}(t) \cong \exp[\sum_{i\in I_n} \lambda_i a_i(t)]$$

in some sense, and we wish to determine arguments λ_i, $i \in I_n$.

We write

$$\tilde{b}_k = \int_T \exp[\sum_{i\in I_n} \lambda_i a_i(t)] a_k(t) dt, \qquad k \in I_n,$$

while

$$b_k = \int_T \bar{x}(t) a_k(t) dt, \qquad k \in I_n,$$

and we denote $\tilde{B}$ and B by the matrices generated in the algorithms using the data $\{\tilde{b}_i\}$ and $\{b_i\}$ respectively.

By the construction of the algorithms, λ can be determined by r which solves a linear system

$$\tilde{B}r = \tilde{b}.$$

But from the input data $\{b_i\}$ and B, we can only obtain $\tilde{r}$, which solves the linear system

$$B\tilde{r} = b.$$

Since B is nonsingular under mild hypotheses, we can obtain $\tilde{r}$ and hence $\tilde{\lambda}$. We now need to estimate the error bounds of $\|\lambda - \tilde{\lambda}\|$ in some gevin norm. From the nonsingularity of the matrix B, it is easy to see that

$$\tilde{r} - r = B^{-1}(b - Br). \tag{4.38}$$

Considering the algebraic case first, we have

$$\tilde{B} = \begin{bmatrix} 1 & \tilde{b}_1 & \tilde{b}_2 & \cdots & \tilde{b}_n \\ 1 & \tilde{b}_2 & \tilde{b}_3 & \cdots & \tilde{b}_{n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \tilde{b}_{n+1} & \tilde{b}_{n+2} & \cdots & \tilde{b}_{2n} \end{bmatrix}, \qquad \tilde{b} = \begin{bmatrix} \tilde{b}_0 \\ 2\tilde{b}_1 \\ \vdots \\ (n+1)\tilde{b}_n \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & b_1 & b_2 & \cdots & b_n \\ 1 & b_2 & b_3 & \cdots & b_{n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & b_{n+1} & b_{n+2} & \cdots & b_{2n} \end{bmatrix}, \qquad b = \begin{bmatrix} b_0 \\ 2b_1 \\ \vdots \\ (n+1)b_n \end{bmatrix}.$$

We assume

$$\bar{x}(t) = \exp[\sum_{i=0}^{n} \lambda_i t^i + \varepsilon_n(t)], \tag{4.39}$$

and

$$\|\varepsilon_n(t)\|_\infty \leq \delta_n, \tag{4.40}$$

for $n = 0, 1, \cdots$.

Note that when $\varepsilon_n(\cdot)$ is differentiable on $[0, 1]$

$$\begin{aligned} b_k &\triangleq \int_0^1 \bar{x}(t) t^k dt \\ &= \frac{1}{k+1}\bar{x}(1) - \frac{1}{k+1}\int_0^1 t^{k+1}\bar{x}(t)(\sum_{i=1}^{n} i\lambda_i t^{i-1} + \varepsilon_n'(t))dt \\ &= \frac{1}{k+1}\bar{x}(1) - \frac{1}{k+1}\sum_{i=1}^{n} i\lambda_i b_{i+k} \\ &\quad -\frac{1}{k+1}\int_0^1 t^{k+1}\bar{x}(t)\varepsilon_n'(t)dt. \end{aligned}$$

Considering the kth component of $(b - Br)$ in (4.38), we have

$$\begin{aligned} (b - Br)_k &= (k+1)b_k + \sum_{i=1}^{n} i\lambda_i b_{k+i} - \exp[\sum_{i=0}^{n} \lambda_i] \\ &= \bar{x}(1)(1 - \exp[-\varepsilon_n(1)] - \varepsilon_n(1)\bar{x}(1) \\ &\quad + \int_0^1 \varepsilon_n(t)((k+1)\bar{x}(t)t^k + t^{k+1}\bar{x}'(t))dt \\ &= \bar{x}(1)[1 - \varepsilon_n(1) - \exp[-\varepsilon_n(1)]] \\ &\quad + \int_0^1 \varepsilon_n(t)((k+1)\bar{x}(t)t^k + t^{k+1}\bar{x}'(t))dt, \end{aligned} \tag{4.41}$$

for $k = 0, 1, \cdots, n$.

Lemma 4.3.9 *Suppose $\bar{x} \in C^1[0,1]$ is of the form in (4.39), $\{a_i(t),\ i \in I_n\}$ are algebraic polynomials $1, t, \cdots, t^n$ on $[0,1]$. r, $\tilde{r}$, B, b are defined as before. Then*

$$\|b - Br\|_\infty \leq [(\frac{e^{\delta_{max}}}{\delta_{max}} + 2)\|\bar{x}\|_\infty + \frac{1}{2}\|\bar{x}'\|_\infty]\delta_n,$$

where

$$\delta_{max} = \max\{\delta_n,\ i = 0, 1, ...\},$$

hence

$$\|\tilde{r} - r\|_\infty \ \leq \ C_1\|B^{-1}\|_\infty \delta_n,$$

where

$$C_1 \triangleq (\frac{e^{\delta_{max}}}{\delta_{max}} + 2)\|\bar{x}\|_\infty + \frac{1}{2}\|\bar{x}'\|_\infty.$$

Proof: First we recall an inequality (proved in [2, Lemma 4.10]),

$$|e^x - 1| \leq \frac{e^M - 1}{M}|x|, \qquad \text{for } |x| \leq M. \tag{4.42}$$

By (4.41), we have

$$\begin{aligned} |(b - Br)_k| \ &\leq \ \bar{x}(1)(e^{\delta_n} - 1) + |\varepsilon_n(1)|\bar{x}(1) \\ &\quad + \delta_n \int_0^1 |(k+1)\bar{x}(t)t^k + t^{k+1}\bar{x}'(t)|dt \\ &\leq \ \|\bar{x}\|_\infty(e^{\delta_n} - 1) + \delta_n\|\bar{x}\|_\infty + \delta_n(\|\bar{x}\|_\infty + \|\bar{x}'\|_\infty \frac{1}{k+2}), \end{aligned} \tag{4.43}$$

and hence by (4.42),

$$\|b - Br\|_\infty \ \leq \ [(\frac{e^{\delta_{max}}}{\delta_{max}} + 2)\|\bar{x}\|_\infty + \frac{1}{2}\|\bar{x}'\|_\infty]\delta_n.$$

The result follows now from (4.38) ■

From Lemma 4.3.9, we have, for $k = 1, 2, \cdots, n$,

$$\begin{aligned} |\lambda_k - \tilde{\lambda}_k| \ &= \ \frac{1}{k}|r_k - \tilde{r}_k| \\ &\leq \ \frac{1}{k}\|r - \tilde{r}\|_\infty \\ &\leq \ \frac{1}{k}\|B^{-1}\|_\infty C_1 \delta_n, \end{aligned} \tag{4.44}$$

for a constant C_1 depending on $\bar{x}$ and δ_{max}, but independent of n.

To estimate $|\lambda_0 - \tilde{\lambda}_0|$, we need the following mean value theorem.

Lemma 4.3.10 *If $g(t) \geq 0$ is integrable, and $f(t) \geq 0$ is continuous on $[0,1]$, then there exists $\hat{t} \in [0,1]$, such that*

$$\int_0^1 f(t)g(t)dt = f(\hat{t}) \int_0^1 g(t)dt.$$

We now give the error bound for $|\lambda_0 - \tilde{\lambda}_0|$. From the algorithm, we know that

$$e^{\tilde{\lambda}_0} = \frac{b_0}{\int_0^1 \exp[\sum_{i=1}^n \tilde{\lambda}_i t^i]dt}.$$

By (4.39) and Lemma 4.3.10, we have

$$\begin{aligned}
\int_0^1 \exp[\sum_{i=1}^n \tilde{\lambda}_i t^i]dt &= \int_0^1 \exp[\sum_{i=1}^n \lambda_i t^i + \varepsilon_n(t)]\exp[\sum_{i=1}^n (\tilde{\lambda}_i - \lambda_i)t^i - \varepsilon_n(t)]dt \\
&= e^{-\lambda_0} \int_0^1 \bar{x}(t)\exp[\sum_{i=1}^n (\tilde{\lambda}_i - \lambda_i)t^i - \varepsilon_n(t)]dt \\
&= e^{-\lambda_0} b_0 \exp[\sum_{i=1}^n (\tilde{\lambda}_i - \lambda_i)\hat{t}^i - \varepsilon_n(\hat{t})],
\end{aligned}$$

for some $\hat{t} \in [0,1]$. Thus

$$e^{\tilde{\lambda}_0} = \frac{e^{\lambda_0}}{\exp[\sum_{i=1}^n (\tilde{\lambda}_i - \lambda_i)\hat{t}^i - \varepsilon_n(\hat{t})]},$$

and

$$\begin{aligned}
|\lambda_0 - \tilde{\lambda}_0| &= |\sum_{i=1}^n (\tilde{\lambda}_i - \lambda_i)\hat{t}^i - \varepsilon_n(\hat{t})| \\
&\leq \sum_{i=1}^n |\tilde{\lambda}_i - \lambda_i| + \delta_n \\
&\leq (\sum_{i=1}^n \frac{1}{i})\|\tilde{r} - r\|_\infty + \delta_n \\
&\leq (C_1\|B^{-1}\|_\infty \sum_{k=1}^n \frac{1}{k} + 1)\delta_n \\
&\leq (C_1\|B^{-1}\|_\infty (1 + \log n) + 1)\delta_n, \qquad (4.45)
\end{aligned}$$

noting that

$$\sum_{k=1}^{n} \frac{1}{k} < 1 + \int_{1}^{n} \frac{1}{x} dx = 1 + \log n.$$

We now have

Theorem 4.3.11 *Suppose* $\log \bar{x} \in C^1[0,1]$, *that the moments are given by*

$$b_k = \int_0^1 \bar{x}(t) t^k dt, \qquad k = 0, 1, \cdots, n$$

and that the estimate density $\tilde{x}_n(t)$ *is computed by Algorithm 4.3.2. Then*

$$\|\tilde{x}_n - \bar{x}\|_\infty \leq \|\bar{x}\|_\infty (\exp(2E_n(C_1 \|B_n^{-1}\|_\infty (1 + \log n) + 1)) - 1),$$

where

$$E_n = \inf_{\lambda \in \mathbb{R}^{n+1}} \{\| \log \bar{x} - \sum_{i=0}^{n} \lambda_i t^i \|_\infty\},$$

and C_1 *is a constant dependent only on* $\bar{x}$.

Proof: By the definition of E_n there exists $\lambda^n \in \mathbb{R}^{n+1}$ such that

$$\log \bar{x} = \sum_{i=0}^{n} \lambda_i^n t^i + \varepsilon_n(t)$$

and

$$\|\varepsilon_n(t)\|_\infty \leq \delta_n = E_n.$$

Using Algorithm 4.3.2, from (4.44) and (4.45), we have

$$|\lambda_k^n - \tilde{\lambda}_k| \leq \frac{1}{k} \|B_n^{-1}\|_\infty C_1 E_n,$$

for $k = 1, 2, \cdots, n$, and

$$|\lambda_0^n - \tilde{\lambda}_0| \leq (C_1 \|B_n^{-1}\|_\infty (1 + \log n) + 1) E_n.$$

Thus we have

$$\begin{aligned}
\|\bar{x}-\tilde{x}_n\|_\infty &\le \|\bar{x}\|_\infty \,\|1-\exp(\sum_{i=0}^{n}(\tilde{\lambda}_i-\lambda_i^n)t^i-\varepsilon_n(t))\|_\infty \\
&\le \|\bar{x}\|_\infty(\exp(\sum_{i=0}^{n}|\tilde{\lambda}_i-\lambda_i^n|+\delta_n)-1) \\
&\le \|\bar{x}\|_\infty(\exp(2E_n(C_1\|B_n^{-1}\|_\infty(1+\log n)+1))-1).
\end{aligned}$$

■

It is clear that $E_n=0$ implies $\|\bar{x}-\tilde{x}_n\|_\infty=0$.

Similarly, in the trigonometric case, we assume $\bar{x}$ is of the form

$$\bar{x}(t)=\exp(\lambda_0+\sum_{k=1}^{n}(\lambda_k\cos kt+\mu_k\sin kt)+\varepsilon_n(t)),$$

and

$$\|\varepsilon_n(t)\|_\infty\le\delta_n,$$

for $n=0,1,\cdots$.

In the same way we proved for Lemma 4.3.9 , using trigonometric angle formulae, we note that,

$$\begin{aligned}
a_k &\triangleq \int_{-\pi}^{\pi}\bar{x}(t)\cos ktdt \\
&= \frac{1}{k}\bar{x}(t)\sin kt|_{-\pi}^{\pi}-\frac{1}{k}\int_{-\pi}^{\pi}\bar{x}'(t)\sin ktdt \\
&= -\frac{1}{k}\int_{-\pi}^{\pi}\bar{x}(t)\sin kt(\sum_{j=1}^{n}(-j\lambda_j\sin jt+j\mu_j\cos jt)+\varepsilon_n'(t))dt \\
&= -\frac{1}{k}\int_{-\pi}^{\pi}\bar{x}(t)(\sum_{j=1}^{n}(-j\lambda_j\sin jt\sin kt+j\mu_j\cos jt\sin kt)+\varepsilon_n'(t)\sin kt)dt \\
&= \frac{1}{2k}\sum_{j=1}^{n}(j\lambda_j(a_{j-k}-a_{j+k})+j\mu_j(b_{j-k}-b_{j+k})) \\
&\quad -\frac{1}{k}\int_{-\pi}^{\pi}\bar{x}(t)\varepsilon_n'(t)\sin ktdt, \qquad (4.46)
\end{aligned}$$

and

$$
\begin{aligned}
b_k &\triangleq \int_{-\pi}^{\pi} \bar{x}(t)\sin ktdt \\
&= -\frac{1}{k}\bar{x}(t)\cos kt|_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi} \bar{x}'(t)\cos ktdt \\
&= \frac{1}{k}\int_{-\pi}^{\pi} \bar{x}(t)\cos kt(\sum_{j=1}^{n}(-j\lambda_j \sin jt + j\mu_j \cos jt) + \varepsilon_n'(t))dt \\
&= \frac{1}{k}\int_{-\pi}^{\pi} \bar{x}(t)(\sum_{j=1}^{n}(-j\lambda_j \sin jt\cos kt + j\mu_j \cos jt\cos kt) + \varepsilon_n'(t)\cos kt)dt \\
&= \frac{1}{2k}\sum_{j=1}^{n}(-j\lambda_j(b_{k+j} - b_{k-j}) + j\mu_j(a_{k-j} + a_{k+j})) \\
&\quad + \frac{1}{k}\int_{-\pi}^{\pi} \bar{x}(t)\varepsilon_n'(t)\cos ktdt, \qquad (4.47)
\end{aligned}
$$

for $k = 1, 2, \cdots, n$. Hence from the periodity of $\bar{x}$,

$$
\begin{aligned}
(b - Br)_k &\triangleq 2ka_k - \sum_{j=1}^{n}(a_{j-k} - a_{j+k})j\lambda_j \\
&\quad + \sum_{j=1}^{n}(-b_{j-k} + b_{j+k})j\mu_j \\
&= -2\int_{-\pi}^{\pi} \bar{x}(t)\varepsilon_n'(t)\sin ktdt \\
&= 2\int_{-\pi}^{\pi} \varepsilon_n(t)(\bar{x}'(t)\sin kt + k\bar{x}(t)\cos kt)dt,
\end{aligned}
$$

and

$$
\begin{aligned}
(b - Br)_{n+k} &\triangleq 2kb_k - \sum_{j=1}^{n}((b_{j-k} - b_{j+k})j\lambda_j \\
&\quad + (a_{j-k} + a_{j+k})j\mu_j) \\
&= 2\int_{-\pi}^{\pi} \bar{x}(t)\varepsilon_n'(t)\cos ktdt \\
&= -2\int_{-\pi}^{\pi} \varepsilon_n(t)(\bar{x}'(t)\cos kt + k\bar{x}(t)\sin kt)dt,
\end{aligned}
$$

for $k = 1, 2, \cdots, n$. Taking supremum norm, we have

$$
\|b - Br\|_\infty \leq 4\pi\delta_n(\|\bar{x}'\|_\infty + k\|\bar{x}\|_\infty). \qquad (4.48)
$$

From Algorithm 4.3.7, we then have

$$|\lambda_k - \tilde{\lambda}_k| \leq \frac{2\pi}{k}\|B^{-1}\|_\infty \delta_n(\|\bar{x}'\|_\infty + k\|\bar{x}\|_\infty), \tag{4.49}$$

and

$$|\mu_k - \tilde{\mu}_k| \leq \frac{2\pi}{k}\|B^{-1}\|_\infty \delta_n(\|\bar{x}'\|_\infty + k\|\bar{x}\|_\infty), \tag{4.50}$$

for $k = 1, 2, \cdots .n$, here B is

$$\begin{bmatrix} A_1 - A_2 & -B_1 - B_2 \\ B_1 - B_2 & A_1 + A_2 \end{bmatrix}$$

constructed in Algorithm 4.3.7.

As to $|\lambda_0 - \tilde{\lambda}_0|$, note that in the algorithm, we have

$$e^{\tilde{\lambda}_0} = \frac{a_0}{\int_{-\pi}^{\pi} \exp(\sum_{j=1}^{n} \tilde{\lambda}_j \cos jt + \tilde{\mu}_j \sin jt)dt}.$$

Since

$$\begin{aligned}
&\int_{-\pi}^{\pi} \exp(\sum_{j=1}^{n} \tilde{\lambda}_j \cos jt + \tilde{\mu}_j \sin jt)dt \\
&= \int_{-\pi}^{\pi} \bar{x}(t)e^{-\lambda_0}\exp(\sum_{j=1}^{n}((\tilde{\lambda}_j - \lambda_j)\cos jt + (\tilde{\mu}_j - \mu_j)\sin jt) - \varepsilon_n(t))dt \\
&= e^{-\lambda_0}\exp(\sum_{j=1}^{n}((\tilde{\lambda}_j - \lambda_j)\cos j\hat{t} + (\tilde{\mu}_j - \mu_j)\sin j\hat{t}) - \varepsilon_n(\hat{t}))a_0 \\
&\qquad\qquad (\text{ by Lemma 4.3.10 for some } \hat{t} \in (-\pi, \pi)).
\end{aligned}$$

Thus

$$\begin{aligned}
|\lambda_0 - \tilde{\lambda}_0| &= |\sum_{j=1}^{n}((\tilde{\lambda}_j - \lambda_j)\cos j\hat{t} + (\tilde{\mu}_j - \mu_j)\sin j\hat{t}) - \varepsilon_n(\hat{t})| \\
&\leq \sum_{j=1}^{n}(|\tilde{\lambda}_j - \lambda_j| + |\tilde{\mu}_j - \mu_j|) + \delta_n.
\end{aligned}$$

Combining this with (4.49) and (4.50), we have

Theorem 4.3.12 *Suppose* $\log \bar{x} \in C^1[-\pi, \pi]$, $\bar{x}$ *is periodic with the period* 2π. *Given* $4n+1$ *moments*

$$\begin{aligned} a_0 &= \int_{-\pi}^{\pi} \bar{x}(t)dt \\ a_k &= \int_{-\pi}^{\pi} \bar{x}(t)\cos kt dt \\ b_k &= \int_{-\pi}^{\pi} \bar{x}(t)\sin kt dt \\ &\qquad k = 1, 2, \cdots, 2n. \end{aligned}$$

Let $\tilde{x}_n(t)$ *be the estimate density constructed from the Algorithm 4.3.7. Then*

$$\|\tilde{x}_n - \bar{x}\|_\infty \le \|\bar{x}\|_\infty(\exp(8\pi E_n\|B^{-1}\|_\infty((1+\log n)\|\bar{x}'\|_\infty + n\|\bar{x}\|_\infty) + 1) - 1),$$

where

$$E_n \triangleq \inf\{\|\log \bar{x} - \lambda_0 - \sum_{j=1}^{n}(\lambda_j \cos jt + \mu_j \sin jt)\|_\infty \mid (\lambda, \mu) \in I\!R^{2n+1}\}.$$

From the error bounds in Theorem 4.3.11 or 4.3.5 we see that the product

$$\|B_n^{-1}\|_\infty \cdot E_n$$

is an overestimate for the rate of the convergence of $\tilde{x}_n$ to $\bar{x}$. From approximation theory, Jackson's Theorem (see [108]) tells us that if

$$\log \bar{x} \in C^r[0, 1],$$

then

$$E_n = o(\frac{1}{n^r}).$$

Moreover, if $\log \bar{x}$ is analytic on $[0, 1]$, then

$$E_n \le Cq^n,$$

where C is a constant and $q < 1$. Unfortunately, we haven't found any theoretical bound for $\|B_n^{-1}\|_\infty$. Numerical results indicate that

$$\|B_n^{-1}\|_\infty \to \infty,$$

(see Table 4.9. and Table 4.10.). and so that when the number of moments gets too large, the computational results may not be reliable due to the accumulation of errors. This difficulty accurs in the numerical computation when we use too many moments. But for the trigonometric case, when the prior density is smooth enough, we can see from Table 4.10. that $\|B_n^{-1}\|_\infty$ appears to be dominated by a polynomial, so that using Jackson's theorems, the convergence of our algorithm for trigonometric polynomial moments may follow. Numerically, as we saw in Section 4.3.4, we have successfully recontructed a somewhat bizarre function $\bar{x}_{10}$ by using 121 moments.

$\|B_n^{-1}\|_\infty$	F_1	F_2	F_3	F_4
n=2	0.320E02	0.469E02	0.498E02	0.477E02
3	0.578E03	0.240E04	0.219E05	0.195E04
4	0.206E05	0.105E06	0.121E08	0.732E05
5	0.608E06	0.450E07	0.107E11	0.363E07
6	0.231E08	0.160E09	0.895E13	0.151E09
7	0.594E09	0.560E10	0.746E16	0.416E10
8	0.223E11	0.209E12	0.866E19	0.152E12
9	0.667E12	0.717E13	0.904E22	0.642E13
10	0.233e14	0.233E15	0.110E26	0.176E15
11	0.675E15	0.844E16	0.179E29	0.840E16
12	0.250E17	0.289E18	0.355E32	0.274E18
13	0.758E18	0.948E19	0.965E35	0.856E19
14	0.257E20	0.326E21	0.451E38	0.326E21
15	0.816E21	0.112E23	—	0.949E22
16	0.289E23	0.372E24	—	0.374E24
17	0.885E24	0.124E26	—	0.123E26
18	0.307E26	0.428E27	—	0.422E27
19	0.989E27	0.144E29	—	0.158E29
20	0.342E29	0.470E30	—	0.477E30

Table 4.9: $\|B_n^{-1}\|_\infty$ for algebraic moments.

In Table 4.9, for functions $[0,1]$

$$F_1 = 2|t-0.5|,$$

$\|B_n^{-1}\|_\infty$	F_1	F_2	F_3	F_4
n=3	0.40166	0.63662	0.932E00	0.12861
5	0.67313	0.63662	0.359E02	0.23679
7	0.99875	1.90986	0.311E04	0.50826
9	1.27501	1.90986	0.339E06	0.82539
11	1.55475	3.81972	0.375E08	0.91805
13	1.84964	3.81972	0.443E10	1.24088
15	2.16509	6.36620	0.667E12	1.77943
17	2.46792	6.36620	0.132E15	2.65173
19	2.77001	9.54940	0.229E17	2.93955
21	3.07217	9.54930	0.495E19	3.23528
23	3.36068	13.36902	0.151E22	4.00094
25	3.65724	13.36902	0.599E24	5.59461
27	3.96081	17.82575	0.253E27	6.09011
29	4.23835	17.82575	0.164E30	6.27491
31	4.53221	22.91831	0.194E33	7.28279
33	4.79253	22.91831	0.156E34	9.72277
35	5.08312	28.64789	0.158E34	10.49603
37	5.33845	28.64789	0.107E35	10.53350

Table 4.10: $\|B_n^{-1}\|_\infty$ for trigonometric moments.

$$
\begin{aligned}
F_2 &= \sin^2 t, \\
F_3 &= \chi_{[0.4,0.6]}, \\
F_4 &= t\sin^2(10t),
\end{aligned}
$$

we compute the corresponding values of $\|B_n^{-1}\|_\infty$. In Table 4.10, we compute the values of $\|B_n^{-1}\|_\infty$ for functions defined on $[0, 2\pi]$

$$
\begin{aligned}
F_1 &= 1.5t, \\
F_2 &= \sin^2 t, \\
F_3 &= 0.8\chi_{[1.4,3.6]}, \\
F_4 &= t\sin^2(2t),
\end{aligned}
$$

in the trigonometric case.

Although the convergence of these algorithms is still unsettled, they often give very good estimates for the problem (P_n), and use much less time than Newton's method, as we can see in Section 4.3.5. If we use the heuristic solution as an initial estimate, then often only a couple of iterations are needed in order to get an almost optimal solution to (P_n).

4.4 Number of nodes in the integration scheme

We know that one of the most time consuming jobs is to compute numerical integrals. Although at each step, we can, and do, evaluate many integrals at the same time, as mentioned in [14], the cost (in time) still depends a lot on the number of integration nodes used in the Gauss quadrature integration scheme. In all our computations, we used 99 nodes for one dimensional cases and 3025 ($= 65^2$) nodes for two dimensional cases. This large number of nodes is not necessary, and we can reduce it without any significant increase in errors. The reason we use this many nodes is to improve drawing pictures using NCARG. More nodes give nicer pictures, especially in two dimensional cases.

It is an accepted fact that the more moments involved in the problem the more nodes required in the integration scheme. Usually the number of the nodes should be at least as many as the number of moments. From our numerical experience, we have found that: to keep a reasonable level accuracy, the number of nodes in the integration scheme should be around two to four times (in one dimensional cases) the number of moments involved in the problem.

In Table 4.11, we give reconstruction errors for the underlying function $\bar{x}_1$ defined in Section 4.2. We use the Newton method to solve the dual problem with the Boltzmann-Shannon entropy and algebraic polynomial moments. We use the stopping criterion as before or iterate up to 55 steps. Both L_1-norm errors and L_∞-norm errors are given in the table when we use varying numbers of nodes (n) and moments (m). To compare the execution time, we use the classical Newton method in which the step length is set to be 1. The right-most column in Table 4.11 gives the execution time (in seconds) per iteration for fixed $m = 25$ and varying amount of nodes. The bottom row shows the execution time (in seconds) for fixed $n = 99$ and varying number of moments. So we can see how the execution time depends on the number of nodes in the integration scheme and the number of moments involved in the problem, respectively. We can observe that the best choice of the number of nodes in the algebraic case is around the double of the number of moments. The *Italicized* data given in the table show unsatisfactory results, while the **Bold** data show the most favorable choices.

L_1-err L_∞-err	m=5	m=10	m=16	m=25	m=37	m=50	Time
n=9	0.1484 0.3966	—	—	—	—	—	—
n=13	**0.1497** **0.3960**	*0.1780* *0.9512*	—	—	—	—	—
n=19	0.1497 0.3960	**0.0622** **0.2609**	*0.1777* *0.9421*	—	—	—	—
n=25	0.1497 0.3960	**0.0620** **0.2593**	0.0620 0.2593	*0.2846* *0.5470*	—	—	0.0180
n=37	0.1497 0.3960	0.0620 0.2593	**0.0243** **0.1346**	0.0417 0.2749	*0.2856* *0.5554*	—	0.0200
n=53	0.1497 0.3960	0.0620 0.2593	**0.0275** **0.1062**	**0.0373** **0.6747**	0.0252 0.1178	*0.1768* *0.9196*	0.0220
n=69	0.1497 0.3960	0.0620 0.2593	0.0362 0.1507	**0.0309** **0.5744**	0.0185 0.1530	0.0247 0.1633	0.0233
n=87	0.1497 0.3960	0.0620 0.2593	0.0284 0.1106	0.0202 0.2095	**0.0162** **0.0786**	**0.0194** **0.1532**	0.0247
n=99	0.1497 0.3960	0.0620 0.2593	0.0375 0.1691	0.0202 0.1504	**0.0186** **0.0998**	**0.0199** **0.1692**	0.0273
Time	0.0040	0.0060	0.0120	0.0273	0.0593	0.1380	—

Table 4.11: Number of nodes vs. number of moments (algebraic case).

In Table 4.12, we give the reconstruction errors for underlying function $\bar{x}_9$ defined in Section 4.3.4. We again use the Newton method to solve the dual problem with the Boltzmann-Shannon entropy but for trigonometric polynomial moments. We use the same stopping criterion as before or iterate up to 55 steps. The right-most column gives the execution time (in seconds) per iteration for fixed $m = 27$ and varying amount of nodes. The bottom row shows the execution time (in seconds) for fixed $n = 199$ and varying number of moments. We can see that in this case the best choice of the number of nodes should be around two to four times of the number of moments.

L_1-err L_∞-err	m=5	m=11	m=19	m=27	m=39	m=55	Time
n=9	0.4771 0.1806	—	—	—	—	—	—
n=19	**0.4586** **0.1625**	*0.1586* *0.0962*	*14.395* *2.7165*	—	—	—	—
n=31	**0.4586** **0.1625**	**0.1338** **0.0784**	0.0849 0.0517	*14.395* *2.7239*	—	—	0.0507
n=45	0.4586 0.1625	**0.1337** **0.0783**	**0.0496** **0.0336**	**0.0421** **0.0237**	*14.395* *2.7247*	—	0.0640
n=61	0.4586 0.1625	0.1337 0.0783	**0.0502** **0.0373**	**0.0403** **0.0295**	**0.0251** **0.0156**	*14.395* *2.7346*	0.0800
n=87	0.4586 0.1625	0.1337 0.0783	**0.0502** **0.0373**	**0.0401** **0.0265**	**0.0246** **0.0162**	0.0251 0.0223	0.1080
n=111	0.4586 0.1625	0.1337 0.0783	0.0502 0.0373	**0.0401** **0.0265**	**0.0243** **0.0160**	**0.0154** **0.0109**	0.1453
n=155	0.4586 0.1625	0.1337 0.0783	0.0502 0.0373	0.0401 0.0265	**0.0243** **0.0160**	**0.0156** **0.0115**	0.1766
n=199	0.4586 0.1625	0.1337 0.0783	0.0502 0.0373	0.0401 0.0265	0.0243 0.0160	**0.0156** **0.0115**	0.2390
Time	0.0280	0.0447	0.1160	0.2390	0.4759	0.9745	—

Table 4.12: Number of nodes vs. number of moments (trigonometric case).

Appendix A

Assumptions

The following assumption are given and used in Chapter 2 and 3.

(A1), (A2): page 17

(A3), (A4), (A5), (A6): page 18

(A7): page 40

(A7': page 41

(A8): page 43

(A8'): page 45

(AF1), (AF2): page 58

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(AT3), (AT4), (AT5): page 77

(AT5'): page 81

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