FUNCTIONAL POSITIVITY AND INVARIANT SUBSPACES OF SEMIGROUPS OF OPERATORS

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Contents

Abstract

The main results in this thesis are about multiplicative semigroups of functionally positive operators and their invariant subspaces.

Let $\mathcal{X}$ be a topological space, and with its Borel structure, a standard Borel space, and m a σ-finite regular Borel measure on $\mathcal{X}$ such that $\mathcal{L}^2(\mathcal{X}, m)$ is of dimension at least two. An operator on $\mathcal{L}^2(\mathcal{X}, m)$ is called (functionally) positive if it maps non-negative functions to non-negative functions. Generally, the algebra generated by all positive operators is not closed in operator norm topology. We introduce a new norm on the algebra and show, using classical methods of functional analysis, that the algebra is a Banach $*$-algebra under the new norm. The spectral aspects of elements of the Banach algebra are discussed.

Suppose $\mathcal{S}$ is a semigroup of positive integral operators on $\mathcal{L}^2(\mathcal{X}, m)$. We show by analyzing the structure of the kernels that $\mathcal{S}$ has a non-trivial invariant subspace if every operator in $\mathcal{S}$ is quasinilpotent. We construct a special kind of bases of the ranges of positive integral idempotent operators consisting of only non-negative functions. Using these bases, We prove that $\mathcal{S}$ has a non-trivial invariant subspace if it contains a non-zero compact operator and $r(AB) \leq r(A)r(B)$ for all A, B in $\mathcal{S}$. Also, we prove that if $\mathcal{S}$ is a semigroup of positive integral operators with the kernels satisfying certain positivity conditions, then there exists a non-trivial standard subspace, *i.e.*, a subspace of the form $\chi_E \mathcal{L}^2(\mathcal{X}, m)$ for some Borel set E in $\mathcal{X}$, invariant under $\mathcal{S}$. We give a non-compact analogue of the Lomonosov - de Pagter result. Let T be an injective positive quasinilpotent operator dominating a non-zero compact positive operator T_0, *i.e.*, $T - T_0$ is positive. Assume $\mathcal{C}$ is a collection of positive operators contained in a norm-closed algebra $\mathcal{A}$ with $\mathcal{A}T \subseteq T\mathcal{A}$. Then there exists a non-trivial standard subspace invariant under $\mathcal{C}$ and T.

Finally, we construct a semigroup of positive nilpotent operators with no non-trivial invariant subspaces.

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Introduction

One of the classical unsolved problems in Operator Theory is the Invariant Subspace Problem: Does every bounded linear operator on an infinite dimensional Hilbert space have a non-trivial invariant subspace? An equally interesting problem is the problem of reducibility of algebras (or, more generally, multiplicative semigroups) of bounded linear operators on an infinite dimensional Hilbert space: What operator algebras (or semigroups) are reducible? By a reducible collection of operators is meant one whose members have a common non-trivial invariant subspace.

Over the years, many important results have been obtained. The most striking ones are spectral theorems for normal operators, Aronszajn-Smith theorem [6] on the existence of invariant subspaces for compact operators on Banach spaces, Lomonosov's theorem [35] on the existence of hyperinvariant subspaces for compact operators, and S. Brown's theorem [12] for subnormal operators. Recently, Brown, Chevreau and Pearcy proved that every contraction on a Hilbert space with spectrum containing the unit circle has a non-trivial invariant subspace (see [11]). However, most of the theorems require the operators to have more than one point in their spectra. In [25], Halmos initiated the study of quasitriangular operators. The concept of quasitriangular operators plays a central role in the proofs of the Aronszajn-Smith theorem. At the end of his paper, Halmos asked: Does every quasitriangular operator have a non-trivial invariant subspace? Surprisingly, C. Apostol, C. Foiaş and D. Voiculescu proved in a series of papers that every non-quasitriangular operator has non-trivial invariant subspaces (see [3]). Thus, the general invariant subspace problem was reduced to the invariant subspace problem for quasitriangular operators. In [9]

Arveson and Feldman proved that every quasinilpotent operator with a cyclic vector is quasitriangular. Therefore, it is worthwhile to examine the existence of invariant subspaces for certain multiplicative semigroups of quasinilpotent operators, especially non-compact quasinilpotent operators.

Several mathematicians have made progress in this direction. It has been proven [41] that a semigroup of quasinilpotent operators is reducible if it contains a non-zero operator in some von Neumann-Schatten class $\mathcal{C}_p$. For the non-compact case, a beautiful theorem was obtained by Andô and Krieger [61, Theorem 136.9]. Let $\mathcal{L}^2(\mathcal{X}, m)$ be a Hilbert space of dimension at least two. Under its natural structure, $\mathcal{L}^2(\mathcal{X}, m)$ is a Banach lattice. It follows from the Andô-Krieger theorem that every quasinilpotent integral operator with non-negative kernel on $\mathcal{L}^2(\mathcal{X}, m)$ must leave $\chi_E \mathcal{L}^2(\mathcal{X}, m)$ invariant for some non-trivial Borel set E. Recently, a number of results in this area have been obtained (see [14], [34] and [46]). However, there are still several interesting problems that have not been solved, including the one posed in [41]: Is a semigroup of compact quasinilpotent operators reducible? In this thesis, we will investigate the reducibility and the existence of so-called *standard* invariant subspaces of certain semigroups of quasinilpotent operators, especially semigroups of integral quasinilpotent operators with non-negative kernels.

A *standard Borel space* is a set $\mathcal{X}$ and a σ-algebra of subsets of $\mathcal{X}$ (called the Borel subsets of $\mathcal{X}$) such that $\mathcal{X}$ is Borel-isomorphic to a Borel subset of some complete separable metric space in its relative Borel structure (see [8, Chapter 3] or [37]). Throughout this thesis, we always assume that $\mathcal{X}$ is a topological space and, with its Borel structure, a standard Borel space. We also assume that m is a σ-finite regular Borel measure on $\mathcal{X}$ such that the Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ is of dimension at least two.

Chapter 1 covers basic aspects of the theory of integral operators and *pseudo-integral operators*, a generalization of integral operators introduced by Arveson in [7]. A number of known results about the algebraic properties of pseudo-integral operators and their kernels are listed for future use. We also introduce some notation and terminology. Most of the material in this chapter comes from Halmos and Sunder

[26], Sourour [57], as well as Zaanen [61].

In Chapter 2, we study the algebra $\mathcal{P}$ of all pseudo-integral operators with absolutely bounded kernels on some Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ of square integrable functions on a finite measure space $(\mathcal{X}, m)$. We indicate that the algebra $\mathcal{P}$ may not be a norm-closed subalgebra of $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ in general; and a new norm $|||\cdot|||$ is introduced on $\mathcal{P}$. Using classical methods of functional analysis, we prove that $(\mathcal{P}, |||\cdot|||)$ is a complex Banach $*$-algebra (Theorem 2.15). The spectral properties of operators as elements of the Banach algebra $(\mathcal{P}, |||\cdot|||)$ are discussed in Section 2.3.

Chapter 3 is on the structure of kernels of positive integral idempotents. In general, if A is an idempotent on a Hilbert space $\mathcal{H}$, then, under a suitable orthonormal basis, A can be represented as a matrix whose upper left corner, corresponding to the compression of A to its range, is the identity matrix of the size of the rank of A. The main results of this chapter (Theorem 3.13 and 3.16) are generalizations of this in the case where A is a positive integral idempotent on $\mathcal{L}^2(\mathcal{X}, m)$. As a result (Corollary 3.17), we can obtain a basis of the range of A consisting of positive elements of $\mathcal{L}^2(\mathcal{X}, m)$. This kind of special bases will be used in Chapter 4 to prove the existence of non-trivial invariant subspaces for certain semigroups of positive integral operators.

Chapter 4 is devoted to the study of reducibility of semigroups of positive operators on $\mathcal{L}^2(\mathcal{X}, m)$. We prove that every semigroup of positive quasinilpotent operators is reducible (Theorem 4.7), and a theorem (Theorem 4.8) which is more general than Theorem 4.7. We also investigate the existence of non-trivial standard invariant subspaces of certain semigroups of positive integral operators, and prove a generalization (Corollary 4.26) of the Andô-Krieger theorem in the special case where the Banach lattice is the functional Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ with its natural lattice structure.

In Chapter 5, we construct a semigroup of positive nilpotent operators on $\mathcal{L}^2([0,1])$ which does not have any invariant subspaces other than $\{0\}$ and $\mathcal{L}^2([0,1])$ itself. The semigroup constructed is discrete (Theorem 5.8), and hence, norm-closed in $\mathcal{B}(\mathcal{L}^2([0,1]))$.

Finally, Chapter 6 consists of two sections. The first section discusses the relation between the Jacobson radicals of operator algebras and the existence of invariant subspaces of the algebras. The main results in this section are Theorems 6.3 and 6.4. The second section studies positive linear mappings between C^*-algebras. We give a sufficient condition that makes a linear mapping between unital C^*-algebras a Jordan homomorphism (Theorem 6.12). The main theorem (Theorem 6.10) answers a question posed in [13].

Chapter 1

Preliminaries

In this chapter, we will discuss some basic aspects of the theory of integral operators and *pseudo-integral operators*, a generalization of integral operators introduced by Arveson in [7]. We list a number of known results that will be used extensively later.

1.1 Integral Operators

Suppose $(\mathcal{X}, m)$ and $(\mathcal{Y}, m')$ are two standard Borel measure spaces with σ-finite regular Borel measures m and m'. The following definitions are from Halmos and Sunder [26]. A *kernel* on $\mathcal{X} \times \mathcal{Y}$ is a complex measurable function on the Cartesian product space $\mathcal{X} \times \mathcal{Y}$. If k is a kernel on $\mathcal{X} \times \mathcal{Y}$, then the measurable function k^* defined by

$$k^*(y, x) = \overline{k}(x, y) \quad (y, x) \in \mathcal{Y} \times \mathcal{X}$$

is a kernel on $\mathcal{Y} \times \mathcal{X}$, and called the *conjugate transpose* of k.

Let k be a kernel on $\mathcal{X} \times \mathcal{Y}$. Suppose k has the property that for all g in $\mathcal{L}^2(\mathcal{Y}, m')$, $k(x, \cdot)g(\cdot) \in \mathcal{L}^1(\mathcal{Y}, m')$ for almost every x in $\mathcal{X}$, and the function f defined by

$$f(x) = \int_{\mathcal{Y}} k(x, y)g(y)m'(dy)$$

is square integrable over $\mathcal{X}$. It was proved in [26, Theorem 3.10] that k actually induces a bounded linear operator $\operatorname{Int} k$ from $\mathcal{L}^2(\mathcal{Y}, m')$ to $\mathcal{L}^2(\mathcal{X}, m)$:

$$(\operatorname{Int} k)g(x) = \int_{\mathcal{Y}} k(x,y)g(y)m'(dy) \quad x \in \mathcal{X} \ a.e.$$

We call kernels that induce bounded linear operators ***bounded kernels***, and operators induced by bounded kernels ***integral operators***. If u is in $\mathcal{L}^2(\mathcal{X}, m)$ and v is in $\mathcal{L}^2(\mathcal{Y}, m')$, then we denote by $u \otimes v$ the measurable function $u(x)\overline{v}(y)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Clearly, $u \otimes v$ is a bounded kernel on $\mathcal{X} \times \mathcal{Y}$ and induces a rank-1 operator. As usual, the rank-1 integral operator induced by $u \otimes v$ is still denoted by $u \otimes v$.

It is easy to see that every function in $\mathcal{L}^2(\mathcal{X} \times \mathcal{Y}, m \times m')$ induces a bounded operator from $\mathcal{L}^2(\mathcal{Y}, m')$ to $\mathcal{L}^2(\mathcal{X}, m)$. The integral operator induced by a kernel in $\mathcal{L}^2(\mathcal{X} \times \mathcal{Y}, m \times m')$ is called *Hilbert-Schmidt operator*.

Suppose $\mathcal{H}$ is an arbitrary Hilbert space. For any positive number p, the Schatten p classes $\mathcal{C}_p$ is the set of all compact operators T on $\mathcal{H}$ with the property that the sequence $\{s_j(T)\}$ of eigenvalues of $(T^*T)^{\frac{1}{2}}$ (counting the multiplicity) is in l^p. It was proved [49] that $\mathcal{C}_p$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$ and a Banach space under the $\mathcal{C}_p$-norm $\|\cdot\|_{\mathcal{C}_p}$ defined by the equation

$$\|T\|_{\mathcal{C}_p} = \left\{ \sum_{j=1}^{\infty} [s_j(T)]^p \right\}^{\frac{1}{p}}.$$

Usually, we call an operator T in $\mathcal{C}_1$ a trace-class operator, and the trace $\operatorname{tr}(T)$ of T is defined to be the sum of all eigenvalues (counting the multiplicity) of T. It is well-known that if $\mathcal{H} = \mathcal{L}^2(\mathcal{X}, m)$, then the $\mathcal{C}_2$ class coincides with the class of Hilbert-Schmidt operators on $\mathcal{L}^2(\mathcal{X}, m)$ (see [49]).

Kernels and integral operators have been extensively studied in [26] and [61]. We list here without proof a number of results from these papers.

Proposition 1.1 [26, Theorem 7.5] *The conjugate transpose k^* of a bounded kernel k is bounded if and only if the adjoint of the induced integral operator is an integral operator, and in that case,* $\operatorname{Int} k^* = (\operatorname{Int} k)^*$.

Proposition 1.2 [26, Corollary 4.4] *If $u_1, u_2, \ldots, u_n$ are in $\mathcal{L}^2(\mathcal{X}, m)$, $v_1, v_2, \ldots, v_n$ are in $\mathcal{L}^2(\mathcal{Y}, m')$ and $k = \sum_{j=1}^n u_j \otimes v_j$, then* $\operatorname{Int} k$ *is a bounded linear operator from $\mathcal{L}^2(\mathcal{Y}, m')$ to $\mathcal{L}^2(\mathcal{X}, m)$ of rank at most n. Conversely, if A is an arbitrary bounded linear operator from $\mathcal{L}^2(\mathcal{Y}, m')$ to $\mathcal{L}^2(\mathcal{X}, m)$ of rank at most n, then $A =$* $\operatorname{Int} k$ *for some kernel k of the form $\sum_{j=1}^n u_j \otimes v_j$.*

Proposition 1.3 [26, Theorem 8.1] *If a bounded kernel k on $\mathcal{X} \times \mathcal{Y}$ induces the zero operator, then $k(x, y) = 0$ for almost every $(x, y) \in \mathcal{X} \times \mathcal{Y}$.*

From the above propositions, it is clear that if A is a finite-rank (integral) operator in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ with a non-negative kernel, then so is A^*.

Definition 1.4 [14, Definition 3.6] A *subspace* of $\mathcal{L}^2(\mathcal{X}, m)$ is a norm-closed linear manifold in $\mathcal{L}^2(\mathcal{X}, m)$. A *standard subspace* of $\mathcal{L}^2(\mathcal{X}, m)$ is a subspace of the form

$$\mathcal{M}_U \equiv \chi_U \mathcal{L}^2(\mathcal{X}, m) = \{f \in \mathcal{L}^2(\mathcal{X}, m) : f = 0 \quad a.e. \quad on \quad U^c\}$$

for some Borel set U in $\mathcal{X}$. The orthogonal projection from $\mathcal{L}^2(\mathcal{X}, m)$ onto $\mathcal{M}_U$ is denoted by P_U.

REMARK. It is easy to see that the union and intersection of any countable set of standard subspaces are still standard subspaces, and so are the complements of standard subspaces. If $\mathcal{L}^2(\mathcal{X}, m)$ is separable, then, as a topological space, it has the Lindelöf property. Consequently, the union and intersection of any set of standard subspaces, countable or uncountable, are still standard subspaces.

Proposition 1.5 [61, Theorem 136.3] *Let U be a Borel set in $\mathcal{X}$. An integral operator $T \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ with non-negative kernel k leaves the standard space $\mathcal{M}_U$ invariant if and only if $k = 0$ a.e. on $U^c \times U$.*

Proposition 1.6 [61, Theorem 135.1] *Let $T \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ be an integral operator with non-negative kernel. Then the spectral radius $r(T)$ of T belongs to the spectrum $\sigma(T)$ of T.*

Proposition 1.7 [61, Theorem 135.2] *Let $T \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ be a compact integral operator with non-negative kernel such that the spectral radius $r(T)$ of T is not zero, then there exists a positive function u in $\mathcal{L}^2(\mathcal{X}, m)$ such that $u \neq 0$ and $Tu = r(T)u$.*

1.2 Kernels of Integral Operators

In this section, we discuss the boundedness of kernels on $\mathcal{X} \times \mathcal{X}$, as well as the product of integral operators. Most of the material in this section comes from Halmos and Sunder [26].

It is easy to check that if h and k are bounded kernels and $h + k$ is their pointwise sum, then $h + k$ is a bounded kernel and $\mathrm{Int}\,(h + k) = \mathrm{Int}\,h + \mathrm{Int}\,k$, and that if k is a bounded kernel and α is a scalar, then αk is bounded and $\mathrm{Int}\,(\alpha k) = \alpha \mathrm{Int}\,k$ where αk is defined by $(\alpha k)(x, y) = \alpha k(x, y)$.

One may expect similar results for the product of integral operators. Unfortunately, the situation is complicated and no general theorem seems to be known about it. Two kernels h and k on $\mathcal{X} \times \mathcal{X}$ are called *multipliable* if $h(x, \cdot)k(\cdot, y)$ belongs to $\mathcal{L}^1(\mathcal{X}, m)$ for almost every $(x, y) \in \mathcal{X} \times \mathcal{X}$. In that case, the *convolution*

$$\int_{\mathcal{X}} h(x, t)k(t, y)m(dt)$$

can be formed for almost every $(x, y) \in \mathcal{X} \times \mathcal{X}$; and it defines a kernel $h{*}k$ on $\mathcal{X} \times \mathcal{X}$. In general, two bounded kernels are not always multipliable, and it is still unknown whether the convolution of two multipliable bounded kernels is necessarily bounded, and whether it necessarily induces the product operator if the convolution is bounded (see [26], p32-33).

Proposition 1.8 *If $A \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ is an integral operator with kernel h and $B \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ is a finite-rank operator with kernel k, $k = \sum_{j=1}^{n} u_j \otimes v_j$ where u_j, v_j are in $\mathcal{L}^2(\mathcal{X}, m)$ for all $j = 1, 2, \ldots, n$, then h and k are multipliable, and the convolution $h{*}k$ of h and k is bounded and induces the operator AB.*

Proof. With no loss of generality (WNLG), we may assume that B is a rank-1 operator with kernel k, $k = u \otimes v$ where u and v are in $\mathcal{L}^2(\mathcal{X}, m)$.

For any $(x, y) \in \mathcal{X} \times \mathcal{X}$,

$$h(x, \cdot)k(\cdot, y) = h(x, \cdot)u(\cdot)\overline{v}(y).$$

Since h is a bounded kernel, $h(x, \cdot)u(\cdot) \in \mathcal{L}^1(\mathcal{X}, m)$ for almost every x in $\mathcal{X}$. Therefore, $h(x, \cdot)k(\cdot, y) \in \mathcal{L}^1(\mathcal{X}, m)$ for almost every $(x, y) \in \mathcal{X} \times \mathcal{X}$. Hence, h and k are multipliable.

The convolution $h*k$ of h and k is given by

$$\begin{aligned} (h*k)(x, y) &= \int_{\mathcal{X}} h(x, t)k(t, y)m(dt) \\ &= \int_{\mathcal{X}} h(x, t)u(t)\overline{v}(y)m(dt) \\ &= (Au)(x)\overline{v}(y). \end{aligned}$$

For any $f \in \mathcal{L}^2(\mathcal{X}, m)$,

$$\begin{aligned} & [\mathrm{Int}\,(h*k)f](x) \\ = & \int_{\mathcal{X}} (h*k)(x, y)f(y)m(dy) \\ = & \int_{\mathcal{X}} (Au)(x)\overline{v}(y)f(y)m(dy) \\ = & (Au)(x) \int_{\mathcal{X}} \overline{v}(y)f(y)m(dy) \\ = & \int_{\mathcal{X}} h(x, t)u(t)m(dt) \int_{\mathcal{X}} \overline{v}(y)f(y)m(dy) \\ = & \int_{\mathcal{X}} h(x, t) \left[\int_{\mathcal{X}} u(t)\overline{v}(y)f(y)m(dy) \right] m(dt) \\ = & \int_{\mathcal{X}} h(x, t)(Bf)(t)m(dt) \\ = & A(Bf)(x) \\ = & [(AB)f](x), \end{aligned}$$

for almost every x in $\mathcal{X}$. This implies that the convolution $h*k$ is a bounded kernel and induces the operator AB. ∎

Let A and B be the same as in the above proposition. Then BA is a finite-rank operator as well, and therefore, an integral operator. But this does not imply that k and h are always multipliable and that the convolution $k*h$ is a bounded kernel. We explain this through the following example.

Recall that the discrete Fourier transform F from $\mathcal{L}^2([0,1])$ to $\mathcal{L}^2(\mathcal{Z})$ is an integral operator induced by ϕ, where ϕ is the kernel on $\mathcal{Z} \times [0,1]$ given by

$$\phi(n,y) = e^{-2\pi n y i} \quad (n,y) \in \mathcal{Z} \times [0,1].$$

The Fourier transform F assigns to each element g in $\mathcal{L}^2([0,1])$ the sequence of its Fourier coefficients; the adjoint F^* assigns to each sequence f in $\mathcal{L}^2(\mathcal{Z})$ the function whose sequence of Fourier coefficients it is. It is a well-known fact [26, Example 7.2] that the transpose ϕ^* of ϕ is not a bounded kernel and F^* is not an integral operator. In fact, if $\{c_n\}$ is in l^2 but not in l^1, then $\sum e^{2\pi n y i} c_n$ is not absolutely summable for every $y \in [0,1]$.

Example 1.9 *There exists an integral operator $A = \mathrm{Int}\, h$ and a rank-1 operator $B = u \otimes u$ on a Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ such that $u \otimes u$ and h are not multipliable kernels.*

Proof. Let $\mathcal{X}$ be the disjoint union $\mathcal{Z} \bigcup_d [0,1]$ of $\mathcal{Z}$ and $[0,1]$ and let m be the measure on $\mathcal{X}$ that is the counting measure on $\mathcal{Z}$ and the Lebesgue measure on $[0,1]$. We identify $\mathcal{L}^2(\mathcal{X}, m)$ with $\mathcal{L}^2(\mathcal{Z}) \oplus \mathcal{L}^2([0,1])$. Let h be the kernel on $\mathcal{X} \times \mathcal{X}$ given by

$$h(x,y) = \begin{cases} \phi(x,y) & \text{if } x \in \mathcal{Z} \text{ and } y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to check that

$$(\mathrm{Int}\, h)\xi = (Fv) \oplus 0$$

for any $\xi = u \oplus v \in \mathcal{L}^2(\mathcal{Z}) \oplus \mathcal{L}^2([0,1])$. Hence h is a bounded kernel.

Choose any sequence $\{c_n\}$ in l^2 but not in l^1 and let $u = \{c_n\} \oplus 0$. Then $u \in \mathcal{L}^2(\mathcal{X}, m)$. For every $(x,y) \in \mathcal{X} \times \mathcal{X}$, we have

$$(u \otimes u)(x,\cdot)h(\cdot,y) = u(x)\overline{u}(\cdot)h(\cdot,y) = u(x)h(\cdot,y)\overline{u}(\cdot)$$

which is not in $\mathcal{L}^1(\mathcal{X}, m)$ since

$$\int |h(t,y)\overline{u}(t)|m(dt) = \sum_n |e^{-2\pi nyi}c_n| = +\infty.$$

Thus $u \otimes u$ and h are not multipliable kernels. ■

Definition 1.10 [26, p.50] A kernel k on $\mathcal{X} \times \mathcal{X}$ is called *absolutely bounded* if $|k|$ is a bounded kernel on $\mathcal{X} \times \mathcal{X}$.

The idea behind the following proposition comes from [46].

Proposition 1.11 *If a kernel k on $\mathcal{X} \times \mathcal{X}$ is dominated by a non-negative bounded kernel h in the sense that $|k(x,y)| \leq h(x,y)$ for almost every (x,y) in $\mathcal{X} \times \mathcal{X}$, then k is bounded and absolutely bounded. Moreover* $\|\mathrm{Int}\, k\| \leq \|\mathrm{Int}\,|k|\| \leq \|\mathrm{Int}\, h\|$.

Proof. For any $f \in \mathcal{L}^2(\mathcal{X}, m)$,

$$|k(x,y)f(y)| \leq h(x,y)|f(y)| = h(x,y)|f|(y) \quad (x,y) \in \mathcal{X} \times \mathcal{X} \text{ a.e.}$$

But h is a bounded kernel and $|f|$ is in $\mathcal{L}^2(\mathcal{X}, m)$, we have that the function $h(x,\cdot)|f|(\cdot)$ is in $\mathcal{L}^1(\mathcal{X}, m)$ for almost every x in $\mathcal{X}$. Therefore, the function $k(x,\cdot)f(\cdot)$ is in $\mathcal{L}^1(\mathcal{X}, m)$ for almost every x in $\mathcal{X}$.

It is clear that the function

$$\int_{\mathcal{X}} k(\cdot,y)f(y)m(dy)$$

is dominated by the function

$$(\mathrm{Int}\, h)|f|(\cdot) = \int_{\mathcal{X}} h(\cdot,y)|f|(y)m(dy).$$

That is

$$|\int_{\mathcal{X}} k(x,y)f(y)m(dy)| \leq \int_{\mathcal{X}} h(x,y)|f|(y)m(dy) \quad x \in \mathcal{X} \text{ a.e.}$$

However, $(\operatorname{Int} h)|f|$ is in $\mathcal{L}^2(\mathcal{X}, m)$, therefore, $\int_{\mathcal{X}} k(\cdot, y)f(y)m(dy)$, as a function on $\mathcal{X}$, is also in $\mathcal{L}^2(\mathcal{X}, m)$. Hence, k is a bounded kernel on $\mathcal{X} \times \mathcal{X}$.

From what we have shown above, we know that, for any $f \in \mathcal{L}^2(\mathcal{X}, m)$,

$$\|(\operatorname{Int} k)f\| \leq \|(\operatorname{Int} |k|)|f|\|,$$

and

$$\|(\operatorname{Int} |k|)f\| \leq \|(\operatorname{Int} h)|f|\|.$$

It follows that

$$\|\operatorname{Int} k\| \leq \|\operatorname{Int} |k|\| \leq \|\operatorname{Int} h\|.$$

■

Corollary 1.12 *All absolutely bounded kernels are bounded kernels.*

Theorem 1.13 [26, Theorem 10.7] *If h and k are absolutely bounded kernels on $\mathcal{X} \times \mathcal{X}$, then h and k are multipliable, and $h*k$ is an absolutely bounded kernel on $\mathcal{X} \times \mathcal{X}$ and* $\operatorname{Int}(h*k) = \operatorname{Int} h \operatorname{Int} k$.

Proof. It follows immediately from Fubini's Theorem. We omit the details. ■

If k is an absolutely bounded kernel on $\mathcal{X} \times \mathcal{X}$, then, from the above theorem, $(\operatorname{Int} k)^n$ is an integral operator induced by the absolutely bounded kernel

$$k^{(n)} = \underbrace{k* \cdots *k}_{n},$$

for all positive integer n.

Corollary 1.14 *If k is a kernel on $\mathcal{X} \times \mathcal{X}$ dominated by a non-negative kernel h, then $r(\operatorname{Int} k) \leq r(\operatorname{Int} h)$, where r denotes spectral radius.*

Proof. It follows from Proposition 1.11 that k is an absolutely bounded kernel on $\mathcal{X} \times \mathcal{X}$. Therefore, for any positive integer n, $(\mathrm{Int}\, k)^n$ is an integral operator induced by the kernel $k^{(n)}$ which is dominated by the kernel $h^{(n)}$. By Proposition 1.11, $\|(\mathrm{Int}\, k)^n\| \leq \|(\mathrm{Int}\, h)^n\|$ for all $n = 1, 2, \cdots$. Thus, $r(\mathrm{Int}\, k) \leq r(\mathrm{Int}\, h)$. ■

1.3 Pseudo-Integral Operators

In this section, we assume further that m is a finite regular Borel measure on $\mathcal{X}$ such that $\mathcal{L}^2(\mathcal{X}, m)$ is of dimension at least two.

The Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ with its natural order structure is a Banach lattice. More explicitly, an element f in $\mathcal{L}^2(\mathcal{X}, m)$ is *lattice positive* (simply, positive) if and only if $f(x) \geq 0$ for almost every x in $\mathcal{X}$. We call an operator T in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ *functionally positive* (simply, positive) if Tf is positive whenever $f \in \mathcal{L}^2(\mathcal{X}, m)$ is positive.

The concept of pseudo-integral operator was introduced by Arveson in [7], and studied by Sourour in [57] and [58]. The following definition comes from [57].

Definition 1.15 [57, Definition 2.1] A bounded linear operator T in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ is called a *pseudo-integral operator* if T is given by the equation

$$(Tf)(x) = \int_{\mathcal{X}} f(y)\mu(x, dy) \quad x \in \mathcal{X} \ a.e.$$

for every f in $\mathcal{L}^2(\mathcal{X}, m)$, where, for almost every x in $\mathcal{X}$, $\mu(x, \cdot)$ is a complex Borel measure on $\mathcal{X}$, and, for every Borel set B in $\mathcal{X}$, the map $x \longmapsto \mu(x, B)$ is assumed to be a Borel function.

The class $\{\mu(x, \cdot) : x \in \mathcal{X}\}$ of measures is called the *kernel* of T.

A kernel $\{\mu(x, \cdot)\}$ is called *absolutely bounded* if $\{|\mu|(x, \cdot)\}$ is the kernel of a bounded operator on $\mathcal{L}^2(\mathcal{X}, m)$.

Clearly, the concept of pseudo-integral operator is a generalization of that of integral operator. Furthermore, the definitions of absolute boundedness of kernels are consistent. The class of pseudo-integral operators is quite large. In fact, for any $\phi \in \mathcal{L}^\infty(\mathcal{X}, m)$, the multiplication operator M_ϕ is a pseudo-integral operator induced by the kernel: $\mu(x, dy) = \phi(x)\delta_x(dy)$, where δ_x is the point mass at x. Particularly, the identity operator I is a pseudo-integral operator. Also if ψ is a measurable map on $\mathcal{X}$ such that the equation $C_\psi f = f \circ \psi$ defines a bounded operator C_ψ on $\mathcal{L}^2(\mathcal{X}, m)$, then C_ψ is a pseudo-integral operator with kernel $\mu(x, dy) = \delta_{\psi(x)}(dy)$ (see [57, p.342]). It is well-known that if $(\mathcal{X}, m)$ is a nonatomic measure space, then neither of them is an integral operator. However, not every bounded operator is a pseudo-integral operator. In [57], Sourour provided three examples to show that if $\mathcal{X}$ is the unit circle and m is the normalized Lebesgue measure, then there exists projections, unitary operators and compact operators on $\mathcal{L}^2(\mathcal{X}, m)$ that are not pseudo-integral operators.

Next, we list several results, to be used later, related to pseudo-integral operators. As before, their proofs are omitted.

In [7], Arveson proved the following disintegration properties for measures, which is essential for analyzing the kernels of pseudo-integral operators.

Lemma 1.16 ([7, p.461]; [57, p.349]) *Let $\mathcal{X}$, $\mathcal{Y}$ be standard Borel spaces, let μ be a finite positive measure on $\mathcal{X} \times \mathcal{Y}$, and let $\mu_1(A) = \mu(A \times \mathcal{Y})$ be the first marginal measure of μ. Then there exists a map $x \longmapsto \mu_0^x$ from $\mathcal{X}$ into the space of all probability measures on $\mathcal{Y}$ such that*

(i) The map $x \longmapsto \mu_0^x(B)$ is a Borel function for every Borel set B in $\mathcal{Y}$.

(ii) $\mu(S) = \int_{\mathcal{X}}\int_{\mathcal{Y}} \chi_S(x, y)\mu_0^x(dy)\mu_1(dx)$ for every Borel set S in $\mathcal{X} \times \mathcal{Y}$.

Proposition 1.17 ([7, Proposition 1.5.3], [57, Proposition 4.2]) *Suppose $(\mathcal{X}, m)$ and $(\mathcal{Y}, m')$ are standard Borel measure spaces with finite regular Borel measures m and m'. Let μ be a bounded complex Borel measure on $\mathcal{X} \times \mathcal{Y}$ such that μ vanishes on marginally null sets (equivalently $|\mu|_1 \ll m, |\mu|_2 \ll m'$, where $|\mu|_1$ and $|\mu|_2$ are* *marginal measures of* $|\mu|$). *Then there exists a map* $x \longmapsto \mu^x$ *of* $\mathcal{X}$ *into the set of all bounded Borel measures on* $\mathcal{Y}$, *and a map* $y \longmapsto \mu_y$ *of* $\mathcal{Y}$ *into the set of all bounded measures on* $\mathcal{X}$, *such that*

(*i*) *For all Borel sets* $A \subseteq \mathcal{X}$ *and* $B \subseteq \mathcal{Y}$, *the maps* $x \longmapsto \mu^x(B)$ *and* $y \longmapsto \mu_y(A)$ *are Borel functions.*

(*ii*) $\mu(dx, dy) = \mu^x(dy)m(dx)$, *i.e., for every Borel set* S *in* $\mathcal{X} \times \mathcal{Y}$,

$$\mu(S) = \iint_{\mathcal{X}\times\mathcal{Y}} \chi_S(x,y)\mu^x(dy)m(dx);$$

and $\mu(dx, dy) = \mu_y(dx)m'(dy)$.

(*iii*) $|\mu|(dx, dy) = |\mu^x|(dy)m(dx) = |\mu_y|(dx)m'(dy)$.

Moreover, μ^x *and* μ_y *are essentially unique.*

It is well-known that an integral operator is positive, *i.e.*, it maps positive elements to positive elements, if and only if its kernel is non-negative. The following theorem not only generalizes this result, but also indicates that the class of pseudo-integral operators is much larger than that of integral operators. The proof can be found in [57].

Theorem 1.18 [57, Theorem 3.1] *Let* T *be an operator on* $\mathcal{L}^2(\mathcal{X}, m)$. *In order for* T *to be a pseudo-integral operator with a positive kernel, it is necessary and sufficient that* T *be a positive operator.*

Recall that an operator T on $\mathcal{L}^2(\mathcal{X}, m)$ is called *order-bounded* if for every positive element $u \in \mathcal{L}^2(\mathcal{X}, m)$, there exists a positive element $v \in \mathcal{L}^2(\mathcal{X}, m)$ such that $|(Tf)(x)| \leq v(x)$ for almost every x in $\mathcal{X}$ whenever $|f(x)| \leq u(x)$ for almost every x in $\mathcal{X}$.

Corollary 1.19 [57, Corollary 3.2] *Let* T *be an operator on* $\mathcal{L}^2(\mathcal{X}, m)$. *The following conditions are equivalent.*

(*i*) *T is a pseudo-integral operator with absolutely bounded kernel.*

(*ii*) *$T = T_1 - T_2 + i(T_3 - T_4)$ for some positive bounded operators T_1, T_2, T_3 and T_4.*

(*iii*) *T is order bounded.*

1.4 Pseudo-Integral Operators with Absolutely Bounded Kernels

Suppose $(\mathcal{X}, m)$ is a standard Borel measure space with finite regular Borel measure m. It follows easily from Corollary 1.19 that the class of pseudo-integral operators with absolutely bounded kernels is a subspace of $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$. We denote it by $\mathcal{P}$. In this section, we will see that $\mathcal{P}$ is actually a $*$-subalgebra of $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$.

The following lemma is obtained by Sourour. It gives a sufficient condition for a measure on $\mathcal{X} \times \mathcal{X}$ to be an absolutely bounded kernel.

Lemma 1.20 [57, Lemma 4.1] *Let μ be a Borel measure on $\mathcal{X} \times \mathcal{X}$ which vanishes on marginally null sets and has the property that the function $h(x, y) = f(y)\overline{g}(x)$ belongs to $\mathcal{L}^1(\mathcal{X} \times \mathcal{X}, |\mu|)$ whenever f and g are in $\mathcal{L}^2(\mathcal{X}, m)$. Then the equation*

$$\langle T_\mu f, g\rangle = \int_{\mathcal{X}\times\mathcal{X}} f(y)\overline{g}(x)\mu(dx, dy)$$

defines a bounded linear operator T_μ on $\mathcal{L}^2(\mathcal{X}, m)$.

REMARK. If a measure μ induces a bounded operator T_μ as in the above lemma, then so does the measure $|\mu|$. Therefore, T_μ is a pseudo-integral operator with an absolutely bounded kernel. In this case, we call the measure μ the absolutely bounded kernel of T_μ (see [57, p.350]).

From now on, T_μ will denote the pseudo-integral operator induced by the absolutely bounded kernel μ.

Theorem 1.21 [57, Theorem 4.3] *If T is a pseudo-integral operator on $\mathcal{L}^2(\mathcal{X}, m)$ with absolutely bounded kernel μ, then so is T^*, and for every f in $\mathcal{L}^2(\mathcal{X}, m)$,*

$$(T^*f)(y) = \int_{\mathcal{X}} f(x)\overline{\mu}_y(dx) \quad y \in \mathcal{X} \text{ a.e.},$$

where $\overline{\mu}(dx, dy) = \overline{\mu}_y(dx)m(dy)$ is the disintegration of the complex conjugate $\overline{\mu}$ of μ.

REMARK. From the above theorem, it is clear that if T is an integral operator with non-negative kernel k, then T^* is also an integral operator, and its kernel is k^*.

We now consider the product of pseudo-integral operators with absolutely bounded kernels. The concept of convolution of absolutely bounded kernels was introduced by Arveson [7]. Suppose T_μ and T_ν are two pseudo-integral operators with absolutely bounded kernels μ and ν respectively. Then the equation

$$(\mu*\nu)(S) = \int_{\mathcal{X}} (\mu_z \times \nu^z)(S)m(dz), \quad \text{for every Borel set } S \subseteq \mathcal{X} \times \mathcal{X}$$

defines a finite Borel measure $\mu*\nu$ on $\mathcal{X} \times \mathcal{X}$ (see [57, p.350]).

Theorem 1.22 ([7, Proposition 1.5.5]; [57, Theorem 4.4]) *If μ and ν are absolutely bounded kernels, then so is $\mu*\nu$, and $T_{\mu*\nu} = T_\mu T_\nu$.*

Theorem 1.23 [57, Theorem 4.5] *The class $\mathcal{P}$ of pseudo-integral operators with absolutely bounded kernels is a selfadjoint algebra containing the identity. The class $\mathcal{I}$ of integral operators with absolutely bounded kernels is a selfadjoint two-sided ideal in $\mathcal{P}$.*

REMARKS. (i) It is easy to see that every Hilbert-Schmidt operator is in $\mathcal{I}$, and therefore, in $\mathcal{P}$.

(ii) Suppose $T_\mu \in \mathcal{P}$ is a pseudo-integral operator with absolutely bounded kernel μ. Then T_μ belongs to $\mathcal{I}$ if and only if

$$\mu(dx, dy) = k(x, y)(m \times m)(dx, dy)$$

for some measurable function k, and T_μ is a Hilbert-Schmidt operator if and only if

$$\mu(dx, dy) = k(x, y)(m \times m)(dx, dy)$$

for some measurable function k in $\mathcal{L}^2(\mathcal{X} \times \mathcal{X}, m \times m)$. In that case, we have that

$$\|T_\mu\|_{\mathcal{C}_2} = \|k\|_{\mathcal{L}^2(\mathcal{X} \times \mathcal{X},\, m \times m)}.$$

(iii) There are compact operators (see [57, Example 2.6]) that are not pseudo-integral operators. Therefore, $\mathcal{P}$ is, in general, not norm-closed in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$.

(iv) There are positive compact operators (see [19, Example 3]) that are the norm-limits of positive finite-rank operators, but not integral operators. Therefore, $\mathcal{I}$ is, in general, not norm-closed in $\mathcal{P}$.

Chapter 2

The Algebra of Pseudo-Integral Operators with Absolutely Bounded Kernels

Suppose $\mathcal{X}$ is a topological space and, with its Borel structure, a standard Borel space, and m is a finite regular Borel measure on $\mathcal{X}$. Consider the collection $\mathcal{P}$ of all pseudo-integral operators with absolutely bounded kernels on $\mathcal{L}^2(\mathcal{X}, m)$. We have shown in Chapter 1 that $\mathcal{P}$ is a subalgebra of $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$, but not norm-closed, in general, in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$. In this chapter, we will define a new norm $|||\cdot|||$ on $\mathcal{P}$ and prove using classical methods of functional analysis that under this new norm $\mathcal{P}$ is a Banach $*$-algebra. We will also discuss some spectral properties of certain operators as elements in $(\mathcal{P}, |||\cdot|||)$.

2.1 A New Norm on $\mathcal{P}$

In this section, we define a new norm $|||\cdot|||$ on $\mathcal{P}$, and examine its relation to the operator norm and Hilbert-Schmidt norm.

Definition 2.1 For any $T \in \mathcal{P}$ with absolutely bounded kernel μ, *i.e.*, $T = T_\mu$, we define the new norm of $T = T_\mu$ as the operator norm of $T_{|\mu|}$. That is

$$|||T||| = |||T_\mu||| \equiv \|T_{|\mu|}\|.$$

Proposition 2.2 $(\mathcal{P}, |||\cdot|||)$ *is a normed space.*

Proof. The proof is straightforward and is omitted. ∎

Recall that the collection of all complex Borel measures on $\mathcal{X} \times \mathcal{X}$ is a Banach space with the norm defined to be the total variation of the measure on $\mathcal{X} \times \mathcal{X}$ (see Dunford and Schwartz [18]).

Proposition 2.3 *If T_μ is in $\mathcal{P}$ with absolutely bounded kernel μ, then*

$$\|\mu\| \equiv |\mu|(\mathcal{X} \times \mathcal{X}) \le m(\mathcal{X})|||T_\mu|||.$$

Proof. For any f and g in $\mathcal{L}^2(\mathcal{X}, m)$,

$$\langle T_{|\mu|}f, g\rangle = \int_{\mathcal{X}\times\mathcal{X}} f(y)\overline{g}(x)|\mu|(dx, dy).$$

Therefore,

$$\begin{aligned}
|\mu|(\mathcal{X} \times \mathcal{X}) &= \int_{\mathcal{X}\times\mathcal{X}} \chi_{\mathcal{X}\times\mathcal{X}}(x,y)|\mu|(dx,dy) \\
&= \int_{\mathcal{X}\times\mathcal{X}} \chi_{\mathcal{X}}(x)\chi_{\mathcal{X}}(y)|\mu|(dx,dy) \\
&= \langle T_{|\mu|}\chi_{\mathcal{X}}, \chi_{\mathcal{X}}\rangle \\
&\le \|T_{|\mu|}\| \, \|\chi_{\mathcal{X}}\|^2 \\
&= m(\mathcal{X})|||T_\mu|||.
\end{aligned}$$

∎

Let $\mathbf{B}$ denote the unit ball of $\mathcal{L}^2(\mathcal{X}, m)$, and $\mathbf{B}^+$ the set consisting of all positive elements of $\mathcal{L}^2(\mathcal{X}, m)$ in $\mathbf{B}$. The following proposition gives several equivalent definitions of the norm $|||\cdot|||$.

Proposition 2.4 *Suppose T_μ is in $\mathcal{P}$. Then,*

$$\begin{aligned} |||T_\mu||| &= \sup\{\|T_{|\mu|}f\| : f \in \mathbf{B}^+\} \\ &= \sup\{\langle T_{|\mu|}f, g\rangle : f \text{ and } g \text{ are in } \mathbf{B}^+\} \\ &= \sup\{\|T_{|\mu|}f\| : f \in \mathbf{B}^+ \text{ is a simple function}\} \\ &= \sup\{\langle T_{|\mu|}f, g\rangle : f,\, g \in \mathbf{B}^+ \text{ are simple functions}\}. \end{aligned}$$

Proof. By the definition of the new norm $|||\cdot|||$, $|||T_\mu||| = \|T_{|\mu|}\|$. Since $T_{|\mu|}$ is a positive operator, we have that, for any $f \in \mathcal{L}^2(\mathcal{X}, m)$,

$$|T_{|\mu|}f|(x) \le (T_{|\mu|}|f|)(x)$$

for almost every x in $\mathcal{X}$. But, $\|f\| = \|\,|f|\,\|$ for every f in $\mathcal{L}^2(\mathcal{X}, m)$, so

$$\begin{aligned} |||T_\mu||| &= \|T_{|\mu|}\| \\ &= \sup\{\|T_{|\mu|}f\| : f \in \mathcal{L}^2(\mathcal{X}, m) \text{ and } \|f\| \le 1\} \\ &= \sup\{\|T_{|\mu|}f\| : f \in \mathcal{L}^2(\mathcal{X}, m) \text{ is positive and } \|f\| \le 1\} \\ &= \sup\{\|T_{|\mu|}f\| : f \in \mathbf{B}^+\}. \end{aligned}$$

For the second equation, it is easy to see that

$$\sup\{\langle T_{|\mu|}f, g\rangle : f \text{ and } g \text{ are in } \mathbf{B}^+\}$$

is less than or equal to $|||T_\mu|||$. Conversely, for any positive element f in $\mathcal{L}^2(\mathcal{X}, m)$, we have $T_{|\mu|}f$ is positive. So if we let

$$g = \frac{T_{|\mu|}f}{\|T_{|\mu|}f\|},$$

then g is in $\mathbf{B}^+$, and $\langle T_{|\mu|}f, g\rangle = \|T_{|\mu|}f\|$. It follows that

$$\sup\{\langle T_{|\mu|}f, g\rangle : f \text{ and } g \text{ are in } \mathbf{B}^+\}$$

is greater than or equal to

$$\sup\{\|T_{|\mu|}f\| : f \in \mathbf{B}^+\},$$

which is equal to $\|\|T_\mu\|\|$. Thus,

$$\sup\{\langle T_{|\mu|}f,\, g\rangle : f \text{ and } g \text{ are in } \mathbf{B}^+\}$$

is equal to $\|\|T_\mu\|\|$.

From what we have proven and the fact that every positive element in $\mathcal{L}^2(\mathcal{X}, m)$ is a norm limit of an increasing sequence of positive simple functions, we have that $\|\|T_\mu\|\|$ is equal to

$$\sup\{\|T_{|\mu|}f\| : f \in \mathbf{B}^+ \text{ is a simple function}\}$$

and

$$\sup\{\langle T_{|\mu|}f,\, g\rangle : f, g \in \mathbf{B}^+ \text{ are simple functions}\}.$$

■

Next, we look at the relation between the new norm and the operator norm, and the relation between the new norm and the Hilbert-Schmidt norm.

Proposition 2.5 *Suppose T_μ is in $\mathcal{P}$. Then,*

(i) $\|T_\mu\| \le \|\|T_\mu\|\|$, and $\|T_\mu\| = \|\|T_\mu\|\|$ if μ is a positive measure.

(ii) $\|\|T_\mu\|\| \le \|T_\mu\|_{\mathcal{C}_2}$ if T_μ is a $\mathcal{C}_2$ operator.

Proof. (i) is obvious.

For (ii), if T_μ is a $\mathcal{C}_2$ operator, then

$$\mu(dx, dy) = k(x,y)(m\times m)(dx, dy)$$

for some $k \in \mathcal{L}^2(\mathcal{X} \times \mathcal{X}, m\times m)$. Therefore,

$$|\mu|(dx, dy) = |k|(x,y)(m\times m)(dx, dy).$$

For any positive elements f and g in $\mathcal{L}^2(\mathcal{X}, m)$,

$$\begin{aligned}
\langle T_{|\mu|}f, g\rangle &= \int_{\mathcal{X}\times\mathcal{X}} f(y)\overline{g}(x)|\mu|(dx, dy) \\
&= \int_{\mathcal{X}\times\mathcal{X}} f(y)g(x)|k|(x,y)(m\times m)(dx, dy) \\
&\leq \left\{\int_{\mathcal{X}\times\mathcal{X}} [f(y)g(x)]^2(m\times m)(dx, dy)\right\}^{\frac{1}{2}} \\
&\qquad \cdot\left\{\int_{\mathcal{X}\times\mathcal{X}} |k|^2(x,y)(m\times m)(dx, dy)\right\}^{\frac{1}{2}} \\
&= \|f\|\,\|g\|\,\|\,|k|\,\|_{\mathcal{L}^2(\mathcal{X}\times\mathcal{X},\, m\times m)} \\
&= \|f\|\,\|g\|\,\|k\|_{\mathcal{L}^2(\mathcal{X}\times\mathcal{X},\, m\times m)} \\
&= \|f\|\,\|g\|\,\|T_\mu\|_{\mathcal{C}_2}.
\end{aligned}$$

Therefore, by Proposition 2.4, $|||T_\mu||| \leq \|T_\mu\|_{\mathcal{C}_2}$. ∎

Proposition 2.6 *If μ is a positive measure and induces a bounded operator T_μ in $\mathcal{P}$, and if ν is a measure on $\mathcal{X}\times\mathcal{X}$ such that $|\nu| \leq \mu$, then ν is an absolutely bounded kernel and induces a bounded operator T_ν in $\mathcal{P}$ with $|||T_\nu||| \leq |||T_\mu|||$.*

Proof. Since μ is a bounded kernel, we have that μ vanishes on all marginally null sets in $\mathcal{X}\times\mathcal{X}$. However, $|\nu| \leq \mu$. So ν also vanishes on all marginally null sets in $\mathcal{X}\times\mathcal{X}$.

For any f ang g in $\mathcal{L}^2(\mathcal{X}, m)$,

$$\begin{aligned}
\left|\int_{\mathcal{X}\times\mathcal{X}} f(y)\overline{g}(x)|\nu|(dx, dy)\right| &\leq \int_{\mathcal{X}\times\mathcal{X}} |f|(y)|g|(x)\mu(dx, dy) \\
&= \langle T_{|\mu|}f, g\rangle \\
&\leq \|T_{|\mu|}\|\,\|f\|\,\|g\|.
\end{aligned}$$

Therefore, the function h given by

$$h(x,y) = f(y)\overline{g}(x)$$

is in $\mathcal{L}^1(\mathcal{X}\times\mathcal{X}, |\nu|)$. By Lemma 1.20, ν induces an operator $T_\nu \in \mathcal{P}$.

It is easy to see that $|||T_\nu||| = \|T_{|\nu|}\| \leq \|T_\mu\| = |||T_\mu|||$. ∎

Suppose T is an operator in $\mathcal{P}$ with absolutely bounded kernel μ. Then μ is a complex measure on $\mathcal{X} \times \mathcal{X}$ such that $|\mu|$ induces a bounded operator $T_{|\mu|}$ on $\mathcal{L}^2(\mathcal{X}, m)$. By the Lebesgue-Radon-Nikodym Theorem (see [50] or [51]), μ has a unique decomposition

$$\mu = \mu_a + \mu_s,$$

where μ_a is absolutely continuous with respect to $m \times m$ (denoted by $\mu_a \ll m \times m$), and μ_s and $m \times m$ are mutually singular (denoted by $\mu_s \perp m \times m$). Consequently, both μ_a and μ_s are dominated by $|\mu|$, *i.e.*, $|\mu_a| \leq |\mu|$ and $|\mu_s| \leq |\mu|$. It follows from Proposition 2.6 that both μ_a and μ_s are absolutely bounded kernels on $\mathcal{X} \times \mathcal{X}$, and that $|||T_{\mu_a}||| \leq |||T_\mu|||$ and $|||T_{\mu_s}||| \leq |||T_\mu|||$.

It is clear from the definition of $\mathcal{I}$ in Theorem 1.23 that

$$\mathcal{I} = \{T \in \mathcal{P} : T = T_\mu \text{ for some kernel } \mu \text{ with } \mu_s = 0\}.$$

Let

$$\mathcal{P}_s = \{T \in \mathcal{P} : T = T_\mu \text{ for some kernel } \mu \text{ with } \mu_a = 0\}.$$

Then, we have the following result.

Theorem 2.7 *For any standard finite measure space $(\mathcal{X}, m)$,*

(i) $\mathcal{I}$ is a closed two-sided ideal in $(\mathcal{P}, |||\cdot|||)$.

(ii) $\mathcal{P}_s$ is a closed subspace in $(\mathcal{P}, |||\cdot|||)$.

(iii) $\mathcal{P} = \mathcal{I} \oplus \mathcal{P}_s$.

Proof. (i) By Theorem 1.23, $\mathcal{I}$ is a two-sided ideal in $\mathcal{P}$.

Suppose $\{T_{\mu_j}\}$ is a sequence in $\mathcal{I}$ that converges to $T_\mu \in \mathcal{P}$ in $(\mathcal{P}, |||\cdot|||)$. Then, by Proposition 2.3,

$$\|\mu_j - \mu\| = |\mu_j - \mu|(\mathcal{X} \times \mathcal{X}) \longrightarrow 0 \quad (j \longrightarrow \infty).$$

Since all T_{μ_j}, $(j = 1, 2, \cdots)$, are in $\mathcal{I}$, we have that all μ_j, $(j = 1, 2, \cdots)$, are absolutely continuous with respect to $m \times m$. Therefore, the measure μ, as the norm limit of $\{\mu_j\}$, is also absolutely continuous with respect to $m \times m$. Thus, T_μ is in $\mathcal{I}$.

(ii) It is easy to see from the definition that $\mathcal{P}_s$ is a linear manifold in $\mathcal{P}$. The proof of the fact that $\mathcal{P}_s$ is closed under the new norm is similar to the proof of (i) since the norm limit of a sequence of measures singular to $m \times m$ remains singular to $m \times m$.

(iii) By the above analysis, $\mathcal{P} = \mathcal{I} + \mathcal{P}_s$. Noticing the fact that $\mu \ll m \times m$ and $\mu \perp m \times m$ together imply $\mu = 0$ for any measure μ on $\mathcal{X} \times \mathcal{X}$, we have that $\mathcal{I} \cap \mathcal{P}_s = \{0\}$. ∎

We conclude this section by providing several interesting examples of operators in $\mathcal{P}$. When the measure space $(\mathcal{X}, m)$ is not purely atomic, the identity operator on $\mathcal{L}^2(\mathcal{X}, m)$ is not an integral operator. In this case, $\mathcal{P}_s \neq \{0\}$. In fact, $\mathcal{P}_s$ contains a large number of multiplication operators M_ϕ with $\phi \in \mathcal{L}^\infty(\mathcal{X}, m)$ and composition operators C_ψ with suitable maps ψ on $\mathcal{X}$. But these operators are clearly not compact. In [57], Sourour asked the following question: If T is a compact pseudo-integral operator, must T be an integral operator? The answer to this question is negative. Consider the best example of nonatomic measure space: the unit interval $[0, 1]$ with the Lebesgue measure. Fremlin provided a method in [19] to construct positive compact operators on $\mathcal{L}^2([0, 1])$ that are not integral operators. The method can be applied to other nonatomic measure spaces.

Example 2.8 *Let $\mathcal{X}$ be the unit interval $[0, 1]$ and m the Lebesgue measure on $\mathcal{X}$. Then there are non-zero positive compact operators in $\mathcal{P}_s$.*

Proof. Using Fremlin's method, we can construct a non-zero positive compact operator T on $\mathcal{L}^2(\mathcal{X}, m)$ that is not an integral operator. Therefore $T = T_\mu$ is in $\mathcal{P}$ for some absolutely bounded kernel μ and the singular part μ_s of μ is non-zero. Thus the operator T_{μ_s} induced by the kernel μ_s is a non-zero positive operator in $\mathcal{P}_s$, and dominated by T_μ, *i.e.*, $T_\mu - T_{\mu_s}$ is positive. However, T_μ is a compact operator. It follows from [1, Theorem 2.3] or [15, Theorem 4.5] that T_{μ_e} is also compact. ∎

Example 2.9 *Let $\mathcal{X}$ be the unit interval $[0,1]$ and m the Lebesgue measure on $\mathcal{X}$. Then there is a non-zero compact operator T in $\mathcal{P}_s$ such that neither T nor $-T$ is positive.*

Proof. From the previous example, it is not difficult to find two non-zero positive compact operators T_1 and T_2, $T_1 \neq T_2$, in $\mathcal{P}_s$ such that neither $T_1 - T_2$ nor $T_2 - T_1$ is positive. ∎

REMARK. As we have pointed out earlier, Fremlin's method of constructing non-zero positive compact operators that are not integral operators can be generalized to other nonatomic measure spaces. Therefore, the above examples can also be generalized to other measure spaces.

2.2 The Completeness of $(\mathcal{P}, \|\!|\cdot|\!\|)$

It follows from Corollary 1.19 that the algebra $\mathcal{P}$ coincides with the algebra generated by positive operators on $\mathcal{L}^2(\mathcal{X}, m)$. In [54] Schaefer studied the algebra generated by positive operators on a general Banach lattice and several results about the algebra were presented. Let $\mathcal{E}$ be an order complete complex Banach lattice. An operator on $\mathcal{E}$ is called *regular* if it is a linear combination of positive operators. Clearly, the set of all regular operators on $\mathcal{E}$ is an operator algebra and we denote it by $\mathcal{B}^r(\mathcal{E})$. Since $\mathcal{E}$ is order complete, an operator on $\mathcal{E}$ is regular if and only if it is order bounded, *i.e.*, maps order bounded subsets to order bounded subsets. Suppose T is a regular operator $\mathcal{E}$. Then, for every positive element x in $\mathcal{E}$, the supremum

$$|T|x = \sup\{|Tz| : |z| \leq x\}$$

is well-defined where $z \rightarrow |z|$ is the modulus function (see [54, Chapter II, Definition 11.1]). It was proved [54, p.234] that $|T|$ can be linearly extended into a positive (bounded) operator on $\mathcal{E}$. Schaefer proved the following equivalent definition of $|T|$.

Theorem 2.10 [54, Chapter IV, Theorem 1.8] *Let $\mathcal{E}$ be an order complete complex Banach lattice. If T is a regular operator on $\mathcal{E}$, then*

$$|T| = \sup_{0 \leq \theta \leq 2\pi} |(\cos\theta)T_1 + (\sin\theta)T_2|$$

where $T = T_1 + iT_2$ is the canonical decomposition of T.

Let $\mathcal{E}$ be an order complete complex Banach lattice. For every T in $\mathcal{B}^r(\mathcal{E})$, let

$$\|T\|_r = \|\,|T|\,\|.$$

We have the following theorem.

Theorem 2.11 [54, Chapter IV, Corollary 2] *Let $\mathcal{E}$ be an order complete complex Banach lattice. Then $(\mathcal{B}^r(\mathcal{E}), |\cdot|)$ is a complex Banach lattice and $\mathcal{B}^r(\mathcal{E})$ is a complex Banach algebra under the norm $\|\cdot\|_r$.*

For the special case where $\mathcal{E} = \mathcal{L}^2(\mathcal{X}, m)$, it is clear that $\mathcal{E}$ is order complete and $\mathcal{B}^r(\mathcal{E}) = \mathcal{P}$. Let T_μ be an arbitrary operator in $\mathcal{P}$. It follows from the definition that $|T_\mu| \leq T_{|\mu|}$.

Proposition 2.12 $|T_\mu| = T_{|\mu|}$ *for any T_μ in $\mathcal{P}$.*

Proof. It suffices to show that $|T_\mu| \geq T_{|\mu|}$. Since $|T_\mu|$ is a positive operator, $|T_\mu| = T_\nu$ for some positive kernel ν. Therefore it suffices to show that $\nu \geq |\mu|$.

Let $T = T_1 + iT_2$ is the canonical decomposition of T_μ and let $\mu = \mu_1 + i\mu_2$ be the canonical decomposition of μ. Then $T_j = T_{\mu_j}$, $(j = 1, 2)$. Fix any $\theta \in [0, 2\pi]$ and consider the measure $(\cos\theta)\mu_1 + (\sin\theta)\mu_2$. For any measurable rectangle $E\times F$, we have

$$|[(\cos\theta)\mu_1 + (\sin\theta)\mu_2](E\times F)| = |\langle[(\cos\theta)T_1 + (\sin\theta)T_2]\chi_F, \chi_E\rangle|,$$

and therefore, is less than or equal to $\langle T_\nu \chi_F, \chi_E\rangle = \nu(E\times F)$ by Theorem 2.10. Thus

$$|(\cos\theta)\mu_1(G) + (\sin\theta)\mu_2(G)| \leq \nu(G)$$

for any finite union G of measurable rectangles and then for any Borel subset G of $\mathcal{X} \times \mathcal{X}$.

For any Borel subset G of $\mathcal{X} \times \mathcal{X}$ and any partition $\{G_j\}$ of G, there exists a sequence of positive numbers $\{\theta_j\}$ in $[0, 2\pi]$ such that

$$\mu(G_j) = (\cos\theta_j)\mu_1(G_j) + (\sin\theta_j)\mu_2(G_j)$$

for all positive integers j. It follows that

$$\sum |\mu(G_j)| = \sum |(\cos\theta_j)\mu_1(G_j) + (\sin\theta_j)\mu_2(G_j)| \leq \sum \nu(G_j) = \nu(G).$$

Hence $|\mu|(G) \leq \nu(G)$ and then $|\mu| \leq \nu$. ∎

It follows immediately from Proposition 2.12 that

$$|||T_\mu||| = \|T_{|\mu|}\| = \|T\|_r$$

for all $T \in \mathcal{P}$. Consequently, $(\mathcal{P}, |||\cdot|||)$ is a Banach algebra by Theorem 2.11.

In this section, we use classical methods of functional analysis to prove that $(\mathcal{P}, |||\cdot|||)$ is a Banach $*$-algebra.

Suppose f is a positive simple function in $\mathcal{L}^2(\mathcal{X}, m)$. Then $f = \sum_{j=1}^{s} \alpha_j \chi_{E_j}$ for some integer $s \geq 1$, where α_j is a non-negative number for all $j = 1, 2, \ldots, s$ and $\{E_j\}_{j=1}^{s}$ is a *Borel partition* of $\mathcal{X}$, *i.e.*, $E_1, E_2, \ldots, E_s$ are pairwise disjoint Borel subsets of $\mathcal{X}$ whose union is equal to $\mathcal{X}$. Thus, $\|f\| = [\sum_{j=1}^{s} \alpha_j^2 m(E_j)]^{\frac{1}{2}}$.

If $f = \sum_{j=1}^{s} \alpha_j \chi_{E_j}$ and $g = \sum_{j=1}^{s'} \beta_j \chi_{F_j}$ are two such positive simple functions, then, for any T_μ in $\mathcal{P}$, we have

$$\begin{aligned} \langle T_{|\mu|} f, g \rangle &= \sum_{j=1}^{s} \sum_{l=1}^{s'} \alpha_j \beta_l \langle T_{|\mu|} \chi_{E_j}, \chi_{F_l} \rangle \\ &= \sum_{j=1}^{s} \sum_{l=1}^{s'} \alpha_j \beta_l |\mu| (F_l \times E_j). \end{aligned}$$

Hence, if $\|f\| \neq 0$, and $\|g\| \neq 0$, then

$$\frac{\sum_{j=1}^{s} \sum_{l=1}^{s'} \alpha_j \beta_l |\mu| (F_l \times E_j)}{[\sum_{j=1}^{s} \alpha_j^2 m(E_j)]^{\frac{1}{2}} [\sum_{l=1}^{s'} \beta_l^2 m(F_l)]^{\frac{1}{2}}} = \frac{\langle T_{|\mu|} f, g \rangle}{\|f\| \, \|g\|} \leq \|T_{|\mu|}\| = |||T_\mu|||.$$

Let Γ be the set consisting of all the pairs $(\{\alpha_j\}_{j=1}^s, \{E_j\}_{j=1}^s)$, where $s \geq 1$ is an integer, all α_j non-negative numbers and $\{E_j\}_{j=1}^s$ a Borel partition of $\mathcal{X}$ such that $\sum_{j=1}^s \alpha_j^2 m(E_j) \neq 0$. We are now ready to prove the following equivalent definition of the norm $|||\cdot|||$. We will use this definition later to show that $|||\cdot|||$ is a complete norm on $\mathcal{P}$.

Lemma 2.13 *Suppose T_μ is in $\mathcal{P}$ with absolutely bounded kernel μ. Then, $|||T_\mu|||$ is the supremum of*

$$\frac{\sum_{j=1}^s \sum_{l=1}^{s'} \alpha_j \beta_l |\mu|(F_l \times E_j)}{[\sum_{j=1}^s \alpha_j^2 m(E_j)]^{\frac{1}{2}} [\sum_{l=1}^{s'} \beta_l^2 m(F_l)]^{\frac{1}{2}}}$$

for all $(\{\alpha_j\}_{j=1}^s, \{E_j\}_{j=1}^s)$ and $(\{\beta_j\}_{j=1}^{s'}, \{E_j\}_{j=1}^{s'})$ in Γ.

Proof. It follows immediately from Proposition 2.4 and the above analysis. ∎

Theorem 2.14 $(\mathcal{P}, |||\cdot|||)$ *is a complex Banach space.*

Proof. It suffices to show that $(\mathcal{P}, |||\cdot|||)$ is complete.

Suppose $\{T_{\mu_j}\}_{j=1}^\infty$ is a Cauchy sequence in $(\mathcal{P}, |||\cdot|||)$. By Propositions 2.3 and 2.5,

$$\|T_{\mu_j} - T_{\mu_l}\| \leq |||T_{\mu_j} - T_{\mu_l}|||,$$

$$\|T_{|\mu_j|} - T_{|\mu_l|}\| \leq \|T_{|\mu_j - \mu_l|}\| = |||T_{\mu_j} - T_{\mu_l}|||,$$

$$\|\mu_j - \mu_l\| = |\mu_j - \mu_l|(\mathcal{X} \times \mathcal{X}) \leq m(\mathcal{X})|||T_{\mu_j} - T_{\mu_l}|||,$$

for all $j, l = 1, 2, \cdots$. Thus, $\{T_{|\mu_j|}\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ with the operator norm, and $\{\mu_j\}_{j=1}^\infty$ is a Cauchy sequence in the Banach space of all complex measures on $\mathcal{X} \times \mathcal{X}$. It follows that there exists an operator T in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ and a Borel measure μ on $\mathcal{X} \times \mathcal{X}$ such that

$$\|T_{|\mu_j|} - T\| \longrightarrow 0 \quad (j \longrightarrow \infty),$$

$$\|\mu_j - \mu\| = |\mu_j - \mu|(\mathcal{X} \times \mathcal{X}) \longrightarrow 0 \quad (j \longrightarrow \infty).$$

But, for every integer j,

$$\| |\mu_j| - |\mu| \| = | |\mu_j| - |\mu| |(\mathcal{X} \times \mathcal{X}) \leq |\mu_j - \mu|(\mathcal{X} \times \mathcal{X}).$$

Hence

$$\| |\mu_j| - |\mu| \| \longrightarrow 0 \quad (j \longrightarrow \infty).$$

Since all $T_{|\mu_j|}$ are positive operators, we have that T, as the norm limit of a sequence of positive operators, is also a positive operator. By Theorem 1.18, T is induced by a positive kernel ν, *i.e.*, $T = T_\nu$. We claim that $\nu = |\mu|$. Indeed, for any measurable rectangle $E \times F$ in $\mathcal{X} \times \mathcal{X}$,

$$\begin{aligned} |\mu|(E \times F) &= \lim_{j \to \infty} |\mu_j|(E \times F) \\ &= \lim_{j \to \infty} \langle T_{|\mu_j|} \chi_F, \chi_E \rangle \\ &= \langle T_\nu \chi_F, \chi_E \rangle \\ &= \nu(E \times F). \end{aligned}$$

Thus, $\nu = |\mu|$ since both ν and $|\mu|$ are Borel measures on $\mathcal{X} \times \mathcal{X}$.

Now, $|\mu|$ is a bounded kernel on $\mathcal{X} \times \mathcal{X}$, therefore, by Proposition 2.6, μ itself is an absolutely bounded kernel on $\mathcal{X} \times \mathcal{X}$ and induces an operator $T_\mu \in \mathcal{P}$.

We complete the proof by showing that

$$|||T_{\mu_j} - T||| \longrightarrow 0 \quad (j \longrightarrow \infty).$$

For any $\epsilon > 0$, there exists an integer $N > 0$ such that

$$|||T_{\mu_j} - T_{\mu_l}||| < \frac{\epsilon}{2} \quad \text{whenever } j, l \geq N.$$

Fix any two non-zero positive simple functions $f = \sum_{j=1}^{s} \alpha_j \chi_{E_j}$ and $g = \sum_{j=1}^{s'} \beta_j \chi_{F_j}$ with $\{E_j\}_{j=1}^{s}$ and $\{F_j\}_{j=1}^{s'}$ two Borel partitions of $\mathcal{X}$. For all integers p and q with $p, q > N$,

$$\frac{\sum_{j=1}^{s} \sum_{l=1}^{s'} \alpha_j \beta_l |\mu_p - \mu|(F_l \times E_j)}{[\sum_{j=1}^{s} \alpha_j^2 m(E_j)]^{\frac{1}{2}} [\sum_{l=1}^{s'} \beta_l^2 m(F_l)]^{\frac{1}{2}}}$$

is less than or equal to

$$\frac{\sum_{j=1}^{s}\sum_{l=1}^{s'}\alpha_j\beta_l|\mu_p-\mu_q|(F_l\times E_j)}{[\sum_{j=1}^{s}\alpha_j^2 m(E_j)]^{\frac{1}{2}}[\sum_{l=1}^{s'}\beta_l^2 m(F_l)]^{\frac{1}{2}}}+\frac{\sum_{j=1}^{s}\sum_{l=1}^{s'}\alpha_j\beta_l|\mu_q-\mu|(F_l\times E_j)}{[\sum_{j=1}^{s}\alpha_j^2 m(E_j)]^{\frac{1}{2}}[\sum_{l=1}^{s'}\beta_l^2 m(F_l)]^{\frac{1}{2}}}.$$

By Lemma 2.13, the first term of the above is less than or equal to $|||T_{\mu_p}-T_{\mu_q}|||$, and therefore, less than $\frac{\epsilon}{2}$ since $p,q>N$. However,

$$|\mu_j-\mu|(\mathcal{X}\times\mathcal{X})\longrightarrow 0 \quad (j\longrightarrow\infty).$$

We can choose q so large that the second term is also less than $\frac{\epsilon}{2}$.

Hence , for any $p>N$,

$$\frac{\sum_{j=1}^{s}\sum_{l=1}^{s'}\alpha_j\beta_l|\mu_p-\mu|(F_l\times E_j)}{[\sum_{j=1}^{s}\alpha_j^2 m(E_j)]^{\frac{1}{2}}[\sum_{l=1}^{s'}\beta_l^2 m(F_l)]^{\frac{1}{2}}}<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

It follows from Lemma 2.13 that

$$|||T_{\mu_p}-T_\mu|||=|||T_{\mu_p-\mu}|||<\epsilon$$

whenever $p>N$. Thus,

$$|||T_{\mu_j}-T|||\longrightarrow 0 \quad (j\longrightarrow\infty).$$

■

Theorem 2.15 *Suppose T and S are in $\mathcal{P}$. Then $|||TS|||\le|||T|||\,|||S|||$ and $|||T^*|||=|||T|||$. Therefore, $(\mathcal{P},|||\cdot|||)$ is a Banach $*$-algebra.*

Proof. Let μ,ν be the absolutely bounded kernels of T and S respectively. Then, by Theorem 1.22, $\mu*\nu$ is the absolutely bounded kernel of TS, where $\mu*\nu$ is given by

$$(\mu*\nu)(G)=\int_{\mathcal{X}}(\mu_z\times\nu^z)(G)m(dz)$$

for every Borel set G in $\mathcal{X} \times \mathcal{X}$. It follows from the essential uniqueness of the disintegration of measures on $\mathcal{X} \times \mathcal{X}$ that

$$\begin{aligned}
|(\mu * \nu)(G)| &\leq \int_{\mathcal{X}} |\mu_z \times \nu^z|(G) m(dz) \\
&\leq \int_{\mathcal{X}} (|\mu_z| \times |\nu^z|)(G) m(dz) \\
&= \int_{\mathcal{X}} (|\mu|_z \times |\nu|^z)(G) m(dz) \\
&= (|\mu| * |\nu|)(G),
\end{aligned}$$

for every Borel set G in $\mathcal{X} \times \mathcal{X}$. Therefore, $|\mu * \nu| \leq |\mu| * |\nu|$ as measures on $\mathcal{X} \times \mathcal{X}$. Hence,

$$|||TS||| = ||T_{|\mu * \nu|}|| \leq ||T_{|\mu| * |\nu|}|| = ||T_{|\mu|} T_{|\nu|}|| \leq ||T_{|\mu|}|| \, ||T_{|\nu|}|| = |||T||| \, |||S|||.$$

By Theorem 1.21, $T_\mu^* = T_{\overline{\mu}}$ where $\overline{\mu}$ is the complex conjugate of μ. It follows from the essential uniqueness of the disintegration of measures that $|\overline{\mu}| = \overline{|\mu|}$. Therefore,

$$|||T^*||| = |||T_\mu^*||| = |||T_{\overline{\mu}}||| = ||T_{|\overline{\mu}|}|| = ||T_{\overline{|\mu|}}|| = ||T_{|\mu|}^*|| = ||T_{|\mu|}|| = |||T|||.$$

■

REMARK. Generally, the Banach $*$-algebra $(\mathcal{P}, ||| \cdot |||)$ is not a C^*-algebra. For example, in the case where $\mathcal{L}^2(\mathcal{X}, m)$ is the two dimensional space C^2, let

$$T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then $T^*T = 2I$ and hence $|||T^*T||| = 2$. However, $|||T|||^2 = 4$. Thus $|||T^*T||| \neq |||T|||^2$, and consequently, $(\mathcal{P}, ||| \cdot |||)$ is not a C^*-algebra.

2.3 Spectral Properties of Elements of $(\mathcal{P}, ||| \cdot |||)$

Having shown that $(\mathcal{P}, ||| \cdot |||)$ is a Banach algebra, we find it interesting to look at some spectral properties of operators as elements of $(\mathcal{P}, ||| \cdot |||)$.

Theorem 2.16 *The set $\mathcal{K} \cap \mathcal{P}$ is a closed two-sided ideal in $(\mathcal{P}, |||\cdot|||)$, where $\mathcal{K}$ is the algebra of all compact operators on $\mathcal{L}^2(\mathcal{X}, m)$.*

Proof. Clearly, $\mathcal{K} \cap \mathcal{P}$ is a two-sided ideal in $\mathcal{P}$. By Proposition 2.5, $\mathcal{K} \cap \mathcal{P}$ is closed in $(\mathcal{P}, |||\cdot|||)$. ∎

Suppose T is in $\mathcal{P}$. Let $\sigma_{\mathcal{P}}(T)$ denote the spectrum of T in the Banach algebra $(\mathcal{P}, |||\cdot|||)$. It is easy to see that $\sigma(T) \subseteq \sigma_{\mathcal{P}}(T)$. Also, we denote $r_{\mathcal{P}}(T)$ the spectral radius of T as an element in the Banach algebra $(\mathcal{P}, |||\cdot|||)$, that is

$$r_{\mathcal{P}}(T) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{P}}(T)\}.$$

When we consider $\mathcal{L}^2(\mathcal{X}, m)$ as a Banach lattice, the spectrum $\sigma_{\mathcal{P}}(\cdot)$ is the same as the *order spectrum* introduced by Schaefer in [55]. The order spectrum of regular operators on Banach lattices was also studied by Arendt and Sourour in [4] and [5]. The main results of this section, Proposition 2.18 and Theorem 2.22, are the special cases of [55, Theorem 3.3] and [5, Theorem 4.4]. The proofs we provide here can be easily understood by those unfamiliar with the theory of Banach lattices.

Proposition 2.17 *If $T \in \mathcal{P}$ is a positive operator, then $r_{\mathcal{P}}(T) = r(T)$.*

Proof. Since T is positive, we have T^p is positive for all positive integers p. By definition, $|||T^p||| = \|T^p\|$ for all positive integers p. Therefore $r_{\mathcal{P}}(T) = r(T)$. ∎

Proposition 2.18 *For every $\phi \in \mathcal{L}^\infty(\mathcal{X}, m)$, $\sigma_{\mathcal{P}}(M_\phi) = \sigma(M_\phi)$.*

Proof. Suppose M_ϕ is invertible in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$. Then its inverse is also an multiplication operator, and hence, is belong to $\mathcal{P}$. It follows that M_ϕ is invertible in $\mathcal{P}$. ∎

Lemma 2.19 *If W is in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ and $\lambda \notin \sigma(W)$, $\lambda \neq 0$, then*

$$(\lambda - W)^{-1} = \frac{1}{\lambda}\left[1 + W(\lambda - W)^{-1}\right].$$

Proof. It suffices to show that $\frac{1}{\lambda}[1+W(\lambda-W)^{-1}]$ is a left inverse of $\lambda - W$. Indeed,

$$\begin{aligned} & \frac{1}{\lambda}\left[1+W(\lambda-W)^{-1}\right](\lambda-W) \\ = & \frac{1}{\lambda}[(\lambda-W)+W] \\ = & I. \end{aligned}$$

■

Proposition 2.20 *Suppose T is a Hilbert-Schmidt operator on $\mathcal{L}^2(\mathcal{X}, m)$.*

(i) If S is a nilpotent operator that belongs to $\mathcal{P}$, then

$$\sigma_{\mathcal{P}}(T+S) = \sigma(T+S).$$

(ii) If S is a quasinilpotent operator and positive, then

$$\sigma_{\mathcal{P}}(T+S) = \sigma(T+S).$$

Proof. (i) Let $\lambda \neq 0$ and $\lambda \notin \sigma(T+S)$. Then $\lambda - (T+S)$ is invertible in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ and

$$\begin{aligned} [\lambda-(T+S)]^{-1} &= [(\lambda-S)-T]^{-1} \\ &= \left\{(\lambda-S)\left[1-(\lambda-S)^{-1}T\right]\right\}^{-1} \\ &= \left[1-(\lambda-S)^{-1}T\right]^{-1}(\lambda-S)^{-1}. \end{aligned}$$

By Lemma 2.19,

$$\left[1-(\lambda-S)^{-1}T\right]^{-1} = I + (\lambda-S)^{-1}T\left[1-(\lambda-S)^{-1}T\right]^{-1},$$

which is in $\mathcal{P}$ since T and then the operator

$$(\lambda-S)^{-1}T\left[1-(\lambda-S)^{-1}T\right]^{-1}$$

is in $\mathcal{C}_2$.

However, there exists a positive integer p such that $S^{p+1} = 0$ because S is a nilpotent operator. Therefore

$$(\lambda - S)^{-1} = \sum_{j=0}^{p} \frac{S^j}{\lambda^{j+1}}.$$

Since S is in $\mathcal{P}$, it follows that $(\lambda - S)^{-1}$ is also in $\mathcal{P}$. Thus, $[\lambda - (T+S)]^{-1}$ is in $\mathcal{P}$. This implies that $\lambda \notin \sigma_{\mathcal{P}}(T+S)$, and hence,

$$\sigma_{\mathcal{P}}(T+S) \subseteq \sigma(T+S) \cup \{0\}.$$

Since $S^{p+1} = 0$, we have $(T+S)^{p+1} \in \mathcal{C}_2$. Therefore, $0 \in \sigma((T+S)^{p+1})$, and hence, $0 \in \sigma(T+S)$. It follows that $\sigma_{\mathcal{P}}(T+S) = \sigma(T+S)$.

(ii) Since S is quasinilpotent and positive, we have $r_{\mathcal{P}}(S) = r(S) = 0$. Therefore, for any non-zero complex number λ,

$$(\lambda - S)^{-1} = \sum_{j=0}^{\infty} \frac{S^j}{\lambda^{j+1}}$$

is in $\mathcal{P}$.

In a way similar to the proof of (i), we can prove that $\lambda - (T+S)$ is invertible in $\mathcal{P}$ for any non-zero complex number $\lambda \notin \sigma(T+S)$. It follows that

$$\sigma_{\mathcal{P}}(T+S) \subseteq \sigma(T+S) \cup \{0\}.$$

However, since S is a quasinilpotent operator, we have that zero is in the essential spectrum $\sigma_e(S)$ of S. Therefore $0 \in \sigma(T+S)$ because T is compact. Thus $\sigma_{\mathcal{P}}(T+S) = \sigma(T+S)$. ∎

We have shown that $\sigma(S) \subseteq \sigma_{\mathcal{P}}(S)$ for all $S \in \mathcal{P}$, and that for certain $S \in \mathcal{P}$, namely nilpotent or positive quasinilpotent operators, $\sigma(S+T) = \sigma_{\mathcal{P}}(S+T)$ for any Hilbert-Schmidt operator T. Naturally, one may wonder how large, compared with $\sigma(S)$, $\sigma_{\mathcal{P}}(S)$ can be for a given S in $\mathcal{P}$. The following results, which generalize the above proposition, provide an answer to this question.

Lemma 2.21 *Let $S \in \mathcal{P}$. Then*

$$\sigma_{\mathcal{P}}(S+T) \subseteq \sigma(S+T) \cup \sigma_{\mathcal{P}}(S)$$

for any Hilbert-Schmidt operator T.

Proof. Fix an arbitrary Hilbert-Schmidt operator T. Suppose λ is a complex number and not in $\sigma(S+T) \cup \sigma_{\mathcal{P}}(S)$. Then $\lambda - (S+T)$ is invertible in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$. It is enough to show that $[\lambda - (S+T)]^{-1}$ is in $\mathcal{P}$.

Since $\lambda \notin \sigma_{\mathcal{P}}(S)$, $\lambda - S$ is invertible in $\mathcal{P}$, *i.e.*, $(\lambda - S)^{-1}$ is in $\mathcal{P}$. Therefore,

$$\begin{aligned} [\lambda - (T+S)]^{-1} &= [(\lambda - S) - T]^{-1} \\ &= \left\{(\lambda - S)\left[1 - (\lambda - S)^{-1}T\right]\right\}^{-1} \\ &= \left[1 - (\lambda - S)^{-1}T\right]^{-1}(\lambda - S)^{-1}. \end{aligned}$$

However, the fact that T is a Hilbert-Schmidt operator implies that $(\lambda - S)^{-1}T$ is also a Hilbert-Schmidt operator. Consequently, $[1 - (\lambda - S)^{-1}T]^{-1} = 1 + D$ for some Hilbert-Schmidt operator D. Thus, $[1 - (\lambda - S)^{-1}T]^{-1}$ is in $\mathcal{P}$, and hence, $[\lambda - (S+T)]^{-1}$ is in $\mathcal{P}$ as well. ■

Theorem 2.22 *For every $S \in \mathcal{P}$,*

$$\sigma_{\mathcal{P}}(S) = \sigma(S) \bigcup \left[\bigcap_{T \in \mathcal{C}_2} \sigma_{\mathcal{P}}(S+T)\right].$$

Proof. Let S be an arbitrary element of $\mathcal{P}$. It follows from Lemma 2.21 that, for every Hilbert-Schmidt operator T,

$$\begin{aligned} \sigma_{\mathcal{P}}(S) &= \sigma_{\mathcal{P}}((S+T) + (-T)) \\ &\subseteq \sigma((S+T) + (-T)) \cup \sigma_{\mathcal{P}}(S+T) \\ &= \sigma(S) \cup \sigma_{\mathcal{P}}(S+T). \end{aligned}$$

Thus,

$$\begin{aligned}\sigma_{\mathcal{P}}(S) &\subseteq \bigcap_{T\in\mathcal{C}_2} [\sigma(S)\cup\sigma_{\mathcal{P}}(S+T)]\\ &= \sigma(S)\bigcup\left[\bigcap_{T\in\mathcal{C}_2}\sigma_{\mathcal{P}}(S+T)\right].\end{aligned}$$

On the other hand, since $\sigma_{\mathcal{P}}(S)\supseteq\sigma(S)$ and $0\in\mathcal{C}_2$, we have

$$\sigma_{\mathcal{P}}(S)\supseteq\sigma(S)\bigcup\left[\bigcap_{T\in\mathcal{C}_2}\sigma_{\mathcal{P}}(S+T)\right].$$

Hence these two sets are equal. ∎

It was proved ([4] and [5, A2]) that, in the case where $\mathcal{X}$ is the unit circle of the complex plane and m is the Lebesgue measure on $\mathcal{X}$, there exists a positive, compact, selfadjoint operator T in $\mathcal{P}$ such that $\sigma_{\mathcal{P}}(T)$ contains the unit circle. Thus $\sigma_{\mathcal{P}}(T)\neq\sigma(T)$. We finish this section with the following question. An affirmative answer to the question will guarantee the existence of non-trivial invariant subspaces, or even the triangularizability, of certain semigroups of positive operators on $\mathcal{L}^2(\mathcal{X},m)$.

Question 1. Is it true that $\sigma_{\mathcal{P}}(S)$ is not a connected set in the complex plane for (i) every compact non-quasinilpotent operator S in $\mathcal{P}$? (ii) every compact positive non-quasinilpotent operator S in $\mathcal{P}$? (iii) every compact positive non-quasinilpotent operator S with the property that the semigroup generated by S is contained in $\mathcal{P}_s$?

Chapter 3

Positive Integral Idempotents

The purpose of this chapter is to study the ranges of positive idempotent integral operators through analyzing their kernels. For a given positive integral idempotent, we construct a basis of its range consisting of positive elements of $\mathcal{L}^2(\mathcal{X}, m)$. This kind of special bases for the ranges of positive integral idempotents will be used in Chapter 4 to obtain a theorem establishing the existence of non-trivial invariant subspaces for certain semigroups of positive integral operators.

In this chapter, we always assume that A is a positive idempotent integral operator with kernel a and the rank of A is equal to s (s could be $+\infty$). We will frequently use the fact that

$$\int_{\mathcal{X}} f(x)m(dx) = 0$$

for a non-negative measurable function f if and only if $f = 0$ *a.e.* on $\mathcal{X}$.

3.1 Positive Integral Idempotents

We first look at some general results about positive integral idempotents. Suppose f is an element in $\mathcal{L}^2(\mathcal{X}, m)$. We say that f is *real* if one of its representations is a real function.

Lemma 3.1 *A^* is also a positive integral idempotent with kernel a^*.*

Proof. Clearly, A^* is an idempotent. It follows immediately from Theorem 1.21 that A^* is a positive integral operator with kernel a^*. ∎

Lemma 3.2 *Suppose A is an idempotent with finite-rank s. If $\{u_j\}_{j=1}^s$ is an orthonormal basis of the range of A, then*

$$A = \sum_{j=1}^{s} u_j \otimes (A^* u_j),$$

or equivalently, the kernel a of A is given by

$$a(x, y) = \sum_{j=1}^{s} u_j(x) \overline{A^* u_j}(y)$$

for almost every (x, y) in $\mathcal{X} \times \mathcal{X}$.

Proof. For any $f \in \mathcal{L}^2(\mathcal{X}, m)$, since $\{u_j\}_{j=1}^s$ is an orthonormal basis of the range of A, we have

$$\begin{aligned} Af &= \sum_{j=1}^{s} \langle Af, u_j \rangle u_j \\ &= \sum_{j=1}^{s} \langle f, A^* u_j \rangle u_j \\ &= \left[\sum_{j=1}^{s} u_j \otimes (A^* u_j) \right] f \end{aligned}$$

Therefore, $A = \sum_{j=1}^{s} u_j \otimes (A^* u_j)$. ∎

Lemma 3.3 *Suppose $U \subseteq \mathcal{X}$ is a measurable set such that $m(U)m(U^c) \neq 0$. If $a = 0$* a.e. *in $U^c \times U$, and a is non-zero on a subset of $U \times U$ of positive measure, then there exists a non-zero element u in $\mathcal{L}^2(\mathcal{X}, m)$ satisfying*

(*i*) *u is a positive element in $\mathcal{L}^2(\mathcal{X}, m)$,* *(ii)* $u = 0$ *a.e. on* U^c,

(iii) u *is in the range of* A, *i.e.,* $Au = u$.

Proof. Recall that P_U is the orthogonal projection of $\mathcal{L}^2(\mathcal{X}, m)$ onto $\mathcal{M}_U \equiv \chi_U \mathcal{L}^2(\mathcal{X}, m)$. We identify $\mathcal{M}_U$ with $\mathcal{L}^2(U, m)$ in the usual way. Therefore, by Proposition 1.5, $P_U \mathcal{L}^2(\mathcal{X}, m)$ is invariant under A, and

$$P_U A \Big|_{\mathcal{L}^2(U,m)} : \quad \mathcal{L}^2(U, m) \longrightarrow \mathcal{L}^2(U, m)$$

is an integral operator with non-negative kernel $a(x, y)$, $(x, y) \in U \times U$. Since a is non-zero on a subset of $U \times U$ of positive measure, $P_U A|_{\mathcal{L}^2(U,m)}$ is a non-zero idempotent. Therefore, there exists a positive element w in $\mathcal{L}^2(U, m)$ such that $P_U A|_{\mathcal{L}^2(U,m)} w \neq 0$. Let

$$w' = P_U A|_{\mathcal{L}^2(U,m)} w.$$

Then

$$P_U A|_{\mathcal{L}^2(U,m)} w' = w'.$$

Define a positive element u in $\mathcal{L}^2(\mathcal{X}, m)$ as follow:

$$u(x) = \begin{cases} w'(x) & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

Clearly, u is non-zero, and satisfies (i), (ii) and (iii). ∎

Lemma 3.4 *Suppose* $u \neq 0$ *is a positive element in* $\mathcal{L}^2(\mathcal{X}, m)$*that belongs to the range of* A. *Fix a non-negative representation of* u *(and still denote it by* u*). If* U *is the measurable subset of* $\mathcal{X}$ *given by*

$$U = \{x \in \mathcal{X} : u(x) \neq 0\},$$

then $a(x, y) = 0$ *for almost every* (x, y) *in* $U^c \times U$.

Proof. Since u is in the range of A, $(Au)(x) = u(x) = 0$ for almost every $x \in U^c$. It follows that, for almost every $x \in U^c$,

$$\int_{\mathcal{X}} a(x,y)u(y)m(dy) = 0.$$

But $a(x,\cdot)$ and $u(\cdot)$ are non-negative functions, and we have that $a(x,y)u(y) = 0$ for almost every y in $\mathcal{X}$. Therefore, $a(x,\cdot)$ is zero almost everywhere on U. Hence, by Fubuni's Theorem, $a(x,y) = 0$ for almost every (x,y) in $U^c \times U$. ∎

Lemma 3.5 *If an element u in the range of A is real, then there exists a positive element h in $\mathcal{L}^2(\mathcal{X},m)$ such that $Ah = 0$, and $u^+ + h$ and $u^- + h$ are in the range of A, where $u^+ = \frac{1}{2}(|u| + u)$ and $u^- = \frac{1}{2}(|u| - u)$ are positive and negative parts of u.*

Proof. By the definitions of u^+ and u^-, we have that

$$u^+ - u^- = u = Au = A(u^+ - u^-) = Au^+ - Au^-,$$

and Au^+ and Au^- are positive. So, if we let

$$h = Au^+ - u^+ = Au^- - u^-,$$

then $Ah = A(Au^+ - u^+) = 0$, and h is positive since u^+ and u^- are the minimum among all pairs of positive elements ϕ and ψ with the property that $u = \phi - \psi$. It follows that

$$A(u^+ + h) = Au^+ + Ah = u^+ + h,$$

and

$$A(u^- + h) = Au^- + Ah = u^- + h.$$

This means that $u^+ + h$ and $u^- + h$ are in the range of A. ∎

Lemma 3.6 *If an element $u \in \mathcal{L}^2(\mathcal{X},m)$ belongs to the range of A, then the real part* $\mathrm{Re}\, u$ *and imaginary part* $\mathrm{Im}\, u$ *of u are also in the range of A.*

Proof. Since A sends positive elements to positive elements in $\mathcal{L}^2(\mathcal{X},m)$, A also sends real elements to real elements. Therefore, $A(\mathrm{Re}\, u) = \mathrm{Re}\, Au = \mathrm{Re}\, u$ and $A(\mathrm{Im}\, u) = \mathrm{Im}\,(Au) = \mathrm{Im}\, u$. It follows that $\mathrm{Re}\, u$ and $\mathrm{Im}\, u$ are in the range of A. ∎

Lemma 3.7 *There exists an orthonormal basis of the range of A consisting of real elements of $\mathcal{L}^2(\mathcal{X}, m)$.*

Proof. Let $\{u_j\}_{j=1}^{s}$ be an arbitrary orthonormal basis of the range of A. For every j, it follows from Lemma 3.6 that both $\operatorname{Re} u_j$ and $\operatorname{Im} u_j$ are in the range of A. Hence, the range of A is the span of the set

$$\{\operatorname{Re} u_j, \operatorname{Im} u_j\}_{j=1}^{s}.$$

Using the Gram-Schmidt process, we can obtain an orthonormal basis of the range of A consisting of real elements of $\mathcal{L}^2(\mathcal{X}, m)$. ∎

3.2 Bases of Ranges of Positive Integral Idempotents

In this section, we will construct, for a given positive integral idempotent, a basis of its range that consists of positive elements of $\mathcal{L}^2(\mathcal{X}, m)$. The construction is based on the analysis of the kernel of the given positive integral idempotent. We will use freely the results about positive integral idempotents proved in the previous section.

Recall that throughout this chapter A will denote a positive integral idempotent induced by kernel a and the rank of A is equal to s (s could be $+\infty$).

Lemma 3.8 *Suppose $T \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ is an integral operator with non-negative kernel k. If h is a positive element in $\mathcal{L}^2(\mathcal{X}, m)$, and $Th = 0$, then $k(x, y) = 0$ for almost every (x, y) in $\mathcal{X} \times U_h$, where U_h is the measurable subset of $\mathcal{X}$ defined as follows: fix a representation of h and*

$$U_h = \{x \in \mathcal{X} : h(x) \neq 0\}.$$

Proof. The proof is similar to the proof of Lemma 3.4. ∎

Lemma 3.9 *If X_0 and X_1 are measurable subsets of $\mathcal{X}$ and $a = 0$ a.e. on both $\mathcal{X}\times X_0$ and $X_0^c\times X_1$, then $a = 0$ a.e. on $\mathcal{X}\times(X_0\cup X_1)$.*

Proof. Since $A^2 = A$, we have that

$$a(x,y) = \int_{\mathcal{X}} a(x,t)a(t,y)m(dt)$$

for almost every (x,y) in $\mathcal{X}\times\mathcal{X}$. Therefore, for almost every $(x,y)\in\mathcal{X}\times X_1$,

$$a(x,y) = \int_{\mathcal{X}} a(x,t)a(t,y)m(dt) = 0.$$

This means that $a = 0$ *a.e.* on $\mathcal{X}\times X_1$ and hence on $\mathcal{X}\times(X_0\cup X_1)$. ∎

Lemma 3.10 *If A is a positive integral idempotent of rank one, then there exist disjoint measurable subsets X_0, X_1 of $\mathcal{X}$ with $m(X_1) > 0$ such that*

(*i*) $a = 0$ a.e. *on* $\mathcal{X}\times X_0$,

(*ii*) $a = 0$ a.e. *on* $(X_0\cup X_1)^c\times\mathcal{X}$,

(*iii*) $a(x,y) > 0$ *for almost every* (x,y) *in* $X_1\times X_0^c$.

Proof. Since $A\neq 0$ is a positive idempotent, there exists a positive element u in $\mathcal{L}^2(\mathcal{X},m)$ such that $\|u\| = 1$ and $Au = u$. It follows from Lemma 3.2 that $a = u\otimes v$ where $v = A^*u$ is a positive element of $\mathcal{L}^2(\mathcal{X},m)$.

Choose non-negative representations of u and v. Then $a(x,y) = u(x)v(y)$ for almost every (x,y) in $\mathcal{X}\times\mathcal{X}$. Let

$$U_u = \{x\in\mathcal{X} : u(x)\neq 0\},$$

$$U_v = \{x\in\mathcal{X} : v(x)\neq 0\},$$

and let $X_0 = U_v^c$, and $X_1 = U_u\cap U_v$. Clearly, X_0 and X_1 are measurable and disjoint. However,

$$\langle u,\, v\rangle = \langle u,\, A^*u\rangle = \langle Au,\, u\rangle = \langle u,\, u\rangle = 1,$$

and we have $m(X_1) = m(U_u \cap U_v) \neq 0$.

It is obvious that, for almost every (x, y) in $\mathcal{X} \times X_0$, $a(x, y) = u(x)v(y) = 0$, and that, for almost every (x, y) in $X_1 \times X_0^c$, $a(x, y) = u(x)v(y) > 0$.

Since

$$X_0 \cup X_1 = U_v^c \cup (U_u \cap U_v) \supseteq U_u,$$

we have $(X_0 \cup X_1)^c \subseteq U_u^c$. Therefore, $a = 0$ *a.e.* on $(X_0 \cup X_1)^c \times \mathcal{X}$. ∎

Theorem 3.11 *Suppose A is a positive integral idempotent with kernel a. If there exists a real element in the range of A with positive and negative parts non-zero in $\mathcal{L}^2(\mathcal{X}, m)$, then there exist pairwise disjoint measurable subsets X_0, X_1 and X_2 of $\mathcal{X}$ such that*

(*i*) $a = 0$ a.e. *on* $\mathcal{X} \times X_0$,

(*ii*) $m(X_1) > 0$ *and* $m(X_2) > 0$,

(*iii*) $a = 0$ a.e. *on* $(X_0 \cup X_j)^c \times X_j$, $(j = 1, 2)$,

(*iv*) *a is non-zero on a subset of $X_j \times X_j$ of positive measure,* $(j = 1, 2)$.

In particular, $\mathcal{M}_{X_0 \cup X_1}$ is a non-trivial standard invariant subspace of A.

Proof. Suppose u is a real element in the range of A with u^+ and u^- non-zero in $\mathcal{L}^2(\mathcal{X}, m)$. By Lemma 3.5, there exists a positive element h in $\mathcal{L}^2(\mathcal{X}, m)$ such that $Ah = 0$, and $u^+ + h$ and $u^- + h$ are in the range of A. Fix representations for u and h. Let

$$X_0 = \{x \in \mathcal{X} : h(x) \neq 0\},$$

$$U_1 = \{x \in \mathcal{X} : u^+(x) \neq 0\},$$

$$U_2 = \{x \in \mathcal{X} : u^-(x) \neq 0\},$$

and let $X_1 = U_1 \backslash X_0$ and $X_2 = U_2 \backslash X_0$. Clearly, X_0, X_1 and X_2 are pairwise disjoint measurable subsets of $\mathcal{X}$.

It follows from Lemma 3.8 that (i) is true.

For (ii), suppose $m(X_1) = 0$. Then $Au^+ = 0$ since $a = 0$ *a.e.* on $\mathcal{X}\times X_0$. It follows that $u^+ + h = A(u^+ + h) = Au^+ + Ah = 0$. This implies that $u^+ = 0$, which contradicts the assumption on u. Thus, $m(X_1) > 0$. Similarly, $m(X_2) > 0$.

For (iii), since $u^+ + h$ is in the range of A, by Lemma 3.4 we have $a = 0$ *a.e.* on $(U_1 \cup X_0)^c \times (U_1 \cup X_0)$. But $U_1 \cup X_0 = X_0 \cup X_1$ and $X_1 \subseteq U_1$. We conclude that $a = 0$ *a.e.* on $(X_0 \cup X_1)^c \times X_1$. Similarly, $a = 0$ *a.e.* on $(X_0 \cup X_2)^c \times X_2$.

Suppose (iv) is not true. Then $a = 0$ *a.e.* on $X_j \times X_j$, $(j = 1 \text{ or } 2)$. Combining with (iii), we have $a = 0$ *a.e.* on $X_0^c \times X_j$. Therefore, it follows from Lemma 3.9 that $a = 0$ *a.e.* on $\mathcal{X}\times X_j$, and hence, on $\mathcal{X}\times U_j$. Thus, $A(u^+ + h) = 0$ or $A(u^- + h) = 0$. This implies that $u^+ = 0$ or $u^- = 0$, which is impossible. Hence a is non-zero on a subset of $X_j \times X_j$ of positive measure, $(j = 1, 2)$.

Finally, since X_1 and X_2 have non-zero measure, $\mathcal{M}_{X_0 \cup X_1}$ is a non-trivial standard subspace of $\mathcal{L}^2(\mathcal{X}, m)$. By the fact that $a = 0$ *a.e.* on $(X_0 \cup X_1)^c \times (X_0 \cup X_1)$, we have that $\mathcal{M}_{X_0 \cup X_1}$ is invariant under A. ∎

Theorem 3.12 *Every positive integral idempotent of rank at least two has a non-trivial standard invariant subspace.*

Proof. Suppose A is a positive integral idempotent of rank at least two. By Theorem 3.11, it is suffice to show that there exists a real element u in the range of A with non-zero positive and negative parts in $\mathcal{L}^2(\mathcal{X}, m)$, *i.e.*, $u^+ \neq 0$ and $u^- \neq 0$.

It follows from Lemma 3.7 that there exists an orthonormal basis of the range of A consisting of real elements of $\mathcal{L}^2(\mathcal{X}, m)$. If one the basis elements has non-zero positive and negative parts, then we are done. Otherwise, we may assume that there are two non-zero orthogonal positive elements in the range of A, and hence, the difference between them is in the range of A with non-zero positive and negative parts. ∎

For any integral operator T from $\mathcal{L}^2(\mathcal{X}, m)$ to another Hilbert space $\mathcal{L}^2(\mathcal{Y}, m')$

with kernel non-negative, let

$$\mathrm{Null}^+(T) = \{g \in \mathcal{L}^2(\mathcal{X}, m) : g \text{ is positive and } Tg = 0\}.$$

We call it the positive null set of T. From Lemma 3.8, It is easy to see that $\mathrm{Null}^+(T) = \{0\}$ if and only if there is no measurable subset E of $\mathcal{X}$ with non-zero measure such that the kernel of T vanishes almost everywhere on $\mathcal{Y}\times E$. In particular, $\mathrm{Null}^+(T) = \{0\}$ if the kernel of T is positive almost everywhere.

Suppose T is an integral operator with kernel non-negative and S is a positive operator. If $\mathrm{Null}^+(T) = \{0\}$, then $TS = 0$ implies that S vanishes on the set of all positive elements, and hence, $S = 0$.

Theorem 3.13 *Suppose A is a positive idempotent of finite rank s. Then there exist pairwise disjoint measurable subsets $X_0', X_0'', X_1, \ldots, X_s$ with union $\mathcal{X}$ such that*

(*i*) $a = 0$ a.e. *on* $\mathcal{X}\times X_0'$, *and on* $X_0''\times\mathcal{X}$,

(*ii*) $a = 0$ a.e. *on* $(X_0'\cup X_j)^c\times X_j$, $(j = 1, 2, \ldots, s)$,

(*iii*) $a(x,y) > 0$ *for almost every* (x,y) *in* $X_j\times X_j$, *and* $a|_{X_j\times X_j}$ *is the kernel of an idempotent of rank one,* $(j = 1, 2, \ldots, s)$.

Proof. We prove this theorem by induction. It follows from Lemma 3.10 that the result is true for $s = 1$.

Suppose $s > 1$, and the result is true for all positive idempotents with rank less than s. Since the range of A is of dimension $s > 1$, we can choose an element u in the range of A such that both u^+ and u^- are non-zero. By Theorem 3.11, there exist pairwise disjoint measurable subsets Y_0, Y_1 and Y_2 of $\mathcal{X}$ with $m(Y_1)$ and $m(Y_2)$ positive such that $a = 0$ *a.e.* on $\mathcal{X}\times Y_0$, $a = 0$ *a.e.* on $(Y_0\cup Y_j)^c\times Y_j$, and a is non-zero on a subset of $Y_j\times Y_j$ of positive measure, $(j = 1, 2)$. Therefore, the restriction $A_1 = P_{Y_0\cup Y_1}A|_{\mathcal{M}_{Y_0\cup Y_1}}$ of A to its standard invariant subspace $\mathcal{M}_{Y_0\cup Y_1}$ is a non-zero idempotent with rank less than s. Repeat this process if the rank of A_1 is great than one. Thus, we may assume that A_1 is of rank one.

Let A_2 be the compression of A to $\mathcal{M}_{(Y_0 \cup Y_1)^c}$, the orthogonal complement of $\mathcal{M}_{Y_0 \cup Y_1}$. Then A_2 is an idempotent of rank $s-1$. By the induction hypothesis, there exist pairwise disjoint measurable subsets $W', W'', X_2, X_3, \ldots, X_s$ of $\mathcal{X}$ whose union is equal to $(Y_0 \cup Y_1)^c$ such that

(*i*) $a = 0$ *a.e.* on $(Y_0 \cup Y_1)^c \times W'$ and on $W'' \times (Y_0 \cup Y_1)^c$, and hence, on $W'' \times \mathcal{X}$,

(*ii*) $a = 0$ *a.e.* on $[(Y_0 \cup Y_1)^c \setminus (W' \cup X_j)] \times X_j$, $(j = 2, 3, \ldots, s)$,

(*iii*) $a(x,y) > 0$ for almost every (x,y) in $X_j \times X_j$, and $a|_{X_j \times X_j}$ is the kernel of an idempotent of rank one, $(j = 2, 3, \ldots, s)$.

It follows that $\mathcal{M}_{Y_0 \cup Y_1 \cup W'}$ is invariant under A, and the restriction of A to it remains an idempotent of rank one. Thus, by redefining Y_1 and Y_2 if necessary, we may assume that $W' = \emptyset$. Applying Lemma 3.10 to the idempotent A_1, we have that there exist pairwise disjoint measurable subsets X_0', X_1 and W of $Y_0 \cup Y_1$, whose union is $Y_0 \cup Y_1$ and $m(X_1) > 0$, such that $a = 0$ *a.e.* on $(Y_0 \cup Y_1) \times X_0'$, and hence, on $\mathcal{X} \times X_0'$, and on $W \times (Y_0 \cup Y_1)$; and $a(x,y) > 0$ for almost every (x,y) in $X_1 \times (X_0' \cup W)$.

Corresponding to the decomposition

$$\mathcal{L}^2(\mathcal{X}, m) = \mathcal{M}_{X_0'} \oplus \mathcal{M}_{X_1} \oplus \mathcal{M}_W \oplus \mathcal{M}_{\cup_{j=2}^s X_j} \oplus \mathcal{M}_{W''},$$

A is of the form

$$A = \begin{pmatrix} 0 & * & * & * & * \\ 0 & A_{22} & A_{23} & A_{24} & * \\ 0 & 0 & 0 & A_{34} & A_{35} \\ & & & A_{44} & * \\ & & 0 & & \\ & & & 0 & 0 \end{pmatrix}.$$

Since $A^2 = A$, we have

$$A_{22}A_{24} + A_{23}A_{34} + A_{24}A_{44} = A_{24}.$$

However, A_{22} is an idempotent. It follows that

$$A_{22}A_{24} + A_{22}A_{23}A_{34} + A_{22}A_{24}A_{44} = A_{22}A_{24}.$$

Thus, $A_{22}A_{23}A_{34} = A_{22}A_{24}A_{44} = 0$ since all operators here are positive integral operators. On the other hand, $\mathrm{Null}^+(A_{22})$ and $\mathrm{Null}^+(A_{23})$ are equal to $\{0\}$ because a is positive almost everywhere on $X_1 \times X_1$ and $X_1 \times W$; and both $\mathrm{Null}^+(A_{44})$ and $\mathrm{Null}^+(A_{44}^*)$ are equal to $\{0\}$ because a is positive almost everywhere on $X_j \times X_j$, $(j = 2, 3, \ldots, s)$. By the earlier comment on the positive null set, we have $A_{34} = 0$ and $A_{24}A_{44} = 0$, and hence, $A_{24} = 0$. Consequently, $A_{35} = 0$ since $A^2 = A$.

Let $X_0'' = W \cup W''$. Then it is easy to see that the pairwise disjoint measurable subsets $X_0', X_0'', X_1, \ldots, X_s$ of $\mathcal{X}$ obtained above satisfy all the requirements. ∎

The technique used above relies on the induction on the rank of A, and therefore, only works for positive integral idempotents of finite rank. However, if we assume that the Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ is separable, then the above result holds for all positive integral idempotents, finite rank as well as infinite rank. We need the following lemmas to prove this.

A *chain* of subspaces of a Hilbert space $\mathcal{H}$ is a family of subspaces of $\mathcal{H}$ that is totally ordered by inclusion. Let Ω be a chain of subspaces of $\mathcal{H}$. For every $\mathcal{M} \in \Omega$, define

$$\mathcal{M}_- = \bigvee\{\mathcal{N} : \mathcal{N} \in \Omega, \mathcal{N} \subseteq \mathcal{M} \text{ but } \mathcal{N} \neq \mathcal{M}\}.$$

We call Ω a continuous chain if $\mathcal{M}_- = \mathcal{M}$ for each $\mathcal{M} \in \Omega$.

The following result, which will be used to prove the main theorem of this chapter, may be of independent interest.

Lemma 3.14 *Suppose T is an integral operator on $\mathcal{L}^2(\mathcal{X}, m)$ with non-negative kernel k. If there exists a continuous chain Ω of standard subspaces of $\mathcal{L}^2(\mathcal{X}, m)$ whose members are invariant under both T and T^*, then $T = 0$.*

Proof. In Hilbert space $\mathcal{L}^2(\mathcal{X} \times \mathcal{X}, m \times m)$, fix a sequence of non-negative measurable functions $\{k_j\}_{j=1}^{\infty}$ with the properties that $k_1 \leq k_2 \leq \cdots \leq k$ and $\{k_j\}_{j=1}^{\infty}$ converges to k almost everywhere on $\mathcal{X} \times \mathcal{X}$. For each positive integer j, let T_j be the integral operator induced by k_j. Then all T_j are Hilbert-Schmidt operators and, by

Proposition 1.5, all elements of Ω are invariant under both T_j and T_j^*, $(j = 1, 2, \cdots)$. Also, by Fubini's Theorem, $\{T_j\}$ converges to T in the weak operator topology. Thus, it suffices to show that $T_j = 0$ for every positive integer j.

Fix an integer j. All elements of Ω are invariant under both T_j and T_j^*, and therefore, are invariant under $T_j^*T_j$. However, Ω is continuous. It follows from Ringrose [49, Theorem 4.3.10] that $T_j^*T_j$ is quasinilpotent. Consequently, $T_j = 0$. ∎

Lemma 3.15 *Suppose $\mathcal{L}^2(\mathcal{X}, m)$ is separable and $A \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m))$ is an integral idempotent with non-negative kernel a. Then there exists a measurable subset X_0 of $\mathcal{X}$ such that $a = 0$ a.e. on $\mathcal{X} \times X_0$ and $\mathrm{Null}^+(A_1) = \{0\}$ where A_1 is the integral operator induced by the non-negative kernel $a|_{X_0^c \times X_0^c}$.*

Proof. Since $\mathcal{L}^2(\mathcal{X}, m)$ is separable, the unit ball of $\mathcal{L}^2(\mathcal{X}, m)$ is weakly metrizable. The intersection of the unit ball of $\mathcal{L}^2(\mathcal{X}, m)$ and $\mathrm{Null}^+(A)$ is closed in the weak topology, and therefore, weakly compact and weakly separable. Let $\{f_j\}$ be a countable weakly dense subset of the intersection. For each j, fix a representation of f_j and let

$$U_j = \{x \in \mathcal{X} : f_j(x) \neq 0\}.$$

Take $X_0 = \cup_j U_j$. Then $a = 0$ *a.e.* on $\mathcal{X} \times X_0$ since $\mathcal{X} \times X_0 = \cup_j \mathcal{X} \times U_j$ and, by Lemma 3.8, $a = 0$ *a.e.* on $\mathcal{X} \times U_j$ for each j.

Suppose $\mathrm{Null}^+(A_1) \neq \{0\}$. Then there exists an positive element $h \in \mathcal{M}_{X_0^c}$, $\|h\| = 1$, such that $A_1 h = 0$. Fix a representation of h and let

$$U_h = \{x \in X_0^c : h(x) \neq 0\}.$$

By Lemma 3.8, $a = 0$ *a.e.* on $X_0^c \times U_h$, and therefore, on $\mathcal{X} \times U_h$ by Lemma 3.9. Thus, $Ah = 0$, and h is in the intersection of the unit ball of $\mathcal{L}^2(\mathcal{X}, m)$ and $\mathrm{Null}^+(A)$. Consequently, h is the weak limit of some subsequence in $\{f_j\}$, which is impossible since h is non-zero and orthogonal to each f_j. ∎

Theorem 3.16 *Suppose $\mathcal{L}^2(\mathcal{X},m)$ is separable and $A \in \mathcal{B}(\mathcal{L}^2(\mathcal{X},m))$ is a positive integral idempotent with kernel a. Then there exists a sequence of pairwise disjoint measurable subsets $X_0', X_0'', X_1, X_2, \cdots$ with union $\mathcal{X}$ such that*

(*i*) $a = 0$ a.e. *on* $\mathcal{X} \times X_0'$, *and on* $X_0'' \times \mathcal{X}$,

(*ii*) $a = 0$ a.e. *on* $(X_0' \cup X_j)^c \times X_j$, $(j = 1, 2, \cdots)$,

(*iii*) $a(x, y) > 0$ *for almost every* (x, y) *in* $X_j \times X_j$, *and* $a|_{X_j \times X_j}$ *is the kernel of an idempotent of rank one,* $(j = 1, 2, \cdots)$.

Proof. By applying Lemma 3.15 to A, we can obtain a measurable subset X_0' of $\mathcal{X}$ such that $a = 0$ *a.e.* on $\mathcal{X} \times X_0'$ and $\mathrm{Null}^+(A_1) = \{0\}$ where A_1 is the integral operator induced by the non-negative kernel $a|_{X_0'^c \times X_0'^c}$. Now A_1^* is also a positive integral idempotent. By Lemma 3.15 again, there exists a measurable subset X_0'' of $X_0'^c$ such that $a^* = 0$ *a.e.* on $X_0'^c \times X_0''$ and $\mathrm{Null}^+(A_2^*) = \{0\}$ where A_2 is the integral operator induced by the non-negative kernel

$$a|_{(X_0' \cup X_0'')^c \times (X_0' \cup X_0'')^c}.$$

Thus $a = 0$ *a.e.* on $\mathcal{X} \times X_0'$ and on $X_0'' \times \mathcal{X}$. Since $\mathrm{Null}^+(A_1) = \{0\}$, we have that $\mathrm{Null}^+(A_2) = \{0\}$.

Consider the operator A_2 defined above. It is the compression of A to the standard space $\mathcal{M}_{(X_0' \cup X_0'')^c}$, and therefore, a positive idempotent. Let $Y = (X_0' \cup X_0'')^c$. Suppose $U \subseteq Y$ is measurable and the standard space $\mathcal{M}_U$ is invariant under A_2. Then, by Proposition 1.5, $a = 0$ *a.e.* on $(Y \backslash U) \times U$. Under the decomposition

$$\mathcal{M}_Y = \mathcal{M}_U \oplus \mathcal{M}_{Y \backslash U},$$

A_2 is of the form

$$A_2 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

Since A_2 is an idempotent, we have

$$A_{11}A_{12} + A_{12}A_{22} = A_{12}.$$

Left multiplying this equation by A_{11}, which is clearly an idempotent, we have

$$A_{11}A_{12} + A_{11}A_{12}A_{22} = A_{11}A_{12}.$$

It follows that $A_{11}A_{12}A_{22} = 0$. However, both $\mathrm{Null}^+(A_2)$ and $\mathrm{Null}^+(A_2^*)$ are $\{0\}$, and hence, so are $\mathrm{Null}^+(A_{11})$ and $\mathrm{Null}^+(A_{22}^*)$. Consequently, $A_{12} = 0$ and $\mathcal{M}_U$ is invariant under both A_2 and A_2^*.

Fix a chain Ω of standard invariant subspaces of A_2 containing both $\{0\}$ and $\mathcal{L}^2(Y, m|_Y)$. By Zorn's Lemma, we may assume that Ω is maximal in the sense that there is no other chain of standard invariant subspaces of A_2 containing Ω properly. For every $\mathcal{M} \in \Omega$, by the remark following the definition of standard subspace (Definition 1.4), $\mathcal{M}_-$ is also a standard subspace. Clearly, $\mathcal{M}_-$ is invariant under A_2 and $\Omega \cup \{\mathcal{M}_-\}$ remains a chain. Thus, $\mathcal{M}_- \in \Omega$. However, every standard subspace which is invariant under A_2 must be also invariant under A_2^*. It follows that every $\mathcal{M} \in \Omega$ is invariant under both A_2 and A_2^*.

Suppose $\mathcal{M} \in \Omega$ and $\mathcal{M}_- \neq \mathcal{M}$. Then $\mathcal{M} \ominus \mathcal{M}_-$ is a standard subspace, *i.e.*, $\mathcal{M} \ominus \mathcal{M}_- = \mathcal{M}_E$ for some measurable subset $E \subseteq Y$. The standard subspace $\mathcal{M}_E$ is invariant under both A_2 and A_2^*. It follows from Proposition 1.5 that $a = 0$ *a.e.* on $(Y \backslash E) \times E$ and $E \times (Y \backslash E)$. The compression $A_{\mathcal{M}}$ of A_2 to $\mathcal{M}_E = \mathcal{M} \ominus \mathcal{M}_-$ is a positive integral idempotent whose kernel is the restriction $a|_{E \times E}$. By the maximality of Ω, $A_{\mathcal{M}}$ has no non-trivial standard invariant subspaces as an operator on $\mathcal{L}^2(E, m)$, and therefore, by Theorem 3.12, is an idempotent of rank one. Since both $\mathrm{Null}^+(A_2)$ and $\mathrm{Null}^+(A_2^*)$ are $\{0\}$, we have that $a(x, y) > 0$ for almost every (x, y) in $E \times E$.

Since $\mathcal{L}^2(\mathcal{X}, m)$ is separable, the set

$$\Omega_1 = \{\mathcal{M} \in \Omega : \mathcal{M}_- \neq \mathcal{M}\}$$

is countable. For each $\mathcal{M} \in \Omega_1$, choose a measurable subset $E_{\mathcal{M}}$ of Y such that

$$\mathcal{M}_{E_{\mathcal{M}}} = \mathcal{M} \ominus \mathcal{M}_-.$$

If $\mathcal{M}$ and $\mathcal{N}$ are two distinct elements of Ω_1, then either $\mathcal{M} \subset \mathcal{N}$ or $\mathcal{M} \supset \mathcal{N}$. Consequently, $m(E_{\mathcal{M}} \cap E_{\mathcal{N}}) = 0$. We may assume, WNLG, that $E_{\mathcal{M}} \cap E_{\mathcal{N}} = \emptyset$ if $\mathcal{M} \neq \mathcal{N}$.

Thus, the set of all $E_{\mathcal{M}}$ with $\mathcal{M} \in \Omega_1$ is countable and can be listed as a pairwise disjoint sequence $X_1, X_2, \cdots$. With the measurable subsets $X_0', X_0'', X_1, X_2, \cdots$ of $\mathcal{X}$, the theorem will be proved if we can show that their union is equal to $\mathcal{X}$, or equivalently,

$$Y = \cup_{j\geq 1} X_j.$$

Therefore, it suffices to show that

$$m(Y \setminus (\cup_{j\geq 1} X_j)) = 0,$$

since we can change, if necessary, one of X_j by a set of measure zero.

Let

$$Y_1 = Y \setminus (\cup_{j\geq 1} X_j),$$

and let A_3 be the restriction of A_2 to $\mathcal{M}_{Y_1}$. Then, since all $\mathcal{M}_{X_j}$ are invariant under both A_2 and A_2^*, A_3 is a positive integral operator whose kernel is $a|_{Y_1 \times Y_1}$. Consider the following chain

$$\Omega_2 = \{\mathcal{N} : \mathcal{N} = \mathcal{M} \cap \mathcal{M}_{Y_1} \text{ for some } \mathcal{M} \in \Omega\}.$$

It is a continuous chain of standard subspaces whose elements are invariant under both A_3 and A_3^*. By Lemma 3.14, we have that $A_3 = 0$. It follows that $\chi_{Y_1} \in \text{Null}^+(A_1)$, and therefore, $m(Y_1) = 0$ since $\text{Null}^+(A_1) = \{0\}$. ∎

Corollary 3.17 *Suppose A is a positive integral idempotent on $\mathcal{L}^2(\mathcal{X}, m)$ with kernel a and its rank is equal to s, $0 \leq s \leq +\infty$, and $\mathcal{L}^2(\mathcal{X}, m)$ is separable if $s = +\infty$. Then there exist pairwise disjoint measurable subsets $\{X_j\}_{j=1}^s$ of $\mathcal{X}$, and an orthonormal set $\{w_j\}_{j=1}^s$ of positive elements in $\mathcal{L}^2(\mathcal{X}, m)$ such that*

(*i*) *$w_j = 0$ a.e. on X_j^c, and $Aw_j = w_j$ a.e. on $\cup_{l=1}^s X_l$ for every j.*

(*ii*) *For every j, let $u_j = Aw_j$. Then $\langle w_j, u_l \rangle = \delta_{jl}$ for every pair of integer j and l. Consequently, $\{u_j\}_{j=1}^s$ is linearly independent.*

(iii) *There exists a set* $\{v_j\}_{j=1}^s$ *of positive elements in* $\mathcal{L}^2(\mathcal{X}, m)$ *such that*

$$A = \sum_{j=1}^{s} u_j \otimes v_j$$

where the series converges in the weak operator topology if $s = +\infty$. *As a result,* $\{u_j\}_{j=1}^s$ *is a basis of the range of* A.

Proof. By Theorem 3.13 or 3.16 if $s = +\infty$, there exist pairwise disjoint measurable subsets X_0', X_0'', and $\{X_j\}_{j=1}^s$ of $\mathcal{X}$, whose union is $\mathcal{X}$, with the properties that $a = 0$ *a.e.* on both $\mathcal{X} \times X_0'$ and $X_0'' \times \mathcal{X}$, and that $a = 0$ *a.e.* on $(X_0' \cup X_j)^c \times X_j$, and $a|_{X_j \times X_j}$ is the kernel of an idempotent of rank one for every j.

Fix an integer j. Since $a|_{X_j \times X_j}$ is the kernel of an idempotent of rank one, there exists a positive element w_j in $\mathcal{L}^2(X_j, m)$ with unit norm such that $a|_{X_j \times X_j} = w_j \otimes w_j'$ for some positive w_j' in $\mathcal{L}^2(X_j, m)$. Define w_j and w_j' to be zero on X_j^c. Then both of them are in $\mathcal{M}_{X_j}$. Since $a = 0$ *a.e.* on $\mathcal{X} \times X_0'$, on $X_0'' \times \mathcal{X}$ and on $(X_0' \cup X_l)^c \times X_l$ for each l, we have that $A w_j = w_j$ *a.e.* on $\cup_{l=1}^s X_l$. Clearly, $\{w_j\}_{j=1}^s$ is an orthonormal set in $\mathcal{L}^2(\mathcal{X}, m)$. Thus (i) has been proven.

It is easy to see that (ii) follows immediately from (i).

For (iii), consider the matrix of A in the following decomposition:

$$\mathcal{L}^2(\mathcal{X}, m) = \mathcal{M}_{X_0'} \oplus \mathcal{M}_{\cup_{j=1}^s X_j} \oplus \mathcal{M}_{X_0''}.$$

The matrix is of the form

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $A^2 = A$, we have $A_{12}A_{22} = A_{12}$, $A_{12}A_{23} = A_{13}$, and $A_{22}A_{23} = A_{23}$. By the definitions of w_j and w_j', the operator $\hat{A}_{22}$ given by

$$\hat{A}_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is equal to $\sum_{j=1}^{s} w_j \otimes w'_j$, where the series converges in the weak operator topology if $s = +\infty$. However, $A\hat{A}_{22}A = A$. Therefore,

$$\begin{aligned} A &= A\left[\sum_{j=1}^{s} w_j \otimes w'_j\right] A \\ &= \sum_{j=1}^{s} (Aw_j) \otimes (A^* w'_j) \\ &= \sum_{j=1}^{s} u_j \otimes v_j, \end{aligned}$$

where $v_j = A^* w'_j$ is a positive element of $\mathcal{L}^2(\mathcal{X}, m)$ for every j since A^* is also a positive operator. ∎

Chapter 4

Semigroups of Positive Operators

A class of bounded operators on a Hilbert space is called *reducible* if there exists a non-trivial subspace of the Hilbert space invariant under every operator in the class. A class of bounded operator on a Hilbert space is called *irreducible* if it is not reducible. Recently, a number of results about the reducibility of semigroups of operators have been obtained. It was proved [41, Theorem 1] that a semigroup $\mathcal{S}$ of quasinilpotent operators on a Hilbert space $\mathcal{H}$ is reducible if it contains an operator other than 0 in some $\mathcal{C}_p$ class. The reduciblity of semigroups of operators represented by matrices with non-negative entries has been studied in [46]. It was proved [46, Theorem 5] that a semigroup $\mathcal{S}$ of compact operators represented by matrices with non-negative entries is reducible if $r(ST) \leq r(S)r(T)$ for every pair S and T in $\mathcal{S}$. The existence of non-trivial stantard invariant subspaces for certain semigroups of positive integral operators on $\mathcal{L}^2(\mathcal{X}, m)$ was studied in [14]. There are some other related results which can found in [34], [45] and [47]. In this chapter, we discuss the reducibility of certain classes of positive operators, especially, semigroups of positive integral operators. We either extend the results mentioned above to more general cases or use them to prove some results about the reducibility of semigroups of positive operators. We are also interested in finding non-trivial standard invariant subspaces for such classes.

4.1 Reducibility of Semigroups of Positive Operators

We first list several known results for future reference. A subset $\mathcal{J}$ of the semigroup $\mathcal{S}$ is called an *ideal* in $\mathcal{S}$ if JS and SJ belong to $\mathcal{J}$ for all $J \in \mathcal{J}$ and $S \in \mathcal{S}$.

Lemma 4.1 [46, Lemma 1] *If a semigroup $\mathcal{S}$ of operators is irreducible, then so is every non-zero ideal $\mathcal{J}$ in $\mathcal{S}$.*

Proof. Suppose $\mathcal{J}$ is a non-zero ideal in $\mathcal{S}$. If $\mathcal{M}$ is a non-trivial invariant subspace of $\mathcal{J}$, then the following two subspaces are invariant under $\mathcal{S}$, and it is easy to verify that at least one of them is non-trivial:

(i) The closed linear span of $\{J\mathcal{M} : J \in \mathcal{J}\}$.

(ii) The intersection of the nullspaces of all J in $\mathcal{J}$.

■

Lemma 4.2 [41, Lemma, p.272] *Suppose $\mathcal{H}$ is an arbitrary Hilbert space and p is a positive number. If $\mathcal{S}$ is a semigroup of operators on $\mathcal{H}$ and $C \in \mathcal{S} \cap \mathcal{C}_p$ is non-zero, and if $\mathcal{S}$ leaves no subspace (other than $\{0\}$) of the nullspace of C invariant, then $\mathcal{S}$ contains a non-zero trace-class operator.*

Proof. Choose an integer n greater than p. We prove the lemma using the fact that any product of n $\mathcal{C}_p$ class operators is a trace-class operator.

For each $x \in \mathcal{H}$, $x \neq 0$, there exists an operator $S \in \mathcal{S}$ such that $CSx \neq 0$, for otherwise the closed linear span of $\{Sx : S \in \mathcal{S}\}$ if $Cx \neq 0$ or of $\{Sx : S \in \mathcal{S}\} \cup \{x\}$ if $Cx = 0$ is a non-trivial invariant subspace of $\mathcal{S}$ contained in the nullspace of C. By applying this procedure n times, we can find n operators $S_1, S_2, \ldots, S_n$ in $\mathcal{S}$ such that $CS_nCS_{n-1}\cdots CS_1x\neq 0$. Therefore, $CS_nCS_{n-1}\cdots CS_1$ is a non-zero operator in $\mathcal{C}_1$ since every CS_j is in $\mathcal{C}_p$, $j=1,2,\ldots,n$. ∎

Theorem 4.3 [41, Theorem 1] *Suppose $\mathcal{H}$ is an arbitrary Hilbert space and p is a positive number. If $\mathcal{S}$ is a semigroup of quasinilpotent operators on $\mathcal{H}$, and if $\mathcal{S}$ contains an operator other than 0 in $\mathcal{C}_p$ class, then $\mathcal{S}$ is reducible.*

Proof. Suppose $\mathcal{S}$ is irreducible. Then, by Lemma 4.2, $\mathcal{S}$ contains a non-zero trace-class operator S. Let $\mathcal{J}$ be the ideal in $\mathcal{S}$ generated by S, and let $\mathcal{A}$ be the algebra generated by $\mathcal{J}$. Clearly, every operator in $\mathcal{J}$ is a trace-class operator, and every operator in $\mathcal{A}$ is a linear combination of operators in $\mathcal{J}$. Since $\mathcal{S}$ consists of quasinilpotent operators, we have that the trace as a function on the trace-class is constantly zero on $\mathcal{J}$, and hence, on $\mathcal{A}$. Consequently, $\mathcal{A}$ is not $\mathcal{C}_1$-dense in $\mathcal{C}_1$. However, by Lemma 4.1, the irreducibility of $\mathcal{S}$ implies that $\mathcal{J}$ is irreducible. Therefore, $\mathcal{A}$ is also irreducible. This contradicts the fact proven in [48] as a consequence of Lomonosov's Lemma [35], that subalgebras of $\mathcal{C}_p$ are reducible unless they are $\mathcal{C}_p$-dense in $\mathcal{C}_p$. ∎

The following two theorems are from [46, Theorem 5], which give two sets of sufficient conditions for a semigroup of compact operators to be reducible.

Theorem 4.4 [46, Theorem 2] *If every member of a semigroup $\mathcal{S}$ is a non-negative scalar multiple of a compact idempotent and $r(ST)\leq r(S)r(T)$ for every pair S and T in $\mathcal{S}$, then $\mathcal{S}$ is reducible.*

Proof. Omitted. ∎

Theorem 4.5 [46, Theorem 5] *Let $\mathcal{S}$ be a semigroup of compact operators represented by matrices with non-negative entries. If $r(ST)\leq r(S)r(T)$ for every pair S and T in $\mathcal{S}$, then $\mathcal{S}$ is reducible.*

Proof. Omitted. ∎

An operator that can be represented by a matrix with non-negative entries may be viewed as an integral operator on some $\mathcal{L}^2(\mathcal{X}, m)$ which is induced by a non-negative kernel with a discrete measure space $(\mathcal{X}, m)$. Therefore, the semigroup $\mathcal{S}$ in the above theorem may be regarded as a semigroup of positive integral operators on some Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ with a discrete measure space $(\mathcal{X}, m)$. We now consider the case where $(\mathcal{X}, m)$ could be any type of measure spaces, discrete or not. Throughout the rest of this chapter, we assume that $\mathcal{S}$ is a multiplicative semigroup of operators on $\mathcal{L}^2(\mathcal{X}, m)$ where $\mathcal{X}$ is a topological space and, with its Borel structure, a standard Borel space and m is a σ-finite regular Borel measure on $\mathcal{X}$ such that the Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ is of dimension at least two.

The following lemma, though not difficult to prove, is interesting. It plays an impotant role in the proofs of main theorems of this chapter.

Lemma 4.6 *Suppose T is a non-zero positive integral operator with kernel k. Then there exists a non-zero positive Hilbert-Schmidt operator T_0 with kernel k_0 such that $k - k_0 \geq 0$ a.e. on $\mathcal{X} \times \mathcal{X}$.*

Proof. For every positive integer j, let

$$G_j = \{(x, y) \in \mathcal{X} \times \mathcal{X} : 0 < k(x, y) \leq j\}.$$

Then all G_j, $(j = 1, 2, \cdots)$, are measurable subsets of $\mathcal{X} \times \mathcal{X}$, and at least one of them, say G_{j_0}, has measure $(m \times m)(G_{j_0}) > 0$ since T is non-zero. Since $(\mathcal{X}, m)$ is a σ-finite measure space, we can choose a measurable subset G_0 of G_{j_0} such that $0 < (m \times m)(G_0) < +\infty$.

Let

$$k_0(x, y) = \chi_{G_0}(x, y) k(x, y) \quad (x, y) \in \mathcal{X} \times \mathcal{X}.$$

Then, k_0 satisfies that $k - k_0 \geq 0$ *a.e.* on $\mathcal{X} \times \mathcal{X}$, and therefore, is a non-negative kernel on $\mathcal{X} \times \mathcal{X}$. By the definition of G_0, we have $k_0 \in \mathcal{L}^2(\mathcal{X} \times \mathcal{X}, m \times m)$. Hence k_0 induces a non-zero positive Hilbert-Schmidt operator T_0 on $\mathcal{L}^2(\mathcal{X}, m)$. ∎

Theorem 4.7 *If $\mathcal{S}$ is a semigroup of positive quasinilpotent integral operators, then $\mathcal{S}$ is reducible.*

Proof. We may assume that $\mathcal{S}$ contains a non-zero operator T. Let k be its kernel. It follows from Lemma 4.6 that there exists a non-zero positive $\mathcal{C}_2$ class operator T_0 with kernel k_0 such that $k - k_0 \geq 0$ *a.e.* on $\mathcal{X} \times \mathcal{X}$.

Let $\mathcal{S}_0$ be the multiplicative semigroup generated by $\mathcal{S} \cup \{T_0\}$. We complete the proof by showing that $\mathcal{S}_0$ is reducible. By Theorem 4.3, it suffices to show that $\mathcal{S}_0$ is a semigroup of quasinilpotent operators since it contains a non-zero $\mathcal{C}_2$ class operator T_0.

Indeed, every element in $\mathcal{S}_0$ has the form $T_0^{q_1} S_1 T_0^{q_2} S_2 \cdots S_{n-1} T_0^{q_n}$, where all S_j are in $\mathcal{S}$ and all q_j are non-negative integers, $(1 \leq j \leq n)$. By Proposition 1.11 and Theorem 1.13, the kernel of $T_0^{q_1} S_1 T_0^{q_2} S_2 \cdots S_{n-1} T_0^{q_n}$ is dominated by the kernel of quasinilpotent operator $T^{q_1} S_1 T^{q_2} S_2 \cdots S_{n-1} T^{q_n}$. It follows from Corollary 1.14 that $T_0^{q_1} S_1 T_0^{q_2} S_2 \cdots S_{n-1} T_0^{q_n}$ is a quasinilpotent operator. ∎

Theorem 4.8 *Suppose $\mathcal{S}$ is a semigroup of compact positive integral operators. If $r(ST) \leq r(S)r(T)$ for every pair S and T in $\mathcal{S}$, then $\mathcal{S}$ is reducible.*

Proof. Since the spectral radius is continuous in norm for compact operators, we may assume that $\mathcal{S}$ is norm-closed and that any non-negative scalar multiple of an operator in $\mathcal{S}$ is still in $\mathcal{S}$. We may also assume that $\mathcal{S}$ contains no quasinilpotent operator other than zero, for otherwise the ideal $\mathcal{J}$ in $\mathcal{S}$ generated by some quasinilpotent operator $S \in \mathcal{S}$, $A \neq 0$, is non-zero and consists of only quasinilpotent operators. By Theorem 4.7, $\mathcal{J}$ is reducible, and therefore, so is $\mathcal{S}$ by Lemma 4.1.

We may assume that $\mathcal{S}$ contains a non-zero operator and thus an operator A with $r(A) = 1$. We claim that there exists a non-zero idempotent of finite rank in $\mathcal{S}$. The proof of this claim is similar to that of [46, Theorem 1]. By the Riesz decomposition theorem [48], A can be represented, under a decomposition of $\mathcal{L}^2(\mathcal{X}, m)$ which may be non-orthogonal, by a matrix of the form

$$A = \begin{pmatrix} U+N & 0 \\ 0 & B \end{pmatrix}$$

where U is a finite unitary matrix commuting with the finite nilpotent matrix N and $r(B) < 1$. Since U is unitary, some subsequence of $\{U^p\}_{p=1}^{\infty}$, say $\{U^{p_i}\}$, approaches the identity matrix of the same size. Therefore, if we can show that $N = 0$, then the norm-limit of sequence $\{A^{p_i}\}$ will be an operator similar to the projection on the range of U, the hence, a non-zero idempotent of finite rank. Suppose $N \neq 0$. Let q be such that $N^q \neq 0$ and $N^{q+1} = 0$. Then

$$(U+N)^p = U^p + \binom{p}{1} U^{p-1}N + \cdots + \binom{p}{q} U^{p-q}N^q$$

for every $p > q$. By taking $p = p_i + q$ and letting i tend to ∞ in the above equation, we have

$$\lim \frac{(U+N)^{p_i+q}}{\binom{p_i+q}{q}} = N^q$$

and

$$\lim \frac{A^{p_i+q}}{\binom{p_i+q}{q}} = \begin{pmatrix} N^q & 0 \\ 0 & 0 \end{pmatrix} \equiv C.$$

It follows that $C \in \mathcal{S}$ is nilpotent and non-zero, which is impossible since $\mathcal{S}$ contains no quasinilpotent operator other than zero. To simplify the notation, we assume that A itself is an idempotent of finite rank.

Suppose A is of rank s. If $s = 1$, then every member of $\mathcal{S}A\mathcal{S}$, which is the ideal in $\mathcal{S}$ generated by A, is a non-negative scalar multiple of a rank-1 idempotent since $\mathcal{S}$ contains no quasinilpotent operators other than 0. Therfore, $\mathcal{S}A\mathcal{S}$ is reducible by Theorem 4.4, and hence, $\mathcal{S}$ is reducible by Lemma 4.1.

For the case where $s > 1$, by Corollary 3.17, there exists an orthonormal set $\{w_1, w_2, \ldots, w_s\}$ in $\mathcal{L}^2(\mathcal{X}, m)$ and a basis $\{u_1, u_2, \ldots, u_s\}$ of the range of A such that

(i) w_j, u_j and v_j are positive, $(j = 1, 2, \ldots, s)$,

(ii) $\langle w_j, u_l \rangle = \delta_{jl}$, $(j, l = 1, 2, \ldots, s)$.

Let $\mathcal{R}$ denote the range of A. Consider the semigroup $A\mathcal{S}A|_{\mathcal{R}}$. If we can show that there exists a non-trivial subspace $\mathcal{M} \subseteq \mathcal{R}$ invariant under every operator in $A\mathcal{S}A|_{\mathcal{R}}$, then the subspace

$$\mathcal{N} = \bigvee \{S\mathcal{M} : S \in \mathcal{S}\}$$

is invariant under $\mathcal{S}$ and is non-trivial since

$$\mathcal{M} = A\mathcal{M} \subseteq \mathcal{N}$$

and

$$S\mathcal{M} = AS\mathcal{M} + (1 - A)S\mathcal{M} \subseteq \mathcal{M} + (1 - A)S\mathcal{M}$$

for all $S \in \mathcal{S}$, so that

$$\mathcal{N} \subseteq \mathcal{M} + (1 - A)\mathcal{L}^2(\mathcal{X}, m).$$

We complete the proof by showing that there indeed exists a non-trivial subspace $\mathcal{M} \subset \mathcal{R}$ invariant under every operator in $A\mathcal{S}A|_{\mathcal{R}}$. By Theorem 4.5, it suffices to show that relative to the basis $\{u_1, u_2, \ldots, u_s\}$, $A\mathcal{S}A|_{\mathcal{R}}$ can be represented by $s \times s$ matrices with non-negative entries, since it is clear that $r(ST) \leq r(S)r(T)$ for every pair S and T in $A\mathcal{S}A|_{\mathcal{R}}$. For any $T \in \mathcal{S}$, suppose relative to the basis $\{u_1, u_2, \ldots, u_s\}$, $ATA|_{\mathcal{R}}$ is represented by the matrix (t_{jl}), that is,

$$ATu_l = \sum_{j=1}^{s} t_{jl} u_j, \quad (l = 1, 2, \ldots, s).$$

However, A and T are positive operators, u_j and w_j are positive for all j by (i) and $\langle w_j, u_l \rangle = \delta_{jl}$ for all j, l by (ii). It follows that

$$\begin{aligned} t_{jl} &= \langle \textstyle\sum_{i=1}^{s} t_{il} u_i, w_j \rangle \\ &= \langle ATu_l, w_j \rangle \\ &\geq 0 \end{aligned}$$

for all $j, l = 1, 2, \ldots, s$. ∎

Suppose $(\mathcal{X}, m)$ is a finite measure space. As in Theorem 2.7, the algebra of all pseudo-integral operators with absolutely bounded kernels can be decomposed into the direct sum

$$\mathcal{P} = \mathcal{I} \oplus \mathcal{P}_s,$$

where $\mathcal{I}$ consists of all integral operators in $\mathcal{P}$, and $\mathcal{P}_s$ of all operators in $\mathcal{P}$ with kernels singular to the product measure $m \times m$. Therefore, corresponding to this direct sum, every operator in $\mathcal{P}$ is the sum of its *integral part* and its *singular part*.

Theorem 4.9 *Suppose $(\mathcal{X}, m)$ is a finite measure space and $\mathcal{S}$ is a semigroup of positive quasinilpotent operators. If $\mathcal{S}$ contains an operator with non-zero integral part, then $\mathcal{S}$ is reducible.*

Proof. Let T be in $\mathcal{S}$ with non-zero integral part T_0 and let $\mathcal{S}_0$ be the semigroup generated by $\mathcal{S} \cup \{T_0\}$. It suffices to show that $\mathcal{S}_0$ is reducible.

Since T is a positive quasinilpotent operator, so is T_0. By an argument similar to that in the proof of Theorem 4.7, we can prove that $\mathcal{S}_0$ is a semigroup of positive quasinilpotent operators. Let $\mathcal{J}$ be the ideal in $\mathcal{S}_0$ generated by the non-zero integral operator T_0. Then $\mathcal{J}$ consists of only integral operators by Theorem 1.23. Therefore $\mathcal{J}$ is reducible by Theorem 4.7 and then so is $\mathcal{S}_0$ by Lemma 4.1. ∎

Theorem 4.10 *Suppose $(\mathcal{X}, m)$ is a finite measure space and $\mathcal{S}$ is a semigroup of compact positive operators with the property that $r(ST) \leq r(S)r(T)$ for every pair S and T in $\mathcal{S}$. If $\mathcal{S}$ contains either a non-quasinilpotent operator or a non-zero integral operator, then $\mathcal{S}$ is reducible.*

Proof. Suppose $\mathcal{S}$ contains a non-quasinilpotent operator T. As in the proof of Theorem 4.8, we may assume that $\mathcal{S}$ is norm-closed and that any non-negative scalar multiple of an operator in $\mathcal{S}$ is still in $\mathcal{S}$. And in particular, we may assume that $r(T) = 1$. Since T is compact, by an argument similar to the one in the proof of Theorem 4.8, the norm limit of a sequence of powers of scaler multiples of T is a non-zero finite-rank operator, and hence, an integral operator. Thus $\mathcal{S}$ contains a non-zero integral operator.

Now, WNLG, we assume that $\mathcal{S}$ contains a non-zero integral operator S. By Theorem 1.23, the ideal $\mathcal{J}$ in $\mathcal{S}$ generated by S is non-zero and consists of only integral operators. It follows from Theorem 4.8 that $\mathcal{J}$ is reducible, and therefore, $\mathcal{S}$ is reducible. ∎

A *maximal subspace chain* is a chain of subspaces of a Hilbert space that is not properly contained in any other chain of subspaces.

Definition 4.11 [41, Definition, p271] A collection of bounded linear operators on a Hilbert space is *(simultaneously) triangularizable* if there exists a maximal subspace chain each of whose members is invariant under all the operators in the collection.

Theorem 4.12 *Suppose $(\mathcal{X}, m)$ is a finite measure space and $\mathcal{A}$ is a norm-closed algebra of compact operators in $\mathcal{P}_s$. Then $\mathcal{A}$ consists of only quasinilpotent operators and hence is triangularizable.*

Proof. Since $\mathcal{A}$ is contained in $\mathcal{P}_s$, it contains no integral operators other than 0, and hence, no finite-rank operators other than 0. Consequently, every operator in $\mathcal{A}$ must be quasinilpotent since $\mathcal{A}$ is a norm-closed algebra of compact operators.

It follows from Lomonosov's results (see [34, theorem 10]) that $\mathcal{A}$ is triangularizable. ∎

4.2 Standard Invariant Subspaces

In this section, we assume further that $\mathcal{X}$ is a locally compact Hausdorff space and that $\mathcal{X}$ is second countable, *i.e.*, has a countable base for its topology. Recall that a standard subspace of $\mathcal{L}^2(\mathcal{X}, m)$ is a subspace of the form

$$\mathcal{M}_U = \chi_U \mathcal{L}^2(\mathcal{X}, m) = \{f \in \mathcal{L}^2(\mathcal{X}, m) : f = 0 \text{ } a.e. \text{ on } U^c\}$$

for some Borel set U in $\mathcal{X}$. The idea of the standard subspace comes from the concept of decomposability for matrices with non-negative entries, as well as the concept of band in the Banach lattice theory (see [54] and [61]). The definition was introduced in [14].

Suppose T is a positive quasinilpotent integral operator on $\mathcal{L}^2(\mathcal{X}, m)$. Then the semigroup generated by T consists of only positive quasinilpotent operators. Theorem 4.7 tells us that T has a non-trivial invariant subspace. But, just from the proof of Theorem 4.7, there is no way we can determine whether the existing non-trivial invariant subspace is a standard subspace or not. When dealing with the case of single operator, the following result, which is a special case of the Andô-Krieger Theorem, obtained by Andô [2] for compact operators and generalized by Krieger [33], is much more powerful. It not only tells us that any positive quasinilpotent integral operator on $\mathcal{L}^2(\mathcal{X}, m)$ has a non-trivial invariant subspace, but also indicates that the invariant subspace is a standard one. The proof of the Andô-Krieger Theorem can be found in [54, p.336] and [61, p.621].

Proposition 4.13 *If T is a positive quasinilpotent integral operator on $\mathcal{L}^2(\mathcal{X}, m)$, then T has a non-trivial standard invariant subspace.*

Proof. See Cororllary 4.26, or [54, p.336], or [61, p.621]. ∎

Remark. It follows from Proposition 1.5 that a positive integral operator on the Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ cannot be quasinilpotent if its kernel is positive almost everywhere on $\mathcal{X} \times \mathcal{X}$.

It was proved in [14] that any semigroup of positive quasinilpotent integral operators on $\mathcal{L}^2(\mathcal{X}, m)$ with lower semicontinuous kernels has non-trivial standard invariant subspaces. The main theorems in this section are generalizations of this result.

For any Borel set E in $\mathcal{X}$, the fact that $\mathcal{X}$ is second countable implies the existence of a maximal open set U in $\mathcal{X}$ with the property that $m(E \cap U) = 0$. Indeed, using the usual set inclusion as a partial order, the collection of all open subsets of $\mathcal{X}$ with the property that their intersections with E have measure zero is a partially ordered set. For any given chain in the collection, the union of all elements in the chain is still an open subset of $\mathcal{X}$ and, because $\mathcal{X}$ is second countable, is equal to the union of countable many elements in the chain. It follows that the union remains in the collection and is the maximum among all elements in the chain. By Zorn's Lemma, there exists a maximal element U in the collect, *i.e.*, U is a maximal open subset of $\mathcal{X}$ with the property that $m(E \cap U) = 0$. Consequently, if we let $E_1 = E \cap U^c$, then $m(E \backslash E_1) = m(E \cap U) = 0$. Suppose V is an open set in $\mathcal{X}$ such that $m(E_1 \cap V) = 0$. Then

$$m(E \cap V) = m([E_1 \cup (E \backslash E_1)] \cap V) = 0.$$

Thus, by the maximality of U, we have that $V \subseteq U$ and hence $E_1 \cap V = \emptyset$.

Lemma 4.14 *For any Borel set $E \subseteq \mathcal{X}$ with $m(E) \neq 0$, there exists a Borel set $E_1 \subseteq E$ with $m(E \backslash E_1) = 0$ such that $m(E_1 \cap V) > 0$ for all open sets V satisfying $E_1 \cap V \neq \emptyset$.*

Proof. It follows from the above analysis. ■

By applying Lemma 4.14 to $\mathcal{X}$ itself and disregarding a subset of measure zero if necessary, we may assume that every non-empty open set in $\mathcal{X}$ has positive measure. We make this assumption throughout the rest of this section.

Lemma 4.15 *Suppose ϕ is a measurable function on $\mathcal{X} \times \mathcal{X}$. If U and V are Borel sets in $\mathcal{X}$, then the following are equivalent:*

(*i*) $\phi = 0$ a.e. *on* $U \times V$.

(*ii*) *For almost every* $x \in U$, $\phi(x, \cdot) = 0$ a.e. *on* V.

(*iii*) *For almost every* $y \in V$, $\phi(\cdot, y) = 0$ a.e. *on* U.

Proof. Since m is σ-finite, we may assume that $m(U)$ and $m(V)$ are finite.

Let

$$D = \{(x,y)\in U\times V : \phi(x,y) \neq 0\},$$

and for every $x\in U$, $y\in V$, let

$$D_x = \{y\in V : \phi(x,y) \neq 0\},$$

$$D^y = \{x\in U : \phi(x,y) \neq 0\}.$$

By Fubini's Theorem,

$$\begin{aligned}\int_{U\times V} \chi_D(x,y)(m\times m)(dx,dy) &= \int_U\int_V \chi_D(x,y)m(dy)m(dx) \\ &= \int_V\int_U \chi_D(x,y)m(dx)m(dy).\end{aligned}$$

Therefore,

$$(m\times m)(D) = \int_U m(D_x)m(dx) = \int_V m(D^y)m(dy).$$

Thus, $(m\times m)(D) = 0$ if and only if $m(D_x) = 0$ for almost every $x\in U$, and if and only if $m(D^y) = 0$ for almost every $y\in V$. ■

Lemma 4.16 *Suppose $\{\phi_\alpha\}$ is a class of measurable functions on $\mathcal{X} \times \mathcal{X}$, and V is a Borel set in $\mathcal{X}$ with $m(V) > 0$. Then there exists a maximal open subset U of $\mathcal{X}$ with the property that $\phi_\alpha = 0$ a.e. on $U\times V$ for all α, and a maximal open subset W of $\mathcal{X}$ with the property that $\phi_\alpha = 0$ a.e. on $V\times W$ for all α.*

Proof. Consider the collection of all open subsets O of $\mathcal{X}$ with the property that $\phi_\alpha = 0$ *a.e.* on $O\times V$ for all α. The collection is non-empty since it contains the empty set.

Use the usual set inclusion as the partial order and choose an arbitrary chain $\mathcal{O}$ in the collection. Let O denote the union of all members of $\mathcal{O}$. Then O is open in $\mathcal{X}$. The fact that $\mathcal{X}$ is second countable implies that O is actually equal to a countable union of members of $\mathcal{O}$. Consequently, $\phi_\alpha = 0$ *a.e.* on $O\times V$ for all α, and hence O is the maximum of all members of $\mathcal{O}$. By Zorn's Lemma, there exists a maximal element U in the collection, *i.e.*, U is a maximal open subset of $\mathcal{X}$ with the property that $\phi_\alpha = 0$ *a.e.* on $U \times V$ for all α.

Similarly, we can obtain a maximal open subset W of $\mathcal{X}$ with the property that $\phi_\alpha = 0$ *a.e.* on $V \times W$ for all α. ∎

Definition 4.17 A function f on $\mathcal{X}$ is said to have *closed zero-set* if the set

$$\{x \in \mathcal{X} : f(x) = 0\}$$

is a closed subset of $\mathcal{X}$.

Suppose f is a non-negative function on $\mathcal{X}$. Then f has closed zero-set if and only if the set given by

$$\{x \in \mathcal{X} : f(x) > 0\}$$

is an open subset of $\mathcal{X}$. We will use this as an equivalent definition of having closed zero-set for non-negative functions.

Theorem 4.18 *Let $\mathcal{S}$ be a semigroup of positive integral operators. Suppose $\mathcal{S}$ satisfies the following conditions:*

(*i*) *for every $S = \operatorname{Int} k_S$ in $\mathcal{S}$, $k_S(\cdot, y)$ has closed zero-set for almost every $y \in \mathcal{X}$,*

(*ii*) *there exists a measurable rectangle $U \times V$ with $m(U)m(V) > 0$ such that $k_S = 0$ a.e. on $U \times V$ for all $S = \operatorname{Int} k_S$ in $\mathcal{S}$,*

(*iii*) *$\mathcal{S}$ has a countable weakly dense subset $\mathcal{S}_0$.*

Then $\mathcal{S}$ has a non-trivial standard invariant subspace.

Proof. By Lemma 4.16, there exists a maximal open set V_0 such that $k_S = 0$ *a.e.* on $U \times V_0$ for all $S = \operatorname{Int} k_S$ in $\mathcal{S}$. Therefore $k_S = 0$ *a.e.* on $U \times (V \cup V_0)$ for all S in

$\mathcal{S}$ by (ii). If either $m(U^c) = 0$ or $m((V \cup V_0)^c) = 0$, then we are done. Thus we may assume that both U^c and $(V \cup V_0)^c$ have non-zero measure.

Let $A = \operatorname{Int} k_A$ be an arbitrary operator in $\mathcal{S}$ with k_A non-zero on a subset of $U^c \times (V \cup V_0)$ of positive measure, and let, for every $y \in \mathcal{X}$,

$$U_y = \{x \in \mathcal{X} : k_A(x, y) \neq 0\}.$$

By Condition (i), there exists a Borel set $X_A \subseteq V \cup V_0$ with $m(X_A) = 0$ such that U_y is open for every y in $(V \cup V_0) \setminus X_A$.

If we can prove that $U_y \subseteq V_0$ for almost every y in $V \cup V_0$, then, by the definition of U_y, we have $k_A = 0$ *a.e.* on $V_0^c \times (V \cup V_0)$. Thus $k_A = 0$ *a.e.* on $(V \cup V_0)^c \times (V \cup V_0)$ for all $A \in \mathcal{S}$. Clearly, $m(V \cup V_0) m((V \cup V_0)^c) > 0$, and therefore, it follows from Proposition 1.5 that $\mathcal{S}$ has a non-trivial standard invariant subspace.

Indeed, for any $S \in \mathcal{S}$, since $k_{SA} = k_S * k_A = 0$ *a.e.* on $U \times (V \cup V_0)$, there exists a Borel set $Y_S \subseteq V \cup V_0$ with $m(Y_S) = 0$ such that

$$(k_S * k_A)(\cdot, y) = 0 \quad a.e. \quad on \quad U$$

for every y in $(V \cup V_0) \setminus Y_A$.

Fix an arbitrary element y in $(V \cup V_0) \setminus [X_A \cup (\cup_{S \in \mathcal{S}_0} Y_S)]$. For any $S \in \mathcal{S}_0$, by the definition of the set Y_S, we have $(k_S * k_A)(\cdot, y) = 0$ *a.e.* on U, *i.e.*,

$$\int_{\mathcal{X}} k_S(x, t) k_A(t, y) m(dt) = 0$$

for almost every $x \in U$. Hence, for almost every $x \in U$, $k_S(x, \cdot) = 0$ *a.e.* on U_y since $k_A(t, y) > 0$ for all $t \in U_y$. By Lemma 4.15, $k_S = 0$ *a.e.* on $U \times U_y$. Therefore Condition (iii) and the fact that

$$\int_{U \times U_y} k_S(x, y)(m \times m)(dx, dy) = \langle S \chi_{U_y}, \chi_U \rangle,$$

imply that $k_S = 0$ *a.e.* on $U \times U_y$ for every $S \in \mathcal{S}$. It follows from the maximality of V_0 that $U_y \subseteq V_0$. Thus, $U_y \subseteq V_0$ for almost every $y \in V \cup V_0$ since $m(X_A) = m(Y_S) = 0$ for all S and then $m(\mathcal{X}_A \cup (\cup_{S \in \mathcal{S}_0} Y_S)) = 0$. ∎

Suppose $\mathcal{T}$ is a collection of operators on $\mathcal{L}^2(\mathcal{X}, m)$. Let

$$\mathcal{T}^* = \{T^* : T \in \mathcal{T}\}.$$

Notice that $(\mathcal{M}_E)^\perp = \mathcal{M}_{E^c}$ for every Borel set E in $\mathcal{X}$. We have the following corollary.

Corollary 4.19 *Let $\mathcal{S}$ be a semigroup of positive integral operators. Suppose $\mathcal{S}$ satisfies the following conditions:*

(*i*) *for every $S = \operatorname{Int} k_S$ in $\mathcal{S}$, $k_S(x, \cdot)$ has closed zero-set for almost every $x \in \mathcal{X}$,*

(*ii*) *there exists a measurable rectangle $U \times V$ with $m(U)m(V) > 0$ such that $k_S = 0$ a.e. on $U \times V$ for all $S = \operatorname{Int} k_S$ in $\mathcal{S}$,*

(*iii*) *$\mathcal{S}$ has a countable weakly dense subset $\mathcal{S}_0$.*

Then $\mathcal{S}$ has a non-trivial standard invariant subspace.

Proof. By Proposition 1.1 or Theorem 1.21, for every $S \in \mathcal{S}$,

$$k_{S^*}(x, y) = k_S(y, x)$$

for almost every $(x, y) \in \mathcal{X} \times \mathcal{X}$. Therefore, it follows immediately from Theorem 4.18 and the above analysis that $\mathcal{S}$ has a non-trivial standard invariant subspace. ∎

Next, we give another set of conditions under which every semigroup of positive integral operators on $\mathcal{L}^2(\mathcal{X}, m)$ will have a non-trivial standard invariant subspace. We are interested in the case where the kernel of integral operator has closed zero-set in both coordinate direction.

Proposition 4.20 *Suppose ϕ and ψ are non-negative Borel functions on $\mathcal{X} \times \mathcal{X}$. If $\phi(x, \cdot)$, $\phi(\cdot, y)$, $\psi(x, \cdot)$ and $\psi(\cdot, y)$ all have closed zero-sets for any x and y in $\mathcal{X}$, then $(\phi * \psi)(x, \cdot)$ and $(\phi * \psi)(\cdot, y)$ also have closed zero-sets for any x and y in $\mathcal{X}$.*

Proof. For any $x \in \mathcal{X}$, if $(\phi * \psi)(x, y_0) > 0$ for some $y_0 \in \mathcal{X}$, *i.e.*,

$$\int_{\mathcal{X}} \phi(x,z)\psi(z,y_0)m(dz) > 0,$$

then the set E given by

$$E = \{z \in \mathcal{X} : \phi(x,z) > 0 \quad \textit{and} \quad \psi(z,y_0) > 0\}$$

is a non-empty open set in $\mathcal{X}$ since both $\phi(x,\cdot)$ and $\psi(\cdot,y_0)$ have closed zero-sets.

Choose any $z_0 \in E$. It follows that the set O given by

$$O = \{y \in \mathcal{X} : \psi(z_0, y) > 0\}$$

is open in $\mathcal{X}$ and contains y_0. Therefore, for any $y \in O$, the set

$$\{z \in \mathcal{X} : \psi(z,y) > 0\}$$

is open and contains z_0. Hence

$$z_0 \in F \equiv E \cap \{z \in \mathcal{X} : \psi(z,y) > 0\},$$

and F is open in $\mathcal{X}$. Consequently, $m(F) > 0$. Thus, for any $y \in O$,

$$\begin{aligned} (\phi * \psi)(x,y) &= \int_{\mathcal{X}} \phi(x,z)\psi(z,y)m(dz) \\ &\geq \int_F \phi(x,z)\psi(z,y)m(dz) \\ &> 0. \end{aligned}$$

By the definition, $(\phi * \psi)(x,\cdot)$ has closed zero-set.

Similarly, for any $y \in \mathcal{X}$, $(\phi * \psi)(\cdot, y)$ has closed zero-set. ∎

With the kernels of operators in $\mathcal{S}$ having closed zero-set in both coordinate direction, we can drop the condition of having a countable weakly dense subset in the statement of Theorem 4.18.

Theorem 4.21 *Let $\mathcal{S}$ be a semigroup of positive integral operators. Suppose $\mathcal{S}$ satisfies the following conditions:*

(*i*) *for every* $S = \operatorname{Int} k_S$ *in* $\mathcal{S}$, $k_S(x,\cdot)$ *and* $k_S(\cdot,y)$ *have closed zero-set for all* x *and* y *in* $\mathcal{X}$,

(*ii*) *there exists a measurable rectangle* $U \times V$ *with* $m(U)m(V) > 0$ *such that* $k_S = 0$ *a.e. on* $U \times V$ *for all* $S = \operatorname{Int} k_S$ *in* $\mathcal{S}$.

Then $\mathcal{S}$ *has a non-trivial standard invariant subspace.*

Proof. By Lemma 4.16, there is a maximal open set U_0 in $\mathcal{X}$ with the property that $k_S = 0$ *a.e.* on $U_0 \times V$ for all $S = \operatorname{Int} k_S$ in $\mathcal{S}$. Hence, by Condition (ii), $k_S = 0$ *a.e.* on $W \times V$ for all $S = \operatorname{Int} k_S$ in $\mathcal{S}$, where $W = U \cup U_0$. Applying Lemma 4.14, we may assume, by disregarding a subset of measure zero from $\mathcal{X}$ if necessary, that $m(O \cap W) > 0$ for all open sets O satisfying $O \cap W \neq \emptyset$, and that $m(O \cap V) > 0$ for all open sets O satisfying $O \cap V \neq \emptyset$.

If $m(W^c) = 0$, then we are done. So we may assume that $m(W^c) > 0$. We complete the proof by showing that $k_S = 0$ *a.e.* on $W \times W^c$ for all $S = \operatorname{Int} k_S$ in $\mathcal{S}$, and consequently, $\mathcal{S}$ has a non-trivial standard invariant subspace by Proposition 1.5.

For any $S = \operatorname{Int} k_S$ in $\mathcal{S}$, and any $x \in W$, $z \in V$, let

$$W_x(S) = \{y \in \mathcal{X} : k_S(x,y) > 0\},$$

$$V^z(S) = \{y \in \mathcal{X} : k_S(y,z) > 0\}.$$

Then, by condition (i), $W_x(S)$ and $V^z(S)$ are open in $\mathcal{X}$ for all $x \in W$ and $z \in V$. Suppose we can prove that $W_x(S) \cap V^z(A) = \emptyset$ for any $x \in W$, $z \in V$, and for any $S = \operatorname{Int} k_S$, $A = \operatorname{Int} k_A$ in $\mathcal{S}$. Fix an arbitray $x \in W$ and an arbitray $S \in \mathcal{S}$. For any $(y,z) \in W_x(S) \times V$, we have that y is in $W_x(S)$ and hence not in $V^z(A)$ for all $A \in \mathcal{S}$. By the definition of the set $V^z(A)$, $k_A(y,z) = 0$ and then $k_A = 0$ *a.e.* on on $W_x(S) \times V$ for all $A \in \mathcal{S}$. It follows from the maximality of U_0 that $W_x(S) \subseteq U_0 \subseteq W$ for all $x \in W$ and $S \in \mathcal{S}$. Thus, by the definition of the set $W_x(S)$, $k_S = 0$ *a.e.* on $W \times W^c$ for all $S = \operatorname{Int} k_S$ in $\mathcal{S}$.

It remains to show that $W_x(S) \cap V^z(A) = \emptyset$ for every $x \in W$, $z \in V$, and for every $S = \operatorname{Int} k_S$, $A = \operatorname{Int} k_A$ in $\mathcal{S}$. Indeed, suppose $W_x(S) \cap V^z(A) \neq \emptyset$ for some $x \in W$,

$z \in V$, and for some $S = \operatorname{Int} k_S$, $A = \operatorname{Int} k_A$ in $\mathcal{S}$, then $m(W_x(S) \cap V^z(A)) > 0$ since we assume that every non-empty open subset of $\mathcal{X}$ has non-zero measure. Therefore

$$\begin{aligned} k_{SA}(x,z) &= (k_S * k_A)(x,z) \\ &= \int_{\mathcal{X}} k_S(x,t) k_A(t,z) m(dt) \\ &\geq \int_{W_x(S) \cap V^z(A)} k_S(x,t) k_A(t,z) m(dt) \\ &> 0. \end{aligned}$$

By Condition (i), the set E given by

$$E = \{y \in \mathcal{X} : k_{SA}(x,y) > 0\}$$

is open in $\mathcal{X}$ and contains z. It follows that $m(E \cap V) > 0$. For each $y \in E \cap V$, again by condition (i), the set given by

$$F^y = \{t \in \mathcal{X} : k_{SA}(t,y) > 0\}$$

is open and contains x. As a result, $m(F^y \cap W) > 0$. Therefore, from Lemma 4.15, k_{SA} is non-zero on a subset of $W \times (E \cap V) \subseteq W \times V$ of positive measure. This contradicts the fact that $k_S = 0$ *a.e.* on $W \times V$ for all $S = \operatorname{Int} k_S$ in $\mathcal{S}$. ∎

Lemma 4.22 *Suppose $\mathcal{S}$ is a semigroup of positive quasinilpotent integral operators. If there exists an operator $T \in \mathcal{S}$ whose kernel k_T is positive almost everywhere on a measurable rectangle $U \times V$ of positive measure, then $k_S = 0$* a.e. *on $V \times U$ for all $S = \operatorname{Int} k_S$ in $\mathcal{S}$.*

Proof. We prove the result by a contradiction.

Suppose there exists an operator $S \in \mathcal{S}$ whose kernel k_S is non-zero on $G \subseteq V \times U$ and $(m \times m)(G) > 0$. For each $x \in V$, let

$$G_x = \{y \in U : (x,y) \in G\}.$$

Clearly, G_x is a Borel set in $\mathcal{X}$ for almost every $x \in V$. By Lemma 4.15, there exists a Borel set $V_0 \subseteq V$ with $m(V_0) > 0$ such that $m(G_x) > 0$ for almost every $x \in V_0$.

For almost every $(x,y)\in V_0\times V_0$,

$$\begin{aligned}(k_S{*}k_T)(x,y) &= \int_{\mathcal{X}} k_S(x,t)k_T(t,y)m(dt)\\ &\geq \int_U k_S(x,t)k_T(t,y)m(dt)\\ &\geq \int_{G_x} k_S(x,t)k_T(t,y)m(dt)\\ &> 0.\end{aligned}$$

Therefore the kernel k_{ST} of $ST \in \mathcal{S}$ is positive almost everywhere on $(x,y)\in V_0\times V_0$. But $k_{ST}|_{V_0\times V_0}$ is non-negative and induces a quasinilpotent integral operator on $\mathcal{L}^2(V_0, m|_{V_0})$ because the operator $ST \in \mathcal{S}$ is quasinilpotent. This contradicts Proposition 4.13 and the remark following its proof. ∎

Corollary 4.23 *Let $\mathcal{S}$ be a semigroup of positive quasinilpotent integral operators. Suppose $\mathcal{S}$ satisfies the following conditions:*

(*i*) *either for every $S = \operatorname{Int} k_S$ in $\mathcal{S}$, $k_S(\cdot,y)$ has closed zero-set for almost every $y \in \mathcal{X}$, or for every $S = \operatorname{Int} k_S$ in $\mathcal{S}$, $k_S(x,\cdot)$ has closed zero-set for almost every $x \in \mathcal{X}$,*

(*ii*) *there exists an operator $T \in \mathcal{S}$ whose kernel k_T is positive almost everywhere on a measurable rectangle $U\times V$ of positive measure,*

(*iii*) *$\mathcal{S}$ has a countable weakly dense subset $\mathcal{S}_0$.*

Then $\mathcal{S}$ has a non-trivial standard invariant subspace.

Proof. It follows immediately from Lemma 4.22, Theorem 4.18 and Corollary 4.19. ∎

Corollary 4.24 *Let $\mathcal{S}$ be a semigroup of positive quasinilpotent integral operators. Suppose $\mathcal{S}$ satisfies the following conditions:*

(*i*) *for every* $S = \operatorname{Int} k_S$ *in* $\mathcal{S}$, $k_S(x,\cdot)$ *and* $k_S(\cdot,y)$ *have closed zero-sets for all* x *and* y *in* $\mathcal{X}$,

(*ii*) *there exists an operator* $T \in \mathcal{S}$ *whose kernel* k_T *is positive almost everywhere on a measurable rectangle* $U \times V$ *of positive measure.*

Then $\mathcal{S}$ *has a non-trivial standard invariant subspace.*

Proof. It follows immediately from Lemma 4.22 and Theorem 4.21. ∎

4.3 An Application of the Lomonosov-Hilden Technique

In 1973, Lomonosov proved one of the best-known results in operator theory [35]: any non-zero compact operator T on an infinite dimensional Hilbert space has a non-trivial hyperinvariant subspace, *i.e.*, a subspace invariant under every operator commuting with T. The proof is simple and involves constructing a (non-linear) map that satisfies the condition of Schauder's fixed point theorem. After being told the result, H. M. Hilden found an even simpler proof that only requires very basic knowledge of functional analysis (see [38] or [48, Corollary 8.25]).

In [43], de Pagter proved by using the Lomonosov-Hilden technique that any positive compact quasinilpotent operator on a Banach lattice of dimension at least two has a non-trivial invariant closed ideal. de Pagter's proof actually proves that any non-zero positive compact quasinilpotent operator T on a Banach lattice of dimension at least two has a non-trivial invariant closed ideal that is also invariant under every positive operator commuting with T. If we view $\mathcal{L}^2(\mathcal{X},m)$, with its natural order, as a Banach lattice, then a subspace of $\mathcal{L}^2(\mathcal{X},m)$ is a closed ideal if and only if it is a standard subspace (see [54] and [61]). The following theorem combines several results in [43]. The idea of the proof is virtually the same, but we state it in a way which can be easily understood by those unfamiliar with the theory of Banach lattices.

Theorem 4.25 [43, Proposition 2 and Proposition 4] *Let T be a positive quasinilpotent operator on $\mathcal{L}^2(\mathcal{X}, m)$. If there exists a non-zero positive compact operator T_0 on $\mathcal{L}^2(\mathcal{X}, m)$ such that $T_0 \leq T$, then T has a non-trivial standard invariant subspace that is also invariant under every positive operator commuting with T.*

Proof. Let

$$\mathcal{T} = \{S \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m)) : S \text{ is positive and } ST = TS\},$$

$$\mathcal{T}_1 = \{S \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m)) : 0 \leq S \leq R \text{ for some } R \in \mathcal{T}\}.$$

Clearly, $\{I, T\} \subseteq \mathcal{T} \subseteq \mathcal{T}_1$, and both $\mathcal{T}$ and $\mathcal{T}_1$ are closed under products and positive linear combinations. Let $\mathcal{A}$ be the algebra generated by $\mathcal{T}_1$. Then every member of $\mathcal{A}$ is actually a linear combination of elements of $\mathcal{T}_1$. Since $I \in \mathcal{T}$, all multiplication operators M_ϕ with $\phi \in \mathcal{L}^\infty(\mathcal{X}, m)$ are in $\mathcal{A}$.

For any $f \in \mathcal{L}^2(\mathcal{X}, m)$, $f \neq 0$, the subspace $\overline{\mathcal{A}f}$ is obviously invariant under $\mathcal{A}$. We claim that $\overline{\mathcal{A}f}$ is a standard subspace of $\mathcal{L}^2(\mathcal{X}, m)$. Indeed, let P be the orthogonal projection on $\overline{\mathcal{A}f}$. Then, by the fact that $\overline{\mathcal{A}f}$ is invariant under $\mathcal{A}$ and $\mathcal{A}$ contains all multiplication operators M_ϕ with $\phi \in \mathcal{L}^\infty(\mathcal{X}, m)$, the projection P commutes with every M_ϕ. However, the collection of all multiplication operators M_ϕ with $\phi \in \mathcal{L}^\infty(\mathcal{X}, m)$ is a maximal abelian selfadjoint algebra. Hence P is a multiplication operator of some characteristic function χ_E where E is a measurable subset of $\mathcal{X}$. It follows that $\overline{\mathcal{A}f} = \mathcal{M}_E$ is a standard subspace of $\mathcal{L}^2(\mathcal{X}, m)$.

Now, $\overline{\mathcal{A}f}$ is a non-zero standard subspace invariant under $\mathcal{T}$ for any non-zero $f \in \mathcal{L}^2(\mathcal{X}, m)$. Therefore, the proof of the theorem will be completed if we can prove that $\overline{\mathcal{A}f} \neq \mathcal{L}^2(\mathcal{X}, m)$ for some non-zero $f \in \mathcal{L}^2(\mathcal{X}, m)$.

Suppose $\overline{\mathcal{A}f} = \mathcal{L}^2(\mathcal{X}, m)$ for all $f \neq 0$. Choose a non-zero positive element h in $\mathcal{L}^2(\mathcal{X}, m)$ such that $T_0 h \neq 0$. Fix an open ball V centering at h such that $0 \notin \overline{V}$ and $0 \notin \overline{T_0 V}$. This can be done by choosing a positive number ϵ small enough such that $\|T_0 h\| - \epsilon \|T_0\| > 0$ (which implies $\|h\| > \epsilon$ automatically), and letting

$$V = \{g \in \mathcal{L}^2(\mathcal{X}, m) : \|g - h\| < \epsilon\}.$$

For any $f \in \overline{T_0V}$, we have $f \neq 0$, and therefore, $\overline{\mathcal{A}f} = \mathcal{L}^2(\mathcal{X}, m)$. Hence $h \in \overline{\mathcal{A}f}$. It follows that there exists an operator $S_f \in \mathcal{A}$ such that $S_f f \in V$. Let V_f be an open neighborhood of f such that $S_f V_f \subseteq V$. Then $\{V_f : f \in \overline{T_0V}\}$ is an open cover of $\overline{T_0V}$ which is a compact set in $\mathcal{L}^2(\mathcal{X}, m)$ since T_0 is compact. Therefore, there exist $f_1, f_2, \ldots, f_p$ in $\overline{T_0V}$ such that

$$\overline{T_0V} \subseteq V_{f_1} \cup V_{f_2} \cup \cdots \cup V_{f_p}.$$

We simply denote V_{f_j} by V_j and S_{f_j} by S_j, $(j = 1, 2, \ldots, p)$.

Since $T_0 h \in T_0 V$, we have $T_0 h \in V_{p_1}$ for some p_1 between 1 and p. It follows that

$$S_{p_1} T_0 h \in S_{p_1} V_{p_1} \subseteq V.$$

Again, we have $T_0 S_{p_1} T_0 h \in T_0 V$, and hence, $T_0 S_{p_1} T_0 h \in V_{p_2}$ for some p_2 between 1 and p, and

$$S_{p_2} T_0 S_{p_1} T_0 h \in S_{p_2} V_{p_2} \subseteq V.$$

Repeating this process, we obtain a sequence $\{p_j\}_{j=1}^{\infty}$ in $\{1, 2, \ldots, p\}$ such that

$$h_j \equiv S_{p_j} T_0 S_{p_{j-1}} T_0 \cdots S_{p_2} T_0 S_{p_1} T_0 h \in V$$

for all positive integer j.

For each j, $1 \leq j \leq p$, $S_j \in \mathcal{A}$, therefore, it follows from the definition of $\mathcal{A}$ that there exists an operator $R_j \in \mathcal{T}$ such that

$$|S_j g| \leq R_j g$$

for all positive $g \in \mathcal{L}^2(\mathcal{X}, m)$. Let

$$\gamma = \max\{\|R_1\|, \|R_2\|, \ldots, \|R_p\|\}.$$

Then, for each j, $(j = 1, 2, \cdots)$.

$$\begin{aligned}
|h_j| &= |S_{p_j} T_0 S_{p_{j-1}} T_0 \cdots S_{p_2} T_0 S_{p_1} T_0 h| \\
&\leq R_{p_j} T_0 R_{p_{j-1}} T_0 \cdots R_{p_2} T_0 R_{p_1} T_0 h \\
&\leq R_{p_j} T R_{p_{j-1}} T \cdots R_{p_2} T R_{p_1} T h \\
&= R_{p_j} R_{p_{j-1}} \cdots R_{p_2} R_{p_1} T^j h.
\end{aligned}$$

Hence,

$$\begin{aligned} \|h_j\| &\leq \gamma^j \|T^j\| \, \|h\| \\ &= \|(\gamma T)^j\| \, \|h\| \longrightarrow 0, \end{aligned}$$

as $j \longrightarrow \infty$ since T is a quasinilpotent operator. This contradicts the fact that $0 \notin \overline{V}$, and the proof is completed. ∎

Corollary 4.26 *If T is a non-zero positive quasinilpotent integral operator, then T has a non-trivial standard invariant subspace which is also invariant under every positive operator commuting with T.*

Proof. By Lemma 4.6, there exists a non-zero positive Hilbert-Schmidt operator T_0 on $\mathcal{L}^2(\mathcal{X}, m)$ such that $T - T_0$ is also positive. The result follows immediately from Theorem 4.25. ∎

REMARK. The above corollary is a generalization of Proposition 4.13, which is a special case of the Andô-Krieger Theorem ([54] and [61]) when the Banach lattice is actually the functional Hilbert space $\mathcal{L}^2(\mathcal{X}, m)$ with its natural lattice structure. We should point out that this result cannot be obtained by simply applying the Andô-Krieger Theorem to the special case.

Next, we give a generalization of the de Pagter Theorem (Theorem 4.25). We first prove the following lemma.

Lemma 4.27 [42, Lemma 4] *Suppose K is an injective operator and $\mathcal{A}$ is a norm-closed algebra of operators on a Hilbert space. If $\mathcal{A}K \subseteq K\mathcal{A}$, then the map Φ on $\mathcal{A}$ defined by*

$$AK = K\Phi(A)$$

is a continuous algebra homomorphism.

Proof. The map Φ is well-defined since K is injective. Clearly, Φ is an algebra homomorphism. To prove that Φ is continuous, it suffices to show that it is a closed map: if $A = \lim A_j$ and $B = \lim \Phi(A_j)$, then

$$AK = \lim A_j K = \lim K\Phi(A_j) = KB,$$

and thus $B = \Phi(A)$. ∎

Theorem 4.28 *Let T be an injective positive quasinilpotent operator on $\mathcal{L}^2(\mathcal{X}, m)$ dominating a non-zero positive compact operator T_0, i.e., $0 \leq T_0 \leq T$. If $\mathcal{T}$ is a collection of positive operators contained in a norm-closed operator algebra $\mathcal{A}$ with $\mathcal{A}T \subseteq T\mathcal{A}$, then T has a non-trivial standard invariant subspace that is also invariant under every operator in $\mathcal{T}$.*

Proof. We may assume that $\mathcal{T}$ contains the identity operator I, for otherwise, we can replace $\mathcal{T}$ by $\mathcal{T} \cup \{I\}$ and $\mathcal{A}$ by the algebra generated by $\mathcal{A} \cup \{I\}$. Also we may assume that $\mathcal{T}$ is closed under products and positive linear combinations.

Let

$$\mathcal{T}_1 = \{S \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}, m)) : 0 \leq S \leq R \text{ for some } R \in \mathcal{T}\}.$$

Clearly, $\mathcal{T}_1$ contains $\mathcal{T}$ and is also closed under products and positive linear combinations. Let $\mathcal{A}_1$ be the algebra generated by $\mathcal{T}_1$. Then every member of $\mathcal{A}_1$ is actually a linear combination of elements of $\mathcal{T}_1$, and all multiplication operators M_ϕ with $\phi \in \mathcal{L}^\infty(\mathcal{X}, m)$ are in $\mathcal{A}_1$.

For any $f \in \mathcal{L}^2(\mathcal{X}, m)$, $f \neq 0$, the subspace $\overline{\mathcal{A}_1 f}$ is obviously invariant under T and every operator in $\mathcal{T}$. As in the proof of Theorem 4.25, we have that $\overline{\mathcal{A}_1 f}$ is a standard subspace of $\mathcal{L}^2(\mathcal{X}, m)$. We complete the proof by showing that $\overline{\mathcal{A}_1 f} \neq \mathcal{L}^2(\mathcal{X}, m)$ for some non-zero $f \in \mathcal{L}^2(\mathcal{X}, m)$.

Suppose $\overline{\mathcal{A}_1 f} = \mathcal{L}^2(\mathcal{X}, m)$ for all $f \neq 0$. Choose a non-zero positive element h in $\mathcal{L}^2(\mathcal{X}, m)$ such that $T_0 h \neq 0$. Fix an open ball V centering at h such that $0 \notin \overline{V}$ and $0 \notin \overline{T_0 V}$. Again, as in the proof of Theorem 4.25, we can obtain, by the compactness of T_0, a positve integer p, a sequence $\{p_j\}_{j=1}^{\infty}$ in $\{1, 2, \ldots, p\}$, and a sequence $\{S_{p_j}\}$ in $\mathcal{A}_1$ such that

$$h_j \equiv S_{p_j} T_0 S_{p_{j-1}} T_0 \cdots S_{p_2} T_0 S_{p_1} T_0 h \in V$$

for every positive ingeger j.

For each j, $1 \leq j \leq p$, $S_j \in \mathcal{A}_1$, therefore, it follows from the definition of $\mathcal{A}_1$ that there exists an operator $R_j \in \mathcal{T}$ such that

$$|S_j g| \leq R_j g$$

for all positive $g \in \mathcal{L}^2(\mathcal{X}, m)$. Let

$$\gamma = \max\{\|R_1\|, \|R_2\|, \ldots, \|R_p\|\},$$

and let Φ be the continuous algebra homomorphism on $\mathcal{A}$ given by

$$AT = T\Phi(A)$$

as in Lemma 4.27. Then, for each j, $(j = 1, 2, \cdots)$,

$$\begin{aligned} |h_j| &= |S_{p_j} T_0 S_{p_{j-1}} T_0 \cdots S_{p_2} T_0 S_{p_1} T_0 h| \\ &\leq R_{p_j} T_0 R_{p_{j-1}} T_0 \cdots R_{p_2} T_0 R_{p_1} T_0 h \\ &\leq R_{p_j} T R_{p_{j-1}} T \cdots R_{p_2} T R_{p_1} T h \\ &= T^j \Phi(\ldots(\Phi(\Phi(\Phi(R_{p_j}) R_{p_{j-1}}) R_{p_{j-2}}) \ldots\ R_{p_1}) h. \end{aligned}$$

Hence,

$$\begin{aligned} \|h_j\| &\leq \|T^j\| \, \|\Phi\|^j \gamma^j \|h\| \\ &= \|(\gamma \|\Phi\| T)^j\| \, \|h\| \longrightarrow 0, \end{aligned}$$

as $j \longrightarrow \infty$ since T is a quasinilpotent operator. This contradicts the fact that $0 \notin \overline{V}$, and the proof is completed. ■

Corollary 4.29 *Suppose $\mathcal{L}^2(\mathcal{X}, m)$ is separable and T is a positive quasinilpotent operator on $\mathcal{L}^2(\mathcal{X}, m)$ with a dense range dominating a non-zero positive compact operator T_0, If $\mathcal{T}$ is a collection of positive operators contained in a norm-closed operator algebra $\mathcal{A}$ with $T\mathcal{A} \subseteq \mathcal{A}T$, then T has a non-trivial standard invariant subspace that is also invariant under every operator in $\mathcal{T}$.*

Proof. With T^* as the operator in Theorem 4.28, $\mathcal{T}^*$ the collection of positive operators, and $\mathcal{A}^*$ the norm-closed algebra, the conditions of Theorem 4.28 are all satisfied. Therefore T^* has a non-trivial standard invariant subspace $\mathcal{M}_E$ for some Borel set E in $\mathcal{X}$ that is also invariant under every operator in $\mathcal{T}^*$. Thus the standard subspace $\mathcal{M}_{E^c}$ is a non-trivial and invariant under T and every operator in $\mathcal{T}$. ∎

Chapter 5

An Irreducible Semigroup of Positive Nilpotent Operators

In Chapter 4, we have proven that certain semigroups of positive quasinilpotent operators are reducible. One may ask: Is every multiplicative semigroup of positive quasinilpotent operators reducible? In [21, Theorem 1], Hadwin et al constructed an irreducible semigroup of nilpotent operators on a Hilbert space such that every operator in the semigroup has nilpotency two. And in [56], Schaefer provided a positive quasinilpotent operator which does not have any non-trivial standard invariant subspaces. It is easy to see that neither of the two examples anwsers the above question. In this chapter, we construct an irreducible semigroup of positive nilpotent operators.

Consider $\mathcal{L}^2([0,1])$ with Lebesgue measure m on $[0,1]$. For every $\alpha \in [0,1]$, we define S_α and T_α in $\mathcal{B}(\mathcal{L}^2([0,1]))$ as follows:

$$(S_\alpha f)(t) = \begin{cases} f(t+\alpha) & \text{if } t \in [0, 1-\alpha] \\ 0 & \text{if } t \in (1-\alpha, 1] \end{cases} \quad (f \in \mathcal{L}^2([0,1])),$$

$$(T_\alpha f)(t) = \begin{cases} 0 & \text{if } t \in [0, \alpha) \\ f(t-\alpha) & \text{if } t \in [\alpha, 1] \end{cases} \quad (f \in \mathcal{L}^2([0,1])).$$

Clearly, S_α and T_α are well-defined bounded linear operators on $\mathcal{L}^2([0,1])$. For convenience, we define $S_\alpha = T_\alpha = 0$ for all $\alpha > 1$. Still, we denote by M_ϕ the multiplication operator corresponding to $\phi \in \mathcal{L}^\infty(\mathcal{X}, m)$.

Lemma 5.1 *For any* $\alpha \in [0,1]$,

(*i*) S_α *and* T_α *are positive operators.*

(*ii*) $S_\alpha^* = T_\alpha$, $S_0 = T_0 = I$, $S_1 = T_1 = 0$.

(*iii*) $S_\alpha T_\alpha = M_{\chi_{[0,1-\alpha]}}$, $T_\alpha S_\alpha = M_{\chi_{[\alpha,1]}}$, *and therefore,* S_α *and* T_α *are partial isometries.*

(*iv*) *If* $\alpha \neq 1$, *then* $\|S_\alpha\| = \|T_\alpha\| = 1$.

Proof. (i) It is obvious that S_α and T_α are positive operators.

(ii) For any $f, g \in \mathcal{L}^2([0,1])$,

$$\begin{aligned}\langle S_\alpha f, g\rangle &= \int_0^1 (S_\alpha f)(t)\overline{g}(t)dt \\ &= \int_0^{1-\alpha} f(t+\alpha)\overline{g}(t)dt \\ &= \int_\alpha^1 f(s)\overline{g}(s-\alpha)ds \\ &= \langle f, T_\alpha g\rangle.\end{aligned}$$

Thus, $S_\alpha^* = T_\alpha$. Clearly, $S_0 = T_0 = I$, $S_1 = T_1 = 0$.

(iii) For any $f, g \in \mathcal{L}^2([0,1])$,

$$\begin{aligned}\langle S_\alpha T_\alpha f, g\rangle &= \langle T_\alpha f, T_\alpha g\rangle \\ &= \int_\alpha^1 f(t-\alpha)\overline{g}(t-\alpha)dt \\ &= \int_0^{1-\alpha} f(s)\overline{g}(s)ds \\ &= \langle M_{\chi_{[0,1-\alpha]}} f, g\rangle.\end{aligned}$$

Therefore, $S_\alpha T_\alpha = M_{\chi_{[0,1-\alpha]}}$. Similarly, $T_\alpha S_\alpha = M_{\chi_{[\alpha,1]}}$.

Since

$$S_\alpha^* S_\alpha = T_\alpha S_\alpha = M_{\chi_{[\alpha,1]}},$$

$$T_\alpha^* T_\alpha = S_\alpha T_\alpha = M_{\chi_{[0,1-\alpha]}}$$

are projections, we have that S_α and T_α are partial isometries.

(iv) It follows immediately from (iii). ∎

Lemma 5.2 *For any $\alpha \in [0,1)$, and any $\phi \in \mathcal{L}^\infty([0,1])$,*

(i) $S_\alpha M_\phi = M_{S_\alpha \phi} S_\alpha$, $T_\alpha M_\phi = M_{T_\alpha \phi} T_\alpha$.

(ii) $M_\phi S_\alpha$, $S_\alpha M_\phi$, $M_\phi T_\alpha$, *and* $T_\alpha M_\phi$ *are all nilpotent operators.*

Proof. (i) For any $f \in \mathcal{L}^2([0,1])$,

$$S_\alpha(\phi f) = (S_\alpha \phi)(S_\alpha f).$$

Therefore,

$$(S_\alpha M_\phi) f = S_\alpha(\phi f) = (S_\alpha \phi)(S_\alpha f) = (M_{S_\alpha \phi} S_\alpha) f.$$

Hence, $S_\alpha M_\phi = M_{S_\alpha \phi} S_\alpha$.

Similarly, $T_\alpha M_\phi = M_{T_\alpha \phi} T_\alpha$.

(ii) It is obvius that $(S_\alpha)^p = S_{p\alpha}$ for any positive integer p. Therefore, it follows from (i) that

$$(M_\phi S_\alpha)^p = M_\phi M_{S_\alpha \phi} \cdots M_{S_{(p-1)\alpha} \phi} S_{p\alpha}$$

for any positive integer p. Hence, $(M_\phi S_\alpha)^p = 0$ for p large enough to satisfy $p\alpha > 1$. Thus. $M_\phi S_\alpha$ is a nilpotent operator.

Similarly, $S_\alpha M_\phi$, $M_\phi T_\alpha$, and $T_\alpha M_\phi$ are all nilpotent operators. ∎

Lemma 5.3 *If α, $\beta \in [0,1]$, then* *(i)* $S_\alpha S_\beta = S_{\alpha+\beta}$ *and* $T_\alpha T_\beta = T_{\alpha+\beta}$.

(ii)

$$S_\alpha T_\beta = \begin{cases} M_{\chi_{[0,1-\alpha]}} T_{\beta-\alpha} & \text{if } \alpha \le \beta \\ M_{\chi_{[0,1-\alpha]}} S_{\alpha-\beta} & \text{if } \alpha > \beta \end{cases},$$

(iii)

$$T_\beta S_\alpha = \begin{cases} M_{\chi_{[\beta,1]}} T_{\beta-\alpha} & \text{if } \alpha \le \beta \\ M_{\chi_{[\beta,1]}} S_{\alpha-\beta} & \text{if } \alpha > \beta \end{cases}.$$

Proof. (i) It is easy to check.

(ii) If $\alpha \le \beta$, then by (i) and Lemma 5.1,

$$S_\alpha T_\beta = S_\alpha T_\alpha T_{\beta-\alpha} = M_{\chi_{[0,1-\alpha]}} T_{\beta-\alpha}.$$

If $\alpha > \beta$, then by (i), Lemma 5.1 and Lemma 5.2,

$$\begin{aligned} S_\alpha T_\beta &= S_{\alpha-\beta} S_\beta T_\beta \\ &= S_{\alpha-\beta} M_{\chi_{[0,1-\beta]}} \\ &= M_{S_{\alpha-\beta}\chi_{[0,1-\beta]}} S_{\alpha-\beta} \\ &= M_{\chi_{[0,1-\alpha]}} S_{\alpha-\beta}. \end{aligned}$$

(iii) The proof is similar to the proof of (ii). ∎

In [57] it was proved that every positive operator S on $\mathcal{L}^2([0,1])$ is a pseudo-integral operator, and that S is determined by a positive finite Borel measure μ_S on $[0,1] \times [0,1]$ by the equation

$$\langle Sf, g \rangle = \int_{[0,1]\times[0,1]} f(y)\overline{g}(x) \mu_S(dx, dy).$$

For any $\alpha \in [0,1]$, let

$$G_\alpha = \{(x,y) \in [0,1] \times [0,1] : y = x + \alpha\},$$

and

$$F_\alpha = \{(x,y) \in [0,1] \times [0,1] : y = x - \alpha\}.$$

It is easy to check that S_α is a pseudo-integral operator determined by μ_α, where μ_α is the positive finite Borel measure defined by the equation

$$\mu_\alpha(E) = m\left(\{x \in [0,1] : (x,y) \in E \cap G_\alpha \text{ for some } y \in [0,1]\}\right).$$

Similarly, T_α is a pseudo-integral operator determined by ν_α, where ν_α is defined by the equation

$$\nu_\alpha(E) = m\left(\{x \in [0,1] : (x,y) \in E \cap F_\alpha \text{ for some } y \in [0,1]\}\right).$$

Next, we construct a multiplicative semigroup of positive nilpotent operators, and prove that the semigroup is irreducible.

Choose an arbitrary irrational number $\theta \in (0,1)$. Let $\mathcal{S}_\theta$ be the multiplicative semigroup generated by the set

$$\{S_a,\ T_{b\theta} : a,\, b \in (0,1) \text{ are rational numbers}\}.$$

Theorem 5.4 *The semigroup $\mathcal{S}_\theta$ consists of nilpotent operators.*

Proof. By (i) of Lemma 5.3, any 'word' in $\mathcal{S}_\theta$ looks like

$$W = S_{a_1}^{p_1} T_{b_1\theta}^{q_1} S_{a_2}^{p_2} T_{b_2\theta}^{q_2} \cdots S_{a_n}^{p_n} T_{b_n\theta}^{q_n}$$

for some integer $n \geq 1$, where a_j, $b_j \in (0,1)$ are rational numbers and p_j, q_j are non-negative integers for all $j = 1, 2, \ldots, n$, and at least one of p_j, q_j is non-zero, $(j = 1, 2, \ldots, n)$.

By Lemma 5.3, W is either 0, or $M_\phi S_{a-b}$ with $a - b > 0$, or $M_\psi T_{b-a}$ with $a - b < 0$, where $a = \sum_{j=1}^n p_j a_j$, $b = \theta \sum_{j=1}^n q_j b_j$, and ϕ and ψ are characteristic functions of some intervals. Clearly, $a \neq b$ since θ is irrational, and thus, by (ii) of Lemma 5.2, W is a nilpotent operator. ∎

Next, we prove that $\mathcal{S}_\theta$ is a discrete and irreducible semigroup of positive nilpotent operators.

Lemma 5.5 *For any $\alpha \in [0,1]$, and any interval $[a,b] \subseteq [0,1]$,*

(i) *Either $M_{\chi_{[a,b]}} S_\alpha = 0$ or $\left\| M_{\chi_{[a,b]}} S_\alpha \right\| = 1$.*

(ii) *Either $M_{\chi_{[a,b]}} T_\alpha = 0$ or $\left\| M_{\chi_{[a,b]}} T_\alpha \right\| = 1$.*

Proof. (i) Since the range of S_α is $\chi_{[0,1-\alpha]}\mathcal{L}^2([0,1])$, the interval $[a',b'] = [a,b] \cap [0,1-\alpha]$ has length $b' - a' > 0$ if $M_{\chi_{[a,b]}} S_\alpha \neq 0$, and

$$M_{\chi_{[a',b']}} S_\alpha = M_{\chi_{[a,b]}} S_\alpha.$$

Clearly, $\left\| M_{\chi_{[a,b]}} S_\alpha \right\| \leq \left\| M_{\chi_{[a,b]}} \right\| \, \| S_\alpha \| = 1$.

Let $f = \chi_{[a'+\alpha, b'+\alpha]}$. Then $\|f\| = \left\| \chi_{[a',b']} \right\| \neq 0$, and $S_\alpha f = \chi_{[a',b']}$. Therefore,

$$\begin{aligned} \left\| (M_{\chi_{[a,b]}} S_\alpha) f \right\| &= \left\| (M_{\chi_{[a',b']}} S_\alpha) f \right\| \\ &= \left\| (M_{\chi_{[a',b']}} S_\alpha) f \right\| \\ &= \left\| M_{\chi_{[a',b']}} \chi_{[a',b']} \right\| \\ &= \left\| \chi_{[a',b']} \right\| \\ &= \|f\|, \end{aligned}$$

and hence, $\left\| M_{\chi_{[a,b]}} S_\alpha \right\| \geq 1$. It follows that $\left\| M_{\chi_{[a,b]}} S_\alpha \right\| = 1$.

(ii) As in the proof of (i), we have that $\left\| M_{\chi_{[a,b]}} T_\alpha \right\| = 1$ if $M_{\chi_{[a,b]}} T_\alpha \neq 0$. ■

Lemma 5.6 *Suppose α, $\beta \in [0,1]$, and E, F are two intervals in $[0,1]$. Then, $\left\| M_{\chi_E} S_\alpha - M_{\chi_F} T_\beta \right\|$ is either equal to 0 or not less than 1.*

Proof. If either $M_{\chi_E} S_\alpha$ or $M_{\chi_F} T_\beta$ is 0, then we are done by Lemma 5.5.

Suppose $M_{\chi_E} S_\alpha \neq 0$ and $M_{\chi_F} T_\beta \neq 0$. Then both $E' = E \cap [0, 1-\alpha]$ and $F' = F \cap [\beta, 1]$ are intervals of length greater than 0. If $\alpha = \beta = 0$, then $S_\alpha = T_\beta = I$, and therefore, the result is obviously true. If $\alpha + \beta > 0$, then by the definition of E', we can choose an interval $[a,b]$ satisfying $0 < b - a < \alpha + \beta$ and

$$[a,b] \subseteq E' + \alpha \subseteq [\alpha, 1].$$

Hence

$$[a,b]-\alpha=[a-\alpha,b-\alpha]\subseteq E'$$

and, because $b-\alpha<a+\beta$,

$$([a,b]-\alpha)\cap([a,b]+\beta)=[a-\alpha,b-\alpha]\cap[a+\beta,b+\beta]=\emptyset.$$

Let $f=\chi_{[a,b]}$. Then $f\neq 0$, and

$$\begin{aligned}
& \left\|(M_{\chi_E}S_\alpha - M_{\chi_F}T_\beta)f\right\|^2 \\
= & \left\|(M_{\chi_{E'}}S_\alpha - M_{\chi_{F'}}T_\beta)f\right\|^2 \\
= & \left\|\chi_{E'}(S_\alpha f) - \chi_{F'}(T_\beta f)\right\|^2 \\
= & \left\|\chi_{E'}\chi_{[a,b]-\alpha} - \chi_{F'}\chi_{[a,b]+\beta}\right\|^2 \\
= & \left\|\chi_{[a,b]-\alpha} - \chi_{F'\cap([a,b]+\beta)}\right\|^2 \\
= & \left\|\chi_{[a,b]-\alpha}\right\|^2 + \left\|\chi_{F'\cap([a,b]+\beta)}\right\|^2 \\
\geq & \left\|\chi_{[a,b]-\alpha}\right\|^2 \\
= & \left\|\chi_{[a,b]}\right\|^2 \\
= & \|f\|^2.
\end{aligned}$$

Thus,

$$\left\|M_{\chi_E}S_\alpha - M_{\chi_F}T_\beta\right\|\geq 1.$$

■

Lemma 5.7 *Suppose α, $\beta\in[0,1]$, and E, F are two intervals in $[0,1]$. Then,*

(*i*) $\left\|M_{\chi_E}S_\alpha - M_{\chi_F}S_\beta\right\|$ *is either equal to 0 or not less than 1,*

(*ii*) $\left\|M_{\chi_E}T_\alpha - M_{\chi_F}T_\beta\right\|$ *is either equal to 0 or not less than 1.*

Proof. (i) If either α or β is 0 or 1, then we are done by Lemma 5.5, Lemma 5.6 and the fact that $S_0=T_0=I$. So we may assume that $0<\alpha\leq\beta<1$. Therefore, by (ii) of Lemma 5.3,

$$\begin{aligned}
& \left\| M_{\chi_E} S_\alpha - M_{\chi_F} S_\beta \right\| \\
= & \left\| M_{\chi_E} S_\alpha - M_{\chi_F} S_\beta \right\| \|T_\alpha\| \\
\geq & \left\| M_{\chi_E} S_\alpha T_\alpha - M_{\chi_F} S_\beta T_\alpha \right\| \\
= & \left\| M_{\chi_E} M_{\chi_{[0,1-\alpha]}} - M_{\chi_F} M_{\chi_{[0,1-\beta]}} S_{\beta-\alpha} \right\| \\
= & \left\| M_{\chi_{E \cap [0,1-\alpha]}} - M_{\chi_{F \cap [0,1-\beta]}} S_{\beta-\alpha} \right\| \\
= & \left\| M_{\chi_{E \cap [0,1-\alpha]}} T_0 - M_{\chi_{F \cap [0,1-\beta]}} S_{\beta-\alpha} \right\|.
\end{aligned}$$

By Lemma 5.6, either

$$\left\| M_{\chi_{E \cap [0,1-\alpha]}} T_0 - M_{\chi_{F \cap [0,1-\beta]}} S_{\beta-\alpha} \right\| \geq 1,$$

or

$$M_{\chi_{E \cap [0,1-\alpha]}} T_0 - M_{\chi_{F \cap [0,1-\beta]}} S_{\beta-\alpha} = 0.$$

Thus, either

$$\left\| M_{\chi_E} S_\alpha - M_{\chi_F} S_\beta \right\| \geq 1,$$

or

$$\begin{aligned}
& M_{\chi_E} S_\alpha - M_{\chi_F} S_\beta \\
= & M_{\chi_{E \cap [0,1-\alpha]}} S_\alpha - M_{\chi_{F \cap [0,1-\beta]}} S_\beta \\
= & (M_{\chi_{E \cap [0,1-\alpha]}} - M_{\chi_{F \cap [0,1-\beta]}} S_{\beta-\alpha}) S_\alpha \\
= & 0.
\end{aligned}$$

(ii) By (ii) of Lemma 5.1, $T_\alpha = (S_\alpha)^*$ and $T_\beta = (S_\beta)^*$. Therefore,

$$\begin{aligned}
& (M_{\chi_E} T_\alpha - M_{\chi_F} T_\beta)^* \\
= & S_\alpha M_{\chi_E} - S_\beta M_{\chi_F} \\
= & M_{S_\alpha \chi_E} S_\alpha - M_{S_\beta \chi_F} S_\beta \\
= & M_{\chi_{E-\alpha}} S_\alpha - M_{\chi_{F-\beta}} S_\beta.
\end{aligned}$$

Thus, (ii) follows immediately from (i). ∎

Theorem 5.8 *The norm-distance between any two distinct elements of $\mathcal{S}_\theta$ is at least 1. Therefore, $\mathcal{S}_\theta$ is discrete, and hence, norm-closed in $\mathcal{B}(\mathcal{L}^2([0,1]))$.*

Proof. From the proof of Theorem 5.4, any element in $\mathcal{S}_\theta$ is of the form $M_{\chi_E} S_\alpha$ or $M_{\chi_F} T_\beta$ where E and F are intervals in $[0,1]$, and $\alpha = a - b\theta$, $\beta = c\theta - d$ are in $[0,1]$ for some rational numbers a, b, c and d. Thus, the result follows immediately from Lemma 5.6 and Lemma 5.7. ∎

Theorem 5.9 *For any $\alpha \in [0,1]$, S_α and T_α are in the weak closure $\overline{\mathcal{S}_\theta}^{\text{WOT}}$ of $\mathcal{S}_\theta$. Consequently, $\overline{\mathcal{S}_\theta}^{\text{WOT}}$ is independent of θ.*

Proof. Clearly, $S_1 = T_1 = 0 \in \mathcal{S}_\theta$.

For any $\alpha \in [0,1)$, choose a decreasing sequence $\{a_j\}$ of rational numbers in $(0,1)$ such that $\lim a_j = \alpha$. We claim that S_α is the weak limit of the sequence $\{S_{a_j}\}$, and hence, $S_\alpha \in \overline{\mathcal{S}_\theta}^{\text{WOT}}$.

We need to show that

$$\langle S_{a_j} f, g\rangle \longrightarrow \langle S_\alpha f, g\rangle \quad (j \longrightarrow \infty)$$

for all f and g in $\mathcal{L}^2([0,1])$. Since $\|S_{a_j}\| = 1$ for all j, and since $C([0,1])$ is dense in $\mathcal{L}^2([0,1])$, it suffices to show that

$$\langle S_{a_j} f, g\rangle \longrightarrow \langle S_\alpha f, g\rangle \quad (j \longrightarrow \infty)$$

for all f and g in $C([0,1])$.

Suppose f and g are in $C([0,1])$. For any positive number $\epsilon > 0$, by the continuity of f, we can find a number $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $x, y \in [0,1]$ and $|x - y| < \delta$. Since $\lim a_j = \alpha$, we can find a positive integer N such that

$$|a_j - \alpha| < \min(\epsilon, \delta)$$

for all j with $j \geq N$.

Therfore, for any j, $j \geq N$,

$$\begin{aligned}
& \left|\langle S_{a_j} f, g\rangle - \langle S_\alpha f, g\rangle\right| \\
= & \left|\int_0^{1-a_j} f(x+a_j)\overline{g}(x)dx - \int_0^{1-\alpha} f(x+\alpha)\overline{g}(x)dx\right| \\
= & \left|\int_0^{1-a_j} f(x+a_j)\overline{g}(x)dx - \int_0^{1-a_j} f(x+\alpha)\overline{g}(x)dx\right. \\
& \qquad \left. + \int_{1-a_j}^{1-\alpha} f(x+\alpha)\overline{g}(x)dx\right| \\
= & \int_0^{1-a_j} |f(x+a_j) - f(x+\alpha)|\,|\overline{g}(x)|dx \\
& \qquad + \int_{1-a_j}^{1-\alpha} |f(x+\alpha)|\,|\overline{g}(x)|dx \\
\leq & \ \epsilon\|g\|_\infty + (a_j - \alpha)\|f\|_\infty\|g\|_\infty \\
\leq & \ \epsilon\|g\|_\infty(1 + \|f\|_\infty).
\end{aligned}$$

Thus,

$$\langle S_{a_j} f, g\rangle \longrightarrow \langle S_\alpha f, g\rangle \quad (j \longrightarrow \infty).$$

Similarly, choosing a decreasing sequence $\{b_j\}$ of rational numbers in $(0,1)$ with $\lim b_j\theta = \alpha$, we can prove that $T_\alpha \in \overline{\mathcal{S}_\theta}^{\mathrm{WOT}}$.

We now prove that $\overline{\mathcal{S}_\theta}^{\mathrm{WOT}}$ is independent of θ. Let θ_1 and θ_2 be two irrational numbers in $(0,1)$. For every $\alpha \in [0,1]$, by what we just proved, S_α and T_α are the weak limits of sequences of operators in $\mathcal{S}_{\theta_1}$. Let W be an arbitrary operator in $\mathcal{S}_{\theta_2}$. To prove that W is in $\overline{\mathcal{S}_{\theta_1}}^{\mathrm{WOT}}$, we may assume that $W \neq 0$. From the proof of Theorem 5.4, W is in the form of $M_{\chi_{[a,b]}} S_\alpha$ or $M_{\chi_{[a,b]}} T_\alpha$ for some interval $[a,b] \subseteq [0,1]$ and some number $\alpha \in [0,1)$. Choose a sequence $\{[a_j, b_j]\}$ of subintervals of $[a,b]$ with the property that $\lim a_j = a$ and $\lim b_j = b$. Then it is easy to check that $M_{\chi_{[a,b]}}$ is the strong limit of the sequence $\{M_{\chi_{[a_j,b_j]}}\}$. However, by (iii) of Lemma 5.1,

$$M_{\chi_{[a_j,b_j]}} = M_{\chi_{[a_j,1]}} M_{\chi_{[0,b_j]}} = T_{a_j} S_{a_j} S_{1-b_j} T_{1-b_j}$$

for every integer j. We can choose $\{[a_j, b_j]\}$ so that all $M_{\chi_{[a_j,b_j]}}$ are in $\mathcal{S}_{\theta_1}$ because $\mathcal{S}_{\theta_1}$ is a semigroup. Thus $M_{\chi_{[a,b]}}$ is the strong limit of a sequence of operators in

$\mathcal{S}_{\theta_1}$. It follows that W is the weak limit of a sequence of operators in $\mathcal{S}_{\theta_1}$, and hence $\mathcal{S}_{\theta_2} \subseteq \overline{\mathcal{S}_{\theta_1}}^{\text{WOT}}$. Consequently, $\overline{\mathcal{S}_{\theta_2}}^{\text{WOT}} \subseteq \overline{\mathcal{S}_{\theta_1}}^{\text{WOT}}$.

Similarly, we have that $\overline{\mathcal{S}_{\theta_1}}^{\text{WOT}} \subseteq \overline{\mathcal{S}_{\theta_2}}^{\text{WOT}}$. Thus $\overline{\mathcal{S}_{\theta_1}}^{\text{WOT}} = \overline{\mathcal{S}_{\theta_2}}^{\text{WOT}}$. ∎

Theorem 5.10 *The semigroup $\mathcal{S}_\theta$ is irreducible.*

proof. Suppose $\mathcal{M}$ is a subspace of $\mathcal{L}^2([0,1])$ invariant under $\mathcal{S}_\theta$. Then $\mathcal{M}$ is also invariant under $\overline{\mathcal{S}_\theta}^{\text{WOT}}$.

Let g be an arbitrary element of $\mathcal{M}^\perp$, the orthogonal complement of $\mathcal{M}$ in $\mathcal{L}^2([0,1])$. For any $f \in \mathcal{M}$ and any $\alpha \in (0,1)$, by (iii) of Lemma 5.1,

$$M_{\chi_{[0,1-\alpha]}} f = S_\alpha T_\alpha f \in \mathcal{M},$$

since both S_α and T_α are in $\overline{\mathcal{S}_\theta}^{\text{WOT}}$. Therefore, $\langle M_{\chi_{[0,1-\alpha]}} f, g\rangle = 0$, or equivalently,

$$\int_0^{1-\alpha} f(x)\overline{g}(x)dx = 0.$$

It follows that

$$f(x)\overline{g}(x) = \overline{f}(x)g(x) = 0$$

for almost every $x \in [0,1]$, and hence $\langle |f|, |g|\rangle = 0$. Thus,

$$\mathcal{M}^\perp \subseteq \{g \in \mathcal{L}^2([0,1]) : \langle |f|, |g|\rangle = 0 \text{ for all } f \in \mathcal{M}\}.$$

On the oth. - hand, it is obvious that

$$\mathcal{M}^\perp \supseteq \{g \in \mathcal{L}^2([0,1]) : \langle |f|, |g|\rangle = 0 \text{ for all } f \in \mathcal{M}\}.$$

Consequently,

$$\mathcal{M}^\perp = \{g \in \mathcal{L}^2([0,1]) : \langle |f|, |g|\rangle = 0 \text{ for all } f \in \mathcal{M}\},$$

and therefore is a standard subspace of $\mathcal{L}^2([0,1])$.

Let E be a Borel set in $[0,1]$ such that

$$\mathcal{M}^\perp = \mathcal{M}_E = \chi_E \mathcal{L}^2([0,1]).$$

For any $\alpha \in [0,1]$, since $\mathcal{M}$ is invariant under S_α, $T_\alpha \in \overline{\mathcal{S}_\theta}^{\text{WOT}}$, we have that $\mathcal{M}_E = \mathcal{M}^\perp$ is invariant under $S_\alpha^* = T_\alpha$ and $T_\alpha^* = S_\alpha$. In particular, $(S_\alpha + T_{1-\alpha})\chi_E \in \mathcal{M}_E$. However,

$$\begin{aligned}(S_\alpha + T_{1-\alpha})\chi_E &= S_\alpha \chi_E + T_{1-\alpha}\chi_E \\ &= \chi_{[0,1-\alpha]}\chi_{E-\alpha} + \chi_{[1-\alpha,1]}\chi_{E+1-\alpha} \\ &= \chi_{E_\alpha},\end{aligned}$$

where E_α is the Borel set in $[0,1]$ given by

$$E_\alpha = \{(E-\alpha)\cap[0,1-\alpha]\} \cup \{(E+1-\alpha)\cap[1-\alpha,1]\}.$$

Thus

$$\chi_{E_\alpha} \in \mathcal{M}_E = \chi_E \mathcal{L}^2([0,1]).$$

However

$$\begin{aligned}m(E_\alpha) &= m(\{(E-\alpha)\cap[0,1-\alpha]\} \cup \{(E+1-\alpha)\cap[1-\alpha,1]\}) \\ &= m((E-\alpha)\cap[0,1-\alpha]) + m((E+1-\alpha)\cap[1-\alpha,1]) \\ &= m(E\cap[\alpha,1]) + m(E\cap[0,\alpha]) \\ &= m(E).\end{aligned}$$

It follows that $\chi_{E_\alpha} = \chi_E$.

We now calculate the Fourier coefficients $\widehat{\chi_{E_\alpha}}(n)$ of χ_{E_α}. For all integers n,

$$\begin{aligned}\widehat{\chi_{E_\alpha}}(n) &= \int_0^1 \chi_{E_\alpha}(t)e^{-2\pi nti}dt \\ &= \int_0^1 \chi_{(E-\alpha)\cap[0,1-\alpha]}(t)e^{-2\pi nti}dt \\ &\qquad + \int_0^1 \chi_{(E+1-\alpha)\cap[1-\alpha,1]}(t)e^{-2\pi nti}dt \\ &= \int_{(E-\alpha)\cap[0,1-\alpha]} e^{-2\pi nti}dt \\ &\qquad + \int_{(E+1-\alpha)\cap[1-\alpha,1]} e^{-2\pi nti}dt \\ &= \int_{E\cap[\alpha,1]} e^{-2\pi n(s-\alpha)i}ds\end{aligned}$$

$$
\begin{aligned}
& + \int_{E \cap [0,\alpha]} e^{-2\pi n(s+1-\alpha)i} ds \\
= \; & e^{2\pi n \alpha i} \int_{E \cap [\alpha,1]} e^{-2\pi n s i} ds \\
& + e^{2\pi n(\alpha-1)i} \int_{E \cap [0,\alpha]} e^{-2\pi n s i} ds \\
= \; & e^{2\pi n \alpha i} \left[\int_{E \cap [\alpha,1]} e^{-2\pi n s i} ds + \int_{E \cap [0,\alpha]} e^{-2\pi n s i} ds \right] \\
= \; & e^{2\pi n \alpha i} \int_E e^{-2\pi n s i} ds \\
= \; & e^{2\pi n \alpha i} \widehat{\mathcal{X}_E}(n).
\end{aligned}
$$

By the fact that $\mathcal{X}_{E_\alpha} = \mathcal{X}_E$ for all $\alpha \in [0,1]$, we have that $\widehat{\mathcal{X}_E}(n) = 0$ for all integers $n \neq 0$. Thus, $\mathcal{X}_E$ is a constant function, and hence, either $m(E) = 0$ or $m(E) = 1$. This implies that $\mathcal{M}^\perp$, and therefore $\mathcal{M}$ itself, is a trivial subpace of $\mathcal{L}^2([0,1])$. ∎

REMARK. The operators S_α and T_α, ($\alpha \in [0,1]$), are so-called *Bishop-type* operators. Some nice properties of the Bishop-type operators can be found in [36] and in the references at the end of [36].

It is easy to see that the index of nilpotence of operators in $\mathcal{S}_\theta$ is not bounded. Hadwin et al [21, Theorem 6] proved that an algebra of nilpotent operators is simultaneously triangularizable if the index of nilpotence is bounded. Thus, it is natural to ask the following question:

Question 2. Is it true that any semigroup of positive nilpotent operators is reducible if the index of nilpotence is bounded?

Chapter 6

Miscellaneous Results

6.1 The Jacobson Radical and Invariant Subspaces

Recently the relation between the Jacobson radical of an operator algebra and the existence of invariant subspaces of the algebra has been studied in several papers (Hadwin et al [22]; Katavolos and Radjavi [32]; Lambrou, Longstaff and Radjavi [34]). Let $\mathcal{A}$ be a linear algebra. A ***representation*** of $\mathcal{A}$ on a vector space $\mathcal{V}$ is an algebra homomorphism from $\mathcal{A}$ to the algebra of all linear transformations on $\mathcal{V}$. A representation π of $\mathcal{A}$ on a vector space $\mathcal{V}$ is (***strictly***) ***irreducible*** if there is no linear manifold of $\mathcal{V}$ other than $\{0\}$ and $\mathcal{V}$ itself invariant under $\pi(\mathcal{A})$. The ***Jacobson radical*** $Rad\mathcal{A}$ of $\mathcal{A}$ is defined to be the intersection of the kernels of the (strictly) irreducible representations of $\mathcal{A}$ (see Aupetit [10] and Jacobson [27]).

Suppose $\mathcal{A}$ is a Banach algebra. It has been shown [10, Appendice I, Theorem 2] that if $\mathcal{A}$ is unital, then

$$Rad\mathcal{A} = \{A \in \mathcal{A} : AB \text{ is quasinilpotent for every } B \in \mathcal{A}\},$$

and therefore, every element in $Rad\mathcal{A}$ is quasinilpotent. These remain true when $\mathcal{A}$

is not unital. Indeed, assume $\mathcal{A}$ is not unital, and let $\tilde{\mathcal{A}}$ be the unitization of $\mathcal{A}$ given by

$$\tilde{\mathcal{A}} = \{(\lambda, A) : A \in \mathcal{A} \text{ and } \lambda \text{ is a complex number}\}$$

with norm defined by $\|(\lambda, A)\| = |\lambda| + \|A\|$. (Usually, we write (λ, A) simply as $\lambda + A$.) Then, $\mathcal{A}$ is a proper closed two-sided ideal of $\tilde{\mathcal{A}}$. First, we claim that $Rad\tilde{\mathcal{A}} \subseteq \mathcal{A}$. To see this, let $\lambda + A$ be an element of $Rad\tilde{\mathcal{A}}$ with $A \in \mathcal{A}$ and λ a complex number. Then $\lambda + A$ is quasinilpotent, and therefore, A is invertible in $\tilde{\mathcal{A}}$ if $\lambda \neq 0$. However, no element of $\mathcal{A}$ can be invertible in $\tilde{\mathcal{A}}$. Thus $\lambda = 0$. Consequently, $Rad\tilde{\mathcal{A}} \subseteq \mathcal{A}$. Secondly, it is clear that any irreducible representation of $\mathcal{A}$ can be extended into an irreducible representation of $\tilde{\mathcal{A}}$. On the other hand, the restriction of any irreducible representation of $\tilde{\mathcal{A}}$ to $\mathcal{A}$ is still irreducible since $\mathcal{A}$ is a two-sided ideal of $\tilde{\mathcal{A}}$. It follows that $Rad\mathcal{A} = Rad\tilde{\mathcal{A}}$. Therefore,

$$Rad\mathcal{A} \supseteq \{A \in \mathcal{A} : AB \text{ is quasinilpotent for every } B \in \mathcal{A}\}.$$

Conversely, if $A \in \mathcal{A}$ and AB is quasinilpotent for every $B \in \mathcal{A}$, then, for any $\lambda + B \in \tilde{\mathcal{A}}$,

$$[A(\lambda + B)]^2 = A[(\lambda + B)A(\lambda + B)]$$

is quasinilpotent, and hence, $A(\lambda + B)$ is also quasinilpotent. This implies that $A \in Rad\tilde{\mathcal{A}}$. Thus,

$$Rad\mathcal{A} = \{A \in \mathcal{A} : AB \text{ is quasinilpotent for every } B \in \mathcal{A}\}.$$

Let $\mathcal{H}$ be an arbitrary Hilbert space of dimension at least two. It is known [41, Theorem 1] that any semigroup $\mathcal{S}$ of quasinilpotent operators on $\mathcal{H}$ is reducible if $\mathcal{S}$ contains an operator other than 0 in the trace class $\mathcal{C}_1$. We are going to examine the relation between the set of all trace class operators in $\mathcal{S}$ and the radical of the norm-closed algebra generated by $\mathcal{S}$ and give a generalization of this result.

Proposition 6.1 *Let $\mathcal{S}$ be a semigroup of quasinilpotent operators on $\mathcal{H}$, and let $\mathcal{A}$ be the norm-closed algebra generated by $\mathcal{S}$. Then $Rad\mathcal{A} \supseteq \mathcal{S} \cap \mathcal{C}_1$.*

Proof. Let $A \in \mathcal{S} \cap \mathcal{C}_1$. We need to show that AB is quasinilpotent for every $B \in \mathcal{A}$. By the continuity of spectral radius of compact operators, it is enough to show that AB is quasinilpotent for every B in the linear span of $\mathcal{S}$.

Suppose $B = \sum_{j=1}^{n} \alpha_j S_j$ where n is a positive integer, S_j is in $\mathcal{S}$ and α_j is a complex number, $(j = 1, 2, \dots, n)$. Then

$$\operatorname{tr}(AB) = \sum_{j=1}^{n} \alpha_j \operatorname{tr}(AS_j) = 0$$

for every such B. Replacing B by $B(AB)^{p-1}$, we have that $\operatorname{tr}((AB)^p) = 0$ for every positive integer p. As in the proof of [47, Theorem 5], we have that AB is a quasinilpotent operator. ∎

Theorem 6.2 *Let $\mathcal{S}$ be a semigroup of quasinilpotent operators on $\mathcal{H}$, and let $\mathcal{A}$ be the norm-closed algebra generated by $\mathcal{S}$. If $Rad\mathcal{A}$ contains a non-zero compact operator, then $\mathcal{S}$ is reducible.*

Proof. It suffices to show that $\mathcal{A}$ is reducible. Fix a compact operator $K \in Rad\mathcal{A}$, $K \neq 0$. If $\mathcal{A}$ is irreducible, then, by Lomonosov's Lemma [48, Lemma 8.22], there exists an operator $A \in \mathcal{A}$ such that 1 is in the spectrum of AK. This contradicts the fact that $K \in Rad\mathcal{A}$. ∎

From the proof of Theorem 6.2, we can see that the existence of non-zero compact operators whose products with all operators in an algebra $\mathcal{A}$ are quasinilpotent is a sufficient condition for $\mathcal{A}$ to be reducible. It turns out that the condition is also necessary.

Theorem 6.3 *Let $\mathcal{S}$ be a set of operators on a Hilbert space, and $\mathcal{A}$ the algebra generated by $\mathcal{S}$. Then the following statements are equivalent:*

(*i*) *$\mathcal{S}$ is reducible.*

(*ii*) *$\mathcal{A}$ is reducible.*

(*iii*) *There exists a rank-1 operator K such that AK is quasinilpotent for all $A \in \mathcal{A}$.*

(*iv*) *There exists a non-zero compact operator K such that AK is quasinilpotent for all $A \in \mathcal{A}$.*

Proof. Clearly, (i) and (ii) are equivalent, and (iii) implies (iv). As in the proof of Theorem 6.2, we have that (iv) implies (ii). Therefore, we only need to show that (ii) implies (iii).

Suppose $\mathcal{M}$ is a non-trivial subspace invariant under $\mathcal{A}$. Let x be a unit vector in $\mathcal{M}$, and y a unit vector in $\mathcal{M}^{\perp}$, and let $K = x \otimes y$. Then, for any $A \in \mathcal{A}$, AK is rank-1. But $\mathrm{tr}(AK) = \mathrm{tr}(Ax \otimes y) = \langle Ax, y \rangle = 0$. Thus, AK is nilpotent. ∎

Let $\mathcal{A}$ be an algebra of operators on $\mathcal{H}$. Katavolos and Radjavi proved [32, Theorem 1] that if $\mathcal{A}$ consists of compact operators, then $\mathcal{A}$ is simultaneously triangularizable if and only if $AB - BA$ is quasinilpotent for all A and B in $\mathcal{A}$. If $\mathcal{A}$ is also norm-closed, then this condition is equivalent to $\mathcal{A}/Rad\mathcal{A}$ being commutative (Murphy [39, Theorem 1]). Hadwin et al [22] asked several questions about the relation between the triangularizability of $\mathcal{A}$ and the commutativity of $\mathcal{A}/Rad\mathcal{A}$ for an algebra $\mathcal{A}$ of not necessarily compact operators. We will answer one of the questions in the case where $\mathcal{A}$ is weakly closed and contains a non-zero essentially unitary C_0 operator. An operator T on $\mathcal{H}$ is called *essentially unitary* if $1 - T^*T$ and $1 - TT^*$ are compact. And a contraction T is called a *C_0 operator* (Sz.-Nagy and Foiaş [60, p123]) if T is completely non-unitary and $\phi(T) = 0$ for some non-zero function $\phi \in H^{\infty}$.

Theorem 6.4 *Let $\mathcal{A}$ be a weakly closed algebra of operators on $\mathcal{H}$ and assume that $\mathcal{A}/Rad\mathcal{A}$ is commutative. If $\mathcal{A}$ contains a non-zero essentially unitary C_0 operator T, then $\mathcal{A}$ is simultaneously triangularizable.*

Proof. We may assume, WNLG, that $\mathcal{A}$ contains the identity operator. By Nordgren [40, Corollary 2], $\mathcal{A}$ contains a sequence of compact operators that converges weakly to the identity operator. Consequently, $\mathcal{A} \cap \mathcal{K}$ is weakly dense in $\mathcal{A}$ where

$\mathcal{K}$ is the algebra of all compact operators on $\mathcal{H}$. The commutativity of $\mathcal{A}/Rad\mathcal{A}$ implies that $AB - BA$ is in $Rad\mathcal{A}$ for any A and B in $\mathcal{A} \cap \mathcal{K}$. Hence, $AB - BA$ is quasinilpotent for any A and B in $\mathcal{A} \cap \mathcal{K}$. It follows from [32, Theorem 1] that $\mathcal{A} \cap \mathcal{K}$ is triangularizable. Thus, $\mathcal{A}$ is triangularizable since $\mathcal{A} \cap \mathcal{K}$ is weakly dense in $\mathcal{A}$. ∎

6.2 Positive Linear Mappings between C*-Algebras

In this section, we will be concerned with the usual notion of positivity in C*-algebra. Let $\mathcal{A}$ be a unital C*-algebra. An element of $\mathcal{A}$ is *positive* if it is self-adjoint with non-negative spectrum. Suppose ϕ is a linear mapping from $\mathcal{A}$ to another unital C*-algebra $\mathcal{B}$. Consider the following conditions on ϕ:

(1) ϕ maps the unit element of $\mathcal{A}$ to the unit element of $\mathcal{B}$,

(2) ϕ maps self-adjoint elements of $\mathcal{A}$ to self-adjoint elements of $\mathcal{B}$, or equivalently, $\phi(A^*) = \phi(A)^*$ for all $A \in \mathcal{A}$,

(2′) ϕ maps positive elements to positive elements,

(3) ϕ maps invertible elements to invertible elements,

(3′) ϕ maps invertible self-adjoint elements to invertible elements.

Depending on which condition ϕ satisfies, we call it (1) *unital*, (2) *self-adjoint*, (2′) *positive*, (3) *invertibility preserving*, and (3′) *invertibility preserving for self-adjoint elements*, respectively. It is obvious that if $\mathcal{A}$ and $\mathcal{B}$ are unital C*-algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (1) and (3′), then ϕ is self-adjoint if and only if it is positive.

Proposition 6.5 [44, Proposition 2.1] *Suppose $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a positive linear mapping. Then ϕ is bounded and $\|\phi\| \leq 2\|\phi(1)\|$.*

Proof. Omitted (see [44, p9]). ∎

Proposition 6.6 [53, Corollary 1] *Suppose $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a unital linear mapping. Then ϕ is positive if and only if $\|\phi\| = 1$.*

Proof. Omitted. (See [53].) ∎

A *Jordan homomorphism* of a C*-algebra $\mathcal{A}$ into another C*-algebra $\mathcal{B}$ is a linear self-adjoint mapping ϕ with the property that $\phi(A^2) = \phi(A)^2$ for every self-adjoint element $A \in \mathcal{A}$. The concept of Jordan homomorphism is from Kaplansky [31]. Jordan homomorphisms have been studied by several mathematicians. It has been shown ([28], [29], [59]) that any Jordan homomorphism is the sum of a $*$-homomorphism and a $*$-anti-homomorphism, and therefore, a $*$-homomorphism if the range is commutative, and that any Jordan homomorphism is a $*$-homomorphism if the domain is commutative. A number of sufficient conditions that a linear mapping be a Jordan homomorphism have been obtained. (See M-D Choi et al [13], Russo [52], and Russo and Dye [53].)

Theorem 6.7 (Russo and Dye [53, Corollary 2]) *Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital linear mapping between unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$. Then ϕ is a Jordan homomorphism if it maps unitary elements of $\mathcal{A}$ to unitary elements of $\mathcal{B}$.*

Proof. Omitted. ∎

Theorem 6.8 (Russo [52, Theorem 2]) *Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear mapping from a von Neumann algebra $\mathcal{A}$ to a unital C*-algebra $\mathcal{B}$. Then ϕ is a Jordan homomorphism if it satisfies condition* (1), (2) *and* (3).

Proof. Omitted. ∎

After proving the above result in [52], Russo asked the following question: Does the result remain true if $\mathcal{A}$ is only a unital C*-algebra? It is Russo's observation that there is no loss of generality in assuming $\mathcal{A}$ to be commutative since only self-adjoint elements are involved.

When the range of the linear mapping is also commutative, the question has been answered positively by Gleason [20] and Kahane and Zelazko [30]. We give a different proof of a special case of the general results obtained in [20] and [30].

Theorem 6.9 (Gleason [20]; Kahane and Zelazko [30]) *Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear mapping from unital* C*-*algebra $\mathcal{A}$ to commutative unital* C*-*algebra $\mathcal{B}$. Then ϕ is a Jordan homomorphism if it satisfies condition* (1), (2) *and* (3).

Proof. We may assume, WNLG, that $\mathcal{B} = C(\mathcal{Y})$ for some compact Hausdorff space $\mathcal{Y}$. We need to show that for any self-adjoint element $A \in \mathcal{A}$,

$$\phi(A^2) = \phi(A)^2.$$

Fix an arbitrary self-adjoint element $A \in \mathcal{A}$, let $C^*(A)$ be the C*-subalgebra generated by A, and let ψ be the restriction of ϕ on $C^*(A)$. Then $C^*(A)$ is commutative and ψ satisfies condition (1), (2) and (3). It suffices to show that ψ is a $*$-homomorphism.

Through the Gelfand transform, we can identity $C^*(A)$ with $C(\mathcal{X})$ where $\mathcal{X} = \sigma(A)$. For any $y \in \mathcal{Y}$, let $\hat{y}$ denote the multiplicative linear functional on $C(\mathcal{Y})$ given by

$$\hat{y}(f) = f(y).$$

Then the composition $\hat{y} \circ \psi$ is a linear functional on $C(\mathcal{X})$ that satisfies condition (1), (2) and (3). It follows from Theorem 6.7, or from M-D Choi et al [13, Theorem 6], that the composition $\hat{y} \circ \psi$ is a multiplicative linear functional, and hence, there exists a unique $x = \theta(y)$ in $\mathcal{X}$ such that

$$(\hat{y} \circ \psi)(g) = g(x) = \hat{x}(g) \quad g \in C(\mathcal{X}).$$

Clearly, the mapping $\theta : \mathcal{Y} \longrightarrow \mathcal{X}$ is well-defined, and

$$\psi(g) = g \circ \theta \quad g \in C(\mathcal{X}).$$

Thus, ψ is a $*$-homomorphism. ∎

Remark. We know that the linear mapping ψ in the above proof is automatically continuous by Proposition 6.5. However, we can prove the continuity of the mapping θ directly using the technique employed by Dunford and Schwartz in proving [18, Theorem IV.6.26].

A topological space $\mathcal{X}$ is called *totally disconnected* if every component in $\mathcal{X}$ is a singleton (Dugundji [17]). Through a careful examination of the proof of Theorem 6.8 in [52], we can see that the assumption of $\mathcal{A}$ being a von Neumann algebra is needed only to ensure that any given self-adjoint element A can be approximated by real linear combinations of commutative orthogonal projections in $\mathcal{A}$ commuting with A. Suppose $\mathcal{X}$ is a compact, totally disconnected, Hausdorff topological space. If f is a real continuous function on $\mathcal{X}$, then $f(\mathcal{X})$ is a totally disconnected subset of the real line. Therefore, f can be approximated by real linear combinations of characteristic functions of mutually disjoint open-and-closed sets in $\mathcal{X}$. Consequently, any linear mapping ϕ from $C(\mathcal{X})$ into a unital C*-algebra $\mathcal{B}$ satisfying condition (1), (2) and (3) is a $*$-homomorphism.

For any bounded Borel function h on the unit circle, the restriction T_h of the multiplication operator M_h to the Hardy space $\mathcal{H}^2$ (consisting of all $\mathcal{L}^2$-functions whose negative Fourier coefficients are 0) is called the *Toeplitz operator* induced by h (see [23]). M-D Choi et al proved [13, Theorem 2] that if $\mathcal{X}$ is a compact Hausdorff space containing a continuous injective image of $[0,1]$, then there exists a linear mapping ϕ from $C(\mathcal{X})$ to $\mathcal{B}(\mathcal{H}^2)$ that satisfies the condition (1), (2) and (3) but is not a Jordan homomorphism. The proof involves Toeplitz operators and is based on the fact (Douglas [16, Corollary 7.28]) that the spectrum $\sigma(T_h)$ for a continuous function h is the range of h together with those points not in the range with respect to which h has non-zero winding number. At the end of [13], M-D Choi et al asked the following question: what is the necessary and sufficient condition on $\mathcal{X}$ that forces all linear mappings from $C(\mathcal{X})$ into a unital C*-algebra satisfying (1), (2) and (3) to be $*$-homomorphisms? The main result of this section is a theorem that answers this question with condition (3) replaced by (3$'$).

Theorem 6.10 *Suppose $\mathcal{X}$ is a compact Hausdorff space. Then all linear mappings* *from $C(\mathcal{X})$ into unital C*-algebras satisfying* (1), (2) *and* (3′) *are* $*$*-homomorphisms if and only if* $\mathcal{X}$ *is totally disconnected.*

We need some preparations to prove the above theorem. First, we prove a generalization of a lemma obtained by Russo [52, Lemma 3].

Lemma 6.11 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras. Suppose $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a linear mapping and satisfies condition* (1), (2) *and* (3′). *Then*

(*i*) *ϕ maps projections into projections.*

(*ii*) *ϕ maps every pair of orthogonal projections into a pair of orthogonal projections.*

Proof. (i) Let $P \in \mathcal{A}$ be a projection. Since ϕ satisfies (1), (2) and (3′), $\phi(P)$ is self-adjoint and $\sigma(\phi(P)) \subseteq \sigma(P) \subseteq \{0,1\}$. Therefore, $\phi(P)$ is a projection.

(ii) It is easy to check that an operator T is a projection if and only if $I - 2T$ is self-adjoint and unitary.

Suppose P and Q are orthogonal projections in $\mathcal{A}$. Then $PQ = QP = 0$. It follows that $U = I - 2P$, $V = I - 2Q$ are self-adjoint and unitary, and that $UV = VU$. Therefore, UV is self-adjoint and unitary. Hence, $\phi(UV)$ is self-adjoint and unitary since $\sigma(\phi(UV)) \subseteq \sigma(UV) \subseteq \{-1,1\}$. However,

$$\phi(UV) = \phi((I-2P)(I-2Q)) = I - 2[\phi(P) + \phi(Q)].$$

It follows that $\phi(P) + \phi(Q)$ is a projection, and thus the projections $\phi(P)$ and $\phi(Q)$ are orthogonal. ∎

The following result is a generalization of [52, Theorem 2]. The proof is essentially the same.

Theorem 6.12 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras. Suppose any self-adjoint element $A \in \mathcal{A}$ can be approximated by real linear combinations of commutative orthogonal projections in $\mathcal{A}$ commuting with A. Then every linear mapping $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ that satisfies condition* (1), (2) *and* (3′) *is a Jordan homomorphism.*

Proof. Suppose $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a linear mapping and satisfies condition (1), (2) and (3′). Then, ϕ is positive, and by Proposition 6.6, $\|\phi\| = 1$.

Let $A \in \mathcal{A}$ be a self-adjoint element and $\|A\| = 1$. For any $\epsilon > 0$, the assumption on $\mathcal{A}$ implies that there exist orthogonal projections $P_1, P_2, \dots, P_n$ in $\mathcal{A}$ commuting with A and real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$\left\| A - \sum_{j=1}^{n} \alpha_j P_j \right\| < \epsilon.$$

By Lemma 6.11, $\phi(P_1), \phi(P_2), \dots, \phi(P_n)$ are orthogonal projections in $\mathcal{B}$. Therefore,

$$\begin{aligned}
&\phi(A)^2 - \phi(A^2) \\
= \;& \phi(A)^2 - \left[\sum_{j=1}^{n} \alpha_j \phi(P_j)\right]^2 + \left[\sum_{j=1}^{n} \alpha_j \phi(P_j)\right]^2 - \phi(A^2) \\
= \;& \phi(A)\left[\phi(A) - \sum_{j=1}^{n} \alpha_j \phi(P_j)\right] + \left[\phi(A) - \sum_{j=1}^{n} \alpha_j \phi(P_j)\right] \sum_{j=1}^{n} \alpha_j \phi(P_j) \\
& + \sum_{j=1}^{n} \alpha_j^2 \phi(P_j) - \phi(A^2) \\
= \;& \phi(A)\phi\left(A - \sum_{j=1}^{n} \alpha_j P_j\right) + \phi\left(A - \sum_{j=1}^{n} \alpha_j P_j\right) \phi\left(\sum_{j=1}^{n} \alpha_j P_j\right) \\
& + \phi\left(\sum_{j=1}^{n} \alpha_j^2 P_j - A^2\right) \\
= \;& \phi(A)\phi\left(A - \sum_{j=1}^{n} \alpha_j P_j\right) + \phi\left(A - \sum_{j=1}^{n} \alpha_j P_j\right) \phi\left(\sum_{j=1}^{n} \alpha_j P_j\right) \\
& + \phi\left(\left[\sum_{j=1}^{n} \alpha_j P_j\right]^2 - A^2\right)
\end{aligned}$$

$$= \phi(A)\phi\left(A - \sum_{j=1}^{n}\alpha_j P_j\right) + \phi\left(A - \sum_{j=1}^{n}\alpha_j P_j\right)\phi\left(\sum_{j=1}^{n}\alpha_j P_j\right)$$
$$+\phi\left((\sum_{j=1}^{n}\alpha_j P_j - A)(\sum_{j=1}^{n}\alpha_j P_j + A)\right).$$

Hence,

$$\begin{aligned}\|\phi(A)^2 - \phi(A^2)\| &\leq \left\|A - \sum_{j=1}^{n}\alpha_j P_j\right\| + \left\|A - \sum_{j=1}^{n}\alpha_j P_j\right\| \left\|\sum_{j=1}^{n}\alpha_j P_j\right\| \\ &\quad + \left\|\sum_{j=1}^{n}\alpha_j P_j - A\right\| \left\|\sum_{j=1}^{n}\alpha_j P_j + A\right\| \\ &= \epsilon + \epsilon(1+\epsilon) + \epsilon(2+\epsilon) \\ &= 2\epsilon(2+\epsilon).\end{aligned}$$

Letting ϵ tend to 0, we have that $\phi(A)^2 = \phi(A^2)$. This implies that ϕ is a Jordan homomorphism. ■

Corollary 6.13 *Suppose $\mathcal{A}$ is a von Neumann algebra and $\mathcal{B}$ is a unital* C*-*algebra. Then every linear mapping $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ that satisfies condition* (1), (2) *and* (3′) *is a Jordan homomorphism.*

Proof. It follows immediately from Theorem 6.12. ■

Corollary 6.14 *Let $\mathcal{B}$ be a unital* C*-*algebra. Suppose $\mathcal{X}$ is a compact, totally disconnected, Hausdorff space. Then every linear mapping $\phi : C(\mathcal{X}) \longrightarrow \mathcal{B}$ that satisfies condition* (1), (2) *and* (3′) *is a Jordan homomorphism, and hence, a $*$-homomorphism.*

Proof. Since the total disconnectness of $\mathcal{X}$ implies that the condition of Theorem 6.12 is satisfied with $\mathcal{A} = C(\mathcal{X})$, the result follows immediately. ■

Theorem 6.15 *Let $\mathcal{X}$ be a compact Hausdorff space. Suppose all linear mappings from $C(\mathcal{X})$ into unital* C*-*algebras that satisfy condition* (1), (2) *and* (3′) *are Jordan homomorphisms. Then $\mathcal{X}$ is totally disconnected.*

Proof. Suppose, on the contrary, there exist two distinct points u and v in the same component $\mathcal{X}_0$ of $\mathcal{X}$. Form the disjoint union $\mathcal{X} \cup_d [0,1]$, and let $\widetilde{\mathcal{X}}$ be the topological space obtained by identifying u with 0 and v with 1. Then $\widetilde{\mathcal{X}}$ is compact and Hausdorff.

Define θ to be the mapping $\theta : C(\mathcal{X}) \longrightarrow C(\widetilde{\mathcal{X}})$ given by

$$(\theta f)(x) = \begin{cases} f(x) & \text{if } x \in \widetilde{\mathcal{X}} \setminus [0,1] \\ (1-x)f(u) + xf(v) & \text{if } x \in [0,1] \end{cases} \qquad f \in C(\mathcal{X}).$$

It is easy to check that θ is well-defined, and that it is linear, unital and positive. For any real $f \in C(\mathcal{X})$, since $\mathcal{X}_0$ is connected, $f(\mathcal{X}_0)$ is a connected subset of the real line, and hence, an interval. Therefore, $u, v \in \mathcal{X}_0$ implies that

$$(1-x)f(u) + xf(v) \in f(\mathcal{X}_0)$$

for all $x \in [0,1]$. Thus, θ satisfies (3′) because

$$\sigma(\theta(f)) = (\theta f)(\widetilde{\mathcal{X}}) = f(\mathcal{X}) = \sigma(f).$$

Fix a continuous surjective mapping h from the unit circle to the unit interval $[0,1]$, and let $\tau : [0,1] \longrightarrow \widetilde{\mathcal{X}}$ be the embedding of $[0,1]$ into $\widetilde{\mathcal{X}}$, i.e.,

$$\tau(x) = x \quad x \in [0,1].$$

Define ψ to be the linear mapping $\psi : C(\widetilde{\mathcal{X}}) \longrightarrow \mathcal{B}(\mathcal{H}^2)$ given by

$$\psi(f) = T_{f \circ \tau \circ h},$$

where $\mathcal{H}^2$ is the Hardy space and $T_{f \circ \tau \circ h}$ is the Toeplitz operator induced by the continuous function $f \circ \tau \circ h$ on the unit circle. Straightforward verifications show that ψ is linear and unital, and that $\psi(1) = 1 = \|\psi\|$. It follows from Proposition 6.6 that ψ is positive.

Let $\phi = \psi \circ \theta$. Then $\phi : C(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{H}^2)$ is linear and satisfies (1), (2) and (3′). We complete the proof by showing that the linear mapping ϕ is not a Jordan homomorphism. Choose any real function $f \in C(\mathcal{X})$ with $f(u) = 0$ and $f(v) = 1$. Such functions do exist by the Tietze Extension Theorem. By the definition, $\phi(f) = T_{(\theta f) \circ \tau \circ h}$. Since, for any $x \in [0,1]$,

$$\theta(f^2)(x) = (1-x)f^2(u) + xf^2(v) = \theta(f)(x),$$

we have that $\phi(f^2) = \phi(f)$. Therefore, it is impossible that $\phi(f^2) = [\phi(f)]^2$, for otherwise we have that $\phi(f) = [\phi(f)]^2$ is a self-adjoint idempotent. It follows that $\phi(f)$ is a projection, and consequently $\sigma(\phi(f)) \subseteq \{0,1\}$. However, $\sigma(\phi(f)) = (\theta(f))([0,1]) = [0,1]$, and we have a contradiction. ∎

The proof of Theorem 6.10. It follows immediately from Corollary 6.14 and Theorem 6.15. ∎

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