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On the Entire Cyclic Cohomology

of Banach Algebras

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Table of Contents

Abstract

In this thesis we study some aspects of A. Connes' entire cyclic cohomology theory. This is a new cohomology theory of de Rham type for Banach algebras. We prove a comparison theorem which shows that the theory can be formulated in terms of the Loday-Quillen-Tsygan bicomplex. This allows us to extend the theory to the non-unital category and is a basis for the rest of the thesis. We improve on the existing formulas for pairing with K-theory and prove stability and additivity results for the theory. Finally, we prove a vanishing theorem for actions of derivations on the theory and deduce the homotopy invariance of the theory.

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Summary of Chapter I

This chapter is mainly expository and should be regarded as an introduction to cyclic cohomology theory with a view towards entire cyclic cohomology. For our purposes in this thesis it is important to be familiar with different formulations of the theory and explicit maps between them. In this chapter we will present three different approaches to the subject. They are based on the *cyclic complex* [3], the *cyclic bicomplex* (also called Loday-Quillen-Tsygan bicomplex) [18] and finally Connes' *(b,B) bicomplex* [3].

Our starting point, in Section 1.1, is a *periodic sequence* relating the Hochschild and cobar complexes of an algebra [19]. This periodic sequence can be naturally interpreted as a bicomplex called the cyclic bicomplex of the given algebra. In Section 1.2 we define the cyclic complex of an algebra as a certain subcomplex of the Hochschild complex and define the cyclic cohomology of the given algebra as the cohomology of this cyclic complex [3]. We then establish Connes' long exact sequence along the lines suggested in Quillen [19]. One merit of this approach is that the operators S and B appear naturally.

In Section 1.3 we introduce the *periodic bicomplex* of an algebra [19] and prove that its cohomology with infinite support vanishes. We then use this result to link the cyclic complex with the cyclic bicomplex approach. Connes' (b,B) bicomplex is introduced in Section 1.4. Here, to get off the ground, we need a technical lemma of Connes which says the E_2 term of the first spectral sequence of the (b,B) bicomplex always vanishes ([3], Lemma 36; the fundamental lemma). We give a straightforward proof of this important fact and use it, in a way similar to Section 1.3, to link the cyclic complex approach to the (b,B) bicomplex approach ([3], Theorem 40).

At the end of this section we prove a certain map, defined in Loday-Quillen [18], from the cyclic complex to the (b,B) bicomplex is a quasi-isomorphism. An extended form of this map and the associated comparison theorem, proved in Chapter II, plays an important role in later chapters. Section 1.5 deals with *periodic cyclic cohomology* as a jumping board for entire cyclic cohomology. We prove that the original definition, obtained by inverting the operator S , coincides with the cohomology of (b,B) ([3], Theorem 40) and the periodic bicomplex. This section ends with an example which illustrates the effect of growth conditions in the context of deRham cohomology.

Summary of Chapter II

This chapter starts with recalling the definition of entire cyclic cohomology of unital Banach algebras from Connes' paper [4]. As it is shown in [4], this theory (like its ancestor, ordinary cyclic cohomology) can be paired with topological K-theory. In Section 2.2, starting from Connes' formula in [4], we will derive a formula for this pairing which unlike the formulas in [4] and [12] does not require any technical conditions to be satisfied by the entire cocycle. In Section 2.3 we prove a *comparison theorem* which shows that the complex of entire cochains in Connes' (b,B) bicomplex is homotopy equivalent to the complex of entire cochains in the periodic (Loday-Quillen-Tsygan) bicomplex. This result allows us to extend the entire cyclic cohomology functor to the category of non-unital Banach algebras and plays an important role for the rest of this thesis In Section 2.4 we show how one can use the above comparison theorem together with a result from Loday-Quillen [18] (extended to the entire case) to prove that inner derivations act trivially on entire cyclic cohomology groups. A Cartan type formula, due to Getzler and Szenes [12] plays an important role here. At the end of this section, we use the above result to prove, via more-or-less standard methods, Morita invariance and additivity theorems for entire cyclic cohomology groups.

Summary of Chapter III

In this chapter we prove that any (continuous) derivation from one Banach algebra into another induces the zero map between entire cyclic cohomology groups. As a result of this we can prove a homotopy invariance theorem for entire cyclic cohomology which shows the theory is invariant under smooth deformations. The proof works in finite dimensional case as well and implies, as a special case, a theorem of Goodwillie [13] which says that derivations act like zero on periodic cyclic cohomology groups. The proofs are however very different and based on entirely different ideas. There is little wonder here since the proof in [13] does not generalize to the infinite dimensional (i.e. entire) case

Our proof is based on the theory of infinite dimensional cycles [4] and makes use of the comparison theorem together with some reductions on the type of the cocycles. In [11] (see also Cuntz' papers [10] and [9] for announcements and earlier versions) Cuntz and Quillen, working in the context of Cuntz algebras QA and traces on them, characterize all those traces (normalized cocycles) which correspond to coboundaries. This is a remarkable result and clearly indicates the power of the Q-approach (Cuntz algebras) to cyclic cohomology. The main result of this chapter can also be deduced from this result of Cuntz and Quillen.

Introduction

The goal of A. Connes' noncommutative geometry ([3], [4], [5]) is to study noncommutative -- or quantum -- spaces and their invariants. Cyclic cohomology is a theory of invariants, of differential geometric nature, for these spaces. Entire cyclic cohomology, which the study of some aspects of it is the subject of this thesis, is an infinite dimensional version of cyclic cohomology. Before discussing the content of the thesis in detail, let us review, in broad terms, the general program of noncommutative geometry.

Let us start with the notion of a noncommutative space first. A key idea towards understanding this concept is the well-known relation between a (classical) space and the (commutative) algebra of functions on that space. This idea is made precise by, for example, Gelfand's theorem in the theory of C*-algebras which shows that the category of locally compact topological spaces is anti-equivalent to the category of commutative C*-algebras; or by the so-called geometrization functor in algebraic geometry which defines an anti-equivalence between the category of affine schemes. Based on these facts one can think of a noncommutative algebra as a first approximate to a new kind of space, a noncommutative space.

On the other hand it is also well known that in quantum mechanics the commutative algebra of classical observables, i.e. functions on the phase space, is replaced by the noncommutative algebra of quantum mechanical observables, i.e. operators on a Hilbert space. In other words, at least one approach to quantization, namely the canonical quantization, amounts to replacing functions by operators and commutative algebras by noncommutative algebras (some extra conditions must of course be satisfied e.g. the Poisson brackets should be preserved etc....).

To sum up, we see that one can think of dropping the commutativity assumption on the algebras involved as a first step in passage from classical, commutative spaces to quantized, noncommutative spaces. We refer the reader to [3] and [5] for some interesting examples.

Now, a general method to find the noncommutative analogue of various notions of geometry and topology suggests itself. Namely, one has first to replace the space X by the commutative algebra A of functions (of a suitable type) on X and then try to define the concept involved in terms of A only and without any reference to the commutativity of A. Based on this new definition one can then try to generalize the concept at hand to an appropriate class of noncommutative algebras. It should however be mentioned that this method has only been partially successful. Moreover in some cases, like deRham cohomology, the right generalization, which is cyclic cohomology, was discovered in an indirect way and through other considerations. We will discuss the cyclic cohomology and its origins later in this introduction. But before that, let us consider two examples of the success and failure of the above mentioned method.

An interesting example of the success of the above mentioned method is K-theory. This theory was originally formulated for topological spaces only. To extend the K-functor (at least in degree zero) to arbitrary algebras one can use a theorem of Serre and Swan which shows that vector bundles on a topological space X are in one-to-one correspondence with finitely generated projective modules on the algebra $A = C(X)$ of continuous functions on X. Based on this result and the definition of the K-theory of X as the Grothendieck group of the semi-group of (isomorphism classes) of vector bundles on X, one can then define the K-theory (in degree zero) of any algebra A to be the Grothendieck group of the semi-group of finitely generated projective modules over A. In this way one obtains an important invariant, namely the topological K-theory, for noncommutative C*-algebras. Closely related to vector bundles and K-theory is Chern-Weil theory of characteristic classes where one constructs invariants of bundles using connection-curvature approach. We refer the reader to [3] and [20] where an analogous noncommutative theory is developed.

For the second example let us discuss a situation where a satisfactory definition of a noncommutative analogue of a classical concept has not been found yet. Ironically, this is concerned with the very notion of a smooth quantum space itself. Recall that von-Neumann algebras (respectively C*-algebras) can be regarded as the noncommutative analogue of measure spaces (respectively topological spaces) and this idea has played an important role in the development of these two fields (see the introduction to [3] for more on this). However, up to now, no definition of a noncommutative geometric space (smooth manifolds, Riemannian manifolds, etc....) has been proposed. For more on this we invite the reader to consult Connes' article [5] where a purely operational definition of a metric (arising from a Riemannian metric) is acheived.

After these rather general remarks on noncommutative geometry we would like to specify the rest of this introduction to a discussion of the cyclic cohomology and in particular entire cyclic cohomology. Cyclic cohomology has two origins. In the work of Tsygan [22] and also Loday and Quillen [18] cyclic cohomology appears as the primitive part of the Lie algebra cohomology of the algebra of matrices over a given algebra. We won't pursue this aspect of cyclic cohomology in this thesis. In the fundamental work of Connes [3], on the other hand, cyclic cohomology appears as a noncommutative analogue of deRham homology of currents for smooth manifolds and in particular as a target space for a chern character map from K-homology. In other words, in Connes' work cyclic cohomology appears as a basic tool of noncommutative geometry. Since K-homology plays an important role both in the definition and applications of cyclic and entire cyclic cohomology, in the next few paragraphs we will briefly explain K-homology and its impact on cyclic cohomology.

Recall that K-homology is the dual of K-theory in the sense that there exists a natural and nontrivial pairing between the two. By general abstract arguments from algebraic topology one knows that such a K-homology functor exists. The important question was - and still is! - to describe the K-homology cycles in a concrete form. By the work of Atiyah; Brown, Douglas and Fillmore [1]; and Kasparov, one can say, roughly speaking, that K-homology cycles on X are represented by abstract elliptic operators on X and while

K-theory classifies vector bundles on a space X, K-homology classifies elliptic operators on X. Moreover, the required pairing now takes the form $\langle[D],[E]\rangle$ = index of the elliptic operator D with coefficients in the vector bundle E. Now, one good thing about this way of formulating the K-homology is that it is operational and hence extends to a non-commutative setup. The corresponding cycles are called Fredholm module by Connes [ ].

More precisely, a (unbounded, odd) Fredholm module over an algebra A is a pair (H,D) where H is a Hilbert space and D is a selfadjoint operator on H together with a representation $\pi: A \longrightarrow \mathcal{L}(H)$ such that (i) $(1+D^2)^{-1}$ is a compact operator and (ii) for all $a \in A$, $[D,\pi(a))]$ is a bounded operator. The Fredholm module is said to be finitely summable if for some p, $(1+D^2)^{-p}$ is a trace class operator. As is discussed in [ ], finite summability, in commutative case, is a characteristic of finite dimensions and one should expect that it fails in infinite dimensional examples. Let D be the Laplacian on flat n-torus. The spectrum of D is easily calculable and one can check that $(1+D^2)^{-p}$ is a trace class operator iff $p > \frac{n}{4}$ and hence the failure of finite summability in infinite dimensions). The question of existence of finitely summable Fredholm modules over highly noncommutative C*algebras is investigated in [5] and very general negative results is obtained. As a simple example, it is easy to see that some naturally defined Fredholm modules over group C*-algebras of discrete groups in which the (word) length function is not of polynomial growth fail to be finitely summable (see [5]). However in all of these examples a weaker summability condition is satisfied, namely the heat operator e^{-tD^2} is trace class for all $t > 0$. These Fredholm modules are called θ-summable by Connes. Remarkably enough, the rigorously defined Wess-Zumino model of A. Jaffe et al. in quantum field theory [14], [15] can be interpreted as defining a θ-summable Fredholm module over the infinite dimensional manifold loopspace of S' (see [5]). The operator D in this model is a Dirac operator on this loop space. A quantum algebra in the sense of Jaffe et al. in [15] is, more or less, a θ-summable Fredholm module.

Next recall that the chern character, in its smooth version, is a natural transformation between K-theory and deRham cohomology theory. In [3] Connes extends this chern character to a noncommutative setup. More precisely for each finitely summable Fredholm module (H,D) over an algebra A a cocycle ch(H,D) in the periodic cyclic cohomology of A is defined. Several definitions of the cyclic cohomology of algebras is given in [3]. In particular, for any algebra A a bicomplex, called Connes or (b,B) bicomplex of A, is defined and the periodic cyclic cohomology of A is defined as the cohomology of this bicomplex. See [3] or chapter I of this thesis for definitions. In [4] it is observed that if, instead of cochains with finite length in the total complex of (b,B) bicomplex, one considers cochains with infinite length which also satisfy a certain growth condition then one obtains a cohomology theory which is an infinite dimensional analogue of periodic cyclic cohomology. This is called entire cyclic cohomology. With this cohomology theory at hand, the definition of chern character can be extended to the infinite dimensional case, i.e. to θ-summable Fredholm modules. Indeed there are two such definitions due to Connes [4] and Jaffe, Lesniewski, Osterwalder [14]. In [6] it is shown that the two cocycles are indeed cohomologous.

There are some major technical differences between cyclic and entire cyclic cohomology groups that makes it much harder to prove general results about the latter. For example, an important role in cyclic cohomology is played by the Connes long exact sequence relating cyclic and Hochschild theory (see [3] or chapter I of this thesis). Since Hochschild cohomology is a derived functor, this makes all the techniques of homological algebra available. In particular in all of the known calculations of cyclic cohomology one first calculates the Hochschild cohomology, using a convenient resolution, and then uses the above result to calculate the cyclic cohomology (see the examples at the end of [3]). There are no long exact sequences or even spectral sequences relating Hochschild and entire cyclic cohomology. This means that new methods and ideas, particularly designed for entire cyclic cohomology are needed and partly explains the lack of a good supply of examples where entire cyclic cohomology is completely known. On the other hand to establish certain properties, like Morita invariance (in a restricted sense) and additivity, it is enough to know that inner derivations act as zero on entire cyclic cohomology. In this thesis we prove the most general result of this sort, namely any continuous derivation act as zero on entire cyclic cohomology. This result is of course the infinitesimal form of homotopy invariance of entire theory under smooth deformations. There are certain technicalities about the use of normalized cochains in entire theory and also non-unital algebras. We prove a comparison theorem which shows the complexes of entire cochains in (b,B) bi-complex is homotopy equivalent to the complex of entire cochains in Loday-Quillen-Tsygan bicomplex. This result proved to be quite useful in getting around the above-mentioned technicalities. A detailed summary of chapters will follow.

Chapter 1

1.1. The Cyclic Bicomplex

In this section we will first introduce the Hochschild and the cobar complex of an algebra. Using an action of cyclic groups on these complexes, we then define chain maps between them resulting in a periodic exact sequence of Hochschild and cobar complexes. This periodic sequence can be interpreted in a natural way as a bicomplex, called the cyclic bicomplex of the algebra. The cyclic bicomplex was first introduced by Loday and Quillen in [18] based on Tsygan's work [22]. There are two important technical facts about this cyclic bicomplex that are used very often in this thesis: The rows of the cyclic bicomplex are exact and, for unital algebras, the odd columns, which are all equal to the cobar complex, are exact too. We will provide the standard proofs. Finally it should be mentioned that in the next section we will only use the periodic sequence, and not the cyclic bicomplex, to prove some basic facts about cyclic cohomology. The cyclic bicomplex will be used in section 1.3 to give an alternative definition of cyclic cohomology.

Let A be an algebra and for $n \geq 0$ let $C^n(A)$ be the space of $(n+1)$-linear functionals on A. When there is no danger of confusion we will simply write C^n for $C^n(A)$. We set $C^n = \{0\}$ for $n < 0$. Elements of C^n are called *n-cochains* or cochains of degree n. Define maps $b, b' : C^n \longrightarrow C^{n+1}$ by

$$b\phi(a^0,\ldots,a^{n+1}) = \sum_{j=0}^{n} (-1)^j \phi(a^0,\ldots,a^j a^{j+1},\ldots,a^{n+1}) + (-1)^{n+1} \phi(a^{n+1} a^0,\ldots,a^n)$$

$$b'\phi(a^0,\ldots,a^{n+1}) = \sum_{j=0}^{n} (-1)^j \phi(a^0,\ldots,a^j a^{j+1},\ldots,a^{n+1}) .$$

We have $b^2 = b'^2 = 0$, i.e. b and b' are coboundary maps. The map b is the *Hochschild coboundary*. We will refer to the cochain complex $C(A) = (C^*(A),b)$ as the *Hochschild complex* of A and to $B(A) = (C^*(A),b')$ as the *cobar complex* of A.

When A is unital the cohomology of the Hochschild complex

$$\longrightarrow C^0 \xrightarrow{b} C^1 \xrightarrow{b} C^2 \longrightarrow$$

is the *Hochschild cohomology* of A with coefficients in the A - A bimodule A^* ($= C^0(A)$). We will use $H^n(A,A^*)$, to denote the cohomology of the Hochschild complex in degree n for A unital or not.

Recall that a cochain complex (C^n,d) is said to be *acyclic* if it admits a *contracting homotopy operator* i.e. a map $s : C^n \to C^{n-1}$ such that $ds + sd = id$.

Lemma 1.1.1. The cobar complex

$$\longrightarrow C^0 \xrightarrow{b'} C^1 \xrightarrow{b'} C^2 \longrightarrow$$

of a unital algebra is acyclic.

Proof. When A is unital there is a contracting homotopy operator $s : C^{n+1} \longrightarrow C^n$ defined by

$$s\phi(a^0,\ldots,a^n) = \phi(1,a^0,\ldots,a^n)$$

which has the property $b's + sb' = 1$; this proves the acyclicity of the cobar complex of a unital algebra. □

Remark. If A is not unital, the cobar complex need not be acyclic. As an example, take any algebra with zero multiplication. The b' operator (as well as b) is then equal to zero and the cobar complex has nonzero cohomology in all degrees. However, there are interesting examples of non-unital algebras where the cobar complex is still acyclic. These algebras are called H-unital by M. Wodzicki and play an important role in his study of excision in cyclic homology [23].

Let λ be the canonical generator of the cyclic group $\mathbb{Z}/n+1$. This group acts on the space of n-cochains C^n by

$$\lambda\phi(a^0,\ldots,a^n) = (-1)^n\phi(a^n,a^0,\ldots,a^{n-1}) .$$

We have $\lambda^{n+1} = 1$. The corresponding norm operator on C^n is defined by

$$N = 1 + \lambda + \ldots + \lambda^n$$

and we have $N(1-\lambda) = (1-\lambda)N = 0$.

Lemma 1.1.2. The periodic chain complex

$$\xrightarrow{1-\lambda} C^n \xrightarrow{N} C^n \xrightarrow{1-\lambda} C^n \xrightarrow{N}$$

is acyclic.

Proof. Proof of this lemma makes use of the characteristic zero hypothesis. Define

$N' : C^n \longrightarrow C^n$ by

$$N' = -\frac{1}{n+1}(1 + 2\lambda + 3\lambda^2 + \ldots + (n+1)\lambda^n) \ .$$

It is easily verified that $(1-\lambda)\, N' + \frac{1}{n+1} N = 1$, so that the complex is acyclic. □

The next lemma records two important facts in cyclic cohomology.

Lemma 1.1.3. The maps $1-\lambda : C(A) \longrightarrow B(A)$ and $N : B(A) \longrightarrow C(A)$ are chain maps. That is, $(1-\lambda)b = b'(1-\lambda)$ and $Nb' = bN$.

Proof. See Connes [3] or Loday-Quillen [18] for the purely combinatorial proof.

□

Combining Lemmas 1.1.2 and 1.1.3 we obtain a *periodic exact sequence*

$$\xrightarrow{N} C(A) \xrightarrow{1-\lambda} B(A) \xrightarrow{N} C(A) \xrightarrow{1-\lambda} \qquad (1.1.4)$$

relating Hochschild and cobar complexes. This exact sequence and especially its by-product, the cyclic bicomplex, to be introduced shortly, plays an important role in the whole subject of cyclic cohomology. For example in the next section we will see how one can use the periodic sequence (1.1.4) to define the S-operation on cyclic cohomology and to obtain the Connes long exact sequence.

Recall that a *bicomplex* $(C^{n,m}, d_1, d_2)$ is defined by a family of linear spaces $C^{n,m}$; $(n,m) \in \mathbb{Z}^2$, and differentials $d_1 : C^{n,m} \to C^{n+1,m}$, $d_2 : C^{n,m} \to C^{n,m+1}$ such that $d_1{}^2 = 0$, $d_2{}^2 = 0$ and $d_1 d_2 + d_2 d_1 = 0$.

By changing the sign of differentials in (1.1.4) appropriately we obtain the cyclic bicomplex, also called the Loday-Quillen-Tsygan bicomplex, of the algebra. We make this precise in:

Definition 1.1.5. Let A be an algebra. The *cyclic bicomplex* of A , $\mathcal{C}_+(A)$, is the following first quadrant double cochain complex

$$\begin{array}{ccccc}
\uparrow b & & \uparrow -b' & & \uparrow b \\
C^2 & \xrightarrow{1-\lambda} & C^2 & \xrightarrow{N} & C^2 \longrightarrow \\
\uparrow b & & \uparrow -b' & & \uparrow b \\
C^1 & \xrightarrow{1-\lambda} & C^1 & \xrightarrow{N} & C^1 \longrightarrow \\
\uparrow b & & \uparrow -b' & & \uparrow b \\
C^0 & \xrightarrow{1-\lambda} & C^0 & \xrightarrow{N} & C^0 \longrightarrow
\end{array} \tag{1.1.6}$$

where the even columns are all equal to the Hochschild complex while the odd columns are equal to the cobar complex, with th sign of the differential changed.

It follows from Lemma 1.1.3 that vertical and horizontal differentials in (1.1.6) anticommute, so that $\mathcal{C}_+(A)$ is a bicomplex. We sometimes simply use $\mathcal{C}_+$ for the cyclic bicomplex of the algebra under consideration if there is no chance of confusion.

1.2. The Cyclic Complex and Cyclic Cohomology

The cyclic cohomology of an algebra was first defined by A. Connes [3] (and independently by Tsygan [22]) as the cohomology of the cyclic complex of the algebra. In this section we introduce this cyclic complex and prove some basic facts about cyclic cohomology. There is an important operation on cyclic cohomology, the so-called S-operation. Following Quillen [19,20] we define this operation using an exact sequence relating the cyclic, Hochschild and bar complexes. This exact sequence is itself a consequence of the periodic exact sequence (1.1.4). We show that up to a scalar factor this definition of S coincides with Connes' definition in [3]. A fundamental result in cyclic cohomology is Connes' long exact sequence relating the cyclic and Hochschild cohomology groups. The proof given here is straightforward and is again based on the periodic sequence (1.1.4). Making the maps in the long exact sequence explicit is the last thing we do in this section. Here one naturally encounters Connes B-operator which is of fundamental importance for cyclic cohomology. A deeper study of B will be undertaken in section 1.4 where we introduce Connes' (b,B) bicomplex.

Let A be an algebra. A cochain $\phi \in C^n(A)$ is called *cyclic* if $(1-\lambda)\,\phi = 0$. Let $CC^n(A)$ be the space of *cyclic n-cochains* on A. It follows from the relation $(1-\lambda)\, b = b'(1-\lambda)$ in Lemma 1.1.3 that if ϕ is cyclic so is $b\phi$. This means that we have a subcomplex $CC(A) = (CC^*(A),b)$ of the Hochschild complex. It is called the *cyclic complex* of A. Using the periodic exact sequence (1.1.4), we obtain the following exact sequence relating the cyclic, Hochschild and cobar complexes of an algebra:

$$0 \longrightarrow CC(A) \xrightarrow{I} C(A) \xrightarrow{1-\lambda} B(A) \xrightarrow{N} CC(A) \longrightarrow 0\ . \tag{1.2.1}$$

Definition 1.2.2. The *cyclic cohomology* $HC^*(A)$ of an algebra A is the cohomology of its cyclic complex $CC(A)$.

Cyclic cohomology is a functor: a homomorphism $f : A \longrightarrow B$ induces a morphism of complexes $f^* : CC(B) \longrightarrow CC(A)$, $f^*(\phi) = \phi \circ f$ for all $\phi \in C^n(B)$, hence linear maps $f^* : HC^*(B) \longrightarrow HC^*(A)$.

There is a fundamental operation of degree 2 on cyclic cohomology called the *S-operation.* Connes' definition of S in [3] is based on his theory of cycles over algebras and their tensor products: using cycles he defines a canonical operator S of degree 2 on cyclic cocycles which induces the operation S on cyclic cohomology. Following Quillen [19] we will give a straight- forward definition of S on the level of cyclic cocycles and show that up to a numerical factor it coincides with Connes' S. Let us recall the exact sequence (1.2.1). Given a class $[\phi] \in HC^n(A)$ we can use exactness to solve the equations

$$N\phi' = \phi \ , \ (1-\lambda)\,\phi'' = b'\phi' \ , \ I\,\tilde{\phi} = b\phi'' \tag{1.2.3}$$

for ϕ', ϕ'' and $\tilde{\phi}$ successively. One defines $S[\phi] = [\tilde{\phi}] \in HC^{n+2}(A)$. It follows easily from the exactness of (1.2.1) that $S : HC^n(A) \longrightarrow HC^{n+2}(A)$ is a well-defined map. A homomorphism $f : A \longrightarrow B$ induces a morphism between the exact sequences (1.2.1) of B and A and it is easy to check that S is natural with respect to such homomophisms, so that S is an operation of degree 2 on cyclic cohomology. This definition of S is similar to the definition of the connecting homomorphism in a long exact sequence associated to a short exact sequence of complexes. Here we are working with a four term exact sequence, hence we have a "connecting" homomorphism of degree 2.

To compare this definition of S with Connes' definition, it is useful to have a formula for S on the level of cyclic cocycles. Let $[\phi] \in HC^{n-1}(A)$. Using the fact that ϕ is cyclic and the formula in the proof of Lemma 1.1.2 it is seen that

$$\phi' = \frac{1}{n}\phi \ , \ \phi'' = N'b'\phi' = \frac{1}{n} N'b'\phi \ , \ \tilde{\phi} = b\phi'' = \frac{1}{n} bN'b'\phi$$

solve the equations (1.2.3), so that

$$S[\phi] = \frac{1}{n}[bN'b'\phi] \in HC^{n+1}(A) \ . \tag{1.2.4}$$

Here $N' : C^n \longrightarrow C^n$ is the operator introduced in Lemma 1.1.2. Let $\phi \in C^{n-1}$ be a cyclic cocycle. Let us define $S\phi \in C^{n+1}$ by

$$S\phi = \frac{1}{n} bN'b'\phi = -\frac{1}{n(n+1)} b(1 + 2\lambda + 3\lambda^2 + \ldots + (n+1)\lambda^n) b'\phi \ . \tag{1.2.5}$$

It is easy to see that $S\phi$ is again a cyclic cocycle. This is our formula for S on the level of cyclic cocycles.

Connes' definition of S in [3] is based on his theory of cycles over algebras. There is no need for us to review this theory and Connes' definition of S in detail. It suffices to say that given any cyclic cocycle $\phi \in C^{n-1}(A)$, one defines a linear functional

$\hat{\phi} : \Omega^{n-1}(A) \longrightarrow \mathbb{C}$ by $\hat{\phi}(a^0da^1...da^{n-1}) = \phi(a^0,...,a^{n-1})$ and $\hat{\phi}(da^1...da^{n-1}) = 0$.

Here $\Omega^{n-1}(A)$ is the space of non-commutative differential forms of degree (n-1) over A . Connes' formula for S is ([3], part II, Prop. 12)

$$S_c\phi = b\psi \ , \tag{1.2.6}$$

where

$$\psi(a^0,...,a^n) = \sum_{k=1}^{n} (-1)^{k+1} \hat{\phi}(a^0da^1...da^{k-1}a^kda^{k+1}...da^n) \ . \tag{1.2.7}$$

Note: The formula in [3] contains a normalization factor of $2\pi i$ which we neglect here.

In order to compare Connes' formula 1.2.6 for S with the above formula (1.2.5), we need some notation and facts.

Let $j : C^{n-1} \longrightarrow C^n$ be the operator defined by

$$j\phi(a^0,...,a^n) = (-1)^n\phi(a^na^0,a^1,...,a^{n-1}) \tag{1.2.8}$$

we have $b = b' + j$ and

$$(\lambda^{-(k+1)}j\lambda^k\phi)(a^0,\ldots,a^n) = (-1)^k\phi(a^0,\ldots,a^ka^{k+1},\ldots,a^n)\ ,\ 0 \le k \le n-1\ . \tag{1.2.9}$$

Since $\lambda^{n+1} = 1$, we can write (1.2.9) as

$$(\lambda^{n-k}j\lambda^k\phi)(a^0,\ldots,a^n) = (-1)^k\phi(a^0,\ldots,a^ka^{k+1},\ldots,a^n)\ . \tag{1.2.10}$$

We are now ready to prove:

Lemma 1.2.11. Let $\phi \in C^{n-1}$ be a cyclic cocycle. Then $(1 + 2\lambda + 3\lambda^2 +\ldots+ (n+1)\lambda^n)\ b'\phi = -\psi$, where ψ is defined from ϕ by (1.2.7).

Proof. Since ϕ is a cyclic cocycle we have $\lambda\phi = \phi$ and $b\phi = 0$, so that $b'\phi = (b-j)\phi = -j\phi$ and

$$-(1 + 2\lambda + 3\lambda^2 +\ldots+ (n+1)\lambda^n)\ b'\phi = \sum_{k=0}^{n} (n+1-k)\ \lambda^{n-k}j\lambda^k\phi\ .$$

Using (1.2.10), the value of this cochain on $(a^0,\ldots,a^n)$ is

$$\sum_{k=0}^{n-1} (-1)^k(n+1-k)\phi(a^0,\ldots,a^ka^{k+1},\ldots,a^n) + (-1)^n\phi(a^na^0,a^1,\ldots,a^{n-1})$$

$$= (n+1)\ \hat{\phi}(a^0a^1da^2\ldots da^n) - n\ \hat{\phi}\ (a^0d(a^1a^2)\ldots da^n) +\ldots$$

$$+ (-1)^n\ \hat{\phi}\ (a^na^0da^1\ \ldots\ da^{n-1}).$$

Using $d(ab) = da\cdot b + a\cdot db$, this last sum simplifies to

$$\hat{\phi}(a^0a^1da^2\ldots da^n) - \hat{\phi}\ (a^0da^1\cdot a^2\cdot da^3\ldots da^n) +\ldots$$

$$+ (-1)^{n+1}\ \hat{\phi}\ (a^0da'\ldots da^{n-1}\cdot a^n) = \psi(a^0,\ldots,a^n)\ .$$

The lemma is proved. □

Combining Lemma 1.2.11 with the definitions of S and S_c we find that

$$S = \frac{1}{n(n+1)}\ S_c\ .$$

Next, we would like to establish Connes' long exact sequence. As mentioned earlier our main tool is the periodic exact sequence (1.1.4) and its various ramifications. Consider first the short exact sequence

$$0 \longrightarrow CC(A) \longrightarrow C(A) \longrightarrow C(A)/CC(A) \longrightarrow 0$$

and its derived long exact sequence

$$\longrightarrow HC^n(A) \longrightarrow H^n(A,A^*) \longrightarrow H^n(C(A)/CC(A)) \longrightarrow HC^{n+1}(A) \longrightarrow \tag{1.2.12}$$

When A is unital we can identify the cohomology groups $H^n(C(A)/CC(A))$ as follows:

Lemma 1.2.13. Let A be a unital algebra. Then $HC^n(A) \simeq H^{n+1}(C(A)/CC(A))$ for all n.

Proof. From the periodic exact sequence (1.1.4) and the definition of the cyclic complex we obtain a short exact sequence

$$0 \longrightarrow C(A)/CC(A) \xrightarrow{1-\lambda} B(A) \xrightarrow{N} CC(A) \longrightarrow 0 \tag{1.2.14}$$

and its derived exact sequence

$$\longrightarrow H^n(C(A)/CC(A)) \longrightarrow H^n(B(A)) \longrightarrow HC^n(A) \longrightarrow H^{n+1}(C(A)/CC(A)) \longrightarrow \tag{1.2.15}$$

The cobar complex of a unital algebra is acyclic, so that $H^n(B(A)) = 0$ for all n; and hence $HC^n(A) \simeq HC^{n+1}(C(A)/CC(A))$ as asserted. $\square$

Combining this lemma with the long exact sequence (1.2.12) we obtain the diagram

$$\begin{array}{ccccccccc}
\longrightarrow HC^n(A) & \longrightarrow & H^n(A,A^*) & \longrightarrow & H^n(C(A)/CC(A)) & \longrightarrow & HC^{n+1}(A) \longrightarrow \\
 & & & & \uparrow & & \\
 & & & & HC^{n-1}(A) & &
\end{array}$$

and hence canonical maps $B : H^n(A,A^*) \longrightarrow HC^{n-1}(A)$ and $S : HC^{n-1}(A) \longrightarrow HC^{n+1}(A)$. Using these maps, the long exact sequence (1.2.12) can be written in the form

$$\longrightarrow HC^n(A) \xrightarrow{I} H^n(A,A^*) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \longrightarrow \qquad (1.2.16)$$

This is the Connes long exact sequence relating the cyclic and Hochschild cohomology groups of a unital algebra A. This sequence is natural: given a unital homomorphism $f : A \longrightarrow B$ there is a chain map from the Connes sequence of B to the Connes sequence of A. This follows easily from the naturality of the long exact sequence associated to a short exact sequence of complexes.

Remark. As we will see in the next few paragraphs the map S that appears in the Connes sequence (1.2.16) is exactly equal to the S we defined earlier in this section. Similarly we will show that the map B is induced by the B-operator so that the long exact sequence (1.2.16) is the same as the one established by Connes in [3] except for some rescaling in S.

The maps I, B, S entering the Connes sequence (1.2.16) can be described as follows. The map I is the easiest of all to describe: I is induced by the canonical inclusion $CC(A) \longrightarrow C(A)$ of cyclic cochains into Hochschild cochains. Let us show the map S in (1.2.16) is the same as the S-operation on cyclic cohomology defined earlier in this section. Indeed S is the composition of the connecting isomorphism $HC^{n-1}(A) \simeq H^n(C(A)/CC(A))$ in Lemma 1.2.13 and the connecting homomorphism $H^n(C(A)/CC(A)) \longrightarrow HC^{n+1}(A)$ from (1.2.12). Hence, S is the composition of two connecting homomorphisms defined from short exact sequences

$$0 \longrightarrow C(A)/CC(A) \xrightarrow{1-\lambda} B(A) \xrightarrow{N} CC(A) \longrightarrow 0$$

$$0 \longrightarrow CC(A) \longrightarrow C(A) \longrightarrow C(A)/CC(A) \longrightarrow 0$$

which is the same as the degree 2 connecting homomorphism defined from the exact sequence (1.2.1), i.e. the S-operation.

To describe the map B we have to go back to Lemma 1.2.13 and describe the inverse of the connecting isomorphism $HC^n(A) \simeq H^{n+1}(C(A)/CC(A))$. Recalling the proof of exactness of the derived long exact sequence of any short exact sequence of complexes, we see that a class $[\phi] \in H^{n+1}(C(A)/CC(A))$, $\phi \in C^{n+1}(A)$, is sent to $[Ns(1-\lambda)\phi] \in HC^n(A)$ under the inverse isomorphism $H^{n+1}(C(A)/CC(A)) \xrightarrow{\sim} HC^n(A)$. Here s is the contracting homotopy operator defined in Lemma 1.1.1. Now, going back to Connes' long exact sequence (1.2.16) and the way B was defined, we have $B[\phi] = [Ns(1-\lambda)\phi] \in HC^{n-1}(A)$, for any Hochschild class $[\phi] \in H^n(A,A^*)$.

The operator $B = Ns(1-\lambda) : C^n(A) \longrightarrow C^{n-1}(A)$ that appeared naturally in the above discussion was first defined by Connes in [3] and is a fundamental operator of cyclic cohomology. In many respects it can be regarded as a noncommutative version of the deRham coboundary operator of differential forms. We note that Connes's approach to B is quite different from the above and is based on bordism of cycles. The next lemma shows B is a boundary map and (up to a sign) a chain map with respect to b.

Lemma 1.2.17. One has $B^2 = bB + Bb = 0$.

Proof. We have $B^2 = Ns(1-\lambda)Ns(1-\lambda) = 0$. Also $bB + Bb = bNs(1-\lambda) + Ns(1-\lambda)b = Nb's(1-\lambda) + Nsb'(1-\lambda) = N(b's + sb')(1-\lambda) = N(1-\lambda) = 0$. Here we have used the observation before Lemma 1.1.2, Lemma 1.1.3 and the proof of Lemma 1.1.1.

□

Let $B_0 : C^n \longrightarrow C^{n-1}$ be the operator defined by $B_0 = s(1-\lambda)$. We have $B = NB_0$. We end this section with the following very useful identity:

$$B_0 b + b'B_0 = 1-\lambda\,. \tag{1.2.18}$$

Indeed, $B_0 b + b'B_0 = s(1-\lambda)b + b's(1-\lambda) = (sb' + b's)(1-\lambda) = 1-\lambda$.

1.3. Cyclic Cohomology via Cyclic Bicomplex

The purpose of this section is to present an alternative definition of cyclic cohomology due to Loday and Quillen [18], to compare this definition with our original definition in section 1.2 and to establish Connes's long exact sequence in this approach. In [18] Loday and Quillen defined cyclic cohomology of an algebra as the (total) cohomology of the cyclic bicomplex of the algebra. One nice thing about this approach is that the operator S introduced in section 1.2 appears naturally as the result of the degree 2 periodicity of cyclic bicomplex and also Connes's long exact sequence can be proved in a straightforward way. However, more important for us are the applications we have in mind of this approach to entire cyclic cohomology. See chapters II and III for more on this.

To compare this definition of cyclic cohomology with the one in section 1.2 based on the cyclic complex, Loday and Quillen construct a chain map between relevant complexes and using a spectral sequence argument they show this chain map is a quasi-isomorphism. For reasons to be explained later in this section we decided to give a direct (i.e. spectral sequence free) proof of this result. In order to do so it is better to introduce the periodic bicomplex of an algebra and to prove a lemma about the vanishing of its cohomology with infinite support. The periodic bicomplex and this lemma are of independent interest later when we study entire cyclic cohomology.

The organization of this section is as follows. We first introduce the periodic bicomplex of an algebra and prove that its cohomology with infinite support vanishes. Next we show that the chain map defined by Loday and Quillen is a quasi-isomorphism. Finally we define the operator S using this approach, compare it to the original definition of S in section 1.2 and indicate a second proof of Connes's long exact sequence. Let us start with

Definition 1.3.1. The *periodic bicomplex* of an algebra A is the following double cochain complex in the upper half plane

$$
\mathcal{C}(A) \qquad
\begin{array}{ccccccc}
 & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & C^2 & \xrightarrow{N} & C^2 & \xrightarrow{1-\lambda} & C^2 & \longrightarrow \\
 & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & \\
\longrightarrow & C^1 & \xrightarrow{N} & C^1 & \xrightarrow{1-\lambda} & C^1 & \longrightarrow \\
 & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & \\
\longrightarrow & C^0 & \xrightarrow{N} & C^0 & \xrightarrow{1-\lambda} & C^0 & \longrightarrow
\end{array}
\quad .
$$

We note that the periodic bicomplex, $\mathcal{C}(A)$, is obtained from the periodic exact sequence 1.1.4 by changing the sign of the differentials suitably. Also, the cyclic bicomplex $\mathcal{C}_+(A)$ is obtained from $\mathcal{C}(A)$ by replacing the negative columns by zero.

Let us recall the definition of the total complex of a bicomplex and, at the same time, various types of cohomologies one can define for a bicomplex. The *total complex* of a bicomplex $\mathcal{C} = (C^{p,q}, d_1, d_2)$ with horizontal and vertical differentials

$d_1 : C^{p,q} \longrightarrow C^{p+1,q}$, $d_2 : C^{p,q} \longrightarrow C^{p,q+1}$, $d_1{}^2 = d_2{}^2 = d_1d_2 + d_2d_1 = 0$, is defined by $\mathrm{Tot}\mathcal{C} = (\bigoplus_{p+q=n} C^{p,q}, \partial)$ where $\partial = d_1 + d_2$ is the total differential. By the cohomology of a bicomplex we mean the cohomology of its total complex. A cochain of degree n in $\mathrm{Tot}\mathcal{C}$ is a string of elements $(\phi_{p,q})_{p+q=n}$; $\phi_{p,q} \in C^{p,q}$ such that $\phi_{p,q} = 0$ for all but a finite number of indices. Such cocnains are said to be of *finite support* or finite length and cohomology of $\mathrm{Tot}\mathcal{C}$ is called cohomology with finite support.

If in the definition of the total complex we use direct products instead of direct sums, we obtain a second complex $\left(\prod_{p+q=n} C^{p,q}, \partial \right)$ where the differential ∂ is the same as above. A cochain of degree n in this complex is a sequence $(\phi_{p,q})_{p+q=n}$; $\phi_{p,q} \in C^{p,q}$. That is, cochains are allowed to have infinite length and the cohomology of this complex is therefore called the cohomology of $\mathfrak{C}$ with *infinite support.* It is to emphasize this difference that the usual cohomology of $\mathfrak{C}$ is sometimes referred to as the cohomology with finite support.

Finally, if $C^{p,q}$'s are normed spaces a third possibility arises. Namely, instead of considering arbitrary cochains $(\phi_{p,q})_{p+q=n}$ with infinite support, we can consider only those cochains with $\|\phi_{p,q}\|$ satisfying certain growth conditions. The entire cyclic cohomology of Banach algebras is an example of such a theory. To study this theory in detail is the object of this thesis.

We can compare these three types of cohomology of a bicomplex with analogous types of de Rham cohomology of noncompact manifolds: one has, for instance, de Rham cohomology with compact support, de Rham cohomology with arbitrary support and L^2-de Rham cohomology.

Techniques from homological algebra, especially spectral sequences, are often useful in the study of cohomology of bicomplexes. This is in part because here one is working with cochains of finite length. However when we consider cohomologies of infinite support or of the third type, spectral sequence arguments are inapplicable and we have to resort to more explicit or totally different methods. For this reason we have decided to avoid spectral sequence arguments altogether, even when they are applicable. As a result of this some of our proofs are longer than usual, but at the same time less technical and more constructive.

Let us consider the periodic bicomplex of an algebra A and its total complex with infinite support

$$\longrightarrow C \xrightarrow{\partial^{ev}} C \xrightarrow{\partial^{odd}} \tag{1.3.2}$$

where $C = \prod_{n\geq 0} C^n(A)$. This is a periodic chain complex of period 2. We have

Proposition 1.3.3. The cohomology of the periodic complex (1.3.2) is zero, i.e. the cohomology with infinite support of the periodic bicomplex of any algebra vanishes.

Proof. Let $\phi = (\phi_n)_{n\geq 0}$ be an even cocycle. We construct a cochain $\psi = (\psi_n)_{n\geq 0}$ such that $\partial^{odd}\psi = \phi$. This will prove that the even dimensional cohomology group is zero. The proof of the odd case is completely similar. Now, $\partial^{odd}\psi = \phi$ is equivalent to

$$N\psi_{2n} + b\psi_{2n-1} = \phi_{2n} \quad \text{and} \quad (1-\lambda)\psi_{2n+1} - b'\psi_{2n} = \phi_{2n+1}\,,\ n \geq 0\,. \tag{1.3.4}$$

First note that there are ψ_0, ψ_1 satisfying (1.3.4) for $n = 0,1$. Indeed, we can simply take $\psi_0 = \phi_0$, then ψ_1 must satisfy $(1-\lambda)\,\psi_1 = b'\psi_0 + \phi_1 = b'\phi_0 + \phi_1$. From the cocycle condition $\partial^{ev}\phi = 0$ it follows that $N(b'\phi_0 + \phi_1) = bN\phi_0 + N\phi_1 =$ $b\phi_0 + N\phi_1 = 0$, so that by Lemma 1.1.2, $b'\phi_0 + \phi_1$ is in the image of $1-\lambda$ and we can take $\psi_1 = N'(b'\phi_0 + \phi_1)$.

In general, assume we have defined $\psi_0,\dots,\psi_{2k+1}$ satisfying equations (1.3.4) for $n = 0,\dots,k$. We prove there are cochains ψ_{2k+2} and ψ_{2k+3} such that $\psi_0,\dots,\psi_{2k+3}$ satisfy (1.3.4) for $n = 0,\dots,k+2$. Note that this finishes the proof. Now, $N\psi_{2k+2} + b\psi_{2k+1} = \phi_{2k+2}$ can be solved for ψ_{2k+2} because

$$\begin{aligned}(1-\lambda)(\phi_{2k+2} - b\psi_{2k+1}) &= (1-\lambda)\,\phi_{2k+2} - b'(1-\lambda)\,\psi_{2k+1}\\ &= (1-\lambda)\,\phi_{2k+2} - b'(b'\psi_{2k} + \phi_{2k+1})\\ &= (1-\lambda)\,\phi_{2k+2} - b'\phi_{2k+1} = 0\end{aligned}$$

by the cocycle condition $\partial^{ev}\phi = 0$. Once ψ_{2k+2} is found, we can find ψ_{2k+3} to satisfy the equation $(1-\lambda)\,\psi_{2k+3} - b'\psi_{2k+2} = \phi_{2k+3}$. This is possible because

$$N(\phi_{2k+3} + b'\psi_{2k+2}) = N\phi_{2k+3} + bN\psi_{2k+2}$$
$$= N\phi_{2k+3} + b(\phi_{2k+2} - b\psi_{2k+1})$$
$$= N\phi_{2k+3} + b\phi_{2k+2} = 0 ,$$

where we used the cocycle condition $\partial^{ev}\phi = 0$ in the last equality. By Lemma 1.1.2, we can take $\psi_{2k+3} = N'(\phi_{2k+3} + b'\psi_{2k+2})$. The proposition is proved. □

Definition 1.3.5. A morphism $\alpha : C \longrightarrow D$ of (cochain) complexes is called a *quasi-isomorphism* if it induces an isomorphism of cohomology groups.

We are now ready to prove that the cohomology of the cyclic bicomplex is canonically isomorphic to the cyclic cohomology.

Let us consider the map $i : CC(A) \longrightarrow \mathrm{Tot}\,\mathcal{C}_+(A)$, $i(\phi) = (0,0,\ldots,\phi)$, $\phi \in C^n(A)$, defined by Loday and Quillen [18]. It is easy to see that i is a cochain map. The following theorem is proved in [18] using a spectral sequence argument. The proof we give here makes no use of spectral sequences, but is much longer. That's the price we have to pay.

Theorem 1.3.6. The map i is a quasi-isomorphism.

Proof. Let us prove the surjectivity first. Let $\phi = (\phi_0,\ldots,\phi_n)$ be an n-cocycle in $\mathrm{Tot}\,(\mathcal{C}_+)$. We can consider ϕ as a cocycle with infinite support. By Proposition 1.3.3 there exists a cochain with infinite support, $\psi = (\psi_k)_{k\geq 0}$, such that $\partial\psi = \phi$. Equating the n-cochains in this identity, we have

$$N\psi_n + b\psi_{n-1} = \phi_n \; . \tag{1.3.7}$$

On the other hand, since $\phi = (\phi_0,\ldots,\phi_n)$ is a cocycle, we have

$$b\phi_n = 0\,. \tag{1.3.8}$$

It follows from (1.3.7) and (1.3.8) that $N\psi_n$ is a cyclic cocycle. Next, we will show that $i[N\psi_n] = [(\phi_0,\dots,\phi_n)]$. Indeed, it follows from $\partial\psi = \phi$ that $(\phi_0,\dots,\phi_n) - (0,\dots,N\psi_n)$ $= \partial(\psi_0,\dots,\psi_{n-1})$, so that $[(\phi_0,\dots,\phi_n)] = i[N\psi_n]$ in $H^n(\mathrm{Tot}\ \mathcal{C}_+)$. Hence i is surjective.

Next we will prove the injectivity. Let $[\phi_n] \in HC^n(A)$ be such that $i[\phi_n] = 0$. This means there is a cochain $\psi = (\psi_0,\dots,\psi_{n-1})$ in $\mathrm{Tot}\ \mathcal{C}_+$ such that $\partial\psi = (0,\dots,\phi_n)$. We will use induction to show there exists a cyclic cochain $\tilde{\psi}_{n-1}$ such that $b\tilde{\psi}_{n-1} = \phi_n$ i.e. $[\phi_n] = 0$. This is quite obvious for $n = 0,1$. Let us assume the assertion is true for $n = 0,\dots,k-1$ and consider a cyclic cocycle ϕ_k such that

$$(0,\dots,\phi_k) = \partial\,(\psi_0,\dots,\psi_{k-1})\,. \tag{1.3.9}$$

It follows from (1.3.9) that $N\psi_{k-2} + b\psi_{k-3} = 0$, so that $N\psi_{k-2}$ is a cyclic cocycle. It also follows from (1.3.9) that

$$(0,\dots,N\psi_{k-2}) = \partial\,(\psi_0,\dots,\psi_{k-3})\,.$$

By the induction hypothesis there exists a cyclic cochain $\tilde{\psi}_{k-3}$ such that $b\tilde{\psi}_{k-3} = N\psi_{k-2}$. Let $\tilde{\psi}_{k-2} = \psi_{k-2} - \frac{1}{k-2}\, b'\tilde{\psi}_{k-3}$. We have $N\tilde{\psi}_{k-2} = N\psi_{k-2} - \frac{1}{k-2}\, Nb'\tilde{\psi}_{k-3} =$ $N\psi_{k-2} - \frac{1}{k-2}\, bN\tilde{\psi}_{k-3} = N\psi_{k-2} - b\,\tilde{\psi}_{k-3} = 0$. Using $\tilde{\psi}_{k-2}$ instead of ψ_{k-2}, we can reduce (1.3.9) to

$$(0,\dots,\phi_k) = \partial\,(0,\dots,\tilde{\psi}_{k-2}, \psi_{k-1})\,. \tag{1.3.10}$$

This is equivalent to

$$b\psi_{k-1} = \phi_k\,,\ (1-\lambda)\,\psi_{k-1} - b'\tilde{\psi}_{k-2} = 0\,,\ N\tilde{\psi}_{k-2} = 0\,. \tag{1.3.11}$$

By Lemma 1.1.2, $N\tilde{\psi}_{k-2} = 0$ implies there exists ψ'_{k-2} such that $\tilde{\psi}_{k-2} = (1-\lambda)\psi'_{k-2}$. We claim $\tilde{\psi}_{k-1} = \psi_{k-1} - b\psi'_{k-2}$ is a cyclic cochain such that $b\tilde{\psi}_{k-1} = \phi_k$. Indeed, $(1-\lambda)\,\tilde{\psi}_{k-1} = (1-\lambda)\,\psi_{k-1} - (1-\lambda)\,b\psi'_{k-2} = (1-\lambda)\,\psi_{k-1} - b'(1-\lambda)\,\psi'_{k-2} = (1-\lambda)\,\psi_{k-1} - b'\tilde{\psi}_{k-2} = 0$ by (1.3.11). This shows $\tilde{\psi}_{k-1}$ is a cyclic cochain. Finally $b\tilde{\psi}_{k-1} = b\psi_{k-1} = \phi_k$ by (1.3.11). This completes the proof of the injectivity and the proposition. □

This proposition shows that the cyclic cohomology of an algebra can be defined as the cohomology of the cyclic bicomplex of the algebra. One immediate application of this approach is an alternative definition of the operator S. In particular, as we will see, the existence of S is a consequence of the degree 2 periodicity of the cyclic bicomplex. We need some notation first. Given a complex C, let $C[2]$ be the complex obtained by shifting C two steps forward:

$$C[2]^n = C^{n-2}\,.$$

There is a degree 2 embedding $\tilde{S}$ of $\operatorname{Tot}\mathcal{C}_+(A)$ into itself

$$\tilde{S} : \operatorname{Tot}\mathcal{C}_+(A)[2] \longrightarrow \operatorname{Tot}\mathcal{C}_+(A)$$

defined by $\tilde{S}\,(\phi_0,\dots,\phi_n) = (\phi_0,\dots,\phi_n,0,0)$. The periodicity of the cyclic bicomplex implies that $\tilde{S}$ is a chain map, so that one has the induced maps $\tilde{S} : H^n(\operatorname{Tot}\mathcal{C}_+) \longrightarrow H^{n+2}(\operatorname{Tot}\mathcal{C}_+)$. We have the following diagram

$$\begin{array}{ccc} H^n(\mathrm{Tot}\,\mathcal{C}_+) & \xrightarrow{\tilde{S}} & H^{n+2}(\mathrm{Tot}\,\mathcal{C}_+) \\ \uparrow i & & \uparrow i \\ HC^n(A) & \xrightarrow{S} & HC^{n+2}(A) \end{array} \tag{1.3.12}$$

Proposition 1.3.13. The diagram (1.3.12) anticommutes.

Proof. Let $[\phi_n] \in HC^n(A)$. Recalling formula (1.2.5) for S, we have $iS[\phi_n] = [(0,\ldots,0,0, \frac{1}{n+1} bN'b'\phi_n)]$ and $\tilde{S}\, i\, [\phi_n] = [(0,\ldots, \phi_n, 0, 0)]$. We have to show

$$(0\ ,\ldots,\ 0,\ 0,\ \frac{1}{n+1} bN'b'\phi_n) + (0\ ,\ldots,\ \phi_n\ , 0, 0) = \partial\psi \tag{1.3.14}$$

for some $(n+1)$ - cochain $\psi = (\psi_0,\ldots,\psi_{n+1})$ in $\mathrm{Tot}\,\mathcal{C}_+$. Let $\psi_k = 0\ , 0 \le k \le n-1$, $\psi_n = \frac{1}{n+1}\phi_n$ and $\psi_{n+1} = \frac{1}{n+1} N'b'\phi_n$. We claim, with this choice of ψ, (1.3.14) holds. Indeed, $b\psi_{n+1} = \frac{1}{n+1} bN'b'\phi_n$. Also, $(1-\lambda)\,\psi_{n+1} - b'\psi_n = \frac{1}{n+1}(1-\lambda)N'b'\phi_n - \frac{1}{n+1} b'\phi_n$. Using the formula $(1-\lambda)\, N' + \frac{1}{n+2} N = 1$, from Lemma 1.1.2, this is equal to $\frac{1}{n+1}(1 - \frac{1}{n+2}N)\, b'\phi_n - \frac{1}{n+1} b'\phi_n = 0$. Finally, $N\psi_n = \frac{1}{n+1} N\phi_n = \phi_n$, since ϕ_n is cyclic. Hence (1.3.14) holds and the proposition is proved. □

The next and last thing we would like to do in this section is a derivation of Connes's long exact sequence starting from the cyclic bicomplex. To do this, let $\mathcal{D}$ be the bicomplex obtained by replacing all, except the first two, columns of the cyclic bicomplex by zero.

$$
\mathcal{D} \qquad
\begin{array}{ccccc}
\uparrow & & \uparrow & & \\
C^2 & \xrightarrow{1-\lambda} & C^2 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \\
b & & -b' & & \\
C^1 & \xrightarrow{1-\lambda} & C^1 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \\
b & & -b' & & \\
C^0 & \xrightarrow{1-\lambda} & C^0 & \longrightarrow & 0
\end{array}
$$

The cobar complex of a unital algebra is acyclic. This implies the following.

Lemma 1.3.15. Let A be a unital algebra. Then the complex $\mathrm{Tot}\,\mathcal{D}$ is quasi-isomorphic to the Hochschild complex of A.

Proof. Define a map $\pi : \mathrm{Tot}\,\mathcal{D} \longrightarrow C(A)$ by $\pi\,(\phi_n,\phi_{n-1}) = \phi_n$. One checks easily that π is a cochain map. We will show that π is a quasi-isomorphism. To prove the surjectivity let $[\phi]$ be a Hochschild class and consider the cochain $(\phi, s(1-\lambda)\,\phi)$ in $\mathrm{Tot}\,\mathcal{D}$. Using the homotopy formula $b's + sb' = 1$ from Lemma 1.1.1, and $b\phi = 0$, we see that $(\phi, s(1-\lambda)\,\phi)$ is a cocycle in $\mathrm{Tot}\,\mathcal{D}$. Since $\pi(\phi, s(1-\lambda)\,\phi) = \phi$, π is surjective. Let us prove the injectivity of π . From $\pi\,[(\phi_n,\phi_{n-1})] = 0$ we have $\phi_n = b\psi_{n-1}$ for some cochain ψ_{n-1} . Let $\psi_{n-2} = s(-\phi_{n-1} + (1-\lambda)\,\psi_{n-1})$. Again, using the above homotopy formula one checks that $\partial(\psi_{n-1}, \psi_{n-2}) = (\phi_n, \phi_{n-1})$ so that, $[(\phi_n, \phi_{n-1})] = 0$ in $\mathrm{Tot}\,\mathcal{D}$. Here ∂ is the total differential of the bicomplex $\mathcal{D}$. The lemma is proved.

$\square$

Now consider the short exact sequence of complexes

$$0 \longrightarrow \mathrm{Tot}\,\mathcal{C}_+[2] \xrightarrow{\tilde{S}} \mathrm{Tot}\,\mathcal{C}_+ \longrightarrow \mathrm{Tot}\,\mathcal{D} \longrightarrow 0 \ . \qquad (1.3.16)$$

Using the canonical isomorphisms $\pi : H^n(\text{Tot}\ \mathcal{D}) \longrightarrow H^n(A,A^*)$ and

$i^{-1} : H^n(\text{Tot}\ \mathcal{C}_+) \longrightarrow HC^n(A)$ from Lemma 1.3.15 and Proposition 1.3.6 respectively,

we see that the derived sequence of (1.3.16) is related to Connes' sequence in the following way

$$\begin{array}{ccccccccc}
\longrightarrow & H^n(\text{Tot}\ \mathcal{C}_+) & \xrightarrow{\tilde{S}} & H^{n+2}(\text{Tot}\ \mathcal{C}_+) & \longrightarrow & H^{n+2}(\text{Tot}\ \mathcal{D}) & \longrightarrow & H^{n+1}(\text{Tot}\ \mathcal{C}_+) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & HC^n(A) & \xrightarrow{S} & HC^{n+2}(A) & \xrightarrow{I} & H^{n+2}(A,A^*) & \xrightarrow{B} & HC^{n+1}(A) & \longrightarrow
\end{array}$$

By Lemma 1.3.13 the squares involving S anticommute. This shows that the canonical shift $\tilde{S}$ in the periodic bicomplex induces the (negative of) S-operation. It is easy to see that the squares involving I commute. Finally note that the squares involving B also commute. Indeed, given a Hochschild cocycle ϕ, we can represent it in $H^*(\text{Tot}\ \mathcal{D})$ by $(s(1-\lambda)\ \phi,\phi)$. The connecting homomorphism in (1.3.16) sends this cocycle to $(0,\ldots,Ns(1-\lambda)\ \phi,0,0)$ in $H^*(\text{Tot}\ \mathcal{C}_+)$. This of course shows that the square commutes i.e. the connecting homomorphism in the first exact sequence is induced by B.

1.4. The (b,B) Bicomplex

We will continue our introduction to cyclic cohomology in this section by first proving a fundamental lemma of Connes in [3] which shows that the b cohomology of the complex $\ker B/\text{Im}\ B$ is 0. Then we introduce the (b,B) bicomplex of an algebra, first defined by Connes in [3], and prove that its cohomology with infinite support is trivial. We prove that the cohomology of the (b,B) bicomplex in the first quadrant is canonically isomorphic to the cyclic cohomology of the algebra. This naturally leads to a third definition of the cyclic cohomology in terms of the (b,B) bicomplex and hence a definition of the S-operation and a proof of the Connes's long exact sequence in this setting. Finally we will study a map between the cyclic and (b,B) bicomplexes defined by Loday and Quillen [18] and using only elementary methdos show that it is a quasi-isomorphism. Throughout this section all algebras are unital (and over a field of characteristic zero) unless stated otherwise.

Let A be an algebra. Let us recall the two differentials $b : C^n(A) \longrightarrow C^{n+1}(A)$ and $B : C^n(A) \longrightarrow C^{n-1}(A)$, introduced earlier in this chapter. Since $B = Ns(1-\lambda)$ we have $\operatorname{Im} B \subset \operatorname{Ker}(1-\lambda)$. The following lemma shows that indeed the equality holds.

Lemma 1.4.1. We have $\operatorname{Im} B = \operatorname{Ker}(1-\lambda)$.

Proof. (Connes [3], part II, Lemma 31). We only have to show $\operatorname{Ker}(1-\lambda) \subset \operatorname{Im} B$. Let $\phi \in CC^n(A)$ be a cyclic cochain, choose a linear functional ϕ_0 on A with $\phi_0(1) = 1$, and let

$$\begin{aligned}\psi(a^0, \ldots, a^{n+1}) = {} & \phi_0(a^0)\, \phi(a^1, \ldots, a^{n+1}) \\ & + (-1)^n \phi((a^0 - \phi_0(a^0)1), a^1, \ldots, a^n)\, \phi_0(a^{n+1}) .\end{aligned}$$

One has $\psi(1, a^0, \ldots, a^n) = \phi(a^0, \ldots, a^n)$ and

$$\begin{aligned}\psi(a^0, \ldots, a^n, 1) & = \phi_0(a^0)\, \phi(a^1, \ldots, a^n, 1) + (-1)^n \phi(a^0, \ldots, a^n) \\ & \quad + (-1)^{n+1} \phi_0(a^0)\, \phi(1, a^1, \ldots, a^n) \\ & = (-1)^n \phi(a^0, \ldots, a^n) .\end{aligned}$$

Thus $B_0\psi = 2\phi$ and $\phi \in \operatorname{Im} B$. □

The next lemma that we would like to prove is the fundamental lemma of part II of [3]. Together with another lemma, it forms the basic technical tool upon which the proof of the long exact sequence and the (b,B) bicomplex approach to the cyclic cohomology in [3] rests. Since we have already established the long exact sequence in section 1.2. We will need this result only to link the cyclic complex with the (b,B) bicomplex. As for the proof, we note that the proof given in [3] is indirect and is based on another technical lemma. The proof presented here is straightforward and is motivated by the proof of Lemma 6 in [4]. To start, note that from the relation $bB + Bb = 0$ in Lemma 1.2.17 it follows that b descends to a differential (of degree -1) on the graded vector space Ker B/Im B .

Lemma 1.4.2. (Fundamental lemma of [3], part II). The b cohomology of the complex Ker B/Im B is trivial.

Proof. Let $[\phi_m]$ be a cocycle in Ker B/Im B . From $B\phi_m = NB_0\phi_m = 0$, using Lemma 1.1.2, it follows that there is a cochain ψ_{m-1}, not necessarily in Ker B , such that $(1-\lambda)\psi_{m-1} = B_0\phi_m$. Let us show that $b\psi_{m-1} - \phi_m \in \mathrm{Im}\, B$. In view of Lemma 1.4.1 this is equivalent to cyclicity of $b\psi_{m-1} - \phi_m$. Using the formula $B_0 b + b'B_0 = 1-\lambda$ from section two, we have

$$(1-\lambda)(b\psi_{m-1}-\phi_m) = (1-\lambda)\, b\psi_{m-1} - B_0 b\phi_m - b'B_0\phi_m .$$

Since $[\phi_m]$ is a cocycle in Ker B/Im B , we have $b\phi_m \in \mathrm{Im}\, B$ and hence $B_0 b\phi_m = 0$, so that the above cochain is equal to

$$= (1-\lambda)\, b\psi_{m-1} - b'(1-\lambda)\, \psi_{m-1} = 0 .$$

Next, let us show that with this choice of ψ_{m-1} we have $b'B_0\psi_{m-1} = 0$. Indeed

$$\begin{aligned} b'B_0\psi_{m-1} &= b's(1-\lambda)\, \psi_{m-1} \\ &= (1-sb')(1-\lambda)\, \psi_{m-1} \\ &= (1-\lambda)\, \psi_{m-1} - s(1-\lambda)\, b\psi_{m-1} \end{aligned}$$

$$= B_0\phi_m - B_0(\phi_m + \phi) \ , \quad \phi \in \mathrm{Im}\, B$$
$$= 0 \ .$$

Now, since $b'B_0\psi_{m-1} = 0$, we can conclude, using Lemma 1.1.1. that there is a cochain ψ_{m-3} such that $B_0\psi_{m-1} = b'\psi_{m-3}$. The cochain $N\psi_{m-3}$ is cyclic and hence by Lemma 1.4.1 there is a cochain ψ_{m-2} such that $B\psi_{m-2} = N\psi_{m-3}$. Let us show $\psi_{m-1} + b\psi_{m-2} \in \mathrm{Ker}\, B$:

$$\begin{aligned} B(\psi_{m-1} + b\psi_{m-2}) &= NB_0\psi_{m-1} + Bb\psi_{m-2} \\ &= Nb'\psi_{m-3} + Bb\psi_{m-2} \\ &= bN\psi_{m-3} - bB\psi_{m-2} \\ &= b(N\psi_{m-3} - B\psi_{m-2}) \\ &= b(0) = 0 \ . \end{aligned}$$

It follows that the cochain $\psi_{m-1} + b\psi_{m-2}$ represents a class in Ker B/Im B. We have

$$b[\psi_{m-1} + b\psi_{m-2}] = b[\psi_{m-1}] = [b\psi_{m-1}] = [\phi_m]$$

where we have used the fact that $b\psi_{m-1} - \phi_m \in \mathrm{Im}\, B$, established at the beginning of the proof. We have shown that every cocycle in Ker B/Im B is a coboundary. The lemma is proved. □

Corollary 1.4.3. Let $\phi \in CC^n(A)$ and $\phi = b\psi$ where $B\psi = 0$. Then $[\phi] = 0$ in $HC^n(A)$.

Proof. From Lemma 1.4.1 we have $b\psi \in \mathrm{Im}\, B$. It follows that ψ is a cocycle in Ker B/Im B so that by Lemma 1.4.2 there exists $\tilde{\psi}$ such that $\psi - b\tilde{\psi} \in \mathrm{Im}\, B$. We have $\phi = b\psi \in b(\mathrm{Im}\, B)$ hence $[\phi] = 0$ in $HC^n(A)$. □

There is an equivalent formulation of Lemma 1.4.2 which is also useful for our purposes in this section. Consider the exact sequence of complexes

$$0 \longrightarrow \operatorname{Im} B \longrightarrow \operatorname{Ker} B \longrightarrow \operatorname{Ker} B/\operatorname{Im} B \longrightarrow 0 \ .$$

By Lemma 1.4.1 $\operatorname{Im} B$ is the cyclic complex $CC(A)$ so the derived long exact sequence is of the form

$$\longrightarrow HC^n(A) \longrightarrow H^n(\ker B) \longrightarrow H^n(\ker B/\operatorname{Im} B) \longrightarrow HC^{n+1}(A) \longrightarrow \ .$$

By Lemma 1.4.2 $H^n(\ker B/\operatorname{Im} B) = 0$ for all n. Hence we have the following:

Corollary 1.4.4. The obvious map from $HC^n(A)$ to $\ker B \cap \ker b/b(\ker B)$ is bijective. □

Definition 1.4.5. The (b,B) bicomplex of a unital algebra A is the following double cochain complex.

$$\beta(A) \qquad \begin{array}{cccccccc}
 & & & & & & \uparrow & \\
 & & & & C^0 & \xrightarrow{b} & C^1 & \longrightarrow \\
 & & & & \uparrow{\scriptstyle B} & & \uparrow{\scriptstyle B} & \\
 & & C^0 & \xrightarrow{b} & C^1 & \xrightarrow{b} & C^2 & \longrightarrow \\
 & & \uparrow{\scriptstyle B} & & \uparrow{\scriptstyle B} & & \uparrow{\scriptstyle B} & \\
C^0 & \xrightarrow{b} & C^1 & \xrightarrow{b} & C^2 & \xrightarrow{b} & C^3 & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}$$

More precisely, $\beta(A) = (\beta^{p,q}, b, B)$ where $\beta^{p,q} = C^{p-q}$, and $b : \beta^{p,q} \longrightarrow \beta^{p+1,q}$, $B : \beta^{p,q} \longrightarrow \beta^{p,q+1}$, for all $(p,q) \in \mathbb{Z}^2$, are the horizontal and vertical differentials.

Note that $\beta^{p,q} = \{0\}$ if $p-q < 0$.

Remark. In [3], Connes defined the (b,B) bicomplex in the same way as above except that instead of the differentials b , B , he considers certain scalar multiples of them. This is to make sure that the S-operation as defined from the (b,B) bicomplex be exactly equal to the S defined in [3], using the theory of cycles. We will see that if we use the above definition of (b,B) bicomplex, the resulting S-operation coincides (up to a sign) with our original definition in section 1.2.

The following proposition is a consequence of Lemma 1.4.2.

Proposition 1.4.6. The cohomology with infinite support of $\beta(A)$ is zero.

Proof. We will prove the even case. Let $\phi = \{\phi_{2n}\}_{n\geq 0}$, $\partial\phi = 0$, be an even cocycle. Let us find a cochain $\psi = \{\psi_{2n+1}\}_{n\geq 0}$ such that $\partial\psi = \phi$, i.e.

$$b\psi_{2n-1} + B\psi_{2n+1} = \phi_{2n} \ , \ n \geq 0 \ . \tag{1.4.7}$$

By Lemma 1.4.1, there is a cochain ψ_1 such that $B\psi_1 = \phi_0$. Assume there are cochains ψ_{2n+1}, $0 \leq n \leq k$, satisfying (1.4.7) for $0 \leq n \leq k$. We can then find $\tilde{\psi}_{2k+1}$ and ψ_{2k+3} such that $\psi_1, \psi_3, \ldots, \tilde{\psi}_{2k+1}, \psi_{2k+3}$ satisfy (1.4.7) for $0 \leq n \leq k+1$. Note that this finishes the proof since we can repeat this argument to find a sequence $\psi = \{\psi_{2n+1}\}_{n\geq 0}$ which satisfies (1.4.7) for all n . Now, to find $\tilde{\psi}_{2k+1}$ and ψ_{2k+3}, consider the cochain $\phi_{2k+2} - b\psi_{2k+1}$. We have

$$B(\phi_{2k+2} - b\psi_{2k+1}) = B\phi_{2k+2} + bB\psi_{2k+1} = B\phi_{2k+2} + b\phi_{2k} = 0 \text{ , and}$$

$$b(\phi_{2k+2} - b\psi_{2k+1}) = b\phi_{2k+2} = -B\phi_{2k+3} \in \operatorname{Im} B \text{ ,}$$

hence $\phi_{2k+2} - b\psi_{2k+1}$ represents a cocycle in $\ker B/\operatorname{Im} B$. It follows from Lemma 1.4.2 that there are ψ'_{2k+1} and ψ_{2k+3} such that

$$B\psi'_{2k+1} = 0 \text{ and } (\phi_{2k+2} - b\psi_{2k+1}) \quad b\psi'_{2k+1} = B\psi_{2k+3} \ . \tag{1.4.8}$$

Let $\tilde{\psi}_{2k+1} = \psi_{2k+1} + \psi'_{2k+1}$. From (1.4.8), we see that $\{\psi_1, \psi_3, ..., \tilde{\psi}_{2k+1}, \psi_{2k+3}\}$ satisfy (1.4.7) for $0 \le n \le k+1$, as we wanted. □

The (b,B) bicomplex is most useful when we study the periodic cyclic cohomology or entire cyclic cohomology of algebras. For the purpose of cyclic cohomology itself, the part of the (b,B) bicomplex which is in the first quadrant is more relevant. We denote this bicomplex by $\beta_+(A)$.

$$\beta_+(A) \qquad \begin{array}{ccccccc} & & & & \uparrow B & & \\ & & C^0 & \xrightarrow{b} & C^1 & \xrightarrow{b} & \\ & & \uparrow B & & \uparrow B & & \\ C^0 & \xrightarrow{b} & C^1 & \xrightarrow{b} & C^2 & \xrightarrow{b} & \end{array}$$

Next we would like to show that the cohomology of $\beta_+(A)$ is canonically isomorphic to the cyclic cohomology of A. To do this, consider the chain map $j : CC(A) \longrightarrow \mathrm{Tot}\,\beta_+(A)$ defined by

$$j(\phi_n) = (0, ..., \phi_n) \quad , \quad \phi_n \in CC^n(A) .$$

The following proposition should be compared with Proposition 1.3.6.

Proposition 1.4.9. The map j is a quasi-isomorphism.

Proof. Let us prove the injectivity first. Let $[\phi_{2n}] \in HC^{2n}(A)$. $j[\phi_{2n}] = 0$ means there exists a cochain $\psi = (\psi_1, ..., \psi_{2n-1}) \in \mathrm{Tot}\,\beta_+(A)$ such that

$$\partial(\psi_1, ..., \psi_{2n-1}) = (0, ..., \phi_{2n}) . \tag{1.4.10}$$

From this we have $b\psi_{2n-1} = \phi_{2n}$. By Corollary 1.4.3, if $B\psi_{2n-1} = 0$ then $[\phi_{2n}] = 0$

in $HC^{2n}(A)$ and we are done. Let us show that there is a cochain $\psi = (\psi_1,...,\psi_{2n-1})$ with $B\psi_{2n-1} = 0$ which satisfies (1.4.10). Indeed, it follows from (1.4.10) that $B\psi_1 = 0$ and $b\psi_1 \in \mathrm{Im}\, B$ so that by Lemma 1.4.2 there are cochains θ_0, θ_2 such that $\psi_1 - b\theta_0 = B\theta_2$. Now let $\tilde{\psi}_3 = \psi_3 + b\theta_2$. We have $b\,\tilde{\psi}_3 = b\psi_3$ and $B\,\tilde{\psi}_3 = B\psi_3 + Bb\theta_2 = -b\psi_1 + b\psi_1 = 0$ and hence $\psi = (0,\ \tilde{\psi}_3, \psi_5,..., \psi_{2n-1})$ satisfies (1.4.10). By repeating this argument we find a cochain of the form $\psi = (0, 0,..., 0,\ \tilde{\psi}_{2n-1})$ satisfying (1.4.10). This proves the injectivity of j. The proof of the surjectivity follows a similar pattern as proof of Proposition 1.3.6. Let $\phi = (\phi_0, \phi_2,..., \phi_{2n})$ be a cocycle in $\mathrm{Tot}\, \beta_+(A)$. From Proposition 1.4.6 we have $\phi = \partial\psi$ where $\psi = (\psi_{2k+1})_{k\geq 0}$ and we consider ϕ as a cocycle of infinite support. From $b\psi_{2n+1} + B\psi_{2n+3} = 0$ we have $b(B\psi_{2n+1}) = -B(b\psi_{2n+1}) = -B^2\psi_{2n+3} = 0$. It follows from Corollary 1.4.4 that $B\psi_{2n+1}$ represents a cyclic cocycle. In other words, there is a cyclic cocycle $\tilde{\phi}_{2n}$ such that $B\psi_{2n+1} - \tilde{\phi}_{2n} = b\psi'_{2n-1}$ where $B\psi'_{2n-1} = 0$. Let $\tilde{\psi}_{2n-1} = \psi_{2n-1} + \psi'_{2n-1}$. We have $b\,\tilde{\psi}_{2n-1} = b\psi_{2n-1} + b\psi'_{2n-1} = b\psi_{2n-1} + B\psi_{2n+1} - \tilde{\phi}_{2n} = \phi_{2n} - \tilde{\phi}_{2n}$ and $B\,\tilde{\psi}_{2n-1} = B\psi_{2n-1} + B\psi'_{2n-1} = B\psi_{2n-1}$. It follows from $\phi = \partial\psi$ that $(\phi_0, \phi_2,..., \phi_{2n} - \tilde{\phi}_{2n}) = \partial(\psi_1,..., \psi_{2n-3},\ \tilde{\psi}_{2n-1})$ so that $j[\tilde{\phi}_{2n}] = [(\phi_0, \phi_2,..., \phi_{2n})]$, and hence the surjectivity of j is proved. □

This proposition shows that the cyclic cohomology of a unital algebra A can be defined as the (total) cohomology of the bicomplex $\beta_+(A)$. Parallel to our discussion in section 1.3 we will now proceed to define the S-operator in terms of the bicomplex $\beta_+(A)$. First we need to find a new formula for S. Let $\phi \in CC^{n-1}(A)$ be a cyclic (n-1)-cocycle. In section 1.2, formula (1.2.5), we defined $S\phi = b(\frac{1}{n}\ N'b'\phi)$. Let $\psi = \frac{1}{n}\ N'b'\phi$.

Then

$$
\begin{aligned}
B\psi &= \frac{1}{n} Ns(1-\lambda)\ N'b'\phi \\
&= \frac{1}{n} Ns[1 - \frac{1}{n+1} N]\ b'\phi \\
&= \frac{1}{n}\ Nsb'\phi - \frac{1}{n(n+1)}\ NsNb'\phi\ .
\end{aligned}
$$

Since $NsNb'\phi = NsbN\phi = 0$ for any cyclic cocycle ϕ, we have

$$
B\psi = \frac{1}{n} Ns\ b'\phi = \frac{1}{n} N(1-b's\ )\ \phi = \phi - \frac{1}{n} bNs\ \ \phi = \phi + bB\theta
$$

for some cochain θ. Now let $\tilde{\psi} = \psi + b\theta$. We then have

$$
B\ \tilde{\psi} = B\psi' + Bb\theta = B\psi - bB\theta = \phi\ .
$$

We can summarize the above calculation as follows: given any cyclic $(n-1)$-cocycle ϕ there is an n-cochain $\tilde{\psi}$ such that

$$
b\ \tilde{\psi} = S\phi \quad \text{and} \quad B\ \tilde{\psi} = \phi\ . \tag{1.4.11}
$$

From the picture of $\beta_+(A)$ it is clear that the canonical degree 2 shift

$$
\tilde{S} : \mathrm{Tot}\ \beta_+[2] \longrightarrow \mathrm{Tot}\ \beta_+
$$

is a cochain map and hence induces a map

$$
\tilde{S} : H^{n-1}(\mathrm{Tot}\ \beta_+) \longrightarrow H^{n+1}(\mathrm{Tot}\ \beta_+)\ .
$$

Using formula (1.4.11) we can easily show that $\tilde{S}$ is the negative of the S-operation. Consider the diagram

$$\begin{array}{ccc} H^{n-1}(\mathrm{Tot}\,\beta_+) & \xrightarrow{\tilde{S}} & H^{n+1}(\mathrm{Tot}\,\beta_+) \\ \uparrow j & & \uparrow j \\ HC^{n-1}(A) & \xrightarrow{S} & HC^{n+1}(A) \end{array}$$

where j is the isomorphism in Proposition 1.4.9.

Lemma 1.4.12. The above diagram anticommutes.

Proof. Let $[\phi] \in HC^{n-1}(A)$. We have

$$(\tilde{S}j + jS)[\phi] = [(0,\ldots, 0, \phi, S\phi)] \ .$$

Let $\tilde{\psi}$ be the cochain in formula (1.4.11). We have $(0,\ldots, \phi, S\phi) = \partial(0,\ldots, 0, \tilde{\psi})$ so that $[(0,\ldots, 0, \phi, S\phi)] = 0$ and hence the diagram anticommutes. □

Remark 1. From the picture of $\beta_+(A)$ it is clear that the Hochschild complex of A , $C(A)$, and $\mathrm{Tot}\,\beta_+$ fit into a short exact sequence

$$0 \longrightarrow \mathrm{Tot}\,\beta_+[2] \xrightarrow{\tilde{S}} \mathrm{Tot}\,\beta_+ \longrightarrow C(A) \longrightarrow 0$$

whose long exact sequence is the Connes long exact sequence (1.2.16) with the sign of S changed. Since the argument is similar to the one in section 1.3, we leave the details to the reader.

Remark 2. Formula (1.4.11) can be used to prove the formula $S = bB^{-1}$ due to Connes ([3] , part II, page 121). We follow Connes's presentation: given $\phi \in CC^{n-1}(A)$, a cyclic cocycle, we can write $\phi = B\psi$ for some ψ . This uniquely defines $b\psi \in \ker b \cap \ker B/b(\ker B)$ and hence an element of $HC^{n+1}(A)$, using the isomorphism in Corollary

1.4.4. Now, to see that this is equal to $S[\phi]$, we can choose $\psi = \tilde{\psi}$ as in formula (1.4.11).

Finally, we want to study a map from Tot $\mathcal{C}_+(A)$ to Tot $\beta_+(A)$. In [18], Loday and Quillen define (the dual of) a map $I : \mathrm{Tot}\, \mathcal{C}_+ \longrightarrow \mathrm{Tot}\, \beta_+$ by $I(\phi)_{2i} = \phi_{2i} + Ns\phi_{2i+1}$ if $\phi = (\phi_0, \ldots, \phi_{2n})$, and $I(\phi)_{2i+1} = \phi_{2i+1} + Ns\phi_{2i+2}$ if $\phi = (\phi_0, \ldots, \phi_{2n+1})$. Using the basic formulas of section 1.1 it is easy to see that I is a chain map. The following proposition is proved in [18] using a spectral sequence argument. Our proof is elementary and is based on two key results in sections 1.3 and 1.4.

Proposition 1.4.13. The map I is a quasi-isomorphism.

Proof. Consider the maps

$$CC(A) \xrightarrow{i} \mathrm{Tot}\, \mathcal{C}_+(A) \xrightarrow{I} \mathrm{Tot}\, \beta_+(A) .$$

Here $i : CC(A) \longrightarrow \mathrm{Tot}\, \mathcal{C}_+(A)$ is the quasi-isomorphism in Proposition 1.3.6. Inspecting the definition of I above shows that $I \circ i$ is the quasi- isomorphism in Proposition 1.4.9. Since i is a quasi-isomorphism, it follows that I is a quasi-isomorphism too. □

1.5. The Periodic Cyclic Cohomology and Beyond

In section 1.2 we defined the S-operator, also called the periodicity operator, on cyclic cohomology groups. In [3] Connes defines a pairing between cyclic cohomology and K-theory of an algebra and shows that this pairing is invariant under the action of S. This, and other facts e.g. the calculation of de Rham homology of a manifold in terms of cyclic cohomology of algebra of smooth functions on the manifold ([3], Theorem 46), shows that as far as K-theory, index theory and in general topology/geometry are concerned the relevant groups are direct limits of cyclic cohomology groups under S. In this section we show that this periodic theory is most naturally defined in terms of either the (b,B) or the cyclic bicomplex and set the ground for the next stage in the development of this theory, namely entire cyclic cohomology.

Definition 1.5.1. The *periodic cyclic cohomology* of an algebra A is the inductive limit of the system $(HC^*(A),S)$.

We denote the periodic cyclic cohomology of A by $HC_{per}(A)$. Since S is of degree 2, this group has a natural $\mathbb{Z}/2$ grading:

$$HC^{ev}_{per}(A) = \varinjlim_{S} HC^{2n}(A) \quad \text{and} \quad HC^{odd}_{per}(A) = \varinjlim_{S} HC^{2n+1}(A) \ .$$

As we saw in Sections 1.3 and 1.4, cyclic cohomology can be defined as the cohomology of the bicomplex $\mathcal{C}_+$ or the bicomplex β_+. The identification of the S-operator as the degree 2 shift in either of these two bicomplexes leads to the following two propositions.

Proposition 1.5.2. The periodic cyclic cohomology of an algebra A is canonically isomorphic to the cohomology of its periodic bicomplex $\mathcal{C}(A)$.

Proof. This is a consequence of Propositions 1.3.6 and 1.3.13. Let us define a map $i : HC^{ev}_{per}(A) \longrightarrow H^{ev}(\mathrm{Tot}\ \mathcal{C})$ by sending a periodic class represented by, say,

$[\phi_{2n}] \in HC^{2n}(A)$ to $(0,...,\phi_{2n},0,0,...)$. By Proposition 1.3.13 i is well defined and by 1.3.6 i is an isomorphism. There is a similar map between odd groups which is similarly proved to be an isomorphism. □

Similarly we have:

Proposition 1.5.3. The periodic cyclic cohomology of a unital algebra A is canonically isomorphic to the cohomology of its (b,B) bicomplex $\beta(A)$. □

The quasi-isomorphism $I : \mathrm{Tot}\ \mathcal{C}_+(A) \longrightarrow \mathrm{Tot}\ \beta_+(A)$ defined in Section 1.4 naturally extends to a map $I : \mathrm{Tot}\ \mathcal{C}(A) \longrightarrow \mathrm{Tot}\ \beta(A)$. Now, this new I is a quasi-isomorphism too. Indeed, using Propositions 1.5.2 and 1.5.3 above, the method of proof of Proposition 1.4.13 extends to show that the above map I is a quasi-isomorphism. We record this in

Proposition 1.5.4. There is a canonical quasi-isomorphism $I : \mathrm{Tot}\ \mathcal{C}(A) \longrightarrow \mathrm{Tot}\ \beta(A)$. □

According to Proposition 1.4.6 the cohomology with infinite support of the (b,B) bicomplex of any algebra is trivial: given any, say even, cocyclce $\phi = (\phi_{2n})_{n\geq 0}$, one can always solve the equations

$$\phi = \partial\psi \tag{1.5.5}$$

to find a cochain $\psi = (\psi_{2n+1})_{n\geq 0}$ that bounds it. The basic observation in [4] is that if A is a normed algebra and we consider only those cochains $\phi = (\phi_m)$ for which the growth of $\|\phi_m\|$ is restricted, then in general we cannot solve (1.5.5) to obtain a ψ which satisfies the same growth conditions. Moreover, in this way one obtains a reasonable theory based on cochains with infinite length. Now, Proposition 1.5.3 shows that such a theory is of the type of periodic cyclic cohomology. We will study this theory, called entire cyclic cohomology, in detail in the next chapters.

We will end this section with a simple example which nicely illustrates the effect of growth conditions in the context of de Rham cohomology of non compact manifolds. We consider three different de Rham complexes for the real line $\mathbb{R}$ and show that the dimension of their cohomology groups (in dimension one) is zero, one and uncountable!

The first complex that we consider is the standard de Rham complex of $\mathbb{R}$

$$0 \longrightarrow \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \longrightarrow 0 \quad .$$

Here $\Omega^i(\mathbb{R})$ is th space of smooth i-forms on $\mathbb{R}$. As is well known the cohomology of this complex in dimension one is trivial: given any 1-form $w = f\,dx$ on $\mathbb{R}$, define the smooth function $g(x) = \int_0^x f(t)\,dt$, then $g \in \Omega^0(\mathbb{R})$ and $dg = w$.

Next, let us consider the de Rham complex of $\mathbb{R}$ with compact support

$$0 \longrightarrow \Omega^0{}_c(\mathbb{R}) \xrightarrow{d} \Omega^1{}_c(\mathbb{R}) \longrightarrow 0$$

where $\Omega^i{}_c(\mathbb{R})$ is the space of smooth i-forms on $\mathbb{R}$ with compact support. Again it is well-known and easy to see that $w \in \Omega^1{}_c(\mathbb{R})$ is exact if and only if $\int_{\mathbb{R}} w = 0$ and in fact the map $[w] \mapsto \int_{\mathbb{R}} w$ defines an isomorphism from $H^1{}_c(\mathbb{R})$ to $\mathbb{R}$, where $H^1{}_c(\mathbb{R}) = \Omega^1{}_c(\mathbb{R})/d\Omega^0{}_c(\mathbb{R})$ is the de Rham cohomology group of $\mathbb{R}$, in dimension one, with compact support.

The last complex that we consider is the so-called bounded de Rham complex defined by Roe (see [21], Section 3). Let M be a Riemannian manifold. Let $\Omega^i{}_\beta(M)$ be the Banach space completion of the space of smooth i-forms which have finite β-norm, where

$$\|w\|_\beta = \sup\{\|w(x)\| + \|dw(x)\| \,;\, x \in M\} \ .$$

Here the norms inside the bracket are induced from the inner products on the tangent spaces. The exterior derivative extends to $\Omega^*{}_\beta(M)$ and we obtain the bounded de Rham complex

$$0 \longrightarrow \Omega^0{}_\beta(M) \xrightarrow{d} \Omega^1{}_\beta(M) \xrightarrow{d} \ \cdots\ \xrightarrow{d} \Omega^n{}_\beta(M) \longrightarrow 0$$

where $n = \dim M$.

Definition 1.5.6. ([21], Section 3). The β-cohomology groups of a Riemannian manifold M are defined to be the groups

$$H^p_\beta(M) = \ker(d : \Omega^p_\beta \longrightarrow \Omega^{p+1}_\beta) \,/\, [\text{closure of } \operatorname{Im}(d : \Omega^{p-1}_\beta \longrightarrow \Omega^p_\beta)] \ .$$

Note the difference with the usual definition of cohomology groups: instead of dividing by exact forms one divides by the closure of exact forms. As a result of this the corresponding cohomology groups are Banach spaces in their own, right.

Now, let us consider the special case of $M = \mathbb{R}$. The following is our main lemma in finding $\dim H^1{}_\beta(\mathbb{R})$ (I am indebted to I. Putnam for discussions which led me to this useful lemma). By $\|\ \|_\infty$ we mean the usual sup-norm.

Lemma 1.5.7. Let f be a bounded function on $\mathbb{R}$ and let λ be a positive real number. Assume for every integer n there is an interval I_n of length n such that either $f(x) \geq \lambda$

for $x \in \cup I_n$ or $f(x) \leq -\lambda$ for $x \in \cup I_n$. Then $\|f-g'\|_\infty \geq \lambda$ for all bounded differentiable functions g on $\mathbb{R}$.

Proof. Assume $f(x) \geq \lambda$ for $x \in \cup I_n$. Let $\varepsilon > 0$ be given and let g be a bounded differentiable function on $\mathbb{R}$. We claim there is a point $x_0 \in \cup I_n$ such that $g'(x_0) \leq \varepsilon$. Indeed, if this is not the case, letting $I_n = [a_n, a_n+n]$, we can write

$$g(a_n+n) - g(a_n) = \int_{a_n}^{a_n+n} g'(t)\,dt \geq \varepsilon \cdot n \quad \text{for all } n.$$

That is $g(a_n+n) \geq g(a_n) + \varepsilon n \geq -\|g\|_\infty + n \cdot \varepsilon$ for all n. This obviously contradicts the boundedness of g. Now, for such a point x_0 we have $\|f-g'\|_\infty \geq |f(x_0) - g'(x_0)| \geq \lambda - \varepsilon$. Since ε is arbitrary we have $\|f-g'\|_\infty \geq \lambda$ as claimed. There is a similar proof for the case where $f(x) \leq -\lambda$ for $x \in \cup I_n$. □

Corollary 1.5.8. Let $w = fdx$ be a smooth 1-form on $\mathbb{R}$ such that f satisfies the conditions of the above lemma. Then w represents a non-trivial cohomology class in $H^1{}_\beta(\mathbb{R})$.

Proof. By the above lemma, we have $\|w-dg\|_\beta = \|f-g'\|_\infty \geq \lambda$ for all $g \in \Omega^0{}_\beta(\mathbb{R}) \cap C^\infty(\mathbb{R})$. This shows that w cannot be approximated by exact forms and hence represents a nontrivial class in $H^1{}_\beta(\mathbb{R})$. □

Next, we utilize this corollary to prove that $H^1{}_\beta(\mathbb{R})$ is infinite dimensional. Fix an integer $N \geq 3$. For each integer $k \geq 1$ divide the interval $[N^k, N^{k+1}]$ into N closed subintervals of equal length $E^k{}_1, \ldots, E^k{}_N$. For each i, $i \leq i \leq N$, choose a bounded smooth function f_i such that $f_i(x) = 0$ for $x \in E^k{}_j$, $j \neq i$, $k \geq 1$ and $f_i(x) = 1$ on a subinteral $I^k{}_i$ of $E^k{}_i$, $k = 1,2,\ldots$. We claim that the differential forms $w_i = f_i\,dx$, $1 \leq i \leq N$, represent linearly independent elements in $H^1{}_\beta(\mathbb{R})$. Indeed, for a sequence of real numbers λ_1 ,..., λ_N where $\lambda_j \neq 0$ for some j , the 1-form $\sum_{i=1}^{N} \lambda_i w_i$ satisfies the conditions of the Corollary 1.5.8 with $\lambda = |\lambda_j|$ and $I_n = I^n{}_j$, and hence represent a nontrivial class in $H^1{}_\beta(\mathbb{R})$. Since the integer N in the above argument is arbitrary, we have, in effect, shown that the (vector space) dimension of $H^1{}_\beta(\mathbb{R})$ is infinite.

Now, as mentioned before, the bounded cohomology groups $H^P{}_\beta(M)$ are Banach spaces. A simple application of Baire category theorem shows that the (vector space) dimension of an infinite dimensional Banach space is uncountable. This fact combined with the above paragraph shows that $H^1{}_\beta(\mathbb{R})$ has uncountable dimension. We summarize these results in:

Proposition 1.5.9. The (vector space) dimension of the bounded cohomology group $H^1{}_\beta(\mathbb{R})$ is uncountable. $\square$

Note that this is in sharp contrast with the first two cases where the cohomology groups were either trivial or 1-dimensional.

Chapter 2

2.1. Definition of Entire Cyclic Cohomology

In this section we give the original definition of the entire cyclic cohomology due to Connes and elaborate an instructive example from [4]. In the last part a trace map is studied which plays an important role for many things to come.

Let A be a unital Banach algebra over $\mathbb{C}$. Instead of arbitrary multilinear functionals on A, we will work only with continuous ones in this chapter. So, for any non-negative integer n, let $C^n = C^n(A)$ be the space of continuous $(n+1)$-linear forms on A. For $n < 0$ we set $C^n = \{0\}$, as usual. The Banach space norm on each C^n is defined by

$$\|\phi\| = \sup\{|\phi(a^0,\ldots,a^n)| \,;\, \|a^i\| \leq 1\} \ . \tag{2.1.1}$$

All of the operators that we defined in Chapter 1, in particular b and B, send a continuous cochain to a continuous one, and are actually bounded. We have the following easy estimates for the norms of some of these operators.

Lemma 2.1.2. For ϕ_n in C^n we have: $\|b\phi_n\| \leq (n+2)\,\|\phi_n\|$, $\|(1-\lambda)\,\phi_n\| \leq 2\|\phi_n\|$, $\|s\phi_n\| \leq \|\phi_n\|$, $\|N\phi_n\| \leq (n+1)\,\|\phi_n\|$ and $\|b'\phi_n\| \leq (n+1)\|\phi_n\|$.

Proof. Obvious. □

Since $B = Ns(1-\lambda)$, from the last three inequalities we have

$$\|B\phi_n\| \leq 2n\|\phi_n\| \ . \tag{2.1.3}$$

Everything we did in Chapter 1 naturally carries over to the category of Banach algebras provided everywhere we use continuous cochains instead of arbitrary ones. Therefore, for a Banach algebra one can define continuous Hochschild, bar and cyclic complexes and continuous Hochschild and cyclic cohomology groups. In particular there is a (b,B) bicomplex $\beta = \beta(A) = (\beta^{n,m},b,B)$ defined for any unital Banach algebra A, where $\beta^{n,m} = C^{n-m}(A)$ for all n, m in $\mathbb{Z}$. By Proposition 1.5.3 in Chapter 1, the cohomology of the bicomplex $\beta(A)$ is isomorphic to the continuous periodic cyclic cohomology of A. On the other extreme, Proposition 1.4.6 shows that the cohomology of the complex of cochains in $\beta(A)$, with infinite support, is zero. The basic observation of [4] is that if we control the growth of $\|\phi_m\|$ in a cochain $(\phi_{2n})_{n\geq 0}$ or $(\phi_{2n+1})_{n\geq 0}$ of the (b,B) bicomplex then we get a nontrivial cohomology theory of deRham type for Banach algebras, i.e. of the type of cyclic cohomology theory. Let $C^{ev} = \{(\phi_{2n})_{n\geq 0}\,;\ \phi_{2n} \in C^{2n}\}$ and $C^{odd} = \{(\phi_{2n+1})_{n\geq 0}\,;\ \phi_{2n+1} \in C^{2n+1}\}$ be the spaces of even and odd cochains with infinite support in $\beta(A)$ and let $\partial = b+B$ be the boundary operator which maps C^{ev} to C^{odd} and C^{odd} to C^{ev}. We shall use the following growth condition:

Definition 2.1.4. (Connes [4]). An even (resp. odd) cochain $(\phi_{2n})_{n\geq 0}$ (resp. $(\phi_{2n+1})_{n\geq 0}$) is called *entire* iff the radius of convergence of the power series $\sum_{n\geq 0} \frac{(2n)!}{n!} \|\phi_{2n}\| z^n$ (resp. $\sum_{n\geq 0} \frac{(2n+1)!}{n!} \|\phi_{2n+1}\| z^n$) is infinite.

The space of even (resp. odd) entire cochains will be denoted by C_ε^{ev}(resp. C_ε^{odd}).

Lemma 2.1.5. If ϕ is an even (or odd) entire cochain, then so is $\partial\phi = (b+B)\,\phi$.

Proof. Let $\partial\phi = \psi$. We have $\psi_m = b\phi_{m-1} + B\phi_{m+1}$ hence by Lemma 2.1.2 and formula (2.1.3), $\|\psi_m\| \leq (m+1)\,\|\phi_{m-1}\| + 2(m+1)\,\|\phi_{m+1}\|$. This shows that ψ is entire if ϕ is. $\square$

The above lemma shows that we have a periodic complex of entire cochains for any Banach algebra A :

$$\longrightarrow C_\varepsilon^{ev} \xrightarrow{\partial} C_\varepsilon^{odd} \xrightarrow{\partial} C_\varepsilon^{ev} \longrightarrow \ . \tag{2.1.6}$$

Definition 2.1.7. Let A be a unital Banach algebra. The *entire cyclic cohomology* of A is the cohomology of the above periodic complex of entire cochains.

We will use $H_\varepsilon^{ev}(A)$ and $H_\varepsilon^{odd}(A)$ to denote the even and odd entire cyclic cohomology groups of A . A finite cochain, $(\phi_m)_{m\geq 0}$; $\phi_m = 0$ for m large enough, is obviously an entire cochain. This implies (using proposition 1.5.3) that there is an obvious map from (continuous) periodic cyclic cohomology groups $HC^*_{per}(A)$ to $H^*_\varepsilon(A)$, $*$ = even or odd.

As an example we calculate the entire cyclic cohomology groups, H_ε^{ev} and H_ε^{odd}, for the simplest Banach algebra i.e. $A = \mathbb{C}$. As we will see this example is instructive in many ways. First note that an (m+1)-linear functional on $\mathbb{C}$ is of the form $\phi_m(a^0,\dots,a^m) = \lambda_m a^0 \dots a^m$, where $\lambda_m = \phi_m(1,\dots,1)$ and $\|\phi_m\| = |\lambda_m|$. A simple calculation then shows that

$$b\phi_{2m} = B\phi_{2m} = 0 \ , \ (b\phi_{2m+1})(a^0,\dots,a^{2m+2}) = \lambda_{2m+1}\, a^0\dots a^{2m+2}$$

and finally

$$(B\phi_{2m+1})(a^0,\dots,a^{2m}) = 2(2m+1)\, \lambda_{2m+1}\, a^0\dots a^{2m} \ .$$

Let E be the space of entire functions on the complex plane. We have the obvious (vector space) isomorphisms $C_\varepsilon^{ev}(\mathbb{C}) \simeq E$, where a sequence $(\lambda_{2n})_{n\geq 0}$ is sent to the entire function $\sum_{n\geq 0} \frac{(2n)!}{n!} \lambda_{2n} Z^n$, and $C_\varepsilon^{odd}(\mathbb{C}) \simeq E$, where $(\lambda_{2n+1})_{n\geq 0}$ is mapped to $\sum_{n\geq 0} \frac{(2n+1)!}{n!} \lambda_{2n+1} Z^n$. From the calculation of b and B in above, we have $\partial(C_\varepsilon^{ev}) = 0$ and $\partial\left(\sum_{n\geq 0} a_n Z^n\right) = 2(Z+1)\left(\sum_{n\geq 0} a_n Z^n\right)$ for an odd cochain $\sum_{n\geq 0} a_n Z^n$ in E.

We, therefore, have an isomorphism of complexes

$$\begin{array}{ccccccccc}
\longrightarrow & C_\varepsilon^{ev}(\mathbb{C}) & \xrightarrow{\partial} & C_\varepsilon^{odd}(\mathbb{C}) & \xrightarrow{\partial} & C_\varepsilon^{ev}(\mathbb{C}) & \longrightarrow & \\
& \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & E & \xrightarrow{0} & E & \xrightarrow{2(Z+1)} & E & \longrightarrow & .
\end{array} \tag{2.1.8}$$

Some elementary analytic function theory reveals that

$$\operatorname{Im}(\partial: C_\varepsilon^{odd} \longrightarrow C_\varepsilon^{ev}) \approx \{f\in E \,;\, f = 2(z+1)g\,,\, g\in E\} = \{f\in E \,;\, f(-1) = 0\}$$

$$\ker(\partial : C_\varepsilon^{odd} \longrightarrow C_\varepsilon^{ev}) \approx \{f\in E \,;\, 2(z+1)f = 0\} = \{0\}\,.$$

Based on this we have

$$H_\varepsilon^{odd}(\mathbb{C}) = \frac{\operatorname{Ker}(\partial : C_\varepsilon^{odd} \to C_\varepsilon^{ev})}{\operatorname{Im}(\partial : C_\varepsilon^{ev} \to C_\varepsilon^{odd})} = \frac{\{0\}}{\{0\}} = \{0\}$$

$$H_\varepsilon^{ev}(\mathbb{C}) = \frac{\operatorname{Ker}(\partial : C_\varepsilon^{ev} \to C_\varepsilon^{odd})}{\operatorname{Im}(\partial : C_\varepsilon^{odd} \to C_\varepsilon^{ev})} = \frac{E}{\{f\in E \,;\, f(-1) = 0\}} \simeq \mathbb{C}$$

where the last isomorphism is defined by evaluation at $-1 : [f] \longrightarrow f(-1)$. We have proved the following proposition in [4].

Proposition 2.1.9. We have $H_\varepsilon^{odd}(\mathbb{C}) = \{0\}$ and $H_\varepsilon^{ev}(\mathbb{C}) = \mathbb{C}$ with isomorphism

$$\text{given by} \quad \sigma((\phi_{2n})_{n\geq 0}) = \sum_0^\infty (-1)^n \frac{(2n)!}{n!} \phi_{2n}(1,\ldots,1) \quad . \qquad \square$$

It is quite instructive to calculate the periodic cyclic cohomology of $\mathbb{C}$ along the same lines. Indeed, all we have to do is to replace the complex of entire cochains $C_\varepsilon^*(\mathbb{C})$ by the complex of finite cochains $C_f^*(\mathbb{C})$, * = even or odd and to replace the space of entire functions E by the space of polynomials in one variable, P. The isomorphism (2.1.8) should then be replaced by

$$\begin{array}{ccccccccc} \longrightarrow & C_f^{ev}(\mathbb{C}) & \xrightarrow{\partial} & C_f^{odd}(\mathbb{C}) & \xrightarrow{\partial} & C_f^{ev}(\mathbb{C}) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & P & \xrightarrow{0} & P & \xrightarrow{2(z+1)} & P & \longrightarrow \end{array}$$

and we obtain $HC_{per}^{odd}(\mathbb{C}) = 0$ and $HC_{per}^{ev}(\mathbb{C}) = \mathbb{C}$. We note that, in this example, the natural map $HC_{per}^{ev} \longrightarrow H_\varepsilon^{ev}$ is an isomorphism. That is so because the cocycle z which is a generator of $HC_{per}^{ev}(\mathbb{C})$ is mapped to z which is also a generator of $H_\varepsilon^{ev}(\mathbb{C})$.

Finally, we should remark that the space of even (or odd) cochains with infinite support in $\beta(\mathbb{C})$ is naturally isomorphic to the space, F, of formal power series in one variable. The isomorphism (2.1.8) is then of the form

$$\begin{array}{ccccccccc} \longrightarrow & C^{ev}(\mathbb{C}) & \xrightarrow{\partial} & C^{odd}(\mathbb{C}) & \xrightarrow{\partial} & C^{ev}(\mathbb{C}) & \longrightarrow \\ & & & & & & \\ \longrightarrow & F & \xrightarrow{0} & F & \quad 2(z+1) \quad & F & \longrightarrow \quad . \end{array}$$

Now, (1+z) is invertible in F so, unlike the other two cases, the cohomology of the above complex is trivial in both even and odd degrees. This is of course a very special case of Proposition 1.4.6 which is true for any algebra. What we have done is to trace this back to a fact about formal power series. Similarly, the "reason" for a nontrivial cohomology in the first two cases is the non-invertibility of $(1+z)$ in E or P.

For q a positive integer and A any algebra, let $M_q(A) = M_q(\mathbb{C}) \otimes A$ be the algebra of $q\times q$ matrices over A. There is an important *trace map* $\mathrm{Tr} : C^n(A) \longrightarrow C^n(M_q(A))$ defined by $\mathrm{Tr}\,\phi = \phi^q$, where

$$\phi^q(\mu^0 \otimes a^0,\dots,\mu^n \otimes a^n) = \mathrm{tr}(\mu^0\dots\mu^n)\ \phi(a^0,\dots,a^n)\ . \tag{2.1.10}$$

Here $\mathrm{tr} : M_q(\mathbb{C}) \longrightarrow \mathbb{C}$ is the usual trace of matrices. As an example, let $\phi \in C^0(A)$ be a linear form on A. Then ϕ^q is the linear form on $M_q(A)$ given by $\phi^q((a_{ij})) = \sum_{i=1}^{q} \phi(a_{ii})$. Indeed, letting $e^{ij} \in M_q(\mathbb{C})$ be the elementary matrices, we have

$$\phi^q((a_{ij})) = \phi^q\left(\sum e^{ij} \otimes a_{ij}\right) = \sum \mathrm{tr}(e^{ij})\ \phi(a_{ij}) = \sum \phi(a_{ii})\ .$$

The following lemma shows that Tr is a chain map in a very strong sense of the word.

Lemma 2.1.11. We have $b\circ\mathrm{Tr} = \mathrm{Tr}\circ b$, $\lambda\circ\mathrm{Tr} = \mathrm{Tr}\circ\lambda$ and $s\circ\mathrm{Tr} = \mathrm{Tr}\circ s$.

Proof. This is a consequence of trace property of tr. We prove only the first relation, since the proof of the rest is quite similar. Let $\phi \in C^n$. We have

$$b(\mathrm{Tr}\,\phi)(\mu^0 \otimes a^0,\dots, \mu^{n+1} \otimes a^{n+1}) =$$

$$\sum_{j=0}^{n} (-1)^j \phi^q(\mu^0 \otimes a^0,\dots, \mu^j \otimes a^j \cdot \mu^{j+1} \otimes a^{j+1},\dots, \mu^{n+1} \otimes a^{n+1}) +$$

$$(-1)^{n+1} \phi^q(\mu^{n+1} \otimes a^{n+1} \cdot \mu^0 \otimes a^0,\dots, \mu^n \otimes a^n)$$

$$= \sum_{j=0}^{n} tr(\mu^0,\dots,\mu^{n+1})\,(-1)^j\,\phi(a^0,\dots,a^j a^{j+1},\dots,a^{n+1}) +$$

$$(-1)^{n+1}\,tr(\mu^{n+1}\mu^0,\dots,\mu^n)\,\phi(a^{n+1}a^0,\dots,a^n)$$

$$= tr(\mu^0,\dots,\mu^{n+1})\,(b\phi)\,(a^0,\dots,a^{n+1})$$

$$= Tr(b\phi)\,(\mu^0\otimes a^0,\dots,\mu^{n+1}\otimes a^{n+1})\,.$$

This proves the equality of both sides as applied to elementary tensors. By multilinearity we have the equality. □

It follows from this lemma that $N\circ Tr = Tr\circ N$ and consequently $B\circ Tr = Tr\circ B$. Hence we have the following

Corollary 2.1.12. The map $Tr : \beta(A) \longrightarrow \beta(M_q(A))$ is a map of bicomplexes.

□

Next, for A a Banach algebra, let us define a Banach algebra norm on $M_q(A)$ by defining $\|(a_{ij})\| = \sum_{i,j=1}^{q} \|a_{ij}\|$ for any $q\times q$ matrix (a_{ij}). This is a convenient norm to work with and is equivalent to other standard norms on $M_q(A)$. We should mention that the concept of an entire cochain on a Banach algebra depends only on the equivalence class of the norm we are using (recall: $\|\cdot\|$, $\|\cdot\|'$ are equivalent if there are constants C and C' such that $\|\cdot\| \le C\|\cdot\|' \le C'\|\cdot\|$) hence the choice of a norm for $M_q(A)$, as long as it is equivalent to the above one, is a matter of convenience.

Lemma 2.1.13. We have $\|\phi^q\| \le \|\phi\|$ for all $\phi \in C^n$.

Proof. We prove this only for $n = 1$. The general case is completely similar. we have, with e^{ij} denoting the elementary matrices

$$|\phi^q((a_{ij}^0), (a_{ij}^1))| = |\phi^q(\Sigma\, e^{ij} \times a_{ij}^0, \Sigma\, e^{k\ell} \times a_{k\ell}^1)| =$$

$$|\,\Sigma\, \mathrm{tr}(e^{ij} \cdot e^{k\ell})\, \phi(a_{ij}^0, a_{k\ell}^1)| \le |\,\Sigma\, \phi(a_{ij}^0, a_{k\ell}^1)| \le$$

$$\|\phi\|\, \Sigma\, \|a_{ij}^0\|\, \|a_{k\ell}^1\| = \|\phi\|\, (\Sigma\, \|a_{ij}^0\|)\, (\Sigma\, \|a_{k\ell}^1\|) =$$

$$\|\phi\|\, \|(a_{ij}^0)\|\, \|(a_{k\ell}^1)\| .$$ □

It follows that if $\phi = (\phi_{2n})_{n\ge0}$ (resp. $\phi = (\phi_{2n+1})_{n\ge0}$) is an even (resp. odd) entire cochain then the cochain $\phi^q = (\phi^q{}_{2n})_{n\ge0}$ (resp. $\phi^q = (\phi^q{}_{2n+1})_{n\ge0}$) on $M_q(A)$ is also entire. This fact combined with the previous corollary proves the following

Lemma 2.1.14. The map $\phi \longrightarrow \mathrm{Tr}\,\phi$ is a morphism of the complexes of entire cochains. □

2.2. Pairing with Topological K-Theory

Many of the applications of (ordinary) cyclic cohomology to questions in analysis and topology depend on the fact that this theory can be paired with K-theory. In [3], through explicit formulas, it is shown how to define the pairings $K_0(A) \otimes HC^{2n}(A) \longrightarrow \mathbb{C}$ and $K_1(A) \otimes HC^{2n+1}(A) \longrightarrow \mathbb{C}$. Here K_0 and K_1 are algebraic K-theory functors and A is an arbitrary algebra. This pairing is then shown to be invariant under the action of the S-operator on cyclic cohomology groups and as a result of this we have pairings between

K-theory and periodic cyclic cohomology groups: $K_*(A) \otimes HC^*_{per}(A) \longrightarrow \mathbb{C}$, $* = 0,1$. Moreover, in topological situations e.g. when A is a Banach algebra, one can show that the same formulas define a pairing between topological K-theory and continuous cyclic cohomology of A .

In this section we are interested in the pairing $K_0(A) \otimes H_\varepsilon^{ev}(A) \longrightarrow \mathbb{C}$ where A is a unital Banach algebra. The existing formulas for this pairing are due to Connes [4] and Getzler and Szenes [12] . In both of these formulas one has to assume that the cocycle satisfies certain technical conditions. In this section we will show that it is possible to derive a formula for this pairing which assumes no conditions on the type of the entire cocycle and generalizes the above two formulas. More precisely we show, using the normalization lemma in [4], that it is possible to express Connes' formula in terms of the original cocycle and we will prove that the resulting formula actually defines a pairing between K_0 and H_ε^{ev} . Let us start with

Definition 2.2.1. (Connes [4]). A cocycle $(\phi_{2n})_{n\geq 0}$ (resp. $(\phi_{2n+1})_{n\geq 0}$) is *normalized* iff $B_0\phi_{2n}$ (resp. $B_0\phi_{2n+1}$) is a cyclic cochain for all n .

The concept of a normalized cocycle is important in entire cyclic cohomology. As is emphasized in [4], only normalized cocycles have a natural interpretation as traces on Cuntz algebras or as infinite dimensional cycles on universal differential graded algebras. We will come to this point later. Part of the usefulness of this notion is because of the following lemma, referred to afterwards as the *normalization lemma.*

Lemma 2.2.2. For every entire cocycle there is a normalized cohomologous entire cocycle.

Proof. See Connes [4], Lemma 6 or the remark after Lemma 3.2.1 in this thesis. □

Now, let $e \in A$ be an idempotent and $\phi = (\phi_{2n})_{n\geq 0}$ a normalized entire cocycle. The following formula is used in [4] to define a pairing between $H_\varepsilon^{ev}(A)$ and $K_0(A)$:

$$\langle \phi, e \rangle = \sum_{n\geq 0} (-1)^n \frac{(2n)!}{n!} \phi_{2n}(e,\ldots,e) \quad . \tag{2.2.3}$$

More generally, if $e \in M_q(A)$ is an idempotent, then one defines

$$\langle \phi, e \rangle = \langle \mathrm{Tr}\, \phi, e \rangle = \sum_{n\geq 0} (-1)^n \frac{(2n)!}{n!} \phi_{2n}^{q}(e,\ldots,e) \quad . \tag{2.2.3}'$$

Note that the series are absolutely convergent thanks to the entire condition on the cocycle ϕ. In order to show that (2.2.3) and (2.2.3)' define the required pairing one has to check the following:

(a) if $\phi = \partial\psi$ where ϕ is normalized and ψ is an odd entire cochain then $\langle \phi, e \rangle = 0$ for all idempotents $e \in A$;

(b) if e_t, $0 \leq t \leq 1$, is an smooth path of idempotents in A then $\langle \phi, e_0 \rangle = \langle \phi, e_1 \rangle$;

(c) additivity: if $e = f \oplus g$ where $f \in M_q(A)$ and $g \in M_p(A)$ then $\langle \phi, e \rangle = \langle \phi, f \rangle + \langle \phi, g \rangle$.

As for the proofs of these statements, we note that (c) follows quite easily from the definition of $\mathrm{Tr} : C^n(A) \longrightarrow C^n(M_q(A))$, (a) follows from Lemma 2.2.6 below and we refer the reader to [4], Theorem 8, for the proof of statement (b) above. We start off by showing that (2.2.3) can be expressed in terms of the original cocycle.

Lemma 2.2.4. Let $\phi = (\phi_{2n})_{n\geq 0}$ be an entire cocycle and $\phi' = (\phi'_{2n})_{n\geq 0}$ a normalized entire cocycle cohomologous to ϕ. If $e \in A$ is an idempotent, then

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!} \phi'_{2n}(e,\ldots,e) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!} \{\phi_{2n}(e,\ldots,e) - \frac{1}{2} B_0\phi_{2n}(e,\ldots,e)\} \;.$$

Proof. By Lemma 7 in [4] or 2.2.6 below the value of the left hand side depends only on the class of the normalized cocycle ϕ' in $H_{\varepsilon}^{ev}(A)$. A careful look at the proof of the normalization lemma reveals that we can choose $\phi'_{2n} = \phi_{2n} - b\tilde{\theta}_{2n-1}$ where $\tilde{\theta}_{2n-1} = N'\theta_{2n-1} = \frac{-1}{2n}(1 + 2\lambda + 3\lambda^2 + ... + 2n\lambda^{2n-1})\,\theta_{2n-1}$ and $\theta_{2n-1} = B_0\,\phi_{2n} - \frac{1}{2n}NB_0\,\phi_{2n}$. Now with this choice we have

$$\phi'_{2n}(e,...,e) = \phi_{2n}(e,...,e) - \tilde{\theta}_{2n-1}(e,...,e)\,.$$

But

$$\tilde{\theta}_{2n-1}(e,...,e) = -\frac{1}{2n}(1 - 2 + 3 - ... + (2n-1) - 2n)\,\theta_{2n-1}(e,...,e)$$

$$= \frac{1}{2}\,\theta_{2n-1}(e,...,e) = \frac{1}{2}\,B_0\phi_{2n}(e,...,e) - \frac{1}{4n}\,B\phi_{2n}(e,...,e)\ .$$

Using the fact that $(\phi_{2n})_{n\geq 0}$ is a cocycle, we have $B\phi_{2n} = -b\phi_{2n-2}$ and hence

$$B\phi_{2n}(e,...,e) = -b\phi_{2n-2}(e,...,e) = 0\ .$$

As a result of this we have

$$\tilde{\theta}_{2n-1}(e,...,e) = \frac{1}{2}\,B_0\phi_{2n}(e,...,e)\ .$$

and finally

$$\phi'_{2n}(e,...,e) = \phi_{2n}(e,...,e) - \frac{1}{2}\,B_0\phi_{2n}(e,...,e)$$

we should remind the reader that this last equality holds only for a particular choice of normalized cocycle $(\phi'_{2n})_{n\geq 0}$, namely the one we defined above. For this choice the two series are equal term by term and hence their sums are equal. Since, as we mentioned at the beginning of the proof, the value of the left hand side depends only on the cohomology class of ϕ' we are done. $\square$

As a result of this lemma it seems natural to define a pairing between any entire cocycle ϕ and an idempotent e by the formula

$$\langle \phi, e\rangle = \sum (-1)^n \frac{(2n)!}{n!} \{\phi_{2n}(e,\ldots,e) - \frac{1}{2} B_0\phi_{2n}(e,\ldots,e)\} \quad . \tag{2.2.5}$$

In order to show that this defines a pairing $H_\varepsilon^{ev}(A) \otimes K_0(A) \longrightarrow \mathbb{C}$ we have to repeat the steps (a), (b), (c) as before. The following lemma is part (a) of this plan.

Lemma 2.2.6. (Compare Lemma 7 in [4]). Let $\phi = (\phi_{2n})_{n\geq 0}$ be an entire cocycle on A. If ϕ is a coboundary, then we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!} \{\phi_{2n}(e,\ldots,e) - \frac{1}{2} B_0\phi_{2n}(e,\ldots,e)\} = 0$$

for any idempotent $e \in A$.

Proof. Let $\psi = (\psi_{2n+1})_{n\geq 0}$ be an (odd) entire cochain such that $\partial\psi = \phi$. We thus have $\phi_{2n} = b\psi_{2n-1} + B\psi_{2n+1}$ for all n. Recalling the formula $b'B_0 + B_0 b = 1 - \lambda$ from section 1.2, we have $B_0\phi_{2n} = B_0(b\psi_{2n-1} + B\psi_{2n-1} = (1-\lambda)\,\psi_{2n-1} - b'B_0\,\psi_{2n-1}$. It follows, since $e^2 = e$, that

$$B_0\phi_{2n}(e,\ldots,e) = 2\psi_{2n-1}(e,\ldots,e) - B_0\psi_{2n-1}(e,\ldots,e) \quad .$$

Next we have

$$\phi_{2n}(e,\ldots,e) = (b\psi_{2n-1} + B\psi_{2n+1})(e,\ldots,e) = \psi_{2n-1}(e,\ldots,e)$$

$$+ (2n+1)\, B_0\psi_{2n+1}(e,\ldots,e) \quad .$$

From these two relations we obtain

$$\phi_{2n}(e,\ldots,e) - \frac{1}{2} B_0\phi_{2n}(e,\ldots,e) = (2n+1)\, B_0\psi_{2n+1}(e,\ldots,e)$$

$$+ \frac{1}{2} B_0\psi_{2n-1}(e,\ldots,e) \quad .$$

Let $\alpha_n = B_0\psi_{2n+1}(e,\dots,e)$. We have

$$\langle \phi,e\rangle = \langle \partial\psi,e\rangle = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!} \{(2n+1)\,\alpha_n + \frac{1}{2}\alpha_{n-1}\} = 0$$

and the lemma is proved. □

Next let e_t , $0 \le t \le 1$, be a smooth path of idempotents. We have to show that $\langle\phi,e_0\rangle = \langle\phi,e_1\rangle$ where the bracket $\langle\ ,\ \rangle$ is defined using (2.2.5). Let ϕ' be a normalized cocycle cohomologous to ϕ . Then by Lemma 2.2.4 $\langle\phi',e_t\rangle = \langle\phi,e_t\rangle$ where the left hand side is defined by formula (2.2.3). Since $\langle\phi',e_0\rangle = \langle\phi',e_1\rangle$ ([4], theorem 8), we have $\langle\phi,e_0\rangle = \langle\phi,e_1\rangle$. This proves (b). Finally, we note that condition (c), namely the additivity of the pairing, follows directly from the definition of Tr .

To summarize: we have shown that formula (2.2.5) defines a pairing between $H_\varepsilon{}^{ev}(A)$ and $K_0(A)$ which is the same as the pairing defined in [4]. We also note that Getzler and Szenes' formula [12], namely $\langle \phi,e\rangle = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!} \phi_{2n}(e - \frac{1}{2}, e,\dots,e)$, which works only for *normal cocycles,* is a special case of formula (2.2.5). This is clear since $B_0\phi_{2n}(e,\dots,e) = \phi_{2n}(1,e,\dots,e)$ if ϕ_{2n} is a normal cochain.

Remark. A cochain $\phi \in C^n$ is called *normal* if $\phi(a^0,a^1,\dots,a^n) = 0$ whenever $a^i = 1$ for some $i > 0$. Loday-Quillen ([18], section 1) call such cochains normalized but we decided to call them normal and use normalized for cocycles which are normalized á la Connes i.e. in the sense of Definition 2.2.1. Note that a normal 0-cochain is just an ordinary 0-cochain. A cochain $\phi = (\phi_{2n})_{n\ge0}$ (resp. $\phi = (\phi_{2n+1})_{n\ge0}$) in (b,B) bicomplex is said to be normal if each component ϕ_m is normal. It is easy to see that the space of normal cochains is closed under the operators b and B and hence normal cochains form a subcomplex of the (b,B) bicomplex which we denote by $\beta(A)_{norm}$.

2.3. A Comparison Theorem

In this section we prove a comparison theorem which shows that entire cyclic cohomology can be formulated in terms of the periodic bicomplex. A careful look at the definition of the periodic bicomplex reveals that there is no need to assume the algebra is unital. As a result of this we can extend the entire cyclic cohomology functor to the non-unital category.

Let A be a unital algebra and let us recall the Loday-Quillen chain map $I : \mathrm{Tot}\, \mathcal{C}(A) \longrightarrow \mathrm{Tot}\, \beta(A)$ defined in Section 1.5 by the formulas

$$(I\phi)_{2n} = \phi_{2n} + Ns\phi_{2n-1} \qquad \text{if } \phi = (\phi_n)_{n\geq 0} \text{ is an even cochain}$$

$$(I\phi)_{2n+1} = \phi_{2n+1} + Ns\phi_{2n+2} \qquad \text{if } \phi = (\phi_n)_{n\geq 0} \text{ is an odd cochain}$$

In [17] C. Kassel defines (the dual of) a map $J : \mathrm{Tot}\, \beta(A) \longrightarrow \mathrm{Tot}\, \mathcal{C}(A)$ by the formulas

$$(J\phi)_{2n} = \phi_{2n} \ , \ (J\phi)_{2n-1} = B_0\phi_{2n} \qquad \text{if } \phi = (\phi_{2n})_{n\geq 0} \text{ is an even cochain}$$

$$(J\phi)_{2n} = B_0\phi_{2n+1} \ , \ (J\phi)_{2n-1} = \phi_{2n-1} \qquad \text{if } \phi = (\phi_{2n+1})_{n\geq 0} \text{ is an odd cochain.}$$

It is easy to see that J is a chain map. The following lemma and its proof is due to Kassel [17].

Lemma 2.3.1. J is a left homotopy inverse to I.

Proof. Define a homotopy operator (of degree 1) $h : \mathrm{Tot}\, \mathcal{C}(A) \longrightarrow \mathrm{Tot}\, \mathcal{C}(A)$ by the formulas

$(h\phi)_{2n} = s\phi_{2n+1}$, $(h\phi)_{2n-1} = 0$ if ϕ is an even cochain

$(h\phi)_{2n} = 0$, $(h\phi)_{2n-1} = s\phi_{2n}$ if ϕ is an odd cochain .

It is easy to check that

$$J \circ I = \mathrm{id} + h\partial + \partial h$$

where ∂ denotes the coboundary of Tot $\mathfrak{C}(A)$. This of course shows that J is a left homotopy inverse to I . □

The question of whether there exists a nice homotopy between $I \circ J$ and identity is crucial for entire cyclic cohomology and is not answered in [17]. In the following lemma we show that such a homotopy indeed exists.

Lemma 2.3.2. J is a right homotopy inverse to I .

Proof. Let us define a homotopy operator (of degree 1) $\tilde{h} : \mathrm{To}\beta(A) \longrightarrow \mathrm{To}\beta\,(A)$ by the formulas

$(\tilde{h}\phi)_{2n} = Ns^2 B_0 \phi_{2n+3}$ if $\phi = (\phi_{2n+1})_{n\geq 0}$ is an odd cochain

$(\tilde{h}\phi)_{2n+1} = Ns^2 B_0 \phi_{2n+4}$ if $\phi = (\phi_{2n})_{n\geq 0}$ is an even cochain .

We have to show that $IoJ = \mathrm{id} + \partial \tilde{h} + \tilde{h}\partial$, where $\partial = b+B$ is the coboundary of Tot $\beta(A)$. Let $\phi = (\phi_{2n+1})_{n\geq 0}$ be an odd cochain in Tot $\beta(A)$. It is easy to see that

$$(IJ\phi)_{2n+1} = NsB_0\, \phi_{2n+3} + \phi_{2n+1} \cdot$$

On the other hand we have

$$((\partial \tilde{h} + \tilde{h}\partial)\phi)_{2n+1} = b(\tilde{h}\phi)_{2n} + B(\tilde{h}\phi)_{2n+2} + Ns^2B_0(\partial\phi)_{2n+4}$$

$$= b(Ns^2B_0\phi_{2n+3}) + B(Ns^2B_0\phi_{2n+5})$$

$$+ Ns^2B_0(b\phi_{2n+3} + B\phi_{2n+5}) \ .$$

But $BN = B_0B = 0$ and hence the above sum is equal to

$$= bNs^2B_0\phi_{2n+3} + Ns^2B_0b\phi_{2n+3}$$

$$= Nb'ssB_0\phi_{2n+3} + Ns^2B_0b\phi_{2n+3}$$

$$= N(1\text{-}sb')sB_0\phi_{2n+3} + Ns^2B_0b\phi_{2n+3}$$

$$= NsB_0\phi_{2n+3} - Ns(1\text{-}sb')s(1\text{-}\lambda)\phi_{2n+3} + Ns^2B_0b\phi_{2n+3}$$

$$= NsB_0\phi_{2n+3} - Ns^2(1\text{-}\lambda)\phi_{2n+3} + Ns^2b's(1\text{-}\lambda)\phi_{2n+3} + Ns^2B_0b\phi_{2n+3}$$

$$= NsB_0\phi_{2n+3} - Ns^2(1\text{-}\lambda)\phi_{2n+3} + Ns^2(1\text{-}sb')(1\text{-}\lambda)\phi_{2n+3} + Ns^2B_0b\phi_{2n+3}$$

$$= NsB_0\phi_{2n+3} - Ns^2sb'(1\text{-}\lambda)\phi_{2n+3} + Ns^2B_0b\phi_{2n+3}$$

$$= NsB_0\phi_{2n+3}$$

where we have used the homotopy formula $b's + sb' = id$ together with the relations $Nb' = bN$ and $b'(1\text{-}\lambda) = (1\text{-}\lambda)b$ all from Section 1.1. This shows that $IJ\phi = (id + \partial\tilde{h} + \tilde{h}\partial)\phi$ for any odd cochain ϕ. The proof for even cochains is completely similar. □

Lemmas 2.3.1 and 2.3.2, combined together, show that the Loday-Quillen map $I : \mathrm{Tot}\, \mathcal{C}(A) \longrightarrow \mathrm{Tot}\, \beta(A)$ is a homotopy equivalence with homotopy inverse J. In particular this shows that I is a quasi-isomorphism i.e. induces an isomorphism between cohomology groups and provides a new proof for propositions 1.4.13 and 1.5.4. The important thing is that this proof, unlike our first one, easily extends to the case of entire cyclic cohomology.

We impose the following growth condition on the norm of cochains in the periodic bicomplex. Let A be a non-unital Banach algebra.

Definition 2.3.3. An even (or odd) cochain $(\phi_n)_{n\geq 0}$ in $\mathcal{C}(A)$ is called *entire* iff

$$\sum_{n\geq 0} \frac{n!}{(\frac{n}{2})!} \|\phi_n\| z^n$$ is an entire function of z.

In the above definition $(\frac{n}{2}) = k$ if $n = 2k$ or $n = 2k+1$. It is easy to see that entire cochains form a subcomplex of the complex of cochains with infinite support in $\mathcal{C}(A)$. It is the cohomology of the entire cochains in $\mathcal{C}(A)$ that we would like to compare to entire cyclic cohomology. Note that the maps I and J send entire cochains to entire cochains and hence are morphisms of the complexes of entire cochains. Similarly the homotopies h and $\tilde{h}$ respect the entire growth conditions and proofs of Lemmas 2.3.1 and 2.3.2 extend word for word to the case of entire cochains. The upshot is the following:

Theorem 2.3.4. (Comparison Theorem). Let A be a unital Banach algebra. The map I is a homotopy equivalence between the complexes of entire cochains in the periodic and (b,B) bicomplexes of A . □

We utilize this comparison theorem to extend the definition of entire cyclic cohomology to non-unital Banach algebras.

Definition 2.3.5. Let A be a non-unital Banach algebra. The entire cyclic cohomology of A is the cohomology of the complex of entire cochains in the periodic bicomplex $\mathfrak{C}(A)$.

We will use the same notation, H^*_ε, to denote this functor. A non-unital continuous homomorphism $f: A \longrightarrow B$ induces a map of bicomplexes from $\mathfrak{C}(B)$ to $\mathfrak{C}(A)$ in an obvious way. Using continuity of f we can check that entire cochains are mapped to entire cochains and hence we get a map $f^*: H^*_\varepsilon(B) \longrightarrow H^*_\varepsilon(A)$.

There are certain obvious compatibility questions that must be addressed. First, let us check that the new functor extends the old one. For this let $f: A \longrightarrow B$ be a unital homomorphism. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}\ \mathfrak{C}(B) & \xrightarrow{f^*} & \mathrm{Tot}\ \mathfrak{C}(A) \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle I} \\ \mathrm{Tot}\ \beta(B) & \xrightarrow{f^*} & \mathrm{Tot}\ \beta(A) \end{array}$$

Combined with the comparison theorem, this shows that the two definitions of H^*_ε coincide on unital Banach algebras.

Next, let $f: A \longrightarrow B$ be a non-unital homomorphism between unital algebras. Then $If^*J : \mathrm{Tot}\ \beta(B) \longrightarrow \mathrm{Tot}\ \beta(A)$ is a morphism of complexes and non-unital homorphisms act on cocycles in (b,B) bicomplex in this way. On the other hand from [4] and [7] it is clear that non-unital homomorphisms act on normalized cocycles in a direct way. Indeed (cf. [4], Proposition 3) normalized cocycles on A correspond to traces or super-traces, depending on parity, on the algebra QA defined by Cuntz (or to infinite dimensional cycles on the universal differential graded algebra ΩA). A non-unital homomorphism $f: A \longrightarrow B$

induces a homomorphism from QA to QB (or ΩA to ΩB). Hence traces on QB (or cycles on ΩB) can be pulled back to traces on QA (or cycles on ΩA). This is the real motivation for the following lemma and its proof. Recall that a cocycle

$\phi = (\phi_{2n})_{n\geq 0}$ (resp. $\phi = (\phi_{2n+1})_{n\geq 0}$) is said to be normalized if, for all $n \geq 0$, $B_0\phi_{2n}$ (resp. $B_0\,\phi_{2n+1}$) is a cyclic cochain.

Lemma 2.3.6. Let $f: A \longrightarrow B$ be a non-unital homomorphism of unital algebras. Let $\phi = (\phi_{2n})_{n\geq 0}$ (resp. $\phi = (\phi_{2n+1})_{n\geq 0}$) be a normalized even (resp. odd) cocycle on B. Then $f^*\phi = (f^*\phi_{2n})_{n\geq 0}$ (resp. $f^*\phi = (f^*\phi_{2n+1})_{n\geq 0}$) is a (normalized) cocycle on A.

Proof. It suffices to prove the even case. Let us first show that $B_0f^*\phi_{2n} = f^*B_0\phi_{2n}$. This is rather surprising since f need not be unital. We have $B_0f^*\phi_{2n} = s(1-\lambda)\, f^*\phi_{2n} = sf^*(1-\lambda)\,\phi_{2n} = sf^*(B_0b + b'B_0)\,\phi_{2n}$. By the cocycle condition for ϕ, we have $B_0b\phi_{2n} = 0$ and hence

$$
\begin{aligned}
B_0f^*\phi_{2n}(a^0,\dots,a^{2n-1}) &= sf^*b'B_0\phi_{2n}(a^0,\dots,a^{2n-1}) \\
&= f^*b'B_0\phi_{2n}(1,a^0,\dots,a^{2n-1}) \\
&= b'B_0\phi_{2n}(f(1), f(a^0),\dots,f(a^{2n-1}))\,.
\end{aligned}
$$

But $bB_0\phi_{2n} = 0$ since ϕ is normalized, so that the above term is equal to

$$
\begin{aligned}
&= -B_0\phi_{2n}(f(a^{2n-1})\, f(1), f(a^0),\dots,f(a^{2n-2})) \\
&= -B_0\phi_{2n}(f(a^{2n-1}), f(a^0),\dots,f(a^{2n-2})) \\
&= B_0\phi_{2n}(f(a^0),\dots,f(a^{2n-1}))
\end{aligned}
$$

$$= f^*B_0\phi_{2n}(a^0,\ldots,a^{2n-1}) .$$

From $B_0f^*\phi_{2n} = f^*B_0\phi_{2n}$ we conclude that $B_0f^*\phi_{2n}$ is cyclic and also $Bf^*\phi_{2n} = NB_0f^*\phi_{2n} = f^*NB_0\phi_{2n} = f^*B\phi_{2n}$. It remains to check the cocycle condition. But this is trivial since $bf^*\phi_{2n} + Bf^*\phi_{2n+2} = f^*(b\phi_{2n} + B\phi_{2n+2}) = 0$. We have shown that $f^*\phi = (f^*\phi_{2n})_{n\geq 0}$ is a normalized cocycle. □

Using above lemma we can derive a simple formula for $f^*: H^*_\varepsilon(B) \longrightarrow H^*_\varepsilon(A)$ when f is non-unital.

Lemma 2.3.7. Let $f: A \longrightarrow B$ be a non-unital homomorphism of unital algebras and let ϕ be a normalized cocycle on B. Then $If^*J\phi$ is cohomologous to the normalized cocycle $f^*\phi$.

Proof. Assume $\phi = (\phi_{2n})_{n\geq 0}$ is an even cocycle. We have

$$\begin{aligned}(If^*J\phi)_{2n} &= f^*\phi_{2n} + Nsf^*B_0\phi_{2n+2}\\ &= f^*\phi_{2n} + NsB_0f^*\phi_{2n+2}\end{aligned}$$

where we have used the proof of Lemma 2.3.6 above. By the same lemma $f^*\phi = (f^*\phi_{2n})_{n\geq 0}$ is a cocycle and hence by Lemma 2.3.2, $NsB_0f^*\phi_{2n+2}$ are the components of a coboundary. This shows that $f^*\phi$ and $If^*J\phi$ are cohomologous. □

Finally, we would like to show that I (and J) send normalized cocycles to normalized cocycles. But first

Definition 2.3.8. (Cuntz [10]). An even (resp. odd) cochain $\phi = (\phi_n)_{n\geq 0}$ in the periodic bicomplex is called *normalized* iff ϕ_{2n+1} (resp. ϕ_{2n}) is a cyclic cochain for all $n \geq 0$.

Now, assume ϕ is a normalized cocycle in $\mathfrak{C}(A)$. It is easy to see that $I\phi$ is a normalized cocycle in $\beta(A)$. Indeed, assuming ϕ is even, we have

$$B_0(I\phi)_{2n} = B_0(\phi_{2n} + Ns\phi_{2n+1}) = B_0\phi_{2n}.$$

By the normalized cocycle condition on ϕ we have

$$(1-\lambda)\phi_{2n+1} = 0\ ,\ (1-\lambda)\phi_{2n} - b'\phi_{2n-1} = 0 \text{ and } N\phi_{2n+1} + b\phi_{2n} = 0\,.$$

Hence $b\phi_{2n} = -(2n+2)\phi_{2n+1}$ so that $b\phi_{2n+1} = 0$. Now, $B_0\phi_{2n} = s(1-\lambda)\phi_{2n} = sb'\phi_{2n-1}$. We thus have

$$\begin{aligned} B_0\phi_{2n}(a^0,\ldots,a^{2n-1}) &= b'\phi_{2n-1}(1,a^0,\ldots,a^{2n-1}) \\ &= -\phi_{2n-1}(a^{2n-1},a^0,\ldots,a^{2n-2}) = \phi_{2n-1}(a^0,\ldots,a^{2n-1}) \end{aligned}$$

where we have used the fact that $b\phi_{2n-1} = 0$. Thus $B_0\phi_{2n} = \phi_{2n-1}$ and is cyclic. We also note that $J\phi$ is normalized if ϕ is normalized.

2.4. The Vanishing of Lie Derivatives and its Consequences

In this section we prove, in full generality, that inner derivations act trivially on the entire cyclic cohomology groups. As a consequence of this we can prove that inner automor- phisms induce the identity morphism on the entire cyclic cohomology groups. This in turn implies Morita invariance and additivity of the theory.

Let A be a unital algebra. At the end of Section 2.2 we defined the notion of normal cochains. A closely related concept is that of a reduced cochain (cf. [18], Section 4). An n-cochain $\phi \in C^n$, $n > 0$, is said to be *reduced* iff $\phi(a^0,\dots,a^n) = 0$ whenever $a^i = 1$ for some $i > 0$. A 0-cochain ϕ is reduced iff $\phi(1) = 0$. Note that for $n > 0$ reduced and normal cochains are the same. It is easily checked that the operators b and B send reduced cochains to reduced cochains and hence we have a subcomplex $\beta(A)_{red}$ of (b,B) bicomplex. The cohomology of this bicomplex is called the *reduced cyclic cohomology* of A ([18], Section 4). If A is a Banach algebra we can consider reduced entire cochains on A and hence the reduced entire cyclic cohomology of A can be defined.

For an algebra A, let $\tilde{A}$ be the algebra obtained by adjoining a unit to A. For every $n \geq 0$, we have an isomorphism of vector spaces $C^n(A) \oplus C^{n-1}(A) \simeq C^n(\tilde{A})_{red}$ where a pair (ϕ_n, ϕ_{n-1}) is sent to the reduced cochain $\tilde{\phi}$ defined by:

$$\tilde{\phi}(\tilde{a}^0,\dots,\tilde{a}^n) = \phi_n(a^0,\dots,a^n) + \lambda_0\phi_{n-1}(a^1,\dots,a^n).$$

We can then define a map $\theta : \mathrm{Tot}\, \mathcal{C}(A) \longrightarrow \mathrm{Tot}\, \beta(\tilde{A})_{red}$ by sending a, say even, cochain $\phi = (\phi_n)_{n\geq 0}$ to $\tilde{\phi} = (\tilde{\phi}_{2n})_{n\geq 0}$ where $\tilde{\phi}_{2n} = \phi_{2n} \oplus \phi_{2n-1}$ and similarly for odd cochains. θ is obviously an isomorphism of vector spaces. The fact that it commutes with coboundary operators is the content of the following important proposition in [18], the proof of which is a direct calculation.

Proposition 2.4.1. The map $\theta : \mathrm{Tot}\, \mathcal{C}(A) \longrightarrow \mathrm{Tot}\, \beta(\tilde{A})_{red}$ is an isomorphism of chain complexes.

Proof. See Loday-Quillen [18], Proposition 4.2. □

By a *derivation* of an algebra A we mean a linear map $\delta : A \longrightarrow A$ which satisfies $\delta(ab) = a\cdot\delta b + \delta a\cdot b$. One should think of derivations as infinitesimal homomorphisms. Homomorphisms act on cochains (by pullback) and this action commutes with many of the operators of the theory. The infinitesimal version of this is the action of derivations on cochains and the corresponding commutation relations (Lemma 2.4.1 below). More precisely, given a derivation δ we can define a map ([3], part II, page 340)

$L_\delta : C^n(A) \longrightarrow C^n(A)$ by

$$L_\delta\phi(a^0,\dots,a^n) = \sum_{i=0}^{n} \phi(a^0,\dots, \delta a^i,\dots,a^n) .$$

The map L_δ is the *Lie derivative* associated to derivation δ . The proof of the following lemma is a straightforward computation and we skip it. As mentioned before it is the infinitesimal version of the corresponding result for homomorphisms.

Lemma 2.4.2. Let $\delta : A \longrightarrow A$ be a derivation of an algebra A . The following commutation relations hold

(i) $bL_\delta = L_\delta b$

(ii) $b'L_\delta = L_\delta b'$

(iii) $\lambda L_\delta = L_\delta \lambda$

(iv) $sL_\delta = L_\delta s$. $\square$

It follows from the above lemma that $BL_\delta = L_\delta B$ and hence L_δ defines a morphism of the (b,B) and periodic bicomplexes. Moreover, it is easy to see that L_δ sends a normal (or reduced) cochain to a cochain of the same type and hence descends to a map of normal (or reduced) bicomplexes.

If A is a Banach algebra and δ is a continuous derivation of A, then it is clear that $L_\delta\phi = (L_\delta\phi_m)$ is an entire cochain if $\phi = (\phi_m)$ is entire. In this way we see that continuous derivations act on the various types of entire cyclic cohomology groups that have been introduced.

Recall the homotopy equivalences I and J in the comparison theorem (Theorem 2.3.4). The following lemma shows that these maps behave well with respect to Lie derivative. The proof is a simple consequence of definitions and Lemma 2.4.2 above.

Lemma 2.4.3. We have $L_\delta I = IL_\delta$ and $JL_\delta = L_\delta J$. □

In [3] part II, Proposition 5, Connes, working with the cyclic complex, shows that inner derivations act trivially on ordinary cyclic cohomology. Later on, Goodwillie [13], working with the bicomplex $\beta(A)_{norm}$, generalized this result to all derivations. However his proof is highly nonconstructive and cannot be extended to the case of entire cyclic cohomology. For our purposes in this chapter it is enough to prove that inner derivations act like zero on entire cyclic cohomology groups. To do so we modify a result of Getzler and Szenes [12] (see also [4], proof of theorem 8) to show that inner derivations act trivially on $\beta(A)_{red}$. We then combine this with Proposition 2.4.1. and comparison Theorem 2.3.4 to prove the desired result.

Recall that the *inner derivation* defined by an element $a \in A$ is the derivation δ_a defined by $\delta_a(b) = [a,b] = ab - ba$ for all $b \in A$. We use L_a to denote the Lie derivative of δ_a.

Proposition 2.4.4. (Getzler and Szenes [12], see also [4], proof of theorem 8). Inner derivations act trivially on the cohomology of $\beta(A)_{norm}$.

Proof. Define an *interior multiplication* operator $i_a : C^n(A) \longrightarrow C^{n-1}(A)$ by

$$i_a\phi(a^0,\ldots,a^{n-1}) = -\phi(a^0,a,a^1,\ldots,a^{n-1}) + \phi(a^0,a^1,a,\ldots,a^{n-1}) \ldots$$
$$+ (-1)^n \phi(a^0,\ldots,a^{n-1},a) .$$

The identity

$$i_a b + b i_a = L_a \qquad (2.4.5)$$

can be verified straightforwardly. Moreover i_a sends normal cochains to normal cochains and we have

$$B i_a + i_a B = 0 \qquad (2.4.6)$$

which is true only for normal cochains. From (2.4.5) and (2.4.6) we have $\partial i_a + i_a \partial = L_a$ where $\partial = b + B$ is the coboundary of $\beta(A)_{norm}$. This of course shows that inner derivations act trivially on the cohomology of $\beta(A)_{norm}$. □

Remark. Let x be a vector field on a smooth manifold. Cartan's formula in differential geometry tells us that $di_x + i_x d = L_x$. Here L_x is the Lie derivative with respect to x, i_x is the interior multiplication with respect to x and d is the exterior derivative of differential forms. We see that the formula $\partial i_a + i_a \partial = L_a$ in the above proof, which is true only for normal cochains, can be regarded as a noncommutative analogue of Cartan's formula.

Corollary 2.4.7. Inner derivations act trivially on the chomology of $\beta(A)_{red}$.

Proof. Note that the operator i_a is not a map of $\beta(A)_{red}$ (for $\phi \in C^1$, $i_a\phi(a^0) = -\phi(a^0,a)$ and hence $i_a\phi(1)$ could be different from zero even if ϕ is reduced). This problem is easily overcome by noticing that $i_a\phi$ is reduced if ϕ is a reduced cocycle. Indeed for $\phi = (\phi_{2n+1})_{n\geq 0}$ a reduced cocycle we have $B\phi_1 = B_0\phi_1 = 0$ and hence $\phi_1(1,a^0) + \phi_1(a^0,1) = 0$. But $\phi_1(a^0,1) = 0$ since ϕ is reduced. It follows that $\phi_1(1,a^0) = 0$ i.e. $i_a\phi_1$ is reduced and hence $i_a\phi$ is reduced. We then have $L_a\phi = \partial i_a\phi$ for any reduced cocycle ϕ. This shows that inner derivations act trivially on the cohomology of $\beta(A)_{red}$. □

Note that $i_a\phi$ is entire if ϕ is entire and hence the proof of the above lemma and its corollary continue to hold when A is a unital Banach algebra.

Next we would like to transport this result to the periodic bicomplex $\mathcal{C}(A)$ using Proposition 2.4.1. We have to make sure that the isomorphism θ behaves well with respect to Lie derivatives. Note that a derivation $\delta : A \longrightarrow A$ extends uniquely to a derivation on $\tilde{A}$ denoted also by δ and defined by $\delta(a + \lambda I) = \delta(a)$.

Lemma 2.4.8. We have $\theta L_\delta = L_\delta\theta$, where θ is the isomorphism in Proposition 2.4.1.

Proof. Let $\phi = (\phi_n)_{n\geq 0}$ be an even cochain in Tot $\mathcal{C}(A)$. We have

$(\theta L_\delta\phi)_{2n}(\tilde{a}^0,\ldots,\tilde{a}^{2n}) = (L_\delta\phi)_{2n}(a^0,\ldots,a^{2n}) + \lambda_0(L_\delta\phi)_{2n-1}(a^1,\ldots,a^{2n})$, and

$$(L_\delta\theta\varphi)_{2n}(\tilde a^0,\dots,\tilde a^{2n}) = \sum_{k=0}^{2n} (\theta\phi)_{2n}(\tilde a^0,\dots, L_\delta \tilde a^k,\dots, \tilde a^{2n})$$

$$= (\theta\phi)_{2n}(L_\delta \tilde a^0, \tilde a^1,\dots, \tilde a^{2n}) + \sum_{k=1}^{2n} (\theta\phi)_{2n}(\tilde a^0,\dots, L_\delta \tilde a^k,\dots, \tilde a^{2n})$$

$$= \phi_{2n}(L_\delta a^0, a^1,\dots,a^{2n}) + \sum_{k=1}^{2n} \{\phi_{2n}(a^0,\dots, L_\delta a^k,\dots, a^{2n})$$

$$+ \lambda_0\phi_{2n-1}(a^1,\dots, L_\delta a^k,\dots, a^{2n})\}$$

$$= \sum_{k=0}^{2n} \phi_{2n}(a^0,\dots, L_\delta a^k,\dots, a^{2n}) + \lambda_0 \sum_{k=1}^{2n} \phi_{2n-1}(a^1,\dots, L_\delta a^k,\dots, a^{2n})$$

$$= L_\delta\, \phi_{2n}(a^0,\dots,a^{2n}) + \lambda_0\, L_\delta\, \phi_{2n-1}(a^1,\dots,a^{2n})$$

$$= (L_\delta\, \phi)_{2n}\,(a^0,\dots,a^{2n}) + \lambda_0(L_\delta\phi)_{2n-1}(a^1,\dots,a^{2n}).$$

Thus, we see that the two sides are equal. □

Now we have a commutative diagram

$$\begin{tikzcd} \mathrm{Tot}\ \mathfrak{C}(A) \arrow[r, "L_a"] \arrow[d, "\theta"'] & \mathrm{Tot}\ \mathfrak{C}(A) \arrow[d, "\theta"] \\ \mathrm{Tot}\ \beta(\tilde A)_{red} \arrow[r, "L_a"] & \mathrm{ToT}\ \beta(\tilde A)_{red} \end{tikzcd} .$$

From Corollary 2.4.7 and the fact that θ is an isomorphism we can conclude

Proposition 2.4.9. Inner derivations act like zero on the entire cyclic cohomology of non-unital Banach algebras. □

When the Banach algebra is unital we know that the entire theory can be formulated in terms of the (b,B) bicomplex. By Lemma 2.4.3 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}\,\beta(A) & \xrightarrow{L_a} & \mathrm{Tot}\,\beta(A) \\ I\uparrow & & \uparrow I \\ \mathrm{Tot}\,\mathcal{C}(A) & \xrightarrow{L_a} & \mathrm{ToT}\,\mathcal{C}(A) \end{array} .$$

By Theorem 2.4.3 (comparison theorem) I is a homotopy equivalence. This proves

Proposition 2.4.10. Let A be a unital Banach algebra. Inner derivations act trivially on the cohomology of entire cochains in $\beta(A)$. □

The fact that inner derivations act trivially on entire cyclic cohomology groups is the infinitesimal manifestation of another fact: inner automorphisms induce the identity morphism on entire cyclic cohomology groups. Integrating the first result yields the second one. We adopt the method of proof of Proposition 5 in [3], part II, where the same results for ordinary cyclic cohomology are proved. We give the full details here since the exposition in [3] is rather brief.

Lemma 2.4.11. The inner automorphism defined by an element of square one induces the identity morphism on entire cyclic cohomology groups.

Proof. Let $\theta(x) = uxu^{-1}$ be the given inner automorphism. Since $u^2 = 1$, we can write $u = -i \exp \frac{\pi i}{2} u$. Consider the family of invertibles $u_t = \exp \frac{\pi i t}{2} u = \cos \frac{\pi t}{2} \cdot 1 + i \sin \frac{\pi t}{2} \cdot u$, $0 \le t \le 1$, and let $\theta_t(x) = u_t x u_t^{-1}$ be the associated family of inner automorphisms. Note that $\theta_0 = \mathrm{id}$ and $\theta_1 = \theta$. Let $\phi = (\phi_n)_{n \ge 0}$ be an, say even, entire cocycle in periodic the bicomplex. We have

$$\frac{d}{dt}\theta_t^{\,*}\phi_n(a^0,\dots,a^n) = \frac{d}{dt}\,\phi_n(\theta_t a^0,\dots,\theta_t a^n)$$

$$= \sum_{j=0}^{n} \phi_n(\theta_t a^0,\dots,\frac{d}{dt}\theta_t a^j,\dots,\theta_t a^n)$$

$$= \frac{\pi i}{2}\sum_{j=0}^{n} \phi_n(\theta_t a^0,\dots,\delta_u\,\theta_t a^j\dots,\theta_t a^n)$$

$$= \frac{\pi i}{2}\,\theta_t^{\,*} L_u\,\phi_n(a^0,\dots,a^n)\ .$$

In short we have: $\frac{d}{dt}\theta_t^{\,*}\phi = \frac{\pi i}{2}\,\theta_t^{\,*}L_u\phi$.

By Proposition 2.4.9 $L_u\,\phi = \partial\psi$ for an entire cochain ψ. Now we can write

$$\theta^*\phi - \phi = \theta_1^*\,\phi - \theta_0^*\,\phi = \int_0^1 \frac{d}{dt}\,\theta_t^*\phi\cdot dt$$

$$= \frac{\pi i}{2}\int_0^1 \theta_t^{\,*}\,\partial\psi\cdot dt$$

$$= \frac{\pi i}{2}\,\partial\int_0^1 \theta_t\ \ \psi\cdot dt$$

$$= \frac{\pi i}{2}\,\partial\overline{\psi}\ .$$

We have to show $\overline{\psi}$ is an entire cochain. Indeed

$$
\begin{aligned}
|\bar{\psi}_n(a^0,\dots,a^n)| &= \left| \int_0^1 \psi_n(\theta_t a^0,\dots,\theta_t a^n)\,dt \right| \\
&\le \|\psi\| \sup_{0\le t\le 1} \prod_{j=0}^{n} \|\theta_t a^j\|
\end{aligned}
$$

but $\|\theta_t a\| = \|\exp\frac{\pi i t}{2} u \cdot a \cdot \exp -\frac{\pi i t}{2} u\| \le (1+\|u\|)^2 \|a\|$ and hence

$\|\bar{\psi}_n\| \le (1+\|u\|)^{2n+2} \|\psi_n\|$. this estimate is enough to show that $\bar{\psi} = (\bar{\psi}_n)$ is an entire cochain. The lemma is proved. □

Next, we can get rid of the technical condition on u by a matrix trick as in [3], part II, Proposition 5.

Proposition 2.4.12. Inner automorphisms induce the identity morphism on entire cyclic cohomology groups.

Proof. Let $\theta(x) = uxu^{-1}$ be the given inner automorphism. Recall the maps $Tr: H_\varepsilon{}^*(A) \longrightarrow H_\varepsilon{}^*(M_q(A))$ and $i^* : H_\varepsilon{}^*(M_q(A)) \longrightarrow H_\varepsilon{}^*(A)$. We have $i^*\circ Tr = id$. Let $\phi \in H_\varepsilon{}^*(A)$. We have $\theta^*\phi - \phi = i^*Tr\,\theta^*\phi - i^*Tr\,\phi$. Let $U = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. We have $U = U_1U_2$ where $U_1 = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, $U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $U_1^2 = U_2^2 = 1$. It is easy to see that $i^*Tr\,\theta^*\phi = i^*\Theta^*Tr\,\phi$, where Θ is the inner automorphism of $M_2(A)$ defined by U . Hence it is enough to prove that $\Theta^* = id$. But this is a consequence of previous lemma since $\Theta^* = \Theta_1{}^*\circ\Theta_2{}^*$ where Θ_1 and Θ_2 are inner automorphisms induced by elements of square 1 . □

Now we turn to the question of Morita invariance of entire cyclic cohomology. Let A be a unital Banach algebra and $i : A \longrightarrow M_q(A)$ the (non-unital) homomorphism that puts A in the upper left corner of $M_q(A)$. Let $i^* : H_\varepsilon{}^*(M_q(A)) \longrightarrow H_\varepsilon{}^*(A)$ be the morphism induced by i. *Morita invariance* (or stability) of entire cyclic cohomology is the verification of the statement: i^* is an isomorphism.

In [18] this has been established, for ordinary cyclic cohomology, by using Morita invariance of Hochschild cohomology. One uses the Connes long exact sequence relating Hochschild and cyclic cohomology groups to deduce the former result from the latter one. This approach doesn't work for entire cyclic cohomology simply because there is no spectral sequence from Hochschild theory to entire theory. However there is an alternative method of proof, used also in algebraic k-theory as well as ordinary cyclic cohomology, which deduces Morita invariance from invariance of the theory under inner automorphisms.

Let $\mathrm{Tr} : \beta(A) \longrightarrow \beta(M_q(A))$ be the trace map introduced in Section 2.1. As we saw there, Tr induces a map between entire cyclic cohomology groups which we will denote by $\mathrm{Tr} : H_\varepsilon{}^*(A) \longrightarrow H_\varepsilon{}^*(M_q(A))$. We obviously have

$$i^* \circ \mathrm{Tr} = \mathrm{id} \quad . \tag{2.4.13}$$

This of course shows that i^* is surjective. Proving the injectivity is harder. For this consider the algebra $A \otimes M_q \otimes M_q$, where $M_q = M_q(\mathbb{C})$. Let $\theta : A \otimes M_q \otimes M_q \longrightarrow A \otimes M_q \otimes M_q$ be the automorphism which interchanges the last two factors. Since all automorphisms of matrix algebras are inner, we know that θ is inner as well.

Lemma 2.4.14. We have $\mathrm{Tr} \circ i^* = \mathrm{id}$

Proof. Consider the pentagon

$$\begin{tikzcd}
& H_\varepsilon^*(A\otimes M_q) \arrow[dl, "i^*"'] \arrow[r, "Tr"] & H_\varepsilon^*(A\otimes M_q\otimes M_q) \arrow[dd, "\theta^*"] \\
H_\varepsilon^*(A) \arrow[dr, "Tr"'] & & \\
& H_\varepsilon^*(A\otimes M_q) & H_\varepsilon^*(A\otimes M_q\otimes M_q) \arrow[l, "i^*"']
\end{tikzcd}$$

where (to simplify the notation) we have used the same symbols to denote different maps. This diagram commutes. More precisely we have $Tr \circ i^* = i^* \circ \theta^* \circ Tr$. To see this let $\phi \in C^n(A\otimes M_q)$. We have

$$Tr\, i^*\phi(a^0\otimes\mu^0,\ldots, a^n\otimes\mu^n) = tr(\mu^0 \ldots \mu^n)\, \phi(a^0\otimes e_{00}, \ldots, a^n\otimes e_{00})$$

and

$$\begin{aligned}
i^*\theta^* Tr\, \phi(a^0\otimes\mu^0, \ldots, a^n \otimes\mu^n) &= \theta^* Tr\, \phi(a^0 \otimes\mu^0 \otimes e_{00}, \ldots, a^n \otimes\mu^n \otimes e_{00}) \\
&= Tr\, \phi(a^0 \otimes e_{00} \otimes\mu^0, \ldots, a^n \otimes e_{00} \otimes\mu^n) \\
&= tr\, (\mu^0, \ldots, \mu^n)\, \phi(a^0 \otimes e_{00}, \ldots, a^n \otimes e_{00}).
\end{aligned}$$

We see that the two sides are equal. Here e_{00} is the elementary matrix with 1 in position (1,1) and 0 elsewhere. Since θ is inner, by Proposition 2.4.12, we have $\theta^* = id$. Using (2.4.13) we have $Tr \circ i^* = i^* \circ \theta^* \circ Tr = i^* \circ Tr = id$. □

The above lemma and formula (2.4.13) conspire to prove

Corollary 2.4.15. (Morita invariance). Let A be a unital Banach algebra. For all $q \geq 1$, the map $i^* : H_\varepsilon^*(M_q(A)) \longrightarrow H_\varepsilon^*(A)$ is an isomorphism with inverse given by Tr. □

Remark. More generally let $1 \leq p \leq q$ and let $i_p : A \longrightarrow M_q(A)$ be the homomorphism which puts A in the (p,p) position in $M_q(A)$. The same proof shows that $i_p^* : H_\varepsilon^*(M_q(A)) \longrightarrow H_\varepsilon^*(A)$ is an isomorphism with inverse given by Tr. As a corollary we have $i_p^* = i^*$ for all $1 \leq p \leq q$.

The next and last thing we do in this section is to establish additivity of entire cyclic cohomology. Let A_1 and A_2 be unital Banach algebras. Consider the obvious homomorphisms $i_k : A_k \longrightarrow A_1 \oplus A_2$ and $\pi_k : A_1 \oplus A_2 \longrightarrow A_k$, $k = 1,2$. These homomorphisms can then be used to define maps $\alpha : H_\varepsilon^*(A_1 \oplus A_2) \longrightarrow H_\varepsilon^*(A_1) \oplus H_\varepsilon^*(A_2)$ and $\beta : H_\varepsilon^*(A_1) \oplus H_\varepsilon^*(A_2) \longrightarrow H_\varepsilon^*(A_1 \oplus A_2)$.

Lemma 2.4.16. We have $\alpha \circ \beta = \mathrm{id}$.

Proof. We have

$$\begin{aligned}\alpha\circ\beta(x,y) &= \alpha(\pi_1^*(x) + \pi_2^*(y)) \\ &= (i_1^*(\pi_1^*(x) + \pi_2^*(y)), i_2^*(\pi_1^*(x) + \pi_2^*(y))) = (x,y)\end{aligned}$$

where we have used the relations $\pi_2 \circ i_1 = 0$, $\pi_1 \circ i_2 = 0$ and $\pi_k \circ i_k = \mathrm{id}$. □

Showing $\beta \circ \alpha = \mathrm{id}$ is harder. consider the algebra of 2×2 matrices over $A_1 \oplus A_2$, $M_2(A_1 \oplus A_2) = M_2(A_1) \oplus M_2(A_2)$, and the automorphism

$$\theta : M_2(A_1 \oplus A_2) \longrightarrow M_2(A_1 \oplus A_2)$$

defined by $\theta(x,y) = (x, \tilde{y})$ where $\tilde{y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that $y \mapsto \tilde{y}$ is an inner automorphism and hence θ is inner too.

Lemma 2.4.17. We have $\beta\circ\alpha = \mathrm{id}$.

Proof. One can easily verify that the following diagram is commutative i.e. $\beta\circ\alpha = i^* \circ \theta^* \circ \mathrm{Tr}$.

$$\begin{array}{ccc} H_\varepsilon^*(A_1 \oplus A_2) & \xrightarrow{\mathrm{Tr}} & H_\varepsilon^*(M_2(A_1 \oplus A_2)) \\ \beta\circ\alpha \downarrow & & \downarrow \theta^* \\ H_\varepsilon^*(A_1 \oplus A_2) & \xleftarrow{i^*} & H_\varepsilon^*(M_2(A_1 \oplus A_2)) \end{array}$$

The automorphism θ is inner and hence by 2.4.12, $\theta^* = \mathrm{id}$. By Morita invariance (Corollary 2.4.15), $i^* \circ \mathrm{Tr} = \mathrm{id}$ and hence $\beta\circ\alpha = \mathrm{id}$. □

Lemmas 2.4.16 and 2.4.17 combined together show that α is an isomorphism with inverse β. This is, of course, the additivity of the theory.

Remark. As far as author knows, the question of Morita invariance for non-unital Banach algebras is open. More precisely, we don't know if, for a non-unital Banach algebra A, the map $i^* : H_\varepsilon^*(M_q(A)) \longrightarrow H_\varepsilon^*(A)$ is an isomorphism. It is probably wrong in general and is true only for a suitable class of Banach algebras containing all (non-unital) C*-algebras. This is suggested by the case of ordinary cyclic cohomology. Indeed, in [24] it is shown that Morita invariance (in ordinary cyclic cohomology) holds for all Banach algebras with a bounded approximate unit. The proof consists of establishing an excision theorem for such algebras (and more generally for H-unital algebras, see [23]) and applying it to the (split) exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{A} & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_q(A) & \longrightarrow & M_q(\tilde{A}) & \longrightarrow & M_q(\mathbb{C}) & \longrightarrow & 0 \end{array} .$$

It is needless to say that the proof of excision theorem in [23] does not generalize to the case of entire cyclic cohomology and one needs a new approach to this question in this case.

Finally, we would like to mention that the problem of additivity of the entire theory, for non-unital Banach algebras, also remains open. Indeed our proof of this property in the unital case makes use of Morita invariance which is not settled yet in the non-unital case.

Chapter 3

3.1. Infinite Dimensional Cycles

In order to penetrate deeper into cyclic cohomology one has to bring in more sophisticated formulations of the theory. For example, as we saw in Chapter 1, ordinary cyclic cohomology of an algebra, which is originally defined as the cohomology of the associated cyclic compleesx, can also be formulated as the (total) cohomology of the (b,B) or periodic bicomplex of the given algebra. This is, more or less, equivalent to a central result for ordinary cyclic cohomology, namely Connes' long exact sequence and is also crucial for the definition of entire cyclic cohomology.

In this section we will quickly review Connes' theory of cycles over algebras and especially look at its infinite dimensional form. The goal is to make a formal analogy between cyclic cohomology and deRham's homology of currents on manifolds. More precisely one wants to interpret a periodic cyclic cocycle as a kind of current - or integral - over the algebra of (non-commutative) differential forms. As we will see in the next section, this will enable us to prove, in full generality, that derivations act trivially on all kinds of cyclic cohomology groups and in particular on entire cyclic cohomology.

By a *differential graded algebra* (DG algebra) we mean a graded algebra $\Omega = \bigoplus_{i\geq 0} \Omega^i$ together with a graded derivation d of degree 1 with $d^2 = 0$. More precisely we have

(1) $\Omega^i \cdot \Omega^j \subset \Omega^{i+j}$

(2) $d\Omega^i \subset \Omega^{i+1}$, $d(\omega_1 \omega_2) = d\omega_1 \cdot \omega_2 + (-1)^{\deg \omega_1} \omega_1 \cdot d\omega_2$, $d^2 = 0$.

An important example of such algebras is of course $(\Omega(M),d)$: the algebra of differential forms on a manifold M together with exterior derivative. Note that this example is (graded) commutative. A non-commutative example is the algebra of matrix valued forms on M with the obvious extension of d .

Now let A be an algebra. We can associate to A its *universal DG algebra*. This is a DG algebra $(\Omega A, d)$ together with a homomorphism $A \longrightarrow \Omega^0 A$, which is universal: given any DG algebra Ω and a homomorphism $A \longrightarrow \Omega^0$, there is a unique morphism of DG algebras $\pi : \Omega A \longrightarrow \Omega$ such that the following diagram commutes

$$\begin{tikzcd} & \Omega A \arrow[dd, dashed, "\pi"] \\ A \arrow[ur] \arrow[dr] & \\ & \Omega \end{tikzcd}$$

A nice way to think about, and construct, the universal DG algebra $(\Omega A, d)$ of an algebra A is as follows. Let ΩA be the graded algebra obtained by adjoining elements $\{da;\, a \in A\}$ to A subject to the relations

$$d(ab) = a\,db + da \cdot b \ , \quad d(a + \lambda b) = da + \lambda db$$

$$\deg a = 0 \quad \text{and} \quad \deg da = 1 \ ,$$

for all $a, b \in A$ and scalars λ. We then have $\Omega^0 A = A$ and $\Omega^n A$, $n \geq 1$, is generated (as a vector space) by elements of the form $a^0\, da^1 \ldots da^n$, $da^1 \ldots da^n$, $a^i \in A$. Note that this is a construction in the non-unital category: ΩA is not unital even if A is unital, and $1da \neq da$. The graded derivation d is defined by

$$d(a^0 da^1 \ldots da^n) = da^0 \ldots da^n \quad \text{and} \quad d(da^1 \ldots da^n) = 0 \ .$$

It is popular to think of $(\Omega A, d)$ as the algebra of non-commutative differential forms on A. This analogy, although sometimes useful, is nevertheless superficial. On a technical level, as many examples suggest, it is more appropriate to think of Hochschild cohomology groups as the space of differential forms on A and to think of the operator B as the right generalization of the exterior derivative.

In [3], Connes introduced the notion of *cycle* as a starting point of (ordinary) cyclic cohomology. More precisely, given an algebra A , an *n-dimensional cycle* on A is a linear form $\int : \Omega A \longrightarrow \mathbb{C}$, supported on $\Omega^n A$, which satisfies the two conditions

$$\int d\omega = 0 \quad \text{and} \quad \int [\omega_1, \omega_2] = 0 \quad \text{(graded commutator).}$$

Given an n-cycle on A one can define its *character* , which is an n-cochain on A , by

$$\phi(a^0,...,a^n) = \int a^0 da^1 \dots da^n .$$

Now, it is easy to check, using the above two conditions for cycles, that the character of an n-cycle is a cyclic n-cocycle, i.e. we have

$$b\phi = 0 \quad \text{and} \quad (1-\lambda)\,\phi = 0 .$$

Conversely, given a cyclic n-cocycle ϕ on A , we can define an n-cycle $\int : \Omega A \longrightarrow \mathbb{C}$ by

$$\int a^0 da^1 \dots da^n = \phi(a^0,...,a^n)\ , \ \int da^1 \dots da^n = 0 \ \text{ and } \int \omega = 0 \ \text{ if } \ \omega \notin \Omega^n A .$$

In this way, we have a canonical one-to-one correspondence between n-dimensional cycles on A and cyclic n-cocycles on A .

Now, by Proposition 1.4.9, a cyclic cocycle on A corresponds to a cocycle of finite length in the (b,B) bicomplex. A natural question that arises is: how to interpret a cocycle of infinite support in (b,B) bicomplex e.g. an entire cocycle, in terms of cycles. The answer is given by the following definition and the proposition that comes after.

Definition 3.1.1. (Connes [4]). An even (resp. odd) *infinite dimensional cycle* over an algebra A is an even (resp. odd) linear form $\int : \Omega A \longrightarrow \mathbb{C}$ such that

$$\int [\omega_1, \omega_2] = (-1)^{\deg \omega_1} \int d\omega_1 \cdot d\omega_2 \qquad (3.1.2)$$

for all ω_1 and ω_2 in ΩA .

In the following proposition the algebra A is unital.

Proposition 3.1.3. (Connes [4]). Let $(\psi_{2n})_{n\geq 0}$, $\psi_{2n} \in C^{2n}$ (resp. $(\psi_{2n+1})_{n\geq 0}$, $\psi_{2n+1} \in C^{2n+1}$) be such that

(a) $b\psi_m = B_0\psi_{m+2} \quad \forall m$,

(b) $B_0\psi_m$ is cyclic $\quad \forall m$.

Then the functional $\int : \Omega A \longrightarrow \mathbb{C}$ defined by

(α) $\int a^0da^1 \dots da^m = \psi_m(a^0,\dots,a^m)$

(β) $\int da^1 \dots da^m = B_0\psi_m(a^1,\dots,a^m)$

(γ) $\int \omega = 0$ if $\deg \omega$ is odd (resp. even),

is an infinite dimensional even (resp. odd) cycle.

Proof. We prove the even case. Note that suffices to check the cycle condition (3.1.2) only for ω_1 of the form a or da, $a \in A$, and arbitrary ω_2. Let us check (3.1.2) for $\omega_1 = a$ and $\omega_2 = a^0da^1 \dots da^{2n}$. We have

$$\begin{aligned}
[\omega_1,\omega_2] &= \omega_1\omega_2 - (-1)^{\deg \omega_1 \cdot \deg \omega_2}\, \omega_2\omega_1 = aa^0da^1\dots da^{2n} - a^0da^1\dots da^{2n}\cdot a\,, \\
&= aa^0da^1 \dots da^{2n} - a^0da^1 \dots d(a^{2n}a) + a^0da^1 \dots d(a^{2n-1}a^{2n})\cdot da - \dots \\
&\quad - a^0a^1da^2 \dots da^{2n} \cdot da\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int [\omega_1,\omega_2] &= -b\psi_{2n}(a^0,\dots,a^{2n},a) = -B_0\psi_{2n+2}(a^0,\dots,a^{2n},a) \\
&= B_0\psi_{2n+2}(a,a^0,\dots,a^{2n}) = \int da\cdot da^0 \dots da^{2n} \\
&= \int d\omega_1\, d\omega_2\,.
\end{aligned}$$

If $\omega_1 = da$ and $\omega_2 = a^0 da^1 \dots da^{2n-1}$, then

$$[\omega_1,\omega_2] = da\cdot a^0da^1 \dots da^{2n-1} + a^0da^1 \dots da^{2n-1}\cdot da$$
$$= d(aa^0)\, da^1 \dots da^{2n-1} - ada^0da^1 \dots da^{2n-1} + a^0da^1 \dots da^{2n-1}da \ .$$

Since $bB_0\psi_m = bb\psi_{m-2} = 0$, we have $b'B_0\psi_{2n}(a^0,\dots,a^{2n-1},a) =$ $-B_0\psi_{2n}(aa^0,a^1,\dots,a^{2n-1})$. Using this, we have

$$\int [\omega_1,\omega_2] = -b'B_0\psi_{2n}(a^0,\dots,a^{2n-1},a) + (1-\lambda)\, \psi_{2n}(a^0,\dots,a^{2n-1},a)\, .$$

Using the identity $b'B_0 + B_0b = 1 - \lambda$ from Section 1.2, we can write

$$\int [w_1,w_2] = B_0b\psi_{2n}(a^0,\dots,a^{2n-1},a) = B_0B_0\psi_{2n+2}(a^0,\dots,a^{2n-1},a) = 0$$

since $B_0\psi_m$ is cyclic.

These two cases are the only non-trivial cases. In view of the comment at the beginning of the proof, the proposition is proved. □

Conversely, given an even (resp. odd) infinite dimensional cycle on A , one can define cochains ψ_m by

$$\psi_m(a^0,\dots,a^m) = \int a^0da^1 \dots da^m\, , \ \ m = \text{even (resp. odd)}$$

and the same proof shows that $\psi = (\psi_m)_{m\geq 0}$ satisfies conditions (a) and (b) of the above proposition.

Finally, we should mention that $b\psi_m = B_0\psi_{m+2}$ is not quite the same as the cocycle condition in (b,B) bicomplex. Define universal constants $\lambda_{2n} = (-1)^n(2n)(2n-2)\dots 2\cdot 1$, $n \geq 0$. Given a normalized even cocycle $\phi = (\phi_{2n})_{n\geq 0}$, define $\psi = (\psi_{2n})_{n\geq 0}$ by

$$\psi_{2n}(a^0,\dots,a^{2n}) = \lambda_{2n}\phi_{2n}(a^0,\dots,a^{2n}) \ .$$

It is easy to see that $\psi = (\psi_{2n})_{n\geq 0}$ satisfies (a) and (b) if and only if $\phi = (\phi_{2n})_{n\geq 0}$ is a normalized cocycle in (b,B) bicomplex. Thus Proposition 3.1.3 defines a one-to-one correspondence between normalized cocycles in $\beta(A)$ and infinite dimensional cycles on A. As it is emphasized in [4], this is the reason for the importance of normalized cocycles: they have a natural geometric interpretation.

The rescaling constants λ_{2n} become important in the next section. For future reference, we note that

$$\frac{\lambda_{2n+2}}{(2n+2)\lambda_{2n}} = -1 \ . \tag{3.1.4}$$

3.2. A Vanishing Theorem

In this section we prove, in full generality, that any continuous derivation of a Banach algebra induces the zero homomorphism on the entire cyclic cohomology groups. We use the language of cycles together with reduced cochains to achieve this.

Let A be a non-unital algebra. Recall from Section 2.3 that an even (resp. odd) periodic cocycle $\phi = (\phi_n)_{n\geq 0}$ is said to be normalized if $(1-\lambda)\,\phi_{2n+1} = 0$ (resp. $(1-\lambda)\phi_{2n} = 0$) for all $n \geq 0$. The following normalization lemma is the analog of normalization Lemma 2.2.2. Its proof however is considerably simpler. This is due to the fact that we work in the periodic bicomplex. Moreover, as we will see, using the comparison theorem, it implies Lemma 2.2.2.

Lemma 3.2.1. Every periodic cocycle is cohomologous to a normalized cocycle.

Proof. Let $\phi = (\phi_n)_{n\geq 0}$ be an even cocycle in $\mathcal{C}(A)$ and let $\theta_{2m-1} = \phi_{2m-1} - \frac{1}{2m} N\phi_{2m-1}$. We have $N\theta_{2m-1} = 0$ and hence by Lemma 1.1.2 there is a cochain $\tilde{\theta}_{2m-1}$ such that $(1-\lambda)\tilde{\theta}_{2m-1} = \theta_{2m-1}$ for all $m \geq 0$. Let us define a cocycle $\phi' = (\phi'_m)$ by the formulas $\phi'_{2m} = \phi_{2m} - b\tilde{\theta}_{2m-1}$ and $\phi'_{2m-1} = \phi_{2m-1} - \theta_{2m-1}$. ϕ' is obviously normalized. Let us show ϕ' is cohomologous to ϕ. To see this define the (odd) cochain $\psi = (\psi_m)_{m\geq 0}$ by $\psi_{2m} = 0$ and $\psi_{2m-1} = \tilde{\theta}_{2m-1}$. We have $(\partial\psi)_{2m} = N\psi_{2m} + b\psi_{2m-1} = b\tilde{\theta}_{2m-1}$ and $(\partial\psi)_{2m-1} = (1-\lambda)\,\psi_{2m-1} - b'\psi_{2m-2}$ $= (1-\lambda)\,\psi_{2m-1} = \theta_{2m-1}$. Here ∂ is the total (odd) coboundary of the periodic bicomplex. It follows that $\phi' = \phi - \partial\psi$ and hence ϕ' is cohomologus to ϕ. There is a similar proof for the odd case. □

Note that if A is a Banach algebra and ϕ is an entire periodic cocycle on A (cf. Definition 2.3.3) then the normalized cocycle ϕ' in the above lemma can be chosen to be entire. Indeed, from the proof of Lemma 1.1.2, it is clear that we can choose $\tilde{\theta}_{2m-1} = \frac{-1}{2m}(1 + 2\lambda + \ldots + 2m\lambda^{2m-1})\,\theta_{2m-1}$. Thus we have

$$\|\tilde{\theta}_{2m-1}\| \leq (m+1)\,\|\theta_{2m-1}\| \leq 2(m+1)\|\phi_{2m-1}\| .$$

It follows that ψ and hence ϕ' are entire cochains.

Remark. Let A be a unital Banach algebra. The homotopy equivalences I and J send a normalized cocycle to a cocycle of the same type (cf. discussion at the end of Section 2.3). This shows that the above lemma combined with the comparison Theorem 2.3.4 proves the normalization lemma of Connes (Lemma 2.2.2): for every entire cocycle in $\beta(A)$ there is a normalized cohomologous entire cocycle. This is another example of the usefulness of the comparison theorem. It is sometimes easier to prove or to see the results in the periodic bicomplex picture. We can then transport these results to the (b,B) bicomplex, without worrying about the entire growth conditions, using the comparison theorem.

Let A be a non-unital Banach algebra and $\delta : A \longrightarrow A$ a continuous derivation. We would like to show the Lie derivative L_δ acts trivially on the entire cyclic cohomology of A. Let us recall the map $\theta : \mathrm{Tot}\, \mathcal{C}(A) \longrightarrow \mathrm{Tot}\, \beta(\tilde{A})_{red}$ defined by

$$(\theta\phi)_{2n}(\tilde{a}^0, \ldots, \tilde{a}^{2n}) = \phi_{2n}(a^0, \ldots, a^{2n}) + \lambda_0\, \phi_{2n-1}(a^1, \ldots, a^{2n})$$

if ϕ is an even cochain, and similarly for odd cochains. By Proposition 2.4.1, θ is an isomorphism of chain complexes.

Lemma 3.2.2. The map θ sends normalized cocycles to cocycles of the same type.

Proof. Recall Defintiion 2.1.1 of a normalized cocyle in the (b,B) bicomplex. For an even normalized periodic cocycle $\phi = (\phi_n)_{n\geq 0}$, we have to show $B_0(\theta\phi)_{2n}$ is a cyclic cochain for all $n \geq 0$. We have

$$\begin{aligned} B_0(\theta\phi)_{2n}(\tilde{a}^0, \ldots, \tilde{a}^{2n-1}) &= (\theta\phi)_{2n}(1, \tilde{a}^0, \ldots, \tilde{a}^{2n-1}) \\ &\quad - (-1)^{2n}(\theta\phi)_{2n}(\tilde{a}^0, \ldots, \tilde{a}^{2n-1}, 1) \\ &= \phi_{2n}(0, a^0, \ldots, a^{2n-1}) + \phi_{2n-1}(a^0, \ldots, a^{2n-1}) - \phi_{2n}(a^0, \ldots, a^{2n-1}, 0) \\ &\quad - \lambda^0 \phi_{2n-1}(a^1, \ldots, a^{2n-1}, 0) \\ &= \phi_{2n-1}(a^0, \ldots, a^{2n-1}). \end{aligned}$$

This shows that $B_0(\theta\phi)_{2n}$ is cyclic and hence $\theta\phi$ is normalized. $\square$

Theorem 3.2.3. Let A be a unital Banach algebra and $\delta : A \longrightarrow A$ a continuous derivation. Let ϕ be a reduced and normalized entire cocycle on A. Then there is a canonically defined reduced and entire cochain ψ on A such that $L_\delta \phi = \partial\psi$, where ∂ is the (total) coboundary of (b,B) bicomplex.

Proof. Let us prove the even case. Let $\phi = (\phi_{2n})_{n\geq 0}$, be the given even cocycle and $\int : \Omega A \longrightarrow \mathbb{C}$ be the corresponding cycle. Let us define an odd cochain

$\overline{\psi} = (\overline{\psi}_{2n+1})_{n\geq 0}$ by

$$\overline{\psi}_{2n+1}(a^0, \dots, a^{2n+1}) = \sum_{j=1}^{2n+1} (-1)^{j-1} \int a^0\, da^1 \dots \delta a^j \dots da^{2n+1}$$
$$= \sum_{j=1}^{2n+1} (-1)^{j-1}\, \psi^j_{2n+1}(a^0, \dots, a^{2n+1}) .$$

We would like to show that $\overline{\psi} = (\overline{\psi}_{2n+1})_{n\geq 0}$ is a reduced cochain and estimate the norms of its components. First note that $\int da^1 \dots da^{2k} = 0$ if $a^i = 1$ for some i. Indeed, in this case,

$$\int da^1 \dots da^{2k} = \lambda_{2k} B_0 \phi_{2k}(a^1, \dots, a^{2k})$$
$$= \lambda_{2k}\{\phi_{2k}(1, a^1, \dots, a^{2k}) - \phi_{2k}(a^1, \dots, a^{2k}, 1)\} = 0$$

in view of the fact that $\phi = (\phi_{2n})_{n\geq 0}$ is reduced. Now we have

$$\int \omega_1 \cdot d1 \cdot \omega_2 = \pm \int \omega_2 \omega_1 d1 \pm \int d\omega_1 \cdot d1 \cdot d\omega_2 = 0$$

for all ω_1 and ω_2 in ΩA. This shows that each cochain ψ^j_{2n+1} is reduced and hence $\overline{\psi} = (\overline{\psi}_{2n+1})_{n\geq 0}$ is reduced. To estimate the norms of $\overline{\psi}_{2n+1}$, $n \geq 0$, note that we can write, using $d(ab) = adb + da \cdot b$, each cochain ψ^j_{2n+1} in terms of ϕ_{2n+1}. This gives us the following estimate

$$\|\psi^j_{2n+1}\| \leq j|\lambda_{2n}| \, \|\phi_{2n}\| \, \|\delta\| \ .$$

Using $\overline{\psi}_{2n+1} = \sum_{j=1}^{2n+1} (-1)^{j-1} \psi^j_{2n+1}$, we obtain

$$\|\overline{\psi}_{2n+1}\| \leq |\lambda_{2n}| \, (n+1)(2n+1) \, \|\phi_{2n}\| \, \|\delta\| \ . \tag{3.2.3}$$

Next, we will calculate $B_0\overline{\psi}_{2n+1}$ and show that it is cyclic. Using the relation $1da^0 = da^0 - d1 \cdot a^0$, we can write

$$\begin{aligned} \int 1da^0 \ldots \delta a^{j-1} \ldots da^{2n} &= \int da^0 \ldots \delta a^{j-1} \ldots da^{2n} - \int d1 \cdot a^0 \ldots \delta a^{j-1} \ldots da^{2n} \\ &= \int da^0 \ldots \delta a^{j-1} \ldots da^{2n} . \end{aligned}$$

It follows that

$$\begin{aligned} B_0\psi^j_{2n+1}(a^0,\ldots,a^{2n}) &= \psi^j_{2n+1}(1,a^0,\ldots,a^{2n}) + \psi^j_{2n+1}(a^0,\ldots,a^{2n},1) \\ &= \int 1da^0 \ldots \delta a^{j-1} \ldots da^{2n} = \int da^0 \ldots \delta a^{j-1} \ldots da^{2n} . \end{aligned}$$

From this we have

$$B_0\overline{\psi}_{2n+1}(a^0,\ldots,a^{2n}) = \sum_{j=1}^{2n+1} (-1)^{j-1} \int da^0 \ldots \delta a^{j-1} \ldots da^{2n} \ . \tag{3.2.4}$$

This is a cyclic cochain. Indeed a straightforward calculation shows

$$B_0\overline{\psi}_{2n+1} = NB_0\psi^{2n+1}_{2n+1} \ . \tag{3.2.5}$$

Let us calculate the Hochschild coboundary of $\overline{\psi}_{2n+1}$. We have, by definition

$$b\psi^j_{2n+1}(a^0,\ldots,a^{2n+2}) = \sum_{k=0}^{2n+1} (-1)^k \psi^j_{2n+1}(a^0,\ldots,a^k a^{k+1},\ldots,a^{2n+2})$$

$$+ (-1)^{2n+2} \psi^j_{2n+1}(a^{2n+2} a^0,\ldots, a^{2n+1}) \ .$$

Using the relations $d(ab) = adb + da\cdot b$ and $\delta(ab) = a\cdot\delta b + \delta a\cdot b$, we see that all the terms in the above sum cancel except

$$= -\int a^0 da^1 \ldots \delta a^j \ldots da^{2n+1}\cdot a^{2n+2} + \int a^{2n+2} a^0 da^1 \ldots \delta a^j \ldots da^{2n+1}$$

$$= -\int d(a^0 da^1 \ldots \delta a^j \ldots da^{2n+1}) \cdot da^{2n+2}$$

$$= -\int da^0\, da^1 \ldots \delta a^j \ldots da^{2n+1}\cdot da^{2n+2}$$

$$+ (-1)^j \int a^0 da^1 \ldots d\delta a^j \ldots da^{2n+1}\cdot da^{2n+2} \ .$$

We thus have

$$b\overline{\psi}_{2n+1}(a^0,\ldots, a^{2n+2}) = \sum_{j=1}^{2n+1} (-1)^j \int da^0 \ldots \delta a^j \ldots da^{2n+1}\cdot da^{2n+2}$$

$$- \sum_{j=1}^{2n+1} \int a^0 da^1 \ldots d\delta a^j \ldots da^{2n+1}\cdot da^{2n+2} \ . \qquad (3.2.6)$$

Now, we have

$$\lambda_{2n+2} L_\delta \phi_{2n+2}(a^0,\ldots,a^{2n+2}) = \lambda_{2n+2} \sum_{j=0}^{2n+2} \phi_{2n+2}(a^0,\ldots,\delta a^j,\ldots,a^{2n+2})$$

$$= \int \delta a^0 da^1 \ldots da^{2n+2} + \int a^0 da^1 \ldots da^{2n+1}\cdot d\delta a^{2n+2} + \sum_{j=1}^{2n+1} \int a^0 da^1 \ldots d\delta a^j \ldots da^{2n+2} \ .$$

Combining this with the formulas (3.2.4) and (3.2.6) we get

$$b\overline{\psi}_{2n+1} = B_0 \overline{\psi}_{2n+3} - \lambda_{2n+2} L_\delta \phi_{2n+2} - \chi_{2n+2} \qquad (3.2.7)$$

where the cochains χ_{2n} , $n \geq 0$, are defined by

$$\chi_{2n}(a^0,\ldots,a^{2n}) = \int d(a^0 da^1 \ldots da^{2n+1}\cdot \delta a^{2n+2}) \ .$$

To proceed further we need a lemma about infinite dimensional cycles.

Lemma 3.2.8. $\int d[\omega_1,\omega_2] = 0$ for all ω_1 and ω_2 in ΩA. Here $[\ ,\ ]$ is the super-bracket.

Proof. Let $\partial_i = \deg \omega_i$. We have

$$\int d[\omega_1,\omega_2] = \int d\omega_1 \cdot \omega_2 + (-1)^{\partial_1} \omega_1 d\omega_2 - (-1)^{\partial_1\partial_2} d\omega_2 \cdot \omega_1 - (-1)^{(\partial_1+1)\partial_2} \omega_2 d\omega_1$$

$$= \int (d\omega_1 \cdot \omega_2 - (-1)^{(\partial_1+1)\partial_2} \omega_2 d\omega_1) + (-1)^{\partial_1} \int (\omega_1 \cdot d\omega_2 - (-1)^{\partial_1(\partial_2+1)} d\omega_2 \cdot \omega_1$$

$$= \int d^2\omega_1 \cdot d\omega_2 + (-1)^{\partial_1} \int d\omega_1 \cdot d^2\omega_2 = 0 \quad \text{since} \quad d^2 = 0 . \qquad \square$$

Now, using the above lemma, we can write χ_{2n} as the coboundary of a suitable cochain and hence bring the equation (3.2.7) to the desired form. Define the cochains ψ^*_{2n+1}, $n \geq 0$, by

$$\psi^*_{2n+1}(a^0,\dots, a^{2n+1}) = \int d(da^0\, da^1 \dots da^{2n} \cdot \delta a^{2n+1}) .$$

Note that ψ^*_{2n+1} is reduced and $B_0\psi^*_{2n+1} = 0$. Indeed, we have

$$B_0\psi^*_{2n+1}(a^0,\dots, a^{2n}) = \int d(d1\, da^0 \dots da^{2n-1} \cdot \delta a^{2n}) = 0 .$$

To estimate $\|\psi^*_{2n+1}\|$, note that

$$\psi^*_{2n+1}(a^0,\dots, a^{2n+1}) = - \lambda_{2n+2}\, B_0\, \phi_{2n+2}(a^0,\dots, a^{2n}, \delta a^{2n+1}) ,$$

and hence

$$\|\psi^*_{2n+1}\| \leq |\lambda_{2n+2}|\, \|\phi_{2n+2}\|\, \|\delta\| . \tag{3.2.9}$$

We also have to calculate the Hochschild coboundary of ψ^*_{2n-1}. After obvioius cancellations, we have

$$b\psi^*_{2n-1}(a^0,\dots,a^{2n}) = \int d(a^0da^1 \dots da^{2n-1}.\ \delta a^{2n} - da^0da^1 \dots da^{2n-2}\ \delta a^{2n-1}.\ a^{2n}$$

$$+ a^{2n}\ da^0 \dots da^{2n-2}.\ \delta a^{2n-1} + da^{2n}.\ a^0da^1 \dots da^{2n-2}.\ \delta a^{2n-1}).$$

By the previous lemma, the two middle terms cancel and we get

$$b\psi^*_{2n-1}(a^0,\dots,a^{2n}) = \int d(a^0da^1 \dots da^{2n-1}.\ \delta a^{2n}) - \int d(a^0da^1 \dots da^{2n-2}.\ \delta a^{2n-1}.\ da^{2n})$$

Thus we have

$$b\psi^*_{2n-1} = \chi_{2n} - b\psi^{2n-1}_{2n-1}$$

$$\text{or}\quad b(\psi^*_{2n-1} + \psi^{2n-1}_{2n-1}) = \chi_{2n}\ .$$

Using this we can write (3.2.7) in the form

$$b(\bar{\psi}_{2n+1} + \psi^*_{2n+1} + \psi^{2n+1}_{2n+1}) = B_0\ \bar{\psi}_{2n+3} - \lambda_{2n+2}\ L_\delta\ \phi_{2n+2}\ . \qquad (3.2.10)$$

Now let $\tilde{\psi}_{2n+1} = \bar{\psi}_{2n+1} + \psi^*_{2n+1} + \psi^{2n+1}_{2n+1}$. We have, since $B_0\ \psi^*_{2n+1} = 0$ and using (3.2.5) ,

$$B\tilde{\psi}_{2n+3} = (2n+3)\ B_0\bar{\psi}_{2n+3} + B_0\bar{\psi}_{2n+3} = (2n+4)\ B_0\bar{\psi}_{2n+3}\ .$$

We can thus write (3.2.10) in form

$$b\tilde{\psi}_{2n+1} = \frac{1}{2n+4}\ B\tilde{\psi}_{2n+3} - \lambda_{2n+3}\ L_\delta\ \phi_{2n+2}\ . \qquad (3.2.11)$$

Finally, if we define the required cochain $\psi = (\psi_{2n+1})_{n\geq 0}$ by

$$\psi_{2n+1} = \lambda^{-1}_{2n+2}\ \tilde{\psi}_{2n+1}\ ,$$

the equation (3.2.11) transforms to

$$\lambda_{2n+2}\ b\psi_{2n+1} - \frac{1}{2n+4}\lambda_{2n+4}\ B\psi_{2n+3} = -\lambda_{2n+2}\ L_\delta\ \phi_{2n+2}$$

or,

$$b\psi_{2n+1} + B\psi_{2n+3} = -L_{\delta}\,\phi_{2n+2}$$

which is the required equation.

We have to show ψ is an entire cochain. Using the estimates (3.2.3) and (3.2.9), we have

$$\|\psi_{2n+1}\| \le |\lambda^{-1}_{2n+1}|\,(\|\bar{\psi}_{2n+1}\| + \|\psi^*{}_{2n+1}\| + \|\psi^{2n+1}_{2n+1}\|$$

$$\le ((n+2)\|\phi_{2n}\| + \|\phi_{2n+2}\|)\,\|\delta\| \ . \qquad (3.2.12)$$

This of course guarantees that ψ is an entire cochain. The theorem is proved. □

Corollary 3.2.13. Let A be a non-unital Banach algebra and $\delta : A \longrightarrow A$ a continuous derivation. Then $L_\delta : H^*_\varepsilon(A) \longrightarrow H^*_\varepsilon(A)$ is the zero homomorphism.

Proof. Since A is non-unital we have to work in the periodic bicomplex. Let ϕ be an entire periodic cocycle. By Lemma 3.2.1 ϕ is cohomologous to a normalized entire cocycle and hence we can assume it is normalized. Now recall the commutative diagram

$$\begin{array}{ccc}
\mathrm{Tot}\ \mathcal{C}(A) & \xrightarrow{L_\delta} & \mathrm{Tot}\ \mathcal{C}(A) \\
\theta \downarrow & & \downarrow \theta \\
\mathrm{Tot}\ \beta(\tilde{A})_{red} & \xrightarrow{L_\delta} & \mathrm{Tot}\ \beta(\tilde{A})_{red}
\end{array}$$

The cocycle $\theta\phi$ is reduced and normalized. By the above theorem there is a reduced entire cochain ψ such that $L_\delta\theta\phi = \partial\psi$. We thus have $L_\delta\phi = \partial\theta^{-1}\psi$ and the corollary is proved. □

In the next section we are going to need explicit estimates for $\|(\theta^{-1}\psi)_n\|$. We have, using (3.2.12),

$$\|(\theta^{-1}\psi)_{2n+1}\| \le \|\psi_{2n+1}\| \le \{(n+2)\|(\theta\phi)_{2n}\| + \|(\theta\phi)_{2n+2}\|\}\ \|\delta\|$$
$$\le (n+2)\Big(\sum_{i=2n-1}^{2n+2} \|\phi_i\|\Big)\|\delta\| \tag{3.2.14}$$

and similarly for $\|(\theta^{-1}\psi)_{2n}\|$.

3.3. A Homotopy Invariance Theorem

To prove the homotopy invariance theorem we have to generalize the notion of Lie derivatives a little further and prove the analogue of Theorem 3.2.3 in this context. So, let A and B be algebras and $f : A \longrightarrow B$ a homomorphism between them. A linear map $\delta : A \longrightarrow B$ is called a *derivation* if

$$\delta(ab) = f(a)\delta(b) + \delta(a)f(b) \quad \text{for all } a,b \text{ in } A .$$

Given such a derivation δ , we can then define a *Lie derivative*

$$L_\delta : C^n(B) \longrightarrow C^n(A)$$

by the formula

$$L_\delta\phi(a^0,\dots, a^n) = \sum_{i=0}^{n} \phi(f(a^0),\dots, \delta a^i,\dots, f(a^n)) .$$

One can easily check that the analogue of Lemma 2.4.2 holds i.e. the map L_δ commutes with the operators b, b', λ and s . As a result of this we obtain a map of bicomplexes

$$L_\delta : \mathcal{C}(B) \longrightarrow \mathcal{C}(A)$$

and also, in case $f : A \longrightarrow B$ is unital,

$$L_\delta : \beta(B) \longrightarrow \beta(A) .$$

The proof of the following theorem is completely similar to the proof of Theorem 3.2.3 with obvious modifications.

Theorem 3.3.1. Let A and B be unital Banach algebras and $f : A \longrightarrow B$ a continuous unital homomorphism with a continuous derivation $\delta : A \longrightarrow B$. Let ϕ be a normalized, reduced and entire cocycle on B. Then there is a canonical reduced and entire cochain ψ on A such that $L_\delta\phi = \partial\psi$.

Proof. Let ϕ be even and let $\int : \Omega B \longrightarrow \mathbb{C}$ be its associated cycle. The definition of ψ is a modification of the corresponding definition in Theorem 3.2.3. For example

$$\psi^j_{2n+1}(a^0,\ldots,a^{2n+1}) = \int f(a^0)df(a^1)\ \ldots\ \delta a^j\ \ldots\ df(a^{2n+1}),\quad 1 \leq j \leq 2n+1$$

and similarly for ψ^*_{2n+1} etc. The proof of Theorem 3.2.3 extends word for word to show that the cochain $\psi = (\psi_{2n+1})_{n\geq 0}$ defined as before by $\psi_{2n+1} = \overline{\psi}_{2n+1} + \psi^*_{2n+1} + \psi^{2n+1}_{2n+1}$ is a reduced and entire cochain which satisfies $L_\delta\phi = -\partial\psi$. Finally, we have the following estimate for $\|\psi_{2n+1}\|$:

$$\|\psi_{2n+1}\| \leq ((n+2)\ \|\phi_{2n}\| + \|\phi_{2n+2}\|)\ \|f\|^{2n+1}\ \|\delta\|. \qquad \square$$

Corollary 3.3.2. Any continuous derivation between Banach algebras acts trivially on entire cyclic cohomology groups.

Proof. Let A and B be the given (non-unital) Banach algebras, $f : A \longrightarrow B$ a homomorphism between them and $\delta : A \longrightarrow B$ a derivation. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Tot}\ \mathcal{C}(A) \arrow[r, "L_\delta"] \arrow[d, "\theta"'] & \mathrm{Tot}\ \mathcal{C}(A) \arrow[d, "\vartheta"] \\
\mathrm{Tot}\ \beta(\widetilde{B})_{red} \arrow[r, "L_\delta"] & \mathrm{Tot}\ \beta(\widetilde{A})_{red}
\end{tikzcd}.$$

We can now apply Theorem 3.3.1. The rest of the proof is similar to the proof of Corollary 3.2.13. □

Given a periodic, say even, cocycle $\phi = (\phi_n)_{n\geq 0}$, using the above corollary, we can write $L_\delta\phi = \partial\theta^{-1}\psi$. Using the estimate in Theorem 3.3.1, we have

$$\|(\theta^{-1}\psi)_{2n+1}\| \leq \|\psi_{2n+1}\| \leq (n+2)\left(\sum_{i=2n-1}^{2n+2} \|\phi_i\|\right) \|f\|^{2r} \quad . \tag{3.3.3}$$

Next, let A and B be (non-unital) Banach algebras and $f_t : A \longrightarrow B$, $0 \leq t \leq 1$, a 1-parameter family of homomorphisms between them. Such a family will be called *smooth* iff

(a) each f_t, $t \in [0,1]$, is continuous with $\|f_t\| \leq M$, i.e. f_t is a uniformly bounded family of continuous homomorphisms,

(b) for all $a \in A$, the map $t \mapsto f_t(a)$ from $[0,1]$ to B is C^1. Moreover, the corresponding family of derivatives $\delta_t : A \longrightarrow B$ is uniformly bounded.

The derivatives $\delta_t : A \longrightarrow B$ are of course defined by

$$\delta_t(a) = \lim_{s\to 0} \frac{f_{t+s}(a) - f_t(a)}{s} \quad .$$

since f_t is a family of homomorphisms, we have

$$\delta_t(ab) = \delta_t(a)\, f_t(b) + f_t(a)\, \delta_t(b) \, .$$

This shows that each δ_t is a derivation with respect to f_t.

Theorem 3.3.4. (Homotopy Invariance). Let $f_t : A \longrightarrow B$, $t \in [0,1]$, be a smooth family of homomorphisms between Banach algebras A and B . Then f_0 and f_1 induce the same map between entire cyclic cohomology groups.

Proof. Let δ_t , $0 \le t \le 1$, be the family of derivations defined by f_t . By our assumptions, there are constants M and N such that $\|f_t\| \le M$ and $\|\delta_t\| \le N$, uniformly in t . Let $\phi = (\phi_n)_{n\ge 0}$ be a, say even, entire cocycle. We can assume ϕ is normalized. We have

$$\begin{aligned} \frac{\partial}{\partial t} f^*_t \phi_n(a^0,\dots, a^n) &= \frac{\partial}{\partial t} \phi_n (f_t (a^0),\dots, f_t (a^n)) \\ &= \sum_{i=0}^{n} \phi_n(f_t (a^0),\dots, \delta_t a^i,\dots, f_t (a^n)) \\ &= L_{\delta_t} \phi_n(a^0,\dots, a^n) \ . \end{aligned}$$

In short, we can write

$$\frac{\partial}{\partial t} f^*_t \phi = L_{\delta_t} \phi \ .$$

Now, by Corollary 3.3.2, for each $t \in [0,1]$, there is a canonical entire cochain $\psi^t = (\psi^t_{2n+1})$ such that $L_{\delta_t} \phi = \partial\psi^t$. We can then write, at least morally

$$f^*_1\phi - f^*_0\phi = \int_0^1 \frac{\partial}{\partial t} f^*_t\phi \cdot dt = \int_0^1 \partial\psi^t \cdot dt = \partial \int_0^1 \psi^t dt \ ,$$

which shows that the difference is a coboundary and hence $f^*_1 = f^*_0$ on the level of cohomology. To make this precise, we have to show that the integral exists and defines an entire cochain. Fix $a^0,\dots, a^n$ in A . Then our proof of Theorem 3.3.1 shows that $\psi^t_n(a^0,\dots, a^n)$ is at least a continuous function of $t \in [0,1]$ (by the smoothness hypothesis, for $a \in A$, $\delta_t(a)$ is a continuous function of t). Thus the integrals $\int_0^1 \psi_n^t \, dt$ exist (in the strong sense) and define the required cochains. To check the entire growth condition, use (3.3.3) to get

$$\|\psi^t_{2n+1}\| \leq (n+2) \left(\sum_{i=2n-1}^{2n+2} \|\phi_i\| \right) \|f_t\|^{2n+1} \|\delta_t\|$$

$$\leq (n+2) \left(\sum_{i=2n-1}^{2n+2} \|\phi_i\| \right) M^{2n+1} \cdot N$$

and similar formula for $\|\psi^t{}_{2n}\|$. In view of the fact that $\phi = (\phi_n)_{n\geq 0}$ is an entire cochain, the above estimate certainly implies that $\left(\int_0^1 \psi_n^t \, dt \right)_{n\geq 0}$ is an entire cochain. $\square$

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