OPTIMAL CONTROL OF PIECEWISE DETERMINISTIC MARKOV PROCESSES

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Contents

Abstract

This thesis describes a complete theory of optimal control of piecewise deterministic Markov processes under weak assumptions. The theory consists of a description of the processes, a nonsmooth stochastic maximum principle as a necessary optimality condition, a generalized Bellman–Hamilton–Jacobi necessary and sufficient optimality condition involving the Clarke generalized gradient, existence results and regularity properties of the value function. The impulse control problem is transformed to an equivalent optimal dynamic control problem. Cost functions are subject only to growth conditions.

Piecewise deterministic Markov processes, termed PDPs for short, are continuous time homogeneous Markov processes consisting of a mixture of deterministic motion and random jumps. PDPs, with stochastic jump processes and deterministic dynamical systems as special cases, include virtually all of the stochastic models of applied probability except diffusions. Their impulse control extends their applicability to discrete event problems such as stochastic scheduling. The processes are controlled by an open loop control depending on the postjump state and the time elapsed since the last jump in the interior of the state space, a feedback control on the boundary of the state space and impulse controls on the entire state space. The expected value of a performance functional of integral type with additional boundary and impulse costs is to be minimized.

The PDP optimal control problem is converted to an infinite horizon discrete-time stochastic optimal control problem and it is shown that the optimal strategy for control of a PDP is to choose after each jump a control function which is an optimal control in a corresponding deterministic control problem where the state of the system is required to stop at the boundary. This deterministic control problem is however non-standard in that the terminal time is not fixed but instead is either infinity or the first time the trajectory reaches the boundary of the state space. As preliminary results, we obtain a nonsmooth maximum principle as a necessary optimality condition and a necessary and sufficient optimality condition in terms of a generalized Bellman–Hamilton–Jacobi equation involving the Clarke generalized gradient for the deterministic problem. The desired results then follow in a straight-forward manner.

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Introduction

Almost all the continuous-time stochastic process models of applied probability consist of some combination of the following:

(a) diffusion

(b) deterministic motion

(c) random jumps.

If one wishes to study a stochastic process with continuous trajectories, there is a general model called a diffusion. For the study of diffusions, there are some standard techniques based on the theory of the Ito calculus and stochastic differential equations. For control of a diffusion, there are some highly developed optimality conditions such as those of the Bellman–Hamilton–Jacobi equation and Pontryagin's maximum principle. The situation is quite different, however, if one wishes to study non-diffusion models—i.e., those involving ingredients (b) and (c). Before the invention of the piecewise deterministic Markov process (abbreviated as PDP) there was no *general* model for non-diffusion processes. The available theory consisted largely of a heterogeneous collection of special models and methodologies appropriate to specific problems. Furthermore, some important problems, of which the capacity expansion model described in §1.5.1 is an instance, fell outside the scope of any available theory of control.

The class of *piecewise deterministic* Markov processes (PDPs), first introduced by Davis (1984), provides a general family of stochastic models covering virtually all non-diffusion applications. Such processes provide a framework for studying optimization problems arising in queueing systems, inventory theory, resource allocation and other areas of the operations research. Stochastic calculus for these processes was developed, and a complete characterization of their extended generators was given by Davis (1984).

The optimal control theory of PDPs has recently been developed for optimal control by Vermes (1985), Soner (1986) and Davis (1986), optimal stopping by Lenhart & Liao (1985), Gugerli (1986), Costa & Davis (1988) and impulse control by Costa & Davis (1988), Gatarek (1988a,b) and Lenhart (1989). Using a modification of Vinter and Lewis's convex duality approach, Vermes (1985) showed the existence of an optimal control and gave a limiting form of the Bellman–Hamilton–Jacobi partial differential equation as a necessary and sufficient optimality condition. Soner (1986) investigated the optimal control of PDPs with state space constraint, i.e., the process whose trajectories have to stay within a given set and characterized the value function as the viscosity solution of the corresponding Bellman–Hamilton–Jacobi equation (BHJ equation). Davis (1986) converted the optimal control problem of PDPs to an infinite horizon discrete-time stochastic optimal control problem. Gugerli (1986) obtained some optimality conditions for the value function of the optimal stopping of PDPs by iteration methods. Lenhart & Liao (1985) characterized the value function of the optimal stopping problem of PDPs as the unique solution to the variational inequality with an integro-differential operator. Lenhart (1989) obtained some existence and uniqueness results for viscosity solutions of the quasi-variational inequalities associated with the optimal impulse control problem of PDPs. Gatarek (1988) obtained similar results to Lenhart and Liao (1985) by using a different approach. Gatarek's technique is to approximate value functions for an optimal stopping (impulse control) problem for a PDP by value functions for Feller piecewise deterministic processes. Costa and Davis (1988) presented a numerical technique for solving the optimal stopping problem of PDPs by discretization of the state space. Applying these results to the impulse control problem, they developed a numerical technique for computing optimal impulse controls for PDPs (1988). Using the framework of

PDPs, some specific problems have been solved by Davis *et al.* (1987) and Dempster and Solel (1987).

Related to PDPs are so-called Markov decision drift processes (MDDP) introduced by van der Duyn Schouten in his 1979 thesis (now available as an Amsterdam Mathematical Centre Tract, van der Duyn Schouten, 1983). The methods of analysis are, however, completely different, being based on time discretization and weak convergence. A slightly generalized MDDP is given by Yushkevich (1983). Yushkevich also studied the problem involving interventions (impulse controls).

In this thesis, we study the control problem of PDPs with full control (i.e. dynamic control plus impulse control). Under fairly general assumptions, we obtain the following main results:

(1) existence of an optimal control

(2) Lipschitz continuity of the value function

(3) necessary and sufficient optimality conditions in terms of a generalized BHJ equation involving Clarke generalized gradients

(4) a nonsmooth Pontryagin maximum principle.

We conclude this introduction with a brief indication of some of the major points in the six chapters that follow.

In Chapter 1, we introduce the concept of PDPs, the infinitesmal generator of a PDP, the control of PDPs and give some examples. We also introduce the definition and properties of the Clarke generalized gradient. For future reference, we state a nonsmooth maximum principle for a standard optimal control problem and give some results on differential inclusions.

In Chapter 2, we study a deterministic control problem where the state of the system is required to stop at the boundary. We show that the value function is a Lipschitz continuous solution of the generalized BHJ equation with a boundary condition and obtain some necessary and sufficient optimality conditions in terms of the generalized BHJ equation. Under a certain regularity assumption, the uniqueness result for the nonsmooth solution of the BHJ equation with a boundary condition is also obtained.

In Chapter 3, we show that the optimal strategy for control of PDPs is to choose after each jump a control function which is an optimal control in the corresponding deterministic optimal control problem with a boundary condition formulated in Chapter 2. Then, by applying the results from Chapter 2, we prove that the value function for the PDP optimal control problem is a Lipschitz continuous solution of the generalized BHJ equation with the boundary condition and obtain necessary and sufficient conditions for optimality in terms of the generalized BHJ equation involving Clarke generalized gradients.

In Chapter 4, we develop a nonsmooth maximum principle for the deterministic optimal control problem with a boundary condition formulated in Chapter 2. By reducing the control problem of a PDP to a family of corresponding deterministic control problems with a boundary condition parametrized by initial states, as in Chapter 3, we derive a nonsmooth maximum principle for control of PDPs.

In Chapter 3 and 4, we discuss only problems with dynamic control. In Chapter 5, we add impulse control. We transform the original process with dynamic control plus impulse control to a process with only dynamic control so that the optimal control theory developed in the previous chapters can be used.

In the previous chapters, we assumed that the cost functions are bounded. In Chapter 6, we extend the results to include the case where the cost functions are those subject only to bounded growth.

Chapter 1

Preliminaries

1.1 Introduction

This chapter basically contains the preliminaries and the preliminary results that will be useful in the development of the following chapters.

1.2 Definition of a Piecewise Deterministic Process

We give first some terminology for ordinary differential equations which will be useful later.

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a Lipschitz continuous function (*cf.* Definition 1.7), then the equation $\dot{\zeta} = f(\zeta)$ has a unique solution defined for all $t \in \mathbb{R}$, i.e., there is a unique function $\zeta : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that for all $z \in \mathbb{R}^n$

$$\begin{aligned} &\zeta(0, z) := z \\ &\frac{d}{dt}\zeta(t, z) = f(\zeta(t, z)). \end{aligned} \tag{1.1}$$

If $W : \mathbb{R}^n \longrightarrow \mathbb{R}$ is an arbitrary sufficiently smooth (C^1) function then

$$\frac{d}{dt}W(\zeta_t) \quad = \quad \nabla W(\zeta_t) f(\zeta_t)$$

$$:= \sum_{i=1}^{n} \frac{\partial W}{\partial x_i}(\zeta_t) f_i(\zeta_t), \tag{1.2}$$

where $\zeta_t = \zeta(t, z)$. Let $\mathcal{X}$ denote the first-order differential operator

$$\begin{aligned} \mathcal{X}W(z) &:= \nabla W(z) f(z) \\ &:= \sum_{i=1}^{n} \frac{\partial W}{\partial x_i}(z) W_i(z). \end{aligned} \tag{1.3}$$

Then (1.2) becomes

$$\begin{aligned} &\frac{d}{dt} W(\zeta_t) = \mathcal{X} W(\zeta_t), \\ &\zeta_0 := z, \end{aligned} \tag{1.4}$$

and this is equivalent to (1.1) in that $\zeta(t, z)$ is the unique function such that (1.4) is satisfied for all smooth W. Equation (1.4) is the "co-ordinate-free" form of the differential equation; $\mathcal{X}$ is a *vector field* and $\zeta(t, z)$ is the *integral curve* of $\mathcal{X}$. It has the *semigroup* property

$$\zeta(t + s, z) = \zeta(t, \zeta(s, z)), \qquad s, t \in \mathbb{R}_+.$$

Formulated in this way the differential equation can take values in some differential manifold or, in particular, in a Euclidean space $\mathbb{R}^n$.

Now we can give a formal definition of a *piecewise deterministic Markov process.* Its state space E^0 is defined as follows. Let K be a countable set and $d : K \longrightarrow \mathbb{N}$(the natural numbers) be a given function. For each $\nu \in K$, M_ν is an open subset of $\mathbb{R}^{d(\nu)}$(or M_ν can be a $d(\nu)$-dimensional manifold). Then the state space is

$$E^0 := \bigcup_{\nu \in K} M_\nu = \{(\nu, \zeta) : \nu \in K, \zeta \in M_\nu\}.$$

Let $\mathcal{E}$ denote the following class of measurable sets in E^0.

$$\mathcal{E} := \{ \bigcup_{\nu \in K} A_\nu : A_\nu \in \mathcal{M}_\nu \},$$

where $\mathcal{M}_\nu$ denotes the Borel sets of M_ν. Then $(E^0, \mathcal{E})$ is a Borel space. The state of the process will be denoted $x := (\nu, \zeta)$. The probability law of $\{\mathrm{x}_t\}$ is determined by the following objects, termed *local characteristics*:

(1) vector fields $(\mathcal{X}_\nu, \nu \in K)$

(2) a jump rate which is a measurable function $\lambda : E^0 \longrightarrow I\!R_+$

(3) a transition measure $Q : \mathcal{E} \times (E^0 \cup \Gamma^*) \longrightarrow [0,1]$ (Γ^* is defined by (1.5) below.)

The vector fields $\mathcal{X}_\nu$ are supposed to be such that for each $z \in M_\nu$ there is a unique integral curve $\phi_\nu(t,z)$ satisfying (1.4) with $\mathcal{X} = \mathcal{X}_\nu$ and $\zeta_t = \phi_\nu(t,z)$. Further, it is supposed that the $\mathcal{X}_\nu$ are conservative, i.e., the integral curves are defined for all $t > 0$ (no " explosions"). We denote by ∂M_ν the boundary of M_ν and by $\partial^* M_\nu$ those boundary points which integral curves of M_ν may reach, i.e.

$$\partial^* M_\nu := \{\zeta \in \partial M_\nu : \phi_\nu(t,z) = \zeta \text{ for some } t > 0 \text{ and } z \in M_\nu\}.$$

Now define

$$\begin{aligned} \partial E &:= \bigcup_{\nu \in K} \partial M_\nu \\ \Gamma^* &:= \bigcup_{\nu \in K} \partial^* M_\nu. \end{aligned} \tag{1.5}$$

For $x := (\nu, z) \in E^0$ we denote by

$$t_*(x) := \inf\{t > 0 : \phi_\nu(t,z) \in \partial^* M_\nu\},$$

the first time a trajectory starting from x hits the boundary of the state space. By convention, $\inf \emptyset := \infty$, so that $t_*(x) = \infty$ means that a trajectory starting from $z \in M_\nu$ never hits the boundary of M_ν.

Finally, we write $\mathcal{X}_\nu W(x)$ instead of the more accurate $\mathcal{X}_\nu W(\nu, \cdot)(\zeta)$ for the action of vector fields $\mathcal{X}_\nu$ on functions $W : E^0 \longrightarrow I\!R$ at $x = (\nu, \zeta) \in E^0$.

As regards the function λ, we suppose that for each $(\nu, z) \in E^0$ there exists $\varepsilon > 0$ such that the function $s \longmapsto \lambda(\nu, \phi_\nu(s,z))$ is integrable for $s \in [0, \varepsilon]$. The transition measure $Q(A;x)$ is a measurable function of x for each fixed $A \in \mathcal{E}$, defined for $x \in E^0 \cup \Gamma^*$, and is a probability measure on $(E^0, \mathcal{E})$ for each $x \in E^0$.

The motion of the process $\{\mathbf{x}_t\}$ starting from $x = (n, z) \in E^0$ can now be constructed in the following way. Define a survivor function $\bar{F}$ by

$$\bar{F}(t) = \begin{cases} \exp(-\int_0^t \lambda(n, \phi_n(s, z))ds) & t < t_*(x) \\ 0 & t \geq t_*(x). \end{cases} \tag{1.6}$$

Realize a random variable $\mathbf{T}_1 = T_1$ such that $P[\mathbf{T}_1 > t] = \bar{F}(t)$. Now realize, independently, an E^0-valued random variable $(\mathbf{N}, \mathbf{Z}) = (N, Z)$ having distribution $Q(\cdot\,; \phi_n(T_1, z))$. The trajectory of $\{\mathbf{x}_t\}$ for $t \leq T_1$ is given by

$$x_t = (\nu_t, \zeta_t) = \begin{cases} (n, \phi_n(t, z)) & t < T_1 \\ (N, Z) & t = T_1. \end{cases}$$

Starting from x_{T_1} we now realize the next inter-jump time $\mathbf{T}_2 - \mathbf{T}_1 = T_2 - T_1$ and post-jump location $\mathbf{x}_{T_2} = x_{T_2}$ in a similar way, and so on. This gives a piecewise deterministic trajectory of $\{\mathbf{x}_t\}$ with jump times $T_1, T_2, \ldots$.

Under the stated local integrability condition on λ,

$$P[\mathbf{T}_1 > 0] = 1$$
$$P[\mathbf{T}_i - \mathbf{T}_{i-1} > 0] = 1 \qquad \text{for } i = 2, 3, \ldots.$$

We will also make the following assumption: Let $\mathbf{N}_t := \sum_i I_{\{t \geq \mathbf{T}_i\}}$ be the number of jumps in $[0, t]$, where

$$I_A(\omega) := \begin{cases} 0 & \omega \notin A \\ 1 & \omega \in A \end{cases}$$

is the indicator function of the event A. Then

$$E\mathbf{N}_t < \infty \qquad \text{for all } t. \tag{1.7}$$

In particular, (1.7) implies that

$$P[\mathbf{T}_i \uparrow \infty, i \uparrow \infty] = 1. \tag{1.8}$$

Since all the random variables in the above algorithm can be generated in the standard way from uniform [0,1] random variables, we have in effect defined a measurable mapping from a countable product of unit interval probability spaces denoted by $(\Omega, \mathcal{B}, P)$, to the space of right-continuous left-limited E^0-valued functions. Thus the probability law P_x of $\{\mathrm{x}_t\}$ starting at $x \in E^0$ is well defined.

As shown by Davis (1984), $\{\mathrm{x}_t\}$ is a strong Markov process.

1.3 The Extended Generator of PDPs

Denote by $B(E^0)$ the set of bounded measurable real-valued functions on E^0 equipped with the essential supremum norm $\|\cdot\|_\infty$. It is well known that the following formula defines a semigroup of operators $\{T_t\}$ on $B(E^0)$: (i.e. T_t has the semigroup property $T_{t+s} = T_t T_s$)

$$T_t W(x) := E_x W(\mathrm{x}_t)$$

where E_x denotes the expectation respect to x_t with initial state x.

The *strong (infinitesimal) generator* of this semigroup is an operator $\tilde{\mathcal{A}}$ acting on a domain of functions $\mathcal{D}(\tilde{\mathcal{A}}) \subset B(E^0)$ such that for $W \in \mathcal{D}(\tilde{\mathcal{A}})$

$$\tilde{\mathcal{A}} W(x) = s\text{-}\lim_{t \downarrow 0} \frac{1}{t}(T_t W(x) - W(x)),$$

where s-lim indicates that the limit is taken with respect to the supremum norm in $B(E^0)$. The important property of the generator for our purposes is the *Dynkin formula*

$$T_t W(x) - W(x) = \int_0^t T_s \tilde{\mathcal{A}} W(x) ds, \tag{1.9}$$

which, from an analytic point of view, is the "fundamental theorem of calculus" for semigroups. Probabilistically, however, the Dynkin formula is equivalent to the statement that the process

$$\mathrm{C}_t^W := W(\mathrm{x}_t) - W(\mathrm{x}_0) - \int_0^t \tilde{\mathcal{A}} W(\mathrm{x}_s) ds$$

is an $\mathcal{F}_t$-martingale, where the natural filtration $\mathcal{F}_t := \sigma\{\mathrm{x}_s : s \leq t\}$ is the σ-algebra generated by $\{\mathrm{x}_s : s \leq t\}$ (i.e. $E[\mathrm{C}_t^W - \mathrm{C}_s^W | \mathcal{F}_s] = 0$). We can regard this property as the minimal connection between an operator $\tilde{\mathcal{A}}$ and the corresponding process $\{\mathrm{x}_t\}$.

For technical reasons it is however convenient to enlarge $\mathcal{D}(\tilde{\mathcal{A}})$ to include those W for which C_t^W is only a *local* $\mathcal{F}_t$-martingale, i.e. possesses the martingale property in a (time) neighbourhood of t. This leads to the following definition of of the extended generator $\mathcal{A}$ of $\{x_t\}$.

Definition 1.1 The *extended generator* of a homogeneous Markov process is an operator $\mathcal{A}$ acting on a class $\mathcal{D}(\mathcal{A})$ of functions $W : E^0 \longrightarrow I\!R$ such that for $W \in \mathcal{D}(\mathcal{A})$ the process

$$C_t^W := W(x_t) - W(x_0) - \int_0^t \mathcal{A}W(x_s)ds$$

is a local martingale. ■

This definition is given by Jacod (1979).

Evidently $\mathcal{A}$ and $\tilde{\mathcal{A}}$ coincide on $\mathcal{D}(\tilde{\mathcal{A}})$ so that $\mathcal{A}$ is an extension of the strong generator $\tilde{\mathcal{A}}$.

Davis (1984) gave an exact characterization of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ as follows.

Proposition 1.1 (Davis 1984, Theorem 5.5, p.367)
A measurable function $W : E^0 \longrightarrow I\!R^1$ belongs to the domain $\mathcal{D}(\mathcal{A})$ of the extended generator $\mathcal{A}$ if and only if the following three conditions are satisfied:-

(i) *For each $(n, z) \in E^0$ the function $t \longmapsto W(n, \phi_n(t, z))$ is absolutely continuous for $t \in [0, t_*(n, z))$.*

(ii) *There exists a sequence σ_n of stopping times (i.e. σ_n is $\mathcal{F}_t$-measurable) such that $P_x[\sigma_n \uparrow \infty, n \uparrow \infty] = 1$ and for each n*

$$E_x \sum_i |W(x_{T_i \wedge \sigma_n}) - W(x_{T_i^- \wedge \sigma_n})| < \infty.$$

(iii) $W(z) := \lim_{t \downarrow 0} W \circ \phi_n(-t, z)$ *exists for all $(n, z) \in \partial E$ and W satisfies the boundary condition*

$$W(x) = \int_{E^0} W(y)Q(dy; x) \quad \text{for } x \in \partial E.$$

For $W \in \mathcal{D}(\mathcal{A})$*,* $\mathcal{A}f$ *is given at* $x = (n, z)$ *by*

$$\mathcal{A}W(x) = \mathcal{X}_n W(x) + \lambda(x) \int_{E^0} [W(y) - W(x)] Q(dy; x) \tag{1.10}$$

where $\mathcal{X}_n$ *is the first-order differential operator given by* (1.3).

■

1.4 The PDP Optimal Control Problem

Optimal dynamic control problems arise when the local characteristics, $\mathcal{X}, \lambda, Q$ of a PDP depend, besides on the state x, on a free control parameter v from a compact set U. The set of admissible controls may be different for interior and boundary states. We assume that $v \in U_0 \subset I\!R^m$ if $x \in E^0$ and $v \in U_\partial \subset I\!R^l$ if $x \in \partial E$. It should be noted that it suffices to take U_0 and U_∂ to be Polish, i.e. compact separable metric, spaces and we shall use such a control space in Chapter 5 where a one-point compactification of a half line is used. We shall distinguish the transition measure $Q_0(dy; x, v)$, for $x \in E^0, v \in U_0$, describing jumps from interior points, from $Q_\partial(dy; x, v)$, for $x \in \partial E, v \in U_\partial$, describing jumps from boundary points. The "usual" class of admissible controls in Markovian problems is that of state feedback controls $u_t = u(x_t)$. Corresponding to u, functions $\mathcal{X}^u, \lambda^u, Q^u$ are defined by

$$\begin{aligned}
\mathcal{X}^u W(x) &:= \nabla W(x) f(x, u(x)) \\
\lambda^u(x) &:= \lambda(x, u(x)) \\
Q^u(A; x) &:= Q(A; x, u(x)).
\end{aligned}$$

Under certain smoothness conditions on $u(\cdot)$, a PDP having local characteristics, $\mathcal{X}^u, \lambda^u, Q^u$, can be constructed as in §1.2. The state feedback control is however not the appropriate choice if we require u to be only measurable since the secondary component ζ_t has to satisfy between jumps the ordinary differential equation

$$\frac{d}{dt}\zeta_t = f(\nu_t, \zeta_t, u(\nu_t, \zeta_t))$$

and this equation may fail to have a unique solution unless u is sufficiently smooth (e.g. Lipschitz continuous). To be able to use controls that are only measurable, we now consider controls in the form $u = (u_0, u_\partial)$ where

$$u_0 : \mathbb{R}_+ \times E^0 \longrightarrow U_0$$
$$u_\partial : \partial E \longrightarrow U_\partial$$

are measurable functions. If T_k is the last jump time before time t, let $\mathbf{z}_t := \mathbf{x}_{T_k}$ denote the *post jump state* and $\tau_t := t - \mathrm{T}_k$ be the *time elapsed since the last jump*. Then the ordinary differential equation

$$\frac{d}{dt}\zeta_t = f(\nu_t, \zeta_t, u_0(\tau_t, z_t)) \qquad t \geq T_k$$

has a unique solution as long as f is Lipschitz in ζ_t, by the *Carathéodory existence and uniqueness theorem.* Augmentation of the process to keep track of $\mathbf{z}_t$ and τ_t as states is also possible (see Davis 1984 and Vermes 1985 for details) but not necessary, since $\mathbf{z}_t$, τ_t can be derived from $\mathbf{x}_t$ and $\{\mathrm{T}_k\}$ in an obvious manner (*cf.* §1.2). Such *piecewise open loop controls* are therefore the "appropriate" ones. Consider a deterministic control problem as an extremal special case of a PDP. The "appropriate" controls should be *open loop controls* depending only on the initial state x_0 (which is the only "post jump state") and the elapsed time t. Therefore it is interesting to see that the control of PDPs involves an intriguing mixture of "deterministic" and "stochastic" features.

The controls we have discussed so far are actions which only affect the infinitesimal generator of the process. Impulse controls are required, however, if one wishes to take actions which can cause an immediate change in the state of the process (i.e., a jump). We shall term the times that a such decision is taken *intervention times* and denote them by $\{\tau_i\}$. We define $\{\tau_i\}$ as a sequence of stopping times. At intervention times, upon applying an impulse control action $v \in U_\delta \subset \mathbb{R}^k$, the state x is moved to the state $\mathbf{y}$ which is a random variable with transition measure $Q_\delta(A; x, v)$.

The performance criterion to be minimized includes a *running cost*

$l_0 : E^0 \times U_0 \longrightarrow I\!R_+$, a *boundary cost* $l_\partial : \partial E \times U_\partial \longrightarrow I\!R_+$ and an *intervention cost* $l_\delta : E \times U_\delta \longrightarrow I\!R_+$. Let $\delta > 0$ be a *discount factor.*

The PDP *optimal* (full) *control problem* is to find a sequence of optimal intervention times $\{\tau_i\}$ and an optimal control (i.e. a triple $u = (u_0, u_\partial, u_\delta)$ of measurable functions: $u_0 : E^0 \times I\!R_+ \longrightarrow U_0, u_\partial : \partial E \longrightarrow U_\partial$ and $u_\delta(\mathrm{x}_{\tau_i})$) such that the expected cost

$$J_x(u) = E_x[\int_0^\infty e^{-\delta t} l_0(\mathrm{x}_t, u_0(\tau_t, \mathrm{z}_t))dt + \sum_{\mathrm{T}_i \neq \tau_i} e^{-\delta \mathrm{T}_i} l_\partial(\mathrm{x}_{T_i^-}, u_\partial(\mathrm{x}_{T_i^-}))1_{(\mathrm{x}_{T_i^-} \in \partial E)}$$
$$+ \sum_i e^{-\delta \tau_i} l_\delta(\mathrm{x}_{\tau_i^-}, u_\delta(\mathrm{x}_{\tau_i^-}))] \qquad (1.11)$$

is minimized.

1.5 Examples

1.5.1 A model for capacity expansion

Capacity expansion is the process of adding facilities of similar type over time to meet a rising demand for their services. Typical examples are electrical power generating stations, water resource facilities, major computer and communication systems and large manufacturing facilities such as blast furnaces and rolling mills in the steel industry. These are large-scale projects, tackled relatively infrequently, and this precludes a short term incremental approach to the planning of expansion. Planning decisions concern timing, scale and location of major projects in the face of uncertain–often highly uncertain–demand forecasts, costs and completion times. Project location decisions are of fundamentally different character to those of timing and scale, and indeed are often predetermined by engineering and /or political considerations. Here we consider optimization of timing and scale with the major emphasis on timing. To be realistic, the mathematical model we consider here incorporates uncertain future demand, non-zero lead time and random cost overruns.

We present a simple model for the capacity expansion problem incorporating these features. The reader is referred to Davis *et al.* (1987) for some extensions in the direction of more realistic models.

It is supposed that the demand for some utility is monotone increasing and can be modelled as a Poisson process with rate μ. This demand is to be met by construction of new identical expansion projects, each of which costs $\$\sigma$ and meets K units of demand when completed. Investment is channelled into the current expansion project at a rate $\$u_t$ per unit time (to be decided upon). The maximum feasible investment rate being a known constant. The current project is complete when investment in it reaches level $\$\ \boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ is a random variable whose distribution function is given by

$$P[\sigma < s] := H(s) := \begin{cases} \int_0^s h(r)dr & s < \Sigma \\ 1 & s \geq \Sigma \end{cases}$$

where $h(\cdot)$ is a bounded continuous function and Σ is a given constant. Thus $\boldsymbol{\sigma} \leq \Sigma$ always and $\boldsymbol{\sigma}$ has a continuous density h on $[0, \Sigma)$. The corresponding *hazard rate* is

$$\begin{aligned} \gamma(s) &:= \lim_{t \downarrow 0} t^{-1} P[\sigma \leq s + t | \sigma > s] \\ &= h(s)/(1 - H(s)) \quad s \in [0, \Sigma). \end{aligned} \tag{1.12}$$

This can be interpreted as the *completion rate* of the project due to constant investment at unit rate. We assume that $\gamma(\cdot)$ is bounded. The costs of successive projects are independent, and the cost of the current project is unknown to the decision-maker (who, however, knows h and Σ) until completion actually takes place. Upon completion, K units of capacity are supplied and further investment beyond this time is investment in the next project. We denote by $\mathbf{c}_t$ the total capacity installed at time t; thus $\mathbf{c}_t = K\mathbf{N}_t$ where $\mathbf{N}_t$ is the number of completed projects.

If the existing capacity does not meet the demand, a penalty (shortage cost) is paid; excess capacity may also be penalized. Let $\mathbf{d}_t$ be the demand process. We denote by $\boldsymbol{v}_t =: \mathbf{d}_t - \mathbf{c}_t$ the *undercapacity process* and by $q(\nu)$ the penalty per unit time paid for undercapacity ν (thus $q(\nu)$ is the shortage cost for $\nu > 0$ and the excess capacity cost for $\nu < 0$). We assume that $q(\nu)$ is non-negative and monotonically increasing as $\nu > 0$ increases or $\nu < 0$ decreases, with at most polynomial growth in both cases. The decision-maker has to choose the investment rate $\mathbf{u}_t$ for $t > 0$ so as to minimize the discounted infinite horizon cost

$$E \int_0^\infty e^{-\delta t}(\mathbf{u}_t + q(v_t))dt.$$

Now we formulate this problem as a PDP optimal control problem. Denote by ζ_t the cumulative investment in the current project. Thus,

$$\frac{d}{dt}\zeta_t = u_t.$$

We now formulate the process $\mathbf{x}_t := (v_t, \zeta_t)$ as a controlled PDP taking values in the state space $E^0 = I\!N \times (-1/2, \Sigma)$. An admissible control is a measurable function $u : I\!R_+ \times E^0 \longrightarrow [0, p]$. For all $\nu \in I\!N$, we define the right hand side of the dynamics as $f(x, v) := v$ for all $x \in E, v \in [0, p]$. Then f determines the vector fields $\mathcal{X}_\nu = \mathcal{X}$. Thus between jumps the secondary component ζ_t satisfies the ordinary differential equation

$$\frac{d}{dt}\zeta_t = f(\nu_t, \zeta_t, u(\tau_t, z_t)) = u(\tau_t, z_t),$$

where z_t is the the most recent post jump state before t, τ_t is the time elapsed since the last jump. For $x := (\nu, \zeta)$, define $\lambda(x, v) := \mu + v\gamma(\zeta)$ where γ is the hazard rate defined by (1.12). If a jump occurs, it may be because a demand increment has arrived, or because a project has been completed, these events having relative probalities $\mu/\lambda, v\gamma/\lambda$ respectively. Thus,

$$\begin{aligned} Q_0[dy; x, v] \quad &:= \quad \frac{\mu}{\lambda}\delta_{(\nu+1,\zeta)}(dy) + \frac{v\gamma}{\lambda}\delta_{(\nu-K,0)}(dy) \\ &\qquad\qquad\qquad \text{for } x = (\nu, \zeta),\ \zeta < \Sigma. \end{aligned}$$

On the other hand, if $\zeta = \Sigma$, then a completion must take place since Σ is the maximum project cost. Thus,

$$Q_\partial[dy; x] := \delta_{(\nu-K,0)}(dy), \qquad \text{for } x = (\nu, \Sigma),$$

where δ_y denotes the one-atom probability measure concentrated on y. Once the control u is chosen, the above specifications determine the probability law of the process $\{\mathbf{x}_t\}$. The expected cost is now

$$J_x(u) = E_x \int_0^\infty e^{-\delta t}[u(\tau_t, \mathbf{z}_t) + q(v_t)]dt$$

where x is the initial state. It is now an optimal PDP control problem to determine the control u which minimizes $J_x(u)$.

1.5.2 Stochastic scheduling

In this example, we will show how to formulate the stochastic scheduling problem as a PDP optimal control problem. This extends the applicability of PDPs to some discrete event systems. For a detailed description, see Dempster and Solel (1987) and Solel (1986).

Consider the following general *precedence constrained stochastic scheduling problem.* A finite number n of *jobs* must be executed on a finite number m of *parallel* identical (i.e. differing speed) *machines.* Jobs are to be processed by machines without *preemption* of running jobs except at *review times* (interventions) to be determined. There is a *precedence* (partial) *order* for processing jobs $\{j : j = i, \ldots, n\}$ and there are *forbidden sets* of jobs which cannot be processed together. The random job *processing requirements* p_j have finite expectations and joint distribution P on $\mathbb{P}^n := \{x \in \mathbb{R}^n : x \geq 0\}$. The *cost function* $k : \mathbb{P}^n \longrightarrow \mathbb{R}_+$ is such that if a job j, $1 \leq j \leq n$, is processed in time $c_j := t_j + p_j$, where t_j denotes the start time of job j, $j = 1, \ldots, n$, then $k(c_1, \cdots, c_n)$ is the induced cost and is usually assumed to be a linearly (polynomially, exponentially) bounded function. Scheduling (i.e. control) is to decide, after jobs finish being processed or at review times, the set of jobs to start (or to take off and process later) and the length t' of the time remaining until the next review time, in case no job finishes being processed within this time, based on process information to date regarding finished, running and unprocessed jobs—start times, processing times and present time—with a view to minimizing expected cost.

The PDP model of this situation involves the state vector

$$x := (w_1, \ldots, w_n, t_1, \ldots, t_n, \delta_1, \ldots, \delta_n, t', m)' \in \mathbb{R}^{3n+2},$$

where prime denotes transpose and for job j, w_j denotes the amount of processing currently received and δ_j denotes the current job state (0 not started, 1 running, 2 finished), t' represents *either* time remaining to the next review or *fictitious time* and m is the current number of jumps completed by the process.

The process evolves in $3^n + 1$ linear manifolds, corresponding to all possibilities of job states and representing the combinatorial complexity of the scheduling problem, and the coffin state $\triangle$.

Real time t is represented only implicitly in terms of *process time* s as $t(s) := t_j(s) + w_j(s)$, where j is any currently running job.

The process jumps whenever there is a job completion or at review times. The dynamics of the process between jumps consists of movement along straight lines:

$$
\begin{aligned}
\dot{w}_j(s) &= \begin{cases} 1 & \text{if } \delta_j = 1 \text{ and } t' > 0 \\ 0 & \text{otherwise} \end{cases} \\
\dot{t}_j(s) &= 0 \\
\dot{\delta}_j(s) &= 0 \\
\dot{t}'(s) &= -1 \\
m(s) &= 0.
\end{aligned}
$$

The drift (vector field) and (job completion) intensity (which is assumed known) is uncontrolled. Therefore there is no interior control and the only control (scheduling) is the boundary control. The admissible controls are those taking account of the precedence and the forbidden sets constraints.

Fictitious time t' is started by job completion or review epochs with a jump to the interior of a suitable interval and runs while real time t is stopped until the boundary of this interval is reached and the control (scheduling) is exerted at these boundary points. By this device two jump states are passed through at each real time jump epoch –one state (involving only a fictitious time change) to review or complete finished jobs and another to start new jobs under the control process. The transition measure can be calculated accordingly.

The PDP control problem for this case is to find an admissible control u (scheduling strategy) so as to minimize the expected cost defined by

$$\begin{aligned} & E[\int_0^\infty l_0(\mathbf{x}_s^u)ds] \\ = \quad & E[\int_{t_0}^{t_0+1} 3k(\mathbf{t}_1+\mathbf{p}_1,\ldots,\mathbf{t}_n+\mathbf{p}_n)(\mathbf{t}'(s)+6)^2 ds \end{aligned}$$

where t_0 is the time the process jumped to $t'=-6$ with $\delta_1 = \ldots = \delta_n = 2$, and $t'(s) = -6-(s-t_0)$ evolves until it reaches $t'=-7$ and then the process jumps to the coffin state $\triangle$ and terminates.

1.6 Basic Concepts of Nonsmooth Analysis

In this section, we introduce some basic concepts on nonsmooth analysis. The reader is referred to Clarke (1983) for further details.

1.6.1 Definition and properties of Clarke generalized gradients

We shall be working in a Banach space X whose norm we shall denote by $\|\cdot\|_X$, and whose open unit ball is denoted by B_X; the closed unit ball is denoted by $\bar{B}_X$.

The Lipschitz condition

Definition 1.2 Let Y be a subset of X. A function $f: Y \longrightarrow I\!R$ is said to be *Lipschitz continuous* (on Y) or to be *Lipschitz* (on Y) with constant L_f provided that, for some nonnegative scalar L_f, one has

$$|f(y)-f(z)| \le L_f\|y-z\|_X$$

for all points y,z in Y. We shall say that f is Lipschitz (with constant L_f) near x if, for some $\varepsilon>0, f$ is Lipschitz continuous (with constant L_f) on the set $x+\varepsilon B_X$ (i.e. within an ε-neighborhood of x). f is said to be *locally Lipschitz* if f is Lipschitz near every interior point of Y. ∎

A function which is Lipschitz near a point need not be differentiable there, nor need it admit directional derivatives in the classical sense.

The generalized directional derivative

Definition 1.3 Let f be Lipschitz near a given point x, and let d be any vector in X. The *generalized directional derivative* of f at x in the direction d, denoted $f^0(x; d)$, is defined as follows:

$$f^0(x;d) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+td) - f(y)}{t},$$

where y is a vector in X and t is a positive scalar. ∎

Definition 1.4 If we use lim inf instead of lim sup in Definition 1.3, we define a *lower* generalized directional derivative and denote it by $f_0(x; d)$. ∎

f^0 has the following basic properties:

Proposition 1.2 (*cf.* Clarke 1983, Proposition 2.1.1, p.25)
Let f be as in Definition 1.3. Then

(a) *the function $d \longmapsto f^0(x; d)$ is positively homogeneous, is subadditive on X and satisfies*

$$|f^0(x;d)| \leq L_f \|d\|_X,$$

where $L_f \geq 0$ is the Lipschitz constant of f,

(b) $f^0(x; -d) = (-f)^0(x; d)$. ∎

The generalized gradient

Definition 1.5 Let f be Lipschitz continuous with constant L_f near x. Denote by X^* the dual space of X. The *generalized gradient* of f at x, denoted $\partial f(x)$, is the subset of X^* given by

$$\partial f(x) := \{\zeta' \in X^* : f^0(x;d) \geq \zeta' d \quad \text{for all} \quad d \in X\},$$

where $'$ denotes that ζ' is a dual object. In particular, if $X = \mathbb{R}^n$, then a dual object ζ' is the transpose of a vector $\zeta \in \mathbb{R}^n$. ∎

We denote by $\|\zeta'\|_*$ the norm in X^*:

$$\|\zeta'\|_* := \sup\{\zeta' d : d \in X, \|d\|_X \leq 1\}$$

and by B_* the open unit ball in X^*.

The following proposition summarizes some basic properties of the generalized gradient:

Proposition 1.3 (*cf.* Clarke 1983, Proposition 2.1.2, p.27)
Let f be Lipschitz with constant L_f near x. Then

(a) *$\partial f(x)$ is a nonempty, convex, weak*-compact subset of X^* and $\|\zeta'\|_* \leq L_f$ for every ζ' in $\partial f(x)$,*

(b) *for every d in X, one has*

$$\begin{aligned} f^0(x;d) &= \max_{\zeta' \in \partial f(x)} \zeta' d \\ f_0(x;d) &= \min_{\zeta' \in \partial f(x)} \zeta' d \end{aligned} \tag{1.13}$$

where $f_0(x;d)$ is the lower directional derivative of f (*cf.* Definition 1.4). ■

Relation to derivatives and subderivatives

If f is smooth, $\partial f(x)$ reduces to the conventional gradient.

If f is continuous and convex, the generalized gradient coincides with the subgradient of the convex analysis.

Basic calculus

Proposition 1.4 (Scalar Multiples, *cf.* Clarke 1983, Proposition 2.3.1, p.38)
Let f be Lipschitz near x. For any scalar s, one has

$$\partial(sf)(x) = s\partial f(x).$$

■

Proposition 1.5 (Local Extrema, *cf.* Clarke 1983, Proposition 2.3.2, p.38)
Let f be Lipschtiz near x. If f attains a local minimum or maximum at x, then $0 \in \partial f(x)$. ■

Proposition 1.6 (Finite Sums, *cf.* Clarke 1983, Proposition 2.3.3, p.38)
If $f_i (i = 1, 2, \ldots, n)$ is a finite family of functions each of which is Lipschitz near x, it follows easily that their sum $f = \sum f_i$ is also Lipschitz near x. The following inclusion holds:

$$\partial(\sum f_i)(x) \subset \sum \partial f_i(x),$$

and the equality holds if all but at most one of the functions f_i are continuously (Gâteaux) differentiable at x. ■

The following proposition is based on Clarke (1983), Theorem 2.3.9, p.42.

Proposition 1.7 (Chain Rule)
Let $f = g \circ h$, where $h : X \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ are given functions. The coordinate functions of h will be denoted $h_i (i = 1, 2, \ldots, n)$. We assume that each h_i is Lipschitz near x and g is Lipschitz near $h(x)$; this implies that f is Lipschitz near x. Let $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \partial g$. One has

$$\partial f(x) \subset \overline{co}\{\sum_{i=1}^{n} \alpha_i \zeta_i' : \zeta_i' \in \partial h_i(x), \alpha' \in \partial g(h(x))\}$$

(where $\overline{co}$ denotes the weak-closed convex hull).* ■

Regularity

It is often the case that calculus formulas for generalized gradients involve inclusions, such as in the finite sums formula. The addition of further hypotheses can serve to sharpen such rules by turning the inclusions to equalities. A class of functions that proves useful in this connection is one called "regular".

Definition 1.6 (Clarke) Regularity
f is said to be (Clarke) *regular* at x provided

(i) for all d, the usual one-sided directional derivative $f'(x;d)$ exists,

(ii) for all d, $f'(x;d) = f^0(x;d)$. ■

Partial generalized gradients

Let $X = X_1 \times X_2$, where X_1, X_2 are Banach spaces, and let $f : X \longrightarrow \mathbb{R}$ be Lipschitz near (x_1, x_2). We denote by $\partial_1 f(x_1, x_2)$ the (partial) generalized gradient of $f(\cdot\,, x_2)$ at x_1, and by $\partial_2 f(x_1, x_2)$ that of $f(x_1, \cdot)$ at x_2. The notation $f_1^0(x_1, x_2; v)$ will represent the generalized directional derivative at x_1 in the direction $v \in X_1$ of the function $f(\cdot, x_2)$. It is a fact that in general neither of the sets $\partial f(x_1, x_2)$ and $\partial_1 f(x_1, x_2) \times \partial_2 f(x_1, x_2)$ need be contained in the other. In some cases, however, we can show that equalities hold between these sets. The following proposition which gives such relations will be needed in the development of Theorem 2.2 and 4.1.

Throughout this thesis, we shall denote by $\|\cdot\|$ the Euclidean norm and in particular by $|\cdot|$ the Euclidean norm in $\mathbb{R}$, i.e. the absolute value.

Proposition 1.8 *If the function $f : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^n \times (0, \infty) \longrightarrow \mathbb{R}$ is such that $f(x_1, x_2) := F(x_1)$, $g(x_1, x_2) := x_2 G(x_1)$, where $x_2 > 0$, and $G(x_1) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous, then the following equalities hold:*

$$\begin{aligned} \partial f(x_1, x_2) &= \partial_1 f(x_1, x_2) \times \{0\} \\ &= \partial F(x_1) \times \{0\} \end{aligned} \tag{1.14}$$

$$\begin{aligned} \partial g(x_1, x_2) &= \partial_1 g(x_1, x_2) \times \partial_2 g(x_1, x_2) \\ &= x_2 \partial G(x_1) \times \{G(x_1)\}. \end{aligned} \tag{1.15}$$

Proof By Definition 1.3, for all $d_1 \in \mathbb{R}^n$, $d_2 \in \mathbb{R}^1$, we have

$$\begin{aligned} f_1^0(x_1, x_2; d_1) &:= \limsup_{\substack{\tilde{x}_1 \to x_1 \\ t \searrow 0}} \frac{f(\tilde{x}_1 + td_1, x_2) - f(\tilde{x}_1, x_2)}{t} \\ &= \limsup_{\substack{\tilde{x}_1 \to x_1 \\ t \searrow 0}} \frac{F(\tilde{x}_1 + td_1) - F(\tilde{x}_1)}{t} \end{aligned}$$

$$
\begin{aligned}
&= F^0(x_1; d_1) \\
f^0(x_1, x_2; d_1, d_2) \; &:= \limsup_{\substack{\tilde{x}_1 \to x_1 \\ \tilde{x}_2 \to x_2 \\ t \downarrow 0}} \frac{f(\tilde{x}_1 + td_1, \tilde{x}_2 + td_2) - f(\tilde{x}_1, \tilde{x}_2)}{t} \\
&= \limsup_{\substack{\tilde{x}_1 \to x_1 \\ t \searrow 0}} \frac{F(\tilde{x}_1 + td_1) - F(\tilde{x}_1)}{t} \\
&= F^0(x_1; d_1) \\
f_2^0(x_1, x_2; d_2) \; &:= \limsup_{\substack{\tilde{x}_2 \to x_2 \\ t \searrow 0}} \frac{f(x_1, \tilde{x}_2 + td_2) - f(x_1, \tilde{x}_2))}{t} \\
&= \limsup_{\substack{\tilde{x}_2 \to x_2 \\ t \searrow 0}} \frac{F(x_1) - F(x_1)}{t} \\
&= 0 \\
f^0(x_1, x_2; 0, d_2) \; &:= \limsup_{\substack{\tilde{x}_1 \to x_1 \\ \tilde{x}_2 \to x_2 \\ t \downarrow 0}} \frac{f(\tilde{x}_1 + t \cdot 0, \tilde{x}_2 + td_2) - f(\tilde{x}_1, \tilde{x}_2)}{t} \\
&= \limsup_{\substack{\tilde{x}_1 \to x_1 \\ t \searrow 0}} \frac{F(\tilde{x}_1) - F(\tilde{x}_1)}{t} \\
&= 0.
\end{aligned}
$$

That is,

$$
\begin{aligned}
f_1^0(x_1, x_2; d_1) &= f_0(x_1, x_2; d_1, d_2) = F^0(x_1; d_1) && (1.16) \\
f_2^0(x_1, x_2; d_2) &= f_0(x_1, x_2; 0, d_2) = 0 && (1.17)
\end{aligned}
$$

from which we now derive equality (1.14).

Indeed, for all $\eta' := ({\eta_1}', \eta_2) \in \partial f(x_1, x_2)$, we have by Definition 1.5 and equality (1.16),

$$
\begin{aligned}
{\eta_1}' d_1 &\leq f^0(x_1, x_2; d_1, 0) \\
&= F^0(x_1; d_1) \quad \text{for all } d_1 \in I\!R^n
\end{aligned}
$$

which impies that ${\eta_1}' \in \partial F(x_1)$ by Definition 1.5.

Similarly, we have by Definition 1.5 and equality (1.17)

$$\eta_2 d_2 \leq f^0(x_1, x_2; 0, d_2) = 0 \quad \text{for all } d_2 \in \mathbb{R}^1$$

which implies that $\eta_2 = 0$.

Conversely, for all $\eta' = (\eta_1', 0)$, where $\eta_1' \in \partial F(x_1)$, we have by Definition 1.5 and equality (1.16)

$$\begin{aligned} \eta_1' d_1 &\leq F^0(x_1; d_1) \\ &= f^0(x_1, x_2; d_1, d_2) \quad \text{for all } d_1 \in \mathbb{R}^n,\ d_2 \in \mathbb{R} \end{aligned}$$

which implies that $\eta' = (\eta_1', 0) \in \partial f(x_1, x_2)$.

Therefore, equality (1.14) holds.

Similarly for $g(x_1, x_2)$ by definition, we have

$$\begin{aligned} g_1^0(x_1, x_2; d_1) &:= \limsup_{\substack{\tilde{x}_1 \to x_1 \\ t \searrow 0}} \frac{x_2 G(\tilde{x}_1 + td_1) - x_2 G(\tilde{x}_1)}{t} \\ &= x_2 G^0(x_1; d_1) \qquad\qquad (\text{since } x_2 > 0) \\ g^0(x_1, x_2; d_1, 0) &:= \limsup_{\substack{\tilde{x}_1 \to x_1 \\ \tilde{x}_2 \to x_2 \\ t \downarrow 0}} \frac{\tilde{x}_2 G(\tilde{x}_1 + td_1) - \tilde{x}_2 G(\tilde{x}_1)}{t} \\ &= \inf_{\substack{\epsilon_1 > 0 \\ \epsilon_2 > 0 \\ \delta > 0}} \sup_{\substack{\|\tilde{x}_1 - x_1\| < \epsilon_1 \\ |\tilde{x}_2 - x_2| < \epsilon_2 \\ t \in (0, \delta)}} \tilde{x}_2 \frac{G(\tilde{x}_1 + td_1) - G(\tilde{x}_1)}{t} \\ &= \inf_{\substack{\epsilon_1 > 0 \\ \delta > 0}} \Big(\inf_{\epsilon_2 > 0} \sup_{|\tilde{x}_2 - x_2| < \epsilon_2} \tilde{x}_2 \Big) \sup_{\substack{\|\tilde{x}_1 - x_1\| < \epsilon_1 \\ t \in (0, \delta)}} \frac{G(\tilde{x}_1 + td_1) - G(\tilde{x}_1)}{t} \end{aligned} \tag{1.18}$$

$$\begin{aligned} &= x_2 \inf_{\substack{\epsilon_1 > 0 \\ \delta > 0}} \sup_{\substack{\|\tilde{x}_1 - x_1\| < \epsilon_1 \\ t \in (0, \delta)}} \frac{G(\tilde{x}_1 + td_1) - G(\tilde{x}_1)}{t} \\ &= x_2 G^0(x_1; d_1) \end{aligned} \tag{1.19}$$

where equality (1.18) follows from the fact that $x_2 > 0$.

$$
\begin{aligned}
g_2^0(x_1, x_2; d_2) \quad &:= \quad \limsup_{\substack{\tilde{x}_2 \to x_2 \\ t \searrow 0}} \frac{(\tilde{x}_2 + td_2)G(x_1) - \tilde{x}_2 G(x_1)}{t} \\
&= \quad G(x_1)d_2 \\
g^0(x_1, x_2; 0, d_2) \quad &:= \quad \limsup_{\substack{\tilde{x}_1 \to x_1 \\ \tilde{x}_2 \to x_2 \\ t \downarrow 0}} \frac{(\tilde{x}_2 + td_2)G(\tilde{x}_1) - \tilde{x}_2 G(\tilde{x}_1)}{t} \\
&= \quad G(x_1)d_2 \qquad \text{(since } G \text{ is continuous)} \\
g^0(x_1, x_2; d_1, d_2) \quad &:= \quad \limsup_{\substack{\tilde{x}_1 \to x_1 \\ \tilde{x}_2 \to x_2 \\ t \downarrow 0}} \frac{(\tilde{x}_2 + td_2)G(\tilde{x}_1 + td_1) - \tilde{x}_2 G(\tilde{x}_1)}{t} \\
&= \quad \limsup_{\substack{\tilde{x}_1 \to x_1 \\ \tilde{x}_2 \to x_2 \\ t \downarrow 0}} \frac{\tilde{x}_2 G(\tilde{x}_1 + td_1) - \tilde{x}_2 G(\tilde{x}_1) + td_2 G(\tilde{x}_1 + td_1)}{t} \\
&= \quad \limsup_{\substack{\tilde{x}_1 \to x_1 \\ \tilde{x}_3 \to x_2 \\ t \downarrow 0}} \frac{\tilde{x}_2 [G(\tilde{x}_1 + td_1) - G(\tilde{x}_1)]}{t} + G(x_1)d_2 \qquad (1.20) \\
&= \quad x_2 G^0(x_1; d_1) + G(x_1)d_2 \qquad \text{(since } x_2 > 0\text{)}
\end{aligned}
$$

where equality (1.20) follows from the fact that G is continuous. That is,

$$
\begin{aligned}
g_1^0(x_1, x_2; d_1) \quad &= \quad g^0(x_1, x_2; d_1, 0) = x_2 G^0(x_1; d_1) \qquad (1.21) \\
g_2^0(x_1, x_2; d_2) \quad &= \quad g^0(x_1, x_2; 0, d_2) = G(x_1)d_2 \qquad (1.22) \\
g^0(x_1, x_2; d_1, d_2) \quad &= \quad x_2 G^0(x_1; d_1) + G(x_1)d_2 \qquad (1.23)
\end{aligned}
$$

from which we now derive equality (1.15).

Indeed, for all $\eta' = ({\eta_1}', \eta_2) \in \partial g(x_1, x_2)$, we have by Definition 1.5 and equality (1.21)

$$
{\eta_1}' d_1 \leq g^0(x_1, x_2; d_1, 0) = x_2 G^0(x_1; d_1) \quad \text{for all } d_1 \in I\!R^n
$$

which implies that ${\eta_1}' \in x_2 \partial G(x_1)$.

Similarly, we have by equality (1.22)

$$\eta_2 d_2 \leq g^0(x_1, x_2; 0, d_2) = G(x_1)d_2 \quad \text{for all } d_2 \in I\!R^1$$

which implies $\eta_2 = G(x_2)$.

Conversely, for all $\eta' = (\eta_1', G(x_1))$, where $\eta_1' \in x_2\partial G(x_1)$, we have by equality (1.23),

$$\begin{aligned} \eta_1' d_1 + G(x_1)d_2 &\leq x_2 G^0(x_1; d_1) + G(x_1)d_2 \\ &= g^0(x_1, x_2; d_1, d_2) \qquad \text{for all } d_1 \in I\!R^n,\ d_2 \in I\!R^1 \end{aligned}$$

which implies $\eta' = (\eta_1', G(x_1)) \in \partial g(x_1, x_2)$.

Therefore. equality (1.15) holds.

■

The case in which X is finite-dimensional

Proposition 1.9 (see e.g. Clarke 1983, Theorem 2.5.1, p.63)
Let f be Lipschitz near x, suppose S is any set of Lebesgue measure 0 in $I\!R^n$ and Ω_f is the set of points at which f fails to be differentiable. Then

$$\partial f(x) = co\{\lim \nabla f(x_i) : x_i \to x, x_i \notin S, x_i \notin \Omega_f\}, \tag{1.24}$$

where coA is the convex hull of set A defined by

$$coA := \{\sum_{i=1}^{n} \lambda_i a_i : a_i \in A, \lambda_i \geq 0, \sum_i \lambda_i = 1\}.$$

■

Remark 1.1 The meaning of (1.24) is the following: consider any sequence x_i converging to x while avoiding both S and points at which f is not differentiable, and such that the sequence $\nabla f(x_i)$ converges; then the convex hull of all such limit points is $\partial f(x)$. ■

The following proposition is based on Clarke (1983), Proposition 2.1.5. p.29:

Proposition 1.10 *Let $f : I\!R^n \longrightarrow I\!R$ be Lipschitz near x, then*

(a) *the generalized gradient of f as a set-valued map $\partial f : I\!R^n \rightrightarrows I\!R^{n'}$ is closed; that is*

$$x_i \to x,\ \zeta_i' \in \partial f(x_i), \zeta_i' \to \zeta' \quad \textit{implies} \quad \zeta' \in \partial f(x),$$

(b) *∂f is upper semicontinuous at x, i.e. for any $\varepsilon > 0$, there is $\delta > 0$ such that, for all $y \in x + \delta B_X$*

$$\partial f(y) \subset \partial f(x) + \varepsilon B_n.$$

where B_X and B_n denote the unit balls of X and $I\!R^n$ respectively. ■

Generalized Jacobians

Definition 1.7 Now consider a vector-valued function $F : X \subset I\!R^n \longrightarrow I\!R^m$. F is said to be *Lipschitz continuous* (on X) with *constant* L_F provided that, for some nonnegative scalar L_F, one has

$$\|F(x) - F(y)\| \leq L_F \|x - y\|_X.$$

We shall say that $F(\cdot)$ is *Lipschitz* (with constant L_F) *near* x if, for some $\varepsilon > 0$, F is Lipschitz continuous on the set $x + \varepsilon B_n$. ■

We shall endow the space of $m \times n$ matrices $I\!R^{m\times n}$ with the *Frobenius* or *Euclidean norm*

$$\|A\|_{m\times n} := (\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}{}^2)^{\frac{1}{2}} \qquad \forall A = (a_{ij}) \in I\!R^{m\times n}.$$

Denote by $JF(y)$ for the usual $m \times n$ Jacobian matrix of partial derivatives whenever y is a point at which the necessary partial derivatives exist.

Definition 1.8 The *generalized Jacobian* of F at x, denoted $\partial F(x)$, is the convex hull of all $m \times n$ matrices Z obtained as the limit of a sequence of the form $JF(x_i)$, where $x_i \to x$ and $x_i \notin \Omega_F$, i.e. the set of points at which F fails to be differentiable. Hence

$$\partial F(x) := co\{\lim JF(x_i) : x_i \to x, x_i \notin \Omega_F\}.$$

■

The following proposition is based on Clarke (1983), Proposition 2.6.2, p.70.

Proposition 1.11 *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be Lipschitz near x with constant L_F. Then we have:*

(a) $\partial F(x)$ *is a nonempty convex compact subset of* $\mathbb{R}^{m\times n'}$.

(b) *The set-valued map* $\partial F : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m\times n'}$ *is closed.*

(c) *The set-valued map ∂F is upper semicontinuous at x, i.e. for any $\varepsilon > 0$, there is a $\delta > 0$, such that*

$$\partial F(y) \subset \partial F(x) + \varepsilon B_{m\times n} \quad \forall y \in x + \delta B_n.$$

(d) $\partial F(x) \subset L_F \bar{B}_{m\times n}$.

(e) *If $m = 1$, then the generalized gradient and the generalized Jacobian coincide.*

Here $B_{m\times n}$ denotes the unit ball of the $m \times n$ matrix space $\mathbb{R}^{m\times n}$ and $\bar{B}_{m\times n}$ denotes the closure of $B_{m\times n}$. ■

The follwing proposition is based on Clarke (1983), Proposition 2.2.1 and Proposition 2.6.5.

Proposition 1.12 (Jacobian Chain Rule)
Let $f = g \circ F$ be the composite of F and g, where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is Lipschitz near x and where $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ is Lipschitz near $F(x)$. Then f is Lipschitz near x and one has

$$\partial f(x) \subset co\{\partial g(F(x))\partial F(x)\}. \tag{1.25}$$

■

1.6.2 A nonsmooth maximum principle

Definition 1.9 Let S, Ω_t be the sets defined by

$$\begin{aligned} S &:= \{t : (t,x) \in \Omega \quad \text{for some } x \in I\!R^n\} \\ \Omega_t &:= \{x : (t,x) \in \Omega\}. \end{aligned}$$

Ω is called a *tube* provided the set S is an interval ($[a,b]$, say) and provided there exists a continuous function ω and a continuous positive function ε on $[a,b]$ such that $\Omega_t = \omega(t) + \varepsilon(t)B_n$ for $t \in [a,b]$. We call such a tube Ω *a tube on* $[a,b]$. ∎

Let $f : I\!R^n \times I\!R^m \longrightarrow I\!R^n$ and $f_0 : I\!R^n \times I\!R^m \longrightarrow I\!R$ be measurable and Lipschitz continuous in $x \in I\!R^n$ uniformly in $u \in U \subset I\!R^m$, with U compact. Let $F : I\!R^n \longrightarrow I\!R$ be locally Lipschitz and C_1 be a closed subset of $I\!R^n$.

Consider the following autonomous deterministic optimal control problem with fixed initial time and free terminal time:

$$\begin{aligned} (P_C) \qquad & \text{minimize} \quad \int_{t_0}^{t_1} f_0(x(t),u(t))dt + F(x(t_1)) \\ & \text{over the class of admissible pairs } (u(\cdot),x(\cdot)) \\ & \text{such that} \quad u : [t_0,t_1] \longrightarrow I\!R^m \text{ is measurable,} \\ & \qquad\qquad u(t) \in U \subset I\!R^m \;\; \forall t \in [t_0,t_1], \\ & \qquad\qquad (t,x(t)) \in \Omega \;\; \forall t \in [t_0,t_1], \\ & \qquad\qquad \dot{x}(t) = f(x(t),u(t)) \;\; \text{a.e. } t \in [t_0,t_1] \\ & \qquad\qquad x(t_0) := x_0 \quad x(t_1) \in C_1. \end{aligned}$$

Define the Hamiltonian function for (P_C)

$$H(x,u;p',r) := p'f(x,u) - rf_0(x,u)$$

for $x \in I\!R^n, u \in I\!R^m, p' \in I\!R^n$ and $r \in I\!R$.

The following nonsmooth *maximum principle* for problem (P_C) is based on Clarke (1983), Theorem 5.2.1, p.211 and Theorem 5.2.3, p.213.

Theorem 1.1 (Maximum Principle)
Let (x^*, u^*) solve the problem (P_C). Then there exists an absolutely continuous function

$$p' : [t_0, t_1] \longrightarrow I\!R^{n'}$$

and a nonegative scalar r, which can be taken as 0 or 1, such that:

(1) *the optimal control function u^* maximizes the Hamiltonian function:*

$$\begin{aligned} &H(x^*(t), u^*(t); p'(t), r) \\ &= \max_{u \in U} H(x^*(t), u; p'(t), r) = 0 \qquad a.e.\ t \in [t_0, t_1], \end{aligned}$$

(2) *the dual variable p' satisfies the adjoint equation in the form of the differential inclusion:*

$$-\dot{p}'(t) \in \partial_x H(x^*(t), u^*(t); p'(t), r) \quad a.e.\ t \in [t_0, t_1], \tag{1.26}$$

(3) *the system (1.26) is subject to the transversality condition:*

$$p'(t_1) \in -r\partial F(x(t_1)) - \rho \partial d_{C_1}(x(t_1)),$$

where ρ is a nonegative scalar and $d_{C_1}(y)$ is the distance function from the point y to the set C_1 defined by $d_{C_1}(y) := \inf\{\|z - y\| : \forall z \in C_1\}$,

(4) *the dual variable satisfies the nontrivality condition:*

$$\|p'\|_\infty + r > 0,$$

where $\|p'\|_\infty := \sup_{t \in [t_0, t_1]} \|p'(t)\|$ is the supremum norm. ■

Remark 1.2 For a fixed initial and terminal time and state problem, i.e. t_1 is fixed and $C_1 := \{c_1\}$, the optimal solution (x^*, u^*) satisfies all conditions of the maximum principle except the transversality condition. ■

1.7 Some Results on Differential Inclusions

In this section, we shall state some definitions and results on the differential inclusions which will be used in Chapter 4. The reader is referred to Aubin and Cellina (1984) for further details.

We start with some standard results on ordinary differential equations (see e.g. Aubin and Cellina 1984, p.119).

Proposition 1.13 (Gronwall Inequality, see e.g. Aubin and Cellina 1984, Proposition 1, p.119)
Let $\alpha,\ \phi : [a,b] \longrightarrow \mathbb{R}^n$ be continuous and $k : [a,b] \longrightarrow \mathbb{R}_+$ be an integrable function. Assume that

$$\phi(t) \leq \alpha(t) + \int_a^t k(s)\phi(s)ds, \quad t \in [a,b].$$

Then

$$\phi(t) \leq \alpha(t) + \int_a^t k(s)\alpha(s)e^{\int_s^t k(u)du}ds.$$

∎

Let $f : [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(t,x) \longmapsto f(t,x)$ be continuous and Lipschitz continuous in x with Lipschitz constant $k(t)$, i.e. there is a nonnegative integrable function $k : [a,b] \longrightarrow [0,\infty)$ such that

$$\|f(t,x) - f(t,y)\| \leq k(t)\|x - y\|.$$

Let $x(\cdot)$ and $y(\cdot)$ be absolute continuous solutions of the differential equation

$$\dot{x} = f(t,x) \tag{1.27}$$

with initial points x_0 and y_0 respectively. Setting $\phi(t) := x(t) - y(t)$ and $\alpha(t) := x_0 - y_0$ in the Gronwall inequality, it is easy to derive the following result on the continuous dependence of solutions on the initial data::

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\|e^{\int_0^t k(s)ds}.$$

For differential inclusions, Aubin and Cellina (1984) gave an analogue of Gronwall's inequality (*cf.* Aubin and Cellina 1984, Theorem 1, p.121) from which a result on continuous dependence of solutions was derived.

First we need the following definition, which is the analogue of that for a Lipschitz continuous function.

Definition 1.10 Let $F : I\!R^{n+1} \rightrightarrows I\!R^{n+1}, (t,x) \longmapsto F(t,x)$ be a set-valued map from $I\!R^{n+1}$ to subsets of $I\!R^{n+1}$. F is a *Lipschitzean map* with *constant* $k(t)$ if there is a nonnegative integrable function $k(t)$ such that

$$d(F(t,x), F(t,y)) \leq k(t)\|x-y\|,$$

where $d(C_1, C_2) := \inf\{\|c_1 - c_2\| : c_1 \in C_1, c_2 \in C_2\}$ is the *distance* between the sets C_1 and C_2. ■

Proposition 1.14 (*cf.* Aubin and Cellina 1984, Theorem 1, p.121)
Let x_0, y_0 be two initial points. Then with any solution $y(\cdot)$ of the differential inclusion

$$\dot{x}(t) \in F(t, x(t))$$

such that $y(0) = y_0$, we can associate a solution $x(\cdot)$ such that $x(0) = x_0$ and

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\| e^{\int_0^t k(s)ds}.$$

■

Definition 1.11 We denote by $\mathcal{F}_\infty(x_0)$ the set of *solutions of the differential inclusion:*

$$\dot{x}(t) \in F(t, x(t)) \quad x(0) := x_0 \tag{1.28}$$

on the interval $[0, \infty)$. ■

Definition 1.12 Define $m(K) := \{k \in K : \|k\| = \min_{l \in K} \|l\|\}$. The solutions of the differential inclusion

$$\dot{x}(t) = m(F(t, x(t)))$$

are called the *minimal norm trajectories* of the set-valued map F. ■

Now we are ready for the following proposition which is the analogue of the continuous dependence of solutions of differential equations on the initial data stated above.

Proposition 1.15 (see e.g. Aubin and Cellina, Theorem 1, p.121)
Let $F : [0,\infty) \times E \rightrightarrows \mathbb{R}^{n+1}$ *be an upper semicontinuous set-valued map from* $[0,\infty)\times \subset \mathbb{R}^{n+1}$ *to convex compact subsets* $F(t,x)$ *of* $\mathbb{R}^{n+1}$. *Suppose* $m(F(\cdot,\cdot))$ *remains in a compact subset of* $\mathbb{R}^n$. *Then for any* $x_0 \in E$ *there exists an absolutely continuous solution of* (1.28) *and* $\mathcal{F}_\infty(x_0)$ *is a compact set in* $\mathbb{R}^n$. ∎

Chapter 2

Necessary and Sufficient Optimality Conditions for a Control Problem with a Boundary Condition

2.1 Introduction

Let the state space E be a bounded connected set in $\mathbb{R}^n$ with interior E^0 and boundary ∂E.

The optimal control problem with a boundary condition we shall study is a *Bolza problem* formulated as follows:

$$
\begin{aligned}
(P_z)\quad & \text{minimize} \quad J(z,u(\cdot)) := \int_0^{t_*^u(z)} e^{-\Lambda_t^u(z)} f_0(x(t),u(t))dt + e^{-\Lambda_{t_*}^u(z)} F(x(t_*^u(z))) \\
& \text{over the class } \Omega_z \text{ of all } \textit{admissible pairs } (x(\cdot),u(\cdot)) \\
& \text{such that} \quad u : [0,t_*^u(z)) \to \mathbb{R}^m \text{ is measurable,} \\
& \qquad\qquad\quad u(t) \in U \subset \mathbb{R}^m \quad \forall t \in [0,t_*^u(z)), \\
& \qquad\qquad\quad \dot{x}(t) = f(x(t),u(t)) \quad \text{a.e. } t \in [0,t_*^u(z)), \\
& \qquad\qquad\quad x(0) := z \in E^0,
\end{aligned}
$$

where $\Lambda_t^u(z) := \int_0^t \bar{\lambda}(x(s), u(s))ds$ and $t_*^u(z)$ is called the *boundary hitting time* of the trajectory $x(t)$ corresponding to control u for initial state z defined by

$$t_*^u(z) := \inf\{t > 0 : \; x(t) \in \partial E\}. \tag{2.1}$$

In the case where the trajectory for initial state z never reaches the boundary of E, $t_*^u(z) = \inf \emptyset = \infty$ by convention. Where there is no confusion, we will simply use $t_*(z)$ or t_* instead of $t_*^u(z)$.

To make sure that J is well defined, we assume that $\underline{\lambda} := \inf_{x \in E^0, v \in U} \bar{\lambda}(x, v) > 0$. Thus even if $t_*^u(z)$ is ∞, the integral converges and in this case the term $e^{-\Lambda_{t_*}^u(z)} F(x(t_*^u(z)))$ vanishes (by virtue of the boundedness of the cost functions assumed below).

Define $\Gamma^* \subset \partial E$ as

$$\Gamma^* := \{z \subset \partial E : \exists t > 0, x \in E^0 \; (x(\cdot), u(\cdot)) \in \Omega_x \text{ s.t. } z = x(t)\} \tag{2.2}$$

the *active boundary* which flows may reach.

We also *assume* the following conditions hold:

(A2.1) the control set U is compact in $I\!R^m$,

(A2.2) $f : E \times U \to I\!R^n$ is continuous, is Lipschitz continuous with constant L_f in $x \in E$ uniformly in $u \in U$ and $|f(x,u)| \le M_f$ for all $(x,u) \in E \times U$,

(A2.3) $f_0 : E^0 \times U \to I\!R_+$ is continuous, is Lipschitz continuous with constant L_{f_0} in $x \in E^0$ uniformly in $u \in U$ and $|f_0(x,u)| \le M_{f_0}$ for all $(x,u) \in E^0 \times U$,

(A2.4) $\bar{\lambda} : E^0 \times U \to I\!R_+$ is continous, is Lipschitz continuous with constant $L_{\bar{\lambda}}$ in $x \in E^0$ uniformly in $u \in U$ and $|\bar{\lambda}(x,u)| \le M_{\bar{\lambda}}$ for all $(x,u) \in E^0 \times U$,

(A2.5) F is defined on ∂E and has a Lipschitz continuous extension to E such that $F : E \to I\!R_+$ is Lipschitz continuous with constant L_F and $|F(x)| \le M_F$ for all $x \in E$,

where M_f, M_{f_0}, $M_{\bar{\lambda}}$, M_F are nonnegative constants.

Remark 2.1 The Lipschitz continuity of $F(\cdot)$ on whole state space is only required to show that the value function is Lipschitz continuous. ■

We define the *value function* for (P_z) as a function $V : E \longrightarrow I\!R_+$ such that

$$\begin{aligned} V(z) &:= \inf_{(x(\cdot),u(\cdot))\in\Omega_z} J(z,u(\cdot)) \qquad\qquad \forall\, z \in E^0 \\ V(z) &:= F(z) \qquad\qquad \forall\, z \in \partial E. \end{aligned}$$

In the case where the value function $V(z)$ is smooth, in the spirit of classic optimal control theory (see e.g. Fleming and Rishel 1975), the following sufficient condition is also necessary.

Proposition 2.1 (Verification Theorem)
Let W be a C^1 solution of the Bellman–Hamilton–Jacobi (BHJ) equation

$$\min_{v\in U}\{\nabla W(z)f(z,v) - \bar{\lambda}(z,v)W(z) + f_0(z,v)\} = 0 \quad \forall z \in E^0 \tag{2.3}$$

with boundary condition

$$W(z) = F(z) \quad \forall z \in \partial E. \tag{2.4}$$

Let (x^*, u^*) be an admissible pair for P_{z_0} such that

$$\begin{aligned} \nabla W(x^*(t))f(x^*(t),u^*(t)) - \bar{\lambda}(x^*(t),u^*(t))W(x^*(t)) + f_0(x^*(t),u^*(t)) &= 0 \\ a.e.\ t \in [0, t_*^{u^*}(z_0)). & \end{aligned}$$

Then (x^*, u^*) is an optimal solution for P_{z_0}. ■

Remark 2.2 It should be noted that the boundary condition (2.4) is in fact only required for $z \in \Gamma^*$. ■

Unfortunately, however, the value function is generally not smooth, so that the above condition is sufficient but far from necessary. For standard control problems, several approaches have been taken in the control theory literature to cope with this difficulty. Boltyanski (1966) (see also Fleming and Rishel 1975) restrict the class of controls so that the value function becomes piecewise smooth. Instead of a C^1 solution, Vinter and Lewis (1978) characterize optimality through a sequence of C^1 subsolutions of the BHJ equation. Crandall and Lions (1983) consider a generalized solution of the BHJ equation called a *viscosity solution*. On the other hand, local conditions for optimality has been given using nonsmooth analysis by Clarke and Vinter (1983).

Using Clarke generalized gradients, the BHJ equation (2.3) can be generalized as follows:

$$\min_{\substack{\xi' \in \partial W(z) \\ v \in U}} \{\xi' f(z,v) - \bar{\lambda}(z,v)W(z) + f_0(z,v)\} = 0 \qquad \forall z \in E^0.$$

It is obvious that when W is C^1, the above equation is the BHJ equation (2.3). Therefore, the *generalized BHJ equation* is a refined sufficient condition. For standard optimal control problems, Clarke and Vinter (1983) showed that this condition is also necessary under a *calmness* assumption (*cf.* Clarke 1983).

In this thesis, we will take the generalized approach to the BHJ equation first introduced by Offin (1978). Under fairly general assumptions, we will provide a necessary and sufficient optimality condition of this kind using simple straightforward proofs.

2.2 Sufficiency of the Generalized BHJ Equation: The Verification Theorem

Theorem 2.1 *Let W be a locally Lipschitz solution of the generalized BHJ equation*

$$\min_{\substack{\xi' \in \partial W(z) \\ v \in U}} \{\xi' f(z,v) - \bar{\lambda}(z,v)W(z) + f_0(z,v)\} = 0 \qquad \forall z \in E^0 \tag{2.5}$$

with boundary condition

$$W(z) = F(z) \qquad \forall z \in \partial E \quad (\forall z \in \Gamma^*). \tag{2.6}$$

Let (x^*, u^*) *be an admissible pair for* (P_{z_0}). *If the following condition is satisfied:*

$$W(z_0) = \int_0^{t_*^{u^*}(z_0)} e^{-\Lambda_t^{u^*}(z_0)} f_0(x^*(t), u^*(t))dt + e^{-\Lambda_{t_*}^{u^*}(z_0)} F(x^*(t_*^{u^*}(z_0))), \tag{2.7}$$

then (x^*, u^*) *solves* (P_{z_0}).

Furthermore, if W *is regular in the sense of Clarke* (*cf.* Definition 1.6), *then the following condition implies condition* (2.7): *there exists* $\xi_t^{*\prime} \in \partial W(x^*(t))$ *for almost all* $t \in [0, t_*(z_0))$ *such that*

$$\xi_t^{*\prime} f(x^*(t), u^*(t)) - \bar{\lambda}(x^*(t), u^*(t))W(x^*(t)) + f_0(x^*(t), u^*(t)) = 0, \tag{2.8}$$

where $\partial W(x^*(t))$ *is the generalized gradient set of* W *at* $x^*(t)$.

Proof Suppose $(y(\cdot), u(\cdot))$ is any admissible pair for (P_{z_0}) and let $\theta(t) := W(y(t))$. Observe that θ is the composition of two Lipschitz continuous maps and hence is Lipschitz continuous. By the chain rule for the generalized gradient (Proposition 1.12), we have

$$\dot{\theta}(t) \in co\{\partial W(y(t))\partial y(t)\}. \tag{2.9}$$

That is,

$$\dot{\theta}(t) \in co\{\xi_t{}' f(y(t), u(t)) : \xi_t{}' \in \partial W(y(t))\} \qquad \text{a.e. } t \in [0, t_*(z_0)). \tag{2.10}$$

Since $W(\cdot)$ is a solution of (2.5), we have $\forall \xi_t' \in \partial W(y(t))$,

$$\xi_t' f(y(t), u(t)) - \bar{\lambda}(y(t), u(t))W(y(t)) + f_0(y(t), u(t)) \geq 0$$
$$t \in [0, t_*(z_0)).$$

Therefore the following inequality holds

$$\frac{dW(y(t))}{dt} - \bar{\lambda}(y(t), u(t))W(y(t)) + f_0(y(t), u(t)) \geq 0$$
$$\text{a.e. } t \in [0, t_*(z_0)). \tag{2.11}$$

Consequently,

$$\begin{aligned}
&\int_0^{t_*^u(z_0)} e^{-\Lambda_t^u(z_0)} f_0(y(t),u(t))dt + e^{-\Lambda_{t_*}^u(z_0)} F(y(t_*^u(z_0))) \\
=\; & \int_0^{t_*^u(z_0)} e^{-\Lambda_t^u(z_0)} f_0(y(t),u(t))dt + e^{-\Lambda_{t_*}^u(z_0)} W(y(t_*^u(z_0))) && (2.12) \\
=\; & \int_0^{t_*^u(z_0)} e^{-\Lambda_t^u(z_0)} f_0(y(t),u(t))dt + e^{-\Lambda_0^u(z_0)} W(y(0)) \\
& + \int_0^{t_*^u(z_0)} e^{-\Lambda_t^u(z_0)} [-\bar{\lambda}(y(t),u(t))W(y(t)) + \frac{dW(y(t))}{dt}]dt && (2.13) \\
=\; & \int_0^{t_*^u(z_0)} e^{-\Lambda_t^u(z_0)} [f_0(y(t),u(t)) - \bar{\lambda}(y(t),u(t))W(y(t)) \\
& + \frac{dW(y(t))}{dt}]dt + W(z_0) \\
\geq\; & W(z_0) && (2.14) \\
=\; & \int_0^{t_*^{u^*}(z_0)} e^{-\Lambda_t^{u^*}(z_0)} f_0(x^*(t),u^*(t))dt + e^{-\Lambda_{t_*}^{u^*}(z_0)} F(x^*(t_*^{u^*}(z_0))) && (2.15)
\end{aligned}$$

The equalities (2.12) and (2.15) hold by virtue of conditions (2.6) and (2.7) respectively. The equality (2.13) follows from an application of the fundamental theorem of calculus to $e^{-\Lambda_{(\cdot)}^u(z_0)}W(y(\cdot))$. The inequality (2.14) holds by virtue of inequality (2.11). Therefore, (x^*,u^*) is an optimal solution to the problem (P_{z_0}).

Now suppose W is (Clarke) regular (Definition 1.6). At t,where $\dot{y}(t)$ exists and $\dot{\theta}(t)$ exists, we have

$$\begin{aligned}
\dot{\theta}(t) &= \frac{dW(y(t))}{dt} \\
&:= \lim_{h\searrow 0} \frac{W(y(t-h)) - W(y(t))}{-h} \\
&= \lim_{h\searrow 0} \frac{W(y(t)-h\dot{y}(t)) - W(y(t))}{-h} && (2.16) \\
&= -W^0(y(t);-\dot{y}(t)) && (2.17) \\
&\leq -\xi_t'(-\dot{y}(t)) \\
&= \xi_t'\dot{y}(t) \\
&= \xi_t' f(y(t),u(t)) \qquad \forall \xi_t' \in \partial W(y(t)), && (2.18)
\end{aligned}$$

where equality (2.16) follows by the Lipschitz continuity of W and equality (2.17) holds by the regularity of W.

If condition (2.8) holds, by (2.18), we have for all $\xi_t^{*\prime} \in \partial W(x^*(t))$,

$$\begin{aligned} \frac{dW(x^*(t))}{dt} &\le \xi_t^{*\prime} f(x^*(t), u^*(t)) \\ &= \bar{\lambda}(x^*(t), u^*(t))W(x^*(t)) - f_0(x^*(t), u^*(t)) \\ & \qquad \text{a.e. } t \in [0, t_*^{u^*}(z_0)). \end{aligned} \tag{2.19}$$

Combining (2.19) with (2.11), we have

$$\frac{dW(x^*(t))}{dt} = \bar{\lambda}(x^*(t), u^*(t))W(x^*(t)) - f_0(x^*(t), u^*(t)) \tag{2.20}$$

$$\text{a.e. } t \in [0, t_*^{u^*}(z_0)). \tag{2.21}$$

Multiplying both sides of (2.21) by $e^{-\Lambda_t^{u^*}(z_0)}$ and integrating from 0 to $t_*^{u^*}(z_0)$, we obtain condition (2.7). ∎

2.3 Necessity of the Generalized BHJ Equation: The Value Function is a Solution.

Theorem 2.2 *Assume in addition to* (A2.1)–(A2.5) *that the following conditions are met:*

(A2.6) $\exists \alpha > 0$, *such that*

$$f(x,v) \cdot n(x) \ge \alpha > 0 \quad \forall x \in \partial E, \quad \forall v \in U, \tag{2.22}$$

where $n(x)$ is the unit outward normal to ∂E at the point x and $\cdot$ denotes the inner product,

(A2.7) $\underline{\lambda} := \inf_{x \in E^0, v \in U} \bar{\lambda}(x,v) > \bar{\lambda}^0_+$,

where $\xi_+ := \max\{\xi, 0\}$ and

$$\bar{\lambda}^0 = \sup_{\substack{y,z \in E^0 \\ v \in U}} \{(y-z)'(f(y,v) - f(z,v))/\|y-z\|^2\},$$

(A2.8) *for every* $z \in E^0$ *there exists an (ordinary, i.e. not a relaxed) optimal solution for* (P_z).

Then the value function V *is a Lipschitz continuous solution of the generalized BHJ equation*

$$\min_{\substack{\xi' \in \partial V(z) \\ v \in U}} \{\xi' f(z,v) - \bar{\lambda}(z,v)V(z) + f_0(z,v)\} = 0 \qquad \forall z \in E^0 \tag{2.23}$$

with boundary condition

$$V(z) = F(z) \qquad \forall\, z \in \partial E \ (\forall z \in \Gamma^*).$$

If (x^*, u^*) *is an optimal solution for* P_{z_0}*, then*

$$\min_{\xi_t' \in \partial V(x^*(t))} \{\xi_t' f(x^*(t), u^*(t)) - \bar{\lambda}(x^*(t), u^*(t))V(x^*(t)) + f_0(x^*(t), u^*(t))\} \;=\; 0 \qquad a.e.\ t \in [0, t_*(z_0)), \tag{2.24}$$

where $t_*(z_0)$ *can be equal to* ∞. ■

Remark 2.3 Condition (A2.6) postulates that when the controlled trajectories get sufficiently close to the boundary, they must hit the boundary by virtue of the requirement (2.22) that on the boundary the corresponding field element makes an acute angle with the unit outward normal. Therefore, this condition could be replaced by any suitable condition: e.g. $\exists$ positive γ and a such that

$$\frac{d}{dt} d_{\partial E}(y(t)) \leq -\gamma \ \text{ for all } t \quad s.t. \quad d_{\Gamma^*}(y(t)) \leq a, \tag{2.25}$$

where $d_C(x)$ is the distance from a point x to a set C and Γ^* is the active boundary defined by (2.2).

It is obvious that condition (A2.7) is implied by the following condition:

$$\exists r > 0 \text{ s.t. } \bar{\lambda}(x,v) \geq L_f + r \qquad \forall x \in \partial E \ \ \forall v \in U, \tag{2.26}$$

where L_f is the Lipschitz constant of function $f(\cdot, v)$.

Condition (A2.8) is implied by the condition that the set $\{(f(x,u)', f_0(x,u))' : u \in U\}$ is convex for any $x \in E^0$ (*cf.* §3.5). ■

Proof Technique Before studying the original Bolza problem we study the corresponding Mayer problem. We show that the value function for the Mayer problem satisfies the generalized BHJ equation. We then transform the original Bolza problem to the Mayer problem and investigate the relationship between value functions. Using this information, we derive the generalized BHJ equation for the original problem from the one for the corresponding Mayer problem. ∎

2.3.1 Lipschitz continuity of the value function

The following proposition is due to Gonzalez and Rofman (1978) (see also Lions 1982, Proposition 1.4, p.39) and the proof technique is based on Gonzalez (1980).

Proposition 2.2 *Under assumptions* (A2.1)-(A2.7), *the value function* $V(z)$ *for* (P_z) *is Lipschitz continuous on* E^0.

Proof Let $(\phi_{(\cdot)}(x), u(\cdot))$ and $(\phi_{(\cdot)}(z), u(\cdot))$ be two admissible pairs for (P_x) and (P_z) respectively and let τ_x, τ_z be their corresponding boundary hitting times respectively (see (2.1)). Define

$$\begin{aligned}
\eta_t(x) &:= \exp[-\int_0^t \bar{\lambda}(\phi_s(x), u(s))ds] \\
\eta_t(z) &:= \exp[-\int_0^t \bar{\lambda}(\phi_s(z), u(s))ds].
\end{aligned}$$

We consider the difference

$$|\Delta J| = |J(x, u(\cdot)) - J(z, u(\cdot))|$$

separately for each of three cases: where $\tau_x = \tau_z = \infty$, where $\tau_x = \tau_z = \tau \neq \infty$ and where $\tau_x \neq \tau_z$.

Case 1. $\tau_x = \tau_z = \infty$

In this case, the difference

$$\begin{aligned}
|\Delta J| &:= \Delta_1 \\
&= |\int_0^\infty \eta_t(x) f_0(\phi_t(x), u(t))dt - \int_0^\infty \eta_t(z) f_0(\phi_t(z), u(t))dt| \\
&\leq \int_0^\infty [\eta_t(x)|f_0(\phi_t(x), u(t)) - f_0(\phi_t(z), u(t))| \\
&\quad + |\eta_t(x) - \eta_t(z)||f_0(\phi_t(z), u(t))|]dt \\
&\leq \int_0^\infty \{e^{-\bar{\lambda}t} L_{f_0} \|\phi_t(x) - \phi_t(z)\| + M_{f_0}|\eta_t(x) - \eta_t(z)|\}dt. \qquad (2.27)
\end{aligned}$$

Condition (A2.7) implies the following inequality:

$$\bar{\lambda}^0 \|x - z\|^2 \geq (x - z)'(f(x, v) - f(z, v)),$$

from which we have

$$\begin{aligned}
\frac{d}{dt}\|\phi_t(x) - \phi_t(z)\|^2 &= 2(\phi_t(x) - \phi_t(z))'(f(\phi_t(x), u(t)) - f(\phi_t(z), u(t))) \\
&\leq 2\bar{\lambda}^0 \|\phi_t(x) - \phi_t(z)\|^2.
\end{aligned}$$

Solving this differential inequality, we have

$$\ln \frac{\|\phi_t(x) - \phi_t(z)\|^2}{\|\phi_0(x) - \phi_0(z)\|^2} \leq 2\bar{\lambda}^0 t.$$

That is,

$$\|\phi_t(x) - \phi_t(z)\|^2 \leq e^{2\bar{\lambda}^0 t}\|x - z\|^2,$$

or

$$\|\phi_t(x) - \phi_t(z)\| \leq e^{\bar{\lambda}^0 t}\|x - z\|. \qquad (2.28)$$

On the other hand,

$$\begin{aligned}
\frac{d}{dt}[\eta_t(x) - \eta_t(z)] &= -\eta_t(x)\bar{\lambda}(\phi_t(x), u(t)) + \eta_t(z)\bar{\lambda}(\phi_t(z), u(t)) \\
&= \bar{\lambda}(\phi_t(x), u(t))[\eta_t(z) - \eta_t(x)] \\
&\quad + \eta_t(z)[\bar{\lambda}(\phi_t(z), u(t)) - \bar{\lambda}(\phi_t(x), u(t))].
\end{aligned}$$

Solving this differential equation, since $\eta_0(x) = \eta_0(z) = 1$, we have

$$
\begin{aligned}
&\eta_t(x) - \eta_t(z) \\
&= e^{-\int_0^t \bar{\lambda}(\phi_l(x)),u(l))dl}\{\int_0^t \eta_s(z)[\bar{\lambda}(\phi_s(z),u(s)) - \bar{\lambda}(\phi_s(x),u(s))]e^{\int_0^s \bar{\lambda}(\phi_l(x),u(l))dl}ds\} \\
&= \int_0^t e^{-\int_0^s \bar{\lambda}(\phi_l(z),u(l))dl}[\bar{\lambda}(\phi_s(z),u(s)) - \bar{\lambda}(\phi_s(x),u(s))]e^{-\int_s^t \bar{\lambda}(\phi_l(x),u(l))dl}ds.
\end{aligned}
$$

Therefore, by virtue of inequality (2.28), we have

$$
\begin{aligned}
|\eta_t(x) - \eta_t(z)| &\leq \int_0^t e^{-\underline{\lambda} s} L_{\bar{\lambda}} \|\phi_s(x) - \phi_s(z)\| e^{-\underline{\lambda}(t-s)} ds \\
&\leq e^{-\underline{\lambda} t} L_{\bar{\lambda}} \int_0^t e^{\bar{\lambda}^0 s} \|x - z\| ds.
\end{aligned}
$$

If $\bar{\lambda}^0 = 0$, the last inequality becomes

$$
|\eta_t(x) - \eta_t(z)| \leq L_{\bar{\lambda}} t e^{-\underline{\lambda} t} \|x - z\|; \tag{2.29}
$$

If $\bar{\lambda}^0 \neq 0$, the last inequality becomes

$$
|\eta_t(x) - \eta_t(z)| \leq L_{\bar{\lambda}}/\bar{\lambda}^0 e^{-\underline{\lambda} t}[e^{\bar{\lambda}^0 t} - 1]\|x - z\|. \tag{2.30}
$$

Substituting (2.28) and (2.29) or (2.30) into (2.27), if $\bar{\lambda}^0 = 0$, we have

$$
\begin{aligned}
|\Delta_1| &\leq \int_0^\infty e^{-\underline{\lambda} t} L_{f_0} \|x - z\| dt + M_{f_0} \int_0^\infty L_{\bar{\lambda}} t e^{-\underline{\lambda} t} \|x - z\| dt \\
&= (L_{f_0}/\underline{\lambda} + M_{f_0} L_{\bar{\lambda}}/\underline{\lambda}^2)\|x - z\|;
\end{aligned} \tag{2.31}
$$

if $\bar{\lambda}^0 \neq 0$, we have

$$
\begin{aligned}
|\Delta_1| &\leq \int_0^\infty e^{-\underline{\lambda} t} L_{f_0} e^{\bar{\lambda}^0 t} \|x - z\| dt + M_{f_0} \int_0^\infty L_{\bar{\lambda}}/\bar{\lambda}^0 \cdot e^{-\underline{\lambda} t}[e^{\bar{\lambda}^0 t} - 1] dt \|x - z\| \\
&= [L_{f_0}/(\underline{\lambda} - \bar{\lambda}^0) + M_{f_0} L_{\bar{\lambda}}/\bar{\lambda}^0[(\underline{\lambda} - \bar{\lambda}^0)^{-1} - \underline{\lambda}^{-1})]\|x - z\|.
\end{aligned} \tag{2.32}
$$

Case 2 $\tau_x = \tau_z = \tau \neq \infty$.

In this case, the difference

$$\begin{aligned} |\Delta J| &\leq \int_0^{\tau} |\eta_s(x) f_0(\phi_s(x), u(s)) - \eta_s(z) f_0(\phi_s(z), u(s))| ds \\ &\quad + |\eta_\tau(x) F(\phi_\tau(x)) - \eta_\tau(z) F(\phi_\tau(z))| \\ &\leq \Delta_1 + \eta_\tau(x) |F(\phi_\tau(x)) - F(\phi_\tau(z))| + |\eta_\tau(x) - \eta_\tau(z)| F(\phi_\tau(z)) \\ &\leq \Delta_1 + e^{-\underline{\lambda}\tau} L_F \|\phi_\tau(x) - \phi_\tau(z)\| + M_F |\eta_\tau(x) - \eta_\tau(z)| \end{aligned} \tag{2.33}$$

Substituting (2.28) and (2.29) or (2.30) into (2.33), if $\bar{\lambda}^0 = 0$, we have

$$\begin{aligned} |\Delta J| &\leq \Delta_1 + e^{-\underline{\lambda}\tau} L_F \|x - z\| + M_F L_{\bar{\lambda}} \tau e^{-\underline{\lambda}\tau} \|x - z\| \\ &\leq (L_{f_0}/\underline{\lambda} + M_{f_0} L_{\bar{\lambda}}/\underline{\lambda}^2 + L_F + M_F L_{\bar{\lambda}} M_1) \|x - z\|, \end{aligned}$$

where the last inequality follows from ineqality (2.31) and the fact that $e^{-\underline{\lambda}\tau} \leq 1$ and there exists a constant $M_1 \geq 0$ for all $t \geq 0$ such that $te^{-\underline{\lambda}t} \leq M_1$; if $\bar{\lambda}^0 \neq 0$, we have

$$\begin{aligned} |\Delta J| &\leq \Delta_1 + e^{-\underline{\lambda}\tau} L_F e^{\bar{\lambda}^0 \tau} \|x - z\| + M_F L_{\bar{\lambda}}/\bar{\lambda}^0 e^{-\underline{\lambda}\tau} (e^{\bar{\lambda}^0 \tau} - 1) \|x - z\| \\ &\leq [L_{f_0}/(\underline{\lambda} - \bar{\lambda}^0) + M_{f_0} L_{\bar{\lambda}}/\bar{\lambda}^0 [(\underline{\lambda} - \bar{\lambda}^0)^{-1} - \underline{\lambda}^{-1}] + L_F + M_F L_{\bar{\lambda}} M_2] \|x - z\|, \end{aligned}$$

where the last inequality follows from inequality (2.32) and the fact that $e^{(\bar{\lambda}^0 - \underline{\lambda})t} \leq 1$ for all $t \geq 0$ and $\bar{\lambda}^{0^{-1}} e^{-\underline{\lambda}t}(e^{\bar{\lambda}^0 t} - 1)$ is bounded by a constant $M_2 \geq 0$.

Case 3 $\tau_x \neq \tau_z$. Because of the symmetry, without loss of generality we may assume that $\tau_x < \tau_z$.

Case 3a. Suppose $\tau_x < \tau_z \neq \infty$. In this case, the difference

$$\begin{aligned} |\Delta J| &\leq \int_0^{\tau_x} |\eta_t(x) f_0(\phi_t(x), u(t)) - \eta_t(z) f_0(\phi_t(z), u(t))| dt \\ &\quad + \int_{\tau_x}^{\tau_z} |\eta_t(z) f_0(\phi_t(z), u(t))| dt + |\eta_{\tau_x}(x) F(\phi_{\tau_x}(x)) - \eta_{\tau_z}(z) F(\phi_{\tau_z}(z))| \\ &\leq \Delta_1 + M_{f_0} e^{-\underline{\lambda}\tau_x} (\tau_z - \tau_x) \\ &\quad + |\eta_{\tau_x}(x) F(\phi_{\tau_x}(x)) - \eta_{\tau_z}(z) F(\phi_{\tau_z}(z))|. \end{aligned} \tag{2.34}$$

Suppose

$$\|x - z\| \leq \frac{ae^{-\bar{\lambda}^0 \tau_x}}{1 + M_f/\gamma}. \tag{2.35}$$

where γ and a are the constants in condition (2.25) whose existence is assured by (A2.6). Then it can be shown that

$$\tau_z - \tau_x \leq e^{\bar{\lambda}^0 \tau_x} \|x - z\|/\gamma. \tag{2.36}$$

Indeed, for all $t \geq \tau_x$, by virtue of inequality (2.28) and the boundedness of the vector field f, we have

$$\begin{aligned} \|\phi_t(z) - \phi_{\tau_x}(x)\| &\leq \|\phi_t(z) - \phi_{\tau_x}(z)\| + \|\phi_{\tau_x}(z) - \phi_{\tau_x}(x)\| \\ &\leq M_f(t - \tau_x) + e^{\bar{\lambda}^0 \tau_x} \|x - z\|. \end{aligned}$$

Define $\sigma := e^{\bar{\lambda}^0 \tau_x} \|x - z\|/\gamma$. By assumption (2.35), we have for all $\tau_x \leq t \leq \tau_x + \sigma$,

$$\begin{aligned} \|\phi_t(z) - \phi_{\tau_x}(x)\| &\leq M_f \sigma + \frac{a}{1 + M_f/\gamma} \\ &\leq M_f \frac{a/\gamma}{1 + M_f/\gamma} + \frac{a}{1 + M_f/\gamma} = a, \end{aligned}$$

which implies $d_{\Gamma^*}(\phi_t(z)) \leq a \quad \forall t \in [\tau_x, \tau_x + \sigma]$. Therefore $\frac{d}{dt} d_{\partial E}(\phi_t(z)) \leq -\gamma \quad \forall \ t \in [\tau_x, \tau_x + \sigma]$ by virtue of inequality (2.25) and this implies the following inequality:

$$\begin{aligned} d_{\partial E}(\phi_t(z)) &\leq d_{\partial E}(\phi_{\tau_x}(z)) + (-\gamma)(t - \tau_x) \\ &\leq \|\phi_{\tau_x}(z) - \phi_{\tau_x}(x)\| - \gamma(t - \tau_x) \\ &\leq e^{\bar{\lambda}^0 \tau_x} \|x - z\| - \gamma(t - \tau_x), \end{aligned} \tag{2.37}$$

where the last inequality follows from inequality (2.28). If we suppose that $\phi_t(z) \in E^0$ for all $t \in [\tau_x, \tau_x + \sigma)$, we obtain by the last inequality that

$$d_{\partial E}(\phi_{\tau_x + \sigma}(z)) \leq e^{\bar{\lambda}^0 \tau_x} \|x - z\| - \gamma e^{\bar{\lambda}^0 \tau_x} \|x - z\|/\gamma = 0,$$

i.e. $\phi_{\tau_x + \sigma}(z) \in \partial E$. Therefore, the trajectory $\phi_{(\cdot)}(z)$ hits the boundary ∂E at $\tau_x + \sigma$. Therefore $\tau_z \leq \tau_x + \sigma := \tau_x + e^{\bar{\lambda}^0 \tau_x} \|x - z\|/\gamma$. Inequality (2.36) follows from the last inequality and the assumption that $\tau_x < \tau_z$.

Since

$$\begin{aligned}
|\eta_{\tau_x}(x) - \eta_{\tau_z}(x)| &= |\int_{\tau_x}^{\tau_z} \eta_s(x)\bar{\lambda}(\phi_s(x), u(s))ds| \\
&\leq e^{-\underline{\lambda}\tau_x} \int_{\tau_x}^{\tau_z} |\bar{\lambda}(\phi_s(x), u(s))|ds \\
&\leq e^{-\underline{\lambda}\tau_x} M_{\bar{\lambda}}(\tau_z - \tau_x), \qquad (2.38)
\end{aligned}$$

we have that

$$\begin{aligned}
&|\eta_{\tau_x}(x)F(\phi_{\tau_x}(x)) - \eta_{\tau_z}(z)F(\phi_{\tau_z}(z))| \\
&\leq \eta_{\tau_x}(x)|F(\phi_{\tau_x}(x)) - F(\phi_{\tau_x}(z))| + |\eta_{\tau_x}(x) - \eta_{\tau_x}(z)||F(\phi_{\tau_x}(z))| \\
&\quad + \eta_{\tau_x}(z)|F(\phi_{\tau_x}(z)) - F(\phi_{\tau_z}(z))| + F(\phi_{\tau_z}(z))|\eta_{\tau_x}(z) - \eta_{\tau_z}(z)| \\
&\leq e^{-\underline{\lambda}\tau_x} L_F e^{\bar{\lambda}^0 \tau_x}\|x - z\| + |\eta_{\tau_x}(x) - \eta_{\tau_x}(z)|M_F \\
&\quad + e^{-\underline{\lambda}\tau_x} L_F M_f(\tau_z - \tau_x) + M_F e^{-\underline{\lambda}\tau_x} M_{\bar{\lambda}}(\tau_z - \tau_x) \\
&\leq L_F\|x - z\| + M_F|\eta_{\tau_x}(x) - \eta_{\tau_x}(z)| \\
&\quad + (L_F M_f + M_F M_{\bar{\lambda}})\|x - z\|/\gamma, \qquad (2.39)
\end{aligned}$$

where the last inequality follows from inequality (2.36) and the fact that $\underline{\lambda} \geq \bar{\lambda}^0$. If $\bar{\lambda}^0 = 0$, by virtue of inequality (2.29), (2.39) becomes

$$\begin{aligned}
&|\eta_{\tau_x}(x)F(\phi_{\tau_x}(x)) - \eta_{\tau_z}(z)F(\phi_{\tau_z}(z))| \\
&\leq L_F\|x - z\| + M_F L_{\bar{\lambda}} \tau_x e^{-\underline{\lambda}\tau_x}\|x - z\| \\
&\quad + (L_F M_f + M_F M_{\bar{\lambda}})\|x - z\|/\gamma \qquad (2.40) \\
&= [L_F + M_F L_{\bar{\lambda}} + (L_F M_f + M_F M_{\bar{\lambda}})/\gamma]\|x - z\| \qquad (2.41)
\end{aligned}$$

Therefore, if $\bar{\lambda}^0 = 0$, by virtue of inequality (2.31) and (2.36), inequality (2.34) becomes

$$\begin{aligned}
|\Delta J| \leq\ & \Delta_1 + M_{f_0} e^{-\underline{\lambda}\tau_x}(\tau_z - \tau_x) \\
& + [L_F + M_F L_{\bar{\lambda}} + (L_F M_f + M_F M_{\bar{\lambda}})/\gamma)]\|x - z\|
\end{aligned}$$

$$\leq [L_{f_0}/\underline{\lambda} + M_{f_0} L_{\bar{\lambda}}/\underline{\lambda}^2 + M_{f_0}/\gamma$$
$$+L_F + M_F L_{\bar{\lambda}} + (L_F M_f + M_F M_{\bar{\lambda}})/\gamma]\|x - z\|$$

If $\bar{\lambda}^0 \neq 0$, by virtue of inequality (2.30), (2.39) becomes

$$\begin{aligned}
& |\eta_{\tau_x}(x)F(\phi_{\tau_x}(x)) - \eta_{\tau_z}(z)F(\phi_{\tau_z}(z))| \\
\leq\ & L_F\|x-z\| + M_F L_{\bar{\lambda}}/\bar{\lambda}^0 \cdot e^{-\underline{\lambda}\tau_x}[e^{\bar{\lambda}^0\tau_x} - 1]\|x-z\| \\
& +(L_F M_f + M_F M_{\bar{\lambda}})/\gamma \cdot \|x-z\| \\
=\ & [L_F + M_F L_{\bar{\lambda}} M_2 + (L_F M_f + M_F M_{\bar{\lambda}})/\gamma]\|x-z\|,
\end{aligned} \tag{2.42}$$

where the last equality follows from $\underline{\lambda}^{-1}e^{-\underline{\lambda}t}(e^{\bar{\lambda}^0 t} - 1) \leq M_2$. Therefore, if $\bar{\lambda}^0 \neq 0$, by virtue of inequality (2.32) and (2.36), inequality (2.34) becomes

$$\begin{aligned}
|\Delta J| \leq\ & \Delta_1 + M_{f_0} e^{-\underline{\lambda}\tau_x}(\tau_z - \tau_x) \\
& +[L_F + M_F L_{\bar{\lambda}} M_2 + (L_F M_f + M_F M_{\bar{\lambda}})\frac{1}{\gamma}]\|x-z\| \\
\leq\ & [L_{f_0}/(\underline{\lambda} - \bar{\lambda}^0) + M_{f_0} L_{\bar{\lambda}}/\bar{\lambda}^0((\underline{\lambda} - \bar{\lambda}^0)^{-1} - \underline{\lambda}^{-1}) + M_{f_0}/\gamma \\
& +L_F + M_F L_{\bar{\lambda}} M_2 + (L_F M_f + M_F M_{\bar{\lambda}})/\gamma]\|x-z\|.
\end{aligned}$$

Now suppose inequality (2.36) fails to hold, i.e. $\tau_z - \tau_x > e^{\bar{\lambda}^0\tau_x}\|x-z\|/\gamma$. Then we have

$$\|x - z\| > \frac{ae^{-\bar{\lambda}^0\tau_x}}{1 + M_f/\gamma}, \tag{2.43}$$

because (2.35) implies (2.36). In this case the difference $|\Delta J|$ is bounded in the following way

$$\begin{aligned}
|\Delta J| \leq\ & \Delta_1 + \int_{\tau_x}^{\tau_z} \eta_t(z)|f_0(\phi_t(z), u(t))|dt + |\eta_{\tau_x}(x)F(\phi_{\tau_x}(x)) - \eta_{\tau_z}(z)F(\phi_{\tau_z}(z))| \\
\leq\ & \Delta_1 + M_{f_0}\int_{\tau_x}^{\infty} e^{-\underline{\lambda}s}ds + 2M_F e^{-\underline{\lambda}\tau_x} \\
\leq\ & \Delta_1 + M_{f_0}e^{-\underline{\lambda}\tau_x}/\underline{\lambda} + 2M_F e^{-\underline{\lambda}\tau_x}.
\end{aligned}$$

But by virtue of (2.43)

$$e^{-\underline{\lambda}\tau_x} \leq e^{-\bar{\lambda}^0\tau_x} < \frac{1 + M_f/\gamma}{a}\|x - z\|.$$

Case 3b. Now suppose $\tau_z = +\infty$, we have

$$\tau_z - \tau_x > e^{\bar{\lambda}^0 \tau_x} \|x - z\| / \gamma$$

which implies

$$\|x - z\| > \frac{a e^{-\bar{\lambda}^0 \tau_x}}{1 + M_f/\gamma}.$$

Therefore,

$$\begin{aligned} |\Delta J| &\leq \Delta_1 + \int_{\tau_x}^{\infty} \eta_t(z) |f_0(\phi_t(z), u(t))| dt + \eta_{\tau_x}(x) |F(\phi_{\tau_x}(x))| \\ &\leq \Delta_1 + M_{f_0} e^{-\underline{\lambda}\tau_x} / \underline{\lambda} + M_F e^{-\underline{\lambda}\tau_x}. \end{aligned}$$

But $e^{-\underline{\lambda}\tau_x} \leq e^{-\bar{\lambda}^0 \tau_x} < \frac{1+M_f/\gamma}{a} \|x - z\|$.

In all cases, there exists a constant L_J such that for all admissible controls $u(\cdot)$, $V(x) \leq J(x, u(\cdot)) \leq J(z, u(\cdot)) + L_J \|x - z\|$. Hence $V(x) \leq \inf_{u(\cdot)} J(z, u(\cdot)) + L_J \|x - z\|$. Therefore, $V(x) - V(z) \leq L_J \|x - z\|$. Since x and z are arbitrary, we have the reverse inequality, i.e.

$$|V(x) - V(z)| \leq L_J \|x - z\|.$$

■

2.3.2 A Mayer problem

We define the Mayer problem of interest to us as follows:

$$\begin{array}{lll} (P_{t_0, x_0}) & \text{minimize} & \Phi(t_*, x(t_*)) \\ & \multicolumn{2}{l}{\text{over the class } \mathcal{F} \text{ of all admissible pairs } (x(\cdot), u(\cdot)) \text{ for } (P_{t_0, x_0})} \\ & \text{such that} & u : [t_0, t_*) \longrightarrow I\!R^m \text{ is measurable,} \\ & & u(t) \in U \subset I\!R^m \quad \forall t \in [t_0, t_*), \\ & & \dot{x}(t) = f(t, x(t), u(t)) \qquad \text{a.e. } t \in [t_0, t_*), \\ & & x(t_0) = x_0 \ \in E^0, \end{array}$$

where $t_* := \inf\{t > 0,\ x(t) \in \partial E\} \leq \infty$ is the boundary hitting time. When $t_* = \infty$, we agree that $\Phi(t_*, x(t_*)) := \lim_{t \to \infty} \Phi(t, x(t))$ (which exists by assumption (A2.10) below).

Assume that the following conditions hold:

(A2.9) the control set U is compact in $I\!R^m$,

(A2.10) $\Phi : I\!R \times E^0 \longrightarrow I\!R_+$ is locally Lipschitz, and for every admissible pair such that $t_* = \infty$ $\lim_{t\to\infty} \Phi(t, x(t))$ exists and is finite,

(A2.11) $f : I\!R \times E^0 \times U \longrightarrow I\!R^n$ is continuous and locally Lipschitz in $(t, x) \in I\!R \times E^0$ uniformly in $u \in U$.

Define the *value function* for $(P_{t_0 x_0})$ as the function $V : [t_0, \infty) \times E^0 \longrightarrow I\!R_+$ given by

$$\begin{aligned} V(s,y) &:= \inf_{(x(\cdot),u(\cdot))\in\mathcal{F}} \Phi(t_*, x(t_*)) \qquad \forall (s,y) \in [t_0,\infty) \times E^0 \\ V(s,y) &:= \Phi(s,y) \qquad\qquad\qquad\qquad \forall (s,y) \in [t_0,\infty) \times \partial E. \end{aligned}$$

Then we have the following lemmas and theorem.

Lemma 2.1 *The value function evaluated along any trajectory corresponding to a control is a nondecreasing function of time. More specifically, for $t_0 \le \tau_1 \le \tau_2 < t_*$ $V(\tau_1, x(\tau_1)) \le V(\tau_2, x(\tau_2))$.* ∎

Lemma 2.2 *The value function evaluated along any optimal trajectory is constant for $t_0 \le t < t_*$.* ∎

We skip the proofs of Lemmas 2.1 and 2.2 since they are exactly the same as the proofs of Theorems 3.1 and 3.2 of Fleming and Rishel (1975).

Theorem 2.3 *Suppose that the value function is locally Lipschitz on $[t_0, \infty) \times E^0$. Let $(s, y) \in (t_0, \infty) \times E^0$. Then $V(s, y)$ satisfies the BHJ partial differential inequality in the following form:*

$$\alpha + \beta' f(s, y, v) \ge 0 \qquad \forall (\alpha, \beta') \in \partial V(s, y), \qquad v \in U. \tag{2.44}$$

If there exists an optimal solution (x^*, u^*) for any $(P_{s,y})$, then the value function $V(s,y)$ is a solution of the following generalized BHJ equation

$$\min_{\substack{(\alpha,\beta')\in\partial V(s,y)\\ v\in U}} \{\alpha + \beta' f(s,y,v)\} = 0 \qquad \forall (s,y) \in (t_0,\infty)\times E^0 \tag{2.45}$$

with boundary condition

$$V(s,y) = \Phi(s,y) \qquad \forall (s,y) \in [t_0,\infty)\times \partial E$$

and

$$\min_{(\alpha,\beta')\in\partial V(t,x^*(t))} \{\alpha + \beta' f(t,x^*(t),u^*(t))\} = 0 \qquad a.e.\ t\in[s,t_*). \tag{2.46}$$

Proof Assume first that $V(s,y)$ is differentiable at $(s,y)\in(t_0,\infty)\times E^0$. Since (s,y) is an interior point of $[t_0,\infty)\times E$, if an arbitrary constant control $v\in U$ is used over an interval $[s,s+k]$ and k is small enough, the solution of the system

$$\begin{aligned} &\dot{x} = f(t,x,v) && \text{a.e. } t\in[s,s+k] \\ &x(s) := y \end{aligned}$$

will lie in $[t_0,\infty)\times E^0$ for $s\le t\le s+k$. Let $\tilde{u}(t)$ be any control for initial point $(s+k, x(s+k))$. Define a control u_k by

$$u_k(t) := \begin{cases} v & s\le t\le s+k \\ \tilde{u}(t) & s+k\le t< t_*. \end{cases}$$

Let x_k denote the solution of the system

$$\begin{aligned} &\dot{x} = f(t,x,v) && \text{a.e. } t\in[s,s+k] \\ &x(s) := y \\ &\dot{y} = f(t,y,\tilde{u}) && \text{a.e. } t\in[s+k,t_*) \\ &y(s+k) := x(s+k). \end{aligned}$$

Let $D^+g(t)$ denote the right derivative of a function $g(t)$. Since u_k is constant on $[s,s+k]$, we have

$$D^+x_k(t) = f(t,x_k(t),u_k(t)^+) \qquad \forall t\in[s,s+k],$$

where $u_k(t)^+$ is the limit from the right of u_k at t. By Lemma 2.1, $V(\cdot, x_k(\cdot))$ is non-decreasing, hence $D^+V(t, x_k(t)) \geq 0$ for any value of t for which this derivative exists. Hence computing this derivative at $t := s$, using the chain rule for differentiation, we have

$$V_s + V_y f(s, y, v) \geq 0 \qquad \forall v \in U. \tag{2.47}$$

Now for any $(s, y) \in (t_0, \infty) \times E^0$, since $V(s, y)$ is locally Lipschitz, we have by Proposition 1.9

$$\partial V(s, y) = co\{\lim \nabla V(s_i, y_i) : \ (s_i, y_i) \to (s, y) \quad (s_i, y_i) \notin S \cup \Omega_V\}, \tag{2.48}$$

where S is a set of measure zero and Ω_V is the set of points at which V is not differentiable. Equation (2.48) implies that any $(\alpha, \beta') \in \partial V(s, y)$ is a convex combination of sequential limits defined by $(\alpha_k, \beta'_k) := \lim\limits_{i \to \infty} \nabla V(s_i^k, y_i^k)$, where $(s_i^k, y_i^k) \notin S \cup \Omega_V$ and $(s_i^k, y_i^k) \to (s, y)$ as $i \to \infty$. Since (s, y) is an interior point, the points (s_i^k, y_i^k) can be chosen to be in the interior of $[t_0, \infty) \times E^0$ and where V is differentiable. Therefore, by virtue of inequality(2.47), we have

$$V_s(s_i^k, y_i^k) + V_y(s_i^k, y_i^k) f(s_i^k, y_i^k, v) \geq 0 \qquad \forall v \in U. \tag{2.49}$$

Taking limits as $i \to \infty$, we have, by the continuity of f and the definition of (α_k, β'_k),

$$\alpha_k + \beta'_k f(s, y, v) \geq 0 \qquad \forall v \in U.$$

Since (α, β') is a convex combination of (α_k, β'_k), inequality (2.44) follows.

To prove the validity of equation (2.45), it suffices to show that the equality

$$\alpha^* + \beta^{*\prime} f(s, y, v^*) = 0. \tag{2.50}$$

holds for certain $(\alpha^*, \beta^{*\prime}) \in \partial V(s, y)$, $v^* \in U$.

First, we can show that any $\xi \in \partial x(s)$ is a convex combination of ξ_k such that

$$\xi_k \in G(s, y) := \{f(s, y, v) : \ \ v \in U\}. \tag{2.51}$$

Indeed, by Proposition 1.9, $\partial x(s) = co\{\lim \dot{x}(s_i) : \ s_i \to s \ \ s_i \notin S \cup \Omega_x\}$. Therefore, any $\xi \in \partial x(s)$ can be expressed as a convex combination of sequential limits ξ_k defined by $\xi_k = \lim_{i\to\infty} \dot{x}(s_i^k)$, where $s_i^k \to s, s_i^k \notin S \cup \Omega_x$. By the differentiablity of $x(\cdot)$ at s_i^k, we have

$$\dot{x}(s_i^k) = f(s_i^k, x(s_i^k), u(s_i^k)). \tag{2.52}$$

By the compactness of U, there exists a convergent subsequence of the sequence $\{u(s_i^k)\}_i$ (denoted by the same symbols) such that $u(s_i^k) \to v^k \in U \quad i \to \infty$. Taking (sub)sequential limits in (2.52) as $s_i^k \to s$, we have $\xi_k = f(s, x(s), v^k)$ by the continuity of f and x and the definition of ξ_k. This implies that $\xi_k \in G(s,y)$.

By Lemma 2.2, the value function V evaluated along the corresponding trajectory x^* is constant. Hence

$$\frac{d}{dt}V(t, x^*(t)) = 0 \qquad t \in (s, t_*).$$

Let $H(t) := (t, x^*(t))$. By the chain rule for Clarke generalized gradients (Proposition 1.12) we have

$$\begin{aligned} 0=\frac{d}{dt}V(t,x^*(t)) \quad &\in \quad co\{\partial V(t,x^*(t))\partial H(t)\} \\ &:= co\{(\alpha_t, \beta_t')\cdot(1,\xi_t') : (\alpha_t,\beta_t') \in \partial V(t,x^*(t)), \xi_t \in \partial x^*(t)\} \\ &= co\{\alpha_t + \beta_t'\xi_t : (\alpha_t,\beta_t') \in \partial V(t,x^*(t)), \xi_t \in \partial x^*(t)\} \qquad (2.53) \\ &\subset co\{\alpha_t + \beta_t'\xi_t : (\alpha_t,\beta_t') \in \partial V(t,x^*(t)), \xi_t \in G(t,x^*(t))\}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.54) \end{aligned}$$

where the last inclusion (2.54) follows from the inclusion (2.51).

Consequently, for all $t \in (s,t_*)$, there are element $(\alpha_t, \beta_t') \in \partial V(t, x^*(t))$ and $v_t \in U$, such that

$$0 = \alpha_t + \beta_t' f(t, x^*(t), v_t). \tag{2.55}$$

By Proposition 1.3, for all $(\alpha_t, \beta_t') \in \partial V(t,x^*(t))$, $s < t < t_*$, $\|(\alpha_t,\beta_t')\| \le L_V$ which implies that the set $\{(\alpha_t,\beta_t') \in \partial V(t,x^*(t)) : s < t < t_*\}$ contains in a compact set. Therefore we can assume (taking subsequences if necessary) that $(\alpha_t, \beta_t') \to (\alpha^*, \beta^{*\prime})$ as $t \to s$. By Proposition 1.10, the generalized gradient of function $V(\cdot)$ as a set-valued map is closed; therefore $(\alpha^*, \beta^{*\prime}) \in \partial V(s, x^*(s)) = V(s,y)$. By compactness of U, we may assume (taking a subsequence if necessary) that $v_t \to v^* \in U$. Taking limits in (2.55), we obtain the desired equality (2.50) by continuity of the function f.

Equation (2.46) follows from inclusion (2.53) and inequality (2.44). ■

2.3.3 Proof of necessity

(P_z) can be equivalently posed as the following problem of Mayer type:

$$
\begin{array}{lll}
(P_{0,z}) & \min & \Phi(\tilde{x}(t_*)) := x_{n+1}(t_*) + x_0(t_*)F(x(t_*)) \\
& \multicolumn{2}{l}{\text{over the class } \tilde{\mathcal{F}} \text{ of all pairs } (\tilde{x}(\cdot), u(\cdot)) \text{ with}} \\
& & \tilde{x}(\cdot) := (x(\cdot)', x_0(\cdot), x_{n+1}(\cdot))' \\
& \text{s.t.} & u : [0, t_*) \longrightarrow \mathbb{R}^m \text{ is measurable} \\
& & u(t) \in U \subset \mathbb{R}^m \quad \forall t \in [0, t_*) \\
& & \dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t), u(t)) \quad \text{a.e. } t \in [0, t_*) \\
& & \tilde{x}(0) = (z, 1, 0) \in E^0 \times (0,1] \times [0, M_{f_0}/\underline{\lambda}],
\end{array}
$$

where $\tilde{f}(\tilde{x}, u) := (f(x,u)', -x_0\bar{\lambda}(x,u), x_0 f_0(x,u))'$.

Define the value function for $(P_{0,z})$ as the function $\tilde{V} : (E \times (0,1] \times [0, M_{f_0}/\underline{\lambda}]) \cup (\partial E \times \{1\} \times \{0\}) \longrightarrow \mathbb{R}_+$ given by

$$
\tilde{V}(\tilde{z}) := \begin{cases} \inf_{(\widetilde{x(\cdot)}, u(\cdot)) \in \tilde{\mathcal{F}}} \Phi(\tilde{x}(t_*)), & \tilde{z} := (z, z_0, z_{n+1}) \in E^0 \times (0,1] \times [0, M_{f_0}/\underline{\lambda}] \\ \Phi(\tilde{z}) = F(z), & \tilde{z} := (z, z_0, z_{n+1}) \in \partial E \times \{1\} \times \{0\}. \end{cases}
$$

Then we have

$$
\begin{aligned}
\tilde{V}(\tilde{z}) &= \inf_{u(\cdot)} \{x_{n+1}(t_*) + x_0(t_*)F(x(t_*))\} \\
&= \inf_{u(\cdot)} \{x_{n+1}(0) + \int_0^{t_*} x_0(t) f_0(x(t), u(t)) dt + x_0(t_*)F(x(t_*))\} \\
&= z_{n+1} + \inf_{u(\cdot)} \{\int_0^{t_*} x_0(t) f_0(x(t), u(t)) dt + x_0(t_*)F(x(t_*))\} \\
&= z_{n+1} + \inf_{u(\cdot)} \{\int_0^{t_*} x_0(0) e^{-\int_0^t \bar{\lambda}(x(l), u(l)) dl} f_0(x(t), u(t)) dt \\
&\quad + x_0(0) e^{-\int_0^{t_*} \bar{\lambda}(x(l), u(l)) dl} F(x(t_*))\}
\end{aligned}
$$

$$
\begin{aligned}
&= z_{n+1} + z_0 \inf_{u(\cdot)}\{\int_0^{t_*} e^{-\int_0^t \bar{\lambda}(x(l),u(l))dl} f_0(x(t),u(t))dt \\
&\quad + e^{-\int_0^{t_*} \bar{\lambda}(x(l),u(l))dl} F(x(t_*))\} \\
&= z_{n+1} + z_0 V(z).
\end{aligned}
$$

By Proposition 1.6, we have

$$
\begin{aligned}
\partial \widetilde{V}(\tilde{z}) &= \partial \widetilde{V}_1(\tilde{z}) + \partial \widetilde{V}_2(\tilde{z}) \\
&= (0,0,1) + \partial \widetilde{V}_2(\tilde{z})
\end{aligned} \tag{2.56}
$$

where $\widetilde{V}_1(\tilde{z}) := z_{n+1}$ and $\widetilde{V}_2(\tilde{z}) := z_0 V(z)$. Define $Q(z, z_0) := z_0 V(z)$. By Proposition 1.8, since $z_0 > 0$, we have

$$
\begin{aligned}
\partial \widetilde{V}_2(\tilde{z}) &= \partial Q(z, z_0) \times \{0\} \\
&= z_0 \partial V(z) \times \{V(z)\} \times \{0\}.
\end{aligned} \tag{2.57}
$$

Combining equalities (2.56) and (2.57), we have

$$
\begin{aligned}
\partial \widetilde{V}(z, z_0, z_{n+1}) &= (0,0,1) + z_0 \partial V(z) \times \{V(z)\} \times \{0\} \\
&= z_0 \partial V(z) \times \{V(z)\} \times \{1\}.
\end{aligned} \tag{2.58}
$$

By Proposition 2.2, $V(z)$ is Lipschitz continuous, therefore $\widetilde{V}(\tilde{z}) = z_{n+1} + z_0 V(z)$ is Lipschitz continuous.

By virtue of assumptions (A2.1)–(A2.5), assumption (A2.9)–(A2.11) are satisfied for problem (P_{0z}). Noticing that $\widetilde{V}(\tilde{z})$ is time independent and applying Theorem 2.3, we conclude that $\widetilde{V}(\tilde{z})$ satisfies the generalized BHJ equation

$$
\min_{\substack{\eta' \in \partial \widetilde{V}(\tilde{z}) \\ v \in U}} \{\eta' \tilde{f}(\tilde{z}, v)\} = 0. \tag{2.59}
$$

Using equation (2.58), any $\eta' \in \partial \widetilde{V}(\tilde{z})$ can be expressed as $\eta' = (z_0 \xi', V(z), 1)$ where $\xi' \in \partial V(z)$. Therefore equation (2.59) is equivalent to the following equation

$$
\min_{\substack{\xi' \in \partial V(z) \\ v \in U}} \{z_0 \xi' f(z,v) - z_0 \bar{\lambda}(z,v) V(z) + z_0 f_0(z,v)\} = 0.
$$

Since $z_0 > 0$, the above equation implies equation (2.23).

The rest of the theorem follows analogously. ∎

2.4 Uniqueness of the Generalized Solution

Following Dempster (1989), we shall need the following definitions.

Definition 2.1 Let function $\phi : E \longrightarrow \mathbb{R}$ be Lipschitz continuous. Its *contingent derivative* at x_0 in *direction* $d_0 \in \mathbb{R}^n$ is defined by

$$\phi_-(x_0; d_0) := \liminf_{\substack{d \to d_0 \\ t \searrow 0}} \frac{\phi(x + td) - \phi(x)}{t}.$$

■

Definition 2.2 Let the function $\phi : E \longrightarrow \mathbb{R}$ be Lipschitz continuous. We say ϕ is *regular* if, and only if, for all $x \in E$ and all $d \in \mathbb{R}^n$

$$\phi_-(x; d) = \min_{\xi' \in \partial\phi(x)} \xi' d. \tag{2.60}$$

■

This definition of regularity is equivalent to $\phi_-(x; d) = \phi_0(x; d)$ (*cf.* Proposition 1.3), where $\phi_0(x; d)$ is Clarke's lower directional derivative (Definition 1.4) and hence generalizes Clarke's notion of regularity (*cf.* Definition 1.6). It is implied by the relation $\partial\phi(x) = \partial_+\phi(x)$, where $\partial_+\phi(x)$ denotes the superdifferential of the theory of viscosity solutions (see e.g. Elliott 1987).

The uniqueness result for a regular Lipschitz continuous solution of the Dirichlet problem with Cauchy data defined by (2.61) is new (Dempster 1989) and turns out to be the nonsmooth extension in this context of a proof of Haar (1928) (see also Courant and Hilbert 1953, pp. 145-147) of the uniqueness of a C^1 solution of the Cauchy problem for a general class of first order nonlinear PDEs.

Under our assumptions we may consider the left hand side of the generalized BHJ equation (2.23) to define an operator $G(\cdot\,; f_0) : C(E^0) \to C(E^0)$ by

$$G(\phi\,; f_0)(z); = \min_{\substack{\xi' \in \partial\phi(z) \\ v \in U}} [\xi' f(z, v) - \bar{\lambda}(z, v)\phi(z) + f_0(z, v)],$$

where $C(M)$ denotes the space of all bounded real-valued continuous functions on the set M equipped with supremum norm.

In terms of this operator the *Dirichlet problem* for the generalized BHJ equation becomes to find a Lipschitz continuous funtion $\phi \in C(E)$ such that

$$G(\phi; f_0)|_{E^0} = 0 \quad \text{and} \quad \phi|_{\partial E} = F. \tag{2.61}$$

Theorem 2.4 *A regular Lipschitz continuous solution ϕ of the Dirichlet problem* (2.61) *is unique.* ■

To see the simple idea of the proof consider the following.

Proposition 2.3 *A solution $\phi \in C^1(E)$, the space of all real-valued C^1 functions on E equipped with the supremum norm, of the Dirichlet problem* (2.61) *is unique.* ■

This result is a simple consequence of the following.

Lemma 2.3 *Let $f_0, g_0 \in C(E^0 \times U)$ and suppose that $\phi, \psi \in C^1(E)$ solve respectively the BHJ equations $G(\phi; f_0) = 0$ and $G(\psi; g_0) = 0$ on E^0 with $\phi|_{\partial E} = \psi|_{\partial E} = F$. Then*

$$\max_{z \in E}[\phi(z) - \psi(z)]_+ \leq \max_{\substack{z \in E^0 \\ v \in U}}[f_0(z,v) - g_0(z,v)]_+ / \underline{\lambda},$$

where $\underline{\lambda} := \inf_{z \in E^0, v \in U} \bar{\lambda}(z,v) > 0$.

Proof Unless $\phi < \psi$ throughout E^0 (in which case the roles of ϕ and ψ may be reversed), a maximum of the $C^1(E)$ function $\phi - \psi$ exists at $z_0 \in E^0$ at which $\nabla\phi(z_0) = \nabla\psi(z_0)$. Now choose $v_0 \in U$ such that

$$\psi(z_0) \;\; = \;\; [\nabla\psi(z_0) f(z_0, v_0) + g_0(z_0, v_0)] / \bar{\lambda}(z_0, v_0) \tag{2.62}$$

and

$$\phi(z_0) \;\; \leq \;\; [\nabla\phi(z_0) f(z_0, v_0) + f_0(z_0, v_0)] / \bar{\lambda}(z_0, v_0) \tag{2.63}$$

and hence

$$\begin{aligned} & [\phi(z_0) - \psi(z_0)]_+ \\ & \leq \;\; \underline{\lambda}^{-1}[(\nabla\phi(z_0) - \nabla\psi(z_0)) f(z_0, v_0) + f_0(z_0, v_0) - g_0(z_0, v_0)]_+ \\ & = \;\; \underline{\lambda}^{-1}[f_0(z_0, v_0) - g_0(z_0, v_0)]_+ \\ & \leq \;\; \underline{\lambda}^{-1} \max_{\substack{z \in E^0 \\ v \in U}}[f_0(z,v) - g_0(z,v)]_+. \end{aligned} \tag{2.64}$$

By symmetry it follows that

$$\|\phi - \psi\|_\infty \leq \underline{\lambda}^{-1} \|f_0 - g_0\|_\infty$$

relative to the supremum norms for $C(E)$ and $C(E^0 \times U)$ respectively. Hence a $C^1(E)$ solution of (2.61) is unique.

■

To extend this result to the uniqueness of a regular Lipschitz continuous solution of (2.61) , we replace (2.62) by

$$\psi(z_0) = [\eta_0' f(z_0, v_0) + g_0(z_0, v_0)]/\bar{\lambda}(z_0, v_0),$$

for $\eta_0{}' \in \arg\min_{\eta' \in \partial\psi(z_0)} \eta' f(z_0, v_0)$, and (2.63) by

$$\phi(z_0) \leq [\xi_0' f(z_0, v_0) + f_0(z_0, v_0)]/\bar{\lambda}(z_0, v_0)],$$

for $\xi_0{}' \in \arg\min_{\xi' \in \partial\phi(z_0)} \xi' f(z_0, v_0)$. Then (2.64) becomes

$$\begin{aligned} & [\phi(z_0) - \psi(z_0)]_+ \\ \leq \; & \underline{\lambda}^{-1} [(\xi_0 - \eta_0)' f(z_0, v_0) + f_0(z_0, v_0) - g_0(z_0, v_0)]_+ \end{aligned}$$

and we are through by virtue of the following.

Lemma 2.4 *For regular Lipschitz continuous solutions ϕ, ψ of (2.61), we have*

$$(\xi_0 - \eta_0)' f(z_0, v_0) \leq 0$$

Proof Since $z_0 \in E^0$ maximizes $\phi - \psi$ over E^0,

$$\psi(z) - \psi(z_0) \geq \phi(z) - \phi(z_0) \tag{2.65}$$

for $z \in E^0$. Consider sequences

$$d_i \to f(z_0, v_0) \qquad t_i \searrow 0$$

such that

$$\liminf_{\substack{d \to f(z_0, v_0) \\ t \searrow 0}} ([\psi(z_0 + td) - \psi(z_0)]/t) = \lim_{i \to \infty} ([\psi(z_0 + t_i d_i) - \psi(z_0)]/t_i),$$

for which from (2.65)

$$t_i^{-1}[\psi(z_0 + t_i d_i) - \psi(z_0)] \geq t_i^{-1}[\phi(z_0 + t_i d_i) - \phi(z_0)].$$

Taking (subsequential) limits (if necessary) it follows that

$$\begin{aligned} \psi_-(z_0; f(z_0, v_0)) &\geq \lim_{i \to \infty} t_i^{-1}[\phi(z_0 + t_i d_i) - \phi(z_0)] \\ &\geq \phi_-(z_0; f(z_0, v_0)). \end{aligned}$$

By regularity (*cf.* (2.60)), $\eta_0' f(z_0, v_0) \geq \xi_0' f(z_0, v_0)$. ■

Chapter 3

Necessary and Sufficient Optimality Conditions for Control of PDPs

3.1 Introduction

Throughout this chapter, for the purpose of exposition, we shall take the interior of the state space E^0 of a PDP to be a open bounded connected subset in $\mathbb{R}^n$ such that $E^0 = \{x \in \mathbb{R}^n : \ \psi(x) > 0\}$ defined by a *boundary function* $\psi \in C^1(\mathbb{R}^n)$ for which $\|\nabla\psi(x)\| > 1$ for $x \in \partial E := \{x \in \mathbb{R}^n : \psi(x) = 0\}$, the boundary of the state space. Denote the state space (once again) by $E := E^0 \cup \partial E$.

For more general cases where the state space is a union of sets in $\mathbb{R}^n$, or even manifolds, all results remain true provided that each component satisfies the above smoothness assumption on the boundary.

Supposing no impulse control action is allowable (to be relaxed in Chapter 5), we make the following *assumptions* throughout this and the next chapter:

(A3.1) The *interior control set* $U_0 \subset \mathbb{R}^m$ and the *boundary control set* $U_\partial \subset \mathbb{R}^l$ are compact.

(A3.2) The vector field $f : E \times U_0 \longrightarrow I\!R^n$ is bounded, continuous and Lipschitz continuous in $x \in E$ uniformly in $u \in U_0$.

(A3.3) The jump rate $\lambda : E^0 \times U_0 \longrightarrow I\!R_+$ is bounded, continuous and Lipschitz continuous in $x \in E^0$ uniformly in $u \in U_0$.

(A3.4) As it was mentioned in §1.4, the transition measure Q may be expressed in terms of $Q_0 := Q|_{E^0} : E^0 \times U_0 \longrightarrow I\!P(E^0)$ and $Q_\partial := Q|_{\partial E} : \partial E \times U_\partial \longrightarrow I\!P(E^0)$. $Q_0 : E^0 \times U_0 \longrightarrow I\!P(E^0)$ is bounded, continuous (where $I\!P(E^0)$ denotes the set of probability measures on E^0 with the topology of (probabilistic) weak convergence) relative to the weak* topology on $I\!P(E^0)$ and Lipschitz continuous in $x \in E^0$ (e.g. for all $\theta \in C(E^0)$ the map $x \longmapsto \int_{E^0} \theta(y) Q_0(dy; x, v)$ is continuous and Lipschitz) uniformly in $v \in U_0$. Q_∂ is defined on $\partial E \times U_\partial$ and has an extension to $E \times U_\partial$ such that the extension $Q_\partial : E \times U_\partial \longrightarrow I\!P(E^0)$ is bounded, continuous and Lipschitz continuous in $x \in E$ uniformly in $u \in U_\partial$.

(A3.5) The set of admissible controls $u := (u_0, u_\partial) \subset \mathcal{C} \subset \mathcal{C}_0 \times \mathcal{C}_\partial$ is defined in terms of the set of interjump *open loop* measurable (deterministic) *control functions*,

$$\mathcal{C}_0 := \{u_0 \in \mathcal{L} : u_0(\tau, z) : I\!R_+ \times E^0 \longrightarrow U_0\},$$

where τ represents the time elapsed since the last jump and z represents the post jump state, and the set of measurable feedback boundary controls

$$\mathcal{C}_\partial := \{u_\partial \in \mathcal{L} : u_\partial : \partial E \longrightarrow U_\partial\},$$

for which $P_x^u\{\lim_{n\to\infty} \mathrm{T}_n = \infty\} = 1$ for all $x \in E^0$, where for initial state x $P_x^u(\cdot)$ is the probability measure (on path space) induced by u and $\mathcal{L}$ denotes the set of all measurable functions between a given domain and range.

(A3.6) The running cost $l_0 : E^0 \times U_0 \longrightarrow I\!R_+$ is bounded, continuous and Lipschitz continuous in $x \in E^0$ uniformly in $u \in U_\partial$. The boundary (jump) cost $l_\partial : \partial E \times U_\partial \longrightarrow I\!R_+$ is continuous and has an extension to $E \times U_\partial$ such that the extension $l_\partial : E \times U_\partial \longrightarrow I\!R_+$ is bounded, continuous and Lipschitz continuous in $x \in E$ uniformly in $v \in U_\partial$.

It should be noted that the requirement of the existence of Lipschitz continuous extensions of Q_∂ and l_∂ to whole state space E is only needed for the Lipschitz continuity of the value function as in the deterministic case (*cf.* Remark 2.1).

As mentioned in §1.4, the use of controls which are only measurable necessitates the open loop nature of the interjump control function $u_0(\cdot, z)$ (or $u_{0(\cdot)}(z)$ in an obvious notation) which is appropriate to the initial condition z and which need only be specified from 0 to the controlled boundary hitting time $t_*^{u_0}(z) \leq \infty$ (*cf.* (2.1)) in case no random jump occurs up to this time elapsed from the last jump time. If a jump occurs before $t_*^{u_0}(z)$ elapses, to $w \in E^0$ say, the control function $u_0(\cdot, w)$ is used next and the corresponding flow $\phi^{u_0}(\cdot, w)$ (or $\phi^{u_0}_{(\cdot)}(w)$) is the unique absolutely continuous solution of the nonautonomous dynamical system determined by the controlled vector field $f(\cdot, u_0(\cdot, w)) : E \times [0, t_*^u(w)) \longrightarrow E$ as

$$\frac{\partial}{\partial \tau}\phi^{u_0}(\tau, w) = f(\phi^{u_0}(\tau, w), u_0(\tau, w)) \qquad \phi^{u_0}(0, w) := w. \tag{3.1}$$

This follows from Caratheodory's existence and uniqueness theorem for first order differential equations of the form (3.1) in $\mathbb{R}^n$ on $[0, t_*^{u_0}(w))$ written in terms of the Lebesgue integral as

$$\phi^{u_0}(\tau, w) := w + \int_0^\tau f(\phi^{u_0}(t, w), u_0(t, w))dt \quad \tau \in [0, t_*^u(w)) \tag{3.2}$$

(see e.g. Fleming and Rishel 1975, p.63).

The PDP optimal control problem(with dynamic control) is to find an admissible control $u = (u_0, u_\partial) \in \mathcal{C}$ so as to minimize the expected *discounted total cost* functional

$$J_x(u) := E_x^u\Big[\int_0^\infty e^{-\delta t} l_0(\mathrm{x}_t^u, u_0(\tau_t^u, \mathrm{z}_t^u))dt + \sum_i e^{-\delta \mathrm{T}_i^u} l_\partial(\mathrm{x}^u_{T_i^-}, u_\partial(\mathrm{x}^u_{T_i^-}))1_{(\mathrm{x}^u_{T_i^-} \in \partial E)}\Big]. \tag{3.3}$$

where E_x^u denotes expectation with respect to P_x^u, $\delta > 0$ is the discount rate and $1_{\{\cdot\}}$ denotes the indicator function of the event $\{\cdot\}$. Henceforth for simplicity we sometimes suppress the notation showing the explicit dependence of process entities on a control policy $u \in \mathcal{C}$.

In this chapter, we aim at developing a necessary and sufficient optimality condition of dynamic programming type for the PDP control problem. First we will show that the optimal control for the PDP control problem is to choose after each jump a control function which is an optimal control in a deterministic optimal control problem with boundary conditions by applying the theory on discrete time stochastic optimal control problems of Bertsekas & Shreve (1978). Then by applying the results in Chapter 2, we will conclude that the value function for the PDP optimal control problem is Lipschitz continuous and that a generalized BHJ equation is a necessary and sufficient condition for optimality.

3.2 Reduction to Discrete Time Stochastic Control

In order to establish the existence of an optimal policy, as mentioned in §3.1, Davis's (1986) idea (originally carried out for the case of no boundary control, i.e. $\mathcal{C}_\partial := \emptyset$) is to reduce the continuous time PDP optimal control problem to an optimal discrete time stochastic control problem by examination of the embedded Markov decision process with the same admissible policies $\mathcal{C}$.

To this end *assume* further that:

(A3.7) There exists $\alpha > 0$ such that for all $x \in \partial E$ and all $v \in U_0$,

$$f(x,v) \cdot n(x) \geq \alpha > 0, \tag{3.4}$$

where $n(x) := \nabla\psi(x)/\|\nabla\psi(x)\|$ is the unit outward normal to $\partial E \subset \mathbb{R}^n$ at the point $x \in \partial E$. (This is actually assumption (A2.6) repeated here in the PDP context.)

Let $\mathcal{U}_0 = L_\infty(\mathbb{R}_+; U_0)$ denote the space of all bounded measurable functions $u_0 : \mathbb{R}_+ \longrightarrow U_0$ with the essential supremum norm. For any $u_0 \in \mathcal{U}_0$, let $\phi_t^u(z)$ be the integral curve of

$$\dot{x}_t = f(x_t, u_0(t)) \quad x_0 := z \quad \text{a.e. } t \in [0, t_*^u(z)),$$

where $t_*^u(z) := \inf\{t > 0 : \phi_t^u(z) \in \partial E\}$ is the boundary hitting time. Define the *cumulative jump rate*

$$\Lambda_t^u(z) := \int_0^t (\lambda(\phi_s^u(z), u_0(s)) + \delta)ds]. \tag{3.5}$$

Expression (3.5) represents the cumulative total jump rate of the *killed process* (see the proof of Proposition 3.3).

Let $\mathcal{U} := \mathcal{U}_0 \times U_\partial$. Define the function $g : E^0 \times \mathcal{U} \longrightarrow I\!R_+$ by

$$g(z,u) := \int_0^{t_*^u(z)} e^{-\Lambda_t^u(z)} l_0(\phi_t^u(z), u_0(t))dt + e^{-\Lambda_{t_*}^u(z)} l_\partial(\phi_{t_*}^u(z), v), \tag{3.6}$$

and the transition measure $\mathcal{Q} : E^0 \times \mathcal{U} \longrightarrow I\!P(E^0)$ by

$$\mathcal{Q}(A; z, u) :=$$
$$\int_0^{t_*^u(z)} Q_0(A; \phi_t^u(z), u_0(t))\lambda(\phi_t^u(z), u_0(t))e^{-\Lambda_t^u(z)}dt + e^{-\Lambda_{t_*}^u(z)} Q_\partial(A; \phi_{t_*}^u(z), v), \tag{3.7}$$

where $A \in \mathcal{E}$ (here the Borel sets of E^0).

The following problem is a well defined *infinite horizon discrete time stochastic control problem* (*cf.* Bertsekas and Shreve 1978):

$$\begin{array}{lll} \text{(DP1)} & \text{minimize} & E_x \sum_0^\infty g(\mathbf{z}_k, \mu_k) \\ & \text{over } \textit{policies } \pi := (\mu_0, \mu_1, \ldots, \mu_N, \ldots) & \\ & \text{such that} & \mu_k(z_0, \mu_0, \ldots, z_{k-1}, \mu_{k-1}, z_k) \in \mathcal{U}_0 \times U_\partial \quad k = 1, 2, \ldots, \end{array}$$

where g is defined by (3.6) and $\mathbf{z}_k$ is the discrete time Markov process with transition measure $\mathcal{Q}$ defined by (3.7). We shall call g the *one step cost function* and $\mathcal{U}_0 \times U_\partial$ the *control space*.

According to Bertsekas and Shreve (1978) (Definition 8.7, p.208), (DP1) belongs to *the lower semicontinuous model* if the state space E^0 is a Borel space, the control space $\mathcal{U}_0 \times U_\partial$ is compact, the transition measure $\mathcal{Q} : E^0 \times \mathcal{U} \longrightarrow I\!P(E^0)$ is continuous and the one step cost function $g : E^0 \times \mathcal{U} \longrightarrow I\!R_+$ is lower semicontinuous and bounded below. Alternatively (see Bertsekas and Shreve 1978, Definition 8.8, p.210), if the state space E^0 and the control space are both Borel spaces, $\mathcal{Q} : E^0 \times \mathcal{U} \longrightarrow I\!P(E^0)$

is continuous and $g : E^0 \times \mathcal{U} \longrightarrow \mathbb{R}_+$ is upper semicontinuous and bounded above, then (DP1) belongs to the *upper semicontinuous model.*

The state space E^0 of our model (defined in §3.1) with the Euclidean topology is a Borel space. Both l_0 and l_∂ are positive and bounded above, so g is bounded both below and above. However the set of admissible interior controls $\mathcal{U}_0 := L_\infty(\mathbb{R}_+; U_0)$ is *not* compact, so that the control space $\mathcal{U} = \mathcal{U}_0 \times U_\partial$ is not compact.

In order to make the nonseparable space $L_\infty(\mathbb{R}_+; U_0)$ compact we must consider $L_\infty(\mathbb{R}_+; \mathbb{P}(U_0))$, the space of probability measure-valued *relaxed* (generalized, mixed) *controls.* Using a relaxed control amounts to randomizing at each time t over U_0 rather than choosing a specific value $u \in U_0$. The space of all ordinary controls $\mathcal{U}_0$ is embedded in the space of all relaxed controls by means of the injection defined by $u(t) \longmapsto \tilde{u}(t) := \delta_{u(t)}(\cdot)$ where δ_ξ denotes the 1-atom probability measure concentrated on ξ.

Now we define a topology on the *relaxed control space* $\tilde{\mathcal{U}}_0 := L_\infty(\mathbb{R}_+; \mathbb{P}(U_0)) \subset L_\infty(\mathbb{R}_+; C^*(U_0))$ as follows: Let $X := L_1(\mathbb{R}_+; C(U_0))$ denote the space of all mappings $h_{(\cdot)} : \mathbb{R}_+ \longrightarrow C(U_0)$ such that $\int_0^\infty \|h_t(\cdot)\|_\infty dt < \infty$. It is a Banach space whose dual space is $X^* = L_\infty(\mathbb{R}_+; C^*(U_0))$ with the pairing

$$(h, \tilde{u}) = \int_0^\infty \int_U h_t(u)\tilde{u}_t(du)dt$$

for $h_{(\cdot)} \in X$ and $\tilde{u}_{(\cdot)} \in X^*$. Let $\|\cdot\|_1$ denote the total variation norm in $C^*(U_0)$. Then the unit ball in X^* is

$$B_* = \{\tilde{u}_{(\cdot)} \in L_\infty(R_+; C^*(U_0)) : \ ess \sup_{t \in R_+} \|\tilde{u}_t(\cdot)\|_1 \le 1\}$$

and this is *weak**-compact by Alaoglu's theorem. The following proposition shows that the set of relaxed controls $\tilde{\mathcal{U}}_0$ is a weak*-closed subset of B_*.

Proposition 3.1 *$\tilde{\mathcal{U}}_0$ is weak*-closed in B_*. That is, for any sequence $\{\tilde{u}^i\} \subset \tilde{\mathcal{U}}_0$, $\tilde{u}^i \to \tilde{u}$ in the weak* topology implies $\tilde{u} \in \tilde{\mathcal{U}}_0$.*

Proof

$$\tilde{\mathcal{U}}_0 = \{\tilde{u} \in L_\infty(\mathbb{R}_+; C^*(U_0)) : \ ess \sup_{t \in R_+} \|\tilde{u}_t(\cdot)\|_1 = 1\}$$

is obviously a subset of B_*

Since B_* is weak*-compact, $\tilde{u}^i \to \tilde{u}$ in the weak* topology implies that $\tilde{u} \in B_*$. By definition, $\tilde{u}^i \to \tilde{u}$ in the weak* topology means that for any $h_{(\cdot)} \in L_1(\mathbb{R}_+; C(U_0))$,

$$(h, \tilde{u}^i) \to (h, \tilde{u}),$$

i.e.

$$\int_0^\infty \int_{U_0} h_t(u)\tilde{u}_t^i(du)dt \to \int_0^\infty \int_{U_0} h_t(u)\tilde{u}_t(du)dt. \tag{3.8}$$

In particular, let $h_t(u) := e^{-t} \in L_1(\mathbb{R}_+; C(U_0))$. Since $\tilde{u}^i \in \tilde{\mathcal{U}}_0$, $\int_{U_0} \tilde{u}_t^i(du) = 1$. Therefore (3.8) implies that

$$\int_0^\infty e^{-t} \int_{U_0} \tilde{u}_t(du)dt = \int_0^\infty e^{-t}dt$$

That is,

$$\int_0^\infty e^{-t}[\int_{U_0} \tilde{u}_t(du) - 1]dt = 0.$$

Since $\tilde{u} \in B_*$ implies $\int_{U_0} \tilde{u}_t(du) \leq 1$, we have

$$\int_{U_0} \tilde{u}_t(du) = \int_{U_0} |\tilde{u}_t(du)| = 1 \quad \text{a.e. } t \in [0, \infty),$$

by standard results in funtional analysis. Hence That is,

$$ess \sup_{t \in \mathbb{R}_+} \|\tilde{u}_t(\cdot)\|_1 = 1$$

and $\tilde{u} \in \tilde{\mathcal{U}}_0$ as required. ■

Thus $\tilde{\mathcal{U}}_0$ is a weak*-compact subset of X^*. We term the relative weak* topology on $\tilde{\mathcal{U}}_0$ the *Young topolgy*, and this is the only topology on $\tilde{\mathcal{U}}_0$ considered hence forth. With the Young topology on $\tilde{\mathcal{U}}_0$ and the Euclidean topology on U_∂, the control space $\tilde{\mathcal{U}}_0 \times U_\partial$ with the product topology is compact.

Redefine, in terms of $\tilde{u} := (\tilde{u}_0, v) \in \tilde{\mathcal{U}}$, a relaxed control function $\tilde{u}_0$ and boundary control action v pair, the flow (3.2) as

$$\phi_\tau^{\tilde{u}}(z) := z + \int_0^\tau \int_{U_0} f(\phi_t^{\tilde{u}}(z), u)\tilde{u}_{0t}(du)dt, \tag{3.9}$$

with boundary hitting time

$$t_*^{\tilde{u}}(z) := \inf\{t > 0 : \phi_t^{\tilde{u}}(z) \in \partial E\}$$

and cumulative jump rate (3.5) as

$$\Lambda_t^{\tilde{u}}(z) := \int_0^t \int_{U_0} \lambda(\phi_s^{\tilde{u}}(z)), u)\tilde{u}_{0s}(du)ds + \delta t.$$

Then the one step cost (*cf.* (3.6)) becomes

$$g(z,\tilde{u}) := \int_0^{t_*^{\tilde{u}}(z)} \int_{U_0} e^{-\Lambda_t^{\tilde{u}}(z)} l_0(\phi_t^{\tilde{u}}(z), u)\tilde{u}_{0t}(du)dt + e^{-\Lambda_{t_*}^{\tilde{u}}(z)} l_\partial(\phi_{t_*}^{\tilde{u}}(z), v) \tag{3.10}$$

with corresponding one step transition measure (*cf.* (3.7)) given by

$$Q(A;z,\tilde{u}) := \int_0^{t_*^{\tilde{u}}(z)} \int_{U_0} Q_0(A;\phi_t^{\tilde{u}}(z), u)\lambda(\phi_t^{\tilde{u}}(z), u)e^{-\Lambda_t^{\tilde{u}}(z)}\tilde{u}_{0t}(du)dt + e^{-\Lambda_{t_*}^{\tilde{u}}}(z)Q_\partial(A;\phi_{t_*}^{\tilde{u}}(z), v). \tag{3.11}$$

To show that the functions defined by $g : E^0 \times \tilde{\mathcal{U}} \longrightarrow I\!R_+$ and $Q : E^0 \times \tilde{\mathcal{U}} \longrightarrow I\!P(E^0)$ are continuous with respect the topology defined on $E^0 \times \tilde{\mathcal{U}}$, we need the following lemma (see e.g. Warga 1972, p.325).

Lemma 3.1 *Under assumption* (A3.2), *the map* $(z, \tilde{u}_{0(\cdot)}) \longmapsto \phi_{(\cdot)}^{\tilde{u}}(z)$ *is continuous from* $I\!R^n \times \tilde{\mathcal{U}}_0$ *to* $C([0,T]; I\!R^n)$ *with respect to the supremum norm on* $C([0,T]; I\!R^n)$ *for any* $T < t_*^u(z) \in I\!R_+$.

Proof To show that the map $(z, \tilde{u}_{0(\cdot)}) \longmapsto \phi_{(\cdot)}^{\tilde{u}}(z)$ is continuous. We must show that if $(z_n, \tilde{u}^n) \in E^0 \times \tilde{\mathcal{U}}_0$ and $(z_n, \tilde{u}^n) \to (z, \tilde{u})$ as $n \to \infty$, then it follows that

$$\phi_t^n \to \phi_t \quad \text{as } n \to \infty,$$

where ϕ_t is the solution of (3.9) and ϕ_t^n is the solution of (3.9) with $(z, \tilde{u})$ replaced by $(z_n, \tilde{u}_n)$.

For simplicity, define

$$\int_0^T \tilde{f}(\phi_t, \tilde{u}_t)dt := \int_0^T \int_{U_0} f(\phi_t, u)\tilde{u}_t(du)dt.$$

Then (3.9) becomes

$$\phi_t = z + \int_0^t \tilde{f}(\phi_s, \tilde{u}_s)ds.$$

Therefore, we have,

$$\begin{aligned}
\|\phi_t^n - \phi_t\| &\leq \|z_n - z\| + |\int_0^t [\tilde{f}(\phi_s^n, \tilde{u}_s^n) - \tilde{f}(\phi_s, \tilde{u}_s)ds| \\
&\leq \|z_n - z\| + \int_0^t |\tilde{f}(\phi_s^n, \tilde{u}_s^n) - \tilde{f}(\phi_s, \tilde{u}_s^n)|ds + |\int_0^t \tilde{f}(\phi_s, \tilde{u}_s^n - \tilde{u}_s)ds| \\
&\leq \|z_n - z\| + L_f \int_0^t \|\phi_s^n - \phi_s\|ds + |\int_0^t \tilde{f}(\phi_s, \tilde{u}_s^n - \tilde{u}_s)ds|. \qquad (3.12)
\end{aligned}$$

Let

$$\begin{aligned}
h_n(t) &:= \int_0^t \tilde{f}(\phi_s, \tilde{u}_s^n - \tilde{u}_s)ds \\
&= \int_0^\infty I_{[0,t]}(s)\tilde{f}(\phi_s, \tilde{u}_s^n - \tilde{u}_s)ds
\end{aligned}$$

Since $\tilde{u}_n \to \tilde{u}$ as $n \to \infty$, we have,

$$h_n(t) \to 0 \quad \text{for all } t \in [0, T]$$

by the general convergence theorem (see e.g. Royden 1963, Proposition 18, p.232).

Now we prove that this convergence is uniform on $[0, T]$ by contradition. Therefore suppose $h_n(t)$ do not converge to zero uniformly for $t \in [0, T]$. Then there must exist $\varepsilon > 0$, $\tau \in [0, T]$ and subsequence $\{t_j\}_{j\in J} \in [0, T]$ with $J \subset \{1, 2, \cdots\}$ such that

$$\lim_{j\in J} t_j = \tau \quad \text{and } |h_j(t_j)| > \varepsilon \ \ \forall\, j \in J. \qquad (3.13)$$

Since h_n is continuous for each $j \in J$, let $\delta > 0$ be such that

$$|h_j(t) - h_j(\tau)| < \varepsilon/2 \ \text{ if } |t - \tau| < \delta$$

and since $h_j(\tau) \to 0$ choose $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$|h_j(\tau)| < \varepsilon/2 \ \text{ and } |t_j - \tau| < \delta.$$

Then for all $j \in J$ such that $j \geq j_0$, we have

$$\begin{aligned} |h_j(t_j)| &\leq |h_j(t_j) - h_j(\tau)| + |h_j(\tau)| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

contradictory to (3.13).

By Gronwall's inequality (Proposition 1.13), from (3.12) we have

$$\begin{aligned} \|\phi_t^n - \phi_t\| &\leq \|z_n - z\| + |h_n(t)| + \int_0^t L_f[\|z_n - z\| + |h_n(s)|]e^{\int_s^t L_f dl} ds \\ &= \|z_n - z\| + |h_n(t)| + L_f C(t)\|z_n - z\| + L_f C(t) \int_0^t |h_n(s)| ds \\ &\leq \|z_n - z\| + C(T)L_f T\|z_n - z\| + C(T)L_f \int_0^t |h_n(s)| ds, \end{aligned}$$

where $C(t) := \int_0^t e^{\int_s^t L_f dl} ds$. We conclude that $\phi_t^n \to \phi_t$ as $n \to \infty$ uniformly in $t \in [0, T]$ from the fact that $h_n(t) \to 0$ uniformly. ■

We now have the following continuity result.

Proposition 3.2 *Suppose conditions* (A3.1)–(A3.7) *hold. Then the maps* $g : E^0 \times \mathcal{U} \longrightarrow I\!R_+$ *and* $Q : E^0 \times \mathcal{U} \longrightarrow I\!P(E^0)$ *are continuous respect to the appropriate topologies.* ■

Davis (1986) proved this result with the assumption of no boundary controls. Using same technique, we shall give a proof allowing boundary controls.

Proof Fix $\widetilde{u}_{0(\cdot)} \in \widetilde{\mathcal{U}}_0$, $\theta(\cdot) \in C(E^0)$ and $z \in E^0$, define functions $\phi_t, \eta_t, I_t^\theta$ and C_t by the following differential equations

$$\dot{\phi}_t = \int_{U_0} f(\phi_t, u)\widetilde{u}_{0t}(du) \qquad \phi_0 := z \tag{3.14}$$

$$\dot{\eta}_t = -\int_{U_0} \lambda(\phi_t, u)\eta_t \widetilde{u}_{0t}(du) \qquad \eta_0 := 1 \tag{3.15}$$

$$\dot{I}_t^\theta = \int_{U_0} \int_{E^0} \theta(y) Q_0(dy; \phi_t, u)\lambda(\phi_t, u)\eta_t \widetilde{u}_{0t}(du) \qquad I_0^\theta := 0 \tag{3.16}$$

$$\dot{C}_t = \int_{U_0} \eta_t l_0(\phi_t, u)\widetilde{u}_{0t}(du) \qquad C_0 := 0. \tag{3.17}$$

To show that Q is continuous we must show that if $(z_n, \tilde{u}_n) \to (z, \tilde{u})$ as $n \to \infty$, then

$$\int_{E^0} \theta(y) Q(dy; z_n, \tilde{u}_n) \to \int_{E^0} \theta(y) Q(dy; z, \tilde{u}) \tag{3.18}$$

for all $\theta \in C(E^0)$.

Let ϕ_t be the solution of (3.14) with boundary hitting time t_* and ϕ_t^n be the solution of (3.14) with $(z_n, \tilde{u}_n)$ replacing $(z, \tilde{u})$ with boundary hitting time t_*^n and let η_t, η_t^n and I_t, I_t^n be the corresponding solutions of (3.15) and (3.16) for $\theta \in C(E^0)$ fixed. Then

$$\int_{E^0} \theta(y) Q(dy; z_n, \tilde{u}_n) = I_{t_*^n}^n + \eta_{t_*^n}^n \int_{E^0} \theta(y) Q_\partial(dy; \phi_{t_*^n}^n, \tilde{u}_n). \tag{3.19}$$

Case a. $t_* = \infty$.

Let $T_\varepsilon(t)$ denote the ε-tube around ϕ up to time t, i.e.

$$T_\varepsilon(t) := \{y \in I\!R^n : \min_{0 \le s \le t} \|\phi_s - y\| < \varepsilon\}.$$

Lemma 3.1 implies that $\{\phi_s^n, 0 \le s \le t\} \subset T_\varepsilon(t)$ for n sufficiently large and hence, since $T_\varepsilon(t) \subset E^0$ for small ε, that $t_*^n \to \infty$ as $n \to \infty$. Thus using (A3.4) the last term in (3.19) converges to zero as $n \to \infty$ and it remains to show that $I_{t_*^n}^n \to I_\infty$.

It follows from the definitions that $\eta_t^n \le e^{-\delta t}$ for all n and that $\eta_t \le e^{-\delta t}$ which together imply that $I_t^n \to I_\infty^n$ as $t \to \infty$ uniformly in n and $I_t \to I_\infty$. Applying the general convergence theorem, we have $I_t^n \to I_t$ as $n \to \infty$ uniformly in t by Lemma 3.1. Hence $I_\infty^n \to I_\infty$ as $n \to \infty$; since for any $T > 0$ we can write

$$|I_\infty^n - I_\infty| \le |I_\infty^n - I_T^n| + |I_T^n - I_T| + |I_T - I_\infty|.$$

We can now choose T (independently of n) and then n to yield $|I_\infty^n - I_\infty| < \varepsilon$ for any $\varepsilon > 0$. Finally, we have

$$|I_{t_*^n}^n - I_\infty| \le |I_{t_*^n} - I_\infty^n| + |I_\infty^n - I_\infty|.$$

The result follows, using again the uniform convergence of I_t^n to I_∞^n and the fact that $t_*^n \to \infty$ as $n \to \infty$.

Case b. $t_* < \infty$

Let $\xi_t := (\phi_t{}', \eta_t, I_t^\theta\ C_t)'$ denote the solution of equations (3.14)–(3.17) and let ξ_t^n denote this solution with $(z_n, \tilde{u}_n)$ replacing $(z, \tilde{u})$. These solutions evolve in the space $\hat{E} := E \times I\!R^3 = \{(x,y) : \hat{\psi}(x,y) > 0\}$, where $\hat{\psi}(x,y) := \psi(x)$, the original boundary function of §3.1, for $x \in I\!R^n, y \in I\!R^3$. It follows from this construction that if $f(x,v)$ satisfies (A3.7) then ξ_t satisfies

$$\liminf_{t \to t_*} \frac{(\xi_{t_*} - \xi_t)'\nu}{t_* - t} > 0, \tag{3.20}$$

where $\nu := \nabla\hat{\psi}(\xi_{t_*})[\|\nabla\hat{\psi}(\xi_{t_*})\|]^{-1}$ is the outward unit normal to $\partial\hat{E}$. Now the right hand side of (3.19) takes the form $h(\xi_{t_*^n}^n, \tilde{u}_n)$ for some continuous function $h(\cdot,\cdot)$, so to establish (3.18) it suffices to prove that $\xi_{t_*^n}^n \to \xi_{t_*}$ as $n \to \infty$. We do this by showing that

$$\mathrm{diam}(\hat{T}_\varepsilon(t_*) \cap \partial\hat{E}) \to 0 \quad \text{as } \varepsilon \downarrow 0 \tag{3.21}$$

where $\hat{T}_\varepsilon(t)$ is an ε-tube around $\{\xi_s, 0 \le s \le t\}$. Applying Lemma 3.1, we have $\xi_t^n \to \xi_t$ as $n \to \infty$ for any t. Therefore for arbitrary t and ε, $\{\xi_s^n, 0 \le s \le t\} \subset \hat{T}_\varepsilon(t)$ for sufficiently large n. Hence (3.21) will imply that eventually t_*^n is finite and that $|\xi_{t_*^n}^n - \xi_{t_*}| < \mathrm{diam}(\hat{T}_\varepsilon(t_*) \cap \partial\hat{E})$. ¿From the hypothesis that $\hat{E}$ has a C^1 boundary we see that (3.21) holds if

$$\mathrm{diam}(\hat{T}_\varepsilon(t_*) \cap \Gamma) \to 0 \qquad \text{as } \varepsilon \downarrow 0, \tag{3.22}$$

where Γ is the tangent hyperplane to $\partial\hat{E}$ at ξ_{t_*}.

Thus it remains to establish (3.22). To this end, using (3.20), take $t_1 \in [0, t_*)$ and $\gamma > 0$ such that for all $t \in [t_1, t_*)$

$$(\xi_{t_*} - \xi_t)'\nu > \gamma(t_* - t). \tag{3.23}$$

Since $\{\xi_s, 0 \le s \le t_1\} \subset \hat{E}^0$, there exists $\varepsilon_1 > 0$ such that $\hat{T}_{\varepsilon_1}(t_1) \subset \hat{E}^0$. Thus for $\varepsilon \in (0, \varepsilon_1]$, $\hat{T}_\varepsilon(t_*) \cap \Gamma = [\hat{T}_\varepsilon(t_*) - \hat{T}_\varepsilon(t_1)] \cap \Gamma$, i.e. we can discard the trajectory prior to t_1. From (A3.2) the velocity of ξ_t is locally bounded, i.e. there exists a constant α such that for $t \in [t_1, t_*)$

$$\|\xi_t - \xi_{t_*}\| < \alpha(t_* - t). \tag{3.24}$$

Now (3.23) and (3.24) imply that the trajectory segment $F = \{\xi_t, t_1 \leq t \leq t_*\}$ is contained in a cone of internal angle $2\cos(\gamma/\alpha)$. Hence if $\varepsilon < \varepsilon_1$, and $y \in \Gamma$ is such that $d_F(y) < \varepsilon$ then $\|y - \xi_{t_*}\| < \alpha\varepsilon/\gamma$, i.e. $diam(\hat{T}_\varepsilon(t_*) \cap \Gamma) < 2\alpha\varepsilon\gamma$. This establishes (3.22) and shows that $\mathcal{Q}$ is continuous.

Continuity of g is proved in an exactly similar manner. ∎

From above discussion, we have seen that the control space $\widetilde{\mathcal{U}}_0 \times U_\partial$ is compact and that g and $\mathcal{Q}$ are continuous. Therefore we conclude that the problem (DP1) with relaxed controls may be defined as:

$$(\widetilde{DP}1) \quad \text{minimize} \quad E_x \sum_{k=0}^{\infty} g(\mathbf{z}_k, \mu_k)$$

$$\text{over relaxed policies } \pi := (\mu_0, \mu_1, \ldots, \mu_N, \ldots)$$

$$\text{such that} \quad \mu_k(z_0, \mu_0, \ldots, z_{k-1}, \mu_{k-1}, z_k) \in \widetilde{\mathcal{U}}_0 \times U_\partial \quad k = 1, 2, \ldots,$$

where $\mathbf{z}_k$ is the discrete time Markov process with the transition measure given by $\mathcal{Q}(A; z, \tilde{u})$ and that belongs both to the classes of upper and of lower semicontinuous models.

In particular, if $\pi := (\mu, \mu, \ldots)$ for $\mu : E^0 \longrightarrow \widetilde{\mathcal{U}}_0 \times U_\partial$, then the policy is called a *stationary (relaxed) policy*. Moreover if $\pi := (\mu, \mu, \ldots)$ for $\mu : E^0 \longrightarrow \mathcal{U}_0 \times U_\partial$, then the policy is called a *simple* stationary policy.

By Corollary 9.17.2 of Bertsekas and Shreve (1978), p.235, there exists a Borel measurable optimal stationary policy for $(\widetilde{DP}1)$, i.e. $\mu : E^0 \longrightarrow \widetilde{\mathcal{U}}$ is Borel measurable. Therefore, from now on, we may consider only stationary polices.

Rewrite the problem $(\widetilde{DP}1)$ with stationary policies as follows:

$$(\widetilde{DP}1) \quad \text{minimize} \quad E_x \sum_{k=0}^{\infty} g(\mathbf{z}_k, \mu(\mathbf{z}_k))$$

$$\text{over stationary policies } \mu = (\mu_0, \mu_\partial) : E^0 \longrightarrow \widetilde{\mathcal{U}}_0 \times U_\partial,$$

where $\mathbf{z}_k$ is the discrete time Markov process with the transition measure given by $\mathcal{Q}(A; z, \mu(z))$.

We now reformulate the PDP optimal c .rol problem of §3.1 as (DP1). In the case of no boundary control Davis (1986) has shown that the PDP optimal control problem can be reformulated as (DP1) by considering the the embedded Markov chain given by sequence of postjump states, i.e. $\{\mathbf{z}_k := \mathbf{x}_{T_k}, k = 0, 1, \ldots\}$. For the case in which boundary controls are allowable, we now prove the following theorem using the same techniques.

Proposition 3.3 *The PDP optimal control problem is equivalent to the following infinite horizon discrete time stochastic decision problem:*

$$(DP2) \qquad \textit{minimize} \qquad E_x \sum_{k=0}^{\infty} g(\mathbf{z}_k, \mu(\mathbf{z}_k))$$

$$\textit{over simple stationary policies } \mu := (\mu_0, \mu_\partial) : E^0 \longrightarrow \mathcal{U}_0 \times U_\partial$$

$$\textit{such that} \qquad \mu(z) := (u_{0t}(z), u_\partial(\phi^{\mathbf{u}}_{t_*}(z)))$$

where $\mathbf{z}_k := \mathbf{x}_{T_k}$, the postjump process, is a discrete time Markov process with the transition measure given by $Q(A; z, \mu(z))$. ∎

Remark 3.1 If we define $(DP2)$ with relaxed control $(\widetilde{DP}2)$ as above, then the PDP optimal control problem with relaxed interior controls is equivalent to $(\widetilde{DP}2)$ by this theorem. ∎

Proof For convenience, we avoid explicit mention of the discount factor by introducing *killing*. We adjoin an isolated point $\triangle$ (termed the *coffin state*) to the state space and define a new process $\hat{\mathbf{x}}_t$ on $E^0_\triangle := E^0 \cup \{\triangle\}$ by

$$\hat{\mathbf{x}}_t = \begin{cases} \mathbf{x}_t & t < \mathbf{T} \\ \triangle & t \geq \mathbf{T} \end{cases}$$

where $\mathbf{T}$ is an independent exponentially distributed (*process*) *killing time* (i.e. $P_x[\mathbf{T} > t] = e^{-\delta t}$ for all $x \in E^0$). The running cost $l_0 : E^0 \times U_0 \longrightarrow \mathbb{R}_+$ is extended to $E^0_\triangle \times U_0$ by setting $l_0(\triangle, u) := 0$ for all $u \in U_0$. The boundary cost $l_\partial : E \times U_\partial \longrightarrow \mathbb{R}^+$ is extended to $E_\triangle \times U_\partial$ by setting $l_\partial(\triangle, u) := 0$ for all $u \in U_\partial$ where $E_\triangle := E^0_\triangle \cup \partial E$.

Let $\mathcal{F}_t$ denote the natural filtration of $\hat{\mathbf{x}}_t$, Then the cost $J_x(u)$ can be written in terms of the killed process $\hat{\mathbf{x}}_t$ as

$$\begin{aligned}
J_x(u) &= E_x[\int_0^\infty l_0(\hat{\mathbf{x}}_t, u_0(\hat{\tau}_t, \hat{\mathbf{z}}_t)dt \\
&\quad + \sum_k l_\partial(\hat{\mathbf{x}}_{T_k^-}, u_\partial(\hat{\mathbf{x}}_{T_k^-}))I_{(\hat{\mathbf{x}}_{T_k^-} \in \partial E)}] \\
&= E_x \sum_{k=0}^\infty E_x[\int_{\mathbf{T}_k}^{\mathbf{T}_{k+1}} l_0(\hat{\mathbf{x}}_t, u_0(\hat{\tau}_t, \hat{\mathbf{z}}_t))dt \\
&\quad + l_\partial(\hat{\mathbf{x}}_{T_{k+1}^-}, u_\partial(\hat{\mathbf{x}}_{T_{k+1}^-}))I_{(\hat{\mathbf{x}}_{T_{k+1}^-} \in \partial E)}|\mathcal{F}_{\mathbf{T}_k}] \\
&= E_x \sum_{k=0}^\infty E_{\hat{\mathbf{x}}_{T_k}}[\int_0^{\mathbf{T}_{k+1}-\mathbf{T}_k} l_0(\hat{\mathbf{x}}_t, u_0(t, \hat{\mathbf{x}}_{T_k}))dt \\
&\quad + l_\partial(\hat{\mathbf{x}}_{(T_{k+1}-T_k)^-}, u_\partial(\hat{\mathbf{x}}_{(T_{k+1}-T_k)^-})I_{(\hat{\mathbf{x}}_{(T_{k+1}-T_k)^-} \in \partial E)}]
\end{aligned} \tag{3.25}$$

where the last equation holds by virtue of the strong Markov property (see §1.2).

Let $\phi_t^u(x)$ be the deterministic flow given by

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\phi_t^u(x) = f(\phi_t^u(x), u_{0t}(x)) \text{ a.e. } t \in [0, t_*^u(x)) \\
&\phi_0^u(x) := x,
\end{aligned}$$

where $t_*^u(x) := \inf\{t > 0 : \phi_t^u(x) \in \partial E\}$. Then by the construction of a PDP we have

$$\begin{aligned}
&E_x[\int_0^{\mathbf{T}_1} l_0(\hat{\mathbf{x}}_t, u_{0t}(x))dt + l_\partial(\hat{\mathbf{x}}_{T_1^-}, u_\partial(\hat{\mathbf{x}}_{T_1^-}))I_{(\hat{\mathbf{x}}_{T_1^-} \in \partial E)}] \\
&= E_x[\int_0^{\mathbf{T}_1} e^{-\delta t} l_0(\mathbf{x}_t, u_{0t}(x))]dt + E_x[e^{-\delta \mathbf{T}_1} l_\partial(\mathbf{x}_{T_1^-}, u_\partial(\mathbf{x}_{T_1^-}))I_{(\mathbf{x}_{T_1^-} \in \partial E)}] \\
&= \int_0^{t_*^u(x)} \int_0^s e^{-\delta t} l_0(\phi_t^u(x), u_{0t}(x))dt \frac{d(1 - P[\mathbf{T}_1 > s])}{ds} ds \\
&\quad + e^{-\delta t_*^u(x)} l_\partial(\phi_{t_*}^u(x), u_\partial(\phi_{t_*}^u(x)))P[\mathbf{T}_1 = t_*^u(x)] \\
&= -\int_0^s e^{-\delta t} l_0(\phi_t^u(x), u_{0t}(x))dt P[\mathbf{T}_1 > s]|_0^{t_*^u(x)} \\
&\quad + \int_0^{t_*^u(x)} e^{-\delta s} l_0(\phi_s^u(x), u_{0s}(x))P[\mathbf{T}_1 > s]ds \\
&\quad + e^{-\delta t_*^u(x)} l_\partial(\phi_{t_*}^u(x), u_\partial(\phi_{t_*}^u(x)))e^{-\int_0^{t_*} \lambda(\phi_t^u(x), u_{0t}(x))dt} \\
&= 0 + \int_0^{t_*^u(x)} e^{-\Lambda_s^u(x)} l_0(\phi_s^u(x), u_{0s}(x))ds + e^{-\Lambda_{t_*}^u(x)} l_\partial(\phi_{t_*}^u(x), u_\partial(\phi_{t_*}^u(x))) \\
&= g(x, \mu(x)).
\end{aligned} \tag{3.26}$$

Similarly, we find that

$$
\begin{aligned}
&P_x[\hat{\mathbf{x}}_{T_1} \in A] && (3.27)\\
&= P_x[\hat{\mathbf{x}}_{T_1} \in A | \mathbf{T}_1 < t_*^u(x)] P_x[\mathbf{T}_1 < t_*^u(x)] \\
&\quad + P[\hat{\mathbf{x}}_{T_1} \in A | \mathbf{T}_1 = t_*^u(x)] P[\mathbf{T}_1 = t_*^u(x)] \\
&= \int_0^{t_*^u(x)} P_x[\hat{\mathbf{x}}_{T_1} \in A | \mathbf{T}_1 = t] \frac{d(1 - P[\mathbf{T}_1 > t])}{dt} dt \\
&\quad + P_x[\hat{\mathbf{x}}_{T_1} \in A | \mathbf{T}_1 = t_*^u(x)] P[\mathbf{T}_1 = t_*^u(x)] \\
&= \int_0^{t_*^u(x)} e^{-\delta t} P_x[\mathbf{x}_{T_1} \in A | \mathbf{T}_1 = t] \lambda(\phi_t^u(x), u_{0t}(x)) e^{-\int_0^t \lambda(\phi_s^u(x), u_{0s}(x)) ds} dt \\
&\quad + e^{-\delta t_*^u(x)} P_x[\hat{\mathbf{x}}_{T_1} \in A | \mathbf{T}_1 = t_*^u(x)] e^{-\int_0^{t_*} \lambda(\phi_s^u(x), u_{0s}(x)) ds} \\
&= \int_0^{t_*^u(x)} e^{-\delta t} Q_0(A; \phi_t^u(x), u_{0t}(x)) \lambda(\phi_t^u(x), u_{0t}(x)) e^{-\int_0^t \lambda(\phi_s^u(x), u_{0s}(x)) ds} dt \\
&\quad + e^{-\delta t_*^u(x)} Q_\partial(A; \phi_{t_*}^u(x), u_\partial(\phi_{t_*}^u(x))) e^{-\int_0^{t_*} \lambda(\phi_s^u(x), u_{0s}(x)) ds} \\
&= \int_0^{t_*^u(x)} Q_0(A; \phi_t^u(x), u_{0t}(x)) \lambda(\phi_t^u(x), u_{0t}(x)) e^{-\Lambda_t^u(x)} dt \\
&\quad + e^{-\Lambda_{t_*}^u(x)} Q_\partial(A; \phi_{t_*}^u(x), u_\partial(\phi_{t_*}^u(x))) \\
&= \mathcal{Q}(A; x, \mu(x)). && (3.28)
\end{aligned}
$$

Let $\mathbf{z}_k := \mathbf{x}_{T_k}$ be the postjump process. By virtue of equality (3.28), $\mathbf{z}_k$ is a discrete time Markov process with the transition measure $\mathcal{Q}$. Due to equality (3.26), the cost (3.25) can be rewritten as

$$J_x(u) = E_x \sum_{k=0}^{\infty} g(\mathbf{z}_k, \mu(\mathbf{z}_k)).$$

■

(DP2) (($\widetilde{DP}2$)) is equivalent to (DP1) (($\widetilde{DP}1$)) if and only if the following result holds. As a consequence, the PDP optimal control problem (with relaxed controls) is equivalent to (DP1) (($\widetilde{DP}1$)).

Proposition 3.4 *If $\mu^* = (\mu_0^*, \mu_\partial^*)$ is an optimal policy for $(\widetilde{DP}1)$, then there exists a feedback function $u_\partial : \partial E \longmapsto U_\partial$ such that $u_\partial(\phi_{t_*}^{\tilde{u}}(x)) = \mu_\partial^*(x)$, i.e. if μ^* is an optimal policy for $(\widetilde{D\,1})$ and $\phi_{t_*}^{\tilde{u}}(x) = \phi_{t_*}^{\tilde{u}}(z)$, then $\mu_\partial^*(x) = \mu_\partial^*(z)$ for all $x, z \in E^0$.*

Proof Suppose the assertion is not true. That is, there exists $x \neq z$ such that $\phi^{\tilde{u}}_{t_*}(x) = \phi^{\tilde{u}}_{t_*}(z)$ and $\mu^*_\partial(x) \neq \mu^*_\partial(z)$. Without loss of generality, we can assume that

$$\begin{aligned} & l_\partial(\phi^{\tilde{u}}_{t_*}(x), \mu^*_\partial(x)) + \int_{E^0} V(y) Q_\partial(dy; \phi^{\tilde{u}}_{t_*}(x), \mu^*_\partial(x)) \\ > \quad & l_\partial(\phi^{\tilde{u}}_{t_*}(x), \mu^*_\partial(z)) + \int_{E^0} V(y) Q_\partial(dy; \phi^{\tilde{u}}_{t_*}(x), \mu^*_\partial(z)). \end{aligned}$$

By Proposition 9.12 of Bertsekas and Shreve (1978), p.227, a stationary (relaxed) policy μ^* is optimal, if and only if for all $x \in E^0$, $\mu^*(x) \in \tilde{\mathcal{U}}_0 \times U_\partial$ solves the following problem

$$(\widetilde{P_x}) \qquad \min_{\tilde{u} \in \tilde{\mathcal{U}}_0 \times U_\partial} \{ g(x, \tilde{u}) + \int_{E^0} V(y) Q(dy; x, \tilde{u}) \}.$$

Let $\bar{\mu}(x) := (\mu^*_0(x), \mu^*_\partial(z))$, for which we have

$$\begin{aligned} & g(x, \bar{\mu}(x)) + \int_{E^0} V(y) Q(dy; x, \bar{\mu}(x)) \\ = \quad & \int_0^{t_*} \int_{U_0} e^{-\Lambda^{\tilde{u}}_t(x)} l_0(\phi^{\tilde{u}}_t(x), u) \tilde{u}_{0t}(du) dt + e^{-\Lambda^{\tilde{u}}_{t_*}(x)} l_\partial(\phi^{\tilde{u}}_{t_*}(x), \mu^*_\partial(z)) \\ & + \int_{E^0} \int_0^{t_*} \int_{U_0} V(y) \lambda(\phi^{\tilde{u}}_t(x), u) e^{-\Lambda^{\tilde{u}}_{t_*}(x)} Q_0(dy; \phi^{\tilde{u}}_t(x), u) \tilde{u}_{0t}(du) dt \\ & + e^{-\Lambda^{\tilde{u}}_{t_*}(x)} \int_{E^0} V(y) Q_\partial(dy; \phi^{\tilde{u}}_t(x), \mu^*_\partial(z)) \\ < \quad & g(x, \mu^*(x)) + \int_{E^0} V(y) Q(dy; x, \mu^*(x)). \end{aligned}$$

Therefore, $\bar{\mu}(x)$ is a better solution than $\mu^*(x)$. This contradicts the fact that $\mu^*(x)$ is optimal for $(\widetilde{P_x})$. ∎

3.3 Regularity Properties of the Value Function

We show first that the value function of the PDP is continuous by showing that this property holds for the value function of the equivalent problem $(\widetilde{DP}1)$.

By Corollary 9.17.2 of Bertsekas and Shreve (1978), p.235, since $(\widetilde{DP}1)$ belongs to the class of lower semicontinuous models, $V(x)$ is lower semicontinuous. By their Proposition 9.21, p.241, for negative upper semicontinuous models, $V(x)$ is upper semicontinuous. Our model does not satisfy the negativity assumption of g but g is bounded above. Suppose $g(x,\tilde{u}) \leq b$, then $g(x,\tilde{u})-b \leq 0$ and the value function using $g(x,\tilde{u}) - b$ as one step cost instead of $g(x,\tilde{u})$ will be upper semicontinuous. Recall from the proof of Proposition 3.3 that the killing time T is distributed exponentially, i.e. $P_x[\mathrm{T} > t] = e^{-\delta t}$. Since $E_x[\mathrm{T}] = \delta^{-1} < \infty$ and $P_x[\lim_{n\to\infty} \mathrm{T}_n = \infty] = 1$, we must have

$$\begin{aligned}
\tilde{J}_x(u) &= E_x \sum_{k=0}^{\infty} \{g(\mathbf{z_k}, \mu(\mathbf{z_k})) - b\} \\
&= E_x \sum_{k=0}^{\infty} g(\mathbf{z_k}, \mu(\mathbf{z_k})) - bE_x(\mathrm{N}) \\
&= J_x(u) - bE_x(\mathrm{N}),
\end{aligned}$$

where N is the jump number at which the process is killed, i.e. $\mathbf{z_N} = \mathbf{x}_T$, the coffin state.

Therefore,

$$\tilde{V}(x) = V(x) - bE_x[\mathrm{N}],$$

where $\tilde{V}(x)$ denotes the value function with $g(x,\tilde{u}) - b$ as the one step cost.

Consequently, by Proposition 9.21 of Bertsekas and Shreve (1978), p.241, $\tilde{V}(x)$ is upper semicontinuous. This implies that $V(x)$ is upper semicontinuous and hence $V(x)$ is continuous.

Next we can show that $V(x)$ is bounded in E^0.

Indeed, suppose $l_0 \leq M_0$, $l_\partial \leq M_\partial$, for any $\in E^0$ and any $u = (u_0, u_\partial)$. Then

$$\begin{aligned}
J_x(u) &= E_x[\int_0^\infty e^{-\delta t} l_0(\mathbf{x}_t, u_0(\tau_t, \mathbf{z}_t))dt + \sum_i e^{-\delta \mathrm{T}_i} l_\partial(\mathbf{x}_{\mathrm{T}_i^-}, u_\partial(\mathbf{x}_{\mathrm{T}_i^-})) I_{(\mathbf{x}_{\mathrm{T}_i^-} \in \partial E)}] \\
&\leq M_0 \int_0^\infty e^{-\delta t} dt + M_\partial E_x[\sum_k e^{-\delta \mathrm{T}_k}] \\
&\leq M_0/\delta + M_\partial/\delta = (M_0 + M_\partial)/\delta.
\end{aligned}$$

3.4 Reduction to the Family of Deterministic Problems

The purpose of this section is to prove the following theorem.

Theorem 3.1 *The PDP optimal (relaxed) control $\tilde{u} = (\tilde{u}_0, u_\partial)$ is equivalent to choosing for each possible postjump state $z \in E^0$, an optimal (relaxed) control function $\tilde{u}_{0(\cdot)}(z)$ in the deterministic control problem with boundary condition (P_z) defined in §2.1 where*

$$f_0(x,v) := l_0(x,v) + \int_{E^0} V(y)Q_0(dy;x,v) \tag{3.29}$$

$$F(x) := \min_{v\in U_\partial}\{l_\partial(x,v) + \int_{E^0} V(y)Q_\partial(dy;x,v)\} \tag{3.30}$$

$$\bar{\lambda}(x,v) := \lambda(x,v) + \delta \tag{3.31}$$

and for each $z \in \partial E$, an optimal feedback control action $u_\partial(z)$ which solves the following optimization problem:

$$\min_{v\in\partial E}\{l_\partial(z,v) + \int_{E^0} V(y)Q_\partial(dy;z,v)\}.$$

Proof By Proposition 9.12 of Berksekas and Shreve (1978), p.227, a stationary policy μ is optimal if and only if for any $x \in E^0$, $\mu(x) \in \tilde{\mathcal{U}}_0 \times U_\partial$ solves the following problem

$$(\widetilde{P}_x) \qquad \min_{\tilde{u}\in\tilde{\mathcal{U}}_0\times U_\partial}\{g(x,\tilde{u}) + \int_{E^0} V(y)Q(dy;x,\tilde{u})\}$$

and $V(x)$ is the corresponding value.

Using the definitions of g (see (3.10)) and Q (see (3.11)), we have

$$\begin{aligned} & g(x,\tilde{u}) + \int_{E^0} V(y)Q(dy;x,\tilde{u}) \\ = & \int_0^{t_*}\int_{U_0} e^{-\Lambda_t^{\tilde{u}}(x)} l_0(\phi_t^{\tilde{u}}(x),u)\tilde{u}_{0t}(du)dt + e^{-\Lambda_{t_*}^{\tilde{u}}(x)} l_\partial(\phi_{t_*}^{\tilde{u}}(x),v) \\ & + \int_{E^0}\int_0^{t_*}\int_{U_0} V(y)Q_0(dy;\phi_t^{\tilde{u}}(x),u)\lambda(\phi_t^{\tilde{u}}(x),u)e^{-\Lambda_t^{\tilde{u}}(x)}\tilde{u}_{0t}(du)dt \end{aligned}$$

$$
\begin{aligned}
&+e^{-\Lambda_{t_*}^{\tilde{u}}(x)}\int_{E^0}V(y)Q_\partial(dy;\phi_{t_*}^{\tilde{u}}(x),v)\\
= &\int_0^{t_*}\int_{U_0}e^{-\Lambda_t^{\tilde{u}}(x)}\{l_0(\phi_t^{\tilde{u}}(x),u)\tilde{u}_{0t}(du)dt\\
&+\int_{E^0}V(y)Q_0(dy;\phi_t^{\tilde{u}}(x),u)\lambda(\phi_t^{\tilde{u}}(x),u)\}\tilde{u}_{0t}(du)dt\\
&+e^{-\Lambda_{t_*}^{\tilde{u}}(x)}\{l_\partial(\phi_{t_*}^{\tilde{u}}(x),v)+\int_{E^0}V(y)Q_\partial(dy;\phi_{t_*}^{\tilde{u}}(x),v)\},
\end{aligned}
$$

where the last equlity follows from Fubini theorem.

Define f_0, F and $\bar{\lambda}$ by (3.29), (3.30) and (3.31) respectively. The desired result follows in a straightforward manner. ∎

3.5 Existence of an Optimal Ordinary Control

We now impose the following *assumption*:

(A3.8) The set

$$N_\theta(x) := \{(f(x,u)',\lambda(x,u),l_0(x,u)+\int_{E^0}\theta(y)Q_0(dy;x,u)\lambda(x,u)) : u\in U_0\}$$

is convex for all $x\in E^0$ and $\theta\in C(E^0)$.

This required only in the interests of clear presentation to obviate the necessity for considering relaxed or generalized control policies in cumbersome detail. The following approximation lemma is given by Young (1969), Lemma 76.1, p.190.

Lemma 3.2 *Let U denote a compact (σ-compact) subset of a Banach space X with norm $\|\cdot\|_X$. Let $f: U \longrightarrow X$ be continuous and take $\delta > 0$. Then there exists a finite number of points $u_i \in U$ and a finite number of continuous functions $h_i : U \longrightarrow [0,1]$ such that $\sum_i h_i = 1$ and*

$$\|f(\cdot)-\textstyle\sum_i f(u_i)h_i(\cdot)\|_\infty < \delta,$$

where $\|g\|_\infty =: \sup_{u\in U}\|g(u)\|_X$. ∎

Using the approximation lemma, it is easy to see that the following proposition (see Young 1969, Theorem 79.1, p.192) holds.

Proposition 3.5 *Let U denote a compact (σ-compact) subset of a Banach space X. Let $f : U \longrightarrow X$ be continuous and μ be a unit Riesz measure on U, i.e. a nonnegative Borel measure satisfying $\int_U 1 d\mu = 1$. Then*

$$\int_U f(u)d\mu \in \overline{co}\{f(u) : u \in U\}.$$

■

Now we are ready for the following existence result.

Theorem 3.2 *Under assumption* (A3.1)–(A3.8), *there exists a simple optimal stationary policy (or equivalently, an ordinary optimal control for the original PDP).*

Proof By Corollary 9.17.2 of Berksekas and Shreve (1978), p.235, there exists an optimal stationary policy for $(\widetilde{DP}1)$. Under our assumption that $N_\theta(x)$ is convex, we shall show that this stationary policy must be a simple one.

Reformulate the problem (P_x) with relaxed controls as a problem of Mayer type.

$$\begin{aligned}
(P_{0x}) \quad & \text{minimize} \quad x_{n+1}(t) + x_0(t_*)F(\phi^{\tilde{u}}_{t_*}(x)) \\
& \text{over all admissible controls } \tilde{u}_{0(\cdot)} \in \tilde{\mathcal{U}}_0 \\
& \text{such that} \quad \dot{\phi}^{\tilde{u}}_t(x) = \int_{U_0} f(\phi^{\tilde{u}}_t(x), u)\tilde{u}_{0t}(du) \qquad \text{a.e. } t \in [0, t_*) \\
& \qquad \dot{x}_0(t) = -x_0(t)\int_{U_0}[\lambda(\phi^{\tilde{u}}_t(x), u) + \delta)\tilde{u}_{0t}(du)] \qquad \text{a.e. } t \in [0, t_*) \\
& \qquad \dot{x}_{n+1}(t) = -x_0(t)\int_{U_0}[l_0(\phi^{\tilde{u}}_t(x), u) \\
& \qquad\qquad + \int_{E^0} V(y)Q_0(dy; \phi^{\tilde{u}}_t(x), u)\lambda(\phi^{\tilde{u}}_t(x), u)]\tilde{u}_{0t}(du) \quad \text{a.e. } t \in [0, t_*) \\
& \qquad \phi^{\tilde{u}}_0(x) := x \\
& \qquad x_0(0) := 1 \\
& \qquad x_{n+1}(0) := 0.
\end{aligned}$$

Define the set

$$M_\theta(x', x_0) := \{f(x, u)', x_0\lambda(x, u), x_0[l_0(x, u) + \int_{E^0} \theta(y)Q_0(dy; x, u)\lambda(x, u)]) : u \in U_0\}.$$

By Proposition 3.5, since every relaxed control $\tilde{u}_0 \in \tilde{\mathcal{U}}_0$ is a unit Riesz measure, we have

$$(\dot{\phi}^{\tilde{u}}_t(x), \dot{x}_0(t), \dot{x}_{n+1}(t)) \in \overline{co}M_V(\phi^{\tilde{u}}_t(x)', x_0). \tag{3.32}$$

For any fixed (x', x_0), $M_\theta(x', x_0)$ is convex and compact by virtue of the convexity of $N_\theta(x)$ (assumption A3.8) and the compactness of U_0. Consequently, $\overline{co}M_\theta(x', x_0) = M_\theta(x', x_0)$. In particular, since $V(x) \in C(E^0)$ by §3.3, we have $\overline{co}M_V(x', x_0) = M_V(x', x_0)$. Consequently, the right hand side of (3.32) is equal to

$$\begin{aligned} &M_V(\phi_t^{\tilde{u}}(x)', x_0(t)) \\ &= \{(f(\phi_t^{\tilde{u}}(x), u)', x_0(t)\lambda(\phi_t^{\tilde{u}}(x), u), x_0(t)[l_0(\phi_t^{\tilde{u}}(x), u) \\ &\quad + \int_{E^0} V(y)Q_0(dy; \phi_t^{\tilde{u}}(x), u)\lambda(\phi_t^{\tilde{u}}(x), u)]) : u \in U_0\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &(\dot{\phi}_t^{\tilde{u}}(x)', \dot{x}_0(t), \dot{x}_{n+1}(t)) \\ &\in \{(f(\phi_t^{\tilde{u}}(x), u)', x_0(t)\lambda(\phi_t^{\tilde{u}}(x), u), x_0(t)[l_0(\phi_t^{\tilde{u}}(x), u) \\ &\quad + \int_{E^0} V(y)Q_0(dy; \phi_t^{\tilde{u}}(x), u)\lambda(\phi_t^{\tilde{u}}(x), u)]) : u \in U_0\}. \end{aligned}$$

By the Fillippov Lemma (see e.g. Young 1969, Corollary 34.7, p.297), we can choose a measurable function $u_{0(\cdot)} : [0, t_*) \longrightarrow U_0$ such that

$$\begin{aligned} &\dot{\phi}_t^{\tilde{u}}(x) = f(\phi_t^{\tilde{u}}(x), u_{0t}) \\ &\dot{x}_0(t) = -x_0(t)[\lambda(\phi_t^{\tilde{u}}(x), u_{0t}) + \delta] \\ &\dot{x}_{n+1} = x_0(t)\{l_0(\phi_t^{\tilde{u}}(x), u_{0t}) + \int_{E^0} V(y)Q_0(dy; \phi_t^{\tilde{u}}(x), u_{0t})\lambda(\phi_t^{\tilde{u}}(x), u_{0t})\}. \end{aligned}$$

This implies that the optimal control can be taken to be ordinary. ■

3.6 Necessary and Sufficient Optimality Conditions for Control of PDPs

We are now in a position to state the main result of this chapter.

First we set out a final *assumption* (*cf.* Gonzalez and Rofman 1978 and A2.7) on the PDP jump rate which ensures the Lipschitz continuity of the value function of the PDP optimal control problem:

(A3.9) The jump rate satisfies

$$\inf_{x \in E^0, u \in U_0} \lambda(x,u) + \delta > \lambda^0_+, \tag{3.33}$$

where $\lambda^0 := \sup_{\substack{x,y \in E^0 \\ u \in U_0}} (x-y)'(f(x,v) - f(y,v))/\|x-y\|^2$.

Theorem 3.3 *Under assumptions* (A3.1)–(A3.9), *there exists an optimal ordinary control policy* $u^* = (u_0^*, u_\partial^*) \in C$ *which solves the PDP optimal control problem with (expected) cost functional* J_z *defined by* (3.3) *for any initial point* $z \in E$ *and the value function* $V(z)$ *defined by*

$$V(z) \quad := \quad \min_{u \in C} J_z(u) \quad \forall z \in E$$

is a Lipschitz continuous solution of the generalized BHJ equation on E^0

$$\min_{\substack{\xi' \in \partial V(z) \\ v \in U_0}} \{\xi' f(z,v) + \lambda(z,v) \int_{E^0} (V(y) - V(z)) Q_0(dy; z, v) - \delta V(z) + l_0(z,v)\} = 0 \tag{3.34}$$

with boundary condition

$$V(z) = \min_{v \in U_\partial} \{l_\partial(z,v) + \int_{E^0} V(y) Q_\partial(dy; z, v)\} \tag{3.35}$$

on ∂E. *If* V *is regular in the sense of* (2.60), *then it is the unique solution of* (3.34),(3.35).

An admissible control $u = (u_0, u_\partial) \in C$ *is optimal if and only if for all* $z \in E^0$

$$\begin{aligned} V(z) &= J_z(u) \\ &= \int_0^{t_*} e^{-\Lambda_t^u(z)} [l_0(\phi_t^u(z), u_{0t}(z)) \\ &\quad + \int_{E^0} V(y) Q_0(dy; \phi_t^u(z), u_{0t}(z)) \lambda(x(t), u_{0t}(z))] dt \\ &\quad + e^{-\Lambda_{t_*}^u(z)} \{l_\partial(\phi_{t_*}^u(x), u_\partial(\phi_{t_*}^u(x))) \\ &\quad + \int_{E^0} V(y) Q_\partial(dy; \phi_{t_*}^u(z), u_\partial(\phi_{t_*}^u(z))\}. \end{aligned} \tag{3.36}$$

and forl all $z \in \partial E$*,*

$$V(z) = l_\partial(z, u_\partial(z)) + \int_{E^0} V(y) Q_\partial(dy; z, u_\partial(z)),$$

where $V(z)$ *is the value function.*

Furthermore, if the value function is (Clarke) regular, then condition (3.36) *can be replaced by the following condition: there exists* $\xi_t'(z) \in \partial V(\phi_t^u(z))$ *such that*

$$\xi_t'(z) f(\phi_t^u(z), u_{0t}(z)) + \lambda(\phi_t^u(z), u_{0t}(z)) \int_{E^0} (V(y) - V(\phi_t^u(z))) Q_0(dy; \phi_t^u(z), u_{0t}(z))$$
$$-\delta V(\phi_t^u(z)) + l_0(\phi_t^u(z), u_{0t}(z)) = 0 \qquad a.e.\ t \in [0, t_*).$$

■

Remark 3.2 As in the deterministic case (Chapter 2), the first term in the generalized BHJ equation shows that the nature of the generalization is that the usual gradient term for a C^1 value function is replaced by the appropriate minimum element of the Clarke generalized gradient of the Lipschitz continuous value function at $z \in E^0$. The remaining terms in this equation are due respectively to interior jumps, discounting and running costs. In the extreme case when $\lambda = 0$, the PDP is reduced to the deterministic control problem with boundary condition and the generalized BHJ equation is reduced to (2.5) for the case when $\bar{\lambda} := \delta$. ■

Proof The existence of an optimal ordinary control has been shown in §3.5. Here we prove the rest of theorem.

By §3.4 and §3.5, the PDP optimal control $u = (u_0, u_\partial)$ is equivalent to choose for each $z \in E^0$ an optimal control function $u_{0(\cdot)}(z)$ in a deterministic control problem with boundary condition (P_z) defined in §2.1 where

$$\begin{aligned} f_0(x,v) &:= l_0(x,v) + \int_{E^0} V(y) Q_0(dy; x, v) \\ F(x) &:= \min_{v \in U_\partial} \{ l_\partial(x,v) + \int_{E^c} V(y) Q_\partial(dy; x, v) \} \\ \bar{\lambda}(x,v) &:= \lambda(x,v) + \delta \end{aligned}$$

and for each $z \in \partial E$, an optimal feedback control action $u_\partial(z)$ which solves the following optimization problem:

$$\min_{v \in \partial E} \{ l_\partial(z,v) + \int_{E^0} V(y) Q_\partial(dy; z, v) \}.$$

Since $V(x) \in C(E^0)$, f_0, F satisfies conditions (A2.3), (A2.5) respectively by virtue of (A3.4) (A3.7) & (A3.6), $\bar{\lambda}$ satisfies conditions (A2.4) &(A2.7) by virtue of (A3.3) and (A3.9) respectively. Therefore (P_x) is a well-defined deterministic optimal control problem with boundary condition (see Chapter 2). Applying results obtained in Chapter 2, we conclude that the value function $V(x)$ is Lipschitz continuous on E^0 and satisfies the generalized BHJ equation

$$\min_{\substack{\xi' \in \partial V(z) \\ v \in U_0}} \{ \xi' f(z,v) - \bar{\lambda}(z,v) V(z) + f_0(z,v) \} = 0 \qquad z \in E^0 \tag{3.37}$$

with boundary condition

$$V(z) = F(z) \qquad z \in \partial E. \tag{3.38}$$

Substituting f_0, F and $\bar{\lambda}$ into equations (3.37) and (3.38), we obtain the generalized BHJ equation for the PDP control problem:

$$\min_{\substack{\xi' \in \partial V(z) \\ v \in U_0}} \{ \xi' f(z,v) + \lambda(z,v) \int_{E^0} (V(y) - V(z)) Q_0(dy; z, v) - \delta V(z) + l_0(z,v) \} = 0$$

$$\forall z \in E^0$$

with the boundary condition

$$V(z) = \min_{v \in U_\partial} \{ l_\partial(z,v) + \int_{E^0} V(z) Q_\partial(dy; z, v) \}$$

$$\forall z \in \partial E \ (\forall z \in \Gamma^*).$$

The rest of the theorem follows analogously.

■

Chapter 4

Maximum Principles

4.1 Introduction

In this chapter, our aim is to develop a nonsmooth maximum principle for control of piecewise deterministic Markov processes under weak assumptions. The PDP optimal control problem considered in this chapter is the one formulated in §3.1 and all assumptions remain in effect.

In Chapter 3 we have shown that the optimal control for the PDP control problem is to choose after each jump a control function which is an optimal control in a deterministic optimal control problem with a boundary condition. Therefore it is obvious that a maximum principle for the PDP control problem will follow once the appropriate one for the control problem with boundary condition is established.

As noted in Chapter 2, this deterministic control problem is however non-standard in that the terminal time t_* is not fixed, but is instead either $+\infty$ or the first time the trajectory reaches the boundary of the state space. In the proof, we will consider separately the case when t_* is finite and when t_* is $+\infty$. A nonsmooth maximum principle developed by Clarke (1983) will be used in the case where t_* is finite, while an infinite horizon nonsmooth maximum principle will be developed for the case where t_* is infinite using some results on differential inclusions of Aubin and Cellina (1984).

4.2 A Maximum Principle for a Control Problem with a Boundary Condition

In this section, we consider the deterministic control problem with boundary condition (P_z) as formulated in §2.1.

Given $x \in E^0$, $v \in U$, $p' \in \mathbb{R}^{n'}$, $q \in \mathbb{R}$, $r \in \mathbb{R}$, define the *Hamiltonian function* for (P_z) as

$$H(x,v;p',q,r) := p'f(x,v) - q\bar{\lambda}(x,v) - rf_0(x,v).$$

The following theorem provides a maximum principle for (P_z).

Theorem 4.1 *Let $(x^*(\cdot),u^*(\cdot))$ be an optimal solution for problem (P_z) and t_* the corresponding boundary hitting time. Then under assumptions* (A2.1)–(A2.5), *there exist:*

(a) *absolutely continuous functions*

$$p' : [0,t_*) \longrightarrow \mathbb{R}^{n'} \qquad q : [0,t_*) \longrightarrow \mathbb{R},$$

(b) *a scalar $r \in \{0,1\}$*

such that:-

(1) *The optimal control $u^*(t)$ maximizes the Hamiltonian pointwise, viz.*

$$\begin{aligned} &\max_{v\in U} H(x^*(t),v;p'(t),q(t),r) \\ &= H(x^*(t),u^*(t);p'(t),q(t),r) \\ &= 0 \qquad a.e \quad t \in [0,t_*). \end{aligned} \tag{4.1}$$

(2) *The dual variables (p',q) satisfy the adjoint equations in the form of the differential inclusions*

$$
\begin{aligned}
-\dot{p}'(t) \quad &\in \quad p'(t)\partial_x f(x^*(t),u^*(t)) \\
&\quad -q(t)\partial_x\bar{\lambda}(x^*(t),u^*(t)) \\
&\quad -r\partial_x f_0(x^*(t),u^*(t)) \\
&\quad -\bar{\lambda}(x^*(t),u^*(t))p'(t) \\
&\qquad\qquad\qquad a.e.\ t\in[0,t_*) \qquad (4.2) \\
-\dot{q}(t) \quad &= \quad -rf_0(x^*(t),u^*(t)) \\
&\quad -q(t)\bar{\lambda}(x^*(t),u^*(t)) \\
&\qquad\qquad\qquad a.e.\ t\in[0,t_*). \qquad (4.3)
\end{aligned}
$$

(3) *The system is subject to the transversality condition: if $t_* < \infty$, then*

$$(p'(t_*), q(t_*)) + r\xi' \in -\rho\partial d_{\partial E}(x^*(t_*)) \times \{0'\} \qquad (4.4)$$

for some scalar $\rho \geq 0$ and $\xi' \in \mathbb{R}^{n+1'}$ with

$$\xi' \in \partial F(x^*(t_*)) \times \{(F(x^*(t_*))\}.$$

(4) *If $t_* < \infty$, then the dual variables satisfy the nontriviality condition*

$$\|p'\|_\infty + \|q\|_\infty + r > 0, \qquad (4.5)$$

where ∂ denotes either the Clarke generalized gradient or the generalized Jacobian, ∂_x denotes the generalized partial derivative with respect to x and $\|\cdot\|_\infty$ is the supremum norm for the spaces of appropriate continuous functions on $[0,t_*)$.

Proof It is convenient to replace the exponential term in the cost by an extra differential relation

$$
\begin{aligned}
\dot{x}_0(t) &= -x_0(t)\bar{\lambda}(x(t),u(t)) \\
x_0(0) &= 1.
\end{aligned}
$$

Problem (P_z) can be equivalently posed as follows:

$$(\bar{P}_z) \qquad \min \quad \int_0^{t_*} x_0(t) f_0(x(t), u(t))dt + x_0(t_*)F(x(t_*))$$

on the class $\bar{\Omega}$ of all pairs $(\bar{x}(\cdot), u(\cdot))$ with

$$\bar{x}(\cdot) := (x(\cdot)', x_0(\cdot))'$$

$$\text{s.t.} \quad \frac{d}{dt}\bar{x}(t) = [f(x(t), u(t))', -x_0(t)\bar{\lambda}(x(t), u(t))]' \quad \text{a.e. } t \in [0, t_*)$$

$$\bar{x}(0) := (z', 1)'$$

$$t_* := \inf\{t > 0 : x(t) \in \partial E\}$$

For an optimal pair $(x^*(\cdot), u^*(\cdot))$ in Ω we denote by $(\bar{x}^*(\ ), u^*(\cdot))$ the corresponding solution for $(\bar{P}_z)$ in the class $\bar{\Omega}$.

Now we divide the analysis into two cases:

(a) the boundary hitting time of the optimal trajectory $x^*(\cdot)$ is finite,

(b) the boundary hitting time of the optimal trajectory $x^*(\cdot)$ is infinite.

Since by assumption (A2.6) any trajectory must hit the boundary of E in such a way that the corresponding vector field element makes an acute angle with the outward pointing unit normal, we can find a tube about the optimal trajectory $x^*(t)$ such that any trajectory in the tube hits the boundary at most once. Therefore, $(\bar{x}(\cdot), u^*(\cdot))$ is the optimal solution of the following problem:

$$(P_C) \qquad \text{minimize} \quad \int_0^{t_*} x_0(t) f_0(x(t), u(t))dt + x_0(t_*)F(x(t_*))$$

on the class $\bar{\Omega}$ of all pairs $(\bar{x}(\cdot), u(\cdot))$ with

$$\bar{x}(\cdot) = (x'(\cdot), x_0(\cdot))'$$

$$\text{s.t.} \quad \frac{d}{dt}\bar{x}(t) = [f(x(t), u(t))', -x_0(t)\bar{\lambda}(x(t), u(t))]' \quad \text{a.e. } t \in [0, t_*)$$

$$x(t) \in T(x^*; \varepsilon)$$

$$\bar{x}(0) := (z', 1)'$$

$$(t_*, \bar{x}(t_*)') \in M,$$

where $M =: [0, \infty) \times \partial E \times [0, 1]$ in case (a), $M := \{\infty\} \times E^0 \times [0, 1]$ in case (b), $T(x^*; \varepsilon)$ is the ε-tube about optimal trajectory x^* defined by

$$T(x^*; \varepsilon) := \{v \in \mathbb{R}^n : \|x^*(t) - v\| < \varepsilon, t \geq 0\},$$

and $\varepsilon > 0$ is sufficiently small to ensure that $T(x^*; \varepsilon) \subset E^0$ for $t \in [0, t_*)$.

Case (a). $t_* < \infty$

In this case the time interval is finite and the endpoint constraint set $[0, \infty) \times \partial E \times [0, 1]$ is closed in $\mathbb{R}^{n+2'}$, the nonsmooth deterministic maximum principle developed by Clarke is applicable. We refer to Theorem 1.1 and identify the data for (P_C) with the corresponding data in the theorem.

The Hamiltonian function for the problem (P_C) is defined as follows:

$$\bar{H}(\bar{x}, v; \bar{p}', \bar{q}, \bar{r}) := \bar{p}' f(x, v) - \bar{q} x_0 \bar{\lambda}(x, v) - \bar{r} x_0 f_0(x, v)$$

for $\bar{x} := (x', x_0)' \in \mathbb{R}^{n+1}$, $v \in U$, $\bar{p}' \in \mathbb{R}^{n'}$, $\bar{q} \in \mathbb{R}$ and $\bar{r} \in \mathbb{R}$.

Applying Theorem 1.1 we have the following maximum principle for the problem (P_C):

There exist

(a) absolutely continuous functions

$$\bar{p}' : [0, t_*) \longrightarrow \mathbb{R}^{n'} \qquad \bar{q} : [0, t_*) \longrightarrow \mathbb{R}$$

(b) a scalar $\bar{r} \in \{0, 1\}$

such that:-

(1) The optimal control function $u^*(t)$ minimizes the Hamiltonian pointwise, *viz.*

$$\begin{aligned} & \max_{v \in U} \bar{H}(\bar{x}^*(t), v; \bar{p}', \bar{q}, \bar{r}) \\ &= \bar{H}(\bar{x}^*(t), u^*(t); \bar{p}', \bar{q}, \bar{r}) \\ &= 0 \qquad\qquad \text{a.e. } t \in [0, t_*). \end{aligned} \tag{4.6}$$

(2) The dual variables $(\bar{p}', \bar{q})$ satisfies the adjoint equations in the form of the differential inclusions

$$-\frac{d}{dt}(\bar{p}'(t), \bar{q}(t)) \in \partial_{\bar{x}}\bar{H}(\bar{x}^*(t), u^*(t); \bar{p}'(t), \bar{q}(t), \bar{r}) \qquad \text{a.e. } t \in [0, t_*). \tag{4.7}$$

(3) The system is subject to the transversality condition

$$(\bar{p}'(t_*), \bar{q}(t_*)) \in -\bar{r}\partial\bar{F}(\bar{x}(t_*)) - \rho\partial d_{\partial E\times[0,1]}(x(t_*), x_0(t_*)) \tag{4.8}$$

where the function $\bar{F}(\bar{x}) := x_0 F(x)$ and ρ is some nonnegative scalar.

(4) The dual variables satisfy the nontriviality condition

$$\|(\bar{p}', \bar{q})\|_\infty + \bar{r} > 0. \tag{4.9}$$

Now we need to rearrange the expressions so that we have a maximum principle for the problem (P_z).

Define $p'(\cdot) := \bar{p}'(\cdot)/x_0^*(\cdot)$, which is well defined since $x_0^*(\cdot) > 0$. We also identify $q(\cdot) := \bar{q}(\cdot)$ and $r := \bar{r}$. It follows that (4.6) implies (4.1) and (4.9) implies (4.5) by the above definitions.

Let $\bar{H}_1(\bar{x}, v; \bar{p}') := \bar{p}' f(x, v)$ and $\bar{H}_2(\bar{x}, v; \bar{q}, \bar{r},) = -x_0(\bar{q}\bar{\lambda}(x, v) + \bar{r} f_0(x, v))$. By Proposition 1.8 we have

$$\begin{aligned}
\partial_{\bar{x}}\bar{H}_1 &= \partial_x \bar{H}_1 \times \{0\} \\
\partial_{\bar{x}}\bar{H}_2 &= \partial_x \bar{H}_2 \times \{-\bar{q}\lambda(x, u) - \bar{r} f_0(x, u)\}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\partial_{\bar{x}}\bar{H} &:= \partial_{\bar{x}}[H_1 + H_2] \\
&\subset \partial_{\bar{x}}\bar{H}_1 + \partial_{\bar{x}}\bar{H}_2 \qquad (4.10) \\
&= (\partial_x \bar{H}_1 + \partial_x \bar{H}_2) \times \{-\bar{q}\lambda(x, u) - \bar{r} f_0(x, u)\}, \qquad (4.11)
\end{aligned}$$

where the inclusion (4.10) follows from the finite sums formula (Proposition 1.6).

Therefore (4.7) implies

$$\begin{aligned} -\frac{d}{dt}\bar{p}'(t) &\in \partial_x \bar{H}_1(\bar{x}^*(t), u^*(t); \bar{p}'(t)) + \partial_x \bar{H}_2(\bar{x}^*(t), u^*(t); \bar{q}(t), \bar{r}) \\ &= \bar{p}'(t)\partial_x f(x^*(t), u^*(t)) - x_0^*(t)[\bar{q}\partial_x \bar{\lambda}(x^*(t), u^*(t)) \\ &\quad + \bar{r}\,\partial_x f_0(x^*(t), u^*(t))] \qquad \text{a.e. } t \in [0, t_*) \qquad (4.12) \end{aligned}$$

and

$$\begin{aligned} -\frac{d}{dt}\bar{q}(t) &= -\bar{q}(t)\bar{\lambda}(x^*(t), u^*(t)) - \bar{r} f_0(x^*(t), u^*(t)) \\ &\qquad \text{a.e. } t \in [0, t_*). \qquad (4.13) \end{aligned}$$

Since $\bar{p}'(t) := x_0^*(t)p'(t)$ by definition, the left hand side of inclusion (4.12) is equal to

$$\begin{aligned} \frac{d}{dt}[x_0^*(t)p'(t)] &= x_0^*(t)\frac{d}{dt}p'(t) + \frac{d}{dt}x_0^*(t) \cdot p'(t) \\ &= x_0^*(t)\frac{d}{dt}p'(t) + x_0^*(t)\bar{\lambda}(x^*(t), u^*(t))p'(t). \end{aligned}$$

Therefore inclusion (4.12) and the definition of $\bar{p}'$ imply (4.2). Equation (4.3) is obtained from equation (4.13) and this definition.

By Proposition 1.8, since $x_0(t_*) > 0$ and F is continuous, we have

$$\partial \bar{F}(\bar{x}(t_*)) = x_0(t_*)\partial F(x(t_*)) \times \{F(x(t_*))\}. \qquad (4.14)$$

Since $\partial d_{C_1 \times C_2}(x_1, x_2) = \partial d_{C_1}(x_1) \times \partial d_{C_2}(x_2)$ (see the corollary of Theorem 2.4.5 of Clarke 1983, p.54), we have

$$\begin{aligned} \partial d_{\partial E \times [0,1]}(x(t_*), x_0(t_*)) &= \partial d_{\partial E}(x(t_*)) \times \partial d_{[0,1]}(x_0(t_*)) \\ &= \partial d_{\partial E}(x(t_*)) \times \{0\}, \qquad (4.15) \end{aligned}$$

where the last equality follows from the fact that $x_0(t_*) \in (0,1)$.

By substituting equalities (4.14) and (4.15) into inclusion (4.8), we have

$$\begin{aligned} (\bar{p}'(t_*), \bar{q}(t_*)) &\in -\bar{r}[x_0(t_*)\partial F(x(t_*)) \times \{F(x(t_*))\}] - \rho\partial d_{\partial E}(x(t_*)) \times \{0\} \\ &= [(-\bar{r}x_0(t_*))\partial F(x(t_*)) - \rho\partial d_{\partial E}(x(t_*))] \times \{-\bar{r}F(x(t_*))\}, \end{aligned}$$

from which we derive the inclusion (4.4).

Case (b). $t_* = \infty$

The situation of this case is not covered derectly by the results employed in the previous case due to the fact that $t_* = +\infty$. For infinite horizon optimal control problems with smooth data, a maximum principle is developed b Carlson and Haurie (1987). Here we develop a nonsmooth maximum principle for infinite horizon optimal control problems.

Take a strictly increasing sequence $\{t_i\}$ in $[0,\infty)$ such that $t_i \to \infty$ as $i \to \infty$.

A collection of deterministic problems $\{(\bar{P}_i)\}$ can be defined as follows:

$$(\bar{P}_i) \qquad \text{minimize} \qquad \int_0^{t_i} x_0(t) f_0(x(t), u(t))dt$$

over the class Ω_i of all pairs $(\bar{x}(\cdot), u(\cdot))$ on $[0, t_i]$ with $\bar{x}(\cdot) := (x'(\cdot), x_0(\cdot))'$

$$\text{s.t.} \qquad \frac{d}{dt}\bar{x}(t) = [f(x(t), u(t))', -x_0(t)\bar{\lambda}(x(t), u(t))]' \quad \text{a.e. } t \in [0, t_*)$$

$$\bar{x}(0) := (z', 1)$$

$$\bar{x}(t_i) := (x^{*\prime}(t_i), x_0^*(t_i))'.$$

Then $\bar{x}^*$ restricted to $[0, t_i]$ is an optimal trajectory for (P_i). Since, if $\bar{x}^*$ is not optimal, then for some $t_i > 0$ and some $(\bar{x}^+, u^+) \in \Omega_i$ one has

$$\int_0^{t_i} x_0^+(t) f_0(x^+(t), u^+(t))dt < \int_0^{t_i} x_0^*(t) f_0(x_0^*(t), u^*(t))dt. \tag{4.16}$$

Now let $(\tilde{x}, \tilde{u})$ be defined by

$$(\tilde{x}(t), \tilde{u}(t)) := \begin{cases} (\bar{x}^+(t), u^+(t)) & \text{for } t \in [0, t_i] \\ (\bar{x}^*(t), u^*(t)) & \text{for } t \in [t_i, \infty). \end{cases}$$

From the optimality of $(\bar{x}^*(t), u^*(t))$, there exists $T > t_i$ so that

$$\begin{aligned} \int_0^T x_0^*(t) f_0(x^*(t), u^*(t))dt &< \int_0^T \tilde{x}_0(t) f_0(\tilde{x}(t), \tilde{u}(t))dt \\ &= \int_0^{t_i} x_0^+(t) f_0(x^+(t), u^+(t))dt + \int_{t_i}^T x_0^*(t) f_0(x^*(t), u^*(t))dt \\ &< \int_0^{t_i} x_0^*(t) f_0(x^*(t), u^*(t))dt + \int_{t_i}^T x_0^*(t) f_0(x^*(t), u^*(t))dt \\ &= \int_0^T x_0^*(t) f_0(x^*(t), u^*(t))dt. \end{aligned}$$

This is a contradiction. Therefore $(\bar{x}^*(t), u^*(t))$ restricted to $[0, t_i]$ is an optimal pair for $(\bar{P}_i)$.

Thus we can again use Clarke's nonsmooth maximum principle. By Theorem 1.2, we conclude that for the problem $(\bar{P}_i)$ there exist absolutely continuous functions

$$\bar{p}_i' : [0, t_i] \longrightarrow I\!R^{n'} \qquad \bar{q}_i : [0, t_i] \longrightarrow I\!R$$

and a scalar $\bar{r}_i \geq 0$ such that

$$\begin{aligned} & \max_{u \in U} \bar{H}(\bar{x}^*(t), u; \bar{p}_i'(t), \bar{q}_i(t), \bar{r}_i) \\ = \;& \bar{H}(\bar{x}^*(t), u^*(t); \bar{p}_i'(t), \bar{q}_i(t), \bar{r}_i) \\ = \;& 0 \qquad\qquad \text{a.e. } t \in [0, t_i] \end{aligned} \tag{4.17}$$

$$-\frac{d}{dt}(\bar{p}_i'(t), \bar{q}_i(t)) \in \partial_{\bar{x}} \bar{H}(\bar{x}^*(t), u(t); \bar{p}_i'(t), \bar{q}_i(t), \bar{r}) \qquad \text{a.e. } t \in [0, t_i] \tag{4.18}$$

$$\|(\bar{p}_i', \bar{q}_i)\|_\infty + \bar{r}_i > 0. \tag{4.19}$$

As in case (a) , we can rearrange the expressions by redefining

$$p_i{}'(\cdot) := \frac{\bar{p}_i'(\cdot)}{x_0^*(\cdot)} \quad q_i(\cdot) := \bar{q}_i(\cdot) \;\; r_i := \bar{r}_i$$

to yield

$$\begin{aligned} -\dot{p}_i'(t) \in \;& p_i'(t)\partial_x f(x^*(t), u^*(t)) \\ & -q_i(t)\partial_x \lambda(x^*(t), u^*(t)) \\ & -r_i \partial_x f_0(x^*(t), u^*(t)) \\ & -\bar{\lambda}(x^*(t), u^*(t))p'(t) \\ & \qquad\qquad \text{a.e. } t \in [0, t_i] \end{aligned} \tag{4.20}$$

$$-\dot{q}_i(t) = -q_i(t)\bar{\lambda}(x^*(t), u^*(t)) - r_i f_0(x^*(t), u^*(t)) \qquad \text{a.e. } t \in [0, t_i] \tag{4.21}$$

$$\|p_i'\|_\infty + \|q_i\|_\infty + r_i > 0. \tag{4.22}$$

By normalization, the condition (4.22) could be equivalently replaced by $\|p_i'\|_\infty + \|q_i\|_\infty + r_i = 1$ so that $\|p_i'(0)\| + |q_i(0)| + r_i$ is bounded. Hence by passing to an appropriate subsequence one may assume that

$$\lim_{i\to\infty} p_i(0) = p(0), \quad \lim_{i\to\infty} q_i(0) = q(0), \quad \lim_{i\to\infty} r_i = r$$

exist.

Define set-valued maps $A(t)$ and $B(t)$ as follows:

$A : t \longmapsto A(t) \subset \mathbb{R}^{(n+1)\times(n+1)}$ such that $\forall a(t) \in A(t)$

$$a(t) = \begin{bmatrix} z_{11}+\bar\lambda & z_{12} & \cdots & z_{1n} & \alpha_1 \\ z_{21} & z_{22}+\bar\lambda & \cdots & z_{2n} & \alpha_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn}+\bar\lambda & \alpha_n \\ 0 & 0 & 0 & 0 & \bar\lambda \end{bmatrix},$$

where $\bar\lambda := \bar\lambda(x^*(t), u^*(t))$, $\alpha := (\alpha_1, \cdots, \alpha_n) \in \partial_x\bar\lambda(x^*(t), u^*(t))$ and $(z_{ij})_{n\times n} \in \partial_x f(x^*(t), u^*(t))$.

$B(\cdot) : t \longmapsto B(t) \subset \mathbb{R}^{n+1}$ such that $\forall b(t) \in B(t)$

$$b(t) = \begin{bmatrix} r\beta_1 \\ \vdots \\ r\beta_n \\ rf_0 \end{bmatrix},$$

where $f_0 := f_0(x^*(t), u^*(t))$, $\beta := (\beta_1, \cdots, \beta_n) \in \partial_x f_0(x^*(t), u^*(t))$.

Rewrite the differential equation (4.3) and the differential inclusion (4.2) as a differential inclusion in the form:

$$\frac{d}{dt}(p'(t), q(t))' \in F(t, (p'(t), q(t))') \text{ a.e. } t > 0, \tag{4.23}$$

where F is a set-valued map from $\mathbb{R}^{n+1}$ into subsets of $\mathbb{R}^{n+1}$ defined by

$$F(t, x) = A(t)x + B(t).$$

The set value map A is compact-valued due to the compactness of generalized gradient sets and generalized Jacobians (see Propositions 1.3 and 1.11). Consequently, the set-valued map F satisfies

$$\begin{aligned} d(F(t,x),F(t,y)) &\leq \|a_t x - a_t y\| \\ &= \|a_t\| \|x-y\| \\ &\leq \max_{a_t \in A(t)} \|a_t\| \cdot \|x-y\| \end{aligned}$$

i.e. F is a Lipschitzean map with Lipschitz constant $k(t) := \max_{a_t \in A(t)} \|a_t\|$ (see Definition 1.10).

Let $q : [0,\infty) \longrightarrow \mathbb{R}$ be the unique absolutely continuous solution of the differential equation (4.3) with the initial data $q(0) = \lim_{i\to\infty} q_i(0)$. One has

$$\lim_{i\to\infty} q_i(t) = q(t) \tag{4.24}$$

due to the continuous dependence of solutions of the differential equation (4.3) with respect to the initial data (see §1.7).

By results on continuous dependence of solutions for differential inclusions of Aubin and Cellina (Proposition 1.14), since $p(0) = \lim_{i\to\infty} p_i(0)$ and $q(0) = \lim_{i\to\infty} q_i(0)$, for each $(p_i'(\cdot), q_i(\cdot))$ of the differential inclusion (4.23) with initial data $(p_i'(0), q_i(0))$, we can associate a solution $y_i'(\cdot)$ of the differential inclusion (4.2) with initial point $p'(0)$ (or equivalently $(y_i'(\cdot), q(\cdot))$ of the differential inclusion (4.23) with initial data $(p'(0), q(0))$ such that

$$\|(y_i'(t), q(t)) - (p_i'(t), q_i(t))\| \leq \|(p'(0), q(0)) - (p_i'(0), q_i(0))\| e^{\int_0^t k(s)ds} \tag{4.25}$$

which implies that

$$\|y_i(t) - p_i(t)\|^2 \leq (\|p(0) - p_i(0)\|^2 + \|q(0) - q_i(0)\|^2) e^{2\int_0^t k(s)ds} - \|q_i(t) - q(t)\|^2. \tag{4.26}$$

Since $\bar{\lambda}$ is bounded, $\bar{\lambda}$, f_0 and f is Lipschitz in x uniformly in u, by Propositions 1.3 and 1.11, it is easy to see that $F(t,x)$ remains in a compact set of $\mathbb{R}^{n+1}$, that is the minimum norm trajectory of (4.23) remains in a compact subset of $\mathbb{R}^{n+1}$.

Since F is convex and compact valued by the convexity and compactness of the generalized gradients and generalized Jacobians (Propositions 1.3 and 1.11 respectively), by Proposition 1.15 the set of all solutions of the differential inclusion (4.23) with initial data $(p'(0), v(0))$ is compact in the uniform convergence (supremum) norm. Therefore, without loss of generality, we may assume there exists a solution $p(t)$ to the differential inclusion (4.4) with initial data $p(0)$ such that

$$\lim_{i\to\infty} y_i'(t) = p'(t). \tag{4.27}$$

We now show that

$$p'(t) = \lim_{i\to\infty} p_i'(t). \tag{4.28}$$

Indeed, we have

$$\|p_i(t) - p(t)\| \quad \leq \quad \|p_i(t) - y_i(t)\| + \|y_i(t) - p(t)\|.$$

Hence for any $\varepsilon > 0$, we can choose an $N \in \mathbb{N}$ so that the right hand side of inequality (4.26) (consequently the first term of the last inequality) is less then or equal to $\varepsilon/2$ for all $i > N$. The second term of the last inequality can be treated in the same way by virtue of (4.27).

Taking limits in (4.17), by virtue of (4.24) and (4.28), we obtain (4.1) due to the linearity of H in r, q and p.

It is obvious that r can be taken as 0 or 1 so the proof is complete. ∎

4.3 A Maximum Principle for the PDP Control Problem

Definition 4.1 The Hamiltonian function for the PDP control problem is defined as follows:

$$\begin{aligned} H(x,u;p',q,r,\theta) \quad &:= \quad p'f(x,u) - q(\lambda(x,u)+\delta) \\ &\quad -r[l_0(x,u) + \lambda(x,u)\int_{E^0}\theta(y)Q_0(dy;x,u)] \end{aligned}$$

for $x \in \mathbb{R}^n$, $u \in U_0$, $p' \in \mathbb{R}^{n'}$, $q, r \in \mathbb{R}$ and $\theta \in C(E^0)$. ∎

Theorem 4.2 (A Nonsmooth Maximum Principle for the PDP Control Problem)

Under assumptions (A3.1)-(A3.8), *let* $u^* = (u_0^*, u_\partial^*)$ *be an ordinary control policy which solves the PDP optimal control problem. For arbitrary* $z \in E^0$, *let* $\phi_t^{u^*}(z)$ *be the corresponding trajectory in* E^0 *on* $[0, t_*^{u^*}(z))$ *with initial point* z. *Then (setting* $\phi_t^*(z) := \phi_t^{u^*}(z)$ *and* $t_* := t_*^{u^*}(z)$*) there exist:*

(a) *absolutely continuous functions*

$$p' : [0, t_*) \longrightarrow \mathbb{R}^{n'} \qquad q : [0, t_*) \longrightarrow \mathbb{R},$$

(b) *A scalar* $r \in \{0, 1\}$

such that:-

(1) *The optimal control function* $u_{0(\cdot)}^*(z)$ *maximizes the Hamiltonian pointwise, viz.*

$$\begin{aligned} & \max_{v \in U_0} H(\phi_t^*(z), v; p'(t), q(t), r, J_z(u^*)) \\ = & \ H(\phi_t^*(z), u_{0t}^*(z); p'(t), q(t), r, J_z(u^*)) \\ = & \ 0 \qquad\qquad a.e. \quad t \in [0, t_*) \end{aligned}$$

(2) *The dual variables* (p', q) *satisfy the adjoint equations in the form of the differential inclusions*

$$\begin{aligned} -\dot{p}'(t) \ \in \ & p'(t)(\partial_x f(\phi_t^*(z), u_{0,t}^*(z)) - [\lambda(\phi_t^*(z), u_{0t}^*(z)) + \delta] I_n) \\ & -q(t)\partial_x \lambda(\phi_t^*(z), u_{0t}^*(z)) \\ & -r\partial_x [l_0(\phi_t^*(z), u_{0t}^*(z)) \\ & +\lambda(\phi_t^*(z), u_{0t}^*(z)) \int_{E^0} J_y(u^*) Q_0(dy; \phi_t^*(u), u_{0t}^*(z))] \\ & \qquad\qquad\qquad\qquad \text{a.e.} \quad t \in [0, t_*) \\ -\dot{q}(t) \ = \ & -q(t)(\lambda(\phi_t^*(z), u_{0t}^*(z)) - r[l_0(\phi_t^*(z), u_{0t}^*(z)) \\ & +\lambda(\phi_t^*(z), u_{0t}^*(z)) \int_{E^0} J_y(u^*) Q_0(dy; \phi_t^*(z), u_{0t}^*(z))] \\ & \qquad\qquad\qquad\qquad \text{a.e.} \quad t \in [0, t_*) \end{aligned}$$

where $J_y(u)$ is the cost corresponding to control u starting from the interior state $y \in E^0$ and $J(u)$ is defined as the function $y \longmapsto J_y(u)$.

(3) *The system is subject to the transversality condition: if $t_* < \infty$, then*

$$(p'(t_*), q(t_*)) + r\xi' \in -\rho\partial d_{\partial E}(\phi_t^*(z)) \times \{0'\} \tag{4.29}$$

for some scalar $\rho \geq 0$ and $\xi' \in I\!R^{n+1'}$ with

$$\xi' \in \partial F(\phi_{t_*}^*(z)) \times \{F(\phi_{t_*}^*(z))\},$$

where

$$F(x) := l_\partial(x, u_\partial^*(x)) + \int_{E^0} J_y(u^*)Q_\partial(dy; x, u_\partial^*(x)).$$

(4) *If $t_* < \infty$, then the dual variables satisfy nontriviality condition*

$$\|p'\|_\infty + \|q\|_\infty + r > 0. \tag{4.30}$$

■

Remark 4.1 For every $z \in E^0$, there is a multiplier function (p', q) which depends Borel measurably on z. Hence corresponding to the optimally controlled PDP $\{\mathbf{x}_t\}$ we may consider the multiplier process $(\mathbf{p}', \mathbf{q})$ as a random process. ■

Proof In §3.4 we have shown that a control (u_0^*, u_∂^*) is optimal if and only if for each $. \in E^0, u_{0(\cdot)}^*(z)$ is an optimal control function in the deterministic optimal control problem with boundary condition (P_z) with the following data:

$$f_0(x, u) := l_0(x, u) + \lambda(x, u)\int_{E^0} V(y)Q_0(dy; x, u) \tag{4.31}$$

$$F(x) := \min_{v \in U_\partial}\{l_\partial(x, v) + \int_{E^0} V(y)Q_\partial(dy; x, v)\} \tag{4.32}$$

$$\bar{\lambda}(x, u) := \lambda(x, u) + \delta \tag{4.33}$$

and for $z \in \partial E$, $u_\partial^*(z)$ solves the following optimization problem:

$$\min_{v \in U_\partial}\{l_\partial(z, v) + \int_{E^0} V(y)Q_\partial(dy; z, v)\}.$$

Notice that we are dealing with a necessary condition. The value function $V(y)$ is known to be equal to the expected cost $J_y(u^*)$. Subsituting $f_0(x,u)$,$F(x)$,$\bar{\lambda}(x,u)$ defined by (4.31), (4.32) and (4.33) respectively into Theorem 4.1, the maximum principle for optimal control of PDFs follows in a straightforward manner. ■

Chapter 5

Optimal Impulse Control of PDPs

5.1 Introduction

In the previous chapters we have developed a control theory for optimal control of PDPs in the absence of impulse controls. In this chapter, we study the PDP optimal control problem with dynamic control plus impulse control as formulated in §1.3.

The impulse control problem for PDPs has been studied in the literature by Costa and Davis (1988), Gatarek (1988a,b) and Lenhart (1988). In their papers, the optimal *PDP impulse control problem* is formulated as follows. At *stopping time* τ, the state is moved from x to $x+\xi \in E^0$ with *impulse* $\xi \in U \subset I\!R^n$ and cost $c(x,\xi)$ is incurred when the impulse ξ is applied while the process is in state x. An *impulse control* (*strategy*) π is a sequence of stopping times and impulses,

$$\pi = \{\tau_1, \xi_1, \tau_2, \xi_2, \cdots\},$$

where $\tau_i \to \infty$ almost surely as $i \to \infty$. The controlled PDP x^π satisfies $x^\pi(\tau_i{}^+) = x^\pi(\tau_i{}^-) + \xi_i$.

The associated *expected cost* to be minimized is

$$J_x(\pi) := E_x[\int_0^\infty e^{-\delta t} l_0(x^\pi(t))dt + \sum_{i=1}^\infty e^{-\delta\tau_i} c(x^\pi(\tau_i^-), \xi_i)].$$

To solve this optimal impulse control problem, Costa and Davis take the *value improvement* approach while the others take the (*quasi-*)*variational inequality* approach.

Since we will relate our approach to the quasi-variational inequality approach in the end of this chapter, we now illustrate this approach. Under certain assumptions, Gatarek (1988a,b) and Lenhart (1989) characterized the value function as a generalized (e.g. Viscosity) solution of the following quasi-variational inequality:

$$(AV - \delta V + l_0) \wedge (MV - V) = 0 \quad \text{ for } x \in E^0$$
$$V(x) = \int_{E^0} V(y) Q_\partial(dy; x)$$

where

$$AV(x) := \nabla V(x) f(x) + \lambda(x) \int_{E^0} (V(y) - V(x)) Q_0(dy; x)$$

and

$$MV(x) := \inf_{v \in U} \{c(x, v) + V(x + v)\}.$$

The approach taken to the optimal PDP impulse control problem in this chapter is different from the ones in the literature in the two aspects: the very general formulation of the problem and the characterization of optimality given. It was first studied in a special case by Dempster and Solel (1987) (see also Solel 1986).

1. Formulation of the problem

By applying an impulse control action v at state x, instead of being moved to state $x + v$, the state x will be moved to state $\mathbf{y}$ which is a random variable with transition measure $Q_\delta(\cdot\,; x, v)$. Since a determined change to state $x + v$ can be considered to be a random variable with distribution $\delta_{\{x+v\}}(\cdot)$, the 1-atom measure concentrated on $x + v$, our problem formulation generalizes the formulation of the PDP optimal impulse control problem considered in the literature. It is similar to the concept of interventions introduced by Yushkevich (1983). Therefore we will use the words *intervention* and impulse control interchangeably. We will also call a stopping time an *intervention epoch* (or *moment*).

Due to the (strong) Markov nature of PDPs and by the structure of stopping times (Davis 1976), for any stopping time τ there exists a sequence of non-negative random variable $\mathbf{r}_n$ such that:

(1) $\mathbf{r}_n$ is $\mathcal{F}_{T_n}$ measurable for $n = 0, 1, 2, \ldots,$

(2) $\tau = \sum I_{\{\mathbf{T}_n < \mathbf{r}_n < \mathbf{T}_{n+1}\}}(\mathbf{T}_n + \mathbf{r}_n) \wedge \mathbf{T}_{n+1},$

where $\mathbf{T}_1, \mathbf{T}_2, \ldots$ is the sequence of jump times of the (controlled) PDP. Consequently, by specifying after each jump (either a process jump or a jump caused by an intervention) a *time remaining to intervene* $t' > 0$ ($\mathbf{r}_n$), then an stopping time $\tau \in (\mathbf{T}_n, \mathbf{T}_{n+1}]$ is either the time when $t' = 0$, providing no process jump has occurred (this corresponds to the case where $T_n + \mathbf{r}_n < T_{n+1}$), or the jump epoch, if a process jump has occurred with $t' > 0$ (this corresponds to the case where $T_n + \mathbf{r}_n > T_{n+1}$).

Therefore, impulse control strategies can be implemented as follows. For each possible prejump state $x \in E^0$ of the process, a time remaining to intervene $t'(x) > 0$ (in the absence of a process jump) is specified which subsequently diminishes with (process) time. Providing no process jump has occurred previously, an impulse control action is applied whenever $t' = 0$ and a decision is made whether or not to intervene at each jump epoch.

Unlike the usual formulation of impulse control problems with no dynamic control, the problem considered here includes not only impulse control but also full dynamic control.

Having implemented interventions in the way we have just described, we can formulate the PDP optimal control problem with dynamic and impulse control as follows.

The PDP optimal control problem with both dynamic and impulse control is to find a *dynamic control (policy)* u as before (i.e. a pair $u := (u_0, u_\partial)$ of measurable functions $u_0 : E^0 \times I\!R_+ \longrightarrow U_0$ and $u_\partial : \partial E \longrightarrow U_\partial$) and an *impulse control (policy)* (u_δ, t'), which specifies for each (pre jump) state $x \in E$ a (post jump) time remaining to intervene $t'(x) > 0$ (i.e. a measurable function $t' : E \longrightarrow (0, \infty]$) and an *intervention control action* $u_\delta(x)$ (i.e. a measurable *intervention control* function $u_\delta : E \longrightarrow U_\delta \subset I\!R^m$, a compact) which influences the (given) *intervention transition* *measure* $Q_\delta : E \times U_\delta \longrightarrow \mathbb{P}(E^0))$ so as to minimize the expected cost

$$J_x(u) = E_x\Big[\int_0^\infty e^{-\delta t} l_0(\mathrm{x}_t^u, u_0(\tau_t^u, \mathrm{z}_t^u))dt + \sum_{\mathrm{T}_i^u \neq \tau_i} e^{-\delta \mathrm{T}_i^u} l_\partial(\mathrm{x}_{T_i^-}^u, u_\partial(\mathrm{x}_{T_i^-}^u)) 1_{(\mathrm{x}_{T_i^-}^u \in \partial E)}$$

$$+ \sum_i e^{-\delta \tau_i} l_\delta(\mathrm{x}_{\tau_i^-}^u, u_\delta(\mathrm{x}_{\tau_i^-}^u))\Big], \tag{5.1}$$

where $\tau_1, \tau_2, \ ..$ denotes the sequence of stopping times corresponding to the impulse control (u_δ, t').

2. Characterization of optimality

The approach used here is to reduce the original PDP control problem with both dynamic control and impulse control to a new PDP control problem with only dynamic control. The new problem is equivalent to the original problem in that they both have the same expected cost, the data for the new problem is obtained from the original problem, and the control strategy (dynamic plus impulse control) of the original problem can be recovered from the corresponding control strategy (dynamic control only) of the new problem. Although the new state space is not bounded, we note that the boundness of the state space was only needed for previously the uniqueness results, so that we can characterize optimality through applying the BHJ necessary and sufficient optimality condition and the maximum principle to the new problem with only dynamic control.

On the other hand, we deal with the nonsmoothness of the value function by using Clarke generalized gradients instead of viscosity solutions.

The results in this chapter are related to the quasi-variational inequality approach by a relation between the value function for the original optimal process and the one for the new problem.

5.2 Reduction to a New Problem with Only Dynamic Control

If we compare a boundary control with an impulse control, we find that they both move a process instantaneously to a new state chosen according to the transition measures Q_∂ and Q_δ respectively. The difference is only in the timing. A boundary control action is applied whenever the process hits the boundary of the state space, while an impulse control action is applied at intervention epochs. To reduce impulse controls to boundary controls, it is sufficient to embed the original process in a new process in such way so that at intervention epochs of the original process the embedded process will hit the boundary of the new state space.

It is obvious that if we let t' be one of the coordinates of state of the new process, the new process will hit the boundary of the new state space when $t' = 0$ since 0 is an end point of the interval $(0, \infty)$. However, in the case when the process jumps while $t' > 0$, i.e. an ordinary interior jump, the problem is how to embed the original process so that the new process will hit the boundary at this time.

The idea here is to use a *fictitious time* construction following Yuskevitch (1983, 1988) and Dempster and Solel (1987). We consider an ordinary interior jump to be an interior jump of the new process. The new process jumps to a state where all the coordinates are kept constant except for which t' is set equal to -5, an interior point of the fictitious time interval $(-6, -4)$. Fictious time then runs backwards until it hits the boundary at $t' = -6$ at which time we decide whether or not to intervene.

To be consistent, we also let fictitious time run after both jump epochs and interventions. We distinguish two kinds of boundary states for the new process, i.e. boundary states at which we always intervene and ones at which we can decide whether or not to intervene. Thus we define the state space for fictitious time as a union of two disjoint time intervals $(-6, -4) \cup (-3, -1)$. In the case when $t' = 0$, the new process will jump to a state where all the coodinates are kept constant except t' which is set equal to -2, an interior point of the fictitious time interval $(-3, -1)$.

When the new process hits the boundary $t' = -3$, an impulse control action is taken.

Due to the use of fictitious time, the *new* process time increases one *unit* for each intervention and each process jump. To calculate the *original* process time, we must therefore keep track both of the number of original process jumps and the number of interventions.

We must also keep track of the postjump state and the time elapsed since the last jump for the original process because interior controls depend on them.

We now give the precise formulation. Define from given controlled process $\mathbf{x}_t$ a *new controlled process* $\hat{\mathbf{x}}_s$ with *state*

$$\hat{x} := (x, z, \tau, t', m, n),$$

where

x is the *state* of the *original* process;

z is the *postjump* state of the *original* process;

τ is the *time elapsed since the last jump* of the *original* process ;

t' is the *time remaining to intervene* or *fictitious time*;

m_s, n_s are respectively number of interventions and the number of original process jumps up to the present *process time* s.

If a strategy under consideration does not specify a next intervention decision time, i.e. we need to take t' to be ∞, we will instead take $t' := -8$. Therefore, the new process $\hat{\mathbf{x}}_s$ evolves in a new *state space* defined as follows:

$$\begin{aligned} \hat{E}^0 &= (E^0 \times E^0 \times T \times T' \times I\!N^2) \\ &\quad \cup (\partial E \times E^0 \times T \times [(-6, -4) \cup (-3, -1)] \times I\!N^2), \end{aligned}$$

where

$$T = (0, \infty);$$
$$T' = (-\infty, -7) \cup (-6, -4) \cup (-3, -1) \cup (0, \infty)$$

The *effective boundary* is

$$\begin{aligned}\hat{\Gamma}^* &= (E^0 \times E^0 \times T \times [\{-6\} \cup \{-3\} \cup \{0\}] \times \mathbb{N}^2) \cup \\ &\quad (\Gamma^* \times E^0 \times T \times [\{-6\} \cup \{-3\} \cup \{0\}] \times \mathbb{N}^2).\end{aligned}$$

Denote by $(\mathbf{x}_t \in E^0\colon\ t \in T)$ the *original* process and $(\hat{\mathbf{x}}_s \in \hat{E}^0 : s \in S)$ the *new* process, where the original time set T is called the *real time* and S is termed the *new process time set.* Then real time t in the new process is represented implicitly in terms of process time s as $t(s) := s - (m_s + n_s + 1)$.

Since the time remaining to intervene and fictitious time t' runs backwards at unit speed and all coordinates but fictitious time are kept constant while fictitious time is running, the dynamics of the new process are as follows:

In $E^0 \times E^0 \times T \times [(-\infty, -7) \cup (0, \infty)] \times \mathbb{N}^2$

$$\begin{aligned}\dot{x}_s &= f(x_s, u_0(\tau_s, z_s))\\ \dot{z}_s &= 0\\ \dot{\tau}_s &= 1\\ \dot{t}'_s &= -1\\ \dot{m}_s &= 0\\ \dot{n}_s &= 0\end{aligned}$$

In $[E^0 \cup \partial E] \times E^0 \times T \times [(-6, -4) \cup (-3, -1)] \times \mathbb{N}^2$

$$\begin{aligned}\dot{x}_s &= 0\\ \dot{z}_s &= 0\\ \dot{\tau}_s &= 0\\ \dot{t}'_s &= -1\\ \dot{m}_s &= 0\\ \dot{n}_s &= 0.\end{aligned}$$

While a trajectory of the original controlled process $\mathbf{x}_t$ starting at x_0 proceeds with time t, the corresponding trajectory of the new controlled process $\hat{\mathbf{x}}_s$ taking values in the state space $\hat{E}$ with the dynamics defined as above proceeds with time s in the following way.

The new process $\hat{\mathbf{x}}_s$ starts at the initial point $(x_0, x_0, 0, -2, 0, 0)$ at time $s :=$

0 and goes in fictitious time to $(x_0, x_0, 0, -3, 0, 0)$ which is a boundary point of $E^0 \times E^0 \times T \times (-3, -1) \times \{0\} \times \{0\}$ at $s = 1$.

Applying an impulse control action u_{δ_0}, the original process jumps to x_0^+ chosen by transition measure $Q_\delta(\cdot\,; x_0, u_{\delta_0})$ and $t_0{}'$ is set. This formulation allows impulse controls to be taken even at time $t = 0$. The new process jumps to $(x_0^+, x_0^+, 0, t_0', 1, 0)$ which is an interior point of $E^0 \times E^0 \times T \times T' \times \{1\} \times \{0\}$ and the new process continues its motion described by the integral curves until one of two possible cases occurs at real time t or process time $s = t + 1$

(i) $t' = 0$,

(ii) $t' > 0$ or $t' < -8$ and it is a jump epoch (either an interior jump or a boundary jump).

In case (i), the new process hits the boundary. It jumps to $(x_{t^-}, x_0^+, \tau_{t^-}, -2, 1, 0) \in E^0 \times E^0 \times T \times (-3, -1) \times \{1\} \times \{0\}$ or $\partial E \times E^0 \times T \times (-3, -1) \times \{1\} \times \{0\}$ depending on whether $x_{t^-} \in E^0$ or $x_{t^-} \in \partial E$.

In case (ii), if $x_{t^-} \in E^0$, the new process has an interior jump to $(x_{t^-}, x_0^+, \tau_{t^-}, -5, 1, 0) \in E^0 \times E^0 \times T \times (-6, -4) \times \{1\} \times \{0\}$. If $x_{t^-} \in \partial E$, the new process hits the boundary. It jumps to $(x_{t^-}, x_0^+, \tau_{t^-}, -5, 1, 0)$ which is an interior point of $\partial E \times E^0 \times T \times (-6, -4) \times \{1\} \times \{0\}$.

In both cases, the new process will continue along the appropriate integral curve until $t' = -3$ in case (i) or $t' = -6$ in case (ii) at which point it will jump using the given control strategy to a new state in which $t' \in (0, \infty)$ or $t' := -8 \in (-\infty, -7)$.

In case (i), the original process jumps under an impulse control action u_δ from x_{t^-} to x_t according to the transition measure $Q_\delta(\cdot\,; x_{t^-}, u_\delta)$. In case (ii), the original process jumps under an impulse control (as in case (i)) or jumps under an ordinary control according to the appropriate transition measure $Q_0(\cdot\,; x_{t^-}, u_0(\tau_{t^-}, x_0^+))$ or $Q_\partial(\cdot\,; x_{t^-}, u_\partial)$ optimally according to relevant remaining expected total cost. In the first instance, a cost $e^{-\delta t} l_\delta(x_{t^-}, u_\delta)$ is incurred, while in the second instance, a cost 0 or $e^{-\delta t} l_\partial(x_{t^-}, u_\partial)$ is incurred as the process enjoyed an interior or a boundary jump.

Note that in all cases, whether or not an intervention is dictated by the control policy, the state variable of the original process jumps to a point in E^0. In case (i), the process restarts again from the interior point $(x_t, x_t, 0, t', 2, 0)$ or $(x_t, x_t, 0, t', 1, 1)$ depending on which action (impulse or not) takes place. Similarly in case (ii), the process restarts again from the interior point $(x_t, x_t, 0, t', 2, 0)$ or $(x_t, x_t, 0, t', 1, 1)$.

To ensure that the new controlled process proceeds in the way described above, it remains to define the control sets $\hat{U}_0$ and $\hat{U}_\partial$, admissable controls $\hat{u} = (\hat{u}_0, \hat{u}_\partial)$, the jump rate $\hat{\lambda}$, and the transition measures $\hat{Q}_0$, $\hat{Q}_\partial$. The new control sets $\hat{U}_0$ and $\hat{U}_\partial$ to be defined below will also be compact.

Since the new process undergoes an interior jump only when $t' > 0$ and it is an interior jump epoch of the original process, the *interior control set* of the new process can be taken to be that of the original process, i.e. $\hat{U}_0 := U_0$. The new *jump rate* is

$$\hat{\lambda}(\hat{x}, u) := \begin{cases} \lambda(x, u) & \text{if } \hat{x} \in E^0 \times E^0 \times T \times [(0, \infty) \cup (-\infty, -7)] \times I\!N^2 \\ 0 & \text{otherwise} \end{cases}$$

When the new process has an interior jump, we expect it jump to the state with all coordinates kept the same except that fictitious time is set to -5. Therefore, the new *interior jump transition measure* is given by

$$\hat{Q}_0(\cdot\,; \hat{x}) = \begin{cases} \delta_{\{x, z, \tau, -5, m, n\}}(\cdot) & \text{if } \hat{x} \in E^0 \times E^0 \times T \times [(-\infty, -7) \cup (0, \infty)] \times I\!N^2 \\ \delta_{\{\hat{x}\}}(\cdot) & \text{otherwise.} \end{cases}$$

The new *boundary control set* is defined as

$$\hat{U}_\partial = (U_0 \cup U_\partial \cup U_\delta) \times U_{t'},$$

where $U_{t'} := [0, \infty) \cup \{-8\}$ is a one point compacification of $[0, \infty)$. It is thus a compact separable metric space. As mentioned in §1.3, this control set is well-defined since all results in Chapters 3 and 4 will hold if we use a such control set instead of a compact subset in Euclidean space.

An admissible *boundary control* is a feedback function $\hat{u}_\partial : \partial\hat{E} \longrightarrow \hat{U}_\partial$ such that

$$\begin{array}{ll} \hat{u}_\partial(\hat{x}) \in U_\delta \times U_{t'} & \text{if } \hat{x} \in E^0 \times E^0 \times T \times \{-3\} \times \{m\} \times \{n\} \\ \hat{u}_\partial(\hat{x}) \in [U_\delta \cup U_\partial] \times U_{t'} & \text{if } \hat{x} \in \partial E \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\} \\ \hat{u}_\partial(\hat{x}) \in [U_\delta \cup \{u_0(\tau, z)\}] \times U_{t'} & \text{if } \hat{x} \in E^0 \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\}. \end{array}$$

The *boundary jump transition measure* $\hat{Q}_\partial$ is defined as follows:

$$\hat{Q}_\partial(\cdot;\hat{x},\hat{u}_\partial) := \begin{cases} \delta_{\{x,z,\tau,-2,m,n\}}(\cdot) & \text{if } \hat{x} \in E \times E^0 \times T \\ & \quad \times\{0\} \times \{m\} \times \{n\}, \\ \delta_{\{x,z,\tau,-5,m,n\}}(\cdot) & \text{if } \hat{x} \in \partial E \times E^0 \times T \times \\ & \quad [(0,\infty) \cup (-\infty,-7)] \\ & \quad \times\{m\} \times \{n\}, \\ \delta_{\{y\}}(dz) Q_\delta(dy; x, u_\delta) \delta_{\{0,t',m+1,n\}}(\cdot) & \text{if } \hat{x} \in E \times E^0 \times T \\ & \quad \times \{-3\} \times \{m\} \times \{n\} \\ & \text{and } \hat{u}_\partial := (u_\delta, t'), \\ \delta_{\{y\}}(dz) Q_0(dy; x, u_0(\tau,z)) \delta_{\{0,t',m,n+1\}}(\cdot) & \text{if } \hat{x} \in E^0 \times E^0 \times T \\ & \quad \times\{-6\} \times \{m\} \times \{n\} \\ & \text{and } \hat{u}_\partial := (u_0(\tau,z), t'), \\ \delta_{\{y\}}(dz) Q_\delta(dy; x, u_\delta) \delta_{\{0,t',m+1,n\}}(\cdot) & \text{if } \hat{x} \in E^0 \times E^0 \times T \\ & \quad \times\{-6\} \times \{m\} \times \{n\} \\ & \text{and } \hat{u}_\partial := (u_\delta, t'), \\ \delta_{\{y\}}(dz) Q_\partial(dy; x, u_\partial) \delta_{\{0,t',m,n+1\}}(\cdot) & \text{if } \hat{x} \in \partial E \times E^0 \times T \\ & \quad \times\{-6\} \times \{m\} \times \{n\} \\ & \text{and } \hat{u}_\partial := (u_\partial, t'), \\ \delta_{\{y\}}(dz) Q_\delta(dy; x, u_\delta) \delta_{\{0,t',m+1,n\}}(\cdot) & \text{if } \hat{x} \in \partial E \times E^0 \times T \\ & \quad \times\{-6\} \times \{m\} \times \{n\} \\ & \text{and } \hat{u}_\partial := (u_\delta, t'). \end{cases}$$

We have now finished the construction of the embedding process.

Next we identify cost functions for the new problem so that it has the same expected total cost as the original problem.

The expected total cost of the original problem is

$$\begin{aligned} J_x(u) \;:=\; & E_x\Big[\int_0^\infty e^{-\delta t} l_0(\mathbf{x}_t^u, u_0(\tau_t^u, \mathbf{z}_t^u))dt \\ & + \sum_{\mathbf{T}_i^u \neq \tau_i} e^{-\delta \mathbf{T}_i^u} l_\partial(\mathbf{x}_{T_i^-}^u, u_\partial(\mathbf{x}_{T_i^-}^u)) 1_{\{\mathbf{x}_{T_i^-}^u \in \partial E\}} \\ & + \sum_i e^{-\delta \tau_i} l_\delta(\mathbf{x}_{\tau_i^-}^u, u_\delta(\mathbf{x}_{\tau_i^-}^u))\Big] \end{aligned}$$

Arrange T_i and τ_i in increasing order and denote the resulting sequence by $\{T_i^*\}$, so that T_i^* is the ith jump epoch of the original process $\mathbf{x}_t$ controlled by both dynamic and impulse control. In terms of new process time, we define the *jump time* as $\hat{T}_i := T_i^* + i + 1$.

Now rewrite the expected total cost of the original problem as follows:

$$\begin{aligned} J_x(u) \;=\; & E_x\Big[\sum_{i=0}^{\infty} \int_{\mathbf{T}_i^*}^{\mathbf{T}_{i+1}^*} e^{-\delta t} l_0(\mathbf{x}_t, u_0(\tau_t, \mathbf{z}_t))dt \\ & + \sum_{\mathbf{T}_i \neq \tau_i} e^{-\delta \mathbf{T}_i} l_\partial(\mathbf{x}_{T_i^-}, u_\partial(\mathbf{x}_{T_i^-})) I_{\{\mathbf{x}_{T_i^-} \in \partial E\}} \\ & + \sum_i e^{-\delta \tau_i} l_\delta(\mathbf{x}_{\tau_i^-}, u_\delta(\mathbf{x}_{\tau_i^-}))\Big]. \end{aligned} \tag{5.2}$$

Setting $t := s - (i+1)$, we have

$$\begin{aligned} & \int_{T_i^*}^{T_{i+1}^*} e^{-\delta t} l_0(x_t, u_0(\tau_t, z_t))dt \\ = \; & \int_{T_i^*+(i+1)}^{T_{i+1}^*+(i+1)} e^{-\delta(s-(i+1))} l_0(x_{s-(i+1)}, u_0(\tau_{s-(i+1)}, z_{s-(i+1)}))ds \\ = \; & \int_{\hat{T}_i}^{\hat{T}_{i+1}-1} e^{-\delta s} e^{\delta(i+1)} l_0(x_{s-(i+1)}, u_0(\tau_{s-(i+1)}, z_{s-(i+1)}))ds. \end{aligned}$$

Therefore, if we define the new *running cost* function $\hat{l}_0$ for $u \in U_0$ as

$$\hat{l}_0(\hat{x}, u) := \begin{cases} e^{\delta(m+n+1)} l_0(x, u) & \text{if } \hat{x} \in E^0 \times E^0 \times T \times [(-\infty, -7) \cup (0, \infty)] \\ & \qquad \times \{m\} \times \{n\} \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\int_{T_i^*}^{T_{i+1}^*} e^{-\delta t} l_0(x_t, u_0(\tau_t, z_t))dt = \int_{\hat{T}_i}^{\hat{T}_{i+1}} e^{-\delta s} \hat{l}_0(\hat{x}_s, \hat{u}_0(\hat{\tau}_s, \hat{z}_s))ds.$$

Consequently, we have

$$\begin{aligned}\sum_{i=0}^{\infty}\int_{T_i^*}^{T_{i+1}^*} e^{-\delta t} l_0(x_t, u_0(\tau_t, z_t))dt] &= \sum_{i=0}^{\infty}\int_{\hat{T}_i}^{\hat{T}_{i+1}} e^{-\delta s}\hat{l}_0(\hat{x}_s, \hat{u}_0(\hat{\tau}_s, \hat{z}_s))ds \\ &= \int_0^{\infty} e^{-\delta s}\hat{l}_0(\hat{x}_s, \hat{u}_0(\hat{\tau}_s, \hat{z}_s))ds. \qquad (5.3)\end{aligned}$$

Similarly, if we define the new *boundary cost* function $\hat{l}_\partial$ as

$$\hat{l}_\partial(\hat{x}, \hat{u}_\partial) := \begin{cases} e^{\delta(m+n+1)} l_\partial(x, u_\partial) & \text{if } \hat{x} \in \partial E \times E^0 \times T \times \{-6\} \times \{m\} \times \{n\} \\ & \text{and } \hat{u}_\partial := (u_\partial, t') \\ e^{\delta(m+n+1)} l_\delta(x, u_\delta) & \text{if } \hat{x} \in \partial E \times E^0 \times T \times [\{-3\} \cup \{-6\}] \times \{m\} \times \{n\} \\ & \text{and } \hat{u}_\partial := (u_\delta, t') \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned}&\sum_{T_i \neq \tau_i} e^{-\delta T_i} l_\partial(x_{T_i^-}, u_\partial(x_{T_i^-})) I_{\{x_{T_i^-} \in \partial E\}} + \sum_i e^{-\delta \tau_i} l_\delta(x_{\tau_i^-}, u_\delta(x_{\tau_i^-}))] \\ &= \sum_i e^{-\delta \hat{T}_i} \hat{l}_\partial(\hat{x}_{\hat{T}_i^-}, \hat{u}_\partial(\hat{x}_{\hat{T}_i^-})) 1_{\{\hat{x}_{\hat{T}_i^-} \in \partial E\}}. \qquad (5.4)\end{aligned}$$

The new PDP optimal control problem is to find an optimal control among all *admissible controls* $\hat{u} := (\hat{u}_0, \hat{u}_\partial) \in \hat{C}$ such that the *expected total cost*

$$\begin{aligned}J_{\hat{x}}(\hat{u}) &:= E_{\hat{x}}[\int_0^{\infty} e^{-\delta s}\hat{l}_0(\hat{\mathbf{x}}_s, \hat{u}_0(\hat{\tau}_s, \hat{\mathbf{z}}_s))ds \\ &\quad + \sum_i e^{-\delta \hat{\mathbf{T}}_i} \hat{l}_\partial(\hat{\mathbf{x}}_{\hat{T}_i^-}, \hat{u}_\partial(\hat{\mathbf{x}}_{\hat{T}_i^-})) 1_{\{\hat{\mathbf{x}}_{\hat{T}_i^-} \in \partial \hat{E}\}}]\end{aligned}$$

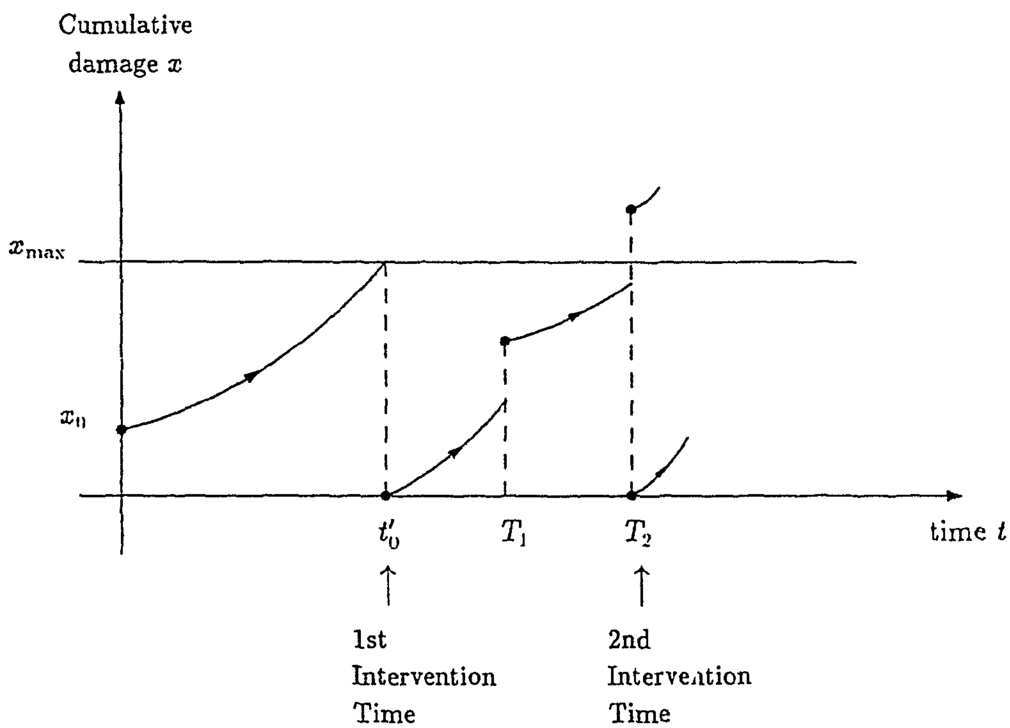
is minimized. Here $\hat{x} := (x_0, x_0, 0, -2, 0, 0)$.

We conclude from equalities (5.2), (5.3) and (5.4) that the expected cost of the new problem is the same as that of the original problem.

We will give an example to illustrate the construction of a new boundary controlled process from an original impulse controlled process.

Example A Repair/Maintenance Model

Suppose x_t repesents the cumulative *degree of damage* to a machine at time t. This increases at rate $f(x)$ when the degree of damage is x, and also discontinuouly due to independent random shocks which occur at Poisson times and have some known distribution function G. The *intervention stategy* is to replace the machine (i.e. set x_t to 0) when the cumulative damage first exceeds some fixed level x_{max}. (Of course, this could happen at a shock time or between shocks, see the figure.) There may or may not be some delay in doing this.



Since there is no dynamic control in this case, we can take the new state space to be $\hat{E} := E^0 \times T' \times I\!N^2$, where T' is defined as above.

While the trajectory of the original impulse controlled process $\mathbf{x}_t$ starting at x_0 proceeds with (real) time t, the corresponding trajectory of the new process $\hat{\mathbf{x}}_s$ taking values in the new state space $\hat{E}$ proceeds with (process) time s in the following way.

The new process starts at the initial state $(x_0, -2, 0, 0)$ at time $s := 0$ and goes in fictitious time to $(x_0, -3, 0, 0)$ which is a boundary point of $E^0 \times (-3, -1) \times \{0\} \times \{0\}$ at $s = 1$.

Set the time remaining to intervene (i.e. the time remaining to replace the machine provided no random shocks have occurred) $t' := t_0'$, the time at which, starting from the initial damage level x_0, the cumulative damage will first exceed $x_{\max}$ at $t = t_0'$, i.e. such that $\int_0^{t_0'} f(x)dx + x_0 = x_{\max}$, providing no random shocks occur before time t_0' and let the impulse control action be equal to zero. This is equivalent to taking a new boundary control $\hat{u}_\partial := (u_\delta, t') := (0, t_0')$. Under this boundary control action, the new process jumps to $(x_0, t_0', 1, 0) \in E^0 \times T' \times \{1\} \times \{0\}$ and, if (as shown) there are no shocks before $t = t_0'$, continues its motion until it reaches the state $(x_{\max}, 0, 1, 0)$ which is a boundary point of $E^0 \times T' \times \{1\} \times \{0\}$. At this boundary point, the new process has an uncontrolled boundary jump to $(x_{\max}, -2, 1, 0) \in E^0 \times (-3, -1) \times \{1\} \times \{0\}$ and goes in fictitious time to $(x_{\max}, -3, 1, 0)$ which is a boundary point of $E^0 \times (-3, -1) \times \{1\} \times \{0\}$. Applying the boundary control action $\hat{u}_\partial := (-x_{\max}, t_1')$, where t_1' satisfies $x_{\max} = \int_0^{t_1'} f(x)dx$, the new process $\hat{x}_t$ jumps to $(0, t_1', 2, 0) \in E^0 \times T' \times \{2\} \times \{0\}$ and continues its motion until it reaches the state $(x_{T_1^-}, t_1' - (T_1 - t_0'), 2, 0)$ at the first jump time $t = T_1$ or $s = T_1 + 2$. The process $\hat{x}_s$ then takes an interior jump to $(x_{T_1}-, -5, 2, 0) \in E^0 \times (-6, -4) \times \{2\} \times \{0\}$ and runs in fictitious time to $(x_{T_1^-}, -6, 2, 0)$ which is a boundary point of $E^0 \times (-6, -4) \times \{2\} \times \{0\}$. Applying the boundary control action $\hat{u}_\partial := t_2'$ (i.e. do not intervene), where $x_{\max} = \int_0^{t_2'} f(x)dx + x_{T_1}$, the new process jumps to $(x_{T_1}, t_2', 2, 1) \in E^0 \times T' \times \{2\} \times \{1\}$, where x_{T_1} is determined by the distribution function G. The new process $\hat{x}_s$ again continues its motion until it reaches the state $(x_{T_2^-}, t_2' - (T_2 - T_1), 2, 1)$ at the second jump time $t = T_2$ or $s = T_2 + 3$, It then takes an interior jump to $(x_{T_2^-}, -5, 2, 1)$ and proceeds in fictitious time to $(x_{T_2^-}, -6, 2, 1)$, a boundary point. Applying the boundary control action $\hat{u}_\partial := (-x_{T_2^-}, t_3')$ (i.e. intervene to replace the machine), where $t_3' := t_1'$ (i.e. $x_{\max} = \int_0^{t_3'} f(x)dx$), the new process jumps to $(0, t_3', 3, 1)$ and restarts again from this interior point.

This example shows that three possible cases can occur (see the figure).

(1) At time t_0', $t' = 0$. We intervene to replace the machine.

(2) At time T_1, a jump epoch, a decision is made not to intervene.

(3) At time T_2, a (second) jump epoch, a decision is made to intervene and replace the machine.

■

5.3 Generalized BHJ Equation for the New Problem

In the last section, we have reduced the original problem with both dynamic and impulse control to an equivalent problem with only dynamic control. Therefore if Q_δ satisfies the same conditions as Q_0 and Q_∂, we can apply all but the uniqueness result (which as stated requires the compactness of the state space not satisfied in this case) obtained in Charpters 3 and 4 to the new problem. We will not state the BHJ necessary and sufficient optimality condition or the maximum principle for the new problem here since they are straightforward but complicated. However, since the value function and the BHJ equation play a very important role in our charactrizations of the optimality, we will set up the generalized BHJ equation for the new problem in this section.

Define the *value function* for the new problem by

$$\hat{V}(\hat{x}) := \min_{\hat{u}\in\hat{C}} J_{\hat{x}}(\hat{u}) \quad \text{for any } \hat{x} \in \hat{E}.$$

Then the *generalized BHJ equation* for the new problem is

$$\min_{\substack{\xi'\in\partial\hat{V}(\hat{z})\\ v\in U_0}} \{\xi'\hat{f}(\hat{z},v) + \hat{\lambda}(\hat{z},v)\int_{\hat{E}^0}(\hat{V}(\hat{y}) - \hat{V}(\hat{z}))\hat{Q}_0(d\hat{y};\hat{z})$$
$$-\delta\hat{V}(\hat{z}) + \hat{l}_0(\hat{z},v)\} = 0 \qquad \forall \hat{z} \in \hat{E}^0 \tag{5.5}$$

with *boundary condition*

$$\hat{V}(\hat{z}) = \min_{\hat{v}\in\hat{U}_\partial}\{\hat{l}_\partial(\hat{z},\hat{v}) + \int_{\hat{E}^0} \hat{V}(\hat{y})\hat{Q}_\partial(d\hat{y};\hat{z},\hat{v})\} \ \ \forall \hat{z}\in\partial\hat{E}. \tag{5.6}$$

Substitute the non-hat counterparts into equations (5.5) and (5.6). Then the BHJ equation (5.5) becomes

$$\min_{\substack{\xi'\in\partial\hat{V}(\hat{x})\\ v\in U_0}} \{\xi'(f(x,v),0,1,-1,0,0) + \lambda(x,v)[\hat{V}(x,z,\tau,-5,m,n) - \hat{V}(\hat{x})]$$
$$-\delta\hat{V}(\hat{x}) + e^{\delta(m+n+1)}l_0(x,v)\} = 0$$
$$\forall \hat{x}\in E^0\times E^0\times T\times[(-\infty,-7)\cup(0,\infty)]\times I\!N^2$$

$$\min_{\xi'\in\partial\hat{V}(\hat{x})}\{\xi'(0,0,0,-1,0,0) - \delta\hat{V}(\hat{x})\} = 0$$
$$\forall \hat{x}\in[E^0\cup\partial E]\times E^0\times T\times[(-6,-4)\cup(-3,-1]\times I\!N^2$$

and the boundary condition (5.6) implies

$$\hat{V}(\hat{x}) = \min_{\substack{v\in U_\delta\\ t'\in U_{t'}}}\{e^{\delta(m+n+1)}l_\delta(x,v) + \int_{E^0}\hat{V}(z,z,0,t',m+1,n)Q_\delta(dz;x,v)\} \tag{5.7}$$
$$\forall \hat{x}\in[E^0\cup\partial E]\times E^0\times T\times\{-3\}\times\{m\}\times\{n\}$$

$$\hat{V}(\hat{x}) = \min_{\substack{u_\partial\in U_\partial\\ u_\delta\in U_\delta\\ t'\in U_{t'}}} \min\{ \quad e^{\delta(m+n+1)}l_\partial(x,u_\partial) + \int_{E^0}\hat{V}(z,z,0,t',m,n+1)Q_\partial(dz;x,u_\partial),$$
$$e^{\delta(m+n+1)}l_\delta(x,u_\delta) + \int_{E^0}\hat{V}(y,y,0,t',m+1,n)Q_\delta(dy;x,u_\delta)\}$$
$$\forall \hat{x}\in\partial E\times E^0\times T\times\{-6\}\times\{m\}\times\{n\}$$

$$\hat{V}(\hat{x}) = \min_{\substack{u_0(z,\tau)\\ u_\delta\in U_\delta\\ t'\in U_{t'}}} \min\{ \quad e^{\delta(m+n+1)}l_\delta(x,u_\delta) + \int_{E^0}\hat{V}(z,z,0,t',m+1,n)Q_\delta(dz;x,u_\delta),$$
$$\int_{E^0}\hat{V}(z,z,0,t',m,n+1)Q_0(dz;x,u_0(\tau,z))\}$$
$$\forall \hat{x}\in E^0\times E^0\times T\times\{-6\}\times\{m\}\times\{n\}\ .$$

Remark 5.1 It follows from the above construction that prior to jumps of the new process

$$\begin{aligned} V(x) &= e^{-\delta(m+n+1)}\hat{V}(x,z,\tau,-3,m,n) \qquad &\text{(intervention)} \\ V(x) &= e^{-\delta(m+n+1)}\hat{V}(x,z,\tau,-6,m,n) \qquad &\text{(jump),} \end{aligned}$$

while postjump we have

$$\begin{aligned} V(z) &= \min_{t'\in U_{t'}} e^{-\delta(m+n+1)}\hat{V}(z,z,0,t',m+1,n) \qquad &\text{(intervention)} \\ V(z) &= \min_{t'\in U_{t'}} e^{-\delta(m+n+1)}\hat{V}(z,z,0,t',m,n+1) \qquad &\text{(jump).} \end{aligned}$$

■

To relate our approach to the quasi-variational approach, assume there is *no* dynamic control, the impulse control action does *not* take place at boundary states, $l_\partial = 0$ and $Q_\delta(\cdot\,;x,v) := \delta_{x+v}(\cdot)$ the one-atom probability measure concentrated on $x+v$. Then the boundary condition for the new embedding problem and Remark 5.1 together imply that if $x \in E^0$ is a state where the BHJ equation is not satisfied by the original problem, then following equality, which implicitly defines the *intervention boundaries* in E^0, must be satisfied,

$$V(x) = \min_{v\in U_\delta}\{l_\delta(x,v) + V(x+v)\},$$

while if $x \in \partial E$, then the following boundary condition must be satisfied,

$$V(x) = \int_{E^0} V(y) Q_\partial(dy;x).$$

This relates our approach to the quasi-variational inequality approach (*cf.* §5.1).

Chapter 6

An Equivalent model with Bounded Costs

6.1 Introduction

In the previous chapters, we have developed a control theory for the PDP optimal control problem with bounded cost functions l_0 and l_∂. Unfortunately, cost functions are in general not bounded. In a lot of cases, for example, capacity expansion problems (*cf.* Davis *et al.* 1986, 1987 or Example 1.5.1) and stochastic scheduling (*cf.* Solel 1986, Dempster and Solel 1987 or Example 1.52.), cost functions are not bounded but are instead subject to certain bounded growth conditions.

In this chapter, we consider a PDP optimal (dynamic) control problem as formulated in §3.1 with cost functions l_0 and l_∂ that are subject to ***bounded growth*** by which we mean that there exists a function $g(x) : E \longrightarrow \mathbb{R}_+$ such that $l_0(x,u) \leq g(x)$ and $l_\partial(x,u) \leq g(x)$ for all x and u. Applying the results of the previous chapter this extends immediately to the PDP optimal control problem with both dynamic and impulse control.

The purpose of this chapter is to construct an equivalent PDP control problem with *bounded* costs using the technique of the *multiplicative functional transformation* as in Davis *et al.* (1986).

6.2 An Equivalent Model with Bounded Costs

As was shown in the proof of Proposition 3.3, a discounted problem with discount rate δ is equivalent to a non-discounted problem with killing rate δ. The killed process is a new PDP on $E^0_\triangle := E^0 \cup \{\triangle\}$ with total jump rate $\lambda(x,u)+\delta$. All functions $\chi(x,u)$ are extended to $E^0_\triangle \times U_0$ or $E_\triangle \times U_\partial$ by setting $\chi(\triangle,u)=0$. A trajectory of the killed process proceeds as the original one until it jumps to the coffin state $\triangle$ and remains at $\triangle$ thereafter. Since the killing rate is a part of the jump rate for the killed process, it can be generalized to depend on the state x and control u just as the original jump rate $\lambda(x,u)$ does.

Definition 6.1 A *multiplicative functional* of a Markov process $\mathbf{x}_t$ is an adapted two-parameter process β^s_t such that

$$\beta^r_t = \beta^r_s \beta^s_t \quad \text{for} \quad r \le s \le t.$$

■

Multiplicative functionals and their properties are described in detail in Dynkin (1965) and Blumenthal and Getoor (1968). An introduction adequate for our purposes here may be found in Davis (1981).

The purpose of this section is to prove the following theorem.

Theorem 6.1 *Suppose there is a C^1 function $g : E \longrightarrow \mathbb{R}_+$ such that the following conditions are satisfied:*

(A6.1) $\qquad l_0(x,v) \le g(x) \qquad$ *for all $x \in E^0$ and $v \in U_0$*

$\qquad\qquad l_\partial(x,v) \le g(x) \qquad$ *for all $x \in \partial E$ and $v \in U_\partial$,*

(A6.2) $\qquad E^u_x[g(\mathbf{x}_t)] \le g(x)$,

(A6.3)

$$(1-\bar{g}(x,v)/g(x))\lambda(x,v)+\delta+1/g(x)\cdot\nabla g(x) f(x,v) \ge 0,$$

where $\bar{g}(x,v) := \int_{E^0} Q_0(dy\,;x,v)$. *Then there is a PDP* $\{\tilde{x}_t\}$ *on* E^0 *with the same vector field* f, *jump rate* $\tilde{\lambda}(x,v) := \lambda(x,v)\bar{g}(x,v)/g(x)$, *killing rate*

$$\psi(x,v) := (1-\bar{g}(x,v)/g(x))\lambda(x,v) + \delta + 1/g(x)\cdot\nabla g(x)f(x,v)$$

and transition measure

$$\int_A \tilde{Q}_0(dy;x,v) := \frac{1}{\bar{g}(x,v)}\int_A g(y)Q_0(dy;x,v).$$

The problem of control of the new process $\{\tilde{x}_t\}$ *with initial state* $x \in E^0$ *and the bounded costs* $\tilde{l}_0(x,v) := l_0(x,v)/g(x)$ *and* $\tilde{l}_\partial(x,v) := l_\partial(x,v)/g(x)$ *is equivalent to the original problem in the sense that its expected costs are equal to* $\frac{1}{g(x)}$ *times the expected costs of the original problem.*

Proof

Step 1. Fix an admissible control u and let P_t^u be the semigroup (i.e. P_t^u has the semigroup property $P_t^uP_s^u = P_{t+s}^u$) of the corresponding controlled killed process $\{x_t\}$ defined for $\phi \in B(E_\triangle^0)$ (or $\phi \in B(E_\triangle)$) by

$$P_t^u\phi(x) := E_x^u[\phi(x_t)],$$

where E_x^u denotes the expectation with respect to the process x_t with initial state x.

Define $\beta_t := g(x_t^u)/g(x)$. Since $\beta_t = \beta_0\beta_t$, it is a one-parameter multiplicative functional of x_t.

Let $\tilde{P}_t^u$ be the semigroup corresponding to β defined by

$$\begin{aligned} \tilde{P}_t^u\phi(x) &:= E_x^u[\phi(x_t)\beta_t] \\ &= \frac{1}{g(x)}E_x^u[\phi(x_t)g(x_t)]. \end{aligned} \tag{6.1}$$

By assumption (A6.2), we have

$$\tilde{P}_t^u I_{E^0} \le 1.$$

Therefore, $\tilde{P}_t^u$ defines a sub-Markovian semigroup.

As is shown in Blumenthal and Getoor (1968), one can construct a transformed process $\{\tilde{\mathrm{x}}_t\}$ satisfying

$$\tilde{P}_t^u \phi(x) = E_x^u \phi(\tilde{\mathrm{x}}_t) \quad \phi \in B(E_\triangle^0) \quad (\text{or } \forall \phi \in B(E_\triangle)) \quad x \in E^0. \tag{6.2}$$

For simplicity, we write $l_0(x,u)$ as $l_0^u(x)$ and $l_\partial(x,u)$ as $l_\partial^u(x)$. It follows from this and (6.1) that for

$$\phi(x) := \tilde{l}_0^u(x) = l_0^u(x)/g(x) \in B(E_\triangle^0)$$

and

$$\phi(x) := \tilde{l}_\partial^u(x) I_{(x\in\partial E)} = l_\partial^u(x)/g(x) I_{(x\in\partial E)} \in B(E_\triangle),$$

we can use the Fubini theorem to yield

$$\begin{aligned}
\tilde{J}_x(u) &:= E_x^u\Big[\int_0^\infty \tilde{l}_0^u(\tilde{\mathrm{x}}_t)dt + \sum_i \tilde{l}_\partial^u(\tilde{\mathrm{x}}_{T_i^-}) I_{(\tilde{\mathrm{x}}_{T_i^-}\in\partial E)}\Big] \\
&= \int_0^\infty E_x^u[\tilde{l}_0^u(\tilde{\mathrm{x}}_t)]dt + \sum_i E_x^u[\tilde{l}_\partial^u(\tilde{\mathrm{x}}_{T_i^-}) I_{(\tilde{\mathrm{x}}_{T_i^-}\in\partial E)}] && (6.3) \\
&= \int_0^\infty \frac{1}{g(x)} E_x^u[\tilde{l}_0^u(\mathrm{x}_t) g(\mathrm{x}_t)]dt \\
&\quad + \sum_i \frac{1}{g(x)} E_x^u[\tilde{l}_\partial^u(\mathrm{x}_{T_i^-}) g(\mathrm{x}_{T_i^-}) I_{(\mathrm{x}_{T_i^-}\in\partial E)}] && (6.4) \\
&= \frac{1}{g(x)} E_x^u\Big[\int_0^\infty \tilde{l}_0^u(\mathrm{x}_t) g(\mathrm{x}_t) dt \\
&\quad + \sum_i \tilde{l}_\partial^u(\mathrm{x}_{T_i^-}) g(\mathrm{x}_{T_i^-}) I_{(\mathrm{x}_{T_i^-}\in\partial E)}\Big] && (6.5) \\
&= \frac{1}{g(x)} E_x^u\Big[\int_0^\infty l_0^u(\mathrm{x}_t) dt \\
&\quad + \sum_i l_\partial^u(\mathrm{x}_{T_i^-}) I_{(\mathrm{x}_{T_i^-}\in\partial E)}\Big] && (6.6) \\
&= \frac{1}{g(x)} J_x(u). && (6.7)
\end{aligned}$$

Here the equalities (6.3) and (6.5) follow from Fubini theorem. The equality (6.4) has on both sides an expression for $\tilde{P}_t^u(\cdot)$ given respectively by equalities (6.1) and (6.2). The equalities (6.6) and (6.7) follow from the definitions of $\tilde{l}_0^u$ and $\tilde{l}_\partial^u$ and $J_x(u)$ respectively.

Since $l_0(x,u) \leq g(x)$ and $l_\partial(x,u) \leq g(x)$ by assumption (A6.2), $\tilde{l}_0(x,u) = l_0(x,u)/g(x) \leq 1$ and $\tilde{l}_\partial(x,u) = l_\partial(x,u)/g(x) \leq 1$. Thus the control problem for $\{\mathbf{x}_t\}$ is replaced by an equivalent one for $\{\tilde{\mathbf{x}}_t\}$ with bounded cost functions.

Step 2. By Davis (1981), the extended generator of the transformed process $\{\tilde{\mathbf{x}}_t\}$ is

$$(\tilde{\mathcal{A}}^v\phi)(x) = \frac{1}{g(x)}\mathcal{A}^v[\phi g](x),$$

where $\mathcal{A}^v\phi$ is the extended generator of the original process $\{\mathbf{x}_t\}$ corresponding to a fixed control action $v \in U_0$. By Proposition 1.1, the extended generator of the original killed process $\{\mathbf{x}_t\}$ is given by:

$$(\mathcal{A}^v\phi)(x) = \nabla\phi(x)f(x,v) + \lambda(x,v)[\int_{E^0}\phi(y)Q_0(dy;x,v) - \phi(x)] - \delta\phi(x).$$

Consequently,

$$\begin{aligned}
(\tilde{\mathcal{A}}^v\phi)(x) &= \frac{1}{g(x)}\mathcal{A}^v[\phi g](x) \\
&= \frac{1}{g(x)}[\phi(x)\nabla g(x)f(x,v) + g(x)\nabla\phi(x)f(x,v) \\
&\quad +\lambda(x,v)(\int_{E^0}\phi(y)g(y)Q_0(dy;x,v) - \phi(x)g(x)) - \delta\phi(x)g(x)] \\
&= \nabla\phi(x)f(x,v) + 1/g(x)\cdot\nabla g(x)f(x,v)\phi(x) \\
&\quad +\lambda(x,v)\bar{g}(x,v)/g(x)[\int_{E^0}\phi(y)g(y)/\bar{g}(x,v)Q_0(dy;x,v) - \phi(x)] \\
&\quad -[(1-\bar{g}(x,v)/g(x))\lambda(x,v) + \delta]\phi(x),
\end{aligned}$$

where $\bar{g}(x,v) = \int_{E^0} g(y)Q_0(dy;x,v)$.

Using

$$\begin{aligned}
\tilde{\lambda}(x,v) &:= \lambda(x,v)\bar{g}(x,v)/g(x) \\
\tilde{Q}_0(dy;x,v) &:= g(y)/\bar{g}(x,v)Q_0(dy;x,v) \\
\psi(x,v) &:= (1-\bar{g}(x,v)/g(x))\lambda(x,v) + \delta + 1/g(x)\cdot\nabla g(x)f(x,v),
\end{aligned}$$

we have that

$$(\tilde{\mathcal{A}}^v\phi)(x) = \nabla\phi(x)f(x,v) + \tilde{\lambda}(x,v)[\int_{E^0}\phi(y)\tilde{Q}_0(dy;x,v) - \phi(x)] - \psi(x,v)\phi(x).$$

Since $\psi(x,v) \geq 0$ by assumption (A6.3), this is the generator of a PDP with vector field f, jump rate $\tilde{\lambda}$, killing rate $\psi(x,v)$ and transition measure $\tilde{Q}_0$.
Consequently, we have the desired result from the uniqueness of the extended generator. ∎

Remark 6.1 If $g(x)$ is equal to a constant, then it is easy to see that the transformed process coincides with the original process. ∎

Bibliography

1. J.P. Aubin and A. Cellina (1984). *Differential Inclusions.* Springer-Verlag, Berlin.

2. D. Berksekas and S.E. Shreve (1978). *Stochastic Optimal Control: The Discrete-time Case.* Academic Press, New York.

3. R.M. Blumenthal and R.K. Getoor (1968). *Markov Processes and Potential Theory.* Academic Press, New York.

4. V.G. Boltyanskii (1966). Sufficient conditions for optimality and justification of the dynamic programming method. *SIAM J. Control and Optimization* **4**, 326-361.

5. D.A. Carlson and A. Haurie (1987). *Infinite Horizon Optimal Control.* Lecture Notes in Economics and Mathematical Systems **290**. Springer-Verlag, Berlin.

6. F.H. Clarke (1983). *Optimization and Nonsmooth Analysis.* John Wiley & Sons, New York.

7. F.H. Clarke and R.B. Vinter (1983). Local optimality conditions and Lipschitzian solutions to the Hamilton-Jacobi equation. *SIAM J. Control and Optimization* **21**, 856-870.

8. O.L.V. Costa and M.H.A. Davis (1988). Approximations for optimal stopping of a piecewise-deterministic process. *Mathematics of Control, Signals, and Systems* **1**, 123-146.

9. O.L.V. Costa and M.H.A. Davis (1989). Impulse control of piecewise-deterministic processes. *Mathematics of Control, Signals, and Systems* **2**, 187-206.

10. R. Courant and D. Hilbert (1962). *Methods of Mathematics Physics*, Vol. II: Partial Differential Equations. Wiley-Interscience, New York.

11. M.C. Crandall and P.-L. Lions (1983). Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.* **227**, 1-42.

12. M.H.A. Davis (1976). The representation of martingales of jump processes. *SIAM J. Control and Optimization* **14**, 623-638.

13. M.H.A. Davis (1981). Pathwise nonlinear filtering. *Stochastic Systems.* M. Hazewinkel and J.C. Willems, eds. Reidel, Dordrecht, 520-528.

14. M.H.A. Davis (1984). Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. *J. Roy. Statist. Soc. Ser. B* **46**, 353-388.

15. M.H.A. Davis (1986). Control of piecewise-deterministic processes via discrete time dynamic programming. *Proceedings of the 3rd Bad Honnef Conference on Stochastic Differential Systems 3-7.6.85.* Lecture Notes in Control and Information Sciences **16**. Springer-Verlag, Berlin, 140-150.

16. M.H.A. Davis, M.A.H. Dempster and D. Vermes (1986). A new approach to optimal capacity expansion under uncertainty. Research report, Dept. of Electrical Eng., Imperial College, London.

17. M.H.A. Davis, M.A.H. Dempster, S.P. Sethi and D. Vermes (1987). Optimal capacity expansion under uncertainty. *Adv. in Appl. Probab.* **19**, 156-176.

18. M.A.H. Dempster (1989). Optimal control of piecewise deterministic Markov processes. To appear in *Proceedings of the Imperial College Workshop on Applied Stochastic Analysis 5-7.4.89.* M.H.A. Davis and R.J. Elliott, eds. Gordon & Breach, London.

19. M.A.H. Dempster and E. Solel (1987). Stochastic scheduling via stochastic control. *Proc. First World Congress Bernoulli Society.* K. Krickeberg and A. Shiryaev, eds. VNU Science Press, Utrecht, 783-788.

20. M.A.H. Dempster and J.J. Ye (1990). A maximum principle for control of piecewise deterministic Markov processes, *Approximation, Optimization and Computing: Theory and Applications.* A.G. Law and C.L. Wang, eds., North-Holland, 235-240.

21. F.A. van der Duyn Schouten (1983). *Markov Decision Drift Processes.* Amsterdam: CWI Tract, Centre for Mathematics and Computer Science.

22. E.B. Dynkin (1965). *Markov Processes.* Springer-Verlag, Berlin.

23. R.J. Elliott (1987). *Viscosity Solutions and Optimal Control.* Research Notes in Maths. **165**. Longman, Harlow, Essex.

24. W.H. Fleming and R.W. Rishel (1975). *Deterministic and Stochastic Optimal Control.* Springer-Verlag, New York.

25. D. Gatarek (1988a). On value functions for impulsive control of piecewise-deterministic processes. *Proc. 6th IFIP Working Conf. on Stoch. Systems & Opt. 12-18.9.88.* Warsaw.

26. D. Gatarek (1988b). Impulsive control of piecewise-deterministic processes. *Stochastics*, to appear.

27. U.S. Gugerli (1986). Optimal stopping of a piecewise-deterministic Markov process. *Stochastics* 19, 221-236.

28. R. Gonzalez and E. Rofman (1978). An algorithm to obtain the maximum solutions of the Hamilton-Jacobi equation, *Optimization Techniques,* J. Store, ed. Springer, Berlin.

29. R. Gonzalez (1980). Sur la resolution de l'equation de Hamilton-Jacobi du contrôle deterministe. Thèse de 3° Cycle, Univ. Paris.

30. A. Haar (1928). Über eindeutigkeit und analytizität der Lösungen partieller Differenzial-gleichungen. *Atti del Congr. Intern. dei Mat., Bologna.* **3**, 5-10.

31. A. Hordijk and F.A. van der Duyn Schouten (1984). Discretization and weak Convergence in Markov decision drift processes. *Math. Oper. Res.* **9**, 112-141.

32. J. Jacod (1979). *Calcul Stochastique et Problèmes de Martingles.* Lecture Notes in Mathematics **714**. Springer-Verlag, Berlin.

33. S.M. Lenhart (1989). Viscosity solutions associated with impulse control problems for piecewise-deterministic processes. *Internat. J. Math. & Math. Sci.* **12**, 145-158.

34. S.M. Lenhart and Y.C. Liao (1985). Integro-differential equations associated with optimal stopping of a piecewise deterministic process. *Stochastics* **15**, 183-208.

35. L.P. Lions (1982). *Generalized Solutions of Hamilton-Jacobi Equations.* Research Notes in Maths. **69**. Pitman, Boston.

36. D.C. Offin (1978). A Hamilton-Jacobi approach to the differential inclusion problem. M.Sc. thesis, Dept. of Maths., University of British Columbia.

37. H.L. Royden (1968). *Real Analysis.* Macmillan, New York.

38. E. Solel (1986). A dynamic approach to stochastic scheduling via stochastic control. Ph.D. Dissertation (TR-86-12), Dept. of Math., Stats. & C.S., Dalhousie University, Halifax, Canada.

39. H.M. Soner (1986). Optimal control with state space constraint I. *SIAM J.Control and Optimization* **24**, 552-561.

40. H.M. Soner (1986). Optimal control with state space constraint II. *SIAM J. Control and Optimization* **24**, 1110-1122.

41. D. Vermes (1985). Optimal control of piecewise deterministic Markov processes. *Stochastics* **14**, 165-208.

42. R.B. Vinter and R.M. Lewis (1978). A necessary and sufficient control of condition for optimality of dynamic programming type, making no a priori assumption on the controls. *SIAM J.Control and Optimization* **16**, 571-583.

43. J. Warga (1972). *Optimal Control of Differential and Functional Equations.* Academic Press, New York.

44. L.C. Young (1969). *Lectures on the Caculus of Variations and Optimal Control Theory.* W.B. Saunders, Philadelphia.

45. A.A. Yushkevich (1983). Continuous-time Markov decision process with intervention. *Stochastics* **9**, 235-274.

46. A.A. Yushkevich (1987). Bellman inequalities in Markov decision deterministic drift processes. *Stochastics* **23**, 25-77.