QUALITATIVE ANALYSIS OF COSMOLOGICAL MODELS

By

Robert Joseph van den Hoogen

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
AT
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
SEPTEMBER, 1995

ISBN 0-612-08718-2

Name Robert Joseph van der Hoogen

Dissertation Abstracts International is arranged by broad, general subject categories. Please select the one subject which most nearly describes the content of your dissertation. Enter the corresponding four-digit code in the spaces provided.

Mathematics — SUBJECT TERM

0405 — SUBJECT CODE

U·M·I

Subject Categories

THE HUMANITIES AND SOCIAL SCIENCES

COMMUNICATIONS AND THE ARTS
- Architecture 0729
- Art History 0377
- Cinema 0900
- Dance 0378
- Fine Arts 0357
- Information Science 0723
- Journalism 0391
- Library Science 0399
- Mass Communications 0708
- Music 0413
- Speech Communication 0459
- Theater 0465

EDUCATION
- General 0515
- Administration 0514
- Adult and Continuing 0516
- Agricultural 0517
- Art 0273
- Bilingual and Multicultural 0282
- Business 0688
- Community College 0275
- Curriculum and Instruction 0727
- Early Childhood 0518
- Elementary 0524
- Finance 0277
- Guidance and Counseling 0519
- Health 0680
- Higher 0745
- History of 0520
- Home Economics 0278
- Industrial 0521
- Language and Literature 0279
- Mathematics 0280
- Music 0522
- Philosophy of 0998
- Physical 0523
- Psychology 0525
- Reading 0535
- Religious 0527
- Sciences 0714
- Secondary 0533
- Social Sciences 0534
- Sociology of 0340
- Special 0529
- Teacher Training 0530
- Technology 0710
- Tests and Measurements 0288
- Vocational 0747

LANGUAGE, LITERATURE AND LINGUISTICS
Language
- General 0679
- Ancient 0289
- Linguistics 0290
- Modern 0291

Literature
- General 0401
- Classical 0294
- Comparative 0295
- Medieval 0297
- Modern 0298
- African 0316
- American 0591
- Asian 0305
- Canadian (English) 0352
- Canadian (French) 0355
- English 0593
- Germanic 0311
- Latin American 0312
- Middle Eastern 0315
- Romance 0313
- Slavic and East European 0314

PHILOSOPHY, RELIGION AND THEOLOGY
- Philosophy 0422

Religion
- General 0318
- Biblical Studies 0321
- Clergy 0319
- History of 0320
- Philosophy of 0322
- Theology 0469

SOCIAL SCIENCES
- American Studies 0323

Anthropology
- Archaeology 0324
- Cultural 0326
- Physical 0327

Business Administration
- General 0310
- Accounting 0272
- Banking 0770
- Management 0454
- Marketing 0338
- Canadian Studies 0385

Economics
- General 0501
- Agricultural 0503
- Commerce-Business 0505
- Finance 0508
- History 0509
- Labor 0510
- Theory 0511
- Folklore 0358
- Geography 0366
- Gerontology 0351

History
- General 0578
- Ancient 0579
- Medieval 0581
- Modern 0582
- Black 0328
- African 0331
- Asia, Australia and Oceania 0332
- Canadian 0334
- European 0335
- Latin American 0336
- Middle Eastern 0333
- United States 0337
- History of Science 0585
- Law 0398

Political Science
- General 0615
- International Law and Relations 0616
- Public Administration 0617
- Recreation 0814
- Social Work 0452

Sociology
- General 0626
- Criminology and Penology 0627
- Demography 0938
- Ethnic and Racial Studies 0631
- Individual and Family Studies 0628
- Industrial and Labor Relations 0629
- Public and Social Welfare 0630
- Social Structure and Development 0700
- Theory and Methods 0344
- Transportation 0709
- Urban and Regional Planning 0999
- Women's Studies 0453

THE SCIENCES AND ENGINEERING

BIOLOGICAL SCIENCES
Agriculture
- General 0473
- Agronomy 0285
- Animal Culture and Nutrition 0475
- Animal Pathology 0476
- Food Science and Technology 0359
- Forestry and Wildlife 0478
- Plant Culture 0479
- Plant Pathology 0480
- Plant Physiology 0817
- Range Management 0777
- Wood Technology 0746

Biology
- General 0306
- Anatomy 0287
- Biostatistics 0308
- Botany 0309
- Cell 0379
- Ecology 0329
- Entomology 0353
- Genetics 0369
- Limnology 0793
- Microbiology 0410
- Molecular 0307
- Neuroscience 0317
- Oceanography 0416
- Physiology 0433
- Radiation 0821
- Veterinary Science 0778
- Zoology 0472

Biophysics
- General 0786
- Medical 0760

EARTH SCIENCES
- Biogeochemistry 0425
- Geochemistry 0996
- Geodesy 0370
- Geology 0372
- Geophysics 0373
- Hydrology 0388
- Mineralogy 0411
- Paleobotany 0345
- Paleoecology 0426
- Paleontology 0418
- Paleozoology 0985
- Palynology 0427
- Physical Geography 0368
- Physical Oceanography 0415

HEALTH AND ENVIRONMENTAL SCIENCES
- Environmental Sciences 0768

Health Sciences
- General 0566
- Audiology 0300
- Chemotherapy 0992
- Dentistry 0567
- Education 0350
- Hospital Management 0769
- Human Development 0758
- Immunology 0982
- Medicine and Surgery 0564
- Mental Health 0347
- Nursing 0569
- Nutrition 0570
- Obstetrics and Gynecology 0380
- Occupational Health and Therapy 0354
- Ophthalmology 0381
- Pathology 0571
- Pharmacology 0419
- Pharmacy 0572
- Physical Therapy 0382
- Public Health 0573
- Radiology 0574
- Recreation 0575
- Speech Pathology 0460
- Toxicology 0383
- Home Economics 0386

PHYSICAL SCIENCES
Pure Sciences
Chemistry
- General 0485
- Agricultural 0749
- Analytical 0486
- Biochemistry 0487
- Inorganic 0488
- Nuclear 0738
- Organic 0490
- Pharmaceutical 0491
- Physical 0494
- Polymer 0495
- Radiation 0754
- Mathematics 0405

Physics
- General 0605
- Acoustics 0986
- Astronomy and Astrophysics 0606
- Atmospheric Science 0608
- Atomic 0748
- Electronics and Electricity 0607
- Elementary Particles and High Energy 0798
- Fluid and Plasma 0759
- Molecular 0609
- Nuclear 0610
- Optics 0752
- Radiation 0756
- Solid State 0611
- Statistics 0463

Applied Sciences
- Applied Mechanics 0346
- Computer Science 0984

Engineering
- General 0537
- Aerospace 0538
- Agricultural 0539
- Automotive 0540
- Biomedical 0541
- Chemical 0542
- Civil 0543
- Electronics and Electrical 0544
- Heat and Thermodynamics 0348
- Hydraulic 0545
- Industrial 0546
- Marine 0547
- Materials Science 0794
- Mechanical 0548
- Metallurgy 0743
- Mining 0551
- Nuclear 0552
- Packaging 0549
- Petroleum 0765
- Sanitary and Municipal 0554
- System Science 0790
- Geotechnology 0428
- Operations Research 0796
- Plastics Technology 0795
- Textile Technology 0994

PSYCHOLOGY
- General 0621
- Behavioral 0384
- Clinical 0622
- Developmental 0620
- Experimental 0623
- Industrial 0624
- Personality 0625
- Physiological 0989
- Psychobiology 0349
- Psychometrics 0632
- Social 0451

Contents

List of Tables

List of Figures

For Suzanne

Acknowledgements

There are so many people that ought to be acknowledged. I must first thank Dr. Alan Coley for his guidance and patience. Without his constant redirection, I surely would not have finished this work. He provided support and advice, for this I must thank him. I also must thank the readers, Dr. Ken Dunn and Dr. Shigui Ruan, and Dr. David Hobill, whose suggestions and criticisms have improved the readability of this thesis.

I also thank my parents Herman and Antonia van den Hoogen for providing much needed support during the years I was in university. I also must thank Mémé Jeanette for all the food, fish, and other things, but most importantly for the *badeleines* (spelling). I also thank Ronnie and Lorraine Boudreau for their help. I cannot forget about all the love and affection from Joshua and Josée. It is they who made me forget about work when I got home.

There are others that deserve some recognition because without them I would not be where I am today. Before I even started thinking about writing theses and completing degrees, when I was in High School, there was a teacher who showed me the magic of mathematics and gave me the interest to pursue an education in the sciences. Her name was Sister Agnes Campbell and it is to her that I attribute my love for the subject. Thank you.

I also must thank Cor and Mary van den Hoogen for the many Sunday dinners and for helping me cope with city life. Knowing that they were only a phone call away gave me peace of mind when I first came to Halifax.

I must acknowledge the efforts of Dr. Walt Finden and Dr. Mo Tak Kiang of Saint Mary's University, who suggested that I apply for a Natural Sciences and Engineering Research Council Postgraduate scholarship, and that I apply to take my Master's degree. Without their faith in me, I would not be where I am today.

I also must thank Gretchen, Paula, Maria, and Ursula for all the help they have given me over the years. I also thank them for allowing me to use the telephone whenever I needed to. I would also like to acknowledge my office-mates; we have shared the same office for five years, I cannot think of a better place in the entire department in which I could work/rest/eat. I thank Gary and Hossein for putting up with me for all these years.

There is one more person I must thank. There is actually no way to repay her the time she has spent waiting for me to finish my degree. It is also to her that I owe my degree, without her support and persistence I would not be finished now. She has made my life/career what it is now. I think John Lennon sings it best

> Woman, I can hardly express
> my mixed emotions and my thoughtlessness.
> Afterall, I am forever in your debt,
> and woman, I will try to express
> my inner feelings and my thankfulness
> for showing me the meaning of success.
>
> Woman, I know you understand
> the little child inside the man.
> Please remember my life is in your hands,
> and woman, hold me close to your heart
> however distant don't keep us apart,
> afterall it was written in the stars.

or more appropriately maybe, in the palm of your hand.

Thank-you Suzanne for everything. I love you.

Robert

Preface

Much of the work in this thesis has appeared, or is about to appear elsewhere in scientific journals or workshop proceedings. Parts of Chapter 2 have appeared in the workshop proceedings *Deterministic Chaos in General Relativity* [1]. Chapter 3 has been published in the Journal of Mathematical Physics [2]. Chapter 4 represents work that is about to be published in Classical and Quantum Gravity [3, 4]. Chapter 5 is a different version of similar work done with Alan Coley and Roy Maartens which will be submitted for publication in the near future. Sections of Chapter 6, written in collaboration with Jesus Ibáñez and Alan Coley, has appeared in Physical Review D [5]. Chapter 7 can be divided into two sections. The first section investigates the properties of Soft Inflation and has appeared in Physical Review D [6]. The second section discusses the oscillatory behaviour found in various cosmological models and is about to appear in the International Journal of Theoretical Physics [7].

Notation

units	$8\pi G = 1$; $c = 1$ (unless otherwise indicated)
latin indices	$a, b, c, \ldots$ run from 0 to 3.
greek indices	$\alpha, \beta, \gamma, \ldots$ run from 1 to 3.
metric signature	$-+++$
$T_{ab,c}$	a comma refers to ordinary partial differentiation.
$T_{ab;c}$	a semicolon refers to covariant differentiation (with respect to the metric tensor g_{ab}).
u^a	represents the fluid velocity.
h_{ab}	$= g_{ab} + u_a u_b$, — the projection tensor (it projects tensors into the three-dimensional space orthogonal to u^a).
θ	volume expansion; $H = \theta/3$, where H is the Hubble parameter.
σ_{ab}	shear tensor
ρ	energy density
p	isotropic pressure
Π	bulk viscous pressure
π_{ab}	shear viscous stress
q_a	heat conduction vector
T	temperature
s	specific entropy or entropy per baryon
n	baryon number density
ζ	bulk viscosity coefficient

η	shear viscosity coefficient
κ	thermal conductivity coefficient
EC(s)	energy condition(s)
WEC	weak energy condition
DEC	dominant energy condition
SEC	strong energy condition
ODE	ordinary differential equation
FRW	Friedmann-Robertson-Walker, — Spatially homogeneous and isotropic cosmological models that have a 6-dimensional, G_6, group of isometries acting on spacelike hypersurfaces.
OSH	Orthogonal Spatially Homogeneous. An OSH cosmological model is a spatially homogeneous and anisotropic model that has a three-dimensional, G_3, group of isometries acting simply transitive on the three dimensional hypersurfaces orthogonal to the fluid flow.
H_n	An n-dimensional Lie group of similarities generated by a homothetic vector and $n-1$ Killing vectors. Each H_n contains the subgroup G_{n-1}.
G_n	A n-dimensional group of isometries generated by n Killing vectors.

Abstract

It is proven that if one desires self-similar asymptotic limit points in spatially homogeneous cosmological models, then dimensionless equations of state are necessary. The converse is also true; it is proven that dimensionless equations of state imply self-similar asymptotic limit points. These results are subsequently used in the investigation of various cosmological models.

Dimensionless equations of state and a set of dimensionless, expansion-normalized variables are used to reduce the dimension of the system describing the evolution of spatially homogeneous imperfect fluid cosmological models. Since the resulting system is an autonomous system of ordinary differential equations, dynamical systems techniques can be used to determine its qualitative behaviour.

In particular, viscous fluid Bianchi type V models with heat conduction are analyzed and compared using both the Eckart and the 'Truncated' Israel-Stewart theories of irreversible thermodynamics, and Friedmann-Robertson-Walker models with bulk viscosity are studied and compared using both the 'Truncated' and the 'Full' Israel-Stewart theories of irreversible thermodynamics. Furthermore, the dynamical system describing the evolution of the viscous fluid isotropic curvature models is given. The qualitative behaviour of the first order Eckart theory can be very different from the qualitative behaviours of the second order Israel-Stewart theories. It is found that in the Eckart theory the anisotropic Bianchi type I and V models always isotropize, however, the same is not true in the second order Israel-Stewart theories where it is shown that they need not isotropize. It is also found that bulk viscous inflation is

possible in all of these theories. Finally, it is demonstrated that there can be more entropy produced in the Truncated Israel-Stewart theory than in the Eckart theory.

The only scalar field models that allow self-similar asymptotic limit points are those in which the potential is either of exponential form or zero. Using the property that the dynamical system describing the spatially homogeneous models can be rewritten in terms of dimensionless variables, a class of spatially homogeneous models are investigated. A general result pertaining to the isotropization and inflation of Bianchi models with an exponential potential is obtained. It is found that the only Bianchi models that can possibly inflate and isotropize when $k^2 > 2$ are those of Bianchi types I, V, VII or IX.

One of the criteria that breaks the self-similarity condition is the existence of a scalar field with a non-exponential potential. In isotropic and spatially homogeneous models with a quadratic potential, it is shown that oscillatory behaviour is possible. A survey of various models exhibiting this oscillatory behaviour is given and examples demonstrating this oscillatory behaviour, are found. Also, a qualitative analysis of a cosmological model arising from a soft inflationary scenario is done and the asymptotic behaviour is determined.

Chapter 1

Introduction

1.1 Relativistic Cosmology

In cosmology, one is primarily interested in studying mathematical models of the universe that agree with astronomical observations. Assuming that the universe is electrically neutral, the dominant force on large scales is that of gravity. At present, there are many theories of gravity to choose from, however, Einstein's General Relativity has proven itself to be an excellent approximation in describing the gravitational dynamics of the solar system. We will extrapolate this observation and assume that General Relativity describes the gravitational interaction on scales larger than that of the solar system, for instance, on scales of galaxy clusters.

In General Relativity, the force of gravity is represented by the curvature of a Lorentzian manifold representing our spacetime. The Einstein field equations which relate curvature and the matter content are,

$$G_{ab} = 8\pi T_{ab},$$

where G_{ab} is the Einstein tensor calculated from the metric tensor, and T_{ab} is the energy momentum tensor which represents the energy and matter contributions. By the contracted Bianchi identities, the energy-momentum tensor satisfies

$$T^{ab}{}_{;b} = 0,$$

which represents the energy and momentum conservation equations.

Recent measurements of the cosmic microwave background temperature indicate that the universe has a very high degree of isotropy [8], which in turn suggests that the universe is also largely spatially homogeneous. Assuming these two symmetries are exact, (isotropy and spatial homogeneity) is called the Cosmological Principle.

On very large scales we can consider galaxy clusters as particles of a gas that fill the universe. We are then able to model this gas as a fluid. Astronomical observations also suggest that the distribution of galaxy clusters is rather isotropic about us and that there is an overall expansion of these clusters [9]. Therefore, (assuming we do not occupy a privileged position in spacetime), there is an average velocity vector at each spacetime point. This velocity vector can be thought of as representing the average velocity of the fluid particles [9]. By far, the most common assumption concerning the nature of the fluid approximation is that it have negligible viscosity and heat conduction, in which case it is called a perfect fluid.

1.1.1 The Standard Model

The Einstein field equations are very difficult to solve, and one usually imposes some symmetries on the spacetime so as to make some progress in solving them. Given that the matter can be described as a perfect fluid, the simplest cosmological models satisfying the Cosmological Principle are obtained when one assumes that there is a six-dimensional G_6 group of isometries acting multiply transitively on spacelike orbits. This results in the spatially homogeneous and isotropic Friedmann-Robertson-Walker (FRW) model, also known as the Standard model. The Standard model has had some success in describing the present day universe. Firstly, the Standard model agrees with the Hubble expansion law. Secondly, since the FRW spacetime is isotropic, the

Standard model agrees with the isotropy observations of the universe. Thirdly, and perhaps the greatest success of the Standard model was the prediction and subsequent discovery of the cosmic microwave background radiation. Fourthly, the Standard model predicts primordial abundances of light elements in accord with observations.

Nonetheless, the Standard model does have its weaknesses. For instance, there is no acceptable theory of galaxy formation. Also, on scales larger than galaxy clusters, we observe the universe to be clumpy with large voids [10] — the Standard model does not allow the formation of such structures. Therefore, it becomes interesting to investigate cosmological models with a richer structure both geometrically and with respect to the matter content.

1.1.2 Spatially Homogeneous Models

It is unrealistic to expect to solve the Einstein field equations in full generality. (They are a coupled non-linear system of partial differential equations.) The Standard model is spatially homogeneous and isotropic, which is, perhaps, too restrictive to describe the true behaviour of the universe. By dropping the isotropy condition of the Standard model, the class of spatially homogeneous but anisotropic perfect fluid models result. A very important subclass of the general spatially homogeneous models are the orthogonal spatially homogeneous (OSH) models. (OSH models are those where the average fluid 4-velocity is orthogonal to the surfaces of homogeneity.) It was shown by Ellis and MacCallum [11] that the Einstein field equations for all OSH models with appropriate equations of state can be written as an autonomous system of ordinary differential equations. The perfect fluid OSH cosmological models have been extensively studied [12, 13, 14, 15, 16, 17, 18, 19] [see also [20, 21] and references therein] using various approaches and techniques. One of the most successful approaches is the use of dynamical systems techniques to determine the qualitative behaviour of the perfect fluid OSH cosmological models.

1.2 Dynamical Systems and Spatially Homogeneous Cosmologies

There have been many groups using dynamical systems theory to study OSH cosmological models. The most elegant, and perhaps, most complete qualitative analysis of the perfect fluid OSH models has been done by Wainwright and Hsu (Bianchi type A) [18] and by Hewitt and Wainwright (Bianchi type B) [19]. In these papers, the Ellis-MacCallum orthonormal tetrad techniques [11] were used to write down the Einstein field equations. In this way, the commutation functions associated with this tetrad basis become the physical variables. The system of ordinary differential equations with respect to these basic physical variables admits a scaling symmetry that allows one to introduce dimensionless variables. The way one chooses the dimensionless variables is not unique; however, since the expansion plays a dominant role in these models, it is natural to choose dimensionless, expansion-normalized variables [18]. A property of the dimensionless variables, is that the equilibrium points of the reduced dynamical system represent self-similar cosmological models, that is, in addition to the three Killing vectors, there exists a homothetic vector (see equation (2.1) on page 13 for a definition of homothetic vector). In addition, the reduced dynamical system is analytic, (more over, it is polynomial). One further advantage of using dimensionless variables in the perfect fluid OSH models is that the phase space of the reduced dynamical system became a compact set in the Bianchi type B models [19] and becomes a closed but unbounded set in the Bianchi type A models [18].

The resulting system of ordinary differential equations permits one to use dynamical systems techniques to determine the qualitative behaviour. (See Appendix A for a review of Dynamical Systems techniques.) One of the aims in this thesis is to extend the ideas used by Wainwright and his collaborators [18, 19] to viscous fluid spatially homogeneous models. Since the dynamical system in the imperfect fluid case becomes significantly more complicated than in the perfect fluid cases previously studied, only special subsets of the OSH viscous fluid models will be investigated here.

1.3 Viscous Fluid Cosmological Models

1.3.1 Motivation

Much of this section has been taken from the recent comprehensive review of viscous fluid cosmology by Ø. Grøn [22].

It was Misner [23, 24, 25] who pointed out that viscosity, and in particular neutrino viscosity, in the early universe may have played a part in the isotropization of the universe. Misner's idea [23, 24, 25] was that after the temperature decreased to about 10^{10} K, the neutrinos would decouple from the rest of the radiation. At this point, the neutrinos were neither collision-free nor collision-dominated and therefore viscosity would be the dominant process. However, Doroshkevich et al. [26], Stewart [27, 28] and Collins and Stewart [29] argued that Misner's suggestion, based on the relativistic fluid approximation, assumed that the initial anisotropy was already very small. Stewart [27, 28] and Collins and Stewart [29] have shown, using statistical mechanical techniques, that if the initial anisotropy is sufficiently large then the universe need not isotropize. The question of isotropization in spatially homogeneous models has also been addressed by Collins and Hawking [30]. They proved that only a set of measure zero of all initial conditions in these models can possibly lead to isotropization (provided the dominant energy condition and the positive pressure criterion are satisfied). However, Belinskii and Khalatnikov [31] have shown that the viscous fluid anisotropic Bianchi type I models (in particular) do isotropize to the future. In all of the above work the Eckart theory was used, and it is not clear whether dissipation can lead to isotropization in anisotropic models using the second order Israel-Stewart theories.

The present entropy per baryon in the universe is of the order of $10^8 - 10^9$. This is an extremely large number. Why is it so large today? Because the universe has been approximately Friedmann-Robertson-Walker from the time of recombination, there has been no significant growth in the entropy per baryon since this time. Therefore, any significant entropy growth must have occurred in the early universe. Weinberg [32] was the first to do some specific entropy growth calculations arising from dissipative effects. Using the Eckart theory, Weinberg [32] showed that bulk viscosity alone can in no way explain the enormous entropy per baryon. On the other hand, in work by Caderni and his coworkers [33, 34, 35], it was shown that shear viscosity can produce significant amounts of entropy. They found a class of models in which the entropy per baryon at the present time could be of the order of 10^9. Maartens [36] illustrated that during a period of bulk viscous inflation an enormous amount of entropy can be generated. Consequently, dissipation may offer a possible explanation for the currently observed high entropy per baryon.

Barrow and Matzner [37] pointed out that there are other possible dissipative mechanisms available in the early universe, including, for example, graviton collisions, quantum particle creation, and mini blackholes. Higher-dimensional superstring cosmological models give rise to bulk viscosity through the conversion of massive string modes to massless string modes [38]. Dissipative effects may also be important with respect to galaxy formation [29] and with respect to the possibility of bulk viscous inflation [22, 39, 40, 41, 42, 43]. In some instances dissipative effects have been known to change the nature of the initial singularity [44] and in other instances have been known to remove it altogether [45, 46].

Therefore, dissipative cosmological models are of interest and should be analyzed in order to discover what properties they may have. A number of important questions include; do dissipative models isotropize[1], do dissipative models sufficiently increase the amount of entropy in the universe, do they cause inflation[2], do they remove the initial singularity. These are all questions that will be addressed in the thesis.

[1] A cosmological model will be said to isotropize if (i) the shear tends to zero and, (ii) in case of imperfect fluid models, the anisotropic stress also tends to zero.

[2] Inflation is defined to occur when the generalized deceleration parameter, $q \equiv -\ddot{l}l/\dot{l}^2$, is negative, where l is the average length scale. $H \equiv \frac{1}{3}\theta = \dot{l}/l$ is the Hubble parameter.

1.3.2 Eckart Theory

Given that the universe can be modelled as a simple fluid, a relativistic theory of irreversible thermodynamics is needed to describe the dissipative effects of the fluid.

In one of the first proposed theories of irreversible thermodynamics [47], it was assumed that there was a linear relationship between the bulk viscous pressure[3] Π and the expansion θ, viscous pressure Π and the expansion θ, and a linear relationship between the heat conduction vector, q^a, and the gradient of the temperature summed with the acceleration $\dot{u}^a$, as well as a linear relationship between the anisotropic stress π_{ab} and the shear σ_{ab}; that is,

$$\begin{aligned} \Pi &= -\zeta\theta, \\ q^a &= -\kappa h^{ab}(T_{,b} + T\dot{u}_b), \\ \pi_{ab} &= -2\eta\sigma_{ab}, \end{aligned} \tag{1.1}$$

where ζ denotes the bulk viscosity coefficient, κ denotes the thermal conductivity, and η denotes the shear viscosity coefficient. Equations (1.1) describe Eckart's theory of irreversible thermodynamics [47]. Eckart's theory is a first order approximation of the viscous pressure Π, heat conduction vector q^a, and the anisotropic stress π_{ab} and is assumed to be valid near equilibrium [47]. With equations of state of the form

$$\zeta = \zeta_o \rho^m, \qquad \eta = \eta_o \rho^n, \tag{1.2}$$

Belinskii and Khalatnikov [31] found that the Bianchi type I models isotropized to the future. The addition of viscosity allowed for a variety of different qualitative behaviours (different from those of the corresponding perfect fluid models). However, Eckart's theory of irreversible thermodynamics [47] suffers from the property that signals in the fluid can propagate faster than the speed of light (i.e., non-causality), and also that the equilibrium states in this theory are unstable (see Hiscock and

[3]Definitions of all terms can be found in the Glossary.

Salmonson [48] and references therein). Therefore, a more complete theory of irreversible thermodynamics is necessary for fully analyzing cosmological models with viscosity.

1.3.3 Israel-Stewart Theory

Among the first to extend the Eckart theory of irreversible thermodynamics were Israel [49] and Israel and Stewart [50, 51]. For a simple fluid, and for small deviations from equilibrium, one may write the entropy flux as

$$S^a = s(\rho, n)N^a + \frac{q^a}{T} - Q^a(\Pi, q^a, \pi^{ab}),$$

where s is the specific entropy, ρ is the energy density, $N^a = nu^a$ is the number density flux, T is the temperature and Q^a is a general four-vector representing the deviations from equilibrium upto and including second order terms. In the first order Eckart theory, the general four vector Q^a is zero. Kinetic theory arguments, however, suggest the contrary, that is, Q^a does not vanish in general. To second order in Π, q^a, and π_{ab}, the most general expression for Q^a is of the form

$$Q^a = -\alpha_0 \Pi q^a - \alpha_1 \pi^{ab} q_b + \frac{1}{2} u^a \left(\beta_0 \Pi^2 + \beta_1 q^b q_b + \beta_2 \pi^{ab} \pi_{ab}\right).$$

Assuming an irrotational fluid flow, the simplest way to satisfy the H-theorem (positive entropy production) is to assume the following set of linear (in Π, q^a, π^{ab}) phenomenological laws for Π, q^a, and π^{ab} [48, 51, 52]:

$$\begin{aligned}
\Pi &= -\zeta \left\{ u^a_{;a} + \beta_0 \dot{\Pi} - \alpha_0 q^a{}_{;a} + \epsilon \left[\frac{1}{2} T \Pi \left(\frac{\beta_0 u^a}{T} \right)_{;a} - \gamma_0 q^a T \left(\frac{\alpha_0}{T} \right)_{;a} \right] \right\}, \\
q_a &= -\kappa T h_a{}^b \left\{ \frac{T_{;b}}{T} + \dot{u}_b + \beta_1 \dot{q}_b - \alpha_0 \Pi_{;b} - \alpha_1 \pi_b{}^c{}_{;c} + \epsilon \left[\frac{1}{2} T q_b \left(\frac{\beta_1 u^c}{T} \right)_{;c} \right. \right. \\
&\qquad \left. \left. -(1-\gamma_0) \Pi T \left(\frac{\alpha_0}{T} \right)_{;b} - (1-\gamma_1) \pi_b{}^c T \left(\frac{\alpha_1}{T} \right)_{;c} \right] \right\}, \qquad (1.3) \\
\pi_{ab} &= -2\eta \left\langle u_{a;b} + \beta_2 \dot{\pi}_{ab} - \alpha_1 q_{b;a} + \epsilon \left[\frac{1}{2} T \pi_{ab} \left(\frac{\beta_2 u^c}{T} \right)_{;c} - \gamma_1 q_b T \left(\frac{\alpha_1}{T} \right)_{;a} \right] \right\rangle,
\end{aligned}$$

where the angle brackets denote the purely spatial trace-free part of the enclosed tensor; that is,

$$\langle A_{ab}\rangle \equiv \frac{1}{2}h_a{}^c h_b{}^d(A_{cd}+A_{dc}-\frac{2}{3}h_{cd}h^{ef}A_{ef})$$

The Israel-Stewart theory has three transport coefficients ζ, κ, and η; three parameters related to the relaxation times, β_0, β_1, and β_2; and two coupling coefficients α_0 and α_1, and two completely arbitrary parameters γ_0 and γ_1. The parameter ϵ is a discrete parameter that takes on values of 1 or 0, thereby effectively switching off certain components of equations (1.3). The variable, T, represents the temperature. We shall refer to equations (1.3) with $\epsilon = 1$ as the full Israel-Stewart equations. Israel and Stewart [49, 50, 51] originally assumed that the divergences and spatial gradients were sufficiently small so that their products with first order quantities were negligible. This class of theories is effectively obtained from the above equations (1.3) by setting ($\epsilon = 0$) and is appropriately named the truncated Israel-Stewart theory. These equations, (1.3), reduce to the Eckart equations (1.1) used in [2, 53, 54, 55] when $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = \beta_2 = 0$.

Belinskii et al. [44] were the first to study cosmological models satisfying the truncated ($\epsilon = 0$) Israel-Stewart theory of irreversible thermodynamics. Using qualitative analysis, Bianchi type I models were investigated. They assumed equations of state of the form [44]

$$\zeta = \zeta_0\rho^m, \quad \eta = \eta_0\rho^n, \quad \beta_0 = \rho^{-1}, \quad \text{and} \quad \beta_2 = \rho^{-1}, \tag{1.4}$$

where m and n are constants and ζ_o and η_o are parameters. The isotropizing effect found in the Eckart models no longer necessarily occurred in the truncated Israel-Stewart models [44]. It was also found that the cosmological singularity still existed but was of a new type, namely one with an accumulated "visco-elastic" energy [44]. Similar to the work done by Belinskii et al. [44], Pavón et al. [56] and Chimento and Jakubi [57] studied the flat Friedmann-Robertson-Walker models. They assumed the same equations of state as Belinskii et al. [44], namely (1.4), but studied the models using slightly different techniques. Chimento and Jakubi [57] also found exact solutions in the exceptional case $m = 1/2$. They found that the future qualitative behaviour of the model was independent of the value of m; however, to the past, "bouncing solutions" and deflationary evolutions are possible [57].

Hiscock and Salmonson [48] analyzed a viscous fluid cosmological model using the full ($\epsilon = 1$) Israel-Stewart theory. Hiscock and Salmonson used equations of state arising from the assumption that the fluid could be modelled as a Boltzmann gas. They concluded that when the Eckart equations, (1.1), or the truncated ($\epsilon = 0$) Israel-Stewart equations, (1.3), were used, inflation could occur, but in the full ($\epsilon = 1$) Israel-Stewart theory, inflation was no longer present. This result led them to conclude that "inflation is a spurious effect produced from using a truncated theory" [48]. However, Zakari and Jou [52] also employed the full ($\epsilon = 1$) Israel-Stewart theory of irreversible thermodynamics, but assumed equations of state of the form (1.4) and found that inflation was present in all three theories (Eckart, truncated ($\epsilon = 0$) Israel-Stewart, full ($\epsilon = 1$) Israel-Stewart). Therefore, it appears that the equations of state chosen determine whether the model will experience bulk-viscous inflation. Romano and Pavón [58] also analyzed Bianchi III models using both the truncated ($\epsilon = 0$) Israel-Stewart theory and the full ($\epsilon = 1$) Israel-Stewart theory. They analyzed the isotropic singular points and concluded that the qualitative behaviour of the models in the two different theories was similar in that the anisotropy of the models died away.

1.4 Remarks

1.4.1 Inflationary Cosmologies

In Chapters 6 and 7 various scalar field cosmological models are studied. In these chapters questions concerning isotropization and inflation are answered. See Chapter 6 and Chapter 7 for a general introduction into scalar field cosmological models.

1.4.2 Divisions

In all, there are four primary parts to the thesis. The first part comprises the Introduction, Chapter 1, and the chapter on Self-Similarity, Chapter 2. The second part (and perhaps the primary theme in the thesis) is the work done on the viscous fluid cosmological models found in Chapters 3–5. The third part is work done in Chapter 6 on the spatially homogeneous models with an exponential potential. The final part consists of the analysis of the inflationary theories undertaken in Chapter 7. The final chapter is a summary of the principal results together with some concluding remarks.

Chapter 2

Self-Similar Asymptotic Solutions of the Einstein Field Equations

2.1 Introduction

Wainwright and Hsu [18] have shown for the orthogonal Bianchi type A perfect fluid cosmological models that the asymptotic limit points are represented by self-similar models. Similarly, Hewitt and Wainwright [19] have shown for the orthogonal Bianchi type B perfect fluid cosmological models that the asymptotic limit points are also represented by self-similar models. Hewitt and Wainwright [59] have also proved that the 'dynamical equilibrium states' are self-similar for the orthogonally transitive G_2 cosmologies. How far can these observations be extended? We propose to generalize these observations to general orthogonal Bianchi models containing imperfect fluid sources.

In section 2.2 we shall define self-similarity. In section 2.3 the Einstein field equations will be set up as a dynamical system using the orthonormal tetrad formalism of Ellis and MacCallum [60]. Provided appropriate equations of state are given, the dynamical system will be shown to admit a symmetry. This symmetry defines new variables under which one of the equations in the system will decouple. It will be shown that the equilibrium points of the reduced system represent self-similar space-times provided $\rho + 3\bar{p} \geq 0$. Conversely, it will also be shown that if self-similarity is assumed, the field equations will then imply these same equations of state for the isotropic pressure $\bar{p}$ and anisotropic stress $\pi_{\alpha\beta}$. In section 2.4 we will include dissipative effects arising from theories of irreversible thermodynamics (for example the Israel-Stewart theory of Irreversible Thermodynamics) and will determine the equations of state required. In section 2.5 the analysis will be further generalized to include a scalar field. In section 2.6 a list of conditions breaking self-similarity will be given and examples illustrating these conditions will be shown. In section 2.7 we shall analyze a class of scalar tensor theories. In section 2.8 we will conclude with a discussion. The notation will be consistent with that used by MacCallum in the Cargése Lectures [20]; in particular, lower case Latin indices range from 0 to 4 and lower case Greek indices range from 1 to 3.

2.2 Self-Similarity

Let $(M, \mathbf{g})$ be a space-time manifold with metric $\mathbf{g}$. Let $L_{\vec{\mathbf{X}}}$ denote the Lie derivative in the direction $\vec{\mathbf{X}}$ and let c be a constant. A vector $\vec{\mathbf{X}}$ that satisfies

$$L_{\vec{\mathbf{X}}}\mathbf{g} = 2c\,\mathbf{g}, \tag{2.1}$$

generates a one parameter family of similarities. If $c = 0$, then $\vec{\mathbf{X}}$ is a Killing vector and if $c \neq 0$, then $\vec{\mathbf{X}}$ is a homothetic vector. The collection of all similarities of a space-time $(M, \mathbf{g})$ forms a Lie group, called the similarity group. A space-time is defined to be self-similar if it admits a homothetic vector, and transitively self-similar if it admits an H_4 [61]; that is, in addition to the homothetic vector, there exist three Killing vectors that act transitively on 3-dimensional hypersurfaces. In order to be consistent with previous work [61], we are using the term self-similarity to characterize the properties of the geometry, rather than characterize the properties of the matter [62]. It is somewhat conventional, however, to define self-similarity with respect to symmetries on the matter [62]. For perfect fluid spacetimes the two definitions are compatible, however, the same cannot be said about imperfect fluid spacetimes. This observation is part of the symmetry inheritance problem. (See Coley and Tupper [62] for more details.)

Hsu and Wainwright [61] state and prove a theorem in which the conditions are given for a simply transitive similarity group H_4 to exist.

Theorem 1. *A spacetime* $(M, \mathbf{g})$ *admits a simply transitive similarity group* H_4 *if and only if there exists an orthonormal frame* $\{\vec{\mathbf{e}}_a\}$ *and a scalar field* t *such that*

$$\gamma^c{}_{ab} = F^c{}_{ab} t^{-1} \tag{2.2}$$

$$\vec{\mathbf{e}}_a(t) = e_a{}^i \frac{\partial}{\partial x^i}(t) = n_a \tag{2.3}$$

where $[\vec{\mathbf{e}}_a, \vec{\mathbf{e}}_b] = \gamma^c{}_{ab}\vec{\mathbf{e}}_c$, *and where* $F^c{}_{ab}$ *and* n_a *are constants.*

Proof. See Hsu and Wainwright [61].

2.3 The Field Equations

Consider a 4-dimensional space-time manifold $(M, \mathbf{g})$. Suppose we also have a timelike congruence of curves through every point p of the manifold M. This congruence of curves defines a tangent vector $\vec{\mathbf{u}}$ at each space-time point p. The covariant derivative of $\vec{\mathbf{u}}$ can be written as

$$u_{a;b} = \frac{\theta}{3} h_{ab} + \sigma_{ab} + \omega_{ab} - \dot{u}_a u_b, \tag{2.4}$$

where $\sigma_{ab} = \sigma_{(ab)}$, $\sigma_{ab}u^b = 0$, $\sigma_a{}^a = 0$, $\omega_{ab}u^b = 0$, $\omega_{ab} = \omega_{[ab]}$, and $\dot{u}_a = u_{a;b}u^b$. If we interpret $\vec{\mathbf{u}}$ as the velocity field of a fluid, then θ is the expansion, σ_{ab} is the shear, ω_{ab} is the vorticity, $\dot{u}_a$ is the acceleration, of neighboring particles in the fluid and $h_{ab} = u_a u_b + g_{ab}$ is the projection tensor [20].

The energy-momentum tensor, T_{ab}, can be decomposed [20] with respect to u_a as follows,

$$T_{ab} = \rho u_a u_b + \bar{p} h_{ab} + q_a u_b + u_a q_b + \pi_{ab}, \tag{2.5}$$

where $q_a u^a = 0$, $\pi_{ab} u^b = 0$, $\pi_a{}^a = 0$ and $\pi_{ab} = \pi_{(ab)}$. In general the quantities ρ, q_a, $\bar{p}$, and π_{ab} have no physical meaning, however, if u^a is the velocity vector of the fluid, then ρ can be interpreted as the energy density, q_a as the energy flux, $\bar{p}$ as the isotropic pressure, and π_{ab} as the anisotropic stress as measured by an observer moving with 4-velocity u^a.

Assuming that the fluid is moving hypersurface orthogonal, $\vec{\mathbf{u}} = \vec{\mathbf{n}}$ ($\vec{\mathbf{n}}$ is the unit normal to the surfaces of homogeneity), the acceleration and vorticity are zero; that is, $\dot{u}_a = \omega_{ab} = 0$. If we parameterize the surfaces by distance along the geodesics normal to the surfaces, then we can consider the surfaces as surfaces of constant time where $n_a = -t_{,a}$.

Using $\vec{\mathbf{u}}$ and the three Killing vectors generating the G_3 group of motions, the Einstein field equations for the orthogonal spatially homogeneous models may be written in terms of an orthonormal tetrad $\{\vec{\mathbf{e}}_a\}$. If $\vec{\mathbf{e}}_0 = \vec{\mathbf{u}}$, then the quantities $\gamma^c{}_{ab}$ defined by the commutator relation $[\vec{\mathbf{e}}_a, \vec{\mathbf{e}}_b] = \gamma^c{}_{ab}\vec{\mathbf{e}}_c$ are spatially independent and are functions of t only. With respect to this basis, the non-zero components of σ_{ab}, q_a, and π_{ab} are respectively $\sigma_{\alpha\beta}$, q_α, and $\pi_{\alpha\beta}$. The quantities $\gamma^c{}_{ab}$ may be written in terms of θ, $\sigma_{\alpha\beta}$, and new variables $n_{\alpha\beta}$ and a_β as follows [20]:

$$\begin{aligned}
\gamma^0_{0\alpha} &= \gamma^0_{\alpha\beta} = 0, \\
\gamma^\alpha_{0\beta} &= -\frac{6}{3}h_{\alpha\beta} - \sigma_{\alpha\beta} + \epsilon_{\alpha\beta\tau}\Omega^\tau, \\
\gamma^\alpha_{\beta\delta} &= \epsilon_{\beta\delta\epsilon}n^{\epsilon\alpha} + \delta_\delta{}^\alpha a_\beta - \delta_\beta{}^\alpha a_\delta.
\end{aligned} \tag{2.6}$$

Furthermore, the basis $\{\vec{\mathbf{e}}_a\}$ can be chosen so that $n_{\alpha\beta} = \mathrm{diag}(n_1, n_2, n_3)$ and $a^\beta = (a, 0, 0)$. The Einstein field equations are [equations (113–121) in MacCallum's Cargèse lectures [20], with $\Lambda = 0$ and $\theta_{ab} = \frac{\theta}{3}h_{ab} + \sigma_{ab}$]

$$\dot{\theta} = -\frac{\theta^2}{3} - 2\sigma^2 - \frac{1}{2}(\rho + 3\bar{p}), \tag{2.7}$$

$$q_\alpha = 3\sigma_\alpha{}^\beta a_\beta - \epsilon_{\alpha\beta\gamma}n^{\gamma\delta}\sigma_\delta{}^\beta, \tag{2.8}$$

$$\begin{aligned}
\dot{\sigma}_{\alpha\beta} &= -\frac{2}{3}\delta_{\alpha\beta}\left(\frac{\theta^2}{3} - \sigma^2 - \rho\right) + \pi_{\alpha\beta} - \theta\sigma_{\alpha\beta} \\
&\quad -2\sigma^\gamma{}_{(\alpha}\epsilon_{\beta)\delta\gamma}\Omega^\delta + 2\epsilon_{\gamma\delta(\alpha}n^\gamma_{\beta)}a^\delta - 2n^\gamma{}_{(\alpha}n_{\beta)\gamma}
\end{aligned}$$

$$+n_{\gamma}{}^{\gamma}n_{\alpha\beta}+\delta_{\alpha\beta}\left(2a_{\gamma}a^{\gamma}+n^{\gamma\delta}n_{\gamma\delta}-\frac{(n_{\alpha}{}^{\alpha})^2}{2}\right). \tag{2.9}$$

The Jacobi identities, $[\vec{\mathbf{e}}_a,[\vec{\mathbf{e}}_b,\vec{\mathbf{e}}_c]]=0$, are

$$\dot{a}_{\alpha} = -\sigma_{\alpha}{}^{\beta}a_{\beta}-\frac{\theta}{3}a_{\alpha}+\epsilon_{\alpha\beta\gamma}a^{\beta}\Omega^{\gamma}, \tag{2.10}$$

$$\dot{n}^{\alpha\beta} = 2n^{\gamma(\alpha}\epsilon^{\beta)}{}_{\gamma\delta}\Omega^{\delta}+2n_{\gamma}{}^{(\alpha}\sigma^{\beta)\gamma}+2n_{\gamma}{}^{(\alpha}\delta^{\beta)\gamma}\frac{\theta}{3}-n^{\alpha\beta}\theta. \tag{2.11}$$

The energy-momentum conservation equations are

$$\dot{\rho} = -(\rho+\bar{p})\theta-\pi_{\alpha\beta}\sigma^{\alpha\beta}+2a^{\alpha}q_{\alpha}, \tag{2.12}$$

$$\dot{q}_{\alpha} = -\epsilon_{\alpha\beta\gamma}q^{\beta}\Omega^{\gamma}-\sigma_{\alpha\beta}q^{\beta}-\frac{4}{3}\theta q_{\alpha}+3a_{\beta}\pi_{\alpha}{}^{\beta}+\pi_{\beta}{}^{\gamma}\epsilon_{\gamma\alpha\delta}n^{\beta\delta}. \tag{2.13}$$

The generalized Friedmann equation is

$$\frac{\theta^2}{3}=\sigma^2+\rho+\frac{1}{2}\left(6a_{\alpha}a^{\alpha}+n^{\alpha\beta}n_{\alpha\beta}-\frac{(n_{\alpha}{}^{\alpha})^2}{2}\right). \tag{2.14}$$

The quantity Ω^{α} is essentially the angular velocity of an observer moving with velocity $\vec{\mathbf{e}}_0$, of the triad $\{\vec{\mathbf{e}}_{\alpha}\}$ with respect to a set of Fermi propagated axes. For models of Bianchi type A, $\Omega^{\alpha}=0$, and for models of Bianchi type B, Ω^{α} is a linear combination of components of the shear tensor, $\sigma_{\alpha\beta}$ [60].

Equations (2.8) and (2.14) are first integrals of the system. Hence, the generalized Friedmann equation (2.14) can be used to define ρ and equation (2.8) can be used to define q_{α} in terms of the remaining variables. It is important to note that both $\rho=\rho(\theta,\sigma_{\alpha\beta},n_{\alpha\beta},a_{\alpha})$ and $q_{\alpha}=q_{\alpha}(\theta,\sigma_{\alpha\beta},n_{\alpha\beta},a_{\alpha})$ are homogeneous functions of degree two in their arguments. The remaining equations (2.7, 2.9, 2.10, 2.11) constitute a dynamical system. The dynamical system (2.7, 2.9, 2.10, 2.11) is invariant under the transformation

$$\begin{array}{lll} \theta\to\lambda\theta, & a_{\alpha}\to\lambda a_{\alpha}, & \bar{p}\to\lambda^2\bar{p}, \\ \sigma_{\alpha\beta}\to\lambda\sigma_{\alpha\beta}, & n_{\alpha\beta}\to\lambda n_{\alpha\beta}, & \\ \pi_{\alpha\beta}\to\lambda^2\pi_{\alpha\beta}, & t\to\lambda^{-1}t. & \end{array} \tag{2.15}$$

This invariance implies that there exists a symmetry in the dynamical system [63] (See also Appendix B). With the following change of variables

$$\begin{aligned} &\Sigma_{\alpha\beta} = \tfrac{\sigma_{\alpha\beta}}{\theta}, \quad A_\alpha = \tfrac{a_\alpha}{\theta}, \quad N_{\alpha\beta} = \tfrac{n_{\alpha\beta}}{\theta}, \\ &\Theta = \ln\theta, \qquad \tfrac{dt}{d\tau} = \tfrac{1}{\theta}, \end{aligned} \tag{2.16}$$

the new evolution equations for $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$ and A_α become independent of the variable Θ. That is, Θ decouples from the dynamical system describing the evolution of $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$ and A_α. The dynamical system can be considered as a reduced dynamical system for $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$ and A_α together with an evolution equation for Θ.

What equations of state for the isotropic pressure $\bar{p}$ and anisotropic stress $\pi_{\alpha\beta}$ are needed to satisfy the conditions that $\bar{p} \to \lambda^2\bar{p}$ and $\pi_{\alpha\beta} \to \lambda^2\pi_{\alpha\beta}$ in equation (2.15)? In order for the dynamical system described by equations (2.7, 2.9, 2.10, 2.11) to have a unique solution, the equations of state must be C^1 functions of their arguments. The equations of state for $\bar{p}$ and $\pi_{\alpha\beta}$ must also be homogeneous functions of degree two, that is,

$$\bar{p}(\lambda\theta, \lambda\sigma_{\alpha\beta}, \lambda n_{\alpha\beta}, \lambda a_\alpha) = \lambda^2\bar{p}(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha), \tag{2.17}$$

and

$$\pi_{\alpha\beta}(\lambda\theta, \lambda\sigma_{\alpha\beta}, \lambda n_{\alpha\beta}, \lambda a_\alpha) = \lambda^2\pi_{\alpha\beta}(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha). \tag{2.18}$$

In the new variables (2.16) the equations of state needed are of the form

$$\begin{aligned} P \equiv \frac{\bar{p}}{\theta^2} &= \frac{\bar{p}(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha)}{\theta^2} \\ &= P\left(\frac{\theta}{\theta}, \frac{\sigma_{\alpha\beta}}{\theta}, \frac{n_{\alpha\beta}}{\theta}, \frac{a_\alpha}{\theta}\right) \\ &= P(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha) \end{aligned} \tag{2.19}$$

and

$$\Pi_{\alpha\beta} \equiv \frac{\pi_{\alpha\beta}}{\theta^2} = \Pi_{\alpha\beta}(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha). \tag{2.20}$$

Thus, any C^1 functions of the dimensionless variables $(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha)$ gives rise to equations of state that satisfy the conditions $\bar{p} \to \lambda^2\bar{p}$ and $\pi_{\alpha\beta} \to \lambda^2\pi_{\alpha\beta}$ in equation (2.15). We note that $P = \bar{p}\theta^{-2}$ and $\Pi_{\alpha\beta} = \pi_{\alpha\beta}\theta^{-2}$ are functions of dimensionless variables and therefore are dimensionless. We shall call the corresponding equations of state (2.19, 2.20) 'dimensionless' equations of state [64].

The equilibrium points of the new reduced dynamical system can be found. At these equilibrium points $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$, and A_α are constant and consequently the equation

$$\frac{\dot{\theta}}{\theta^2} = -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P) \tag{2.21}$$

may be integrated where $P = P(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha)$. [Note that equation (2.14) is used to define $\rho\theta^{-2}$ in terms of the other variables.] The solution to (2.21) is

$$\theta = \begin{cases} \theta_o t^{-1} & \text{if } -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P) \neq 0 \\ \\ \theta_o & \text{if } -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P) = 0 \end{cases} \tag{2.22}$$

at the equilibrium point. If the condition $\rho + 3\bar{p} \geq 0$ is satisfied at the equilibrium points then $\theta = \theta_o t^{-1}$ and the remaining physical variables may also be integrated to yield $\sigma_{\alpha\beta} = (\sigma_{\alpha\beta})_o t^{-1}$, $n_{\alpha\beta} = (n_{\alpha\beta})_o t^{-1}$, and $a_\alpha = (a_\alpha)_o t^{-1}$, where the subscript '$o$' denotes constant values. These solutions imply that the commutation coefficients $\gamma^c{}_{ab}$ are functions of t^{-1} [see equations (2.6)]. Therefore, using Theorem 1 stated in section 2.2, the equilibrium points of the reduced system represent transitively self-similar cosmological models. However, the equilibrium points of the reduced dynamical system also represent the asymptotic limit points of the Einstein field equations.

Conversely, assume that the asymptotic limit points are self-similar then Theorem 1 in section 2.2 implies that the commutation functions $\gamma^c{}_{ab} \propto t^{-1}$. Therefore $\gamma^\alpha_{0\alpha} = -\theta \propto t^{-1}$, and similarly the physical variables $\sigma_{\alpha\beta}$, $n_{\alpha\beta}$, and a_α are also functions of t^{-1} [see equations (2.6)]. Equations (2.7–2.14) then imply that the isotropic pressure $\bar{p}$ and anisotropic stress $\pi_{\alpha\beta}$ are functions of t^{-2}; that is, $\bar{p}(t) = p_o t^{-2}$ and $\pi_{\alpha\beta}(t) = (\pi_{\alpha\beta})_o t^{-2}$. [Since θ, $\sigma_{\alpha\beta}$, $n_{\alpha\beta}$, $a_\alpha \propto t^{-1}$, from equation (2.14) we see that $\rho \propto t^{-2}$ and substituting into (2.7), we obtain the result $\bar{p} \propto t^{-2}$. Similarly for $\pi_{\alpha\beta}$ using equation (2.9).]

If the pressure $\bar{p}$ has an equation of state of the form

$$\bar{p} = \bar{p}(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha), \tag{2.23}$$

then

$$\begin{aligned}\bar{p}(t) &= \bar{p}(\theta(t), \sigma_{\alpha\beta}(t), n_{\alpha\beta}(t), a_\alpha(t)),\\ &= \bar{p}((\theta)_o t^{-1}, (\sigma_{\alpha\beta})_o t^{-1}, (n_{\alpha\beta})_o t^{-1}, (a_\alpha)_o t^{-1}).\end{aligned} \tag{2.24}$$

Thus, it follows that

$$\begin{aligned}\bar{p}(\lambda^{-1}t) &= \bar{p}\Big(\lambda((\theta)_o t^{-1}), \lambda((\sigma_{\alpha\beta})_o t^{-1}), \lambda((n_{\alpha\beta})_o t^{-1}), \lambda((a_\alpha)_o t^{-1})\Big),\\ &= \bar{p}(\lambda\theta, \lambda\sigma_{\alpha\beta}, \lambda n_{\alpha\beta}, \lambda a_\alpha).\end{aligned} \tag{2.25}$$

But $\bar{p}(\lambda^{-1}t) = \lambda^2 p_o t^{-2} = \lambda^2 \bar{p}(t)$, thus the equation of state for $\bar{p}$ is of the form $\bar{p}(\lambda\theta, \lambda\sigma_{\alpha\beta}, \lambda n_{\alpha\beta}, \lambda a_\alpha) = \lambda^2\bar{p}(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha)$. The result is similar for $\pi_{\alpha\beta}$. Therefore, assuming that the spacetime is transitively self-similar, the equations of state for the isotropic pressure $\bar{p}$ and anisotropic stress $\pi_{\alpha\beta}$ must be homogeneous functions of degree two of the variables $(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha)$. The previous two results are summarized in the following theorem.

Theorem 2. *Let there be a G_3 group of isometries acting transitively on a 3-dimensional hypersurface, and assume that the fluid is moving hypersurface orthogonal. Then the asymptotic limit points of the Einstein field equations are transitively self-similar if and only if the equations of state for the isotropic pressure $\bar{p}$ and anisotropic stress $\pi_{\alpha\beta}$ are homogeneous functions of degree two; that is,*

$$\begin{aligned}\bar{p}(\lambda\theta, \lambda\sigma_{\alpha\beta}, \lambda n_{\alpha\beta}, \lambda a_\alpha) &= \lambda^2\bar{p}(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha)\\ \pi_{\alpha\beta}(\lambda\theta, \lambda\sigma_{\alpha\beta}, \lambda n_{\alpha\beta}, \lambda a_\alpha) &= \lambda^2\pi_{\alpha\beta}(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha)\end{aligned} \tag{2.26}$$

provided $\rho + 3\bar{p} \geq 0$.

Note: The condition $\rho + 3\bar{p} \geq 0$ is a sufficient condition and not necessary. The equilibrium points are self-similar as long as $-\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P) \neq 0$ at the equilibrium point. Also, the condition $\rho + 3\bar{p} \geq 0$ is equivalent to the SEC (strong energy condition) if the energy momentum tensor, T_{ab}, is diagonal.

2.4 Equations of State in Theories of Irreversible Thermodynamics

We have already determined that the isotropic pressure $\bar{p}$ and the anisotropic stress $\pi_{\alpha\beta}$ should be invariant under the transformation (2.15) so that the asymptotic limit points will be represented by self-similar cosmological models. We will now use this information to determine the equations of state that will be needed to complete various theories of irreversible thermodynamics if the requirement that the asymptotic limit points be represented by self-similar cosmological models is assumed.

In imperfect fluid models with bulk viscosity, the isotropic pressure, $\bar{p}$, has two components, the thermodynamic pressure, p, and the bulk viscous pressure, Π, that is $\bar{p} = p + \Pi$. A second order theory of irreversible thermodynamics was proposed by Israel and Stewart [49, 50] to model viscous effects in a simple fluid. The evolution equations for the bulk viscous pressure, Π, the heat conduction vector, q_a, and the anisotropic stress, π_{ab} are given implicitly in equations (1.3).

2.4.1 First Order Eckart Theory

The first order Eckart theory [47] is obtained by setting $\alpha_1 = \alpha_2 = \beta_0 = \beta_1 = \beta_2 = 0$, in equation (1.3). In the Eckart theory we need three equations of state for the three transport coefficients ζ, κ, and η. In order for the system (1.3) to be invariant under the transformation (2.15), we must have

$$\Pi \to \lambda\Pi, \qquad \zeta \to \lambda\zeta, \qquad \kappa \to \lambda\kappa, \qquad \eta \to \lambda\eta. \tag{2.27}$$

Now if equations of state of the form

$$\begin{aligned} \zeta &= \zeta(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha), \\ \kappa &= \kappa(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha), \\ \eta &= \eta(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha), \end{aligned} \tag{2.28}$$

are assumed such that they satisfy (2.27) then the corresponding dimensionless equations of state are

$$
\begin{aligned}
\frac{\zeta}{\theta} &= \bar{\zeta}(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha), \\
\frac{\kappa}{\theta} &= \bar{\kappa}(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha), \\
\frac{\eta}{\theta} &= \bar{\eta}(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha).
\end{aligned}
\tag{2.29}
$$

The dynamical system describing the evolution of the spatially homogeneous models in dimensionless variables (2.16) is independent of the variable $\Theta = \ln\theta$ and hence the system has decoupled into a reduced system and an evolution equation for Θ. At the equilibrium points of the reduced dynamical system, $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$, and A_α, are constant and consequently the equation

$$
\frac{\dot{\theta}}{\theta^2} = -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P - 3\bar{\zeta}), \tag{2.30}
$$

may be integrated where $P = P(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha)$ and $\bar{\zeta} = \bar{\zeta}(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha)$. [Note that $\rho\theta^{-2}$ is given by equation (2.14).] The solution to (2.30) is

$$
\theta = \begin{cases} \theta_o t^{-1} & \text{if } -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P - 3\bar{\zeta}) \neq 0, \\ \theta_o & \text{if } -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P - 3\bar{\zeta}) = 0, \end{cases} \tag{2.31}
$$

at the equilibrium point. If the condition $\rho + 3\bar{p} \geq 0$ (note $\bar{p} = p + \Pi = p - \zeta\theta$) is satisfied at the equilibrium points then $\theta = \theta_o t^{-1}$ and the remaining physical variables may also be integrated to yield $\sigma_{\alpha\beta} = (\sigma_{\alpha\beta})_o t^{-1}$, $n_{\alpha\beta} = (n_{\alpha\beta})_o t^{-1}$, and $a_\alpha = (a_\alpha)_o t^{-1}$, where the subscript '$o$' denotes constant values. These solutions imply that the commutation coefficients $\gamma^c{}_{ab}$ are functions of t^{-1} [see equations (2.6)]. Therefore, using Theorem 1 stated in section 2.2, the equilibrium points of the reduced system represent transitively self-similar cosmological models.

We can deduce at the equilibrium points of the reduced system that $p = p_o t^{-2}$, $\zeta = \zeta_o t^{-1}$, $\kappa = \kappa_o t^{-1}$, $\eta = \eta_o t^{-1}$, and $\rho = \rho_o t^{-2}$. Therefore, at the equilibrium points of the reduced system, the equations of state are of the form:

$$
\begin{aligned}
p &\propto \rho, & \zeta &\propto \rho^{1/2}, \\
\kappa &\propto \rho^{1/2}, & \eta &\propto \rho^{1/2}.
\end{aligned}
\tag{2.32}
$$

We can now conclude that if we choose any dimensionless equations of state for p, ζ, κ, and η, then, asymptotically, as the cosmological models approach an equilibrium point, the equations of state will have the above asymptotic form (2.32).

2.4.2 Second Order Israel-Stewart Theories

The second order Israel-Stewart theory is described by equations (1.3). In addition to the three transport coefficients (ζ, κ, η) also found in the Eckart theory, there are five other quantities $(\beta_0, \beta_1, \beta_2, \alpha_0, \alpha_1)$ that require equations of state. In order for the system (1.3) to be invariant under the transformation (2.15) we find that equation (2.27) and

$$\beta_i \to \lambda^{-2}\beta_i \quad (i = 0, 1, 2), \qquad \alpha_j \to \lambda^{-2}\alpha_j \quad (j = 1, 2) \tag{2.33}$$

must be satisfied. Therefore, if the equations of state for the above quantities satisfy (2.27) and (2.33) then it is possible to define new dimensionless variables $y \equiv \Pi/\theta^2$ and $z_{\alpha\beta} \equiv \pi_{\alpha\beta}/\theta^2$ and use the dimensionless variables (2.16) such that the dimension of the system governing this model is reduced by one.

The most general equations of state are of the form:

$$\begin{aligned}
\zeta &= \zeta(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha, \Pi, \pi_{\alpha\beta}), \\
\kappa &= \kappa(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha, \Pi, \pi_{\alpha\beta}), \\
\eta &= \eta(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha, \Pi, \pi_{\alpha\beta}), \\
\beta_i &= \beta_i(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha, \Pi, \pi_{\alpha\beta}), \qquad (i = 0, 1, 2) \\
\alpha_j &= \alpha_j(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha, \Pi, \pi_{\alpha\beta}). \qquad (j = 0, 1)
\end{aligned} \tag{2.34}$$

If the equations of state satisfy equations (2.27) and (2.33) then the above equations of state (2.34) can be written in terms of dimensionless variables (2.16), viz;

$$\begin{aligned}
\frac{\zeta}{\theta} &= \bar{\zeta}(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha, y, z_{\alpha\beta}), \\
\frac{\kappa}{\theta} &= \bar{\kappa}(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha, y, z_{\alpha\beta}),
\end{aligned}$$

$$
\begin{aligned}
\frac{\eta}{\theta} &= \bar{\eta}(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha, y, z_{\alpha\beta}), && && (2.35)\\
\beta_i\theta^2 &= \bar{\beta}_i(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha, y, z_{\alpha\beta}), && (i = 0,1,2)\\
\alpha_i\theta^2 &= \bar{\alpha}_j(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha, y, z_{\alpha\beta}). && (j = 0,1)
\end{aligned}
$$

Using the same arguments found in the previous subsection we find at the equilibrium points of the dynamical system describing the model that

$$
\frac{\dot{\theta}}{\theta^2} = -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P + 3y), \tag{2.36}
$$

may be integrated where $P = P(\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha, y, z_{\alpha\beta})$ and $\rho\theta^{-2}$ is given by equation (2.14). The solution to (2.36) is

$$
\theta = \begin{cases} \theta_o t^{-1} & \text{if } -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P + 3y) \neq 0, \\ \theta_o & \text{if } -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \frac{1}{2}(\rho\theta^{-2} + 3P + 3y) = 0, \end{cases} \tag{2.37}
$$

at the equilibrium point. Therefore, using Theorem 1 stated in section 2.2, the equilibrium points of the reduced system represent transitively self-similar cosmological models provided $\rho\theta^{-2} + 3P + 3y \geq 0$. Note, this condition is satisfied if one assumes the strong energy condition [65].

We deduce that at the equilibrium points of the reduced system, $p = p_o t^{-2}$, $\zeta = \zeta_o t^{-1}$, $\kappa = \kappa_o t^{-1}$, $\eta = \eta_o t^{-1}$, $\beta_i = (\beta_i)_o t^2$, $\alpha_i = (\alpha_i)_o t^2$, and $\rho = \rho_o t^{-2}$. Therefore, at the equilibrium points of the reduced system the equations of state are of the form:

$$
\begin{aligned}
p &\propto \rho, & \zeta &\propto \rho^{1/2}, \\
\kappa &\propto \rho^{1/2}, & \eta &\propto \rho^{1/2}, \\
\beta_i &\propto \rho^{-1}, & \alpha_j &\propto \rho^{-1}.
\end{aligned} \tag{2.38}
$$

We can now conclude that if we choose any dimensionless equations of state for p, ζ, κ, η, β_i, and α_j, then, asymptotically, as the cosmological models approach an equilibrium point, the equations of state will have the above asymptotic form (2.38).

2.5 Scalar Field

We have seen in the previous sections that self-similarity plays a role in the spatially homogeneous fluid spacetimes. This property of self-similar asymptotic limit points may also be present in some scalar field cosmologies. In this section we shall determine which scalar field models allow self-similar asymptotic limit points. The addition of a scalar field ϕ with potential $V(\phi)$ is equivalent to

$$\begin{aligned} \rho &= \rho_f + \rho_\phi; \qquad \rho_\phi \equiv \frac{1}{2}\dot{\phi}^2 + V(\phi), && (2.39) \\ \bar{p} &= p_f + p_\phi; \qquad p_\phi \equiv \frac{1}{2}\dot{\phi}^2 - V(\phi), && (2.40) \\ \pi_{\alpha\beta} &= {}_f\pi_{\alpha\beta}, \end{aligned}$$

where a subscript f denotes the usual fluid components. With the introduction of the field ϕ, one more equation is needed to complete the system, namely the Klein-Gordon equation

$$\ddot{\phi} = -\theta\dot{\phi} - \frac{\partial V(\phi)}{\partial \phi}. \tag{2.41}$$

The conditions $\bar{p} \to \lambda^2 \bar{p}$ and $\pi_{\alpha\beta} \to \lambda^2 \pi_{\alpha\beta}$, necessary for the symmetry to exist in the dynamical system, (see section 2.3), imply that $\dot{\phi} \to \lambda\dot{\phi}$ and $V(\phi) \to \lambda^2 V(\phi)$. Furthermore, since $\dot{\phi} \to \lambda\dot{\phi}$ is a linear transformation, it follows that $\phi \to \phi + \phi_o$, but this is incompatible with $V(\phi) \to \lambda^2 V(\phi)$, except when $V(\phi) \equiv 0$ or $V(\phi) = \Lambda e^{\kappa\phi}$ $(\kappa \neq 0)$ in which case $\phi_o = \frac{2}{\kappa}\ln\lambda$. Therefore, if $V(\phi) = \Lambda e^{\kappa\phi}$ $(\kappa \neq 0)$ a symmetry will exist. If $\Lambda \neq 0$ then the potential permits power law inflation while if $\Lambda = 0$ the scalar field is massless.

The existence of this symmetry in the dynamical system again implies that there exists a transformation of variables (See Appendix B). Using the variables in equation (2.16) and the new variables $\Phi = \dot{\phi}\theta^{-1}$ and $\Psi^2 = e^{\kappa\phi}\theta^{-2}$, the dynamical system is transformed such that one of the equations decouples. The equilibrium points of the new reduced dynamical system can be found. At these equilibrium points $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$,

A_α, Ψ and Φ are constant. Thus, at these equilibrium points the equation

$$\frac{\dot{\theta}}{\theta^2} = -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \Phi^2 - \Lambda\Psi^2 - \frac{1}{2}(\rho_f\theta^{-2} + 3P_f), \tag{2.42}$$

may be integrated. [Note that equation (2.14) can be used to define $\rho_f\theta^{-2}$ in terms of the other variables.] The solution to (2.42) is

$$\theta = \begin{cases} \theta_o t^{-1} & \text{if } -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \Phi^2 - \Lambda\Psi^2 - \frac{1}{2}(\rho_f\theta^{-2} + 3P_f) \neq 0, \\ \theta_o & \text{if } -\frac{1}{3} - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - \Phi^2 - \Lambda\Psi^2 - \frac{1}{2}(\rho_f\theta^{-2} + 3P_f) = 0, \end{cases} \tag{2.43}$$

at the equilibrium point. If the condition $\rho_f + 3p_f \geq 0$, is satisfied then the remaining physical variables may also be integrated to yield $\sigma_{\alpha\beta} = (\sigma_{\alpha\beta})_o t^{-1}$, $n_{\alpha\beta} = (n_{\alpha\beta})_o t^{-1}$, $a_\alpha = (a_\alpha)_o t^{-1}$ and $\dot{\phi} = (\dot{\phi})_o t^{-1}$. These solutions imply that the commutation coefficients $\gamma^c{}_{ab}$ are functions of t^{-1}. Therefore, using Theorem 1 stated in section 2.2 , the equilibrium points represent transitively self-similar cosmological models in the case of a zero or an exponential potential. See Chapter 6 for an explicit example of a cosmological model containing a scalar field with an exponential potential.

2.6 Non-Self-Similar Asymptotic Models

The self-similarity of the asymptotic limit points of the Einstein field equations is not a robust property. For example, self-similarity is broken if any of the following conditions are satisfied:

A. The equations of state are not of the form (2.26).

B. The existence of a scalar field with a non-exponential potential (e.g., $V(\phi) = \lambda_n\phi^{2n}$).

C. The existence of a cosmological constant Λ.

In addition, self-similarity may be broken if :

D. The condition $\rho + 3\bar{p} \geq 0$ is not satisfied.

As an illustration, let us consider a Friedmann-Robertson-Walker (FRW) model with a perfect fluid source and a scalar field ϕ with potential $V(\phi) = \lambda_n \phi^{2n}$. Since the $k = 0$ flat FRW model occurs as the asymptotic limit of the $k = \pm 1$ (positive and negative curvature) FRW models, we shall further simplify our examples by assuming $k = 0$. The solutions examined here are exact $k = 0$ FRW solutions and are not asymptotic solutions.

The Einstein field equations are [see equations (2.39), (2.40)]

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \frac{1}{2}(\rho_f + \rho_\phi + 3p_f + 3p_\phi), \tag{2.44}$$

$$\frac{\theta^2}{3} = \rho_f + \rho_\phi, \tag{2.45}$$

$$\dot{\rho_f} = -\theta(\rho_f + p_f), \tag{2.46}$$

$$\dot{\rho_\phi} = -\theta(\rho_\phi + p_\phi). \tag{2.47}$$

Note that equation (2.47) is equivalent to the Klein-Gordon equation (2.41).

Equations of State and Non-zero Potential.

Let us demonstrate that dimensionless equations of state imply self-similarity. Consider a dimensionless equation of state of the form

$$\frac{p}{\theta^2} = (\gamma - 1)\frac{\rho}{\theta^2}; \qquad \gamma = \textit{constant}. \tag{2.48}$$

Substituting (2.48) and (2.45) into (2.44) we obtain

$$\frac{\dot{\theta}}{\theta^2} = -\frac{\gamma}{2}. \tag{2.49}$$

The solution is $\theta \sim t^{-1}$, which implies that the model is self-similar.

However, now assume $\rho_f = p_f = 0$ and a potential $V(\phi) = \frac{1}{2}m^2\phi^2$. This is equivalent to having a non-dimensionless equation of state for p_ϕ, viz.,

$$\frac{p_\phi}{\theta^2} = \frac{\rho_\phi}{\theta^2} - m^2\frac{\phi^2}{\theta^2}. \tag{2.50}$$

Substituting (2.50) into (2.44) and (2.45) we obtain

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \dot{\phi}^2 + m^2\phi^2, \tag{2.51}$$

$$\frac{1}{3}\theta^2 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2. \tag{2.52}$$

Taking linear combinations of (2.51) and (2.52) we obtain

$$\dot{\theta} + \theta^2 = 2m^2\phi^2, \tag{2.53}$$

$$\dot{\theta} - \frac{1}{3}\theta^2 = -2\dot{\phi}^2. \tag{2.54}$$

If we assume that the solution is self-similar, then $\theta \sim t^{-1}$ and $\dot{\theta} \sim t^{-2}$. Equations (2.53) and (2.54) then imply that both $\phi \sim t^{-1}$ and $\dot{\phi} \sim t^{-1}$. However, this is a contradiction and thus the solution cannot be self-similar.

Cosmological Constant.

In a similar manner one may include a cosmological constant in the model by setting $\dot{\phi} = 0$, whence $\rho_\phi = V_o$ and $p_\phi = -V_o$. If we consider the vacuum case in which $p_f = \rho_f = 0$, equation (2.45) may be simplified to give $\theta = \sqrt{3V_o}$. This solution represents the de Sitter model which is not self-similar. Note, in this case $p/\theta^2 = -V_o/\theta^2$, hence the equation of state for the pressure p is not dimensionless.

$\boldsymbol{\rho + 3\bar{p} \geq 0}$.

One may also consider a simple imperfect fluid model with bulk viscosity by putting $\rho_\phi = p_\phi = 0$ and $p_f = p - \zeta\theta$, where p is the thermodynamic pressure and ζ is the bulk viscosity coefficient. [This is an example from the Eckart theory, see previous section 2.5.] If we assume the equations of state

$$\frac{\zeta}{\theta} = \zeta_o \frac{\rho_f}{\theta^2} \quad \text{and} \quad \frac{p}{\theta^2} = (\gamma - 1)\frac{\rho_f}{\theta^2}, \tag{2.55}$$

then the equation of state for $p_f\theta^{-2}$ is dimensionless. Substituting (2.55) and (2.45) into (2.44) we find that

$$\dot{\theta} = -\frac{1}{2}(\gamma - \zeta_o)\theta^2. \qquad (2.56)$$

The solution for $\gamma \neq \zeta_o$ is $\theta \sim t^{-1}$, which is self-similar. However, for $\gamma = \zeta_o$ the solution is $\theta = \theta_o$ (a constant), which is not self-similar. In this case

$$\begin{aligned} \rho + 3\bar{p} &= \rho_f + 3p - 3\zeta\theta, \\ &= \rho_f + 3\rho_f(\gamma - 1) - 3\gamma\rho_f, \\ &= -2\rho_f, \end{aligned} \qquad (2.57)$$

whence $\rho + 3\bar{p} \ngeq 0$ if the energy density, ρ_f, is non-negative.

2.7 Scalar Tensor Theories

We have shown that, in General Relativity, the asymptotic limits of the Einstein field equations are self-similar provided that the equations of state for the isotropic pressure $\bar{p}$ and the anisotropic stress π_{ab} are of the form (2.26) and $\rho + 3\bar{p} \geq 0$. Building upon this result, a general class of modified theories of gravity will be studied to find under what conditions one might expect to have self-similar asymptotic limits in gravitational theories other than General Relativity.

The class of scalar-tensor theories under consideration has an action of the form

$$S = \int d^4x\sqrt{-\det g_{ab}}\left\{-f(\phi)R - \frac{1}{2}\phi_{;c}\phi^{;c} - V(\phi) + L_m\right\}, \qquad (2.58)$$

where L_m is the Lagrangian of matter. A number of theories can be cast in this form. For example, a) the classic Brans-Dicke theory is obtained by setting $f(\phi) = \zeta\phi^2$ and $V(\phi) = 0$, and b) the non-minimally coupled scalar field theories are achieved by setting $f(\phi) = -\frac{1}{16\pi G} + \frac{1}{2}\epsilon\phi^2$.

Varying the action S in a FRW background we obtain the following dynamical system

$$\dot{\rho} = -\theta(\rho + \bar{p}),$$

$$
\begin{aligned}
\dot{\phi} &= \Phi, \\
\dot{\Phi} &= -\theta\Phi - \frac{f(\phi)}{3f'(\phi)^2 - f(\phi)}\Big\{-V'(\phi) + \frac{f'(\phi)}{2f(\phi)}\Big((6f''(\phi)-1)\Phi^2 \\
&\qquad\qquad +4V(\phi) + \rho - 3\bar{p}\Big)\Big\}, \qquad (2.59) \\
\dot{\theta} &= -\frac{1}{3}\theta^2 + \frac{f'(\phi)}{f(\phi)}\theta\Phi \\
&- \frac{1}{4(3f'(\phi)^2 - f(\phi))}\Big\{(2 - 6f''(\phi))\Phi^2 - 2V(\phi) + \rho + 3\bar{p} \\
&\qquad\qquad +6f'(\phi)V'(\phi) - 6\frac{f'(\phi)^2}{f(\phi)}\Big(\rho + \frac{1}{2}\Phi^2 + V(\phi)\Big)\Big\},
\end{aligned}
$$

with first integral

$$
\frac{3k}{a^2} = -\frac{1}{3}\theta^2 - \theta\Phi\frac{f'(\phi)}{f(\phi)} - \frac{1}{2f(\phi)}\Big(\rho + \frac{1}{2}\Phi^2 + V(\phi)\Big), \qquad (2.60)
$$

where k takes on values $+1, 0, -1$ which represents the closed, flat and open models respectively and where $\theta = 3\frac{\dot{a}}{a}$.

The system (2.59) is invariant under the transformation

$$
\begin{aligned}
&\theta \to \lambda\theta, \qquad \rho \to \lambda^2\rho, \qquad \bar{p} \to \lambda^2\bar{p}, \\
&\Phi \to \lambda\Phi, \qquad \phi \to \phi,
\end{aligned} \qquad (2.61)
$$

if and only if $V(\phi) = 0$ and the equation of state for p has the form (2.26). Therefore, in the case $V(\phi) = 0$, new dimensionless variables can be defined as follows:

$$
x = \frac{3\rho}{\theta^2}, \qquad \psi = \phi, \qquad \Psi = \frac{\Phi}{\theta}, \qquad \Theta = \ln\theta, \qquad \frac{dt}{d\tau} = \frac{1}{\theta}. \qquad (2.62)
$$

The evolution equations for x, ψ, and Ψ are now independent of Θ. Hence, the equations decouple and the qualitative behaviour of the system can be determined from the reduced system of equations. A dimensionless equation of state of the form $\bar{p}/\theta^2 = (\gamma - 1)x/3$ is assumed for $\bar{p}$ (this reduces to the usual γ-law equation of state) in which case the equations become:

$$
\frac{dx}{d\tau} = -x\left(\gamma + 2\frac{\dot{\theta}}{\theta^2}\right), \qquad (2.63)
$$

$$\begin{aligned}\frac{d\psi}{d\tau} &= \Psi, & (2.64)\\ \frac{d\Psi}{d\tau} &= -\Psi - \frac{f'(\psi)}{2(3f'(\psi)^2 - f(\psi))}\left\{(6f''(\psi)-1)\Psi^2 + (4-3\gamma)\frac{x}{3}\right\} - \Psi\frac{\dot{\theta}}{\theta^2}, & (2.65)\end{aligned}$$

where

$$\begin{aligned}\frac{\dot{\theta}}{\theta^2} &= -\frac{1}{3} + \frac{f'(\psi)}{f(\psi)}\Psi - \frac{1}{4(3f'(\psi)^2 - f(\psi))}\left\{(2-6f''(\psi))\Psi^2 + (3\gamma-2)\frac{x}{3}\right.\\ &\qquad \left. -6\frac{f'(\psi)^2}{f(\psi)}\left(\frac{1}{3}x + \frac{1}{2}\Psi^2\right)\right\}. & (2.66)\end{aligned}$$

The equilibrium points are found by setting each derivative to zero. Equation (2.64) implies that at any equilibrium point $\Psi = 0$. From equation (2.63) either $x = 0$ or $\frac{\dot{\theta}}{\theta^2} = -\gamma/2$. If $x = 0$, then from equation (2.66), we see that $\frac{\dot{\theta}}{\theta^2} = -1/3$. Hence in either case the equilibrium points of the system (2.63–2.66) represent self-similar cosmological models if $\gamma \neq 0$.

In summary, if $V(\phi) = 0$ and the equation of state for the pressure $p = (\gamma - 1)\rho$, then the dynamical system describing the Friedmann-Robertson-Walker model in a Scalar tensor theory of gravity allows one equation to decouple when dimensionless variables are used. In these new dimensionless variables the explicit forms for $\frac{\dot{\theta}}{\theta^2}$ were found at all the equilibrium points. If $\gamma \neq 0$ then $\theta \sim t^{-1}$ at all of the equilibrium points, and thus these equilibrium points represent self-similar cosmological models. This conclusion is independent of the particular scalar tensor theory, that is, it is independent of the form of $f(\phi)$, but this result does require that $V(\phi) \equiv 0$. [Unless $f'(\phi) \equiv 0$ in which case the theory reduces to general relativity with a scalar field (see section 2.4).]

2.8 Conclusions

The Einstein field equations with an imperfect fluid source were investigated using techniques from dynamical systems theory and employing methods from the theory of symmetries of differential equations. We assumed a G_3 group of isometries acting transitively on 3-dimensional spacelike hypersurfaces; that is, the models under consideration were the spatially homogeneous Bianchi models. Also, we assumed that the fluid flow is moving hypersurface orthogonal. The equations of state for the pressure p and anisotropic stress $\pi_{\alpha\beta}$ were assumed to be homogeneous functions of degree two in the variables $(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha)$, so that a symmetry in the field equations could be used to define new variables. The field equations in these new variables allowed one equation to decouple. The equilibrium points of the reduced system were shown to represent self-similar cosmological models; provided $\rho + 3p \geq 0$. Also, it was shown that if the spacetime is transitively self-similar then the equations of state for the pressure $\bar{p}$ and anisotropic stress $\pi_{\alpha\beta}$ are homogeneous functions of degree two in the variables $(\theta, \sigma_{\alpha\beta}, n_{\alpha\beta}, a_\alpha)$, whence the resulting equations of state in the new variables (2.16) are 'dimensionless'.

In the literature on cosmological models, the most utilized equation of state for the thermodynamic pressure p is the barotropic γ-law equation of state

$$p = (\gamma - 1)\rho. \tag{2.67}$$

For the spatially homogeneous models, the energy density defined by the Friedmann equation (2.14) is a homogeneous function of degree two. Thus, employing the theorem in section 2.3, self-similar asymptotic solutions are to be expected for $\gamma \gtrsim 2/3$ (this is the condition for $\rho + 3p \geq 0$). Hence, as is most common in the literature, if a γ-law equation of state is assumed in the spatially homogeneous perfect fluid models, then the asymptotic solutions are generally going to be self-similar.

Self-similarity is not only found in perfect fluid models, but it is also a feature of viscous fluid models. In both the Eckart and the Israel-Stewart theories of irreversible thermodynamics, it was shown that if the the equations of state are dimensionless and if the matter satisfies $\rho + 3\bar{p} \geq 0$ then the equilibrium points of the system describing the model represent self-similar cosmological models. It was also shown that if the equations of state are dimensionless then asymptotically they had the form $p \propto \rho$, $\eta \propto \rho^{1/2}$, $\kappa \propto \rho^{1/2}$, $\zeta \propto \rho^{1/2}$, $\beta_i \propto \rho^{-1}$, and $\alpha_j \propto \rho^{-1}$, where $(i = 0, 1, 2)$ and $(j = 0, 1)$.

This result is helpful in determining appropriate equations of state since primarily they are a priori assumptions. Thus, we see that dimensionless equations of state lead naturally to the above asymptotic forms and therefore to self-similar solutions. (See Chapters 3, 4, 5.)

The Einstein field equations containing an imperfect fluid and a scalar field were also investigated. It was shown that in the case of a scalar field with a non-exponential potential, the field equations in general did not admit a symmetry that allowed one to define new dimensionless variables. However, in the case of a massless scalar field or an exponential potential the field equations admitted a symmetry that did allow new variables to be defined. The dynamical system in the new variables led to one equation decoupling. The equilibrium points of the reduced dynamical system were shown to be transitively self-similar if the condition $\rho + 3\bar{p} \geq 0$ is satisfied. (See Chapter 6.)

The asymptotic behaviour of a class of scalar-tensor theories of gravity was also analyzed. If the action is of the form (2.58) where f and V are arbitrary functions of ϕ [66], then the asymptotic states will not, in general, be self-similar. However, if $V(\phi) \equiv 0$ (this case includes the Brans-Dicke theory of gravity), then for isotropic and spatially homogeneous perfect fluid models, a symmetry exists. With a change of variables and provided the equations of state for $p = (\gamma - 1)\rho$, the asymptotic limits can be shown to be self-similar independent of the form of $f(\phi)$.

It should be stated clearly that asymptotic self-similarity is not a generic property of cosmological models. For instance, with the existence of a scalar field with a non-zero potential, a cosmological constant, or non-'dimensionless' equations of state, the self-similarity may be broken. (See Chapter 7.)

In closing, we note that the dynamical system (2.7, 2.9, 2.10, 2.11) admits a symmetry, and this symmetry allows one to define new variables. However, the choice of variables (2.16) made here is by no means the only choice. Any one of the original dynamical variables may be chosen so that it may decouple from the rest. This sort of analysis may be extended to other cosmological models, in particular the orthogonal

G_2 cosmological models [59] and the tilting (non-orthogonal) G_3 models [67].

Chapter 3

Bianchi type V Imperfect Fluid Cosmological Models

3.1 Introduction

The purpose of this chapter and the following chapters 4 and 5 is to further extend the analysis in [12, 13, 14, 15, 16, 17, 18, 19] which analyzed spatially homogeneous perfect fluid models to imperfect fluid models. In particular, we shall investigate the effects of viscosity and heat conduction in the spatially homogeneous and anisotropic Bianchi type V model. Bianchi type V models are of interest because they are generalizations of the negative curvature FRW models and are sufficiently complex to allow viscous processes and heat conduction. In the following sections, the dissipative effects are described by Eckart's theory of irreversible thermodynamics [47]. The Einstein field equations are derived in section 3.2. Introducing dimensionless variables and assuming a set of dimensionless equations of state, the system describing the Bianchi type V viscous fluid cosmological models with heat conduction reduces to a two-dimensional system of autonomous ordinary differential equations. A complete qualitative analysis is done in section 3.3.2 and exact solutions and asymptotic behaviour is discussed in section 3.3.4.

3.2 The Bianchi type V Model

The diagonal form of the Bianchi type V metric is given by:

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 e^{2x} dy^2 + c(t)^2 e^{2x} dz^2. \tag{3.1}$$

The energy-momentum tensor considered in this work is due to an imperfect fluid that includes bulk viscosity, shear viscosity, and heat conduction, viz.,

$$T_{ab} = (\rho + \bar{p}) u_a u_b + \bar{p} g_{ab} + \pi_{ab} + q_a u_b + q_b u_a, \tag{3.2}$$

where u^a is the fluid 4-velocity (which will be assumed to be co-moving), ρ is the energy density, the quantity $\bar{p}$ is defined to be $\bar{p} = p + \Pi$, where p is the thermodynamic pressure, Π is the bulk viscous pressure, π_{ab} is the shear viscous stress, and q_a is the heat conduction vector such that $q_a u^a = 0$ (which implies for a co-moving fluid, along with the field equations, that the only non-zero component of q_a is q_1).

The Einstein field equations and the energy conservation equations in terms of the expansion(θ) and shear(σ) are:

$$\begin{aligned}
\dot{\theta} &= -2\sigma^2 - \frac{1}{3}\theta^2 - \frac{1}{2}(\rho + 3p + 3\Pi), && (3.3)\\
\dot{\rho} &= -\theta(\rho + p + \Pi) - \frac{2}{a^2} q_1 - \frac{1}{3}\Big(\sigma_1(2\Pi_1 - \Pi_2) + \sigma_2(2\Pi_2 - \Pi_1)\Big), && (3.4)\\
\dot{\sigma}_1 &= -\theta\sigma_1 + \Pi_1, && (3.5)\\
\dot{\sigma}_2 &= -\theta\sigma_2 + \Pi_2, && (3.6)
\end{aligned}$$

$$\begin{aligned}
\theta^2 &= 3\sigma^2 + 3\rho - \frac{3}{2}\,{}^3R, && (3.7)\\
q_1 &= -\sigma_1 - \sigma_2, && (3.8)
\end{aligned}$$

where $\sigma^2 = \frac{1}{3}(\sigma_1 + \sigma_2)^2 - \sigma_1\sigma_2$. We have used the property that both $\sigma^a{}_a = 0$ and $\pi^a{}_a = 0$ to define new shear variables $\sigma_1 = \sigma^1{}_1 - \sigma^2{}_2$ and $\sigma_2 = \sigma^1{}_1 - \sigma^3{}_3$ and new anisotropic stress variables $\Pi_1 = \pi^1{}_1 - \pi^2{}_2$ and $\Pi_2 = \pi^1{}_1 - \pi^3{}_3$ in an attempt to simplify the system.

Now the system of equations (3.3–3.8), is invariant under the mapping (see Chapter 2, equation (2.15)

$$\begin{array}{lll} \theta \to \lambda\theta, & \sigma_1 \to \lambda\sigma_1, & \sigma_2 \to \lambda\sigma_2, \\ p \to \lambda^2 p, & \rho \to \lambda^2\rho, & \Pi \to \lambda^2\Pi, \\ \Pi_1 \to \lambda^2\Pi_1, & \Pi_2 \to \lambda^2\Pi_2, & t \to \lambda^{-1}t. \end{array} \tag{3.9}$$

This invariance implies that there exists a symmetry [63] in the system and hence a change of variables such that one of the equations can be made to decouple from the system.

We define new dimensionless variables x, Σ_1, Σ_2, y, z_1, z_2 and a new time variable Ω as follows:

$$x \equiv \frac{3\rho}{\theta^2}, \qquad \Sigma_1 \equiv \frac{2\sqrt{3}\sigma_1}{\theta}, \qquad \Sigma_2 \equiv \frac{2\sqrt{3}\sigma_2}{\theta}, \qquad y \equiv \frac{9\Pi}{\theta^2},$$

$$z_1 \equiv \frac{\sqrt{3}\Pi_1}{2\theta^2}, \qquad z_2 \equiv \frac{\sqrt{3}\Pi_2}{2\theta^2}, \quad \text{and} \quad \frac{d\Omega}{dt} = -\frac{1}{3}\theta. \tag{3.10}$$

The variable x measures the dynamical importance of the matter content, Σ_1 and Σ_2 ($=\beta_1$ and β_2 resp. in Ref. [68]), measures the rate of shear (anisotropy) in terms of the expansion, and we define $\ell \equiv e^{\Omega}$, where where ℓ is the average length scale of the universe (i.e., $\theta = 3\frac{\dot{\ell}}{\ell}$).

The Einstein field equations (3.3–3.8) written in terms of the above dimensionless variables (3.10) are:

$$\begin{aligned} \frac{dx}{d\Omega} &= x(1-2q) + 9\frac{p}{\theta^2} + y + \Sigma_1(2z_1 - z_2) + \Sigma_2(2z_2 - z_1) \\ &\qquad -\frac{1}{4\sqrt{3}}(\Sigma_1 + \Sigma_2)(4 - 4x - \Sigma^2), \end{aligned} \tag{3.11}$$

$$\frac{d\Sigma_1}{d\Omega} = \Sigma_1(2-q) - 12z_1, \tag{3.12}$$

$$\frac{d\Sigma_2}{d\Omega} = \Sigma_2(2-q) - 12z_2, \tag{3.13}$$

$$\frac{q_1}{\theta} = -\frac{1}{2\sqrt{3}}(\Sigma_1 + \Sigma_2) \tag{3.14}$$

where $\Sigma^2 \equiv \frac{1}{3}(\Sigma_1 + \Sigma_2)^2 - \Sigma_1\Sigma_2$. The quantity q is the generalized dimensionless deceleration parameter given by

$$q \equiv \frac{-\ddot{\ell}\,\ell}{\dot{\ell}^2} = \frac{1}{2}\left(x + y + 9\frac{p}{\theta^2} + \Sigma^2\right). \tag{3.15}$$

Finally, from the Friedmann equation, (3.7), we obtain the dimensionless Friedmann equation

$$4 - 4x - \Sigma^2 = -6\,{}^3R\theta^{-2}, \tag{3.16}$$

where, in the Bianchi type V model studied here ${}^3R = -6a(t)^{-2}$, which results in the following inequality

$$4 - 4x - \Sigma^2 = \frac{36}{a^2\theta^2} \geq 0, \tag{3.17}$$

The interior of the parabola $4 = \Sigma^2 + 4x$ in the phase space represents models of Bianchi type V, while the parabola itself represents models of Bianchi type I. There are other physical constraints that may be imposed, namely the energy conditions [65], which will place bounds on the variables x, Σ_1, Σ_2, y, z_1, and z_2. A full list of the energy conditions is given in Appendix C. In the present work we shall always assume that $x \geq 0$, which states that the energy density be non-negative, which is a necessary condition of the weak energy condition (WEC) [69].

In order to complete the system of equations (3.11–3.13) describing an anisotropic viscous cosmological model, we require equations for the dimensionless bulk viscous pressure y and the dimensionless anisotropic stresses z_1 and z_2 (or equivalently, we require equations for the viscous pressure Π and the anisotropic stresses π_1 and π_2). In the next section and in the following chapters we will analyze various scenarios concerning viscous fluid cosmological models in which the equations for the bulk viscous pressure Π and the anisotropic stress π_1 and π_2 will be described by the Eckart theory (section 3.3), the truncated Israel-Stewart theory (Chapter 4), and the full Israel-Stewart theory, (Chapter 5).

3.3 Eckart Theory

3.3.1 The Equations

We will assume in this section that the bulk viscous pressure and the shear viscous stress can be approximated by the Eckart relations [47]

$$\Pi = -\zeta\theta, \qquad \pi_{ab} = -2\eta\sigma_{ab}, \tag{3.18}$$

where ζ is the bulk viscosity coefficient and η is the shear viscosity coefficient. These relationships are expected to be satisfied near equilibrium. Written in terms of the dimensionless variables (3.10), equation (3.18) yields algebraic equations for y, z_1, and z_2, of the form

$$y = -9\left(\frac{\zeta}{\theta}\right), \qquad z_1 = -\frac{1}{2}\left(\frac{\eta}{\theta}\right)\Sigma_1, \qquad z_2 = -\frac{1}{2}\left(\frac{\eta}{\theta}\right)\Sigma_2, \tag{3.19}$$

which can be directly substituted into the system (3.11–3.13). Therefore, we require equations of state for the thermodynamic pressure p and the transport coefficients ζ and η.

There are very few known equations of state[1], hence, they must be put forward as assumptions a priori. In the perfect fluid case with gamma-law equation of state for the pressure p, the asymptotic limit points of the Einstein field equations are represented by known self-similar cosmological models. We shall require that the same is true for the imperfect fluid models. Therefore, employing the conclusions reached in Chapter 2, dimensionless equations of state should to be introduced. To conform with previous usage of dimensionless equations of state, we follow the lead set forth by Coley [64, 70] and assume:

$$\begin{aligned} \frac{p}{\theta^2} &= p_o x^\ell, \\ \frac{\zeta}{\theta} &= \zeta_o x^m, \\ \frac{\eta}{\theta} &= \eta_o x^n, \end{aligned} \tag{3.20}$$

[1] E.g., the equations of state are known for a Boltzmann gas.

where p_o, ζ_o, and η_o are positive constants, and ℓ, m, and n are constant parameters (x is the dimensionless density parameter defined earlier). In the models under consideration, θ is strictly positive, thus equations (3.20) are well defined. Equations of state (3.20) are phenomenological in nature and are no less appropriate than the equations of state used by Belinskii et al. [31, 44]. The most commonly used equation of state for the pressure is the barotropic equation of state $p = (\gamma - 1)\rho$, hence $p_o = \frac{1}{3}(\gamma - 1)$ and $\ell = 1$ (where $1 \leq \gamma \leq 2$ is necessary for local mechanical stability and for the speed of sound in the fluid to be no greater than the speed of light). Employing these dimensionless equations of state (3.20) and substituting equation (3.19) into the system (3.11–3.13), a three dimensional system of autonomous ordinary differential equations results:

$$\frac{dx}{d\Omega} = x\Big((3\gamma - 2)(1 - x) - \Sigma^2\Big) - 9\zeta_o x^m (1 - x) - 3\eta_o x^n \Sigma^2 \qquad \qquad -\frac{\Sigma_1 + \Sigma_2}{4\sqrt{3}}(4 - 4x - \Sigma^2), \tag{3.21}$$

$$\frac{d\Sigma_1}{d\Omega} = -\frac{\Sigma_1}{2}\Big((3\gamma - 2)x - 4 - 9\zeta_o x^m - 12\eta_o x^n + \Sigma^2\Big), \tag{3.22}$$

$$\frac{d\Sigma_2}{d\Omega} = -\frac{\Sigma_2}{2}\Big((3\gamma - 2)x - 4 - 9\zeta_o x^m - 12\eta_o x^n + \Sigma^2\Big), \tag{3.23}$$

where the physical phase space $\Re$ is defined to be

$$4 \geq \Sigma^2 + 4x, \tag{3.24}$$

$$x \geq 0, \tag{3.25}$$

where $\Sigma^2 \equiv \frac{1}{3}(\Sigma_1 + \Sigma_2)^2 - \Sigma_1\Sigma_2$. Equations (3.22) and (3.23) can be integrated to obtain $\Sigma_1 = k\Sigma_2$ for any constant k. The shear squared Σ^2 can now be written as

$$\Sigma^2 \equiv \frac{1}{3}(k^2 - k + 1){\Sigma_2}^2 \equiv \frac{k^2 - k + 1}{3k^2}{\Sigma_1}^2. \tag{3.26}$$

In this way we have reduced the three dimensional system (x, Σ_1, Σ_2) to a one parameter family of two dimensional systems in variables (x, Σ). Each value of k represents a different surface in the 3-dimensional phase-space. As k ranges from $-\infty$ to ∞,

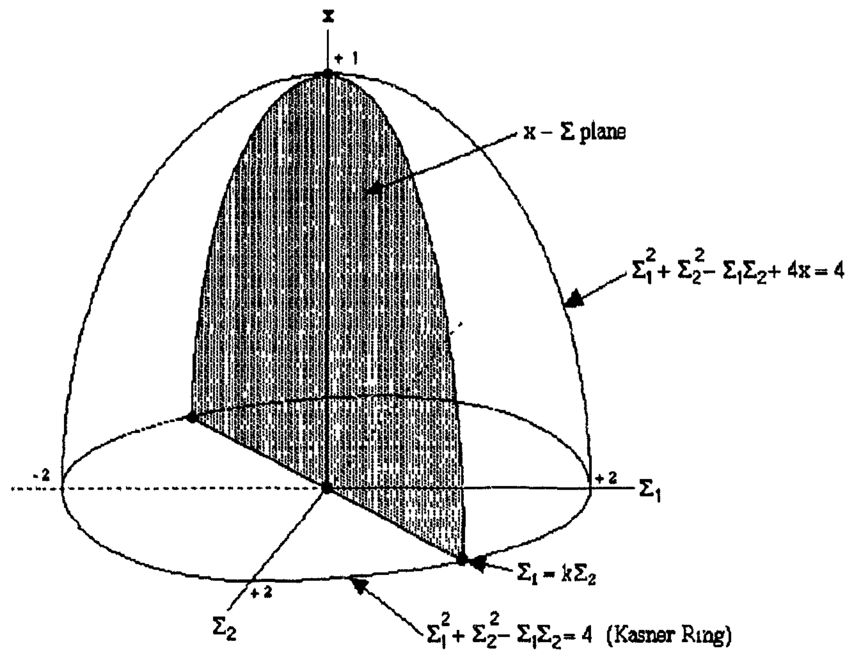


Figure 3.1: *The interior of the three-dimensional paraboloid represents the physical phase space. The two-dimensional shaded parabola represents a particular $x - \Sigma$ plane for a unique value of the parameter k. As we let k range through $-\infty$ to ∞, the two-dimensional $x - \Sigma$ plane will rotate around and cover the entire three-dimensional phase space. For each value of k the plane intersects the Kasner ring of equilibrium points at two points, allowing us to isolate the equilibrium points.*

the one parameter family of 2-dimensional surfaces will rotate around the entire 3-dimensional phase space. Hence the 3-dimensional phase portrait is the union of all the 2-dimensional phase portraits. See Figure 3.1.

In order to simplify the analysis, we define the new parameter

$$C = \frac{k+1}{\sqrt{k^2 - k + 1}} \,. \tag{3.27}$$

The parameter C then ranges between -1 and 2 (see reference [68]). When $C = 0$ we have no heat conduction and thus we have a imperfect fluid with viscosity (see Burd and Coley [55]), and when $C = 2$, we have the system analyzed by Coley and Dunn [54]. By choosing various values of the constants ζ_o, η_o, m, n, and C, the complete qualitative structure of the system in question was determined in my Master's Thesis [68].

The qualitative behaviour of the Bianchi Type V imperfect fluid cosmological models was analyzed thoroughly in [68] (That is, eigenvalues, eigenvectors, and phase portraits are given in [68]). The sections below containing the qualitative analysis and the phase portraits are shortened adaptations of the reults in [68]. These results are then used and expanded upon in the following sections.

3.3.2 Qualitative Analysis

The system we are analyzing which describes a viscous fluid Bianchi type V cosmological model satisfying the Eckart theory of irreversible thermodynamics is:

$$\begin{aligned}
\frac{dx}{d\Omega} &= x\Big((3\gamma-2)(1-x)-\Sigma^2\Big)-9\zeta_o x^m(1-x)-3\eta_o x^n\Sigma^2 \\
&\qquad\qquad -\frac{C}{4}\Sigma(4-4x-\Sigma^2), \\
\frac{d\Sigma}{d\Omega} &= -\frac{\Sigma}{2}\Big((3\gamma-2)x-4-9\zeta_o x^m-12\eta_o x^n+\Sigma^2\Big). \qquad (3.28)
\end{aligned}$$

Information about the stability and other properties of the equilibrium points is summarized in Tables (3.1), (3.2), and (3.3) and some appropriate phase portraits are given in the figures. In the following analysis the order of the coordinates is (x, Σ).

$\mathbf{m = n > 1}$

If $\gamma \neq 2$, the point $(0, 2)$ is generally a stable two-tangent node, unless $C = (3\gamma-2)/2$ in which case the point degenerates to a one-tangent node. When $C = 2$ the point is degenerate but the single sector in $\Re$ is found to be hyperbolic in nature. Finally, if $\gamma = 2$ the point behaves like a stable node. (See also Table A in Ref. [68].)

The point $(0, -2)$ (for $\gamma \neq 2$), is generally a stable two-tangent node, unless $C = -(3\gamma-2)/2$, in which case the point degenerates to a one-tangent node. There is also a degenerate case when $\gamma = 2$ in which case the point behaves like a stable node.

The point $(0,0)$ is generally an unstable two-tangent node unless $\gamma = 4/3$, in which case the point degenerates to a one-tangent node if $C \neq 0$ and a stellar node for $C = 0$.

The point $(1,0)$ will take on a variety of different natures depending on the sign of $\Psi_1 = 9\zeta_o - (3\gamma - 2)$. If $\Psi_1 < 0$ the point is a saddle point. When $\Psi_1 = 0$ the point (1,0) is degenerate, but with a change to polar coordinates we find that the point is saddle-like. When $\Psi_1 > 0$ the point is generally an unstable two-tangent node unless $\Psi_2 = 9\zeta_o - (3\gamma - 2) - 4 - 12\eta_o = 0$, in which case the point degenerates to a stellar node.

When $\Psi_1 > 0$ we have a fifth equilibrium point $(\bar{x}, 0)$ where $\bar{x} = (\frac{9\zeta_o}{3\gamma-2})^{\frac{1}{1-m}}$. The point $(\bar{x}, 0)$ is found to be a saddle point.

m = n = 1

The equilibrium point $(0,2)$ is generally a stable two-tangent node, unless $\Psi_3 + C = 0$ whence the point degenerates to a one-tangent node [where $\Psi_3 = \frac{1}{2}(9\zeta_o - (3\gamma - 2) + 12\eta_o)$]. There also exists a degenerate case when $C = 2$ which is found to be hyperbolic in $\Re$. (See also Table B in Ref. [68].)

The equilibrium point $(0,-2)$ is generally a stable two-tangent node, unless $\Psi_3 - C = 0$ whence the point degenerates to a one-tangent node. There also exists a degenerate case when $C = 2$ which is found to be hyperbolic in $\Re$.

The point $(0,0)$ has a variety of different natures depending on the sign of Ψ_1. If $\Psi_1 > 0$, the point is a saddle point. When $\Psi_1 = 0$ the point $(0,0)$ actually becomes an equilibrium point on a non-isolated line of equilibria, $\Sigma = 0$, and will be discussed later. When $\Psi_1 < 0$, the point $(0,0)$ is generally an unstable two-tangent node unless $\Psi_4 = 9\zeta_o - (3\gamma - 2) + 2 = 0$ in which case the point degenerates to a one-tangent node for $C \neq 0$ and is a stellar node for $C = 0$.

The point $(1,0)$, in this case, has the same qualitative behavior as in the case $m=n>1$ except in the degenerate case when $\Psi_1=0$ where it becomes part of the line of equilibrium points.

In the case when $\Psi_1=0$, we have a non-isolated line of equilibria $(x,0)$, where $0\leq x\leq 1$. The line is an attractor and the slope of the trajectories as $\Sigma\to 0$ is

$$\lim_{\Sigma\to 0}\frac{d\Sigma}{dx}=\frac{-2}{C}(1+3\eta_o)(1-x)^{-1}.$$

If $C<0$, the slope of the trajectories as $\Sigma\to 0$ is always positive, if $C=0$ the slope of the trajectories becomes infinite and the trajectories cross the line at right angles, and if $C>0$ the slope of the trajectories is negative.

m = n = 1/2

In this case, there are at most five equilibrium points in $\Re$. For the points $(0,0)$, $(0,2)$, and $(0,-2)$, the system becomes non-analytic. By transforming to the variable u and time coordinate τ ($u^2=x$; $\frac{d\Omega}{d\tau}=u$), these points can be analyzed using analytic methods. All three points become degenerate, and by a change to polar coordinates, the qualitative behaviour of the equilibrium points is determined.

The point (0,0) has invariant rays $\theta=0$ and $\theta=\theta^*$ where $\tan\theta^*=\frac{-9\zeta_o}{C}$. We find from the analysis that $dr/d\tau<0$ along the invariant ray $\theta=0$, and $dr/d\tau>0$ along the invariant ray $\theta=\theta^*$, thus each sector is hyperbolic. (See also Table C in Ref. [68].)

The point $(0,2)$ has invariant rays $\theta=0$ and $\theta=\theta^*$ where $\tan\theta^*=\frac{9\zeta_o+12\eta_o}{2C}$. The region $\Re$ in the new coordinates is now bounded by $(\Sigma-2)(\Sigma+2)+4u^2=4$, so the invariant ray $\theta=0$ corresponds to the trajectory along the boundary. If $C>0$, the single sector in $\Re$ is hyperbolic. If $C=0$, then $\theta^*=-\pi/2$, which corresponds to the $u=x=0$ boundary where $dr/d\tau<0$; hence the trajectories are attracted to the point along the eigendirection $x=0$. If $C<0$, then $\Re$ is divided into two sectors. One can show that $dr/dt<0$ along the invariant ray $\theta=\theta^*$. The trajectories are attracted to the point along the eigendirection corresponding to the invariant ray $\theta = \theta^*$.

The point $(0,-2)$ has invariant rays $\theta = 0$ and $\theta = \theta^*$ where $\tan\theta^* = \frac{9\zeta_o+12\eta_o}{2C}$. If $C < 0$, the single sector in $\Re$ is hyperbolic. If $C = 0$, then $\theta^* = -\pi/2$ which corresponds to the $u = x = 0$ boundary where $dr/d\tau < 0$; hence the trajectories are attracted to the point along the eigendirection $x = 0$. If $C > 0$, then $\Re$ is divided into two sectors. One can show that $dr/d\tau < 0$ along the invariant ray $\theta = \theta^*$. The trajectories are attracted to the point along the eigendirection corresponding to the invariant ray $\theta = \theta^*$.

The point $(1,0)$ has the same character as it did in the previous two cases except in the degenerate case when $\Psi_1 = 0$. In the degenerate case, changing to polar coordinates, the two sectors in $\Re$ are found to be parabolic in nature, hence, the point behaves like an unstable node.

When $\Psi_1 < 0$ there is a fifth equilibrium point $(\bar{x},0)$, where $\bar{x} = (\frac{9\zeta_o}{3\gamma-2})^2$. The point $(\bar{x},0)$ is an unstable two-tangent node, with the main eigendirection along the x-axis.

m = n = 0

In this case there are two separate situations depending upon whether $\zeta_o = 0$ or $\zeta_o \neq 0$. If $\zeta_o = 0$, there are two equilibrium points, $(0,0)$ and $(1,0)$. The point $(1,0)$ is a saddle point. The point $(0,0)$ is generally an unstable two-tangent node, unless $\Psi_5 = (3\gamma-2) - 2 - 6\eta_o = 0$, whence the point degenerates to a one-tangent node for $C \neq 0$ and to a stellar node for $C = 0$. (See also Table D in Ref. [68].)

If $\zeta_o \neq 0$, we have at most two equilibrium points depending on the sign of Ψ_1. If $\Psi_1 < 0$, there are two equilibrium points, $(1,0)$ and $(\bar{x},0)$, where $\bar{x} = \frac{9\zeta_o}{3\gamma-2}$. In this case, the point $(1,0)$ is a saddle point. The point $(\bar{x},0)$ is generally an unstable two-tangent node, unless $\Psi_6 = 9\zeta_o - (3\gamma-2) + 2 + 6\eta_o = 0$, whence the point degenerates to an unstable one-tangent node for $C \neq 0$, and to a stellar node for $C = 0$. If

$\Psi_1 = 0$, we have only one equilibrium point. The point $(1,0)$ becomes degenerate, but by changing to polar coordinates and using higher order terms in the variable r, we find that the point acts like an unstable node. If $\Psi_1 > 0$, the point $(1,0)$ is again the only equilibrium point where the qualitative behavior is the same as in the previous cases for $\Psi_1 > 0$.

$\zeta_o = \eta_o = 0$

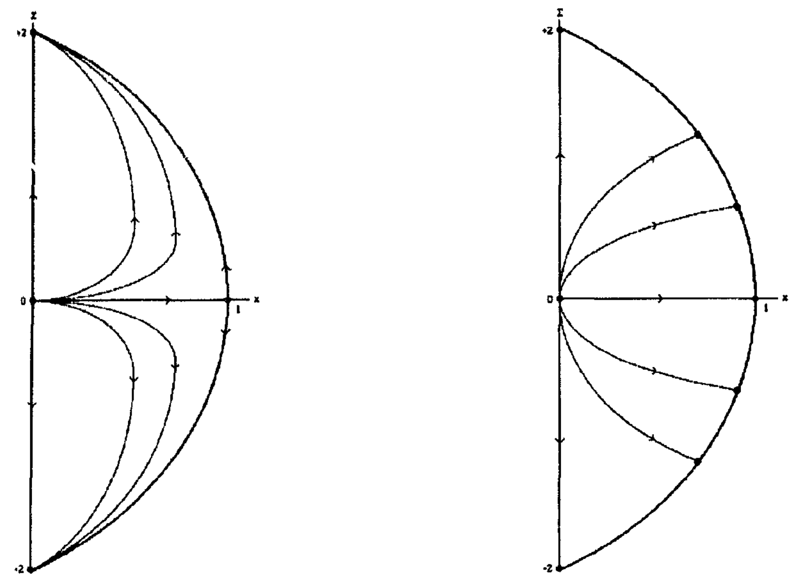
For $\gamma \neq 2$ there are four equilibrium points in $\Re$. The points $(0,2)$, $(0,-2)$, and $(0,0)$ behave in the same manner as the points in the case $m = n > 1$ for $\gamma \neq 2$, and will not be summarized here. The point $(1,0)$ is a saddle when $\gamma \neq 2$. (See also Table E in Ref. [68].)

However, in the case $\gamma = 2$ every point on the boundary $\Sigma^2 + 4x = 4$ becomes a non-isolated line of equilibria. The system of equations can be solved explicitly when $\gamma = 2$. The solution is given by the line $\Sigma = 0$ and the family of parabolas $A\Sigma^2 - C\Sigma + 2x = 0$ $(\Sigma \neq 0)$, where A is an arbitrary constant depending on initial conditions.

3.3.3 Phase Portraits

In the case of the perfect fluid Bianchi type V model with $C = 0$ and $\zeta_o = \eta_o = 0$, we note that all trajectories remain in $\Re$ for all time. The models evolve from the Kasner singularities at $(0,2)$ and $(0,-2)$ towards the Milne model at $(0,0)$. In this case there also exists exceptional trajectories; there are two trajectories along the boundary of $\Re$ that evolve from the Kasner points towards the FRW model at $(1,0)$ (these represent Bianchi I perfect fluid models), and one trajectory that evolves from the matter dominated FRW model at $(1,0)$ towards the Milne model at $(0,0)$ (See Figures 3.2).

In the case of the imperfect fluid Bianchi type V model with viscosity and zero

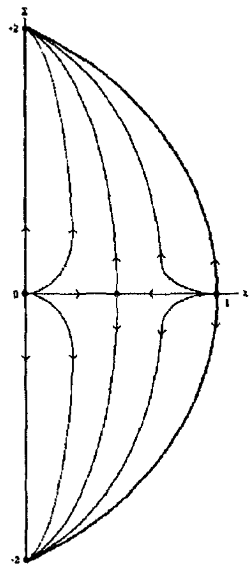


$(a)\ C = 0,\ \zeta_o = \eta_o = 0,\ 1 \leq \gamma < 2$ $(b)\ C = 0,\ \zeta_o = \eta_o = 0,\ \gamma = 2$

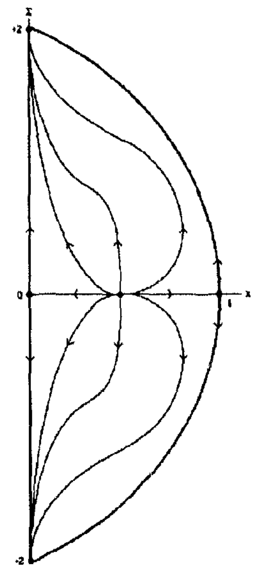
Figure 3.2: *The phase portraits describe the behavior of the perfect fluid Bianchi type V models with no heat conduction or viscosity in the case* $\zeta_o = \eta_o = 0$ *and* $C = 0$. *In all figures, the arrows refer to increasing* Ω*-time or decreasing t-time.*

heat conduction ($C = 0$), there are two cases depending upon whether $m = n = 0$ or not. If $m = n \neq 0$, equation (3.28) implies that $x = 0$ is an invariant set, hence no trajectories can cross the Σ-axis. Therefore, all trajectories remain in $\Re$ for all time. The behavior of the phase portraits depends critically on the sign of Ψ_1 as well as the parameters m and n. The models evolve from the Kasner singularities at $(0,2)$ and $(0,-2)$ towards one of the isotropic models either at $(0,0)$, $(\bar{x},0)$ or $(1,0)$ depending on the sign of Ψ_1. There exists exceptional trajectories emitting from $(0,2)$ and $(0,-2)$ towards the FRW models either at $(1,0)$ or $(\bar{x},0)$. There also exists exceptional trajectories on the x-axis that remain on the axis for all time (see Figures 3.2a, 3.3).

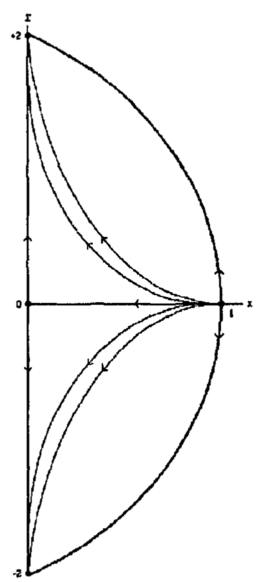
In the case $m = n = 0$ and $C = 0$, depending on the value of Ψ_1, we find in all cases that the models start at some finite time t_o and evolve towards one of the



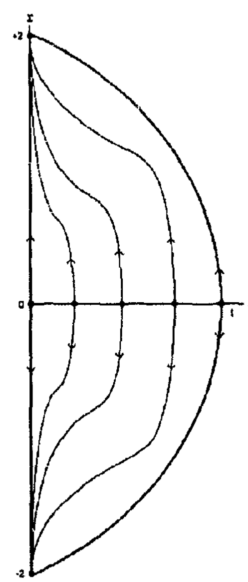
(*a*) $C = 0,\ m = n > 1,\ \Psi_1 > 0$



(*b*) $C = 0,\ m = n = \frac{1}{2},\ \Psi_1 < 0$

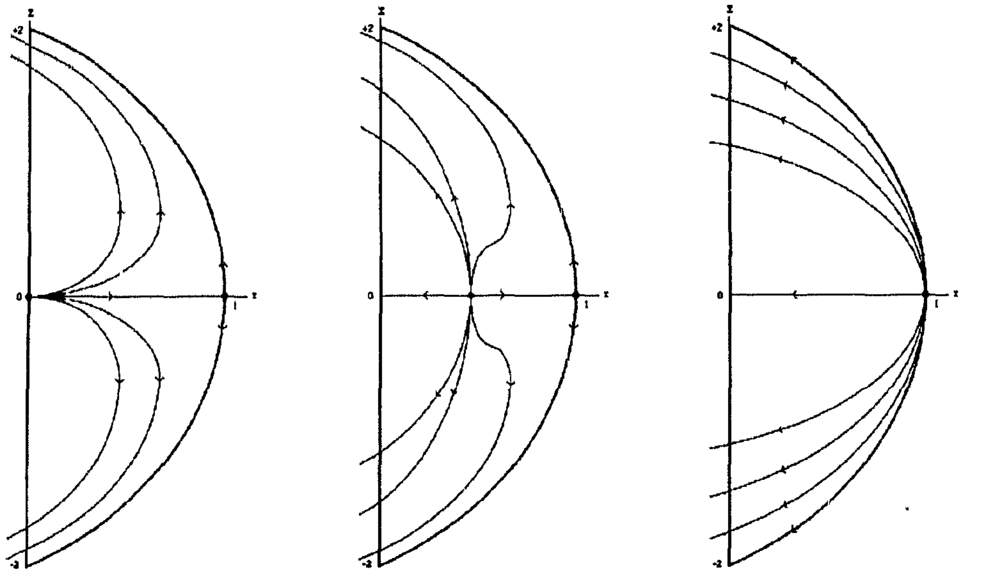


(c) $C = 0,\ m = n = 1,\ \Psi_1 > 0$



(*d*) $C = 0,\ m = n = 1,\ \Psi_1 = 0$

Figure 3.3: *The phase portraits describe the behavior of the Bianchi type V models with viscosity and no heat conduction in the case* $m = n \neq 0$ *and* $C = 0$.



(a) $C = 0,\ m = n = 0,$ $\zeta_o = 0,\ \Psi_1 < 0$

(b) $C = 0,\ m = n = 0,$ $\zeta_o \neq 0,\ \Psi_1 < 0$

(c) $C = 0,\ m = n = 0,$ $\zeta_o \neq 0,\ \Psi_1 \geq 0$

Figure 3.4: *The phase portraits describe the behavior of the Bianchi type V models with viscosity and no heat conduction in the case $m = n = 0$ and $C = 0$.*

isotropic models. Note that the initial Big Bang singularity is avoided in this case (see Figures 3.4).

With the introduction of heat conduction, $x = 0$ is no longer an invariant set, hence trajectories may leave $\Re$, and consequently the WEC is violated. Equations (3.28) are invariant under the transformation $\Sigma \rightarrow -\Sigma$, $C \rightarrow -C$. The phase portraits for $C > 0$ are reflections over the x-axis of the phase portraits for $C < 0$. Therefore, in the remainder of the analysis only the case $C < 0$ will be considered. The specific case $C = 2$ is done by Coley and Dunn [54] (see Figures 2 in Coley and Dunn [54]).

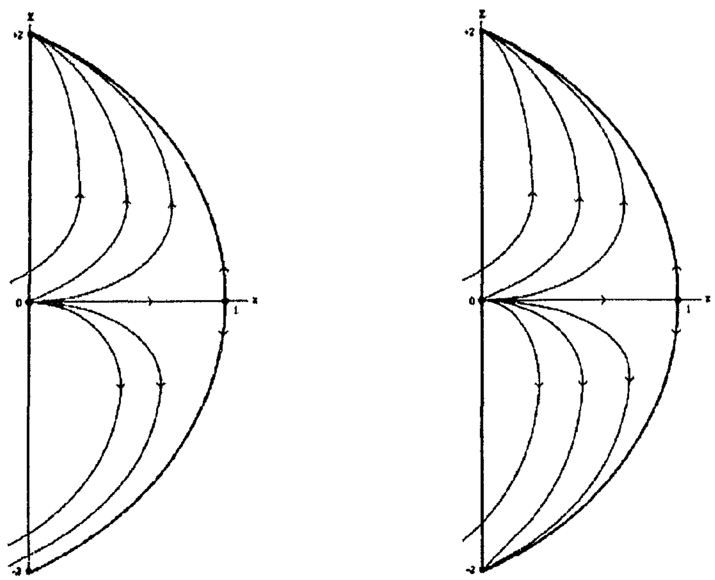
Let us investigate what happens when we have a perfect fluid with heat conduction (i.e., no viscosity). Assuming that the WEC is satisfied for all time, for $1 \leq \gamma < \frac{4}{3}$, the positive Σ quadrant has the same qualitative behavior as in the perfect fluid case. However, when $\gamma \geq \frac{4}{3}$ the WEC is violated for all trajectories at some finite time (except for those exceptional trajectories which are qualitatively the same as in the perfect fluid case). In the negative Σ quadrant, for $C \leq \frac{-(3\gamma-2)}{2}$, all trajectories violate the WEC at some finite time (except again for those exceptional trajectories) and for $C > \frac{-(3\gamma-2)}{2}$, the qualitative behavior is the same as in the perfect fluid case (see Figures 3.5).

Let us consider an imperfect fluid with both viscosity and heat conduction. In the case $m = n = 0$, all trajectories violate the WEC at some finite time and can only describe late time asymptotic behavior, we find in general that the qualitative behavior is the same as if we had viscosity and no heat (see Figures 3.4). But when $\zeta_o = 0$ and $\Psi_5 \geq 0$ there is a slight difference in behaviour; in the positive Σ quadrant all trajectories violate the WEC, while the negative quadrant is the same as if we had no heat (see Figure 3.6).

When $m = n = 1/2$, there are two different phase portraits depending on the sign of Ψ_1. If $\Psi_1 < 0$ and $C < 0$ there is a fifth equilibrium point at $(\bar{x}, 0)$. The positive Σ quadrant is the same as in the case where we just had viscosity (Figure 3.4), but in the negative Σ quadrant all trajectories violate the WEC at some finite time (see Figure 3.7a). If $\Psi_1 \geq 0$ and $C < 0$, the positive Σ quadrant is the same as in the case where there was just viscosity (Figure 3.3c), but in the negative Σ quadrant all trajectories violate the WEC at some finite time (see Figure 3.7b).

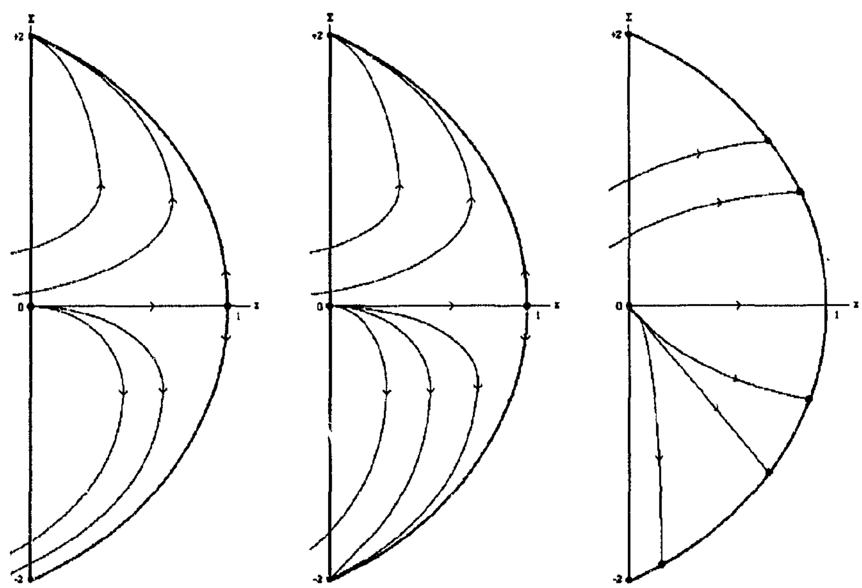
In the case $m = n = 1$, only the degenerate case when $\Psi_1 = 0$ is qualitatively different (to those already discussed). In this case, for $C < 0$, the positive Σ quadrant is similar to that with just viscosity (Figure 3.3d) but trajectories in the negative Σ quadrant will violate the WEC at some finite time (see Figure 3.8).

In the case $m = n > 1$, $C < 0$ and $\Psi_1 \leq 0$, the qualitative behavior is the same as that for other cases (see Figures 3.2a, 3.5 (a-d)). However, in the case $\Psi_1 > 0$, there are different possibilities. Again there exists a fifth equilibrium point. When $1 \leq \gamma < 4/3$ and $C < 0$, the positive Σ quadrant is similar to the case when no heat was present (Figure 3.2a). In the negative Σ quadrant, however, some or all trajectories will violate the WEC at some finite time (see Figures 3.9a, 3.9b). When



(a) $C \leq -\frac{3\gamma-2}{2}$, $\zeta_o = \eta_o = 0$, $1 \leq \gamma < \frac{4}{3}$

(b) $-\frac{3\gamma-2}{2} < C < 0$, $\zeta_o = \eta_o = 0$, $1 \leq \gamma < \frac{4}{3}$

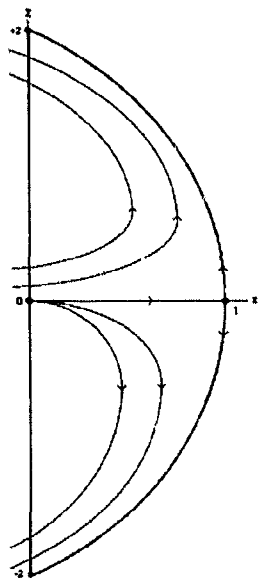


(c) $C = -1$, $\zeta_o = \eta_o = 0$, $\gamma = \frac{4}{3}$

(d) $-1 < C < 0$, $\zeta_o = \eta_o = 0$, $\frac{4}{3} \leq \gamma < 2$

(e) $C < 0$, $\zeta_o = \eta_o = 0$, $\gamma = 2$

Figure 3.5: *The phase portraits describe the behavior of the Bianchi type V models with heat conduction and no viscosity in the case* $\zeta_o = \eta_o = 0$ *and* $C \neq 0$.



(a) $C < 0,\ m = n = 0,\ \zeta_o = 0,\ \Psi_5 \geq 0$

Figure 3.6: *The phase portrait describes the behavior of the Bianchi type V models with heat conduction and viscosity in the case $m = n = 0$ and $C < 0$.*

$\gamma < 4/3$ and $C < 0$ in the negative Σ quadrant some or all trajectories will violate the WEC at some finite time, while in the positive Σ quadrant the only physically realistic models evolve from the point $(0, 2)$ towards $(1, 0)$ whence the SEC is violated (see Figures 3.9c, 3.9d).

3.3.4 Asymptotic Behaviour: Exact Solutions

Equation (3.28) implies that there exists three invariant sets $\Re_- = \{(x, \Sigma)|\Sigma < 0\}$, $\Re_o = \{(x, \Sigma)|\Sigma = 0\}$ and $\Re_+ = \{(x, \Sigma)|\Sigma > 0\}$. We shall discuss what happens in each invariant set and determine the exact solutions corresponding to each equilibrium point. We will also show that almost all of these solutions represent space-times which are transitively self-similar (that is, there exists a homothetic vector field in addition to the three Killing vector fields).

Table 3.1: Stability of the equilibrium points for different values of the parameters (with respect to Ω-time).

			$(0,0)$	$(\bar{x},0)$	$(1,0)$	$(0,2)$	$(0,-2)$
$m=n>1$	$\Psi_1>0$	$\gamma\neq 2$	$R-N_2$[a]	S	$R-N_2$[b]	$A-N_2$[c d]	$A-N_2$[e]
		$\gamma=2$	$R-N_2$	S	$R-N_2$[b]	$A-N^*$[d]	$A-N^*$
	$\Psi_1=0$	$\gamma\neq 2$	$R-N_2$[a]		S^*	$A-N_2$[c d]	$A-N_2$[e]
		$\gamma=2$	$R-N_2$		S^*	$A-N^*$[d]	$A-N^*$
	$\Psi_1<0$	$\gamma\neq 2$	$R-N_2$[a]		S	$A-N_2$[c d]	$A-N_2$[e]
		$\gamma=2$	$R-N_2$		S	$A-N^*$[d]	$A-N^*$
$m=n=1$	$\Psi_1>0$		S		$R-N_2$[b]	$A-N_2$[f d]	$A-N_2$[g]
	$\Psi_1=0$		$***$	$***$	$***$	$A-N_2$[f d]	$A-N_2$[g]
	$\Psi_1<0$		$R-N_2$[h]		S	$A-N_2$[f d]	$A-N_2$[g]
$m=n=1/2$	$\Psi_1>0$		S^*		$R-N_2$[b]	$A-N^*$[i]	$A-N^*$[j]
	$\Psi_1=0$		S^*		$R-N^*$	$A-N^*$[i]	$A-N^*$[j]
	$\Psi_1<0$		S^*	$R-N_2$	S	$A-N^*$[i]	$A-N^*$[j]
$m=n=0$		$\zeta_o=0$	$R-N_2$[k l]		S		
	$\Psi_1>0$	$\zeta_o\neq 0$			$R-N_2$[b]		
	$\Psi_1=0$	$\zeta_o\neq 0$			$R-N^*$		
	$\Psi_1<0$	$\zeta_o\neq 0$		$R-N_2$[m]	S		
$\zeta_o=\eta_o=0$		$\gamma\neq 2$	$R-N_2$[a]		S	$A-N_2$[c d]	$A-N_2$[e]
		$\gamma=2$	$R-N_2$		$***$	$***$	$***$

[a] If $\gamma=4/3$ for $C\neq 0$ the point becomes a $R-N_1$, and for $C=0$ the point becomes a $R-SN$.

[b] If $\Psi_2=0$ the point becomes a $R-SN$.

[c] If $C=(3\gamma-2)/2$ the point becomes a $A-N_1$.

[d] If $C=2$ the point becomes a S^*.

[e] If $C=-(3\gamma-2)/2$ the point becomes a $A-N_1$.

[f] If $C=-\Psi_3$ the point becomes a $A-N_1$.

[g] If $C=\Psi_3$ the point becomes a $A-N_1$.

[h] If $\Psi_4=0$ for $C\neq 0$ the point becomes a $R-N_1$, and for $C=0$ the point becomes a $R-SN$.

[i] If $C>0$ the point becomes a S^*.

[j] If $C<0$ the point becomes a S^*.

[k] If $C\neq 0$ and $\Psi_5=0$ the point becomes a $R-N_1$.

[l] If $C=0$ and $\Psi_5\geq 0$ the point becomes a $R-SN$.

[m] If $\Psi_6=0$ for $C\neq 0$ the point becomes a $R-N_1$ and for $C=0$ the point becomes a $R-SN$.

Table 3.2: Definition of the quantities used in Table 3.1.

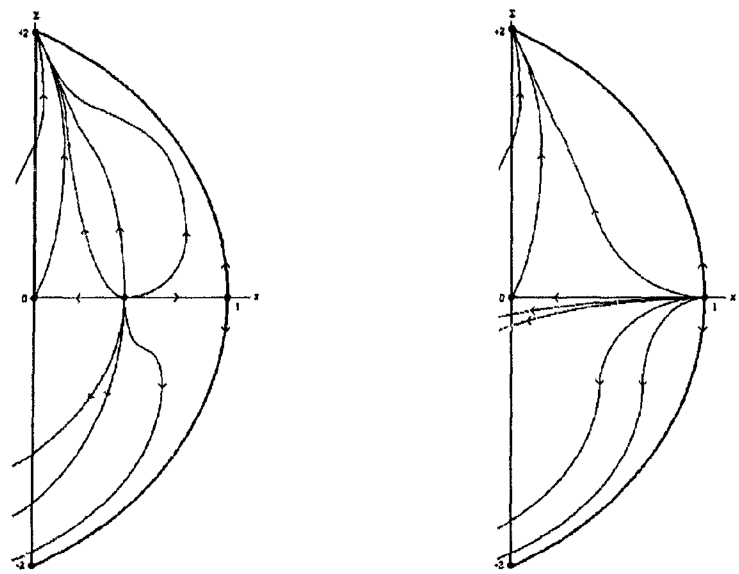
Conditional Quantities	
Ψ_1	$9\zeta_o - (3\gamma - 2)$
Ψ_2	$9\zeta_o - (3\gamma - 2) - 4 - 12\eta_o$
Ψ_3	$\frac{1}{2}(9\zeta_o - (3\gamma - 2) + 12\eta_o)$
Ψ_4	$9\zeta_o - (3\gamma - 2) + 2$
Ψ_5	$(3\gamma - 2) - 2 - 6\eta_o$
Ψ_6	$9\zeta_o - (3\gamma - 2) + 2 + 6\eta_o$

Table 3.3: Notation used to describe the equilibrium points in Table 3.1.

Notation	
R	Repelling
A	Attracting
S	Saddle Point
S^*	Saddle-like [a]
N_1	One-tangent Node
N_2	Two-tangent Node
N^*	Node-like [b]
SN	Stellar Node
$***$	Line Equilibria
No entry	Not Singular

[a] Degenerate point that has the qualitative nature of a saddle-point in the region of interest.

[b] Degenerate point that has the qualitative nature of a two-tangent node in the region of interest.



(a) $C<0,\ m=n=\frac{1}{2},\ \Psi_1<0$ (b) $C<0,\ m=n=\frac{1}{2},\ \Psi_1\geq 0$

Figure 3.7: *The phase portraits describe the behavior of the Bianchi type V models with heat conduction and viscosity in the case* $m=n=\frac{1}{2}$ *and* $C<0$.

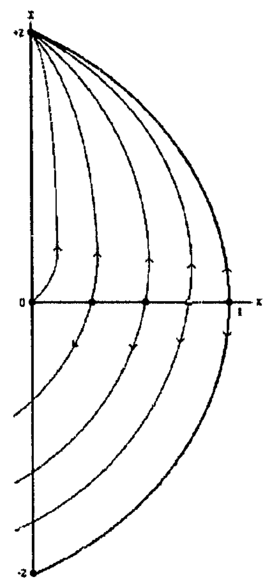
In the set $\Re_-$, there exists only one isolated equilibrium point, $(0,-2)$. It lies on the boundary $\Sigma^2+4x=4$ where ${}^3R=0$, and hence the solution is of Bianchi type I. The Kasner coefficients are:

$$\begin{aligned} p_1 &= \tfrac{1}{3}\Big(1-\tfrac{(k+1)}{\sqrt{k^2-k+1}}\Big), \\ p_2 &= \tfrac{1}{3}\Big(1-\tfrac{(1-2k)}{\sqrt{k^2-k+1}}\Big), \\ p_3 &= \tfrac{1}{3}\Big(1-\tfrac{(k-2)}{\sqrt{k^2-k+1}}\Big). \end{aligned} \tag{3.29}$$

The solution is given by

$$ds^2=-dt^2+t^{2p_1}dx^2+t^{2p_2}dy^2+t^{2p_3}dz^2, \tag{3.30}$$

$$\begin{aligned} &\theta=t^{-1}, && \zeta=0, \\ &\rho=0, && \eta=0, \\ &\sigma=-(\sqrt{3}t)^{-1}, && q_1=0, \\ &\Pi=0, && \pi_{ab}=0. \end{aligned} \tag{3.31}$$



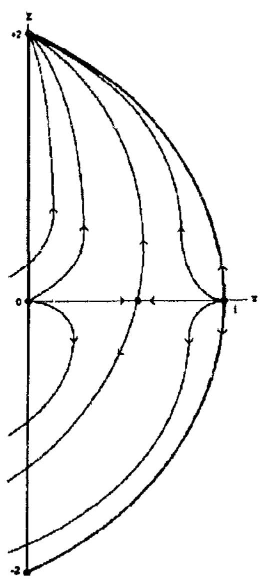
(a) $C \leq \Psi_3,\ m = n = 1,\ \Psi_1 = 0$

Figure 3.8: *The phase portrait describes the behavior of the Bianchi type V models with heat conduction and viscosity for the degenerate case $m = n = 1$ and $\Psi_1 = 0$.*

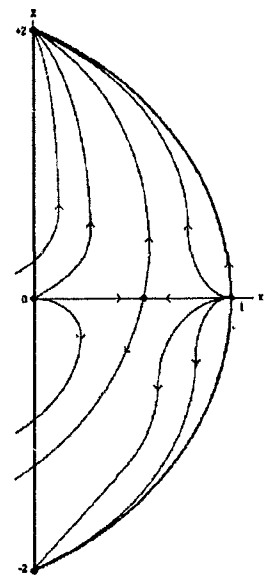
The space-time is transitively self-similar [61] with homothetic vector

$$\mathbf{X} = \mathrm{t}\frac{\partial}{\partial \mathrm{t}} + (1 - \mathrm{p}_1)\mathrm{x}\frac{\partial}{\partial \mathrm{x}} + (1 - \mathrm{p}_2)\mathrm{y}\frac{\partial}{\partial \mathrm{y}} + (1 - \mathrm{p}_3)\mathrm{z}\frac{\partial}{\partial \mathrm{z}}. \tag{3.32}$$

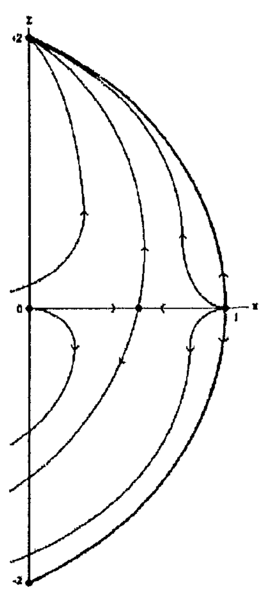
From a dynamical systems point of view, the equilibrium point $(0, -2)$ is always a repellor in $t - time$ (even when the sector in $\Re_-$ is hyperbolic in nature, since trajectories are repelled in $t - time$ along an eigendirection that is not in $\Re_-$), which implies that this point represents an initial singularity. The singularity is generally of *cigar* type, but in the case when $C = 1$ the singularity is of *pancake* type [71]. For particular values of the parameters, there exist trajectories that start at $(0, -2)$ and leave $\Re_-$ after a finite time t_o. There also exist trajectories that start from the equilibrium point $(0, -2)$ and remain in $\Re_-$ for all time; these models expand from the Kasner singularity towards one of the isotropic models located on the x-axis (i.e., these models isotropize as $t \to \infty$). In some cases, there exist trajectories that enter



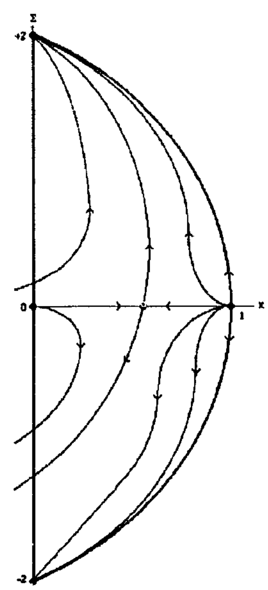
$(a)\ C \leq -\frac{3\gamma-2}{2},\ m = n > 1,$
$\Psi_1 > 0,\ 1 \leq \gamma < \frac{4}{3}$



$(b)\ -\frac{3\gamma-2}{2} < C < 0,\ m = n > 1,$
$\Psi_1 > 0,\ 1 \leq \gamma < \frac{4}{3}$



$(c)\ C = -1,\ m = n > 1,$
$\Psi_1 > 0,\ \gamma = \frac{4}{3}$



$(d)\ -1 < C < 0,\ m = n > 1,$
$\Psi_1 > 0,\ \gamma = \frac{4}{3}$

Figure 3.9: *The phase portraits describe the behavior of the Bianchi type V models with heat conduction and viscosity in the case* $m = n > 1$ *and* $C < 0$.

$\Re_-$ at some finite time t_o; these models evolve towards one of the isotropic models and can only represent late time behavior.

In the case $m = n = 0$, there are no equilibrium points in the invariant set $\Re_-$. All trajectories enter $\Re_-$ at some finite time and evolve towards one of the isotropic models. Hence these models can only describe late time behavior.

In the case $\zeta_o = \eta_o = 0$ with $\gamma = 2$, there is a non-isolated line of equilibria on the boundary $\Sigma^2 + 4x = 4$. The equilibrium points represent stiff perfect fluid Bianchi I solutions (3.30) with coefficients p_i:

$$\begin{aligned} p_1 = &\ \tfrac{1}{3}\Big(1 - \tfrac{(k+1)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\Big), \\ p_2 = &\ \tfrac{1}{3}\Big(1 - \tfrac{(1-2k)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\Big), \\ p_3 = &\ \tfrac{1}{3}\Big(1 - \tfrac{(k-2)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\Big), \end{aligned} \tag{3.33}$$

$$\begin{aligned} &\theta = t^{-1}, && \zeta = 0, \\ &\rho = \tfrac{x_o}{3}t^{-2}, && \eta = 0, \\ &\sigma = -\sqrt{\tfrac{(1-x_o)}{3}}\,t^{-1}, && q_1 = 0, \\ &\Pi = 0, && \pi_{ab} = 0. \end{aligned} \tag{3.34}$$

where the parameter x_o is bounded by $0 \leq x_o \leq 1$. The space-time is transitively self-similar [61] with homothetic vector (3.32) with the p_i now defined by (3.33). For $C \leq 0$ all trajectories remain in $\Re_-$ for all time and evolve towards the isotropic model at $(0,0)$. For $C > 0$ all trajectories leave $\Re_-$ after some finite time hence they may only describe early time behavior.

In the invariant set $\Re_o$ there exists either 1, 2, 3, or a non-isolated line of equilibria. Points in this set represent negatively curved (i.e, $x < 1$) or flat (i.e., $x = 1$) FRW models with at most bulk viscosity.

The point $(1,0)$ is an equilibrium point in all cases. The point lies on the boundary $\Sigma^2 + 4x = 4$, hence, the point represents a flat FRW model. It is a saddle-point for $\Psi_1 \equiv 9\zeta_o - (3\gamma - 2) < 0$. However, when $\Psi_1 > 0$ the point becomes an attracting-node and represents a late-time attractor, [but note that in this case the SEC (strong energy condition) (see Appendix C) is violated and the corresponding asymptotic solution may not be physically acceptable]. In the case $\Psi_1 = 0$, the point is saddle-like for $m = n > 1$ and node-like in the remaining cases. The solution corresponding to this equilibrium point depends upon whether $\gamma = 3\zeta_o$ or not.

For $\gamma \neq 3\zeta_o$ the solution is

$$ds^2 = -dt^2 + (t)^{\frac{4}{3(\gamma-3\zeta_o)}}(dx^2 + dy^2 + dz^2), \tag{3.35}$$

$$\begin{aligned} \theta &= \tfrac{2}{(\gamma-3\zeta_o)}t^{-1}, & \zeta &= \tfrac{2\zeta_o}{(\gamma-3\zeta_o)}t^{-1}, \\ \rho &= \tfrac{4}{3(\gamma-3\zeta_o)^2}t^{-2}, & \eta &= \tfrac{2\eta_o}{(\gamma-3\zeta_o)}t^{-1}, \\ \sigma &= 0, & q_1 &= 0, \\ \Pi &= \tfrac{-4\zeta_o}{(\gamma-3\zeta_o)^2}t^{-2}, & \pi_{ab} &= 0. \end{aligned} \tag{3.36}$$

The space-time (3.35) admits the homothetic vector (3.32) with $p_1 = p_2 = p_3 = \frac{2}{3\gamma-9\zeta_o}$, hence the space-time is transitively self-similar [61].

For $\gamma = 3\zeta_o$ the solution is

$$ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2), \tag{3.37}$$

$$\begin{aligned} \theta &= 3H, & \zeta &= 9\zeta_o H^2, \\ \rho &= 3H^2, & \eta &= 9\eta_o H^2, \\ \sigma &= 0, & q_1 &= 0, \\ \Pi &= -27\zeta_o H^3, & \pi_{ab} &= 0, \end{aligned} \tag{3.38}$$

where H is a constant. The space-time (3.37) does not admit a homothetic vector, hence, the spacetime is not self-similar [72].

When the point $(1,0)$ is a saddle, models start from the matter dominated singularity at $(1,0)$ and evolve towards either the Milne model at $(0,0)$ or the FRW model at $(\bar{x},0)$. However, if the point is an attracting node, all models evolve towards the point $(1,0)$.

The point $(\bar{x},0)$ represents a negatively curved FRW model with bulk viscosity where $\bar{x} = \left(\frac{9\zeta_o}{3\gamma-2}\right)^{\frac{1}{1-m}}$. The space-time is self-similar [61] and the corresponding exact solution is

$$ds^2 = -dt^2 + (1-\bar{x})^{-1}(t)^2(dx^2 + e^{2x}dy^2 + e^{2x}dz^2), \tag{3.39}$$

$$
\begin{aligned}
&\theta = 3t^{-1}, &&\zeta = 3\zeta_o \bar{x}^m t^{-1},\\
&\rho = 3\Sigma t^{-2}, &&\eta = 3\eta_o \bar{x}^m t^{-1},\\
&\sigma = 0, &&q_1 = 0,\\
&\Pi = -9\zeta_o \bar{x}^m t^{-2}, &&\pi_{ab} = 0,
\end{aligned}
\tag{3.40}
$$

with homothetic vector

$$
\mathbf{X} = t\frac{\partial}{\partial t}. \tag{3.41}
$$

When the equilibrium point $(\bar{x}, 0)$ is a saddle, models start from $(\bar{x}, 0)$ and evolve towards either the Milne model at $(0,0)$ or the FRW model at $(1,0)$ (however this latter model violates the SEC). However, if the point is a node, the solution is a late time asymptotic attractor (except in the degenerate case when there is a non-isolated line of equilibria).

The point $(0,0)$ represents an empty cosmological model, commonly known as the Milne model. The space-time is transitively self-similar [61] with homothetic vector (3.41). The solution is

$$
ds^2 = -dt^2 + (t)^2(dx^2 + e^{2x}dy^2 + e^{2x}dz^2), \tag{3.42}
$$

$$
\begin{aligned}
&\theta = 3t^{-1}, &&\zeta = 0,\\
&\rho = 0, &&\eta = 0,\\
&\sigma = 0, &&q_1 = 0,\\
&\Pi = 0, &&\pi_{ab} = 0.
\end{aligned}
\tag{3.43}
$$

(Note, there is one exception to the above solution; if $m = n = 0$ and $\zeta_o = 0$, then $\eta = 3\eta_o t^{-1}$, but $\pi_{ab} = 0$.) When the point $(0,0)$ is an attracting node, the matter dominated singularities at $(\bar{x}, 0)$ or $(1,0)$ evolve towards the Milne model at $(0,0)$. However, when the point $(0,0)$ is a saddle, the Milne model evolves towards one of the other isotropic models.

In the set $\Re_+$ there exists only one isolated equilibrium point, $(0,2)$. It lies on the boundary $\Sigma^2 + 4x = 4$ where ${}^3R = 0$, hence, the solution is of Bianchi type I. The solution is given by equations (3.30) and (3.31) [except that $\sigma = (\sqrt{3}t)^{-1}$], with the p_i defined as:

$$\begin{aligned} p_1 &= \tfrac{1}{3}\Big(1 + \tfrac{(k+1)}{\sqrt{k^2-k+1}}\Big), \\ p_2 &= \tfrac{1}{3}\Big(1 + \tfrac{(1-2k)}{\sqrt{k^2-k+1}}\Big), \\ p_3 &= \tfrac{1}{3}\Big(1 + \tfrac{(k-2)}{\sqrt{k^2-k+1}}\Big). \end{aligned} \tag{3.44}$$

The space-time is transitively self-similar with homothetic vector (3.32) with the p_i defined by (3.44). From a dynamical systems point of view, the equilibrium point $(0,2)$ is always a repellor in $t-time$ (even when the sector in $\Re_+$ is hyperbolic in nature, since trajectories are repelled in $t-time$ along an eigendirection that is not in $\Re_+$), which implies that this point represents an initial singularity. The singularity is generally of *cigar* type, but in the case when $k=1$ or $C=2$ (the LRS case), the singularity is of *pancake* type [71]. For particular values of the parameters, there exist trajectories that start at $(0,2)$ and leave $\Re_+$ at some finite time t_o. There also exist trajectories that start from the equilibrium point $(0,2)$ and remain in $\Re_+$ for all time; these models expand from the Kasner singularity towards one of the isotropic models located on the x-axis (i.e., these models isotropize as $t \to \infty$). In some cases, there exist trajectories that enter $\Re_+$ at some finite time t_o; these models evolve towards one of the isotropic models and can only represent late time behavior.

In the case $m = n = 0$, there are no equilibrium points in the invariant set $\Re_+$. All trajectories enter $\Re_+$ at some finite time and evolve towards one of the isotropic models. Hence these models can only describe late time behavior.

In the case $\zeta_o = \eta_o = 0$ and $\gamma = 2$ there is a non-isolated line of equilibria on the boundary $\Sigma^2 + 4x = 4$. The equilibrium points represent stiff perfect fluid Bianchi I solutions (3.30) with coefficients p_i:

$$\begin{aligned} p_1 &= \tfrac{1}{3}\Big(1 + \tfrac{(k+1)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\Big), \\ p_2 &= \tfrac{1}{3}\Big(1 + \tfrac{(1-2k)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\Big), \\ p_3 &= \tfrac{1}{3}\Big(1 + \tfrac{(k-2)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\Big), \end{aligned} \tag{3.45}$$

$$\begin{array}{ll} \theta = t^{-1}, & \zeta = 0, \\ \rho = \frac{x_o}{3} t^{-2}, & \eta = 0, \\ \sigma = \sqrt{\frac{(1-x_o)}{3}} t^{-1}, & q_1 = 0, \\ \Pi = 0, & \pi_{ab} = 0, \end{array} \tag{3.46}$$

where the parameter x_o is bounded by $0 \leq x_o \leq 1$. The space-time is transitively self-similar with homothetic vector (3.32) with the p_i defined by (3.45). For $C \geq 0$ all trajectories remain in $\Re_+$ for all time and evolve towards the isotropic model at $(0,0)$. For $C < 0$ all trajectories leave $\Re_+$ after some finite time hence they may only describe early time behavior.

3.3.5 Invariant Curves and First Integrals

An implicit function determining the integral curves in the phase portrait for the perfect fluid case $\zeta_o = \eta_o = 0$ and in the case $m = n = 1$, $C = 0$ can be constructed. We use the method of algebraic invariant curves to construct an algebraic first integral using Darboux's theorem (see [73] and references therein). An algebraic invariant curve, $Q_i = 0$, is a curve in the phase space such that $\dot{Q}_i = r_i Q_i$, where r_i is a polynomial of the phase space variables. The following are invariant curves of the system (3.28),

$$\begin{aligned} Q_1 &= x, \\ Q_2 &= \Sigma, \\ Q_3 &= \Sigma^2 + 4x - 4. \end{aligned} \tag{3.47}$$

Calculating $\dot{Q}_i = r_i Q_i$ for $i = 1, 2, 3$ above, we find

$$\begin{aligned} r_1 &= (3\gamma - 2 - 9\zeta_o)(1 - x) - \Sigma^2(1 + 3\eta_o), \\ r_2 &= -\frac{1}{2}[(3\gamma - 2 - 9\zeta_o - 12\eta_o)x + \Sigma^2 - 4], \\ r_3 &= -[(3\gamma - 2 - 9\zeta_o)x + \Sigma^2]. \end{aligned} \tag{3.48}$$

Using Darboux's Theorem, an algebraic first integral Q can be found by setting $Q = Q_1^{\alpha_1} Q_2^{\alpha_2} Q_3^{\alpha_3}$ and then determining what values of α_i satisfy the equation $\dot{Q} = 0$. For the above set of invariant curves, (3.47), we have that

$$\dot{Q} = Q\Big(\alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3\Big) = 0. \tag{3.49}$$

This yields a homogeneous system of equations for the constants α_i. A solution is

$$\begin{aligned} \alpha_1 &= \alpha_1, \\ \alpha_2 &= [9\zeta_o - (3\gamma - 2)]\alpha_1/2, \\ \alpha_3 &= [3(\gamma - 2) - 9\zeta_o - 12\eta_o]\alpha_1/4. \end{aligned} \tag{3.50}$$

Therefore without loss of generality we choose $\alpha_1 = 4$ in which case the first integral is of the form

$$Q(x, \Sigma) = x^4 \Sigma^{18\zeta_o - 2(3\gamma - 2)} (\Sigma^2 + 4x - 4)^{3(\gamma - 2) - 9\zeta_o - 12\eta_o}, \tag{3.51}$$

such that the level sets $Q(x, \Sigma) = Q_o$ a constant, are integral curves of the system, (3.28), in the case $m = n = 1$ and $C = 0$. It is noteworthy to point out that the first integral reduces to the perfect fluid form simply by setting $\zeta_o = \eta_o = 0$ in equation (3.51).

This technique, however, fails whenever the system (3.28) is not polynomial or when one cannot find enough algebraic invariant curves. Nevertheless, this technique has yielded another first integral in addition to the Friedmann equation which may aid us in determining an exact solution of the Einstein field equations for spatially homogeneous cosmologies.

3.4 Conclusions

By using geometric techniques from dynamical systems theory we have been able to determine the qualitative behavior of a class of spatially homogeneous cosmological models that contain viscous matter and heat conduction.

With the introduction of viscosity [satisfying the Eckart theory (3.18)] into the fluid, the qualitative behavior of the models differ from that of the perfect fluid models (for example, in some instances an additional equilibrium point is even created). In particular, it is the nature of Ψ_1 and hence ζ_o that affects the global behavior of the models, while the values of C and η_o change only the local behavior in a neighborhood of an equilibrium point. With the introduction of heat conduction, solutions that violate the WEC (weak energy condition) at some finite time t_o in the past (and, in some instances, in the future) arise.

With the introduction of bulk viscosity, the deceleration parameter q, defined by equation (3.15) may become negative. A negative q indicates that there exists a region of phase space with an accelerated expansion; that is, inflation occurs. For $\Psi_1 > 0$, in all cases, there exists some region of phase space such that $q < 0$, which implies that all models must inflate at some time in their evolution. For $m = n \leq 1$ and $\Psi_1 > 0$, inflation occurs as $t \to \infty$. For $m = n > 1$, the models may inflate for all time t, or up to some finite time t_o. For $\Psi_1 \leq 0$ some models may inflate. In the perfect fluid case, inflation occurs, assuming an equation of state $p = (\gamma - 1)\rho$, when $\gamma < \frac{2}{3}$ [74]. With the addition of bulk viscosity, the fluid effectively acts like a perfect fluid with an equation of state $p = (\gamma_{eff} - 1)\rho$ where $\gamma_{eff} = \gamma - \zeta\theta\rho^{-1}$. Several authors [42, 75, 76] have investigated whether a non-vanishing bulk viscosity could drive an inflationary phase in the early universe. Bulk viscosity can only act as a source for inflation if the SEC is violated. Models that include bulk viscosity and in which the SEC is violated have also been studied [45] since the initial singularity can, in a sense, be eliminated.

Except for the exceptional trajectories located on the x-axis (as well as the stiff perfect fluid case, $\gamma = 2$, with $\zeta_o = \eta_o = 0$), all models that satisfy the WEC for all time start their evolution from the Kasner singularities. Assuming all EC are satisfied, all models (except the exceptional trajectories) either evolve towards the Milne model at $(0,0)$ or the FRW model at the point $(\bar{x}, 0)$. If the EC are not satisfied, then the FRW model at $(1,0)$ also becomes a late time attractor. In either case, models start from an anisotropic state and isotropize towards a flat or negatively curved FRW model. In all models the shear viscous stress, π_{ab}, is asymptotically zero both to the past and to the future. We can also conclude that the bulk viscosity had little affect on the initial singularities at $(0, \pm 2)$ in that the bulk viscous pressure is zero at the initial singularity. However the bulk viscosity did influence the type of singularity (cigar/barrel/pancake) and whether the matter was dynamically important or unimportant initially. The bulk viscosity also determines the final asymptotic state and may cause the model to experience a period of inflation.

Using the dimensionless equations of state, all asymptotic states represent self-similar cosmological models (unless $\gamma = 3\zeta_o$ whence the point $(1, 0)$ is no longer self-similar). This shows that the past asymptotic behavior of the imperfect fluid Bianchi V model without a cosmological constant is represented by self-similar solutions, and if the EC are satisfied the future asymptotic states are also self-similar.

We have shown in this chapter that by using dimensionless variables and dimensionless equations of state, the Einstein field equations reduce to a system of autonomous ordinary differential equations. All models, that satisfy the WEC for all time, isotropize. Including viscosity (and heat conduction) in the models allow for processes such as inflation and the removal of the initial singularity. These models are sufficiently simple to allow us to analyze them qualitatively. By considering better approximations to the bulk viscous pressure, Π, and the shear viscous stress, π_{ab}, more physically realistic models may be analyzed using similar techniques, which may lead to interesting and different qualitative behaviour. (See Chapters 4 and 5.)

Chapter 4

Causal Viscous Fluid Cosmological Models (Truncated Theory)

4.1 Introduction

In the previous chapter and in papers [53, 54, 55] it was assumed that the viscous effects in the fluid could be described by Eckart's theory of irreversible thermodynamics [see equation (3.18)]. However, Eckart's theory of irreversible thermodynamics [47] suffers from the property that signals in the fluid can propagate faster than the speed of light (i.e., non-causality), and, that the equilibrium states in the Eckart theory are unstable (see Hiscock and Salmonson [48] and references therein). Therefore, a more complete and satisfactory theory of irreversible thermodynamics is necessary for fully analyzing cosmological models with viscosity. One such theory is the truncated Israel-Stewart theory [49, 50, 51].

The intent of this chapter will be to build upon the foundation laid by Belinskii et al. [44], Pavon et al. [56] and Chimento and Jakubi [57], and investigate viscous fluid cosmological models satisfying the truncated Israel-Stewart theory of irreversible thermodynamics [equations (1.3) with $\epsilon = 0$]. We shall study the new "visco-elastic" singularity found in [44] and we shall determine whether bulk-viscous inflation is possible. We will also determine if there is a qualitative difference between these models and the models studied in Chapter 3 where the Eckart equation (3.18) was assumed. In this chapter we shall analyze qualitatively a class of anisotropic Bianchi type V and Bianchi type I cosmological models in section 4.3, in addition to the isotropic FRW cosmological models investigated in section 4.2, thereby extending the analysis in Chapter 3 to causal theories.

4.2 Friedmann-Robertson-Walker Models

4.2.1 The Equations

In this section we assume that the spacetime is spatially homogeneous and isotropic and that the fluid is moving orthogonal to the spatial hypersurfaces. The energy-momentum tensor is an imperfect fluid with non-zero bulk viscosity (that is there is no heat conduction, $q_a = 0$, and no anisotropic stress, $\pi_{ab} \equiv 0$). The dimensionless Einstein field equations are equations (3.11), (3.15), and (3.16) with $\Sigma_1 = \Sigma_2 = 0$ and $z_1 = z_2 = 0$. They reduce to the following three equations:

$$\frac{dx}{d\Omega} = x(1-2q) + 9\frac{p}{\theta^2} + y, \tag{4.1}$$

where q, the generalized dimensionless deceleration parameter, is given by

$$q = \frac{1}{2}\left(x + y + 9\frac{p}{\theta^2}\right). \tag{4.2}$$

Finally, from equation (3.16), we obtain

$$4 - 4x = -6\,{}^3R\theta^{-2}, \tag{4.3}$$

where θ is the expansion and ${}^3R$ is the curvature of the spatial hypersurfaces. If the curvature is negative, i.e., ${}^3R < 0$, then the FRW model is open. If ${}^3R \equiv 0$, then the model is flat. If ${}^3R > 0$ then the FRW model is closed. Assuming that the energy density, ρ, is non-negative, it is easily seen from (3.7) that in the open and flat FRW models the expansion is always non-negative, i.e. $\theta \geq 0$, but for the closed FRW models the expansion may become negative. (Great care must be taken in this case because the dimensionless quantities that we will be using become ill-defined at $\theta = 0$.)

In order to close the system given by (4.1–4.3), we need an equation for the dimensionless viscous pressure y, (hence, for Π). Using the truncated Israel-Stewart theory, we can obtain an evolution equation for Π by solving (1.3) [with $\epsilon = 0$] for $\dot{\Pi}$,

$$\dot{\Pi} = -\frac{\Pi}{\beta_0 \zeta} - \frac{1}{\beta_0}\theta. \tag{4.4}$$

Unlike in the Eckart theory where we ended up with an algebraic equation for y [see equation (3 19)], we have the ordinary differential equation

$$\frac{dy}{d\Omega} = y[\left(\frac{\theta}{\zeta_0}\right)\left(\frac{3}{\beta_0 \theta^2}\right) - 2 - y - (3\gamma - 2)x] + 9\left(\frac{3}{\beta_0 \theta^2}\right). \tag{4.5}$$

In order to complete the system of equations we need to specify equations of state for the quantities p, β_0, and ζ. In principle equations of state can be derived from kinetic theory, but in practice one must specify phenomenological equations of state which may or may not have any physical foundations. Following Coley [70, 64], we introduce dimensionless equations of state of the form

$$\begin{aligned} \frac{p}{\theta^2} &= p_o x^{\ell}, \\ \frac{\zeta}{\theta} &= \zeta_0 x^{m}, \\ \frac{3}{\beta_0 \theta^2} &= a x^{r_1}, \end{aligned} \tag{4.6}$$

where p_o, ζ_0, and a are positive constants, and ℓ, m, and r_1 are constant parameters (x is the dimensionless density parameter defined earlier). In the models under consideration, θ is positive in the open and flat FRW models, thus equations (4.6) are well defined. In the closed FRW model the expansion could become zero, in which case these equations of state break down. However, we can utilize these equations to model the asymptotic behaviour at early times, i.e., when $\theta > 0$. The most commonly used equation of state for the pressure is the barotropic equation of state $p = (\gamma - 1)\rho$, whence $p_o = \frac{1}{3}(\gamma - 1)$ and $\ell = 1$ (where $1 \leq \gamma \leq 2$).

If we define a new constant $b = a/\zeta_0$, then using equation (4.6), equations (4.1), (4.2) and (4.5) reduce to

$$\begin{aligned} \frac{dx}{d\Omega} &= (1-x)[(3\gamma-2)x+y], \\ \frac{dy}{d\Omega} &= -y[2+y+(3\gamma-2)x]+bx^{r_1-m}y+9ax^{r_1}. \end{aligned} \tag{4.7}$$

Also, from the Friedmann equation, (4.3), we obtain

$$1 - x = -\frac{3}{2}\,{}^3R\theta^{-2}. \tag{4.8}$$

Thus, the line $x = 1$ divides the phase space into three invariant sets, $x < 1$, $x = 1$, and $x > 1$. If $x = 1$, then the model is necessarily a flat FRW model, if $x < 1$ then the model is necessarily an open FRW model, and if $x > 1$ the model is necessarily a closed FRW model.

The equilibrium points of the above system all represent self-similar cosmological models, except in the case $\gamma = 3\zeta_0$. If $\gamma \neq 3\zeta_0$, the behaviour of the equations of state, equation (4.6), at the equilibrium points, is independent of the parameters m and r_1; namely the behaviour is

$$\zeta \propto \rho^{\frac{1}{2}}, \quad \text{and} \quad \beta_0 \propto \rho^{-1}. \tag{4.9}$$

Therefore, natural choices for m and r_1 are respectively 1/2, 1. We note that in the exceptional case $\gamma = 3\zeta_0$, there is a singular point $\{x = 1, y = -3\gamma\}$ which represents a de Sitter solution and is not self-similar. (This is also the case in the Eckart theory as was analyzed in Chapter 3.)

To further motivate the choice of the parameter r_1, we consider the velocity of a viscous pulse in the fluid [52],

$$v = \left(\frac{1}{\rho\beta_0}\right)^{1/2}, \tag{4.10}$$

where $v = 1$ corresponds to the speed of light. Using (3.10) and equations (4.6), we obtain

$$v = (ax^{r_1-1})^{1/2}. \tag{4.11}$$

Now, if $r_1 = 1$ then not only do we obtain the correct asymptotic behaviour of the equation of state for the quantity β_0 but we are also allowed to choose $a < 1$ since then the velocity of a viscous pulse is less than the velocity of light for any value of the density parameter x. Thus in the remainder of this analysis we shall choose $r_1 = 1$. In order for the system of differential equations (4.7) to remain continuous everywhere, we also assume $m \leq 1$.

4.2.2 Qualitative Analysis

$\mathbf{m = r_1 = 1}$

We now study the specific case when $m = r_1 = 1$. In this case there are three singular points,

$$(0,0), \qquad (1, y^-), \quad \text{and} \quad (1, y^+), \tag{4.12}$$

where

$$y^- = \frac{b - 3\gamma}{2} - \frac{\sqrt{(b-3\gamma)^2 + 36a}}{2} \quad \text{and} \quad y^+ = \frac{b - 3\gamma}{2} + \frac{\sqrt{(b-3\gamma)^2 + 36a}}{2}. \tag{4.13}$$

The point $(0,0)$ has eigenvalues

$$\frac{3\gamma - 4 + b}{2} - \frac{1}{2}\sqrt{(b-3\gamma)^2 + 36a}, \quad \frac{3\gamma - 4 + b}{2} + \frac{1}{2}\sqrt{(b-3\gamma)^2 + 36a}. \tag{4.14}$$

This point is either a saddle or a source depending on the value of the parameter, B_1;

$$B_1 = (2 - b)(3\gamma - 2) + 9a. \tag{4.15}$$

If $B_1 > 0$, then the point is a saddle point and if $B_1 < 0$, then the point is a source. If $B_1 = 0$ (the bifurcation value), then the point is degenerate (discussed later).

The point $(1, y^-)$ has eigenvalues

$$\sqrt{(b-3\gamma)^2 + 36a}, \quad -\frac{3\gamma - 4 + b}{2} + \frac{1}{2}\sqrt{(b-3\gamma)^2 + 36a}. \tag{4.16}$$

If $B_1 < 0$, then the point $(1, y^-)$ is a saddle point and if $B_1 > 0$, then the point $(1, y^-)$ is a source. If $B_1 = 0$ (the bifurcation value), then the point $(1, y^-)$ is degenerate (discussed later).

The singular point $(1, y^+)$ has eigenvalues

$$-\sqrt{(b-3\gamma)^2+36a}, \quad -\frac{3\gamma-4+b}{2}-\frac{1}{2}\sqrt{(b-3\gamma)^2+36a}. \tag{4.17}$$

This singular point is a sink for $\gamma > 2/3$ (See also Table 4.1 for details).

In addition to the invariant set $x = 1$, there exist two other invariant sets. These are straight lines, $y = m_\pm x$, where

$$m_\pm = \frac{(b-3\gamma) \pm \sqrt{(b-3\gamma)^2+36a}}{2}. \tag{4.18}$$

The invariant line $y = m_+ x$ passes through the singular points $(0,0)$ and $(1, y^+)$ while the line $y = m_- x$ passes through the singular points $(0,0)$, and $(1, y^-)$. These invariant sets represent the eigendirections at each of the singular points [see also section 4.2.3].

In order to sketch a complete phase portrait, we also need to calculate the vertical isoclines which occur whenever $dx/d\Omega = 0$. From (4.7) we can see that this occurs either when $x = 1$ or when $y = -(3\gamma - 2)x$. This straight line passes through the origin, and through the singular point $(1, y^-)$ if $B_1 = 0$. If $B_1 > 0$ then the vertical isocline has a negative slope which is greater than the slope of the slope of the invariant line $y = m_- x$, [i.e., $m_- < -(3\gamma - 2)$], and when $B_1 < 0$ the vertical isocline has a negative slope which is less than the slope of the invariant line $y = m_- x$ [i.e., $m_- > -(3\gamma - 2)$].

To complete the analysis of this model we need to analyze the poir 's at infinity. We do this by first converting to polar coordinates and then compactifying the radial coordinate. We change to polar coordinates via

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \theta = \tan^{-1}\frac{y}{x}, \tag{4.19}$$

and we derive evolution equations for r and θ. We essentially compactify the phase space by changing our radial coordinate r and our time Ω as follows,

$$\bar{r} = \frac{r}{1+r} \qquad \text{and} \qquad \frac{d\Omega}{d\tau} = (1-\bar{r}), \tag{4.20}$$

that is, the plane $\mathbb{R}^2$ is mapped to the interior of the unit circle, with the boundary of this circle representing points at infinity of $\mathbb{R}^2$. We have (for $r_1 = 1$ and general m)

$$\begin{aligned}
\frac{d\bar{r}}{d\tau} &= (1-\bar{r})\Big\{\bar{r}(1-\bar{r})[(3\gamma-2)\cos^2\theta - 2\sin^2\theta + (1+9a)\cos\theta\sin\theta] \\
&\qquad -\bar{r}^2[(3\gamma-2)\cos\theta + \sin\theta] + \bar{r}^{2-m}(1-\bar{r})^m[b\sin^2\theta\cos^{1-m}\theta]\Big\}, \qquad (4.21) \\
\frac{d\theta}{d\tau} &= (1-\bar{r})\Big\{9a\cos^2\theta - \sin^2\theta - 3\gamma\cos\theta\sin\theta \\
&\qquad + b\bar{r}^{1-m}(1-\bar{r})^{m-1}\sin\theta\cos^{2-m}\theta\Big\}. \qquad (4.22)
\end{aligned}$$

We easily conclude that if $m = 1$ (or any $m > 0$), then the entire circle, $\bar{r} = 1$, is singular. Therefore, we have a non-isolated set of singular points at infinity. To determine their stability we look at the sign of $d\bar{r}/d\tau$ as $\bar{r} \to 1$. In this case we see

$$\left.\frac{d\bar{r}}{d\tau}\right|_{\bar{r}\approx 1} \approx (3\gamma-2)\cos\theta + \sin\theta \tag{4.23}$$

which implies that points above the line $y = -(3\gamma-2)x$ are repellors, while those points which lie below the line are attractors.

For completeness, we would also like to determine the qualitative behaviour of the system at the bifurcation value $B_1 = 0$ where the singular points are $(1, b-2)$ and the line of singular points $y = -(3\gamma-2)x$. (Note that since $B_1 = 0$, $b-2 > 0$.) Fortunately we are able to completely integrate the equations in this case to find

$$|b-2-y| = k|1-x|, \tag{4.24}$$

where k is an integration constant. We see that all trajectories are straight lines that pass through the point $(1, b-2)$. It is straightforward to see that the line of singular points are repellors while the point $(1, b-2)$ is an attractor. We are now able to sketch complete phase portraits (See Figures 4.1, 4.2 and 4.3).

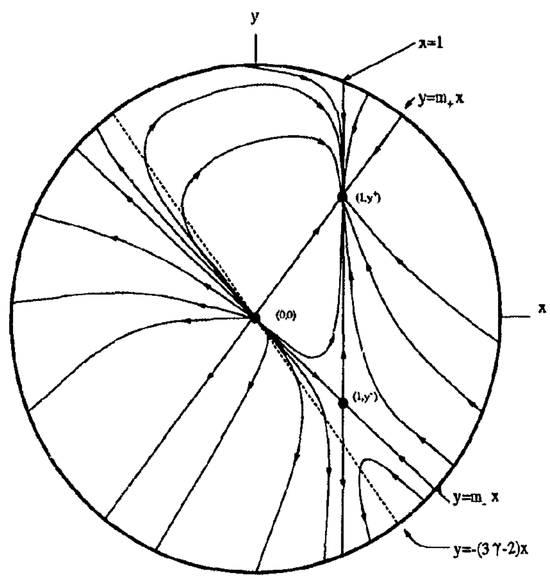


Figure 4.1: *The phase portrait describes the qualitative behavior of the FRW models with bulk viscous pressure in the case $m = r_1 = 1$ and $B_1 < 0$. The arrows in the figure denote increasing Ω-time ($\Omega \to \infty$) or decreasing t-time ($t \to 0^+$).*

$\mathbf{m = 1/2}$ and $\mathbf{r_1 = 1}$

This is a case of particular interest since it represents the asymptotic behaviour of the FRW models for any m and r_1 (since at the singular points the viscosity coefficient behaves like $\zeta \propto \rho^{1/2}$ and the relaxation time like $\beta_0 \propto \rho^{-1}$). Note that the physical phase space is defined for $x \geq 0$, but the system is not differentiable at $x = 0$. In this case there are four singular points,

$$(0,0), \qquad (\bar{x}, \bar{y}), \qquad (1, y^-), \quad \text{and} \quad (1, y^+), \tag{4.25}$$

where

$$\bar{x} = \frac{(9a + 2(3\gamma - 2))^2}{b^2(3\gamma - 2)^2}, \qquad \bar{y} = -(3\gamma - 2)\bar{x}, \tag{4.26}$$

and y^+ and y^- are given be equation (4.13).

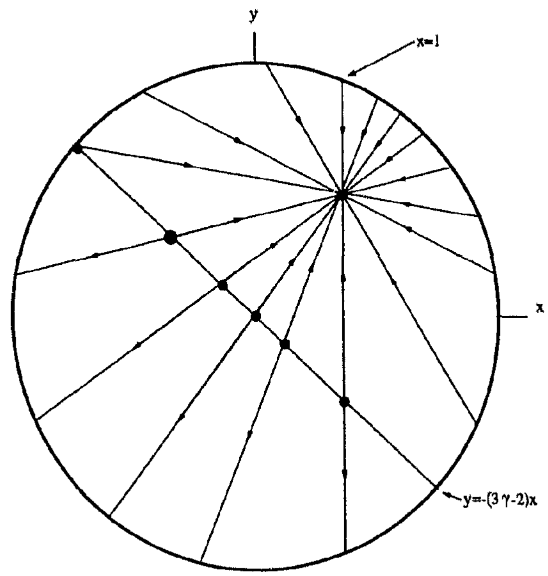


Figure 4.2: *The phase portrait describes the qualitative behavior of the FRW models with bulk viscous pressure in the case $m = r_1 = 1$ and $B_1 = 0$. The arrows in the figure denote increasing Ω-time ($\Omega \to \infty$) or decreasing t-time ($t \to 0^+$).*

The dynamical system, (4.7) is not differentiable at the singular point $(0,0)$. We can circumvent this problem by changing variables to $u^2 = x$ and a new time variable τ defined by $d\Omega/d\tau = u$. The system then becomes

$$\frac{du}{d\tau} = (1-u^2)[(3\gamma-2)u^2 + y], \tag{4.27}$$

$$\frac{dy}{d\tau} = u[9au^2 - 2y - y^2 - (3\gamma-2)u^2 + byu]. \tag{4.28}$$

In terms of the new variables the system is differentiable at the point $u = 0, y = 0$, but one of the eigenvalues is zero, hence the point is not hyperbolic. Therefore in order to determine the stability of the point we change to polar coordinates, and find that the point has some saddle-like properties; however, the true determination of the stability is difficult. [We investigate the nature of this singular point numerically. The integration and plotting was done using Maple V release 3. From the qualitative

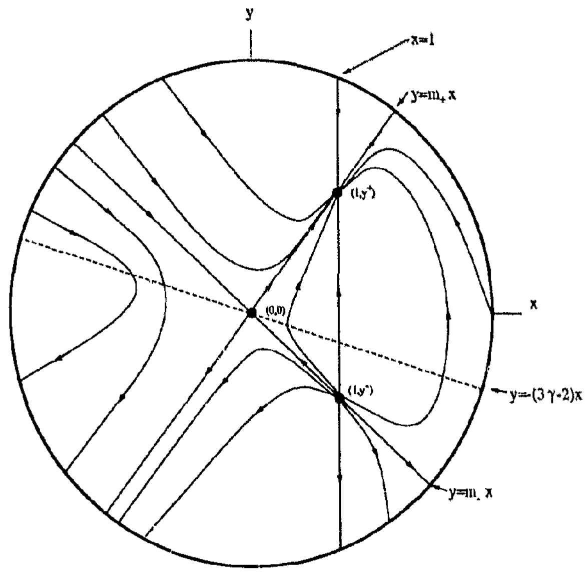


Figure 4.3: *The phase portrait describes the qualitative behavior of the FRW models with bulk viscous pressure in the case $m = r_1 = 1$ and $B_1 > 0$. The arrows in the figure denote increasing Ω-time ($\Omega \to \infty$) or decreasing t-time ($t \to 0^+$).*

analysis we find that the behaviour depends on the parameter B_1. In the first of these two plots we choose $\gamma = 1$, $a = 1/9$, and $b = 4$, so that $B_1 = -1 < 0$ (see Figure 4.4). In the second plot we choose $\gamma = 1$, $a = 1/9$, and $b = 2$ so that $B_1 = 1 > 0$ (see Figure 4.5). From the numerical plots we can conclude that the point $(0,0)$ has a saddle-point like nature, in agreement with preliminary remarks.]

The singular point $(\bar{x}, \bar{y})$ has eigenvalues

$$\frac{(3\gamma-2)^2+9a}{2(3\gamma-2)} \pm \frac{1}{2}\sqrt{\left(\frac{(3\gamma-2)^2+9a}{(3\gamma-2)}\right)^2 - 2(1-\bar{x})[9a+2(3\gamma-2)]}. \qquad (4.29)$$

This singular point varies both its position in phase space with its stability depending upon whether $\bar{x}$ is less than, equal to or greater than one. If $B_1 > 0$, then $\bar{x} > 1$ and the point is a saddle point. If $B_1 < 0$, then $\bar{x} < 1$ and the point is a source. Finally, if $B_1 = 0$ (the bifurcation value), then $\bar{x} = 1$ and the point is degenerate (discussed

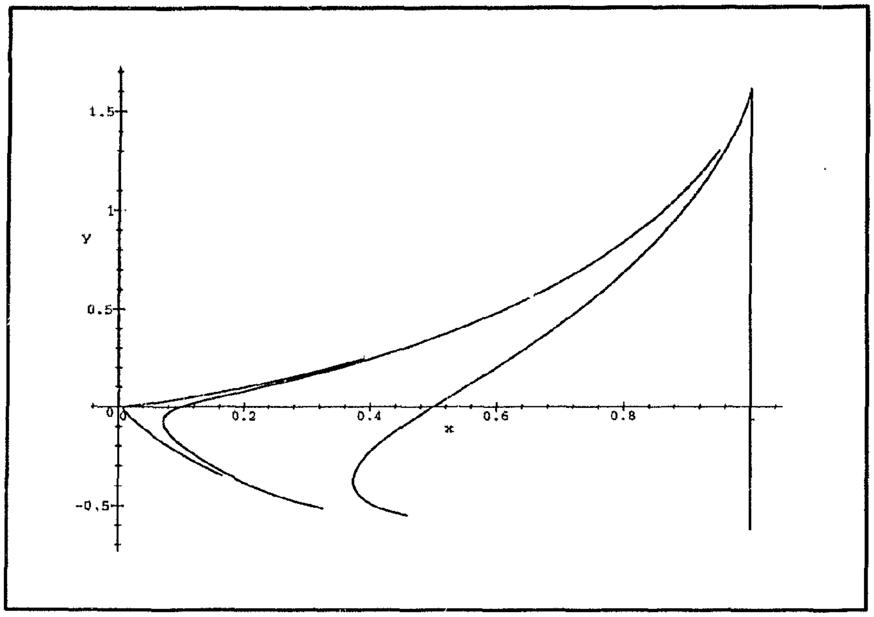


Figure 4.4: *The phase portrait describes the qualitative behavior of the FRW models in a neighborhood of the equilibrium point* $(0,0)$ *in the case* $m = 1/2$ *and* $B_1 = -1 < 0$.

later).

The stability of the points $(1, y^-)$ and $(1, y^+)$ is the same as in the previous case, see equations (4.16) and (4.17) for their eigenvalues and the corresponding text. (See also Table 4.1 for details).

The vertical isoclines occur at $x = 1$ and $y = -(3\gamma - 2)x$. This straight line is easily seen to pass through the origin and the point $(\bar{x}, -(3\gamma - 2)\bar{x})$. If $B_1 > 0$, then the vertical isocline lies below the point $(1, y^-)$ and if $B_1 < 0$, the vertical isocline lies above the point $(1, y^-)$. Finally if $B_1 = 0$ (the bifurcation value), the vertical isocline passes through the point $(1, y^-)$.

From an analysis similar to that in the previous subsection, we conclude that there is a non-isolated set of singular points at infinity. Their qualitative behaviour is the same in this case $m = 1/2$ as in the previous case $m = 1$; namely, points which lie

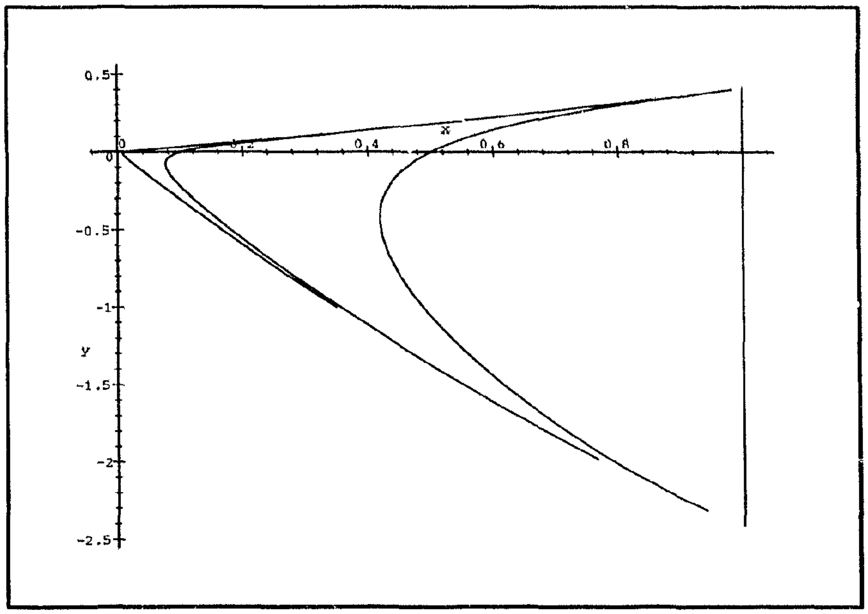


Figure 4.5: *The phase portrait describes the qualitative behavior of the FRW models in a neighborhood of the equilibrium point* $(0,0)$ *in the case* $m = 1/2$ *and* $B_1 = 1 > 0$.

above the line $y = -(3\gamma - 2)x$ are repellors, while those points which lie below the line are attractors.

At the bifurcation value $B_1 = 0$, the points $(1, y^-)$ and $(\bar{x}, -(3\gamma - 2)\bar{x})$ come together; consequently these points undergo a saddle-node bifurcation as B_1 passes through the value 0. The singular point is no longer hyperbolic, but the qualitative behaviour near the singular point can be determined from the fact that we know the nature of the bifurcation. Hence the singular point is a repelling node in one sector and a saddle in the others. A complete phase portrait is sketched in Figures 4.6, 4.7 and 4.8.

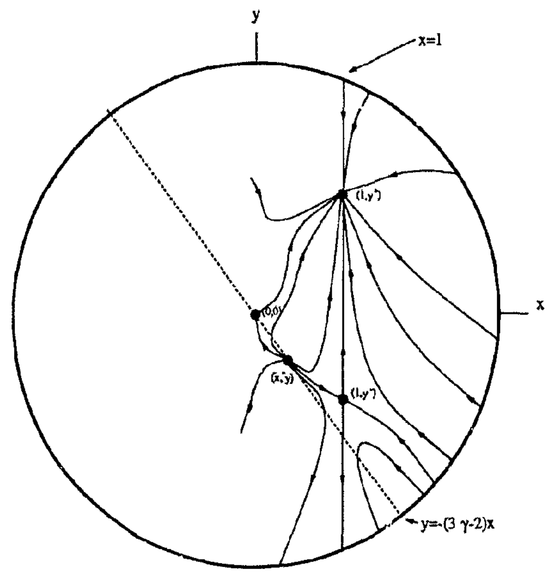


Figure 4.6: *The phase portrait describes the qualitative behavior of the FRW models with bulk viscous pressure in the case $m = 1/2$ and $r_1 = 1$ with $B_1 < 0$. The arrows in the figure denote increasing Ω-time ($\Omega \to \infty$) or decreasing t-time ($t \to 0^+$).*

4.2.3 Discussion

Exact Solutions

The exact solution of the Einstein field equations at each of the singular points represent the asymptotic solutions (both past and future) of FRW models with a causal viscous fluid source. The solution at each of the singular points represents a self similar cosmological model except in one isolated case [see the singular point $(1, y^-)$].

At the singular point $(0,0)$ we have (after a re-coordinatization)

$$\theta(t) = 3t^{-1}, \qquad a(t) = a_o t,$$
$$\rho(t) = 0, \qquad \Pi(t) = 0,$$

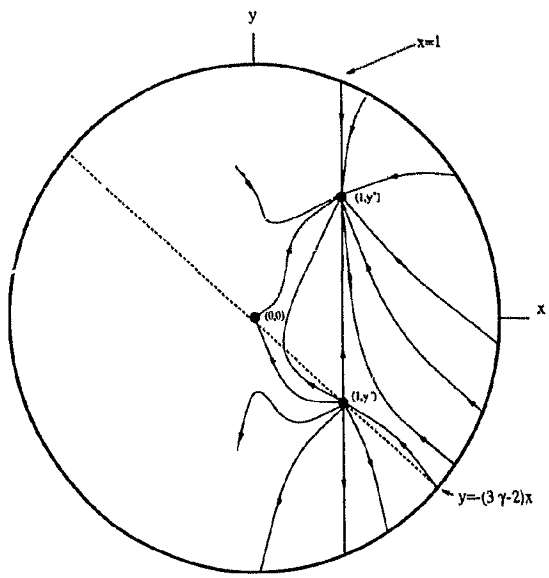


Figure 4.7: *The phase portrait describes the qualitative behavior of the FRW models with bulk viscous pressure in the case $m = 1/2$ and $r_1 = 1$ with $B_1 = 0$. The arrows in the figure denote increasing Ω-time ($\Omega \to \infty$) or decreasing t-time ($t \to 0^+$).*

which represents the standard vaccuum Milne model.

The singular point $(1, y^+)$ represents a flat FRW model with a solution (after a re-coordinatization)

$$\begin{aligned} \theta(t) &= 3A^+ t^{-1}, & a(t) &= a_o t^{A^+}, \\ \rho(t) &= 3(A^+)^2 t^{-2}, & \Pi(t) &= y^+(A^+)^2 t^{-2}, \end{aligned}$$

where $A^+ = 2/(3\gamma + y^+) > 0$.

The singular point $(1, y^-)$ represents a flat FRW model. If $\gamma \neq 3\zeta_0$ then the solution is

$$\begin{aligned} \theta(t) &= 3A^-(t - t_o)^{-1}, & a(t) &= a_o|t - t_o|^{A^-}, \\ \rho(t) &= 3(A^-)^2(t - t_o)^{-2}, & \Pi(t) &= y^+(A^-)^2(t - t_o)^{-2}, \end{aligned}$$

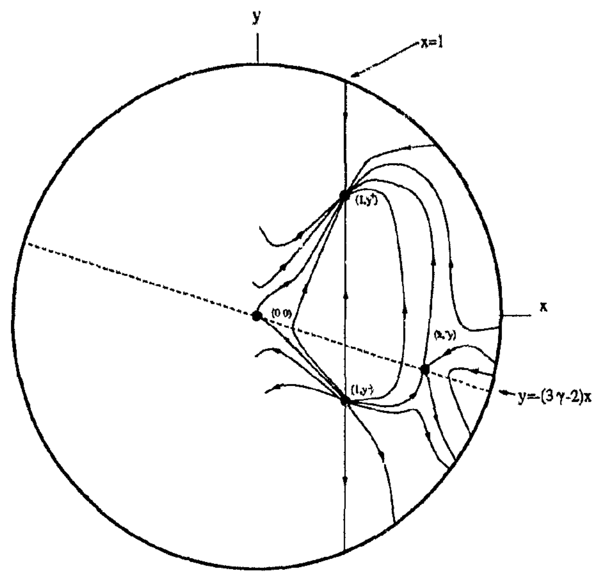


Figure 4.8: *The phase portrait describes the qualitative behavior of the FRW models with bulk viscous pressure in the case $m = 1/2$ and $r_1 = 1$ with $B_1 > 0$. The arrows in the figure denote increasing Ω-time ($\Omega \to \infty$) or decreasing t-time ($t \to 0^+$).*

where $A^- = 2/(3\gamma + y^-)$. (Note that in this case we cannot simply change coordinates to remove the constants of integration.) The sign of A^- depends on the sign of $\gamma - 3\zeta_0$. If $\gamma > 3\zeta_0$ then $A^- > 0$, and if $\gamma < 3\zeta_0$ then $A^- < 0$. Thus if $A^- < 0$ then θ is positive only in the interval $0 \leq t \leq t_o$, hence we can see that after a finite time t_o θ, ρ, and a all approach infinity. (We will see later that the WEC is violated in this case). If $A^- > 0$ then we can re-coordinatize the time t so as to remove the constant of integration, t_o, and the absolute value signs in the solution for $a(t)$. If $\gamma = 3\zeta_0$ then $A^- = 0$ and the solution is the de Sitter model with (after a re-coordinatization)

$$\theta(t) = 3H_o, \qquad a(t) = a_o e^{H_o t},$$
$$\rho(t) = 3{H_o}^2, \qquad \Pi(t) = {H_o}^2 y^-.$$

This exceptional solution is the only one that is not self-similar. It can be noted here that this is precisely the same situation that occurred in the Eckart models studied

Table 4.1: Qualitative nature of the equilibrium points of the dynamical system, (4.7), for different values of the parameter B_1[a] and $r_1 = 1$ (with respect to Ω-time).

	$(0,0)$	$(1,y^-)$[b]	$(1,y^+)$[c]	$(\bar{x},\bar{y})$[d]
$m = 1$ and $B_1 < 0$	source	saddle	sink	
$m = 1$ and $B_1 = 0$	source [e]	source [e]	sink	
$m = 1$ and $B_1 > 0$	saddle	source	sink	
$m = 1/2$ and $B_1 < 0$	saddle	saddle	sink	source
$m = 1/2$ and $B_1 = 0$	saddle	saddle-node [f]	sink	saddle-node [f]
$m = 1/2$ and $B_1 > 0$	saddle	source	sink	saddle

[a] $B_1 = (6-b)(\gamma-2) + 3a$.
[b] $y^- = \left(b - 3\gamma - \sqrt{(b-3\gamma)^2 + 36a}\right)/2$.
[c] $y^+ = \left(b - 3\gamma + \sqrt{(b-3\gamma)^2 + 36a}\right)/2$.
[d] $\bar{x} = (9a + 2(3\gamma-2))^2/b^2(3\gamma-2)^2, \qquad \bar{y} = -(3\gamma-2)\bar{x},$
[e] These points are part of the non-isolated line singularity $y = -(3\gamma-2)x$.
[f] This is the situation when the points $(1, y^-)$ and $(\bar{x},\bar{y})$ coalesce.

in [55].

The singular point $(\bar{x},\bar{y})$ represents either an open, flat, or closed model depending on the value of the parameter B_1. The solution in all cases is (after a re-coordinatization)

$$\theta(t) = 3t^{-1}, \qquad a(t) = a_o t,$$
$$\rho(t) = 3\bar{x}t^{-2}, \qquad \Pi(t) = \bar{y}t^{-2}.$$

Energy Conditions

The Energy conditions are given explicitly in Appendix C. Since the energy momentum tensor is diagonal in this case the energy conditions simplify immmensely. In the particular model under investigation here, the WEC in dimensionless variables becomes

$$x \geq 0 \quad \text{and} \quad y \geq -3\gamma x. \tag{4.30}$$

The dominant energy (DEC) becomes

$$\text{WEC and} \quad y \leq 3(\gamma - 2)x. \tag{4.31}$$

The strong energy condition (SEC) becomes

$$\text{WEC and} \quad y \geq -(3\gamma - 2)x. \tag{4.32}$$

If we assume that the WEC is satisfied throughout the evolution of these models then we find that there are five distinct situations. If $\gamma < 3\zeta_0$, then $B_1 > 0$ and the line $y = -3\gamma x$ intersects the line $x = 1$ at a point $y > y^-$. If $\gamma = 3\zeta_0$, then $B_1 > 0$ and the line $y = -3\gamma x$ intersects the line $x = 1$ at the point $y = y^-$. If $\gamma > 3\zeta_0$ then B_1 can be of any sign or zero, but the line $y = -3\gamma x$ intersects the line $x = 1$ at a point $y < y^-$. If the WEC condition is assumed to be satisfied throughout the evolution of these models then the possible asymptotic behaviour of the models is greatly restricted.

Asymptotic Behaviour

The qualitative behaviour depends on the values of B_1 and m. If the parameter m is different from unity then there is an additional singular point. This property is also present in the Eckart models studied in Chapter 3 and in [55]. The value $m = 1$ corresponds to the case when the dynamical system, (4.7), is polynomial (the only other value of m that exhibits this property is $m = 0$). The value $m = 1/2$ is of particular interest as it represents the asymptotic behaviour of all the viscous fluid FRW models, and also, this is the case when the equation of state for ζ is independent of θ (i.e., $\zeta \propto \rho^{1/2}$). The parameter, B_1, plays a role similar to the parameter $\Psi_1 = 9\zeta_0 - (3\gamma - 2)$ found in the previous chapter 3. The value of the parameter B_1 determines the stability and global behaviour of the system.

One of the goals of this section is to determine the generic behaviour of the system of equations (4.7). Using the above energy conditions, and in particular the

WEC, and the phase-portraits (Figures 4.1–4.8), we can determine the generic and exceptional behaviour of all the viscous fluid models satisfying the WEC. We are primarily interested in the generic asymptotic behaviour of the FRW model with viscosity: If we consider the dynamical system, (4.7), as $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$ where $\mathbf{X} = (x, y, a, b, \gamma) \subset \mathbb{R}^5$, x, y are the variables, and a, b, γ are the free parameters then generic behaviour occurs in sets of non-zero measure with respect to the set $\mathbb{R}^5$ (except for the flat models in which case the state space is a subset of $\mathbb{R}^4$). For example, the case $B_1 = 0$ is a set of measure zero with respect to the set $\mathbb{R}^5$. All behaviour is summarized in Table 4.2.

Typically, if $B_1 < 0$, then the open models evolve from the big-bang visco-elastic singularity at $(1, y^+)$ and evolve to the Milne model at $(0,0)$ [if $m = 1$] or the non-vacuum open model at $(\bar{x}, \bar{y})$ [if $m = 1/2$]. If $B_1 < 0$, then the closed models evolve from the big-bang visco-elastic singularity at $(1, y^+)$ to points at infinity. A visco-elastic singularity is a singularity in which a significant portion of the initial total energy is in the form of viscous elastic energy, that is $\Pi \gg 1$. These particular points at infinity correspond to the points where $\theta = 0$ (point of maximum expansion) and the various dimensionless variables breakdown.

Typical behaviour of models with $B_1 > 0$ depends upon the sign of $\gamma - 3\zeta_0$. If $\gamma < 3\zeta_0$, then all trajectories for the open models will violate the WEC and if $\gamma > 3\zeta_0$, then the open models evolve from the big-bang visco-elastic singularity at $(1, y^+)$, become open models and then evolve towards an inflationary flat FRW model at the point $(1, y^-)$. Concerning the closed models when $B_1 > 0$, if $\gamma < 3\zeta_0$ then models evolve from the big-bang visco-elastic singularity at $(1, y^+)$ to points at infinity. However, if $\gamma > 3\zeta_0$, then the closed models again evolve from the big-bang visco-elastic singularity at $(1, y^+)$ but now have two different typical behaviours. There is a class of models which approach points at infinity and do not inflate and there is a class of models which evolve towards the inflationary flat FRW model at the point $(1, y^-)$.

The flat FRW models consist of a subset of measure zero of the total state space

Table 4.2: Asymptotic behaviour of the FRW models with bulk viscosity satisfying the WEC with $r_1 = 1$ (with respect to t-time).

parameters	m	models	generic behaviour	exceptional behaviour[a]
$\gamma < 3\zeta_0$, $B_1 > 0$	$m = 1, 1/2$	open		$(1, y^+) \to (0,0)$
		flat	$(1, y^+) \to \infty$	
		closed	$(1, y^+) \to \infty$	$(\bar{x}, \bar{y}) \to \infty$, $(1, y^+) \to (\bar{x}, \bar{y})$[b]
$\gamma \geq 3\zeta_0$, $B_1 > 0$	$m = 1, 1/2$	open	$(1, y^+) \to (1, y^-)$	$(1, y^+) \to (0,0)$, $(0,0) \to (1, y^-)$
		flat	$(1, y^+) \to \infty$	
			$(1, y^+) \to (1, y^-)$	
		closed	$(1, y^+) \to \infty$	$(\bar{x}, \bar{y}) \to (1, y^-)$
			$(1, y^+) \to (1, y^-)$	$(\bar{x}, \bar{y}) \to \infty$, $(1, y^+) \to (\bar{x}, \bar{y})$[b]
$\gamma > 3\zeta_0$, $B_1 = 0$	m=1	open	$(1, y^+) \to (x_o, y_o)$[c]	
		flat	$(1, y^+) \to \infty$	
			$(1, y^+) \to (1, y^-)$	
		closed	$(1, y^+) \to \infty$	
			$(1, y^+) \to (x_o, y_o)$[c]	
	m=1/2	open	$(1, y^+) \to (1, y^-)$	$(1, y^+) \to (0,0)$, $(0,0) \to (1, y^-)$
		flat	$(1, y^+) \to \infty$	
			$(1, y^+) \to (1, y^-)$	
		closed	$(1, y^+) \to \infty$	$(1, y^-) \to \infty$
$\gamma > 3\zeta_0$, $B_1 < 0$	m=1	open	$(1, y^+) \to (0,0)$	$(1, y^-) \to (0,0)$
		flat	$(1, y^+) \to \infty$	
			$(1, y^+) \to (1, y^-)$	
		closed	$(1, y^+) \to \infty$	$(1, y^-) \to \infty$
	m=1/2	open	$(1, y^+) \to (\bar{x}, \bar{y})$	$(1, y^+) \to (0,0)$, $(1, y^-) \to (\bar{x}, \bar{y})$
		flat	$(1, y^+) \to \infty$	
			$(1, y^+) \to (1, y^-)$	
		closed	$(1, y^+) \to \infty$	$(1, y^-) \to \infty$

[a] These are exceptional trajectories and do not represent typical or generic behaviour.

[b] In the case $m = 1/2$.

[c] $y_o = -(3\gamma - 2)x_o$.

$\mathbb{R}^5$. The flat models are of a special interest however, in that the flat models represent the past asymptotic behaviour of both the open and closed models. If $B_1 < 0$, then the flat models evolve from the visco-elastic singularity at $(1, y^+)$ to points at infinity or to the flat model located at $(1, y^-)$. If $B_1 > 0$, and $\gamma < 3\zeta_0$, then the flat models evolve from the big-bang visco-elastic singularity at $(1, y^+)$ to points at infinity. And if $B_1 > 0$ and $\gamma > 3\zeta_0$, then models evolve from the visco-elastic singularity at $(1, y^+)$ to points at infinity (non-inflationary) or to the inflationary model at the point $(1, y-)$.

Note, that if the WEC is dropped (i.e., let $\gamma < 3\zeta_0$), then a class of very interesting models occurs. There will exist models that will evolve from the visco-elastic singularity at $(1, y^+)$ with $\theta > 0$ and $\dot{\theta} < 0$ start inflating at some point t_i and at a finite time after t_i will start expanding at increasing rates, that is $\dot{\theta} > 0$, and will eventually evolve towards the point $(1, y^-)$. (This is the special case mentioned in the previous subsection.) What this means in terms of the open and flat models is that they will expand with decreasing rates of expansion, start to inflate, and then continue to expand with increasing rates of expansion. For the closed models, the models will expand with decreasing rates of expansion, start to inflate, and then continue to expand with increasing rates of expansion, these models will not recollapse.

First Integrals

We will use Darboux's theorem [73] to find an algebraic first integral of the system in the case $m = r_1 = 1$ by first finding a number of a algebraic invariant curves. The following are invariant curves of the system:

$$\begin{aligned} Q_1 &= x - 1, \\ Q_2 &= y - m_- x, \\ Q_3 &= y - m_+ x, \end{aligned}$$

where

$$m_\pm = \frac{(b - 3\gamma) \pm \sqrt{(b - 3\gamma)^2 + 36a}}{2}. \tag{4.33}$$

Calculating $\dot{Q}_i = r_i Q_i$, we find

$$\begin{aligned} r_1 &= -[y + (3\gamma - 2)x], \\ r_2 &= -[y + (3\gamma - 2)x + (2 - b + m_-)], \\ r_3 &= -[y + (3\gamma - 2)x + (2 - b + m_+)]. \end{aligned}$$

Using Darboux's Theorem, an algebraic first integral Q can be found by setting $Q = Q_1^{\alpha_1} Q_2^{\alpha_2} Q_3^{\alpha_3}$ and then determining what values of α_i satisfy the equation $\dot{Q} = 0$. Solving the resulting algebraic system we find the following algebraic first integral of the dynamical system, (4.7), in the case $m = r_1 = 1$:

$$Q = (x-1)^{\alpha_1}(y - m_- x)^{\alpha_2}(y - m_+ x)^{\alpha_3} = K \text{ (constant)} \tag{4.34}$$

where α_1 is a free parameter and α_2 and α_3 must satisfy

$$\begin{aligned} \alpha_2 &= \left(-\frac{1}{2} - \frac{b + 3\gamma - 4}{\sqrt{(b - 3\gamma)^2 + 36a}} \right) \alpha_1, \\ \alpha_3 &= \left(-\frac{1}{2} + \frac{b + 3\gamma - 4}{\sqrt{(b - 3\gamma)^2 + 36a}} \right) \alpha_1. \end{aligned} \tag{4.35}$$

This first integral determines the integral curves of the phase portraits in Figures 4.1, 4.2 and 4.3, where the value K determines which integral curve(s) is being described. For example, if $K = 0$, the integral curves are $x = 1$ and $y = m_\pm x$. Also, we can see that if $b = 4 - 3\gamma$, $\alpha_1 = -2$ and $K = 1$ then the integral curve describes an ellipse; however, these closed curves necessarily pass through the points $(1, y^+)$ and $(1, y^-)$, thereby nullifying the possible existence of closed orbits.

4.2.4 Summary of the Isotropic Models

The only models that can possibly satisfy the WEC and inflate are those models with $\gamma > 3\zeta_0$ and $B_1 > 0$. Therefore, we can conclude that bulk-viscous inflation is possible in the truncated Israel-Stewart models. However, in the models studied by Hiscock and Salmonson [48] inflation did not occur (note, the equations of state assumed in [48] are derived from assuming that the universe could be modelled as a Boltzmann gas), while inflation does occur in the models studied by Zakari and Jou [52] who utilized different equations of state. In our truncated model we choose dimensionless equations of state and find that inflation is sometimes possible. The question of which equations of state are most appropriate remains unanswered, and clearly the possibility of inflation depends critically upon the equations of state utilized [52].

This work improves over previous work on viscous cosmology using the non-causal and unstable first order thermodynamics of Eckart [47] and differs from the work of Belinskii et al. [44] in that dimensionless equations of state are utilized. From the previous discussion we can conclude that the visco-elastic singularity at the point $(1, y^+)$ is a dominant feature in our truncated models. This singular point remains the typical past asymptotic attractor for various values of the parameters m and B_1. This agrees with the results of Belinskii et al. [44]. The future asymptotic behaviour depends upon both the values of m and B_1. If $B_1 < 0$, then the open models tend to the Milne model at $(0,0)$ $[m = 1]$ or to the open model at $(\bar{x}, \bar{y})$ $[m = 1/2]$, and if $B_1 > 0$, then the open models tend to the inflationary model at the point $(1, y^-)$ or are unphysical. If $B_1 < 0$, then the closed models tend to points at infinity, and if $B_1 > 0$, then the closed models tend to the inflationary model at the point $(1, y^-)$. The future asymptotic behaviour of the flat models is that they either tend to points at infinity or tend to the point $(1, y^-)$, in agreement with the exact solution given in [57].

Belinskii et al. [44] utilized the physical variables θ, ρ, and Π in their analysis and assumed non-dimensionless equations of state, and they found a singularity in which the expansion was zero but the metric coefficients were neither infinite nor zero; the authors passed over this observation stating that in a more realistic theory this undesirable asymptotic behaviour would not occur. We note, by using dimensionless variables and and a set of dimensionless equations of state, all asymptotic behaviour in our models can be represented by either a de Sitter model or by an exact power-law solution (i.e., the length scale $a(t) \propto t^\alpha$ for some constant α). Therefore the un-desirable behaviour observed in [44] does not occur in our analysis.

The behaviour of the Eckart models in Chapter 3 and in Burd and Coley [55] with $\Psi_1 \equiv 9\zeta_0-(3\gamma-2) > 0$ is very similar to the behaviour of the truncated Israel-Stewart models studied here in the case $B_1 < 0$. This result also agrees with the conclusions of Zakari and Jou [52]. However, when $B_1 > 0$ various new possibilities can occur; for instance, there exist open and closed models that asymptotically approach a flat FRW model both to the past and to the future. Interestingly enough this is also the case in which the future asymptotic endpoint is an inflationary attractor. This type of behaviour does not occur in the Eckart theory.

4.3 Anisotropic Models

4.3.1 Introduction

Recently, Romano and Pavón [58, 77] have studied anisotropic cosmological models in a causal theory of irreversible thermodynamics, analyzing the stability of the isotropic equilibrium points in the Bianchi type I and III models. They also assumed equations of state of the form (1.4). However, they concluded that any initial anisotropy dies away rapidly but the shear viscous stress need not vanish, hence neither the de Sitter models nor the Friedmann models are attractors.

In this section we shall analyze qualitatively a class of anisotropic cosmological models arising from the use of the truncated Israel-Stewart equations, thereby expanding the analysis in section 4.2 to anisotropic models and extending the analysis in Chapter 3 and in [53, 54, 55] to causal theories. We will analyze both the Bianchi type V and the Bianchi type I models, which are simple generalizations of the open and flat FRW models.

4.3.2 The Equations

The dimensionless Einstein field equations are given by equations (3.11-3.16), where in the Bianchi V case studied here, ${}^3R = -6a(t)^{-2}$. In order to close the system we need equations for y, z_1, and z_2 and therefore for Π, Π_1, and Π_2.

Assuming that Π, Π_1, and Π_2 can be described by the truncated Israel-Stewart theory [49, 51], the evolution equations for Π, Π_1, and Π_2 in this particular model are derived from (1.3) [with $\epsilon = 0$],

$$\begin{aligned}
\dot{\Pi} &= -\frac{\Pi}{\beta_0\zeta} - \frac{1}{\beta_0}\left(\theta + \frac{2}{9}\alpha_0(\sigma_1+\sigma_2)(\theta^2 - 3\sigma^2 - 3\rho)\right), \\
\dot{\Pi}_1 &= -\frac{\Pi_1}{2\eta\beta_2} - \frac{1}{\beta_2}\left(\sigma_1 - \frac{1}{9}\alpha_1(\sigma_1+\sigma_2)(\theta^2 - 3\sigma^2 - 3\rho)\right), \qquad (4.36)\\
\dot{\Pi}_2 &= -\frac{\Pi_2}{2\eta\beta_2} - \frac{1}{\beta_2}\left(\sigma_2 - \frac{1}{9}\alpha_1(\sigma_1+\sigma_2)(\theta^2 - 3\sigma^2 - 3\rho)\right).
\end{aligned}$$

Note that the heat conduction q_1 is completely determined by the shear via equation (3.8); thus the equation for q^a in equations (1.3) is not needed for the determination of the asymptotic behaviour of the models. The corresponding evolution equations for y, z_1 and z_2 are:

$$\begin{aligned}
\frac{dy}{d\Omega} &= y\left[\left(\frac{\theta}{\zeta}\right)\left(\frac{3}{\beta_0\theta^2}\right) - 2 - 2q\right] + 9\left(\frac{3}{\beta_0\theta^2}\right) \\
&\quad + \left(\frac{3}{\beta_0\theta^2}\right)\left(\frac{\alpha_0\theta^2}{4\sqrt{3}}\right)(\Sigma_1+\Sigma_2)(4 - 4x - \Sigma^2), \\
\frac{dz_1}{d\Omega} &= z_1\left[2\left(\frac{\theta}{\eta}\right)\left(\frac{3}{4\beta_2\theta^2}\right) - 2 - 2q\right] + \Sigma_1\left(\frac{3}{4\beta_2\theta^2}\right) \\
&\quad - \left(\frac{3}{4\beta_2\theta^2}\right)\left(\frac{\alpha_1\theta^2}{36}\right)(\Sigma_1+\Sigma_2)(4 - 4x - \Sigma^2), \qquad (4.37)\\
\frac{dz_2}{d\Omega} &= z_2\left[2\left(\frac{\theta}{\eta}\right)\left(\frac{3}{4\beta_2\theta^2}\right) - 2 - 2q\right] + \Sigma_2\left(\frac{3}{4\beta_2\theta^2}\right) \\
&\quad - \left(\frac{3}{4\beta_2\theta^2}\right)\left(\frac{\alpha_1\theta^2}{36}\right)(\Sigma_1+\Sigma_2)(4 - 4x - \Sigma^2).
\end{aligned}$$

In order to complete the system of equations we need to specify equations of state for the quantities p, ζ, η, β_0, β_2, α_0 and α_1. In principle, equations of state can be derived from kinetic theory, but in practice one must specify phenomenological equations of state which may or may not have any physical foundations. We shall introduce dimensionless equations of state similar to that used in section 4.2 of the form:

$$
\begin{aligned}
&\frac{p}{\theta^2} = p_o x^\ell, & & \\
&\frac{\zeta}{\theta} = \zeta_o x^m, & &\frac{\eta}{\theta} = \eta_o x^n, \\
&\frac{3}{\beta_0\theta^2} = a_1 x^{r_1}, & &\frac{3}{4\beta_2\theta^2} = a_2 x^{r_2}, \\
&\frac{\alpha_0\theta^2}{4\sqrt{3}} = d_1 x^{p_1}, & &\frac{\alpha_1\theta^2}{36} = d_2 x^{p_2},
\end{aligned}
\tag{4.38}
$$

where p_o, ζ_o, η_o, a_i and d_i $(1 \le i \le 2)$ are positive constants, and ℓ, m, n, r_i and p_i $(1 \le i \le 2)$ are constant parameters (x is the dimensionless density parameter defined earlier). In the models under consideration θ is strictly positive, thus equations (4.38) are well defined.

We define new constants $b_1 = a_1/\zeta_o$, $b_2 = a_2/\eta_o$, $c_1 = a_1 d_1$ and $c_2 = a_2 d_2$. Augmenting the system of equations (3.11–3.13), and (4.37), and employing the equations of state (4.38), the following system results:

$$
\begin{aligned}
\frac{dx}{d\Omega} &= x(1-2q) + 9p_o x^\ell + y + \Sigma_1(2z_1 - z_2) + \Sigma_2(2z_2 - z_1) \\
&\qquad - \frac{1}{4\sqrt{3}}(\Sigma_1 + \Sigma_2)(4 - 4x - \Sigma^2), && (4.39) \\
\frac{d\Sigma_1}{d\Omega} &= \Sigma_1(2-q) - 12z_1, && (4.40) \\
\frac{d\Sigma_2}{d\Omega} &= \Sigma_2(2-q) - 12z_2, && (4.41) \\
\frac{dy}{d\Omega} &= y(b_1 x^{r_1 - m} - 2 - 2q) + 9a_1 x^{r_1} \\
&\qquad + c_1 x^{p_1 + r_1}(\Sigma_1 + \Sigma_2)(4 - 4x - \Sigma^2), && (4.42) \\
\frac{dz_1}{d\Omega} &= z_1(2b_2 x^{r_2 - n} - 2 - 2q) + a_2 x^{r_2}\Sigma_1 \\
&\qquad - c_2 x^{p_2 + r_2}(\Sigma_1 + \Sigma_2)(4 - 4x - \Sigma^2), && (4.43) \\
\frac{dz_2}{d\Omega} &= z_2(2b_2 x^{r_2 - n} - 2 - 2q) + a_2 x^{r_2}\Sigma_2 \\
&\qquad - c_2 x^{p_2 + r_2}(\Sigma_1 + \Sigma_2)(4 - 4x - \Sigma^2), && (4.44)
\end{aligned}
$$

where $\Sigma^2 \equiv \frac{1}{3}(\Sigma_1+\Sigma_2)^2-\Sigma_1\Sigma_2$ and the dimensionless deceleration parameter is given by

$$q=\frac{1}{2}\left(x+y+9p_o x^l+\Sigma^2\right). \tag{4.45}$$

From (3.14) we have

$$\Sigma_1+\Sigma_2=-2\sqrt{3}\frac{q_1}{\theta}, \tag{4.46}$$

and finally, from the Friedmann equation, (3.16), we obtain the inequality

$$4-4x-\Sigma^2=\frac{36}{a^2\theta^2}\geq 0. \tag{4.47}$$

The interior of the parabola $4=\Sigma^2+4x$ in the phase space represents models of Bianchi type V, while the parabola itself represents models of Bianchi type I. There are other physical constraints that may be imposed, namely the energy conditions [65], which will place bounds on the variables x, y, Σ_1, Σ_2, z_1, and z_2. A full list of the energy conditions is given in Appendix C. We shall always assume that $x\geq 0$, which states that the energy density be non-negative, which is a necessary condition of the weak energy condition (WEC) [69].

The equilibrium points of the above system all represent self-similar cosmological models, except for those equilibrium points that satisfy $\dot{\theta}/\theta^2=-(q+1)/3=0$. If $q\neq -1$, the nature of the equations of state (4.38) at the equilibrium points, is independent of the parameters l, m, n, r_1, r_2, p_1, and p_2, and is given by

$$p\propto\rho,\qquad \zeta\propto\rho^{\frac{1}{2}},\qquad \eta\propto\rho^{\frac{1}{2}},$$

$$\beta_0\propto\rho^{-1},\qquad \beta_2\propto\rho^{-1},\qquad \alpha_0\propto\rho^{-1},\quad\text{and}\quad \alpha_1\propto\rho^{-1}. \tag{4.48}$$

Therefore natural choices for l, m, n, r_1, r_2, p_1 and p_2 are respectively 1, 1/2, 1/2, 1, 1, -1, -1. We note, if there exists a equilibrium point with $q=-1$, then it necessarily represents a de Sitter type solution which is not self-similar.

The most commonly used equation of state for the pressure is the barotropic equation of state $p=(\gamma-1)\rho$, whence from (4.38) $p_o=\frac{1}{3}(\gamma-1)$ and $l=1$ (where $1\leq\gamma\leq 2$ is necessary for local mechanical stability and for the speed of sound in the fluid to be no greater than the speed of light). In addition, $l = 1$ reflects the asymptotic behaviour of the equation of state for p.

Using analogous arguments found in the previous section we set $r_1 = 1$, and therefore the requirement that viscous pulses travel at sub-light speed translates to the condition $0 < a_1 < 1$. In this way the parameter a_1 has a physical interpretation as the square of the speed of a viscous pulse in the fluid. Therefore, in the remainder of this analysis we shall choose $r_1 = r_2 = 1$. In an effort to keep the system polynomial and therefore tractable we shall choose $m = n = 1$ and $p_1 = p_2 = -1$.

Using these particular values for m, n, r_1, r_2, p_1, and p_2, we can easily show that all equilibrium points are self-similar except in the case $\gamma = 3\zeta_o$, whence the equilibrium point $(x, \Sigma_1, \Sigma_2, y, z_1, z_2) = (1, 0, 0, -3\gamma, 0, 0)$ represents a de Sitter model. This is precisely the same as in in Chapter 3 where the Eckart theory was employed.

The full six-dimensional system (4.39–4.47) is very difficult to analyze completely, so various physically interesting subsystems are investigated. The case of zero heat conduction implies, via equations (4.46) that $\Sigma_1 + \Sigma_2 = 0$. In addition, adding equations (4.40) and (4.41) we deduce that $z_1 + z_2 = 0$, in which case the resulting system is four-dimensional (see Section 4.3.3). The case of non-zero heat conduction with zero anisotropic stress is a three dimensional system and is discussed in Section 4.3.4.

4.3.3 Qualitative Analysis — Zero Heat Conduction

In the case of zero heat conduction, $q_1 = 0$, the field equations imply that $\Sigma_1 + \Sigma_2 = 0$ and $z_1 + z_2 = 0$. Since, $\Sigma^2 = {\Sigma_1}^2$, we shall drop the subscripts on Σ and z; that is, $\Sigma \equiv \Sigma_1 = -\Sigma_2$ and $z \equiv z_1 = -z_2$. The system of equations (4.39–4.47) then becomes:

$$\begin{aligned} x' &= x(3\gamma - 2 - 2q) + y + 6z\Sigma, \\ \Sigma' &= \Sigma(2 - q) - 12z, \\ y' &= y(b_1 - 2 - 2q) + 9a_1 x, \end{aligned} \tag{4.49}$$

$$z' = z(2b_2 - 2 - 2q) + a_2 x\Sigma,$$

where

$$q = \frac{1}{2}\left((3\gamma - 2)x + y + \Sigma^2\right), \tag{4.50}$$

and the physical phase space is

$$4 - 4x - \Sigma^2 \geq 0 \qquad \text{and} \qquad x \geq 0. \tag{4.51}$$

There exists three obvious and physically motivated invariant sets of the system. They are $\mathcal{FRW} := \{(x, \Sigma, y, z) | \Sigma = z = 0\}$, $\mathcal{BI} := \{(x, \Sigma, y, z) | 4 - 4x - \Sigma^2 = 0, \text{and} \Sigma \neq 0\}$, and $\mathcal{BV} := \mathcal{BI}^c \cap \mathcal{FRW}^c$ (where subscript c denotes the complement) which represents the Bianchi type V models. The set $\mathcal{FRW}$ represents the spatially homogeneous and isotropic negative and flat curvature FRW models and the set $\mathcal{BI}$ represents the Bianchi type I models. There are possibly eleven different equilibrium points of the system (4.49) at finite values. The equilibrium points lying in the set $\mathcal{FRW}$ are

$$(0, 0, 0, 0), \qquad (1, 0, y^-, 0), \qquad (1, 0, y^+, 0), \tag{4.52}$$

where

$$y^\pm = \frac{b_1 - 3\gamma}{2} \pm \frac{1}{2}\sqrt{(b_1 - 3\gamma)^2 + 36a_1}. \tag{4.53}$$

Also, if $B_1 = 0$ then there is a non-isolated line of equilibrium points that passes through the points $(0, 0, 0, 0)$ and $(1, 0, y^-, 0)$, where B_1 is,

$$B_1 = (3\gamma - 2)(2 - b_1) + 9a_1. \tag{4.54}$$

These points represent open ($x = 0$) and flat ($x = 1$) FRW models. There are possibly six equilibrium points lying in the $\mathcal{BI}$ invariant set. They are

$$(0, -2, 0, 0), \qquad (0, +2, 0, 0) \tag{4.55}$$

which represent Kasner models, and

$$\begin{gathered}(\bar{x}^+, +\bar{\Sigma}^+, \bar{y}^+, +\bar{z}^+), \qquad (\bar{x}^+, -\bar{\Sigma}^+, \bar{y}^+, -\bar{z}^+) \\ (\bar{x}^-, +\bar{\Sigma}^-, \bar{y}^-, +\bar{z}^-), \qquad (\bar{x}^-, -\bar{\Sigma}^-, \bar{y}^-, -\bar{z}^-)\end{gathered} \tag{4.56}$$

where

$$
\begin{aligned}
\bar{x}^{\pm} &= \frac{(\bar{q}^{\pm}-2)}{6a_2}(b_2-1-\bar{q}^{\pm}), \\
(\bar{\Sigma}^{\pm})^2 &= 4-4\bar{x}^{\pm}, \\
\bar{y}^{\pm} &= \frac{(\bar{q}^{\pm}-2)}{2a_2}\Big(4a_2+(2-\gamma)(b_2-1-\bar{q}^{\pm})\Big), \\
\bar{z}^{\pm} &= \frac{-\bar{\Sigma}^{\pm}}{12}(\bar{q}^{\pm}-2),
\end{aligned}
$$

and $\bar{q}^{\pm}$ is given by

$$
\bar{q}^{\pm} = \frac{(S_1+2S_2) \pm \sqrt{(S_1-2S_2)^2+96a_1a_2}}{4(2-\gamma)}, \tag{4.57}
$$

where

$$
\begin{aligned}
S_1 &= (2-\gamma)(b_1-2)+3a_1, &\quad (4.58)\\
S_2 &= (2-\gamma)(b_2-1)+4a_2. &\quad (4.59)
\end{aligned}
$$

The remaining two equilibrium points lie in the $\mathcal{BV}$ invariant set. They are

$$
\begin{aligned}
&\left(\frac{(1-b_2)}{3a_2}, +\sqrt{\frac{(1-b_2)B_1}{3a_2(2-b_1)}}, \frac{3a_1(1-b_2)}{a_2(2-b_1)}, +\frac{1}{6}\sqrt{\frac{(1-b_2)B_1}{3a_2(2-b_1)}}\right), \\
&\left(\frac{(1-b_2)}{3a_2}, -\sqrt{\frac{-(1-b_2)B_1}{3a_2(2-b_1)}}, \frac{3a_1(1-b_2)}{a_2(2-b_1)}, -\frac{1}{6}\sqrt{\frac{-(1-b_2)B_1}{18a_2(2-b_1)}}\right),
\end{aligned} \tag{4.60}
$$

where B_1 is given by equation (4.54).

The stability of each of these equilibrium points is very difficult to determine in general. However, one question that can be asked is whether these anisotropic models generally isotropize; that is, "Does there exist a stable (t-time) equilibrium point in the set $\mathcal{FRW}$? (see the following subsection). We shall also analyze the model when there is zero anisotropic stress ($z=0$) in order to determine the effects that bulk viscous pressure may have on the models (see two subsections ahead). We shall also analyze the effect of anisotropic stress in an anisotropic model with zero bulk viscous pressure ($y=0$). (see three subsections ahead).

Stability of Isotropic Singular Points

In this subsection we are going to resolve the stability of the isotropic equilibrium points, that is, those equilibrium points lying in the set $\mathcal{FRW}$. We want to determine if there exists a stable (t-time) equilibrium point in the future. In Ω-time this translates to showing that there exists a source in the set $\mathcal{FRW}$.

The equilibrium point $(0,0,0,0)$ represents the Milne model, and has eigenvalues

$$\begin{aligned} 2, \qquad 2(b_2-1), \tfrac{1}{2}\left\{(3\gamma+b_1-4)+\sqrt{(3\gamma+b_1-4)^2+4B_1}\right\}, \\ \tfrac{1}{2}\left\{(3\gamma+b_1-4)-\sqrt{(3\gamma+b_1-4)^2+4B_1}\right\}, \end{aligned} \tag{4.61}$$

where B_1 is given by equation (4.54). The bifurcation values are $b_2 = 1$ and $B_1 = 0$. If $B_1 = 0$ then there exists a non-isolated line of equilibrium points. The stability of this point is summarized in Table 4.3.

The equilibrium point $(1,0,y^-,0)$ represents a flat viscous fluid FRW model and has eigenvalues

$$\begin{aligned} -\frac{1}{2}\left\{(b_1+3\gamma-4)-\sqrt{(b_1+3\gamma-4)^2+4B_1}\right\}, \qquad \sqrt{(b_1+3\gamma-4)^2+4B_1}, \\ \frac{1}{4}\left\{B_3+\sqrt{B_3{}^2-8B_2}\right\}, \qquad \frac{1}{4}\left\{B_3-\sqrt{B_3{}^2-8B_2}\right\}, \end{aligned} \tag{4.62}$$

where

$$B_2 = (2b_2-3\gamma-y^-)(6-3\gamma-y^-)+24a_2, \tag{4.63}$$

$$B_3 = 4b_2-3(3\gamma-2)-3y^-. \tag{4.64}$$

The stability of this point is summarized in Table 4.4.

The equilibrium point $(1,0,y^+,0)$ represents a flat viscous fluid FRW model and has eigenvalues

$$\begin{aligned} -\frac{1}{2}\left\{(b_1+3\gamma-4)+\sqrt{(b_1+3\gamma-4)^2+4B_1}\right\}, \qquad -\sqrt{(b_1+3\gamma-4)^2+4B_1}, \\ \frac{1}{4}\left\{B_5+\sqrt{B_5{}^2-8B_4}\right\}, \qquad \frac{1}{4}\left\{B_5-\sqrt{B_5{}^2-8B_4}\right\}, \end{aligned} \tag{4.65}$$

Table 4.3: Stability of the equilibrium point $(0,0,0,0)$ where $\dim(W^s)$ is the dimension of the stable manifold with respect to Ω-time.

$\mathrm{sgn}(B_1)$	$\mathrm{sgn}(b_2 - 1)$	type	$\dim(W^s)$
+	+	Saddle	1
+	−	Saddle	2
−	+	Source	0
−	−	Saddle	1

Table 4.4: Stability of the equilibrium point $(1,0,y^-,0)$ where $\dim(W^s)$ is the dimension of the stable manifold with respect to Ω-time.

$\mathrm{sgn}(B_1)$	$\mathrm{sgn}(B_2)$	$\mathrm{sgn}(B_3)$	type	$\dim(W^s)$
+	−		Saddle	1
+	+	+	Source	0
−	−		Saddle	2
−	+	+	Saddle	1
−	+	−	Saddle	3

where

$$B_4 = (2b_2 - 3\gamma - y^+)(6 - 3\gamma - y^+) + 24a_2, \tag{4.66}$$

$$B_5 = 4b_2 - 3(3\gamma - 2) - 3y^+. \tag{4.67}$$

The stability of this point is summarized in Table 4.5.

From the stability analysis of these equilibrium points we can conclude that there exists a range of parameter values such that one of the equilibrium points in the set $\mathcal{FRW}$ is a source (sink in t-time). If $B_1 < 0$ and $b_2 > 1$ then the point $(0,0,0,0)$ is a source — this result is similar to the observation in Chapter 3 using the Eckart theory when $m = n = 1$ and $\Psi_1 = 9\zeta_o - (3\gamma - 2) < 0$. If $B_1 > 0$, $B_2 > 0$, and $B_3 > 0$ then the point $(1, 0, y^-, 0)$ is a source. However, if either of these two conditions are not satisfied then the anisotropic models will not tend to an isotropic FRW model to the future (t-time). Therefore there is a set of parameter values having non-zero measure such that the models will not isotropize. Romano and Pavón [77, 58] remarked that the anisotropy dies away quickly in the anisotropic models but does not tend to an FRW or de Sitter model since the anisotropic stress does not tend to zero. The same result is true here for some range of parameter values. If $b_2 < 1$ and $B_1 < 0$ then the models all isotropize but the anisotropic stress does not tend to zero and therefore the model does not asymptotically approach an FRW model.

Zero Anisotropic Stress

In order to observe the effects of bulk viscous pressure in the model we set $\Pi_1 = \Pi_2 = 0$, and $\eta = 0$. In the model under consideration here this amounts to setting $z = 0$ and $a_2 = 0$ in system (4.49). The resulting system is three-dimensional and has the form

$$\begin{aligned} x' &= x(3\gamma - 2 - 2q) + y, \\ \Sigma' &= \Sigma(2 - q), \\ y' &= y(b_1 - 2 - 2q) + 9a_1 x, \end{aligned} \tag{4.68}$$

where

$$q = \frac{1}{2}\left((3\gamma - 2)x + y + \Sigma^2\right), \tag{4.69}$$

and the physical phase space is

$$4 - 4x - \Sigma^2 \geq 0, \qquad \text{and} \qquad x \geq 0. \tag{4.70}$$

In this case there exist four invariant sets of particular interest. Similar to the previous analysis we have the set $\mathcal{FRW} := \{(x, \Sigma, y) | \Sigma = 0\}$ and $\mathcal{BI} := \{(x, \Sigma, y) | 4 - 4x - \Sigma^2 = 0 \text{and} \Sigma \neq 0\}$. The Bianchi type V invariant set can be subdivided into two disjoint sets, $\mathcal{BV}^+ := \mathcal{BI}^c \cap \mathcal{FRW}^c \cap \{(x, \Sigma, y) | \Sigma > 0\}$ and $\mathcal{BV}^- := \mathcal{BI}^c \cap \mathcal{FRW}^c \cap \{(x, \Sigma, y) | \Sigma < 0\}$. Due to the symmetry in the equations (reflection through the $\Sigma = 0$ plane) the qualitative behaviour in the set $\Sigma < 0$ is equivalent to that in the set $\Sigma > 0$; henceforth (and without loss of generality), we shall only concern ourselves with the part of the phase space with $\Sigma \geq 0$.

The equilibrium points of the system (4.68) are

$$(0, 2, 0), \quad (0, 0, 0), \quad (1, 0, y^-), \quad (1, 0, y^+), \tag{4.71}$$

where $y^{\pm}$ is given by equation (4.53).

There is only one equilibrium point in the invariant set $\Sigma > 0$. The equilibrium point $(0, 2, 0)$ is in the set $\mathcal{BI}$ and has eigenvalues

$$\begin{aligned} -4, \quad & \frac{b_1 + 3\gamma - 12}{2} - \frac{1}{2}\sqrt{(b_1 - 3\gamma)^2 + 36a_1}, \\ & \frac{b_1 + 3\gamma - 12}{2} + \frac{1}{2}\sqrt{(b_1 - 3\gamma)^2 + 36a_1}. \end{aligned} \tag{4.72}$$

This equilibrium point is either a saddle or a sink depending on the value of the parameter

$$B_6 = (2 - \gamma)(b_1 - 6) + 3a_1. \tag{4.73}$$

If $B_6 > 0$, then the point is a saddle with a 2-dimensional stable manifold. If $B_6 < 0$, then the point is a sink, and if $B_6 = 0$, then the point is degenerate (discussed later). The solution at this equilibrium point is a Kasner model. The stability of this point is summarized in Table 4.7.

The equilibrium point $(0,0,0)$ has eigenvalues

$$2, \quad \frac{1}{2}\left\{(b_1+3\gamma-4)+\sqrt{(b_1+3\gamma-4)^2+4B_1}\right\},$$
$$\frac{1}{2}\left\{(b_1+3\gamma-4)-\sqrt{(b_1+3\gamma-4)^2+4B_1}\right\}. \tag{4.74}$$

This point is either a saddle or a source depending on the value of the parameter B_1. If $B_1 > 0$, then the point is a saddle with a 1-dimensional stable manifold. If $B_1 < 0$, then the point is a source, and if $B_1 = 0$ the point becomes degenerate (discussed later). The stability of this point is summarized in Table 4.7.

The equilibrium point $(1,0,y^-)$ has eigenvalues

$$-\frac{1}{2}\left\{(b_1+3\gamma-4)-\sqrt{(b_1+3\gamma-4)^2+4B_1}\right\}, \qquad \sqrt{(b_1+3\gamma-4)^2+4B_1},$$
$$-\frac{1}{4}\left\{(b_1+3\gamma-12)-\sqrt{(b_1-3\gamma)^2+36a_1}\right\}. \tag{4.75}$$

This point is either a saddle or a source depending on the value of the parameter B_1. If $B_1 < 0$, then the point is a saddle point with a 1-dimensional stable manifold. If $B_1 > 0$, then the point is a source, and if $B_1 = 0$ the point becomes degenerate (discussed later). The stability of this point is summarized in Table 4.7.

The equilibrium point $(1,0,y^+)$ has eigenvalues

$$-\frac{1}{2}\left\{(b_1+3\gamma-4)+\sqrt{(b_1+3\gamma-4)^2+4B_1}\right\}, \quad -\sqrt{(b_1+3\gamma-4)^2+4B_1},$$
$$-\frac{1}{4}\left\{(b_1+3\gamma-12)+\sqrt{(b_1-3\gamma)^2+36a_1}\right\}. \tag{4.76}$$

This equilibrium point is either a saddle or a sink depending on the parameter B_6. If $B_6 < 0$, then the point is a saddle with a 2-dimensional stable manifold. If $B_6 > 0$, then the point is a sink, and if $B_6 = 0$ then the point is degenerate (discussed later). The stability of this point is summarized in Table 4.7.

The bifurcations in this model occur at $B_1 = 0$ and $B_6 = 0$. If $B_1 = 0$ then there exists a line of equilibrium points passing through the points $(0,0,0)$ and $(1,0,y^-)$. This line can be shown to have some saddle-like properties. In particular, if $B_1 = 0$ then the points $(1,0,y^-)$ and $(0,0,0)$ experience a saddle-node bifurcation. If $B_6 = 0$

then the curve $y = 3(2-\gamma)x$, $\Sigma^2 = 4-4x$ which lies in the set $\mathcal{BI}$ is a non-isolated line of equilibria. This observation is analogous to the case $\gamma = 2$ in perfect fluid Bianchi type V models in Chapter 3. In particular, if $B_6 = 0$ then the points $(1, 0, y^+)$ and $(0, 2, 0)$ experience a saddle-node bifurcation.

All information about the equilibrium points is summarized in Table 4.7. It is very easily seen that if $B_6 > 0$ then the solutions tend to an isotropic model both to the past and to the future (in t-time), while if $B_6 < 0$, then solutions only tend to an isotropic model to the future. In the region of phase space where the DEC is satisfied, it can be shown that $\Sigma' > 0$ in that set. Therefore in this region there are no periodic orbits. Thus, all models that initially are in the set where the DEC is satisfied isotropize to the future ($\Omega \to -\infty$ or $t \to \infty$). Note the difference in the result here and the result in the previous subsection. If there is a 'non-zero' anisotropic stress then there is a range of parameter values such that models will not isotropize, and if there is 'zero' anisotropic stress then all models will isotropize to the future. Therefore we can conclude that in the truncated Israel-Stewart theory the anisotropic stress plays a dominant role in determining the future evolution of the anisotropic models. This result is contrary to the observations in the previous chapter based upon the Eckart theory where the anisotropic stress played only a minor role and did not determine the the future evolution of the models.

Zero Bulk Viscous Pressure

As we have seen in the previous subsection, anisotropic stress plays a dominant role in the evolution of the anisotropic cosmological models. To further analyze the effects of anisotropic stress on the evolution of an anisotropic model we shall set $\Pi \equiv 0$ and $\zeta \equiv 0$. This translates into setting $y = 0$ and $a_1 = 0$ in equations (4.49). In order to illustrate the possible influence anisotropic stress may have on an anisotropic model we further restrict ourselves to the set $\mathcal{BI} := \{(x, \Sigma, z) | 4 - 4x - \Sigma^2 = 0\}$. The resulting system is planar and lends itself easily to a complete qualitative analysis.

Consequently the system under consideration is

$$\begin{aligned}\Sigma' &= \frac{3}{8}(2-\gamma)\Sigma(4-\Sigma^2) - 12z,\\ z' &= (2b_2 - 3\gamma)z - \frac{3}{4}(2-\gamma)z\Sigma^2 + \frac{a_2}{4}\Sigma(4-\Sigma^2).\end{aligned} \tag{4.77}$$

The equilibrium points are $(0,0)$, $(+2,0)$, $(-2,0)$, (Σ^+, z^+), and (Σ^-, z^-) where

$$\Sigma^{\pm} = \pm\sqrt{4 + \frac{8}{3(2-\gamma)^2}B_7}, \qquad z^{\pm} = \frac{(2-\gamma)}{32}\Sigma^{\pm}(4 - \Sigma^{\pm 2}), \tag{4.78}$$

where

$$B_7 = (2-\gamma)(b_2 - 3) + 4a_2. \tag{4.79}$$

The point $(0,0)$ represents a flat FRW model; the eigenvalues of this point are

$$\frac{1}{4}\left\{B_8 \pm \sqrt{{B_8}^2 - 48[B_7 + \frac{3}{2}(2-\gamma)^2]}\right\}, \tag{4.80}$$

where

$$B_8 = 4b_2 - 3(3\gamma - 2). \tag{4.81}$$

If $B_7 + \frac{3}{2}(2-\gamma)^2 < 0$, then the point $(0,0)$ is a saddle point. If $B_7 + \frac{3}{2}(2-\gamma)^2 > 0$, then the stability of the point $(0,0)$ depends on the parameter B_8. If $B_8 > 0$ the point $(0,0)$ is a source and if $B_8 < 0$ the point $(0,0)$ is a sink. Bifurcations of this point occur when $B_7 = -\frac{3}{2}(2-\gamma)^2$ and $B_8 = 0$ and are discussed later.

The points $(\pm 2, 0)$ represent Kasner models. The eigenvalues are

$$\frac{1}{2}\left\{(2b_2 + 3\gamma - 12) \pm \sqrt{(2b_2 + 3\gamma - 12)^2 + 24B_7}\right\}. \tag{4.82}$$

If $B_7 > 0$, the points $(\pm 2, 0)$ are saddle points. If $B_7 < 0$, then the points $(\pm 2, 0)$ are sinks. The bifurcation that occurs when $B_7 = 0$ is discussed later.

The points $(\Sigma^{\pm}, z^{\pm})$ only exist when $B_7 + \frac{3}{2}(2-\gamma)^2 > 0$. The eigenvalues are

$$\begin{aligned}\frac{1}{2}\Bigg\{&\left(2b_2 + 3\gamma - 12 - \frac{5}{(2-\gamma)}B_7\right)\\ &\pm\sqrt{\left(2b_2 + 3\gamma - 12 - \frac{5}{(2-\gamma)}B_7\right)^2 - 24B_7\left(1 + \frac{2}{3(2-\gamma)^2}B_7\right)}\Bigg\}.\end{aligned} \tag{4.83}$$

If $-\frac{3}{2}(2-\gamma)^2 < B_7 < 0$, then the points $(\Sigma^{\pm}, z^{\pm})$ are saddle points. If $B_7 > 0$, then the points $(\Sigma^{\pm}, z^{\pm})$ are sinks. The bifurcation values $B_7 = -\frac{3}{2}(2-\gamma)^2$ and $B_7 = 0$ are discussed later. The stability of all equilibrium points is summarized in Table 4.6.

Knowing the equilibrium points and their eigenvalues only reveals the local behaviour of the system (4.77). The determination of some of the global properties requires investigating the existence or non-existence of periodic orbits and analyzing points at infinity.

The existence of periodic orbits is difficult to prove. However, with the aid of Dulac's criterion [78] (see Appendix A), we are able to prove the non-existence of periodic orbits for a range of parameter values. Taking the divergence of the system (4.77) we can see that

$$\nabla \cdot f = \frac{1}{2}B_8 - \frac{15}{8}(2-\gamma)\Sigma^2. \tag{4.84}$$

Therefore, if $B_8 < 0$ then there do not exist any periodic orbits.

To analyze the points at infinity we first change to polar coordinates $r^2 = \Sigma^2 + z^2$ and $\theta = \tan^{-1}(z/\Sigma)$ and then we compactify the phase space through the following transformations:

$$\bar{r} = \frac{r}{1+r}, \qquad \bar{\theta} = \theta, \qquad \frac{d\Omega}{d\bar{t}} = (1-\bar{r})^2, \tag{4.85}$$

in which case the evolution equations for $\bar{r}$ and $\bar{\theta}$ become:

$$\begin{aligned} \frac{d\bar{r}}{d\bar{t}} &= (1-\bar{r})^3\bar{r}\left\{\frac{3}{2}(2-\gamma)\cos^2\bar{\theta} + (2b_2 - 3\gamma)\sin^2\bar{\theta} + (a_2 - 12)\cos\bar{\theta}\sin\bar{\theta}\right\} \\ &- (1-\bar{r})\bar{r}^3\frac{\cos^2\bar{\theta}}{4}\left\{\frac{3}{2}(2-\gamma)\cos^2\bar{\theta} + 3(2-\gamma)\sin^2\bar{\theta} + a_2\cos\bar{\theta}\sin\bar{\theta}\right\}, \quad (4.86) \\ \frac{d\bar{\theta}}{d\bar{t}} &= (1-\bar{r})^2\left\{(2b_2 - \frac{3}{2}\gamma - 3)\cos\bar{\theta}\sin\bar{\theta} + a_2\cos^2\bar{\theta} + 12\sin^2\bar{\theta}\right\} \\ &\qquad -\bar{r}^2\frac{\cos^3\bar{\theta}}{4}\left\{\frac{3}{2}(2-\gamma)\sin\bar{\theta} + a_2\cos\bar{\theta}\right\}. \quad (4.87) \end{aligned}$$

The points at $r = \infty$ are mapped to the unit circle $\bar{r} = 1$. Hence the equilibrium points at infinity are those points on the unit circle $\bar{r} = 1$ where $\frac{d\bar{\theta}}{dt} = 0$. The equilibrium points are thus

$$(1, \frac{\pi}{2}), \qquad (1, -\frac{\pi}{2}), \qquad (1, \theta^*), \quad \text{and} \quad (1, \theta^* + \pi), \tag{4.88}$$

where $\tan\theta^* = -2a_2/3(2-\gamma)$. In order to determine the stability of these equilibrium points we need to study the values of $\frac{d\bar{r}}{d\bar{t}}$ and $\frac{d\bar{\theta}}{d\bar{t}}$ in a neighborhood of each of the equilibrium points. We find that the points $(1,\theta^*)$ and $(1,\theta^*+\pi)$ are saddle-points while the points $(1,\pm\pi/2)$ are sources.

To obtain a complete picture of the qualitative behaviour of the model we must discuss the various bifurcations that occur. A bifurcation occurs at $B_7 = -\frac{3}{2}(2-\gamma)^2$, in which case the point $(0,0)$ undergoes a pitchfork bifurcation [79] to create the two new equilibrium points $(\Sigma^\pm, z^\pm)$ and its stability is transferred to them. When $B_7 > -\frac{3}{2}(2-\gamma)^2$, the point $(0,0)$ experiences an Andronov-Hopf bifurcation at $B_8 = 0$ [79]. Therefore, it can be shown that there exists a $\delta > 0$ such that for every $B_8 \in (0,\delta)$ there exists a periodic orbit. In addition, the periodic orbit is an attractor. A third bifurcation occurs at $B_7 = 0$ when the points $(\Sigma^\pm, z^\pm)$ and $(\pm 2, 0)$ undergo a transcritical bifurcation [79] in which they exchange stability. The stability of all of the equilibrium points (finite and infinite) is given in Table 4.6.

Let us now discuss the qualitative properties of this model. If $B_7 < -\frac{3}{2}(2-\gamma)^2$, then all trajectories evolve from the equilibrium points at $(\pm 2, 0)$ representing Kasner models to points at infinity. These models are generally unsatisfactory since the WEC (which implies $\Sigma^2 \leq 4$ for the Bianchi I models here), is broken eventually. However, there do exist two exceptional trajectories for which the WEC is satisfied always. These are the trajectories that lie on the unstable manifold of the equilibrium point $(0,0)$ which describe models that have a Kasner-like behaviour in the past (t-time) and isotropize to the future toward the point $(0,0)$. A phase portrait of this model is given in Figure 4.9.

If $-\frac{3}{2}(2-\gamma)^2 < B_7 < 0$ and $B_8 < 0$ then there exist two classes of generic behaviour. One class of trajectories evolve from the isotropic equilibrium point $(0,0)$ and evolve to points at infinity. The second class of trajectories evolve from the equilibrium points $(\pm 2, 0)$, which represent Kasner models, and evolve to points at infinity. Both of these classes of trajectories describe models that fail to isotropize, and describe models that will eventually violate the WEC. If $-\frac{3}{2}(2-\gamma)^2 < B_7 < 0$

Table 4.5: Stability of the equilibrium point $(1, 0, y^+, 0)$ where $\dim(W^s)$ is the dimension of the stable manifold with respect to Ω-time.

$\mathrm{sgn}(B_4)$	$\mathrm{sgn}(B_5)$	type	$\dim(W^s)$
$-$		Saddle	3
$+$	$+$	Saddle	2
$+$	$-$	Sink	4

Table 4.6: Stability of the equilibrium points with respect to Ω-time, at both finite and infinite values, for the Bianchi type I anisotropic model with anisotropic stress and zero viscous pressure.

B_7	$\mathrm{sgn}(B_8)$	$(0,0)$	$(\pm 2, 0)$	$(\Sigma^{\pm}, z^{\pm})$	$(0, \pm\infty)$[a]	$(\pm\infty, \mp\infty)$[b]	P.O.[c]
$B_7 < \bar{B}$[d]		saddle	sink		source	saddle	
$\bar{B} < B_7 < 0$	$-$	sink	sink	saddle	source	saddle	
$\bar{B} < B_7 < 0$	$+$	source	sink	saddle	source	saddle	sink
$B_7 > 0$	$-$	sink	saddle	sink	source	saddle	
$B_7 > 0$	$+$	source	saddle	sink	source	saddle	sink

[a] These are the points at infinity corresponding to $(\bar{r} = 1, \bar{\theta} = \pm\pi/2)$.

[b] These are the points at infinity corresponding to $(\bar{r} = 1, \bar{\theta} = \theta^*)$ and $(\bar{r} = 1, \bar{\theta} = \theta^* + \pi)$.

[c] P.O. = Periodic Orbit

[d] $\bar{B} \equiv -\frac{3}{2}(2 - \gamma)^2$

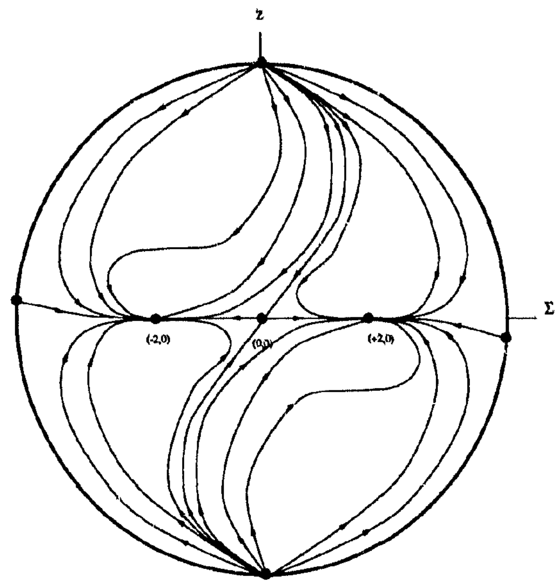


Figure 4.9: *The phase portrait describes the qualitative behavior of the anisotropic Bianchi type I models with anisotropic stress and zero bulk viscous pressure in the case* $B_7 < -\frac{3}{2}(2-\gamma)^2$. *The arrows in the figure denote increasing* Ω*-time* $(\Omega \to \infty)$ *or decreasing t-time* $(t \to 0^+)$.

and $B_8 < 0$, then there exists three sets of exceptional trajectories. One set is the stable manifolds of the points $(\Sigma^\pm, z^\pm)$ which represent models that start at $(\Sigma^\pm, z^\pm)$ and evolve to points at infinity, hence the WEC will be violated. There do exist trajectories describing models that satisfy the WEC for all time, namely the unstable manifolds of the point $(\Sigma^\pm, z^\pm)$. One set of these trajectories start at the equilibrium point $(\pm 2, 0)$ which represent the Kasner models and evolve to the point $(\Sigma^\pm, z^\pm)$. The other set of trajectories evolve from the isotropic equilibrium point $(0, 0)$ to the equilibrium points $(\Sigma^\pm, z^\pm)$. In this case there are no models which isotropize. A phase portrait of this model is given in Figure 4.10.

If $-\frac{3}{2}(2-\gamma)^2 < B_7 < 0$ and $B_8 > 0$ then there exist three classes of models. One class of trajectories evolves from the periodic orbit to the isotropic equilibrium point $(0,0)$. This class of models is interesting in that the past singularity has an oscillatory nature, that is, both the dimensionless shear Σ and the dimensionless anisotropic stress z tend to a closed periodic orbit in the past (t-time). This class of trajectories also represent models that isotropize and represent models in which the WEC is satisfied always. The second class of trajectories are those which evolve from the periodic orbit to points at infinity. This class of trajectories represent models that will not satisfy the WEC at some point in the future. The third class of trajectories is the same as the second class of trajectories in the case $-\frac{3}{2}(2-\gamma)^2 < B_7 < 0$. Again there exist three sets of exceptional trajectories. The stable manifolds of the points $(\Sigma^{\pm}, z^{\pm})$ represent models that will eventually violate the WEC. The unstable manifolds of the points $(\Sigma^{\pm}, z^{\pm})$ represent either models that start at the Kasner-like equilibrium point $(\pm 2, 0)$ and evolve to the point $(\Sigma^{\pm}, z^{\pm})$ or represent models that start from the periodic orbit and evolve to the point $(\Sigma^{\pm}, z^{\pm})$. The phase portrait in this case is very similar to that of Figure 4.11.

If $B_7 > 0$ and $B_8 < 0$, then the behaviour of the trajectories is very similar to the behaviour of the trajectories in the case $-\frac{3}{2}(2-\gamma)^2 < B_7 < 0$ and $B_8 < 0$. The difference stems from the fact that the points $(\pm 2, 0)$ are now saddles and the points $(\Sigma^{\pm}, z^{\pm})$ are now sinks. The phase portrait in this case is very similar to that of Figure 4.10.

If $B_7 > 0$ and $B_8 > 0$, then the behaviour of the trajectories is very similar to the behaviour of the trajectories in the case $-\frac{3}{2}(2-\gamma)^2 < B_7 < 0$ and $B_8 > 0$. The difference stems from the fact that the points $(\pm 2, 0)$ are now saddles and $(\Sigma^{\pm}, z^{\pm})$ are now sinks. A phase portrait of this model is given in Figure 4.11.

In conclusion, the general behaviour of these models is unsatisfactory in that the WEC is violated eventually, except in the case $B_8 > 0$ and $-\frac{3}{2}(2-\gamma)^2 < B_7$ where there exists a set of models (of non-zero measure) that will always satisfy the WEC. These are the models represented by the trajectories which start at the periodic orbit and isotropize to the point $(0,0)$ to the future (t-time). There also exist models which satisfy the WEC always, but these are the models represented by the unstable

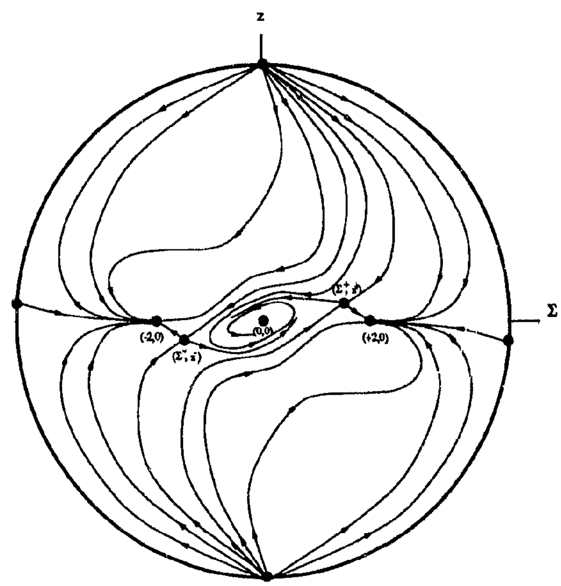


Figure 4.10: *The phase portrait describes the qualitative behavior of the anisotropic Bianchi type I models with anisotropic stress and zero bulk viscous pressure in the case* $-\frac{3}{2}(2-\gamma)^2 < B_7 < 0$ *and* $B_8 < 0$. *The arrows in the figure denote increasing* Ω*-time* ($\Omega \to \infty$) *or decreasing t-time* ($t \to 0^+$).

manifolds of the saddle-points.

Clearly the anisotropic stress in the truncated-Israel-Stewart theory plays a very dominant role in the evolution of the anisotropic models. This is in contrast to what was found in Chapter 3 using the Eckart theory where it was found that anisotropic stress played a very minor role in determining the asymptotic behaviour. However, if $B_8 < 0$, then all models are generically unsatisfactory in that the WEC will be violated. If $B_8 > 0$, then there does exist a set of satisfactory models where the WEC will always be satisfied. It is also interesting to note briefly the existence of a periodic orbit, this type of behaviour is not seen in the Eckart models. With the existence of this periodic orbit, the past attractor which this periodic orbit represents, has a oscillatory character to it, in that the dimensionless shear (and therefore the

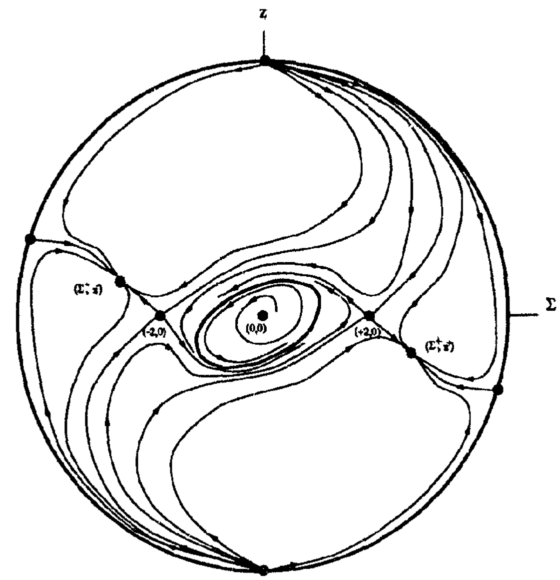


Figure 4.11: *The phase portrait describes the qualitative behavior of the anisotropic Bianchi type I model with anisotropic stress and zero bulk viscous pressure in the case* $B_7 > 0$ *and* $B_8 > 0$. *The closed elliptical orbit close to the center represents the periodic orbit. The arrows in the figure denote increasing* Ω*-time* ($\Omega \to \infty$) *or decreasing t-time* ($t \to 0^+$).

dimensionless density) and the dimensionless anisotropic stress will have an oscillatory nature.

4.3.4 Qualitative Analysis — Non-Zero Heat Conduction

In this section we will study the effects of heat conduction on the models. For simplicity we will assume that the anisotropic stress is zero. Although $\Pi_{ab} \equiv 0$, the bulk viscosity is still present, therefore, in a sense, we are investigating the effect heat conduction will have on the viscous models with bulk viscosity. Under these assumptions, the system (4.39–4.46) reduces to a one-parameter family (that is, in addition to the parameters arising from the equations of state) of three-dimensional systems analogous to the parameterization used in Chapter 3. We label this new parameter $-1 \leq C = (k+1)/\sqrt{k^2 - k + 1} \leq 2$, which is a function of the integration constant, k, that appears when equations (4.40) and (4.41) are integrated. The value $C = 0$ corresponds to the case in which there is zero heat conduction, and $C = 2$ corresponds to the case in which the model is locally rotationally symmetric (LRS). We define a new shear variable

$$\Sigma \equiv \frac{\Sigma_1 + \Sigma_2}{\sqrt{3}C}, \tag{4.89}$$

whence the system (4.39–4.47) becomes

$$\begin{aligned} x' &= x(3\gamma - 2 - 2q) + y - C\frac{\Sigma}{4}(4 - 4x - \Sigma^2), \\ \Sigma' &= \Sigma(2 - q), \\ y' &= y(b_1 - 2 - 2q) + 9a_1 x + \sqrt{3}Cc_1\Sigma(4 - 4x - \Sigma^2), \end{aligned} \tag{4.90}$$

where

$$q = \frac{1}{2}\left((3\gamma - 2)x + y + \Sigma^2\right), \tag{4.91}$$

and the physical phase space is

$$4 - 4x - \Sigma^2 \geq 0 \qquad \text{and} \qquad x \geq 0. \tag{4.92}$$

There exists three physically interesting invariant sets in the phase space of the system, namely, $\mathcal{FRW} := \{(x, \Sigma, y) | \Sigma = 0\}$, $\mathcal{BI} := \{(x, \Sigma, y) | 4 - 4x - \Sigma^2 = 0, \text{and} \Sigma \neq 0\}$, and $\mathcal{BV} := \mathcal{BI}^c \cap \mathcal{FRW}^c$ (where subscript c denotes the complement) which represents Bianchi type V models. As before, the set $\mathcal{FRW}$ represents the spatially homogeneous and isotropic negative and flat curvature FRW models and the set $\mathcal{BI}$ represents the Bianchi type I models.

There are six different equilibrium points of the system. The equilibrium points lying in the set $\mathcal{FRW}$ are:

$$(0,0,0), \qquad (1,0,y^-), \qquad (1,0,y^+), \tag{4.93}$$

where $y^{\pm}$ is given by equation (4.53). Also, if $B_1 = 0$ then there is a non-isolated line of equilibria that passes through the points $(0,0,0)$ and $(1,0,y^-)$, where B_1 is given by equation (4.54). These points represent open ($x = 0$) and flat ($x = 1$) FRW models. The eigenvalues of the linearization in a neighborhood of each of the isotropic equilibrium points are the same as in the case with zero heat conduction (see equations (4.74), (4.75), and (4.76) for the eigenvalues of $(0,0,0)$, $(1,0,y^-)$, and $(1,0,y^+)$, respectively, and the appropriate parts of Table 4.7).

The equilibrium points located in the set $\mathcal{BI}$ are

$$(0,-2,0), \qquad \text{and} \qquad (0,+2,0). \tag{4.94}$$

The eigenvalues of the linearization about the point $(0,-2,0)$ are similar to those in the case with zero heat conduction [see equation (4.72)]; indeed, only the first eigenvalue is different, namely, instead of $\lambda_1 = -4$, we now have $\lambda_1 = -4 - 2C$, which is very easily seen to be negative definite. Therefore the stability of the point $(0,-2,0)$ is the same as in the previous case with zero heat conduction. Similarly, for the eigenvalues of the linearization about the point $(0,+2,0)$, only the first eigenvalue is different, namely, instead of $\lambda_1 = -4$ we now have $\lambda_1 = -4+2C$, which is negative definite for $C \neq 2$. Therefore, if $C \neq 2$, the stability of the point $(0,+2,0)$ is the same as in the case with zero heat conduction. The case $C = 2$ is discussed below.

The sixth equilibrium point is $(\bar{x}, \bar{\Sigma}, \bar{y})$, where

$$\begin{aligned}
\bar{x} &= \frac{4(C^2-4)(b_1 - 6 + 4\sqrt{3}c_1)}{C^2[16\sqrt{3}c_1 + (3\gamma-2)(b_1-6) - 9a_1]}, \\
\bar{\Sigma} &= \frac{4}{C}, \\
\bar{y} &= \frac{12(C^2-4)[4\sqrt{3}c_1(2-\gamma) - 3a_1]}{C^2[16\sqrt{3}c_1 + (3\gamma-2)(b_1-6) - 9a_1]}.
\end{aligned} \tag{4.95}$$

However this last equilibrium point lies outside the region of phase space defined by equations (4.92) for $C \neq 2$.

If $C = 2$, then a 'transcritical' bifurcation occurs, the points $(\bar{x}, \bar{\Sigma}, \bar{y})$ and $(0,+2,0)$ coalesce and become a single point. The stability of this point cannot be determined via linearization. If $(3\gamma-2)(6-b_1)+9a_1-16\sqrt{3}c_1=0$, then there is a line of equilibrium points $y+(3\gamma-2)x=0$, $\Sigma=2$. The stability of the equilibrium point is very difficult to determine analytically (even with the use of center manifold theory [79]). However, numerical experiments in addition to some analysis show that the equilibrium point has some of the same behaviour as in the case with $C\neq 2$ (e.g., if $B_6>0$ the point is a saddle and if $B_6<0$ then the point has both saddle-like and sink-like behaviour).

The stability of the equilibrium points $(0,0,0)$, $(1,0,y^-)$, $(1,0,y^+)$, $(0,-2,0)$, and $(0,+2,0)$ are the same as in the case with zero heat conduction. The heat conduction does not determine the stability of the equilibrium points that lie in the physical phase space (4.92) but does play a role in determining their eigendirections. This is similar to the situation in which the bulk viscous pressure is absent whence the model reduces to one that was analyzed in Chapter 3; there the addition of heat conduction did not change the stability of the equilibrium points but did allow the models to violate the WEC by rotating the principal eigendirections [68].

4.3.5 Summary of the Anisotropic Models

This work improves over previous work in Chapter 3 and in [53, 54, 55] on viscous cosmology using the non-causal first-order thermodynamics of Eckart [47] in that a causal theory of irreversible thermodynamics has been utilized. Also, this work enhances the analysis of anisotropic viscous cosmologies in [44, 58, 77] because more than just the isotropic equilibrium points have been analyzed. The present work also generalizes the analysis of causal viscous FRW models in section 4.2.

Again we have seen that the equilibrium points of the dynamical system describing the evolution of an anisotropic viscous fluid cosmological model are, in general, self-similar. In the case in which $m=n=1$, $r_i=1$ and $p_i=-1$ $(i=1,2)$ all equilibrium points are self-similar except in the case in which $\gamma=3\zeta_o$ when there exists a equilibrium point that represents a de Sitter model which is not self-similar.

We have found that in the case of zero heat conduction the anisotropic models need not isotropize (that is, there exists a range of parameter values and initial conditions such that the models will not isotropize). The parameter b_2, which is the parameter related to the relaxation time of the anisotropic stress, plays a major role in determining the stability of the isotropic models. In the special case of zero anisotropic stress we have shown that all models isotropize. The addition of anisotropic stress on an anisotropic Bianchi type I model reveals some of the effects that anisotropic stress has on an anisotropic model. For instance, anisotropic stress generically causes models to increase their anisotropy and eventually violate the weak energy condition. Anisotropic stress in some instances causes the creation of a periodic orbit. This periodic orbit represents a past attractor in which the dimensionless quantities ρ/θ^2 and σ/θ are approximately periodic. It is interesting to note that it is only when this periodic orbit is present that there exist trajectories (in the interior) which represent models that will isotropize and satisfy the weak energy condition.

The models with heat conduction analyzed here had no anisotropic stress but did have bulk viscosity. Consequently, we have investigated whether any qualitative changes arise from the inclusion of heat conduction. From our analysis the addition of heat conduction in the model did not change the stability of the equilibrium points, and hence the asymptotic states of the models. However, the inclusion of heat conduction did affect the dynamics in a neighborhood of each of the equilibrium points in that some of the eigendirections have changed.

4.4 Conclusions

In this chapter we have employed the truncated Israel-Stewart theory which is a causal and stable second order relativistic theory of irreversible thermodynamics. The asymptotic behaviour of the Eckart models can be significantly different than the asymptotic behaviour of the truncated Israel-Stewart models. Therefore, one must exercise caution when drawing conclusions about the dynamics of the universe when using the Eckart theory.

It is possible that the truncated theory is applicable in the very early universe, however, it is known that such a truncated theory could result in some pathological behaviour, (e.g., in the temperature [36]). Hence, this work should be considered as a first step in the analysis of the full Israel-Stewart theory [48, 52, 51, 80]. This chapter provides a foundation for the analysis of viscous cosmological models using the Full Israel-Stewart theory which will be presented in the following chapter.

Table 4.7: Stability of the equilibrium points with respect to Ω-time (both with and without heat conduction).

$\mathrm{sgn}(B_6)$	$\mathrm{sgn}(B_1)$	$(0,0,0)$	$(1,0,y^-)$	$(1,0,y^+)$	$(0,2,0)$
$+$	$-$	source	saddle	sink	saddle
$+$	$+$	saddle	source	sink	saddle
$-$	$-$	source	saddle	saddle	sink
$-$	$+$	saddle	source	saddle	sink

Chapter 5

Causal Viscous Fluid Cosmological Models (Full Theory)

5.1 Introduction

In Chapter 4 spatially homogeneous viscous fluid cosmological models were investigated using the truncated Israel-Stewart [49, 50, 81] theory of irreversible thermodynamics to model the viscous effects. Although the truncated theory is a causal and stable second order relativistic theory of thermodynamics, the truncated version of the theory is known to give rise to very different behaviour than the full Israel-Stewart theory [49, 50, 81]; in fact, Maartens [36] argues that in many cases the truncated theory will lead to pathological behaviour (for example, in the behaviour of the temperature). Therefore, although it can be argued that the truncated theory might be better formulated in terms of a redefinition of equilibrium values and that the truncated theory may be applicable in particular circumstances, a neglection of the divergence terms in Chapter 4 can only be regarded as a first step in the study of dissipative processes in the universe utilizing the full (non-truncated) theory.

5.2 Friedmann-Robertson-Walker Models

5.2.1 The Equations

The dimensionless Einstein field equations for a spatially homogeneous and isotropic cosmological model are given by equations (4.1–4.3). Analogous to the analysis in the truncated Israel-Stewart theory, we require an equation for the dimensionless viscous pressure y and therefore for Π. Here, we shall assume that the bulk viscous pressure, Π, obeys the evolution equation (1.3) [48, 52],

$$\Pi = -\zeta\theta - \tau\dot{\Pi} - \frac{\epsilon}{2}\tau\Pi\left[\theta + \frac{\dot{\beta}_0}{\beta_0} - \frac{\dot{T}}{T}\right], \tag{5.1}$$

where $\zeta \geq 0$ is the bulk viscosity coefficient, $\beta_0 \geq 0$ is a relaxation coefficient for transient bulk viscous effects, and $T \geq 0$ is the temperature. Equation (5.1) with $\epsilon = 1$ arises as a simple solution of the H-theorem (positive entropy production) [36]. The truncated theory effectively arises by setting $\epsilon = 0$, and it is the vanishing of the term in square brackets in equation (5.1) that is the effective source of the pathological behaviour of T [36]. Rewriting equation (5.1) with respect to the dimensionless variables (3.10) we derive

$$\frac{dy}{d\Omega} = y\left[\left(\frac{\theta}{\zeta}\right)\left(\frac{3}{\beta_0\theta^2}\right) - 2 - 2q\right] + 9\left(\frac{3}{\beta_0\theta^2}\right) + \frac{\epsilon}{2}y\left[3 - \frac{\beta_0'}{\beta_0} + \frac{T'}{T}\right]. \tag{5.2}$$

Equations of state for p, ζ and β_0 and a temperature law for T are needed in order for the system of equations (4.1–4.3) and (5.2) to be closed. Unlike Belinski et al. [44] in which ζ, β_0 (and T) are assumed to be proportional to powers of ρ [36], we shall adopt dimensionless equations of state (4.6) where p/θ^2, ζ/θ and $\beta_0\theta^2$ are proportional to powers of the dimensionless density parameter x, namely,

$$\frac{p}{\theta^2} = \frac{1}{3}(\gamma - 1)x, \qquad \frac{\zeta}{\theta} = \zeta_o x^m, \quad \text{and} \quad \frac{3}{\beta_0\theta^2} = a x^{r_1}, \tag{5.3}$$

where m and r_1 are constants which are assumed to be non-negative and ζ_o and a are positive parameters. Clearly the equations of state employed will determine the qualitative properties of the models [36, 48, 52, 44]. Equations of state (5.3), which ensure that the asymptotic limit points represent self-similar models, are phenomenological in nature and are no less appropriate than the equations of state used by Belinskii et al. [44]. We note that the equations of state chosen by Belinskii et al. [44] and those above coincide when $m = 1/2$, $r_1 = 1$. For simplicity we shall assume that $r_1 = m$, and define $b \equiv a/\zeta_o$. (Note, the parameters a and b used here are precisely those used in Chapter 4.)

Finally, using equation (5.3), equations (4.1–4.3) and (5.2) become:

$$
\begin{aligned}
x' &= (1-x)[(3\gamma-2)x+y], &(5.4)\\
y' &= y[b-2-y-(3\gamma-2)x]+9ax^m+\frac{\epsilon}{2}y[\Psi], &(5.5)
\end{aligned}
$$

where

$$\Psi \equiv 3+2\frac{\theta'}{\theta}+m\frac{x'}{x}+\frac{T'}{T}, \tag{5.6}$$

and

$$1-x=-\frac{3}{2}\,{}^3R\theta^{-2}. \tag{5.7}$$

Thus, the line $x = 1$ divides the phase space into three invariant sets, $x < 1$, $x = 1$, and $x > 1$. If $x = 1$, then the model is necessarily a flat FRW model, if $x < 1$ then the model is necessarily an open FRW model, and if $x > 1$ the model is necessarily a closed FRW model.

Equations (5.4) and (5.5) constitute a plane autonomous system of ODEs for x and y. In the truncated theory $\epsilon = 0$, whence the final term in equation (5.5) is absent and there is no need to specify an equation for T. From this point on, we shall set $\epsilon = 1$, and adopt the following temperature law:

$$T = T_0 x^p \theta^{2q} e^{r\Omega}, \tag{5.8}$$

where p, q, and r are constant. Consequently

$$
\begin{aligned}
\Psi &= (3+r)+2(1+q)\frac{\theta'}{\theta}+(m+p)\frac{x'}{x},\\
&= c_0+c_1y+c_2x+c_3\frac{y}{x},
\end{aligned}
$$

where

$$\begin{aligned} c_0 &= 5 + r + 2q + (3\gamma - 2)(m + p), \\ c_1 &= 1 + q - m - p, \\ c_2 &= (3\gamma - 2)c_1, \\ c_3 &= m + p. \end{aligned}$$

Relation (5.8) supports a variety of different interesting temperature laws, including, for example,

1. $T = T_0 x^p \theta^{1/2}$ $(r = 0,\ q = 1/4)$ Represents a dimensionless equation of state, assuming that the matter is pure radiation.

2. $T = T_0 x^p \theta^{2p}$ $(r = 0,\ q = 2p)$ Represents the equation of state $T = T_0 \rho^p$ which is usually assumed when studying dissipative cosmology.

5.2.2 Qualitative Analysis

The flat case

All of the FRW models are governed by equations (5.4) and (5.5) together with equation (5.7). We note from (5.4) that $x = 1$ is an invariant set, where from (5.7) we see that this set represents the flat FRW models. Let us study this physically important zero curvature case first. When $x = 1$, we have that

$$\Psi = (c_0 + c_2) + (c_1 + c_3)y, \tag{5.9}$$

whence equation (5.5) becomes

$$y' = \frac{(q-1)}{2}y^2 + \frac{1}{2}[2b + 3 + r + 3\gamma(q-1)]y + 9a. \tag{5.10}$$

That is, the equations governing the evolution of the flat FRW viscous fluid models reduce to a single autonomous ODE in y. If $q \neq 1$, then equation (5.10) is a Riccati equation with constant coefficients and its solutions can be found in implicit form. If $q=1$, then equation (5.10) is linear with the solution

$$y = C_0 e^{\frac{1}{2}(2b+3+r)\Omega} - \frac{18a}{2b+3+r}.$$

The analysis of (5.10) is straightforward. There are three possibilities concerning the number of equilibrium points at finite values. There are either two, one, or zero equilibrium points depending upon the values of the parameters q and $B_9 \equiv \frac{1}{4}[2b+3+r+3\gamma(q-1)]^2 - 18a(q-1)$.

If $B_9 > 0$ and $q \neq 1$, then there are two equilibrium points at

$$y^{\pm} = \frac{-\frac{1}{2}[2b+3+r+3\gamma(q-1)] \pm \sqrt{B_9}}{q-1}, \tag{5.11}$$

where one is a sink and the other is a source (with respect to the invariant set $x=1$, not the full set of all FRW models). If $B_9 > 0$ and $q < 1$, then one root is positive and the other negative. See Figure 5.1 for phase portrait. If $B_9 > 0$, $q > 1$, and $2b+3+r+3\gamma(q-1) > 0$, then both roots are negative. See Figure 5.2 for phase portrait. If $B_9 > 0$, $q > 1$, and $2b+3+r+3\gamma(q-1) < 0$, then both roots are positive.

There are two isolated cases when there is only a single equilibrium point, namely when $q = 1$ or $B_9 = 0$. If $q = 1$, then there is a single equilibrium point at

$$y = \frac{-18a}{2b+3+r}.$$

If $q = 1$ and $2b+3+r > 0$, then this point is a source. See Figure 5.3 for a phase portrait. If $q = 1$ and $2b+3+r < 0$, then this point is a sink. See Figure 5.4 for a phase portrait. If $B_9 = 0$, then necessarily $q > 1$ and the equilibrium point has coordinate

$$y = \frac{-\frac{1}{2}[2b+3+r+3\gamma(q-1)]}{q-1}.$$

If $B_9 = 0$ and $2b+3+r+3\gamma(q-1) > 0$, then this point has a negative y coordinate. If $B_9 = 0$ and $2b+3+r+3\gamma(q-1) < 0$, then this point has a positive y coordinate. In each case the equilibrium point is nonlinear and has a stable and an unstable direction. See Figure 5.5 for a phase portrait. If $B_9 < 0$, then there are no equilibrium points at finite values. Evaluating y' at $y = 0$ reveals the direction of the flow. See Figure 5.6 for a phase portrait.

Two conclusions can be made quite easily concerning the flat FRW models. One is that the behaviour of the flat models depends critically upon the value of the parameter q found in the temperature law (5.8). The second is that the behaviour of the flat models with $q < 1$ using the full (non-truncated) theory is qualitatively the same as the qualitative behaviour in the truncated theory (see Chapter 4).

The general curvature case

Let us now return to the general curvature case $x \neq 1$ [see equations (5.4) and (5.5)]. Equation (5.5) can be written as

$$y' = y\left[\left(b - 2 + \frac{c_0}{2}\right) + x(3\gamma - 2)\left(\frac{c_1}{2} - 1\right) + y\left(\frac{c_1}{2} - 1\right) + \frac{c_3}{2}yx^{-1}\right] + 9ax^m. \quad (5.12)$$

From equation (5.4), the equilibrium points $(\bar{x}, \bar{y})$ not lying in the invariant set $x = 1$ satisfy $\bar{y} = -(3\gamma - 2)\bar{x}$, and from (5.12) we obtain

$$9a\bar{x}^m - \frac{1}{2}(3\gamma - 2)(2b + r + 2q + 1)\bar{x} = 0 \quad (5.13)$$

For $m > 0$, there exists a singular point at the origin $(0,0)$. (Note, however, that the system of ODEs as given by equations (5.4) and (5.12) is not defined at $x = 0$ except when $c_3 = 0$, therefore the point $(0,0)$ may not be a well defined equilibrium point of the system. Changing to polar coordinates, it can be shown that this singular point is saddle-like in nature (hyperbolic sectors) if $m < 1$. If $m > 1$ then the point $(0,0)$ has parabolic and hyperbolic sectors.

If $m \neq 1$, then there is a second equilibrium point at

$$(\bar{x}, \bar{y}) = \left(\left[\frac{(3\gamma - 2)}{18a}(2b + r + 2q + 1)\right]^{1/m-1}, -(3\gamma - 2)\bar{x}\right). \quad (5.14)$$

If $B_{10} = (3\gamma - 2)(2b + r + 2q + 1) - 18a > 0$, then $\bar{x} > 1$, and when $m < 1$, this point is a saddle. If $B_{10} < 0$, then $\bar{x} < 1$, and when $m > 1$, this equilibrium point is again a saddle. There is a variety of other possible behaviours.

5.2.3 Summary

The qualitative behaviour of the flat FRW models has been determined completely. If $q < 1$ (necessarily $B_9 > 0$), then the flat models evolve from the visco-elastic singularity represented by the point $y = y^+$ with corresponding solution (after recoordination)

$$a(t) = t^{2/(y^+ + 3\gamma)}, \qquad \theta(t) = \frac{6}{y^+ + 3\gamma} t^{-1},$$
$$\rho(t) = \frac{12}{(y^+ + 3\gamma)^2} t^{-2}, \qquad \Pi(t) = \frac{4y^+}{(y^+ + 3\gamma)^2} t^{-2}, \tag{5.15}$$

towards either points at infinity or to the point $y = y^-$ which has solution

$$a(t) = |t - t_0|^{2/(y^- + 3\gamma)}, \qquad \theta(t) = \frac{6}{y^- + 3\gamma}(t - t_0)^{-1}$$
$$\rho(t) = \frac{12}{(y^- + 3\gamma)^2}(t - t_0)^{-2}, \qquad \Pi(t) = \frac{4y^-}{(y^- + 3\gamma)^2}(t - t_0)^{-2}. \tag{5.16}$$

[Note if $y^- + 3\gamma > 0$, then the solution (5.16) can be recoordinatized such that $t_0 = 0$.]

The Israel-Stewart theory of irreversible thermodynamics is a linear approximation of the true dynamics of a viscous fluid. The theory can only describe processes near equilibrium and therefore the magnitude of the bulk viscous pressure π should be less than the equilibrium thermodynamical pressure p, that is,

$$|\Pi| < (\gamma - 1)\rho. \tag{5.17}$$

[Note: this condition can be relaxed if the bulk viscosity is a result of particle creation.] In dimensionless variables, equation (5.17) becomes

$$|y| < 3(\gamma - 1)x. \tag{5.18}$$

One can easily verify that solution (5.15) satisfies (5.18) if $(y^+ + 3\gamma) < 3(2\gamma - 1)$. It can also be shown that solution (5.16) satisfies (5.18) if $y^- + 3\gamma > 3$.

If $q = 1$ and $2b + 3 + r > 0$, then the trajectories describing the models evolve from points at infinity to the point $y = -18a/(2b + 3 + r)$ which has a solution similar to

(5.16). If $q = 1$ and $2b + 3 + r < 0$ then the trajectories describing the models evolve from point $y = -18a/(2b + 3 + r)$ which has a solution of the form (5.15) to points at infinity.

If $B_9 = 0$ (necessarily $q > 1$) then there are trajectories that describe models which evolve from the point $y = -\frac{1}{2}[2b + 3 + r + 3\gamma(q - 1)]/(q - 1)$ [which has a solution of the form (5.15) if $y + 3\gamma > 0$] to points at infinity. There are also trajectories that describe models that evolve from points at infinity to the point $y = -\frac{1}{2}[2b + 3 + r + 3\gamma(q - 1)]/(q - 1)$. If $B_9 < 0$, then the trajectory describing the flat models evolves from $y = +\infty$ to $y = -\infty$. This trajectory describes models that evolve from a big-bang singularity and after a finite time start to inflate. After inflating for a finite period of time, the models will start expanding at increasing rates (i.e., $\dot{\theta} > 0$) and will evolve to $y = -\infty$, which represents models with an infinite negative viscous pressure.

The behaviour of the viscous fluid FRW models where the bulk viscous pressure satisfies the full Israel-Stewart theory of irreversible thermodynamics has been analyzed. The stability of the singular point $(0, 0)$, representing the Milne model depends upon the value of m. The equilibrium point $(\bar{x}, \bar{y})$ can represent either an open, flat or closed FRW model depending upon the value of the parameter B_{10}. Exact determination of the nature of this particular singular point is extremely difficult; however, a partial result is possible—if $B_{10}(1 - m) > 0$ then the equilibrium point is a saddle.

Maartens [36] has derived the evolution of the specific entropy, s, to be

$$\dot{s} = -\frac{\theta\Pi}{nT}, \tag{5.19}$$

where n is the number density. Using the number density conservation law ($\dot{n} + n\theta = 0$), equation (5.19) can be translated to dimensionless variables

$$s' = (3T_o n_o)^{-1} y x^{-p} \theta^{2(1-q)} e^{-(3+r)\Omega}, \tag{5.20}$$

Requiring $\dot{s} > 0$ or equivalently $s' < 0$ indicates that y should be constrained to be less than zero, i.e., $y < 0$. Other possible constraints on p, q , and r can be found by imposing further conditions on the entropy evolution, e.g., imposing that it increase but at decreasing rates, i.e. $\ddot{s} < 0$.

It can be concluded that the behaviour of the FRW models in which the bulk viscous pressure satisfies the full Israel-Stewart theory can be qualitatively quite different from the behaviour of the truncated models. On the other hand, it can be argued that the physically meaningful case is when $q < 1$, in which case the qualitative behaviour of the flat FRW models using the truncated and the full Israel-Stewart theory is similar. The equations of state utilized for the temperature (5.8) and for the bulk viscosity coefficient ζ play major roles in determining the dynamics of the models.

A complete analysis of the asymptotic behaviours of these viscous fluid models, depending on the (many) free parameters in the model $(a, b, \gamma, m, p, q, r)$ and utilizing the energy conditions can be made. The next step in this research programme, however, is to attempt to use results from kinetic theory in order to motivate physically plausible equations of state, or, at the very least, to limit the form of the phenomenological equations of state used.

5.3 Isotropic Curvature Models

5.3.1 Introduction

In Chapter 3, a simple Bianchi type V model was investigated where the viscous effects were modelled by the first-order non-causal Eckart theory of irreversible thermodynamics [47]. In Chapter 4, isotropic Friedmann-Robertson-Walker and anisotropic Bianchi type I and type V models were analyzed where the viscous effects were described by the second-order causal truncated Israel-Stewart theory of irreversible thermodynamics [49, 50] where it was found that anisotropy led to some interesting behaviour. In the previous section 5.2, an isotropic Friedmann-Robertson-Walker model

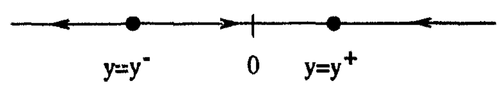


Figure 5.1: *The phase portrait describes the qualitative behavior of the isotropic viscous fluid FRW models in the case $B_9 > 0$ and $q < 1$. The arrows in all the figures denote increasing Ω-time ($\Omega \to \infty$) or decreasing t-time ($t \to 0^+$).*

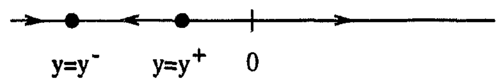


Figure 5.2: *The case $B_9 > 0$ and $q > 1$.*

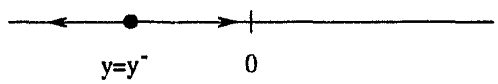


Figure 5.3: *The case $q = 1$ and $2b + 3 + r > 0$.*

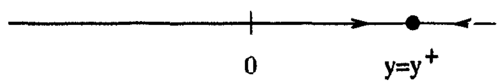


Figure 5.4: *The case $q = 1$ and $2b + 3 + r < 0$.*

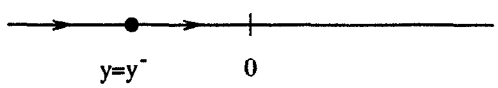


Figure 5.5: *The case $B_9 = 0$ and $2b + 3 + r + 3\gamma(q - 1) > 0$.*

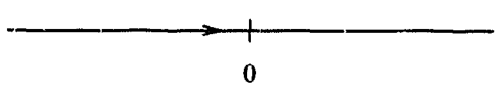


Figure 5.6: *The case $B_9 < 0$.*

was studied in which the bulk viscous pressure satisfies the full (non-truncated) Israel-Stewart theory of irreversible thermodynamics [48, 51, 52]. The obvious next step in this line of research is to analyze the anisotropic generalizations of the Friedmann-Robertson-Walker models employing the full Israel-Stewart theory of irreversible thermodynamics to describe the viscous effects.

In this section, to simplify the mathematics and derive a tractable system of equations we shall assume zero heat conduction. We will investigate viscous effects in a class of spatially homogeneous and anisotropic cosmological models, in particular, the class of isotropic curvature models. The isotropic curvature models are described by the property that the three dimensional Ricci curvature tensor is isotropic; that is, ${}^3R_{\alpha\beta} = \frac{1}{3}{}^3R\delta_{\alpha\beta}$ on the spatial hypersurfaces [20] (hence, the spatial hypersurfaces have constant curvature [20]). This class of models contains the Bianchi type I (${}^3R = 0$) and Bianchi type V (${}^3R < 0$) models and a special case of the Bianchi type IX (${}^3R > 0$) models. In essence, the isotropic curvature models (Bianchi V, Bianchi I, Bianchi IX) can be regarded as the simplest anisotropic generalizations of the Friedmann-Robertson-Walker (open, flat, closed) models.

5.3.2 The Equations

The dimensionless Einstein field equations are for the isotropic curvature models equivalent to equations (3.11–3.16). If the model is of Bianchi type V then there is an additional constraint that must be satisfied, namely $\Sigma_1 + \Sigma_2 = 0$. In order to close the system of equations we need relationships for the dimensionless viscous pressure y and the dimensionless anisotropic stresses z_1 and z_2. Therefore, we require equations for Π, Π_1, and Π_2.

Assuming that the universe can be modelled as a simple fluid , the non-truncated Israel-Stewart equations for the bulk viscous pressure, Π, and the anisotropic stresses,

Π_1, and Π_2, are given by (1.3) [48, 51, 52]:

$$\begin{aligned}
\Pi &= -\zeta\left(\theta + \beta_0\dot{\Pi} + \frac{\epsilon}{2}\beta_0\Pi\left[\theta + \frac{\dot{\beta}_0}{\beta_0} - \frac{\dot{T}}{T}\right]\right), \\
\Pi_1 &= -2\eta\left(\sigma_1 + \beta_2\dot{\pi}_1 + \frac{\epsilon}{2}\beta_2\Pi_1\left[\theta + \frac{\dot{\beta}_2}{\beta_2} - \frac{\dot{T}}{T}\right]\right), \qquad (5.21)\\
\Pi_2 &= -2\eta\left(\sigma_2 + \beta_2\dot{\pi}_2 + \frac{\epsilon}{2}\beta_2\Pi_2\left[\theta + \frac{\dot{\beta}_2}{\beta_2} - \frac{\dot{T}}{T}\right]\right).
\end{aligned}$$

The variable T represents the temperature, ζ is the bulk viscosity coefficient, η is the shear viscosity coefficient, β_0 and β_2 are proportional to the relaxation times. The above equations reduce to the Eckart equations used in Chapter 3 and in [53, 54, 55] when $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = \beta_2 = 0$, and reduce to the truncated equations used in Chapter 4 when $\epsilon = 0$.

Rewriting the above system (5.21) in terms of the dimensionless variables (3.10) and augmenting the result to equations (3.11–3.13) we obtain the following system:

$$\begin{aligned}
\frac{dx}{d\Omega} &= x(1-2q) + 9\frac{p}{\theta^2} + y + \Sigma_1(2z_1 - z_2) + \Sigma_2(2z_2 - z_1) \\
\frac{d\Sigma_1}{d\Omega} &= \Sigma_1(2-q) - 12z_1, \\
\frac{d\Sigma_2}{d\Omega} &= \Sigma_2(2-q) - 12z_2, \qquad (5.22)\\
\frac{dy}{d\Omega} &= y\left[\left(\frac{\theta}{\zeta}\right)\left(\frac{3}{\beta_0\theta^2}\right) - 2 - 2q\right] + 9\left(\frac{3}{\beta_0\theta^2}\right) + \frac{\epsilon}{2}y\left[3 - \frac{\beta_0'}{\beta_0} + \frac{T'}{T}\right], \\
\frac{dz_1}{d\Omega} &= z_1\left[2\left(\frac{\theta}{\eta}\right)\left(\frac{3}{4\beta_2\theta^2}\right) - 2 - 2q\right] + \Sigma_1\left(\frac{3}{4\beta_2\theta^2}\right) + \frac{\epsilon}{2}z_1\left[3 - \frac{\beta_2'}{\beta_2} + \frac{T'}{T}\right], \\
\frac{dz_2}{d\Omega} &= z_2\left[2\left(\frac{\theta}{\eta}\right)\left(\frac{3}{4\beta_2\theta^2}\right) - 2 - 2q\right] + \Sigma_2\left(\frac{3}{4\beta_2\theta^2}\right) + \frac{\epsilon}{2}z_2\left[3 - \frac{\beta_2'}{\beta_2} + \frac{T'}{T}\right],
\end{aligned}$$

where $\Sigma^2 \equiv \frac{1}{3}(\Sigma_1 + \Sigma_2)^2 - \Sigma_1\Sigma_2$, the dimensionless deceleration parameter

$$q = \frac{1}{2}\left(x + y + 9\frac{p}{\theta^2} + \Sigma^2\right), \qquad (5.23)$$

and where $'$ denotes differentiation with respect to Ω. Finally, the Friedmann equation (3.16) is

$$4 - 4x - \Sigma^2 = -6\frac{{}^3R}{\theta^2}. \qquad (5.24)$$

The interior of the parabola $4 = \Sigma^2 + 4x$ in the phase space represents models of Bianchi type V. The parabola itself represents models of Bianchi type I, and the exterior of the parabola represents models of Bianchi type IX. Presently, we shall always assume that $x \geq 0$, which states that the energy density in the rest frame of the matter is non-negative. This is a necessary condition for the fulfillment of the WEC [69].

Note also that if ${}^3R < 0$ then the model is of Bianchi type V and the extra constraint $\Sigma_1 + \Sigma_2 = 0$ must be satisfied. Imposing this constraint on the Bianchi type I and IX models leads to a sub-class of the isotropic curvature models, which, since $\Sigma = \Sigma_1 = -\Sigma_2$ and $\Pi = \Pi_1 = -\Pi_2$, is governed by the four-dimensional system of ordinary differential equations:

$$\begin{aligned}
\frac{dx}{d\Omega} &= x(1-2q) + 9\frac{p}{\theta^2} + y + 6\Sigma z \\
\frac{d\Sigma}{d\Omega} &= \Sigma(2-q) - 12z, \\
\frac{dy}{d\Omega} &= y\left[\left(\frac{\theta}{\zeta}\right)\left(\frac{3}{\beta_0\theta^2}\right) - 2 - 2q\right] + 9\left(\frac{3}{\beta_0\theta^2}\right) + \frac{\epsilon}{2}y\left[3 - \frac{\beta_0'}{\beta_0} + \frac{T'}{T}\right], \\
\frac{dz}{d\Omega} &= z\left[2\left(\frac{\theta}{\eta}\right)\left(\frac{3}{4\beta_2\theta^2}\right) - 2 - 2q\right] + \Sigma\left(\frac{3}{4\beta_2\theta^2}\right) + \frac{\epsilon}{2}z\left[3 - \frac{\beta_2'}{\beta_2} + \frac{T'}{T}\right],
\end{aligned} \tag{5.25}$$

where q is given by equation (5.23). In order to close the system, a set of equations of state must again be adopted. These equations of state can be chosen phenomenologically in the same way as in the previous chapter [see Chapter 4, equation (4.38)]. However, the full Israel-Stewart theory requires a temperature law for T.

5.3.3 Future Work

Future work will include (i) an attempt at using relativistic kinetic theory to place limits on the form of the phenomenological equations of state and (ii) an extensive qualitative analysis of the isotropic curvature models using these physically motivated equations of state, thereby expanding the analysis in section 5.2 to anisotropic models and extending the analysis in Chapter 3 and in [53, 54, 55] to causal theories.

5.4 Conclusions

The most obvious observation that one can make regarding this chapter is that the qualitative behaviour of the flat FRW models, using the truncated Israel-Stewart theory, and the qualitative behaviour of the flat models, using the full Israel-Stewart theory, can be significantly different. This difference however, appears as a result of the equation of state for the temperature. The primary conclusion that can be drawn from all these results is that the equations of state play a primary role in determining the behaviour of the viscous fluid cosmological models. Which equations of state are most appropriate remains to be seen, however, from a mathematical perspective, the dimensionless equations of state used throughout this thesis offer a convenient way of reducing the dimensionality of the problem. It is hoped that through some analysis of kinetic theory of relativistic fluids that the form of the equations of state can be found explicitly or for our matters at least asymptotically.

Chapter 6

Spatially Homogeneous Cosmologies with an Exponential Potential

6.1 Introduction

We shall study cosmological models containing a self-interacting scalar field with an exponential potential. Scalar field cosmology is of importance in the study of inflation, an idea originally proposed by Guth [82], in which the universe underwent a period of accelerated expansion (see, for example, Olive [83]). Models with an exponential scalar field potential arise naturally in alternative theories of gravity, such as, for example, theories based on the Brans-Dicke theory (for example, extended inflation [84, 85], and hyper-extended inflation [86]), in the Salam-Sezgin model of $N = 2$ super-gravity coupled to matter [87], and in theories undergoing dimensional reduction to an effective four dimensional theory [88]. In addition, other theories of gravity, such as, for example, quadratic Lagrangian theories, are known to be conformally equivalent to general relativity plus a scalar field with potentials of exponential-type [89, 90]. Cosmologies of this type have been studied by a number of authors, including Burd and Barrow [91], Kitada and Maeda [92, 93] and Feinstein and Ibáñez [94].

Our aim here is to perform a qualitative analysis on a class of Bianchi cosmologies containing a scalar field with an exponential potential. Since the potential is an exponential function, the governing differential equations exhibit a symmetry [63] which allows expansion-normalized variables to be defined. The resulting phase space is compact (except in the Bianchi IX case in which the phase space is closed but unbounded), and the differential equation for the expansion decouples from the other equations. Therefore the reduced system of ordinary differential equations can be analyzed by using standard geometric (dynamical systems) techniques [18, 54, 55]. In particular, we wish to study qualitatively whether the models inflate and/or isotropize, thereby determining the applicability of the so-called cosmic no-hair theorem in homogeneous scalar field cosmologies with an exponential potential. Essentially the cosmic no-hair conjecture asserts that inflation is typical in a wide class of scalar field cosmologies. Because inflation in scalar field models with an exponential potential is of power-law type [91] which is weaker than in conventional exponential inflation (for which no-hair theorems exist [95]), it is not obvious that there exists a cosmic no-hair theorem for these models. In addition, Feinstein and Ibáñez [94] have found exact homogeneous solutions (of Bianchi types III and VI) which neither inflate nor isotropize; this work will determine the relevance of these exact solutions and investigate whether their qualitative properties are generic.

Cosmological models with a minimally coupled scalar field have a stress-energy tensor given by

$$T_{ab} = \phi_{;a}\phi_{;b} - g_{ab}\left(\frac{1}{2}\phi_{;c}\phi^{;c} + V(\phi)\right), \tag{6.1}$$

where for a homogeneous scalar field $\phi = \phi(t)$, so that $\phi_{;c}\phi^{;c} = -\dot{\phi}^2$ (where an over-dot denotes differentiation with respect to the proper time). In this case we can formally treat the stress-energy tensor as a perfect fluid with velocity vector $u^a = \phi^{;a}/\sqrt{-\phi_{;a}\phi^{;a}}$, where the energy density and the pressure are given by

$$\rho_\phi \equiv E = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{6.2}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{6.3}$$

In the models under consideration, the potential of the scalar field is given by

$$V(\phi) = \Lambda e^{k\phi}, \tag{6.4}$$

where Λ (> 0) and k are constants.

As noted earlier, a number of authors have studied such cosmological models. Homogeneous and isotropic, FRW models were studied using phase-plane methods by Halliwell [88] [see also, for example, Olive [83]]. Homogeneous but anisotropic models have been studied by Burd and Barrow [91] [Bianchi models of type I and III (and Kantowski-Sachs models); exact solutions and a discussion of their stability], Lidsey [96] [Bianchi type I], Aguirregabiria et al. [97] [Bianchi type I; exact solutions and qualitative analysis for all k], Feinstein and Ibáñez [94] [Bianchi types III and VI; exact solutions for all k] and Kitada and Maeda [92, 93] [all Bianchi types; qualitative analysis for models with $k^2 < 2$, including standard matter satisfying the various energy conditions].

This chapter is organized as follows. In section 6.2 we shall discuss general qualitative features of homogeneous scalar field cosmologies with an exponential potential, such as for example, whether they isotropize or inflate and what is the relevance of the Feinstein-Ibáñez solutions [94]. In addition, we will show that all equilibrium points in the 'reduced' dynamical system correspond to self-similar cosmological models. In section 6.3 we will perform a qualitative analysis of a particular class of Bianchi models, and in doing so illustrate these general properties. In section 6.4 we conclude with a discussion.

6.2 Isotropization and the Cosmic No-Hair Theorem

6.2.1 Background

Wald's [95] result that "all initially expanding homogeneous models, with ordinary matter satisfying both the strong and dominant energy conditions and with a positive cosmological constant, asymptotically approach the isotropic de Sitter space-time", was one of the first versions of the cosmic no-hair theorem that was actually proven. Since then a number of extended 'cosmic no-hair theorems' have been proven for Bianchi models. In particular, and essentially using Wald's approach, Kitada and Maeda [92, 93] have proven that for $k^2 < 2$, all initially expanding anisotropic models containing a scalar field with an exponential potential (and ordinary matter satisfying the energy conditions) locally approach an isotropic, power-law inflationary (FRW) solution (in the Bianchi type IX case the models must also satisfy the condition that the ratio of the effective vacuum energy to the maximum three curvature is larger than some critical value). In the special case $k = 0$, the theorem essentially reduces to Wald's result [95], and the unique attractor is the exponential inflationary de Sitter solution.

In related work, Heusler [98] proved that all Bianchi models with ordinary matter satisfying the usual energy conditions and containing a scalar field with a positive, convex potential (with a local minimum such that $V(\phi_o) = 0$; for example, $V(\phi) = \frac{1}{2}m\phi^2$), *can only approach isotropy* at infinite times if the underlying isometry group is a Lie subgroup of the G_6 group of isometries describing the FRW model, that is, if the underlying Lie group is one of Bianchi types I, V, VII, or IX. This work extended (by including scalar fields) the famous result of Collins and Hawking [30] that only a subclass of measure zero in the space of all homogeneous models can approach isotropy. Here, we shall extend Heusler's result to the case of a scalar field with an exponential potential of the form (6.4) with $k^2 > 2$. In this case, the scalar field

ϕ is generally not bounded and $\phi V'(\phi) \geq V(\phi)$ is only satisfied when ϕ is positive; therefore, the conditions in Heusler's main theorem are not met. However, Heusler's Proposition 1 (where now $\theta \to 0$ and $V \to 0$ as $t \to \infty$ if there exists a time t_o with $\theta(t_o) \geq 0$) and Proposition 2 (which gives necessary conditions in order for a homogeneous model which is not among the Bianchi types admitted by an FRW model to isotropize), are both true in the case of an exponential potential. In our calculation below we effectively replace Heusler's Proposition 3 with an analogous result on the behaviour of V/E in the case of an exponential potential.

6.2.2 Equations

From the Einstein field equations describing the spatially homogeneous models, we have the Raychaudhuri equation governing the evolution of the expansion

$$\dot{\theta} = -2\sigma^2 - \frac{1}{3}\theta^2 - \dot{\phi}^2 + V(\phi), \tag{6.5}$$

and the generalized Friedmann equation

$$\theta^2 = 3\sigma^2 + \frac{3}{2}\dot{\phi}^2 + 3V(\phi) - \frac{3}{2}P, \tag{6.6}$$

where σ is the shear scalar, P is the scalar curvature of the homogeneous hypersurfaces, which is always negative or zero except in the Bianchi IX case [95], and $V(\phi)$ is given by equation (6.4). The Klein-Gordon equation for the scalar field with an exponential potential is then

$$\ddot{\phi} + \theta\dot{\phi} + kV(\phi) = 0. \tag{6.7}$$

Defining the new variable ψ as

$$\psi = \dot{\phi} + \frac{k}{3}\theta, \tag{6.8}$$

and using equations (6.5) and (6.6), the Klein-Gordon equation can be rewritten as

$$\dot{\psi} + \theta\psi + \frac{k}{3}P = 0. \tag{6.9}$$

We now introduce new expansion-normalized variables and a new time variable as follows:

$$\beta = \sqrt{3}\frac{\sigma}{\theta}, \qquad \frac{dt}{d\Omega} = \frac{3}{\theta},$$
$$\Psi = \frac{\sqrt{6}}{2}\frac{\dot{\phi}}{\theta}, \qquad \Phi = \sqrt{3\Lambda}\frac{e^{k\phi/2}}{\theta}. \tag{6.10}$$

With these definitions and using equations (6.5) and (6.6), equation (6.7) can be rewritten as

$$\Psi' = -\Psi(2 - 2\beta^2 - 2\Psi^2 + \Phi^2) - \frac{\sqrt{6}k}{2}\Phi^2, \tag{6.11}$$
$$\Phi' = -\Phi(-1 - 2\beta^2 - 2\Psi^2 + \Phi^2 - \frac{\sqrt{6}k}{2}\Psi), \tag{6.12}$$

where $'$ denotes differentiation with respect the new time Ω. The equilibrium points of the system have either $\Phi = \Psi = 0$, which corresponds to the massless scalar field case, or $\beta^2 + \Psi^2 = 1, \Phi = 0$, which represents the Kasner-like initial (line) singularity, or else (and in all cases of interest here) obey the following relation

$$\Phi^2 + \Psi^2 = -\frac{\sqrt{6}}{k}\Psi. \tag{6.13}$$

In terms of these new expansion-normalized variables, the energy density of the scalar field (6.2) can be written as

$$\frac{E}{\theta^2} = \frac{1}{3}(\Psi^2 + \Phi^2), \tag{6.14}$$

and we have that

$$\Psi = -\frac{k}{\sqrt{6}} + \frac{\sqrt{3}}{\sqrt{2}}\frac{\psi}{\theta}. \tag{6.15}$$

Hence, at the equilibrium points we obtain

$$\frac{E}{\theta^2} = -\frac{\sqrt{6}}{3k}\Psi = \frac{1}{3}\left(1 - \frac{3}{k}\frac{\psi}{\theta}\right), \tag{6.16}$$
$$\frac{V}{E} = \frac{\Phi^2}{\Psi^2 + \Phi^2} = 1 - \frac{k^2}{6} + \frac{k}{2}\frac{\psi}{\theta}. \tag{6.17}$$

6.2.3 Isotropization

Following Heusler [98], the necessary conditions for the anisotropic and homogeneous models, which are not admitted by the FRW models[1], that contain a homogeneous scalar field, to isotropize are:

$$\beta = 0, \tag{6.18}$$

and (Heusler's Proposition 2 [98])

$$\begin{aligned} \frac{E}{\theta^2} &\rightarrow \frac{1}{3}, \quad \text{as} \quad t \rightarrow \infty \qquad (6.19) \\ \left\langle \frac{V}{E} \right\rangle &\geq \frac{2}{3}, \qquad (6.20) \end{aligned}$$

where $\langle\ \rangle$ denotes an appropriate time average [Heusler [98], equation (20)].

Now, using equation (6.16), equation (6.19) implies that as $t \rightarrow \infty$

$$\frac{\psi}{\theta} \rightarrow 0. \tag{6.21}$$

Substituting equation (6.21) into (6.17) we can compute $\langle V/E \rangle$, viz.,

$$\left\langle \frac{V}{E} \right\rangle = \left\langle 1 - \frac{k^2}{6} \right\rangle = 1 - \frac{k^2}{6} \tag{6.22}$$

Hence, the necessary condition, equation (6.20), for isotropization to occur implies that

$$1 - \frac{k^2}{6} \geq \frac{2}{3} \quad \Rightarrow \quad k^2 \leq 2. \tag{6.23}$$

Therefore, we have shown that if $k^2 > 2$ and if the model is of Bianchi types II, III, IV, VI or VIII then it *cannot* isotropize. Another way, if $k^2 > 2$ and the model is not of Bianchi types I, V, VII, or IX, then it *cannot* isotropize. Like Heusler [98], we have not completely generalized the Collins and Hawking [30] result that only a subclass of homogeneous models of measure zero can isotropize, since we have not explicitly investigated Bianchi models of types VII_h and IX.

[1]These are the Bianchi type II, III, IV, VI and VIII models.

The following question consequently arises; what is the future asymptotic behaviour of the models when $k^2 > 2$. This question will be partially addressed in section 6.3.

6.2.4 Inflation

For inflation to occur we must have a negative deceleration parameter q, i.e.,

$$q = 2\beta^2 + 2\Psi^2 - \Phi^2 < 0, \tag{6.24}$$

so that, using equations (6.13), (6.15) and (6.24), at the equilibrium points the solution will inflate if

$$(k^2 - 2) - 3k\frac{\psi}{\theta} < 0. \tag{6.25}$$

Therefore, from equations (6.18) and (6.21), for models to inflate and isotropize k^2 must be less than two, a well known result [88, 92, 93].

We have shown that $k^2 \leq 2$ is a necessary condition for the spatially homogeneous models under consideration to isotropize, and when $k^2 < 2$ these models will also inflate. Note, we have not proven that all such models with $k^2 \leq 2$ isotropize (although we shall explicitly demonstrate that this is the case for a subclass of Bianchi models in section III). However, the no-hair theorem of Kitada and Maeda [92, 93], described in section 6.2.1, shows precisely this; namely, that for $k^2 < 2$ the isotropic, power-law inflationary FRW solution is the unique attractor for any initially expanding Bianchi model. In addition, these authors also showed [93] that in these models anisotropies always enhance inflation in models with non-positive spatial curvature (over their isotropic counterparts) and generally enhance inflation in models of Bianchi type IX (however; see the detailed discussion in Kitada and Maeda [93], pp 720–721).

6.2.5 Remarks

First, we note that in our investigation we have not included ordinary matter (satisfying the usual energy conditions). However, matter could easily be included in precisely the same way as in Heusler [98] and Kitada and Maeda [92, 93]. One would just require that the matter satisfy the DEC and the SEC.

Second, for an exponential potential the equation describing the evolution of the expansion (6.5) written in terms of the new expansion-normalized variables (6.10) decouples from the 'reduced dynamical system' and can be written as

$$\frac{\dot{\theta}}{\theta^2} = -\frac{2}{3}\beta^2 - \frac{1}{3} - \frac{2}{3}\Psi^2 + \frac{1}{3}\Phi^2. \tag{6.26}$$

Consequently at the equilibrium points of the reduced system β, Ψ and Φ are constants, therefore we must have that

$$\theta = \theta_o t^{-1}, \tag{6.27}$$

whence the corresponding cosmological models are necessarily self-similar in that they admit a homothetic vector [61] (except in the degenerate case $k = 0$ in which the right-hand side of equation (6.5) can be zero and the corresponding model is the de Sitter space-time which is not self-similar).

In particular, the isotropic, power-law inflationary (FRW) attracting solutions (in the case $k^2 < 2$), are self-similar models and the Feinstein-Ibáñez [94] solutions (in the case $k^2 > 2$) are also self-similar.

6.3 A Class of Anisotropic Cosmological Models

6.3.1 The Equations

The diagonal form of the Bianchi type VI_h metric is given by:

$$ds^2 = -dt^2 + a(t)^2dx^2 + b(t)^2e^{2mx}dy^2 + c(t)^2e^{2x}dz^2, \tag{6.28}$$

where $m = h - 1$. If $m = 1$ then the metric is of Bianchi type V, if $m = 0$ then the metric is of Bianchi type III, and if $m = -1$ then the metric is of Bianchi type VI_o. Thus we are considering a 1-parameter (m) class of Bianchi models which include Bianchi types III ($m = 0$), V ($m = 1$), VI_0 ($m = -1$), and VI_h (all other m). In addition, the Bianchi type I models can also be included as they are on the boundary.

The expansion scalar, which determines the volume behavior of the fluid, is given by

$$\theta = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}. \tag{6.29}$$

The shear tensor, σ_{ab}, determines the distortion arising in the fluid flow leaving the volume invariant. The nonzero components of the shear tensor are

$$\begin{aligned}
\sigma_{11} &= \frac{a}{3}\left(2\frac{\dot{a}}{a} - \frac{\dot{b}}{b} - \frac{\dot{c}}{c}\right), \\
\sigma_{22} &= \frac{b\mathrm{e}^{2mx}}{3}\left(2\frac{\dot{b}}{b} - \frac{\dot{a}}{a} - \frac{\dot{c}}{c}\right), \\
\sigma_{33} &= \frac{c\mathrm{e}^{2x}}{3}\left(2\frac{\dot{c}}{c} - \frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right),
\end{aligned} \tag{6.30}$$

and the shear scalar, $\sigma^2 = \frac{1}{2}\sigma^{ab}\sigma_{ab}$, is given by

$$\sigma^2 = \frac{1}{3}\left[\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{b}}{b}\right)^2 + \left(\frac{\dot{c}}{c}\right)^2 - \frac{\dot{a}\dot{b}}{ab} - \frac{\dot{a}\dot{c}}{ac} - \frac{\dot{b}\dot{c}}{bc}\right]. \tag{6.31}$$

In the case under consideration here, there is no rotation and no acceleration.

The Einstein field equations with a homogeneous scalar field having an exponential potential (6.4) are:

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = -\dot{\phi}^2 + \Lambda\mathrm{e}^{k\phi}, \tag{6.32}$$

$$\frac{\dot{a}}{a}(1+m) - m\frac{\dot{b}}{b} - \frac{\dot{c}}{c} = 0, \tag{6.33}$$

$$\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} - \frac{m^2+1}{a^2} = \Lambda\mathrm{e}^{k\phi}, \tag{6.34}$$

$$\frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} - \frac{m^2+m}{a^2} = \Lambda\mathrm{e}^{k\phi}, \tag{6.35}$$

$$\frac{\ddot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} - \frac{m+1}{a^2} = \Lambda\mathrm{e}^{k\phi}. \tag{6.36}$$

From the above equations one obtains the generalized Friedmann equation [see equation (6.6)]

$$\theta^2 = 3\sigma^2 + \frac{3}{2}\dot{\phi}^2 + 3\Lambda e^{k\phi} + \frac{3}{a^2}(m^2 + m + 1). \tag{6.37}$$

Note that the quantity $m^2 + m + 1 \geq 3/4 > 0$. The Raychaudhuri equation is [see equation 6.5)]

$$\dot{\theta} = -2\sigma^2 - \frac{1}{3}\theta^2 - \dot{\phi}^2 + \Lambda e^{k\phi}. \tag{6.38}$$

The evolution equation for the shear is

$$\dot{\sigma} = -\sigma\theta + \frac{(1-m)}{3\sqrt{3}\sqrt{m^2+m+1}}\left(\theta^2 - 3\sigma^2 - \frac{3}{2}\dot{\phi}^2 - 3\Lambda e^{k\phi}\right). \tag{6.39}$$

The Klein-Gordon equation for the scalar field is [see equation (6.7)]

$$\ddot{\phi} = -\theta\dot{\phi} - k\Lambda e^{k\phi}. \tag{6.40}$$

The above system of equations (6.37)-(6.40) is invariant under the transformation (see section 2.5),

$$\begin{array}{lll} \theta \to \lambda\theta & \dot{\phi} \to \lambda\dot{\phi} & \phi \to \phi + \frac{2}{k}\ln\lambda \\ \sigma \to \lambda\sigma & t \to \lambda^{-1}t & \end{array} \tag{6.41}$$

This invariance implies that there exists a symmetry in the dynamical system (6.37)-(6.40) [63]. With the change of variables given by equation (6.10), the evolution equations for β, Ψ and Φ become independent of the variable θ. That is, θ decouples from the dynamical system describing the evolution of β, Ψ and Φ. The dynamical system can be considered as a reduced dynamical system for β, Ψ and Φ together with an evolution equation for θ (see equations below).

The system of differential equations in the expansion-normalized variables becomes:

$$\begin{aligned} \frac{d\beta}{d\Omega} &= \beta(q-2) + \frac{1-m}{\sqrt{m^2+m+1}}\left(1 - \beta^2 - \Psi^2 - \Phi^2\right) \\ \frac{d\Psi}{d\Omega} &= \Psi(q-2) - \frac{\sqrt{6}k}{2}\Phi^2 \\ \frac{d\Phi}{d\Omega} &= \Phi(1+q) + \frac{\sqrt{6}k}{2}\Psi\Phi \end{aligned} \tag{6.42}$$

$$\frac{d\theta}{d\Omega} = -\theta(1+q)$$

where the deceleration parameter q, is defined to be

$$q = 2\beta^2 + 2\Psi^2 - \Phi^2 \tag{6.43}$$

The phase space [determined by equation (6.37)] is the region defined by

$$\beta^2 + \Psi^2 + \Phi^2 \leq 1, \tag{6.44}$$

which describes the interior of a sphere. The sphere itself is the phase space for the Bianchi type I models. We also note that the above system is invariant under the transformation $\Phi \to -\Phi$, hence without loss of generality we restrict ourselves to the set defined by equation (6.44) and $\Phi \geq 0$.

Inflation, in the context of this paper, is defined to occur whenever the deceleration parameter q is negative, that is $q < 0$. We easily see from equation (6.43) the inflationary regime describes a cone inside the sphere.

6.3.2 Qualitative Analysis

Equilibrium Points

The equilibrium point

$$\left\{\beta = \frac{1-m}{2\sqrt{m^2+m+1}}, \Psi = 0, \Phi = 0\right\}, \tag{6.45}$$

satisfies the boundary condition for all m, and when $m = -1$ the point is part of the non-isolated line of equilibrium points $\beta^2 + \Psi^2 = 1$ (which will be discussed later). The inflationary condition $q < 0$ is never satisfied, hence this point is non-inflationary. The linearized system in a neighborhood of the equilibrium point has the following eigenvalues:

$$\lambda_1 = \frac{-3(m+1)^2}{2(m^2+m+1)},$$

$$\begin{aligned}\lambda_2 &= \frac{-3(m+1)^2}{2(m^2+m+1)}, \\ \lambda_3 &= \frac{3(m^2+1)}{2(m^2+m+1)}.\end{aligned} \tag{6.46}$$

It is easily seen that this point is a saddle point with a 2-dimensional stable manifold. The exact solution corresponding to this point is that of a vacuum Bianchi type VI_h model or its degeneracies (that is if $m = 0$ it is type III, and if $m = 1$ it is of type V), with line element (after a re-coordinatization)

$$ds^2 = -dt^2 + a_o{}^2(t^{2p_1}dx^2 + t^{2p_2}e^{2mx}dy^2 + t^{2p_3}e^{2x}dz^2), \tag{6.47}$$

where

$$p_1 = 1, \qquad p_2 = \frac{m^2+m}{m^2+1}, \qquad p_3 = \frac{m+1}{m^2+1}, \tag{6.48}$$

such that $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2$.

The equilibrium point

$$\left\{\beta = 0, \Psi = -\frac{\sqrt{6}k}{6}, \Phi = \frac{\sqrt{6}}{6}\sqrt{6-k^2}\right\}, \tag{6.49}$$

does not exist if $k^2 > 6$ and is part of the non-isolated line of equilibrium points $\beta^2 + \Psi^2 = 1$ when $k^2 = 6$. The point lies on the boundary of the phase space $\beta^2 + \Psi^2 + \Phi^2 = 1$, hence it is of Bianchi type I. The point is inflationary if

$$q = \frac{k^2-2}{2} < 0; \tag{6.50}$$

that is, the point represents an inflationary model if $k^2 < 2$. The linearized system in a neighborhood of the equilibrium point has the following eigenvalues:

$$\begin{aligned}\lambda_1 &= \frac{k^2-6}{2}, \\ \lambda_2 &= \frac{k^2-6}{2}, \\ \lambda_3 &= k^2-2.\end{aligned} \tag{6.51}$$

If $k^2 < 2$ the point is a sink, and if $2 < k^2 < 6$ the the point is a saddle point. (The behaviour when $k^2 = 2$ or $k^2 = 6$, the bifurcation values, will be discussed later.) For

$k \neq 0$ the exact solution corresponding to this equilibrium point is that of a flat FRW model with line element given by (after a re-coordinatization)

$$ds^2 = -dt^2 + t^{4/k^2}(dx^2 + dy^2 + dz^2), \tag{6.52}$$

and if $k = 0$ then the exact solution is the de Sitter model. The scalar field is given by

$$\phi = \phi_o - \frac{2}{k} \ln t. \tag{6.53}$$

The equilibrium point

$$\left\{ \begin{array}{rcl} \beta & = & -\dfrac{(k^2-2)}{2} \dfrac{(m-1)\sqrt{m^2+m+1}}{[(k^2-2)(m^2+m+1)+3(m^2+1)]}, \\ \Psi & = & -\dfrac{\sqrt{6}k}{2} \dfrac{(m^2+1)}{[(k^2-2)(m^2+m+1)+3(m^2+1)]}, \\ \Phi & = & \dfrac{\sqrt{6}}{2} \dfrac{\sqrt{m^2+1}\sqrt{[(k^2-2)(m+1)^2+4(m^2+1)]}}{[(k^2-2)(m^2+m+1)+3(m^2+1)]} \end{array} \right\} \tag{6.54}$$

can be shown (after much algebra) to satisfy the boundary condition if $k^2 \geq 2$ and satisfies the inflationary condition $q < 0$ if $k^2 < 2$, which reveals that the point is inflationary if $k^2 < 2$ and is non-inflationary if $k^2 > 2$. The linearized part of the system in a neighborhood of the equilibrium point has the following eigenvalues:

$$\begin{aligned} \lambda_1 &= -\frac{3}{2}\left\{ \frac{4(m^2+1)+(k^2-2)(m+1)^2}{(k^2-2)(m^2+m+1)+3(m^2+1)} \right\}, \\ \lambda_2 &= -\frac{3}{4}\left\{ \frac{(k^2-2)(m+1)^2+4(m^2+1)}{(k^2-2)(m^2+m+1)+3(m^2+1)]} \right. \\ &\quad \left. + \frac{\sqrt{[(k^2-2)(m+1)^2+4(m^2+1)][4(m^2+1)-(k^2-2)(7m^2-2m+7)]}}{(k^2-2)(m^2+m+1)+3(m^2+1)} \right\}, \\ \lambda_3 &= -\frac{3}{4}\left\{ \frac{(k^2-2)(m+1)^2+4(m^2+1)}{(k^2-2)(m^2+m+1)+3(m^2+1)} \right. \\ &\quad \left. - \frac{\sqrt{[(k^2-2)(m+1)^2+4(m^2+1)][4(m^2+1)-(k^2-2)(7m^2-2m+7)]}}{(k^2-2)(m^2+m+1)+3(m^2+1)} \right\}. \end{aligned}$$

It can be shown that if $k^2 > 2$ then all three eigenvalues are negative, hence the equilibrium point represents a stable node. It is also interesting to note that if $k^2 >$

$2+4(m^2+1)/(7m^2-2m+7)$, then the point is a focus (i.e., the solution oscillates in a neighborhood of the equilibrium point). The behaviour of the system at the bifurcation value $k^2 = 2$ will be discussed later. The exact solution corresponding to this point is that of a Bianchi type VI_h or its degeneracies (that is if $m = 0$ it is type III and if $m = 1$ it is of type V), with line element (after a re-coordinatization)

$$ds^2 = -dt^2 + {a_o}^2(t^{2p_1}dx^2 + t^{2p_2}e^{2mx}dy^2 + t^{2p_3}e^{2x}dz^2), \tag{6.55}$$

where

$$\begin{aligned} p_1 &= 1, \\ p_2 &= \frac{2}{k^2}\left(1 + \frac{(k^2-2)(m^2+m)}{2(m^2+1)}\right), \\ p_3 &= \frac{2}{k^2}\left(1 + \frac{(k^2-2)(m+1)}{2(m^2+1)}\right). \end{aligned} \tag{6.56}$$

The scalar field in this case is given by

$$\phi = \phi_o - \frac{2}{k}\ln t. \tag{6.57}$$

This solution was first given by Feinstein and Ibáñez [94].

Boundaries

The qualitative behaviour on the boundaries can also help to determine the behaviour in the interior of the phase space. In this situation, each boundary set $\Phi = 0$, $\beta^2 + \Psi^2 + \Phi^2 = 1$ is an invariant set. The invariant set $\Phi = 0$ represents models with a massless scalar field or zero potential. Presumably, this invariant set will represent the behaviour of the system as the scalar field ϕ tends to minus infinity. The remaining system of equations for β and Ψ can be directly integrated to yield

$$\Psi = C\left(2\beta - \frac{(1-m)}{\sqrt{m^2+m+1)}}\right). \tag{6.58}$$

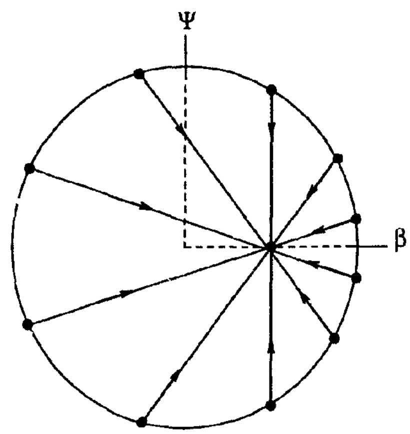


Figure 6.1: *Phase portrait in the invariant set* $\Phi = 0$.

These are straight lines emanating from the equilibrium point, (6.45), directed inwards, and thus in the two dimensional invariant set $\Phi = 0$ the point is a sink. However, in the three dimensional phase space, the point is a saddle point, and thus we can conclude that the invariant set $\Phi = 0$ is the 2-dimensional stable manifold. Also, it is easy to conclude that the outer ring described ' y $\beta^2 + \Psi^2 = 1$ is a source (see Figure 6.1).

We can alsc analyze the invariant set $\beta^2 + \Psi^2 + \Phi^2 = 1$ which represents the Bianchi type I models with a scalar field and an exponential potential. Again the system of equations can be integrated and the solutions are found to be straight lines emanating from the ring of equilibrium points given by $\beta^2 + \Psi^2 = 1$ and evolving to the equilibrium point, (6.49), if $k^2 < 6$. In the full three dimensional phase space for $2 < k^2 < 6$ this point is a saddle, and thus we can conclude that the invariant set $\beta^2 + \Psi^2 + \Phi^2 = 1$ is the 2-dimensional stable manifold. However, if $k^2 > 6$, then part of the ring of equilibrium points $\beta^2 + \Psi^2 = 1$ becomes a sink and the rest remains a source (see Figures 6.2 and 6.3). Also, in the full three dimensional phase space, the ring of equilibrium points $(\beta^2 + \Psi^2 = 1,\ \Phi = 0)$ for $k^2 < 6$ is a global source, and for $k^2 > 6$ we find that some of the ring acts like a source and the remaining part of the ring behaves like a saddle. The solution at the equilibrium points $(\beta_o, \pm\sqrt{1-\beta_o^2}, 0)$

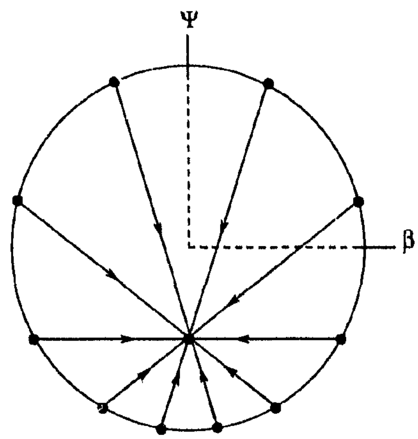


Figure 6.2: *Projection of the phase portrait in the invariant set* $\beta^2 + \Psi^2 + \Phi^2 = 1$ *with* $k^2 < 6$.

has the form

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{6.59}$$

where

$$\begin{aligned} p_1 &= \frac{1}{3}\left(1 + \frac{(1-m)\beta_o}{\sqrt{m^2+m+1}}\right), \\ p_2 &= \frac{1}{3}\left(1 - \frac{(2+m)\beta_o}{\sqrt{m^2+m+1}}\right), \\ p_3 &= \frac{1}{3}\left(1 + \frac{(1+2m)\beta_o}{\sqrt{m^2+m+1}}\right), \end{aligned} \tag{6.60}$$

where $-1 \leq \beta_o \leq 1$. Note that $p_1 + p_2 + p_3 = 1$ but $p_1^2 + p_2^2 + p_3^2 = \frac{1}{3}(1 + 2\beta_o)$, hence these points are not Kasner models but are Kasner-like.

Bifurcation Values

We shall now concern ourselves with the bifurcation values. If $k^2 = 0$, it is easily determined that the critical points and the qualitative behaviour is the same as in the case $0 < k^2 < 2$. However, the exact solutions are different. (Note that the $k^2 = 0$ case corresponds to the situation of when there is a positive cosmological constant.)

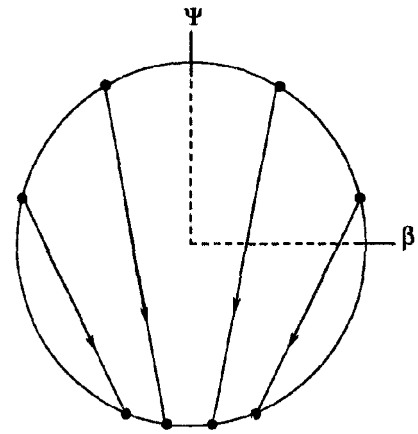


Figure 6.3: *Phase portrait in the invariant set* $\beta^2 + \Psi^2 + \Phi^2 = 1$ *with* $k^2 > 6$.

At the bifurcation value of $k^2 = 2$, we find that the equilibrium points, (6.49) and equation (6.55), coalesce to become a single equilibrium point. The linearized system at this point has a zero eigenvalue. However, by using polar coordinates we can see that the point is a sink, hence the qualitative behaviour of the system is the same as in the case $0 < k^2 < 2$. Thus, we conclude that the equilibrium point, (6.49), undergoes a saddle-node bifurcation at $k^2 = 2$. At the bifurcation value of $k^2 = 6$, the equilibrium point, (6.49), now becomes part of the line of equilibrium points $(\beta^2 + \Psi^2 = 1,\ \Phi = 0)$. This particular point remains a saddle point and the rest of the ring of equilibrium points remain sources, however as the value of k^2 is increased past 6, more and more of the ring starts to behave like a saddle point. Thus, in some extended sense of the definition, the ring of equilibrium points $(\beta^2 + \Psi^2 = 1,\ \Psi = 0)$ starts to undergo a saddle-node bifurcation at $k^2 = 6$.

6.3.3 Remarks

The behaviour of the models depend critically on the values of k and somewhat on the parameter m. The parameter m determines which Bianchi model we are considering. However, the parameter k has a profound affect on the qualitative behaviour of the models. For $0 \leq k^2 < 2$ all trajectories (that is, all models Bianchi I's, III's, V's and the $\mathrm{VI_h}$'s) except for a set of measure zero, evolved from the ring of equilibria $\beta^2 + \Psi^2 = 1$, $\Phi = 0$, representing the Kasner models and evolved towards the isotropic and power-law inflationary model located at equation (6.49). For $k^2 = 2$ all trajectories evolved from the ring of equilibrium points $\beta^2 + \Psi^2 = 1$, $\Phi = 0$ and evolved towards the isotropic model located at equation (6.49), however, these models need not inflate. For $2 < k^2$ all trajectories in the Bianchi III and $\mathrm{VI_h}$ cases evolved from some portion of the ring $\beta^2 + \Psi^2 = 1$ representing the Kasner models and evolved to the equilibrium point, (6.55), which represents the Feinstein-Ibáñez solution [94] which is neither isotropic nor inflationary. However, in the Bianchi I and Bianchi V cases, for $2 < k^2 < 6$ all trajectories evolved from some portion of the ring and isotropized, but they need not inflate. For $6 < k^2$ the Bianchi V models remain to isotropize while the Bianchi I models fail to do so. Hence the condition given be Kitada and Maeda [93] for the Cosmic No-Hair conjecture to hold follows here. We also see that if $k^2 > 6$ that the Cosmic No-hair conjecture cannot in general be satisfied.

6.4 Discussion

We have shown in section 6.2 a result that extends the analysis of Heusler [98] to potentials that are exponential functions. Namely, if $V(\phi) = \Lambda e^{k\phi}$ and $k^2 > 2$ then the spatially homogeneous models cannot isotropize to the future unless they are of Bianchi types I, V, VII_h or IX. We have also demonstrated that if $k^2 > 6$ then the Bianchi type I model does not isotropize to the future, however, this set of models represents a set of measure zero, with respect to the set of all spatially homogeneous models, the most general being Bianchi types VII_h, VIII, and IX.

The investigation started here is not a complete qualitative analysis of the spatially homogeneous cosmological models with an exponential potential. It does, however, illustrate how the dimension of the problem can be reduced through the use of dimensionless variables. In addition, this analysis puts into perspective the role played by the Feinstein-Ibáñez solutions [94]. These solutions are neither isotropic nor inflationary and therefore created an interesting problem as to whether these solutions violated the cosmic no-hair theorems proven Kitada and Maeda [92, 93] for $k^2 < 2$. We show that the Feinstein-Ibáñez solutions only become physical when $k^2 > 2$ and therefore do not violate the Cosmic No-Hair theorems of Kitada and Maeda [92, 93]. It is also interesting to note that these solutions also represent the generic future asymptotic behaviour of the Bianchi types III and VI models when $k^2 > 2$.

Chapter 7

Qualitative Analysis of Inflationary Theories

7.1 Soft Inflation

7.1.1 Introduction

Inflationary cosmology was originally investigated in the hope that some outstanding problems in cosmology might be solved. To date, however, there is no fully acceptable model for the source of inflation. In a recent paper [89], a Soft Inflationary scenario was proposed in which the matter content is described by two coupled scalar fields, one of which has a decaying potential and the other which serves as the inflaton during the expansion [89, 90]. Inflation, with two scalar fields, has been considered previously in [99, 100, 101]. The effect of the decaying exponential potential in Soft Inflation, however, is to reduce the rate of inflation in a manner similar to that in Extended Inflation [84, 85]. As the inflaton rolls down a flat plateau the second scalar field evolves on the exponential potential resulting in power-law inflation [88, 102, 103, 104]. The advantages of Soft Inflation are: (i) when the inflaton is of New Inflation-type [105, 106] the fine tuning of initial conditions is lessened and density perturbations are suppressed, and (*ii*) when the inflaton is of Chaotic Inflation-type [105, 107], the restrictions placed upon the coupling parameter are reduced considerably. Thus, Soft Inflation allows the constraints placed on previous models to be loosened.

In this chapter we shall show that the field equations governing soft inflation can be written as a dynamical system allowing us to analyze, in a mathematically rigourous manner, the evolution of the model and the corresponding asymptotic behaviours.

The action under investigation is

$$S = \int d^4x\sqrt{-g}\Big(\frac{R}{2\kappa^2} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}(\nabla\psi)^2 - e^{-\beta\kappa\phi}V(\psi)\Big), \tag{7.1}$$

where $\kappa^2 = 8\pi G$, ϕ and ψ(the inflaton) are scalar fields, $V(\psi)$ is a potential and β is the coupling constant. Variation of the action in a flat FRW universe yields the following set of non-linear second order ordinary differential equations:

$$\ddot{\phi} + 3H\dot{\phi} - \beta\kappa e^{-\beta\kappa\phi}V(\psi) = 0, \tag{7.2}$$

$$\ddot{\psi} + 3H\dot{\psi} + e^{-\beta\kappa\phi}\frac{dV(\psi)}{d\psi} = 0, \tag{7.3}$$

where the constraint equation is

$$H^2 = \frac{\kappa^2}{3}\Big(\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\dot{\psi})^2 + e^{-\beta\kappa\phi}V(\psi)\Big), \tag{7.4}$$

and where an overdot denotes differentiation with respect to time and $H = \frac{\dot{a}}{a}$ is the Hubble parameter where a is the length scale.

Berkin et al. [89] have found a unique stable power-law inflationary solution as the field $\phi \rightarrow +\infty$. The solution is given by

$$\begin{aligned} \kappa\phi &= \kappa\phi_c + (2/\beta)\ln(t/\kappa), \\ a &= a_o(t/t_o)^{2/\beta^2}, \\ f(\psi) &= f(\psi_o) - (1 - \beta^2/6)ln(a/a_o), \end{aligned} \tag{7.5}$$

where for a general potential V we have

$$\begin{aligned} e^{\beta\kappa\phi_c} &\equiv \frac{\beta^4\kappa^4 V(\psi)}{12(1-\beta^2/6)}, \\ f(\psi) &\equiv \kappa^2\int d\psi\frac{V}{V'}. \end{aligned}$$

We wish to investigate whether this solution is generic. This can be done using qualitative techniques of dynamical systems theory.

First we define new independent and dependent variables

$$\begin{aligned} \frac{dt}{d\tau} &= e^{\frac{\beta\kappa\phi}{2}}, \\ \Phi &= \phi' = \dot{\phi}e^{\frac{\beta\kappa\phi}{2}}, \\ \Psi &= \psi' = \dot{\psi}e^{\frac{\beta\kappa\phi}{2}}, \end{aligned} \tag{7.6}$$

where $'$ denotes differentiation with respect to the new time τ. Calculating both Φ' and Ψ', the resulting equations form a four dimensional autonomous system of ordinary differential equations:

$$\phi' = \Phi, \tag{7.7}$$

$$\psi' = \Psi, \tag{7.8}$$

$$\Phi' = \frac{\beta\kappa}{2}\Phi^2 - \frac{\sqrt{6}\kappa}{2}\Phi\Big(\Phi^2 + \Psi^2 + 2V(\psi)\Big)^{\frac{1}{2}} + \beta\kappa V(\psi), \tag{7.9}$$

$$\Psi' = \frac{\beta\kappa}{2}\Psi\Phi - \frac{\sqrt{6}\kappa}{2}\Psi\Big(\Phi^2 + \Psi^2 + 2V(\psi)\Big)^{\frac{1}{2}} - \frac{dV(\psi)}{d\psi}, \tag{7.10}$$

where the constraint equation is

$$H^2 = e^{-\beta\kappa\phi}\frac{\kappa^2}{6}\Big(\Phi^2 + \Psi^2 + 2V(\psi)\Big). \tag{7.11}$$

We observe that the equilibrium points (defined by $\phi' = \psi' = \Phi' = \Psi' = 0$) at finite values are given by

$$\Phi = 0, \qquad \Psi = 0, \qquad V(\psi) = 0, \qquad \frac{dV(\psi)}{d\psi} = 0. \tag{7.12}$$

We note that the equations (3.5)–(3.7) in Berkin and Maeda [90] hold at all finite equilibrium points in the full system which implies that the slow-roll approximation employed by Berkin and Maeda [90] and the full system are compatible close to the equilibrium points. Further analysis depends on the chosen form of the potential $V(\psi)$. In the next section we shall consider potentials arising from Chaotic inflation.

7.1.2 Chaotic Inflation

In Chaotic Inflation we choose the potential $V(\psi) = \frac{\lambda_n}{n}\psi^n$ where n is even and λ_n is constant [90]. In particular, here we consider $n = 2$. From (7.12), we see that for finite values of ϕ, we have a non-isolated line of equilibria along the ϕ-axis $(\phi_o, 0, 0, 0)$. Linearizing about this line, we find that all eigenvalues have $\Re e(\lambda) = 0$, hence, all equilibrium points are 'nonlinear'. We note that the system (ψ, Φ, Ψ) is independent of ϕ. Thus, for each $\phi = \phi_o$ we need only consider a three dimensional system to determine the qualitative behavior. Progress is achieved by converting to cylindrical-coordinates:

$$\begin{aligned} \Psi &= r\cos\theta, \\ \psi &= \frac{r}{\sqrt{\lambda_2}}\sin\theta, \\ \Phi &= z. \end{aligned} \tag{7.13}$$

The inverse transformation is

$$\begin{aligned} r^2 &= \Psi^2 + \lambda_2\psi^2, \\ \theta &= \tan^{-1}\left(\frac{\sqrt{\lambda_2}\psi}{\Psi}\right), \\ z &= \Phi. \end{aligned} \tag{7.14}$$

The equations then become (hereafter dropping the suffix on λ for convenience),

$$r' = r\cos^2\theta\frac{\kappa}{2}\left(\beta z - \sqrt{6}(z^2+r^2)^{\frac{1}{2}}\right), \tag{7.15}$$

$$\theta' = \sqrt{\lambda} - \cos\theta\sin\theta\frac{\kappa}{2}\left(\beta z - \sqrt{6}(z^2+r^2)^{\frac{1}{2}}\right), \tag{7.16}$$

$$z' = z\frac{\kappa}{2}\left(\beta z - \sqrt{6}(z^2+r^2)^{\frac{1}{2}}\right) + \frac{\beta\kappa}{2}r^2\sin^2\theta. \tag{7.17}$$

It can be shown that if $\beta < \sqrt{6}$ then

$$\beta z - \sqrt{6}(z^2+r^2)^{\frac{1}{2}} \leq 0, \tag{7.18}$$

and therefore $r' \leq 0$ everywhere. We define the compact set $S_1 = \{(r,\theta,z)|r \leq \epsilon, -1 \leq z \leq 1\}$, where $\epsilon^2 + 1 = \frac{6}{\beta^2}$. On the boundary $r = \epsilon$, for $(-1 \leq z \leq 1)$,

$r' \leq 0$. On the boundary $z = -1$, for $(0 \leq r \leq \epsilon)$, it is easily seen that $z' \geq 0$. On the boundary $z = 1$, for $(0 \leq r \leq \epsilon)$, after some algebra it can be seen that $z' \leq 0$. Hence the set S_1 is a positively invariant compact set in $\mathbb{R}^3$, i.e., it is a trapping set. We can choose $V = r$ as a Lyapunov function. Since $r' \leq 0$ everywhere, $V' = r' \leq 0$ everywhere in S. From the Global Lyapunov Theorem [108, 79], we have

$$\forall a \in S_1, \quad \omega(a) \subseteq W = \{x \in S | \dot{V}(x) = 0\}, \tag{7.19}$$

where

$$W = \left\{(r = 0), (\theta = \frac{\pi}{2}), (\theta = -\frac{\pi}{2}), (r = 0, z = 0)\right\} \tag{7.20}$$

but the omega-limit set of $\{a\}$, $\omega(a)$, is the union of complete orbits. The only whole orbit in W is the equilibrium point $\{r = 0, z = 0\}$. Therefore, the $\omega(a)$ for any point $\{a\}$ in the trapping set S_1 is the equilibrium point. Therefore, for each $\phi = \phi_o$ the equilibrium point $(\phi_o, 0, 0, 0)$ is a sink.

Let us now consider the conditions for these equilibrium points to be inflationary. Using the fact that $\dot{a} = a \cdot H$ and the appropriate coordinate transformations we calculate

$$\frac{\ddot{a}}{a} = \mathrm{e}^{-\beta\kappa\phi} \frac{\kappa^2}{3} (\frac{\lambda}{2}\psi^2 - \Phi^2 - \Psi^2). \tag{7.21}$$

For inflation to occur $\frac{\ddot{a}}{a}$ must be greater than zero. Hence, the condition for which inflation takes place is given by

$$\frac{\lambda}{2}\psi^2 - \Phi^2 - \Psi^2 > 0. \tag{7.22}$$

This inequality describes the interior of a cone aligned along the ψ axis, so any orbits inside the cone will experience an acceleration in their expansion. We note that the apex of the cone is at the equilibrium point (see Figure 7.1).

Let us define the compact set $S_2 = \{(r, \theta, z) | r \leq \epsilon, -\epsilon \leq z \leq \epsilon\}$, where $\epsilon = \frac{2\sqrt{\lambda}}{\kappa(\beta + \sqrt{12})} > 0$. After some algebra it can be shown that $\theta' > 0$ inside S_2. Along with the fact that $r' \leq 0$, this shows that orbits spiral around the equilibrium point infinitely many times in a sufficiently small neighborhood of the equilibrium point. Thus, for any point $\{a\}$ in the inflationary regime, we can show that the orbit through $\{a\}$ will eventually leave the inflationary regime.

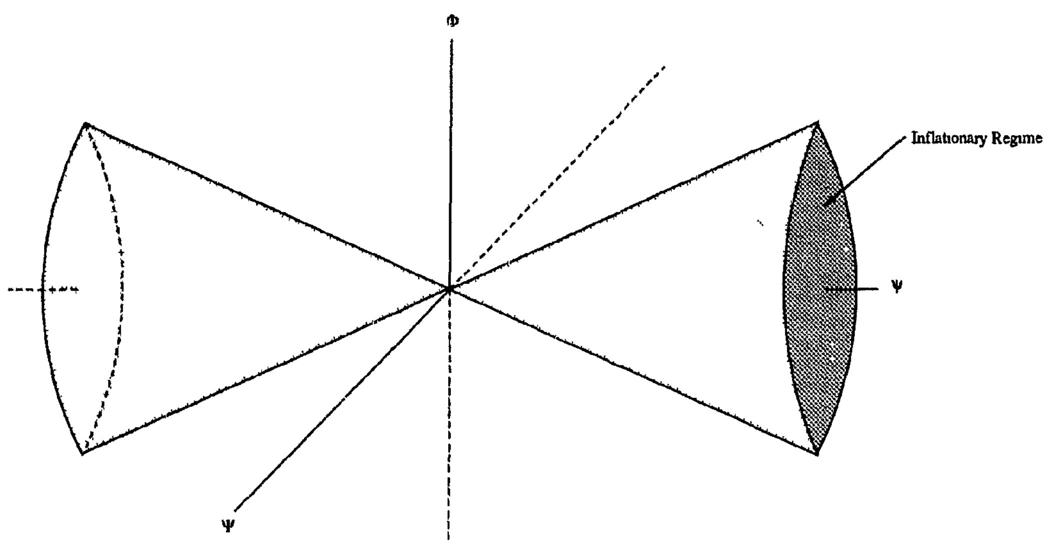


Figure 7.1: *The cone describes the inflationary regime defined by equation* (7.22).

We can show that if $\beta < \sqrt{2}$ then every orbit (except the orbit $r = 0$) that enters the inflationary regime will do so infinitely many times as it spirals its way to the equilibrium point. We choose $S_3 = \{(r, \theta, z) | r \leq \epsilon, |z| \leq \frac{r}{\sqrt{2}}\}$ where $\epsilon > 0$. The set S_3 is a compact set that contains part of the inflationary region in such a way that at $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ the inflationary cone is bounded by S_3. On the boundary $r = \epsilon$, we have that $r' \leq 0$. The system of differential equations, equations (7.15–7.17) defines a vector field $\vec{v}$ on the surfaces $z = \frac{\pm r}{\sqrt{2}}$ with inward normals (with respect to S_3) $\vec{n}_+$ and $\vec{n}_-$. It can be shown that if $\beta < \sqrt{2}$, that $\vec{n} \cdot \vec{v} \geq 0$, so the angle between $\vec{n}$ and $\vec{v}$ is less than 90° which implies that the vector field $\vec{v}$ is directed into S_3, that is trajectories are flowing into the set S_3. Hence, for $\beta < \sqrt{2}$ the set S_3 is a trapping set, and thus any orbit that enters S_3 must also enter the inflationary regime infinitely many times as it spirals its way to the equilibrium point.

We are also interested in the behavior of the field ϕ at infinity[1]. By making use

[1] *Soft Inflation* [89, 90] (and for that matter, Extended Inflation [84, 85]) occurs as $\phi \to \infty$ (with the scale factor inflating as a power-law) – see equation (7 5)

of a Poincaré-like transformation (and a new time transformation) given by:

$$x = \frac{1}{\phi}, \qquad u = \frac{\psi}{\phi}, \qquad v = \frac{\Phi}{\phi}, \qquad w = \frac{\Psi}{\phi}, \qquad \left(\frac{d\tau}{d\overline{\tau}} = \frac{1}{\phi}\right), \tag{7.23}$$

the transformed set of equations become:

$$\dot{x} = -vx^2, \tag{7.24}$$

$$\dot{u} = x(w - uv), \tag{7.25}$$

$$\dot{v} = \frac{\beta\kappa}{2}v^2 - \frac{\sqrt{6}\kappa}{2}v(v^2 + w^2 + \lambda u^2)^{\frac{1}{2}} - xv^2 + \frac{\beta\kappa}{2}\lambda u^2, \tag{7.26}$$

$$\dot{w} = \frac{\beta\kappa}{2}vw - \frac{\sqrt{6}\kappa}{2}w(v^2 + w^2 + \lambda u^2)^{\frac{1}{2}} - xvw - \lambda ux. \tag{7.27}$$

We are interested in the equilibrium points on the hypersurface $x = 0$ (that is, as $\phi \to +\infty$, $x \to 0^+$). We note that the set $x = 0$ is an invariant set. Thus, the problem becomes less difficult because $x = 0$ divides the phase space into three invariant sets.

In the set $x = 0$ we find that the critical values depend on the value of the parameter β and are given by:

$$\begin{aligned}
&u = u_o, \quad v = v_o = \sqrt{\tfrac{\lambda}{6-\beta^2}}\beta\,|u_o|, \quad w = 0, \quad \beta < \sqrt{6}, \\
&u = 0, \quad v = v_o, \quad w = 0, \quad \beta = \sqrt{6}, \\
&u = 0, \quad v = v_o = \sqrt{\tfrac{6}{\beta^2-6}}\,|w_o|, \quad w = w_o, \quad \beta > \sqrt{6}.
\end{aligned} \tag{7.28}$$

Note that in each case the equilibrium points are again non-isolated. Motivated by Berkin et al. [89], hereafter, we shall consider the case $\beta < \sqrt{6}$.

Linearizing the system about the non-isolated line of equilibria, we find that the eigenvalues are:

$$\lambda_1 = 0, \qquad \lambda_2 = \lambda_3 = -\frac{\kappa}{2}\sqrt{\lambda(6 - \beta^2)}\,|u_o| < 0. \tag{7.29}$$

There exists a center manifold which is tangent to the eigenvector associated with the zero eigenvalue namely $v = w = 0$, the u-axis is a center manifold for all u_o. The nonlinear system is thus topologically equivalent in a neighborhood of the equilibrium point to the linear system restricted to the center manifold [109]. We also note that $u = u_o$ is a 2-dimensional invariant set and thus we have effectively foliated the phase space and now need only consider the planar system (v, w) (with parameter u_o). For $u_o \neq 0$, the equilibrium point is hyperbolic and so (by the Hartman-Grobman Theorem [10]) the non-linear system is topologically equivalent to the linear system which is an attracting stellar node. For $u_o = 0$, the equations can be integrated exactly and the same behavior results. Thus inside the invariant set $x = 0$ the non-isolated line of equilibria is a sink.

We note, $\dot{x} = -vx^2$ and that the line of equilibria has positive v coordinate. Hence, in any sufficiently small neighborhood of the line $\dot{x} < 0$, so in the set $x > 0$ the orbits are attracted to the line. However, in the set $x < 0$ orbits are repelled away from the line. Thus as $\phi \to +\infty$ the line is a sink and as $\phi \to -\infty$ the line is a source.

We next consider whether these equilibrium points at positive infinity are inflationary. In the appropriate coordinates we have

$$\frac{\ddot{a}}{a} = e^{-\beta\kappa\phi} x^{-2} \left(\frac{\lambda}{2} u^2 - v^2 - w^2 \right). \tag{7.30}$$

We note that in a neighborhood of the equilibrium point the condition

$$\frac{\lambda}{2} u^2 - v^2 - w^2 > 0, \tag{7.31}$$

must hold true if inflation is to occur. Equation (7.31) represents a cone along the u-axis. If in any $u = u_o$ stable manifold we substitute the coordinate values of the equilibrium point into the condition (7.31) we find that in order to have inflation $\beta < \sqrt{2}$ (which is precisely the same condition given in Berkin et al. [89] to guarantee power-law inflation). Thus there is a neighborhood about the line of equilibria such that it is a stable attractor and for $\beta < \sqrt{2}$ it is also inflationary.

7.1.3 Summary of Soft Inflation

We have used the geometrical techniques of dynamical systems theory to investigate the generic behavior of the differential equations resulting from Soft Inflation. We found that as the field $\phi \to +\infty$ there exists no unique equilibrium point which can act as an attractor, but a one-dimensional submanifold of equilibrium points; consequently the nature of this submanifold needed further analysis. For the Chaotic Inflation case, we found that for finite values of ϕ_o, there does not exist an asymptotically stable inflationary solution, but if $\beta < \sqrt{2}$ then there exists trajectories that enter the inflationary regime infinitely many times. Also, as the field $\phi \to +\infty$, for $\beta < \sqrt{2}$ there exist regions $U \subset \mathbb{R}^4$ such that for any initial point in U the orbit asymptotically approaches a stable equilibrium point evolving through some inflationary regime as it approaches the equilibrium point; hence the solution given by Berkin et al. [89] is representative of a class of solutions in which the model undergoes power-law inflation as $\phi \to +\infty$.

7.2 Oscillatory Behaviour in Inflationary Theories

7.2.1 Introduction

Cosmological models with an oscillatory behaviour are investigated. This analysis is motivated, in part, by observations from the deep narrow-cone pencil beam surveys of Broadhurst et al. [110] which find an apparent regular galaxy distribution with a characteristic scale of $128h^{-1}$ Mpc. This apparent periodicity in the galaxy distribution suggests that the universe may have an oscillatory nature. The analysis in this section is also related to work of Morikawa [111, 112] in which the oscillatory behaviour in the Hubble parameter was used to model these observations. The cosmological models proposed by Morikawa are not physical in the sense that they do not agree with all astronomical observations [113]. However, we want to stress the fact that oscillatory behaviour is a general feature of general relativity with a scalar field in particular, and of scalar-tensor theories in general (in which oscillatory behaviour is found in the effective gravitational constant as well as in the Hubble parameter). It is this oscillatory behaviour in the cosmological models that may account for the apparent periodicity in the Broadhurst et al. observations.

In section 7.1.2, it was shown that the oscillating behaviour of the scalar field in the soft-inflationary model manifests itself in the Hubble parameter. It is known that in the standard Friedmann-Robertson-Walker (FRW) model with a single scalar field with potential $V(\phi) = \frac{\lambda}{2}\phi^2$ that the scalar field undergoes oscillatory behaviour [114, 115, 116]. We further expand this result in section 7.2.3 by analyzing an FRW model containing a perfect fluid source and a scalar field. Again an approximate solution is found whereby the Hubble parameter is assumed a priori to have an oscillatory nature. In section 7.2.4, we argue (using the conformal equivalence between general relativity with minimally coupled scalar fields and scalar-tensor theories of gravity), that this oscillatory behaviour is a general property of cosmological models arising from scalar-tensor theories of gravity. In the final section we make some concluding remarks and briefly comment upon the question of whether the oscillatory nature observed in flat FRW models persists in non-flat models and the related question of chaotic behaviour in closed models.

7.2.2 Soft Inflation

Asymptotic Solution

In a neighborhood of the stable equilibrium point ($r = 0$, $z = 0$) an approximate solution may be found. For small r, z, equation (7.16) yields $\theta' = \sqrt{\lambda}$, which may be integrated to yield $\theta = \sqrt{\lambda}\tau$ (after normalization). Assuming $z = \alpha r$ (α constant), equations (7.15) and (7.17) are consistent if $\beta^2 < 6$ and

$$\alpha = \frac{\beta}{\sqrt{6-\beta^2}}. \tag{7.32}$$

(Note, the assumption $z = \alpha r$ is only good for $z \geq 0$, as α must be positive.) The evolution equation for r becomes

$$r' = -\frac{\kappa\sqrt{6-\beta^2}}{2} r^2 \cos^2\left(\sqrt{\lambda}\tau\right), \tag{7.33}$$

which is integrated to yield a solution for r:

$$\frac{1}{r} - \frac{1}{r_o} = \frac{\kappa\sqrt{6-\beta^2}}{4}\left[\tau + \frac{1}{2\sqrt{\lambda}}\sin\left(2\sqrt{\lambda}\tau\right)\right]. \tag{7.34}$$

From equations (7.34) and (7.11) we then obtain (after re-normalizing τ)

$$H_\tau = \frac{4}{6-\beta^2}\left[\tau + \frac{1}{2\sqrt{\lambda}}\sin\left(2\sqrt{\lambda}(\tau - \tau_o)\right)\right]^{-1}, \tag{7.35}$$

where

$$\tau_o = \frac{4}{r_o\kappa\sqrt{6-\beta^2}}.$$

We note the presence of a trigonometric term in the Hubble parameter. It is precisely this term that leads to the oscillatory behavior of the cosmological models.

7.2.3 Friedmann-Robertson-Walker Model with a Scalar Field

Qualitative Analysis

We will use a qualitative analysis to show that this oscillatory type of behavior is possible for an isotropic and spatially homogeneous universe containing a single classical scalar field and non-interacting matter. We shall then adopt the 'Ellis-inverse method' [117], to obtain an ad-hoc potential $V(\phi)$ corresponding to the desired behavior of the scale factor a.

The field equations of general relativity for an FRW model with a homogeneous scalar field with potential $V(\phi)$ and a matter field in the form of a non-interacting co-moving perfect fluid with equation of state $p = (\gamma - 1)\mu$ are:

$$\begin{aligned} H^2 + \frac{k}{a^2} &= \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \mu\right), \\ 3\dot{H} + 3H^2 &= \kappa^2\left(V(\phi) - \dot{\phi}^2 - \frac{1}{2}(3\gamma - 2)\mu\right), \end{aligned} \tag{7.36}$$

where $k = \pm 1, 0$, and $\kappa^2 = 8\pi G$. The separate conservation laws are:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{7.37}$$

and

$$\dot{\mu} + 3\gamma H\mu = 0; \qquad \mu = Ma^{-3\gamma}, \tag{7.38}$$

where $V' = \frac{dV}{d\phi}$ and M is a constant. The weak and dominant energy conditions require $M \geq 0$, $0 \leq \gamma \leq 2$, and $V \geq 0$.

In particular, in the qualitative analysis we shall assume that $k = 0$ and that the potential $V(\phi)$ has the form, $V(\phi) = \frac{1}{2}\lambda\phi^2$. By defining the new cylindrical-coordinates (r, θ, z) as follows,

$$\dot{\phi} = r\cos\theta, \qquad \phi = \frac{r}{\sqrt{\lambda}}\sin\theta, \qquad \mu = \frac{1}{2}z^2, \tag{7.39}$$

the following autonomous system of ODEs result:

$$\dot{r} = -\frac{\sqrt{6}}{2}\kappa r(r^2 + z^2)^{1/2}\cos^2\theta, \tag{7.40}$$

$$\dot{\theta} = \sqrt{\lambda} + \frac{\sqrt{6}}{2}\kappa(r^2 + z^2)^{1/2}\cos\theta\sin\theta, \tag{7.41}$$

$$\dot{z} = -\frac{\sqrt{6}}{4}\kappa\gamma z(r^2 + z^2)^{1/2}. \tag{7.42}$$

The equilibrium point at finite values of the variables is given by $(r = 0,\ z = 0)$. This equilibrium point is easily seen to be a sink, because $\dot{r} \leq 0$, and for $z > 0$, $\dot{z} < 0$ and for $z < 0$, $\dot{z} > 0$. As both r and z approach zero the dominant part of equation (7.41) is the first constant term, thus θ will monotonically increase as $t \to \infty$. Hence, the equilibrium point $(r = 0,\ z = 0)$ is a stable focus for the $k = 0$ flat FRW models.

Asymptotic Solution

A simple Painlevé analysis using the ARS algorithm [118, 119] indicates that the system of equations (7.36),(7.37), and (7.38) (for general k and a potential of the form $V(\phi) = \frac{1}{2}\lambda\phi^2$) does not have the Painlevé property, which is conjectured to be a necessary condition for integrability [118, 119], and hence an exact solution may not exist. Consequently, we shall seek an approximate solution.

Again, in a neighborhood of the equilibrium point ($r = 0,\ z = 0$), an approximate solution may be found. Using the 'Ellis-inverse method', and in analogy with (7.35), we assume that the Hubble parameter is of the following form

$$H = \alpha\left(\frac{1}{t} - \frac{\beta}{t^2}\sin(bt)\right), \tag{7.43}$$

where α, β and b are constants and we choose units so that $8\pi G = \kappa^2 = 1$. From equation (7.43) we have

$$\begin{aligned} a &= a_o t^{\alpha}\left(1 + \frac{\alpha\beta}{bt^2}\cos(bt) + O(\frac{1}{t^3})\right), &(7.44)\\ \dot{H} &= -\frac{\alpha}{t^2}\Big(1 + b\beta\cos(bt)\Big) + O(\frac{1}{t^3}), &(7.45)\end{aligned}$$

where $O(\frac{1}{t^3})$ denotes terms of order $\frac{1}{t^3}\cdot$(trigonometric functions). From (7.38) we have

$$\mu = mt^{-2} + O(\frac{1}{t^4}), \tag{7.46}$$

where $m \equiv M{a_o}^{-3\gamma}$ and we have chosen $\alpha \equiv \frac{2}{3\gamma}$.

From equations (7.36) we find that

$$\dot{\phi}^2 = 2\frac{k}{a^2} - 2\dot{H} - \gamma m t^{-2}. \tag{7.47}$$

Hence-forward we shall assume that $k = 0$, whence

$$\begin{aligned} \dot{\phi}^2 &= \frac{1}{t^2}\Big([2\alpha - \gamma m] + 2\alpha b\beta\cos(bt)\Big), \\ &= \frac{\delta^2}{t^2}\Big(1 + \cos(bt)\Big), &(7.48)\end{aligned}$$

where $\delta^2 \equiv 2\alpha - \gamma m = 2\alpha b\beta$ (which serves to define m). Thus

$$\dot{\phi} = \frac{\sqrt{2}\delta}{t}\cos\left(\frac{b}{2}t\right), \tag{7.49}$$

and therefore

$$\phi = \phi_o + 2\frac{\sqrt{2}\delta}{bt}\sin\left(\frac{b}{2}t\right) + O(\frac{1}{t^2}), \tag{7.50}$$

which serves to define $t = t(\phi)$.

Finally, from equations (7.36) and (7.50), we obtain the potential $V(\phi)$:

$$\begin{aligned} V(\phi) &= \dot{H} + 3H^2 + \frac{1}{2}(\gamma - 2)\mu, \\ &= \left(3\alpha^2 - \alpha - \alpha b\beta + \frac{1}{2}(\gamma - 2)m\right)\frac{1}{t^2} + \frac{2\alpha b\beta}{t^2}\sin^2\left(\frac{b}{2}t\right), \\ &= \delta^2\frac{(1-\gamma)}{\gamma}\frac{1}{t(\phi)^2} + \frac{b^2}{8}(\phi - \phi_o)^2. \end{aligned} \tag{7.51}$$

To leading order, equations (7.36) and equation (7.37) are satisfied. In general, we obtain the desired behavior for $H(t)$ [viz. (7.43)] for a single scalar field with potential (7.51) which has a quadratic part and an 'additional part'. Note, however, that in the case $\gamma = 1$ the first term in (7.51) is absent; that is, the potential $V(\phi)$ will be a simple quadratic function of the scalar field and the energy density μ will be of the form $\mu = mt^{-2}$, where $m = \frac{4}{3}(1 - b\beta)$.

7.2.4 Scalar-Tensor Theories

In the previous two sections we have demonstrated that in two cosmological models the late-time asymptotic behaviour (that is, as $t \to \infty$) of the Hubble parameter contains trigonometric contributions. We will argue that this oscillatory behaviour is a rather general property in the class of scalar-tensor theories of gravity.

The action in the so-called Bergmann-Wagoner theories of gravity [120] can be written in the form

$$S = \int d^4x\sqrt{-g}\left\{\phi R - \frac{w(\phi)}{\phi}(\nabla\phi)^2 - 2\phi\lambda(\phi) + L_m\right\}, \tag{7.52}$$

where L_m is the Lagrangian due to matter and other non-gravitational fields. Action (7.52) is equivalent (up to field redefinitions) to,

$$S = \int d^4x\sqrt{-g}\left\{f(\Phi)R - \frac{1}{2}(\nabla\Phi)^2 - V(\Phi) + L_m\right\}, \tag{7.53}$$

where

$$\phi = f(\Phi), \qquad w(\phi) = \frac{f(\Phi)}{2f'(\phi)^2}, \quad \text{and} \quad \lambda(\phi) = \frac{V(\Phi)}{2f(\Phi)}. \tag{7.54}$$

One of the well known examples of a scalar-tensor theory of gravity is the subclass where

$$f(\Phi) = \frac{\zeta\Phi^2}{16\pi} \quad \text{and} \quad V(\Phi) \equiv 0,$$

or equivalently

$$w(\phi) = \frac{2\pi}{\zeta} \quad \text{and} \quad \lambda(\phi) \equiv 0,$$

which results in the standard Brans-Dicke theory of gravity. The benefits of using either the action (7.52) or the action (7.53) are discussed in [121].

We note that if $L_m \equiv 0$, then both (7.52) and (7.53) can be recast into the form of general relativity minimally coupled to a scalar field through conformal transformations and field redefinitions (see [122, 123] and references within). If $L_m = -\frac{1}{2}(\nabla\psi)^2 - V(\psi)$, whence the matter is due to a second scalar field, then (7.52) and (7.53) may be recast into the form of a soft-inflationary scenario again through a conformal transformation and field redefinitions [90]. It was precisely these two cases that were investigated in the previous two sections of this paper. In sections 7.1.2 and 7.2.3 we observed that asymptotically the Hubble parameter contained trigonometric contributions. Therefore, in scalar-tensor theories, the conformally related Hubble parameter might also be expected to contain trigonometric contributions in general.

However, these conformal transformations may lead to problems, such as, for example, the metric may change signature [122], or the conformal transformation may become singular at the equilibrium points of the field equations [123]. Therefore, general results using qualitative theory are problematic. Consequently, we shall simply demonstrate the genericity of the oscillatory behaviour of these models with the above action (7.52) or (7.53), by briefly discussing previous work done without utilizing a conformal transformation.

Walliser [123] has studied the field equations resulting from the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{6}h(\phi)R + g(\phi)(\nabla\phi)^2 - V(\phi) + L_m\right\},$$

which is a further generalization of the the action (7.53), in a flat Robertson-Walker background. He has found that at finite values, there is a equilibrium point that is a stable focus, and near this equilibrium point the oscillatory behaviour in the variables manifests itself in both the Hubble parameter and the effective gravitational constant. Romero and Barros [124] have investigated a class of vacuum Brans-Dicke models. They have found that for appropriate values of the parameter w, the late time asymptotic behaviour may be oscillatory in nature, whence the effective gravitational constant will also oscillate. (Note, in this case the potential $\lambda(\phi) \equiv 0$.) Scalar-tensor theories of gravity based upon the action (7.53) with a non-minimal coupling function $f(\Phi)$ of the form

$$f(\Phi) = \frac{1}{16\pi \mathrm{G}} - \frac{1}{2}\zeta\Phi^2, \tag{7.55}$$

have also been investigated recently [111, 112, 125, 126]. Barroso et al. [125] have shown that for finite values, the equilibrium points exhibit an oscillatory nature. It is precisely these models that Morikawa [111, 112], investigated in an attempt to model the periodic distribution of galaxies.

However, oscillatory behaviour is found not only in scalar-tensor theories of gravity, but also in more general theories of gravity such as, for example, theories in which derivatives of the scalar field are non-minimally coupled to the curvature R via an action of the form

$$S = \int d^4x\sqrt{-g}\left\{\left[\frac{1}{16\pi \mathrm{G}} - \zeta f(\Phi) - \eta(\nabla\Phi)^2\right]R - (\nabla\Phi)^2 + V(\Phi) + L_m\right\},$$

where ζ and η are constants and where $f(\Phi)$ is an arbitrary function of Φ [127, 128], and modified theories of gravity with an action of the form

$$S = \int d^4x\sqrt{-g}\left\{F(R,\ R_{\mu\nu}R^{\mu\nu},\ C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta},\ \ldots)\right\},$$

where F is an arbitrary function of its arguments [129]. This indicates that oscillatory behaviour in alternative theories of gravity may be a generic property.

It is also interesting to note that the deceleration parameter q, defined by

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{1}{H^2}(\dot{H} + H^2), \tag{7.56}$$

changes sign periodically. A negative q indicates that there exists a region of phase space with an accelerated expansion; that is, inflation occurs. In the soft inflation case of section 7.1.2 it was shown in [130] and [6] that for $\beta^2 < 2$ the model must undergo periods of both accelerated and decelerated expansion. For the asymptotic solution (7.35) it is easy to see that q has the form

$$q = \frac{2-\beta^2}{4} + \frac{6-\beta^2}{4}\cos(2\sqrt{\lambda}(\tau - \tau_o)), \tag{7.57}$$

which has an oscillatory nature. For the asymptotic solution (7.43) the deceleration parameter is given by

$$q = \frac{1-\alpha}{\alpha} + \frac{b\beta}{\alpha}\cos(bt), \tag{7.58}$$

which is again easily seen to have an oscillatory behaviour [130].

The oscillatory behaviour in (both the Hubble parameter H and) the deceleration parameter q implies that the universe expands faster in some stages (and slower in others) than its average value (which is the same as in the 'non-oscillating' case) [66]. Maeda [131] has studied such oscillatory models with regards to structure formation and found a significant enhancement in the growth of density perturbations, which perhaps further motivates the study of such models. In other work Futamase and Maeda [126] studied scalar-tensor theories with a non-minimal coupling function of the form (7.55) and found that there exist severe restrictions on the parameter ζ in order for inflation to occur, and suggested that adding a second minimally coupled scalar field might give rise to a more realistic model.

7.2.5 Summary of Oscillatory Behaviour in Inflationary Theories

It was proposed by Morikawa [111, 112] that an oscillating Hubble parameter may be responsible for the apparent periodic distribution of galaxies [110]. However, Hill et al. [113] have argued that such oscillations in the Hubble parameter are not consistent with the observations of Broadhurst et al. within the standard FRW model of general relativity. In addition, they argue that Morikawa's models are not physically viable.

However, according to Hill et al. [113], an oscillating gravitational constant G is one of the most viable candidates for generating an apparent periodicity in the distribution of galaxies. If the gravitational constant G is allowed to vary with respect to time in the standard FRW models, then oscillations in the Hubble parameter induce oscillations in the gravitational constant G. In this case, the resulting model nearly agrees with the Broadhurst et al. results and the shortfalls may be due to errors in determining the quantity $\dot{G}/G$ from the Viking experiment [113].

Morikawa [111, 112], Hill et al. [113] and Steinhardt [66], have suggested that one way in which the gravitational constant G may vary is to introduce a scalar field that is non-minimally coupled to the curvature R in the Lagrangian (in other words, to introduce a scalar-tensor theory of gravity). In Morikawa [111, 112] and Hill et al. [113] the non-minimal coupling function has the form (7.55), and consequently the effective gravitational constant, G_{eff}, is given by

$$16\pi G_{\text{eff}} = \left(\frac{1}{16\pi G} - \frac{1}{2}\zeta\Phi^2\right)^{-1}, \tag{7.59}$$

and thus varies with time. This class of theories constitute a subclass of the larger class of scalar-tensor theories of gravity governed by (7.53) in which the effective gravitational constant varies with time according to

$$16\pi G_{\text{eff}} = f(\Phi)^{-1}, \tag{7.60}$$

[or, equivalently, $16\pi G_{\text{eff}} = \phi^{-1}$ from (7.52)]. Therefore, these more general scalar-tensor theories of gravity may give rise to physically acceptable cosmological models.

Both of the models that we have analyzed are flat FRW models and the discussion has focussed upon such zero-curvature cosmologies. It is of interest to ask whether the oscillatory nature of the $k = 0$ FRW models is stable to perturbations in the curvature; that is, whether this oscillatory behaviour persists in the $k \neq 0$ models. Belinskii et al. [114, 115, 116] have studied the FRW models with a minimally coupled scalar field and a potential of the form $V(\phi) = \frac{1}{2}\lambda\phi^2$ using qualitative analysis. They found that for the $k = 0$ and $k = -1$ models the oscillatory behaviour is a general feature; however, in the $k = +1$ case they noted the existence of a closed chain of trajectories which hints at the possibility of the system having periodic orbits. Hawking [132] has shown that there do indeed exist periodic orbits in such models and, in particular, a countable infinite set of periodic orbits without singularity. Furthermore, Page [133] has shown that there exists a discrete set of non-periodic orbits without a equilibrium point. These two properties suggest the possibility of chaos in the closed models.

We note that the Painlevé analysis discussed in section 7.2.3 concerning the integrability of the minimally coupled FRW model also suggests chaos. Recently, Calzetta [134] has studied various cosmological models using Melnikov's method. In particular, Calzetta analyzed a class of scalar-tensor theories with $f(\Phi)$ of the form (7.55) with $\zeta = 1/6$ (the conformally coupled case) and with a potential of the form $V(\Phi) = \frac{1}{2}\lambda\Phi^2$ and found, using both Melnikov's method and numerical techniques, that the $k = +1$ FRW models exhibit chaotic behaviour (which in turn, suggests the non-integrability of the models). Clearly it is of interest to study whether closed FRW models with a potential of the form $V(\Phi) = \frac{1}{2}\lambda\Phi^2$ exhibit chaotic behaviour in other theories of gravity.

7.3 Conclusions

In closing, we have studied two inflationary models using geometric techniques from dynamical systems theory. In section 7.1.2 the primary result is: as the field

$\phi \to +\infty$, for $\beta < \sqrt{2}$ there exist regions $U \subset \mathbb{R}^4$ such that for any initial point in U the orbit asymptotically approaches a stable equilibrium point evolving through some inflationary regime as it approaches the equilibrium point; hence the solution given by Berkin et al. [89] is representative of a class of solutions in which the model undergoes power-law inflation as $\phi \to +\infty$. A secondary result (and much less important) was the observation that the soft-inflationary model exhibited an oscillatory behaviour in its expansion towards the future.

In section 7.2 both the soft-inflationary model and the FRW model with a scalar field exhibited some oscillatory behaviour. It was shown that the Hubble parameter contained trigonometric contributions asymptotically. Using the fact that these two theories are conformally equivalent to particular scalar-tensor theories of gravity (up to field redefinitions), we have argued that such oscillatory behaviour is a general property of models in all scalar-tensor theories of gravity. We also remarked that this oscillatory behaviour is found not only in general relativity and in scalar-tensor theories of gravity but also in other alternative theories of gravity. With the deep narrow-cone pencil-beam red-shift surveys exhibiting an apparent oscillatory behaviour in the observed universe [110], these oscillatory cosmological models merit further investigation.

Chapter 8

Conclusions

8.1 General Remarks

8.1.1 Self-Similarity

Cosmological models with various matter sources have been investigated using techniques from dynamical systems theory and employing methods from the theory of symmetries of differential equations. In particular, spatially homogeneous models were analyzed, primarily since the set of equations describing the evolution of such cosmological models reduce to a set of autonomous ordinary differential equations, susceptible to a qualitative analysis. In addition, if one assumes a set of dimensionless equations of state, then a symmetry in the system allows one to define new dimensionless variables that permits one differential equation to decouple (in essence, reducing the dimensionality of the problem). In dimensionless variables, the equilibrium points of the reduced dynamical system always represent self-similar cosmological models (provided $\dot{\theta}/\theta^2 \neq 0$ at the equilibrium point). One then uses this property of asymptotic self-similarity as an ansatz to derive equations of state needed for various theories of irreversible thermodynamics. In addition, it was demonstrated that the only scalar field cosmological models that can be asymptotically self-similar are those with an exponential potential ($V \sim e^{k\phi}$) or with a massless scalar field ($V \equiv 0$). It was argued that self-similar asymptotic limit points are to be expected in various Scalar-Tensor theories of gravity in which $V(\phi) \equiv 0$ (e.g., this includes the standard Brans-Dicke models). Furthermore, it was shown that self-similarity is not a robust property in that the existence of non-dimensionless equations of state, a cosmological constant, or a non-exponential potential can destroy this property.

8.1.2 Viscous Fluid Models

Viscous fluid cosmological models were investigated using various theories of irreversible thermodynamics to describe the viscous effects. In particular, a rather comprehensive analysis was completed for the Eckart and the Truncated Israel-Stewart theories of irreversible thermodynamics and a partial analysis was done using the Full Israel-Stewart theory.

Past Asymptotic Behaviour In the anisotropic Bianchi type I and V models, if the Eckart theory is employed, then the past asymptotic attractor is always a Kasner model, while if the truncated theory is employed, then the past asymptotic attractor need not always be a Kasner model (e.g., sometimes there exists a periodic orbit or a non-vacuum model representing the past asymptotic attractor.)

Concerning the isotropic models, if the energy conditions are satisfied then the past asymptotic attractor in both the Eckart and the Truncated Israel-Stewart theory is represented by a flat FRW model representing a Big-Bang singularity. In the Truncated theory, however, the nature of the initial singularity is different than that of the standard Big-Bang singularity in that the bulk viscous pressure makes up a significant portion of the initial total energy.

Future Asymptotic Behaviour In the Eckart theory the anisotropic Bianchi type I and V models with zero heat conduction isotropize to the future. In the case of non-zero heat conduction, the anisotropic Bianchi type V model isotropizes provided the weak energy condition is satisfied. In the Truncated Israel-Stewart theory this result no longer remains true since there exists sets of non-zero measure of parameter values in which the models do not isotropize. In those models which do not isotropize to the future, the shear viscous stress becomes arbitrarily large.

For the negatively curved FRW models, the future asymptotic state is represented by a Milne model or a negatively curved FRW model with viscosity, unless the models experience a period of inflation, in which case the models evolve toward a power-law inflationary attractor (or an exponential inflationary attractor if $\gamma = 3\zeta_0$).

Entropy Production Using the Eckart theory, Weinberg [32] showed that bulk viscosity could not have produced the presently observed high entropy per baryon. Fustero and Pavon [135] have done calculations which imply that there is more entropy produced if one uses the Truncated Israel-Stewart theory instead of the Eckart theory. In the viscous fluid models studied here, one finds that the total change in entropy in a co-moving volume is given as

$$\Sigma = snR^3,$$

where s is the specific entropy, n is the baryon number density and R is the scale factor of the universe. It follows from

$$\dot{n} + n\theta = 0, \qquad \text{and} \qquad \dot{s} = -\frac{\theta\Pi}{nT},$$

that the total change of entropy in a comoving volume between times $t_0 < t < t_1$ is

$$\Sigma(t_1) - \Sigma(t_0) = -\int_{t_0}^{t_1} \frac{\theta\Pi R^3}{T}\mathrm{d}t.$$

Converting to dimensionless variables and assuming

$$T = T_0 x^p \theta^{2q},$$

for the temperature T, we find that in the case of the flat FRW models (note, $x = 1$ for flat FRW models)

$$\Sigma(t_1) - \Sigma(t_0) = -\frac{1}{9T_0}\int_{t_0}^{t_1} y\theta^{3-2q} R^3 \mathrm{d}t.$$

It was shcwn previously that there exists an attracting equilibrium point in each of the Eckart, Truncated Israel-Stewart, and the Full Israel-Stewart thermodynamical theories. At these equilibrium points, $y = \bar{y}$ is a negative constant and $\bar{y} + 3\gamma > 0$ is necessary for the weak energy condition to be satisfied. For the

Eckart theory: $\bar{y} = y_E \equiv -9\zeta_0,$

Truncated Israel-Stewart theory: $\bar{y} = y_T \equiv \frac{1}{2}(b - 3\gamma) - \frac{1}{2}\sqrt{(b-3\gamma)^2 + 36a},$

Full Israel-Stewart theory: $\bar{y} = y_F \equiv \dfrac{\frac{1}{2}[2b + 3 + 3\gamma(q-1)] - \sqrt{B_9}}{(1-q)},$

where B_9 is given by equation (??). The solution at each of these points is given by

$$R(t) = R_0 t^{2/(\bar{y}+3\gamma)}, \qquad \theta(t) = \frac{6}{\bar{y}+3\gamma} t^{-1},$$
$$\rho(t) = \frac{12}{(\bar{y}+3\gamma)^2} t^{-2}, \qquad \Pi(t) = \frac{4\bar{y}}{(\bar{y}+3\gamma)^2} t^{-2}, \tag{8.1}$$

whence

$$\Sigma(t_1) - \Sigma(t_0) = -\bar{y}\frac{R_0^3}{9T_0}\left(\frac{2}{\bar{y}+3\gamma}\right)^{3-2q} \frac{1}{2q - 2 + 6/(\bar{y}+3\gamma)} \left({t_1}^{2q-2+6/(\bar{y}+3\gamma)} - {t_0}^{2q-2+6/(\bar{y}+3\gamma)}\right),$$

except if $2q - 2 + 6/(\bar{y}+3\gamma) = 0$ whence $\Sigma(t_1) - \Sigma(t_0)$ is proportional to $\ln(t_1/t_0)$. We shall compare the amount of entropy generated as $t_1 \to \infty$ in each of these theories.

Let Σ_E, Σ_T and Σ_F represent the total entropy in a comoving volume for $t_1 \gg t_0$ in each of the Eckart, Truncated Israel-Stewart, and Full Israel-Stewart theories respectively. Then

$$\frac{\Sigma_E}{\Sigma_T} = C_{E/T} t_1^{\frac{6(y_T - y_E)}{(y_E+3\gamma)(y_T+3\gamma)}},$$

where $C_{E/T}$ is a positive constant depending upon the various parameters, in the model and a similar expression also holds for Σ_E/Σ_F. In the limit as $t_1 \to \infty$ we find that

$$\frac{\Sigma_E}{\Sigma_T} < 1 \text{ whenever } y_T - y_E < 0 \text{ which implies that } \gamma - 3\zeta_0 > 0,$$

$$\frac{\Sigma_E}{\Sigma_F} < 1 \text{ whenever } y_F - y_E < 0 \text{ which implies that } \gamma - 3\zeta_0 > 1/(1-q).$$

The condition $\gamma - 3\zeta_0 > 0$ is the weak energy condition for the Eckart models. We can conclude that there is more entropy produced in the Truncated theory than in the Eckart theory (in agreement with the result of Fustero and Pavon [135]). Comparisons of the entropy production in the the Full theory and the Eckart and the Truncated theories does not yield a definitive result.

Summary

There exist different qualitative behaviours between the Eckart theory and the Truncated Israel-Stewart theory. The striking difference between the Eckart and the Truncated Israel-Stewart theory is the fact that the anisotropic stress can play a very dominant role in determining the future asymptotic behaviour of the truncated Israel-Stewart theory, while in the Eckart theory the anisotropic stress plays a very minor role and does not affect the global dynamics. One of the similarities between the Eckart theory and the truncated Israel-Stewart theory is that the addition of heat conduction in both theories does not change the global dynamics or the stability of the equilibrium points. The question of whether inflation can occur is also addressed in both theories and it is found that *bulk viscous inflation can* indeed *occur in both theories* depending upon the particular equations of state chosen.

There can exist different qualitative behaviours between the Truncated and the Full Israel-Stewart theories. Preliminary results concerning the Full Israel-Stewart theory indicate that the asymptotic behaviour depends crucially upon the temperature law chosen for T. There do exist parameter values such that the Full Israel-Stewart model has the same qualitative behaviour as the Truncated Israel-Stewart theory, at least for simple FRW models. If $r = 0$ and $p = q$, then the assumed equation of state for the temperature is $T = T_0\rho^q$. It can be argued that the most physically plausible range for the parameter q is $q < 1$. It is in this case that the qualitative behaviour of the Truncated and the Full Israel-Stewart theories is similar. Again *bulk viscous inflation is possible in the Full Israel-Stewart theory.*

8.1.3 Exponential Potential

The key idea in the analysis of the cosmological models with an exponential potential is the fact that new variables can be defined so that the dimension of the problem can again be reduced. The main result is that if $V(\phi) = \Lambda e^{k\phi}$ and $k^2 > 2$, then the spatially homogeneous models cannot isotropize unless the underlying Lie group is one of Bianchi types I, V, VII, or IX. Another interesting result is that the non-isotropic non-inflationary Feinstein-Ibáñez Bianchi type VI_h solution [94] represents the future asymptotic solution for all Bianchi type VI_h models if $k^2 > 2$. In addition, we have illustrated how the techniques from dynamical systems theory and how dimensionless variables can be applied to analyze this particular inflationary model.

8.1.4 Inflationary Models

The use of dynamical systems theory has proven itself to be very powerful in the analysis of other inflationary scenarios as well. In the Soft Inflationary scenario, in which the potential of the inflaton field is quadratic, it was proven that the solution given by Berkin et al. [89] is the future asymptotic attractor of the governing autonomous system of ordinary differential equations. Qualitative analysis was also used to illustrate that large classes of cosmological models exhibit some sort of oscillatory behaviour.

8.2 Future Work

Throughout this thesis dynamical systems techniques have been used extensively to determine various properties of different cosmological models. These techniques are ideally suited for the type of questions we wish to ask concerning the possible past, intermediate, and future asymptotic behaviours of the models. The methods, techniques, and analysis employed in this thesis will provide an excellent foundation for future research concerning the qualitative properties of cosmological models.

One conclusion that can be drawn from our analysis is that the equations of state play a fundamental role in determining the behaviour of the viscous fluid cosmological models. Which equations of state are most appropriate is unclear; however, from a mathematical perspective, the dimensionless equations of state used throughout this thesis offer a convenient way of reducing the dimensionality of the problem and making the analysis more tractable. The primary weakness in the analysis of these viscous fluid models is the fact that the equations of state used are phenomenological in nature. It is clear, however, that the qualitative behaviour in each of the three theories of irreversible thermodynamics can be quite different. Indeed, one can conclude that the first order Eckart theory does not accurately approximate the higher order Israel-Stewart theories. Future work will include both an attempt to use relativistic kinetic theory to determine (or at least place limits on) the form of the phenomenological equations of state to be used in the Full Israel-Stewart theory, and then an extensive qualitative analysis of the isotropic curvature models (discussed in Chapter 5) using more physically motivated equations of state.

In this thesis we have employed the Eckart and the Truncated Israel-Stewart theory of irreversible thermodynamics to describe the viscous effects. It was originally assumed that the Eckart theory and the Truncated Israel-Stewart theory would at least be applicable in the very early universe; however, the Eckart theory is non-causal and the Truncated Israel-Stewart theory suffers from a pathological behaviour in the temperature [36]. Hence, this work should only be considered as a first step in the analysis of the Full Israel-Stewart theory. Moreover, by considering better approximations for the bulk viscous pressure, Π, and the shear viscous stress, π_{ab}, than in the Full Israel-Stewart theory, perhaps more physically realistic models may be analyzed using similar techniques. The analysis of the isotropic curvature models using the Full Israel-Stewart theory will be of great interest. In a preliminary investigation, we have found that there appear to exist open sets (of initial conditions and parameter values) of non-zero measure for which these models isotropize (and opens sets of non-zero measure which do not). The results depend on the values of the parameters used in the equations of state, and consequently the question of isotropization in viscous fluid models is as yet unresolved.

Concerning the cosmological models with an exponential potential, there are a number of problems that merit further investigation. Future work will consist of a comprehensive analysis of the Bianchi type VII_h models, with particular focus on the question of whether these models isotropize if $k^2 > 2$. Another problem, and perhaps a more important one, is to extend the analysis done in this thesis to the inhomogeneous G_2 cosmological models containing a scalar field with an exponential potential, thereby generalizing the Hewitt and Wainwright [59]. analysis of perfect fluid G_2 models.

Appendix A

Dynamical Systems Review

A.1 Preliminary Definitions

This review comes from two primary sources. A set of dynamical systems notes prepared by John Wainwright which appeared in the workshop proceedings *Deterministic Chaos in General Relativity* [136] and out of the first chapter of Stephen Wiggins book *Introduction to Applied Nonlinear Dynamical Systems and Chaos* [79].

Definition 1 *An equilibrium solution of the DE $\dot{x} = f(x)$ is a point $\bar{x} \in \mathbb{R}^n$ such that*

$$f(\bar{x}) = 0.$$

Once an equilibrium solution is found, it becomes of interest to determine the behaviour of solutions of the DE in a neighborhood of the equilibrium solution.

Definition 2 *Let $\bar{x} \in \mathbb{R}^n$ be an equilibrium point of the DE $\dot{x} = f(x)$, and let $u = x - \bar{x}$, then the nonlinear DE $\dot{x} = f(x)$ has an associated linear DE*

$$\dot{u} = \mathrm{D}f(\bar{x})u$$

which is called the linearization of the DE $\dot{x} = f(x)$ at the equilibrium point $\bar{x}$.

Definition 3 *Let $\bar{x}$ be a equilibrium point of the DE $\dot{x} = f(x)$. Then $\bar{x}$ is called a hyperbolic equilibrium point if none of eigenvalues of $\mathrm{D}f(\bar{x})$ have zero real parts.*

A.2 The Flow of a Non-Linear DE

Definition 4 *Let $x(t) = \psi_a(t)$ be a solution of the DE $\dot{x} = f(x)$ with initial condition $x(0) = a$. The flow $\{g^t\}$ is defined in terms of the solution function $\psi_a(t)$ of the DE by*

$$g^t a = \psi_a(t).$$

Definition 5 *The orbit through a, denoted by $\gamma(a)$ is defined to be*

$$\gamma(a) = \left\{x \in \mathbb{R}^n | x = g^t a, \text{ for all } t \in \mathbb{R}\right\}$$

Orbits are classified as *point orbits*, *periodic orbits*, and *non-periodic orbits.*

Definition 6 *An ω-limit set of a point a, $\omega(a)$, is the set of points in $\mathbb{R}^n$ which are approached along the orbit through a with increasing time.*

Definition 7 *Given a DE $\dot{x} = f(x)$ in $\mathbb{R}^n$, a set $S \subseteq \mathbb{R}^n$ is called an invariant set for the DE if for any point $a \in S$, the orbit through a lies entirely in S, that is $\gamma(a) \subseteq S$.*

In order to determine an ω-limit set, it is helpful to know that an orbit enters a bounded set S and never leaves it. Such a set is called a trapping set.

Definition 8 *Given a DE $\dot{x} = f(x)$ in $\mathbb{R}^n$, with flow $\{g^t\}$, a subset $S \subset \mathbb{R}^n$ is said to be a trapping set of the DE if it satisfies*

1. *S is a closed and bounded set,*

2. *$a \in S$ implies $g^t a \in S$ for all $t \geq 0$.*

The usefulness of trapping sets lies in this result; if S is a trapping set of a DE $\dot{x} = f(x)$, then for all $a \in S$, the ω-limit set $\omega(a)$ is non-empty and is contained in S.

Definition 9

1. *The equilibrium point $\bar{x}$ of a DE $\dot{x} = f(x)$ is* <u>*stable*</u> *if for all neighborhoods U of $\bar{x}$, there exists a neighborhood V of $\bar{x}$ such that $\mathrm{g}^t V \subseteq U$ for all $t \geq 0$ where g^t is the flow of the DE.*

2. *The equilibruim point $\bar{x}$ of a DE $\dot{x} = f(x)$ is* <u>*asymptotically stable*</u> *if it is stable and if, in addition, for all $x \in V$, $\lim_{t\to\infty} ||\mathrm{g}^t x - \bar{x}|| = 0$.*

Theorem 1 (Lyapunov Stability) *Let $\bar{x}$ be an equilibrium point of the DE $\dot{x} = f(x)$ in $\mathbb{R}^n$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that $V(\bar{x}) = 0$, $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, where U is a neighborhood of $\bar{x}$.*

1. *If $\dot{V}(x) < 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is asymptotically stable.*

2. *If $\dot{V} \leq 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is stable.*

3. *If $\dot{V}(x) > 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is unstable.*

Proof. [See Wainwright [136].] □

A function $V : \mathbb{R}^n \to \mathbb{R}$ which satisfies $V(\bar{x}) = 0$, $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, and $\dot{V}(x) \leq 0$ (respectively < 0) for all $x \in U - \{\bar{x}\}$, is called a Lyapunov function (respectively, a strict Lyapunov function for the equilibrium point $\bar{x}$).

Theorem 2 (Criterion for Asymptotic Stability) *Let $\bar{x}$ be an equilibrium point of the DE $\dot{x} = f(x)$ in $\mathbb{R}^n$. If all eigenvalues of the derivative matrix $\mathrm{D}f(\bar{x})$ satisfy $\Re e(\lambda) < 0$, then the equilibrium point $\bar{x}$ is asymptotically stable.*

Proof. [See Wiggins [79], page 13.] □

A.3 The Hartman-Grobman Theorem

Theorem 3 (Hartman-Grobman) *Let $\bar{x}$ be a hyperbolic equilibrium point of the DE $\dot{x} = f(x)$ in $\mathbb{R}^n$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1. Then there is a homeomorphism which maps orbits of the linear flow $e^{tDf(\bar{x})}$ onto orbits of the non-linear flow g^t in a neighborhood of the equilibrium point $\bar{x}$, preserving the parameter t.*

Proof. [See Hartman [137], pages 244-250.] □

A hyperbolic fixed point $\bar{x}$, is called a saddle if not all of the eigenvalues of the associated linearization are of the same sign. $\bar{x}$ is called a source if the eigenvalues are all positive, and a sink if they are all negative.

The following theorem follows from the Hartmann-Grobman theorem.

Theorem 4 (Stable Manifold Theorem) *Let $\bar{x}$ be a equilibrium point of $\dot{x} = f(x)$ in $\mathbb{R}^n$, where f is of class C^2, and let E^s be the stable subspace of the linearization at $\bar{x}$, that is the subspace spanned by the eigenvectors corresponding to the eigenvalues with $\Re\mathrm{e}(\lambda) < 0$. Then there exists a neighborhood U of $\bar{x}$ such that the local stable manifold $W^s(\bar{x}, U)$ is a smooth (C^1) manifold that is tangent to E^s at $\bar{x}$.*

A.4 Periodic Orbits and Limit Sets in the Plane

Theorem 5 (Dulac's Criterion) *If $D \subseteq \mathbb{R}^2$ is a simply connected open set and* $\mathrm{div}(Bf) = \frac{\partial}{\partial x_1}(Bf_1) + \frac{\partial}{\partial x_2}(Bf_2) > 0$, (or < 0) *for all $x \in D$ where B is a C^1 function, then the DE $\dot{x} = f(x)$ where $f \in C^1$ has no periodic orbit which is contained in D.*

Proof. Based on Green's Theorem. □

Comment: The function $B(x_1, x_2)$ is called a Dulac function for the DE in the set D.

The second criterion for excluding periodic orbits, which is valid in $\mathbb{R}^n$, $n \geq 2$, follows from the observation that if a function $V(x)$ is monotone decreasing along an orbit of a DE, then that orbit cannot be periodic.

Theorem 6 *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. If $\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0$ on a subset $D \subseteq \mathbb{R}^n$, then any periodic orbit of the DE $\dot{x} = f(x)$ which lies in D, belongs to the subset $\left\{x | \dot{V}(x) = 0\right\} \cap D$.*

Theorem 7 *Consider a DE $\dot{x} = f(x)$ in $\mathbb{R}^2$. Let $a \in \mathbb{R}^2$ be an initial point such that $\{g^t a | t \geq 0\}$ lies in a closed bounded subset $K \subset \mathbb{R}^2$. If K contains only a finite number of equilibrium points then one of the following holds:*

1. *$\omega(a)$ is an equilibrium point*
2. *$\omega(a)$ is a periodic orbit*
3. *$\omega(a)$ is a cycle graph*[1].

Proof. The proof is based on the fundamental lemma of ω-limit sets in $\mathbb{R}^2$. [See Hale [138], page 230, and Lefshetz [139], page 129]. □

Comment: This theorem does not generalize to DEs in $\mathbb{R}^n$, $n \geq 3$, or to DEs on the 2–torus. Indeed, the problem of describing all possible ω-limit sets in $\mathbb{R}^n$, $n \geq 3$, is presently unsolved.

A.5 Bifurcations of Equilibria

Consider a DE in $\mathbb{R}^n$ of the form $\dot{x} = f(x, \mu)$ where μ is a real parameter. Bifurcation theory, as applied to DEs, is the study of how the portrait of the orbits change as μ varies.

Theorem 8 (Hopf) *Consider the DE $\dot{x} = f(x, \mu)$ in $\mathbb{R}^2$, where $f \in C^3$. Suppose $f(0, \mu) = 0$ for all $\mu \in I \subset \mathbb{R}$, and that $Df(0, \mu)$ has eigenvalues $\alpha(\mu) + i\beta(\mu)$. If*

[1]A cycle graph is a union of two or more whole orbits, e.g., a homoclinic orbit.

H1: *there exists* $\mu_0 \in I$ *such that* $\alpha(\mu_0) = 0$, $\beta(\mu_0) \neq 0$, $\alpha'(\mu_0) \neq 0$

H2: *the equilibrium point* $x = 0$ *is not a nonlinear center when* $\mu = \mu_0$

then

C: *there exists a* $\delta > 0$ *such that for each* $\mu \in (\mu_0, \mu_0 + \delta)$ *or* $\mu \in (\mu_0 - \delta, \mu_0)$*, the DE has a unique periodic orbit (when restricted to a sufficiently small neighborhood of* $x = 0$*).*

Proof. [See Hopf [140, 141], vol. 94 , pages 1-22 and vol. 95, pages 3-22.] □

Appendix B

Homogeneous Differential Equations

Consider an autonomous system of second order homogeneous equations

$$\frac{dx_i}{dt} \equiv \dot{x}_i = F_i(x_1, x_2, \dots, x_n), \qquad i = 1 \dots n \tag{B.1}$$

that is $F_i(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^2 F(x_1, x_2, \dots, x_n)$. This system is invariant under the transformation

$$x_i \to \lambda x_i, \qquad t \to \lambda^{-1} t. \tag{B.2}$$

There exists new variables such that the n-dimensional system (B.1) can be reduced by one dimension. Define

$$\begin{aligned} r_i &= \frac{x_i}{x_n}, \qquad i = 1 \dots n-1 \\ s &= \ln x_n, \\ \tau &= \frac{t}{x_n}. \end{aligned} \tag{B.3}$$

Then

$$\frac{dr_i}{d\tau} = \frac{dr_i}{dt}\frac{dt}{d\tau},$$

$$
\begin{aligned}
&= \left(\frac{\dot{x}_i}{x_n} - \frac{x_i}{{x_n}^2}\dot{x}_n\right)\frac{1}{x_n}, \\
&= \frac{F_i(x_1, x_2, \ldots, x_n)}{{x_n}^2} - \frac{x_i}{x_n}\frac{F_n(x_1, x_2, \ldots, x_n)}{{x_n}^2}, \\
&= F_i(r_1, r_2, \ldots, r_{n-1}, 1) - r_i F_n(r_1, r_2, \ldots, r_{n-1}, 1),
\end{aligned} \tag{B.4}
$$

and

$$
\begin{aligned}
\frac{ds}{d\tau} &= \frac{ds}{dt}\frac{dt}{d\tau}, \\
&= \left(\frac{\dot{x}_n}{x_n}\right)\frac{1}{x_n}, \\
&= \frac{F_n(x_1, x_2, \ldots, x_n)}{{x_n}^2}, \\
&= F_n(r_1, r_2, \ldots, r_{n-1}, 1).
\end{aligned} \tag{B.5}
$$

Therefore one can easily see that the original system (B.1) rewritten in the new variables (B.3) decouples to an $n-1$ system (B.4) for r_i and 1 equation for s where s is given by the integral of (B.5),

$$
s = \int F_n(r_1, r_2, \ldots, r_{n-1})d\tau. \tag{B.6}
$$

Appendix C

Energy Conditions

For the imperfect fluid cosmological models studied in Chapters 3-5, the energy conditions can be formulated with respect to the eigenvalues of the energy momentum tensor [69]. The weak energy condition (WEC) states that $T_{ab}W^aW^b \geq 0$ for any timelike vector W^a [65]. In the models under investigation the WEC, written in dimensionless variables (3.10), becomes

$$\begin{aligned} 3(2-\gamma)x - y - 2\sqrt{3}(z_1+z_2) + 9\Delta &\geq 0, \\ 3\gamma x + y + 2\sqrt{3}(z_1 - 5z_2) + 9\Delta &\geq 0, \\ 3\gamma x + y + 2\sqrt{3}(z_2 - 5z_1) + 9\Delta &\geq 0, \end{aligned}$$

where

$$\Delta \equiv \frac{1}{18}\sqrt{[6\gamma x + 2y + 6\sqrt{3}(z_1+z_2)]^2 - 3(\Sigma_1+\Sigma_2)^2(4 - 4x - \Sigma^2)}. \tag{C.1}$$

Before we proceed, it must be stated that the eigenvalues of the energy momentum tensor must be real [69] and therefore the quantity under the root sign in (C.1) must be positive.

The dominant energy condition (DEC) states that for every timelike vector W_a, $T_{ab}W^aW^b \geq 0$ and $T^{ab}W_a$ is non-spacelike [65]. In the models under investigation the

DEC becomes

$$\begin{aligned}
0 &\leq 3(2-\gamma)x - y - 2\sqrt{3}(z_1+z_2), \\
0 &\leq 3\gamma x + y + 2\sqrt{3}(z_1 - 5z_2) + 9\Delta \leq 6(2-\gamma)x - 2y - 4\sqrt{3}(z_1+z_2) + 18\Delta, \\
0 &\leq 3\gamma x + y + 2\sqrt{3}(z_2 - 5z_1) + 9\Delta \leq 6(2-\gamma)x - 2y - 4\sqrt{3}(z_1+z_2) + 18\Delta.
\end{aligned}$$

The strong energy condition (SEC) states that $T_{ab}W^aW^b - \frac{1}{2}T_a{}^aW^bW_b \geq 0$ for any timelike vector W^a [65]. In the models under investigation the SEC becomes

$$\text{WEC and} \quad 6(\gamma-1)x + 2y - 2\sqrt{3}(z_1+z_2) + 9\Delta \geq 0.$$

Bibliography

[1] A. A. Coley and R. J. van den Hoogen, in *Deterministic Chaos in General Relativity NATO ASI 332B,* edited by D. Hobill and A. Burd and A. Coley, Plenum, New York, (1994).

[2] A. A. Coley and R. J. van den Hoogen, *J. Math. Phys.*, **35**, 4117–4144, (1994).

[3] A. A. Coley and R. J. van den Hoogen, *Class. Quantum Grav.*, **12**, 1977–1994, (1995).

[4] R. J. van den Hoogen and A. A. Coley, *Class. Quantum Grav.*, **12**, 2335–2354, (1995).

[5] J. Ibáñez, R. J. van den Hoogen, and A. A. Coley, *Phys. Rev. D*, **51**, 928–930, (1995).

[6] G. Abolghasem, A. Burd, A. Coley, and R. van den Hoogen, *Phys. Rev. D*, **48**, 557–561, (1993).

[7] A. A. Coley and R. J. van den Hoogen, *Int. J. Theor. Phys.*, accepted, (1995).

[8] S. Borgani, *Phys. Rep.*, **251**, 1–152, (1995).

[9] G. F. R. Ellis, in *Proceedings of the International School of Physics, Course XLVII, General Relativity and Cosmology,* edited by B. K. Sachs, Academic Press, New York, (1971).

[10] W. Saunders *et al.*, *Nature*, **349**, 32, (1991).

[11] G. F. R. Ellis and M. A. H. MacCallum, *Comm. Math. Phys.*, **12**, 108, (1969).

[12] L. P. Grischuk, A. G. Doreshkevich, and I. D. Novikov, *Sov. Phys. JETP*, **28**, 1210, (1969).

[13] T. V. Ruzmaikina and A. A. Ruzmaikin, *Sov. Phys. JETP*, **29**, 934, (1969).

[14] C. B. Collins and G. F. R. Ellis, *Phys. Rep.*, **56**, 65, (1979).

[15] C. B. Collins, *Commun. Math. Phys.*, **23**, 137, (1971).

[16] M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies*, Princeton University Press, Princeton, (1975).

[17] O. I. Bogoyavlensky, *Methods in the Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynmaics*, Springer-Verlag, New York, (1985).

[18] J. Wainwright and L. Hsu, *Class. Quantum Grav.*, **6**, 1409, (1989).

[19] C. G. Hewitt and J. Wainwright, *Class. Quantum Grav.*, **10**, 99, (1993).

[20] M. A. H. MacCallum, in *Cargèse Lectures in Physics,* edited by E. Schatzman, Gordon and Breach, New York, (1973).

[21] M. A. H. MacCallum, in *General Relativity; An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel, Cambridge University Press, Cambridge, (1979).

[22] Ø. Grøn, *Astrophys. Space Sci.*, **173**, 191–225, (1990).

[23] C. W. Misner, *Nature*, **214**, 40, (1967).

[24] C. W. Misner, *Phys. Rev. Lett.*, **19**, 533, (1967).

[25] C. W. Misner, *Astrophys. J.*, **151**, 431, (1968).

[26] A. G. Doroshkevich, Y. B. Zeldovich, and I. D. Novikov, *Sov. Phys. JETP*, **26**, 408, (1968).

[27] J. M. Stewart, *Astrophys. Lett.*, **2**, 133, (1968).

[28] J. M. Stewart, *Monthly Notices Roy. Astron. Soc.*, **145**, 347, (1969).

[29] C. B. Collins and J. M. Stewart, *Monthly Notices Roy. Astron. Soc.*, **153**, 419, (1971).

[30] C. B. Collins and S. W. Hawking, *Astrophys. J.*, **180**, 317, (1973).

[31] V. A. Belinskii and I. M. Khalatnikov, *Sov. Phys. JETP*, **42**, 205–210, (1976).

[32] S. Weinberg, *Gravitation and Cosmology*, Wiley, New York, (1972).

[33] N. Caderni and R. Fabbri, *Nuovo Cimento Lett.*, **20**, 185, (1977).

[34] N. Caderni and R. Fabbri, *Nuovo Cimento B*, **44**, 228, (1978).

[35] N. Caderni, R. Fabbri, L. J. van den Horn, and T. J. Siskens, *Phys. Lett. A*, **66**, 251, (1978).

[36] R. Maartens, *Class. Quantum Grav.*, **12**, 1455, (1995).

[37] J. D. Barrow and R. A. Matzner, *Monthly Notices Roy. Astron. Soc.*, **181**, 719, (1977).

[38] S. Myung and B. H. Cho, *Mod. Phys. Letter A*, **1**, 37, (1986).

[39] L. Diósi, B. Keszthelyi, B. Lukács, and G. Paál, *Acta Phys. Pol. B*, **15**, 909, (1984).

[40] L. Diósi, B. Keszthelyi, B. Lukács, and G. Paál, *Phys. Lett. B*, **157**, 23, (1985).

[41] J. D. Barrow, *Phys. Lett. B*, **186**, 335, (1986).

[42] J. A. S. Lima and R. Portugal, *Phys. Rev. D*, **37**, 2755, (1988).

[43] I. Waga, R. C. Falcao, and R. Chaudra, *Phys. Rev. D*, **33**, 1839, (1986).

[44] V. A. Belinskii, E. S. Nikomarov, and I. M. Khalatnikov, *Sov. Phys. JETP*, **50**, 213–221, (1979).

[45] G. Murphy, *Phys. Rev. D*, **8**, 4231, (1973).

[46] M. Heller, Z. Klimek, and L. Suszycki, *Astrophys. and Space Sci.*, **20**, 205, (1973).

[47] C. Eckart, *Phys. Rev.*, **58**, 919–924, (1940).

[48] W. A. Hiscock and J. Salmonson, *Phys. Rev. D*, **43**, 3249–3258, (1991).

[49] W. Israel, *Ann. Phys.*, **100**, 310–331, (1976).

[50] W. Israel and J. M. Stewart, *Proc. R. Soc. Lond. A.*, **365**, 43–52, (1979).

[51] W. Israel and J. M. Stewart, *Ann. Phys.*, **118**, 341–372, (1979).

[52] M. Zakari and D. Jou, *Phys. Rev. D*, **48**, 1597–1601, (1993).

[53] G. Abolghasem and A. A. Coley, *Int. J. Theor. Phys.*, **33**, 695–713, (1994).

[54] A. Coley and K. Dunn, *J. Math. Phys.*, **33**, 1772–1779, (1992).

[55] A. Burd and A. Coley, *Class. Quantum Grav.*, **11**, 83–105, (1994).

[56] D. Pavón, J. Bafaluy, and D. Jou, *Class. Quantum Grav.*, **8**, 347–360, (1991).

[57] L. P. Chimento and A. S. Jakubi, *Class. Quantum Grav.*, **10**, 2047–2058, (1993).

[58] V. Romano and D. Pavón, *Phys. Rev. D*, **50**, 2572–2580, (1994).

[59] C. G. Hewitt and J. Wainwright, *Class. Quantum Grav.*, **7**, 2295, (1990).

[60] G. F. R. Ellis and M. A. H. MacCallum, *Commun. Math. Phys.*, **12**, 108, (1969).

[61] L. Hsu and J. Wainwright, *Class. Quantum Grav.*, **3**, 1105, (1986).

[62] A. Coley and B. Tupper, *Class. Quantum Grav.*, **7**, 1961, (1990).

[63] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, (1989).

[64] A. A. Coley, *J. Math. Phys.*, **31**, 1698–1703, (1990).

[65] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge Univ. Press, Cambridge, (1973).

[66] P. J. Steinhardt, in *Proceedings of the Sixth Marcel Grossmann Meeting on General Relativity,* edited by H. Sato and T. Nakamura, World Scientific, Singapore, (1991).

[67] A. R. King and G. F. R. Ellis, *Commun. Math. Phys.*, **31**, 209, (1973).

[68] R. J. van den Hoogen, Qualitative Analysis of Bianchi V Imperfect Fluid Cosmological Models, Master's thesis, Dalhousie University, Halifax, Nova Scotia, (1991).

[69] C. A. Kolassis, N. O. Santos, and D. Tsoubelis, *Class. Quantum Grav.*, **5**, 1329–1338, (1988).

[70] A. A. Coley, *Gen. Rel. Grav.*, **22**, 3–18, (1990).

[71] M. A. H. MacCallum, *Commun. Math. Phys.*, **20**, 57, (1971).

[72] R. Maartens and S. D. Maharaj, *Class. Quantum Grav.*, **3**, 1005, (1986).

[73] D. Schomliuk, *Trans. Amer. Soc.*, **338**, 799–841, (1993).

[74] A. Coley and J. Wainwright, *Class. Quantum Grav.*, **9**, 651, (1992).

[75] T. Pacher, J. A. Stein-Schabes, and M. S. Turner, *Phys. Rev. D*, **36**, 1603, (1987).

[76] B. Cheng, *Phys. Lett. A*, **160**, 329, (1991).

[77] V. Romano and D. Pavón, *Phys. Rev. D*, **47**, 1396–1403, (1993).

[78] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier, *Qualitative Theory of Second-Order Dynamic Systems,* Wiley, New York, (1973).

[79] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag, New York, (1990).

[80] D. Pavón, D. Jou, and J. Casas-Vázquez, *Ann. Inst. H. Poincaré A*, **36**, 79–88, (1982).

[81] W. Israel and J. M. Stewart, *Ann. Phys.*, **118**, 341, (1979).

[82] A. H. Guth, *Phys. Rev. D*, **23**, 347, (1981).

[83] K. A. Olive, *Phys. Rep.*, **190**, 307, (1990).

[84] D. La and P. J. Steinhardt, *Phys. Rev. Lett.*, **62**, 376, (1989).

[85] D. La, P. J. Steinhardt, and E. W. Bertschinger, *Phys. Lett. B*, **231**, 231, (1989).

[86] P. J. Steinhardt and F. S. Accetta, *Phys. Rev. Lett.*, **220**, 2740, (1990).

[87] A. Salam and E. Sezgin, *Phys. Lett. B*, **147**, 47, (1984).

[88] J. J. Halliwell, *Phys. Lett. B*, **185**, 341, (1987).

[89] A. L. Berkin, K. Maeda, and J. Yokoyama, *Phys. Rev. Lett.*, **65**, 141, (1990).

[90] A. L. Berkin and K. Maeda, *Phys. Rev. D*, **44**, 1691–1704, (1991).

[91] A. B. Burd and J. D. Barrow, *Nucl. Phys. B*, **308**, 929, (1988).

[92] Y. Kitada and K. Maeda, *Phys. Rev. D*, **45**, 1416, (1992).

[93] Y. Kitada and K. Maeda, *Class. Quantum Grav.*, **10**, 703, (1993).

[94] A. Feinstein and J. Ibáñez, *Class. Quantum Grav.*, **10**, 93, (1993).

[95] R. M. Wald, *Phys. Rev. D*, **28**, 2118, (1983).

[96] J. Lidsey, *Class. Quantum Grav.*, **9**, 1239, (1992).

[97] J. M. Aguirrebiria, A. Feinstein, and J. Ibáñez, *Phys. Rev. D*, **48**, 4662, (1993).

[98] M. Heusler, *Phys. Lett. B*, **255**, 33, (1991).

[99] L. A. Kofman, A. D. Linde, and A. A. Starobinsky, *Phys. Lett. B*, **157**, 36, (1985).

[100] A. A. Starobinsky, *Sov. Phys. JETP*, **42**, 152, (1985).

[101] J. Silk and M. Turner, *Phys. Rev. D*, **35**, 419, (1986).

[102] L. F. Abbott and M. B. Wise, *Nucl. Phys. B*, **244**, 541, (1984).

[103] F. Lucchin and S. Matarrese, *Phys. Rev. D*, **32**, 1316, (1985).

[104] J. Yokoyama and K. Maeda, *Phys. Lett. B*, **207**, 31, (1988).

[105] A. D. Linde, *Phys. Lett. B*, **108**, 389, (1982).

[106] A. Albrecht and P. J. Steinhardt, *Phys. Rev. Lett.*, **48**, 1220, (1982).

[107] A. D. Linde, *Phys. Lett. B*, **138**, 160, (1990).

[108] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York, (1983).

[109] D. K. Arrowsmith and C. M. Place, *An Introduction to Dynamical Systems*, Cambridge University Press, Cambridge, (1990).

[110] T. J. Broadhurst, R. S. Ellis, D. C. Koo, and A. S. Szalay, *Nature*, **343**, 726–728, (1990).

[111] M. Morikawa, *Astrophys. J.*, **362**, L37–L39, (1990).

[112] M. Morikawa, *Astrophys. J.*, **369**, 20–29, (1990).

[113] C. T. Hill, P. J. Steinhardt, and M. S. Turner, *Phys. Lett. B*, **252**, 343–348, (1990).

[114] V. A. Belinskii, L. P. Grishchuk, I. M. Khalatnikov, and Y. B. Zel'dovich, *Phys. Lett. B*, **155**, 232–236, (1985).

[115] V. A. Belinskii, L. P. Grishchuk, Y. B. Zel'dovich, and I. M. Khalatnikov, *Sov. Phys. JETP*, **62**, 195–203, (1985).

[116] V. A. Belinskii and I. M. Khalatnikov, *Sov. Phys. JETP*, **66**, 441–449, (1987).

[117] G. F. R. Ellis and M. S. Madsen, *Class. Quantum Grav.*, **8**, 667, (1991).

[118] M. J. Ablowitz, A. Ramani, and H. Segur, *J. Math. Phys.*, **21**, 715–721, (1980).

[119] A. Ramani, B. Grammaticos, and T. Bountis, *Phys. Rep.*, **180**, 159–245, (1989).

[120] C. M. Will, *Theory and Experiment in Gravitational Physics*, Cambridge University Press, New York, (1981).

[121] A. R. Liddle and D. Wands, *Phys. Rev. D*, **45**, 2665, (1992).

[122] G. Magnano and L. M. Sokolowski, On the Physical Equivalence between Nonlinear Gravity Theories and a General-Relativistic Self-Gravitating Scalar Field, gr-qc 9312008.

[123] D. Walliser, *Nucl. Phys. B*, **378**, 150–174, (1992).

[124] C. Romero and A. Barros, *Gen. Rel. Grav.*, **25**, 491–502, (1993).

[125] A. Barroso *et al.*, *Phys. Lett. B*, **275**, 264–272, (1992).

[126] T. Futamase and K. Maeda, *Phys. Rev. D*, **39**, 399–404, (1989).

[127] L. Amendola, *Phys. Lett. B*, **301**, 175–182, (1993).

[128] L. Amendola, D. Bellisai, and F. Occhionero, *Phys. Rev. D*, **47**, 4267–4272, (1993).

[129] R. Brandenberger, V. Mukhanov, and A. Sornborger, *Phys. Rev. D*, **48**, 1629–1642, (1993).

[130] R. van den Hoogen, in *Proceedings of the Fifth Canadian Conference on General Relativity and Relativistic Astrophysics,* edited by R.B. Mann and R.G. McLenaghan, World Scientific, Singapore, (1994).

[131] K. Maeda, in *Proceedings of the Sixth Marcel Grossman Meeting on General Relativity,* edited by H. Sato and T. Nakamura, World Scientific, Singapore, (1991).

[132] S. W. Hawking, in *Relativity, Groups and Topology II,* edited by B. Dewitt and R. Stora, North-Holland, Amsterdam, (1984).

[133] D. N. Page, *Class. Quantum Grav.*, **1**, 417–427, (1984).

[134] E. Calzetta, in *Deterministic Chaos in General Relativity NATO ASI 332B,* edited by D. Hobill and A. Burd and A. Coley, Plenum, New York, (1994).

[135] X. Fustero and D. Pavón, *Nuovo Cimento Lett.*, **35**, 427–429, (1982).

[136] J. Wainwright, in *Deterministic Chaos in General Relativity NATO ASI 332B,* edited by D. Hobill and A. Burd and A. Coley, Plenum, New York, (1994).

[137] P. Hartman, *Ordinary Differential Equations*, Wiley, New York, (1982).

[138] J. Hale, *Ordinary Differential Equations*, Wiley, New York, (1969).

[139] S. Lefschetz, *Differential Equations: Geometric Theory,* Interscience, New York, (1957).

[140] E. Hopf, *Berichte Math-Phys. KI Sächs Akad. Wiss. Leipzig.*, **94**, 1, (1942).

[141] E. Hopf, *Berichte Math-Phys. KI Sächs Akad. Wiss. Leipzig.*, **95**, 3, (1942).