Stability of asymmetric spike solutions to the Gierer-Meinhardt system

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In this paper, we study the spectra of asymmetric spike solutions to the Gierer-Meinhardt system. It has previously been shown that the spectra of such solutions may be determined by finding the generalized eigenvalues of matrices, which are determined by the positions of the spikes and various parameters from the system. We will examine the spectra of asymmetric solutions near the point at which they bifurcate off of a symmetric branch. We will confirm that all such solutions are unstable in a neighborhood of the bifurcation point and we derive an explicit expression for the leading order terms of the critical eigenvalues. © 2007 American Institute of Physics. [DOI: 10.1063/1.2751391]

In Ref. 1 asymmetric steady-state spike solutions to a two-component reaction diffusion system are constructed. These solutions are composed of spikes of two predetermined heights arranged arbitrarily. It has been shown that such solutions may persist for long times, but are ultimately unstable. In this paper we perform a detailed analysis of the spectra of all such patterns as they bifurcate off of a stable symmetric spike solution branch.

I. INTRODUCTION

Since Turing first proposed the existence of spatial patterns in reaction-diffusion systems, a wide range of behaviors has been uncovered. In particular, much work has gone into the study of pattern formation for the Gierer-Meinhardt system of two equations of reaction-diffusion type. The scaled Gierer-Meinhardt equations are given in (1.1), in which a represents a slowly diffusing activator and h a quickly diffusing inhibitor. Much of the early analytical results take the limit as the inhibitor diffusivity tends to infinity. This system is called the shadow system and there are a variety of results for spike formation on the boundary4–6 and interior spike pattern solutions.7,8 When the diffusivity of the inhibitor is order one, the two equations are strongly coupled and results for this case have revealed many more possible behaviors. The existence and stability of symmetric spike solutions are considered in Refs. 9 and 10. The dynamics of multipike solutions in a one-dimensional (1D) domain are considered in Ref. 11. A detailed analysis of spikes in a two-dimensional (2D) domain is explored in Ref. 12.

Among the more unexpected results is the formation of steady-state solutions with asymmetric spike patterns. These patterns consist of spikes of two different heights ordered arbitrarily. The detailed construction of these solutions can be found in Refs. 13 and 1, where it is shown that asymmetric spike solutions to (1.1) exist. In Ref. 14 it is shown that periodic asymmetric spike solutions to the Gierer-Meinhardt equations are unstable when posed on R. It is shown that a consequence of this fact is that asymmetric spike solutions are unstable when posed on finite domains as well. In this paper, we will examine the spectra of the asymmetric solutions near the point of bifurcation off a symmetric branch. We will confirm that, sufficiently close to this point, all such solutions are unstable and explicitly compute the leading order term of the critical eigenvalues. A result of this calculation shows that the operator resulting from a linearization about an asymmetric spike solution with k1 small spikes will result in exactly k1 positive eigenvalues in its spectrum.

We will consider a scaled version of the system,

\begin{equation}
  a_i = e^2 a_{xx} - a + \frac{a^m}{h^s}, \quad -1 < x < 1, \quad t > 0,
\end{equation}

\begin{equation}
  0 = Dh_{xx} - \mu h + \frac{1}{e} \frac{a^m}{h^s}, \quad -1 < x < 1, \quad t > 0,
\end{equation}

\begin{equation}
  a_i(\pm 1, t) = h_i(\pm 1, t) = 0.
\end{equation}

Here \( a(x, t), h(x, t), e \ll 1, D > 0, \) and \( \mu > 0 \) represent the scaled activator concentration, inhibitor concentration, activator diffusivity, inhibitor diffusivity, and inhibitor decay rate. The exponents \( p, q, m, \) and \( s \) are assumed to satisfy

\begin{equation}
  p > 1, \quad q > 0, \quad m > 0, \quad s > 0, \quad 0 < \frac{p-1}{q} < \frac{m}{s+1}.
\end{equation}

The remainder of this paper will proceed as follows: In Sec. II we will give a brief review of the construction and stability results of Ref. 1. In Sec. III we find the leading order corrections to the heights and locations of asymmetric spikes near the bifurcation off the symmetric branch [see (3.9)]. In Sec. IV we find that the stability of an asymmetric spike profile is determined by the signs of the eigenvalues of a matrix. This matrix will be dependent on the arrangement and number of spikes (see proposition 4.1). In Sec. V we will show that to determine the stability, we only need to consider the eigenvalues of a simple diagonal matrix, with entries of \( \pm 1 \). It then follows that all asymmetric patterns will be unstable near the bifurcation point. Furthermore, the number of unstable eigenvalues (counting multiplicity) will be equal to...
the number of small spikes in the asymmetric pattern (see proposition 5.1). Finally, in Sec. VI, we conclude.

II. REVIEW OF ASYMMETRIC SPIKE RESULTS

In this section, we will briefly review the results of Ref. I. First, we will show that (1.1) admits solutions with spikes of at most two different heights in arbitrary arrangements. The solutions are constructed using the method of matched asymptotic expansions. We will then consider the linear stability of a given profile by examining the spectrum of the operator resulting from a linearization of (1.1) about the profile. Two types of eigenvalues, which we will refer to as the large and small eigenvalues, must be considered. The large eigenvalues determine the stability of the profile on an $O(1)$ time scale. It has been shown that if $D$ is below a critical value, then the profile will be stable with respect to these eigenvalues. The small eigenvalues act on a much slower time scale and unstable asymmetric solutions can persist for times of duration $O(e^{-2})$. To determine the small eigenvalues of the operator, one needs to solve for the eigenvalues of a matrix. In Ref. I this is done numerically on a case by case basis.

The calculations considered in this section are very involved. We include some of the main results and methods for completeness, but the subtle details are omitted. References I and 11 have the complete calculations with all the details.

We will begin by considering a single spike on a domain of undetermined length. Define $\ell$ as half the length of the “support” for a spike, and construct a one-spike equilibrium solution to

\[ a_x + ax + \frac{a^2}{h^2} - \ell < x < \ell, \quad t > 0, \]  

\[ 0 = Dh_{xx} - \mu h + \frac{1}{\epsilon} h^2, \quad -\ell < x < \ell, \quad t > 0, \]

\[ a_s(\pm \ell, t) = h_s(\pm \ell, t) = 0. \]  

We expect the solution to (2.1a) to be a single sharp spike centered at $x=0$ and exponentially small elsewhere. Thus we can approximate the last term in (2.1b) by a scaled Dirac delta function. We then find that the height $h(\ell)$ at the ends of the support interval is given by

\[ h(\ell) = \left( \frac{2 \sqrt{\mu D}}{\int_{-\ell}^{\ell} u_r(y) dy} \right)^{1/2} b(\ell \sqrt{\mu D}), \]  

(2.2)

where $u_r(y)$ is the unique positive solution to

\[ u_r'' - u_r + u^p_r = 0, \quad \infty < y < \infty \quad u_r'(0) = 0 \]  

(2.3)

\[ u_r \to 0 \text{ as } |y| \to \infty; \quad u_r(0) > 0, \]

and $u_r$ is given by

\[ u_r(y) = \left( \frac{p + 1}{2} \right)^{1/(p-1)} \left( \cosh \left( \frac{(p - 1)y}{2} \right) \right)^{-2/(p-1)}. \]

We refer to $u_r$ as the canonical spike solution.

In (2.2) the function $b(\ell)$, for $\ell > 0$, and the exponent $r$, are defined by

\[ b(\zeta) = \frac{\tanh \zeta}{\cosh \zeta}; \quad r = \left( \frac{mq}{p - 1} \right)^{1/(s + 1)}. \]  

(2.4)

It can be shown that $b(\zeta)$ has a unique maximum at $\zeta = \zeta_c$, where

\[ \zeta_c = \text{arcsinh} \frac{1}{r + 1} \]  

and thus for each $\zeta \in (0, \zeta_c)$ there is a unique $\tilde{\zeta} \in (\zeta_c, \infty)$ such that $b(\zeta) = b(\tilde{\zeta})$. Defining $\tilde{\ell}$ so that $b(\ell \sqrt{\mu D}) = b(\tilde{\ell} \sqrt{\mu D})$, it follows from (2.2) that $h(\ell) = h(\tilde{\ell})$. It is thus possible to glue spike solutions of (2.1) together to form asymmetric $k$-spike solutions. Because each $\ell$ determines a unique $\tilde{\ell}$, it follows that the height of each spike in every asymmetric pattern is
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one of two predetermined values. For \( r=1 \) the plot of \( b(z) \) is given in Fig. 1.

The equality \( b(z)=b(\bar{z}) \) for \( 0<z<z_c<\bar{z} \) establishes a function \( f(z)=\bar{z} \) between \( z \) and \( \bar{z} \). For any \( r>0 \), \( f(z) \) is convex on \((0,z_c)\), \( f'(z)<-1 \) on \((0,z_c)\) and \( f'(z_c)=-1 \).

For any asymmetric spike pattern, there are exactly two support lengths \( 2\ell \) and \( 2\bar{\ell} \). For \( k_1 \) small spikes and \( k_2=k \), large spikes, the support lengths must fit in the interval \([-1,1]\),

\[
\sum_{j=1}^{k} 2\ell_j = 2 \sqrt{\frac{D}{\mu}} (k_1z + k_2\bar{z}) = 2,
\]

where

\[
\ell_j = \begin{cases} 
\ell & \text{if } j \text{ is for a small spike} \\
\bar{\ell} & \text{if } j \text{ is for a large spike.}
\end{cases}
\]

This asymmetric pattern will exist if both \( k_1z + k_2\bar{z} = \sqrt{\mu D} \)

and \( f(z)=\bar{z} \).

In Ref. 1 it is shown that there are four combinations of \( k_1, k_2, D, \) and \( D_m \) for which asymmetric spike patterns exist:

(i) Exactly one solution: \( k_1 < k_2, \quad D < D_m \); \hfill (2.7a)

(ii) Exactly one solution: \( k_1 = k_2, \quad D < D_m \); \hfill (2.7b)

(iii) Exactly one solution: \( k_1 > k_2, \quad D < D_m \); \hfill (2.7c)

(iv) Exactly two solutions: \( k_1 > k_2, \quad D_m > D > D_m \); \hfill (2.7d)

where

\[
D_m = \frac{\mu}{k_1 z_c^2}
\]

and \( D_{m_1} \) is defined as the tangency solution of the system

\[
-k_1 \frac{z}{k_2} + \frac{1}{k_2} \sqrt{\frac{\mu}{D}} = f(z), \quad -k_1 \frac{z}{k_2} = f'(z).
\]  

(2.9)

When \( D=D_m \), then \( z=\bar{z}=z_c \), and the solution is \( k \) equal height and equally spaced spikes, the “symmetric” case, which is analyzed in Ref. 11.

The equilibrium result in Ref. 1 states that, for \( r>0 \), \( D<D_m \), and \( \epsilon \rightarrow 0 \), there exists an asymmetric equilibrium solution \((a_e,h_e)\) to (1.1) of the form

\[
a_e(x) \sim \sum_{j=1}^{k} [h_{e_j}]^{q(p-1)} u(x_j) \left( \frac{x-x_j}{\epsilon} \right),
\]

(2.10)

where \( x_j \) is the location of the center of the \( j \)th spike, the value of \( h \) at \( x_j \) satisfies

\[
h_{e_j} = \left( \frac{2\sqrt{\mu D} \tanh(\ell_j \sqrt{\mu D})}{\int_{-\ell}^{\ell} [u(y)]^p dy} \right)^r,
\]

(2.11)

and the equilibrium \( h_e \) is

\[
h_e(x) \sim \sum_{j=1}^{k} 2\sqrt{\mu D} \tanh(\ell_j \sqrt{\mu D}) h_{e_j} G(x;x_j),
\]

(2.12)

where \( G(x;x_j) \) satisfies

\[
DG_{xx} - \mu G = -\delta(x-x_j), \quad -1 < x < 1,
\]

(2.13a)

\[
G_e(\pm1;x_j) = 0.
\]

(2.13b)

For \( k_1=k_2=2 \) and \( r=1 \) a plot of the activator \( a_e(x) \) and inhibitor \( h_e(x) \) is given in Fig. 2.

To study the stability of the solutions \( a_e \) and \( h_e \), substitute

\[
a(x,t) = a_e(x) + e^{\lambda t} \phi(x), \quad h(x,t) = h_e(x) + e^{\lambda t} \eta(x)
\]

(2.14)

into (1.1), where \( \phi \ll 1 \) and \( \eta \ll 1 \). This leads to the eigenvalue problem

\[
L_e \phi - q \frac{a_e'}{h_{e1}^p} \eta = \lambda \phi, \quad -1 < x < 1,
\]

(2.15a)
The conditions for which $\text{Re}(\lambda) < 0$ in (2.21) can be obtained by using a key result of Wei.\textsuperscript{15}
Thus to find the small eigenvalues of $O$ with the largest real part and assume that condition (2.24) holds. Let $e_m$ be the maximum eigenvalue of the tridiagonal matrix $\bar{E}$ defined by $\bar{E} = C^{-1}B$. Then, $\text{Re}(\lambda_0) > 0$ when

$$e_m > 1 + \frac{1}{r}.$$  

(2.29)

Also, $\text{Re}(\lambda_0) < 0$ when $e_m < 1 + r^{-1}$. Here $r$ is defined in (2.4).

To determine the sign of these eigenvalues, we must consider the higher order terms.

The analysis of the small eigenvalues also involves reducing (2.15) to a matrix eigenvalue problem. If we differentiate $u(x)$ with respect to $x$, we find that $L_0 u = u$. Thus to find the small eigenvalues of (2.15), we consider eigenfunctions of the form

$$\phi = \phi_0 + \varepsilon \phi_1 + \cdots, \quad \eta = \varepsilon \eta_0 + \cdots,$$  

(2.30a)

where

$$\phi_0 = \sum_{j=1}^k c_j \mu_j \left( \frac{x-x_j}{\varepsilon} \right), \quad \phi_1 = \sum_{j=1}^k c_j \phi_0 \left( \frac{x-x_j}{\varepsilon} \right)$$  

(2.30b)

and the $c_j$ are arbitrary coefficients. Both Refs. 1 and 11 show that $\eta_0$ satisfies

$$D \eta_{\text{last}} - \mu \eta_0 = -e^{-2m} \frac{\alpha_{j-1}}{\varepsilon} \left( \phi_0 + \varepsilon \phi_1 \right) + e^{-1} \frac{\alpha_m}{\varepsilon} \phi_0,$$

$$-1 < x < 1.$$  

(2.31)

Since $\phi_0$ is a linear combination of $u_j(x)$, it follows that the term multiplied by $\phi_0$ in (2.31) behaves like a dipole. References 1 and 11 also show that $\phi_0$ is continuous across $x = x_j$ and has the form of a spike. This implies that the term in (2.31) proportional to $\phi_0$ behaves like a linear combination of $\delta(x-x_j)$ when $\varepsilon \ll 1$, and is of the same order in $\varepsilon$ as the dipole term proportional to $\phi_0$. Thus it is necessary to approximate the eigenfunction for $\phi$ to both the $O(1)$ and $O(\varepsilon)$ terms in order to calculate an eigenvalue of order $O(\varepsilon^2)$.

Substitute (2.30) into (2.15) to obtain the following result:

**Proposition 2.4.** The eigenvalues of $O(\varepsilon^2)$ for (2.15) satisfy

$$\lambda \int_{-\infty}^{\infty} [u_n(x)]^2 dx - \frac{\varepsilon^2 q}{p+1} \int_{-\infty}^{\infty} [u_n(x)]^{p+1} dx \left( \eta_{\text{last}} - \frac{\varepsilon \mu}{D} \right),$$

$$j = 1, \ldots, k.$$  

(2.32)

Here $\langle \eta_{\text{last}} \rangle$ is to be calculated from

$$D \eta_{\text{last}} - \mu \eta_0 = 0; \quad -1 < x < 1; \quad \eta_0(\pm 1) = 0.$$  

(2.33a)

$$[D \eta_j] = -2 \sqrt{\mu} D \eta_j \tan an(\gamma) \eta_j,$$

$$D \eta_{\text{last}} = 2 \sqrt{\mu} D \eta_{\text{last}} \tan an(\eta_{\text{last}}),$$

(2.33b)

$$\tilde{s} = s - \frac{q m}{p - 1},$$  

(2.33c)

where, defining $\tilde{\xi}(x_j)$ as the one-sided limits of $\xi(x)$ as $x \rightarrow x_j$, $\tilde{\xi}_j = (\xi(x_j) + \xi(x_j - \varepsilon))/2$, and $\tilde{\xi}_j = \xi(x_j - \varepsilon) - \xi(x_j)$. To convert (2.32) and (2.33) to a matrix eigenvalue problem, define the matrices $D$, $PB$, $P g B g$, $Q$, and $B g$ as follows:

$$D = \tilde{s}D^2 (K + \tilde{S})^{-1},$$

(2.34)

where $K$ is the diagonal matrix of eigenvalues of $C^{-1}B$,

$$\begin{pmatrix}
  c_1 & d_1 & 0 & \cdots & 0 & 0 \\
  -d_1 & c_2 & \ddots & \ddots & \ddots & \ddots \\
  0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & \cdots & c_{k-1} & d_{k-1} & c_k
\end{pmatrix},$$

(2.35)

where

$$c_1 = \tanh z_1 - \coth (z_1 + z_2), \quad c_k = \coth (z_k + z_{k-1}) - \tanh z_k,$$

(2.36a)

$$c_\ell = \coth (z_\ell + z_{\ell-1}) - \coth (z_\ell + z_{\ell+1}), \quad \ell = 2, \ldots, k - 1,$$

(2.36b)

$$d_\ell = \csch (z_\ell + z_{\ell+1}), \quad \ell = 1, \ldots, k - 1.$$  

(2.36c)

The tridiagonal matrix $PB g$ is defined to be the same as $PB$ except that (2.36a) is replaced by

$$\tilde{c}_1 = \coth z_1 - \coth (z_1 + z_2); \quad \tilde{c}_k = \coth (z_k + z_{k-1}) - \coth z_k.$$  

(2.37)

The matrix $Q$ is defined to be the matrix whose columns are the orthonormal eigenvectors of $C^{-1}B$. The tridiagonal matrix $B g$ is defined to be the same as $B$ in (2.28) except that (2.28b) is replaced by

$$\tilde{c}_j = \coth (z_1 + z_2) + \coth z_1; \quad \tilde{c}_k = \coth (z_k + z_{k-1}) + \coth z_k.$$  

(2.38)

(see Ref. 1), to obtain the following result:

**Proposition 2.5.** For $\varepsilon \ll 1$, the eigenvalues of (2.15) of order $\lambda = O(\varepsilon^2)$ satisfy

$$\lambda_j \sim \frac{\varepsilon^2 q \mu}{D(p+1)} \left[ \int_{-\infty}^{\infty} [u_n(x)]^{p+1} dx \right] \left( \frac{1}{\omega_j} - 1 \right), \quad j = 1, \ldots, k,$$

(2.39)

where $\omega_j$ is an eigenvalue of the generalized eigenvalue problem

$$C^{-1}B g u = \omega (I + R) u.$$  

(2.40a)
Here $R$ is given by

\[ R = -PBQD^{-1}P_s B_s. \]  

(2.40b)

The eigenvector $\phi$ is given by (2.30), where

\[ c_j = H^{-1} c B_s u_j \]  

(2.40c)

and $u_j$ is an eigenvector of (2.40a).

For a symmetric $k$-spike pattern, $\omega_j$ can be calculated analytically from (2.40), since $C$ is a constant multiple of the identity matrix, and $B_s$ and $R$ were found to have exactly the same eigenspace. This analysis was done in Sec. 4.2 of Ref. 11; and the following result was given in proposition 11 of Ref. 11.

**Proposition 2.6 (from Ref. 11).** Consider a symmetric $k$-spike equilibrium solution where $z_1 = z_2 = \cdots = z_k = z_c$. Then for $\epsilon \ll 1$, the eigenvalues $\lambda$ of (2.15) of $O(\epsilon^2)$ are all real, and they are negative when

\[ D < D_m = \frac{\mu}{k^2 z_c^2}, \]  

(2.41)

where $z_c$ is given in (2.5). When $D > D_m$, then $k-1$ small eigenvalues are positive. When $D = D_m$, $\lambda = 0$ is a small eigenvalue of algebraic multiplicity $k-1$. Furthermore, $D_m < D_k$, where $D_k$ is the largest value of $D$ for which the symmetric branch is stable with respect to the large $O(1)$ eigenvalues.

Define $\omega_{0,j}$ as $\omega_j$ in (2.39) when $D = D_m$. From proposition 2.6, it follows that $k-1$ of the $\omega_{0,j}$'s are equal to one, and from Ref. 11 it is possible to compute that the remaining $\omega_{0,j}$ is two. Thus we can label the $\omega_{0,j}$ so that

\[ \omega_{0,j} = 1, \quad j = 1, \ldots, k-1, \]  

(2.42a)

\[ \omega_{0,k} = 2. \]  

(2.42b)

As in Ref. 1, define $\omega^*$ by

\[ \omega^* = \min(\omega_j) \quad \text{such that } \omega_j > 0 \quad \text{for } j = 1, \ldots, k. \]  

(2.43)

From proposition 2.5, it follows that an asymmetric small spike pattern will be unstable when $\omega^* < 1$, or when

\[ \omega^* - \omega_{0,j} < 0, \quad j = 1, \ldots, k-1. \]  

(2.44)

From proposition 2.6, the symmetric branch will be stable with respect to both the large and small eigenvalues when $D < D_m$. This paper will consider asymmetric patterns with $D$ near $D_m$. Such solutions are near a symmetric solution, for which analytic results exist in Ref. 11, and are thus amenable to perturbation methods.

### III. Heights and Positions Near the Bifurcation

We begin the analysis by approximating $D$ near $D_m$,

\[ D = D_m \pm \delta, \]  

(3.1)

where the bifurcation parameter $\delta \ll 1$ and the sign in (3.1) depend on the case in (2.7).

Using the fact that $z = z_c = \bar{z}$ at $D = D_m$ we form asymptotic expansions for $z$ and $\bar{z}$,

\[ z = z_c + z_1 \nu_1(\delta) + z_2 \nu_2(\delta) + \cdots, \]  

(3.2a)

\[ \bar{z} = z_c + \bar{z}_1 \nu_1(\delta) + \bar{z}_2 \nu_2(\delta) + \cdots, \]  

(3.2b)

where

\[ 1 \gg \nu_1(\delta) \gg \nu_2(\delta) \gg \cdots. \]  

(3.3)

Expanding $b(z) = b(\bar{z})$ about $z = z_c = \bar{z}$ and using $b'(z_c) = 0$, gives

\[ \begin{array}{l}
\frac{1}{b''(z_c)} (z_c^3 + 2z_c^2 \nu_1 + z_c \nu_2 + \nu_1^2 + \nu_2) \\
+ \frac{1}{6} b'''(z_c) (z_c^3 \nu_1^2 + 2z_c \nu_1^3 + \nu_2^2) \\
+ \frac{1}{6} \nu_2 \nu_1^2 (z_c^3 + 2z_c \nu_1^3 + \nu_2^2) + \cdots.
\end{array} \]  

(3.4)

From (3.3), the lowest order terms in (3.4) are the $\nu_1^2$ terms. Equating these gives $z_1^2 = z_2^2$. Because $0 < z < z_c < \bar{z}$, it follows that $z_1 < 0 < z_2$. We define

\[ \alpha = \bar{z}_1 = -z_1 > 0. \]  

(3.5)

Thus (3.2) can be written as

\[ z = z_c - \alpha \nu_1(\delta) + z_2 \nu_2(\delta) + \cdots, \]  

(3.6a)

\[ \bar{z} = z_c + \alpha \nu_1(\delta) + \bar{z}_2 \nu_2(\delta) + \cdots. \]  

(3.6b)

Note that all that will be required for the main stability result is that $\alpha > 0$, and this is satisfied by (3.5). However, we continue with this calculation for completeness.

Next we expand the condition $k_1 z + k_2 \bar{z} = \sqrt{\mu/D}$ about $\delta = 0$,

\[ k_1 z + k_2 \bar{z} = \sqrt{\frac{\mu}{D_m}} = \sqrt{\frac{\mu}{2}} D_m^{-3/2} \delta + \cdots, \quad D = D_m \pm \delta. \]  

(3.7)

Substituting (3.6) into (3.7) and using $(k_1 + k_2) z_c = \sqrt{\mu/D_m}$ gives

\[ (k_2 - k_1) \alpha \nu_1 + (k_1 z_c + k_2 \bar{z}_2) \nu_2 = \sqrt{\frac{\mu}{2}} D_m^{-3/2} \delta + \cdots, \]  

(3.8)

\[ D = D_m \pm \delta. \]

For the case (2.7a), with $D = D_m - \delta$, (3.8) gives the leading order correction as

\[ (k_2 - k_1) \alpha \nu_1(\delta) = \sqrt{\frac{\mu}{2}} D_m^{-3/2} \delta. \]

Thus we can set

\[ \alpha = \frac{1}{2(k_2 - k_1)} \sqrt{\frac{\mu}{D_m^3}} \nu_1(\delta) = \delta. \]  

(3.9)

Each of the other three cases will involve $\nu_2$, $z_2$, and $\bar{z}_2$. To find a relation among these, we equate terms in (3.4) of equal order. *A priori*, it is not known whether $\nu_2 \gg \nu_1^2$, $\nu_2 \sim \nu_1^2$, or $\nu_2 \ll \nu_1^2$. However, an assumption other than $\nu_2 \sim \nu_1^2$ will lead to $z_2 = \bar{z}_2 = 0$ for $i > 1$ and satisfying (3.8) will no longer be possible. Thus we set $\nu_2 = \nu_1^2$ and use (3.5) and (3.4) to get
\[
\tilde{z}_2 = -z_2 - \alpha^2 b''(z_2) = -z_2 + \alpha^2 \frac{2r + 1}{3 \sqrt{r(1 + r)}}.
\]  

(3.10)

For the case (2.7b) the leading order term on the left-hand side of (3.8) is \(O(v_2)\). Thus we require \(v_2 \sim \delta^0\) and \(v_1 \sim \delta^{1/2}\). Substituting (3.10) into (3.8) with \(D = D_m - \delta\) gives the leading order correction as

\[
\alpha = \tilde{c} \pm \sqrt{\tilde{c}^2 - \tilde{c} \sqrt{\frac{\mu}{k_1 - k_2}} \frac{v_2(\delta)}{v_1(\delta)} = \delta}. \quad (3.19)
\]

Here, both values of \(\alpha\) will be positive. Thus for cases (3.18) and (3.19) we may choose \(v_1(\delta) = \delta\) and then it follows that \(v_2(\delta) = \delta^2\).

Now define

\[
S_\ell = \begin{cases} 
-1 & \text{if } z_\ell = z, \text{ the scaled support for a small spike,} \\
+1 & \text{if } z_\ell = \bar{z}, \text{ the scaled support for a large spike.}
\end{cases}
\]

(3.20)

Then we can write (3.6) as

\[
z_\ell = z_c + S_\ell \alpha v_1(\delta) + O(v_2(\delta)), \quad \ell = 1, \ldots, k.
\]

(3.21)

where \(\alpha\) and \(v_1(\delta)\) are defined above for each of the four cases.

**IV. CORRECTION TO THE SMALL EIGENVALUES**

In this section, we will find the leading order correction to the small eigenvalues of (2.39). We begin by expanding (2.40a) in a \(\delta\) asymptotic series. The \(O(\delta)\) term will force a solvability condition on the corrections to the eigenvalues. This solvability condition will itself be in the form of an eigenvalue problem.

Let

\[
A = (I + R)^{-1}(C^{-1}B_{\delta}),
\]

(4.1a)

\[
A_0 = (I + R_{\delta})^{-1}(C_0^{-1}B_{\delta0}),
\]

(4.1b)

where the zero subscript denotes the \(\delta = 0\) case. Let \(\omega, u\) and \(\omega_0, u_0\) be the eigenpairs of \(A u = \omega u\) and \(A_0 u_0 = \omega_0 u_0\), respectively. Define \(U\) as the \(k \times k\) matrix with columns \(u_0, \ldots, u_{k-1}\) and order the columns so that the first \(k-1\) columns are the eigenvectors associated with the eigenvalue \(\omega_0 = 1\). We expand \(\omega\) and \(A\) as follows:

\[
\omega = \omega_0 + \delta \omega_1 + O(\delta^2),
\]

(4.2a)

\[
A = A_0 + \delta A_1 + O(\delta^2),
\]

(4.2b)

Because \(\omega_0 = 1\) has multiplicity \(k-1\), define \(b\) and \(u_1\) so that

\[
u = \langle u_1, \ldots, u_{k-1}, 0 \rangle.
\]

(4.3)

For \(\omega_0 = 1\), we have \(b' = (b_1, \ldots, b_{k-1}, 0)\), and for \(\omega_0 = 2, b' = (0, \ldots, 0, b_k)\).

Substituting (4.2) and (4.3) into \(A u = \omega u\), the first-order correction terms satisfy

\[
(A_0 - \omega_0 f) u_1 = - (A_1 - \omega_1 f) \nu.
\]

(4.4)

Because, for every \(j\),

\[
\langle (A_0 - \omega_0 f) u_1, u_{0,j} \rangle = \langle u_1, (A_0 - \omega_0 f) u_{0,j} \rangle = 0,
\]

(4.5)

where \(\langle a, b \rangle\) is the inner product of \(a\) and \(b\), it follows that \((A_1 - \omega_1 f) \nu\) is orthogonal to each \(u_{0,j}\). For each \(\omega_1\) and \(b\), this gives \(k\) equations,

\[
u_{0,j} A_1 \nu = \omega_1 u_{0,j}^T \nu, \quad j = 1, \ldots, k.
\]

(4.6)

which could be written (using \(U^T U = I\)) as follows:
Equation (4.7) is a standard eigenvalue problem with \(k\) eigenpairs. Write \(\omega_{1,j}\) for the \(j\)th \(\omega_1\). For \(\delta \to 0\), it follows from (4.2a) that the condition (2.44) will be met if

\[
\omega_{1,j} < 0 \quad \text{for at least one } j \text{ such that } j = 1, \ldots, k - 1.
\]

To expand \(A\), we define

\[
\mathcal{R} = \mathcal{R}_0 + \delta \mathcal{R}_1 + \cdots,
\]

and substitute into (4.1) to give

\[
A = A_0 + \delta (I + \mathcal{R}_0)^{-1}[(C^{-1}B_g)_1 - \mathcal{R}_1 A_0] + \cdots.
\]

The eigenvalues of \(A_0\) are given by (2.42). Define the \(k \times k\) matrix,

\[
\Omega_0 = \text{diag} \omega_{0,j}.
\]

Then the set of \(k\) equations \(A_j \mu_{0,j} = \omega_{0,j} \mu_{0,j}\) for \(j = 1, \ldots, k\) can be written \(A_j \mu = \mathcal{U} \Omega_0\), and (4.11) can be written

\[
(\mathcal{U} + \mathcal{R}_0)^{1/2} \Omega_0 + \delta (I + \mathcal{R}_0)^{-1} \mathcal{R}_1 \Omega_0.
\]

Substitute this into (4.7), multiply on the left by \(\mathcal{U} (I + \mathcal{R}_0) \mathcal{U}\), to get

\[
[\mathcal{U} (I + \mathcal{R}_0) \mathcal{U} - \mathcal{U} \mathcal{R}_1 \mathcal{U} \Omega_0] \mu = \omega_1 [\mathcal{U} (I + \mathcal{R}_0) \mathcal{U} - \mathcal{U} \mathcal{R}_1 \mathcal{U} \Omega_0] \mu = \omega_1 (I + \mathcal{U} \mathcal{R}_1 \mathcal{U} \Omega_0) \mu.
\]

As derived in Ref. 11 the eigenvectors of \(R_0\) are \(u_0\), so that \(\mathcal{U} \mathcal{R}_1 \mathcal{U} \Omega_0 = \Sigma\), where, as shown in Ref. 11,

\[
\Sigma_{jj} = \begin{cases} 
1 - \cos(\pi j/k) & \text{for } j = 1, \ldots, k - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus (4.14) becomes

\[
[\mathcal{U} (I + \mathcal{R}_0) \mathcal{U} - \mathcal{U} \mathcal{R}_1 \mathcal{U} \Omega_0] \mu = \omega_1 (I + \Sigma) \mu.
\]

We can readily divide the eigenspace of (4.16) into two subspaces. For the first subspace \(\mathcal{V}_1\), \(b\) is of the form \((b_1, \ldots, b_{k-1}, 0)^T\). For the second subspace \(\mathcal{V}_2\), \(b\) is of the form \((0, \ldots, 0, b_1)^T\).

From (4.8) and (4.16) we obtain:

**Proposition 4.1.** For \((D_m - D) \to 0\), the asymmetric solution to (1.1) given by (2.10) and (2.12) will be unstable with respect to the small eigenvalues if at least one of the eigenvalues from \(\mathcal{V}_1\) of

\[
(I + \Sigma)^{-1} [\mathcal{U} (C^{-1}B_g) \mathcal{U} - \mathcal{U} \mathcal{R}_1 \mathcal{U} \Omega_0]
\]

is negative.

The eigenvalues associated with \(\mathcal{V}_2\) will have no effect on the stability of the profile. This is due to the fact that \(\omega_{0,k}\) is 2 and thus \(\lambda_k\) in (2.39) is negative for \(\delta \ll 1\). Thus for the remainder of this paper, we will only consider \(\mathcal{V}_1\).

**V. PROFILE INSTABILITY**

In this section we prove the main result of the paper. We will explicitly evaluate the entries of the matrix given in (4.17). We do this for the case (2.7a). However, the conclusions about stability from this case will also hold for the other cases in (2.7), since it is only the definitions \(a\) and \(v_1(\delta)\) in (3.21) which will change, and \(\alpha\) is always positive.

The details of the calculation will proceed as follows: In Sec. VA we expand the matrices in (4.17) and simplify using properties of the symmetric case. In Sec. VB we expand the entries of the matrices for the asymmetric case about \(z = z_c\), in a Taylor series using (3.21) in (2.28b), (2.28c), and (2.36)–(2.38), with \(\alpha\) and \(v_1(\delta)\) given by (3.9). In Sec. VC we will show that the signs of the eigenvalues of (4.17) may be determined by the number of small spikes. The presence of \(k_1\) small spikes leads to \(k_1\) eigenvectors in \(\mathcal{V}_1\) with negative eigenvalues. Thus we confirm that an asymmetric spike solution must always be unstable relative to the small eigenvalues near the bifurcation point.

**A. Expansion of \(\mathcal{U} \mathcal{R}_1 \mathcal{U}\)**

Here we will find a key simplification, which will allow us to complete the characterization of the spectrum of (4.17). The chief difficulty is due to the presence of the matrix \(Q\) in (2.40b). The matrix \(Q\) will be expanded as follows:

\[
Q = Q_0 + \delta Q_1 + O(\delta^2).
\]

The columns \(v_1\) of \(Q_1\) are the corrections to the eigenvectors of \(C^{-1}B\). If we expand the eigenvalue problem \(C^{-1}Bv = \lambda v\), the leading order corrections to \(\lambda\) and \(v\), \(\lambda_1\) and \(v_1\), must satisfy \((C^{-1}_0 B_0 - \lambda_0 I) v_1 = -\mathcal{U} (C^{-1}B) v_0\). Because \(C^{-1}_0 B_0 - \lambda_0 I\) is singular, it is difficult to solve explicitly for the \(v_1\)'s. However, we show it is possible to replace expressions containing \(Q_1\) with expressions that we may evaluate in general.

Substituting expansions of \(PB = P_0 B_0 + \delta P_1 B_1, D = D_0 + \delta D_1, P_0 B_0 = (P_0 B_0^T) + \delta (P_0 B_0^T)\), and (5.1.1) into (2.40b), we find

\[
\mathcal{U} \mathcal{R}_1 \mathcal{U} = -\mathcal{U} (\mathcal{P}_1 B_1 \mathcal{D}_1) \mathcal{Q}_0^{-1} (\mathcal{P}_1 B_1 \mathcal{D}_1) \mathcal{U} - \mathcal{U} (\mathcal{P}_1 B_1 \mathcal{D}_1) \mathcal{Q}_0^{-1} (\mathcal{P}_1 B_1 \mathcal{D}_1) \mathcal{U}
\]

\[
= -\mathcal{U} \mathcal{P}_1 B_1 \mathcal{D}_1 \mathcal{Q}_0^{-1} (\mathcal{P}_1 B_1 \mathcal{D}_1) \mathcal{U} + \mathcal{U} \mathcal{P}_1 B_1 \mathcal{D}_1 \mathcal{Q}_0^{-1} (\mathcal{P}_1 B_1 \mathcal{D}_1) \mathcal{U}
\]

\[
= -\mathcal{U} (\mathcal{P}_1 B_1 \mathcal{D}_1) \mathcal{Q}_0^{-1} (\mathcal{P}_1 B_1 \mathcal{D}_1) \mathcal{U}.
\]

In Ref. 11 the superdiagonal matrix \(M\) is defined as

\[
M_{j,j+1} = 2 \sin(\pi j/k), \quad j = 1, \ldots, k - 1
\]

and is shown to relate to \(P_0^T B_0^T\) and \(PB\) as follows:
\( (\mathcal{P}_x \mathcal{B}_y) \mathcal{U} = \frac{\csc h(2z_c)}{2D} Q_0 \mathcal{M}' \)

and

\[
\mathcal{U}(\mathcal{P}B)_0 = -\frac{\csc h(2z_c)}{2D} M Q_0^\prime.
\] (5.1.4)

Substituting (5.1.4) into (5.1.2) (and using \(Q_0^\prime = Q_0^{-1}\)),

\[
\mathcal{U} \mathcal{R}_i \mathcal{U} = -\frac{\csc h(2z_c)}{2D} \left[ \mathcal{U}(\mathcal{P}B)_0 Q_0 \mathcal{D}_0 \mathcal{M}' \right.
\]

\[
\left. - M D_0 Q_0^{-1}(\mathcal{P}_x \mathcal{B}_y) \mathcal{U} \right] + \left( \frac{\csc h^2(2z_c)}{4D^2} [M(Q_0^0 Q_1 D_0 \right.
\]

\[
\left. - D_0 Q_0^0 Q_1) \mathcal{M}' + M D_1 \mathcal{M}'] \right].
\] (5.1.5)

Consider first the \( M D_1 \mathcal{M}' \) term in (5.1.5). To determine \( D_1 \), we expand the diagonal matrix \( D \) defined in (2.34). Using (2.4) and (2.33c) we can write \( \bar{s} \) in terms of \( r \),

\[
\bar{s} = -\frac{1+r}{r}.
\] (5.1.6)

Substitute (5.1.6) into (2.34) to give

\[
D = -\left( \frac{1+r}{r} \right) D^2 \left( \mathcal{K} - \left( \frac{1+r}{r} \right) I \right)^{-1}.
\] (5.1.7)

Since we are able to ignore the common \( D^2 \) factor when computing \( \mathcal{R} \) [the matrix \( \mathcal{R} \) is defined in (2.40b)], and \( D \),

defined in (2.34), is of the form \( D^2 M_1 \), where \( M_1 \) is a \( k \times k \) matrix, which does not depend on \( D \). Also, (2.35)-(2.37) give \( PB \) and \( P_x B_y \) as \((1/D)M_2 \) and \((1/D)M_3 \) for \( k \times k \) matrices \( M_2 \) and \( M_3 \), whose entries are hyperbolic trigonometric functions of the \( z_j \)’s. Thus \( \mathcal{R} = -M_2 Q_1^0 M_3^0 \) \( Q_0^0 Q_1^0 \mathcal{M}_3 \), which does not depend on \( D \) we expand \( \mathcal{K} = \mathcal{K}_0 + \delta \mathcal{K}_1 + O(\delta^2) \) to yield

\[
D = -\left( \frac{1+r}{r} \right) D^2 \left( \mathcal{K}_0 - \left( \frac{1+r}{r} \right) I \right)^{-1} + \delta \left( \frac{1+r}{r} \right)
\]

\[
\times D^2 \left( \mathcal{K}_0 - \left( \frac{1+r}{r} \right) I \right)^{-1} \mathcal{K}_1 \left( \mathcal{K}_0 - \left( \frac{1+r}{r} \right) I \right)^{-1} + O(\delta^2).
\] (5.1.8)

The diagonal matrix \( \mathcal{K}_0 \) is the matrix of eigenvalues of \((C^{-1}B)_0 = C_0^{-1}B_0 \), where \( C_0^{-1} = \frac{1}{2} \) coth \( z_j I \), and \( \mathcal{K}_1 \) is the diagonal matrix of the lowest order corrections to the eigenvalues of \((C^{-1}B)_0 \).

Proposition 2 in Ref. 11 gives the \( j \)th eigenvalue of \( B_0 \),

\[
\kappa_j = 2 \coth 2z_c - 2 \csc h 2z_c \cos (\pi(j - 1)/k)
\]

\[
= \frac{1 + 2r \cos (\pi(j - 1)/k)}{\sqrt{1 + r}}, \quad j = 1, \ldots, k,
\] (5.1.9)

where the second equality uses (2.5). Thus

\[
D_0 = -\left( \frac{1+r}{r} \right) D^2 \left( \mathcal{K}_0 - \left( \frac{1+r}{r} \right) I \right)^{-1}
\]

\[
= \text{diag} \left( \frac{2D^2(1+r)}{1 + \cos[\pi(j - 1)/k]} \right).
\] (5.1.10)

Since \( \mathcal{K}_1 \) is also diagonal,

\[
\left[ \left( \mathcal{K}_0 - \left( \frac{1+r}{r} \right) I \right)^{-1} \mathcal{K}_1 \left( \mathcal{K}_0 - \left( \frac{1+r}{r} \right) I \right)^{-1} \right]_{jj}
\]

\[
= \frac{4r^2}{(1 + \cos[\pi(j - 1)/k])^2} (\mathcal{K}_1)_{jj},
\] (5.1.11)

and the leading order correction to D is

\[
D_1 = \text{diag} \left( \frac{4r(1+r)D^2}{(1 + \cos[\pi(j - 1)/k])^2} (\mathcal{K}_1)_{jj} \right). \quad (5.1.12)
\]

In the \( M D_1 \mathcal{M}' \) term, \( D_1 \) is diagonal and \( M_{nn} = 0 \) unless \( m = n + 1 \), so that

\[
(M D_1 \mathcal{M}')_{ij} = \sum_{p,q} M_{ip}(D_1)_{pq} M_{qj}
\]

\[
= \begin{cases} M_{i,j+1}(D_1)_{i+1,j+1}M_{i+1,j} & \text{for } i = j, \\
0 & \text{otherwise}. \end{cases}
\] (5.1.13)

Thus \( M D_1 \mathcal{M}' \) is diagonal and

\[
(M D_1 \mathcal{M}')_{jj} = (M_{j,j+1})^2(D_1)_{j+1,j+1}
\]

\[
= 4 \sin^2 \left( \frac{\pi j}{k} \right) (D_1)_{j+1,j+1},
\] (5.1.14)

Substituting (5.1.12) into (5.1.14) gives

\[
(M D_1 \mathcal{M}')_{ij} = \begin{cases} 16r(1+r)D^2 \sin^2 \left( \frac{\pi j}{k} \right) (K_1)_{j+1,j+1} & \text{for } i = j, \quad i,j = 1, \ldots, k - 1, \\
0 & \text{otherwise}. \end{cases}
\] (5.1.15)

Now we consider the \( M(Q_0^0 Q_1 D_0 - D_0 Q_0^0 Q_1) \mathcal{M}' \) term in (5.1.5). Substituting \( \mathcal{K} = \mathcal{K}_0 + \delta \mathcal{K}_1 + \cdots \) and \( Q = Q_0^0 + \mathcal{Q}_1 + \cdots \) into \( C^{-1}B_0 = C_0 \mathcal{K} \) gives

\[
C_0^{-1}B_0 Q_1 - Q_1 \mathcal{K}_0 = Q_0 \mathcal{K}_1 - (C^{-1}B)_1 Q_0,
\] (5.1.16)

so that

\[
Q_0^0 C_0^{-1}B_0 Q_1 - Q_0^0 \mathcal{K}_0 = \mathcal{K}_1 - Q_0^0 (C^{-1}B)_1 Q_0.
\] (5.1.17)

Since \( \mathcal{K}_0 \) is the (diagonal) matrix of eigenvalues of \( C_0^{-1}B_0 \),

\[
C_0^{-1}B_0 = \mathcal{K}_0 Q_0^0 Q_0' \quad \text{or} \quad Q_0^0 C_0^{-1}B_0 = \mathcal{K}_0 Q_0^0.
\] (5.1.18)

Substituting (5.1.18) into (5.1.17) gives

\[
\mathcal{K}_0 Q_0^0 \mathcal{K}_1 - Q_0^0 \mathcal{K}_0 = \mathcal{K}_1 - Q_0^0 (C^{-1}B)_1 Q_0.
\] (5.1.19)

Solve (5.1.10) for \( \mathcal{K}_0 \),

\[
\mathcal{K}_0 = \frac{1+r}{r} (I - D^2 D_0^{-1}),
\] (5.1.20)

and substitute into the left-hand side of (5.1.19),
\[ K_0 Q'_0 Q_1 - Q'_0 Q_1 K_0 = - \left( \frac{1 + r}{r} \right)^2 D'(Q'_0 Q_1) - D'(Q'_0 Q_1) D_0 \].

(5.1.21)

Now multiply (5.1.21) on both the left and the right by \( D_0 \),
\[ D_0 (K_0 Q'_0 Q_1 - Q'_0 Q_1 K_0) D_0 = - \left( \frac{1 + r}{r} \right)^2 D_0 (Q'_0 Q_1) D_0 - D_0 (Q'_0 Q_1). \]

(5.1.22)

Substitute (5.1.19) into (5.1.22) and write the result as
\[ Q'_0 Q_1 D_0 - D_0 Q'_0 Q_1 = \left( \frac{r}{(1 + r)^2} \right)^2 D_0 (Q'_0 (C^{-1} B)_1 Q_0 - K_1) D_0, \]

(5.1.23)

which replaces the term with \( Q_1 \) by matrices which can be evaluated. Multiplying (5.1.23) on the left by \( M \) and on the right by \( M' \) gives
\[
(M(Q'_0 Q_1 D_0 - D_0 Q'_0 Q_1 M')_{ij}) = \left( \frac{r}{(1 + r)^2} \right)^2 M_{i,j+1}(D_0)_{i+1,j+1} (Q'_0 (C^{-1} B)_1 Q_0 - K_1)_{i+1,j+1}.
\]

(5.1.24)

From a solvability condition, which arises from expanding \( C^{-1} B = \lambda Q_1 \),
\[
(K_1)_{i,i} = (Q'_0 (C^{-1} B)_1 Q_0)_{i,i} \quad \text{and} \quad (K_1)_{i,j} = 0 \quad \text{for} \quad i \neq j.
\]

(5.1.25)

Substituting the expressions (5.1.3), (5.1.10), and (5.1.25) into (5.1.24) we get
\[
(M(Q'_0 Q_1 D_0 - D_0 Q'_0 Q_1 M')_{ij}) = \begin{cases} 
0 & \text{for} \quad i = j, \\
16(r(1 + r)^2)^2 (\sin(\pi i/k) \sin(\pi j/k)) \left[ \frac{1 + \cos(\pi i/k)}{1 + \cos(\pi j/k)} \right] (Q'_0 (C^{-1} B)_1 Q_0)_{i+1,j+1} & \text{otherwise,}
\end{cases}
\]

(5.1.26)

From (2.5) it follows that \( \text{csch}^2(2\pi z) / (4 D^2) = (16r(1 + r)^2)^{-1} \). Substitute this, Eqs. (5.1.15), (5.1.25), and (5.1.26), into the second line of (5.1.5) to get
\[
\text{csch}^2(2\pi z) / (4 D^2) = \left[ M(Q'_0 Q_1 D_0 - D_0 Q'_0 Q_1 M' + M D_0 M')_{ij} \right]
\]
\[
\left[ \frac{\sin(\pi i/k) \sin(\pi j/k)}{1 + \cos(\pi i/k)(1 + \cos(\pi j/k))} (Q'_0 (C^{-1} B)_1 Q_0)_{i+1,j+1}, \right]
\]
\[
i, j = 1, \ldots, k - 1, \quad i = k, \quad j = k.
\]

(5.1.27)

For the \( U'(PB)_1 Q_0 D_0 M' - M D_0 (P B g)_1 U \) term in (5.1.5), because \( D_0 \) is diagonal and \( M \) is superdiagonal,
\[
(U'(PB)_1 Q_0 D_0 M')_{ij} = \sum_{p,q} U'_{pq} [(PB)_1 g_{0,j+1} (D_0)_{i+1,j+1} M_{i,j+1}]
\]
\[
= \sum_{p,j} M_{i+1,j+1} (D_0)_{i+1,j+1} (P B g)_1 u_{0,i+1,j+1}
\]
\[
= \begin{cases} 
(D_0)_{i+1,j+1} M_{i,j+1} u_{0,i+1,j+1} & \text{if} \quad i = 1, \ldots, k - 1, \quad j = 1, \ldots, k, \\
0 & \text{if} \quad i = k, \quad j = k.
\end{cases}
\]

(5.1.28)

and
\[
(M D_0 Q'_0 (P B g)_1 U)_{ij}
\]
\[
= \sum_{r,s} M_{i+1,r+1} (D_0)_{i+1,r+1} (P B g)_1 u_{0,i+1,r+1}
\]
\[
= \begin{cases} 
M_{i+1,r+1} u_{0,i+1,r+1} (P B g)_1 u_{0,i+1,r+1} & \text{if} \quad i = 1, \ldots, k - 1, \quad j = 1, \ldots, k, \\
0 & \text{if} \quad i = k, \quad j = k,
\end{cases}
\]

(5.1.29)

where the \( \ell \)th column of \( U \) is \( u_{0,\ell} \) and the \( \ell \)th column of \( Q_0 \) is \( v_{0,\ell} \).

Substituting (5.1.3) and (5.1.10) for \( M \) and \( D_0 \) gives
\[
\text{csch}^2(2\pi z) / (4 D^2) = \left[ U'(PB)_1 Q_0 D_0 M' - M D_0 Q'_0 (P B g)_1 U \right]_{ij}
\]
\[
= \begin{cases} 
D_{i,j+1} \left[ \frac{\sin(\pi i/k)}{1 + \cos(\pi i/k)} \right] v_{0,i+1,j} - \left[ \frac{\sin(\pi j/k)}{1 + \cos(\pi j/k)} \right] u_{0,j} (PB)_1 v_{0,j+1} & \text{if} \quad i = 1, \ldots, k - 1, \\
D_{i,j+1} \left[ \frac{\sin(\pi i/k)}{1 + \cos(\pi i/k)} \right] v_{0,i+1,j} & \text{if} \quad i = k, \quad j = 1, \ldots, k - 1, \\
D_{i,j+1} \left[ \frac{\sin(\pi i/k)}{1 + \cos(\pi i/k)} \right] u_{0,j} (PB)_1 v_{0,j+1} & \text{if} \quad i = k, \quad j = 1, \ldots, k - 1, \\
0 & \text{if} \quad i = j = k.
\end{cases}
\]

(5.1.30)
Thus, the matrix $UJR_uU$ may be written as the sum of the matrices computed in (5.1.30) and (5.1.27).

### B. Evaluation of $Q_0(c^{-1}B_1)Q_0$, $U(c^{-1}B_g)U$, $U(PB)Q_0$, and $Q_0(P_g B_g)U$

In this section, we will find explicit expressions for the matrix $UJ(c^{-1}B_1)JR_u$ and the three matrices $Q_0(c^{-1}B_1)Q_0$, $UJ(PB)Q_0$, and $Q_0(P_g B_g)JR_u$, which are used to evaluate $UJR_uU$. These matrices will be calculated by substituting (3.21) in (2.19), (2.28b), (2.28c), and (2.36)–(2.38) and expanding about $\delta=0$. The arrangement of spikes enters the calculation by way of (3.20).

The expansions of $c^{-1}B$ and $c^{-1}B_g$ are given by

$$c^{-1}B = (c^{-1}B)_0 + \delta[(c^{-1}B)_1 + c^{-1}(B)_0] + O(\delta^2),$$  
(5.2.1)

$$c^{-1}B_g = (c^{-1}B_g)_0 + \delta[(c^{-1}B_g)_1 + c^{-1}(B_g)_0] + O(\delta^2),$$  
(5.2.2)

where

$$(c^{-1}B)_1 = C_0^{-1}B_1 + C_1^{-1}B_0,$$  
(5.2.3)

$$(c^{-1}B_g)_1 = C_0^{-1}(B_g)_1 + C_1^{-1}(B_g)_0.$$  
(5.2.4)

From (2.19), (3.20), and (3.21), expanding about $\delta=0$ gives

$$C_{jj} = 1 + \delta r_c \cosh z_c = 2 \delta r_s \cosh^2 z_c$$  
(5.2.5)

Define the diagonal matrix $S$ by $S_{jj} = s_j$ and use (2.5) to get

$$C_0^{-1} = \frac{1}{2} \left( \frac{1+r}{r} \right) I, \quad C_1^{-1} = -\frac{\alpha}{2r} S$$  
(5.2.6)

and

$$(c^{-1}B)_1 = \frac{1}{2} \left[ \frac{1+r}{r} B_1 - \frac{\alpha}{2r} SB_0 \right],$$  
(5.2.7)

$$(c^{-1}B_g)_1 = \frac{1}{2} \left[ \frac{1+r}{r} (B_g)_1 - \frac{\alpha}{2r} S(B_g)_0 \right].$$  
(5.2.8)

Writing $v_{0,p}$, $\kappa_p$, and $u_{0,p}$, $\xi_p$, respectively, for the $p^{th}$ eigenpair of $B_0$ and $(B_g)_0$,

$$(Q_0(c^{-1}B)Q_0)_{i+1,j+1} = \frac{1}{2} \left[ \frac{1+r}{r} v_{0,i+1}B_1 v_{0,j+1} \right.$$  
$$- \frac{\alpha \kappa_{j+1}}{2r} v_{0,i+1} S v_{0,j+1} \right]$$  
(5.2.9)

and

$$(U(c^{-1}B_g)U)_{ij} = \frac{1}{2} \left[ \frac{1+r}{r} u_{0,i}(B_g)_1 u_{0,j} \right.$$  
$$- \frac{\alpha \xi_j}{2r} u_{0,i} S u_{0,j} \right].$$  
(5.2.10)

The entries for the matrices $B$ and $B_g$ are given in (2.28b), (2.28c), and (2.38). The expansion of these entries about $\delta=0$ is given in the Appendix. Denoting the leading order correction by the subscript $\ell$, the entries for the matrices $(B_g)_1$ and $B_1$ are

$$c_{\ell,1} = \alpha^* \left( \frac{r}{1+r} \right) (-s_{\ell-1} - 2s_{\ell} - s_{\ell+1}), \quad \ell = 2, \ldots, k-1,$$  

$$c_{1,1} = \alpha^* \left( \frac{r}{1+r} \right) (-s_1 - 2s_2 - s_3),$$  

$$c_{k,1} = \alpha^* \left( \frac{r}{1+r} \right) (-s_k - 2s_{k-1} - s_{k+1}),$$  

$$c_{1,1} = \alpha^* \left( \frac{r}{1+r} \right) (-s_1 - 2s_2 - s_3),$$  

$$c_{k,1} = \alpha^* \left( \frac{r}{1+r} \right) (-s_k - 2s_{k-1} - s_{k+1}),$$  

$$c_{d,1} = \alpha^* \left( \frac{r}{1+r} \right) (s_1 + s_{\ell+1})(1 + 2r), \quad \ell = 1, \ldots, k-1,$$  

where $\alpha^* = \frac{\alpha}{4r^2(1+r)}$ and the form of the matrices $B_1$ or $(B_g)_1$ is given in (2.28a).

Following the same steps, we get the leading order corrections to $PB$ and $P_g B_g$,

$$c_{\ell,1} = \alpha^* \left( \frac{r}{1+r} \right) (s_{\ell+1} - s_{\ell-1}), \quad \ell = 2, \ldots, k-1,$$  

$$c_{1,1} = \alpha^* \left( \frac{r}{1+r} \right) (s_1 + s_{\ell+1}),$$  

$$c_{k,1} = \alpha^* \left( \frac{r}{1+r} \right) (-s_k - s_{\ell+1}),$$  

$$c_{d,1} = \alpha^* \left( \frac{r}{1+r} \right) (-s_1 - s_2 - s_3),$$  

where the form of the matrices $(PB)_1$ or $(P_g B_g)_1$ is given in (2.35).

We will compute the $i,j$ components of $Q_0B_1Q_0$, $U(c^{-1}B_1)JR_u$, $UJ(PB)Q_0$, and $Q_0(P_g B_g)JR_u$ in terms of $v_0$ and $u_0$, the columns of $Q_0$ and $U$, respectively. The calculations for these four are similar and are facilitated by defining the superdiagonal matrix,
\[
T = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

(5.2.14)

From (5.2.11), (5.2.13), and (5.2.14), it is straightforward to check that

\[
\sqrt{\frac{1 + r}{r} \alpha} B_1 = -2S - TST - \alpha TST
\]

\[
+ (1 + 4r)(2S - STT - STT)
\]

\[
+ (1 + 2r)(ST + TS + TS + ST),
\]

(5.2.15a)

\[
\sqrt{\frac{1 + r}{r} 2D \alpha} (PB)_1 = TST - TST + (1 + 4r)(STT - STT)
\]

\[
+ (1 + 2r)(ST - TS + TS + ST),
\]

(5.2.15b)

\[
\sqrt{\frac{1 + r}{r} 2D \alpha} (PB)_2 = TST - TST
\]

\[
- (3 + 4r)(STT - STT)
\]

\[
+ (1 + 2r)(ST - TS + TS + ST),
\]

(5.2.15c)

In computing \(v_{0,j}^p, B_1 v_{0,j}^p, u_{0,j}^p(B_1) u_{0,j}, u_{0,j}^p(PB)_1 v_{0,j}^p, \) and \(u_{0,i}^p(PB)_2(B_1) u_{0,i}^p,\) it follows from (5.2.15) that we will evaluate terms such as \(T v_{0,j+1}, T v_{0,j+1}^p, T u_{0,j},\) and \(T u_{0,j+1}.\)

Propositions 2 and 9 of Ref. 11 give the components of \(v_0\) and \(u_0.\)

\[
u_{p,\ell} = (v_{0,p})_\ell = \frac{1}{\sqrt{k^{2}}} \sin \left( \pi p \left( \ell - 1 \right) / 2 \right), \quad p = 1, \ldots, k-1,
\]

(5.2.16a)

\[
u_{k,\ell} = (u_{0,k})_\ell = \frac{1}{\sqrt{k^{2}}} (-1)^{\ell+1},
\]

(5.2.16b)

\[
u_{p+1,\ell} = (v_{0,p+1})_\ell = \sqrt{\frac{2}{k}} \cos \left( \pi p \left( \ell - 1 \right) / 2 \right), \quad p = 1, \ldots, k-1,
\]

(5.2.16c)

Using standard trigonometric identities and (5.2.16) it follows that (for \(j = 1, \ldots, k - 1\))

\[
v_{j+1,\ell+1} = \cos \frac{\pi j}{k} v_{j+1,\ell} - \sin \frac{\pi j}{k} u_{j,\ell}, \quad \ell = 1, \ldots, k - 1,
\]

(5.2.16d)

\[
v_{j+1,\ell-1} = \cos \frac{\pi j}{k} v_{j+1,\ell} + \sin \frac{\pi j}{k} u_{j,\ell}, \quad \ell = 2, \ldots, k,
\]

(5.2.16e)

\[
u_{j,\ell+1} = \cos \frac{\pi j}{k} u_{j,\ell} + \sin \frac{\pi j}{k} v_{j+1,\ell}, \quad \ell = 1, \ldots, k - 1,
\]

(5.2.16f)

\[
u_{j,\ell-1} = \cos \frac{\pi j}{k} u_{j,\ell} - \sin \frac{\pi j}{k} v_{j+1,\ell}, \quad \ell = 2, \ldots, k.
\]

(5.2.16g)

At \(\ell = k\)

\[
\cos \frac{\pi j}{k} v_{j+1,\ell} - \sin \frac{\pi j}{k} u_{j,\ell} = v_{j+1,k}
\]

and

\[
\cos \frac{\pi j}{k} u_{j,\ell} + \sin \frac{\pi j}{k} v_{j+1,\ell} = -u_{j,k},
\]

(5.2.16h)

and at \(\ell = 1\)

\[
\cos \frac{\pi j}{k} v_{j+1,\ell} + \sin \frac{\pi j}{k} u_{j,\ell} = v_{j+1,1}
\]

and

\[
\cos \frac{\pi j}{k} u_{j,\ell} - \sin \frac{\pi j}{k} v_{j+1,\ell} = -u_{j,1}.
\]

(5.2.16i)

Use (5.2.17) and (5.2.18) to write

\[
T v_{0,j+1} = \cos \frac{\pi j}{k} v_{0,j+1} - \sin \frac{\pi j}{k} u_{0,j} - v_{j+1,k} (I - TT),
\]

(5.2.17a)

\[
T v_{0,j+1} = \cos \frac{\pi j}{k} v_{0,j+1} + \sin \frac{\pi j}{k} u_{0,j} - v_{j+1,1} (I - TT),
\]

(5.2.17b)

\[
T u_{0,j} = \cos \frac{\pi j}{k} u_{0,j} + \sin \frac{\pi j}{k} v_{0,j+1} + u_{j,k} (I - TT),
\]

(5.2.17c)
where $i,j=1,\ldots,k-1$. Then, since $(I-T'T)\mathbf{S}_j = (\mathbf{S}_j v_{j+1,1},0,\ldots,0)'$, and using (5.2.17)

$$
T\mathbf{u}_{i,j} = \cos \frac{\pi j}{k} u_{i,j} - \sin \frac{\pi j}{k} v_{0,j+1} + u_{j,1}(I - T'T),
$$

(5.2.19d)

and $i,j=1,\ldots,k-1$. Then, since $(I-T'T)\mathbf{S}_j = (\mathbf{S}_j v_{j+1,1},0,\ldots,0)'$, and using (5.2.17)

$$
\mathbf{v}_{i+1,1}^{T} \mathbf{S} \mathbf{v}_{0,j+1} = \left( \cos \frac{\pi i}{k} v_{0,j+1} + \sin \frac{\pi i}{k} u_{i,j} \right) \left( \frac{\cos \pi j}{k} v_{0,j+1} + \sin \frac{\pi j}{k} u_{i,j} \right)
- s_i v_{i+1,1} v_{j+1,1},
$$

(5.2.20a)

$$
\mathbf{v}_{i+1,1}^{T} \mathbf{S} \mathbf{v}_{0,j+1} = \left( \cos \frac{\pi i}{k} v_{0,j+1} - \sin \frac{\pi i}{k} u_{i,j} \right) \left( \cos \frac{\pi j}{k} v_{0,j+1} - \sin \frac{\pi j}{k} u_{i,j} \right)
- s_i v_{i+1,1} v_{j+1,1},
$$

(5.2.20b)

$$
\mathbf{u}_{i,j}^{T} \mathbf{S} \mathbf{u}_{0,j} = \left( \cos \frac{\pi i}{k} u_{i,j} - \sin \frac{\pi i}{k} v_{0,j+1} \right) \left( \cos \frac{\pi j}{k} u_{0,j} + \sin \frac{\pi j}{k} v_{0,j+1} \right)
- s_i u_{i,1} u_{j,1},
$$

(5.2.20c)

$$
\mathbf{u}_{i,j}^{T} \mathbf{S} \mathbf{u}_{0,j} = \left( \cos \frac{\pi i}{k} u_{i,j} + \sin \frac{\pi i}{k} v_{0,j+1} \right) \left( \cos \frac{\pi j}{k} u_{0,j} + \sin \frac{\pi j}{k} v_{0,j+1} \right)
- s_i u_{i,1} u_{j,1},
$$

(5.2.20d)

$$
\mathbf{u}_{i,j}^{T} \mathbf{S} \mathbf{u}_{0,j} = \left( \cos \frac{\pi i}{k} u_{i,j} - \sin \frac{\pi i}{k} v_{0,j+1} \right) \left( \cos \frac{\pi j}{k} v_{0,j+1} + \sin \frac{\pi j}{k} u_{i,j} \right)
+ s_i u_{i,1} v_{j+1,1},
$$

(5.2.20e)

$$
\mathbf{u}_{i,j}^{T} \mathbf{S} \mathbf{u}_{0,j} = \left( \cos \frac{\pi i}{k} u_{i,j} + \sin \frac{\pi i}{k} v_{0,j+1} \right) \left( \cos \frac{\pi j}{k} v_{0,j+1} - \sin \frac{\pi j}{k} u_{i,j} \right)
- s_i u_{i,1} v_{j+1,1},
$$

(5.2.20f)

From (5.2.16) it follows that, for $i=1,\ldots,k-1$,

$$
v_{i+1,1} = -v_{i+1,1} \quad \text{and} \quad u_{i,1} = -u_{i,1}.
$$

(5.2.21)

Use the facts that the terms on the left-hand sides in (5.2.20) are scalars and so are equal to their transposes to get

$$
\mathbf{v}_{i+1,1}^{T} (\mathbf{T}_j + \mathbf{T}^T) \mathbf{v}_{0,j+1} = 2 \cos \frac{\pi i}{k} \cos \frac{\pi j}{k} v_{0,i+1} u_{i,j} + 2 \sin \frac{\pi i}{k} \sin \frac{\pi j}{k} u_{i,j} v_{0,i+1} - (s_1 + s_k) u_{i+1,1} v_{j+1,1},
$$

(5.2.22a)

$$
\mathbf{u}_{i,j}^{T} (\mathbf{T}_j + \mathbf{T}^T) \mathbf{u}_{0,j} = 2 \sin \frac{\pi i}{k} \sin \frac{\pi j}{k} v_{0,i+1} u_{i,j} + 2 \cos \frac{\pi i}{k} \cos \frac{\pi j}{k} u_{i,j} u_{0,i} - (s_1 + s_k) u_{i+1,1} u_{j+1,1},
$$

(5.2.22b)

$$
\mathbf{u}_{i,j}^{T} (\mathbf{T}_j - \mathbf{T}^T) \mathbf{v}_{0,j+1} = 2 \sin \frac{\pi i}{k} \sin \frac{\pi j}{k} v_{0,i+1} u_{i,j} - 2 \sin \frac{\pi i}{k} \cos \frac{\pi j}{k} v_{0,i+1} u_{i,j} + (s_1 - s_k) u_{i+1,1} v_{j+1,1},
$$

(5.2.22c)

$$
\mathbf{v}_{i,j+1}^{T} (\mathbf{T}_j - \mathbf{T}^T) \mathbf{v}_{0,j+1} = 2 \sin \frac{\pi i}{k} \sin \frac{\pi j}{k} v_{0,i+1} u_{i,j} - 2 \cos \frac{\pi i}{k} \cos \frac{\pi j}{k} v_{0,i+1} u_{i,j} + (s_1 - s_k) u_{i+1,1} u_{j+1,1},
$$

(5.2.22d)

For the terms multiplied by $(1+4r)$ and $-(3+4r)$,

$$
\mathbf{v}_{i,j+1}^{T} (2\mathbf{S} - \mathbf{T}^T - \mathbf{T}^T) \mathbf{v}_{0,j+1} = (s_1 + s_k) v_{i+1,1} v_{j+1,1},
$$

(5.2.23a)

$$
\mathbf{u}_{i,j}^{T} (2\mathbf{S} - \mathbf{T}^T - \mathbf{T}^T) \mathbf{u}_{0,j} = (s_1 + s_k) u_{i+1,1} u_{j+1,1},
$$

(5.2.23b)

$$
\mathbf{v}_{i,j+1}^{T} (\mathbf{S}^T - \mathbf{T}^T) \mathbf{v}_{0,j+1} = (s_1 - s_k) v_{i+1,1} u_{j+1,1},
$$

(5.2.23c)

$$
\mathbf{u}_{i,j}^{T} (\mathbf{S}^T - \mathbf{T}^T) \mathbf{u}_{0,j} = (s_1 - s_k) u_{i+1,1} u_{j+1,1},
$$

(5.2.23d)

For the terms multiplied by $(1+2r)$, we have, for example,

$$
\mathbf{v}_{i,j+1}^{T} \mathbf{S} \mathbf{v}_{0,j+1} = \left( \cos \frac{\pi i}{k} v_{0,i+1} + \sin \frac{\pi i}{k} u_{0,i} \right) \mathbf{S}_{0,j+1}
- s_i v_{i+1,1} v_{j+1,1},
$$

(5.2.24a)

$$
\mathbf{v}_{i,j+1}^{T} \mathbf{S} \mathbf{v}_{0,j+1} = \left( \cos \frac{\pi i}{k} v_{0,i+1} - \sin \frac{\pi i}{k} u_{0,i} \right) \mathbf{S}_{0,j+1}
- s_i v_{i+1,1} v_{j+1,1},
$$

(5.2.24b)

$$
\mathbf{v}_{i,j+1}^{T} \mathbf{S} \mathbf{v}_{0,j+1} = \left( \cos \frac{\pi j}{k} v_{0,j+1} - \sin \frac{\pi j}{k} u_{0,j} \right) \mathbf{S}_{0,j+1}
- s_i v_{i+1,1} v_{j+1,1},
$$

(5.2.24c)
\[ v_{0,i+1}^{l}(ST)v_{0,j+1} = v_{0,i+1}^{l}S \left( \cos \frac{\pi j}{k} v_{0,j+1} - \sin \frac{\pi j}{k} u_{0,j} \right) 
- s_{k} v_{0,i+1}^{l} u_{0,j}, \] (5.2.24d)

Similar expressions involving the \( u_{0} \) can be derived. Adding the expressions in (5.2.24), or their counterparts involving (5.2.21),

\[ v_{0,i+1}^{l}(ST+TS+TS+ST)v_{0,j+1} = 2 \left( \cos \frac{\pi i}{k} + \cos \frac{\pi j}{k} \right) v_{0,i+1}^{l} S v_{0,j+1} 
- 2(s_{1} + s_{k}) u_{i+1,j+1}, \] (5.2.25a)

\[ u_{0,j}^{l}(ST+TS+TS+ST)u_{0,j} = 2 \left( \cos \frac{\pi i}{k} + \cos \frac{\pi j}{k} \right) u_{0,j}^{l} S u_{0,j} + 2(s_{1} + s_{k}) u_{i+1,j+1}, \] (5.2.25b)

\[ u_{0,j}^{l}(-ST-TS+TS+ST)v_{0,j+1} = 2 \sin \frac{\pi i}{k} v_{0,i+1}^{l} v_{0,j+1} + 2 \sin \frac{\pi j}{k} u_{0,j}^{l} S u_{0,j} 
- 2(s_{1} - s_{k}) u_{i+1,j+1}, \] (5.2.25c)

\[ v_{0,j+1}^{l}(-ST-TS+TS+ST)u_{0,j} = -2 \sin \frac{\pi j}{k} v_{0,i+1}^{l} S v_{0,j+1} - 2 \sin \frac{\pi i}{k} u_{0,j}^{l} S u_{0,j} 
+ 2(s_{1} - s_{k}) u_{i+1,j+1}, \] (5.2.25d)

Substitute (5.2.22), (5.2.23), and (5.2.25) into (5.2.15) to get

\[ \frac{1}{2} \sqrt{1 + r} v_{0,i+1}^{l} E_{l} v_{0,j+1} = -\alpha \left[ \left( \frac{1}{1 + \cos \frac{\pi j}{k}} - (1 + 2r) \right) \left( \cos \frac{\pi i}{k} \right) \right] 
+ \sin \frac{\pi j}{k} u_{0,j}^{l} S u_{0,j} \right], \] (5.2.26)

\[ \frac{1}{2} \sqrt{1 + r} u_{0,j}^{l} (E_{l}) u_{0,i} = -\alpha \left[ \left( \frac{1}{1 + \cos \frac{\pi i}{k}} - (1 + 2r) \right) \left( \cos \frac{\pi j}{k} \right) \right] 
+ \sin \frac{\pi j}{k} u_{0,j}^{l} S u_{0,j} \right], \] (5.2.27)

\[ (\mathcal{L}(C^{-1}B_{1})u_{ij}) = \frac{1}{2} \sqrt{1 + r} \left[ \sin \frac{\pi j}{k} \right] u_{0,j}^{l} (P_{S}B_{k}) u_{0,i} 
- \frac{\sin \left( \pi j/k \right)}{1 + \cos \left( \pi j/k \right)} u_{0,j}^{l} (P_{S}B_{k}) u_{0,i} \right] \]
\[ \times \left( \sin \left( \pi j/k \right) \sin \left( \pi i/k \right) \right) \times \left( 1 + \cos \left( \pi j/k \right) \right) \times \left( 1 + \cos \left( \pi i/k \right) \right) \] (5.3.1)

where, from Ref. 11 it can be shown that \( \xi_{j} = \kappa_{j+1} \), and where \( \kappa_{j} \) is given in (5.19).

From (5.1.5), (5.1.27), and (5.1.30),

\[ (\mathcal{L}(\mathcal{R}_{1}u_{ij}) = D \sqrt{1 + r} \left[ \sin \left( \pi i/k \right) \right] u_{0,j}^{l} (P_{S}B_{k}) u_{0,i} 
- \frac{\sin \left( \pi j/k \right)}{1 + \cos \left( \pi j/k \right)} u_{0,j}^{l} (P_{S}B_{k}) u_{0,i} \right] \]
\[ \times \left( \sin \left( \pi i/k \right) \sin \left( \pi j/k \right) \right) \times \left( 1 + \cos \left( \pi i/k \right) \right) \times \left( 1 + \cos \left( \pi j/k \right) \right) \]
(5.3.2)

Noting that \( v_{0,j+1}^{l}(C^{-1}B_{1}) v_{0,j+1}^{l} = (Q_{0}^{l}C^{-1}B_{1}) Q_{0}^{l} u_{ij} \) we substitute (5.2.28), (5.2.29), (5.2.9), and (5.1.9) into (5.3.2) to get
Thus we can use 
\[ U_t \mathcal{R}_t U \] 
so that 
\[ (A^{-1}B)_{ii} = \frac{\sin(\pi i k)}{(1 + \cos(\pi i k))(1 - \cos(\pi i k) + 2r)}, \]  
for \( i \neq k \).

Then, since diagonal matrices commute, by (5.3.5) and (5.3.8),
\[ (I + \Sigma)^{-1}[\mathcal{U}(C^t B^t) \mathcal{U} - \mathcal{U} \mathcal{R}_t \mathcal{U}] = 8r^* A^{-1}Q_{ij}^t S Q_{ij}^t A B. \]  
Substitute (5.3.9) into (5.9) to get
\[ 8r^* Q_{ij}^t S Q_{ij}^t B(\mathcal{A}) = \omega_j \mathcal{A} \]  
Now we will demonstrate that it always is possible to find a set of \( k \)-independent vectors \( z_i \) such that for \( k_i \) of the vectors \( \omega_i \) in (5.10) is negative and for \( k_i \) of the vectors \( \omega_i \) is positive. This will imply that the spectrum of (2.15) will have exactly \( k_i \) small positive eigenvalues and exactly \( k_i \) small negative eigenvalues. We begin by noting that for \( x \in \mathbb{R}^k \), \( Q_{ij}^t Bx \) will be in the space spanned by the first \( k \) columns of \( Q_{ij}^t \). This space may also be described as the \( k \)-dimensional subspace \( \mathcal{W} \) of \( \mathbb{R}^k \), which is orthogonal to the vector \((1, \ldots, 1)\). With this fact, we only need to find a \( k_i \) dimensional subspace of \( \mathcal{W} \) on which \( \mathcal{S} \) is negative definite and a \( k_i \)-1 dimensional subspace on which it is positive definite.

We begin the construction with the two subspaces \( \mathcal{W}^+ \) and \( \mathcal{W}^- \) for which \( x^* \mathcal{S} x < 0 \) and \( x^* \mathcal{S} x > 0 \), respectively. The subspace we are ultimately interested in is defined by
\[ \mathcal{T} = \text{proj}_{\mathcal{W}^+}(\text{proj}_{\mathcal{W}^-}(\mathcal{W}^+))). \]  
Here \( \text{proj}_{\mathcal{W}^+}(S) \) is the projection of the space \( S \) onto \( \mathcal{W}^+ \) and \( \text{proj}(S) \) is the subspace orthogonal to \( S \). Note that \( \dim(\text{proj}_{\mathcal{W}^+} (\mathcal{W}^-)) = k_i - 1 \) and \( \dim(\text{proj}_{\mathcal{W}^-} (\mathcal{W}^+)) = k_i - 1 \), and thus \( \dim(S) = k_i \). We define the index sets \( I^\pm \) by \( \mathcal{W}^\pm = \text{span}(e_i)_{i \in I^\pm} \) and \( \mathcal{W} = \text{span}(e_i)_{i \in I} \) (note that the standard basis vectors \( e_i \) are eigenvectors of \( S \)).

Define \( i^* = \max_{i \in I^+} \). Then the \( k_i - 1 \) vectors given by \( \{e_i - e_i\}_{i \in I^*, i \neq j} \) form a basis of \( \text{proj}_{\mathcal{W}^+}(\mathcal{W}^-) \). For the basis of \( \text{proj}_{\mathcal{W}^-}(\mathcal{W}^+) \), define the vector \( v_j \) by
\[ v_j = \begin{cases} 1 & i \in I^*, \\ 0 & i \in I. \end{cases} \]  
Then \( \text{proj}_{\mathcal{W}^-}(\mathcal{W}^+) \) is spanned by the \( k_i + 1 \) vectors \( \{e_j\}_{j \in I, j \neq i} \). The subspace \( T \) is spanned by the \( k_i \) vectors \( z_i \), with components defined by
\[ (z_i)_j = \begin{cases} 1 & j \in I^*, \\ -k_i & j = i, \\ 0 & \text{otherwise}. \end{cases} \]  
where \( i \in I^* \). Finally, \( z_i^* \mathcal{S} z_i = -k_i^2 - k_i^2 \), which will always be negative if \( k_i > 1 \). For the case \( k_i = 1 \), we can construct \( z_i \) as follows: Set \( (z_i)_j = 1 \) for the one value of \( j \in I^* \). We can then place \( m \) negative ones and \( m - 1 \) positive ones in the remaining places provided that \( m > 1 \). We may always form a basis in this manner and for any such basis vector \( x \), \( x^* \mathcal{S} x = -2m + 2 < 0 \). The only case left to consider is \( k_i = k_i = 1 \). In this case, the results of Ref. 14 imply the solution is unstable.
TABLE I. Estimates of the leading order corrections, $\delta_1$, to the eigenvalues of the generalized eigenvalue problem (2.40a) in $V_2$ as a function of $\delta$, compared to the corresponding eigenvalues $\omega_1$ in (5.3.10).

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>$\delta = .0001$</th>
<th>$\delta = .00001$</th>
<th>$\delta = .000001$</th>
<th>$\omega_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - 1, 1, 1$ pattern</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3.393$</td>
<td>$3.396$</td>
<td>$3.396$</td>
<td>$3.396$</td>
<td></td>
</tr>
<tr>
<td>$35.13$</td>
<td>$35.30$</td>
<td>$35.33$</td>
<td>$35.33$</td>
<td></td>
</tr>
<tr>
<td>$1 - 1, 1, -1, 1$ pattern</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-61.14$</td>
<td>$-60.59$</td>
<td>$-60.52$</td>
<td>$-60.52$</td>
<td></td>
</tr>
<tr>
<td>$16.79$</td>
<td>$16.87$</td>
<td>$16.88$</td>
<td>$16.88$</td>
<td></td>
</tr>
<tr>
<td>$285.0$</td>
<td>$292.8$</td>
<td>$294.5$</td>
<td>$294.6$</td>
<td></td>
</tr>
<tr>
<td>$1 - 1, 1, 1, -1, 1, 1$ pattern</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-400.8$</td>
<td>$-376.1$</td>
<td>$-373.7$</td>
<td>$-373.5$</td>
<td></td>
</tr>
<tr>
<td>$-171.0$</td>
<td>$-166.5$</td>
<td>$-166.1$</td>
<td>$-166.1$</td>
<td></td>
</tr>
<tr>
<td>$31.97$</td>
<td>$32.60$</td>
<td>$32.66$</td>
<td>$32.67$</td>
<td></td>
</tr>
<tr>
<td>$338.0$</td>
<td>$356.5$</td>
<td>$358.4$</td>
<td>$358.6$</td>
<td></td>
</tr>
<tr>
<td>$1005.0$</td>
<td>$1132.0$</td>
<td>$1147.0$</td>
<td>$1148.0$</td>
<td></td>
</tr>
</tbody>
</table>

with respect to the small eigenvalues, and the remaining small eigenvalue, corresponding to the eigenvector in $V_2$, must always be negative.

Thus there is a set of $k_1$ independent vectors on which the eigenvalues of (5.3.10) must be negative and a set of $k_2 - 1$ independent vectors on which they must be positive. We summarize the results in this section in the following proposition:

**Proposition 5.1**. Given an asymmetric steady-state solution (2.10) and (2.12) with $k_1$ small spikes sufficiently close to the bifurcation, the eigenvalue problem (2.15) will possess exactly $k_1$ small positive eigenvalues counting multiplicity.

Table I reports a check of these conclusions. For some specific spike patterns and for $r = 1$ and $\mu = 1$, we compute the eigenvalues $\omega_1$ in (5.3.10) using Maple™ (Ref. 16). We compare them to estimates, $\hat{\omega}_1$, of corrections to $\omega_1$ in (2.40a). We compute these estimates as follows. From (2.40a) we use (4.2a) to approximate $\omega_1$ as

\[ \hat{\omega}_1 = \frac{\omega - \omega_0}{\delta}. \]  

(5.3.14)

We use Maple™ to compute $\omega$ from (2.40a). The values of $\omega_0$ are given in (2.42). Following the notation (3.20) we write spike patterns as a sequence of $+1$’s and $-1$’s. For example, “$-1, 1, 1, 1, -1, 1$” is a pattern with five spikes, with one small spike at the left end and another small spike next to a big spike at the right end of the $[-1, 1]$ interval. As $\delta \to 0$ we should get that $\hat{\omega}_1 \to \omega_1$. Since we are interested only in the $\omega_1$ for $\omega_0 = 1$, we report only those $k-1$ values. As expected $k_1$ of these $k-1$ values are negative.

VI. CONCLUSIONS

We have confirmed that all asymmetric spike solutions (1.1) are unstable as they bifurcate off of a stable symmetric branch. Moreover, the signs of the eigenvalues are determined by the eigenvalues of the diagonal matrix $S$. Thus if there are $k_1$ small spikes, $S$ will always have $k_1$ negative eigenvalues and there must be exactly $k_1$ positive eigenvalues for (2.15). Using (5.3.10) we can compute the leading order terms for the critical eigenvalues for any given pattern.

In this paper, we have restricted our analysis to the Gierer-Meinhardt system, however, similar asymmetric patterns have been observed in many other reaction-diffusion systems such as the Gray-Scott,\cite{13} and with minor modifications, our result should carry over to all such systems. The key features of (1.1) needed to perform this analysis are the relations between the stability of a spike profile and the matrix eigenvalue problem (2.40a). This relationship is a result of the inner and outer matching of multispikes solutions and similar relations should hold for reaction diffusion equations of activator and inhibitor type with a slowly diffusing inhibitor.

APPENDIX: THE CORRECTIONS $B_1$ TO $B_4$

By (2.28c) and (3.20)

\[
c_1 = \coth(z_{\ell-1} + z_{\ell}) + \coth(z_{\ell} + z_{\ell+1})
= \coth(2z_{\ell} + \delta\alpha(s_{\ell-1} + s_{\ell})) + \coth(2z_{\ell} + \delta\alpha(s_{\ell} + s_{\ell+1}))
= 2\coth(2z_{\ell}) - \delta\alpha \csch^2(2z_{\ell})(s_{\ell-1} + 2s_{\ell} + s_{\ell+1}) + O(\delta^3)
= \frac{1 + 2r}{\sqrt{r(1 + r)}} + O\left(\frac{\alpha(s_{\ell-1} - 2s_{\ell} - s_{\ell+1})}{4r(1 + r)}\right) + O(\delta^3),
\]

(5.3.10)

\[c_1 = \frac{1 + 2r}{\sqrt{r(1 + r)}} + \delta\alpha^* \sqrt{\frac{r}{1 + r}}(s_{\ell-1} - 2s_{\ell} - s_{\ell+1}) + O(\delta^3), \quad \ell = 2, \ldots, k - 1 \]  

(A.1)

and

\[
d_1 = -\csch(z_{\ell} + z_{\ell+1})
= -\csch(2z_{\ell} + \delta\alpha(s_{\ell} + s_{\ell+1}))
= -\csch(2z_{\ell}) + \delta\alpha(s_{\ell} + s_{\ell+1})\csch(2z_{\ell})\coth(2z_{\ell}) + O(\delta^3)
= -\frac{1}{2\sqrt{r(1 + r)}} + \delta\alpha(s_{\ell} + s_{\ell+1}) + \frac{1 + 2r}{4r(1 + r)} + O(\delta^3)
= -\frac{1}{2\sqrt{r(1 + r)}} + \delta\alpha^* \sqrt{\frac{r}{1 + r}}(s_{\ell} + s_{\ell+1})(1 + 2r) + O(\delta^3).
\]

(A.2)

The conversion from hyperbolic functions of $z$ to algebraic functions of $r$ is accomplished by using (2.5), and $\alpha^*$ is defined in (5.2.12).

From (2.28b),

\[
c_1 = \coth(z_{\ell} + z_{\ell+1}) + \tanh z_{\ell}
= \coth(2z_{\ell} + \delta\alpha(s_{\ell} + s_{\ell+1})) + \tanh z_{\ell} + \delta\alpha s_{\ell}
= \coth(2z_{\ell}) + \tanh z_{\ell} + \delta\alpha(-s_{\ell} + s_{\ell+1})\csch^2(2z_{\ell})
+ s_{\ell} \sech^2 z_{\ell} + O(\delta^3)
\]

(B.1)
\[
\begin{align*}
\frac{1 + 2r}{2\sqrt{r(1 + r)}} + \sqrt{\frac{r}{1 + r}} + \delta \alpha \left( -\frac{S_1 + S_2}{4r(1 + r)} + \frac{S_1}{1 + r} \right) + O(\delta^2) \\
= \frac{1 + 4r}{2\sqrt{r(1 + r)}} + \delta \alpha \left( \frac{S_1(4r - 1) - S_2}{4r(1 + r)} \right) + O(\delta^2) \\
= \frac{1 + 4r}{2\sqrt{r(1 + r)}} + \delta \alpha \sqrt{\frac{r}{1 + r}} \left( S_1(4r - 1) - S_2 \right) + O(\delta^2) \\
\end{align*}
\]

and

\[
\begin{align*}
c_k &= \coth(z_k + z_{k-1}) + \tanh z_k \\
&= \frac{1 + 4r}{2\sqrt{r(1 + r)}} + \delta \alpha \sqrt{\frac{r}{1 + r}} \left( S_1(4r - 1) - S_{k-1} \right) + O(\delta^2).
\end{align*}
\]

The corrections to \( B_y, P_B, \) and \( P_s B_k \) are computed in a similar fashion.


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